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Nonlinear Mean Value Properties related to the p-Laplacian

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José González Llorente

"No matter what happens now. You shouldn't be afraid.
Because I know today has been the most perfect day I've ever seen."

Radiohead, *Videotape*.

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A Aurora y a Victoria.

Contents

Introduction

This dissertation focuses on the study of functions satisfying certain nonlinear mean value properties related to the p-Laplace equation. The starting motivation is the classical mean value property for harmonic functions which plays a relevant role in Geometric Function Theory and is indeed the fundamental key of the interplay between classical potential theory, probability and Brownian motion.

In this introduction we summarize the results obtained during our research and we briefly explain the context in which they have been developed. The detailed proofs –as well as the background needed to understand them– can be found throughout the three chapters in which the dissertation is divided.

The results in this memory are essentially part of several papers: Chapter 1 corresponds to the article [AL2], while Chapter 2 contains the results in [AL1] and [AL3]. We want to make special emphasis in the fact that the results in Section 2.5 were partially proved in [AL1], but the proofs presented here rely in some results obtained later in [AL3] and are much simpler and direct than in the original article. The papers [AL1], [AL2] and [AL3] have been done in collaboration and under the supervision of J.G. Llorente, while Chapter 3 describes the results in the article [AHP] in collaboration with J. Heino and M. Parviainen.

Harmonic functions and the mean value property

Well known results due to Gauss and Koebe established the connection between harmonicity and the mean value property. More precisely, in the nineteenth century, Gauss showed ([Gau]) that a harmonic function u in a domain $\Omega \subset \mathbb{R}^n$ satisfies the *mean value property*,

$$\fint_{B(x,r)} u \, d\mathcal{L} = u(x) \tag{1}$$

for each $x \in \Omega$ and all $0 < r < \operatorname{dist}(x, \partial\Omega)$, where $\mathcal{L} = \mathcal{L}^n$ denotes the n-dimensional Lebesgue measure.

A natural question raised by the analysts of the early XX century was the so-called *converse mean value property* that asks under what conditions the mean value property (1) implies harmonicity. The first converse result is a theorem due to Koebe stating that if (1) holds for a continuous function $u \in C(\Omega)$ and for each admissible radius, then u is harmonic in Ω (see [Koe]). More precisely, Koebe's proof actually shows that the conclusion holds under weaker assumptions: if u is a continuous function in $\Omega \subset \mathbb{R}^n$ and for each $x \in \Omega$ there is a sequence of radii $r_k \to 0$ such that (1) holds for $r = r_k$, then u is harmonic in Ω.

Another possible extension comes from the formula

$$\Delta u(x) = 2(n+2) \lim_{r \to 0} \frac{1}{r^2} \left[\fint_{B(x,r)} u \, d\mathcal{L} - u(x) \right], \tag{2}$$

which follows essentially from averaging the second order Taylor's expansion of u at a neighborhood of $x \in \Omega$. This suggests that we can define the laplacian without using derivatives. In that sense, several works due to Blaschke, Privaloff and Zaremba ([Bla], [Pri] and [Zar], respectively) characterized harmonicity in the following terms: a continuous function u is harmonic in Ω if and only if it satisfies the so-called *asymptotic mean value property*,

$$\fint_{B(x,r)} u \, d\mathcal{L} = u(x) + o(r^2) \qquad (r \to 0)$$

for each $x \in \Omega$.

The results mentioned above require some sort of asymptotic behavior of the continuous functions in order to ensure harmonicity. On the other hand, an alternative generalization is the so-called *restricted mean value property* that asks how many radii in (1) are enough to guarantee harmonicity. In particular, we focus our attention on the single radius case: under which conditions on the function u, the domain Ω and the radius function $\varrho = \varrho(x)$ the restricted mean value property

$$\fint_{B(x,\varrho(x))} u \, d\mathcal{L} = u(x), \tag{3}$$

for all $x \in \Omega$ implies harmonicity? One of the most remarkable results in this direction says that if Ω is bounded, $u \in C(\overline{\Omega})$ and for each $x \in \Omega$ there is a radius $\varrho = \varrho(x)$ with $0 < \varrho \leq \operatorname{dist}(x, \partial\Omega)$ for which (3) holds, then u is harmonic in Ω. This result was first proved by Volterra for regular domains in [Vol] and later by Kellogg, who removed the regularity assumption, in [Kel]. In short, this means that (under appropriate hypothesis) one just needs a single radius at each point to ensure harmonicity.

Moreover, for $\Omega = \mathbb{B}$ the unit ball of $\mathbb{R}^n$, Littlewood asked ([Lit]) if the converse mean value property with an single radius $\varrho(x)$ at each point is also true when the function u is assumed to be continuous and bounded in Ω instead of continuous up to the boundary (as it was proved by Volterra and Kellogg). This question remained open for several decades until it was positively solved by Hansen and Nadirashvili in the 90's (see [HN1, HN2]). Furthermore, they showed that the converse is false when $n = 2$ if we consider *spherical* mean values

$$\fint_{\partial B(x,\varrho(x))} u \, d\mathcal{L}^{n-1} = u(x), \tag{4}$$

instead of (1) for each $x \in \mathbb{B} \subset \mathbb{R}^2$. Indeed, they proved that there exists a bounded continuous function u in the unit disk satisfying (4) that fails to be harmonic. We refer the reader to the paper [NT] for a detailed description of this and other results regarding the mean value property.

Asymptotic nonlinear mean value properties and the p-laplacian

During the last years, some efforts have been devoted to understand what sort of mean value property should be related to the p-laplacian and the ∞-laplacian in the same way as it happens for the classical laplacian. If $1 < p < \infty$, the *p-laplacian* is the divergence form operator given by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\,\nabla u).$$

The equation $\Delta_p u = 0$ appears as the Euler-Lagrange equation for the p-norm of the gradient, and its weak solutions in $W^{1,p}_{\text{loc}}$ are called *p-harmonic* by analogy with the case $p = 2$, for which we recover the classical laplacian. Suppose that $u \in C^2$ and that $\nabla u \neq 0$. Then direct computation gives

$$\frac{\Delta_p u}{|\nabla u|^{p-2}} = \Delta u + (p-2)\Delta_\infty u, \tag{5}$$

which is sometimes called the *normalized* p-laplacian and is denoted by $\Delta_p^{\mathrm{N}} u$. So, at least in the smooth case and away from the critical points, the p-laplacian can be understood as a sort of linear combination of the laplacian and the so-called *∞-laplacian*[1],

$$\Delta_\infty u := \langle \mathrm{D}^2 u \cdot \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \rangle = \frac{1}{|\nabla u|^2} \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i,x_j},$$

which is a non-divergence form operator and the choice of its name is justified by the fact that it is achieved –in a certain way– as limit of the p-laplacian when $p \to \infty$. Indeed, from equation (5), we see that

$$\Delta_p u = 0 \iff \frac{\Delta u}{p-2} + \Delta_\infty u = 0,$$

and taking limits as $p \to \infty$ we get $\Delta_\infty u = 0$. Viscosity solutions of the equation $\Delta_\infty u = 0$ are called *∞-harmonic* and they were first studied by Aronsson in the 60's in connection to the problem of finding optimal Lipschitz extensions (see [Aro1, Aro2]). Furthermore, for $u \in C^2$ and $\nabla u \neq 0$, it turns out that we can write $\Delta_\infty u$ asymptotically in terms of mean values similarly to (2),

$$\Delta_\infty u(x) = 2 \lim_{r\to 0} \frac{1}{r^2} \left[\frac{1}{2} \left(\sup_{B(x,r)} u + \inf_{B(x,r)} u \right) - u(x) \right]. \tag{6}$$

[1]Some authors call this the *normalized* ∞-laplacian while they define $\Delta_\infty u = \langle \mathrm{D}^2 u \cdot \nabla u, \nabla u \rangle$. Since for our purposes it is not needed to distinguish between this two definitions, we prefer to state it in this way.

Combination of formulas (2) and (6) together with (5) suggests the following tentative *asymptotic p-mean value property* related to the p-laplacian:

$$\frac{p-2}{n+p} \cdot \frac{1}{2}\left(\sup_{B(x,r)} u + \inf_{B(x,r)} u \right) + \frac{n+2}{n+p} \fint_{B(x,r)} u \, d\mathcal{L} = u(x) + o(r^2) \quad (r \to 0). \tag{7}$$

It turns out that, for a function $u \in C^2$ such that $\nabla u(x) \neq 0$, the asymptotic expansion (7) holds if and only if $\Delta_p u(x) = 0$. However, since p-harmonic functions are defined as weak solutions of $\Delta_p u = 0$, they are not C^2 in general. In fact, as it was proved by Ural'tseva ([Ura]) and by Lewis ([Lew]), p-harmonic functions belong to $C^{1,\gamma}_{\mathrm{loc}}$ for some $\gamma = \gamma(n,p) \in (0,1)$. Despite this fact, the authors in [MPR1] showed that any continuous function satisfying (7) is p-harmonic.

On the other hand, the converse is much more delicate. Indeed, it is known that p-harmonic functions satisfy (7) in a weak (viscosity) sense (see [JLM] together with [MPR1]). Thus, the following is an interesting question to pose: do p-harmonic functions satisfy (7)? This is an open question for dimension $n \geq 3$, while, for the planar case, Lindqvist and Manfredi ([LM]) answered the question positively for $1 < p < 9.52\ldots$ We dedicate Chapter 1 to show that, in fact, this property holds in the plane for the whole range $1 < p < \infty$. This is the first original result presented in this thesis and can be found in [AL2].

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a domain and let $1 < p < \infty$. Then a function $u \in C(\Omega)$ is p-harmonic in Ω if and only if the asymptotic expansion*

$$\frac{p-2}{p+2} \cdot \frac{1}{2}\left(\sup_{B(x,r)} u + \inf_{B(x,r)} u \right) + \frac{4}{p+2} \fint_{B(x,r)} u \, d\mathcal{L} = u(x) + o(r^2)$$

holds as $r \to 0$ for each $x \in \Omega$.

One of the main advantages of working with 2-dimensional PDE's is that there are more available tools –due to the complex structure of the plane– than in the general n-dimensional space. This is the case of our proof of Theorem 1.2, which relies in a fact showed by Bojarski and Iwaniec in the 80's ([BI]): the complex gradient $f(x+iy) = u_z = \frac{1}{2}(u_x - iu_y)$ of a p-harmonic function u is a quasiregular mapping. Therefore, we can deal with p-harmonic functions from a new perspective not available in higher dimensions.

The restricted mean value property and p-harmonious functions

Motivated by the asymptotic expansion (7) for p-harmonic functions, one may ask whether p-harmonic functions satisfy an analogous version of (3) for $p \neq 2$ that we call *restricted* *p-mean value propety* (or simply *p-mean value property*) and is stated as follows,

$$\frac{\alpha}{2}\left(\sup_{B_x} u + \inf_{B_x} u\right) + (1-\alpha)\fint_{B_x} u \, d\mathcal{L} = u(x), \tag{8}$$

for each $x \in \Omega$, where

$$\alpha = \frac{p-2}{n+p}, \tag{9}$$

and $B_x = \overline{B}(x, \varrho(x))$ with a a certain choice of the radii $0 < \varrho(x) \leq \mathrm{dist}(x, \partial\Omega)$.

Unfortunately, for $p \neq 2$, p-harmonic functions do not satisfy this restricted version of the mean value property and a new class of functions arises when we consider solutions of (8) for $p \neq 2$. Such solutions are called *p-harmonious functions*[2]. As one may expect, the definition of p-harmonious functions depends on the choice of an *admissible radius function* $\varrho(x)$, that is, a positive function ϱ in Ω such that $B_x := \overline{B}(x, \varrho(x))$ is contained in Ω for each $x \in \Omega$. A remarkable case happens when we set $p = \infty$. Then $\alpha = 1$ and the second term in the left-hand side of (8) cancels out, so there is no need of a measure in the space and we can talk about solutions of (8) in a more general context of metric spaces. A result in this direction was obtained by Le-Gruyer and Archer in [LA] when $\Omega \subset \mathbb{X}$ is a domain in a metrically convex compact metric space $(\mathbb{X}, d)$. Assuming the 1-Lipschitz regularity of the admissible radius function $\varrho(x)$, the authors showed that, for any given continuous function on the boundary $f \in C(\partial\Omega)$, the Dirichlet problem

$$\begin{cases} u(x) = \dfrac{1}{2}\left(\sup\limits_{B_x} u + \inf\limits_{B_x} u\right) & \text{for } \ x \in \Omega, \\ u(x) = f(x) & \text{for } \ x \in \partial\Omega, \end{cases}$$

has a unique solution $u \in C(\overline{\Omega})$ which was called the *harmonious extension of f*.

In Chapter 2, we deal with functions satisfying (8) in the more general setting of a metric measure space $(\mathbb{X}, d, \mu)$. Thus, for metric measure spaces other than $\mathbb{R}^n$ with the euclidean distance and the Lebesgue measure, it does not make sense to talk about the p-mean value property since the link (9) between p and α is missing. Instead of that, we construct an analogous version of (8) as follows: let $(\mathbb{X}, d, \mu)$ be a proper metric measure space, $\Omega \subset \mathbb{X}$ a bounded domain and ϱ an admissible radius function in Ω. For $\alpha \in \mathbb{R}$, a function $u \in C(\overline{\Omega})$ is said to satisfy the *α-mean value property* in Ω (with respect to the admissible radius function ϱ) if

$$\mathcal{T}_\alpha u = u, \tag{10}$$

where $\mathcal{T}_\alpha$ is the operator defined for functions $u \in L^\infty(\Omega)$ by

$$\mathcal{T}_\alpha u(x) := \frac{\alpha}{2}\left(\sup_{B_x} u + \inf_{B_x} u\right) + (1-\alpha)\fint_{B_x} u \, d\mu$$

[2]The '*p-harmonious*' term was originally used in [MPR2] for denoting (not necessarily continuous) solutions of (8) when the admissible radius functions is constant.

for each $x \in \Omega$. We sometimes refer to functions satisfying this property as *generalized p-harmonious functions*. The choice of this name comes from the particular case where $\mathbb{X} = \mathbb{R}^n$, d is de euclidean distance and $\mu = \mathcal{L}$ is the Lebuesgue measure, in which case the connection between p-harmonious functions and the α-mean value property has already been pointed out above.

For a bounded domain $\Omega \subset \mathbb{X}$, the main goal in this part of the dissertation is to provide Hölder and Lipschitz regularity estimates for continuous functions satisfying (10) in Ω, basically depending on the regularity of the admissible radius function ϱ and the choice of the measure μ. However, the existence part is not discussed here (see Section 2.5 where the existence of solutions is proven in $\mathbb{R}^n$ with the euclidean distance and the Lebesgue measure). In order to obtain these regularity results, we need to impose a regularity assumption on the measure that has received an increasing attention over the past few years: a metric measure space $(\mathbb{X}, d, \mu)$ satisfies the *δ-annular decay property* for $\delta \in (0, 1]$ if there exists a constant $D \geq 1$ such that

$$\mu(B(x,R) \setminus B(x,r)) \leq D \left(\frac{R-r}{R} \right)^{\delta} \mu(B(x,R)),$$

for each $x \in \mathbb{X}$ and $0 < r \leq R$. This property was introduced in manifolds by Colding and Minicozzi ([CM]) and, independently, in metric spaces by Buckley ([Buc]). See also [BBL] for a local version. It is easy to check that the δ-annular decay property implies the doubling property. Conversely, in [Buc], it is proved in particular that a geodesic metric space $(\mathbb{X}, d, \mu)$ with a doubling measure μ satisfies a δ-annular decay property for some $\delta \in (0, 1]$, only depending on the doubling constant.

It is noteworthy to mention that the case $\alpha = 0$ is interesting enough by itself. In fact, harmonicity in a metric measure space in connection to the mean value property was introduced in [GG] and [AGG] in the following way: a locally integrable function in a domain $\Omega \subset \mathbb{X}$ is said *strongly harmonic* in Ω if it satisfies the 0-mean value property in any ball compactly contained in Ω. In particular, the authors in [AGG] proved that, if $(\mathbb{X}, d, \mu)$ is a metric measure space satisfying the δ-annular decay property for some $\delta \in (0, 1]$, then every bounded and strongly harmonic function u in a domain $\Omega \subset \mathbb{X}$ is locally δ-Hölder continuous in Ω and, if $\delta = 1$, then u is locally Lipschitz continuous in Ω.

Our first main regularity result is stronger than in [AGG] in the sense that we do not require the function u to satisfy the 0-mean value property at each $B(x, r)$ with $0 < r \leq \operatorname{dist}(x, \partial\Omega)$ in order to obtain regularity estimates, and just a single radius $\varrho(x)$ is needed at each $x \in \Omega$, provided that ϱ is a Hölder or Lipschitz continuous function in Ω.

Theorem 2.20. *Let $(\mathbb{X}, d, \mu)$ be a proper metric measure space satisfying the δ-annular decay property for some $\delta \in (0, 1]$. Suppose that $\Omega \subset \mathbb{X}$ is a bounded domain and ϱ is a γ-Hölder continuous admissible radius function in Ω for some $\gamma \in (0, 1]$. Then any $u \in L^{\infty}(\Omega)$ verifying* *the* 0*-mean value property in* Ω *(w.r.t.* ϱ*),*

$$\fint_{B_x} u\, d\mu = u(x),$$

for each $x \in \Omega$*, is locally* $\gamma\delta$*-Hölder continuous in* Ω*. In particular, if* $\gamma = \delta = 1$ *then* u *is locally Lipschitz continuous in* Ω*.*

Furthermore, we also obtain an analogous result for the general case $\alpha \neq 0$. Note that we need to introduce certain rigid control of the radius function in order to get the regularity estimates.

Theorem 2.27. *Let* $(\mathbb{X}, d, \mu)$ *be a proper geodesic metric measure space satisfying the* δ*-annular decay property for some* $\delta \in (0,1]$ *and let* $\Omega \subset \mathbb{X}$ *be a bounded domain. Suppose that* ϱ *is a Lipschitz admissible radius function in* Ω *with Lipschitz constant* $L \geq 1$ *such that*

$$\lambda \operatorname{dist}(x, \partial\Omega) \leq \varrho(x) \leq \varepsilon \operatorname{dist}(x, \partial\Omega),$$

for all $x \in \Omega$*, where*

$$0 < \lambda \leq \varepsilon < 1 - L\,|\alpha|\,.$$

Then any $u \in C(\overline{\Omega})$ *verifying the* α*-mean value property* (10) *in* Ω *(w.r.t.* ϱ*) is locally* δ*-Hölder continuous in* Ω*. In particular, if* $\delta = 1$ *then* u *is locally Lipschitz continuous in* Ω*.*

In Section 2.5 we return to the original setting $\mathbb{X} = \mathbb{R}^n$ with d the euclidean distance and $\mu = \mathcal{L}$ the Lebesgue measure and we present a proof of the existence and uniqueness of solutions of the Dirichlet problem for p-harmonious functions. The techniques applied in this section are much more direct than in [AL1], where we originally proved this theorem in a more particular case.

Our approach to existence for the Dirichlet problem relies on the equicontinuity of the iterates $\{\mathcal{T}_\alpha^k u\}_k$ in $\overline{\Omega}$, where u is any continuous extension of the boundary data. While the local equicontinuity in Ω can be obtained in the general metric measure space setting (under appropriate restrictions on the radius function), boundary equicontinuity is more delicate and we have been able to prove it when $\mathbb{X} = \mathbb{R}^n$ with $\mu = \mathcal{L}$ the Lebesgue measure. One of the main peculiarities of the method employed in Section 2.5 is that we require the domain $\Omega \subset \mathbb{R}^n$ to be bounded and *strictly convex*, that is, for each $x, y \in \partial\Omega$ the open segment joining x and y is entirely contained in Ω.

Theorem 2.29. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded and strictly convex domain,* $\alpha \in [0,1)$ *and* ϱ *a continuous admissible radius function satisfying*

$$\lambda \operatorname{dist}(x, \partial\Omega) \leq \varrho(x) \leq \varepsilon \operatorname{dist}(x, \partial\Omega),$$

for all $x \in \Omega$*, where*

$$0 < \lambda \leq \varepsilon < 1 - \alpha.$$

Then, for any $f \in C(\partial\Omega)$, there exists a unique continuous function $u \in C(\overline{\Omega})$ such that

$$\begin{cases} \mathcal{T}_\alpha u = u & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Moreover, for any $u_0 \in C(\overline{\Omega})$ such that $u_0\big|_{\partial\Omega} \equiv f$, the sequence of continuous functions $\{u_k\}_k$ given recursively by

$$u_k = \mathcal{T}_\alpha u_{k-1}$$

converges uniformly to u. In addition, if ϱ is Lipschitz continuous with constant $L \geq 1$ and

$$0 < \lambda \leq \varepsilon < 1 - L\alpha,$$

then u is locally Lipschitz continuous in Ω.

Tug-of-war games and the normalized $p(x)$-laplacian

Restricted mean value properties also appear in the context of some stochastic differential games known as *Tug-of-war* games. The importance of such games lies in their connection with the p-laplacian and the ∞-laplacian, which was introduced in the influential papers [PS] and [PSSW]. Broadly speaking, a Tug-of-war game is a two-players, zero-sum game: two players (say Player I and Player II) are in contest and the total earnings of one are the losses of the other. Hence, Player I, plays trying to maximize his expected outcome, while Player II is trying to minimize Player I's outcome.

The game can be described as follows: given a bounded domain $\Omega \subset \mathbb{R}^n$ and a fixed constant $\varepsilon > 0$, a token is placed at some initial point $x_j \in \Omega$. Then, with probability $\alpha \in [0,1]$, a fair coin is tossed and the winner of the toss is allowed to choose the new position $x_{j+1} \in B(x_j, \varepsilon)$ of the game token, and with probability $1-\alpha$, the token is placed at a random point x_{j+1} in $B(x_j, \varepsilon)$. Note that, it may happen that the ball $B(x_j, \varepsilon)$ is not entirely contained in Ω. For that reason, the domain Ω needs to be extended to the larger domain

$$\Omega_\varepsilon := \{x \in \mathbb{R}^n \ : \ \text{dist}(x, \Omega) \leq \varepsilon\}.$$

This step is repeated until the game token exits Ω for the first time. Then, Player II pays to Player I the amount given by $F(x_\tau)$, where x_τ is the position of the token at this time and F is a pay-off function defined in $\overline{\Omega}_\varepsilon$. Let us denote by u the expected outcome for Player I. Then, starting from $x_j \in \Omega$, Player I wins the toss and chooses the new token position $x_{j+1} \in B(x_j, \varepsilon)$ maximizing u with probability $\alpha/2$, that is,

$$x_{j+1} \in B(x_j, \varepsilon) \quad \text{s.t.} \quad u(x_{j+1}) = \sup_{B(x_j,\varepsilon)} u \quad \text{(with probability } \alpha/2).$$

On the other hand, with the same probability, Player II wins the toss and chooses the new token position

$$x_{j+1} \in B(x_j, \varepsilon) \quad \text{s.t.} \quad u(x_{j+1}) = \inf_{B(x_j,\varepsilon)} u \quad \text{(with probability } \alpha/2).$$

Otherwise,

$$x_{j+1} \text{ is randomly chosen in } B(x_j, \varepsilon) \quad \text{(with probability } 1-\alpha).$$

Summing up all the possible outcomes, by conditional probability, the expected outcome for Player I starting from x_j is a function satisfying a *dynamic programming principle* which reads as

$$u(x_j) = \frac{\alpha}{2}\left(\sup_{B(x_j,\varepsilon)} u + \inf_{B(x_j,\varepsilon)} u\right) + (1-\alpha) \fint_{B(x_j,\varepsilon)} u \, d\mathcal{L}$$

and corresponds essentially to the functional equation (8) with $\alpha \in [0,1]$ and $B_x = B(x, \varepsilon)$ for each $x \in \Omega$.

Questions such as the existence, uniqueness and properties of solutions of (8) in the extended domain Ω_ε with constant radii $\varrho(x) = \varepsilon > 0$ were studied in several papers (see [MPR2], [MPR3], [LPS1] and [LPS2]). Indeed, the authors in [MPR2] showed that, if $\Omega \subset \mathbb{R}^n$ is bounded and satisfies some regularity assumption, then there exists a unique function u_ε satisfying (8) and having F as boundary values (in the extended sense). Furthermore, $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$, where u is the unique p-harmonic function solving the Dirichlet problem

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ u(x) = F(x) & \text{on } \partial\Omega. \end{cases}$$

By (9) and since $\alpha \in [0,1]$, the Tug-of-war introduced above is related to the p-laplacian only when $p \geq 2$. In order to avoid this inconvenient and cover also the case in which $1 < p < 2$, a slightly different type of Tug-of-war game was proposed in [KMP]: the *Tug-of-war with orthogonal noise*. The game is defined as in the previous case, but the token is moved to a new position according to a different rule: starting from $x_j \in \Omega$, a fair coin is tossed and the winner of the toss chooses a direction $|\nu| = \varepsilon$. With probability $\alpha \in [0,1]$, the token will be placed at the position $x_{j+1} = x_j + \nu$, otherwise, with probability $1-\alpha$, the new token position will be chosen randomly in the $(n-1)$-dimensional ball of radius ε centered at x_j and orthogonal to ν, that is $x_{j+1} = x_j + \nu'$, where $|\nu'| < \varepsilon$ and $\nu' \perp \nu$. This step is repeated until the game token exits Ω for the first time as in the original Tug-of-war game and Player II pays to Player I the quantity $F(x_\tau)$. Hence, it turns out that the expected outcome u for Player I satisfies a different mean value property which reads as follows:

$$\begin{aligned} u(x) = \frac{1}{2}\Bigg[&\sup_{|\nu|=\varepsilon}\left(\alpha u(x+\nu) + (1-\alpha)\fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h)\right) \\ &+ \inf_{|\nu|=\varepsilon}\left(\alpha u(x+\nu) + (1-\alpha)\fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h)\right)\Bigg], \end{aligned} \tag{11}$$

with $\alpha \in [0,1]$, where $B_\varepsilon^\nu := B(0,\varepsilon)\cap\nu^\perp$ denotes the $(n-1)$-dimensional ball orthogonal to ν and $\mathcal{L}^{n-1}$ is the $(n-1)$-dimensional Lebesgue measure (see Section 3.1 for a detailed description of the game). The authors in [KMP] showed that, asymptotically, this mean value property is related to the p-laplacian in the same way that in (7) but, in this case, the relation the between coefficients α and p is given by

$$\alpha = \frac{p-1}{n+p}, \tag{12}$$

and thus $1 < p < \infty$ if and only if $\alpha \in (0,1)$. Questions such as existence and uniqueness of continuous solutions of (11) were obtained in [Har] introducing some correction near the boundary.

Chapter 3 concerns this new type of Tug-of-war games with space dependent probabilities, that is, instead of a constant probability α in (11), a probability $\alpha(x)$ depending on $x \in \Omega$ is given. Moreover, we also include an additional boundary correction term in the rules of the game. The existence and uniqueness of solutions in this case can be obtained following the same ideas in [Har] for the constant $\alpha(x)$ case:

Theorem ([AHP, Theorem 3.7]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\alpha : \overline{\Omega} \to (0,1)$ a continuous function. Then, for $\varepsilon > 0$ and any $F \in C(\Omega_\varepsilon)$, there exists a unique continuous function $u_\varepsilon \in C(\overline{\Omega})$ satisfying*

$$\begin{aligned} u_\varepsilon(x) = \frac{1-\delta(x)}{2}\Bigg[&\sup_{|\nu|=\varepsilon}\left(\alpha(x) u_\varepsilon(x+\nu) + (1-\alpha(x))\fint_{B_\varepsilon^\nu} u_\varepsilon(x+h)\, d\mathcal{L}^{n-1}(h)\right) \\ &+ \inf_{|\nu|=\varepsilon}\left(\alpha(x) u_\varepsilon(x+\nu) + (1-\alpha(x))\fint_{B_\varepsilon^\nu} u_\varepsilon(x+h)\, d\mathcal{L}^{n-1}(h)\right)\Bigg] \\ &+ \delta(x)F(x) \end{aligned} \tag{13}$$

for each $x \in \Omega$, where $\delta(x) := \min\left\{0, 1-\varepsilon^{-1}\operatorname{dist}(x, \mathbb{R}^n\setminus\Omega)\right\}$.

The main result in Chapter 3 is an asymptotic Hölder regularity estimate for functions u_ε satisfying (13) and the statement reads as follows:

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $\alpha : \overline{\Omega} \to (0,1)$ a continuous function. For $\varepsilon > 0$ and any $F \in C(\Omega_\varepsilon)$, let $u_\varepsilon \in C(\overline{\Omega})$ be the unique function satisfying (13) for each $x \in \Omega$.*

Then u is asymptotically Hölder continuous for some exponent $\gamma > 0$, that is,

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C(|x-y|^\gamma + \varepsilon^\gamma),$$

for each pair of points x and y contained in a ball $B(z,R) \subset B(z,2R) \subset \Omega$ and some constant $C = C(\alpha, n, R, \gamma) > 0$.

Moreover, we obtained boundary estimates that allows us to to control the continuity of the solutions near de boundary of the domain. For that purpose, we need to ask some condition on the geometry of the domain in order to obtain these estimates. In particular, we need the domain $\Omega \subset \mathbb{R}^n$ to satisfy the so called *boundary regularity condition*: there are universal constants $r_0, s \in (0,1)$ such that, for each $r \in (0, r_0]$ and $y \in \partial\Omega$ there exists a ball

$$B(z, sr) \subset B(y,r) \setminus \Omega$$

for some $z \in B(y,r) \setminus \Omega$.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the boundary regularity condition and $\alpha : \overline{\Omega} \to (0,1)$ a continuous function. For $\varepsilon > 0$ and any $F \in C(\Omega_\varepsilon)$, let $u_\varepsilon \in C(\overline{\Omega})$ be the unique function satisfying (13) for each $x \in \Omega$. Let $\eta > 0$, then there is a constant $\overline{r} > 0$ such that for all $r \in (0, \overline{r}]$ there exist constants $k \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for any $y \in \mathbb{R}^n$ satisfying* $\operatorname{dist}(y, \partial\Omega) < \varepsilon$ *it holds*

$$|u_\varepsilon(x_0) - F(y)| < \eta$$

for each $0 < \varepsilon < \varepsilon_0$ and $x_0 \in B(y, 4^{1-k}r) \cap \overline{\Omega}_\varepsilon$.

Finally, as a consequence of these two results, it can be shown that the value function of the game u_ε converges to a continuous viscosity solution u of the *normalized $p(x)$-laplacian*,

$$\Delta^{\mathrm{N}}_{p(x)} u(x) := \Delta u(x) + (p(x) - 2)\Delta_\infty u(x).$$

Theorem ([AHP, Theorem 6.2]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the boundary regularity condition and $\alpha : \overline{\Omega} \to (0,1)$ a continuous function. Let $\varepsilon_0 > 0$ and $F \in C(\Omega_{\varepsilon_0})$. Let u_ε denote the unique continuous solution to (13) with $0 < \varepsilon \leq \varepsilon_0$. Then, there exists a sequence $\varepsilon_j \to 0$ such that $\{u_{\varepsilon_j}\}_j$ converges uniformly to $u \in C(\overline{\Omega})$ a viscosity solution of the Dirichlet problem for the normalized $p(x)$-laplacian,*

$$\begin{cases} \Delta^{\mathrm{N}}_{p(x)} u(x) = 0 & \textit{for } \ x \in \Omega, \\ u(x) = F(x) & \textit{for } \ x \in \partial\Omega, \end{cases}$$

where $p : \overline{\Omega} \to (1, \infty)$ is a continuous function depending on $\alpha(x)$ as in (12). Moreover, the function u is locally γ-Hölder continuous in Ω.

Chapter 1

The asymptotic mean value property for p-harmonic functions in the plane

As we noted in the Introduction, in this chapter we deal with *p-harmonic* functions in $\mathbb{R}^n$, which are defined as weak solutions in $W^{1,p}_{\text{loc}}$ of the *p-Laplace* equation $\Delta_p u = 0$. In addition, if $u \in C^2(\Omega)$ such that $\nabla u \neq 0$, then u is p-harmonic if and only if satisfies

$$\Delta u + (p-2)\Delta_\infty u = 0, \tag{1.1}$$

where Δ_∞ stands for the *∞-laplacian* given by

$$\Delta_\infty u := \langle \mathrm{D}^2 u \cdot \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \rangle. \tag{1.2}$$

Let $u \in C^2(\Omega)$. It is well known that we can express the laplacian of u asymptotically in terms of mean values as follows:

$$\Delta u(x) = 2(n+2) \lim_{r\to 0} \frac{1}{r^2} \left[\fint_{B(x,r)} u \, d\mathcal{L} - u(x) \right], \tag{1.3}$$

for each $x \in \Omega$. Moreover, this formula allows us to characterize harmonic functions by replacing $\Delta u(x) = 0$ in (1.3). In order to obtain an analogous expression for p-harmonic functions in C^2 and in view of (1.1), first we need an asymptotic expression for the ∞-laplacian. This can be done by considering the *midrange average* of u on $B(x,r)$, which is the quantity given by

$$\frac{1}{2} \left(\sup_{B(x,r)} u + \inf_{B(x,r)} u \right).$$

Proposition 1.1. *Let $u \in C^2(\Omega)$ and $x \in \Omega$ such that $\nabla u(x) \neq 0$. Then*

$$\Delta_\infty u(x) = 2 \lim_{r\to 0} \frac{1}{r^2} \left[\frac{1}{2} \left(\sup_{B(x,r)} u + \inf_{B(x,r)} u \right) - u(x) \right]. \tag{1.4}$$

The proof of formula (1.4) can be found in [MPR1] but we describe it here for the sake of completeness.

Proof. The Taylor's expansion of u in a neighborhood of x reads as follows:

$$u(x + r\,\zeta) = u(x) + r\langle \nabla u(x), \zeta\rangle + \frac{r^2}{2}\langle \mathrm{D}^2 u(x)\cdot\zeta, \zeta\rangle + o(r^2), \tag{1.5}$$

for any $|\zeta| \leq 1$ as $r \to 0$. For each small enough $r > 0$, choose $|\zeta_{\max}(r)|, |\zeta_{\min}(r)| \leq 1$ so that

$$\sup_{B(x,r)} u = \sup_{|\zeta|\leq 1} u(x + r\zeta) = u(x + r\,\zeta_{\max}(r)),$$
$$\inf_{B(x,r)} u = \inf_{|\zeta|\leq 1} u(x + r\zeta) = u(x + r\,\zeta_{\min}(r)).$$

In particular,

$$\frac{1}{2}\left(\sup_{B(x,r)} u + \inf_{B(x,r)} u\right) \leq \frac{1}{2}\big(u(x + r\,\zeta_{\max}(r)) + u(x - r\,\zeta_{\max}(r))\big),$$

and replacing (1.5) the first order term vanishes and we get

$$\frac{1}{2}\left(\sup_{B(x,r)} u + \inf_{B(x,r)} u\right) \leq u(x) + \frac{r^2}{2}\langle \mathrm{D}^2 u(x)\cdot\zeta_{\max}(r), \zeta_{\max}(r)\rangle + o(r^2).$$

Following an analogous argument for $\zeta_{\min}(r)$, we obtain

$$\frac{1}{2}\left(\sup_{B(x,r)} u + \inf_{B(x,r)} u\right) \geq u(x) + \frac{r^2}{2}\langle \mathrm{D}^2 u(x)\cdot\zeta_{\min}(r), \zeta_{\min}(r)\rangle + o(r^2).$$

Since $\nabla u(x) \neq 0$, by a standard argument with Lagrange multipliers, it turns out that

$$\lim_{r\to 0}\zeta_{\max}(r) = -\lim_{r\to 0}\zeta_{\min}(r) = \frac{\nabla u(x)}{|\nabla u(x)|}.$$

Thus, replacing in the previous inequalities and after recalling (1.2) we get the expansion

$$\frac{1}{2}\left(\sup_{B(x,r)} u + \inf_{B(x,r)} u\right) = u(x) + \frac{r^2}{2}\Delta_\infty u(x) + o(r^2).$$

Finally, rearranging terms and taking limits as $r \to 0$ we obtain (1.4). □

In consequence, combining (1.3) and (1.4) as in (1.1), it can be shown that if $u \in C^2$ such that $\nabla u(x) \neq 0$, then $\Delta_p u(x) = 0$ if and only if the asymptotic expansion

$$\frac{p-2}{n+p}\cdot\frac{1}{2}\left(\sup_{B(x,r)} u + \inf_{B(x,r)} u\right) + \frac{n+2}{n+p}\fint_{B(x,r)} u\, d\mathcal{L} = u(x) + o(r^2) \tag{1.6}$$

holds as $r \to 0$. However, since p-harmonic functions are not C^2 in general, it is not clear if (1.6) should be true for any p-harmonic function. In any case, recalling the results from [JLM] and [MPR1], it follows that if u is continuous and satisfies (1.6), then u is p-harmonic.

As for the converse, when $n = 2$, Lindqvist and Manfredi proved ([LM]) that every planar p-harmonic function with $1 < p < 9.52\ldots$ satisfies (1.6) even at those points where u does not have continuous second derivatives. The main result of this chapter (that can be also found in [AL2]) is an improvement of the previous result by Lindqvist and Manfredi and it states that, in fact, this is also true for $1 < p < \infty$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a domain and let $1 < p < \infty$. Then a function $u \in C(\Omega)$ is p-harmonic in Ω if and only if the asymptotic expansion*

$$\frac{p-2}{p+2} \cdot \frac{1}{2}\left(\sup_{B(x,r)} u + \inf_{B(x,r)} u \right) + \frac{4}{p+2} \fint_{B(x,r)} u \, d\mathcal{L} = u(x) + o(r^2) \tag{1.7}$$

holds as $r \to 0$ at each $x \in \Omega$.

Due to the complex structure of the plane ($\mathbb{R}^2 \approx \mathbb{C}$), we can identify each point $(x, y) \in \mathbb{R}^2$ with a complex number $z = x + iy \in \mathbb{C}$. Therefore, instead of working with the gradient ∇u of a p-harmonic function u in $\Omega \subset \mathbb{R}^2$, we use the complex gradient of u, ∂u, which is defined by

$$\partial u = u_z = \frac{1}{2}(u_x - iu_y).$$

From [BI], it turns out that $\partial u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a quasiregular mapping. Indeed, the authors in [BI] proved that the p-harmonic equation $\Delta_p u = 0$ in the plane can be rewritten as the (quasi-linear) Beltrami equation

$$\overline{\partial} f = \frac{2-p}{2p} \left[\frac{\overline{f}}{f}\, \partial f + \frac{f}{\overline{f}}\, \overline{\partial f} \right] \tag{1.8}$$

in Ω, where $f = \partial u$ and $\overline{\partial}$ denotes the complex derivative with respect to $\overline{z}$, that is,

$$\overline{\partial} = \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

By quasiregularity, it turns out that the set of critical points $S = \{z \in \Omega \ : \ \partial u(z) = 0\}$ consists of isolated points (unless u is constant) and ∂u is real-analytic outside S (see [BI] and [IM]). In consequence, it is enough to prove that a p-harmonic function satisfies (1.7) at a critical point. We will then focus on the local behavior of a p-harmonic function around a critical point of multiplicity $n \in \mathbb{N}$ (not to be confused with the dimension of the space, which in this chapter is fixed and equal to 2).

For simplicity, in what follows we assume without loss of generality that $\Omega \subset \mathbb{C}$ is a domain containing the origin and u is a p-harmonic function in Ω with only one critical point of order $n \in \mathbb{N}$ at $z = 0$. The main idea in the proof of Theorem 1.2 consists on giving an asymptotic expansion for the p-harmonic function u in a neighborhood of the critical point $z = 0$,

$$u(z) = u(0) + \mathfrak{U}(z) + \mathcal{O}(|z|^\gamma),$$

where $\gamma = \gamma(n, p) > 2$ and $\mathfrak{U}$ is a p-harmonic function in the plane (Proposition 1.11) whose symmetry properties imply that

$$\frac{p-2}{p+2} \cdot \frac{1}{2}\left(\sup_{|z|<r} \mathfrak{U}(z) + \inf_{|z|<r} \mathfrak{U}(z) \right) + \frac{4}{p+2} \fint_{|z|<r} \mathfrak{U}(z)\, dz = 0,$$

for small enough $r > 0$ (Lemma 1.12). Then equation (1.7) will follow.

1.1 The hodographic representation of u

As in [LM], we follow the so-called *hodograph method*, which was first proposed by Bers and Lavrentiev in order to study non-linear problems in hydrodynamics. The method consists on performing a change of variables in (1.8) in such a way that the independent variable is the inverse of f and the resulting equation is linear with variable coefficients. Our proof of Theorem 1.2 exploits the power series expansion of the complex gradient in the hodographic plane that was obtained in [IM].

Let u be a p-harmonic function with a critical point of multiplicity $n \in \mathbb{N}$ at the origin and let

$$f := \partial u = \frac{1}{2}(u_x - iu_y)$$

be the complex gradient of u, which is a quasiregular mapping satisfying (1.8). However, the inverse of f may not be well defined in Ω. For that reason, in order to follow the hodograph method, we recall the Stoilow factorization theorem (see, for example, [AIM]). Then, since f is quasiregular and has a zero of order $n \in \mathbb{N}$ at $z = 0$, it turns out that there exists a quasiconformal mapping (and, thus, invertible) χ such that

$$f(z) = (\chi(z))^n, \tag{1.9}$$

in a neighborhood of the origin.

We denote by $H = \chi^{-1}$ the inverse of χ. Note that, by quasiconformality, H is also quasiconformal in a neighborhood of the origin. If we perform the *hodograph change of variables*

$$\begin{cases} \xi = \chi(z), \\ z = H(\xi), \end{cases} \tag{1.10}$$

then (1.9) can be rewritten as

$$\xi^n = f(H(\xi))$$

in a neighborhood of $\xi = 0$. Note that the left-hand side in the previous equation is an holomorphic function, then

$$\begin{cases} \partial(\xi^n) = n\,\xi^{n-1}, \\ \overline{\partial}(\xi^n) = 0, \end{cases}$$

and the chain rule yields

$$\begin{bmatrix} n\,\xi^{n-1} \\ 0 \end{bmatrix} = \begin{bmatrix} \partial H & \overline{\overline{\partial} H} \\ \overline{\partial} H & \overline{\partial H} \end{bmatrix} \cdot \begin{bmatrix} \partial f \\ \overline{\partial} f \end{bmatrix}.$$

Let $J_H(\xi)$ denote the determinant of the matrix in the previous equation, that is, $J_H(\xi) = |\partial H(\xi)|^2 - \left|\overline{\partial} H(\xi)\right|^2$. Since H is quasiconformal, it turns out that $J_H(\xi) > 0$ for every ξ in a neighborhood of the origin and we can solve the system,

$$\begin{bmatrix} \partial f \\ \overline{\partial} f \end{bmatrix} = J_H^{-1} \begin{bmatrix} \overline{\partial H} & -\overline{\overline{\partial} H} \\ -\overline{\partial} H & \partial H \end{bmatrix} \cdot \begin{bmatrix} n\,\xi^{n-1} \\ 0 \end{bmatrix}.$$

In particular,

$$\begin{cases} \partial f = n\,J_H^{-1}\,\xi^{n-1}\,\overline{\partial H}, \\ \overline{\partial} f = -n\,J_H^{-1}\,\xi^{n-1}\,\overline{\partial} H. \end{cases} \tag{1.11}$$

Replacing (1.9), (1.10) and (1.11) in (1.8) and rearranging terms we obtain

$$\overline{\partial} H = \frac{p-2}{2p}\left[\frac{\xi}{\overline{\xi}}\,\partial H + \left(\frac{\overline{\xi}}{\xi}\right)^n \overline{\partial H}\right] \tag{1.12}$$

in a neighborhood of $\xi = 0$.

The authors in [IM] obtained a series representation for solutions of (1.12) in a neighborhood of the origin which reads as follows,

$$z = H(\xi) = \left(\frac{\xi}{|\xi|}\right)^{-n} \sum_{k=n+1}^{\infty} |\xi|^{\lambda_k}\left[A_k\left(\frac{\xi}{|\xi|}\right)^k + \epsilon_k\,\overline{A_k}\left(\frac{\overline{\xi}}{|\xi|}\right)^k\right], \tag{1.13}$$

where $A_k \in \mathbb{C}$, $A_{n+1} \neq 0$ are constants satisfying

$$\sum_{k=n+1}^{\infty} k\,|A_k|^2 < \infty \tag{1.14}$$

and

$$\epsilon_k = \frac{\lambda_k + n - k}{\lambda_k + n + k}, \quad \lambda_k = \frac{1}{2}\left(\sqrt{4k^2(p-1) + n^2(p-2)^2} - np\right). \tag{1.15}$$

From (1.15) it is easy to check that

$$0 < \lambda_k < \frac{k^2 - n^2}{n} \quad \text{and} \quad |\epsilon_k| < \frac{k-n}{k+n}. \tag{1.16}$$

Equation (1.13) can be interpreted as the *hodographic representation* of the point $z = x + iy$ near the origin. Note that, also from [IM], it follows that if H is given by (1.13) for certain coefficients A_k, ϵ_k and λ_k satisfying (1.14) and (1.15) then there exists a p-harmonic function u (with a critical point of order $n \in \mathbb{N}$ at $z = 0$) such that (1.9) holds locally around the origin.

Let us denote by $\widetilde{u}$ the *hodographic representation of* u, i.e.,

$$\widetilde{u}(\xi) := (u \circ H)(\xi). \tag{1.17}$$

The following result characterizes the hodographic representation of all p-harmonic functions with a critical point at $z = 0$.

Proposition 1.3. *u is a p-harmonic function with a critical point of order $n \in \mathbb{N}$ at $z = 0$ if and only if its hodographic representation $\widetilde{u} = (u \circ H)$ has the following power series expansion in a neighborhood of $\xi = 0$:*

$$\widetilde{u}(\xi) = u(0) + \sum_{k=n+1}^{\infty} \mu_k \, |\xi|^{n+\lambda_k} \, \mathfrak{Re} \left\{ A_k \left(\frac{\xi}{|\xi|} \right)^k \right\}, \tag{1.18}$$

where

$$\mu_k = \frac{4\lambda_k}{\lambda_k + n + k}$$

and the coefficients A_k, ϵ_k and λ_k satisfy (1.14) and (1.15). Moreover, $0 \le \mu_k < 4\Big(1 - \dfrac{n}{k}\Big)$.

Proof. Let u be p-harmonic with a critical point of order n at the origin and let $z = H(\xi)$ be the hodographic representation. We can split $H(\xi)$ into its real and imaginary parts, i.e., $H(\xi) = \widetilde{z}(\xi) = \widetilde{x}(\xi) + i\widetilde{y}(\xi)$. By (1.9) together with (1.10), $\partial u(z) = \xi^n$. Hence, in polar coordinates $\xi = \rho e^{i\vartheta}$, this equation reads as

$$\begin{cases} u_x = 2\rho^n \cos(n\vartheta), \\ u_y = -2\rho^n \sin(n\vartheta). \end{cases}$$

Then, we compute $\widetilde{u}_\rho$ using the chain rule in (1.17):

$$\widetilde{u}_\rho = (u \circ H)_\rho = u_x \, \widetilde{x}_\rho + u_y \, \widetilde{y}_\rho = 2\rho^n \left[\widetilde{x}_\rho \cos(n\vartheta) - \widetilde{y}_\rho \sin(n\vartheta) \right],$$

the expression in brackets being equal to $\mathfrak{Re}\left\{e^{in\vartheta}H_\rho\right\}$. In polar coordinates, (1.13) reads as

$$H(\rho e^{i\vartheta}) = e^{-in\vartheta}\sum_{k=n+1}^{\infty}\rho^{\lambda_k}\varphi_k(\vartheta),$$

where $\varphi_k(\vartheta) = A_k e^{ik\vartheta} + \epsilon_k \overline{A_k} e^{-ik\vartheta}$ for each $k = n+1, n+2, \ldots$ Therefore, replacing this in $\widetilde{u}_\rho$ we get

$$\widetilde{u}_\rho = 2\rho^n \mathfrak{Re}\left\{e^{in\vartheta}H_\rho\right\} = 2\sum_{k=n+1}^{\infty}\lambda_k(1+\epsilon_k)\rho^{n+\lambda_k-1}\mathfrak{Re}\left\{A_k e^{ik\vartheta}\right\}.$$

Integrating with respect to ρ and recalling (1.15) we get (1.18). The bound on μ_k follows from (1.16). Since this argument can be reverted, the proof is completed. □

Given any p-harmonic function u with a critical point of order $n \in \mathbb{N}$ at $z = 0$, it follows from Proposition 1.3 that

$$\widetilde{u}(\xi) = u(0) + \widetilde{\mathfrak{U}}(\xi) + \mathcal{O}(|\xi|^{n+\lambda_{n+2}}),$$

where $\widetilde{\mathfrak{U}}(\xi)$ is the first term in the power series expansion of $\widetilde{u}$, (1.18),

$$\widetilde{\mathfrak{U}}(\xi) = \mu_{n+1}\,|\xi|^{n+\lambda_{n+1}}\,\mathfrak{Re}\left\{A_{n+1}\left(\frac{\xi}{|\xi|}\right)^{n+1}\right\}. \tag{1.19}$$

In particular, by Proposition 1.3, $\widetilde{\mathfrak{U}}(\xi)$ is the hodographic representation of a p-harmonic function $\mathfrak{U}(z)$ with a critical point of order $n \in \mathbb{N}$. More precisely,

$$\mathfrak{U}(z) = (\widetilde{\mathfrak{U}} \circ \mathcal{A}^{-1})(z), \tag{1.20}$$

where $\mathcal{A}(\xi)$ is a quasiconformal mapping defined as the first term in the power series expansion of the function $z = H(\xi)$ associated to the p-harmonic function u, i.e.,

$$\mathcal{A}(\xi) = \left[A_{n+1}\left(\frac{\xi}{|\xi|}\right)^{n+1} + \epsilon_{n+1}\overline{A_{n+1}}\left(\frac{\overline{\xi}}{|\xi|}\right)^{n+1}\right]\left(\frac{\xi}{|\xi|}\right)^{-n}|\xi|^{\lambda_{n+1}}. \tag{1.21}$$

Remark 1.4. From now on, we can assume without loss of generality that $u(0) = 0$.

1.2 Quantitative injectivity estimates for $\mathcal{A}$

For simplicity, we will use hereafter the notations $a \lesssim b$ (resp. $a \approx b$) to indicate that $a \leq Cb$ (resp. $C^{-1}a \leq b \leq Ca$) for some positive constant C independent of a and b.

Lemma 1.5. *The following estimates hold in a neighborhood of $\xi = 0$:*

$$\left|\widetilde{u}(\xi) - \widetilde{\mathfrak{U}}(\xi)\right| \lesssim |\xi|^{n+\lambda_{n+2}}, \tag{1.22}$$

$$|H(\xi) - \mathcal{A}(\xi)| \lesssim |\xi|^{\lambda_{n+2}}, \tag{1.23}$$

$$|\mathcal{A}(\xi)| \approx |H(\xi)| \approx |\xi|^{\lambda_{n+1}}. \tag{1.24}$$

Proof. From (1.14) and (1.16), we get in particular that the sequence $\{A_k\}_k$ is bounded and that $|\epsilon_k| < 1$ for all k. Since $\{\lambda_k\}_k$ is increasing, (1.22), (1.23) and (1.24) follow from the estimate

$$\sum_{k=n+2}^{\infty} |\xi|^{\lambda_k} = \mathcal{O}(|\xi|^{\lambda_{n+2}}). \tag{1.25}$$

Now an elementary computation shows that there is $C = C(p) > 0$ such that $\lambda_k - \lambda_{n+2} \geq C(k-(n+2))$ for all $k \geq n+2$. This implies (1.25) and proves the lemma. □

Now, we study the behavior of $\mathcal{A}$ and we give an injectivity estimate. For this purpose, we will use the following elementary lemma, whose proof is omitted.

Lemma 1.6. *Let $\tau > 0$, $\lambda > 0$ and $t \in \mathbb{R}$. Then for all $k \in \mathbb{N}$,*

$$\left|\tau e^{ikt} - 1\right| \leq k\left|\tau e^{it} - 1\right|. \tag{1.26}$$

Furthermore, if $\Lambda > 1$ and if $\Lambda^{-1} \leq \tau \leq \Lambda$ then there is a constant $C = C(\lambda, \Lambda) > 0$ such that

$$\left|\tau^{\lambda} e^{it} - 1\right| \geq C\tau^{\lambda-1}\left|\tau e^{it} - 1\right|. \tag{1.27}$$

Lemma 1.7. *The mapping $\mathcal{A} : \mathbb{C} \to \mathbb{C}$ is bijective and satisfies*

$$|\mathcal{A}(\xi) - \mathcal{A}(\zeta)| \geq C\left||\xi|^{\lambda_{n+1}-1}\xi - |\zeta|^{\lambda_{n+1}-1}\zeta\right|, \tag{1.28}$$

where $C = (1-(2n+1)\,|\epsilon_{n+1}|)\,|A_{n+1}|$.

Proof. First, we observe that (1.16) for $k = n+1$ implies that $0 < \lambda_{n+1} < 2 + \frac{1}{n}$ and that

$$|\epsilon_{n+1}| < \frac{1}{2n+1}.$$

We show first that $\mathcal{A}$ is surjective. We write $\lambda \equiv \lambda_{n+1}$, $\epsilon \equiv \epsilon_{n+1}$ and $A \equiv A_{n+1}$. Then

$$\mathcal{A}(\rho e^{i\vartheta}) = \rho^{\lambda} e^{i\vartheta}\left(A + \epsilon\overline{A}e^{-i2(n+1)\vartheta}\right).$$

Assume, for simplicity, that $A = 1$. Then we can write

$$\mathcal{A}(\rho e^{i\vartheta}) = \rho^{\lambda} m(\vartheta) e^{i\phi(\vartheta)},$$

where

$$\phi(\vartheta) = \vartheta + \arg(1 + \epsilon\, e^{-i2(n+1)\vartheta})$$

and

$$m(\vartheta) = \left|1 + \epsilon\, e^{-i2(n+1)\vartheta}\right| = \sqrt{1 + \epsilon^2 + 2\epsilon\cos(2(n+1)\vartheta)}. \tag{1.29}$$

To prove that $\mathcal{A}$ is surjective, let $w = se^{it} \in \mathbb{C}$ such that $w \neq 0$ (if $w = 0$ it is obvious that $\mathcal{A}(0) = 0$). Since $\phi(0) = 0$ and $\phi(2\pi) = 2\pi$, by continuity we can pick $k \in \mathbb{Z}$ and $\vartheta \in [0, 2\pi]$ such that $t + 2k\pi \in [0, 2\pi]$ and $\phi(\vartheta) = t + 2k\pi$. Then $e^{i\phi(\vartheta)} = e^{it}$. For that ϑ, choose $\rho > 0$ so that

$$\rho = \left(\frac{s}{m(\vartheta)}\right)^{1/\lambda}.$$

Then we have shown that $\mathcal{A}(\rho e^{i\vartheta}) = w$ so the surjectiveness of $\mathcal{A}$ follows. To finish the proof of the lemma, it is enough to prove (1.28), which is a quantitative form of injectiveness. By (1.21),

$$\begin{aligned} |\mathcal{A}(\xi) - \mathcal{A}(\zeta)| \geq\ & |A_{n+1}| \left| |\xi|^\lambda \frac{\xi}{|\xi|} - |\zeta|^\lambda \frac{\zeta}{|\zeta|} \right| \\ & - |A_{n+1}|\, |\epsilon| \left| |\xi|^\lambda \left(\frac{\overline{\xi}}{|\xi|}\right)^{2n+1} - |\zeta|^\lambda \left(\frac{\overline{\zeta}}{|\zeta|}\right)^{2n+1} \right|. \end{aligned} \tag{1.30}$$

Now apply (1.26) with $\tau = \left|\frac{\xi}{\zeta}\right|^\lambda$, $e^{it} = \frac{\xi/\zeta}{|\xi/\zeta|}$ and $k = 2n+1$, and multiply both sides of the inequality by $|\zeta|^\lambda$. Then

$$\left| |\xi|^\lambda \left(\frac{\xi}{|\xi|}\right)^{2n+1} - |\zeta|^\lambda \left(\frac{\zeta}{|\zeta|}\right)^{2n+1} \right| \leq (2n+1) \left| |\xi|^\lambda \frac{\xi}{|\xi|} - |\zeta|^\lambda \frac{\zeta}{|\zeta|} \right|.$$

Replacing this expression in (1.30) we obtain

$$|\mathcal{A}(\xi) - \mathcal{A}(\zeta)| \geq |A_{n+1}|\,(1 - (2n+1)\,|\epsilon|) \left| |\xi|^\lambda \frac{\xi}{|\xi|} - |\zeta|^\lambda \frac{\zeta}{|\zeta|} \right|.$$

so the proof is finished. □

Lemma 1.8. *Let $\Lambda > 1$. Then there is a constant $C = C(n, p, \Lambda, |A_{n+1}|) > 0$ such that for any $\xi, \zeta \in \mathbb{C}$ with $\Lambda^{-1}|\zeta| \leq |\xi| \leq \Lambda|\zeta|$ we have*

$$|\mathcal{A}(\xi) - \mathcal{A}(\zeta)| \geq C\,|\xi|^{\lambda_{n+1}-1}\,|\xi - \zeta|.$$

Proof. Apply (1.27) with $\tau = \left|\frac{\xi}{\zeta}\right|$ and $e^{it} = \frac{\xi/\zeta}{|\xi/\zeta|}$ and multilply by $|\zeta|^\lambda$ to obtain

$$\left| |\xi|^\lambda \frac{\xi}{|\xi|} - |\zeta|^\lambda \frac{\zeta}{|\zeta|} \right| \geq C\,|\xi|^{\lambda-1}\,|\xi - \zeta|. \tag{1.31}$$

Then the lemma follows from (1.28) together with (1.31). □

1.3 The perturbation method

Given ξ in the hodographic plane, set $z = H(\xi)$, $\zeta = \mathcal{A}^{-1}(z)$ and $w = H(\zeta)$. Then

$$\begin{cases} \xi = \chi(z), \\ \zeta = \chi(w) = \mathcal{A}^{-1}(H(\xi)). \end{cases}$$

Since

$$|z| = |\mathcal{A}(\zeta)| \approx |H(\zeta)| = |w|$$

by (1.24), it follows from quasiconformality ([Ahl]) that

$$|\xi| = |\chi(z)| \approx |\chi(w)| = |\zeta|\,.$$

We recall the p-harmonic functions

$$\begin{aligned} u(z) &= (\widetilde{u} \circ \chi)(z), \\ \mathfrak{U}(z) &= (\widetilde{\mathfrak{U}} \circ \mathcal{A}^{-1})(z), \end{aligned}$$

where $\widetilde{\mathfrak{U}}$ is given by (1.19).

Lemma 1.9. *Let $\Lambda > 1$. There is a constant $C = C(n, p, \Lambda, |A_{n+1}|) > 0$ such that for any $\xi, \zeta \in \mathbb{C}$ with $\Lambda^{-1} |\zeta| \leq |\xi| \leq \Lambda |\zeta|$ then*

$$\left|\widetilde{\mathfrak{U}}(\xi) - \widetilde{\mathfrak{U}}(\zeta)\right| \leq C\,|\xi|^n\,|\mathcal{A}(\xi) - \mathcal{A}(\zeta)|\,.$$

Proof. From (1.19), the fact that $0 \leq \mu_k < 4$ if $k \geq n$ and direct computation it follows that

$$\left|\widetilde{\mathfrak{U}}(\xi) - \widetilde{\mathfrak{U}}(\zeta)\right| \leq C\,|A_{n+1}|\,|\xi|^{n+\lambda_{n+1}-1}\,|\xi - \zeta|\,,$$

where $C = C(n, \Lambda) > 0$. Then the conclusion follows from Lemma 1.8.

□

Corollary 1.10. *Let $\xi, \zeta \in \mathbb{C}$ such that $\mathcal{A}(\zeta) = H(\xi)$. Then the following estimate holds in a neighborhood of $\xi = 0$:*

$$\left|\widetilde{\mathfrak{U}}(\xi) - \widetilde{\mathfrak{U}}(\zeta)\right| \lesssim |\xi|^{n+\lambda_{n+2}}\,.$$

Proof. Use the fact that $|\xi| \approx |\zeta|$, Lemma 1.9 and estimate (1.23). □

Now we are ready to prove the following singular expansion of a p-harmonic function.

Proposition 1.11. *Let u be a p-harmonic function with a critical point of order n at $z = 0$ and $u(0) = 0$. Then u can be written as*

$$u(z) = \mathfrak{U}(z) + \mathcal{O}(|z|^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}})$$

in a neighborhood of $z = 0$.

Proof. By (1.17) and (1.20) we can write

$$u(z) - \mathfrak{U}(z) = \widetilde{u}(\xi) - \widetilde{\mathfrak{U}}(\zeta) = \widetilde{\mathfrak{U}}(\xi) - \widetilde{\mathfrak{U}}(\zeta) + \widetilde{u}(\xi) - \widetilde{\mathfrak{U}}(\xi).$$

By (1.22), (1.24) and Corollary 1.10 we get

$$|u(z) - \mathfrak{U}(z)| \lesssim |\xi|^{n+\lambda_{n+2}} \approx |H(\xi)|^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}} = |z|^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}},$$

so the proof is finished. □

1.4 Proof of Theorem 1.2

As before, we will write λ, ϵ and μ instead of λ_{n+1}, ϵ_{n+1} and μ_{n+1}, respectively. We can assume without loss of generality that $A_{n+1} = 1$. Then

$$\mathcal{A}(\rho e^{i\vartheta}) = \rho^{\lambda} e^{-in\vartheta}(e^{i(n+1)\vartheta} + \epsilon e^{-i(n+1)\vartheta})$$

and $\left|\mathcal{A}(\rho e^{i\vartheta})\right| = \rho^{\lambda} m(\vartheta)$, where $m(\vartheta)$ is given by (1.29). Furthermore

$$\widetilde{\mathfrak{U}}(\rho e^{i\vartheta}) = \mu \rho^{n+\lambda} \cos((n+1)\vartheta).$$

Denote by $D_r = B(0,r)$ the open disc centered at 0 with radius $r > 0$ and define the *hodographic disc* $\widetilde{D_r}$ as $\mathcal{A}^{-1}(D_r)$. Then, a point $\rho e^{i\vartheta}$ of the hodographic plane belongs to $\widetilde{D_r}$ if and only if $\left|\mathcal{A}(\rho e^{i\vartheta})\right| < r$ and $\widetilde{D_r}$ can be described using in polar coordinates as

$$\widetilde{D_r} = \left\{ \rho e^{i\vartheta} \ : \ \rho < \left(\frac{r}{m(\vartheta)}\right)^{1/\lambda} \right\}.$$

Now we define the function $J(\zeta)$ as the absolute value of the jacobian of $\mathcal{A}(\zeta)$. Computing $J(\zeta)$ in polar coordinates we get

$$J(\rho e^{i\vartheta}) = \lambda \rho^{2(\lambda-1)} \left(1 - (2n+1)\epsilon^2 - 2n\epsilon \cos(2(n+1)\vartheta)\right). \tag{1.32}$$

(Observe that, since $|\epsilon| < (2n+1)^{-1}$, the expression in the right hand side of (1.32) is positive).

Lemma 1.12. *The p-harmonic function $\mathfrak{U}(z)$ given by (1.20) satisfies the following properties, for small enough $r > 0$:*

$$\sup_{D_r} \mathfrak{U} + \inf_{D_r} \mathfrak{U} = 0, \tag{1.33}$$

$$\int_{D_r} \mathfrak{U}(z)\, dz = 0. \tag{1.34}$$

Proof. By (1.20), we need to study the behavior of $\widetilde{\mathfrak{U}}(\xi)$ in $\widetilde{D_r}$. Then, (1.33) is a direct consequence of the symmetries of $\widetilde{D_r}$. To show (1.34), observe that, by a change of variables,

$$\int_{D_r} \mathfrak{U}(z)\,dz = \int_{\widetilde{D_r}} \widetilde{\mathfrak{U}}(\zeta)J(\zeta)\,d\zeta$$

and using polar coordinates we get

$$\int_{D_r} \mathfrak{U}(z)\,dz = \mu\lambda \int_0^{2\pi}\int_0^{\rho(\vartheta)} \rho^{n+3\lambda-1}\cos((n+1)\vartheta)j(\vartheta)\,d\rho\,d\vartheta, \tag{1.35}$$

where

$$\rho(\vartheta) = \left(\frac{r}{m(\vartheta)}\right)^{1/\lambda}, \quad j(\vartheta) = 1-(2n+1)\epsilon^2 - 2n\epsilon\cos(2(n+1)\vartheta)$$

and $m(\vartheta)$ is given by (1.29). Now (1.34) follows directly from (1.35) and the symmetry properties of $m(\vartheta)$ and $j(\vartheta)$. □

Lemma 1.13. *The inequality*

$$\frac{n+\lambda_{n+2}}{\lambda_{n+1}} > 2 \tag{1.36}$$

holds for each $1 < p < \infty$ and each $n \geq 1$.

Proof. From (1.15) and some computation it follows that inequality (1.36) is equivalent to

$$n(p+2)\sqrt{n^2p^2+16(n+1)(p-1)} > n^2p^2 + (-2n^2+8n)p - (2n^2+8n). \tag{1.37}$$

Now we distinguish two cases. If $n = 1$ then (1.37) becomes

$$(p+2)\sqrt{p^2+32(p-1)} > p^2+6p-10.$$

If the right hand is negative then the inequality follows. Otherwise, squaring the previous inequality we get

$$2p^3+7p^2+10p-19 > 0.$$

which holds for each $p > 1$ since the left-hand side is increasing in p and vanishes for $p = 1$. This proves (1.37) when $n = 1$.

Now assume $n \geq 2$ and observe that $\sqrt{n^2p^2+16(n+1)(p-1)} \geq np$ for each $p > 1$. Then (1.37) would follow if

$$n^2p(p+2) > n^2p^2 + (-2n^2+8n)p - (2n^2+8n),$$

which is equivalent to

$$(2n-4)p+n+4 > 0,$$

and holds trivially if $n \geq 2$. This finishes the proof of the lemma. □

Proof of Theorem 1.2. As stated at the beginning of this chapter, we only need to prove that planar p-harmonic functions satisfy (1.7) since the other implication is already clear. We also discussed there that (1.7) need only to be checked at a critical point. Therefore, we can assume that $x = 0$, $u(0) = 0$ and that 0 is a critical point of u.

Let $r > 0$ be small enough. By Proposition 1.11 and Lemma 1.12,

$$\fint_{D_r} u(z)\,dz = \mathcal{O}\left(r^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}}\right),$$

and

$$\frac{1}{2}\left(\sup_{D_r} u + \inf_{D_r} u\right) = \mathcal{O}\left(r^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}}\right).$$

Finally, combine both equations and divide by r^2 to obtain that for any $\alpha \in \mathbb{R}$

$$\frac{1}{r^2}\left[\alpha\left(\frac{1}{2}\sup_{D_r} u + \frac{1}{2}\inf_{D_r} u\right) + (1-\alpha)\fint_{D_r} u(z)\,dz\right] = \mathcal{O}\left(r^{\frac{n+\lambda_{n+2}}{\lambda_{n+1}}-2}\right).$$

By Lemma 1.13 the exponent of r in the right-hand side is strictly positive. Therefore, taking limits as $r \to 0$, we show that (1.7) holds at the origin and we conclude the proof. $\square$

Chapter 2

The restricted mean value property and p-harmonious functions

Let $(\mathbb{X}, d, \mu)$ denote a proper metric space endowed with a Borel positive measure μ such that $0 < \mu(B) < \infty$ for any ball $B \subset \mathbb{X}$. This chapter is concerned with functions satisfying the *α-mean value property*,

$$\frac{\alpha}{2}\left(\sup_{B_x} u + \inf_{B_x} u\right) + (1-\alpha)\fint_{B_x} u \, d\mu = u(x), \tag{2.1}$$

where B_x denotes the closed ball $\overline{B}(x, \varrho(x))$ with $0 < \varrho(x) \leq \operatorname{dist}(x, \partial\Omega)$ for each x in a given bounded domain $\Omega \subset \mathbb{X}$. For the sake of simplicity, if $u \in C(\overline{\Omega})$ and $x \in \Omega$, we denote by $\mathcal{S}u(x)$ and $\mathcal{M}u(x)$ the *midrange* and the *average* of u on B_x, respectively. That is, the operators given by

$$\mathcal{S}u(x) := \frac{1}{2}\left(\sup_{B_x} u + \inf_{B_x} u\right), \tag{2.2}$$

$$\mathcal{M}u(x) := \fint_{B_x} u \, d\mu, \tag{2.3}$$

for each $x \in \Omega$. Additionally, in what follows, we define $\mathcal{T}_\alpha$ as the combination of these two operators,

$$\mathcal{T}_\alpha u(x) := \alpha\, \mathcal{S}u(x) + (1-\alpha)\mathcal{M}u(x), \tag{2.4}$$

where $\alpha \in \mathbb{R}$. Therefore, a function $u \in C(\overline{\Omega})$ satisfying the α-mean value property (2.1) can be seen as a fixed point of $\mathcal{T}_\alpha$ in $C(\overline{\Omega})$, that is, u is a solution of the functional equation

$$\mathcal{T}_\alpha u = u.$$

It is easy to check that, defined in this way, $\mathcal{T}_\alpha$ is non-expansive in $L^\infty(\Omega)$ for $\alpha \in [0,1]$, that is,

$$\|\mathcal{T}_\alpha u - \mathcal{T}_\alpha v\|_\infty \leq \|u - v\|_\infty, \tag{2.5}$$

for every $u, v \in L^\infty(\Omega)$. Indeed, by the properties of the supremum and infimum,

$$|\mathcal{S}u(x) - \mathcal{S}v(x)| \leq \frac{1}{2}\left|\sup_{B_x} u - \sup_{B_x} v\right| + \frac{1}{2}\left|\inf_{B_x} u - \inf_{B_x} v\right| \leq \sup_{B_x} |u(x) - v(x)|,$$

for each $x \in \Omega$, and thus (2.5) follows for $\alpha = 1$. For $\alpha = 0$, the non-expansiveness follows immediately by the linearity of the integral in (2.3). Thus, by (2.4) we get the result for all $\alpha \in [0, 1]$.

This chapter is divided into two different parts. In the first part, we present the results obtained in [AL3] (although some previous results in this direction were already obtained in [AL1] in the case $\mathbb{X} = \mathbb{R}^n$, with the euclidean distance, μ doubling and ϱ 1-Lipschitz continuous) where we provide *a priori* Hölder and Lipschitz regularity estimates for the class of functions satisfying the α-mean value property (2.1). Such functions are also called *generalized p-harmonious functions* in analogy to the (n-dimensional) euclidean case, where the coefficient α is related to p by

$$\alpha = \frac{p-2}{n+p}.$$

Regarding the functional equation $\mathcal{T}_\alpha u = u$, we distinguish two cases: $\alpha = 0$ and $\alpha \neq 0$. For the first case, $\mathcal{T}_0$ corresponds to the operator $\mathcal{M}$ defined in (2.3), and the estimates obtained for this operator depend essentially on the choice of the measure μ and the radius function ϱ instead of the function u itself (see Subsections 2.2.2 and 2.2.3). Thus, the estimates in this case are much more direct than in the general case $\alpha \neq 0$ in which the appearance of the operator $\mathcal{S}$ plays an important role. In Lemma 2.10, we adapt an argument due to Le-Gruyer and Archer for obtaining continuity estimates for $\mathcal{S}$ (see [LA]) and we combine it with the results for $\mathcal{M}$ in order to get an estimate for $\mathcal{T}_\alpha$. However, the estimate obtained in this way is not enough for establishing a regularity result for solutions of $\mathcal{T}_\alpha u = u$. For that reason, in Sections 2.3 and 2.4 we perform an iteration method that gives better estimates.

Note that we need to assume the existence of continuous solutions in order to obtain the estimates conducting to the regularity results. For that reason, the second part of this chapter (which corresponds with Section 2.5) is concerned with the existence and uniqueness of solutions in $C(\overline{\Omega})$ of the Dirichlet problem

$$\begin{cases} \mathcal{T}_\alpha u = u & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

for a fixed continuous boundary data $f \in C(\partial\Omega)$. We can reformulate this as a fixed point problem, that is, to find a continuous function satisfying $\mathcal{T}_\alpha u = u$ among all continuous extensions u of f to the whole domain. Given a function $f \in C(\partial\Omega)$, we denote by $\mathcal{K}_f$ the collection of all norm-preserving continuous extensions of f to $\overline{\Omega}$, that is,

$$\mathcal{K}_f := \left\{ u \in C(\overline{\Omega}) \ : \ u\big|_{\partial\Omega} \equiv f \ \text{ and } \ \|u\|_{\infty,\Omega} = \|f\|_{\infty,\partial\Omega} \right\}. \tag{2.6}$$

Therefore, by the non-expansiveness of the operator (2.5) and the fact that $\mathcal{T}_\alpha u\big|_{\partial\Omega} \equiv u\big|_{\partial\Omega}$ for every $u \in C(\overline{\Omega})$, it turns out that $\mathcal{T}_\alpha$ maps $\mathcal{K}_f$ into $\mathcal{K}_f$. Moreover, it is easy to check that $\mathcal{K}_f$ is a closed subset of $C(\overline{\Omega})$.

From these assumptions, it is not difficult to establish the uniqueness of solutions, meaning that, if there exists a fixed point of $\mathcal{T}_\alpha$ in $\mathcal{K}_f$, then it must be unique. Indeed, it can be immediately deduced from a comparison principle for generalized p-harmonious functions (see Section 2.5.1).

On the other hand, existence of fixed points is more delicate and we prove it for the case $\mathbb{X} = \mathbb{R}^n$ with $\mu = \mathcal{L}$ is the Lebesgue measure. The key point in our proof of existence of solutions requires the sequence $\{\mathcal{T}_\alpha^k u_0\}_k$ to be equicontinuous in $\overline{\Omega}$ for any $u_0 \in \mathcal{K}_f$ (see Section 2.5.2). Equicontinuity at interior points of Ω can be deduced from Section 2.3 even in the general case of metric measure spaces, but we cannot deduce equicontinuity on the boundary from these results. For that reason, we postpone the analysis of boundary equicontinuity until Section 2.5.3 and we establish it in bounded and strictly convex domains on $\mathbb{R}^n$ following an idea due to Javaheri (see [Jav]).

2.1 Preliminary facts

2.1.1 Metric spaces and admissible radius functions

We start recalling some basic concepts and definitions that will be useful in this chapter.

Definition 2.1. Let $(\mathbb{X}, d)$ be a metric space. We say that $(\mathbb{X}, d)$ is *proper* if every closed and bounded subset of $\mathbb{X}$ is compact. $(\mathbb{X}, d)$ is a *geodesic space* if for any two points $x, y \in \mathbb{X}$ there is a curve connecting x and y whose length is equal to $d(x, y)$.

Given any subset $G \subset \mathbb{X}$, we denote by $\operatorname{dist}(x, G)$ the infimum of all distances $d(x, y)$ where $y \in G$. Moreover, if G is bounded, let $\ell(G)$ be the largest distance to the boundary for points in G:

$$\ell(G) := \sup_{x \in G} \{\operatorname{dist}(x, \partial G)\} \leq \frac{1}{2} \operatorname{diam} G. \tag{2.7}$$

Definition 2.2. A *modulus of continuity* in a bounded domain $\Omega \subset \mathbb{X}$ is a non-decreasing continuous function $\omega : [0, \operatorname{diam} \Omega] \to [0, \infty)$ such that $\omega(0) = 0$. We will often require ω to be concave too. If $G \subset \Omega$ and $u \in C(\overline{G})$, we will denote by $\omega_{u,G}$ a concave modulus of continuity such that

$$|u(x) - u(y)| \leq \omega_{u,G}(d(x, y)) \tag{2.8}$$

for all $x, y \in G$.

Definition 2.3. Let $(\mathbb{X}, d)$ be a proper metric space and $\Omega \subset \mathbb{X}$ a bounded and open domain. We say that a non-negative function $\varrho \in C(\overline{\Omega})$ is an *admissible radius function in* Ω if

$$0 < \varrho(x) \leq \operatorname{dist}(x, \partial\Omega)$$

for each $x \in \Omega$, and $\varrho(x) = 0$ if and only if $x \in \partial\Omega$. Whenever $G \Subset \Omega$, we define

$$\varrho_G := \inf_G \varrho > 0. \tag{2.9}$$

Also, we introduce the following notation for closed balls in Ω with radii given by ϱ:

$$B_x := \overline{B}(x, \varrho(x)),$$

for each $x \in \Omega$. Since the balls B_x with $x \in G$ are not necessarily contained in G, we define $\widetilde{G}$ as the union of all balls B_x with centers in G,

$$\widetilde{G} := \bigcup_{x \in G} B_x. \tag{2.10}$$

Remark 2.4. We will hereafter make use of some of the concepts introduced in this subsection (like the family of balls $\{B_x \,:\, x \in \Omega\}$ and the operator on sets $\widetilde{(\cdot)}$) without any explicit mention of their dependence on the choice of the admissible radius function ϱ, which is assumed to be fixed.

Following the notation in (2.8), we denote by $\omega_{\varrho,\Omega}$ a concave modulus of continuity for ϱ in Ω. Since $|\varrho(x) - \varrho(y)| \leq \ell(\Omega) < \operatorname{diam}\Omega$ for each $x, y \in \Omega$, we can also assume that $\omega_{\varrho,\Omega}(\operatorname{diam}\Omega) \leq \operatorname{diam}\Omega$ (otherwise, we just replace $\omega_{\varrho,\Omega}(t)$ by $\min\{\omega_{\varrho,\Omega}(t), \operatorname{diam}\Omega\}$, which is also a concave modulus of continuity for ϱ). As we will see in the next sections, a distinguished case occurs when the admissible radius function is Lipschitz, that is,

$$|\varrho(x) - \varrho(y)| \leq L\, d(x, y),$$

for each $x, y \in \Omega$ and some $L > 0$, in which case we can simply take $\omega_{\varrho,\Omega}(t) = Lt$. For technical reasons, we need to define another concave modulus of continuity for ϱ (that will be denoted by $\widehat{\omega}_\varrho$) as follows: if $\omega_{\varrho,\Omega}(t) \leq t$ for all $t \in [0, \operatorname{diam}\Omega]$ then we set $\widehat{\omega}_\varrho(t) := t$. Otherwise, we define

$$\widehat{\omega}_\varrho(t) := \frac{\operatorname{diam}\Omega}{\omega_{\varrho,\Omega}(\operatorname{diam}\Omega)}\, \omega_{\varrho,\Omega}(t). \tag{2.11}$$

Note that, defined in this way, $\widehat{\omega}_\varrho(t)$ is a concave modulus of continuity for ϱ in Ω satisfying

$$\max\{t, \omega_{\varrho,\Omega}(t)\} \leq \widehat{\omega}_\varrho(t) \leq \operatorname{diam}\Omega = \widehat{\omega}_\varrho(\operatorname{diam}\Omega) \tag{2.12}$$

for each $t \in [0, \operatorname{diam}\Omega]$. Consequently, successive compositions of ω_ϱ with itself will produce a sequence of continuous functions $\widehat{\omega}_\varrho^{(k)} : [0, \operatorname{diam}\Omega] \to [0, \operatorname{diam}\Omega]$ given by

$$\widehat{\omega}_\varrho^{(k)}(t) := \widehat{\omega}_\varrho\big(\widehat{\omega}_\varrho^{(k-1)}(t)\big),$$

for $k \in \mathbb{N}$, where $\widehat{\omega}_\varrho^{(0)}(t) = t$.

In this setting, we can define the operator $\mathcal{S}$ as in (2.2) and we can obtain the desired estimates for it (see Lemma 2.10).

2.1.2 Metric measure spaces

Definition 2.5. A *metric measure space* $(\mathbb{X}, d, \mu)$ is a metric space endowed with a Borel positive regular measure μ. From now on, we will only consider measures μ such that $0 < \mu(B) < \infty$ for every ball $B \subset \mathbb{X}$.

Definition 2.6. Let $(\mathbb{X}, d, \mu)$ be a metric measure space. We say that μ is *doubling* (equivalently, $(\mathbb{X}, d, \mu)$ is a *doubling metric measure space*) if there exists a constant $D_\mu \geq 1$ such that

$$\mu\left(B(x, 2r)\right) \leq D_\mu \, \mu(B(x, r))$$

for any $x \in \mathbb{X}$ and each $r > 0$.

In order to get estimates for the operator $\mathcal{M}$, we will make use of the following property, which will play a central role in what follows and is closely related to the doubling property.

Definition 2.7. Let $\delta \in (0, 1]$. A metric measure space $(\mathbb{X}, d, \mu)$ satisfies the *δ-annular decay property* if there exists a constant $D_\delta \geq 1$ such that

$$\mu\left(B(x, R) \setminus B(x, r)\right) \leq D_\delta \left(\frac{R - r}{R}\right)^\delta \mu(B(x, R)), \tag{2.13}$$

for each $x \in \mathbb{X}$ and $0 < r \leq R$. For $\delta = 1$, this property is also known as the *strong annular decay property*.

Example 2.8. As a canonical example, $\mathbb{R}^n$ endowed with the euclidean distance and $\mathcal{L} = \mathcal{L}^n$ the Lebesgue n-dimensional measure satisfies the 1-annular decay property. Indeed,

$$\frac{\mathcal{L}(B(x, R) \setminus B(x, r))}{\mathcal{L}(B(x, R))} = \frac{R^n - r^n}{R^n} \leq n \, \frac{R - r}{R},$$

for each $x \in \mathbb{R}^n$ and $0 < r \leq R$.

It is easy to check that the δ-annular decay property implies the doubling property. Conversely, in [Buc] it is proved in particular that a geodesic metric space $(\mathbb{X}, d, \mu)$ with a doubling measure μ satisfies a δ-annular decay condition for some $\delta \in (0, 1]$, where δ only depends on the doubling constant. (See also Lemma 2.1 in [AL1] where this implication is proven in $\mathbb{R}^n$).

In addition, we will also use the following definition when studying the continuity properties of the operator $\mathcal{M}$.

Definition 2.9. We say that a (Borel, regular) measure μ in a metric space $\mathbb{X}$ is *ring-continuous* if, for each $x \in \mathbb{X}$ the function

$$r \longmapsto \mu(B(x, r))$$

is continuous in $(0, +\infty)$.

From (2.13), one can easily deduce that a measure μ satisfying the δ-annular decay property for some $\delta \in (0,1]$ is also ring-continuous. However, the converse is not true and one can find examples of ring-continuous measures that do not satisfy this property (see Example 2 in [AGG]).

2.2 Continuity and regularity estimates

2.2.1 Continuity of $\mathcal{S}$

The following lemma was proven in [LA] when the admissible radius function is 1-Lipschitz, that is

$$|\varrho(x) - \varrho(y)| \leq d(x,y),$$

for every $x, y \in \Omega$. Note that, since the operator $\mathcal{S}$ does not depend on any measure, we state it in the context of a metric space $(\mathbb{X}, d)$.

Lemma 2.10. *Let $(\mathbb{X}, d)$ be a geodesic metric space and let ϱ be a continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$. Then, for any $u \in C(\overline{\Omega})$, any compact subset $K \subset \Omega$ and each $x, y \in K$ we have*

$$|\mathcal{S}u(x) - \mathcal{S}u(y)| \leq \omega_{u,\widetilde{K}} \left(\widehat{\omega}_\varrho(d(x,y))\right),$$

where $\widetilde{K}$, $\omega_{u,\widetilde{K}}$ and $\widehat{\omega}_\varrho$ are as in (2.10), (2.8) and (2.11), respectively. We have, in particular

$$\omega_{\mathcal{S}u,K}(t) \leq \omega_{u,\widetilde{K}} \left(\widehat{\omega}_\varrho(t)\right). \tag{2.14}$$

Proof. Recalling the definition of $\mathcal{S}u$, (2.2), and the elementary formulas

$$\begin{aligned} \sup_{i\in I} x_i - \sup_{j\in J} y_j &= \sup_{i\in I} \inf_{j\in J} (x_i - y_j), \\ \inf_{i\in I} x_i - \inf_{j\in J} y_j &= \sup_{j\in J} \inf_{i\in I} (x_i - y_j), \end{aligned}$$

we can write

$$\mathcal{S}u(x) - \mathcal{S}u(y) = \frac{1}{2} \sup_{s\in B_x} \inf_{t\in B_y} (u(s) - u(t)) + \frac{1}{2} \sup_{t\in B_y} \inf_{s\in B_x} (u(s) - u(t)). \tag{2.15}$$

Note that it may happen that $B_x \not\subset K$ or $B_y \not\subset K$. However, by (2.10), the inclusion $B_x \cup B_y \subset \widetilde{K}$ holds. Then,

$$\sup_{s\in B_x} \inf_{t\in B_y} (u(s) - u(t)) \leq \sup_{s\in B_x} \inf_{t\in B_y} \omega_{u,\widetilde{K}}(d(s,t)) \leq \omega_{u,\widetilde{K}} \left(\sup_{s\in B_x} \inf_{t\in B_y} d(s,t) \right).$$

Replacing this term (the other term is analogous) in (2.15) and using that $\omega_{u,\widetilde{K}}$ is concave, we get

$$\mathcal{S}u(x) - \mathcal{S}u(y) \leq \omega_{u,\widetilde{K}} \left(\frac{1}{2} \sup_{s\in B_x} \inf_{t\in B_y} d(s,t) + \frac{1}{2} \sup_{t\in B_y} \inf_{s\in B_x} d(s,t) \right).$$

Thus, we need to show that, for any $x, y \in \Omega$,

$$\frac{1}{2} \sup_{s \in B_x} \inf_{t \in B_y} d(s,t) + \frac{1}{2} \sup_{t \in B_y} \inf_{s \in B_x} d(s,t) \leq \widehat{\omega}_\varrho(d(x,y)). \tag{2.16}$$

From [LA, p.282], since $(\mathbb{X}, d)$ is geodesic by assumption, we get:

$$\begin{aligned} \sup_{t \in B_y} \inf_{s \in B_x} d(s,t) &\leq \max \{d(x,y) + \varrho(x) - \varrho(y), 0\}, \\ \sup_{s \in B_x} \inf_{t \in B_y} d(s,t) &\leq \max \{d(x,y) + \varrho(y) - \varrho(x), 0\}. \end{aligned} \tag{2.17}$$

Finally, (2.16) follows from (2.17) and (2.12). Therefore, this together with (2.15) finishes the proof. □

2.2.2 Continuity of $\mathcal{M}$

We will first look at the continuity and regularity of the function

$$x \longmapsto \mathcal{M}u(x) = \fint_{B_x} u \, d\mu$$

where an admissible radius function ϱ in a domain $\Omega \subset \mathbb{X}$, a measure μ and a bounded continuous function u in Ω are given. The following Lemma is a preliminary result in this direction. Before that, we introduce some notation: given two subsets $A, B \subset \mathbb{X}$, we denote by $A \triangle B = (A \setminus B) \cup (B \setminus A)$ the symmetric difference of A and B. If $A, B, C \subset \mathbb{X}$, it follows that

$$A \triangle B = (A \triangle C) \triangle (C \triangle B) \subset (A \triangle C) \cup (C \triangle B).$$

If $A, B \subset \mathbb{X}$ are two measurable subsets, then

$$|\mu(A) - \mu(B)| \leq \mu(A \triangle B),$$

and, from the triangle inequality,

$$\mu(A \triangle B) \leq \mu(A \triangle C) + \mu(C \triangle B). \tag{2.18}$$

Lemma 2.11. *Let $(\mathbb{X}, d, \mu)$ be a metric measure space. If B_1 and B_2 are two balls contained in $\mathbb{X}$, then*

$$\left| \fint_{B_1} u \, d\mu - \fint_{B_2} u \, d\mu \right| \leq 2 \|u\|_\infty \frac{\mu(B_1 \triangle B_2)}{\max \{\mu(B_1), \mu(B_2)\}},$$

for each $u \in L^\infty(\mathbb{X})$.

Proof. We can assume that $\mu(B_1) \geq \mu(B_2)$, then

$$\mu(B_1) \left(\fint_{B_1} u \, d\mu - \fint_{B_2} u \, d\mu \right) = \int_{B_1} u \, d\mu - \int_{B_2} u \, d\mu + (\mu(B_2) - \mu(B_1)) \fint_{B_2} u \, d\mu,$$

and estimating this, we obtain

$$\begin{aligned}\mu(B_1)\left|\fint_{B_1} u\,d\mu - \fint_{B_2} u\,d\mu\right| &\leq \left|\int_{B_1} u\,d\mu - \int_{B_2} u\,d\mu\right| + \|u\|_\infty |\mu(B_2)-\mu(B_1)| \\ &\leq \int_{B_1\triangle B_2} |u|\,\,d\mu + \|u\|_\infty \mu(B_1\triangle B_2) \\ &\leq 2\,\|u\|_\infty \mu(B_1\triangle B_2). \qquad \square\end{aligned}$$

The following corollary follows from Lemma 2.11 and the non-expansivenes of $\mathcal{M}$ in $L^\infty(\Omega)$.

Corollary 2.12. *Let $(\mathbb{X}, d, \mu)$ be a metric measure space. Let $\Omega \subset \mathbb{X}$ be a domain and ϱ an admissible radius function in Ω. Then, for each $u \in L^\infty(\Omega)$ and all $x, y \in \Omega$ we have*

$$\left|\mathcal{M}^k u(x) - \mathcal{M}^k u(y)\right| \leq 2\,\|u\|_\infty \frac{\mu(B_x \triangle B_y)}{\max\{\mu(B_x), \mu(B_y)\}}. \tag{2.19}$$

The importance of Corollary 2.12 lies in the fact that the continuity of $\mathcal{M}u$ can be derived from the continuity of the function

$$x \longmapsto \mu(B_x) = \mu(B(x, \varrho(x)))$$

without any dependence of the function u. To see that, consider any $x, y \in \Omega$ and $r_1, r_2 > 0$ and recall (2.18). Then

$$\mu(B(x,r_1)\triangle B(y,r_2)) \leq \mu(B(x,r_1)\triangle B(x,r_2)) + \mu(B(x,r_2)\triangle B(y,r_2)). \tag{2.20}$$

Now suppose that μ is ring-continuous (see Definition 2.9). Since $B(x, r_1) \subset B(x, r_2)$ or $B(x, r_2) \subset B(x, r_1)$, the first term in the right hand side of (2.20) is equal to

$$|\mu(B(x,r_1)) - \mu(B(x,r_2))|\,,$$

so it is controlled by the ring-continuity of μ. On the other hand, for the second term we recall the following result due to Gaczkowski and Górka:

Lemma ([GG, Theorem 2.1])**.** *Let $(\mathbb{X}, d, \mu)$ be a metric measure space such that μ is ring-continuous. Then for each $x \in \mathbb{X}$ and each $r > 0$,*

$$\lim_{y\to x} \mu\left(B(x,r)\triangle B(y,r)\right) = 0.$$

Moreover, the function $x \mapsto \mu(B(x,r))$ is continuous (w.r.t. d) for each fixed $r > 0$.

Therefore, replacing $r_1 = \varrho(x)$ and $r_2 = \varrho(y)$ in (2.20) we get the following proposition.

Proposition 2.13. *Let $(\mathbb{X}, d, \mu)$ be a metric measure space such that μ is ring-continuous. Suppose that $\Omega \subset \mathbb{X}$ is a domain and ϱ is a continuous admissible radius function in Ω. Then, $\mathcal{M} : L^\infty(\Omega) \to C(\Omega)$.*

Remark 2.14. By definition, the continuous admissible radius function ϱ vanishes on the boundary of the domain Ω, thus $\mu(B_x)$ tends to zero as x approaches the boundary of Ω. In consequence, estimates obtained from (2.19) are local, that is, they only make sense on compact subsets $K \subset \Omega$.

2.2.3 Equicontinuity and regularity estimates for $\mathcal{M}^k$

Let $\Omega \subset \mathbb{X}$ be a given domain in a metric measure space $(\mathbb{X}, d, \mu)$ and let $K \subset \Omega$ be a compact subset. We will construct moduli of continuity $\mathcal{W}_{\mu,K}$ depending on μ, ϱ and K such that

$$\frac{\mu(B_x \triangle B_y)}{\max\{\mu(B_x), \mu(B_y)\}} \leq \frac{1}{2}\mathcal{W}_{\mu,K}(d(x,y)), \tag{2.21}$$

for every $x, y \in K$. Hence, by (2.19), we would have

$$\left|\mathcal{M}^k u(x) - \mathcal{M}^k u(y)\right| \leq \|u\|_\infty \mathcal{W}_{\mu,K}(d(x,y)), \tag{2.22}$$

for each $k \in \mathbb{N}$.

In this subsection, we mainly obtain two different types of estimates: (local) equicontinuity and regularity. In order to show equicontinuity of the sequence of iterates $\{\mathcal{M}^k\}_k$ one just needs the admissible radius function ϱ to be continuous (Theorem 2.18), while for the regularity of solutions of $\mathcal{M}u = u$, we have to ask some extra regularity to ϱ (Theorem 2.20).

We first assume that ϱ is Lipschitz continuous.

Lemma 2.15. *Let $(\mathbb{X}, d, \mu)$ be a metric measure space satisfying the δ-annular decay property (2.13) for some $\delta \in (0,1]$ and $D_\delta \geq 1$. Suppose that ϱ is a Lipschitz continuous admissible radius function in a domain $\Omega \subset \mathbb{X}$ for some Lipschitz constant $L \geq 1$. Then, for any compact set $K \subset \Omega$ and each $x, y \in K$ we have*

$$\frac{\mu(B_x \triangle B_y)}{\max\{\mu(B_x), \mu(B_y)\}} \leq 4\, L\, D_\delta \left(\frac{d(x,y)}{\varrho_K}\right)^\delta. \tag{2.23}$$

Proof. Since ϱ is Lipschitz by assumption, $|\varrho(x) - \varrho(y)| \leq L\, d(x,y)$. Then:

– If $d(x,y) > \dfrac{\varrho_K}{2L}$, then $D_\delta \left(\dfrac{2L\, d(x,y)}{\varrho_K}\right)^\delta > 1$, and

$$\begin{aligned} \mu(B_x \triangle B_y) &\leq 2 \max\{\mu(B_x), \mu(B_y)\} \\ &< 2\, D_\delta \left(\frac{2L\, d(x,y)}{\varrho_K}\right)^\delta \max\{\mu(B_x), \mu(B_y)\}. \end{aligned}$$

– If $d(x,y) \le \dfrac{\varrho_K}{2L}$ then, since $L \ge 1$, we get that $|\varrho(x)-\varrho(y)| \le \varrho_K/2$ and, in particular, $\varrho(y) \ge \varrho(x)/2$ and $\varrho(x) \ge \varrho(y)/2$. As a consequence, the following inclusions hold:

$$B_x \setminus B_y \subset B_x \setminus B(x, \varrho(y) - d(x,y)),$$
$$B_y \setminus B_x \subset B_y \setminus B(y, \varrho(x) - d(x,y)).$$

Thus, by the δ-annular decay property (2.13) and the fact that $\varrho(x), \varrho(y) \ge \varrho_K$ for every $x, y \in K$, we obtain

$$\mu(B_x \setminus B_y) \le D_\delta \left(\frac{\varrho(x) - \varrho(y) + d(x,y)}{\varrho_K} \right)^\delta \max\left\{\mu(B_x), \mu(B_y)\right\},$$
$$\mu(B_y \setminus B_x) \le D_\delta \left(\frac{\varrho(y) - \varrho(x) + d(x,y)}{\varrho_K} \right)^\delta \max\left\{\mu(B_x), \mu(B_y)\right\}.$$

Using the Lipschitz assumption on ϱ and adding these two quantities we get the estimate

$$\mu(B_x \triangle B_y) \le 2\, D_\delta \left(\frac{(L+1)\, d(x,y)}{\varrho_K} \right)^\delta \max\left\{\mu(B_x), \mu(B_y)\right\},$$

which implies (2.23). □

Remark 2.16. Note that if x, y are as in the statement of Lemma 2.15, then only the pointwise inequality $|\varrho(x) - \varrho(y)| \le L\, d(x,y)$ is really used in the proof.

Lemma 2.17. *Let $(\mathbb{X}, d, \mu)$ be a proper metric measure space satisfying the δ-annular decay property (2.13) for some $\delta \in (0,1]$ and $D_\delta \ge 1$. Suppose that ϱ is a continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$. Then, for any compact set $K \subset \Omega$ and each x,$y \in K$ we have*

$$\frac{\mu(B_x \triangle B_y)}{\max\left\{\mu(B_x), \mu(B_y)\right\}} \le C \left(\frac{\widehat{\omega}_\varrho(d(x,y))}{\varrho_K} \right)^\delta, \tag{2.24}$$

where $C = C(D_\delta, \mu) > 0$ and $\widehat{\omega}_\varrho$ is as in (2.11).

Proof. For each pair of points $x, y \in K$, we need to distinguish two cases depending on the values of $|\varrho(x) - \varrho(y)|$.

– If $|\varrho(x) - \varrho(y)| \le d(x,y)$, this case was already studied in Lemma 2.15 with $L = 1$, then (2.24) follows from (2.23) and (2.12).

– Otherwise, $|\varrho(x) - \varrho(y)| > d(x,y)$. We can assume that

$$d(x,y) < \varrho(x) - \varrho(y), \tag{2.25}$$

since the other case is analogous. Then $B_y \subset B_x$ and

$$B_x \triangle B_y = B_x \setminus B_y \subset B(y, \varrho(x) + d(x,y)) \setminus B(y, \varrho(y)).$$

Consequently, the δ-annular decay property (2.13) yields

$$\mu(B_x \triangle B_y) \le D_\delta \left(\frac{\varrho(x) - \varrho(y) + d(x,y)}{\varrho(x) + d(x,y)} \right)^\delta \mu(B(y, \varrho(x) + d(x,y))). \tag{2.26}$$

On the other hand, since the δ-annular decay property implies that μ is doubling with some constant $D_\mu \ge 1$, using the inclusion $B(y, \varrho(x) + d(x,y)) \subset B(y, 2\varrho(x))$, it turns out that

$$\mu(B(y, \varrho(x) + d(x,y))) \le D_\mu^2 \, \mu(B_x).$$

Therefore, replacing this in (2.26) we get

$$\mu(B_x \triangle B_y) \le D_\mu^2 \, D_\delta \left(\frac{\varrho(x) - \varrho(y) + d(x,y)}{\varrho(x) + d(x,y)} \right)^\delta \mu(B_x).$$

Since $d(x,y) \ge 0$, $\varrho(x) \ge \varrho_K$, $\mu(B_x) \ge \mu(B_y)$ and (2.25),

$$\mu(B_x \triangle B_y) \le D_\mu^2 \, D_\delta \left(2 \, \frac{\varrho(x) - \varrho(y)}{\varrho_K} \right)^\delta \max \left\{ \mu(B_x), \mu(B_y) \right\}.$$

Recalling (2.12) the proof is complete. $\square$

Recalling (2.22) we have shown the following theorem.

Theorem 2.18. *Let $(\mathbb{X}, d, \mu)$ be a proper metric measure space satisfying the δ-annular decay property (2.13) for some $\delta \in (0,1]$ and $D_\delta \ge 1$. Suppose that ϱ is a continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$. Then, for any $u \in L^\infty(\Omega)$, any compact set $K \subset \Omega$, any $x, y \in K$ and each $k \in \mathbb{N}$ we have*

$$\left| \mathcal{M}^k u(x) - \mathcal{M}^k u(y) \right| \le C \, \|u\|_\infty \left(\frac{\widehat{\omega}_\varrho(d(x,y))}{\varrho_K} \right)^\delta, \tag{2.27}$$

where $C = C(D_\delta, \mu) > 0$. In particular, the sequence $\{\mathcal{M}^k u\}_k$ is locally uniformly equicontinuous in Ω.

In particular, if ϱ is γ-Hölder continuous with $\gamma \in (0,1]$, then we can assume directly that $\widehat{\omega}_\varrho(t) = L \, t^\gamma$ and we get the following corollary of Theorem 2.18.

Corollary 2.19. *Let $(\mathbb{X}, d, \mu)$ be a proper metric measure space satisfying the δ-annular decay property (2.13) for some $\delta \in (0,1]$ and $D_\delta \ge 1$. Suppose that ϱ is a γ-Hölder continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$, for some $\gamma \in (0,1]$. Then, for any $u \in L^\infty(\Omega)$, any compact set $K \subset \Omega$, any $x, y \in K$ and each $k \in \mathbb{N}$ we have*

$$\left| \mathcal{M}^k u(x) - \mathcal{M}^k u(y) \right| \le \|u\|_\infty \, \mathcal{W}_{\mu,K}(d(x,y)),$$

where $\mathcal{W}_{\mu,K} : [0, \operatorname{diam} \Omega] \to \mathbb{R}$ is given by

$$\mathcal{W}_{\mu,K}(t) = C \, \varrho_K^{-\delta} \, t^{\gamma \delta}, \tag{2.28}$$

with $C = C(D_\delta, D_\mu, L)$ and $L > 0$ is the Hölder coefficient of ϱ. In particular, the operator $\mathcal{M}$ sends $L^\infty(\Omega)$ to the space $C^{0,\gamma\delta}_{\mathrm{loc}}(\Omega)$ of locally $\gamma\delta$-Hölder continuous functions in Ω,

$$\mathcal{M} : L^\infty(\Omega) \to C^{0,\gamma\delta}_{\mathrm{loc}}(\Omega).$$

and the sequence $\{\mathcal{M}^k u\}_k$ is locally uniformly equicontinuous in Ω.

Moreover, we immediately deduce a regularity result for bounded solutions of $\mathcal{M}u = u$ which can be stated as follows.

Theorem 2.20. *Let $(\mathbb{X}, d, \mu)$ be a proper metric measure space satisfying the δ-annular decay property for some $\delta \in (0, 1]$. Suppose that $\Omega \subset \mathbb{X}$ is a bounded domain and ϱ is a γ-Hölder continuous admissible radius function in Ω for some $\gamma \in (0, 1]$. Then any $u \in L^\infty(\Omega)$ verifying the 0-mean value property in Ω with respect to ϱ (that is, $\mathcal{M}u = u$) is locally $\gamma\delta$-Hölder continuous in Ω. In particular, if $\gamma = \delta = 1$ then u is locally Lipschitz continuous in Ω.*

2.3 Iteration of $\mathcal{T}_\alpha$

We now focus our attention on the case $\alpha \neq 0, 1$. The first step consists on giving a continuity estimate for $\mathcal{T}_\alpha$ which can be easily derived from the analogous estimates for $\mathcal{S}$ and $\mathcal{M}$ in the previous sections. More specifically, the following is a direct consequence of Proposition 2.13 and Lemma 2.10.

Proposition 2.21. *Let $(\mathbb{X}, d, \mu)$ be a proper geodesic metric measure space with μ ring-continuous. Suppose that $\Omega \subset \mathbb{X}$ is a bounded domain and let ϱ be a continuous admissible radius function in Ω. Then $\mathcal{T}_\alpha : C(\overline{\Omega}) \to C(\overline{\Omega})$ for any $\alpha \in \mathbb{R}$.*

As in the case $\alpha = 0$ in which $\mathcal{T}_\alpha$ reduces to $\mathcal{M}$, to go beyond this result we need to take into consideration stronger hypothesis on the measure μ.

Lemma 2.22. *Let $(\mathbb{X}, d, \mu)$ be a proper geodesic metric measure space and let $\Omega \subset \mathbb{X}$ be a bounded domain. Suppose that ϱ is an admissible radius function in Ω and assume that, for every compact set $K \subset \Omega$, a modulus of continuity $\mathcal{W}_{\mu,K}$ is given satisfying (2.21). Then, if $|\alpha| \leq 1$, and $u \in C(\overline{\Omega})$, the estimate*

$$\omega_{\mathcal{T}_\alpha u, K}(t) \leq |\alpha|\, \omega_{u,\widetilde{K}}\left(\widehat{\omega}_\varrho(t)\right) + (1-\alpha)\, \|u\|_\infty\, \mathcal{W}_{\mu,K}(t), \tag{2.29}$$

holds for all $t \in [0, \operatorname{diam}\Omega]$.

Proof. Let $x, y \in K$. Then, recalling the definition of the operator $\mathcal{T}_\alpha$, (2.4), we get

$$\left|\mathcal{T}_\alpha u(x) - \mathcal{T}_\alpha u(y)\right| \leq |\alpha| \left|\mathcal{S}u(x) - \mathcal{S}u(y)\right| + (1-\alpha)\left|\mathcal{M}u(x) - \mathcal{M}u(y)\right|,$$

and (2.29) is obtained by taking into consideration the estimates (2.14) and (2.22). □

However, estimate (2.29) is not good enough by itself to infer further regularity for the solutions of $\mathcal{T}_\alpha u = u$. For that reason, the key point for this section is the iteration of formula (2.29). Note that, in order to obtain better estimates for $\mathcal{T}_\alpha u$ on the compact set K via iteration of the operator, we need to control u on $\widetilde{K} \supset K$, where $\widetilde{K}$ is given by (2.10). Thus, when iterating (2.29), we need to guarantee some control on the sequence of sets given by successive application of the $\widetilde{(\cdot)}$ operation over the compact set K. For that reason, we need to assume that the domain $\Omega \subset \mathbb{X}$ is bounded and we impose the following restriction on ϱ:

$$\lambda\, \mathrm{dist}(x, \partial\Omega)^\beta \leq \varrho(x) \leq \varepsilon\, \mathrm{dist}(x, \partial\Omega), \tag{2.30}$$

for each $x \in \Omega$, where $0 < \lambda \leq \ell(\Omega)^{1-\beta}\varepsilon$, $0 < \varepsilon < 1$ and $\beta \geq 1$, with $\ell(\Omega)$ given by (2.7). We also introduce the following exhaustion of Ω:

$$K_m := \{x \in \Omega \ : \ \mathrm{dist}(x, \partial\Omega) \geq (1-\varepsilon)^m\},$$

for $m \in \mathbb{N}$, where ε is the constant appearing in (2.30). Hence, $K_1 \subset K_2 \subset \cdots \Subset \Omega$ and $\lim_{m\to\infty} K_m = \Omega$ in the sense that, for every $x \in \Omega$, there exists large enough $m_0 = m_0(x) \in \mathbb{N}$ such that $x \in K_m$ for all $m \geq m_0$. Moreover, by (2.10) and (2.30), it is easy to check that

$$\widetilde{K_m} \subset K_{m+1},$$

for $m \in \mathbb{N}$. From (2.30), we can also control from below the values of ϱ on K_m:

$$\varrho_{K_m} \geq \lambda\big(\inf_{K_m} \mathrm{dist}(x, \partial\Omega)\big)^\beta \geq \lambda(1-\varepsilon)^{m\beta}, \tag{2.31}$$

where ϱ_{K_m} is as in (2.9). Replacing K by K_m in (2.29) and iterating it we can control the oscillation of $\mathcal{T}_\alpha^k$, for $k \in \mathbb{N}$, as the next lemma shows.

Lemma 2.23. *Let $(\mathbb{X}, d, \mu)$ be a proper geodesic metric measure space, $\Omega \subset \mathbb{X}$ a bounded domain and let ϱ be a continuous admissible radius function in Ω. Suppose that, for every compact set $K \subset \Omega$, a modulus of continuity $\mathcal{W}_{\mu,K}$ is given satisfying (2.21). Then, for $|\alpha| \leq 1$ and $u \in C(\overline{\Omega})$, the estimate*

$$\begin{aligned} \omega_{\mathcal{T}_\alpha^k u, K_m}(t) \leq\ & |\alpha|^k\, \omega_{u, K_{m+k}}\big(\widehat{\omega}_\varrho^{(k)}(t)\big) \\ & + (1-\alpha)\, \|u\|_\infty \sum_{j=0}^{k-1} |\alpha|^j\, \mathcal{W}_{\mu, K_{m+j}}\big(\widehat{\omega}_\varrho^{(j)}(t)\big) \end{aligned} \tag{2.32}$$

holds for each $m, k \in \mathbb{N}$ and every $t \in [0, \mathrm{diam}\,\Omega]$.

Proof. Since $\widetilde{K_m} \subset K_{m+1}$, we get from (2.29)

$$\omega_{\mathcal{T}_\alpha u, K_m}(t) \leq |\alpha|\, \omega_{u, K_{m+1}}\left(\widehat{\omega}_\varrho(t)\right) + (1-\alpha)\, \|u\|_\infty\, \mathcal{W}_{\mu, K_m}(t)$$

for each $t \in [0, \mathrm{diam}\,\Omega]$. Now, iteration of this inequality gives (2.32). $\square$

To get local equicontinuity of the sequence $\{\mathcal{T}_\alpha^k u\}_k$, we need to add some extra condition that controls the convergence of the series in (2.32).

Lemma 2.24. *Let $(\mathbb{X}, d, \mu)$ be a proper geodesic metric measure space with a continuous admissible radius function ϱ in a bounded domain $\Omega \subset \mathbb{X}$. Suppose that, for every compact sect $K \subset \Omega$, a modulus of continuity $\mathcal{W}_{\mu,K}$ is given satisfying (2.22). Assume also that*

$$|\alpha| \limsup_{j\to\infty} \left(\mathcal{W}_{\mu,K_j}(\operatorname{diam}\Omega)\right)^{1/j} < 1. \tag{2.33}$$

Then for any $u \in C(\overline{\Omega})$, the sequence $\{\mathcal{T}_\alpha^k u\}_k$ is locally uniformly equicontinuous in Ω.

Proof. Fix $m \in \mathbb{N}$. Regarding the first term in the right-hand side of (2.32) we note that, since $\widehat{\omega}_\varrho(t) \leq \operatorname{diam}\Omega$ for each $t \in [0, \operatorname{diam}\Omega]$, then

$$|\alpha|^k \omega_{u,K_{m+k}}\left(\widehat{\omega}^{(k)}(t)\right) \leq |\alpha|^k \omega_{u,\Omega}(\operatorname{diam}\Omega) \xrightarrow[k\to\infty]{} 0.$$

Thus,

$$\left\{t \longmapsto |\alpha|^k \omega_{u,\Omega}\left(\widehat{\omega}_\varrho^{(k)}(t)\right)\right\}_k \xrightarrow[k\to\infty]{} 0 \tag{2.34}$$

uniformly in $[0, \operatorname{diam}\Omega]$ as $k \to \infty$. Consequently there exists a common modulus of continuity $\mathcal{F}_1$ for the sequence (2.34). Now we focus on the series in (2.32). Note that

$$\mathcal{W}_{\mu,K_{m+j}}\left(\widehat{\omega}_\varrho^{(j)}(t)\right) \leq \mathcal{W}_{\mu,K_{m+j}}(\operatorname{diam}\Omega)$$

for all $t \in [0, \operatorname{diam}\Omega]$. Then, since

$$\limsup_{j\to\infty} \left(\mathcal{W}_{\mu,K_{m+j}}(\operatorname{diam}\Omega)\right)^{1/j} = \limsup_{j\to\infty} \left(\mathcal{W}_{\mu,K_{m+j}}(\operatorname{diam}\Omega)\right)^{1/(m+j)},$$

it follows from (2.33) that

$$|\alpha| \limsup_{j\to\infty} \left(\mathcal{W}_{\mu,K_{m+j}}(\operatorname{diam}\Omega)\right)^{1/j} < 1,$$

so the root test implies that the series

$$\sum_{j=0}^{\infty} |\alpha|^j \mathcal{W}_{\mu,K_{m+j}}\left(\widehat{\omega}_\varrho^{(j)}(t)\right) < \infty$$

converges uniformly in $[0, \operatorname{diam}\Omega]$. In particular, there exists another modulus of continuity for the series, say $\mathcal{F}_2$. Summarizing:

$$\omega_{\mathcal{T}_\alpha^k u, K_m}(t) \leq \mathcal{F}_1(t) + (1-\alpha)\|u\|_\infty \mathcal{F}_2(t).$$

Since m is arbitrary and the right-hand side of the previous inequality does not depend on $k \in \mathbb{N}$, the proof is finished. $\square$

In particular, if we assume that the measure μ satisfies the δ-annular decay property (2.13) we get the following theorem.

Theorem 2.25. *Let $(\mathbb{X}, d, \mu)$ be a proper geodesic metric measure space satisfying the δ-annular decay property (2.13) for some $\delta \in (0, 1]$. Let $|\alpha| < 1$ and suppose that ϱ is a continuous admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$ satisfying (2.30) with $0 < \lambda \leq \ell(\Omega)^{1-\beta}\varepsilon$ and $\ell(\Omega)$ given by (2.7). Assume also that*

$$\begin{aligned} &0 < \varepsilon < 1 - |\alpha|, \\ &1 \leq \beta < \frac{\log \dfrac{1}{|\alpha|}}{\log \dfrac{1}{1-\varepsilon}}. \end{aligned} \tag{2.35}$$

Then, for any $u \in C(\overline{\Omega})$, the sequence of iterates $\{\mathcal{T}_\alpha^k u\}_k$ is locally uniformly equicontinuous in Ω.

Proof. We only need to check that the assumptions in Lemma 2.24 are satisfied. By Theorem 2.18, for any compact set $K \subset \Omega$, we can choose $\mathcal{W}_{\mu,K}$ as in (2.27) for any compact set $K \subset \Omega$. Thus, after replacing K by K_j and t by $\operatorname{diam} \Omega$ and recalling that $\widehat{\omega}_\varrho(\operatorname{diam} \Omega) = \operatorname{diam} \Omega$, we get,

$$\big(\mathcal{W}_{\mu,K_j}(\operatorname{diam} \Omega)\big)^{1/j} = \Big(C(\operatorname{diam} \Omega)^\delta\Big)^{1/j} \varrho_{K_j}^{-\delta/j},$$

and by (2.31),

$$\big(\mathcal{W}_{\mu,K_j}(\operatorname{diam} \Omega)\big)^{1/j} \leq \left(\frac{C(\operatorname{diam} \Omega)^\delta}{\lambda^\delta}\right)^{1/j} (1-\varepsilon)^{-\delta\beta}.$$

Taking limits we get

$$\limsup_{j\to\infty} \big(\mathcal{W}_{\mu,K_j}(\operatorname{diam} \Omega)\big)^{1/j} \leq (1-\varepsilon)^{-\delta\beta}.$$

On the other hand, by (2.35) we have $|\alpha| < 1 - \varepsilon \leq (1-\varepsilon)^{\delta\beta}$ so condition (2.33) follows and the sequence $\{\mathcal{T}_\alpha^k u\}_k$ is locally uniformly equicontinuous in Ω by Lemma 2.24. $\square$

2.4 Regularity of solutions

In this section we give a priori regularity results for functions $u \in C(\overline{\Omega})$ satisfying the α-mean value property (2.1) (that is, solutions of the functional equation $\mathcal{T}_\alpha u = u$) with respect to an admissible radius function in a bounded domain $\Omega \subset \mathbb{X}$. When $\alpha = 0$, then $\mathcal{T}_0 = \mathcal{M}$ and the regularity of such solutions was already obtained in Theorem 2.20. However, the case $\alpha \neq 0$ is more delicate and stronger assumptions on the radius function ϱ are needed, as we have already seen in Section 2.3.

We focus our attention on inequality (2.32). Since the continuous function u is assumed to be a fixed point of the operator $\mathcal{T}_\alpha$, after replacing $\mathcal{T}_\alpha^k u$ by u, we are allowed to pass to the limit when $k \to \infty$. Then (2.32) becomes

$$\omega_{u,K_m}(t) \leq (1-\alpha)\, \|u\|_\infty \sum_{j=0}^{\infty} |\alpha|^j\, \mathcal{W}_{\mu,K_{m+j}}\big(\widehat{\omega}_\varrho^{(j)}(t)\big), \tag{2.36}$$

for $t \in [0, \operatorname{diam}\Omega]$, where $m \in \mathbb{N}$ is fixed. Therefore, the series in (2.36) will provide the information about the regularity of the solution u. The following is the main regularity result of this chapter.

Theorem 2.26. *Let $(\mathbb{X}, d, \mu)$ be a proper geodesic metric measure space satisfying the δ-annular decay condition for some $\delta \in (0,1]$ and let $\Omega \subset \mathbb{X}$ be a bounded domain. Suppose that ϱ is a Lipschitz admissible radius function in Ω with Lipschitz constant $L \geq 1$ such that*

$$\lambda \operatorname{dist}(x, \partial\Omega)^\beta \leq \varrho(x) \leq \varepsilon \operatorname{dist}(x, \partial\Omega),$$

for all $x \in \Omega$, where $0 < \lambda \leq \ell(\Omega)^{1-\beta}\varepsilon$ and $\ell(\Omega)$ is given by (2.7). Assume also that

$$\begin{aligned} |\alpha| &< L^{-1}, \\ 0 < \ \varepsilon \ &< 1 - L\,|\alpha|\,, \end{aligned}$$

and choose β so that

$$1 \ \leq \ \beta < \frac{\log \dfrac{1}{L\,|\alpha|}}{\log \dfrac{1}{1-\varepsilon}}. \tag{2.37}$$

Then any $u \in C(\overline{\Omega})$ verifying the α-mean value property in Ω with respect to ϱ (that is, $\mathcal{T}_\alpha u = u$) is locally δ-Hölder continuous in Ω. In particular, if $\delta = 1$ then u is locally Lipschitz continuous in Ω.

Proof. By assumption, ϱ is Lipschitz continuous with constant $L \geq 1$, therefore we have $\widehat{\omega}_\varrho(t) = \min\{Lt, \operatorname{diam}\Omega\}$. Iterating we get the inequality $\widehat{\omega}_\varrho^{(j)}(t) \leq L^j t$ for each $t \in [0, \operatorname{diam}\Omega]$ and each $j \in \mathbb{N}$. Moreover, since μ satisfies the δ-annular decay property (2.13), from (2.28) together with (2.31) we get

$$\mathcal{W}_{\mu,K_{m+j}}(t) \leq \frac{C\, t^\delta}{\lambda^\delta (1-\varepsilon)^{(m+j)\beta\,\delta}}$$

for some constant $C = C(D_\delta, D_\mu, L) \geq 1$. Replacing all this in (2.36) we obtain the following estimate:

$$\omega_{u,K_m}(t) \leq \frac{C(1-\alpha)\, \|u\|_\infty}{\lambda^\delta (1-\varepsilon)^{m\,\beta\,\delta}} \left(\sum_{j=0}^{\infty} \left(\frac{L^\delta\, |\alpha|}{(1-\varepsilon)^{\beta\,\delta}} \right)^j \right) t^\delta.$$

Now observe that (2.37) implies the convergence of the above series and, consequently, the desired Hölder regularity estimate. □

In the particular case that $\beta = 1$, we obtain the following Theorem as a corollary of Theorem 2.26.

Theorem 2.27. *Let $(\mathbb{X}, d, \mu)$ be a proper geodesic metric measure space satisfying the δ-annular decay condition for some $\delta \in (0, 1]$ and let $\Omega \subset \mathbb{X}$ be a bounded domain. Suppose that ϱ is a Lipschitz admissible radius function in Ω with Lipschitz constant $L \geq 1$ such that*

$$\lambda \operatorname{dist}(x, \partial\Omega) \leq \varrho(x) \leq \varepsilon \operatorname{dist}(x, \partial\Omega),$$

for all $x \in \Omega$, where

$$0 < \lambda \leq \varepsilon < 1 - L\,|\alpha|\,.$$

Then any $u \in C(\overline{\Omega})$ verifying the α-mean value property in Ω with respect to ϱ (that is, $\mathcal{T}_\alpha u = u$) is locally δ-Hölder continuous in Ω. In particular, if $\delta = 1$ then u is locally Lipschitz continuous in Ω.

2.5 The Dirichlet problem for p-harmonious functions

Let $\mathbb{X} = \mathbb{R}^n$ with d the euclidean distance and $\mu = \mathcal{L}$ the Lebesgue measure. Given a bounded domain $\Omega \subset \mathbb{R}^n$ and any fixed continuous boundary data $f \in C(\partial\Omega)$, in this section we show existence and uniqueness of fixed points of $\mathcal{T}_\alpha$ in $\mathcal{K}_f$, where $\mathcal{K}_f$ stands for the set of all norm-preserving continuous extensions of f to $\overline{\Omega}$ defined in (2.6).

Uniqueness of solutions is easily deduced from a comparison principle for fixed points of $\mathcal{T}_\alpha$ which also holds for general metric measure spaces (see Proposition 2.31). On the other hand, our proof of existence of fixed points relies on the equicontinuity of the sequence of iterates $\{\mathcal{T}_\alpha^k u\}_k$ in $\overline{\Omega}$ for any $u \in \mathcal{K}_f$. Indeed, in Theorem 2.33 we show that the equicontinuity of the iterates in the clausure of the domain implies existence of solutions even in the case of a metric measure space $(\mathbb{X}, d, \mu)$.

Consequently, since the local equicontinuity has been already studied in Section 2.3, the main issue in this section is to show the equicontinuity of the sequence $\{\mathcal{T}_\alpha^k u\}_k$ at each point of the boundary. With that purpose, Section 2.5.3 describes the adaptation of a clever argument due to Javaheri ([Jav]) for the operator $\mathcal{M}$ in euclidean domains, as is done in [AL1]. Unfortunately, this method cannot be employed in the context of a general metric measure space. Moreover, as one may expect, the argument adapted from [Jav] needs some extra hypothesis on the geometry of the domain, more precisely, (in addition to boundedness) we require Ω to be *strictly convex*, that is, for each $x, y \in \partial\Omega$ the open segment connecting x and y is entirely contained in Ω (see Theorem 2.34). This is the only part of the proof where strict convexity is used.

To summarize, we state the main theorem of this section, which is consequence of the results of this and previous sections.

Theorem 2.28. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and strictly convex domain, $\alpha \in [0,1)$ and ϱ a continuous admissible radius function in Ω satisfying*

$$\lambda\, \mathrm{dist}(x,\partial\Omega)^{\beta} \le \varrho(x) \le \varepsilon\, \mathrm{dist}(x,\partial\Omega)$$

with $0<\lambda\le \ell(\Omega)^{1-\beta}\varepsilon$ and $\ell(\Omega)$ is given by (2.7). Assume also that

$$0<\varepsilon<1-\alpha,$$
$$1\le\beta<\frac{\log\dfrac{1}{\alpha}}{\log\dfrac{1}{1-\varepsilon}}.$$

Let $\mathcal{T}_\alpha$ the operator (2.4) with $\mu=\mathcal{L}$ the Lebesgue measure. Then, for any $f\in C(\partial\Omega)$, there exists a unique fixed point $v=\mathcal{T}_\alpha v$ in $\mathcal{K}_f$. Moreover, for any $u\in\mathcal{K}_f$, the sequence of iterates $\{\mathcal{T}_\alpha^k u\}_k$ converges uniformly to v.

Taking $\beta=1$ we obtain the following as a corollary of the previous result and Theorem 2.27.

Theorem 2.29. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and strictly convex domain, $\alpha \in [0,1)$ and ϱ a continuous admissible radius function satisfying*

$$\lambda\, \mathrm{dist}(x,\partial\Omega) \le \varrho(x) \le \varepsilon\, \mathrm{dist}(x,\partial\Omega)$$

for all $x\in\Omega$, where

$$0<\lambda\le\varepsilon<1-\alpha.$$

Then, for any $f\in C(\partial\Omega)$, there exists a unique fixed point $v=\mathcal{T}_\alpha v$ in $\mathcal{K}_f$. Moreover, for any $u\in\mathcal{K}_f$, the sequence of iterates $\{\mathcal{T}_\alpha^k u\}_k$ converges uniformly to v. In addition, if ϱ is Lipschitz continuous with constant $L\ge 1$ and

$$0<\lambda\le\varepsilon<1-L\alpha,$$

then u is locally Lipschitz continuous in Ω.

Remark 2.30. The results in this section were originally proved in [AL1] when the admissible radius function is 1-Lipschitz, that is,

$$|\varrho(x)-\varrho(y)|\le|x-y|\,,$$

for each $x,y\in\Omega$. The proof of existence of solutions (Lemma 2.32 and Theorem 2.33) is much more powerful than in [AL1] since we show the convergence of the sequence of iterates $\{\mathcal{T}_\alpha^k u_0\}_k$ for a fixed point directly, while in [AL1] we employed a different argument based on a technical result for non-expansive mappings in Banach spaces due to Ishikawa (see [Ish] and [GK]). This method requires the aid of an auxiliary operator $\mathcal{H}_\alpha=\frac{1}{2}(\mathcal{I}+\mathcal{T}_\alpha)$ (whose fixed points coincide with the fixed points of $\mathcal{T}_\alpha$) and consists on studying the convergence of its iterates.

2.5.1 Uniqueness of solutions

The uniqueness part follows from the next comparison principle.

Proposition 2.31. *Let $(\mathbb{X}, d, \mu)$ be a metric measure space endowed with a Borel positive regular measure μ such that $0 < \mu(B) < \infty$ for every ball $B \subset \mathbb{X}$. Let $\Omega \subset \mathbb{X}$ be a bounded domain, ϱ an admissible radius function in Ω and $\alpha \in [0, 1)$. Suppose that u and v are fixed points of $\mathcal{T}_\alpha$ in $C(\overline{\Omega})$ satisfying that $u \le v$ on $\partial\Omega$. Then $u \le v$ in Ω.*

Proof. The argument is standard in comparison results. Let u and v as in the statement of the proposition. Let $m = \max\limits_{\overline{\Omega}}(u - v)$. We will show that $m \le 0$. Suppose, on the contrary, that $m > 0$. Then, by monotonicity, the inequalities

$$\mathcal{S}u \le m + \mathcal{S}v \quad \text{and} \quad \mathcal{M}u \le m + \mathcal{M}v \tag{2.38}$$

hold. On the other hand, define

$$A := \{x \in \Omega \ : \ (u - v)(x) = m\},$$

which is a nonempty and closed subset of Ω, and take any $y \in A$. We will see that $B_y \subset A$ and thus A is also open. Indeed, since u and v are fixed points of $\mathcal{T}_\alpha$ and $u(y) = m + v(y)$,

$$\alpha\mathcal{S}u(y) + (1 - \alpha)\mathcal{M}u(y) = \alpha(m + \mathcal{S}v(y))(1 - \alpha)(m + \mathcal{M}v(y)),$$

and by (2.38) we must have in particular that $\mathcal{M}u(y) = m + \mathcal{M}v(y)$. Therefore,

$$\fint_{B_y} (v + m - u) \, d\mu = 0.$$

The integrand in this equation is continuous and non-negative so by continuity and the hypothesis on μ it follows that $v + m - u \equiv 0$ in B_y. This proves that A is open. Therefore, by connectedness $A = \Omega$ and $v - u \equiv m > 0$ in Ω, which contradicts the assumption $u \le v$ on $\partial\Omega$. Then $m \le 0$ and the proposition follows. $\square$

2.5.2 Equicontinuity in $\overline{\Omega}$ implies existence

Let $\Omega \subset \mathbb{X}$ be a bounded domain and consider $f \in C(\partial\Omega)$ any continuous boundary data. In order to show the existence of fixed points for $\mathcal{T}_\alpha$ in $\mathcal{K}_f$, as well as the convergence of the sequence of iterates to a fixed point, we will make use of the following technical result, which can be stated in the more general context of Banach spaces.

Lemma 2.32. *Let $(X, \|\cdot\|)$ be a Banach space, $\emptyset \ne K \subset X$ any closed subset and $T : K \to K$ a non-expansive operator. For $x \in K$, suppose that the sequence of iterates $\{T^k x\}_k$ has a limit point $y \in K$. Let*

$$C := \lim_{k\to\infty} \|T^{k+1}x - T^k x\| \ge 0. \tag{2.39}$$

Then,

$$\|T^{\ell+1}y - T^\ell y\| = C, \tag{2.40}$$

for every $\ell = 0, 1, 2, \ldots$ In particular, if $C = 0$, the sequence $\{T^k x\}_k$ is said to be asymptotically regular *and then $y \in K$ is a fixed point of T.*

Proof. Since $y \in K$ is a limit point of $\{T^k x\}_k$, there exists a subsequence $\{T^{k_j} x\}_j$ converging to y. The non-expansiveness of T implies that the sequence $\{\|T^{k+1}x - T^k x\|\}_k$ is non-increasing, thus the constant $C \geq 0$ given by (2.39) is well-defined. Now, for any $\ell = 0, 1, 2, \ldots$, the triangle inequality and the non-expansiveness of T yield

$$\begin{aligned}\|T^{\ell+1}y - T^{\ell}y\| &\leq \|T^{k_j+\ell+1}x - T^{\ell+1}y\| + \|T^{k_j+\ell}x - T^{\ell}y\| + \|T^{k_j+\ell+1}x - T^{k_j+\ell}x\| \\ &\leq 2\|T^{k_j}x - y\| + \|T^{k_j+1}x - T^{k_j}x\|,\end{aligned}$$

for each $j \in \mathbb{N}$. Taking limits as $j \to \infty$, we get $\|T^{\ell+1}y - T^{\ell}y\| \leq C$. On the other hand,

$$\begin{aligned}\|T^{k_j+\ell+1}x - T^{k_j+\ell}x\| &\leq \|T^{k_j+\ell+1}x - T^{\ell+1}y\| + \|T^{k_j+\ell}x - T^{\ell}y\| + \|T^{\ell+1}y - T^{\ell}y\| \\ &\leq 2\|T^{k_j}x - y\| + \|T^{\ell+1}y - T^{\ell}y\|.\end{aligned}$$

Again, taking limits we obtain the reversed inequality and we get (2.40). Finally, if $C = 0$, choosing $\ell = 0$ in (2.40) we get $Ty = y$, so $y \in K$ is a fixed point of T. $\square$

For the next result, we assume that the sequence of iterates is equicontinuous in $\overline{\Omega}$ and we use Lemma 2.32 to show that this sequence converges uniformly to a fixed point.

Theorem 2.33. *Let $(\mathbb{X}, d, \mu)$ be a metric measure space endowed with a Borel positive regular measure μ such that $0 < \mu(B) < \infty$ for every ball $B \subset \mathbb{X}$. Let $\Omega \subset \mathbb{X}$ be a bounded domain, ϱ an admissible radius function in Ω and $\alpha \in [0, 1)$. For a fixed function $f \in C(\partial\Omega)$, suppose that the sequence of iterates $\{\mathcal{T}_\alpha^k u\}_k$ is equicontinuous in $\overline{\Omega}$ for each $u \in \mathcal{K}_f$. Then:*

1. *For any $u \in \mathcal{K}_f$, the sequence of iterates $\{\mathcal{T}_\alpha^k u\}_k$ is asymptotically regular, that is,*

$$\lim_{k\to\infty} \|\mathcal{T}_\alpha^{k+1}u - \mathcal{T}_\alpha^k u\| = 0.$$

2. *There exists a unique function $v \in \mathcal{K}_f$ such that $\mathcal{T}_\alpha v = v$.*

3. *The sequence of iterates converges uniformly in $\overline{\Omega}$ to the fixed point $v \in \mathcal{K}_f$ for each $u \in \mathcal{K}_f$.*

Proof. Since $\{\mathcal{T}_\alpha^k u\}_k$ is assumed to be equicontinuous in $\overline{\Omega}$, by Arzelà-Ascoli's theorem, there exists at least one subsequence converging uniformly to a function $v \in \mathcal{K}_f$. Let

$$C := \lim_{k\to\infty} \|\mathcal{T}_\alpha^{k+1}u - \mathcal{T}_\alpha^k x\|_\infty \geq 0.$$

By definition (2.6), $\mathcal{K}_f$ is a closed subset of $C(\overline{\Omega})$ and $\mathcal{T}_\alpha : \mathcal{K}_f \to \mathcal{K}_f$, thus Lemma 2.32 yields that

$$\|\mathcal{T}_\alpha^{\ell+1}v - \mathcal{T}_\alpha^{\ell}v\|_\infty = C, \tag{2.41}$$

for every $\ell = 0, 1, 2, \ldots$ To see that $C = 0$, we argue by contradiction: suppose that $C > 0$ and choose a large enough $\ell \in \mathbb{N}$ so that

$$\ell > 2\,\frac{\|f\|_\infty}{C}. \tag{2.42}$$

Since $\mathcal{T}_\alpha^{\ell+1}v - \mathcal{T}_\alpha^{\ell}v$ is a continuous function vanishing on $\partial\Omega$, we can choose an interior point $x_0 \in \Omega$ such that

$$|\mathcal{T}_\alpha^{\ell+1}v(x_0) - \mathcal{T}_\alpha^{\ell}v(x_0)| = C.$$

We assume that $\mathcal{T}_\alpha^{\ell+1}v(x_0) - \mathcal{T}_\alpha^{\ell}v(x_0) = C$ since otherwise the proof goes in an analogous way. Recalling the definition of $\mathcal{T}_\alpha$ and $\mathcal{M}$, (2.4) and (2.3), respectively, it turns out that

$$C = \alpha\left[\mathcal{S}(\mathcal{T}_\alpha^{\ell}v)(x_0) - \mathcal{S}(\mathcal{T}_\alpha^{\ell-1}v)(x_0)\right] + (1-\alpha)\fint_{B_{x_0}} (\mathcal{T}_\alpha^{\ell}v - \mathcal{T}_\alpha^{\ell-1}v)\,d\mu. \tag{2.43}$$

For $\alpha \neq 0$, using (2.41), we know that

$$\fint_{B_{x_0}} (\mathcal{T}_\alpha^{\ell}v - \mathcal{T}_\alpha^{\ell-1}v)\,d\mu \le C,$$

then, replacing in (2.43), rearranging terms and dividing by α,

$$C \le \mathcal{S}(\mathcal{T}_\alpha^{\ell}v)(x_0) - \mathcal{S}(\mathcal{T}_\alpha^{\ell-1}v)(x_0) \le \|\mathcal{S}\mathcal{T}_\alpha^{\ell}v - \mathcal{S}\mathcal{T}_\alpha^{\ell-1}v\|_\infty.$$

From the non-expansiveness of the operator $\mathcal{S}$ and (2.41), we get

$$\mathcal{S}(\mathcal{T}_\alpha^{\ell}v)(x_0) - \mathcal{S}(\mathcal{T}_\alpha^{\ell-1}v)(x_0) = C,$$

and, as a consequence of (2.43),

$$\fint_{B_{x_0}} (\mathcal{T}_\alpha^{\ell}v - \mathcal{T}_\alpha^{\ell-1}v)\,d\mu = C,$$

which is also true when $\alpha = 0$ (just replace it in (2.43)). Together with (2.41) and since $\mu(B_{x_0}) > 0$, this implies that the function in the previous averaged integral is equal to C in B_{x_0}. In particular,

$$\mathcal{T}_\alpha^{\ell}v(x_0) - \mathcal{T}_\alpha^{\ell-1}v(x_0) = C,$$

so we can repeat this argument iteratively until we finally reach that

$$\mathcal{T}_\alpha^{\ell}v(x_0) = v(x_0) + \ell C.$$

Recalling that $\mathcal{T}_\alpha : \mathcal{K}_f \to \mathcal{K}_f$ together with (2.6) and (2.42), we obtain the desired contradiction,

$$\|f\|_\infty \ge \mathcal{T}_\alpha^{\ell}v(x_0) = v(x_0) + \ell C > -\|f\|_\infty + 2\|f\|_\infty = \|f\|_\infty\,,$$

and thus $C = 0$. This proves not only that the sequence of iterates is asymptotically regular, but the existence of a fixed point $v \in \mathcal{K}_f$ for $\mathcal{T}_\alpha$ (see Lemma 2.32). Finally, to see that the iterates actually converge uniformly to the fixed point, suppose on the contrary that there are $\varepsilon > 0$ and a subsequence $\{\mathcal{T}_\alpha^{k_j} u\}_j$ such that

$$\|\mathcal{T}_\alpha^{k_j} u - v\| \geq \varepsilon$$

for each $j \in \mathbb{N}$. We can assume that this subsequence converges uniformly to a function $w \in \mathcal{K}_f$ (otherwise, by equicontinuity and Arzelà-Ascoli's theorem we can take a further subsequence converging uniformly to w). In particular,

$$\|w - v\|_\infty \geq \varepsilon. \tag{2.44}$$

However, following the same reasoning, it turns out that w is also a fixed point for $\mathcal{T}_\alpha$ in $\mathcal{K}_f$ and, by uniqueness (Section 2.5.1) it turns out that $w = v$ in $\overline{\Omega}$, which contradicts (2.44). □

2.5.3 Boundary equicontinuity in the euclidean case

In this section, let $\mathbb{X} = \mathbb{R}^n$, d the euclidean distance and $\mu = \mathcal{L}$ the Lebesgue measure. In what follows, we will assume that $\Omega \subset \mathbb{R}^n$ is a bounded and strictly convex domain (that is, for each pair of points $x, y \in \partial\Omega$, the open segment connecting them is contained in Ω) and ϱ is a continuous admissible radius function in Ω. Here, the operator $\mathcal{T}_\alpha$ is defined as usual. Then, we show the following result:

Theorem 2.34. *Let $\alpha \in [0, 1)$. Then, for any $u \in C(\overline{\Omega})$, the sequence of iterates $\{\mathcal{T}_\alpha^k u\}_k$ is equicontinuous at each point of $\partial\Omega$.*

As we mentioned at the beginning of this section, in order to show Theorem 2.34, we adapt an argument from [Jav]. The idea is to show that the graph of each iterate $\mathcal{T}_\alpha^k u$ is contained in the convex hull of the graph of u for any $u \in C(\overline{\Omega})$. Then, together with the strict convexity of the domain Ω, this property imposes a control on the behavior of the iterates near the boundary in such a way that the equicontinuity at each point in $\partial\Omega$ follows.

Once the boundary equicontinuity has been proved, we are in conditions of showing the existence result for strictly convex domains in $\mathbb{R}^n$:

Proof of Theorem 2.28. Uniqueness of solutions follows directly from the comparison principle for generalized p-harmonious functions (Proposition 2.31), while the existence of solutions will follow from Theorem 2.33 once the equicontinuity of $\{\mathcal{T}_\alpha^k u\}$ in $\overline{\Omega}$ has been provided for any continuous extension $u \in \mathcal{K}_f$. Indeed, equicontinuity at points in $\partial\Omega$ is proven in Theorem 2.34, while for equicontinuity at interior points we recall Theorem 2.25. □

First, we introduce some basic notation. For $u \in C(\overline{\Omega})$, we define the graph of u as the set

$$G_u := \left\{(x, u(x)) \ : \ x \in \overline{\Omega}\right\} \subset \Omega \times \mathbb{R} \subset \mathbb{R}^{n+1}.$$

Moreover, we denote by $\text{co}(A)$ the convex hull of a set A, that is, the smallest convex set containing A.

Proposition 2.35. *Let $\alpha \in [0, 1]$. If $u \in C(\overline{\Omega})$, then*

$$G_{\mathcal{T}_\alpha^k u} \subset \text{co}(G_u), \tag{2.45}$$

for each $k \in \mathbb{N}$.

Proof. Note that, to obtain (2.45), it is enough to prove that the inclusion

$$G_{\mathcal{T}_\alpha u} \subset \text{co}(G_u) \tag{2.46}$$

holds for each function $u \in C(\overline{\Omega})$. Indeed, if (2.46) holds, then

$$\text{co}(G_{\mathcal{T}_\alpha(\mathcal{T}_\alpha^{k-1}u)}) \subset \text{co}(G_{\mathcal{T}_\alpha^{k-1}u})$$

for every $k \in \mathbb{N}$ and we easily get (2.45) after applying this inclusion iteratively. Moreover, since α is assumed to be between 0 and 1, by definition of $\mathcal{T}_\alpha$ (2.4), each point in the graph of $\mathcal{T}_\alpha u$ is convex combination of a point in $G_{\mathcal{S}u}$ with a point in $G_{\mathcal{M}u}$, more precisely,

$$(x, \mathcal{T}_\alpha u(x)) = \alpha\,(x, \mathcal{S}u(x)) + (1-\alpha)\,(x, \mathcal{M}u(x))$$

for each $x \in \overline{\Omega}$, and thus we just need to prove (2.46) for the extreme cases $\alpha = 0$ and $\alpha = 1$, that is,

$$G_{\mathcal{S}u} \subset \text{co}(G_u), \tag{2.47}$$

$$G_{\mathcal{M}u} \subset \text{co}(G_u). \tag{2.48}$$

Fix $x \in \Omega$. To see (2.47), since u is continuous, we can select points ξ_{max} and ξ_{min} in $B_x = \overline{B}(x, \varrho(x))$ such that

$$\sup_{B_x} u = u(\xi_{\text{max}}),$$
$$\inf_{B_x} u = u(\xi_{\text{min}}),$$

together with their reflections with respect to x, $\xi'_{\text{max}} = 2x - \xi_{\text{max}}$ and $\xi'_{\text{min}} = 2x - \xi_{\text{min}}$, then

$$u(\xi_{\text{min}}) + u(\xi'_{\text{min}}) \leq \sup_{B_x} u + \inf_{B_x} u \leq u(\xi_{\text{max}}) + u(\xi'_{\text{max}}),$$

and by continuity, there exists $\xi \in B_x$ such that

$$\mathcal{S}u(x) = \frac{u(\xi) + u(\xi')}{2}.$$

where $\xi' = 2x - \xi \in B_x$. Consequently,

$$(x, \mathcal{S}u(x)) = \frac{1}{2}(\xi, u(\xi)) + \frac{1}{2}(\xi', u(\xi')),$$

which is a convex combination of points in G_u, and (2.47) follows.

We prove now (2.48). Since $\zeta \in B_x$ if and only if $2x - \zeta \in B_x$ and $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}^n$, then

$$\int_{B_x} u(\zeta)\, d\zeta = \int_{B_x} u(2x-\zeta)\, d\zeta$$

and we can write

$$\mathcal{M}u(x) = \fint_{B_x} \frac{u(\zeta)+u(2x-\zeta)}{2}\, d\zeta.$$

Therefore, by continuity, there exists $\xi \in B_x$ so that

$$\mathcal{M}u(x) = \frac{u(\xi)+u(\xi')}{2},$$

and (2.48) follows as in the previous case. $\square$

Lemma 2.36. *Let $u \in C(\overline{\Omega})$. Then,*

$$\mathrm{co}(G_u) \cap (\{\xi\} \times \mathbb{R}) = \{(\xi, u(\xi))\},$$

for each $\xi \in \partial\Omega$.

Proof. Fix any $\xi \in \partial\Omega$ and suppose that $t \in \mathbb{R}$ is given such a way that the point (ξ, t) is contained $\mathrm{co}(G_u)$. We need to show that this forces t to be equal to $u(\xi)$. By Carathéodory's theorem, (ξ, t) can be written as a convex combination of (at most) $n+2$ points in G_u, that is,

$$(\xi, t) = \sum_{i=1}^{n+2} \lambda_i (x_i, u(x_i))$$

where $x_1, x_2, \ldots, x_{n+2} \in \overline{\Omega}$ and $\lambda_1, \lambda_2, \ldots, \lambda_{n+2} \geq 0$ satisfy $\lambda_1 + \cdots + \lambda_{n+2} = 1$. We deduce that the convex combination in the previous equation must be trivial in the sense that, say, $\lambda_1 = 1$ and $\lambda_2 = \cdots = \lambda_{n+2} = 0$. Otherwise, $\xi \in \partial\Omega$ would be a proper convex combination of points in $\overline{\Omega}$, which contradicts the strict convexity of $\overline{\Omega}$. Then $x_1 = \xi$ and $t = u(\xi)$ for the choices of the parameters above. $\square$

Proof of Theorem 2.34. We proceed by contradiction: suppose that there exists a continuous function $u \in C(\overline{\Omega})$ such that the sequence of iterates $\{\mathcal{T}_\alpha^k u\}_k$ is not equicontinuous at certain $\xi \in \partial\Omega$. Then there is a small enough $\varepsilon > 0$, an increasing sequence of integers $\{k_j\}_j \subset \mathbb{N}$ and a sequence of points $\{x_j\}_j \subset \overline{\Omega}$ satisfying $x_j \to \xi$ and

$$|\mathcal{T}_\alpha^{k_j} u(x_j) - u(\xi)| = |\mathcal{T}_\alpha^{k_j} u(x_j) - \mathcal{T}_\alpha^{k_j} u(\xi)| \geq \varepsilon$$

for each $j \in \mathbb{N}$. We can assume (otherwise we could take a further subsequence) that $\mathcal{T}_\alpha^{k_j} u(x_j) \to t \in \mathbb{R}$ as $j \to \infty$ and that

$$|t - u(\xi)| \geq \frac{\varepsilon}{2}. \tag{2.49}$$

By Proposition 2.35,

$$(x_j, \mathcal{T}_\alpha^{k_j} u(x_j)) \in \operatorname{co}(G_u)$$

for each $j \in \mathbb{N}$, and since $\operatorname{co}(G_u)$ is a closed set, taking limits we get

$$(\xi, t) \in \operatorname{co}(G_u).$$

Then the contradiction follows from the (2.49) together with Lemma 2.36. Therefore $\{\mathcal{T}_\alpha^k u\}_k$ is equicontinuous at each point of $\partial\Omega$. $\square$

Chapter 3

Tug-of-war games and the normalized $p(x)$-Laplacian

In this chapter we study a tug-of-war game with orthogonal noise and space dependent probabilities. One of the key tools in studying the tug-of-war games is the dynamic programming principle (DPP), which, in this case, reads as

$$u(x) = \frac{1-\delta(x)}{2}\left[\sup_{|\nu|=\varepsilon}\left(\alpha(x)u(x+\nu)+\beta(x)\fint_{B_\varepsilon^\nu} u(x+h)\,d\mathcal{L}^{n-1}(h)\right)\right.$$
$$\left.+\inf_{|\nu|=\varepsilon}\left(\alpha(x)u(x+\nu)+\beta(x)\fint_{B_\varepsilon^\nu} u(x+h)\,d\mathcal{L}^{n-1}(h)\right)\right]$$
$$+\,\delta(x)F(x),$$

with a given boundary cut-off function δ, a continuous boundary function F and continuous probability functions $\alpha(x), \beta(x)$. Here, B_ε^ν denotes the $(n-1)$-dimensional ball orthogonal to ν and $\mathcal{L}^{n-1}$ stands for the $(n-1)$-dimensional Lebesgue measure. This formula can be understood as some sort of mean value property.

The results in this chapter are contained in [AHP]. However, in this memory we do not describe all the results of this article and we mainly focus on giving a detailed proof of the (asymptotic) regularity estimates for solutions of the DPP. We organize this chapter in three sections: first, in Section 3.1 we describe the game introduced above which turns out to be very helpful for showing existence and uniqueness of solutions of DPP. Later, in Section 3.2, we show that these solutions satisfy some locally asymptotic Hölder continuity estimate (Theorem 3.2). Finally, Section 3.3 is devoted to give boundary estimates for the solutions using barrier arguments.

3.1 The two-player tug-of-war game

First we introduce some notation that will be useful to describe the game and throughout the following sections. Given a bounded domain $\Omega \subset \mathbb{R}^n$ and $\varepsilon > 0$, we define the following open sets:

$$
\begin{aligned}
I_\varepsilon &= \{x \in \Omega \;:\; \operatorname{dist}(x, \partial\Omega) < \varepsilon\}, \\
O_\varepsilon &= \left\{x \in \mathbb{R}^n \setminus \overline{\Omega} \;:\; \operatorname{dist}(x, \partial\Omega) < \varepsilon\right\},
\end{aligned}
$$

ant the extended domain $\Omega_\varepsilon := \overline{\Omega} \cup O_\varepsilon$. Also, for brevity, the compact boundary strip of the game domain is denoted by $\Gamma_\varepsilon := \overline{I}_\varepsilon \cup \overline{O}_\varepsilon$. Let F be a continuous boundary function $F \in C(\Gamma_\varepsilon)$. For measurability reasons, we need to introduce a boundary correction function $\delta : \overline{\Omega}_\varepsilon \to [0,1]$ which is given by

$$
\delta(x) = \min\left\{0, 1 - \varepsilon^{-1} \operatorname{dist}(x, O_\varepsilon)\right\}. \tag{3.1}
$$

Then, $\delta(x) = 0$ if and only if $x \in \Omega \setminus I_\varepsilon$. Suppose that $\alpha, \beta : \overline{\Omega} \to (0,1)$ are continuous functions such that

$$
\alpha(x) + \beta(x) = 1
$$

for all $x \in \overline{\Omega}$. In addition, we define the bounds

$$
0 < \alpha_{\min} := \inf_{x \in \Omega} \alpha(x) \leq \sup_{x \in \Omega} \alpha(x) =: \alpha_{\max} < 1, \tag{3.2}
$$

and, in consequence, $\beta_{\min} := 1 - \alpha_{\max}$ and $\beta_{\max} := 1 - \alpha_{\min}$. We denote by B_ε^ν the open ball of radius ε in the $(n-1)$-dimensional hyperplane $\nu^\perp$ orthogonal to $0 \neq \nu \in \mathbb{R}^n$, that is,

$$
B_\varepsilon^\nu := B(0, \varepsilon) \cap \nu^\perp := \{y \in \mathbb{R}^n \;:\; |y| < \varepsilon \text{ and } \langle y, \nu \rangle = 0\}.
$$

Let us consider a game involving two players (say P_{I} and P_{II}). A token is placed at a starting point $x_0 \in \Omega$. Suppose that, after $j = 0, 1, 2, \ldots$ movements, the token is at a point $x_j \in \Omega$. Then,

- if $x_j \in \Omega \setminus I_\varepsilon$, then P_{I} and P_{II} decide their possible movements ν_{j+1}^{I} and ν_{j+1}^{II}, respectively, with $|\nu_{j+1}^{\mathrm{I}}| = |\nu_{j+1}^{\mathrm{II}}| = \varepsilon$. A fair coin is tossed and if $i \in \{\mathrm{I}, \mathrm{II}\}$ and P_i wins the toss, we have two possibilities:
  - with probability $\alpha(x_j)$, the token is moved to $x_{j+1} = x_j + \nu_{j+1}^i$, and
  - with probability $\beta(x_j)$, the token is moved to a point $x_{j+1} \in x_j + B_\varepsilon^{\nu_{j+1}^i}$ uniformly random;
- if $x_j \in I_\varepsilon$,
  - the game ends with probability $\delta(x_j)$ and then, P_{II} pays P_{I} the amount given by $F(x_j)$, and
  - with probability $1 - \delta(x_j)$, the players play a game as in the previous case $x_j \in \Omega \setminus I_\varepsilon$.
- if $x_j \in O_\varepsilon$, then the game ends and P_{II} pays P_{I} the amount given by $F(x_j)$.

Let τ denote the time when the game ends, and denote by $x_\tau \in \Gamma_\varepsilon$ the position where the game ends. Then, P_{II} pays P_{I} the quantity $F(x_\tau)$. We define a *history* of the game as the vector $(x_0, x_1, \ldots, x_j)$ describing the positions of the token at each step after j repetitions. A *strategy* is a sequence of Borel measurable functions that gives the next game position given the history of the game. Therefore, we define $S_i := (S_i^j)_{j=1}^{\infty}$ with

$$S_i^j : \{x_0\} \times \bigcup_{k=1}^{j-1} (\Omega_\varepsilon)^k \longrightarrow \partial B(0, \varepsilon),$$

for all $j \in \mathbb{N}$ and with both $i \in \{\mathrm{I}, \mathrm{II}\}$. Given a starting point $x_0 \in \Omega$ and strategies $S_{\mathrm{I}}, S_{\mathrm{II}}$, we define a probability measure $\mathbb{P}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}}$ on the natural product σ-algebra of the space of all game trajectories.

Since $\alpha_{\max} < 1$, it can be proven that the game described above ends almost surely in finite time ($\tau < \infty$) regardless of the strategies S_{I} and S_{II} (see Section 2 in [AHP] for a detailed proof and [Har] for the case in which $\alpha(x)$ is constant). Therefore, for all starting points $x_0 \in \Omega$, we can define a *value function* for P_{I} and for P_{II} by

$$\begin{cases} u_{\mathrm{I}}(x_0) = \sup\limits_{S_{\mathrm{I}}} \inf\limits_{S_{\mathrm{II}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}}[F(x_\tau)], \\ u_{\mathrm{II}}(x_0) = \inf\limits_{S_{\mathrm{II}}} \sup\limits_{S_{\mathrm{I}}} \mathbb{E}^{x_0}_{S_{\mathrm{I}}, S_{\mathrm{II}}}[F(x_\tau)]. \end{cases} \tag{3.3}$$

Moreover, it turns out that these value functions satisfy the dynamic programming principle,

$$\begin{aligned} u(x) = \frac{1-\delta(x)}{2} \Bigg[& \sup_{|\nu|=\varepsilon} \left(\alpha(x) u(x+\nu) + (1-\alpha(x)) \fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h) \right) \\ & + \inf_{|\nu|=\varepsilon} \left(\alpha(x) u(x+\nu) + (1-\alpha(x)) \fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h) \right) \Bigg] \\ & + \delta(x) F(x), \end{aligned} \tag{3.4}$$

where we denote

$$\fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h) := \frac{1}{\mathcal{L}^{n-1}(B_\varepsilon^\nu)} \int_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h),$$

with $\mathcal{L}^{n-1}$ the $(n-1)$-dimensional Lebesgue measure.

As we have mentioned, our main interest in this chapter is to study the regularity of solutions of (3.4), for that reason, we introduce the auxiliary function

$$\mathcal{A}u(x, \nu) := \alpha(x) u(x+\nu) + \beta(x) \fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h) \tag{3.5}$$

and the operator

$$\mathcal{R}u(x) := \frac{1-\delta(x)}{2}\left[\sup_{|\nu|=\varepsilon}\mathcal{A}u(x,\nu)+\inf_{|\nu|=\varepsilon}\mathcal{A}u(x,\nu)\right]+\delta(x)F(x) \tag{3.6}$$

for all $x\in\overline{\Omega}_\varepsilon$ and any continuous functions $u\in C(\overline{\Omega}_\varepsilon)$. By using this operator, we can identify the solutions to (3.4) with the fixed points of $\mathcal{R}$. Note that, despite the fact that $\alpha(x)$ and $\beta(x)$ are not defined in the outside strip $\overline{O}_\varepsilon$, (3.6) is well-defined by setting $\mathcal{R}u(x)=F(x)$ for all $x\in\overline{O}_\varepsilon$. Similarly, we set $\delta(x)F(x)=0$ for all $x\in\Omega\setminus I_\varepsilon$.

Defined in this way, $\mathcal{R}:C(\overline{\Omega}_\varepsilon)\to C(\overline{\Omega}_\varepsilon)$ and, if $u,v\in C(\overline{\Omega}_\varepsilon)$ satisfy $u\le v$ then $\mathcal{R}u\le\mathcal{R}v$, that is, the operator $\mathcal{R}$ is monotone. Next, we state the existence and uniqueness of solutions of (3.4).

Theorem 3.1. *Let $\varepsilon>0$ and let $F:\Gamma_\varepsilon\to\mathbb{R}$ be a continuous function. Then, there exists a continuous function $u_\varepsilon:\overline{\Omega}_\varepsilon\to\mathbb{R}$ with the boundary data F such that it satisfies the dynamic programming principle* (3.4)*. Moreover, this function is unique and is the value function of the game, i.e., $u_\varepsilon=u_{\mathrm{I}}=u_{\mathrm{II}}$ with u_{I} and u_{II} defined in* (3.3)*.*

In the following, we give a brief idea of the proof. For a detailed proof of this result, we refer the reader to Section 3 in [AHP] (see also [Har] for the case in which the probability function $\alpha(x)$ is constant).

Sketch of the proof. The idea of the proof is to show the existence of a lower and an upper semicontinuous solutions of (3.4),

$$\mathcal{R}\underline{u}=\underline{u}\quad\text{and}\quad\mathcal{R}\overline{u}=\overline{u},$$

respectively, by iterating the operator $\mathcal{R}$ defined in (3.6). In fact, these functions can be defined as the following pointwise limits:

$$\begin{cases}\underline{u}(x):=\lim\limits_{k\to\infty}\mathcal{R}^k(\inf F),\\ \overline{u}(x):=\lim\limits_{k\to\infty}\mathcal{R}^k(\sup F),\end{cases}$$

where $(\inf F)$ and $(\sup F)$ are understood as constant functions in $\overline{\Omega}_\varepsilon$. Then, it can be shown that every measurable solution u satisfying (3.4) is bounded between these two semicontinuous functions, that is, $\underline{u}\le u\le\overline{u}$ for every measurable function u such that $\mathcal{R}u=u$. Finally, by using the tug-of-war game defined above, it turns out that both semicontinuous solutions are, in fact, the same. From this, existence and uniqueness of solutions follows at the same time. □

Once the existence and uniqueness of continuous solutions of $\mathcal{R}u=u$ have been established, in Section 3.2 we give a local asymptotic regularity estimate for the solutions of (3.4) in $\Omega\setminus I_\varepsilon$. This is the main result of this chapter and can be stated as follows.

Theorem 3.2. *Let $x, y \in B(0,R)$ with $B(0,2R) \subset \Omega$ and*

$$0 < \gamma < \frac{\alpha_{\min}}{\alpha_{\max}} - \kappa \tag{3.7}$$

for arbitrary small $\kappa \in (0, \alpha_{\min}/\alpha_{\max})$ with $\alpha_{\min}, \alpha_{\max}$ defined in (3.2). Then, if u satisfies (3.4), we have

$$|u(x) - u(y)| \leq C\frac{|x-y|^\gamma}{R^\gamma} + C\frac{\varepsilon^\gamma}{R^\gamma} \tag{3.8}$$

with $C := C(\alpha_{\min}, \alpha_{\max}, n, R, \sup_{B_{2R}} u, \gamma)$ and $0 < \varepsilon < 1$.

Note that, by the definition of the operator $\mathcal{R}$ and the boundary cut-off function (3.1), the dynamic programming principle in $\Omega \setminus I_\varepsilon$ reduces to the equation

$$\begin{aligned} u(x) = \frac{1}{2}\Bigg[& \sup_{|\nu|=\varepsilon}\left(\alpha(x)u(x+\nu) + \beta(x)\fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h)\right) \\ & + \inf_{|\nu|=\varepsilon}\left(\alpha(x)u(x+\nu) + \beta(x)\fint_{B_\varepsilon^\nu} u(x+h)\, d\mathcal{L}^{n-1}(h)\right)\Bigg]. \end{aligned} \tag{3.9}$$

Therefore, we need to control the continuity of the solutions near de boundary of the domain. For that purpose, in Section 3.3, we obtain boundary estimates by using barrier arguments. As one may expect, we will need to ask some condition on the geometry of the domain in order to obtain these estimates. In particular, we need the domain $\Omega \subset \mathbb{R}^n$ to satisfy the so called *boundary regularity condition*: there are universal constants $r_0, s \in (0,1)$ such that, for each $r \in (0, r_0]$ and $z \in \partial\Omega$ there exists a ball

$$B(y, sr) \subset B(z,r) \setminus \Omega$$

for some $y \in B(z,r) \setminus \Omega$.

Theorem 3.3. *Consider $\Omega \subset \mathbb{R}^n$ a bounded domain satisfying the boundary regularity condition. Let $\eta > 0$ and let u be the solution of (3.4) with continuous boundary data F. Then, there is a constant $\overline{r} \in (0, r_0]$ such that for all $r \in (0, \overline{r}]$ there exist constants $k \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for any $z \in \Gamma_\varepsilon$ it holds*

$$|u(x_0) - F(z)| < \eta$$

for all $0 < \varepsilon < \varepsilon_0$ and $x_0 \in B(z, 4^{1-k}r) \cap \overline{\Omega}_\varepsilon$.

As a consequence of Theorems 3.2 and 3.3, and in view of the Arzelà-Ascoli's theorem, there exists a continuous function u on $\overline{\Omega}$ with the boundary values F and a subsequence $\{\varepsilon_k\}_k$ such that $u_{\varepsilon_k} \to u$ uniformly on $\overline{\Omega}$ as $k \to \infty$. Moreover, it turns out that this function is a weak solution to the *normalized homogeneous $p(x)$-Laplace equation*

$$\begin{aligned} \Delta_{p(x)}^{\mathrm{N}} u(x) &:= \Delta u(x) + \big(p(x)-2\big)\Delta_\infty u(x) \\ &= \Delta_1^{\mathrm{N}} u(x) + \big(p(x)-1\big)\Delta_\infty u(x) \\ &= 0, \end{aligned} \tag{3.10}$$

where the relation between $\alpha(x)$ and $p(x)$ is given by

$$\alpha(x) = \frac{p(x)-1}{n+p(x)}.$$

Thus, $p : \overline{\Omega} \to (1, +\infty)$ is uniformly continuous and

$$1 < p_{\min} := \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) =: p_{\max} < \infty.$$

Since equation (3.10) is in a non-divergence form, the weak solutions are defined via viscosity theory. Furthermore, by the estimate obtained in Theorem 3.2, it turns out that this solution is locally γ-Hölder continuous. We state this as a theorem and we direct the reader to Theorem 6.2 in [AHP] for the proof.

Theorem 3.4 ([AHP, Theorem 6.2])**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the boundary regularity condition, and let u_ε denote the unique continuous solution to (3.4) with $\varepsilon > 0$ and with a continuous boundary function $F : \Gamma_\varepsilon \to \mathbb{R}$. Then, there are a function $u : \overline{\Omega} \to \mathbb{R}$ and a subsequence $\{\varepsilon_i\}_i$ such that u_{ε_i} converges uniformly to u in $\overline{\Omega}$ and the function u is a viscosity solution to (3.10) with the boundary data F. Moreover, u is locally γ-Hölder continuous for some $\gamma \in (0,1)$.*

3.2 Local regularity

The regularity result is based on a method established by Luiro and Parviainen in [LP]. The method consists of several steps:

– First, we choose a comparison function f having the desired regularity properties. Then, the idea is to analyze two different cases separately. At a small scale, we need to control the effects arising from the discretization. At a bigger scale, the key term of the comparison function is $C\,|x-y|^\gamma$ with $x, y \in \mathbb{R}^n, 0 < \gamma < 1$ and $C > 0$ big enough.

– In the second step, we aim to prove that the error $u(x) - u(y) - f(x,y)$, where u is the solution to (3.9), is smaller in $(B_1 \times B_1) \setminus T$ than in $(B_2 \times B_2) \setminus (B_1 \times B_1 \setminus T)$ with both sets belonging to $\mathbb{R}^{2n}$, where we used the notation B_R for the ball $B(0,R)$ and the set T is the set of points $(x,y) \in \mathbb{R}^{2n}$ such that $x = y$. Then, we thrive for a contradiction by assuming that the error is bigger in $(B_1 \times B_1) \setminus T$.

– As a final step, we get a contradiction by using a multidimensional dynamic programming principle for the comparison function f. In the proof below, intuition based on suitable strategies is helpful even though we are not using stochastic arguments.

Proof of Theorem 3.2. By using a scaling $x \mapsto Rx$, we can assume that $R = 1$. In addition by translation, it is enough to consider the claim (3.8) in the case $y = -x$. For simplicity, we assume $\sup_{B_2\times B_2}(u(x) - u(y)) \leq 1$.

Given $C > 1$, let $N \in \mathbb{N}$ be such that

$$N \geq \frac{100C}{\gamma}.$$

Then, we define the following functions in $\mathbb{R}^{2n}$

$$\begin{aligned} f_1(x,y) &= C\,|x-y|^{\gamma} + |x+y|^2\,, \\ f_2(x,y) &= \begin{cases} C^{2(N-i)}\varepsilon^{\gamma} & \text{if } (x,y) \in A_i, \\ 0 & \text{if } |x-y| > N\frac{\varepsilon}{10}, \end{cases} \\ f(x,y) &= f_1(x,y) - f_2(x,y), \end{aligned}$$

with $A_i = \left\{(x,y) \in \mathbb{R}^{2n} \ : \ (i-1)\frac{\varepsilon}{10} < |x-y| \leq i\frac{\varepsilon}{10}\right\}$ for $i = 0, 1, \dots, N$. The function f_2 is called an *annular step function* and it is needed to control the small scale jumps. Note that we have $\sup f_2 = C^{2N}\varepsilon^{\gamma}$ reached on

$$T := A_0 = \left\{(x,y) \in \mathbb{R}^{2n} \ : \ x = y\right\}.$$

It holds that $f_1 \geq 1$ in $(B_2 \times B_2) \setminus (B_1 \times B_1)$. Here, we need the term $|x+y|^2$ in the function f_1, because

$$|x+y|^2 = 2\,|x|^2 + 2\,|y|^2 - |x-y|^2 \geq 3$$

for all $x, y \in (B_2 \times B_2) \setminus (B_1 \times B_1)$ such that $|x-y| \leq 1$. Therefore, together with $u(x) - u(y) \leq 1$ in $B_2 \times B_2$ and $u(x) - u(y) = 0$ in T, we have

$$u(x) - u(y) - f(x,y) \leq \sup f_2 = C^{2N}\varepsilon^{\gamma}, \tag{3.11}$$

if $(x,y) \in T$ or $(x,y) \in (B_2 \times B_2) \setminus (B_1 \times B_1)$. We have to show that this inequality is also true in $(B_1 \times B_1) \setminus T$. Thriving for a contradiction, write

$$M := \sup_{(x,y) \in B_1 \times B_1 \setminus T} (u(x) - u(y) - f(x,y))$$

and suppose that $M > C^{2N}\varepsilon^{\gamma}$. By (3.11), this is equivalent to

$$M = \sup_{(x,y) \in B_2 \times B_2} (u(x) - u(y) - f(x,y)). \tag{3.12}$$

For all $\eta > 0$, we choose a pair of points $(x,y) \in (B_1 \times B_1) \setminus T$ such that

$$M \leq u(x) - u(y) - f(x,y) + \frac{\eta}{2}. \tag{3.13}$$

Then by (3.9), we have

$$u(x) - u(y) \leq \frac{1}{2} \sup_{\nu_x, \nu_y} (\mathcal{A}u(x,\nu_x) - \mathcal{A}u(y,\nu_y)) + \frac{1}{2} \inf_{\nu_x, \nu_y} (\mathcal{A}u(x,\nu_x) - \mathcal{A}u(y,\nu_y)), \tag{3.14}$$

where $\mathcal{A}u$ is the auxiliary function defined in (3.5).

Given $|\nu_x| = |\nu_y| = \varepsilon$, let $P_{\nu_y,-\nu_x}$ denote any rotation that sends ν_y to $-\nu_x$. By recalling $\alpha(x)+\beta(x)=1$ for $x\in\Omega$, we can decompose the difference $\mathcal{A}u(x,\nu_x)-\mathcal{A}u(y,\nu_y)$. For simplicity, we may assume that $\alpha(x)\ge\alpha(y)$. Thus, we get

$$
\begin{aligned}
\mathcal{A}u(x,\nu_x)-\mathcal{A}u(y,\nu_y) &= \alpha(y)\left[u(x+\nu_x)-u(y+\nu_y)\right] \\
&\quad + \beta(x)\fint_{B_\varepsilon^{\nu_y}}\left[u(x+P_{\nu_y,-\nu_x}h)-u(y+h)\right]\,d\mathcal{L}^{n-1}(h) \\
&\quad + (\alpha(x)-\alpha(y))\left[u(x+\nu_x)-\fint_{B_\varepsilon^{\nu_y}} u(y+h)\,d\mathcal{L}^{n-1}(h)\right].
\end{aligned}
\tag{3.15}
$$

Next, we use the counter assumption (3.12) to estimate each of the terms in (3.15) from above. Consequently, we can estimate

$$
u(x)-u(y)\le M+f(x,y)
$$

for all $x,y\in B_2$. Then, we define

$$
\begin{aligned}
G(f,x,y,\nu_x,\nu_y) := {}& \alpha(y)f(x+\nu_x,y+\nu_y) \\
&+ \beta(x)\fint_{B_\varepsilon^{\nu_y}} f(x+P_{\nu_y,-\nu_x}h,y+h)\,d\mathcal{L}^{n-1}(h) \\
&+ (\alpha(x)-\alpha(y))\fint_{B_\varepsilon^{\nu_y}} f(x+\nu_x,y+h)\,d\mathcal{L}^{n-1}(h).
\end{aligned}
\tag{3.16}
$$

Thus, we have

$$
\mathcal{A}u(x,\nu_x)-\mathcal{A}u(y,\nu_y)\le M+G(f,x,y,\nu_x,\nu_y). \tag{3.17}
$$

By taking the supremum, we obtain

$$
\sup_{\nu_x,\nu_y}\left(\mathcal{A}u(x,\nu_x)-\mathcal{A}u(y,\nu_y)\right)\le M+\sup_{\nu_x,\nu_y}G(f,x,y,\nu_x,\nu_y). \tag{3.18}
$$

On the other hand, choose $|\varrho_x|=|\varrho_y|=\varepsilon$ such that

$$
\inf_{\nu_x,\nu_y}G(f,x,y,\nu_x,\nu_y)\ge G(f,x,y,\varrho_x,\varrho_y)-\eta.
$$

This together with (3.17) yields

$$
\begin{aligned}
\inf_{\nu_x,\nu_y}\left(\mathcal{A}u(x,\nu_x)-\mathcal{A}u(y,\nu_y)\right) &\le \mathcal{A}u(x,\varrho_x)-\mathcal{A}u(y,\varrho_y) \\
&\le M+G(f,x,y,\varrho_x,\varrho_y) \\
&\le M+\inf_{\nu_x,\nu_y}G(f,x,y,\nu_x,\nu_y)+\eta.
\end{aligned}
$$

Therefore, by applying this inequality and (3.18) to (3.14) we get

$$u(x) - u(y) \le M + \frac{1}{2}\left[\sup_{\nu_x,\nu_y} G(f,x,y,\nu_x,\nu_y) + \inf_{\nu_x,\nu_y} G(f,x,y,\nu_x,\nu_y)\right] + \frac{\eta}{2}.$$

Combining this with (3.13), we need to show

$$\sup_{\nu_x,\nu_y} G(f,x,y,\nu_x,\nu_y) + \inf_{\nu_x,\nu_y} G(f,x,y,\nu_x,\nu_y) < 2f(x,y).$$

This inequality follows from Proposition 3.5 below. Consequently, the equation (3.11) holds in $B_2 \times B_2$. □

Proposition 3.5. *Let f and T be as at the beginning of the proof of Theorem 3.2, and fix $(x,y) \in B_1 \times B_1 \setminus T$. In addition, let G be as in (3.16). Then, it holds that*

$$\sup_{\nu_x,\nu_y} G(f,x,y,\nu_x,\nu_y) + \inf_{\nu_x,\nu_y} G(f,x,y,\nu_x,\nu_y) < 2f(x,y).$$

The main part of the section is to show this estimate for G. This is done in several steps below.

Proof of Proposition 3.5

Let $V \subset \mathbb{R}^n$ be the space spanned by $x - y \neq 0$. We denote the orthogonal complement of V by $V^\perp$, i.e.,

$$V^\perp := \{z \in \mathbb{R}^n \;:\; \langle z, x-y\rangle = 0\}.$$

Given any $z \in \mathbb{R}^n$, we can decompose

$$z = z_V \frac{x-y}{|x-y|} + z_{V^\perp},$$

where $z_V \in \mathbb{R}$ is the scalar projection of z onto V and $z_{V^\perp} \in V^\perp$, respectively. For the decomposed point it holds

$$\begin{aligned} z_V &= \langle z, \frac{x-y}{|x-y|}\rangle, \\ |z_{V^\perp}| &= \sqrt{|z|^2 - z_V^2}\,. \end{aligned}$$

By using this notation, the second order Taylor's expansion of f_1 is

$$\begin{aligned} f_1(&x+h_x, y+h_y) - f_1(x,y) \\ &= C\gamma\,|x-y|^{\gamma-1}\,(h_x-h_y)_V + 2\langle x+y, h_x+h_y\rangle \\ &\quad + \frac{1}{2}C\gamma\,|x-y|^{\gamma-2}\left\{(\gamma-1)(h_x-h_y)_V^2 + |(h_x-h_y)_{V^\perp}|^2\right\} \\ &\quad + |h_x+h_y|^2 + \mathcal{E}_{x,y}(h_x,h_y), \end{aligned} \tag{3.19}$$

where $\mathcal{E}_{x,y}(h_x, h_y)$ is the error term. In the above, we used the calculations

$$\langle \nabla f_1(x,y), (h_x^\top, h_y^\top) \rangle = C\gamma \left|x-y\right|^{\gamma-2} \langle x-y, h_x - h_y \rangle + 2\langle x+y, h_x+h_y \rangle$$

and

$$\mathrm{D}^2 f_1 = \begin{bmatrix} A & -A \\ -A & A \end{bmatrix} + 2 \begin{bmatrix} \mathrm{I} & \mathrm{I} \\ \mathrm{I} & \mathrm{I} \end{bmatrix}$$

with

$$A := C\gamma \left|x-y\right|^{\gamma-2} \left\{ (\gamma-2) \frac{x-y}{|x-y|} \otimes \frac{x-y}{|x-y|} + \mathrm{I} \right\}.$$

The matrix I stands for the $n \times n$ identity matrix, and we denote the tensor product of two vectors by $\otimes$, i.e., $h \otimes s := hs^\top$ for column vectors $h, s \in \mathbb{R}^n$. By recalling the elementary formula $h^\top(s \otimes s)h = \langle h, s \rangle^2$ for all $h, s \in \mathbb{R}^n$, we get (3.19).

By Taylor's theorem, the error term satisfies

$$|\mathcal{E}_{x,y}(h_x, h_y)| \leq C\left|(h_x^\top, h_y^\top)\right|^3 (|x-y| - 2\varepsilon)^{\gamma-3},$$

if $|x-y| > 2\varepsilon$. With the choice $N \geq \frac{100C}{\gamma}$ and if $|x-y| > \frac{N}{10}\varepsilon$, we can estimate

$$\begin{aligned} |\mathcal{E}_{x,y}(h_x, h_y)| &\leq C(2\varepsilon)^3 \left(\frac{|x-y|}{2} \right)^{\gamma-3} \\ &\leq 64C\varepsilon^2 \left|x-y\right|^{\gamma-2} \frac{\varepsilon}{|x-y|} \\ &\leq 10 \left|x-y\right|^{\gamma-2} \varepsilon^2, \end{aligned} \tag{3.20}$$

because $|h_x|, |h_y| \leq \varepsilon$. Therefore, to prove the result, we distinguish two separate cases. In the first case, we have $|x-y| \leq \frac{N}{10}\varepsilon$ and in the second case, we have $|x-y| > \frac{N}{10}\varepsilon$.

Proof of Proposition 3.5: Case $|x-y| \leq N\frac{\varepsilon}{10}$

In this case, we do not utilize the formula (3.19). We use concavity and convexity estimates for the terms in f_1 and the properties of the annular step function f_2. For $x, y \in B_1$ and $|h_x|, |h_y| < \varepsilon < 1$, it holds

$$|f_1(x+h_x, y+h_y) - f_1(x,y)| \leq 2C\varepsilon^\gamma + 16\varepsilon \leq 3C\varepsilon^\gamma$$

for $C > 16$. Consequently by (3.16), we have

$$\sup_{h_x, h_y} G(f_1, x, y, h_x, h_y) \leq f_1(x,y) + 3C\varepsilon^\gamma.$$

Together with $f_2 \geq 0$, these estimates yield

$$\sup_{h_x, h_y} G(f, x, y, h_x, h_y) \leq f_1(x,y) + 3C\varepsilon^\gamma. \tag{3.21}$$

Find $i \in \{1, 2, \ldots, N\}$ such that $(i-1)\frac{\varepsilon}{10} < |x-y| \leq i\frac{\varepsilon}{10}$ and choose $|\nu_x|, |\nu_y| < \varepsilon$ such that $(x+\nu_x, y+\nu_y) \in A_{i-1}$. Then for $C > 1$ large enough, we can estimate

$$\begin{aligned}\sup_{h_x,h_y} G(f_2, x, y, h_x, h_y) &\geq G(f_2, x, y, \nu_x, \nu_y) \\ &\geq \alpha(y) f_2(x+\nu_x, y+\nu_y) \\ &= \alpha(y) C^{2(N-i+1)} \varepsilon^\gamma \\ &= \alpha(y)\left(C^2 - \frac{2}{\alpha(y)}\right) C^{2(N-i)}\varepsilon^\gamma + 2f_2(x,y) \\ &> 6C\varepsilon^\gamma + 2f_2(x,y),\end{aligned}$$

where we use $f_2 \geq 0$ in the second inequality and $\alpha(y) > \alpha_{\min} > 0$ for all $y \in \Omega$ in the last inequality. Therefore, by $f = f_1 - f_2$ and (3.21) it holds

$$\begin{aligned}\inf_{h_x,h_y} G(f, x, y, h_x, h_y) &\leq \sup_{h_x,h_y} G(f_1, x, y, h_x, h_y) - \sup_{h_x,h_y} G(f_2, x, y, h_x, h_y)) \\ &\leq f_1(x,y) - 2f_2(x,y) - 3C\varepsilon^\gamma.\end{aligned}$$

Combining this inequality with (3.21), we get

$$\sup_{h_x,h_y} G(f, x, y, h_x, h_y) + \inf_{h_x,h_y} G(f, x, y, h_x, h_y) < 2f(x,y).$$

Hence, the proof of the case is complete.

Proof of Proposition 3.5: Case $|x-y| > N\frac{\varepsilon}{10}$

In this case, $f_2(x,y) = 0$ and hence $f \equiv f_1$. We apply (3.19) to get the result. For $\eta > 0$, let ν_x, ν_y be such that

$$\sup_{h_x,h_y} G(f, x, y, h_x, h_y) \leq G(f, x, y, \nu_x, \nu_y) + \eta.$$

Therefore for any $|\varrho_x|, |\varrho_y| \leq \varepsilon$, we get the following inequality

$$\begin{aligned}&\sup_{h_x,h_y} G(f, x, y, h_x, h_y) + \inf_{h_x,h_y} G(f, x, y, h_x, h_y) \\ &\qquad\leq G(f, x, y, \nu_x, \nu_y) + G(f, x, y, \varrho_x, \varrho_y) + \eta. \qquad (3.22)\end{aligned}$$

By (3.20) and $|h_x|, |h_y| \leq \varepsilon$, the last two terms in (3.19) are bounded above by

$$(4 + 10|x-y|^{\gamma-2})\varepsilon^2.$$

We denote

$$E := E(f, x, y, \gamma, \varepsilon) := f(x,y) + (4 + 10|x-y|^{\gamma-2})\varepsilon^2,$$

and recall the notation $P_{h,s}$ denoting the rotation sending h to s for any vectors $|h| = |s|$ in $\mathbb{R}^n$. By (3.22) and (3.16), it suffices to study

$$\begin{aligned}
\textbf{[I]} :=\, & G(f,x,y,\nu_x,\nu_y) + G(f,x,y,\varrho_x,\varrho_y) - 2E \\
=\, & \alpha(y)\left[f(x+\nu_x, y+\nu_y) + f(x+\varrho_x, y+\varrho_y) - 2E\right] \\
& + \beta(x)\Bigg[\fint_{B_\varepsilon^{\nu_y}} f(x+P_{\nu_y,-\nu_x}h, y+h)\, d\mathcal{L}^{n-1}(h) \\
& \qquad + \fint_{B_\varepsilon^{\varrho_y}} f(x+P_{\varrho_y,-\varrho_x}h, y+h)\, d\mathcal{L}^{n-1}(h) - 2E\Bigg] \\
& + (\alpha(x)-\alpha(y))\Bigg[\fint_{B_\varepsilon^{\nu_y}} f(x+\nu_x, y+h)\, d\mathcal{L}^{n-1}(h) \\
& \qquad + \fint_{B_\varepsilon^{\varrho_y}} f(x+\varrho_x, y+h)\, d\mathcal{L}^{n-1}(h) - 2E\Bigg].
\end{aligned} \tag{3.23}$$

For simplicity, we decompose the previous expression into three terms to be examined separately,

$$\textbf{[I]} = \alpha(y)\textbf{[II]} + \beta(x)\textbf{[III]} + (\alpha(x)-\alpha(y))\textbf{[IV]}. \tag{3.24}$$

Then, by (3.19), we have

$$\begin{aligned}
\textbf{[II]} \leq\, & C\gamma\,|x-y|^{\gamma-1}\left[(\nu_x-\nu_y)_V + (\varrho_x-\varrho_y)_V\right] \\
& + 2\langle x+y, (\nu_x+\nu_y) + (\varrho_x+\varrho_y)\rangle \\
& + \frac{1}{2}C\gamma\,|x-y|^{\gamma-2}\Big\{(\gamma-1)\left[(\nu_x-\nu_y)_V^2 + (\varrho_x-\varrho_y)_V^2\right] \\
& \qquad + \left[\left|(\nu_x-\nu_y)_{V^\perp}\right|^2 + \left|(\varrho_x-\varrho_y)_{V^\perp}\right|^2\right]\Big\}.
\end{aligned} \tag{3.25}$$

Note that the first order terms in **[III]** vanishes when we integrate over the ball. Therefore, we can estimate

$$\begin{aligned}
\textbf{[III]} \leq\, & \frac{1}{2}C\gamma\,|x-y|^{\gamma-2} \cdot \\
& \cdot\Bigg\{\fint_{B_\varepsilon^{\nu_y}} \left[(\gamma-1)(h-P_{\nu_y,-\nu_x}h)_V^2 + \left|(h-P_{\nu_y,-\nu_x}h)_{V^\perp}\right|^2\right] d\mathcal{L}^{n-1}(h) \\
& + \fint_{B_\varepsilon^{\varrho_y}} \left[(\gamma-1)(h-P_{\varrho_y,-\varrho_x}h)_V^2 + \left|(h-P_{\varrho_y,-\varrho_x}h)_{V^\perp}\right|^2\right] d\mathcal{L}^{n-1}(h)\Bigg\}.
\end{aligned} \tag{3.26}$$

In addition, it holds

$$\begin{aligned}\textbf{[IV]} \leq C\gamma\,|x-y|^{\gamma-1}\,(\nu_x+\varrho_x)_V + 2\langle x+y, \nu_x+\varrho_x\rangle \\ + \frac{1}{2}C\gamma\,|x-y|^{\gamma-2}\cdot \\ \cdot\left\{ ⨍_{B_\varepsilon^{\nu_y}} \left[(\gamma-1)(\nu_x-h)_V^2 + |(\nu_x-h)_{V^\perp}|^2\right]\, d\mathcal{L}^{n-1}(h) \right. \\ \left. + ⨍_{B_\varepsilon^{\varrho_y}} \left[(\gamma-1)(\varrho_x-h)_V^2 + |(\varrho_x-h)_{V^\perp}|^2\right]\, d\mathcal{L}^{n-1}(h) \right\}.\end{aligned} \tag{3.27}$$

We distinguish between two cases depending on the value of $(\nu_x-\nu_y)_V^2$ and fix $\tau_0 < \tau < 1$ with $0<\tau_0<1$ to be defined later.

a) Case $|(\nu_x-\nu_y)_V| \geq (\tau+1)\varepsilon$**:** In this case, we choose $\varrho_x = -\nu_x$ and $\varrho_y=-\nu_y$. Replacing these vectors in the inequalities **[II]**, **[III]** and **[IV]** and by symmetry, we obtain

$$\begin{aligned}\textbf{[II]} &\leq C\gamma\,|x-y|^{\gamma-2}\left[(\gamma-1)(\nu_x-\nu_y)_V^2 + |(\nu_x-\nu_y)_{V^\perp}|^2\right], \\ \textbf{[III]} &\leq C\gamma\,|x-y|^{\gamma-2} ⨍_{B_\varepsilon^{\nu_y}} \left|(h-P_{\nu_y,-\nu_x}h)_{V^\perp}\right|^2\, d\mathcal{L}^{n-1}(h), \\ \textbf{[IV]} &\leq C\gamma\,|x-y|^{\gamma-2}\left[(\gamma-1)⨍_{B_\varepsilon^{\nu_y}}(\nu_x-h)_V^2\, d\mathcal{L}^{n-1}(h) + ⨍_{B_\varepsilon^{\nu_y}} |(\nu_x-h)_{V^\perp}|^2\, d\mathcal{L}^{n-1}(h)\right].\end{aligned}$$

We used $\gamma-1<0$ and the choice $P_{\nu_y,-\nu_x} = P_{-\nu_y,\nu_x}$ in the estimate for **[III]**. By assumption, it holds $(\nu_x-\nu_y)_V^2 \geq (\tau+1)^2\varepsilon^2$ implying

$$|(\nu_x-\nu_y)_{V^\perp}|^2 \leq \left[4-(\tau+1)^2\right]\varepsilon^2.$$

Thus, we need to obtain uniform bounds for the terms $(\nu_x-h)_V^2$, $|(\nu_x-h)_{V^\perp}|^2$ and $\left|(h-P_{\nu_y,-\nu_x}h)_{V^\perp}\right|^2$ for $h\in B_\varepsilon^{\nu_y}$.

The assumption $|(\nu_x-\nu_y)_V| \geq (\tau+1)\varepsilon$, together with $|\nu_x|, |\nu_y| \leq \varepsilon$ and Pythagoras' theorem, implies

$$\begin{cases} \tau\varepsilon \leq |(\nu_x)_V| \leq \varepsilon, \quad 0 \leq |(\nu_x)_{V^\perp}| \leq \sqrt{1-\tau^2}\,\varepsilon, \\ \tau\varepsilon \leq |(\nu_y)_V| \leq \varepsilon, \quad 0 \leq |(\nu_y)_{V^\perp}| \leq \sqrt{1-\tau^2}\,\varepsilon. \end{cases} \tag{3.28}$$

Moreover, the same facts yield

$$|(\nu_x+\nu_y)_V| \leq (1-\tau)\varepsilon \quad \text{and} \quad |(\nu_x+\nu_y)_{V^\perp}| \leq 2\sqrt{1-\tau^2}\,\varepsilon.$$

By combining these and using Pythagoras' theorem, we get

$$|\nu_x + \nu_y| < \sqrt{8}\,\sqrt{1-\tau}\,\varepsilon, \tag{3.29}$$

since $\tau < 1$. Let $h \in B_\varepsilon^{\nu_y}$. Then, we have

$$0 = \langle h, \nu_y \rangle = h_V(\nu_y)_V + \langle h_{V^\perp}, (\nu_y)_{V^\perp} \rangle$$

implying

$$h_V = -\frac{\langle h_{V^\perp}, (\nu_y)_{V^\perp} \rangle}{(\nu_y)_V}.$$

In addition by applying this equality together with (3.28) and $|h_{V^\perp}| \leq |h| \leq \varepsilon$, we obtain

$$|h_V| \leq \frac{\varepsilon}{\tau}\sqrt{1-\tau^2}.$$

Consequently, we get the estimates

$$(\nu_x - h)_V \geq \left(\tau - \frac{\sqrt{1-\tau^2}}{\tau}\right)\varepsilon \tag{3.30}$$

and

$$|(\nu_x - h)_{V^\perp}| \leq |(\nu_x)_{V^\perp}| + |h_{V^\perp}| \leq \left(1 + \sqrt{1-\tau^2}\right)\varepsilon. \tag{3.31}$$

We can assume that τ_0 is close enough to 1 guaranteeing the positivity of the quantity $\tau - \tau^{-1}\sqrt{1-\tau^2}$. In order to obtain the last estimate needed, we recall that $P_{\nu_y,-\nu_x}$ is any rotation sending the vector ν_y to $-\nu_x$. In particular, we choose a rotation satisfying

$$\left|h - P_{\nu_y,-\nu_x}h\right| \leq \left|\nu_y - P_{\nu_y,-\nu_x}\nu_y\right| = |\nu_y + \nu_x|$$

for every $|h| \leq \varepsilon$. Hence by recalling (3.29), we get

$$\left|(h - P_{\nu_y,-\nu_x}h)_{V^\perp}\right|^2 \leq 8(1-\tau)\varepsilon^2. \tag{3.32}$$

By replacing the estimates (3.30), (3.31) and (3.32) in **[II]**, **[III]** and **[IV]**, we can calculate

$$\begin{aligned}
\textbf{[II]} &\leq C\gamma\,|x-y|^{\gamma-2}\,\varepsilon^2\left[(\gamma-1)(\tau+1)^2 + 4 - (\tau+1)^2\right],\\
\textbf{[III]} &\leq C\gamma\,|x-y|^{\gamma-2}\,\varepsilon^2\left[8(1-\tau)\right],\\
\textbf{[IV]} &\leq C\gamma\,|x-y|^{\gamma-2}\,\varepsilon^2\left[(\gamma-1)\left(\tau - \frac{\sqrt{1-\tau^2}}{\tau}\right)^2 + \left(1+\sqrt{1-\tau^2}\right)^2\right].
\end{aligned}$$

In addition by (3.24), we get

$$\textbf{[I]} \leq \textbf{[V]} \cdot C\gamma\,|x-y|^{\gamma-2}\,\varepsilon^2,$$

where **[V]** is equal to

$$(\gamma-1)\left[\alpha(y)(\tau+1)^2+(\alpha(x)-\alpha(y))\left(\tau-\frac{\sqrt{1-\tau^2}}{\tau}\right)^2\right] \\ +(\alpha(x)-\alpha(y))(1+\sqrt{1-\tau^2}\,)^2+\alpha(y)\left[(4-(\tau+1)^2)\right]+\beta(x)\,8(1-\tau).$$

The assumption on γ in (3.7) implies that we can choose $\tau_0 := \tau_0(\kappa) < 1$ close enough to 1 such that the previous expression is negative, i.e.,

$$\begin{aligned}\textbf{[V]} &< (\gamma-1)\big(4\alpha(y)+\alpha(x)-\alpha(y)\big)+\alpha(x)-\alpha(y)+\kappa\alpha_{\max}\\ &< 4\big(\gamma\alpha_{\max}-\alpha_{\min}\big)+\kappa\alpha_{\max}\\ &< 0.\end{aligned}$$

Now, by recalling (3.23), we have

$$G(f,x,y,\nu_x,\nu_y)+G(f,x,y,\omega_x,\omega_y)-2f(x,y) \\ \leq 8\varepsilon^2+(20+\textbf{[V]}\cdot C\gamma)\,|x-y|^{\gamma-2}\,\varepsilon^2.$$

By choosing $C>1$ large enough, we obtain

$$(20+\textbf{[V]}\cdot C\gamma)\,|x-y|^{\gamma-2}\,\varepsilon^2 < -10^8\,|x-y|^{\gamma-2}\,\varepsilon^2 < -10^7\varepsilon^2.$$

This estimate yields

$$G(f,x,y,\nu_x,\nu_y)+G(f,x,y,\omega_x,\omega_y)-2f(x,y) < 0.$$

b) Case $|(\nu_x-\nu_y)_V| \leq (\tau+1)\varepsilon$**:** In this case, the first order terms in (3.19) imply the result. By choosing $\varrho_x = -\varepsilon\frac{x-y}{|x-y|}$ and $\varrho_y = \varepsilon\frac{x-y}{|x-y|}$ in V and utilizing these in (3.25), (3.26) and (3.27), we get

$$\begin{aligned}\textbf{[II]} &\leq C\gamma\,|x-y|^{\gamma-1}\,[(\nu_x-\nu_y)_V-2\varepsilon]+2\langle x+y,\nu_x+\nu_y\rangle\\ &\quad+\frac{1}{2}C\gamma\,|x-y|^{\gamma-2}\left\{(\gamma-1)\left[(\nu_x-\nu_y)_V^2+4\varepsilon^2\right]+|(\nu_x-\nu_y)_{V^\perp}|^2\right\},\end{aligned}$$

$$\begin{aligned}\textbf{[III]} &\leq \frac{1}{2}C\gamma\,|x-y|^{\gamma-2}\cdot\\ &\quad\cdot\left\{\fint_{B_\varepsilon^{\nu_y}}\left[(\gamma-1)(h-P_{\nu_y,-\nu_x}h)_V^2+\left|(h-P_{\nu_y,-\nu_x}h)_{V^\perp}\right|^2\right]\,d\mathcal{L}^{n-1}(h)\right\}\end{aligned}$$

and

$$\textbf{[IV]} \leq C\gamma\,|x-y|^{\gamma-1}\,[(\nu_x)_V-\varepsilon]+2\langle x+y,\nu_x-\varepsilon\frac{x-y}{|x-y|}\rangle$$
$$+\frac{1}{2}C\gamma\,|x-y|^{\gamma-2}\cdot\left\{\fint_{B_\varepsilon^{\nu_y}}\left[(\gamma-1)(\nu_x-h)_V^2+|(\nu_x-h)_{V^\perp}|^2\right]\,d\mathcal{L}^{n-1}(h)\right.$$
$$\left.+(\gamma-1)\varepsilon^2+\fint_{B_\varepsilon^{x-y}}|h|^2\,d\mathcal{L}^{n-1}(h)\right\}.$$

The second order terms in these inequalities can be estimated above by

$$3C\gamma\,|x-y|^{\gamma-2}\,\varepsilon^2.$$

In addition, we deduce that $(\nu_x-\nu_y)_V<\left[1+\left(\frac{\tau+1}{2}\right)^2\right]\varepsilon$. Therefore, we have

$$\textbf{[II]} \leq C\gamma\,|x-y|^{\gamma-1}\left[\left(\frac{\tau+1}{2}\right)^2-1\right]\varepsilon+4\,|x+y|\,\varepsilon+3C\gamma\,|x-y|^{\gamma-2}\,\varepsilon^2,$$
$$\textbf{[III]} \leq 3C\gamma\,|x-y|^{\gamma-2}\,\varepsilon^2,$$
$$\textbf{[IV]} \leq 4\,|x+y|\,\varepsilon+3C\gamma\,|x-y|^{\gamma-2}\,\varepsilon^2.$$

By combining all these and recalling (3.23) and (3.24), we get

$$\begin{aligned}
&G(f,x,y,\nu_x,\nu_y)+G(f,x,y,\varrho_x,\varrho_y)-2f(x,y)\\
&\qquad\leq C\alpha(y)\gamma\,|x-y|^{\gamma-1}\left[\left(\frac{\tau+1}{2}\right)^2-1\right]\varepsilon+4\alpha(x)\,|x+y|\,\varepsilon+8\varepsilon^2\\
&\qquad\quad+(20+3C\gamma)\,|x-y|^{\gamma-2}\,\varepsilon^2\\
&\qquad\leq C\alpha(y)\gamma\,|x-y|^{\gamma-1}\left[\left(\frac{\tau+1}{2}\right)^2-1\right]\varepsilon+8\varepsilon^2+\left(\gamma+\frac{2}{C}\right)\gamma\,|x-y|^{\gamma-1}\,\varepsilon\\
&\qquad\leq \gamma\,|x-y|^{\gamma-1}\,\varepsilon\left\{\gamma+1+C\alpha_{\min}\left[\left(\frac{\tau+1}{2}\right)^2-1\right]\right\}+8\varepsilon^2.
\end{aligned}$$

As in the previous case, we can choose the constant $C>1$ large enough to ensure the negativity of the previous equation. Thus, the proof is complete.

3.3 Regularity near the boundary

In this section, provided some regularity on the boundary of the set, we show that the value function of the game is also asymptotically continuous near the boundary. The proof is based on finding a suitable barrier function and a strategy for the other player so that the process under the barrier function is a super- or submartingale depending on the form of the function. Then, the result follows by analyzing the barrier function and iterating the argument.

Lemma 3.6. *Let $r > 0$ and $y \in \mathbb{R}^n$ and let $v(x) = a\,|x-y|^\sigma + b$, for each $x \in \mathbb{R}^n$ such that $|x-y| > r$, where $\sigma < 0$, $a < 0$ and $b \geq 0$. Then, there is a constant $C > 0$ such that*

$$\sup_{|\nu|=\varepsilon} \mathcal{A}v(x,\nu) \leq \mathcal{A}v\left(x, \varepsilon\,\frac{x-y}{|x-y|}\right) + C\varepsilon^3 \tag{3.33}$$

and

$$\mathcal{A}v\left(x, \varepsilon\,\frac{x-y}{|x-y|}\right) + \mathcal{A}v\left(x, -\varepsilon\,\frac{x-y}{|x-y|}\right) < 2v(x) + a\sigma\,|x-y|^{\sigma-2}\,(\alpha(x)(\sigma-1)+\beta(x))\,\varepsilon^2 + C\varepsilon^3 \tag{3.34}$$

for all $\varepsilon > 0$ and $x \in \mathbb{R}^n$ such that $|x-y| > r$, where $\mathcal{A}v$ stands for the auxiliary function defined in (3.5).

Proof. Since v is real-analytic in $\mathbb{R}^n \setminus \overline{B}(y,r)$, we can write the second order Taylor's expansion of v in a neighborhood of any $x \in \mathbb{R}^n$ such that $|x-y| > r$,

$$v(x+h) = v(x) + \frac{1}{2}a\sigma\,|x-y|^{\sigma-2}\left(2\langle x-y,h\rangle + |h|^2 + (\sigma-2)\frac{\langle x-y,h\rangle^2}{|x-y|^2}\right) + \mathcal{O}(|h|^3)$$

as $h \to 0$. Thus, for any given$|\nu| = \varepsilon$,

$$v(x+\nu) = v(x) + \frac{1}{2}a\sigma\,|x-y|^{\sigma-2}\left(2\langle x-y,\nu\rangle + \varepsilon^2 + (\sigma-2)\frac{\langle x-y,\nu\rangle^2}{|x-y|^2}\right) + \mathcal{O}(\varepsilon^3),$$

and averaging over B_ε^ν the first order term vanishes and we get

$$\begin{aligned} &⨍_{B_\varepsilon^\nu} v(x+h)\,d\mathcal{L}^{n-1}(h) \\ &\qquad = v(x) + \frac{1}{2}a\sigma\,|x-y|^{\sigma-2}\left(\frac{n-1}{n+1}\,\varepsilon^2 + (\sigma-2)⨍_{B_\varepsilon^\nu}\frac{\langle x-y,h\rangle^2}{|x-y|^2}\,d\mathcal{L}^{n-1}(h)\right) + \mathcal{O}(\varepsilon^3), \end{aligned}$$

where the identity

$$⨍_{B_\varepsilon^\nu} |h|^2\,d\mathcal{L}^{n-1}(h) = \frac{n-1}{n+1}$$

has been used above. Since $a\sigma(\sigma-2) < 0$, we can estimate

$$⨍_{B_\varepsilon^\nu} v(x+h)\,d\mathcal{L}^{n-1}(h) \leq v(x) + \frac{1}{2}a\sigma\,|x-y|^{\sigma-2}\,\frac{n-1}{n+1}\,\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

Note that the equality is reached when $\nu = \pm\varepsilon\,\frac{x-y}{|x-y|}$. Recalling (3.5),

$$\mathcal{A}v\,(x,\nu) \leq v(x) + \frac{1}{2}a\sigma\,|x-y|^{\sigma-2}\left[\,2\alpha(x)\langle x-y,\nu\rangle + \left(\alpha(x)+\beta(x)\frac{n-1}{n+1}\right)\varepsilon^2 \right.$$
$$\left. + \,\alpha(x)(\sigma-2)\frac{\langle x-y,\nu\rangle^2}{|x-y|^2}\right] + \mathcal{O}(\varepsilon^3).$$

Thus, Taylor's formula proves the equation (3.33).

Next, we show (3.34). Replacing $h = \pm\varepsilon\,\frac{x-y}{|x-y|}$ in the Taylor's expansion of $v(x+h)$ we get

$$v\left(x \pm \varepsilon\,\frac{x-y}{|x-y|}\right) = v(x) + \frac{1}{2}a\sigma\,|x-y|^{\sigma-2}\left(\pm 2\,|x-y|\,\varepsilon + (\sigma-1)\varepsilon^2\right) + \mathcal{O}(\varepsilon^3).$$

On the other hand, since B_ε^ν denotes exactly the same set for each $\nu \in \operatorname{span}\{x-y\}$, then $\langle x-y,h\rangle = 0$ for every $h \in B_\varepsilon^{x-y}$ and

$$⨍_{B_\varepsilon^{x-y}} v(x+h)\,d\mathcal{L}^{n-1}(h) = v(x) + \frac{1}{2}a\sigma\,|x-y|^{\sigma-2}\,\frac{n-1}{n+1}\,\varepsilon^2 + \mathcal{O}(\varepsilon^3).$$

By (3.5) we obtain

$$\mathcal{A}v\left(x, \pm\varepsilon\,\frac{x-y}{|x-y|}\right)$$
$$= v(x) + \frac{1}{2}a\sigma\,|x-y|^{\sigma-2}\left[\pm 2\alpha(x)\,|x-y|\,\varepsilon + \left(\alpha(x)(\sigma-1)+\beta(x)\frac{n-1}{n+1}\right)\varepsilon^2\right] + \mathcal{O}(\varepsilon^3).$$

Therefore, since the coefficient of the $\beta(x)$ term is positive and strictly less than 1, (3.34) follows for a large enough $C > 0$. □

Lemma 3.7. *Given a bounded domain $\Omega \subset \mathbb{R}^n$ satisfying the boundary regularity condition with some constants $r_0, s \in (0,1)$, for $z \in \partial\Omega$ and $r \in (0,r_0]$, let $B(y,sr) \subset B(z,r)\setminus\Omega$. If u is the solution of (3.4) with $0 < \varepsilon < \varepsilon_0 < 1$ and continuous boundary data $F \in C(\Gamma_\varepsilon)$, then for each $\eta > 0$ there exists $k \in \mathbb{N}$ such that*

$$\left|u(x_0) - \sup_{B(y,4r)\cap\Gamma_\varepsilon} F\right| < \eta$$

for each $0 < \varepsilon < \varepsilon_0$ and each $x_0 \in B(z,4^{1-k}r)\cap\overline{\Omega}_\varepsilon$.

Proof. We only show that

$$u(x_0) - \sup_{B(y,4r)\cap\Gamma_\varepsilon} F < \eta, \tag{3.35}$$

since the other case can be shown following an analogous argument. The idea is to find a suitable barrier function so that by Lemma 3.6, if P_{I} pulls towards the point $y \in B(z,r)\setminus\Omega$, the game process inside the barrier function is a supermartingale. Then, recalling the properties of the barrier function, we get the result by iteration.

Let $s \in (0,1)$ given by the boundary regularity condition and fix

$$\theta = \frac{s^\sigma - 2^\sigma}{s^\sigma - 4^\sigma} \in (0,1)$$

where

$$\sigma = -2\,\frac{\beta_{\max}}{\alpha_{\min}} < 0. \tag{3.36}$$

We can assume without loss of generality that F is continuous in Γ_1 (otherwise, we just consider any continuous extension of F to Γ_1). We choose a large enough $k \in \mathbb{N}$ (depending on $\eta > 0$) in such a way that the oscillation of F on Γ_1 is controlled as follows,

$$\theta^k \left(\sup_{\Gamma_1} F - \inf_{\Gamma_1} F \right) < \eta.$$

In particular, since $b_U := \sup_{\Gamma_\varepsilon} F \leq \sup_{\Gamma_1} F$ and $b_{4r} := \sup_{B(y,4r)\cap\Gamma_\varepsilon} F \geq \inf_{\Gamma_1} F$, we have

$$\theta^k \left(b_U - b_{4r}\right) < \eta. \tag{3.37}$$

Next, we define the function

$$v_k(x) = a\,|x-y|^\sigma + b$$

in the annulus $4^{1-k}sr < |x-y| < 4^{2-k}$, where the constants $a \leq 0$ and $b \geq 0$ are chosen such that

$$v_k(x) = \begin{cases} b_{4r} + \theta^{k-1}\left(b_U - b_{4r}\right) & \text{if } \ |x-y| = 4^{2-k}r, \\ b_{4r} & \text{if } \ |x-y| = 4^{1-k}sr. \end{cases} \tag{3.38}$$

If $b_U = b_{4r}$, it turns out that $a = 0$ and the proof is clear. Otherwise, suppose that $b_U > b_{4r}$, then $a < 0$. We extend the function v_k to the set $\mathbb{R}^n \setminus \overline{B}(y, 4^{1-k}sr - 2\varepsilon)$.

Note that the boundary regularity condition with $4^{1-k}r$ instead of r implies the following inclusion,

$$\Omega \cap B(z, 4^{1-k}r) \subset \Omega \cap B(y, 2\cdot 4^{1-k}r) \setminus \overline{B}(y, 4^{1-k}sr).$$

Thus, considering any $x_0 \in B(z, 4^{1-k}r) \cap \Omega$, then

$$x_0 \in \Omega \cap B(y, 2\cdot 4^{1-k}r) \setminus \overline{B}(y, 4^{1-k}sr).$$

By using the boundary values (3.38), we can choose the constants a and b in the function v_k such that

$$v_k(x_0) \leq b_{4r} + \theta^k\left(b_U - b_{4r}\right). \tag{3.39}$$

Thus, we just need to show that $u(x_0) \leq v(x_0)$. For that purpose, consider the game starting from x_0. We construct a strategy S_{II}^* for P_{II} as follows: if P_{II} wins the m-th toss, then the game is played by pulling towards the point y, that is, the displacement of the game token will be equal to $-\varepsilon\frac{x_m-y}{|x_m-y|}$. Also, fix any other strategy for P_{I} and denote it by S_{I}. By using (3.33) and (3.34) from Lemma 3.6 we can estimate

$$
\begin{aligned}
&\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}^*}\left[v_k(x_{m+1}) \,|\, x_0,\dots,x_m\right] \\
&\qquad \leq \frac{1-\delta(x_m)}{2}\left[\sup_{|\nu|=\varepsilon}\mathcal{A}v_k(x_m,\nu)+\mathcal{A}v_k\left(x_m,-\varepsilon\frac{x_m-y}{|x_m-y|}\right)\right] \\
&\qquad\quad +\delta(x_m)F(x_m) \\
&\qquad \leq \frac{1-\delta(x_m)}{2}\left[2v_k(x_m)+a\sigma\left|x_m-y\right|^{\sigma-2}\left(\alpha(x_m)(\sigma-1)+\beta(x_m)\right)\varepsilon^2+2C\varepsilon^3\right] \\
&\qquad\quad +\delta(x_m)F(x_m)
\end{aligned}
$$

for each $m\in\mathbb{N}$ and some large enough $C>0$. Note that, by the choice of the exponent σ in (3.36), $\alpha(x_m)(\sigma-1)+\beta(x_m)<-1$. In addition, we have

$$
a\sigma\left|x_m-y\right|^{\sigma-2} > a\sigma\left(\operatorname{diam}\Omega+1\right)^{\sigma-2} > 0.
$$

Thus, by choosing $\varepsilon_0 := \varepsilon_0(\alpha_{\min},r,\Omega,k)>0$ small enough, we can ensure that

$$
\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}^*}\left[v_k(x_{m+1}) \,|\, x_0,\dots,x_m\right] \leq \left(1-\delta(x_m)\right)v_k(x_m)+\delta(x_m)F(x_m)\leq v_k(x_m)
$$

for all $\varepsilon<\varepsilon_0$. In consequence, the process $M_m := v_k(x_m)$ is a supermartingale when P_{II} uses the strategy S_{II}^* and P_{I} uses any strategy S_{I}.

Define a boundary function $F_{v_k}:\Gamma_\varepsilon\to\mathbb{R}$ such that

$$
F_{v_k} = v_k\big|_{\Gamma_\varepsilon}.
$$

By Theorem 3.1, we have $u=u_{\mathrm{I}}$ where u_{I} is the value function for P_{I}, (3.3). Since $F\leq F_{v_k}$, $(M_m)_{m=1}^\infty$ is a supermartingale, F_{v_k} is bounded and $\tau<\infty$ almost surely, we can estimate with the help of the optimal stopping theorem

$$
\begin{aligned}
u(x_0) &= \sup_{S_{\mathrm{I}}}\inf_{S_{\mathrm{II}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}}\left[F(x_\tau)\right]\leq\sup_{S_{\mathrm{I}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}^*}\left[F(x_\tau)\right] \\
&\leq \sup_{S_{\mathrm{I}}}\mathbb{E}^{x_0}_{S_{\mathrm{I}},S_{\mathrm{II}}^*}\left[F_{v_k}(x_\tau)\right]\leq v_k(x_0).
\end{aligned}
$$

Hence, by (3.37) and (3.39), we finally get (3.35). □

Proof of Theorem 3.3. As in the proof of Lemma 3.7, we only need

$$u(x_0) - F(z) < \eta$$

and the other case follows by an analogous argument. First, assume that $z \in \partial\Omega$. Lemma 3.7 implies that, for any $r \in (0, r_0]$, there are constants $k \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that

$$u(x_0) - \sup_{B(y,4r)\cap\Gamma_\varepsilon} F < \frac{\eta}{10}$$

for all $\varepsilon < \varepsilon_0$ and $x_0 \in B(z, 4^{1-k}r) \cap \overline{\Omega}_\varepsilon$. Let $z^* \in \overline{B}(y, 4r) \cap \Gamma_\varepsilon$ be such that

$$\sup_{B(y,4r)\cap\Gamma_\varepsilon} F < F(z^*) + \frac{\eta}{10}.$$

The boundary function F is continuous on the compact set Γ_ε, so there is a modulus of continuity $\omega_F = \omega_{F,\Gamma_\varepsilon}$ for the function F. Thus, we can estimate

$$u(x_0) - F(z) = [u(x_0) - F(z^*)] + [F(z^*) - F(z)] < \frac{\eta}{5} + \omega_F(|z^* - z|).$$

By the boundary regularity condition, y is some point in $B(z, r) \setminus \Omega$. Therefore, this together with $z^* \in \overline{B}(y, 4r) \cap \Gamma_\varepsilon$ implies $|z^* - z| < 5r$. Thus, we choose small enough $\overline{r} > 0$ such that

$$\omega_F(5r) < \frac{\eta}{10}$$

holds for each $r \in (0, \overline{r}]$. This yields that, for any $r < \overline{r}$,

$$u(x_0) - F(z) < \frac{\eta}{2}$$

for each $0 < \varepsilon < \varepsilon_0$ and $x_0 \in B(z, 4^{1-k}r) \cap \overline{\Omega}_\varepsilon$.

Next, assume that $z \notin \partial\Omega$ and pick a point $z_b \in \partial\Omega$ such that $z \in B(z_b, \varepsilon)$. We choose small enough $\varepsilon_0 > 0$ so that

$$\omega_F(\varepsilon_0) < \frac{\eta}{2}.$$

This implies that

$$|F(z) - F(z_b)| \leq \omega_F(|z - z_b|) < \frac{\eta}{2}$$

for each $0 < \varepsilon < \varepsilon_0$. Since $z_b \in \partial\Omega$, we can use the estimates above to get the result. $\square$

Open problems and further remarks

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < p < \infty$. In Chapter 1 we recalled that if u is a continuous function in Ω such that the asymptotic expansion

$$\frac{p-2}{n+p} \cdot \frac{1}{2}\left(\sup_{B(x,r)} u + \inf_{B(x,r)} u \right) + \frac{n+2}{n+p} \fint_{B(x,r)} u \, d\mathcal{L} = u(x) + o(r^2) \tag{14}$$

holds as $r \to 0$ at each $x \in \Omega$, then u is p-harmonic (see [MPR1]). In Theorem 1.2 we showed that the converse is also true when $n = 2$. However, for $n \geq 3$ it is not known if p-harmonic functions satisfy (14) or not, and we have to understand the asymptotic expansion in a viscosity sense in order to obtain a result in this direction (see [MPR1] together with [JLM]). This sets out the following problem.

Problem. If u is a p-harmonic function in $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, does the asymptotic expansion (14) hold as $r \to 0$ at each $x \in \Omega$?

As we mentioned, the main difference between the planar case and the higher dimensional case is the complex structure available when $n = 2$. In particular, in our proof of Theorem 1.2, we recalled that the complex gradient $f = \partial u$ of a planar p-harmonic function u is a quasiregular mapping satisfying the Beltrami equation

$$\overline{\partial} f = \frac{2-p}{2p} \left[\frac{\overline{f}}{f} \, \partial f + \frac{f}{\overline{f}} \, \overline{\partial f} \right] \tag{15}$$

(see [BI]). As an immediate consequence of this, the principle of *Unique Continuation* follows for p-harmonic functions in the plane. Also, making use of this technique, Manfredi was able to prove the *Strong Comparison Principle* in two dimensions (see [Man]). It is noteworthy to mention that this two principles are not know in higher dimensions. But the most valuable example of the use of equation (15) can be found in [IM]. In this article, Iwaniec and Manfredi obtained the sharpest Hölder exponent $\gamma = \gamma(p)$ for the gradients of p-harmonic functions. This improved the previous regularity results by Ural'tseva and Lewis when $n = 2$ ([Ura, Lew]). Concerning the $p = \infty$ case, it has been recently proved by Evans and Savin that the gradients of ∞-harmonic functions in the plane are locally Hölder continuous for some non-explicit exponent ([ESa]). It is easy to check that the exponents $\gamma(p)$ converge to $\frac{1}{3}$ when $p \to \infty$. However, the estimates obtained in [IM] do not allow to ensure that this is the optimal exponent for the regularity of the gradients of ∞-harmonic functions. On the other hand, the existence of ∞-harmonic functions in that class of regularity and the lack of counterexamples lead us to recall the following conjecture, which also sets out a challenging problem to be faced in a future research:

Conjecture. Planar ∞-harmonic functions are of class $C^{1,\frac{1}{3}}_{\text{loc}}$.

The question of the regularity of p-harmonic functions is one of the major topics in the analysis of PDE's, and some partial results have been obtained in the past few decades, like the results by Ural'tseva and Lewis mentioned above. For $p = \infty$, the problem is much harder and the techniques employed are different from the finite p case. In the general n-dimensional case, Evans and Smart showed in [ESm] that ∞-harmonic functions are everywhere differentiable, but the following question remains unanswered:

Problem. Are n-dimensional ∞-harmonic functions of class $C^{1,\gamma}_{\text{loc}}$ for some $\gamma > 0$ when $n \geq 3$? If so, which is the explicit exponent $\gamma = \gamma(n)$?

Most of the higher dimensional problems that have been mentioned would definitely require the development of new techniques. For that reason, in the recent years, the restricted mean value properties related to the p-laplacian and the p-harmonious functions have received an increasing attention. In Chapter 2 of this dissertation we have presented regularity results for p-harmonious functions in the context of a metric measure space $(\mathbb{X}, d, \mu)$, that is, continuous functions u satisfying

$$\mathcal{T}_\alpha u = u, \tag{16}$$

for $|\alpha| < 1$, where $\mathcal{T}_\alpha$ is the operator defined in (2.4). However, we do not cover the case $\alpha = 1$ which is related to ∞-harmonious functions. The first approach to this particular case was due to Le Gruyer and Archer ([LA]) where they showed existence and uniqueness to the Dirichlet problem for functions satisfying (16) with $\alpha = 1$. Thus, as far as the author knowledge, the following question remains unanswered.

Problem. Let $(\mathbb{X}, d)$ be a geodesic metric space and $\Omega \subset \mathbb{X}$ a bounded domain. Let ϱ be an admissible radius function in Ω and $u \in C(\Omega)$ a function satisfying (16) with $\alpha = 1$. What can we say about the regularity of u?

Moreover, in Section 2.5 we have solved the Dirichlet problem for p-harmonious functions when $p \geq 2$ (that is, $\alpha \in [0, 1)$) in strictly convex domains of $\mathbb{R}^n$ with the Lebesgue measure (Theorem 2.29). A natural question to ask is if we can drop the strict convexity of the domain $\Omega \subset \mathbb{R}^n$ from the assumptions in Theorem 2.29 and still have existence of solutions. On the other hand, it turns out that our method cannot be extended for the case $1 < p < 2$ (corresponding with negative values of the coefficient α in (16)) since we need to require the non-expansiveness of the operator $\mathcal{T}_\alpha$. Furthermore, the question of existence of generalized p-harmonious functions in $(\mathbb{X}, d, \mu)$ is pretty much open. In particular, by Theorem 2.33, it turns out that we just need to provide boundary equicontinuity in order to obtain existence of the Dirichlet problem

$$\begin{cases} \mathcal{T}_\alpha u = u & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{17}$$

for any given $f \in C(\partial\Omega)$.

Problem. Let $(\mathbb{X}, d, \mu)$ be a proper geodesic metric measure space and $|\alpha| < 1$. Suppose that $\Omega \subset \mathbb{X}$ is a bounded domain and ϱ an admissible radius function in Ω. Then, under which conditions on $\mathbb{X}$, d, μ, Ω, α and ϱ can we ensure that the sequence of iterates $\{\mathcal{T}_\alpha^k u\}_k$ is equicontinuous on $\partial\Omega$ for any $u \in C(\overline{\Omega})$? In particular, if $\mathbb{X} = \mathbb{R}^n$ with d the euclidean distance and $\mu = \mathcal{L}$ the Lebesgue measure, does the problem (17) has a continuous solution for more general domains? Moreover, what if $\alpha \in (-1, 0)$?

On the other hand, as we have seen in this memory, the restricted mean value property appears also in the Tug-of-war games. In Chapter 3 we have shown that the (unique) solution $u_\varepsilon \in C(\Omega)$ to

$$\begin{aligned} u_\varepsilon(x) = \frac{1-\delta(x)}{2} \Bigg[& \sup_{|\nu|=\varepsilon} \left(\alpha(x) u_\varepsilon(x+\nu) + (1-\alpha(x)) \fint_{B_\varepsilon^\nu} u_\varepsilon(x+h)\, d\mathcal{L}^{n-1}(h) \right) \\ & + \inf_{|\nu|=\varepsilon} \left(\alpha(x) u_\varepsilon(x+\nu) + (1-\alpha(x)) \fint_{B_\varepsilon^\nu} u_\varepsilon(x+h)\, d\mathcal{L}^{n-1}(h) \right) \Bigg] \\ & + \delta(x) F(x), \end{aligned}$$

is asymptotically γ-Hölder continuous for some $\gamma \in (0, 1)$. Moreover, by taking a subsequence, it turns out that $\{u_\varepsilon\}_{\varepsilon > 0}$ converges uniformly to a viscosity solution of

$$\begin{cases} \Delta_{p(x)}^{\mathrm{N}} u(x) = 0 & \text{for } x \in \Omega, \\ u(x) = F(x) & \text{for } x \in \partial\Omega, \end{cases} \tag{18}$$

when $\varepsilon \to 0$ for some continuous $p : \overline{\Omega} \to (1, \infty)$ depending on $\alpha(x)$. Thus, $u \in C^{0,\gamma}_{\mathrm{loc}}(\Omega)$. However, since it is not known whether the viscosity solution of (18) is unique or not, we cannot deduce that all solutions of (18) are locally Hölder continuous when $1 < p_{\min} < 2$. Hence, this motivates the following problem:

Problem. Let u and v be viscosity solutions of (18) where $p : \overline{\Omega} \to (1, \infty)$ is a continuous function such that $1 < p_{\min} < 2$. Do u and v agree in Ω?

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