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Introduction

This work is devoted to the study of upper and lower estimates of norms and best approximations of the generalized Liouville–Weyl derivatives via the best approximations of functions themselves. We also study estimates of moduli of smoothness of the generalized Liouville–Weyl derivatives via moduli of smoothness of functions themselves.

Let the series

$$\frac{a_0(f)}{2}+\sum_{\nu=1}^{\infty}(a_\nu(f)\cos\nu x+b_\nu(f)\sin\nu x) \tag{1.1.1}$$

be the Fourier series of $f\in L_1$. The transformed Fourier series of (1.1.1) is given by

$$\sigma(f,\lambda,\beta):=\sum_{\nu=1}^{\infty}\lambda_\nu\left[a_\nu\cos(\nu x+\frac{\pi\beta}{2})+b_\nu\sin(\nu x+\frac{\pi\beta}{2})\right],$$

where $\beta\in\mathbb{R}$ and $\lambda=\{\lambda_n\}$ is a sequence of positive numbers.

We call the function $\varphi(x)\sim\sigma(f,\lambda,\beta)$ the Liouville–Weyl derivative (or $(\lambda,\beta)-$derivative of the function f) and denote it by $f^{(\lambda,\beta)}$. As an important example, for $\lambda_n=n^r, r>0, \beta=r$, we have $f^{(\lambda,\beta)}=f^{(r)}$ and for $\lambda_n=n^r, r>0, \beta=r+1$ we have $f^{(\lambda,\beta)}=\pm\tilde{f}^{(r)}$, where $f^{(r)}$ is the fractional derivative in the sense of Weyl and $\tilde{f}^{(r)}$ is the r-fractional derivative of the conjugate function $\tilde{f}$. For more historical details concerning the Liouville–Weyl derivatives, see the papers [7], [22], [38],[55], [73].

Let $E_n(f)_p$ be the best approximation of a function $f\in L_p$ by trigonometric polynomials of degree at most n, i.e.,

$$E_n(f)_p=\inf_{\alpha_k,\beta_k\in\mathbb{R}}\left\|f(x)-\sum_{k=0}^{n}(\alpha_k\cos kx+\beta_k\sin kx)\right\|_p.$$

Denote by $\omega_\alpha(f,\delta)_p$ the modulus of smoothness of fractional order $\alpha, \alpha>0$, of the function $f\in L_p$, i.e.,

$$\omega_\alpha(f,\delta)_p=\sup_{|h|\leqslant\delta}\|\Delta_h^\alpha(f)\|_p,$$

where

$$\Delta_h^\alpha(f)=\sum_{\nu=0}^{\infty}(-1)^\nu\binom{\alpha}{\nu}f(x+(\alpha-\nu)h)$$

is a difference of fractional order $\alpha > 0$ of a function $f \in L_p$ at the point x with increment h.

The study of the question of the existence of the r-th derivative of the function f was started by Bernstein [4] in 1912. He proved the following result for $p = \infty$ (for $1 \leqslant p < \infty$, see [8]):
If $f \in L_p,\ 1 \leqslant p \leqslant \infty$, and $\sum\limits_{k=0}^{\infty}(k+1)^{r-1}E_k(f)_p < \infty$, $r \in \mathbb{N}$, then

$$\|f^{(r)}\|_p \leqslant C(r)\sum_{k=0}^{\infty}(k+1)^{r-1}E_k(f)_p.$$

Later, Stechkin [60] obtained the following inequality for the best approximations of $f^{(r)}$ for $p = \infty$ (for $1 \leqslant p < \infty$, see [37]):

$$E_n(f^{(r)})_p \leqslant C(r,p)\Big(n^r E_n(f)_p + \sum_{k=n+1}^{\infty} k^{r-1}E_k(f)_p\Big), \quad r, n \in \mathbb{N}.$$

The corresponding estimate for moduli of smoothness was obtained by Johnen and Scherer [29] (see also [8, pp.178-179]):

$$\omega_k\Big(f^{(r)}, \frac{1}{n}\Big)_p \leqslant C(k,r,p)\sum_{\nu=n+1}^{\infty} \nu^{r-1}\omega_{k+r}\Big(f, \frac{1}{\nu}\Big)_p, \ \ 1 \leqslant p \leqslant \infty,\ r, k, n \in \mathbb{N}.$$

These investigations have been further developed by Timan, Ditzian, Simonov, Potapov, Tikhonov and others.
In [54] and [55], the authors considered the generalized derivatives in the sense of Liouville–Weyl and extended mentioned above results as follows.
Theorem A1. *Let $1 < p < \infty,\ \theta = \min(2, p),\ \beta \in \mathbb{R}$, and $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ be non-decreasing sequence of positive numbers satisfying Δ_2-condition, i.e., $\lambda_{2n} \leqslant \lambda_n$. If $f \in L_p$ and*

$$\sum_{n=1}^{\infty}(\lambda_{n+1}^{\theta} - \lambda_n^{\theta})E_n^{\theta}(f)_p < \infty,$$

then there exists a function $f^{(\lambda,\beta)} \in L_p$ with the Fourier series $\sigma(f, \lambda, \beta)$ and

$$\|f^{(\lambda,\beta)}\|_p \lesssim \left\{\lambda_1^{\theta}E_0^{\theta}(f)_p + \sum_{n=1}^{\infty}(\lambda_{n+1}^{\theta} - \lambda_n^{\theta})E_n^{\theta}(f)_p\right\}^{\frac{1}{\theta}},$$

$$E_n(f^{(\lambda,\beta)}) \lesssim \left\{ \lambda_n^\theta E_n^\theta(f)_p + \sum_{k=n+1}^{\infty} (\lambda_{k+1}^\theta - \lambda_k^\theta) E_k^\theta(f)_p \right\}^{\frac{1}{\theta}}.$$

Moreover, if $\alpha > 0$ *and* $\{\frac{\lambda_n}{n^r}\}$ *is decreasing for some* $r > 0$, *then*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{n}\Big)_p \lesssim \left\{ n^{-\alpha\theta} \sum_{k=1}^{n} \Big(\frac{\lambda_k^\theta}{k^{r\theta}} - \frac{\lambda_{k+1}^\theta}{(k+1)^{r\theta}} \Big) k^{(r+\alpha)\theta} \omega_{\alpha+r}^\theta \Big(f, \frac{1}{k}\Big)_p \right.$$
$$\left. + \sum_{k=n+2}^{\infty} (\lambda_{k+1}^\theta - \lambda_k^\theta) \omega_{\alpha+r}^\theta \Big(f, \frac{1}{k}\Big)_p + \lambda_{n+1}^\theta \omega_{\alpha+r}^\theta \Big(f, \frac{1}{n+1}\Big)_p \right\}^{\frac{1}{\theta}}.$$

Moreover, the corresponding estimates in the limiting cases ($p = 1, \infty$) were also obtained. It was assumed in this case that $\{\lambda_n\}$ is convex or concave.

Our main goal in Section 1 is to extend results mentioned above for a larger class of the Liouville–Weyl derivatives. For this purpose we consider a more general class of sequences $\{\lambda_n\}$. We replace the monotonicity condition on $\{\lambda_n\}$ and $\{\Delta\lambda_n\}$ by general monotonicity.

Let us recall the definition of general monotone sequences (GM) introduced in [69].

Definition 1.4.1 *A sequence* $\lambda := \{\lambda_n\}_{n=1}^\infty$ *of real numbers is said to be general monotone, written* $\lambda \in GM$, *if the relation*

$$\sum_{k=n}^{2n} |\lambda_k - \lambda_{k+1}| \leqslant C|\lambda_n|$$

holds for all integer n, *where the constant* C *is independent of* n.

In subsection 1.6, we proved the following generalization of Theorem A1.

Theorem 1.6.1 *Let* $1 < p < \infty$, $\theta = \min(2, p)$, $\lambda = \{\lambda_n\}_{n=1}^\infty \in GM$, $\alpha \in \mathbb{R}_+$, *and* $r \in \mathbb{R}_+ \cup \{0\}$. *If* $f \in L_p$ *and the series*

$$\sum_{n=1}^{\infty} |\lambda_{n+1}^\theta - \lambda_n^\theta| E_n^\theta(f)_p$$

converges, then there exists a function $f^{(\lambda,\beta)} \in L_p$ *with the Fourier series* $\sigma(f, \lambda, \beta)$ *and*

$$\|f^{(\lambda,\beta)}\|_p \lesssim \left\{ \lambda_1^\theta E_0^\theta(f)_p + \sum_{n=1}^{\infty} |\lambda_{n+1}^\theta - \lambda_n^\theta| E_n^\theta(f)_p \right\}^{\frac{1}{\theta}},$$

$$E_n(f^{(\lambda,\beta)})_p \lesssim \left\{ \lambda^\theta_{[n/2]} E^\theta_{[n/2]}(f)_p + \sum_{k=[n/4]}^{\infty} |\lambda^\theta_{k+1} - \lambda^\theta_k| E^\theta_k(f)_p \right\}^{\frac{1}{\theta}},$$

and

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{n}\Big)_p \lesssim \left\{ n^{-\alpha\theta} \sum_{k=1}^{n} \Big| \frac{\lambda^\theta_k}{k^{r\theta}} - \frac{\lambda^\theta_{k+1}}{(k+1)^{r\theta}} \Big| k^{(r+\alpha)\theta} \omega^\theta_{\alpha+r}\Big(f, \frac{1}{k}\Big)_p \right.$$
$$\left. + \sum_{k=n+1}^{\infty} |\lambda^\theta_{k+1} - \lambda^\theta_k| \omega^\theta_{\alpha+r}\Big(f, \frac{1}{k}\Big)_p + \lambda^\theta_n \omega^\theta_{\alpha+r}\Big(f, \frac{1}{n}\Big)_p \right\}^{\frac{1}{\theta}}.$$

In subsection 1.7 we obtain the corresponding inequalities for $p = 1$ and $p = \infty$.

In subsection 1.8 we study estimates from below of norms and best approximations of the generalized Liouville–Weyl derivatives. We obtain the following result.

Theorem 1.8.1 *Let* $1 < p < \infty, \max(p, 2) \leqslant \tau < \infty, \lambda = \{\lambda_n\}_{n=1}^\infty$ *be a sequence of positive numbers such that* $\lambda \in GM$. *Assume that* $\{\lambda_n\}_{n=1}^\infty$ *satisfies the additional condition*

$$\Big(\sum_{k=1}^{n} |\lambda^\tau_{2^k} - \lambda^\tau_{2^{k-1}}| \Big)^{\frac{1}{\tau}} \leqslant C|\lambda_{2^n}|$$

for all integer n, *where the constant* C *is independent of* n.

If for $f \in L_p$ *there exists a function* $f^{(\lambda,\beta)} \in L_p$, *with the Fourier series* $\sigma(f, \lambda, \beta)$ $(\beta \in \mathbb{R})$, *then*

$$\|f^{(\lambda,\beta)}\|_p \gtrsim \left\{ \lambda^\tau_1 E^\tau_0(f)_p + \sum_{\nu=1}^{\infty} |\lambda^\tau_{2^\nu} - \lambda^\tau_{2^{\nu-1}}| E^\tau_{2^\nu-1}(f)_p \right\}^{\frac{1}{\tau}},$$

$$E_{2^m-1}(f^{(\lambda,\beta)})_p \gtrsim \left\{ \lambda^\tau_{2^{m-1}} E^\tau_{2^m-1}(f)_p + \sum_{\nu=m}^{\infty} |\lambda^\tau_{2^\nu} - \lambda^\tau_{2^{\nu-1}}| E^\tau_{2^\nu-1}(f)_p \right\}^{\frac{1}{\tau}}.$$

In the second section, we study sharp Ul'yanov-type inequalities for moduli of smoothness of fractional order. The (p, q)-inequalities between moduli of smoothness, nowadays called Ul'yanov-type inequalities. The first result of this type was obtained by Ul'yanov [85] in 1968:

$$\omega(f,\delta)_q \lesssim \left(\int\limits_0^\delta \left(t^{-\theta}\omega(f,t)_p\right)^{q_1} \frac{dt}{t} \right)^{1/q_1},$$

where

$$1 \leqslant p < q \leqslant \infty, \quad \theta = \frac{1}{p} - \frac{1}{q}, \quad q_1 = \begin{cases} q, & q < \infty, \\ 1, & q = \infty. \end{cases}$$

Here $\omega(f,\delta)_p = \omega_1(f,\delta)_p$ is the modulus of continuity and $\omega_k(f,\delta)_p$ is the modulus of smoothness of order $k \in \mathbb{N}$.

This result was extended by DeVore, Riemenschneider, and Sharpley for moduli of smoothness of an integer order and the K-functionals [9] (see also [20, 21]). Similar estimates for moduli of smoothness of the derivatives of a function was obtained by Ditzian and Tikhonov. Recently, a sharp form of Ulyanov-type inequalities was extensively investigated by Simonov, Tikhonov, Trebels, among other authors [19, 34, 36, 56, 68, 83].

Our main goal is to improve results of Tikhonov and Trebels [73] by considering a more general class of sequences $\{\lambda_n\}$. We also consider all limiting cases separately:

(i) $p = 1 < q < \infty$;

(ii) $1 < p < q = \infty$;

(iii) $p = 1 < q = \infty$.

In subsection 2.1 we survey some known results concerning Ul'yanov-type inequalities for moduli of smoothness. The corresponding result concerning inequalities of different metrics for the best approximations can be found in subsection 2.2. In subsection 2.3 we give some necessary lemmas. In subsection 2.4 we obtain the main results for non-limiting case. In particular, we obtain the following.

Theorem 2.4.1 *Let $f \in L_p$, $1 < p < q < \infty$, $\theta = 1/p - 1/q$. Let $\lambda = \{\lambda_n\}_{n=1}^\infty \in GM$. Then, for any $\alpha > 0$,*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \left(\sum_{m=n}^\infty \Big(2^{m(\theta+\rho)} \Lambda_{2^m} \omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_p\Big)^q \right)^{1/q},$$

where

$$\Lambda_n := \max_{1 \leqslant k \leqslant n} \frac{|\lambda_{2^k}|}{2^{k\rho}}, \qquad \rho > 0.$$

In subsection 2.5 we obtain the sharp Ul'yanov-type inequalities for the first limiting case: $p = 1 < q < \infty$.

Theorem 2.5.1 *Let $f \in L_p$, $1 = p < q < \infty$, $\theta = 1 - 1/q$. Let $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in GM$. Then, for any $\alpha > 0$ and $0 < \varepsilon \leqslant \min(\rho, \theta)$,*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \left(\sum_{m=n}^{\infty}\Big(2^{m(\theta+\rho)}\Lambda_{2^m}\omega_{\alpha+\rho+\theta-\varepsilon}\Big(f, \frac{1}{2^m}\Big)_1\Big)^q\right)^{1/q},$$

where

$$\Lambda_{2^n} := 2^{-n\frac{\varepsilon}{2}} \max_{1\leqslant k\leqslant n} \frac{|\lambda_{2^k}|}{2^{k(\rho-\frac{\varepsilon}{2})}}.$$

Theorem 2.5.2 *Let $f \in L_p$, $1 = p < q < \infty$, $\theta = 1 - 1/q$, and $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in GM$. Then, for any $\alpha > 0$,*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \left(\sum_{m=n}^{\infty}\Big(2^{m(\theta+\rho)}\Lambda_{2^m}\omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_1\Big)^q\right)^{1/q},$$

where

$$\Lambda_N := \Big(\sum_{k=1}^{N} \frac{|\lambda_k|^q}{k^{(q\rho+1)}}\Big)^{\frac{1}{q}}, \quad \rho > 0.$$

In subsections 2.6 and 2.7, we obtain the sharp Ul'yanov-type inequalities for the cases $1 < p < q = \infty$ and $p = 1,\ q = \infty, \theta = 1$, correspondingly.
In subsection 2.8 we consider estimates for the L_q-best approximations of the generalized Liouville–Weyl derivatives via the L_p-best approximations of functions themselves.

The third section is devoted to the study of estimates of the angle best approximations of the generalized Liouville–Weyl derivatives by the angle approximation of functions themselves in two-dimensional case. We consider the generalized Liouville–Weyl derivatives in place of the classical Weyl mixed derivatives. Our main goal is to prove analogues of Theorems 1.6.1 and 1.8.1 mentioned above in two-dimensional case. In subsection 3.1 we give necessary notation and known results. Useful lemmas are given in subsection 3.2.

Let $Y_{m_1,m_2}(f)_p$ be the best approximation by a two-dimensional angle of the function $f \in L_p(\mathbb{T}^2)$, i.e.,

$$Y_{m_1,m_2}(f)_p = \inf_{T_{m_1,\infty}, T_{\infty,m_2}} \|f - T_{m_1,\infty} - T_{\infty,m_2}\|_p,$$

where the function $T_{m_1,\infty}(x_1, x_2) \in L_p(\mathbb{T}^2)$ is a trigonometric polynomial of order at most m_1 in x_1, and the function $T_{\infty,m_2}(x_1, x_2) \in L_p(\mathbb{T}^2)$ is a trigonometric polynomial of order at most m_2 in x_2.

By $\sigma(f)$ we will denote the Fourier series of a function $f \in L_p(\mathbb{T}^2)$, that is

$$\sigma(f) \equiv \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} (a_{n_1 n_2} \cos n_1 x_1 \cos n_2 x_2 + b_{n_1 n_2} \cos n_1 x_1 \sin n_2 x_2 + \\ + c_{n_1 n_2} \sin n_1 x_1 \cos n_2 x_2 + d_{n_1 n_2} \sin n_1 x_1 \sin n_2 x_2),$$

where for the sake of brevity we set $\cos(0 \cdot t) = \frac{1}{2}$.

The transformed Fourier series of $\sigma(f)$ is given by

$$\sigma(f, \lambda, \beta_1, \beta_2) \equiv \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \lambda_{n_1,n_2} (a_{n_1 n_2} \cos(n_1 x_1 + \beta_1 \pi/2) \cos(n_2 x_2 + \beta_2 \pi/2) \\ + b_{n_1 n_2} \cos(n_1 x_1 + \beta_1 \pi/2) \sin(n_2 x_2 + \beta_2 \pi/2) \\ + c_{n_1 n_2} \sin(n_1 x_1 + \beta_1 \pi/2) \cos(n_2 x_2 + \beta_2 \pi/2) \\ + d_{n_1 n_2} \sin(n_1 x_1 + \beta_1 \pi/2) \sin(n_2 x_2 + \beta_2 \pi/2)),$$

where $\beta_1, \beta_2 \in \mathbb{R}$ and $\lambda = \{\lambda_{n_1 n_2}\}_{n_1,n_2 \in \mathbb{N}}$ is a sequence of real numbers.

By analogy with the one-dimensional case we call the function $\varphi(x_1, x_2) \sim \sigma(f, \lambda, \beta_1, \beta_2)$ the $(\lambda, \beta_1, \beta_2)-$ mixed derivative of the function f (or Liouville–Weyl derivative) and denote it by $f^{(\lambda,\beta_1,\beta_2)}(x_1, x_2)$.

Definition 3.1.1 *A sequence $\lambda = \{\lambda_{n_1 n_2}\}_{n_1,n_2 \in N}$ is said to be general monotone, written $\lambda \in GM^2$, if the relations*

$$\sum_{k_1=n_1}^{2n_1} |\lambda_{k_1,n_2} - \lambda_{k_1+1,n_2}| \leqslant C|\lambda_{n_1,n_2}|,$$

$$\sum_{k_2=n_2}^{2n_2} |\lambda_{n_1,k_2} - \lambda_{n_1,k_2+1}| \leqslant C|\lambda_{n_1,n_2}|,$$

$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} |\lambda_{k_1,k_2} - \lambda_{k_1+1,k_2} - \lambda_{k_1,k_2+1} + \lambda_{k_1+1,k_2+1}| \leqslant C|\lambda_{n_1,n_2}|$$

hold for all integers n_1 and n_2, where the constant C is independent of n_1 and n_2.

In subsection 3.3 similarly to one-dimensional inequalities given by Theorem 1.6.1, we obtain estimates of the angle approximations of $(\lambda, \beta_1, \beta_2)$-derivatives by angle approximation of functions themselves.

Theorem 3.3.1 *Let $1 < p < \infty,\ 0 < \theta \leqslant \min(p, 2),\ \lambda = \{\lambda_{n_1,n_2}\}_{n_1,n_2 \in N}$ be a sequence of positive numbers such that $\lambda \in GM^2$, $\alpha_i \in \mathbb{R}_+, r_i \in \mathbb{R}_+ \cup \{0\}$, and $\beta_i \in \mathbb{R}$ $(i = 1, 2)$.*

If for $f \in L_p^0(\mathbb{T}^2)$ the series

$$\sum_{n_1=1}^{\infty} |\lambda^{\theta}_{n_1+1,1} - \lambda^{\theta}_{n_1,1}| Y^{\theta}_{n_1,0}(f)_p + \sum_{n_2=1}^{\infty} |\lambda^{\theta}_{1,n_2+1} - \lambda^{\theta}_{1,n_2}| Y^{\theta}_{0,n_2}(f)_p$$
$$+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} |\lambda^{\theta}_{k_1+1,k_2+1} - \lambda^{\theta}_{k_1+1,k_2} - \lambda^{\theta}_{k_1,k_2+1} + \lambda^{\theta}_{k_1,k_2}| Y^{\theta}_{k_1,k_2}(f)_p$$

converges, then there exists a function $f^{(\lambda,\beta_1,\beta_2)} \in L_p^0(\mathbb{T}^2)$, with the Fourier series $\sigma(f, \lambda, \beta_1, \beta_2)$, and

$$Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p \lesssim \Big\{ \lambda^{\theta}_{2^{m_1-1},2^{m_2-1}} Y^{\theta}_{2^{m_1}-1,2^{m_2}-1}(f)_p$$
$$+ \sum_{\nu_1=m_1}^{\infty} |\lambda^{\theta}_{2^{\nu_1},2^{m_2-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1}}| Y^{\theta}_{2^{\nu_1}-1,2^{m_2}-1}(f)_p$$
$$+ \sum_{\nu_2=m_2}^{\infty} |\lambda^{\theta}_{2^{m_1-1},2^{\nu_2}} - \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1}}| Y^{\theta}_{2^{m_1}-1,2^{\nu_2}-1}(f)_p$$
$$+ \sum_{\nu_1=m_1}^{\infty} \sum_{\nu_2=m_2}^{\infty} |\lambda^{\theta}_{2^{\nu_1},2^{\nu_2}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2}}$$
$$- \lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}}| Y^{\theta}_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_p \Big\}^{\frac{1}{\theta}}.$$

In subsection 3.4 we obtain the corresponding estimates from below.

1 Liouville–Weyl derivatives, best approximations, and moduli of smoothness. (L_p, L_p) inequalities

1.1 Notation

In this section we will give needed definitions and introduce some notation. Let $L_p = L_p[0, 2\pi]$ $(1 \leqslant p < \infty)$ be the space of 2π-periodic measurable functions that satisfy

$$\|f\|_p = \left(\int\limits_0^{2\pi} |f(x)|^p dx \right)^{1/p} < \infty.$$

Let $L_\infty \equiv C[0, 2\pi]$ be the space of 2π-periodic continuous functions with

$$\|f\|_p = \max_{x \in [0,1]} |f(x)|, \quad p = \infty.$$

Let the series

$$\frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x) \tag{1.1.1}$$

be the Fourier series of $f \in L_1$. The Fourier coefficients of a function f are given by the formulas

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_0^{2\pi} f(x) dx, \\ a_\nu &= \frac{1}{\pi} \int_0^{2\pi} f(x) \cos \nu x dx, \quad \nu \in \mathbb{N}, \\ b_\nu &= \frac{1}{\pi} \int_0^{2\pi} f(x) \sin \nu x dx, \quad \nu \in \mathbb{N}. \end{aligned}$$

Throughout the paper n denotes an integer number. Let $S_n(f)$ denote the n-th partial sum of (1.1.1), $V_n(f)$ denote the de la Vallée-Poussin sum and $K_n(x)$ be the Fejér kernel, i.e.,

$$S_n(f) = \sum_{\nu=0}^{n} A_\nu(x) = \frac{a_0(f)}{2} + \sum_{\nu=1}^{n} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x), \tag{1.1.2}$$

$$V_n(f) = \frac{1}{n} \sum_{\nu=n}^{2n-1} S_\nu(f), \quad V_0(f) = S_0(f),$$

$$K_n(x) = \frac{1}{n+1}\sum_{\nu=0}^{n}\Big(\frac{1}{2}+\sum_{m=1}^{\nu}\cos mx\Big).$$

The Fourier series of the r-th derivatives of f is given as follow

$$f^{(r)}(x) \sim \sum_{\nu=1}^{\infty}\nu^r\Big(a_\nu(f)\cos(\nu x+\frac{r\pi}{2})+b_\nu(f)\sin(\nu x+\frac{r\pi}{2})\Big),$$

where $r\in\mathbb{N}$.

We will use the following definition of fractional differentiation, which was introduced by Weyl, see [52]. Let f be a 2π-periodic integrable function and $\alpha>0$. Then the function $f^{(\alpha)}$ satisfying

$$f^{(\alpha)}(x) \sim \sum_{\nu=1}^{\infty}\nu^\alpha\Big(a_\nu(f)\cos(\nu x+\frac{\alpha\pi}{2})+b_\nu(f)\sin(\nu x+\frac{\alpha\pi}{2})\Big) \qquad (1.1.3)$$

is called the Weyl fractional derivatives of order α.

Recall that the conjugate series to (1.1.1) is given by

$$\sum_{\nu=1}^{\infty}(a_\nu\sin\nu x-b_\nu\cos\nu x).$$

The transformed Fourier series of (1.1.1) is defined by

$$\sigma(f,\lambda,\beta) := \sum_{\nu=1}^{\infty}\lambda_\nu\left[a_\nu\cos(\nu x+\frac{\pi\beta}{2})+b_\nu\sin(\nu x+\frac{\pi\beta}{2})\right], \qquad (1.1.4)$$

where $\beta\in\mathbb{R}$ and $\lambda=\{\lambda_n\}$ is a sequence of positive numbers.

We call the function $\varphi(x)\sim\sigma(f,\lambda,\beta)$ the Liouville–Weyl derivative (or $(\lambda,\beta)-$derivative of the function f) and denote it by $f^{(\lambda,\beta)}$. As an important example, for $\lambda_n=n^r, r>0, \beta=r$, we have $f^{(\alpha,\beta)}=f^{(r)}$ and for $\lambda_n=n^r, r>0, \beta=r+1$ we have $f^{(\alpha,\beta)}=\pm\tilde{f}^{(r)}$, where $f^{(r)}$ is the fractional derivative in the sense of Weyl (see (1.1.3)) and $\tilde{f}^{(r)}$ is the r-fractional derivative of the conjugate function $\tilde{f}$.

Let $E_n(f)_p$ be the best approximation of a function $f\in L_p$ by trigonometric polynomials of degree at most n, i.e.,

$$E_n(f)_p = \inf_{\alpha_k,\beta_k\in\mathbb{R}}\left\|f(x)-\sum_{k=0}^{n}(\alpha_k\cos kx+\beta_k\sin kx)\right\|_p.$$

Throughout the work we use the notation $F \lesssim G$, with $F, G \geqslant 0$, for the estimate $F \leqslant CG$, where C is a positive constant, independent of essential quantities in F and G. Moreover $F \asymp G$ means that $F \lesssim G \lesssim F$ (in this case we say that F is equivalent to G). Moreover, C denotes positive constants not depending on essential parameters which may be different in different formulas.

1.2 Moduli of smoothness

First, we give the definition of the modulus of smoothness $\omega_\alpha(f, \delta)_p$ of order α, $\alpha \in \mathbb{N}$. We define

$$\omega_\alpha(f, \delta)_p = \sup_{|h| \leqslant \delta} \|\Delta_h^\alpha f\|_p, \tag{1.2.5}$$

where $\Delta_h f(x) = f(x+h) - f(x)$, $\quad \Delta_h^\alpha = \Delta_h \Delta_h^{\alpha-1}$.

Let us now define the difference of fractional order $\alpha > 0$ of function $f \in L_p$ at the point x with increment h by

$$\Delta_h^\alpha(f) = \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\alpha}{\nu} f(x + (\alpha - \nu)h), \tag{1.2.6}$$

where $\binom{\alpha}{\nu} = 1$ for $\nu = 0$, $\binom{\alpha}{\nu} = \alpha$ for $\nu = 1$, and

$$\binom{\alpha}{\nu} = \frac{\alpha(\alpha-1)\cdots(\alpha-\nu+1)}{\nu!} \qquad \text{for} \quad \nu \geqslant 2.$$

Since $\left|\binom{\alpha}{\nu}\right| \leqslant C(\alpha)\nu^{-\alpha-1}$, $\nu \in \mathbb{N}$, then the following series

$$C(\alpha) := \sum_{\nu=0}^{\infty} \left|\binom{\alpha}{\nu}\right|$$

converges for all $\alpha > 0$. Therefore, $\Delta_h^\alpha f \in L_p$ for any $f \in L_p$, $1 \leqslant p \leqslant \infty$.

The main properties of fractional moduli of smoothness are similar to those of the classical moduli. First, we recall some basic properties of fractional differences.

Lemma 1.2.1. *[7, 52, 66] Let $f, f_1, f_2 \in L_p$, $1 \leqslant p \leqslant \infty$, and $\alpha, \beta > 0$. Then*

(a) $\triangle_h^\alpha(f_1 + f_2) = \triangle_h^\alpha f_1 + \triangle_h^\alpha f_2$;

(b) $\triangle_h^\alpha(\triangle_h^\beta f) = \triangle_h^{\alpha+\beta} f$;

(c) $\|\triangle_h^\alpha f\|_p \lesssim \|f\|_p$.

Denote by $\omega_\alpha(f,\delta)_p$ the moduli of smoothness of fractional order $\alpha, \alpha > 0$, of the function $f \in L_p$, i.e.,

$$\omega_\alpha(f,\delta)_p = \sup_{|h|\leqslant\delta} \|\Delta_h^\alpha(f)\|_p,$$

see [7, 52, 66]. Note that if $\alpha \in \mathbb{N}$ this definition coincides with (1.2.5).

The following properties of moduli of smoothness are well known.

Lemma 1.2.2. *[7, 49, 72] Let $f, f_1, f_2 \in L_p$, $1 < p < \infty$, $\alpha > 0$. Then*

(a) *$\omega_\alpha(f,\delta)_p$ is nondecreasing nonnegative function of δ, defined on $(0,\infty)$, with $\lim\limits_{\delta\to 0+} \omega_\alpha(f,\delta)_p = 0$;*

(b) *$\omega_\alpha(f_1+f_2,\delta)_p \leqslant \omega_\alpha(f_1,\delta)_p + \omega_\alpha(f_2,\delta)_p$;*

(c) *If $0 < \delta_2 \leqslant \delta_1 \leqslant \pi$, then*

$$\frac{\omega_\alpha(f,\delta_1)_p}{\delta_1^\alpha} \lesssim \frac{\omega_\alpha(f,\delta_2)_p}{\delta_2^\alpha};$$

(d) *If $\lambda > 1$, then*

$$\omega_\alpha(f,\lambda\delta)_p \lesssim \lambda^\alpha \omega_\alpha(f,\delta)_p.$$

For any $p \geqslant 1$ and $\alpha > 0$, introduce the periodic Sobolev space by

$$W_p^\alpha := \Big\{ g \in L_p : \;\; g^{(\alpha)} \in L_p, \;\; g^{(\alpha)} \sim \sigma(g, \{\lambda_n = n^\alpha\}, \alpha) \Big\},$$

see (1.1.4). It is known [7, 34] that the modulus of smoothness is equivalent to the corresponding K-functional, that is,

$$\omega_\alpha\Big(f, \frac{1}{2^n}\Big)_p \;\asymp\; K\Big(f, \frac{1}{2^n}, L_p, W_p^\alpha\Big), \;\; \alpha > 0, \tag{1.2.7}$$

where $1 \leqslant p \leqslant \infty$ and

$$K\Big(f, \frac{1}{2^n}, L_p, W_p^\alpha\Big) := \inf_{g\in W_p^\alpha} \Big(\|f-g\|_p + 2^{-\alpha n}\|g^{(\alpha)}\|_p \Big).$$

The following lemma will play an important role in the proofs of our main results.

Lemma 1.2.3. *[55] Let $\alpha > 0$. If $\varphi \in L_p$, $1 < p < \infty$, then*

$$\omega_\alpha\Big(\varphi, \frac{1}{n}\Big)_p \lesssim \|\varphi - S_n(\varphi)\|_p + n^{-\alpha}\|S_n^{(\alpha)}(\varphi)\|_p \lesssim \omega_\alpha\Big(\varphi, \frac{1}{n}\Big)_p. \qquad (1.2.8)$$

If $\varphi \in L_p$, $1 \leqslant p \leqslant \infty$, then

$$\omega_\alpha\Big(\varphi, \frac{1}{n}\Big)_p \lesssim \|\varphi - V_n(\varphi)\|_p + n^{-\alpha}\|V_n^{(\alpha)}(\varphi)\|_p \lesssim \omega_\alpha\Big(\varphi, \frac{1}{n}\Big)_p. \qquad (1.2.9)$$

We mention that (1.2.8) and (1.2.9) are realization results for a modulus of smoothness of fractional order. Note that for $\alpha \in \mathbb{N}$ these results follow from [13]. To prove realization results, one needs the following lemma.

Lemma 1.2.4. *[7, 66] Let $T_n(x) = \sum\limits_{k=1}^{n} \big(a_k \cos kx + b_k \sin kx\big)$ be a trigonometric polynomial of degree n, $1 \leqslant p \leqslant \infty$, and $\alpha > 0$. Then*

(a) $\|T_n^{(\alpha)}\|_p \lesssim n^\alpha \|\triangle^\alpha_{\pi/n} T_n\|_p$;

(b) *for all h such that $0 < |h| \leqslant \pi/n$, there holds $\|\triangle^\alpha_h T_n\|_p \lesssim n^{-\alpha}\|T_n^{(\alpha)}\|_p$.*

1.3 History of the question

Direct and inverse inequalities in approximation theory

One of the main problems of the constructive theory of functions is to find a relationship between differential properties of functions and their structural or constructive characteristics. Direct and inverse approximation theorems answer this question. Direct theorems for L_p, $1 \leqslant p \leqslant \infty$, deal with estimates of the best approximation in terms of moduli of smoothness:

$$E_n(f)_p \leqslant C(p,\alpha)\omega_\alpha\Big(f, \frac{1}{n}\Big)_p, \quad n, \alpha \in \mathbb{N}, \qquad (1.3.10)$$

$$E_n(f)_p \leqslant \frac{C(p,\alpha,r)}{n^r}\omega_\alpha\Big(f^{(r)}, \frac{1}{n}\Big)_p \quad n, \alpha, r \in \mathbb{N}. \qquad (1.3.11)$$

Inequality (1.3.10) was obtained by Jackson [27] for $\alpha = 1, p = \infty$. Moreover, Akhiezer [1] proved it for $\alpha = 2, 1 \leqslant p \leqslant \infty$. In the case $\alpha \geqslant 3$ and $p = \infty$, (1.3.10) was proved by Stechkin [59]. For $\alpha \geqslant 3$ and $1 \leqslant p < \infty$ see [8], [78]. The second inequality was obtained in [78], also see [8], [67], [87].

Inverse theorems for $L_p,\ 1 \leqslant p \leqslant \infty$, are written as follows: for $n, \alpha, r \in \mathbb{N}$, we have

$$\omega_\alpha\Big(f, \frac{1}{n}\Big)_p \leqslant \frac{C(p,\alpha)}{n^\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha-1} E_\nu(f)_p, \tag{1.3.12}$$

$$\omega_\alpha\Big(f^{(r)}, \frac{1}{n}\Big)_p \leqslant C(p,r,\alpha)\Bigg(\frac{1}{n^\alpha} \sum_{\nu=0}^{n} (\nu+1)^{\alpha+r-1} E_\nu(f)_p + \sum_{\nu=n+1}^{\infty} \nu^{r-1} E_\nu(f)_p\Bigg). \tag{1.3.13}$$

Inequality (1.3.12) was proved by Salem [51] for $\alpha = 1, p = \infty$; for $\alpha \in \mathbb{N}$ see [8], [78]. Inequality (1.3.13) was obtained by Stechkin [59] for $p = \infty$, for other cases see [8], [78]. Later on, inequalities (1.3.10) - (1.3.13) were extended by Timan [79], [80], [81]. Note that direct and inverse inequalities (1.3.10)-(1.3.13) influenced substantially the further research [8], [78].

Let us now discuss several well-known inequalities related to direct and inverse theorems. Further, we present the most important inequalities for norms, best approximations, and moduli of smoothness of the r-th derivative in terms of those of the function f itself.

The following result was proved by Bernstein [4] for $p = \infty$ (for $1 \leqslant p < \infty$, see [8]):

If $f \in L_p,\ 1 \leqslant p \leqslant \infty$, and $\sum_{k=0}^{\infty} (k+1)^{r-1} E_k(f)_p < \infty$, $r \in \mathbb{N}$, then

$$\|f^{(r)}\|_p \leqslant C(r) \sum_{k=0}^{\infty} (k+1)^{r-1} E_k(f)_p. \tag{1.3.14}$$

Marcinkiewicz [42] and Besov [5] obtained an improvement of this inequality for $1 < p < \infty$:

$$\|f^{(r)}\|_p \leqslant C(r,p)\Big(\sum_{k=0}^{\infty} (k+1)^{\theta r-1} E_k^\theta(f)_p\Big)^{\frac{1}{\theta}}, \quad \theta = \min(2,p).$$

Later on, Stechkin [60] for $p = \infty$ and Konyushkov [37] for $1 < p < \infty$ obtained the following inequality for the best approximations of $f^{(r)}$:

$$E_n(f^{(r)})_p \leqslant C(r,p)\Big(n^r E_n(f)_p + \sum_{k=n+1}^{\infty} k^{r-1} E_k(f)_p\Big), \quad r, n \in \mathbb{N}. \tag{1.3.15}$$

Inequality (1.3.15) was extended by Timan [82] for the case $1 < p < \infty$ as follows

$$E_n(f^{(r)})_p \leqslant C(r,p)\Bigg(n^r E_n(f)_p + \Big(\sum_{k=n+1}^{\infty} k^{\theta r-1} E_k^{\theta}(f)_p\Big)^{\frac{1}{\theta}}\Bigg),$$

where $\theta = \min(2,p),\ r,n \in \mathbb{N}$.
Moreover, Stechkin [60] proved the similar result for $\tilde{f}^{(r)}$:

$$E_n(\tilde{f}^{(r)})_p \leqslant C(r)\Big(n^r E_n(f)_p + \sum_{k=n+1}^{\infty} k^{r-1} E_k(f)_p\Big), \quad r,n \in \mathbb{N}, \tag{1.3.16}$$

where $p = 1$ or $p = \infty$.
Corresponding estimates for moduli of smoothness was obtained by Johnen and Scherer [29] (see also [8, pp.178-179]):

$$\omega_k\Big(f^{(r)},\frac{1}{n}\Big)_p \leqslant C(k,r,p)\sum_{\nu=n+1}^{\infty} \nu^{r-1}\omega_{k+r}\Big(f,\frac{1}{\nu}\Big)_p, \quad 1 \leqslant p \leqslant \infty,\ r,k,n \in \mathbb{N}. \tag{1.3.17}$$

From the result by Bari and Stechkin [61] one has

$$\omega_k\Big(\tilde{f}^{(r)},\frac{1}{n}\Big)_p \leqslant C(k,r)\Big(\frac{1}{n^k}\sum_{\nu=1}^{n} \nu^{k+r-1}E_\nu(f)_p + \sum_{\nu=n+1}^{\infty} \nu^{r-1}E_\nu(f)_p\Big),$$

where $p = 1,\infty$ and $r,k,n \in \mathbb{N}$.
Moreover, in light of Jackson's inequality, we have the following estimate:

$$\omega_k\Big(\tilde{f}^{(r)},\frac{1}{n}\Big)_p \leqslant C(k,r)\Big(\frac{1}{n^k}\sum_{\nu=1}^{n} \nu^{k+r-1}\omega_{k+r}\Big(f,\frac{1}{\nu}\Big)_p + \sum_{\nu=n+1}^{\infty} \nu^{r-1}\omega_{k+r}\Big(f,\frac{1}{\nu}\Big)_p\Big), \tag{1.3.18}$$

where $p = 1,\infty$ and $r,k,n \in \mathbb{N}$.
Inequality (1.3.17) was extended by Ditzian and Tikhonov [11] for the case of $0 < p < \infty$ as follows

$$\omega_k\Big(f^{(r)},\frac{1}{n}\Big) \leqslant C(k,r,p)\Big(\sum_{\nu=n+1}^{\infty} \nu^{r\theta-1}\omega_{k+r}^{\theta}\Big(f,\frac{1}{\nu}\Big)_p\Big)^{\frac{1}{\theta}}, \quad \theta = \min(2,p),\ r,k,n \in \mathbb{N}, \tag{1.3.19}$$

see also [55].

Let us now mention estimates of $E_n(f^{(r)})_p$ from below. The following inequality was proved for $r \in \mathbb{N}$ in [8] and for $r > 0$ in [54]:

$$n^r E_n(f)_p \leqslant C(r) E_n(f^{(r)})_p, \quad 1 \leqslant p \leqslant \infty.$$

Simonov and Tikhonov [54, 55] obtained estimates from below of the best approximations of $f^{(r)}$:

$$\Big(n^{r\tau} E_n^\tau(f)_p + \sum_{k=n+1}^{\infty} k^{\tau r-1} E_k^\tau(f)_p \Big)^{\frac{1}{\tau}} \leqslant C(r) E_n(f^{(r)})_p,$$

where $1 < p < \infty, \tau = \max(2, p)$. For the case $0 < p < 1$, see [35].

The problem of multipliers for trigonometric polynomials

Let us consider the following problem. Let $\mathbb{T}_n$ be the space of trigonometric polynomials of degree of most n, i.e.,

$$t_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \big(a_k \cos kx + b_k \sin kx \big) \in \mathbb{T}_n \tag{1.3.20}$$

with the norm $\|t_n(x)\|_\infty = \max_x |t_n(x)|$. By $\Lambda = \{\lambda_k\}_{k=0}^\infty$ we define a multiplier sequence. We set

$$\tau_n(x) = \tau_n(x, t_n) = \lambda_0 \frac{a_0}{2} + \sum_{k=1}^{n} \lambda_k \big(a_k \cos kx + b_k \sin kx \big)$$

and

$$\widetilde{\tau}_n(x) = \widetilde{\tau}_n(x, t_n) = \sum_{k=1}^{n} \lambda_k \big(a_k \sin kx - b_k \cos kx \big),$$

where a_k and b_k are the coefficients of the polynomial (1.3.20). Several authors investigated the behavior of the following two expressions:

$$M_n = M_n(\Lambda) = \sup_{\|t_n\|_\infty \leqslant 1} \|\tau_n(t_n)\|_\infty$$

and

$$\widetilde{M}_n = \widetilde{M}_n(\Lambda) = \sup_{\|t_n\|_\infty \leqslant 1} \|\widetilde{\tau}_n(t_n)\|_\infty.$$

By definition, the numbers M and $\widetilde{M}_n$ are the smallest constants for which inequalities

$$\|\tau_n(t_n)\| \leqslant M_n \|t_n\|,$$

and

$$\|\widetilde{\tau}_n(t_n)\| \leqslant \widetilde{M}_n\|t_n\|$$

are valid for any polynomial $t_n \in \mathbb{T}_n$. The problem of estimating the constants M_n and $\widetilde{M}_n$ is called the problem of multipliers for trigonometric polynomials. One of the first result related to the problem of multipliers for trigonometric polynomials was the result Bernstein for $\lambda = k$ ([4]). Szego [65] and Fejer [18] proved that $M_n = \widetilde{M}_n = \lambda_n$ in the case $\lambda_0 = 0, \quad \Delta\lambda_k = \lambda_k - \lambda_{k-1} \geqslant 0\ (k = 1, 2, ..., n)$, and $\Delta^2\lambda_k = \lambda_k - 2\lambda_{k-1} + \lambda_{k-2} \geqslant 0\ (k = 2, ..., n)$.
Later on, Sokolov [58] obtained the following estimate

$$M_n \leqslant \frac{1}{n}\Big(\lambda_n + 2\sum_{k=1}^{n-1}\lambda_k\Big),$$

where λ_n satisfies the following conditions:

$$\begin{aligned} \lambda_0 = 0,\ \Delta\lambda_k \geqslant 0\ (k = 1, 2, ..., n), \\ \Delta^2\lambda_k \leqslant 0\ (k = 2, ..., n). \end{aligned} \tag{1.3.21}$$

In the case when conditions (1.3.21) hold, the next result was proved by Stechkin [61] in 1950:

$$\widetilde{M}_n(\Lambda) \sim P_n = \sum_{k=1}^{n}\frac{\lambda_k}{k}.$$

We will extend the above mentioned results in Lemma 1.4.5 below.

Our next problem is closely related to embeddings of function spaces, see [23]-[26], [45]-[47],[74], [75], and [84].

Inequalities for best approximation and moduli of smoothness of Liouville – Weyl derivatives

In [54] and [55], the authors considered the generalized derivatives in the sense of Liouville–Weyl and extended mentioned above results as follows. Recall that $\sigma(f, \lambda, \beta)$ is defined by (1.1.4).

Theorem A1. *Let $1 < p < \infty$, $\theta = \min(2, p)$, $\beta \in \mathbb{R}$, and $\lambda = \{\lambda_n\}_{n=1}^{\infty}$ be non-decreasing sequence of positive numbers satisfying Δ_2-condition, i.e., $\lambda_{2n} \leqslant \lambda_n$. If $f \in L_p$ and*

$$\sum_{n=1}^{\infty}(\lambda_{n+1}^{\theta} - \lambda_n^{\theta})E_n^{\theta}(f)_p < \infty,$$

then there exists a function $f^{(\lambda,\beta)} \in L_p$ *with the Fourier series* $\sigma(f,\lambda,\beta)$ *and*

$$\|f^{(\lambda,\beta)}\|_p \lesssim \left\{\lambda_1^\theta E_0^\theta(f)_p + \sum_{n=1}^{\infty}(\lambda_{n+1}^\theta - \lambda_n^\theta)E_n^\theta(f)_p\right\}^{\frac{1}{\theta}}, \tag{1.3.22}$$

$$E_n(f^{(\lambda,\beta)}) \lesssim \left\{\lambda_n^\theta E_n^\theta(f)_p + \sum_{k=n+1}^{\infty}(\lambda_{k+1}^\theta - \lambda_k^\theta)E_k^\theta(f)_p\right\}^{\frac{1}{\theta}}. \tag{1.3.23}$$

Moreover, if $\alpha > 0$ *and* $\{\frac{\lambda_n}{n^r}\}$ *is decreasing for some* $r > 0$, *then*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{n}\Big)_p \lesssim \Bigg\{n^{-\alpha\theta}\sum_{k=1}^{n}\Big(\frac{\lambda_k^\theta}{k^{r\theta}} - \frac{\lambda_{k+1}^\theta}{(k+1)^{r\theta}}\Big)k^{(r+\alpha)\theta}\omega_{\alpha+r}^\theta\Big(f, \frac{1}{k}\Big)_p + \sum_{k=n+2}^{\infty}(\lambda_{k+1}^\theta - \lambda_k^\theta)\omega_{\alpha+r}^\theta\Big(f, \frac{1}{k}\Big)_p + \lambda_{n+1}^\theta\omega_{\alpha+r}^\theta\Big(f, \frac{1}{n+1}\Big)_p\Bigg\}^{\frac{1}{\theta}}. \tag{1.3.24}$$

In the limiting cases $p = 1$ or $p = \infty$ one has to assume additional conditions on the sequence $\{\lambda_n\}$. The following theorem holds [54, 55].

Theorem A2. *Let* $p = 1, \infty$, $\beta \in \mathbb{R}$ *and* $\lambda = \{\lambda_n\}_{n=1}^\infty$ *be non-decreasing sequence of positive numbers satisfying* Δ_2-*condition and such that* $\Delta\lambda_n \leqslant C\Delta\lambda_{2n}$ *and* $\Delta^2\lambda_n \geqslant 0$ *(or* $\leqslant 0$*).*

If $f \in L_p$ *and*

$$\Big|\cos\frac{\pi\beta}{2}\Big|\sum_{n=1}^{\infty}(\lambda_{n+1} - \lambda_n)E_n(f)_p + \Big|\sin\frac{\pi\beta}{2}\Big|\sum_{n=1}^{\infty}\lambda_n\frac{E_n(f)_p}{n} < \infty,$$

then there exists a function $f^{(\lambda,\beta)} \in L_p$ *with the Fourier series* $\sigma(f,\lambda,\beta)$ *and*

$$\|f^{(\lambda,\beta)}\|_p \lesssim \Big|\cos\frac{\pi\beta}{2}\Big|\sum_{n=1}^{\infty}(\lambda_{n+1} - \lambda_n)E_n(f)_p + \Big|\sin\frac{\pi\beta}{2}\Big|\sum_{n=1}^{\infty}\lambda_n\frac{E_n(f)_p}{n}, \tag{1.3.25}$$

$$E_n(f^{(\lambda,\beta)})_p \lesssim \lambda_n E_n(f)_p + \Big|\cos\frac{\pi\beta}{2}\Big|\sum_{\nu=n+1}^{\infty}(\lambda_{\nu+1} - \lambda_\nu)E_\nu(f)_p + \Big|\sin\frac{\pi\beta}{2}\Big|\sum_{\nu=n+1}^{\infty}\lambda_\nu\frac{E_\nu(f)_p}{\nu}. \tag{1.3.26}$$

Moreover, if $\alpha > 0$ *and the sequence* $\{\lambda_n\}_{n=1}^\infty$ *is such that* $\{\frac{\lambda_n}{n^\rho}\}$ *is decreasing for some* $\rho > 0$ *and, for some* $\tau > 0$*, there holds* $\Delta^2\Big(\frac{\lambda_n}{n^r}\Big) \geqslant 0$ *with* $r = \rho + \tau sign\Big|\sin\frac{(\beta-\rho)\pi}{2}\Big|$*, then*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{n}\Big) \lesssim \Big|\cos\frac{\pi\beta}{2}\Big| \sum_{\nu=n+2}^{\infty} (\lambda_{\nu+1} - \lambda_\nu)\omega_{\alpha+r}\Big(f, \frac{1}{\nu}\Big)_p$$
$$+ \Big|\sin\frac{\pi\beta}{2}\Big| \sum_{\nu=n+2}^{\infty} \frac{\lambda_\nu}{\nu}\omega_{\alpha+r}\Big(f, \frac{1}{\nu}\Big)_p + \lambda_{n+1}\omega_{\alpha+r}(f, \frac{1}{n+1})_p$$
$$+ n^{-\alpha}\sum_{\nu=1}^{n}\Big(\frac{\lambda_\nu}{\nu^\rho} - \frac{\lambda_{\nu+1}}{(\nu+1)^\rho}\Big)\nu^{\alpha+\rho}\omega_{\alpha+\rho}\Big(f, \frac{1}{\nu}\Big)_p. \tag{1.3.27}$$

In particular, (1.3.22) and (1.3.25) generalize (1.3.14), while (1.3.23) and (1.3.26) generalize (1.3.15) and (1.3.16). In addition, (1.3.17), (1.3.18), and (1.3.19) follow from (1.3.24) and (1.3.27).

Our main goal in this section (see subsection 1.6 and 1.7) is to prove analogues of Theorems A1 and A2 by considering a more general class of sequences $\{\lambda_n\}$. We replace the monotonicity condition on $\{\lambda_n\}$ and $\{\Delta\lambda_n\}$ by general monotonicity.

1.4 General monotone sequences and their properties

In this subsection, we present some definition of monotone type sequences. In particular, we will give the definition of general monotone sequences introduced in [69] and their basic properties. First, we recall the definition of monotone sequences:

$$M = \Big\{\lambda = \{\lambda_n\}_{n\in\mathbb{N}} \;:\; \lambda_n \in \mathbb{R},\; \lambda_n > 0,\; \lambda_n \downarrow 0, \text{ i.e., } \lambda_n \geqslant \lambda_{n+1} \geqslant \cdots \to 0\Big\}.$$

The concept of quasi-monotone sequence was introduced in [57, 63] as follows:

$$QM = \Big\{\lambda = \{\lambda_n\}_{n\in\mathbb{N}} \;:\; \lambda_n \in \mathbb{R},\; \exists\, \tau \geqslant 0 \text{ such that } \frac{\lambda_n}{n^\tau}\downarrow\Big\}.$$

Now we give the definition of a more general class of O-regularly varying quasi-monotone sequences (see [62]):

$$ORVQM = \Big\{\lambda = \{\lambda_n\}_{n\in\mathbb{N}} \ : \ \lambda_n \in \mathbb{R},\, \exists\ \mu_n \geqslant 0, \mu_n \nearrow, \mu_{2n} \leqslant C\mu_n \quad \text{such that } \frac{\lambda_n}{\mu_n}\downarrow \Big\}.$$

Leindler in [39] defined another class of sequences named as sequences of rest bounded variation (denoted by $RBVS$), keeping some properties of decreasing sequences:

$$RBVS = \Big\{\lambda = \{\lambda_n\}_{n\in\mathbb{N}} \ : \ \lambda_n \in \mathbb{R} \ \ \lambda_n \to 0, \ \ \sum_{\nu=n}^{\infty} |\lambda_{\nu+1} - \lambda_\nu| \leqslant C|\lambda_n| \ \ \forall n \in \mathbb{N}\Big\}.$$

In particular, $\{\lambda_n\} \in RBVS$ implies that $|\lambda_k| \leqslant C|\lambda_n|$ for any $n \leqslant k$.
The classes QM (or $ORVQM$) and $RBVS$ are not comparable (see [40], [69]). It is clear that $M \subset ORVQM \cap RBVS$.

Recently, Tikhonov [69, 70] introduced the following new class of sequences, which contains all classes of sequences mentioned earlier.

Definition 1.4.1. *A sequence $\lambda := \{\lambda_n\}_{n=1}^{\infty}$ of real numbers is said to be general monotone, written $\lambda \in GM$, if the relation*

$$\sum_{k=n}^{2n} |\lambda_k - \lambda_{k+1}| \leqslant C|\lambda_n|$$

holds for all integer n, where the constant C is independent of n.

In what follows we will consider only non-negative GM-sequences. The class GM contains monotone or quasi-monotone sequences. Moreover, in [69], the author showed that $ORVQM \cup RBVS \subsetneq GM$. We summarize various generalizations of the monotone conditions in the following Figure 1.

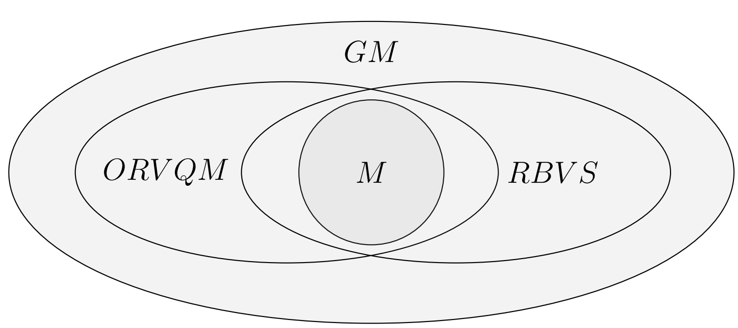

Figure 1: Relationships between different generalized classes of monotonic sequences.

The following characterization of general monotone sequences is known.

Lemma 1.4.1. *[69] $\{\lambda_n\} \in GM$ if and only if there exists $C > 0$ such that*

1. $|\lambda_k| \leqslant C|\lambda_n| \quad \text{for} \quad n \leqslant k \leqslant 2n,$
2. $\sum\limits_{k=n}^{N} |\Delta\lambda_k| \leqslant C\Big(|\lambda_n| + \sum\limits_{k=n+1}^{N} \frac{|\lambda_k|}{k}\Big) \quad \text{for any} \quad n < N.$

The following multiplier property holds true.

Lemma 1.4.2. *[41] If $\lambda = \{\lambda_n\} \in GM$ and $\eta = \{\eta_n\} \in GM$ then $\lambda\eta = \{\lambda_n\eta_n\} \in GM$.*

Note that any GM-sequence satisfies Δ_2-condition, that is, $\lambda_{2n} \leqslant C\lambda_n$, see Lemma 1.4.1.

Clearly any monotone sequence belongs to the GM class but the reverse is not always true. Some examples one can find in [41] and [69]. Below we give examples of $\{\lambda_n\}$ such that $\{\Delta\lambda_n\} \in GM$ and $\{\Delta\frac{\lambda_n}{n^r}\} \in GM$ but $\{\lambda_n\}$ is not monotone or convex. These examples are related to Theorems A1 and A2 and Theorems 1.6.1 and 1.7.1 below.

Example 1.4.1. *Let us define*

$$\lambda_n = \begin{cases} n^{\alpha+1}, & 2^k < n < 2^{k+1}, \\ n^{\alpha+1} + c, & n = 2^k, \end{cases}$$

where $\alpha > 1$ and $c > 2^{k(\alpha+1)}$. It is easy to check that

(1). $\{\lambda_n\} \in GM$,

(2). $\{\Delta\lambda_n\} \in GM$,

(3). $\{\frac{\lambda_n}{n^r}\} \in GM$, where $r > 0$,

(4). $\left\{\Delta\frac{\lambda_n}{n^r}\right\} \in GM$, where $r > 0$.

Example 1.4.2. *Let $\{\lambda_n\}$ be non-decreasing sequence satisfying the condition $\lambda_{2n} \leqslant C\lambda_n$. We define*

$$\left\{\lambda'_n\right\} := \begin{cases} \lambda_n, \quad 2^k \leqslant n \leqslant 2^k + 2^{k-1} - 1, \\ \lambda_{2^k+2^{k-1}+n-1} = \lambda_{2^k+2^{k-1}-n}, \quad 1 \leqslant n \leqslant 2^{k-1}, k \in \mathbb{N}. \end{cases}$$

It is easy to see that $\{\lambda'_n\} \in GM$.

Remark 1.4.1. *One can also define another class of general monotone sequences (see [69]):*

$$\sum_{k=n}^{2n} |\lambda_k - \lambda_{k+1}| \leqslant C|\lambda_{2n}|,$$

where the constant C is independent of n. We denote the class of such sequences by $GM^{\uparrow}$.

The following lemma provides a characterization of sequences from $GM^{\uparrow}$. The proof follows the same line as the proof of Lemma 1.4.1 (see [69]).

Lemma 1.4.3. *A sequences $\{\lambda_n\} \in GM^{\uparrow}$ if and only if there exists $C > 0$ such that*

1. $|\lambda_n| \leqslant C|\lambda_k| \quad$ *for* $\quad n \leqslant k \leqslant 2n,$
2. $\sum\limits_{k=n}^{N} |\Delta\lambda_k| \leqslant C\Big(|\lambda_N| + \sum\limits_{k=n+1}^{N} \frac{|\lambda_k|}{k}\Big) \quad$ *for any* $\quad n < N.$

Note that any non-decreasing sequence belongs to $GM^{\uparrow}$.

To treat the case $p = 1$ or $p = \infty$ in subsection 1.7, we will need the following two lemmas.

Lemma 1.4.4. *If $\Big\{\Delta\lambda_n\Big\} \in GM$, then*

$$\sum_{m=2^n}^{2^{n+2}-2} |\Delta^2\lambda_{m+2}|(m+1) + (2^{n+2}-1)|\Delta\lambda_{2^{n+2}-1}|$$

$$+ (2^{n+1}+2)|\Delta\lambda_{2^n}| \leqslant C \sum_{m=2^{n-1}}^{2^n} |\Delta\lambda_m|.$$

Proof. Using the property of GM sequences, we have the following inequality

$$\sum_{m=2^n}^{2^{n+2}-2} |\Delta^2\lambda_{m+2}|(m+1) + (2^{n+2}-1)|\Delta\lambda_{2^{n+2}-1}| + (2^{n+1}+2)|\Delta\lambda_{2^n}|$$

$$\leqslant C\Big((2^{n+2}-1)|\Delta\lambda_{2^n+2}| + (2^{n+2}-1)|\Delta\lambda_{2^n}| + (2^{n+1}+2)|\Delta\lambda_{2^n}|\Big)$$

$$\leqslant C2^n|\Delta\lambda_{2^n}| \leqslant C \sum_{m=2^{n-1}}^{2^n} |\Delta\lambda_m|,$$

which proves the statement of the lemma. □

For a given sequence $\{\lambda_n\}_{n=1}^{\infty}$ define $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ and $\Delta^2\lambda_n = \Delta(\Delta\lambda_n) = \lambda_n - 2\lambda_{n+1} + \lambda_{n+2}$.

Lemma 1.4.5. *Let $p = 1$ or $p = \infty$ and $\Big\{\Delta\lambda_n\Big\} \in GM$. Let*

$$T_n(x) = \sum_{\nu=1}^{n}(a_\nu \cos \nu x + b_\nu \sin \nu x),$$
$$T_n(\lambda, x) = \sum_{\nu=1}^{n} \lambda_\nu(a_\nu \cos \nu x + b_\nu \sin \nu x).$$

Then the following inequality holds

$$\|T_{2^{n+2}}(\lambda, x) - T_{2^n}(\lambda, x)\|_p \leqslant C\Big(\sum_{k=2^{n-1}}^{2^n} |\Delta\lambda_k| + |\lambda_{2^{n+2}}|\Big) \|T_{2^{n+2}}(f) - T_{2^n}(f)\|_p .$$

Proof. Applying the Abel transformation twice, we get

$$\Big\|T_{2^{n+2}}(\lambda, f) - T_{2^n}(\lambda, f)\Big\|_p = \left\|\frac{1}{\pi}\int_{-\pi}^{\pi}\Big(T_{2^{n+2}}(f) - T_{2^n}(f)\Big)(x+u) \sum_{\nu=2^n}^{2^{n+2}}(\lambda_\nu \cos \nu u)du\right\|_p$$
$$\leqslant \left\|\frac{1}{\pi}\int_{-\pi}^{\pi}\Big(T_{2^{n+2}}(f) - T_{2^n}(f)\Big)(x+u)\Big(\sum_{\nu=2^n}^{2^{n+2}-1} (\lambda_\nu - \lambda_{\nu+1}) \sum_{m=0}^{\nu} \cos mu + \lambda_{2^{n+2}} \sum_{m=0}^{2^{n+2}} \cos mu - \lambda_{2^n} \sum_{m=0}^{2^n-1} \cos mx\Big) du\right\|_p$$

$$\leqslant \left\|\frac{1}{\pi}\int_{-\pi}^{\pi} \Big(T_{2^{n+2}}(f) - T_{2^n}(f)\Big)(x+u)\Bigg(\sum_{\nu=2^n}^{2^{n+2}-2} \Delta^2\lambda_\nu \sum_{k=0}^{\nu}\sum_{m=0}^{k} \cos mu + \Delta\lambda_{2^{n+2}-1} \sum_{k=0}^{2^{n+2}-1} \sum_{m=0}^{k} \cos mu - \Delta\lambda_{2^n} \sum_{k=0}^{2^n}\sum_{m=0}^{k} \cos mu + \lambda_{2^{n+2}} \sum_{m=0}^{2^{n+2}-1} \cos mu - \lambda_{2^n} \sum_{m=0}^{2^n-1} \cos mu\Bigg) du\right\|_p =: I.$$

Using the definition of the Fejér kernel, we have

$$I = \left\| \frac{1}{\pi} \int_{-\pi}^{\pi} \Big(T_{2^{n+2}}(f) - T_{2^n}(f) \Big)(x+u) \Bigg(\sum_{\nu=2^n}^{2^{n+2}-2} \Delta^2 \lambda_\nu K_\nu(u)(\nu+1) \right.$$

$$+ \sum_{\nu=2^n}^{2^{n+2}-2} \Delta^2 \lambda_\nu \frac{(\nu+1)}{2} + \Delta\lambda_{2^{n+2}-1} K_{2^{n+2}-1}(u) 2^{n+2} + \Delta\lambda_{2^{n+2}-1} 2^{n+1} -$$

$$- \Delta\lambda_{2^n} K_{2^n}(u)(2^n+1) - \Delta\lambda_{2^n} \frac{2^n+1}{2}$$

$$\left. + \lambda_{2^{n+2}} \sum_{m=0}^{2^{n+2}-1} \cos mu - \lambda_{2^n} \sum_{m=0}^{2^n-1} \cos mu \Bigg) du \right\|_p$$

$$\leqslant \Big\| T_{2^{n+2}}(f) - T_{2^n}(f) \Big\|_p \Bigg(2 \sum_{\nu=2^n}^{2^{n+2}-2} |\Delta^2 \lambda_\nu| (\nu+1)$$

$$+ |\Delta\lambda_{2^{n+2}-1}| 2^{n+3} + 2|\Delta\lambda_{2^n}|(2^n+1) \Bigg)$$

$$+ \Big\| |\lambda_{2^{n+2}}| \Big(T_{2^{n+2}}(f) - T_{2^n}(f) \Big) \Big\|_p .$$

By Lemma 1.4.4,

$$I \leqslant C \Big(\sum_{k=2^{n-1}}^{2^n} |\Delta\lambda_k| + |\lambda_{2^{n+2}}| \Big) \, \| T_{2^{n+2}}(f) - T_{2^n}(f) \|_p \, ,$$

completing the proof. □

The following result provides a simple multiplier property of general monotone sequences.

Lemma 1.4.6. *Let* $\{a_n\} \in GM$. *Then* $\{n^\alpha a_n\} \in GM$, $\alpha \in \mathbb{R}$.

The proof follows from Lemma 1.4.2.

Lemma 1.4.7. *Let* $\lambda = \{\lambda\}_{m=1}^{\infty} \in GM$, *then*

$$|\lambda_{2^m}| \lesssim 2^{m\rho} \Lambda_{2^m},$$

where $\Lambda_{2^m} := \Big(\sum_{k=1}^{2^m} \frac{|\lambda_k|^q}{k^{q\rho+1}} \Big)^{\frac{1}{q}}$ *and* $\rho, q > 0$.

Proof. Using properties of GM sequences, we get

$$2^{m\rho}|\lambda_{2^m}|(2^{-m\rho q})^{\frac{1}{q}} \asymp 2^{m\rho}|\lambda_{2^m}|\Big(\sum_{k=2^{m-1}}^{2^m}\frac{1}{k^{q\rho+1}}\Big)^{\frac{1}{q}}$$
$$\lesssim 2^{m\rho}\Big(\sum_{k=2^{m-1}}^{2^m}\frac{|\lambda_k|^q}{k^{q\rho+1}}\Big)^{\frac{1}{q}} \lesssim 2^{m\rho}\Big(\sum_{k=1}^{2^m}\frac{|\lambda_k|^q}{k^{q\rho+1}}\Big)^{\frac{1}{q}} = 2^{m\rho}\Lambda_{2^m}.$$

□

1.5 Auxiliary results

In this subsection we give some useful Lemmas that will be used in the proof our main results.

Lemma 1.5.1. *[55] Let* $p = 1, \infty$. *If* $T_{2^n,2^{n+1}}(x) = \sum\limits_{\nu=2^n}^{2^{n+1}}(c_\nu \cos\nu x + d_\nu \sin\nu x)$, *then*

$$\|\widetilde{T}_{2^n,2^{n+1}}(x)\|_p \lesssim \|T_{2^n,2^{n+1}}(x)\|_p \lesssim \|\widetilde{T}_{2^n,2^{n+1}}(x)\|_p.$$

Lemma 1.5.2. *[66] Let* $1 \leqslant p \leqslant \infty$ *and* $\alpha > 0$. *If* $T_n(x) = \sum\limits_{\nu=1}^{n}(c_\nu \cos\nu x + d_\nu \sin\nu x)$, *then*

$$n^{-\alpha}\|T_n^{(\alpha)}(x)\|_p \lesssim \omega_\alpha(T_n, \frac{1}{n})_p.$$

The following result is the well-known Marcinkiewicz multiplier theorem.

Lemma 1.5.3. *[88, Ch.XV] Let* $f \in L_p$, $1 < p < \infty$, *and* (1.1.1) *be the Fourier series of* f. *If* $\lambda = \{\lambda\}_{n=1}^\infty$ *satisfies the following conditions*

$$|\lambda_\nu| \leqslant M,$$
$$\sum_{s=2^\nu}^{2^{\nu+1}-1}|\lambda_s - \lambda_{s+1}| \leqslant M \ \ (\nu = 0, 1, 2, ...),$$

then there exists a function $\varphi \in L_p$ *with the Fourier series* $\sigma(f, \lambda, \beta)$ *and*

$$\|\varphi\|_p \leqslant C(M, p)\|f\|_p.$$

The following result is the well-known Hardy–Littlewood fractional integration theorem.

Lemma 1.5.4. *[88, Ch.XII] Suppose that* $1 < p < q < \infty, \quad \theta = \frac{1}{p} - \frac{1}{q}$, *and* $T_n(x) = \sum\limits_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)$, *then*

$$\|T_n(x)\|_q \lesssim \|T_n(x)^{(\theta)}\|_p.$$

Lemma 1.5.5. *(Minkowskii inequality, [43]). Let* $1 \leqslant p < \infty$ *and* $a_{\nu k} \geqslant 0$. *Then*

$$(a) \quad \Big(\sum_{k=1}^{\infty} \Big(\sum_{\nu=1}^{k} a_{\nu k}\Big)^p\Big)^{\frac{1}{p}} \leqslant \sum_{\nu=1}^{\infty} \Big(\sum_{k=\nu}^{\infty} a_{\nu k}^p\Big)^{\frac{1}{p}},$$

$$(b) \quad \Big(\sum_{k=1}^{\infty} \Big(\sum_{\nu=k}^{\infty} a_{\nu k}\Big)^p\Big)^{\frac{1}{p}} \leqslant \sum_{\nu=1}^{\infty} \Big(\sum_{k=1}^{\nu} a_{\nu k}^p\Big)^{\frac{1}{p}}.$$

Lemma 1.5.6. *[43]. For a function* $f(u, y)$, *defined on measurable set* $E = E_1 \times E_2 \subset \mathbb{R}_n$, *where* $x = (u, y), u = (x_1, ..., x_m), y = (x_{m+1}, ..., x_n)$, *the following inequality*

$$\Big(\int\limits_{E_1} \big|\int\limits_{E_2} f(u, y)dy\big|^p du\Big)^{\frac{1}{p}} \leqslant \int\limits_{E_2} \Big(\int\limits_{E_1} \big|f(u, y)\big|^p du\Big)^{\frac{1}{p}} dy$$

holds for those $p \geqslant 1$ *for which the right part of this inequality is finite.*

Lemma 1.5.7. *[88]. Let* $f \in L_p, 1 < p < \infty, n \in \mathbb{N} \cup \{0\} (i = 1, 2)$. *Then*

$$\|S_n(f)\|_p \leqslant C\|f\|_p,$$

$$C_1\|f - S_n(f)\|_p \leqslant E_n(f)_p \leqslant C_2\|f - S_n(f)\|_p.$$

We denote $\Delta_0 = A_0(x), \Delta_m = \sum\limits_{n=2^{m-1}}^{2^m-1} A_n(x), \ m \in \mathbb{N}$.

Lemma 1.5.8. *[88].*

(a). Let $f \in L_p, 1 < p < \infty$. *Then*

$$\Big(\int\limits_0^{2\pi} \Big(\sum_{\nu=0}^{\infty} \Delta_\nu^2\Big)^{\frac{p}{2}} dx\Big)^{\frac{1}{p}} \leqslant C(p)\|f\|_p.$$

(b). Let $1 < p < \infty$. If (1.1.1) *satisfies the following inequality*

$$I_p = \Big(\int_0^{2\pi}\int_0^{2\pi}\Big(\sum_{\nu=0}^{\infty}\Delta_\nu^2\Big)^{\frac{p}{2}}dx\Big)^{\frac{1}{p}} < \infty.$$

Then (1.1.1) *is the Fourier series of a function $f(x) \in L_p$ and $\|f\|_p \leqslant C(p)I_p$.*

We will also need the following technical lemma.

Lemma 1.5.9. *Let $\alpha \in \mathbb{R}_+, \beta \in \mathbb{R}, \rho \in \mathbb{R}_+ \cup \{0\}$. Let $f \in L_p$, $1 \leqslant p \leqslant \infty$ and $S_n(f)$ be the n-th partial sum of Fourier series of f. In particular,*

$$S_n(f^{(\lambda,\beta)}) = \sum_{m=1}^{n}\lambda_m\bigg(a_m\cos\Big(mx+\frac{\pi\beta}{2}\Big)+b_m\sin\Big(mx+\frac{\pi\beta}{2}\Big)\bigg).$$

Then the following inequality holds

$$\|S_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_p \leqslant \Big|\cos\frac{\pi(\beta-\rho)}{2}\Big|\Big\|S_{2^n}^{(\alpha+\rho)}(f^{(\frac{\lambda_n}{n^\rho},0)})\Big\|_p$$
$$+\Big|\sin\frac{\pi(\beta-\rho)}{2}\Big|\Big\|\widetilde{S}_{2^n}^{(\alpha+\rho)}(f^{(\frac{\lambda_n}{n^\rho},0)})\Big\|_p.$$

Proof. We have

$$S_{2^n}^{(\alpha)}(f^{(\lambda,\beta)}) = \bigg(\sum_{m=1}^{n}\lambda_m\Big[a_m\cos\big(mx+\frac{\pi\beta}{2}\big)+b_m\sin\big(mx+\frac{\pi\beta}{2}\big)\Big]\bigg)^{(\alpha)}$$
$$= \bigg(\sum_{m=1}^{n}\lambda_m\Big[a_m\cos\big(mx+\frac{\pi(\beta-\rho)}{2}+\frac{\pi\rho}{2}\big)+b_m\sin\big(mx+\frac{\pi(\beta-\rho)}{2}+\frac{\pi\rho}{2}\big)\Big]\bigg)^{(\alpha)}$$
$$= \bigg(\Big(\sum_{m=1}^{n}\frac{\lambda_m}{m^\rho}m^\rho\Big[a_m\cos\big(mx+\frac{\pi\rho}{2}\big)+b_m\sin\big(mx+\frac{\pi\rho}{2}\big)\Big]\Big)\cos\frac{\pi(\beta-\rho)}{2}$$
$$-\Big(\sum_{m=1}^{n}\frac{\lambda_m}{m^\rho}m^\rho\Big[a_m\sin\big(mx+\frac{\pi\rho}{2}\big)-b_m\cos\big(mx+\frac{\pi\rho}{2}\big)\Big]\Big)\sin\frac{\pi(\beta-\rho)}{2}\bigg)^{(\alpha)} =: I.$$

Applying the definition of the fractional derivative, we have

$$\|I\|_p = \Bigg\| \left(\sum_{m=1}^{n} \frac{\lambda_m}{m^\rho} \Big[a_m \cos mx + b_m \sin mx \Big] \right)^{(\alpha+\rho)} \cos \frac{\pi(\beta-\rho)}{2}$$

$$- \left(\sum_{m=1}^{n} \frac{\lambda_m}{m^\rho} \Big[a_m \sin mx - b_m \cos mx \Big] \right)^{(\alpha+\rho)} \sin \frac{\pi(\beta-\rho)}{2} \Bigg\|_p$$

$$\leqslant \left| \cos \frac{\pi(\beta-\rho)}{2} \right| \left\| S_{2^n}^{(\alpha+\rho)}(f^{(\frac{\lambda_n}{n^\rho},0)}) \right\|_p + \left| \sin \frac{\pi(\beta-\rho)}{2} \right| \left\| \widetilde{S}_{2^n}^{(\alpha+\rho)}(f^{(\frac{\lambda_n}{n^\rho},0)}) \right\|_p.$$

The proof is now complete. □

1.6 Estimates of best approximations and moduli of smoothness for generalized Liouville - Weyl derivatives

In this subsection, we obtain estimates of norms, best approximations, and moduli of smoothness in L_p, $1 < p < \infty$, of the generalized Liouville–Weyl derivatives via the best approximation and moduli of smoothness of the function itself. The main results of this section were published in [30]. For the sake of convenience, we assume that $[\xi] = 1$ for $0 < \xi < 1$.

Theorem 1.6.1. *Let* $1 < p < \infty$, $\theta = \min(2,p)$, $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in GM$, $\alpha \in \mathbb{R}_+$, *and* $r \in \mathbb{R}_+ \cup \{0\}$. *If* $f \in L_p$ *and the series*

$$\sum_{n=1}^{\infty} |\lambda_{n+1}^{\theta} - \lambda_n^{\theta}| E_n^{\theta}(f)_p \tag{1.6.28}$$

converges, then there exists a function $f^{(\lambda,\beta)} \in L_p$ *with the Fourier series* $\sigma(f,\lambda,\beta)$ *and*

$$\|f^{(\lambda,\beta)}\|_p \lesssim \left\{ \lambda_1^{\theta} E_0^{\theta}(f)_p + \sum_{n=1}^{\infty} |\lambda_{n+1}^{\theta} - \lambda_n^{\theta}| E_n^{\theta}(f)_p \right\}^{\frac{1}{\theta}}, \tag{1.6.29}$$

$$E_n(f^{(\lambda,\beta)})_p \lesssim \left\{ \lambda_{[n/2]}^{\theta} E_{[n/2]}^{\theta}(f)_p + \sum_{k=[n/4]}^{\infty} |\lambda_{k+1}^{\theta} - \lambda_k^{\theta}| E_k^{\theta}(f)_p \right\}^{\frac{1}{\theta}}, \tag{1.6.30}$$

and

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{n}\Big)_p \lesssim \Big\{n^{-\alpha\theta}\sum_{k=1}^{n}\Big|\frac{\lambda_k^\theta}{k^{r\theta}}-\frac{\lambda_{k+1}^\theta}{(k+1)^{r\theta}}\Big|k^{(r+\alpha)\theta}\omega_{\alpha+r}^\theta\Big(f,\frac{1}{k}\Big)_p + \sum_{k=n+1}^{\infty}|\lambda_{k+1}^\theta-\lambda_k^\theta|\omega_{\alpha+r}^\theta\Big(f,\frac{1}{k}\Big)_p+\lambda_n^\theta\omega_{\alpha+r}^\theta\Big(f,\frac{1}{n}\Big)_p\Big\}^{\frac{1}{\theta}}. \tag{1.6.31}$$

Remark 1.6.2. *In fact, we will obtain a sharper inequality than* (1.6.30)*:*

$$E_{2^m-1}(f^{(\lambda,\beta)})_p \lesssim \Big\{\lambda_{2^m}^\theta E_{2^m-1}^\theta(f)_p + \sum_{\nu=m+1}^{\infty}|\lambda_{2^{\nu-1}}^\theta-\lambda_{2^{\nu-2}}^\theta|E_{2^{\nu-1}-1}^\theta(f)_p\Big\}^{\frac{1}{\theta}}.$$

Remark 1.6.3. *Since any decreasing sequence or any increasing sequence satisfying the Δ_2-condition belongs to the GM class, Theorem 1.6.1 extends Theorem A1.*

Analogues of Theorems 1.6.1 for the class $GM^\uparrow$ can be written as follows.

Theorem 1.6.1′**.** *Let $1<p<\infty$, $\theta=\min(2,p)$, $\lambda=\{\lambda_n\}_{n=1}^\infty\in GM^\uparrow$, $\alpha\in\mathbb{R}_+$, $r\geqslant 0$. If $f\in L_p$ and*

$$\sum_{n=1}^{\infty}|\lambda_{2^n}^\theta-\lambda_{2^{n-1}}^\theta|E_{2^{n-1}}^\theta(f)_p<\infty,$$

then there exists a function $f^{(\lambda,\beta)}\in L_p$ with the Fourier series $\sigma(f,\lambda,\beta)$ and

$$\|f^{(\lambda,\beta)}\|_p \lesssim \Big\{\lambda_1^\theta E_0^\theta(f)_p+\sum_{n=1}^{\infty}|\lambda_{2^n}^\theta-\lambda_{2^{n-1}}^\theta|E_{2^{n-1}}^\theta(f)_p\Big\}^{\frac{1}{\theta}},$$

$$E_{2^n}(f^{(\lambda,\beta)})_p \lesssim \Big\{\lambda_{2^n}^\theta E_{2^{n-1}}^\theta(f)_p+\sum_{k=n}^{\infty}|\lambda_{2^k}^\theta-\lambda_{2^{k-1}}^\theta|E_{2^{k-1}}^\theta(f)_p\Big\}^{\frac{1}{\theta}},$$

and

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_p \lesssim \Big\{2^{-n\alpha\theta}\sum_{\nu=1}^{n}2^{(r+\alpha)\nu\theta}\Big|\frac{\lambda_{2^{\nu+1}}^\theta}{2^{(\nu+1)r\theta}}-\frac{\lambda_{2^\nu}^\theta}{2^{\nu r\theta}}\Big|\omega_{\alpha+r}^\theta\Big(f,\frac{1}{2^\nu}\Big)_p + \sum_{\nu=n+2}^{\infty}|\lambda_{2^\nu}^\theta-\lambda_{2^{\nu-1}}^\theta|\omega_{\alpha+r}^\theta\Big(f,\frac{1}{2^\nu}\Big)_p+\lambda_{2^n}^\theta\omega_{\alpha+r}^\theta\Big(f,\frac{1}{2^n}\Big)_p\Big\}^{\frac{1}{\theta}}.$$

Proof of Theorem 1.6.1. Let $f \in L_p$ and series (1.6.28) be convergent. We use the following inequality

$$|\lambda^{\theta}_{2^{n-1}}| \leqslant |\lambda^{\theta}_{1}| + \sum_{\nu=2}^{n} |\lambda^{\theta}_{2^{\nu-1}} - \lambda^{\theta}_{2^{\nu-2}}|. \tag{1.6.32}$$

Recall that $\Delta_1 = A_1(f,x),\ \Delta_n = \sum\limits_{2^{n-1}}^{2^n-1} A_\nu(f,x),\ n = 2,3,...$ Using inequality (1.6.32) and the inequality

$$C_1(\alpha)(u_1^\alpha + u_2^\alpha) \leqslant (u_1+u_2)^\alpha \leqslant C_2(\alpha)(u_1^\alpha + u_2^\alpha)$$

for $\alpha > 0$ and $u_1, u_1 \geqslant 0$, we get

$$\begin{aligned} I_1 &= \left\{ \int_0^{2\pi} \left[\sum_{n=1}^{\infty} |\lambda^2_{2^{n-1}}| \Delta_n^2 \right]^{\frac{p}{2}} dx \right\}^{\frac{1}{p}} \\ &\leqslant \left\{ \int_0^{2\pi} \left[|\lambda^2_1|\Delta_1^2 + \sum_{n=2}^{\infty} \Delta_n^2 \left[\lambda_1^\theta + \sum_{\nu=2}^{n} |\lambda^{\theta}_{2^{\nu-1}} - \lambda^{\theta}_{2^{\nu-2}}| \right]^{\frac{2}{\theta}} \right]^{\frac{p}{2}} dx \right\}^{\frac{1}{p}} \\ &\lesssim \left\{ \int_0^{2\pi} \left\{ \left[\sum_{n=1}^{\infty} |\lambda^2_1|\Delta_n^2 \right]^{\frac{\theta}{2}} + \left[\sum_{n=2}^{\infty} \left(\sum_{\nu=2}^{n} \Delta_n^\theta |\lambda^{\theta}_{2^{\nu-1}} - \lambda^{\theta}_{2^{\nu-2}}| \right)^{\frac{2}{\theta}} \right]^{\frac{\theta}{2}} \right\}^{\frac{p}{\theta}} dx \right\}^{\frac{1}{p}}. \end{aligned}$$

Further, applying again Minkowski's inequality (Lemma 1.5.5) for $\frac{2}{\theta} \geqslant 1$, we have

$$\begin{aligned} I_1 &\lesssim \left\{ \int_0^{2\pi} \left\{ \lambda_1^\theta \left[\sum_{n=1}^{\infty} \Delta_n^2 \right]^{\frac{\theta}{2}} + \sum_{\nu=2}^{\infty} \left[\sum_{n=\nu}^{\infty} \left[\Delta_n^\theta |\lambda^{\theta}_{2^{\nu-1}} - \lambda^{\theta}_{2^{\nu-2}}| \right]^{\frac{2}{\theta}} \right]^{\frac{\theta}{2}} \right\}^{\frac{p}{\theta}} dx \right\}^{\frac{1}{p}} \\ &\lesssim \left\{ \left(\int_0^{2\pi} \left[\lambda_1^\theta \left(\sum_{n=1}^{\infty} \Delta_n^2 \right)^{\frac{\theta}{2}} \right]^{\frac{p}{\theta}} dx \right)^{\frac{\theta}{p}} \right. \\ &\left. + \left(\int_0^{2\pi} \left[\sum_{\nu=2}^{\infty} |\lambda^{\theta}_{2^{\nu-1}} - \lambda^{\theta}_{2^{\nu-2}}| \left(\sum_{n=\nu}^{\infty} \Delta_n^2 \right)^{\frac{\theta}{2}} \right]^{\frac{p}{\theta}} dx \right)^{\frac{\theta}{p}} \right\}^{\frac{1}{\theta}}. \end{aligned}$$

Using Minkowski's inequality with $\frac{p}{\theta} \geqslant 1$, the Littlewood-Paley theorem ([88, Ch.15]), and the following inequality $\|f - S_n(f)\|_p \lesssim E_n(f)_p$ ([8, p. 207]), we obtain

$$I_1 \lesssim \left\{ \lambda_1^\theta E_0^\theta(f)_p + \sum_{\nu=1}^{\infty} |\lambda_{2^\nu}^\theta - \lambda_{2^{\nu-1}}^\theta| E_{2^\nu-1}^\theta(f)_p \right\}^{\frac{1}{\theta}}. \tag{1.6.33}$$

Then, convergence of the series (1.6.28) implies $I_1 < \infty$. Hence, by the Littlewood-Paley theorem, there exists a function $g \in L_p$ with the Fourier series

$$\sum_{n=1}^{\infty} |\lambda_{2^{n-1}}| \Delta_n \tag{1.6.34}$$

and $\|g\|_p \lesssim I_1$. We rewrite series (1.6.34) in the form of $\sum\limits_{n=1}^{\infty} \gamma_n A_n(f, x)$, where $\gamma_1 := |\lambda_1|$, $\gamma_\nu := |\lambda_{2^{n-1}}|$ for $2^{n-1} \leqslant \nu \leqslant 2^n - 1$ $(n = 2, 3, ...)$. Consider

$$\sum_{n=1}^{\infty} \lambda_n A_n(f, x) = \sum_{n=1}^{\infty} \gamma_n \Lambda_n A_n(f, x), \tag{1.6.35}$$

where $\Lambda_1 := 1$, $\Lambda_\nu := \frac{|\lambda_\nu|}{\gamma_n} = \frac{|\lambda_\nu|}{|\lambda_{2^{n-1}}|}$ for $2^{n-1} \leqslant \nu \leqslant 2^n - 1$ $(n = 2, 3, ...)$. Since the sequence $\Lambda = \{\Lambda_\nu\}_{\nu=1}^{\infty}$ satisfies the conditions of the Marcinkiewicz theorem (see Lemma 1.5.3), then series (1.6.35) is the Fourier series of a function $f^{(\lambda,\beta)} \in L_p$ and

$$\|f^{(\lambda,\beta)}\|_p \lesssim \|g\|_p.$$

Further, we estimate the right-hand side of (1.6.33) as follows

$$\begin{aligned} \|f^{(\lambda,\beta)}\|_p &\lesssim \left\{ \lambda_1^\theta E_0^\theta(f)_p + \sum_{\nu=1}^{\infty} |\lambda_{2^\nu}^\theta - \lambda_{2^{\nu-1}}^\theta| E_{2^\nu-1}^\theta(f)_p \right\}^{\frac{1}{\theta}} \\ &\lesssim \left\{ \lambda_1^\theta E_0^\theta(f)_p + \sum_{\nu=1}^{\infty} E_{2^\nu-1}^\theta(f)_p \sum_{n=2^{\nu-1}}^{2^\nu-1} |\lambda_{n+1}^\theta - \lambda_n^\theta| \right\}^{\frac{1}{\theta}} \\ &\lesssim \left\{ \lambda_1^\theta E_0^\theta(f)_p + \sum_{n=1}^{\infty} |\lambda_{n+1}^\theta - \lambda_n^\theta| E_n^\theta(f)_p \right\}^{\frac{1}{\theta}}. \end{aligned}$$

Thus, (1.6.29) has been proved.

In the same way let us verify (1.6.30). Now we use the inequality $E_n(f^{(\lambda,\beta)})_p \leqslant \|f^{(\lambda,\beta)} - S_n(f^{(\lambda,\beta)})\|_p$. Applying inequality (1.6.32) and

Minkowski's inequality, we get

$$\left\{\int\limits_0^{2\pi}\left[\sum_{k=m+1}^{\infty}|\lambda^2_{2^{k-1}}|\Delta_k^2\right]^{\frac{p}{2}}dx\right\}^{\frac{1}{p}}$$

$$\leqslant\left\{\int\limits_0^{2\pi}\left[\sum_{k=m+1}^{\infty}\left(\lambda^\theta_{2^m}+\sum_{\nu=m+1}^{k}|\lambda^\theta_{2^{\nu-1}}-\lambda^\theta_{2^{\nu-2}}|\right)^{\frac{2}{\theta}}\Delta_k^2\right]^{\frac{p}{2}}dx\right\}^{\frac{1}{p}}$$

$$\leqslant\left\{\lambda^\theta_{2^m}\left(\int\limits_0^{2\pi}\left[\sum_{k=m+1}^{\infty}\Delta_k^2\right]^{\frac{p}{2}}dx\right)^{\frac{\theta}{p}}\right.$$

$$\left.+\sum_{\nu=m+1}^{\infty}|\lambda^\theta_{2^{\nu-1}}-\lambda^\theta_{2^{\nu-2}}|\left(\int\limits_0^{2\pi}\left[\sum_{k=\nu}^{\infty}\Delta_k^2\right]^{\frac{p}{2}}dx\right)^{\frac{\theta}{p}}\right\}^{\frac{1}{\theta}}=I_2.$$

By the Littlewood-Paley theorem, we have

$$I_2\lesssim\left\{\lambda^\theta_{2^m}E^\theta_{2^m-1}(f)_p+\sum_{\nu=m+1}^{\infty}|\lambda^\theta_{2^{\nu-1}}-\lambda^\theta_{2^{\nu-2}}|E^\theta_{2^{\nu-1}-1}(f)_p\right\}^{\frac{1}{\theta}}$$

$$\lesssim\left\{\lambda^\theta_{[\frac{n}{2}]}E^\theta_{[\frac{n}{2}]}(f)_p+\sum_{k=[\frac{n}{4}]}^{\infty}|\lambda^\theta_{k+1}-\lambda^\theta_k|E^\theta_k(f)_p\right\}^{\frac{1}{\theta}},\qquad(1.6.36)$$

for the given $m\in\mathbb{N}$. Now we choose n such that $2^m\leqslant n\leqslant 2^{m+1}$. Then two previous inequalities imply (1.6.30).

Finally, let us prove (1.6.31). Taking into account Lemma 1.2.3, we get

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{n}\Big)_p\lesssim\|f^{(\lambda,\beta)}-S_n(f^{(\lambda,\beta)})\|_p+n^{-\alpha}\|S_n^{(\alpha)}(f^{(\lambda,\beta)})\|_p.\qquad(1.6.37)$$

Let us estimate the first term. Applying Jackson's inequality to estimate (1.6.36), we obtain

$$\|f^{(\lambda,\beta)}-S_n(f^{(\lambda,\beta)})\|_p\lesssim\left\{\lambda^\theta_{[n/2]}\omega^\theta_{\alpha+r}\Big(f,\frac{2}{n}\Big)_p+\sum_{s=[n/4]}^{\infty}|\lambda^\theta_{s+1}-\lambda^\theta_s|\omega^\theta_{\alpha+r}\Big(f,\frac{1}{s}\Big)_p\right\}^{\frac{1}{\theta}}.\qquad(1.6.38)$$

To estimate the second term in (1.6.37), we use the same reasoning as in (1.6.29) and the following inequality

$$2^{-sr\theta}|\lambda^\theta_{2^s}|\leqslant 2^{-mr\theta}|\lambda^\theta_{2^m}|+\sum_{\nu=s}^{m-1}|2^{-\nu r\theta}\lambda^\theta_{2^\nu}-2^{-(\nu+1)r\theta}\lambda^\theta_{2^{\nu+1}}|.$$

Using this inequality, we obtain

$$I_3 = \left\{ \int\limits_0^{2\pi} \left[\sum_{s=1}^{m} |\lambda_{2^{s-1}}^2| 2^{2\alpha(s-1)} \Delta_s^2 \right]^{\frac{p}{2}} dx \right\}^{\frac{1}{p}}$$

$$\leqslant \left\{ \int\limits_0^{2\pi} \left[\sum_{s=1}^{m} \Bigg(2^{-mr\theta} \lambda_{2^m}^{\theta} + \right.\right.$$

$$\left.\left. + \sum_{\nu=s-1}^{m-1} |2^{-\nu r\theta} \lambda_{2^\nu}^{\theta} - 2^{-(\nu+1)r\theta} \lambda_{2^{\nu+1}}^{\theta}| \Bigg)^{\frac{2}{\theta}} 2^{2(s-1)(\alpha+r)} \Delta_s^2 \right]^{\frac{p}{2}} dx \right\}^{\frac{1}{p}}.$$

By Minkowski's inequality,

$$I_3 \leqslant \left\{ \left(\int\limits_0^{2\pi} \left[2^{-mr\theta} \lambda_{2^m}^{\theta} \left[\sum_{s=1}^{m} 2^{2(s-1)(\alpha+r)} \Delta_s^2 \right]^{\frac{\theta}{2}} \right]^{\frac{p}{\theta}} dx \right)^{\frac{\theta}{p}} \right.$$

$$\left. + \sum_{\nu=0}^{m-1} |2^{-\nu r\theta} \lambda_{2^\nu}^{\theta} - 2^{-(\nu+1)r\theta} \lambda_{2^{\nu+1}}^{\theta}| \left(\int\limits_0^{2\pi} \left[\sum_{s=1}^{\nu+1} 2^{2(s-1)(\alpha+r)} \Delta_s^2 \right]^{\frac{p}{2}} dx \right)^{\frac{\theta}{p}} \right\}^{\frac{1}{\theta}} = I_4.$$

By the Littlewood–Paley theorem, we have

$$\left(\int\limits_0^{2\pi} \left[\sum_{s=1}^{m} 2^{2(s-1)(\alpha+r)} \Delta_s^2 \right]^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \lesssim \|S_{2^m-1}^{(\alpha+r)}(f)\|_p.$$

From Lemma 1.5.2, we obtain

$$\begin{aligned} \|S_{2^m-1}^{(\alpha+r)}(f)\|_p &\lesssim 2^{m(\alpha+r)} \omega_{\alpha+r}\Big(S_{2^m-1} \pm f, \frac{1}{2^m-1}\Big)_p \\ &\lesssim 2^{m(\alpha+r)} \Big(\omega_{\alpha+r}\Big(f, \frac{1}{2^m-1}\Big)_p + \|S_{2^m-1} - f\|_p \Big) \\ &\lesssim 2^{m(\alpha+r)} \Big(\omega_{\alpha+r}\Big(f, \frac{1}{2^m-1}\Big)_p + E_{2^m}(f)_p \Big) \\ &\lesssim 2^{m(\alpha+r)} \omega_{\alpha+r}\Big(f, \frac{1}{2^m-1}\Big)_p. \end{aligned}$$

Therefore,

$$I_4 \lesssim \left\{ 2^{-mr\theta} \lambda_{2^m}^{\theta} 2^{m(\alpha+r)\theta} \omega_{\alpha+r}^{\theta}\Big(f, \frac{1}{2^m-1}\Big)_p \right.$$

$$\left. + \sum_{\nu=0}^{m-1} |2^{-\nu r\theta} \lambda_{2^\nu}^{\theta} - 2^{-(\nu+1)r\theta} \lambda_{2^{\nu+1}}^{\theta}| 2^{(\nu+1)(\alpha+r)\theta} \omega_{\alpha+r}^{\theta}\Big(f, \frac{1}{2^{\nu+1}-1}\Big)_p \right\}^{\frac{1}{\theta}}.$$

Using monotonicity properties of moduli of smoothness and the formula

$$\frac{\lambda_{2^\nu}^\theta}{2^{\nu\theta r}}-\frac{\lambda_{2^{\nu+1}}^\theta}{2^{(\nu+1)\theta r}}=\sum_{k=2^\nu}^{2^{\nu+1}-1}\Big(\frac{\lambda_k^\theta}{k^{\theta r}}-\frac{\lambda_{k+1}^\theta}{(k+1)^{\theta r}}\Big),$$

we have

$$I_4\lesssim\Bigg\{2^{\alpha m\theta}\lambda_{2^m}^\theta\omega_{\alpha+r}^\theta\Big(f,\frac{1}{2^m-1}\Big)_p$$
$$+\,2^{(\alpha+r)\theta}\sum_{k=1}^{2^m-1}|k^{-r\theta}\lambda_k^\theta-(k+1)^{-r\theta}\lambda_{k+1}^\theta|k^{(\alpha+r)}\omega_{\alpha+r}^\theta\Big(f,\frac{1}{k}\Big)_p\Bigg\}^{\frac{1}{\theta}}.$$

Combining estimates for I_3 and I_4, we get

$$n^{-\alpha}\|S_n^{(\alpha)}(f^{(\lambda,\beta)})\|_p\lesssim\Bigg\{\lambda_{[n/2]}^\theta\omega_{\alpha+r}^\theta\Big(f,\frac{2}{n}\Big)_p$$
$$+\,n^{-\alpha\theta}\sum_{\nu=1}^{n}|\nu^{-r\theta}\lambda_\nu^\theta-(\nu+1)^{-r\theta}\lambda_{\nu+1}^\theta|\nu^{(r+\alpha)\theta}\omega_{\alpha+r}^\theta\Big(f,\frac{1}{\nu}\Big)_p\Bigg\}^{\frac{1}{\theta}}.\tag{1.6.39}$$

From (1.6.38) and (1.6.39), we have

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{n+1}\Big)_p$$
$$\lesssim\Bigg\{n^{-\alpha\theta}\sum_{k=1}^{n}k^{(r+\alpha)\theta}|k^{-r\theta}\lambda_k^\theta-(k+1)^{-r\theta}\lambda_{k+1}^\theta|\omega_{\alpha+r}^\theta\Big(f,\frac{1}{k}\Big)_p\tag{1.6.40}$$
$$+\sum_{k=[n/4]}^{\infty}|\lambda_{k+1}^\theta-\lambda_k^\theta|\omega_{\alpha+r}^\theta\Big(f,\frac{1}{k}\Big)_p+\lambda_{[n/2]}^\theta\omega_{\alpha+r}^\theta\Big(f,\frac{2}{n}\Big)_p\Bigg\}^{\frac{1}{\theta}}.$$

Now we note that

$$\sum_{k=[n/4]}^{n}|\lambda_{k+1}^\theta-\lambda_k^\theta|\omega_{\alpha+r}^\theta\Big(f,\frac{1}{k}\Big)_p$$
$$\leqslant\sum_{k=[n/4]}^{n}\Bigg(\frac{\lambda_{k+1}^\theta}{(k+1)^{r\theta}}k^{r\theta-1}+k^{r\theta}\Big|\frac{\lambda_k^\theta}{k^{r\theta}}-\frac{\lambda_{k+1}^\theta}{(k+1)^{r\theta}}\Big|\Bigg)\omega_{\alpha+r}^\theta\Big(f,\frac{1}{k}\Big)_p$$
$$\leqslant\lambda_{[n/4]}^\theta\omega_{\alpha+r}^\theta\Big(f,\frac{4}{n}\Big)_p+\sum_{k=[n/4]}^{n}\Big|\frac{\lambda_k^\theta}{k^{r\theta}}-\frac{\lambda_{k+1}^\theta}{(k+1)^{r\theta}}\Big|k^{r\theta}\omega_{\alpha+r}^\theta\Big(f,\frac{1}{k}\Big)_p.$$

Thus, we have

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{n}\Big)_p \lesssim \Big\{ n^{-\alpha\theta} \sum_{k=1}^{n} k^{(r+\alpha)\theta} |k^{-r\theta}\lambda_k^\theta - (k+1)^{-r\theta}\lambda_{k+1}^\theta| \omega_{\alpha+r}^\theta\Big(f, \frac{1}{k}\Big)_p$$

$$+ \sum_{k=n+1}^{\infty} |\lambda_{k+1}^\theta - \lambda_k^\theta| \omega_{\alpha+r}^\theta\Big(f, \frac{1}{k}\Big)_p + \lambda_{[n/4]}^\theta \omega_{\alpha+r}^\theta\Big(f, \frac{1}{n}\Big)_p \Big\}^{\frac{1}{\theta}}.$$

Nothing that

$$\lambda_{\frac{n}{4}}^\theta \leqslant \Big(\frac{n}{4}\Big)^{r\theta} \sum_{k=[n/4]}^{n-1} \Big| \frac{\lambda_k^\theta}{k^{r\theta}} - \frac{\lambda_{k+1}^\theta}{(k+1)^{r\theta}} \Big| + \frac{\lambda_n^\theta}{n^{r\theta}} \Big(\frac{n}{4}\Big)^{r\theta},$$

we arrive at

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{n}\Big)_p \lesssim \Big\{ n^{-\alpha\theta} \sum_{k=1}^{n} k^{(r+\alpha)\theta} |k^{-r\theta}\lambda_k^\theta - (k+1)^{-r\theta}\lambda_{k+1}^\theta| \omega_{\alpha+r}^\theta\Big(f, \frac{1}{k}\Big)_p$$

$$+ \sum_{k=n}^{\infty} |\lambda_{k+1}^\theta - \lambda_k^\theta| \omega_{\alpha+r}^\theta\Big(f, \frac{1}{k}\Big)_p + \lambda_n^\theta \omega_{\alpha+r}^\theta\Big(f, \frac{1}{n}\Big)_p \Big\}^{\frac{1}{\theta}}.$$

The proof of Theorem 1.6.1 is complete. □

1.7 Estimates for transformed Fourier series in the limiting cases $p = 1$ and $p = \infty$

In this subsection we obtain estimates of norms, best approximations, and moduli of smoothness of the generalized Liouville–Weyl derivatives in the limiting cases $p = 1$ and $p = \infty$.

Theorem 1.7.1. *Let $p = 1$ or $p = \infty$ and $\{\Delta\lambda_n\} \in GM$. If $f \in L_p$ and the series*

$$\sum_{n=1}^{\infty} \Big(\sum_{k=2^{n-1}}^{2^n} |\Delta\lambda_k| + |\lambda_{2^{n+2}}| \Big) E_{2^n}(f)_p \tag{1.7.41}$$

converges, then there exists a function $f^{(\lambda,\beta)} \in L_p$ with the Fourier series $\sigma(f, \lambda, \beta)$ such that

$$\|f^{(\lambda,\beta)}\|_p \lesssim |\lambda_1| E_0(f)_p + \sum_{n=1}^{\infty} \Big(\sum_{k=2^{n-1}}^{2^n} |\Delta\lambda_k| + |\lambda_{2^{n+2}}| \Big) E_{2^{n-1}}(f)_p \tag{1.7.42}$$

and

$$E_{2^{n+1}}(f^{(\lambda,\beta)})_p \lesssim \sum_{s=n}^{\infty}\Big(\sum_{k=2^{s-1}}^{2^s}|\Delta\lambda_k| + |\lambda_{2^{s+2}}|\Big)E_{2^{s-1}}(f)_p. \qquad (1.7.43)$$

Moreover, if $\{\Delta\frac{\lambda_n}{n^\rho}\}\in GM$, $\alpha\in\mathbb{R}_+$ *and* $\rho\in\mathbb{R}_+\cup\{0\}$, *then*

$$\begin{aligned}\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_p &\lesssim \sum_{s=n}^{\infty}\Big(\sum_{k=2^{s-1}}^{2^s}|\Delta\lambda_k| + |\lambda_{2^{s+2}}|\Big)\omega_{\alpha+r}\Big(f,\frac{1}{2^s}\Big)_p\\ &+ 2^{-n\alpha}\sum_{\mu=1}^{2^{n+2}-1}\Big|\frac{\lambda_\mu}{\mu^\rho}-\frac{\lambda_{\mu+1}}{(\mu+1)^\rho}\Big|\mu^{\alpha+\rho}\omega_{\alpha+\rho}\Big(f,\frac{1}{\mu}\Big)_p \qquad (1.7.44)\\ &+ 2^{-n\alpha}\Big|\sin\frac{\pi(\beta-\rho)}{2}\Big|\sum_{k=1}^{n}|\lambda_{2^{n+1}}|2^{n\alpha}\omega_{\alpha+\rho}\Big(f,\frac{1}{2^n}\Big)_p.\end{aligned}$$

Remark 1.7.4. *Note that the conditions* $\{\Delta\lambda_n\}\in GM$, $\{\Delta\frac{\lambda_n}{n^\rho}\}\in GM$ *in Theorem 1.7.1 are much weaker than the corresponding conditions on* $\{\lambda_n\}$ *in Theorem A2.*

The following analogues of Theorems 1.7.1 for $GM^\uparrow$ can be written as follows.

Theorem 1.7.1′ *Let* $p=1$ *or* $p=\infty$, $\{\Delta\lambda_n\}\in GM^\uparrow$, $\{\Delta\frac{\lambda_n}{n^\rho}\}\in GM^\uparrow$, $\alpha\in\mathbb{R}_+$, *and* $\rho\in\mathbb{R}_+\cup\{0\}$. *If* $f\in L_p$ *and the series*

$$\sum_{n=1}^{\infty}\Big(\sum_{k=2^{n-1}}^{2^n}|\Delta\lambda_k| + |\lambda_{2^{n+2}}|\Big)E_{2^n}(f)_p$$

converges, then there exists a function $f^{(\lambda,\beta)}\in L_p$ *with the Fourier series* $\sigma(f,\lambda,\beta)$ *and*

$$\|f^{(\lambda,\beta)}\|_p \lesssim |\lambda_1|E_0(f)_p + \sum_{n=1}^{\infty}\Big(\sum_{k=2^{n-1}}^{2^n}|\Delta\lambda_k| + |\lambda_{2^{n+2}}|\Big)E_{2^n}(f)_p,$$

$$E_{2^{n+1}}(f^{(\lambda,\beta)})_p \lesssim \sum_{s=n}^{\infty}\Big(\sum_{k=2^{s-1}}^{2^s}|\Delta\lambda_k| + |\lambda_{2^{s+2}}|\Big)E_{2^s}(f)_p,$$

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_p \lesssim \sum_{s=n}^{\infty}\Big(\sum_{k=2^{s-1}}^{2^s}|\Delta\lambda_k| + |\lambda_{2^{s+2}}|\Big)\omega_{\alpha+r}\Big(f,\frac{1}{2^s}\Big)_p$$

$$+ 2^{-n\alpha}\sum_{\mu=1}^{2^{n+2}-1}\Big|\frac{\lambda_\mu}{\mu^\rho}-\frac{\lambda_{\mu+1}}{(\mu+1)^\rho}\Big|\mu^{\alpha+\rho}\omega_{\alpha+\rho}\Big(f,\frac{1}{\mu}\Big)_p$$

$$+ 2^{-n\alpha}\Big|\sin\frac{\pi(\beta-\rho)}{2}\Big|\sum_{k=1}^{n}|\lambda_{2^{n+1}}|2^{n\alpha}\omega_{\alpha+\rho}\Big(f,\frac{1}{2^n}\Big)_p.$$

Proof of Theorem 1.7.1. We consider the series

$$\cos\frac{\pi\beta}{2}V_1(\lambda,f)-\sin\frac{\pi\beta}{2}\widetilde{V}_1(\lambda,f)+$$
$$\sum_{n=1}^{\infty}\Big\{\cos\frac{\pi\beta}{2}\Big(V_{2^n}(\lambda,f)-V_{2^{n-1}}(\lambda,f)\Big)-\sin\frac{\pi\beta}{2}\Big(\widetilde{V}_{2^n}(\lambda,f)-\widetilde{V}_{2^{n-1}}(\lambda,f)\Big)\Big\}, \tag{1.7.45}$$

where $V_1(\lambda,f):=\lambda_1A_1(f,x)$ and

$$V_n=\sigma(\lambda,V_n(f))=\sum_{m=1}^{n}\lambda_mA_m(f,x)+\sum_{m=n+1}^{2n-1}\lambda_m\Big(1-\frac{m-n}{n}\Big)A_m(f,x)\ (n\geqslant 2).$$

Let $M>N>0$. Applying the Abel transformation twice, we get

$$J:=\Big\|\sum_{n=N}^{M}\Big[\Big|\cos\frac{\pi\beta}{2}\Big|\Big(V_{2^{n+1}}(\lambda,f)-V_{2^n}(\lambda,f)\Big)$$
$$-\Big|\sin\frac{\pi\beta}{2}\Big|\Big(\widetilde{V}_{2^{n+1}}(\lambda,f)-\widetilde{V}_{2^n}(\lambda,f)\Big)\Big]\Big\|_p$$
$$=\Big\|\sum_{n=N}^{M}\Big\{\Big|\cos\frac{\pi\beta}{2}\Big|\Big[\frac{1}{\pi}\int_{-\pi}^{\pi}\Big(V_{2^{n+1}}(f)-V_{2^n}(f)\Big)(x+u)\sum_{\nu=2^n}^{2^{n+2}}(\lambda_\nu\cos\nu u)du\Big]$$
$$-\Big|\sin\frac{\pi\beta}{2}\Big|\Big[\frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\widetilde{V}_{2^{n+1}}(f)-\widetilde{V}_{2^n}(f)\Big)(x+u)\sum_{\nu=2^n}^{2^{n+2}}(\lambda_\nu\cos\nu u)du\Big]\Big\}\Big\|_p.$$

By Lemmas 1.4.5 and 1.5.1 and $\|f-V_n(f)\|\leqslant CE_n(f)$, we obtain

$$J\lesssim\sum_{n=N}^{M}\Big(\sum_{k=2^{n-1}}^{2^n}|\Delta\lambda_k|+|\lambda_{2^{n+2}}|\Big)\|V_{2^{n+1}}(f)-V_{2^n}(f)\|_p$$
$$\lesssim\sum_{n=N}^{M}\Big(\sum_{k=2^{n-1}}^{2^n}|\Delta\lambda_k|+|\lambda_{2^{n+2}}|\Big)E_{2^n}(f)_p.$$

Then the convergence of series in (1.7.41) implies that there exists $f^{(\lambda,\beta)} \in L_p$ such that the series (1.7.45) converges to $f^{(\lambda,\beta)}$ in L_p. Moreover, the Fourier series of $f^{(\lambda,\beta)}$ is $\sigma(f,\alpha,\beta)$. This can be shown as in the proof of Theorem 1 (ii) in [55, pp.1379-1380].

Let us prove (1.7.43). Applying again Lemmas 1.4.5 and using the properties of $\{\lambda_n\}$ and the inequality $\|f - V_m(f)\|_p \leqslant C E_m(f)_p$, we obtain

$$
\begin{aligned}
J &= \left\| \sum_{s=N}^{M} \left[V_{2^{s+1}}(f^{(\lambda,\beta)}) - V_{2^s}(f^{(\lambda,\beta)}) \right] \right\|_p \\
&\lesssim \sum_{s=N}^{M} \left\| V_{2^{s+1}}(f) - V_{2^s}(f) \right\|_p \sum_{k=2^{s-1}}^{2^s} |\Delta\lambda_k| + \left\| \sum_{s=N}^{M} \lambda_{2^{s+2}} (V_{2^{s+1}}(f) - V_{2^s}(f)) \right\|_p \\
&\lesssim \sum_{s=N}^{M} \Big(\sum_{k=2^{s-1}}^{2^s} |\Delta\lambda_k| + |\lambda_{2^{s+2}}| \Big) E_{2^s}(f)_p.
\end{aligned}
$$

Hence,

$$
\left\| f^{(\lambda,\beta)} - V_{2^n}(f^{(\lambda,\beta)}) \right\|_p \lesssim \sum_{s=n}^{\infty} \Big(\sum_{k=2^{s-1}}^{2^s} |\Delta\lambda_k| + |\lambda_{2^{s+2}}| \Big) E_{2^s}(f)_p. \qquad (1.7.46)
$$

Therefore, we have

$$
E_{2^n}(f^{(\lambda,\beta)})_p \lesssim \sum_{s=n}^{\infty} \Big(\sum_{k=2^{s-1}}^{2^s} |\Delta\lambda_k| + |\lambda_{2^{s+2}}| \Big) E_{2^s}(f)_p. \qquad (1.7.47)
$$

In order to prove (1.7.44), we use inequality (1.2.9) from Lemma 1.2.3. The estimate of $\left\| f^{(\lambda,\beta)} - V_{2^n}(f^{(\lambda,\beta)}) \right\|_p$ follows from (1.7.46). Now let us estimate second term of (1.2.9). By Lemma 1.5.9, we have

$$
\begin{aligned}
\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_p \leqslant \Big| \cos \frac{\pi(\beta-\rho)}{2} \Big| \left\| V_{2^n}^{(\alpha+\rho)}(f^{(\frac{\lambda_n}{n^\rho},0)}) \right\|_p \\
+ \Big| \sin \frac{\pi(\beta-\rho)}{2} \Big| \left\| \widetilde{V}_{2^n}^{(\alpha+\rho)}(f^{(\frac{\lambda_n}{n^\rho},0)}) \right\|_p.
\end{aligned}
$$

Applying Abel transformation twice, we get

$$\begin{aligned}
&\left\|V_{2^n}^{(\alpha+\rho)}(f^{(\frac{\lambda_n}{n^\rho},0)})\right\|_p \\
&= \left\|\sum_{k=0}^{n-1}\left(V_{2^{k+1}}^{(\alpha+\rho)}(f^{(\frac{\lambda_k}{k^\rho},0)}) - V_{2^k}^{(\alpha+\rho)}(f^{(\frac{\lambda_k}{k^\rho},0)})\right) + V_1^{(\alpha+\rho)}(f^{(\lambda_1,0)})\right\|_p \\
&= \left\|\sum_{k=0}^{n-1}\frac{1}{\pi}\int_{-\pi}^{\pi}(V_{2^{k+1}}^{(\alpha+\rho)} - V_{2^k}^{(\alpha+\rho)})(x+u)\sum_{\nu=2^k}^{2^{k+2}}\frac{\lambda_\nu}{\nu^\rho}\cos\nu u du + \lambda_1 V_1^{(\alpha+\rho)}(f)\right\|_p \\
&\lesssim \sum_{k=0}^{n-1}\|V_{2^{k+1}}^{(\alpha+\rho)} - V_{2^k}^{(\alpha+\rho)}\|_p\left(\sum_{j=2^k}^{2^{k+2}-1}\left|\Delta^2(\frac{\lambda_j}{j^\rho})\right|(j+1) + 2^{k+2}\left|\Delta\frac{\lambda_{2^{k+2}}}{2^{(k+2)\rho}}\right| + 2^k\left|\Delta\frac{\lambda_{2^k}}{2^{k\rho}}\right|\right) \\
&+ \left\|\sum_{k=0}^{n-1}\frac{\lambda_{2^{k+2}}}{2^{(k+2)\rho}}(V_{2^{k+1}}^{(\alpha+\rho)} - V_{2^k}^{(\alpha+\rho)})(f) + \lambda_1 V_1^{(\alpha+\rho)}(f)\right\|_p =: J_1 + J_2.
\end{aligned}$$

To obtain the upper estimate of J_1, we will use the properties of GM sequences (see Lemma 1.4.1)

$$\begin{aligned}
J_1 &\lesssim \sum_{k=0}^{n-1}\|V_{2^{k+1}}^{(\alpha+\rho)} - V_{2^k}^{(\alpha+\rho)}\|_p\left(2^{k+2}\sum_{j=2^k}^{2^{k+2}-1}\left|\Delta^2(\frac{\lambda_j}{j^\rho})\right| + 2^{k+2}\left|\Delta\frac{\lambda_{2^{k+2}}}{2^{(k+2)\rho}}\right| + 2^k\left|\Delta\frac{\lambda_{2^k}}{2^{k\rho}}\right|\right) \\
&\lesssim \sum_{k=0}^{n-1}\|V_{2^{k+1}}^{(\alpha+\rho)} - V_{2^k}^{(\alpha+\rho)}\|_p 2^k\left|\Delta\frac{\lambda_{2^k}}{2^{k\rho}}\right| \\
&\lesssim \sum_{k=0}^{n-1}\|V_{2^{k+1}}^{(\alpha+\rho)} - V_{2^k}^{(\alpha+\rho)}\|_p\sum_{\nu=[2^{k-1}]}^{2^k}\left|\Delta\frac{\lambda_\nu}{\nu^{k\rho}}\right| \\
&\lesssim \sum_{\nu=1}^{2^{n-1}}\left|\frac{\lambda_\nu}{\nu^\rho} - \frac{\lambda_{\nu+1}}{(\nu+1)^\rho}\right|\nu^{\alpha+\rho}\omega_{\alpha+\rho}\left(f,\frac{1}{\nu}\right)_p,
\end{aligned}$$

where in the last step we have used Lemma 1.2.3 and monotonicity properties of modulus of smoothness. Now we estimate J_2:

$$\begin{aligned}
J_2 \leqslant &\left\|\sum_{k=0}^{n-1}(V_{2^{k+1}}^{(\alpha+\rho)} - V_{2^k}^{(\alpha+\rho)})(f)\left(\sum_{\nu=k}^{n-1}\left(\frac{\lambda_{2^{\nu+2}}}{2^{(\nu+2)\rho}} - \frac{\lambda_{2^{\nu+3}}}{2^{(\nu+3)\rho}}\right) + \frac{\lambda_{2^{n+2}}}{2^{(n+2)\rho}}\right)\right. \\
&\left. + \lambda_1 V_1^{(\alpha+\rho)}(f)\right\|_p.
\end{aligned}$$

Changing the order of summation, we derive

$$J_2 \leqslant \Big\| \frac{\lambda_{2^{n+2}}}{2^{(n+2)\rho}} (V_{2^n}^{(\alpha+\rho)} - V_1^{(\alpha+\rho)})(f)$$
$$+ \sum_{\nu=0}^{n-1} \Big(\frac{\lambda_{2^{\nu+2}}}{2^{(\nu+2)\rho}} - \frac{\lambda_{2^{\nu+3}}}{2^{(\nu+3)\rho}} \Big) (V_{2^{\nu+1}}^{(\alpha+\rho)} - V_1^{(\alpha+\rho)})(f) + \lambda_1 V_1^{(\alpha+\rho)}(f) \Big\|_p$$
$$\lesssim \sum_{\nu=0}^{n-1} \Big| \frac{\lambda_{2^{\nu+2}}}{2^{(\nu+2)\rho}} - \frac{\lambda_{2^{\nu+3}}}{2^{(\nu+3)\rho}} \Big| \Big\| V_{2^{\nu+1}}^{(\alpha+\rho)}(f) \Big\|_p$$
$$+ \Big| \frac{\lambda_{2^{n+2}}}{2^{(n+2)\rho}} \Big| \Big\| V_{2^n}^{(\alpha+\rho)} \Big\|_p + |\lambda_1| \Big\| V_1^{(\alpha+\rho)}(f) \Big\|_p$$
$$\lesssim \sum_{\mu=1}^{2^{n+2}-1} \Big| \frac{\lambda_\mu}{\mu^\rho} - \frac{\lambda_{\mu+1}}{(\mu+1)^\rho} \Big| \mu^{\alpha+\rho} \omega_{\alpha+\rho}\Big(f, \frac{1}{\mu}\Big)_p + |\lambda_{2^{n+2}}| 2^{n\alpha} \omega_{\alpha+\rho}\Big(f, \frac{1}{2^n}\Big)_p.$$

Here we used that $|\lambda_1| \leqslant \sum\limits_{\mu=1}^{2^{n+2}-1} \Big| \frac{\lambda_\mu}{\mu^\rho} - \frac{\lambda_{\mu+1}}{(\mu+1)^\rho} \Big| + \Big| \frac{\lambda_{2^{n+2}}}{2^{(n+2)\rho}} \Big|$ and the fact that $\frac{\omega_{\alpha+\rho}(f,t)}{t^{\alpha+\rho}}$ is almost decreasing (see Lemma 1.2.2). To estimate $\Big\| \widetilde{V}_{2^n}^{(\alpha+\rho)}(f^{(\frac{\lambda_n}{n^\rho},0)}) \Big\|_p$, we will need the following lemma, which follows applying the Abel transformation.

Lemma 1.7.1. *Let* $\mu = \{\mu_n\}_{n=1}^\infty$. *Let*

$$T_n(x) = \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)$$

and

$$T_n(\mu, x) = \sum_{\nu=1}^{n} \mu_\nu (a_\nu \cos \nu x + b_\nu \sin \nu x).$$

For any $M > N+1,\ M \in \mathbb{N},\ \ N \in \mathbb{N} \cup \{0\}$, *we have*

$$T_M(\mu, x) - T_n(\mu, x) = \mu_M (T_M(x) - T_N(x))$$
$$+ \frac{1}{\pi} \int_{-\pi}^{\pi} (T_M - T_N)(x+u) \Big[\sum_{j=N+1}^{M-2} (\mu_j - 2\mu_{j+1} + \mu_{j+2})(j+1) K_j(u)$$
$$+ (\mu_{M-1} - \mu_M) M K_{M-1}(u) \Big] du = \mu_M (T_M(x) - T_N(x))$$
$$+ \frac{1}{\pi} \int_{-\pi}^{\pi} (T_M - T_N)(x+u) \Big[\sum_{j=N+1}^{M-2} \Delta^2(\mu_j)(j+1) K_j(u) + \Delta(\mu_{M-1}) M K_{M-1}(u) \Big] du,$$

where

$$K_n(x) = \frac{1}{n+1}\sum_{\nu=0}^{n}\Big(\frac{1}{2}+\sum_{m=1}^{\nu}\cos mx\Big) = \frac{1}{n+1}\sum_{\nu=0}^{n} D_\nu(x).$$

Using Lemma 1.7.1 with $\mu = \{\mu_n = \frac{\lambda_n}{n^\rho}\},\ M = 2^{k+2},\ N = 2^k$, we have

$$\begin{aligned}
&\|\widetilde{V}_{2^n}^{(\alpha+\rho)}(f^{(\{\frac{\lambda_n}{n^\rho}\},0)})\|_p \\
&= \left\|\sum_{k=0}^{n-1}\Big(\widetilde{V}_{2^{k+1}}^{(\alpha+\rho)}(f^{(\{\frac{\lambda_n}{n^\rho}\},0)}) - \widetilde{V}_{2^k}^{(\alpha+\rho)}(f^{(\{\frac{\lambda_n}{n^\rho}\},0)})\Big) + \widetilde{V}_1^{(\alpha+\rho)}(f^{(\{\frac{\lambda_n}{n^\rho}\},0)})\right\|_p
\end{aligned}$$

$$\begin{aligned}
&= \Bigg\|\lambda_1\widetilde{V}_1^{(\alpha+\rho)}(f)(x) + \sum_{k=0}^{n-1}\Bigg[\frac{\lambda_{2^{k+2}}}{2^{(k+2)\rho}}\Big(\widetilde{V}_{2^{k+1}}^{(\alpha+\rho)}(f)(x) - \widetilde{V}_{2^k}^{(\alpha+\rho)}(f)(x)\Big) \\
&\qquad + \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\widetilde{V}_{2^{k+1}}^{(\alpha+\rho)}(f) - \widetilde{V}_{2^k}^{(\alpha+\rho)}(f)\Big)(x+u) \\
&\times\Big(\sum_{j=2^k+1}^{2^{k+2}-2}\Delta_2\Big(\frac{\lambda_j}{j^\rho}\Big)(j+1)K_j(u) + \Delta\Big(\frac{\lambda_{2^{k+2}-1}}{(2^{k+2}-1)^\rho}\Big)2^{k+2}K_{2^{k+2}-1}(u)\Big)du\Bigg]\Bigg\|_p \\
&\leqslant \sum_{k=0}^{n-1}\left|\frac{\lambda_{2^{k+2}}}{2^{(k+2)\rho}}\right|\left\|\widetilde{V}_{2^{k+1}}^{(\alpha+\rho)}(f)(x) - \widetilde{V}_{2^k}^{(\alpha+\rho)}(f)(x)\right\|_p \\
&+ \left\|\lambda_1\widetilde{V}_1^{(\alpha+\rho)}(f)(x)\right\|_p + \sum_{k=0}^{n-1}\left\|\widetilde{V}_{2^{k+1}}^{(\alpha+\rho)}(f) - \widetilde{V}_{2^k}^{(\alpha+\rho)}(f)\right\|_p \\
&\times\left(\sum_{j=2^k+1}^{2^{k+2}-2}\left|\Delta_2\Big(\frac{\lambda_j}{j^\rho}\Big)\right|(j+1) + \left|\Delta\Big(\frac{\lambda_{2^{k+2}-1}}{(2^{k+2}-1)^\rho}\Big)\right|2^{k+2}\right) =: J_3 + J_4 + J_5.
\end{aligned}$$

To estimate J_3, we first note that by Lemma 1.5.1,

$$\left\|\widetilde{V}_{2^{k+1}}(f) - \widetilde{V}_{2^k}(f)\right\|_p \lesssim \left\|V_{2^{k+1}}(f) - V_{2^k}(f)\right\|_p.$$

Then, making use of Lemma 1.4.1, we get

$$\begin{aligned}
J_3 &\lesssim \sum_{k=0}^{n-1}\frac{\lambda_{2^{k+2}}}{2^{(k+2)\rho}}\left\|V_{2^{k+1}}^{(\alpha+\rho)}(f)(x) - V_{2^k}^{(\alpha+\rho)}(f)(x)\right\|_p \\
&\lesssim \sum_{k=0}^{n-1}\frac{\lambda_{2^{k+2}}}{2^{(k+2)\rho}}2^{k(\rho+\alpha)}\omega_{\alpha+\rho}(f,\frac{1}{2^k})_p.
\end{aligned}$$

Further, we have

$$J_4 + J_5 \lesssim \left\|\lambda_1 V_1^{(\alpha+\rho)}(f)(x)\right\|_p + \sum_{k=0}^{n-1} \left\|\left\{V_{2^{k+1}}^{(\alpha+\rho)}(f) - V_{2^k}^{(\alpha+\rho)}(f)\right\}(x)\right\|_p$$

$$\times \left[\sum_{j=2^k+1}^{2^{k+2}-2} \left|\Delta_2\Big(\frac{\lambda_j}{j^\rho}\Big)\right|(j+1) + \left|\Delta\Big(\frac{\lambda_{2^{k+2}-1}}{(2^{k+2}-1)^\rho}\Big)\right|2^{k+2}\right]$$

$$\lesssim \left\|\lambda_1 V_1^{(\alpha+\rho)}(f)(x)\right\|_p + \sum_{k=0}^{n-1} \left\|\left\{V_{2^{k+1}}^{(\alpha+\rho)}(f) - V_{2^k}^{(\alpha+\rho)}(f)\right\}(x)\right\|_p 2^k \left|\Delta \frac{\lambda_{2^k}}{2^{k\rho}}\right|$$

$$\lesssim \sum_{\mu=1}^{2^{n-1}} \left|\frac{\lambda_\mu}{\mu^\rho} - \frac{\lambda_{\mu+1}}{(\mu+1)^\rho}\right| \mu^{\alpha+\rho} \omega_{\alpha+\rho}\Big(f, \frac{1}{\mu}\Big)_p.$$

Collecting estimates of J_1, J_2, J_3, J_4, and J_5, we obtain that

$$\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_p \lesssim \sum_{\mu=1}^{2^{n+2}-1} \left|\frac{\lambda_\mu}{\mu^\rho} - \frac{\lambda_{\mu+1}}{(\mu+1)^\rho}\right| \mu^{\alpha+\rho} \omega_{\alpha+\rho}\Big(f, \frac{1}{\mu}\Big)_p$$

$$+ |\lambda_{2^{n+2}}| 2^{n\alpha} \omega_{\alpha+\rho}\Big(f, \frac{1}{2^n}\Big)_p + \Big|\sin\frac{\pi(\beta-\rho)}{2}\Big| \sum_{k=0}^{n-1} |\lambda_{2^{k+2}}| 2^{k\alpha} \omega_{\alpha+\rho}\Big(f, \frac{1}{2^k}\Big)_p. \tag{1.7.48}$$

Combining (1.7.47) and (1.7.48) and taking into account Jackson's inequality, we arrive at (1.7.44). The proof of Theorem 1.7.1 is now complete. □

1.8 Estimates from below of best approximations for generalized Liouville - Weyl derivatives

Theorem 1.8.1. *Let $1 < p < \infty$, $\max(p, 2) \leqslant \tau < \infty$, and $\lambda = \{\lambda_n\}_{n=1}^\infty$ be a sequence of positive numbers such that $\lambda \in GM$. Let the sequence $\{\lambda_n\}_{n=1}^\infty$ satisfy the additional condition*

$$\Big(\sum_{k=1}^n |\lambda_{2^k}^\tau - \lambda_{2^{k-1}}^\tau|\Big)^{\frac{1}{\tau}} \leqslant C|\lambda_{2^n}| \tag{1.8.49}$$

for all integer n, where the constant C is independent of n.

If for $f \in L_p$ there exists a function $f^{(\lambda,\beta)} \in L_p$, with the Fourier series $\sigma(f,\lambda,\beta)$ $(\beta \in \mathbb{R})$, then

$$\|f^{(\lambda,\beta)}\|_p \gtrsim \Big(\lambda_1^\tau E_0^\tau(f)_p + \sum_{\nu=1}^{\infty} |\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau| E_{2^\nu-1}^\tau(f)_p\Big)^{\frac{1}{\tau}}$$

and

$$E_{2^m-1}(f^{(\lambda,\beta)})_p \gtrsim \Big(\lambda_{2^{m-1}}^\tau E_{2^m-1}^\tau(f)_p + \sum_{\nu=m}^{\infty} |\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau| E_{2^\nu-1}^\tau(f)_p\Big)^{\frac{1}{\tau}}.$$

Remark 1.8.1. *Note that if $\{\lambda_n\}$ satisfies the condition $\{\lambda_n^\tau\} \in GM$, then* (1.8.49) *follows from the condition*

$$\sum_{k=1}^{n} \lambda_{2^k} \lesssim \lambda_{2^n}. \tag{1.8.50}$$

Indeed

$$\sum_{k=1}^{n} |\lambda_{2^k}^\tau - \lambda_{2^{k+1}}^\tau| \leqslant \sum_{k=1}^{n} \sum_{s=2^k}^{2^{k+1}-1} |\lambda_s^\tau - \lambda_{s+1}^\tau| \lesssim \sum_{k=1}^{n} \lambda_{2^k}^\tau \lesssim \lambda_{2^n}^\tau.$$

The last inequality is equivalent to (1.8.50). For more information concerning condition (1.8.50) see [76, 77].

Proof of Theorem 1.8.1. We consider the series

$$\sum_{n=1}^{\infty} \lambda_{2^{n-1}} \Delta_n = \sum_{n=1}^{\infty} \Lambda_\nu \lambda_\nu A_\nu(x), \tag{1.8.51}$$

where $\Lambda_\nu = \frac{\lambda_{2^{n-1}}}{\lambda_\nu}$ for $2^{n-1} \leqslant \nu \leqslant 2^n - 1$ $(n = 1, 2, ...)$. Taking into account that $\lambda \in GM$, we have $|\lambda_k| \leqslant C|\lambda_n|$ for $n \leqslant k \leqslant 2n$.

We will show that the following inequality follows from (1.8.49)

$$C_1 \lambda_s \leqslant \lambda_k \leqslant C_2 \lambda_s, \qquad k \leqslant s \leqslant 2k. \tag{1.8.52}$$

The left-hand side estimate follows from the GM condition. To prove the right-hand side, let $2^n \leqslant k < 2^{n+1}$. Then $2^{n+1} \leqslant 2k < 2^{n+2}$ and

$$\lambda_k^\tau \leqslant C \lambda_{2^n}^\tau = C\Big(\lambda_{2^{n+2}}^\tau + \sum_{m=n}^{n+1} (\lambda_{2^m}^\tau - \lambda_{2^{m+1}}^\tau)\Big)$$

$$\leqslant C\Big(\lambda_{2^{n+2}}^\tau + \sum_{m=1}^{n+2} |\lambda_{2^m}^\tau - \lambda_{2^{m-1}}^\tau|\Big) \leqslant C \lambda_{2^{n+2}}^\tau \leqslant C \lambda_{2k}^\tau.$$

Thus, $\lambda_k \leqslant C\lambda_{2k} \leqslant C\lambda_s$ and (1.8.52) follows. Let us now show that (1.8.52) implies that $\{1/\lambda_n\} \in GM$, that is,

$$\sum_{k=n}^{2n} \Big|\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\Big| \leqslant \frac{C}{\lambda_n}.$$

Indeed,

$$C\lambda_n \geqslant \sum_{k=n}^{2n} |\lambda_k - \lambda_{k+1}| = \sum_{k=n}^{2n} \lambda_k\lambda_{k+1}\Big|\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\Big| \geqslant C\lambda_n^2 \sum_{k=n}^{2n} \Big|\frac{1}{\lambda_k} - \frac{1}{\lambda_{k+1}}\Big|.$$

We also note that

$$1)\ |\Lambda_\nu| = \Big|\frac{\lambda_{2^{n-1}}}{\lambda_\nu}\Big| \leqslant C\Big|\frac{\lambda_{2^{n-1}}}{\lambda_{2^{n-1}}}\Big| \leqslant M,$$

2) for $n = 1$,

$$\sum_{\nu=2^{n-1}}^{2^n-1} |\Lambda_\nu - \Lambda_{\nu+1}| = \Big|\frac{\lambda_1}{\lambda_1} - \frac{\lambda_2}{\lambda_2}\Big| \leqslant M,$$

3) for $n = 2, 3, ...,$

$$\sum_{\nu=2^{n-1}}^{2^n-1} |\Lambda_\nu - \Lambda_{\nu+1}| = \sum_{\nu=2^{n-1}}^{2^n-2} |\Lambda_\nu - \Lambda_{\nu+1}| + |\Lambda_{2^n-1} - \Lambda_{2^n}|$$

$$= \lambda_{2^{n-1}} \sum_{\nu_1=2^{n-1}}^{2^n-2} \Big|\frac{1}{\lambda_\nu} - \frac{1}{\lambda_{\nu+1}}\Big| + \Big|\frac{\lambda_{2^{n-1}}}{\lambda_{2^n-1}} - \frac{\lambda_{2^n}}{\lambda_{2^n}}\Big| \leqslant M.$$

Since the sequence $\{\Lambda_n\}_{n=1}^{\infty}$ satisfies the conditions of Lemma 1.5.3, then the series (1.8.51) is the Fourier series of a function $g(x) \in L_p$, and $\|g\|_p \leqslant C(p)\|f^{(\lambda,\beta)}\|_p$.

Applying Lemmas 1.5.6 and 1.5.8, we have

$$I := \lambda_1^\tau E_0^\tau(f)_p + \sum_{\nu=1}^{\infty} |\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau| E_{2^\nu-1}^\tau(f)_p$$

$$\lesssim \lambda_1^\tau E_0^\tau(f)_p + \sum_{\nu=1}^{\infty} |\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau| \Big(\int_0^{2\pi} \Big[\sum_{n=\nu+1}^{\infty} \Delta_n^2\Big]^{\frac{p}{2}} dx\Big)^{\frac{\tau}{p}}$$

$$\lesssim \lambda_1^\tau E_0^\tau(f)_p + \Big(\int_0^{2\pi} \Big\{\sum_{\nu=1}^{\infty} |\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau| \Big[\sum_{n=\nu+1}^{\infty} \Delta_n^2\Big]^{\frac{\tau}{2}}\Big\}^{\frac{p}{\tau}} dx\Big)^{\frac{\tau}{p}}$$

$$\lesssim \lambda_1^\tau E_0^\tau(f)_p + \Big(\int\limits_0^{2\pi} \Big\{ \sum_{\nu=1}^\infty \Big[\sum_{n=\nu}^\infty |\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau|^{\frac{2}{\tau}} \Delta_{n+1}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{p}{\tau}} dx\Big)^{\frac{\tau}{p}}.$$

Using condition (1.8.49), Lemma 1.5.5 with $\tau \geqslant 2$ and Lemma 1.5.8, we get

$$I \lesssim \lambda_1^\tau E_0^\tau(f)_p + \Big(\int\limits_0^{2\pi} \Big\{ \sum_{n=1}^\infty \Big\{ \sum_{\nu=1}^n \Big[|\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau|^{\frac{2}{\tau}} \Delta_{n+1}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{p}{2}} dx\Big)^{\frac{\tau}{p}}$$

$$= \lambda_1^\tau E_0^\tau(f)_p + \Big(\int\limits_0^{2\pi} \Big\{ \sum_{n=1}^\infty \Delta_{n+1}^2 \Big\{ \sum_{\nu=1}^n |\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau| \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{p}{2}} dx\Big)^{\frac{\tau}{p}}$$

$$\lesssim \lambda_1^\tau E_0^\tau(f)_p + \Big(\int\limits_0^{2\pi} \Big\{ \sum_{n=1}^\infty \Delta_{n+1}^2 \lambda_{2^n}^2 \Big\}^{\frac{p}{2}} dx\Big)^{\frac{\tau}{p}}$$

$$= \lambda_1^\tau E_0^\tau(f)_p + \Big(\int\limits_0^{2\pi} \Big\{ \sum_{n=2}^\infty \Delta_n^2 \lambda_{2^{n-1}}^2 \Big\}^{\frac{p}{2}} dx\Big)^{\frac{\tau}{p}}$$

$$\lesssim \lambda_1^\tau \Big(\int\limits_0^{2\pi} \Big\{ \sum_{n=1}^\infty \Delta_n^2 \Big\}^{\frac{p}{2}} dx\Big)^{\frac{\tau}{p}} + \Big(\int\limits_0^{2\pi} \Big\{ \sum_{n=2}^\infty \Delta_n^2 \lambda_{2^{n-1}}^2 \Big\}^{\frac{p}{2}} dx\Big)^{\frac{\tau}{p}}$$

$$\lesssim \lambda_1^\tau \Big(\int\limits_0^{2\pi} \Big\{ \Delta_1^2 \Big\}^{\frac{p}{2}} dx\Big)^{\frac{\tau}{p}} + \lambda_1^\tau \Big(\int\limits_0^{2\pi} \Big\{ \sum_{n=2}^\infty \Delta_n^2 \Big\}^{\frac{p}{2}} dx\Big)^{\frac{\tau}{p}}$$

$$+ \Big(\int\limits_0^{2\pi} \Big\{ \sum_{n=2}^\infty \Delta_n^2 \lambda_{2^{n-1}}^2 \Big\}^{\frac{p}{2}} dx\Big)^{\frac{\tau}{p}}.$$

Now we use the fact that

$$\lambda_1^\tau \leqslant \lambda_{2^{n-1}}^\tau + \sum_{\nu=1}^{n-1} |\lambda_{2^{\nu-1}}^\tau - \lambda_{2^\nu}^\tau| \leqslant C \lambda_{2^{n-1}}^\tau, \quad n = 2, 3, ...$$

We arrive at

$$I \lesssim \Big(\int\limits_0^{2\pi} \Big[\sum_{n=1}^\infty \Delta_n^2 \lambda_{2^{n-1}}^2 \Big]^{\frac{p}{2}} dx\Big)^{\frac{\tau}{p}} \lesssim \|g\|_p^\tau \lesssim \|f^{(\lambda,\beta)}\|_p^\tau.$$

Thus, we have shown that

$$\|f^{(\lambda,\beta)}\|_p \gtrsim \Big(\lambda_1^\tau E_0^\tau(f)_p + \sum_{\nu=1}^{\infty} |\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau| E_{2^\nu-1}^\tau(f)_p\Big)^{\frac{1}{\tau}}.$$

Now we estimate $E_{2^m-1}(\varphi)_p$ from below as follows:

$$E_{2^m-1}(f^{(\lambda,\beta)})_p \gtrsim \Big\{\int_0^{2\pi}\Big[\sum_{k=m}^{\infty}\lambda_{2^k}^2\Delta_k^2\Big]^{\frac{p}{2}}dx\Big\}^{\frac{1}{p}}.$$

Using Lemmas 1.5.6 and 1.5.8, we obtain

$$\begin{aligned}
J :&= \Big(\lambda_{2^{m-1}}^\tau E_{2^m-1}^\tau(f)_p + \sum_{\nu=m}^{\infty}|\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau| E_{2^\nu-1}^\tau(f)_p\Big)^{\frac{1}{\tau}} \\
&\lesssim \Big(\lambda_{2^{m-1}}^\tau\Big(\int_0^{2\pi}\Big[\sum_{n=m}^{\infty}\Delta_{n+1}^2\Big]^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}} \\
&+ \sum_{\nu=m}^{\infty}|\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau|\Big(\int_0^{2\pi}\Big[\sum_{n=\nu}^{\infty}\Delta_{n+1}^2\Big]^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}}\Big)^{\frac{1}{\tau}} \\
&\lesssim \Big(\lambda_{2^m}^\tau\Big(\int_0^{2\pi}\Big[\sum_{n=m}^{\infty}\Delta_{n+1}^2\Big]^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}} \\
&+ \Big(\int_0^{2\pi}\Big\{\sum_{\nu=m}^{\infty}|\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau|\Big[\sum_{n=\nu}^{\infty}\Delta_{n+1}^2\Big]^{\frac{\tau}{2}}\Big\}^{\frac{p}{\tau}}dx\Big)^{\frac{\tau}{p}} \\
&= \Big(\lambda_{2^{m-1}}^\tau\Big(\int_0^{2\pi}\Big[\sum_{n=m}^{\infty}\Delta_{n+1}^2\Big]^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}} \\
&+ \Big(\int_0^{2\pi}\Big\{\Big\{\sum_{\nu=m}^{\infty}\Big[\sum_{n=\nu}^{\infty}|\lambda_{2^\nu}^\tau - \lambda_{2^{\nu-1}}^\tau|^{\frac{2}{\tau}}\Delta_{n+1}^2\Big]^{\frac{\tau}{2}}\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}}\Big)^{\frac{1}{\tau}}.
\end{aligned}$$

Further, applying Lemma 1.5.5 with $\tau \geqslant 2$, we have

$$J \lesssim \Big(\lambda_{2^{m-1}}^\tau\Big(\int_0^{2\pi}\Big[\sum_{n=m}^{\infty}\Delta_{n+1}^2\Big]^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\Big\{\sum_{n=m}^{\infty}\Big\{\sum_{\nu=m}^{n}\Big[|\lambda_{2^\nu}^\tau-\lambda_{2^{\nu-1}}^\tau|^{\frac{2}{\tau}}\Delta_{n+1}^2\Big]^{\frac{\tau}{2}}\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}}\Big)^{\frac{1}{\tau}}$$

$$=\Big(\lambda_{2^{m-1}}^\tau\Big(\int_0^{2\pi}\Big[\sum_{n=m}^{\infty}\Delta_{n+1}^2\Big]^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\Big\{\sum_{n=m}^{\infty}\Delta_{n+1}^2\Big\{\sum_{\nu=m}^{n}|\lambda_{2^\nu}^\tau-\lambda_{2^{\nu-1}}^\tau|\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}}\Big)^{\frac{1}{\tau}}$$

$$\lesssim\Big(\lambda_{2^{m-1}}^\tau\Big(\int_0^{2\pi}\Big[\sum_{n=m}^{\infty}\Delta_{n+1}^2\Big]^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}}+\Big(\int_0^{2\pi}\Big\{\sum_{n=m}^{\infty}\Delta_{n+1}^2\lambda_{2^n}^2\Big\}^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}}\Big)^{\frac{1}{\tau}}$$

$$=\Big(\lambda_{2^{m-1}}^\tau\Big(\int_0^{2\pi}\Big[\sum_{n=m+1}^{\infty}\Delta_n^2\Big]^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}}+\Big(\int_0^{2\pi}\Big\{\sum_{n=m+1}^{\infty}\Delta_n^2\lambda_{2^{n-1}}^2\Big\}^{\frac{p}{2}}dx\Big)^{\frac{\tau}{p}}\Big)^{\frac{1}{\tau}}.$$

Further, taking into account that

$$\lambda_{2^{m-1}}^\tau=\sum_{\nu=m}^{n-1}(\lambda_{2^{\nu-1}}^\tau-\lambda_{2^\nu}^\tau)+\lambda_{2^{n-1}}^\tau\leqslant\sum_{\nu=m}^{n-1}|\lambda_{2^{\nu-1}}^\tau-\lambda_{2^\nu}^\tau|+\lambda_{2^{n-1}}^\tau\leqslant C\lambda_{2^{n-1}}^\tau,$$

we derive that

$$J\lesssim\Big(\int_0^{2\pi}\Big\{\sum_{n=m+1}^{\infty}\Delta_n^2\lambda_{2^{n-1}}^2\Big\}^{\frac{p}{2}}dx\Big)^{\frac{1}{p}}$$

$$=\Big\{\int_0^{2\pi}\Big[\sum_{k=m}^{\infty}\lambda_{2^k}^2\Delta_{k+1}^2\Big]^{\frac{p}{2}}dx\Big\}^{\frac{1}{p}}\lesssim E_{2^m-1}(f^{(\lambda,\beta)})_p.$$

This gives

$$\Big(\lambda_{2^{m-1}}^\tau E_{2^m-1}^\tau(f)_p+\sum_{\nu=m}^{\infty}|\lambda_{2^\nu}^\tau-\lambda_{2^{\nu-1}}^\tau|E_{2^\nu-1}^\tau(f)_p\Big)^{\frac{1}{\tau}}\lesssim E_{2^m-1}(f^{(\lambda,\beta)})_p,$$

completing the proof. $\square$

2 Sharp Ul'yanov inequalities for Liouville–Weyl derivatives. (L_p, L_q) inequalities

In this section, we study sharp (p,q)-inequalities of Ul'yanov type for moduli of smoothness of fractional order. In more detail, we obtain estimates of the best approximation and the modulus of smoothness of the generalized Liouville–Weyl derivatives. We give examples showing the sharpness of these inequalities. Main results of this section were published in [31, 33].

2.1 Ul'yanov type inequalities for moduli of smoothness

Ul'yanov [85] proved the following (p,q)-inequalities between moduli of continuity

$$\omega(f,\delta)_q \lesssim \left(\int_0^\delta \left(t^{-\theta}\omega(f,t)_p\right)^{q_1} \frac{dt}{t}\right)^{1/q_1},$$

where

$$1 \leqslant p < q \leqslant \infty, \quad \theta = \frac{1}{p} - \frac{1}{q},$$

and

$$q_1 = \begin{cases} q, & q < \infty, \\ 1, & q = \infty. \end{cases}$$

Here $\omega(f,\delta)_p = \omega_1(f,\delta)_p$ is the modulus of continuity and the modulus of smoothness of order $k \in \mathbb{N}$ is given in subsection 1.2.

The following (p,q) estimate for moduli of smoothness (of an integer order) is due to DeVore, Riemenschneider, and Sharpley [9] and Gol'dman [20, 21]:

$$\omega_k(f,\delta)_q \lesssim \left(\int_0^\delta \left(t^{-\theta}\omega_k(f,t)_p\right)^{q_1} \frac{dt}{t}\right)^{1/q_1}.$$

Similar estimates for moduli of smoothness of a function $f^{(\rho)}$ was proved by Z. Ditzian and S. Tikhonov [11], [12]

$$\omega_k(f^{(\rho)},\delta)_q \lesssim \left(\int_0^\delta \left(t^{-\theta-\rho}\omega_{k+\rho}(f,t)_p\right)^{q_1} \frac{dt}{t}\right)^{1/q_1}, \qquad (2.1.1)$$

where $\rho \in \mathbb{N} \cup \{0\}$ and $0 < p < q \leqslant \infty$.
It is easy to see that (2.1.1) does not give, in general, the sharp estimate. Take, for example, $f \in C^\infty$. Then

$$\omega_k(f^{(\rho)}, \delta)_q \asymp \delta^k \lesssim \left(\int\limits_0^\delta (t^{-\theta-\rho} t^{k+\rho})^q \frac{dt}{t} \right)^{\frac{1}{q}} \asymp \delta^{k-\theta}.$$

Recently, several authors (see, e.g., [56], [68], [83]) studied the sharp form of Ul'yanov-type inequalities given by

$$\omega_\alpha(f^{(\rho)}, \delta)_q \lesssim \left(\int\limits_0^\delta \left(t^{-\theta-\rho} \omega_{\alpha+\rho+\theta}(f,t)_p\right)^q \frac{dt}{t} \right)^{1/q},$$

where $\rho > 0$, $\quad 1 < p < q < \infty$, $\quad \theta = \frac{1}{p} - \frac{1}{q}$, $\quad \alpha > 0$, and $f^{(\rho)}$ is ρ-fractional derivative in the sense of Weyl of the function f. Note that this estimate provides sharper estimate than (2.1.1). See also [10, 19, 34, 45, 49].

S. Tikhonov and W. Trebels [73] investigated the sharp Ul'yanov inequality for the Liouville–Weyl derivatives. Let us recall this definition.

Let the Fourier series of $f \in L_1$ be given by

$$f(x) \sim \sum_{\nu \in \mathbb{Z}} \hat{f}_\nu e^{i\nu x},$$

with $\hat{f}_\nu$ being the Fourier coefficients. Then the generalized Liouville–Weyl derivatives $D^\lambda f$ can be defined as follows (if the right-hand side is the Fourier series of an integrable function):

$$D^\lambda f \sim \sum_{\nu \in \mathbb{Z}} \lambda_\nu \hat{f}_\nu e^{i\nu x}, \quad \lambda_\nu = \lambda(|\nu|),$$

where $\lambda : [0, \infty) \to [0, \infty)$ is an increasing (non-decreasing) function satisfying the following conditions:

(A_1) there exists some $\rho \geqslant 0$ such that $t^{-\rho}\lambda(t)$ is increasing;

(A_1') there is slowly varying, increasing function $\xi : [1, \infty) \to [c, \infty)$, $c > 0$, such that $t^{-\rho}\lambda(t)\xi(t)$ is increasing for some $\rho \geqslant 0$;

(A_2) there exists some $\sigma > 0$ such that, with ρ from (A_1), $t^{-\rho-\sigma}\lambda(t)$ is decreasing (i.e. non-increasing);

(A_3) there exists some $\varepsilon > 0$ such that, $t^{-\rho-\varepsilon}\lambda(t)$ is increasing (where $\rho \geqslant 0$);

(A_4) λ is convex or λ' is locally absolutely continuous and $t|\lambda''(t)| \lesssim \lambda'(t),\;\; t>0$.

The following three theorems provide the sharp Ul'yanov inequalities for the Liouville–Weyl derivatives in the cases $1<p<q<\infty$ and $p=1<q<\infty$.

Theorem B1. [73] *Let $f\in L_p$, $1<p<q<\infty$, $\theta=1/p-1/q$. Let λ satisfy conditions (A_1) and (A_2). Then, for any $\alpha>0$,*

$$\omega_\alpha(D^\lambda f,\delta)_q \lesssim \left(\int\limits_0^\delta \left(t^{-\theta}\lambda\Big(\frac{1}{t}\Big)\omega_{\alpha+\rho+\theta}(f,t)_p\right)^q\frac{dt}{t}\right)^{1/q}, \qquad 0<\delta<1.$$

Theorem B2. [73] *Under the hypotheses of theorem B1 with condition (A_1) being replaced by (A_1'), we obtain*

$$\omega_\alpha(D^\lambda f,\delta)_q \lesssim \left(\int\limits_0^\delta \left(t^{-\theta}\lambda\Big(\frac{1}{t}\Big)\xi\Big(\frac{1}{t}\Big)\omega_{\alpha+\rho+\theta}(f,t)_p\right)^q\frac{dt}{t}\right)^{1/q}, \quad \alpha>0,\;\; 0<\delta<1.$$

Theorem B3. [73] *Let $f\in L_1$, $1<q<\infty$, $\theta=1-1/q$.*

(i) If λ satisfies conditions (A_1)–(A_3). Then

$$\omega_\alpha(D^\lambda f,\delta)_q \lesssim \left(\int\limits_0^\delta \left(t^{-\theta}\lambda\Big(\frac{1}{t}\Big)\omega_{\alpha+\rho+\theta}(f,t)_1\right)^q\frac{dt}{t}\right)^{1/q}, \quad \alpha>0, \quad 0<\delta<1.$$

(ii) If λ satisfies conditions (A_1), (A_2), and (A_4), then

$$\omega_\alpha(D^\lambda f,\delta)_q \lesssim \left(\int\limits_0^\delta \left(t^{-\theta}\lambda\Big(\frac{1}{t}\Big)\omega_{\alpha+\rho+\theta}(f,t)_1\right)^q\frac{|\ln t|dt}{t}\right)^{1/q}, \quad \alpha>0, \quad 0<\delta<\frac{1}{2}.$$

To the best of our knowledge, the Ulyanov type inequalities for the generalized Liouville–Weyl derivatives were not studied in the case $p=1$ and $q=\infty$. However, the following theorem holds for the classical Weyl derivatives.

Theorem B4. *Let $f\in L_1$, $\theta=1$, and $\rho\geqslant 0$. We have*

$$\omega_\alpha(f^{(\rho)},\delta)_\infty \lesssim \int\limits_0^\delta t^{-\rho-\theta}\omega_{\alpha+\rho+\theta}(f,t)_1\frac{dt}{t}, \quad \alpha>0, \quad 0<\delta<1.$$

Theorem B4 immediately follows from the sharp Ulyanov inequality ([68])

$$\omega_\alpha(f,\delta)_\infty \lesssim \int\limits_0^\delta t^{-\theta}\omega_{\alpha+1}(f,t)_1\frac{dt}{t}, \quad \alpha>0, \quad 0<\delta<1,$$

and the following (p,p) estimate ($1\leqslant p\leqslant\infty$) for the Weyl derivatives ([53], [55, Cor. 2], for a simple proof see also [71, Lemma 2.12]):

$$\omega_\alpha(f^{(\rho)},\delta)_p \lesssim \int\limits_0^\delta t^{-\rho}\omega_{\alpha+\rho}(f,t)_p\frac{dt}{t}, \quad \alpha,\rho>0, \quad 0<\delta<1.$$

Our main objectives are two-fold. The first goal is to prove analogues of Theorems B1, B2, B3, and B4 by considering a more general class of sequences $\{\lambda_n\}$. We replace the monotonicity condition on $\{\lambda_n\}$ by the general monotonicity. Our second goal is to study in detail all limiting cases:

(i) $p=1<q<\infty$;

(ii) $1<p<q=\infty$;

(iii) $p=1<q=\infty$.

2.2 Inequalities of different metrics for the best approximations

The following inequality establishes the relationship between the best approximations of a function in different L_p metrics. It was proved by Konyushkov and Stechkin (see [37, Theorem 2]). One has

$$E_k(f)_q \lesssim k^\theta E_k(f)_p + \sum_{m=k}^{\infty} m^{\theta-1}E_m(f)_p, \tag{2.2.2}$$

where $1\leqslant p<q\leqslant\infty$ and $\theta=\frac{1}{p}-\frac{1}{q}$.
Inequality (2.2.2) was extended by Ul'yanov [86] for the case of $1\leqslant p<q<\infty$ as follows:

$$E_k(f)_q \lesssim k^\theta E_k(f)_p + \left(\sum_{m=k}^{\infty} m^{q\theta-1}E_m^q(f)_p,\right)^{1/q}, \quad 1\leqslant p<q<\infty.$$

Later, Timan [82] obtained the following inequality for the best approximations of $f^{(r)}$:

$$E_k(f^{(r)})_q \lesssim k^{r+\theta}E_k(f)_p + \left(\sum_{m=k}^{\infty} m^{rq+q\theta-1}E_m^q(f)_p,\right)^{1/q}, \quad 1\leqslant p<q<\infty, \quad r\in\mathbb{N}.$$

An estimate for best approximations involving generalized derivatives was proved by Szalay [64]:

$$E_k(f^{(\lambda)})_q \lesssim k^{\theta}|\lambda_k|E_k(f)_p + \sum_{m=k}^{\infty} m^{\theta}\rho_m E_m(f)_p,$$

where $k \in \mathbb{N}$ and $\rho_m = \max(|\lambda_m - \lambda_{m+1}|, m^{-1}|\lambda_n|)$. Here $f^{(\lambda)}$ defined by (1.1.4) with $\beta = 0$. It is assumed in [64] that $\{\lambda_n\}_{n=1}^{\infty}$ is monotonically increasing, convex (or concave) sequence.

2.3 Some necessary lemmas

To prove our main result we will use several auxiliary results. The first one is the well-known Nikol'skii inequality for trigonometric polynomials.

Lemma 2.3.1. *[8] Let $0 < p \leqslant q \leqslant \infty$, $\theta = \frac{1}{p} - \frac{1}{q}$, and*

$$T_n(x) = \frac{a_0}{2} + \sum_{\nu=1}^{n}(a_\nu \cos \nu x + b_\nu \sin \nu x).$$

Then

$$\|T_n(x)\|_q \lesssim n^{\theta}\|T_n(x)\|_p.$$

Lemma 2.3.2. *[68] For any $T_n(x) = \sum\limits_{\nu=1}^{n}(a_\nu \cos \nu x + b_\nu \sin \nu x)$, we have*

$$\|T_n(x)\|_\infty \leqslant \|(T_n(x))'\|_1.$$

We will also need the following lemma which was proved in [12].

Lemma 2.3.3. *Let $f \in L_p$, $0 < p < q \leqslant \infty$, $\theta = \frac{1}{p} - \frac{1}{q}$, and ψ_σ be a near best trigonometric approximant of f, i.e., $\|\psi_\sigma - f\|_{L_p} \leqslant CE_\sigma(f)_p$, where C does not depend on σ. Then*

$$\Big\|\sum_{l=1}^{m}(\psi_{\sigma 2^l} - \psi_{\sigma 2^{l-1}})\Big\|_q \lesssim \left(\sum_{l=1}^{m}\Big((\sigma 2^l)^{\theta}E_{\sigma 2^{l-1}}(f)_p\Big)^{q_1}\right)^{1/q_1},$$

where $q_1 = \begin{cases} q, & q < \infty, \\ 1, & q = \infty. \end{cases}$

The following result is the Hardy–Littlewood theorem for function with general monotone Fourier coefficients.

Lemma 2.3.4. *[17, 69] Let $\lambda = \{\lambda\}_{m=1}^{\infty} \in GM$. Let the Fourier series of f be given by $\sum_{k=1}^{\infty} \lambda_k \cos kx$. A sufficient condition that the function f should belong to L_p, $1 < p < \infty$, is that $\sum_{k=1}^{\infty} |\lambda_k|^p k^{p-2} < \infty$. Moreover,*

$$\|f\|_p \lesssim \Big(\sum_{k=1}^{\infty} |\lambda_k|^p k^{p-2} \Big)^{\frac{1}{p}}.$$

If, additionally, $\{\lambda_n\}_{n=1}^{\infty}$ is a non-negative sequences, then

$$\|f\|_p \asymp \Big(\sum_{k=1}^{\infty} \lambda_k^p k^{p-2} \Big)^{\frac{1}{p}}.$$

2.4 New results in the case $1 < p < q < \infty$

For this non–limiting case we obtain the following generalization of Theorems B1 and B2.

Theorem 2.4.1. *Let $f \in L_p$, $1 < p < q < \infty$, $\theta = 1/p - 1/q$. Let $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in GM$. Then, for any $\alpha > 0$,*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \left(\sum_{m=n}^{\infty} \Big(2^{m(\theta+\rho)} \Lambda_{2^m} \omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_p \Big)^q \right)^{1/q},$$

where

$$\Lambda_n := \max_{1 \leqslant k \leqslant n} \frac{|\lambda_{2^k}|}{2^{k\rho}}, \qquad \rho > 0.$$

Remark 2.4.1. *Note that if for $f \in L_p$ the series*

$$\sum_{m=1}^{\infty} \Big(2^{m(\theta+\rho)} \Lambda_{2^m} \omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_p \Big)^q$$

converges, then there exists a function $f^{(\lambda,\beta)} \in L_q$ with the Fourier series $\sigma(f, \lambda, \beta)$. Similar remarks are valid for all further theorems.

For the sake of convenience, we formulate Ul'yanov's inequalities for sums. However, the corresponding integral form holds as in Theorems B1–B4.

Remark 2.4.2. *Using the fact that $\omega_\alpha(f, \delta)_p \asymp \omega_\alpha(f, 2\delta)_p$, we note that under conditions of Theorem 2.4.1, one has*

$$\omega_\alpha(f^{(\lambda,\beta)}, \delta)_q \lesssim \left(\int_0^\delta \Big(t^{-\theta-\rho} \Lambda_{[\frac{1}{t}]} \omega_{\alpha+\rho+\theta}(f, t)_p \Big)^q \frac{dt}{t} \right)^{1/q}, \quad 0 < \delta < 1.$$

Remark 2.4.3. *Note that for any* $\{\lambda_n\}_{n=1}^{\infty} \in GM$, *we have*

$$\max_{1\leqslant k\leqslant n} \frac{|\lambda_{2^k}|}{2^{k\rho}} \asymp \max_{1\leqslant s\leqslant 2^n} \frac{|\lambda_s|}{s^\rho}.$$

This holds from property 1 in Lemma 1.4.1.

Remark 2.4.4. *Theorem 2.4.1 generalizes Theorem B1 and B2.*

The proof of this remark will be given below.

Example 2.4.1. *For* $1 < p < q < \infty$, $\theta = 1/p - 1/q$, *Theorem 2.4.1 implies that*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \left(\sum_{m=n}^{\infty} \Big(2^{m(\theta+\rho)} m^\gamma \omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_p\Big)^q\right)^{1/q}, \qquad (2.4.3)$$

where $\lambda_n = n^\rho \ln^\gamma(n+1)$, $\gamma \geqslant 0$, *and*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \left(\sum_{m=n}^{\infty} \Big(2^{m(\theta+\rho)} \omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_p\Big)^q\right)^{1/q}, \qquad (2.4.4)$$

where $\lambda_n = \frac{n^\rho}{\ln^\gamma(n+1)}$, $\gamma \geqslant 0$.

Note that inequality (2.4.3) also follows from Theorem B1.

Remark 2.4.5. *Note that inequality* (2.4.4), *for* $\lambda_n = \frac{n^\rho}{\ln^\gamma(n+1)}$, $\gamma \geqslant 0$, *cannot be improved as follows.*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \left(\sum_{m=n}^{\infty} \Big(\frac{2^{m(\theta+\rho)}}{m^\gamma} \omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_p\Big)^q\right)^{1/q}.$$

See Example 3.2 in [73].

Proof of Theorem 2.4.1. We start with the realization result (Lemma 1.2.3)

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \|f^{(\lambda,\beta)} - S_{2^n}(f^{(\lambda,\beta)})\|_q + 2^{-n\alpha}\|S_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_q =: I_1 + I_2.$$

First, we estimate I_1 using Lemma 2.3.3 to obtain

$$\|S_{2^l}(f^{(\lambda,\beta)}) - S_{2^n}(f^{(\lambda,\beta)})\|_q \lesssim \left(\sum_{m=n}^{l-1} 2^{m\theta q}\|S_{2^{m+1}}(f^{(\lambda,\beta)}) - S_{2^m}(f^{(\lambda,\beta)})\|_p^q\right)^{1/q}.$$

By the Marcinkiewicz multiplier theorem, the definition of GM sequences, and the fact that $|\lambda_k| \lesssim |\lambda_{2^m}|,\ 2^m \leqslant k < 2^{m+1}$, we have

$$\|S_{2^{m+1}}(f^{(\lambda,\beta)}) - S_{2^m}(f^{(\lambda,\beta)})\|_p \lesssim |\lambda_{2^m}| \|S_{2^{m+1}}(f) - S_{2^m}(f)\|_p.$$

Let us denote $\Lambda_{2^m} := \max\limits_{1\leqslant k\leqslant m} \frac{|\lambda_{2^k}|}{2^{k\rho}}$. It is clear that $|\lambda_{2^m}| \leqslant 2^{m\rho}\Lambda_{2^m}$. Then

$$\begin{aligned}|\lambda_{2^m}| \|S_{2^{m+1}}(f) - S_{2^m}(f)\|_p &\leqslant 2^{m\rho}\Lambda_{2^m}\|S_{2^{m+1}}(f) - S_{2^m}(f) \pm f\|_p \\ &\lesssim 2^{m\rho}\Lambda_{2^m} E_{2^m}(f)_p.\end{aligned}$$

Applying Jackson's inequality, we obtain

$$\begin{aligned}I_1 = \|f^{(\lambda,\beta)} - S_{2^n}(f^{(\lambda,\beta)})\|_p &\lesssim \left(\sum_{m=n}^{\infty} 2^{m(\theta+\rho)q}\Lambda_{2^m}^q E_{2^m}^q(f)_p\right)^{1/q} \\ \lesssim \left(\sum_{m=n}^{\infty} 2^{m(\theta+\rho)q}\Lambda_{2^m}^q \omega_{\alpha+\rho+\theta}(f, \frac{1}{2^m})_p^q\right)^{1/q}.\end{aligned}$$

To estimate I_2, we use the Hardy-Littlewood fractional integration theorem

$$\|S_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_q \lesssim \|S_{2^n}^{(\alpha+\theta)}(f^{(\lambda,\beta)})\|_p.$$

Nothing that for any $\{\lambda_n\}_{n=1}^{\infty} \in GM$, we have that $\{\frac{\lambda_n}{n^\rho}\}_{n=1}^{\infty} \in GM$ for any $\rho \in \mathbb{R}$ (see Lemma 1.4.6), and applying Lemma 1.5.3, we get

$$\|S_{2^n}^{(\alpha+\theta)}(f^{(\lambda,\beta)})\|_p \lesssim \max_{1\leqslant k\leqslant n} \frac{|\lambda_{2^k}|}{2^{k\rho}} \|S_{2^n}^{(\alpha+\theta+\rho)}(f)\|_p.$$

Using Lemma 1.2.3, we obtain

$$\begin{aligned}I_2 &\lesssim 2^{-n\alpha} \max_{1\leqslant k\leqslant n} \frac{|\lambda_{2^k}|}{2^{k\rho}} \|S_{2^n}^{(\alpha+\theta+\rho)}(f)\|_p \\ &\lesssim 2^{n(\theta+\rho)} \max_{1\leqslant k\leqslant n} \frac{|\lambda_{2^k}|}{2^{k\rho}} \omega_{\alpha+\rho+\theta}(f, \frac{1}{2^n})_p \\ &= 2^{n(\theta+\rho)}\Lambda_{2^n}\omega_{\alpha+\rho+\theta}(f, \frac{1}{2^n})_p.\end{aligned}$$

Collecting estimates for I_1 and I_2, we have

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \left(\sum_{m=n}^{\infty}\Big(2^{m(\theta+\rho)}\Lambda_{2^m}\omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_p\Big)^q\right)^{1/q},$$

completing the proof.

□

Proof of Remark 2.4.4. We show only a nontrivial part that Theorem 2.4.1 extends Theorem B2. Let conditions (A_1') and (A_2) hold. Then $\lambda_{2n} \lesssim \lambda_n$. Assume that $\xi(t)$ is an increasing slowly varying function. Then the function $t^{-\rho}\xi(t)$ is strictly decreasing for t large enough. Suppose that $\frac{\lambda_n}{n^\rho}\xi(n)$ is increasing. Then we have that $\{\lambda_n\} \in GM$. Indeed, this follows from Lemma 1.4.2, since the conditions of λ imply that $\left\{\frac{\lambda_n \xi(n)}{n^\rho}\right\} \in GM$ and $\left\{\frac{n^\rho}{\xi(n)}\right\} \in GM$. Here we have used an obvious fact that any increasing sequence b_n such that $b_{2n} \lesssim b_n$ is general monotone.

Now we remark that

$$\Lambda_n = \max_{1\leqslant k\leqslant n} \frac{\lambda_{2^k}}{2^{k\rho}}\xi(2^k) = \frac{\lambda_{2^n}}{2^{n\rho}}\xi(2^n).$$

Then Theorem 2.4.1 gives

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \left(\sum_{m=n}^{\infty} \Big(2^{m(\theta+\rho)}\Lambda_{2^m}\omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_p\Big)^q\right)^{1/q}$$

$$= \left(\sum_{m=n}^{\infty} \Big(2^{m\theta}\lambda_{2^m}\xi(2^m)\omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_p\Big)^q\right)^{1/q},$$

the latter is equivalent to the statement of Theorem B2.

□

2.5 New results in the case $p = 1 < q < \infty$

The next results provide the sharp Ul'yanov type inequality for the first limiting case and, in particular, generalize Theorem B3.

Theorem 2.5.1. *Let $f \in L_p$, $1 = p < q < \infty$, $\theta = 1 - 1/q$. Let $\lambda = \{\lambda_n\}_{n=1}^\infty \in GM$. Then, for any $\alpha > 0$ and $0 < \varepsilon \leqslant \min(\rho, \theta)$,*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \left(\sum_{m=n}^{\infty} \Big(2^{m(\theta+\rho)}\Lambda_{2^m}\omega_{\alpha+\rho+\theta-\varepsilon}\Big(f, \frac{1}{2^m}\Big)_1\Big)^q\right)^{1/q},$$

where

$$\Lambda_{2^n} := 2^{-n\frac{\varepsilon}{2}} \max_{1\leqslant k\leqslant n} \frac{|\lambda_{2^k}|}{2^{k(\rho-\frac{\varepsilon}{2})}}.$$

Remark 2.5.1. *It is clear that Theorem B3 (i) follows from Theorem 2.5.1.*

Proof of Theorem 2.5.1. We consider the de la Vallée-Poussin polynomials $V_1(f) := A_1(f,x)$ and

$$V_n(f)(x) = \sum_{m=1}^{n} A_m(f,x) + \sum_{m=n+1}^{2n-1} \Big(1 - \frac{m-n}{n}\Big) A_m(f,x) \ (n \geqslant 2),$$

where $A_m(f,x)$ are given by (1.1.2). We start by applying realization inequality (1.2.9) from Lemma 1.2.3

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \|f^{(\lambda,\beta)} - V_{2^n}(f^{(\lambda,\beta)})\|_q + 2^{-n\alpha}\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_q =: I_1 + I_2. \tag{2.5.5}$$

Let $0 < \varepsilon \leqslant \min(\rho, \theta)$. To estimate I_1, we use Lemma 2.3.3

$$\|V_{2^l}(f^{(\lambda,\beta)}) - V_{2^n}(f^{(\lambda,\beta)})\|_q \lesssim \left(\sum_{m=n}^{l-1} 2^{m\theta_0 q}\|V_{2^{m+1}}(f^{(\lambda,\beta)}) - V_{2^m}(f^{(\lambda,\beta)})\|_{p_0}^q\right)^{1/q},$$

where $1 < p_0 < q < \infty$ is chosen such that

$$\theta_0 = \frac{1}{p_0} - \frac{1}{q} = \theta - \frac{\varepsilon}{2}.$$

By Lemmas 1.4.1 and 1.5.3, we get

$$\|V_{2^{m+1}}(f^{(\lambda,\beta)}) - V_{2^m}(f^{(\lambda,\beta)})\|_{p_0} \lesssim |\lambda_{2^m}|\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_{p_0}.$$

Denoting

$$\Lambda_{2^n} := 2^{-n\varepsilon/2} \max_{1\leqslant k\leqslant n} \frac{|\lambda_{2^k}|}{2^{k(\rho-\varepsilon/2)}},$$

we have

$$|\lambda_{2^m}|\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_{p_0} \lesssim 2^{m\rho}\Lambda_{2^m}\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_{p_0}.$$

Applying Lemma 2.3.1, we obtain

$$2^{m\rho}\Lambda_{2^m}\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_{p_0} \lesssim 2^{m\rho}\Lambda_{2^m}2^{m(1-\frac{1}{p_0})}\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_1 =: I_3.$$

It is well known that $V_n(f)$ is the L_p near-best approximant of f, that is,

$$\|f - V_n(f)\|_p \lesssim E_n(f)_p \equiv \inf_{T\in\mathbb{T}_n} \|f - T\|_p,$$

where $\mathbb{T}_n$ is the set of all trigonometric polynomials of degree at most n. Then

$$I_3 \lesssim 2^{m(\rho+1-\frac{1}{p_0})}\Lambda_{2^m}\|V_{2^{m+1}}(f) - V_{2^m}(f) \pm f\|_1 \lesssim 2^{m(\rho+1-\frac{1}{p_0})}\Lambda_{2^m}E_{2^m}(f)_1.$$

Using Jackson's inequality implies

$$I_1 \lesssim \left(\sum_{m=n}^{\infty} \left(2^{m(\theta_0+\rho+\theta-\theta_0)} \Lambda_{2^m} E_{2^m}(f)_1 \right)^q \right)^{1/q}$$
$$\lesssim \left(\sum_{m=n}^{\infty} \left(2^{m(\theta+\rho)} \Lambda_{2^m} \omega_{\alpha+\rho+\theta-\varepsilon}(f, \frac{1}{2^m})_p \right)^q \right)^{1/q}.$$

Now let us estimate the second term in (2.5.5). By Lemma 1.5.4, we have

$$\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_q \lesssim \|V_{2^n}^{(\alpha+\theta_0)}(f^{(\lambda,\beta)})\|_{p_0}.$$

Note that if $\{\lambda_n\} \in GM$, then $\{n^{\xi}\lambda_n\} \in GM, \quad \xi \in \mathbb{R}$. Again applying Lemmas 1.4.1 and 1.5.3, we obtain

$$I_2 \lesssim 2^{-n\alpha} \|V_{2^n}^{(\alpha+\theta_0)}(f^{(\lambda,\beta)})\|_{p_0}$$
$$\lesssim 2^{-n\alpha} \max_{1\leqslant k\leqslant n} \frac{|\lambda_{2^k}|}{2^{k(\rho-\varepsilon/2)}} \|V_{2^n}^{(\alpha+\theta_0+\rho-\varepsilon/2)}(f)\|_{p_0}$$
$$= 2^{n(\varepsilon/2-\alpha)} \Lambda_{2^n} \|V_{2^n}^{(\alpha+\theta_0+\rho-\varepsilon/2)}(f)\|_{p_0}.$$

Therefore,

$$I_2 \lesssim 2^{n(\varepsilon/2-\alpha)} \Lambda_{2^n} 2^{n(\theta-\theta_0)} \|V_{2^n}^{(\alpha+\theta_0+\rho-\varepsilon/2)}(f)\|_1$$
$$\lesssim 2^{n(\varepsilon/2-\alpha)} \Lambda_{2^n} 2^{n(\theta-\theta_0)} 2^{n(\alpha+\theta_0+\rho-\varepsilon/2)} \omega_{\alpha+\rho+\theta-\varepsilon/2-\varepsilon/2}(f, \frac{1}{2^n})_1$$
$$= 2^{n(\theta+\rho)} \Lambda_{2^n} \omega_{\alpha+\rho+\theta-\varepsilon}(f, \frac{1}{2^n})_1.$$

Here, the first inequality follows from Nikol'skii's inequality given by Lemma 2.3.1 and the second from Lemma 1.5.2. Collecting estimates for I_1 and I_2, we obtain

$$\omega_\alpha(f^{(\lambda,\beta)}, \frac{1}{2^n})_q \lesssim \left(\sum_{m=n}^{\infty} \left(2^{m(\theta+\rho)} \Lambda_{2^m} \omega_{\alpha+\rho+\theta-\varepsilon}\left(f, \frac{1}{2^m}\right)_1 \right)^q \right)^{1/q},$$

which completes the proof.

□

In the next result, unlike Theorem 2.5.1, we deal with $\omega_{\alpha+\rho+\theta}(f,t)_1$ obtaining the sharp Ul'yanov inequality.

Theorem 2.5.2. *Let $f \in L_p$, $1 = p < q < \infty$, $\theta = 1 - 1/q$, and $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in GM$. Then, for any $\alpha > 0$,*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \Bigg(\sum_{m=n}^{\infty}\Big(2^{m(\theta+\rho)}\Lambda_{2^m}\omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_1\Big)^q\Bigg)^{1/q},$$

where

$$\Lambda_N := \Big(\sum_{k=1}^{N}\frac{|\lambda_k|^q}{k^{(q\rho+1)}}\Big)^{\frac{1}{q}}, \quad \rho > 0.$$

Remark 2.5.2. *Since the conditions (A_1),(A_2), and (A_4) imply that $\{\lambda_n = \lambda(n)\} \in GM$ and*

$$\Lambda_{2^m} = \Bigg(\sum_{k=1}^{2^m}\Big(\frac{|\lambda_k|}{k^\rho}\Big)^q\frac{1}{k}\Bigg)^{\frac{1}{q}} \lesssim \frac{|\lambda_{2^m}|}{2^{m\rho}}m^{\frac{1}{q}},$$

then Theorem 2.5.2 sharpens Theorem B3 (ii).

Example 2.5.1 and Remark 2.5.3 below show that the inequalities in Theorems 2.5.1 and 2.5.2 are not comparable.

Example 2.5.1. *Let $1 = p < q < \infty$, $\theta = 1 - 1/q$, $\rho > 0$, $\gamma \in \mathbb{R}$, and $\lambda_n = n^\rho \ln^\gamma(n+1)$. By Theorem 2.5.1, we have*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \Bigg(\sum_{m=n}^{\infty}\Big(2^{m(\theta+\rho)}m^\gamma\omega_{\alpha+\rho+\theta-\varepsilon}\Big(f, \frac{1}{2^m}\Big)_1\Big)^q\Bigg)^{1/q}, \tag{2.5.6}$$

and, by Theorem 2.5.2, we have

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \begin{cases} \Bigg(\sum\limits_{m=n}^{\infty}\Big(2^{m(\theta+\rho)}m^{\gamma+\frac{1}{q}}\omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_1\Big)^q\Bigg)^{1/q}, & \gamma > -\frac{1}{q}, \\ \Bigg(\sum\limits_{m=n}^{\infty}\Big(2^{m(\theta+\rho)}\omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_1\Big)^q \ln m\Bigg)^{1/q}, & \gamma = -\frac{1}{q}, \\ \Bigg(\sum\limits_{m=n}^{\infty}\Big(2^{m(\theta+\rho)}\omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_1\Big)^q\Bigg)^{1/q}, & \gamma < -\frac{1}{q}. \end{cases} \tag{2.5.7}$$

Remark 2.5.3. *We note that inequality* (2.5.6) *for $\gamma \in \mathbb{R}$ and inequality* (2.5.7) *for $\gamma \geqslant 0$ also follows from Theorem B3 (see [73, Corollary 1.5]). Using $\{\Lambda_n\}$ allows us to prove* (2.5.7) *for any choice of $\gamma \in \mathbb{R}$. Moreover, in [73] it was shown that estimates* (2.5.6) *and* (2.5.7) *are not comparable.*

Proof of Theorem 2.5.2. The proof follows the same lines as those in Theorem 2.5.1. We consider the de la Vallée-Poussin sum of the function $f^{(\lambda,\beta)}$. As above, we use Lemma 1.2.3

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_q \lesssim \|f^{(\lambda,\beta)} - V_{2^n}(f^{(\lambda,\beta)})\|_q + 2^{-n\alpha}\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_q =: I_1 + I_2.$$

Now we apply the following (p,q)-inequality:

$$\|V_{2^l}(f^{(\lambda,\beta)}) - V_{2^n}(f^{(\lambda,\beta)})\|_q \lesssim \left(\sum_{m=n}^{l-1} 2^{m\theta_0 q}\|V_{2^{m+1}}(f^{(\lambda,\beta)}) - V_{2^m}(f^{(\lambda,\beta)})\|_{p_0}^q\right)^{1/q},$$

where $1 < p_0 < q < \infty$, $\theta_0 = \frac{1}{p_0} - \frac{1}{q}$. Further,

$$\|V_{2^{m+1}}(f^{(\lambda,\beta)}) - V_{2^m}(f^{(\lambda,\beta)})\|_{p_0} \lesssim |\lambda_{2^m}|\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_{p_0}.$$

Set

$$\Lambda_{2^m} := \Big(\sum_{k=1}^{2^m} \frac{|\lambda_k|^q}{k^{q\rho+1}}\Big)^{\frac{1}{q}}.$$

By Lemma 1.4.7, we have $\lambda_{2^m} \lesssim 2^{m\rho}\Lambda_{2^m}$. Then

$$|\lambda_{2^m}|\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_{p_0} \lesssim 2^{m\rho}\Lambda_{2^m}\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_{p_0}.$$

Using Lemma 2.3.1, we get

$$2^{m\rho}\Lambda_{2^m}\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_{p_0} \lesssim 2^{m\rho}\Lambda_{2^m}2^{m(1-\frac{1}{p_0})}\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_1 =: I_3.$$

Since V_n is the L_p near-best approximant of f, we obtain that

$$I_3 \lesssim 2^{m(\rho+1-\frac{1}{p_0})}\Lambda_{2^m}\|V_{2^{m+1}}(f) - V_{2^m}(f) \pm f\|_1 \lesssim 2^{m(\rho+1-\frac{1}{p_0})}\Lambda_{2^m}E_{2^m}(f)_1.$$

Combining these estimates and Jackson's inequality, we derive that

$$\begin{aligned} I_1 &\lesssim \left(\sum_{m=n}^{\infty}\Big(2^{m(\theta_0+\rho+\theta-\theta_0)}\Lambda_{2^m}E_{2^m}(f)_1\Big)^q\right)^{1/q} \\ &\lesssim \left(\sum_{m=n}^{\infty}\Big(2^{m(\theta+\rho)}\Lambda_{2^m}\omega_{\alpha+\rho+\theta}(f,\frac{1}{2^m})_1\Big)^q\right)^{1/q}. \end{aligned}$$

Let us estimate I_2. Applying Young's convolution inequality ([88]) we have

$$\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_q \lesssim \Big\| \sum_{m=1}^{2^n} \lambda_m m^\alpha A_m(f,x) + \sum_{m=2^n+1}^{2^{n+1}-1} \lambda_m \Big(1 - \frac{m-2^n}{2^n}\Big) m^\alpha A_m(f,x) \Big\|_q$$

$$= \Big\| \Big(\sum_{m=1}^{2^{n+1}-1} \frac{\lambda_m}{m^{\rho+\theta}} \cos mx \Big) * \Big(\sum_{m=1}^{2^n} m^{\alpha+\rho+\theta} A_m(f,x)$$

$$+ \sum_{m=2^n+1}^{2^{n+1}-1} \Big(1 - \frac{m-2^n}{2^n}\Big) m^{\alpha+\rho+\theta} A_m(f,x) \Big) \Big\|_q$$

$$\leqslant \Big\| \sum_{m=1}^{2^{n+1}-1} \frac{\lambda_m}{m^{\rho+\theta}} \cos mx \Big\|_q \|V_{2^n}^{(\alpha+\rho+\theta)}(f,x)\|_1.$$

Using the fact that $\{\lambda_n\}_{n=1}^\infty \in GM$ implies that $\{\frac{\lambda_n}{n^{\rho+\theta}}\}_{n=1}^\infty \in GM$, Lemma 2.3.4 yields

$$\Big\| \sum_{m=1}^{2^{n+1}-1} \frac{\lambda_m}{m^{\rho+\theta}} \cos mx \Big\|_q \lesssim \Big(\sum_{\nu=1}^{2^{n+1}-1} \Big(\frac{|\lambda_\nu|}{\nu^{\rho+\theta}} \Big)^q \nu^{q-2} \Big)^{\frac{1}{q}} = \Big(\sum_{\nu=1}^{2^{n+1}-1} \frac{|\lambda_\nu|^q}{\nu^{q\rho+1}} \Big)^{\frac{1}{q}} = \Lambda_{2^{n+1}-1}.$$

Taking into account that $\lambda_n \lesssim \lambda_k, \quad k \leqslant n \leqslant 2k$, we can estimate $\Lambda_{2^{m+1}-1} \lesssim \Lambda_{2^m}$. Finally, Lemma 1.5.2 implies

$$I_2 \lesssim 2^{-n\alpha} \Lambda_{2^n} \|V_{2^n}^{(\alpha+\rho+\theta)}(f,x)\|_1 \lesssim 2^{n(\rho+\theta)} \Lambda_{2^n} \omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^n}\Big)_1.$$

Collecting estimates for I_1 and I_2, we get

$$\omega_\alpha(f^{(\lambda,\beta)}, \frac{1}{2^n})_q \lesssim \left(\sum_{m=n}^{\infty} \Big(2^{m(\theta+\rho)} \Lambda_{2^m} \omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_1 \Big)^q \right)^{1/q},$$

completing the proof.

□

2.6 New results in the case $1 < p < q = \infty$

In the following theorems we investigate the Ulyanov-type inequality for another limiting case: $1 < p < q = \infty$. As in the previous subsection we obtain two main theorems.

Theorem 2.6.1. *Let $f \in L_p$, $1 < p < q = \infty$, $\theta = 1/p$, and $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in GM$. Then, for any $\alpha > 0$ and $0 < \varepsilon \leqslant \min(\rho, \theta)$,*

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_\infty \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)} \Lambda_{2^m} \omega_{\alpha+\rho+\theta-\varepsilon}\Big(f, \frac{1}{2^m}\Big)_p,$$

where

$$\Lambda_{2^n} := 2^{-n\varepsilon/2} \max_{1\leqslant k\leqslant n} \frac{|\lambda_{2^k}|}{2^{k(\rho-\varepsilon/2)}}.$$

Proof of Theorem 2.6.1. The proof is the same as that of Theorem 2.5.1. We use Lemma 1.2.3 for $q = \infty$ to get

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_\infty \lesssim \|f^{(\lambda,\beta)} - V_{2^n}(f^{(\lambda,\beta)})\|_\infty + 2^{-n\alpha}\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_\infty =: I_1 + I_2.$$

Observing that

$$\lambda_n \lesssim \Lambda_n := 2^{-n\varepsilon/2} \max_{1\leqslant k\leqslant n} \frac{|\lambda_{2^k}|}{2^{k(\rho-\varepsilon/2)}},$$

we use Lemmas 2.3.3 and 1.5.3 to derive

$$\begin{aligned}
\|V_{2^l}(f^{(\lambda,\beta)}) - V_{2^n}(f^{(\lambda,\beta)})\|_\infty &\lesssim \sum_{m=n}^{l-1} 2^{m\theta} \|V_{2^{m+1}}(f^{(\lambda,\beta)}) - V_{2^m}(f^{(\lambda,\beta)})\|_p, \\
&\lesssim \sum_{m=n}^{l-1} 2^{m\theta} |\lambda_{2^m}| \|V_{2^{m+1}}(f) - V_{2^m}(f)\|_p \\
&\lesssim \sum_{m=n}^{l-1} 2^{m\theta} 2^{m\rho} \Lambda_{2^m} \|V_{2^{m+1}}(f) - V_{2^m}(f)\|_p.
\end{aligned}$$

Then

$$I_1 \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)} \Lambda_{2^m} \omega_{\alpha+\rho+\theta-\varepsilon}(f, \frac{1}{2^m})_p.$$

By Lemmas 2.3.1, 1.5.4, and 1.5.3, we choose q_0 such that

$$\theta_0 = \frac{1}{p} - \frac{1}{q_0} = \theta - \frac{\varepsilon}{2}$$

and we estimate

$$
\begin{aligned}
I_2 = 2^{-n\alpha}\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_\infty &\lesssim 2^{-n\alpha}2^{n\frac{1}{q_0}}\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_{q_0}\\
&\lesssim 2^{-n\alpha}2^{n\frac{1}{q_0}}\|V_{2^n}^{(\alpha+\theta_0)}(f^{(\lambda,\beta)})\|_p\\
&\lesssim 2^{n(\frac{1}{q_0}-\alpha)}\max_{1\leqslant k\leqslant n}\frac{|\lambda_{2^k}|}{2^{k(\rho-\varepsilon/2)}}\|V_{2^n}^{(\alpha+\theta_0+\rho-\varepsilon/2)}(f)\|_p\\
&\lesssim 2^{n(\varepsilon/2-\alpha+\frac{1}{q_0})}\Lambda_n 2^{n(\alpha+\theta_0+\rho-\varepsilon/2)}\omega_{\alpha+\rho+\theta-\varepsilon/2-\varepsilon/2}(f,\frac{1}{2^n})_p\\
&= 2^{n(\theta+\rho)}\Lambda_n\omega_{\alpha+\rho+\theta-\varepsilon}(f,\frac{1}{2^n})_p.
\end{aligned}
$$

Combining the obtained estimates, we finally arrive at

$$
\omega_\alpha(f^{(\lambda,\beta)},\frac{1}{2^n})_q \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)}\Lambda_m\omega_{\alpha+\rho+\theta-\varepsilon}\Big(f,\frac{1}{2^m}\Big)_p.
$$

□

The next result is the sharp Ul'yanov inequality for $1<p<q=\infty$.

Theorem 2.6.2. *Let $f\in L_p$, $1<p<q=\infty$, $\theta=1/p$, and $\lambda=\{\lambda_n\}_{n=1}^\infty\in GM$. Then, for any $\alpha>0$,*

$$
\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_\infty \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)}\Lambda_{2^m}\omega_{\alpha+\rho+\theta}\Big(f,\frac{1}{2^m}\Big)_p,
$$

where

$$
\Lambda_N := \Big(\sum_{k=1}^{N}\frac{|\lambda_k|^{p'}}{k^{p'\rho+1}}\Big)^{\frac{1}{p'}},\quad \rho>0.
$$

Example 2.6.1. *Let $f\in L_p$, $1<p<q=\infty$, $\theta=1/p$, $\rho>0$, $\gamma\in\mathbb{R}$, and $\lambda_n=n^\rho\ln^\gamma(n+1)$. Then, by Theorem 2.6.1, we have*

$$
\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_\infty \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)}m^\gamma\omega_{\alpha+\rho+\theta-\varepsilon}\Big(f,\frac{1}{2^m}\Big)_p.
$$

Theorem 2.6.2 yields

$$
\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_\infty \lesssim \begin{cases}
\sum\limits_{m=n}^{\infty} 2^{m(\theta+\rho)}m^{\gamma+\frac{1}{p'}}\omega_{\alpha+\rho+\theta}\Big(f,\frac{1}{2^m}\Big)_p, & \gamma>-\frac{1}{p'},\\
\sum\limits_{m=n}^{\infty} 2^{m(\theta+\rho)}\Big(\ln m\Big)^{\frac{1}{p'}}\omega_{\alpha+\rho+\theta}\Big(f,\frac{1}{2^m}\Big)_p, & \gamma=-\frac{1}{p'},\\
\sum\limits_{m=n}^{\infty} 2^{m(\theta+\rho)}\omega_{\alpha+\rho+\theta}\Big(f,\frac{1}{2^m}\Big)_p, & \gamma<-\frac{1}{p'}.
\end{cases}
\tag{2.6.8}
$$

Remark 2.6.1. *Note that letting formally $\rho = \gamma = 0$ in* (2.5.7) *and* (2.6.8) *gives*

$$\omega_\alpha(f, \frac{1}{2^n})_q \lesssim \left(\sum_{m=n}^{\infty} \left(2^{m\theta} m^{\max\left(\frac{1}{q}, \frac{1}{p'}\right)} \omega_{\alpha+\theta}\Big(f, \frac{1}{2^m}\Big)_p \right)^{q_1} \right)^{\frac{1}{q_1}},$$

for $1 < p < q = \infty$ or $1 = p < q < \infty$, $q_1 = \begin{cases} q, & q < \infty, \\ 1, & q = \infty, \end{cases}$ which is the the sharp Ul'yanov inequality proved in [68].

Proof of Theorem 2.6.2. Let $1 < p < q = \infty$. Applying Lemma 1.2.3, we get

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_\infty \lesssim \|f^{(\lambda,\beta)} - V_{2^n}(f^{(\lambda,\beta)})\|_\infty + 2^{-n\alpha}\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_\infty =: I_1 + I_2,$$

where $V_{2^n}(f^{(\lambda,\beta)})$ is the de la Vallée-Poussin sum of the function $f^{(\lambda,\beta)}$. As in the proof of Theorem 2.6.1, we have

$$\begin{aligned}
&\|V_{2^l}(f^{(\lambda,\beta)}) - V_{2^n}(f^{(\lambda,\beta)})\|_\infty \lesssim \sum_{m=n}^{l-1} 2^{m\theta}\|V_{2^{m+1}}(f^{(\lambda,\beta)}) - V_{2^m}(f^{(\lambda,\beta)})\|_p \\
&\lesssim \sum_{m=n}^{l-1} 2^{m\theta}|\lambda_{2^m}|\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_p \\
&\lesssim \sum_{m=n}^{l-1} 2^{m(\theta+\rho)}\Lambda_{2^m}\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_p,
\end{aligned}$$

where $\Lambda_{2^m} := \left(\sum_{k=1}^{2^m} \frac{|\lambda_k|^{p'}}{k^{p'\rho+1}} \right)^{\frac{1}{p'}}$ and the last inequality follows from Lemma 1.4.7. We know that $V_n(f)$ is the L_p near-best approximant of f. By Jackson's inequality, we get

$$\begin{aligned}
I_1 &\lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)}\Lambda_{2^m}E_{2^m}(f)_p \\
&\lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)}\Lambda_{2^m}\omega_{\alpha+\rho+\theta}(f, \frac{1}{2^m})_p.
\end{aligned}$$

We handle I_2 similarly. Applying Young's convolution inequality and Lemma 2.3.4, we have

$$\begin{aligned}
\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_\infty &\lesssim \Big\| \sum_{m=1}^{2^{n+1}-1} \frac{\lambda_m}{m^{\rho+\theta}} \cos mx \Big\|_{p'} \|V_{2^n}^{(\alpha+\rho+\theta)}(f, x)\|_p \\
&\lesssim \Lambda_{2^{n+1}}\|V_{2^n}^{(\alpha+\rho+\theta)}(f, x)\|_p.
\end{aligned}$$

Remark that $\Lambda_{2^{n+1}} \lesssim \Lambda_{2^n}$. Finally, by Lemma 1.5.2, we obtain

$$I_2 \lesssim 2^{-n\alpha}\Lambda_{2^n}\|V_{2^n}^{(\alpha+\rho+\theta)}(f,x)\|_p \lesssim 2^{n(\rho+\theta)}\Lambda_{2^n}\omega_{\alpha+\rho+\theta}\Big(f,\frac{1}{2^n}\Big)_p.$$

Collecting estimates for I_1 and I_2, we derive

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_q \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)}\Lambda_{2^m}\omega_{\alpha+\rho+\theta}\Big(f,\frac{1}{2^m}\Big)_p.$$

□

2.7 New results in the case $1 = p, q = \infty$

Our two main results here are Theorem 2.7.1 and 2.7.2 below.

Theorem 2.7.1. *Let $f \in L_p$, $p = 1$, $q = \infty$, $\theta = 1$, and $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in GM$. Then, for any $\alpha > 0$ and $0 < \varepsilon \leqslant \min(\rho,\theta)$,*

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_\infty \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)}\Lambda_{2^m}\omega_{\alpha+\rho+\theta-\varepsilon}\Big(f,\frac{1}{2^m}\Big)_1,$$

where

$$\Lambda_{2^n} := 2^{-n\varepsilon/2}\max_{1\leqslant k\leqslant n}\frac{|\lambda_{2^k}|}{2^{k(\rho-\varepsilon/2)}}.$$

Proof of Theorem 2.7.1. Let $p = 1 < q = \infty$. The method of proof is the same as above. Lemma 1.2.3 with $q = \infty$ gives

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_\infty \lesssim \|f^{(\lambda,\beta)} - V_{2^n}(f^{(\lambda,\beta)})\|_\infty + 2^{-n\alpha}\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_\infty =: I_1 + I_2.$$

Applying Lemma 2.3.3 and 1.5.3, we obtain for $\theta_0 = 1/p_0$

$$\|V_{2^l}(f^{(\lambda,\beta)}) - V_{2^n}(f^{(\lambda,\beta)})\|_\infty \lesssim \sum_{m=n}^{l-1} 2^{m\theta_0}\|V_{2^{m+1}}(f^{(\lambda,\beta)}) - V_{2^m}(f^{(\lambda,\beta)})\|_{p_0}$$

$$\lesssim \sum_{m=n}^{l-1} 2^{m\theta_0}|\lambda_{2^m}|\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_{p_0},$$

where $p_0 > 1$. We set $\Lambda_{2^n} := 2^{-n\varepsilon/2}\max_{1\leqslant k\leqslant n}\frac{|\lambda_{2^k}|}{2^{k(\rho-\varepsilon/2)}}$. Using Nikol'skii's inequality, we get

$$|\lambda_{2^m}|\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_{p_0} \leqslant 2^{m\rho}\Lambda_{2^m}\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_{p_0}$$
$$\lesssim 2^{m\rho}\Lambda_{2^m}2^{m(\theta-\theta_0)}\|V_{2^{m+1}}(f) - V_{2^m}(f)\|_1.$$

Finally, we estimate I_1 as follows:

$$I_1 \lesssim \sum_{m=n}^{\infty} 2^{m(\rho+\theta)} \Lambda_{2^m} \omega_{\alpha+\rho+\theta-\varepsilon}\Big(f, \frac{1}{2^m}\Big)_1.$$

Arguing as above, we estimate I_2. Applying Nikol'skii's inequality with $1 < p_0 < q_0 < \infty$, $\theta_0 = \frac{1}{p_0} - \frac{1}{q_0} = \theta - \frac{\varepsilon}{2}$ and then Hardy-Littlewood fractional integration theorem and Lemma 1.5.3, we obtain

$$\begin{aligned} I_2 = 2^{-n\alpha} \|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_\infty &\lesssim 2^{-n\alpha} 2^{n\frac{1}{q_0}} \|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_{q_0} \\ &\lesssim 2^{-n\alpha} 2^{n\frac{1}{q_0}} \|V_{2^n}^{(\alpha+\theta_0)}(f^{(\lambda,\beta)})\|_{p_0} \\ &\lesssim 2^{n(\frac{1}{q_0}-\alpha)} \max_{1\leqslant k\leqslant n} \frac{|\lambda_{2^k}|}{2^{k(\rho-\varepsilon/2)}} \|V_{2^n}^{(\alpha+\theta_0+\rho-\varepsilon/2)}(f)\|_{p_0}. \end{aligned}$$

Using again Nikol'skii's inequality gives

$$\begin{aligned} I_2 &\lesssim 2^{n(\varepsilon/2-\alpha+\frac{1}{q_0})} \Lambda_{2^n} 2^{(1-\frac{1}{p_0})n} \|V_{2^n}^{(\alpha+\theta_0+\rho-\varepsilon/2)}(f)\|_1 \\ &\lesssim 2^{n(\varepsilon/2-\alpha+\theta-\theta_0)} \Lambda_{2^n} 2^{n(\alpha+\theta_0+\rho-\varepsilon/2)} \omega_{\alpha+\rho+\theta-\varepsilon/2-\varepsilon/2}(f, \frac{1}{2^n})_1 \\ &= 2^{n(\theta+\rho)} \Lambda_{2^n} \omega_{\alpha+\rho+\theta-\varepsilon}(f, \frac{1}{2^n})_1. \end{aligned}$$

The last inequality follows from Lemma 1.5.2. Collecting estimates for I_1 and I_2, we have

$$\omega_\alpha(f^{(\lambda,\beta)}, \frac{1}{2^n})_\infty \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)} \Lambda_{2^m} \omega_{\alpha+\rho+\theta-\varepsilon}\Big(f, \frac{1}{2^m}\Big)_1.$$

□

Now we state the sharp Ul'yanov inequality in the case $p = 1,\ q = \infty$, and $\theta = 1$.

Theorem 2.7.2. *Let $\alpha \in \mathbb{R}_+$ and $\rho \in \mathbb{R}_+ \cup \{0\}$. Let $f \in L_1$, $\lambda = \{\lambda_n\}_{n=1}^\infty \in GM$, $\{\Delta\lambda_n\} \in GM$, and $\{\Delta\frac{\lambda_n}{n^\rho}\} \in GM$. We have*

$$\begin{aligned} \omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_\infty \lesssim & \sum_{m=n}^{\infty} 2^m |\lambda_{2^m}| \omega_{\alpha+\rho+1}\Big(f, \frac{1}{2^m}\Big)_1 \\ & + 2^{-n\alpha} \sum_{k=1}^{2^{n+2}-1} \Big|\frac{\lambda_k}{k^\rho} - \frac{\lambda_{k+1}}{(k+1)^\rho}\Big| k^{\rho+\alpha+1} \omega_{\alpha+\rho+1}\Big(f, \frac{1}{k}\Big)_1 \\ & + 2^{-n\alpha} \Big|\sin\frac{\pi(\beta-\rho)}{2}\Big| \sum_{k=1}^{n} |\lambda_{2^k}| 2^{k(\alpha+1)} \omega_{\alpha+\rho+1}\Big(f, \frac{1}{2^k}\Big)_1. \end{aligned} \quad (2.7.9)$$

Proof of Theorem 2.7.2. Let $1 = p$ and $q = \infty$. Using Lemma 1.2.3, we get

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_\infty \lesssim \Big\|f^{(\lambda,\beta)} - V_{2^n}(f^{(\lambda,\beta)})\Big\|_\infty + 2^{-n\alpha}\Big\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\Big\|_\infty =: I_1 + I_2.$$

Taking into account Lemma 2.3.3 for $q = \infty, \quad q_1 = 1, \quad p_0 > 1$ and $\sigma = 2^n,\ L = l - n$, we obtain

$$\Big\|V_{2^l}(f^{(\lambda,\beta)}) - V_{2^n}(f^{(\lambda,\beta)})\Big\|_\infty \lesssim \sum_{m=n}^{l-1} 2^{m\frac{1}{p_0}}\Big\|V_{2^{m+1}}(f^{(\lambda,\beta)}) - V_{2^m}(f^{(\lambda,\beta)})\Big\|_{p_0}.$$

By the Marcinkiewicz multiplier theorem and properties of GM sequences, we have

$$\Big\|V_{2^{m+1}}(f^{(\lambda,\beta)}) - V_{2^m}(f^{(\lambda,\beta)})\Big\|_{p_0} \lesssim |\lambda_{2^m}|\Big\|V_{2^{m+1}}(f) - V_{2^m}(f)\Big\|_{p_0}.$$

Nikol'skii inequality implies

$$|\lambda_{2^m}|\Big\|V_{2^m}(f) - V_{2^m}(f)\Big\|_{p_0} \lesssim |\lambda_{2^m}|2^{m\left(1-\frac{1}{p_0}\right)}\Big\|V_{2^{m+1}}(f) - V_{2^m}(f)\Big\|_1.$$

Applying Jackson's inequality, we obtain

$$I_1 \lesssim \sum_{m=n}^{\infty} 2^m|\lambda_{2^m}|E_{2^m}(f)_1 \lesssim \sum_{m=n}^{\infty} 2^m|\lambda_{2^m}|\omega_{\alpha+\rho+1}\Big(f, \frac{1}{2^m}\Big)_1.$$

To estimate I_2, we first use Lemma 2.3.2

$$\|V_{2^n}^{(\alpha)}(f^{(\lambda,\beta)})\|_\infty \lesssim \|V_{2^n}^{(\alpha+1)}(f^{(\lambda,\beta)})\|_1.$$

Further using (1.7.48), we have

$$\begin{aligned}\|V_{2^n}^{(\alpha+1)}(f^{(\lambda,\beta)})\|_1 \lesssim & \sum_{\mu=1}^{2^{n+2}-1}\Big|\frac{\lambda_\mu}{\mu^\rho} - \frac{\lambda_{\mu+1}}{(\mu+1)^\rho}\Big|\mu^{\alpha+\rho+1}\omega_{\alpha+\rho+1}\Big(f, \frac{1}{\mu}\Big)_1 \\ & + |\lambda_{2^{n+2}}|2^{n(\alpha+1)}\omega_{\alpha+\rho+1}\Big(f, \frac{1}{2^n}\Big)_1 \\ & + \Big|\sin\frac{\pi(\beta-\rho)}{2}\Big|\sum_{k=0}^{n-1}|\lambda_{2^{k+2}}|2^{k(\alpha+1)}\omega_{\alpha+\rho+1}\Big(f, \frac{1}{2^k}\Big)_1.\end{aligned}$$

By properties of GM sequences and collecting estimates of I_1 and I_2, we obtain

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_\infty \lesssim \sum_{m=n}^{\infty} 2^m|\lambda_{2^m}|\omega_{\alpha+\rho+1}\Big(f,\frac{1}{2^m}\Big)_1$$
$$+\,2^{-n\alpha}\sum_{k=1}^{2^{n+2}-1}\Big|\frac{\lambda_k}{k^\rho}-\frac{\lambda_{k+1}}{(k+1)^\rho}\Big|k^{\rho+\alpha+1}\omega_{\alpha+\rho+1}\Big(f,\frac{1}{k}\Big)_1$$
$$+\,2^{-n\alpha}\Big|\sin\frac{\pi(\beta-\rho)}{2}\Big|\sum_{k=1}^{n}|\lambda_{2^k}|2^{k(\alpha+1)}\omega_{\alpha+\rho+1}\Big(f,\frac{1}{2^k}\Big)_1.$$

The proof of Theorem 2.7.2 is now complete. □

Let us apply Theorems 2.7.1 and 2.7.2 for power–logarithmic derivatives.

Example 2.7.1. *Let $f\in L_1,\ \ \theta=1,\ \ $and*

$$\lambda_n=n^\rho\ln^\gamma(n+1),\quad \rho>0,\ \gamma\in\mathbb{R}.$$

Theorem 2.7.1 implies that

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_\infty \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)}m^\gamma\omega_{\alpha+\rho+\theta-\varepsilon}\Big(f,\frac{1}{2^m}\Big)_1.$$

On the other hand, Theorem 2.7.2 implies

(i) if $\gamma=0,\ \ \beta=\rho$,

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_\infty \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)}\omega_{\alpha+\rho+\theta}\Big(f,\frac{1}{2^m}\Big)_1, \tag{2.7.10}$$

(ii) if $\gamma=0$,

$$\omega_\alpha\Big(f^{(\lambda,\beta)},\frac{1}{2^n}\Big)_\infty \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)}\omega_{\alpha+\rho+\theta}\Big(f,\frac{1}{2^m}\Big)_1 \tag{2.7.11}$$
$$+\,2^{-n\alpha}\Big|\sin\frac{\pi(\beta-\rho)}{2}\Big|\sum_{k=1}^{n}2^{k(\alpha+\theta+\rho)}\omega_{\alpha+\rho+\theta}\Big(f,\frac{1}{2^k}\Big)_1,$$

(iii) if $\gamma \neq 0$,

$$\omega_\alpha\Big(f^{(\lambda,\beta)}, \frac{1}{2^n}\Big)_\infty \lesssim \sum_{m=n}^{\infty} 2^{m(\theta+\rho)} m^\gamma \omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^m}\Big)_1$$
$$+ 2^{-n\alpha} \sum_{k=1}^{n} 2^{k(\rho+\theta+\alpha)} k^{\gamma-1} \omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^k}\Big)_1$$
$$+ 2^{-n\alpha} \Big| \sin \frac{\pi(\beta-\rho)}{2} \Big| \sum_{k=1}^{n} 2^{k(\alpha+\theta+\rho)} k^\gamma \omega_{\alpha+\rho+\theta}\Big(f, \frac{1}{2^k}\Big)_1.$$

Proof. The proofs of (2.7.10) and (2.7.11) are clear. The part (iii) follows from Theorem 2.7.2 and the fact that

$$\sum_{k=2^n}^{2^{n+1}-1} |\ln^\gamma k - \ln^\gamma (k+1)| k^{\rho+\alpha+1} \omega_{\alpha+\rho+1}(f, \frac{1}{k})_1 \asymp n^{\gamma-1} 2^{n(\rho+\alpha+1)} \omega_{\alpha+\rho+1}(f, \frac{1}{2^n})_1.$$

□

Remark 2.7.1. *Note that inequality* (2.7.10) *and* (2.7.11) *imply the following estimates:*

$$\omega_\alpha(f^{(\rho)}, \delta)_\infty \lesssim \int_0^\delta t^{-\theta-\rho} \omega_{\alpha+\rho+1}(f,t)_1 \frac{dt}{t}$$

and

$$\omega_\alpha(\tilde{f}^{(\rho)}, \delta)_\infty \lesssim \int_0^\delta t^{-\theta-\rho} \omega_{\alpha+\rho+1}(f,t)_1 \frac{dt}{t} + t^\alpha \int_\delta^1 t^{-\theta-\rho-\alpha} \omega_{\alpha+\rho+1}(f,t)_1 \frac{dt}{t}.$$

Here the first inequality is given in Theorem B4 and the second one is new.

2.8 Estimates for the best approximations

In this subsection we obtain estimates for the L_q−best approximations of the generalized Liouville–Weyl derivatives via the L_p−best approximations of the function themselves.

Theorem 2.8.1. *Let* $f \in L_p$, $1 < p < q \leqslant \infty$, $\theta = 1/p - 1/q$. *Let* $\lambda = \{\lambda_n\}_{n=1}^\infty \in GM$. *Then, for any* $\alpha > 0$,

$$E_{2^n}(f^{(\lambda,\beta)})_q \lesssim \left(\sum_{m=n}^{\infty} \Big(2^{m\theta} |\lambda_{2^m}| E_{2^m}(f)_p \Big)^{q_1} \right)^{1/q_1}, \qquad (2.8.12)$$

where $q_1 = \begin{cases} q, & q < \infty, \\ 1, & q = \infty. \end{cases}$

Proof of Theorem 2.8.1. Let $1 < p < q \leqslant \infty$. By Lemmas 2.3.3 and 1.5.3, we have

$$\|S_{2^l}(f^{(\lambda,\beta)}) - S_{2^n}(f^{(\lambda,\beta)})\|_q \lesssim \left(\sum_{m=n}^{l-1} \left(2^{m\theta}\|S_{2^{m+1}}(f^{(\lambda,\beta)}) - S_{2^m}(f^{(\lambda,\beta)})\|_p\right)^{q_1}\right)^{1/q_1}$$

$$\lesssim \left(\sum_{m=n}^{l-1} \left(2^{m\theta}|\lambda_{2^m}|\|S_{2^{m+1}}(f) - S_{2^m}(f)\|_p\right)^{q_1}\right)^{1/q_1}$$

$$\lesssim \left(\sum_{m=n}^{\infty} \left(2^{m\theta}|\lambda_{2^m}|E_{2^m}(f)_p\right)^{q_1}\right)^{1/q_1},$$

where $q_1 = \begin{cases} q, & q < \infty, \\ 1, & q = \infty. \end{cases}$ Using the inequality $E_n(f^{(\lambda,\beta)})_q \lesssim \|f^{(\lambda,\beta)} - S_n(f^{(\lambda,\beta)})\|_q$ and the fact that $\|f^{(\lambda,\beta)} - S_{2^n}(f^{(\lambda,\beta)})\|_q \leqslant \lim_{l\to\infty} \|S_{2^l}(f^{(\lambda,\beta)}) - S_{2^n}(f^{(\lambda,\beta)})\|_q$, we obtain

$$E_{2^n}(f^{(\lambda,\beta)})_q \lesssim \left(\sum_{m=n}^{\infty} \left(2^{m\theta}|\lambda_{2^m}|E_{2^m}(f)_p\right)^{q_1}\right)^{1/q_1}.$$

□

Theorem 2.8.2. *Let $f \in L_p$, $1 = p < q \leqslant \infty$, $\theta = 1 - 1/q$. Let $\lambda = \{\lambda_n\}_{n=1}^{\infty} \in GM$. Then, for any $\alpha > 0$,*

$$E_{2^{n+1}-1}(f^{(\lambda,\beta)})_q \lesssim \left(\sum_{m=n}^{\infty} \left(2^{m\theta}|\lambda_{2^m}|E_{2^m}(f)_p\right)^{q_1}\right)^{1/q_1}, \qquad (2.8.13)$$

where $q_1 = \begin{cases} q, & q < \infty, \\ 1, & q = \infty. \end{cases}$

Proof of Theorem 2.8.2. The proof follows from estimate I_1 in Theorem 2.7.2 when $1 < q \leqslant \infty$. $\|f^{(\lambda,\beta)} - V_{2^n}(f^{(\lambda,\beta)})\|_q \leqslant \lim_{l\to\infty} \|V_{2^l}(f^{(\lambda,\beta)}) - V_{2^n}(f^{(\lambda,\beta)})\|_q$, applying the inequality $E_{2^{n+1}-1}(f^{(\lambda,\beta)})_q \lesssim \|f^{(\lambda,\beta)} - V_{2^n}(f^{(\lambda,\beta)})\|_q$, we get

$$E_{2^{n+1}-1}(f^{(\lambda,\beta)})_q \lesssim \left(\sum_{m=n}^{\infty} \left(2^{m\theta}|\lambda_{2^m}|E_{2^m}(f)_1\right)^{q_1}\right)^{1/q_1}.$$

□

Remark 2.8.1. *Note that some estimates of the best approximation of* $f^{(\lambda,\beta)}$ *in* L_p *in terms of the best approximation of* f *in* L_p *were obtained in the paper [54].*

3 Liouville–Weyl derivatives of double trigonometric series

3.1 Some notations, known results, and goals

Let $L_p(\mathbb{T}^2), 1 < p < \infty$, be the space of measureable functions of two variables that are $2\pi-$periodic in each variable and such that

$$\|f\|_p = \left(\int\limits_0^{2\pi}\int\limits_0^{2\pi} |f(x_1,x_2)|^p dx_1 dx_2\right)^{\frac{1}{p}} < \infty.$$

L_p^0 - the set of functions $f \in L_p$ such that

$$\int\limits_0^{2\pi} f(x_1,x_2)dx_2 = 0 \text{ a.e. } x_1$$

and

$$\int\limits_0^{2\pi} f(x_1,x_2)dx_1 = 0 \text{ a.e. } x_2.$$

Let $f^{(\rho_1,\rho_2)}$ be a derivative in the sense of Weyl of the function $f(x_1,x_2)$ of order $\rho_1(\rho_1 \geqslant 0)$ with respect to x_1 and of order $\rho_2(\rho_2 \geqslant 0)$ with respect to x_2.

In the paper [48], the following result was proved.

Theorem C. *Let* $1 < p < \infty, 0 < \theta \leqslant \min(2,p), \max(2,p) \leqslant \tau < \infty,$ *and* $r_1, r_2, \beta_1, \beta_2$ *be positive numbers.*

A. Let $f \in L_p^0(\mathbb{T}^2)$ *and*

$$J_1(f,\theta) := \Big(\int\limits_0^1\int\limits_0^1 t_1^{-r_1\theta-1} t_2^{-r_2\theta-1} \omega^{\theta}_{r_1+\beta_1,r_2+\beta_2}(f,t_1,t_2)_p dt_1 dt_2\Big)^{\frac{1}{\theta}} < \infty,$$

then the mixed derivative of f *in the sense of Weil* $f^{(r_1,r_2)} \in L_p^0(\mathbb{T}^2)$*, and*

$$\|f^{(r_1,r_2)}\|_p \leqslant C J_1(f,\theta).$$

B. Let $f \in L_p^0(\mathbb{T}^2)$ *be such that* $f^{(r_1,r_2)} \in L_p^0(\mathbb{T}^2)$*, then*

$$J_1(f,\tau) \leqslant C\|f^{(r_1,r_2)}\|_p.$$

Here $\omega_{l_1,l_2}(f,t_1,t_2)_p$ is the mixed modulus of smoothness of a function $f \in L_p(\mathbb{T}^2)$ of orders l_1 and l_2 with respect to variables x_1 and x_2, correspondingly, that is,

$$\omega_{l_1,l_2}(f,\delta_1,\delta_2)_p = \sup_{|h_1|\leqslant\delta_1,\ |h_2|\leqslant\delta_2} \|\Delta_{h_1}^{l_1}(\Delta_{h_2}^{l_2}(f))\|_p.$$

The difference of order $l_1 > 0$ with respect to the variable x_1 and the difference of order $l_2 > 0$ with respect to the variable x_2 are defined as follows:

$$\Delta_{h_1}^{l_1}(f) = \sum_{k_1=0}^{\infty}(-1)^{k_1} C_{l_1}^{k_1} f(x_1 + (l_1 - k_1)h_1, x_2)$$

and

$$\Delta_{h_2}^{l_2}(f) = \sum_{k_2=0}^{\infty}(-1)^{k_2} C_{l_2}^{k_2} f(x_1, x_2 + (l_2 - k_2)h_2).$$

Let $Y_{m_1,m_2}(f)_p$ be the best approximation by a two-dimensional angle of the function $f \in L_p(\mathbb{T}^2)$, i.e.,

$$Y_{m_1,m_2}(f)_p = \inf_{T_{m_1,\infty},T_{\infty,m_2}} \|f - T_{m_1,\infty} - T_{\infty,m_2}\|_p,$$

where the function $T_{m_1,\infty}(x_1,x_2) \in L_p(\mathbb{T}^2)$ is a trigonometric polynomial of degree at most m_1 in x_1, and the function $T_{\infty,m_2}(x_1,x_2) \in L_p(\mathbb{T}^2)$ is a trigonometric polynomial of degree at most m_2 in x_2.

The direct and inverse theorems between best approximations by two-dimensional angle and mixed moduli of smoothness are well known [44, 45]. The Jackson inequality reads as follows: if $f \in L_p^0(\mathbb{T}^2)$, $1 < p < \infty$, then

$$Y_{m_1,m_2}(f)_p \lesssim \omega_{\alpha_1,\alpha_2}\Big(f; \frac{1}{m_1}, \frac{1}{m_2}\Big)_p. \tag{3.1.1}$$

The inverse inequality states that if $f \in L_p^0(\mathbb{T}^2)$, $1 < p < \infty$, then

$$\omega_{\alpha_1,\alpha_2}\Big(f; \frac{1}{m_1}, \frac{1}{m_2}\Big)_p \lesssim \frac{1}{m_1^{\alpha_1} m_2^{\alpha_2}} \sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}(k_1+1)^{\alpha_1-1}(k_2+1)^{\alpha_2-1} Y_{k_1,k_2}(f)_p. \tag{3.1.2}$$

Using these estimates and applying Hardy's inequalities, we easily obtain that

$$J_1(f,s) \asymp \Big(\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(k_1+1)^{r_1 s-1}(k_2+1)^{r_2 s-1} Y_{k_1,k_2}^s(f)_p\Big)^{\frac{1}{s}}$$

for any $s > 0$, where as usual $F_1(f,r,s,p) \asymp F_2(f,r,s,p)$ means that there exist positive constants C_1 and C_2, independent of f such that

$$C_1 \cdot F_1(f,r,s,p) \leqslant F_2(f,r,s,p) \leqslant C_2 \cdot F_1(f,r,s,p).$$

The goal of this section is to extend theorem C in the following respects. First, we consider the generalized Liouville–Weyl derivatives in place of the classical mixed Weyl derivatives. Second, similarly to one-dimensional inequalities (1.6.29) and (1.6.30), we obtain estimates of the angle approximations of these derivatives by angle approximation of functions themselves. By $\sigma(f)$ we will denote the Fourier series of a function $f \in L_p(\mathbb{T}^2)$, that is,

$$\sigma(f) \equiv \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}(a_{n_1n_2}\cos n_1x_1\cos n_2x_2 + b_{n_1n_2}\cos n_1x_1\sin n_2x_2 +$$
$$+ c_{n_1n_2}\sin n_1x_1\cos n_2x_2 + d_{n_1n_2}\sin n_1x_1\sin n_2x_2),$$

where for the sake of brevity we assume that $\cos(0\cdot t) = \frac{1}{2}$.

The transformed Fourier series of $\sigma(f)$ is given by

$$\sigma(f,\lambda,\beta_1,\beta_2) \equiv \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\lambda_{n_1,n_2}(a_{n_1n_2}\cos(n_1x_1+\beta_1\pi/2)\cos(n_2x_2+\beta_2\pi/2)$$
$$+ b_{n_1n_2}\cos(n_1x_1+\beta_1\pi/2)\sin(n_2x_2+\beta_2\pi/2)$$
$$+ c_{n_1n_2}\sin(n_1x_1+\beta_1\pi/2)\cos(n_2x_2+\beta_2\pi/2)$$
$$+ d_{n_1n_2}\sin(n_1x_1+\beta_1\pi/2)\sin(n_2x_2+\beta_2\pi/2)),$$

where $\beta_1,\beta_2 \in \mathbb{R}$ and $\lambda = \{\lambda_{n_1n_2}\}_{n_1,n_2\in\mathbb{N}}$ is a sequence of real numbers.

By analogy with the one-dimensional case we call the function $\varphi(x_1,x_2) \sim \sigma(f,\lambda,\beta_1,\beta_2)$ the $(\lambda,\beta_1,\beta_2)-$ mixed derivative of the function f (or Liouville–Weyl derivative) and denote it by $f^{(\lambda,\beta_1,\beta_2)}(x_1,x_2)$. For example, if $\lambda_{n_1n_2} = n_1^{r_1}n_2^{r_2},\ r_i \geqslant 0,\ \beta_i = r_i\ (i=1,2)$, then $f^{(\lambda,\beta_1,\beta_2)} = f^{(r_1,r_2)}$, where $f^{(r_1,r_2)}$ is the mixed derivative of f in the sense of Weyl. Note that, for any β_1 and β_2, $\|f^{(\lambda,\beta_1,\beta_2)}\|_p \asymp \|f^{(\lambda,0,0)}\|_p$, $1<p<\infty$.

We recall [69] that a sequence $\lambda := \{\lambda_n\}_{n=1}^{\infty}$ is said to be general monotone, written $\lambda \in GM$, if the relation

$$\sum_{k=n}^{2n}|\lambda_k - \lambda_{k+1}| \leqslant C|\lambda_n|$$

holds for all integer n, where the constant C is independent of n.
Similarly, one can introduce the class GM^2, where 2 stands for dimension. See [14, 15, 16].

Definition 3.1.1. *A sequence $\lambda = \{\lambda_{n_1 n_2}\}_{n_1,n_2 \in N}$ is said to be general monotone, written $\lambda \in GM^2$, if the relations*

$$\sum_{k_1=n_1}^{2n_1} |\lambda_{k_1,n_2} - \lambda_{k_1+1,n_2}| \leqslant C|\lambda_{n_1,n_2}|,$$
$$\sum_{k_2=n_2}^{2n_2} |\lambda_{n_1,k_2} - \lambda_{n_1,k_2+1}| \leqslant C|\lambda_{n_1,n_2}|,$$
$$\sum_{k_1=n_1}^{2n_1} \sum_{k_2=n_2}^{2n_2} |\lambda_{k_1,k_2} - \lambda_{k_1+1,k_2} - \lambda_{k_1,k_2+1} + \lambda_{k_1+1,k_2+1}| \leqslant C|\lambda_{n_1,n_2}|$$

hold for all integers n_1 and n_2, where the constant C is independent of n_1 and n_2.

3.2 Auxiliary results

In this section we state several useful lemmas that will be used in the proof our main result. First, we introduce some notation.

Let the series

$$\sigma(f) := \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} (a_{n_1 n_2} \cos n_1 x_1 \cos n_2 x_2 + b_{n_1 n_2} \cos n_1 x_1 \sin n_2 x_2$$
$$+ c_{n_1 n_2} \sin n_1 x_1 \cos n_2 x_2 + d_{n_1 n_2} \sin n_1 x_1 \sin n_2 x_2) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} A_{n_1 n_2}(x_1, x_2). \tag{3.2.3}$$

be the Fourier series of $f \in L_p^0(\mathbb{T}^2)$.

We denote

$$\Delta_{m_1 m_2} := \sum_{n_1=2^{m_1-1}}^{2^{m_1}-1} \sum_{n_2=2^{m_2-1}}^{2^{m_2}-1} A_{n_1 n_2}(x_1, x_2), \quad m_1, m_2 = 1, 2, ...$$

Let $S_{m_1,\infty}(f), S_{\infty,m_2}(f), S_{m_1,m_2}(f)$ denote the partial sums of Fourier series of $f(x_1,x_2)$, i.e.,

$$S_{m_1,\infty}(f) = \frac{1}{\pi}\int_0^{2\pi} f(x_1+t_1,x_2)D_{m_1}(t_1)dt_1,$$

$$S_{\infty,m_2}(f) = \frac{1}{\pi}\int_0^{2\pi} f(x_1,x_2+t_2)D_{m_2}(t_2)dt_2,$$

$$S_{m_1,m_2}(f) = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x_1+t_1,x_2+t_2)D_{m_1}(t_1)D_{m_2}(t_2)dt_1dt_2,$$

where $D_m(t) = \frac{\sin(m+\frac{1}{2})t}{2\sin\frac{t}{2}},\ m\in\mathbb{N}\cup\{0\}$.
From [69], it follows that if $\{\lambda_{n_1n_2}\}\in GM^2$, then

$$|\lambda_{k_1,k_2}| \leqslant C|\lambda_{n_1,n_2}| \ \text{ for } \ n_1\leqslant k_1\leqslant 2n_1, \ \ n_2\leqslant k_2\leqslant 2n_2.$$

This implies that the condition

$$\sum_{k_1=n_1}^{2n_1}\sum_{k_2=n_2}^{2n_2}|\lambda_{k_1,k_2}-\lambda_{k_1+1,k_2}-\lambda_{k_1,k_2+1}+\lambda_{k_1+1,k_2+1}| \leqslant C(|\lambda_{n_1,n_2}|+|\lambda_{2n_1,2n_2}|)$$

is equivalent to the condition

$$\sum_{k_1=n_1}^{2n_1}\sum_{k_2=n_2}^{2n_2}|\lambda_{k_1,k_2}-\lambda_{k_1+1,k_2}-\lambda_{k_1,k_2+1}+\lambda_{k_1+1,k_2+1}| \leqslant C|\lambda_{n_1,n_2}|.$$

Lemma 3.2.1. *A sequences $\{\lambda_{n_1n_2}\}\in GM^2$ satisfies the following properties: there exists $C>0$ such that*

(i) $|\lambda_{k_1,k_2}| \leqslant C|\lambda_{n_1,n_2}|$ *for* $n_1\leqslant k_1\leqslant 2n_1,\ n_2\leqslant k_2\leqslant 2n_2,$

(ii) $\sum\limits_{k_1=n_1}^{N_1}|\lambda_{k_1,n_2}-\lambda_{k_1+1,n_2}| \leqslant C\Big(|\lambda_{n_1n_2}|+\sum\limits_{k_1=n_1+1}^{N_1}\frac{|\lambda_{k_1n_2}|}{k_1}\Big)$ *for* $n_1<N_1,$

(iii) $\sum\limits_{k_2=n_2}^{N_2}|\lambda_{n_1,k_2}-\lambda_{n_1,k_2+1}| \leqslant C\Big(|\lambda_{n_1n_2}|+\sum\limits_{k_2=n_2+1}^{N_2}\frac{|\lambda_{n_1k_2}|}{k_2}\Big)$ *for* $n_2<N_2,$

(iv) $\sum\limits_{k_1=n_1}^{N_1}\sum\limits_{k_2=n_2}^{N_2}|\lambda_{k_1,k_2}-\lambda_{k_1+1,k_2}-\lambda_{k_1,k_2+1}+\lambda_{k_1+1,k_2+1}|\leqslant$
$\leqslant C\Big(|\lambda_{n_1n_2}|+\sum\limits_{k_1=n_1+1}^{N_1}\frac{|\lambda_{k_1n_2}|}{k_1}+\sum\limits_{k_2=n_2+1}^{N_2}\frac{|\lambda_{n_1k_2}|}{k_2}+\sum\limits_{k_1=n_1+1}^{N_1}\sum\limits_{k_2=n_2+1}^{N_2}\frac{|\lambda_{k_1k_2}|}{k_1k_2}\Big)$ *for* $n_1<N_1, n_2<N_2$.

Proof. The properties $(i),(ii)$, and (iii) follow from Lemma 1.4.1. Let us prove the property (iv). Let $l_1\in\mathbb{N}\cup\{0\}, l_2\in\mathbb{N}\cup\{0\}$ such that $2^{l_1}n_1\leqslant N_1<2^{l_1+1}n_1, 2^{l_2}n_2\leqslant N_2<2^{l_2+1}n_2$. Then, by definition, we have

$$
\begin{aligned}
A&=\sum_{k_1=n_1}^{N_1}\sum_{k_2=n_2}^{N_2}|\lambda_{k_1,k_2}-\lambda_{k_1+1,k_2}-\lambda_{k_1,k_2+1}+\lambda_{k_1+1,k_2+1}|\\
&\leqslant\sum_{s_1=0}^{l_1}\sum_{s_2=0}^{l_2}\sum_{k_1=2^{s_1}n_1}^{2^{s_1+1}n_1-1}\sum_{k_2=2^{s_2}n_2}^{2^{s_2+1}n_2-1}|\lambda_{k_1,k_2}-\lambda_{k_1+1,k_2}-\lambda_{k_1,k_2+1}+\lambda_{k_1+1,k_2+1}|\\
&\leqslant C\sum_{s_1=0}^{l_1}\sum_{s_2=0}^{l_2}|\lambda_{2^{s_1}n_1,2^{s_2}n_2}|.
\end{aligned}
$$

Using (i), we get

$$
\begin{aligned}
A&\leqslant C\big(|\lambda_{n_1,n_2}|+\sum_{s_1=1}^{l_1}|\lambda_{2^{s_1}n_1,n_2}|+\sum_{s_2=1}^{l_2}|\lambda_{n_1,2^{s_2}n_2}|+\sum_{s_1=1}^{l_1}\sum_{s_2=1}^{l_2}|\lambda_{2^{s_1}n_1,2^{s_2}n_2}|\big)\\
&\leqslant C\big(|\lambda_{n_1,n_2}|+\sum_{s_1=1}^{l_1}|\lambda_{2^{s_1}n_1,n_2}|\sum_{k_1=2^{s_1-1}n_1+1}^{2^{s_1}n_1}\frac{1}{k_1}+\sum_{s_2=1}^{l_2}|\lambda_{n_1,2^{s_2}n_2}|\sum_{k_2=2^{s_2-1}n_2+1}^{2^{s_2}n_2}\frac{1}{k_2}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{s_1=1}^{l_1}\sum_{s_2=1}^{l_2}|\lambda_{2^{s_1}n_1,2^{s_2}n_2}|\sum_{k_1=2^{s_1-1}n_1+1}^{2^{s_1}n_1}\sum_{k_2=2^{s_2-1}n_2+1}^{2^{s_2}n_2}\frac{1}{k_1}\frac{1}{k_2})\\
&\leqslant C\Big(|\lambda_{n_1n_2}|+\sum_{k_1=n_1+1}^{N_1}\frac{|\lambda_{k_1n_2}|}{k_1}+\sum_{k_2=n_2+1}^{N_2}\frac{|\lambda_{n_1k_2}|}{k_2}+\sum_{k_1=n_1+1}^{N_1}\sum_{k_2=n_2+1}^{N_2}\frac{|\lambda_{k_1k_2}|}{k_1k_2}\Big).
\end{aligned}
$$

□

Lemma 3.2.2. *Let $f \in L_p^0(\mathbb{T}^2)$ have the Fourier series* (3.2.3)*. We consider*

$$\sigma(\varphi) \equiv \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}(a_{n_1n_2}\cos(n_1x_1+\beta_1\pi/2)\cos(n_2x_2+\beta_2\pi/2)$$
$$+ b_{n_1n_2}\cos(n_1x_1+\beta_1\pi/2)\sin(n_2x_2+\beta_2\pi/2)$$
$$+ c_{n_1n_2}\sin(n_1x_1+\beta_1\pi/2)\cos(n_2x_2+\beta_2\pi/2)$$
$$+ d_{n_1n_2}\sin(n_1x_1+\beta_1\pi/2)\sin(n_2x_2+\beta_2\pi/2)),$$

where $\beta_1, \beta_2 \in \mathbb{R}$. Then

$$C_1(p)\|f\|_p \leqslant \|\varphi\|_p \leqslant C_2(p)\|f\|_p.$$

The proof is analogous to the proof of Lemma 5.2.6 of [49] and it uses boundedness of the conjugate operator in L_p, $1 < p < \infty$.

Lemma 3.2.3. *[43, p. 125] Let $a_n \geqslant 0, 0 < \alpha \leqslant \beta < \infty$. Then*

$$\Big(\sum_{\nu=1}^{\infty} a_\nu^\beta\Big)^{\frac{1}{\beta}} \leqslant \Big(\sum_{\nu=1}^{\infty} a_\nu^\alpha\Big)^{\frac{1}{\alpha}}.$$

Lemma 3.2.4. *[45, 49] Let $f \in L_p(\mathbb{T}^2), 1 < p < \infty$, and $m_i \in \mathbb{N}\cup\{0\}(i = 1,2)$. Then*

$$\|f - S_{m_1,\infty}(f) - S_{\infty,m_2}(f) + S_{m_1,m_2}(f)\|_p \asymp Y_{m_1,m_2}(f)_p.$$

Lemma 3.2.5. *[43]*

(a). Let $1 < p < \infty$ and (3.2.3) *be the Fourier series of $f \in L_p^0(\mathbb{T}^2)$. Then*

$$\Big(\int_0^{2\pi}\int_0^{2\pi}\Big(\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\Delta_{m_1m_2}^2\Big)^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{1}{p}} \leqslant C(p)\|f\|_p.$$

(b). Let $1 < p < \infty$. If (3.2.3) *satisfies the following inequality*

$$I_p = \Big(\int_0^{2\pi}\int_0^{2\pi}\Big(\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\Delta_{m_1m_2}^2\Big)^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{1}{p}} < \infty,$$

then (3.2.3) *is the Fourier series of a function $f(x_1,x_2) \in L_p(\mathbb{T}^2)$ and*

$$\|f\|_p \leqslant C(p)I_p.$$

Lemma 3.2.6. *[43] Let $f \in L_p^0(\mathbb{T}^2), 1 < p < \infty$, and (3.2.3) be the Fourier series of f. If $\{\lambda_{n_1 n_2}\}$ satisfies the following conditions*

(i) $|\lambda_{n_1,n_2}| \leqslant M$,

(ii) $\sum\limits_{m_1=2^{n_1-1}}^{2^{n_1}-1} |\lambda_{m_1,n_2} - \lambda_{m_1+1,n_2}| \leqslant M, \sum\limits_{m_2=2^{n_2-1}}^{2^{n_2}-1} |\lambda_{n_1,m_2} - \lambda_{n_1,m_2+1}| \leqslant M$,

(iii) $\sum\limits_{m_1=2^{n_1-1}}^{2^{n_1}-1} \sum\limits_{m_2=2^{n_2-1}}^{2^{n_2}-1} |\lambda_{m_1,m_2} - \lambda_{m_1+1,m_2} - \lambda_{m_1,m_2+1} + \lambda_{m_1+1,m_2+1}| \leqslant M$

for all $n_1, n_2 = 1, 2, ...$

Then the series

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \lambda_{n_1,n_2} A_{n_1 n_2}(x_1, x_2)$$

is the Fourier series of a function $\varphi(f, \lambda) \in L_p^0(\mathbb{T}^2)$ and $\|\varphi\|_p \leqslant C\|f\|_p$, where the constant C is independent of f.

3.3 Upper estimates of angle best approximations of generalized Liouville–Weyl derivatives

Our main result reads as follows.

Theorem 3.3.1. *Let $1 < p < \infty,\ 0 < \theta \leqslant \min(p, 2),\ \lambda = \{\lambda_{n_1,n_2}\}_{n_1,n_2 \in N}$ be a sequence of positive numbers such that $\lambda \in GM^2$, $\alpha_i \in \mathbb{R}_+, r_i \in \mathbb{R}_+ \cup \{0\}$, and $\beta_i \in \mathbb{R}$ $(i = 1, 2)$.*

If for $f \in L_p^0(\mathbb{T}^2)$ the series

$$\begin{aligned} &\sum_{n_1=1}^{\infty} |\lambda_{n_1+1,1}^{\theta} - \lambda_{n_1,1}^{\theta}| Y_{n_1,0}^{\theta}(f)_p + \sum_{n_2=1}^{\infty} |\lambda_{1,n_2+1}^{\theta} - \lambda_{1,n_2}^{\theta}| Y_{0,n_2}^{\theta}(f)_p \\ &+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} |\lambda_{k_1+1,k_2+1}^{\theta} - \lambda_{k_1+1,k_2}^{\theta} - \lambda_{k_1,k_2+1}^{\theta} + \lambda_{k_1,k_2}^{\theta}| Y_{k_1,k_2}^{\theta}(f)_p \end{aligned} \tag{3.3.4}$$

converges, then there exists a function $f^{(\lambda,\beta_1,\beta_2)} \in L_p^0(\mathbb{T}^2)$, with the Fourier series $\sigma(f, \lambda, \beta_1, \beta_2)$, and

$$\|f^{(\lambda,\beta_1,\beta_2)}(x_1,x_2)\|_p \lesssim \left\{\lambda_{1,1}^\theta \|f\|_p^\theta + \sum_{n_1=1}^{\infty} |\lambda_{n_1+1,1}^\theta - \lambda_{n_1,1}^\theta| Y_{n_1,0}^\theta(f)_p \right.$$

$$+ \sum_{n_2=1}^{\infty} |\lambda_{1,n_2+1}^\theta - \lambda_{1,n_2}^\theta| Y_{0,n_2}^\theta(f)_p$$

$$\left. + \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} |\lambda_{n_1,n_2}^\theta - \lambda_{n_1+1,n_2}^\theta - \lambda_{n_1,n_2+1}^\theta + \lambda_{n_1+1,n_2+1}^\theta| Y_{n_1,n_2}^\theta(f)_p \right\}^{\frac{1}{\theta}},$$

$$\|f^{(\lambda,\beta_1,\beta_2)}(x_1,x_2)\|_p \lesssim \left\{\lambda_{1,1}^\theta \|f\|_p^\theta + \sum_{\nu_1=1}^{\infty} |\lambda_{2^{\nu_1},1}^\theta - \lambda_{2^{\nu_1-1},1}^\theta| Y_{2^{\nu_1}-1,0}^\theta(f)_p \right.$$

$$+ \sum_{\nu_2=1}^{\infty} |\lambda_{1,2^{\nu_2}}^\theta - \lambda_{1,2^{\nu_2-1}}^\theta| Y_{0,2^{\nu_2}-1}^\theta(f)_p$$

$$\left. + \sum_{\nu_1=1}^{\infty}\sum_{\nu_2=1}^{\infty} |\lambda_{2^{\nu_1},2^{\nu_2}}^\theta - \lambda_{2^{\nu_1-1},2^{\nu_2}}^\theta - \lambda_{2^{\nu_1},2^{\nu_2-1}}^\theta + \lambda_{2^{\nu_1-1},2^{\nu_2-1}}^\theta| Y_{2^{\nu_1}-1,2^{\nu_2}-1}^\theta(f)_p \right\}^{\frac{1}{\theta}},$$

and

$$Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)}(x_1,x_2))_p \lesssim \left\{\lambda_{2^{m_1-1},2^{m_2-1}}^\theta Y_{2^{m_1}-1,2^{m_2}-1}^\theta(f)_p \right.$$

$$+ \sum_{\nu_1=m_1}^{\infty} |\lambda_{2^{\nu_1},2^{m_2-1}}^\theta - \lambda_{2^{\nu_1-1},2^{m_2-1}}^\theta| Y_{2^{\nu_1}-1,2^{m_2}-1}^\theta(f)_p$$

$$+ \sum_{\nu_2=m_2}^{\infty} |\lambda_{2^{m_1-1},2^{\nu_2}}^\theta - \lambda_{2^{m_1-1},2^{\nu_2-1}}^\theta| Y_{2^{m_1}-1,2^{\nu_2}-1}^\theta(f)_p$$

$$+ \sum_{\nu_1=m_1}^{\infty}\sum_{\nu_2=m_2}^{\infty} |\lambda_{2^{\nu_1},2^{\nu_2}}^\theta - \lambda_{2^{\nu_1-1},2^{\nu_2}}^\theta$$

$$\left. -\lambda_{2^{\nu_1},2^{\nu_2-1}}^\theta + \lambda_{2^{\nu_1-1},2^{\nu_2-1}}^\theta| Y_{2^{\nu_1}-1,2^{\nu_2}-1}^\theta(f)_p \right\}^{\frac{1}{\theta}}.$$

Proof of Theorem 3.3.1. Let the series (3.3.4) be convergent and $f \in L_p^0(\mathbb{T}^2)$. We use the following inequality

$$\lambda_{2^{n_1-1},2^{n_2-1}}^\theta = \lambda_{1,2^{n_2-1}}^\theta + \sum_{m_1=2}^{n_1} (\lambda_{2^{m_1-1},2^{n_2-1}}^\theta - \lambda_{2^{m_1-2},2^{n_2-1}}^\theta)$$

$$= \lambda_{1,1}^\theta + \sum_{m_2=2}^{n_2} (\lambda_{1,2^{m_2-1}}^\theta - \lambda_{1,2^{m_2-2}}^\theta) + \sum_{m_1=2}^{n_1} (\lambda_{2^{m_1-1},1}^\theta - \lambda_{2^{m_1-2},1}^\theta)$$

$$+\sum_{m_1=2}^{n_1}\sum_{m_2=2}^{n_2}(\lambda^\theta_{2^{m_1-1},2^{m_2-1}}-\lambda^\theta_{2^{m_1-2},2^{m_2-1}}-\lambda^\theta_{2^{m_1-1},2^{m_2-2}}+\lambda^\theta_{2^{m_1-2},2^{m_2-2}})$$

$$\leqslant \lambda^\theta_{1,1}+\sum_{m_2=2}^{n_2}|\lambda^\theta_{1,2^{m_2-1}}-\lambda^\theta_{1,2^{m_2-2}}|+\sum_{m_1=2}^{n_1}|\lambda^\theta_{2^{m_1-1},1}-\lambda^\theta_{2^{m_1-2},1}| \tag{3.3.5}$$

$$+\sum_{m_1=2}^{n_1}\sum_{m_2=2}^{n_2}|\lambda^\theta_{2^{m_1-1},2^{m_2-1}}-\lambda^\theta_{2^{m_1-2},2^{m_2-1}}-\lambda^\theta_{2^{m_1-1},2^{m_2-2}}+\lambda^\theta_{2^{m_1-2},2^{m_2-2}}|.$$

Let us denote $\Delta_{n_1,n_2}=\sum\limits_{\nu_1=2^{n_1-1}}^{2^{n_1}-1}\sum\limits_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}A_{\nu_1,\nu_2}(f,x_1,x_2)$ $(n_1,n_2=1,2,...)$; see (3.2.3).

We will use several times the simple fact that, for any $\theta>0$, one has

$$C_1(\theta)(j_1^\theta+j_2^\theta)\leqslant(j_1+j_2)^\theta\leqslant C_2(\theta)(j_1^\theta+j_2^\theta),\quad j_1,j_2\geqslant 0.$$

Using (3.3.5), we derive

$$I_1:=\Big\{\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\lambda^2_{2^{n_1-1},2^{n_2-1}}\Delta^2_{n_1,n_2}\Big]^{\frac{p}{2}}dx_1dx_2\Big\}^{\frac{1}{p}}$$

$$=\Big\|\Big[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\lambda^2_{2^{n_1-1},2^{n_2-1}}\Delta^2_{n_1,n_2}\Big]^{\frac12}\Big\|_p$$

$$=\Big\|\Big[\lambda^2_{1,1}\Delta^2_{1,1}+\sum_{n_1=2}^{\infty}\lambda^2_{2^{n_1-1},1}\Delta^2_{n_1,1}+\sum_{n_2=2}^{\infty}\lambda^2_{1,2^{n_2-1}}\Delta^2_{1,n_2}$$

$$+\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\lambda^2_{2^{n_1-1},2^{n_2-1}}\Delta^2_{n_1,n_2}\Big]^{\frac12}\Big\|_p$$

$$\leqslant\Big\|\Big(\lambda^2_{1,1}\Delta^2_{1,1}+\sum_{n_1=2}^{\infty}\Delta^2_{n_1,1}\Big[\lambda^\theta_{1,1}+\sum_{\nu_1=2}^{n_1}|\lambda^\theta_{2^{\nu_1-1},1}-\lambda^\theta_{2^{\nu_1-2},1}|\Big]^{\frac{2}{\theta}}$$

$$+\sum_{n_2=2}^{\infty}\Delta^2_{1,n_2}\Big[\lambda^\theta_{1,1}+\sum_{\nu_2=2}^{n_2}|\lambda^\theta_{1,2^{\nu_2-1}}-\lambda^\theta_{1,2^{\nu_2-2}}|\Big]^{\frac{2}{\theta}}$$

$$+\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta^2_{n_1,n_2}\Big[\lambda^\theta_{1,1}+\sum_{\nu_2=2}^{n_2}|\lambda^\theta_{1,2^{\nu_2-1}}-\lambda^\theta_{1,2^{\nu_2-2}}|+\sum_{\nu_1=2}^{n_1}|\lambda^\theta_{2^{\nu_1-1},1}-\lambda^\theta_{2^{\nu_1-2},1}|$$

$$+\sum_{\nu_1=2}^{n_1}\sum_{\nu_2=2}^{n_2}|\lambda^\theta_{2^{\nu_1-1},2^{\nu_2-1}}-\lambda^\theta_{2^{\nu_1-2},2^{\nu_2-1}}-\lambda^\theta_{2^{\nu_1-1},2^{\nu_2-2}}+\lambda^\theta_{2^{\nu_1-2},2^{\nu_2-2}}|\Big]^{\frac{2}{\theta}}\Big)^{\frac12}\Big\|_p$$

$$\lesssim \Big\|\Big(\lambda_{1,1}^2\Delta_{1,1}^2+\lambda_{1,1}^2\sum_{n_1=2}^{\infty}\Delta_{n_1,1}^2+\lambda_{1,1}^2\sum_{n_2=2}^{\infty}\Delta_{1,n_2}^2+\lambda_{1,1}^2\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta_{n_1,n_2}^2$$
$$+\sum_{n_1=2}^{\infty}\Delta_{n_1,1}^2\Big[\sum_{\nu_1=2}^{n_1}|\lambda_{2^{\nu_1-1},1}^{\theta}-\lambda_{2^{\nu_1-2},1}^{\theta}|\Big]^{\frac{2}{\theta}}$$
$$+\sum_{n_2=2}^{\infty}\Delta_{1,n_2}^2\Big[\sum_{\nu_2=2}^{n_2}|\lambda_{1,2^{\nu_2-1}}^{\theta}-\lambda_{1,2^{\nu_2-2}}^{\theta}|\Big]^{\frac{2}{\theta}}$$
$$+\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta_{n_1,n_2}^2\Big[\sum_{\nu_2=2}^{n_2}|\lambda_{1,2^{\nu_2-1}}^{\theta}-\lambda_{1,2^{\nu_2-2}}^{\theta}|+\sum_{\nu_1=2}^{n_1}|\lambda_{2^{\nu_1-1},1}^{\theta}-\lambda_{2^{\nu_1-2},1}^{\theta}|$$
$$+\sum_{\nu_1=2}^{n_1}\sum_{\nu_2=2}^{n_2}|\lambda_{2^{\nu_1-1},2^{\nu_2-1}}^{\theta}-\lambda_{2^{\nu_1-2},2^{\nu_2-1}}^{\theta}-\lambda_{2^{\nu_1-1},2^{\nu_2-2}}^{\theta}+\lambda_{2^{\nu_1-2},2^{\nu_2-2}}^{\theta}|\Big]^{\frac{2}{\theta}}\Big)^{\frac{1}{2}}\Big\|_p.$$

This implies that

$$I_1\lesssim \Big\|\Big\{\lambda_{1,1}^2\Big(\Delta_{1,1}^2+\sum_{n_1=2}^{\infty}\Delta_{n_1,1}^2+\sum_{n_2=2}^{\infty}\Delta_{1,n_2}^2+\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta_{n_1,n_2}^2\Big)$$
$$+\sum_{n_1=2}^{\infty}\Delta_{n_1,1}^2\Big[\sum_{\nu_1=2}^{n_1}|\lambda_{2^{\nu_1-1},1}^{\theta}-\lambda_{2^{\nu_1-2},1}^{\theta}|\Big]^{\frac{2}{\theta}}$$
$$+\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta_{n_1,n_2}^2\Big[\sum_{\nu_1=2}^{n_1}|\lambda_{2^{\nu_1-1},1}^{\theta}-\lambda_{2^{\nu_1-2},1}^{\theta}|\Big]^{\frac{2}{\theta}}$$
$$+\sum_{n_2=2}^{\infty}\Delta_{1,n_2}^2\Big[\sum_{\nu_2=2}^{n_2}|\lambda_{1,2^{\nu_2-1}}^{\theta}-\lambda_{1,2^{\nu_2-2}}^{\theta}|\Big]^{\frac{2}{\theta}}$$
$$+\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta_{n_1,n_2}^2\Big[\sum_{\nu_2=2}^{n_2}|\lambda_{1,2^{\nu_2-1}}^{\theta}-\lambda_{1,2^{\nu_2-2}}^{\theta}|\Big]^{\frac{2}{\theta}}$$
$$+\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta_{n_1,n_2}^2\Big[\sum_{\nu_1=2}^{n_1}\sum_{\nu_2=2}^{n_2}|\lambda_{2^{\nu_1-1},2^{\nu_2-1}}^{\theta}$$
$$-\lambda_{2^{\nu_1-2},2^{\nu_2-1}}^{\theta}-\lambda_{2^{\nu_1-1},2^{\nu_2-2}}^{\theta}+\lambda_{2^{\nu_1-2},2^{\nu_2-2}}^{\theta}|\Big]^{\frac{2}{\theta}}\Big\}^{\frac{1}{2}}\Big\|_p$$
$$\lesssim \lambda_{1,1}\Big\|\Big[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Delta_{n_1,n_2}^2\Big]^{\frac{1}{2}}\Big\|_p$$

$$+\Big\|\Big(\sum_{n_1=2}^{\infty}\sum_{n_2=1}^{\infty}\Delta^2_{n_1,n_2}\Big[\sum_{\nu_1=2}^{n_1}|\lambda^{\theta}_{2^{\nu_1-1},1}-\lambda^{\theta}_{2^{\nu_1-2},1}|\Big]^{\frac{2}{\theta}}\Big)^{\frac12}\Big\|_p$$

$$+\Big\|\Big(\sum_{n_1=1}^{\infty}\sum_{n_2=2}^{\infty}\Delta^2_{n_1,n_2}\Big[\sum_{\nu_2=2}^{n_2}|\lambda^{\theta}_{1,2^{\nu_2-1}}-\lambda^{\theta}_{1,2^{\nu_2-2}}|\Big]^{\frac{2}{\theta}}\Big)^{\frac12}\Big\|_p$$

$$+\Big\|\Big(\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta^2_{n_1,n_2}\Big[\sum_{\nu_1=2}^{n_1}\sum_{\nu_2=2}^{n_2}|\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}}$$

$$-\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}}-\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}}+\lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}|\Big]^{\frac{2}{\theta}}\Big)^{\frac12}\Big\|_p$$

$$=: J_1+J_2+J_3+J_4.$$

Let us estimate J_1. Applying Lemma 3.2.5, we have $J_1 \leqslant C\lambda_{11}\|f\|_p<\infty$. Now we estimate J_2 given by

$$J_2=\Big\{\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=2}^{\infty}\sum_{n_2=1}^{\infty}\Delta^2_{n_1,n_2}\Big[\sum_{\nu_1=2}^{n_1}|\lambda^{\theta}_{2^{\nu_1-1},1}-\lambda^{\theta}_{2^{\nu_1-2},1}|\Big]^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1dx_2\Big\}^{\frac1p}.$$

Using Minkowski's inequality and Lemma 1.5.5 (a) for $\frac{2}{\theta}\geqslant 1$, we derive

$$\sum_{n_2=1}^{\infty}\sum_{n_1=2}^{\infty}\Delta^2_{n_1,n_2}\Big[\sum_{\nu_1=2}^{n_1}|\lambda^{\theta}_{2^{\nu_1-1},1}-\lambda^{\theta}_{2^{\nu_1-2},1}|\Big]^{\frac{2}{\theta}}$$

$$=\sum_{n_1=2}^{\infty}\Big(\Big(\sum_{n_2=1}^{\infty}\Big[\sum_{\nu_1=2}^{n_1}|\Delta_{n_1,n_2}|^{\theta}|\lambda^{\theta}_{2^{\nu_1-1},1}-\lambda^{\theta}_{2^{\nu_1-2},1}|\Big]^{\frac{2}{\theta}}\Big)^{\frac{\theta}{2}}\Big)^{\frac{2}{\theta}}$$

$$\leqslant\sum_{n_1=2}^{\infty}\Big(\sum_{\nu_1=2}^{n_1}\Big[\sum_{n_2=1}^{\infty}|\Delta_{n_1,n_2}|^2|\lambda^{\theta}_{2^{\nu_1-1},1}-\lambda^{\theta}_{2^{\nu_1-2},1}|^{\frac{2}{\theta}}\Big]^{\frac{\theta}{2}}\Big)^{\frac{2}{\theta}}$$

$$=\Big(\Big(\sum_{n_1=2}^{\infty}\Big\{\sum_{\nu_1=2}^{n_1}\Big[\sum_{n_2=1}^{\infty}|\Delta_{n_1,n_2}|^2|\lambda^{\theta}_{2^{\nu_1-1},1}-\lambda^{\theta}_{2^{\nu_1-2},1}|^{\frac{2}{\theta}}\Big]^{\frac{\theta}{2}}\Big\}^{\frac{2}{\theta}}\Big)^{\frac{\theta}{2}}\Big)^{\frac{2}{\theta}}$$

$$\leqslant\Big(\sum_{\nu_1=2}^{\infty}\Big\{\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}|\Delta_{n_1,n_2}|^2|\lambda^{\theta}_{2^{\nu_1-1},1}-\lambda^{\theta}_{2^{\nu_1-2},1}|^{\frac{2}{\theta}}\Big\}^{\frac{\theta}{2}}\Big)^{\frac{2}{\theta}}$$

$$=\Big(\sum_{\nu_1=2}^{\infty}|\lambda^{\theta}_{2^{\nu_1-1},1}-\lambda^{\theta}_{2^{\nu_1-2},1}|\Big(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=1}^{\infty}|\Delta_{n_1,n_2}|^2\Big)^{\frac{\theta}{2}}\Big)^{\frac{2}{\theta}}.$$

Applying this estimate, we obtain

$$J_2 \leqslant \Big\{ \int_0^{2\pi} \int_0^{2\pi} \Big[\Big\{ \sum_{\nu_1=2}^{\infty} |\lambda^{\theta}_{2^{\nu_1-1},1} - \lambda^{\theta}_{2^{\nu_1-2},1}| \Big(\sum_{n_1=\nu_1}^{\infty} \sum_{n_2=1}^{\infty} |\Delta_{n_1,n_2}|^2 \Big)^{\frac{\theta}{2}} \Big\}^{\frac{2}{\theta}} \Big]^{\frac{p}{2}} dx_1 dx_2 \Big\}^{\frac{1}{p}}$$

$$= \Big(\Big\{ \int_0^{2\pi} \int_0^{2\pi} \Big(\sum_{\nu_1=2}^{\infty} |\lambda^{\theta}_{2^{\nu_1-1},1} - \lambda^{\theta}_{2^{\nu_1-2},1}| \Big(\sum_{n_1=\nu_1}^{\infty} \sum_{n_2=1}^{\infty} |\Delta_{n_1,n_2}|^2 \Big)^{\frac{\theta}{2}} \Big)^{\frac{p}{\theta}} dx_1 dx_2 \Big\}^{\frac{\theta}{p}} \Big)^{\frac{1}{\theta}}.$$

Further, using Minkowski's inequality for $\frac{p}{\theta} \geqslant 1$, Lemmas 3.2.4 and 3.2.5, we have

$$J_2 \leqslant \Big(\sum_{\nu_1=2}^{\infty} |\lambda^{\theta}_{2^{\nu_1-1},1} - \lambda^{\theta}_{2^{\nu_1-2},1}| \Big\{ \int_0^{2\pi} \int_0^{2\pi} \Big(\Big(\sum_{n_1=\nu_1}^{\infty} \sum_{n_2=1}^{\infty} |\Delta_{n_1,n_2}|^2 \Big)^{\frac{\theta}{2}} \Big)^{\frac{p}{\theta}} dx_1 dx_2 \Big\}^{\frac{\theta}{p}} \Big)^{\frac{1}{\theta}}$$

$$= \Big(\sum_{\nu_1=2}^{\infty} |\lambda^{\theta}_{2^{\nu_1-1},1} - \lambda^{\theta}_{2^{\nu_1-2},1}| \Big\| \Big(\sum_{n_1=\nu_1}^{\infty} \sum_{n_2=1}^{\infty} |\Delta_{n_1,n_2}|^2 \Big)^{\frac{1}{2}} \Big\|_p^{\theta} \Big)^{\frac{1}{\theta}}$$

$$\lesssim \Big(\sum_{\nu_1=1}^{\infty} |\lambda^{\theta}_{2^{\nu_1},1} - \lambda^{\theta}_{2^{\nu_1-1},1}| Y^{\theta}_{2^{\nu_1}-1,0}(f)_p \Big)^{\frac{1}{\theta}}.$$

Thus, we obtain

$$J_2 \lesssim \Big(\sum_{\nu_1=1}^{\infty} |\lambda^{\theta}_{2^{\nu_1},1} - \lambda^{\theta}_{2^{\nu_1-1},1}| Y^{\theta}_{2^{\nu_1}-1,0}(f)_p \Big)^{\frac{1}{\theta}}.$$

From (3.3.4), it follows that $J_2 < \infty$.

Let us estimate

$$J_3 = \Big\{ \int_0^{2\pi} \int_0^{2\pi} \Big[\sum_{n_1=1}^{\infty} \sum_{n_2=2}^{\infty} \Delta^2_{n_1,n_2} \Big[\sum_{\nu_2=2}^{n_2} |\lambda^{\theta}_{1,2^{\nu_2-1}} - \lambda^{\theta}_{1,2^{\nu_2-2}}| \Big]^{\frac{2}{\theta}} \Big]^{\frac{p}{2}} dx_1 dx_2 \Big\}^{\frac{1}{p}}.$$

We use Lemmas 1.5.6 and 1.5.5 (a) with $\frac{2}{\theta} \geqslant 1$ to get

$$\sum_{n_1=1}^{\infty} \sum_{n_2=2}^{\infty} \Delta^2_{n_1,n_2} \Big[\sum_{\nu_2=2}^{n_2} |\lambda^{\theta}_{1,2^{\nu_2-1}} - \lambda^{\theta}_{1,2^{\nu_2-2}}| \Big]^{\frac{2}{\theta}}$$

$$\leqslant \sum_{n_2=2}^{\infty} \Big(\sum_{\nu_2=2}^{n_2} \Big(\sum_{n_1=1}^{\infty} \Big[|\Delta_{n_1,n_2}|^{\theta} |\lambda^{\theta}_{1,2^{\nu_2-1}} - \lambda^{\theta}_{1,2^{\nu_2-2}}| \Big]^{\frac{2}{\theta}} \Big)^{\frac{\theta}{2}} \Big)^{\frac{2}{\theta}}$$

$$= \Big(\Big[\sum_{n_2=2}^{\infty} \Big\{ \sum_{\nu_2=2}^{n_2} \Big(\sum_{n_1=1}^{\infty} |\Delta_{n_1,n_2}|^2 |\lambda^{\theta}_{1,2^{\nu_2-1}} - \lambda^{\theta}_{1,2^{\nu_2-2}}|^{\frac{2}{\theta}} \Big)^{\frac{\theta}{2}} \Big\}^{\frac{2}{\theta}} \Big]^{\frac{\theta}{2}} \Big)^{\frac{2}{\theta}}$$

$$\leqslant \Big(\sum_{\nu_2=2}^{\infty} |\lambda^{\theta}_{1,2^{\nu_2-1}} - \lambda^{\theta}_{1,2^{\nu_2-2}}| \Big(\sum_{n_2=\nu_2}^{\infty} \sum_{n_1=1}^{\infty} |\Delta_{n_1,n_2}|^2 \Big)^{\frac{\theta}{2}} \Big)^{\frac{2}{\theta}}.$$

Using this and Lemma 1.5.6 with $\frac{p}{\theta} \geqslant 1$, we obtain that

$$J_3 = \Big\{ \int_0^{2\pi} \int_0^{2\pi} \Big[\sum_{n_1=1}^{\infty} \sum_{n_2=2}^{\infty} \Delta^2_{n_1,n_2} \Big[\sum_{\nu_2=2}^{n_2} |\lambda^{\theta}_{1,2^{\nu_2-1}} - \lambda^{\theta}_{1,2^{\nu_2-2}}| \Big]^{\frac{2}{\theta}} \Big]^{\frac{p}{2}} dx_1 dx_2 \Big\}^{\frac{1}{p}}$$

$$\leqslant \Big\{ \int_0^{2\pi} \int_0^{2\pi} \Big[\Big(\sum_{\nu_2=2}^{\infty} |\lambda^{\theta}_{1,2^{\nu_2-1}} - \lambda^{\theta}_{1,2^{\nu_2-2}}| \Big(\sum_{n_2=\nu_2}^{\infty} \sum_{n_1=1}^{\infty} |\Delta_{n_1,n_2}|^2 \Big)^{\frac{\theta}{2}} \Big)^{\frac{2}{\theta}} \Big]^{\frac{p}{2}} dx_1 dx_2 \Big\}^{\frac{1}{p}}$$

$$\leqslant \Big(\sum_{\nu_2=2}^{\infty} |\lambda^{\theta}_{1,2^{\nu_2-1}} - \lambda^{\theta}_{1,2^{\nu_2-2}}| \Big\{ \int_0^{2\pi} \int_0^{2\pi} \Big(\sum_{n_2=\nu_2}^{\infty} \sum_{n_1=1}^{\infty} |\Delta_{n_1,n_2}|^2 \Big)^{\frac{p}{2}} dx_1 dx_2 \Big\}^{\frac{\theta}{p}} \Big)^{\frac{1}{\theta}}.$$

By Lemmas 3.2.4 and 3.2.5, we have

$$J_3 \lesssim \Big(\sum_{\nu_2=1}^{\infty} |\lambda^{\theta}_{1,2^{\nu_2}} - \lambda^{\theta}_{1,2^{\nu_2-1}}| Y^{\theta}_{0,2^{\nu_2}-1}(f)_p \Big)^{\frac{1}{\theta}} < \infty,$$

provided that (3.3.4) holds.

To estimate J_4, we first obtain the upper estimate of the following sum. Applying Lemmas 1.5.5 and 1.5.6 twice for $\frac{2}{\theta} \geqslant 1$, we get

$$\sum_{n_1=2}^{\infty} \sum_{n_2=2}^{\infty} \Delta^2_{n_1,n_2} \Big[\sum_{\nu_1=2}^{n_1} \sum_{\nu_2=2}^{n_2} |\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}| \Big]^{\frac{2}{\theta}}$$

$$\leqslant \sum_{n_1=2}^{\infty} \Big(\sum_{\nu_2=2}^{\infty} \Big\{ \sum_{n_2=\nu_2}^{\infty} \Big[\Big(\sum_{\nu_1=2}^{n_1} |\Delta_{n_1,n_2}|^{\theta} |\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}}$$

$$- \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}| \Big) \Big]^{\frac{2}{\theta}} \Big\}^{\frac{\theta}{2}} \Big)^{\frac{2}{\theta}}$$

$$\leqslant \sum_{n_1=2}^{\infty} \Big(\sum_{\nu_2=2}^{\infty} \sum_{\nu_1=2}^{n_1} \Big\{ \sum_{n_2=\nu_2}^{\infty} \Big[|\Delta_{n_1,n_2}|^{\theta} |\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}}$$

$$- \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}| \Big]^{\frac{2}{\theta}} \Big\}^{\frac{\theta}{2}} \Big)^{\frac{2}{\theta}}$$

$$= \Big(\Big(\sum_{n_1=2}^{\infty}\Big\{\sum_{\nu_2=2}^{\infty}\sum_{\nu_1=2}^{n_1}\Big[\sum_{n_2=\nu_2}^{\infty}|\Delta_{n_1,n_2}|^2|\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}}$$

$$- \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}|^{\frac{2}{\theta}}\Big]^{\frac{\theta}{2}}\Big\}^{\frac{2}{\theta}}\Big)^{\frac{\theta}{2}}\Big)^{\frac{2}{\theta}}$$

$$\leqslant \Big(\sum_{\nu_2=2}^{\infty}\Big\{\sum_{n_1=2}^{\infty}\Big[\sum_{\nu_1=2}^{n_1}\Big(\sum_{n_2=\nu_2}^{\infty}|\Delta_{n_1,n_2}|^2|\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}}$$

$$- \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}|^{\frac{2}{\theta}}\Big)^{\frac{\theta}{2}}\Big]^{\frac{2}{\theta}}\Big\}^{\frac{\theta}{2}}\Big)^{\frac{2}{\theta}}$$

$$\leqslant \Big(\sum_{\nu_2=2}^{\infty}\sum_{\nu_1=2}^{\infty}\Big[\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}|\Delta_{n_1,n_2}|^2|\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}}$$

$$- \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}|^{\frac{2}{\theta}}\Big]^{\frac{\theta}{2}}\Big)^{\frac{2}{\theta}}$$

$$= \Big(\sum_{\nu_2=2}^{\infty}\sum_{\nu_1=2}^{\infty}|\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}}$$

$$- \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}|\Big(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}|\Delta_{n_1,n_2}|^2\Big)^{\frac{\theta}{2}}\Big)^{\frac{2}{\theta}}.$$

Hence, Lemma 1.5.6 with $\frac{p}{\theta} \geqslant 1$ implies that

$$J_4 = \Big\{\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta^2_{n_1,n_2}\Big[\sum_{\nu_1=2}^{n_1}\sum_{\nu_2=2}^{n_2}|\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}} -$$

$$- \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}|\Big]^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1dx_2\Big\}^{\frac{1}{p}}$$

$$\leqslant \Big\{\int_0^{2\pi}\int_0^{2\pi}\Big[\Big(\sum_{\nu_2=2}^{\infty}\sum_{\nu_1=2}^{\infty}|\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}}$$

$$- \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}|\Big(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}|\Delta_{n_1,n_2}|^2\Big)^{\frac{\theta}{2}}\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1dx_2\Big\}^{\frac{1}{p}}$$

$$= \Big(\Big\{\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{\nu_2=2}^{\infty}\sum_{\nu_1=2}^{\infty}|\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}}$$

$$- \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}|\Big(\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}|\Delta_{n_1,n_2}|^2\Big)^{\frac{\theta}{2}}\Big]^{\frac{p}{\theta}}dx_1dx_2\Big\}^{\frac{\theta}{p}}\Big)^{\frac{1}{\theta}}$$

$$\leqslant \Big(\sum_{\nu_2=2}^{\infty} \sum_{\nu_1=2}^{\infty} |\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}}$$

$$+ \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}| \Big\{ \int_0^{2\pi}\int_0^{2\pi} \Big[\Big(\sum_{n_1=\nu_1}^{\infty} \sum_{n_2=\nu_2}^{\infty} |\Delta_{n_1,n_2}|^2 \Big)^{\frac{\theta}{2}} \Big]^{\frac{p}{\theta}} dx_1 dx_2 \Big\}^{\frac{\theta}{p}} \Big)^{\frac{1}{\theta}}$$

$$= \Big(\sum_{\nu_2=2}^{\infty} \sum_{\nu_1=2}^{\infty} |\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}}$$

$$+ \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}| \Big\{ \int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{n_1=\nu_1}^{\infty} \sum_{n_2=\nu_2}^{\infty} |\Delta_{n_1,n_2}|^2 \Big]^{\frac{p}{2}} dx_1 dx_2 \Big\}^{\frac{\theta}{p}} \Big)^{\frac{1}{\theta}}.$$

By Lemmas 3.2.4 and 3.2.5, we obtain

$$J_4 \lesssim \Big(\sum_{\nu_2=1}^{\infty} \sum_{\nu_1=1}^{\infty} |\lambda^{\theta}_{2^{\nu_1},2^{\nu_2}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2}} - \lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}}| Y^{\theta}_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_p \Big)^{\frac{1}{\theta}}.$$

By (3.3.4), we have $J_4 < \infty$.

Collecting estimates of J_1, J_2, J_3 and J_4, we get $I_1 < \infty$. Hence, by Lemma 3.2.5 (b), there exists a function $g(x_1,x_2) \in L^0_p$, with the Fourier series

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \lambda_{2^{n_1-1},2^{n_2-1}} \Delta_{n_1,n_2}, \tag{3.3.6}$$

and

$$\|g\|_p \leqslant C(p) I_1. \tag{3.3.7}$$

We rewrite series (3.3.6) in the form of

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \gamma_{n_1,n_2} A_{n_1 n_2}(x_1,x_2),$$

where $(n_1, n_2 = 2, 3, ...)$

$$\gamma_{1,1} = \lambda_{1,1}, \ \ \gamma_{1,\nu_2} = \lambda_{1,2^{n_2-1}} \ \text{ for } \ 2^{n_2-1} \leqslant \nu_2 \leqslant 2^{n_2} - 1,$$
$$\gamma_{\nu_1,1} = \lambda_{2^{n_1-1},1} \ \text{ for } \ 2^{n_1-1} \leqslant \nu_1 \leqslant 2^{n_1} - 1,$$
$$\gamma_{\nu_1,\nu_2} = \lambda_{2^{n_1-1},2^{n_2-1}} \ \text{ for } \ 2^{n_2-1} \leqslant \nu_2 \leqslant 2^{n_2} - 1, 2^{n_1-1} \leqslant \nu_1 \leqslant 2^{n_1} - 1.$$

Now we consider the following series

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\lambda_{n_1,n_2}A_{n_1n_2}(x_1,x_2)=\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\gamma_{n_1,n_2}\Lambda_{n_1,n_2}A_{n_1n_2}(x_1,x_2), \qquad (3.3.8)$$

where

$$\Lambda_{1,1}=1,\ \ \Lambda_{1,\nu_2}=\frac{\lambda_{1,\nu_2}}{\gamma_{1,\nu_2}}=\frac{\lambda_{1,\nu_2}}{\lambda_{1,2^{n_2-1}}}\ \text{ for }\ 2^{n_2-1}\leqslant\nu_2\leqslant 2^{n_2}-1\ \ (n_2=2,3,...),$$

$$\Lambda_{\nu_1,1}=\frac{\lambda_{\nu_1,1}}{\gamma_{\nu_1,1}}=\frac{\lambda_{\nu_1,1}}{\lambda_{2^{n_1-1},1}}\ \text{ for }\ 2^{n_1-1}\leqslant\nu_1\leqslant 2^{n_1}-1\ \ (n_1=2,3,...),$$

$$\Lambda_{\nu_1,\nu_2}=\frac{\lambda_{\nu_1,\nu_2}}{\gamma_{\nu_1,\nu_2}}=\frac{\lambda_{\nu_1,\nu_2}}{\lambda_{2^{n_1-1},2^{n_2-1}}}\ \text{ for }\ 2^{n_1-1}\leqslant\nu_1\leqslant 2^{n_1}-1,$$

$$2^{n_2-1}\leqslant\nu_2\leqslant 2^{n_2}-1\ \ (n_1,n_2=2,3,...).$$

Let us show that $\{\Lambda_{n_1,n_2}\}$ satisfies the conditions of Lemma 3.2.6. Taking into account that $\{\lambda_{n_1,n_2}\}=\lambda\in GM^2$, we have

1) $|\Lambda_{n_1,n_2}|\leqslant M$,

2) for $n_1=n_2=1$,

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1}|\Lambda_{\nu_1,1}-\Lambda_{\nu_1+1,1}|=\left|1-\frac{\lambda_{2,1}}{\lambda_{2,1}}\right|\leqslant M,$$

$$\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}|\Lambda_{1,\nu_2}-\Lambda_{1,\nu_2+1}|\leqslant M,$$

3) for $n_2=1,\ \ n_1=2,3,...$,

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1}|\Lambda_{\nu_1,n_2}-\Lambda_{\nu_1+1,n_2}|=\frac{1}{\lambda_{2^{n_1-1},1}}\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2}|\lambda_{\nu_1}-\lambda_{\nu_1+1,1}|+\left|\frac{\lambda_{2^{n_1}-1,1}}{\lambda_{2^{n_1-1},1}}-1\right|\leqslant M,$$

4) for $n_1=1,\ n_2=2,3,...$,

$$\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}|\Lambda_{1,\nu_2}-\Lambda_{1,\nu_2+1}|\leqslant M,$$

5) for $n_2=2,3,...$ such that $2^{m_2-1}\leqslant n_2\leqslant 2^{m_2}-1\ \ (m_2=2,3,...)$,

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1}|\Lambda_{\nu_1,n_2}-\Lambda_{\nu_1+1,n_2}|\leqslant\frac{1}{\lambda_{2^{n_1-1},2^{m_2-1}}}\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2}|\lambda_{\nu_1,n_2}-\lambda_{\nu_1+1,n_2}|$$

$$+\frac{\lambda_{2^{n_1}-1,n_2}}{\lambda_{2^{n_1-1},2^{m_2-1}}}+\frac{\lambda_{2^{n_1},n_2}}{\lambda_{2^{n_1},2^{m_2-1}}}\leqslant M,$$

6) for $n_1=2,3,...$ such that $2^{m_1-1}\leqslant n_1\leqslant 2^{m_1}-1 (m_1=2,3,...)$,

$$\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}|\Lambda_{n_1,\nu_2}-\Lambda_{n_1,\nu_2+1}|\leqslant M,$$

7) for $n_1=n_2=1$,

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1}\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}|\Lambda_{\nu_1,\nu_2}-\Lambda_{\nu_1+1,\nu_2}-\Lambda_{\nu_1,\nu_2+1}+\Lambda_{\nu_1+1,\nu_2+1}|\leqslant M,$$

8) for $n_1=1,\ n_2=2,3,...$,

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1}\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}|\Lambda_{\nu_1,\nu_2}-\Lambda_{\nu_1+1,\nu_2}-\Lambda_{\nu_1,\nu_2+1}+\Lambda_{\nu_1+1,\nu_2+1}|$$

$$\leqslant\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}|\Lambda_{1,\nu_2}-\Lambda_{1,\nu_2+1}|+\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}|\Lambda_{2,\nu_2}-\Lambda_{2,\nu_2+1}|\leqslant M,$$

9) for $n_2=1,\ n_1=2,3,...$,

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1}\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}|\Lambda_{\nu_1,\nu_2}-\Lambda_{\nu_1+1,\nu_2}-\Lambda_{\nu_1,\nu_2+1}+\Lambda_{\nu_1+1,\nu_2+1}|\leqslant M,$$

10) for $n_1,n_2=2,3,...$,

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1}\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1}|\Lambda_{\nu_1,\nu_2}-\Lambda_{\nu_1+1,\nu_2}-\Lambda_{\nu_1,\nu_2+1}+\Lambda_{\nu_1+1,\nu_2+1}|$$

$$=\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-2}\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2}|\Lambda_{\nu_1,\nu_2}-\Lambda_{\nu_1+1,\nu_2}-\Lambda_{\nu_1,\nu_2+1}+\Lambda_{\nu_1+1,\nu_2+1}|$$

$$+\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-2}|\Lambda_{2^{n_1}-1,\nu_2}-\Lambda_{2^{n_1},\nu_2}-\Lambda_{2^{n_1}-1,\nu_2+1}+\Lambda_{2^{n_1},\nu_2+1}|$$

$$+\sum_{\nu_1=2^{n_1-2}}^{2^{n_1}-2}|\Lambda_{\nu_1,2^{n_2}-1}-\Lambda_{\nu_1+1,2^{n_2}-1}-\Lambda_{\nu_1,2^{n_2}}+\Lambda_{\nu_1+1,2^{n_2}}|$$

$$+|\Lambda_{2^{n_1}-1,2^{n_2}-1}-\Lambda_{2^{n_2},2^{n_2}-1}-\Lambda_{2^{n_1}-1,2^{n_2}}+\Lambda_{2^{n_1},2^{n_2}}|$$

$$=\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-2}\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2}\left|\frac{\lambda_{\nu_1,\nu_2}}{\lambda_{2^{n_1-1},2^{n_2-1}}}-\frac{\lambda_{\nu_1+1,\nu_2}}{\lambda_{2^{n_1-1},2^{n_2-1}}}-\frac{\lambda_{\nu_1,\nu_2+1}}{\lambda_{2^{n_1-1},2^{n_2-1}}}+\frac{\lambda_{\nu_1+1,\nu_2+1}}{\lambda_{2^{n_1-1},2^{n_2-1}}}\right|$$

$$+\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-2}\left|\frac{\lambda_{2^{n_1}-1,\nu_2}}{\lambda_{2^{n_1-1},2^{n_2-1}}}-\frac{\lambda_{2^{n_1},\nu_2}}{\lambda_{2^{n_1},2^{n_2-1}}}-\frac{\lambda_{2^{n_1}-1,\nu_2+1}}{\lambda_{2^{n_1-1},2^{n_2-1}}}+\frac{\lambda_{2^{n_1},\nu_2+1}}{\lambda_{2^{n_1},2^{n_2-1}}}\right|$$

$$+\sum_{\nu_1=2^{n_1-2}}^{2^{n_1}-2}\Big|\frac{\lambda_{\nu_1,2^{n_2}-1}}{\lambda_{2^{n_1-1},2^{n_2-1}}}-\frac{\lambda_{\nu_1+1,2^{n_2}-1}}{\lambda_{2^{n_1-1},2^{n_2-1}}}-\frac{\lambda_{\nu_1,2^{n_2}}}{\lambda_{2^{n_1-1},2^{n_2}}}+\frac{\lambda_{\nu_1+1,2^{n_2}}}{\lambda_{2^{n_1-1},2^{n_2}}}\Big|$$

$$+\Big|\frac{\lambda_{2^{n_1}-1,2^{n_2}-1}}{\lambda_{2^{n_1-1},2^{n_2-1}}}-\frac{\lambda_{2^{n_1},2^{n_2}-1}}{\lambda_{2^{n_1},2^{n_2-1}}}-\frac{\lambda_{2^{n_1}-1,2^{n_2}}}{\lambda_{2^{n_1-1},2^{n_2}}}+\frac{\lambda_{2^{n_1},2^{n_2}}}{\lambda_{2^{n_1},2^{n_2}}}\Big|$$

$$\leqslant C\frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{2^{n_1-1},2^{n_2-1}}}+\frac{1}{\lambda_{2^{n_1-1},2^{n_2-1}}}\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2}|\lambda_{\nu_1,2^{n_2}-1}-\lambda_{\nu_1+1,2^{n_2}-1}|$$

$$+\frac{1}{\lambda_{2^{n_1-1},2^{n_2}}}\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2}|\lambda_{\nu_1,2^{n_2}}-\lambda_{\nu_1+1,2^{n_2}}|+\frac{1}{\lambda_{2^{n_1-1},2^{n_2-1}}}\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-2}|\lambda_{2^{n_1}-1,\nu_2}-\lambda_{2^{n_1}-1,\nu_2+1}|$$

$$+\frac{1}{\lambda_{2^{n_1},2^{n_2-1}}}\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-2}|\lambda_{2^{n_1},\nu_2}-\lambda_{2^{n_1},\nu_2+1}|\leqslant M.$$

Since the sequence $\{\Lambda_{n_1n_2}\}_{n_1=1,n_2=1}^{\infty,\infty}$ satisfies the conditions of Lemma 3.2.6, then the series (3.3.8) is the Fourier series of a function $f^{(\lambda,\beta_1,\beta_2)}(x_1,x_2)\in L_p$ and $\|f^{(\lambda,\beta_1,\beta_2)}\|_p\leqslant C(p,\lambda)\|g\|_p$.

Taking into account (3.3.7) and the estimates of J_1,J_2,J_3 and J_4, we have

$$\|f^{(\lambda,\beta_1,\beta_2)}\|_p\lesssim\Bigg\{\lambda_{1,1}^\theta\|f\|_p^\theta+\sum_{\nu_1=1}^{\infty}|\lambda^\theta_{2^{\nu_1},1}-\lambda^\theta_{2^{\nu_1-1},1}|Y^\theta_{2^{\nu_1}-1,0}(f)_p+$$

$$+\sum_{\nu_2=1}^{\infty}|\lambda^\theta_{1,2^{\nu_2}}-\lambda^\theta_{1,2^{\nu_2-1}}|Y^\theta_{0,2^{\nu_2}-1}(f)_p$$

$$+\sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty}|\lambda^\theta_{2^{\nu_1},2^{\nu_2}}-\lambda^\theta_{2^{\nu_1-1},2^{\nu_2}}-\lambda^\theta_{2^{\nu_1},2^{\nu_2-1}}+\lambda^\theta_{2^{\nu_1-1},2^{\nu_2-1}}|Y^\theta_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_p\Bigg\}^{\frac{1}{\theta}}$$

$$\lesssim\Bigg\{\lambda^\theta_{1,1}\|f\|^\theta_p+\sum_{\nu_1=1}^{\infty}Y^\theta_{2^{\nu_1}-1,0}(f)_p\sum_{n_1=2^{\nu_1-1}}^{2^{\nu_1}-1}|\lambda^\theta_{n_1+1,1}-\lambda^\theta_{n_1,1}|$$

$$+\sum_{\nu_2=1}^{\infty}Y^\theta_{0,2^{\nu_2}-1}(f)_p\sum_{n_2=2^{\nu_2-1}}^{2^{\nu_2}-1}|\lambda^\theta_{1,n_2+1}-\lambda^\theta_{1,n_2}|$$

$$+\sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty}Y^\theta_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_p$$

$$\times \sum_{n_1=2^{\nu_1-1}}^{2^{\nu_1}-1} \sum_{n_2=2^{\nu_2-1}}^{2^{\nu_2}-1} |\lambda^\theta_{n_1,n_2} - \lambda^\theta_{n_1+1,n_2} - \lambda^\theta_{n_1,n_2+1} + \lambda^\theta_{n_1+1,n_2+1}| \Bigg\}^{\frac{1}{\theta}}$$

$$\lesssim \Bigg\{ \lambda^\theta_{1,1} \|f\|^\theta_p + \sum_{n_1=1}^{\infty} |\lambda^\theta_{n_1+1,1} - \lambda^\theta_{n_1,1}| Y^\theta_{n_1,0}(f)_p$$

$$+ \sum_{n_2=1}^{\infty} |\lambda^\theta_{1,n_2+1} - \lambda^\theta_{1,n_2}| Y^\theta_{0,n_2}(f)_p$$

$$+ \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} |\lambda^\theta_{n_1,n_2} - \lambda^\theta_{n_1+1,n_2} - \lambda^\theta_{n_1,n_2+1} + \lambda^\theta_{n_1+1,n_2+1}| Y^\theta_{n_1,n_2}(f)_p \Bigg\}^{\frac{1}{\theta}}.$$

Let us estimate $Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p$. Using Lemma 3.2.4, we get

$$Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p \leqslant C\| f^{(\lambda,\beta_1,\beta_2)}$$
$$- S_{\infty,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)}) - S_{2^{m_1}-1,\infty}(f^{(\lambda,\beta_1,\beta_2)}) + S_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})\|_p.$$

We consider the series (cf. (3.3.8))

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \lambda_{n_1,n_2} A^*_{n_1 n_2}(x_1,x_2) = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \gamma_{n_1,n_2}\Lambda_{n_1,n_2} A^*_{n_1 n_2}(x_1,x_2),$$

where $A^*_{n_1,n_2}(x_1,x_2) = 0$, if $n_1 \leqslant 2^{m_1}-1$ and $n_2 \leqslant 2^{m_2}-1$, also $A^*_{n_1,n_2}(x_1,x_2) = A_{n_1,n_2}(x_1,x_2)$ otherwise.

Since the sequence $\{\Lambda_{n_1,n_2}\}$ satisfies the conditions of Lemma 3.2.6, then

$$\Big\| \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \lambda_{n_1,n_2} A^*_{n_1 n_2}(x_1,x_2) \Big\|_p \leqslant C \Big\| \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \lambda_{2^{n_1-1},2^{n_2-1}} \Delta^*_{n_1,n_2} \Big\|_p,$$

where $\Delta^*_{n_1,n_2} = 0$, if $n_1 \leqslant m_1$ or $n_2 \leqslant m_2$, $\Delta^*_{n_1,n_2} = \Delta_{n_1,n_2}$ otherwise.

By Lemma 3.2.5, we have

$$Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p \lesssim \Bigg\{ \int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{k_1=m_1+1}^{\infty} \sum_{k_2=m_2+1}^{\infty} \lambda^2_{2^{k_1-1},2^{k_2-1}} \Delta^2_{k_1,k_2} \Big]^{\frac{p}{2}} dx_1 dx_2 \Bigg\}^{\frac{1}{p}}. \tag{3.3.9}$$

It is easy to see that

$$\lambda^{\theta}_{2^{k_1-1},2^{k_2-1}} = \lambda^{\theta}_{2^{m_1-1},2^{m_2-1}} + \sum_{\nu_2=m_2+1}^{k_2} (\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2}})$$

$$+ \sum_{\nu_1=m_1+1}^{k_1} (\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1}})$$

$$+ \sum_{\nu_1=m_1+1}^{k_1} \sum_{\nu_2=m_2+1}^{k_2} \left[\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}\right].$$

Substituting this estimate in (3.3.9), we derive

$$Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p \lesssim \Bigg\{ \int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{k_1=m_1+1}^{\infty} \sum_{k_2=m_2+1}^{\infty} \Big(\lambda^{\theta}_{2^{m_1-1},2^{m_2-1}}$$

$$+ \sum_{\nu_2=m_2+1}^{k_2} |\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2}}| + \sum_{\nu_1=m_1+1}^{k_1} |\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1}}|$$

$$+ \sum_{\nu_1=m_1+1}^{k_1} \sum_{\nu_2=m_2+1}^{k_2} |\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}}$$

$$- \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}|\Big)^{\frac{2}{\theta}} \Delta^2_{k_1,k_2}\Big]^{\frac{p}{2}} dx_1 dx_2 \Bigg\}^{\frac{1}{p}}.$$

Then

$$Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p \lesssim \lambda_{2^{m_1-1},2^{m_2-1}} \Big\{ \int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{k_1=m_1+1}^{\infty} \sum_{k_2=m_2+1}^{\infty} \Delta^2_{k_1,k_2}\Big]^{\frac{p}{2}} dx_1 dx_2 \Big\}^{\frac{1}{p}}$$

$$+ \Big\{ \int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{k_1=m_1+1}^{\infty} \sum_{k_2=m_2+1}^{\infty} \Delta^2_{k_1,k_2} \Big(\sum_{\nu_1=m_1+1}^{k_1} |\lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{m_2-1}}|\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}} dx_1 dx_2 \Big\}^{\frac{1}{p}}$$

$$+ \Big\{ \int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{k_1=m_1+1}^{\infty} \sum_{k_2=m_2+1}^{\infty} \Delta^2_{k_1,k_2} \Big(\sum_{\nu_2=m_2+1}^{k_2} |\lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-2}}|\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}} dx_1 dx_2 \Big\}^{\frac{1}{p}}$$

$$+ \Big\{ \int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{k_1=m_1+1}^{\infty} \sum_{k_2=m_2+1}^{\infty} \Delta^2_{k_1,k_2} \Big(\sum_{\nu_1=m_1+1}^{k_1} \sum_{\nu_2=m_2+1}^{k_2} |\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}}$$

$$- \lambda^\theta_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^\theta_{2^{\nu_1-2},2^{\nu_2-2}}|\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}} dx_1dx_2\Big\}^{\frac{1}{p}} =: L_1+L_2+L_3+L_4.$$

We estimate L_1 as J_1 to get

$$\begin{aligned} L_1 &\lesssim \lambda_{2^{m_1-1},2^{m_2-1}}\Big\{\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\Delta^2_{k_1,k_2}\Big]^{\frac{p}{2}}dx_1dx_2\Big\}^{\frac{1}{p}} \\ &\lesssim \lambda_{2^{m_1-1},2^{m_2-1}}Y_{2^{m_1}-1,2^{m_2}-1}(f)_p. \end{aligned}$$

To estimate L_2, we have

$$\begin{aligned} &\Big\{\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\Delta^2_{k_1,k_2}\Big(\sum_{\nu_1=m_1+1}^{k_1}|\lambda^\theta_{2^{\nu_1-1},2^{m_2-1}}-\lambda^\theta_{2^{\nu_1-2},2^{m_2-1}}|\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1dx_2\Big\}^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{\nu_1=m_1+1}^{\infty}|\lambda^\theta_{2^{\nu_1-1},2^{m_2-1}}-\lambda^\theta_{2^{\nu_1-2},2^{m_2-1}}|\Big\{\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\Delta^2_{k_1,k_2}\Big]^{\frac{p}{2}}dx_1dx_2\Big\}^{\frac{\theta}{p}}\Big)^{\frac{1}{\theta}} \\ &\lesssim \Big(\sum_{\nu_1=m_1+1}^{\infty}|\lambda^\theta_{2^{\nu_1-1},2^{m_2-1}}-\lambda^\theta_{2^{\nu_1-2},2^{m_2-1}}|Y^\theta_{2^{\nu_1-1}-1,2^{m_2}-1}(f)_p\Big)^{\frac{1}{\theta}}. \end{aligned}$$

Similarly, we obtain the estimate for L_3:

$$\begin{aligned} &\Big\{\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\Delta^2_{k_1,k_2}\Big(\sum_{\nu_2=m_2+1}^{k_2}|\lambda^\theta_{2^{m_1-1},2^{\nu_2-1}}-\lambda^\theta_{2^{m_1-1},2^{\nu_2-2}}|\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1dx_2\Big\}^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{\nu_2=m_2+1}^{\infty}|\lambda^\theta_{2^{m_1-1},2^{\nu_2-1}}-\lambda^\theta_{2^{m_1-1},2^{\nu_2-2}}|Y^\theta_{2^{m_1}-1,2^{\nu_2-1}-1}(f)_p\Big)^{\frac{1}{\theta}}. \end{aligned}$$

Finally, we estimate L_4 as follows

$$\begin{aligned} L_4 =&\Big\{\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1+1}^{\infty}\sum_{k_2=m_2+1}^{\infty}\Delta^2_{k_1,k_2}\Big(\sum_{\nu_1=m_1+1}^{k_1}\sum_{\nu_2=m_2+1}^{k_2}|\lambda^\theta_{2^{\nu_1-1},2^{\nu_2-1}}-\lambda^\theta_{2^{\nu_1-2},2^{\nu_2-1}} \\ &- \lambda^\theta_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^\theta_{2^{\nu_1-2},2^{\nu_2-2}}|\Big)^{\frac{2}{\theta}}\Big]^{\frac{p}{2}}dx_1dx_2\Big\}^{\frac{1}{p}} \\ &\lesssim \Big(\sum_{\nu_1=m_1+1}^{\infty}\sum_{\nu_2=m_2+1}^{\infty}|\lambda^\theta_{2^{\nu_1-1},2^{\nu_2-1}}-\lambda^\theta_{2^{\nu_1-2},2^{\nu_2-1}} \end{aligned}$$

$$- \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}| \int\limits_0^{2\pi}\int\limits_0^{2\pi} \Big[\sum_{k_1=\nu_1}^{\infty}\sum_{k_2=\nu_2}^{\infty} \Delta^2_{k_1,k_2}\Big]^{\frac{p}{2}} dx_1dx_2\Big\}^{\frac{\theta}{p}}\Big)^{\frac{1}{\theta}}$$

$$\lesssim \Big(\sum_{\nu_1=m_1+1}^{\infty}\sum_{\nu_2=m_2+1}^{\infty} |\lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-1}}$$

$$- \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-2}} + \lambda^{\theta}_{2^{\nu_1-2},2^{\nu_2-2}}|Y^{\theta}_{2^{\nu_1-1}-1,2^{\nu_2-1}-1}(f)_p\Big)^{\frac{1}{\theta}}.$$

Taking into account the estimates for L_1, L_2, L_3, and L_4, we derive

$$Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p \lesssim \Big\{ \lambda^{\theta}_{2^{m_1-1},2^{m_2-1}} Y^{\theta}_{2^{m_1}-1,2^{m_2}-1}(f)_p$$

$$+ \sum_{\nu_1=m_1}^{\infty} |\lambda^{\theta}_{2^{\nu_1},2^{m_2-1}} - \lambda^{\theta}_{2^{\nu_1-1},2^{m_2-1}}| Y^{\theta}_{2^{\nu_1}-1,2^{m_2}-1}(f)_p$$

$$+ \sum_{\nu_2=m_2}^{\infty} |\lambda^{\theta}_{2^{m_1-1},2^{\nu_2}} - \lambda^{\theta}_{2^{m_1-1},2^{\nu_2-1}}| Y^{\theta}_{2^{m_1}-1,2^{\nu_2}-1}(f)_p$$

$$+ \sum_{\nu_1=m_1}^{\infty}\sum_{\nu_2=m_2}^{\infty} |\lambda^{\theta}_{2^{\nu_1},2^{\nu_2}} - \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2}}$$

$$- \lambda^{\theta}_{2^{\nu_1},2^{\nu_2-1}} + \lambda^{\theta}_{2^{\nu_1-1},2^{\nu_2-1}}| Y^{\theta}_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_p \Big\}^{\frac{1}{\theta}}.$$

The proof is now complete.

□

3.4 Estimates from below of angle best approximations of generalized Liouville–Weyl derivatives

Recall that a sequence $\lambda = \{\lambda_{n_1 n_2}\}_{n_1,n_2\in N}$ is such that $\frac{1}{\lambda} \in GM^2$ if the relations

$$\sum_{k_1=n_1}^{2n_1} \Big|\frac{1}{\lambda_{k_1,n_2}} - \frac{1}{\lambda_{k_1+1,n_2}}\Big| \leqslant C\Big|\frac{1}{\lambda_{n_1,n_2}}\Big|,$$

$$\sum_{k_2=n_2}^{2n_2} \Big|\frac{1}{\lambda_{n_1,k_2}} - \frac{1}{\lambda_{n_1,k_2+1}}\Big| \leqslant C\Big|\frac{1}{\lambda_{n_1,n_2}}\Big|,$$

$$\sum_{k_1=n_1}^{2n_1}\sum_{k_2=n_2}^{2n_2}\Big|\frac{1}{\lambda_{k_1,k_2}}-\frac{1}{\lambda_{k_1+1,k_2}}-\frac{1}{\lambda_{k_1,k_2+1}}+\frac{1}{\lambda_{k_1+1,k_2+1}}\Big|\leqslant C\Big|\frac{1}{\lambda_{n_1,n_2}}\Big|$$

hold for all integers n_1 and n_2, where the constant C is independent of n_1 and n_2 (see Definition 3.1.1). Note that the last condition can be equivalently written as follows:

$$\sum_{k_1=n_1}^{2n_1}\sum_{k_2=n_2}^{2n_2}\Big|\frac{1}{\lambda_{k_1,k_2}}-\frac{1}{\lambda_{k_1+1,k_2}}-\frac{1}{\lambda_{k_1,k_2+1}}+\frac{1}{\lambda_{k_1+1,k_2+1}}\Big|\leqslant C\Big(\Big|\frac{1}{\lambda_{n_1,n_2}}\Big|+\Big|\frac{1}{\lambda_{2n_1,2n_2}}\Big|\Big).$$

Our main result in this subsection is the following analogue of Theorem 1.8.1 in two-dimensional case.

Theorem 3.4.1. *Let $1<p<\infty$, $\max(p,2)\leqslant\tau<\theta$, $\alpha_i\in\mathbb{R}_+$, $r_i\in\mathbb{R}_+\cup\{0\}$, $\beta_i\in\mathbb{R}(i=1,2)$, $\lambda=\{\lambda_{n_1,n_2}\}_{n_1,n_2\in N}$ be a sequence of positive numbers such that $\frac{1}{\lambda}\in GM^2$, and also satisfy the additional conditions*

$$\begin{aligned}
&\Big(\sum_{k_2=1}^{n_2}|\lambda^\tau_{2^{n_1-1},2^{k_2}}-\lambda^\tau_{2^{n_1-1},2^{k_2-1}}|\Big)^{\frac{1}{\tau}}\leqslant C|\lambda_{2^{n_1-1},2^{n_2}}|,\\
&\Big(\sum_{k_1=1}^{n_1}|\lambda^\tau_{2^{k_1},2^{n_2-1}}-\lambda^\tau_{2^{k_1-1},2^{n_2-1}}|\Big)^{\frac{1}{\tau}}\leqslant C|\lambda_{2^{n_1},2^{n_2-1}}|,\\
&\Big(\sum_{k_1=1}^{n_1}\sum_{k_2=1}^{n_2}|\lambda^\tau_{2^{k_1},2^{k_2}}-\lambda^\tau_{2^{k_1},2^{k_2-1}}-\lambda^\tau_{2^{k_1-1},2^{k_2}}+\lambda^\tau_{2^{k_1-1},2^{k_2-1}}|\Big)^{\frac{1}{\tau}}\leqslant C|\lambda_{2^{n_1},2^{n_2}}|,
\end{aligned}\tag{3.4.10}$$

where n_1 and n_2 integers and the constant C is independent of n_1 and n_2. If $f\in L^0_p(\mathbb{T}^2)$ and there exists a function $f^{(\lambda,\beta_1,\beta_2)}\in L^0_p(\mathbb{T}^2)$, with the Fourier series $\sigma(f,\lambda,\beta_1,\beta_2)$, then

$$\begin{aligned}
\|f^{(\lambda,\beta_1,\beta_2)}\|_p\gtrsim\Bigg\{&\lambda^\tau_{1,1}\|f\|^\tau_p+\sum_{\nu_1=1}^{\infty}|\lambda^\tau_{2^{\nu_1},1}-\lambda^\tau_{2^{\nu_1-1},1}|Y^\tau_{2^{\nu_1}-1,0}(f)_p\\
&+\sum_{\nu_2=1}^{\infty}|\lambda^\tau_{1,2^{\nu_2}}-\lambda^\tau_{1,2^{\nu_2-1}}|Y^\tau_{0,2^{\nu_2}-1}(f)_p\\
&+\sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty}|\lambda^\tau_{2^{\nu_1},2^{\nu_2}}-\lambda^\tau_{2^{\nu_1-1},2^{\nu_2}}-\lambda^\tau_{2^{\nu_1},2^{\nu_2-1}}+\lambda^\tau_{2^{\nu_1-1},2^{\nu_2-1}}|Y^\tau_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_p\Bigg\}^{\frac{1}{\tau}}
\end{aligned}$$

and

$$Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p \gtrsim \Bigg\{\lambda^{\tau}_{2^{m_1-1},2^{m_2-1}} Y^{\tau}_{2^{m_1}-1,2^{m_2}-1}(f)_p$$

$$+\sum_{\nu_1=m_1}^{\infty} |\lambda^{\tau}_{2^{\nu_1},2^{m_2-1}} - \lambda^{\tau}_{2^{\nu_1-1},2^{m_2-1}}| Y^{\tau}_{2^{\nu_1}-1,2^{m_2}-1}(f)_p$$

$$+\sum_{\nu_2=m_2}^{\infty} |\lambda^{\tau}_{2^{m_1-1},2^{\nu_2}} - \lambda^{\tau}_{2^{m_1-1},2^{\nu_2-1}}| Y^{\tau}_{2^{m_1}-1,2^{\nu_2}-1}(f)_p$$

$$+\sum_{\nu_1=m_1}^{\infty}\sum_{\nu_2=m_2}^{\infty} |\lambda^{\tau}_{2^{\nu_1},2^{\nu_2}} - \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2}} - \lambda^{\tau}_{2^{\nu_1},2^{\nu_2-1}} + \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2-1}}| Y^{\tau}_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_p \Bigg\}^{\frac{1}{\tau}}.$$

Proof of Theorem 3.4.1. We consider the series

$$\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \lambda_{2^{n_1-1},2^{n_2-1}} \Delta_{n_1,n_2} = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \Lambda_{\nu_1,\nu_2}\lambda_{\nu_1,\nu_2}A_{\nu_1,\nu_2}(x_1,x_2), \qquad (3.4.11)$$

where $\Lambda_{\nu_1,\nu_2} = \frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{\nu_1,\nu_2}}$ for $2^{n_i-1} \leqslant \nu_i \leqslant 2^{n_i}-1$ $(n_i = 1,2,...), i = 1,2.$
Taking into account that $\frac{1}{\lambda} \in GM^2$, we list some properties of the sequence $\{\Lambda_{\nu_1,\nu_2}\}$:

1) $|\Lambda_{\nu_1,\nu_2}| = \Big|\dfrac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{\nu_1,\nu_2}}\Big| \leqslant C\Big|\dfrac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{2^{n_1-1},2^{n_2-1}}}\Big| \leqslant M,$

2) for $n_1 = n_2 = 1$,

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1} \Big|\Lambda_{\nu_1,n_2} - \Lambda_{\nu_1+1,n_2}\Big| = \Big|\frac{\lambda_{1,1}}{\lambda_{1,1}} - \frac{\lambda_{2,1}}{\lambda_{2,1}}\Big| \leqslant M,$$

$$\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1} \Big|\Lambda_{n_1,\nu_2} - \Lambda_{n_1,\nu_2+1}\Big| \leqslant M,$$

3) for $n_2 = 1,\ n_1 = 2,3,...,$

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1} \Big|\Lambda_{\nu_1,n_2} - \Lambda_{\nu_1+1,n_2}\Big| = \lambda_{2^{n_1-1},1} \sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2} \Big|\frac{1}{\lambda_{\nu_1,1}} - \frac{1}{\lambda_{\nu_1+1,1}}\Big|$$

$$+\Big|\frac{\lambda_{2^{n_1-1},1}}{\lambda_{2^{n_1}-1,1}} - \frac{\lambda_{2^{n_1},1}}{\lambda_{2^{n_1},1}}\Big| \leqslant M,$$

4) for $n_1 = 1,\ n_2 = 2, 3, ...,$

$$\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1} \left|\Lambda_{1,\nu_2} - \Lambda_{1,\nu_2+1}\right| \leqslant M,$$

5) for $n_2 = 2, 3, ... : 2^{m_2-1} \leqslant n_2 \leqslant 2^{m_2} - 1\ (m_2 = 2, 3, ...)$,

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1} \left|\Lambda_{\nu_1,n_2} - \Lambda_{\nu_1+1,n_2}\right|$$

$$= \sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2} \left|\frac{\lambda_{2^{n_1-1},2^{m_2-1}}}{\lambda_{\nu_1,n_2}} - \frac{\lambda_{2^{n_1-1},2^{m_2-1}}}{\lambda_{\nu_1+1,n_2}}\right| + \left|\frac{\lambda_{2^{n_1-1},2^{m_2-1}}}{\lambda_{2^{n_1}-1,n_2}} - \frac{\lambda_{2^{n_1},2^{m_2-1}}}{\lambda_{2^{n_1},n_2}}\right|$$

$$\leqslant \lambda_{2^{n_1-1},2^{m_2-1}} \sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2} \left|\frac{1}{\lambda_{\nu_1,n_2}} - \frac{1}{\lambda_{\nu_1+1,n_2}}\right| + \frac{\lambda_{2^{n_1-1},2^{m_2-1}}}{\lambda_{2^{n_1}-1,n_2}} + \frac{\lambda_{2^{n_1},2^{m_2-1}}}{\lambda_{2^{n_1},n_2}} \leqslant M,$$

6) for $n_1 = 2, 3, ... : 2^{m_1-1} \leqslant n_1 \leqslant 2^{m_1} - 1\ (m_1 = 2, 3, ...)$,

$$\sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1} \left|\Lambda_{n_1,\nu_2} - \Lambda_{n_1,\nu_2+1}\right| \leqslant M,$$

7) for $n_1 = n_2 = 1$,

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1} \sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1} \left|\Lambda_{\nu_1,\nu_2} - \Lambda_{\nu_1+1,\nu_2} - \Lambda_{\nu_1,\nu_2+1} + \Lambda_{\nu_1+1,\nu_2+1}\right|$$

$$= \left|1 - \frac{\lambda_{2,1}}{\lambda_{2,1}} - \frac{\lambda_{1,2}}{\lambda_{1,2}} + \frac{\lambda_{2,2}}{\lambda_{2,2}}\right| \leqslant M,$$

8) for $n_1 = 1,\ n_2 = 2, 3, ...,$

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1} \sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1} \left|\Lambda_{\nu_1,\nu_2} - \Lambda_{\nu_1+1,\nu_2} - \Lambda_{\nu_1,\nu_2+1} + \Lambda_{\nu_1+1,\nu_2+1}\right|$$

$$\leqslant \sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1} \left|\Lambda_{1,\nu_2} - \Lambda_{1,\nu_2+1}\right| + \sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1} \left|\Lambda_{2,\nu_2} - \Lambda_{2,\nu_2+1}\right| \leqslant M,$$

9) for $n_2 = 1,\ n_1 = 2, 3, ...,$

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1} \sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1} \left|\Lambda_{\nu_1,\nu_2} - \Lambda_{\nu_1+1,\nu_2} - \Lambda_{\nu_1,\nu_2+1} + \Lambda_{\nu_1+1,\nu_2+1}\right|$$

$$= \sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1} \left|\Lambda_{\nu_1,1} - \Lambda_{\nu_1+1,1} - \Lambda_{\nu_1,2} + \Lambda_{\nu_1+1,2}\right| \leqslant M,$$

10) for $n_1, n_2 = 2, 3, ...,$

$$\sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-1} \sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-1} \left|\Lambda_{\nu_1,\nu_2} - \Lambda_{\nu_1+1,\nu_2} - \Lambda_{\nu_1,\nu_2+1} + \Lambda_{\nu_1+1,\nu_2+1}\right|$$

$$= \sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-2} \sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2} \left|\frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{\nu_1,\nu_2}} - \frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{\nu_1+1,\nu_2}} - \frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{\nu_1,\nu_2+1}} + \frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{\nu_1+1,\nu_2+1}}\right|$$

$$+ \sum_{\nu_1=2^{n_1-2}}^{2^{n_1}-2} \left|\frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{\nu_1,2^{n_2}-1}} - \frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{\nu_1+1,2^{n_2}-1}} - \frac{\lambda_{2^{n_1-1},2^{n_2}}}{\lambda_{\nu_1,2^{n_2}}} + \frac{\lambda_{2^{n_1-1},2^{n_2}}}{\lambda_{\nu_1+1,2^{n_2}}}\right|$$

$$+ \sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-2} \left|\frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{2^{n_1}-1,\nu_2}} - \frac{\lambda_{2^{n_1},2^{n_2-1}}}{\lambda_{2^{n_1},\nu_2}} - \frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{2^{n_1}-1,\nu_2+1}} + \frac{\lambda_{2^{n_1},2^{n_2-1}}}{\lambda_{2^{n_1},\nu_2+1}}\right|$$

$$+ \left|\frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{2^{n_1}-1,2^{n_2}-1}} - \frac{\lambda_{2^{n_1},2^{n_2-1}}}{\lambda_{2^{n_1},2^{n_2}-1}} - \frac{\lambda_{2^{n_1-1},2^{n_2}}}{\lambda_{2^{n_1}-1,2^{n_2}}} + \frac{\lambda_{2^{n_1},2^{n_2}}}{\lambda_{2^{n_1},2^{n_2}}}\right|$$

$$\leqslant C\frac{\lambda_{2^{n_1-1},2^{n_2-1}}}{\lambda_{2^{n_1-1},2^{n_2-1}}} + \lambda_{2^{n_1-1},2^{n_2-1}} \sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2} \left|\frac{1}{\lambda_{\nu_1,2^{n_2}-1}} - \frac{1}{\lambda_{\nu_1+1,2^{n_2}-1}}\right|$$

$$+ \lambda_{2^{n_1-1},2^{n_2}} \sum_{\nu_1=2^{n_1-1}}^{2^{n_1}-2} \left|\frac{1}{\lambda_{\nu_1,2^{n_2}}} - \frac{1}{\lambda_{\nu_1+1,2^{n_2}}}\right|$$

$$+ \lambda_{2^{n_1-1},2^{n_2-1}} \sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-2} \left|\frac{1}{\lambda_{2^{n_1}-1,\nu_2}} - \frac{1}{\lambda_{2^{n_1}-1,\nu_2+1}}\right|$$

$$+ \lambda_{2^{n_1},2^{n_2-1}} \sum_{\nu_2=2^{n_2-1}}^{2^{n_2}-2} \left|\frac{1}{\lambda_{2^{n_1},\nu_2}} - \frac{1}{\lambda_{2^{n_1},\nu_2+1}}\right| \leqslant M.$$

Since the sequence $\{\Lambda_{n_1,n_2}\}_{n_1=1,n_2=1}^{\infty,\infty}$ satisfies the conditions of Lemma 3.2.6, then the series (3.4.11) is the Fourier series of a function $g(x_1, x_2) \in L_p$, and $\|g\|_p \leqslant C(p)\|f^{(\lambda,\beta_1,\beta_2)}\|_p$.
Applying Lemmas 1.5.6 and 3.2.5, we get

$$\lambda_{1,1}^\tau\|f\|_p^\tau + \sum_{\nu_1=1}^{\infty} |\lambda_{2^{\nu_1},1}^\tau - \lambda_{2^{\nu_1-1},1}^\tau| Y_{2^{\nu_1}-1,0}^\tau(f)_p + \sum_{\nu_2=1}^{\infty} |\lambda_{1,2^{\nu_2}}^\tau - \lambda_{1,2^{\nu_2-1}}^\tau| Y_{0,2^{\nu_2}-1}^\tau(f)_p$$

$$+ \sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty} |\lambda_{2^{\nu_1},2^{\nu_2}}^\tau - \lambda_{2^{\nu_1-1},2^{\nu_2}}^\tau - \lambda_{2^{\nu_1},2^{\nu_2-1}}^\tau + \lambda_{2^{\nu_1-1},2^{\nu_2-1}}^\tau| Y_{2^{\nu_1}-1,2^{\nu_2}-1}^\tau(f)_p$$

$$
\begin{aligned}
&\lesssim \lambda_{1,1}^{\tau}\|f\|_p^{\tau} + \sum_{\nu_1=1}^{\infty} |\lambda_{2^{\nu_1},1}^{\tau} - \lambda_{2^{\nu_1-1},1}^{\tau}| \Big(\int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{n_1=\nu_1+1}^{\infty} \Delta_{n_1,1}^2 \Big]^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} \\
&+ \sum_{\nu_2=1}^{\infty} |\lambda_{1,2^{\nu_2}}^{\tau} - \lambda_{1,2^{\nu_2-1}}^{\tau}| \Big(\int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{n_2=\nu_2+1}^{\infty} \Delta_{1,n_2}^2 \Big]^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} \\
&+ \sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty} |\lambda_{2^{\nu_1},2^{\nu_2}}^{\tau} - \lambda_{2^{\nu_1-1},2^{\nu_2}}^{\tau} - \lambda_{2^{\nu_1},2^{\nu_2-1}}^{\tau} + \lambda_{2^{\nu_1-1},2^{\nu_2-1}}^{\tau}| \\
&\times \Big(\int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{n_1=\nu_1+1}^{\infty}\sum_{n_2=\nu_2+1}^{\infty} \Delta_{n_1,n_2}^2 \Big]^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} \\
&\lesssim \lambda_{1,1}^{\tau}\|f\|_p^{\tau} + \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{\nu_1=1}^{\infty} |\lambda_{2^{\nu_1},1}^{\tau} - \lambda_{2^{\nu_1-1},1}^{\tau}| \Big[\sum_{n_1=\nu_1+1}^{\infty} \Delta_{n_1,1}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{p}{\tau}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} \\
&+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{\nu_2=1}^{\infty} |\lambda_{1,2^{\nu_2}}^{\tau} - \lambda_{1,2^{\nu_2-1}}^{\tau}| \Big[\sum_{n_2=\nu_2+1}^{\infty} \Delta_{1,n_2}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{p}{\tau}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} \\
&+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty} |\lambda_{2^{\nu_1},2^{\nu_2}}^{\tau} - \lambda_{2^{\nu_1-1},2^{\nu_2}}^{\tau} - \lambda_{2^{\nu_1},2^{\nu_2-1}}^{\tau} + \lambda_{2^{\nu_1-1},2^{\nu_2-1}}^{\tau}| \\
&\times \Big[\sum_{n_1=\nu_1+1}^{\infty}\sum_{n_2=\nu_2+1}^{\infty} \Delta_{n_1,n_2}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{p}{\tau}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} \\
&= \lambda_{1,1}^{\tau}\|f\|_p^{\tau} + \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \Big\{ \sum_{\nu_1=1}^{\infty} \Big[\sum_{n_1=\nu_1}^{\infty} |\lambda_{2^{\nu_1},1}^{\tau} - \lambda_{2^{\nu_1-1},1}^{\tau}|^{\frac{2}{\tau}} \Delta_{n_1+1,1}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} \\
&+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \Big\{ \sum_{\nu_2=1}^{\infty} \Big[\sum_{n_2=\nu_2}^{\infty} |\lambda_{1,2^{\nu_2}}^{\tau} - \lambda_{1,2^{\nu_2-1}}^{\tau}|^{\frac{2}{\tau}} \Delta_{1,n_2+1}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} \\
&+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{\nu_2=1}^{\infty} \Big\{ \Big\{ \sum_{\nu_1=1}^{\infty} \Big[\sum_{n_1=\nu_1}^{\infty} \Big\{ \sum_{n_2=\nu_2}^{\infty} |\lambda_{2^{\nu_1},2^{\nu_2}}^{\tau} - \lambda_{2^{\nu_1-1},2^{\nu_2}}^{\tau} \\
&- \lambda_{2^{\nu_1},2^{\nu_2-1}}^{\tau} + \lambda_{2^{\nu_1-1},2^{\nu_2-1}}^{\tau}|^{\frac{2}{\tau}} \Delta_{n_1+1,n_2+1}^2 \Big\} \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{\tau}{2}} \Big\}^{\frac{p}{\tau}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} = J_1.
\end{aligned}
$$

Using Lemmas 1.5.5 and 1.5.6 for $\tau \geqslant 2$, we obtain

$$J_1 \lesssim \lambda_{1,1}^\tau \|f\|_p^\tau + \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{n_1=1}^{\infty} \Big\{ \sum_{\nu_1=1}^{n_1} \Big[|\lambda_{2^{\nu_1},1}^\tau - \lambda_{2^{\nu_1-1},1}^\tau|^{\frac{2}{\tau}} \Delta_{n_1+1,1}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}$$

$$+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{n_2=1}^{\infty} \Big\{ \sum_{\nu_2=1}^{n_2} \Big[|\lambda_{1,2^{\nu_2}}^\tau - \lambda_{1,2^{\nu_2-1}}^\tau|^{\frac{2}{\tau}} \Delta_{1,n_2+1}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}$$

$$+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{\nu_2=1}^{\infty} \Big\{ \sum_{n_1=1}^{\infty} \Big\{ \sum_{\nu_1=1}^{n_1} \Big[\Big\{ \sum_{n_2=\nu_2}^{\infty} |\lambda_{2^{\nu_1},2^{\nu_2}}^\tau - \lambda_{2^{\nu_1-1},2^{\nu_2}}^\tau$$
$$- \lambda_{2^{\nu_1},2^{\nu_2-1}}^\tau + \lambda_{2^{\nu_1-1},2^{\nu_2-1}}^\tau|^{\frac{2}{\tau}} \Delta_{n_1+1,n_2+1}^2 \Big\}\Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{\tau}{2}} \Big\}^{\frac{p}{\tau}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}.$$

Note that the last term is equal to

$$\Big(\int_0^{2\pi}\int_0^{2\pi} \Big[\Big\{ \sum_{\nu_2=1}^{\infty} \Big\{ \sum_{n_1=1}^{\infty} \Big\{ \sum_{\nu_1=1}^{n_1} \Big[\sum_{n_2=\nu_2}^{\infty} |\lambda_{2^{\nu_1},2^{\nu_2}}^\tau - \lambda_{2^{\nu_1-1},2^{\nu_2}}^\tau$$
$$- \lambda_{2^{\nu_1},2^{\nu_2-1}}^\tau + \lambda_{2^{\nu_1-1},2^{\nu_2-1}}^\tau|^{\frac{2}{\tau}} \Delta_{n_1+1,n_2+1}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big]^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}$$
$$\lesssim \Big(\int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{n_1=1}^{\infty} \Big\{ \sum_{\nu_2=1}^{\infty} \Big\{\Big\{ \sum_{\nu_1=1}^{n_1} \Big[\sum_{n_2=\nu_2}^{\infty} |\lambda_{2^{\nu_1},2^{\nu_2}}^\tau - \lambda_{2^{\nu_1-1},2^{\nu_2}}^\tau$$
$$- \lambda_{2^{\nu_1},2^{\nu_2-1}}^\tau + \lambda_{2^{\nu_1-1},2^{\nu_2-1}}^\tau|^{\frac{2}{\tau}} \Delta_{n_1+1,n_2+1}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big]^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}.$$

Applying Minkowskii inequality, we derive the following estimate of the latter:

$$\Big(\int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{n_1=1}^{\infty} \Big\{ \sum_{\nu_2=1}^{\infty} \Big\{ \sum_{n_2=\nu_2}^{\infty} \Big\{ \sum_{\nu_1=1}^{n_1} \Big[|\lambda_{2^{\nu_1},2^{\nu_2}}^\tau - \lambda_{2^{\nu_1-1},2^{\nu_2}}^\tau$$
$$- \lambda_{2^{\nu_1},2^{\nu_2-1}}^\tau + \lambda_{2^{\nu_1-1},2^{\nu_2-1}}^\tau|^{\frac{2}{\tau}} \Delta_{n_1+1,n_2+1}^2 \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big]^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}.$$

Thus,

$$J_1 \lesssim \lambda_{1,1}^\tau \|f\|_p^\tau + \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{n_1=1}^{\infty} \Delta_{n_1+1,1}^2 \Big\{ \sum_{\nu_1=1}^{n_1} |\lambda_{2^{\nu_1},1}^\tau - \lambda_{2^{\nu_1-1},1}^\tau| \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=1}^{\infty}\Delta^2_{1,n_2+1}\Big\{\sum_{\nu_2=1}^{n_2}|\lambda^\tau_{1,2^{\nu_2}}-\lambda^\tau_{1,2^{\nu_2-1}}|\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=1}^{\infty}\Big\{\sum_{\nu_2=1}^{\infty}\Big\{\sum_{n_2=\nu_2}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big\{\sum_{\nu_1=1}^{n_1}|\lambda^\tau_{2^{\nu_1},2^{\nu_2}}-\lambda^\tau_{2^{\nu_1-1},2^{\nu_2}}$$

$$-\lambda^\tau_{2^{\nu_1},2^{\nu_2-1}}+\lambda^\tau_{2^{\nu_1-1},2^{\nu_2-1}}|\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{\tau}{2}}\Big\}^{\frac{2}{\tau}}\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}=J_2.$$

By Lemma 1.5.5 and taking into account (3.4.10), we have

$$J_2\lesssim\lambda^\tau_{1,1}\|f\|^\tau_p+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_1=1}^{\infty}\Delta^2_{n_1+1,1}\Big\{\sum_{\nu_1=1}^{n_1}|\lambda^\tau_{2^{\nu_1},1}-\lambda^\tau_{2^{\nu_1-1},1}|\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=1}^{\infty}\Delta^2_{1,n_2+1}\Big\{\sum_{\nu_2=1}^{n_2}|\lambda^\tau_{1,2^{\nu_2}}-\lambda^\tau_{1,2^{\nu_2-1}}|\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big\{\sum_{\nu_2=1}^{n_2}\sum_{\nu_1=1}^{n_1}|\lambda^\tau_{2^{\nu_1},2^{\nu_2}}-\lambda^\tau_{2^{\nu_1-1},2^{\nu_2}}$$

$$-\lambda^\tau_{2^{\nu_1},2^{\nu_2-1}}+\lambda^\tau_{2^{\nu_1-1},2^{\nu_2-1}}|\Big\}^{\frac{2}{\tau}}\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$\lesssim\lambda^\tau_{1,1}\|f\|^\tau_p+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_1=1}^{\infty}\Delta^2_{n_1+1,1}\lambda^2_{2^{n_1},1}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=1}^{\infty}\Delta^2_{1,n_2+1}\lambda^2_{1,2^{n_2}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Delta^2_{n_1+1,n_2+1}\lambda^2_{2^{n_1},2^{n_2}}\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}=J_3.$$

Using Lemma 3.2.5 (a), we obtain

$$J_3\lesssim\lambda^\tau_{1,1}\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Delta^2_{n_1,n_2}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_1=2}^{\infty}\Delta_{n_1,1}^2\lambda_{2^{n_1-1},1}^2\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=2}^{\infty}\Delta_{1,n_2}^2\lambda_{1,2^{n_2-1}}^2\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta_{n_1,n_2}^2\lambda_{2^{n_1-1},2^{n_2-1}}^2\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}.$$

Similarly to the estimates of J in the proof of Theorem 1.8.1, we derive

$$J_3\lesssim\lambda_{1,1}^{\tau}\Big[\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\Delta_{1,1}^2\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_1=2}^{\infty}\Delta_{n_1,1}^2\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=2}^{\infty}\Delta_{1,n_2}^2\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta_{n_1,n_2}^2\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}\Big]$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_1=2}^{\infty}\Delta_{n_1,1}^2\lambda_{2^{n_1-1},1}^2\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=2}^{\infty}\Delta_{1,n_2}^2\lambda_{1,2^{n_2-1}}^2\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=2}^{\infty}\sum_{n_2=2}^{\infty}\Delta_{n_1,n_2}^2\lambda_{2^{n_1-1},2^{n_2-1}}^2\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$\lesssim\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty}\Delta_{n_1,n_2}^2\lambda_{2^{n_1-1},2^{n_2-1}}^2\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}\lesssim\|g\|_p^{\tau}\lesssim\|f^{(\lambda,\beta_1,\beta_2)}\|_p^{\tau}.$$

Thus,

$$\|f^{(\lambda,\beta_1,\beta_2)}\|_p\gtrsim\Bigg\{\lambda_{1,1}^{\tau}\|f\|_p^{\tau}+\sum_{\nu_1=1}^{\infty}|\lambda_{2^{\nu_1},1}^{\tau}-\lambda_{2^{\nu_1-1},1}^{\tau}|Y_{2^{\nu_1}-1,0}^{\tau}(f)_p$$

$$+\sum_{\nu_2=1}^{\infty}|\lambda_{1,2^{\nu_2}}^{\tau}-\lambda_{1,2^{\nu_2-1}}^{\tau}|Y_{0,2^{\nu_2}-1}^{\tau}(f)_p$$

$$+\sum_{\nu_2=1}^{\infty}\sum_{\nu_1=1}^{\infty}|\lambda^{\tau}_{2^{\nu_1},2^{\nu_2}}-\lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2}}-\lambda^{\tau}_{2^{\nu_1},2^{\nu_2-1}}+\lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2-1}}|Y^{\tau}_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_p\Bigg\}^{\frac{1}{\tau}}.$$

Now we estimate $Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p$ from below. By Lemma 3.2.5 , we have

$$Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p\gtrsim\Big\{\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{k_1=m_1}^{\infty}\sum_{k_2=m_2}^{\infty}\lambda^2_{2^{k_1},2^{k_2}}\Delta^2_{k_1+1,k_2+1}\Big]^{\frac{p}{2}}dx_1dx_2\Big\}^{\frac{1}{p}}.$$

We consider the following series

$$\begin{aligned}J_4=\Big(&\lambda^{\tau}_{2^{m_1-1},2^{m_2-1}}Y^{\tau}_{2^{m_1}-1,2^{m_2}-1}(f)_p\\
&+\sum_{\nu_1=m_1}^{\infty}|\lambda^{\tau}_{2^{\nu_1},2^{m_2-1}}-\lambda^{\tau}_{2^{\nu_1-1},2^{m_2-1}}|Y^{\tau}_{2^{\nu_1}-1,2^{m_2}-1}(f)_p\\
&+\sum_{\nu_2=m_2}^{\infty}|\lambda^{\tau}_{2^{m_1-1},2^{\nu_2}}-\lambda^{\tau}_{2^{m_1-1},2^{\nu_2-1}}|Y^{\tau}_{2^{m_1}-1,2^{\nu_2}-1}(f)_p\\
&+\sum_{\nu_1=m_1}^{\infty}\sum_{\nu_2=m_2}^{\infty}|\lambda^{\tau}_{2^{\nu_1},2^{\nu_2}}-\lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2}}-\lambda^{\tau}_{2^{\nu_1},2^{\nu_2-1}}+\lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2-1}}|Y^{\tau}_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_p\Big)^{\frac{1}{\tau}}.\end{aligned}$$

Applying Lemmas 1.5.6 and 3.2.5, we get

$$\begin{aligned}J_4\lesssim\Big(&\lambda^{\tau}_{2^{m_1-1},2^{m_2-1}}\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=m_1}^{\infty}\sum_{n_2=m_2}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}\\
&+\sum_{\nu_1=m_1}^{\infty}|\lambda^{\tau}_{2^{\nu_1},2^{m_2-1}}-\lambda^{\tau}_{2^{\nu_1-1},2^{m_2-1}}|\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=\nu_1}^{\infty}\sum_{n_2=m_2}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}\\
&+\sum_{\nu_2=m_2}^{\infty}|\lambda^{\tau}_{2^{m_1-1},2^{\nu_2}}-\lambda^{\tau}_{2^{m_1-1},2^{\nu_2-1}}|\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=m_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}\\
&+\sum_{\nu_1=m_1}^{\infty}\sum_{\nu_2=m_2}^{\infty}|\lambda^{\tau}_{2^{\nu_1},2^{\nu_2}}-\lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2}}-\lambda^{\tau}_{2^{\nu_1},2^{\nu_2-1}}+\lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2-1}}|\end{aligned}$$

$$\times \Big(\int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{n_1=\nu_1}^{\infty} \sum_{n_2=\nu_2}^{\infty} \Delta^2_{n_1+1,n_2+1} \Big]^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} \Big)^{\frac{1}{\tau}}$$

$$\lesssim \Big(\lambda^{\tau}_{2^{m_1-1},2^{m_2-1}} \Big(\int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{n_1=m_1}^{\infty} \sum_{n_2=m_2}^{\infty} \Delta^2_{n_1+1,n_2+1} \Big]^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}$$

$$+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{\nu_1=m_1}^{\infty} |\lambda^{\tau}_{2^{\nu_1},2^{m_2-1}} - \lambda^{\tau}_{2^{\nu_1-1},2^{m_2-1}}| \Big[\sum_{n_1=\nu_1}^{\infty} \sum_{n_2=m_2}^{\infty} \Delta^2_{n_1+1,n_2+1} \Big]^{\frac{\tau}{2}} \Big\}^{\frac{p}{\tau}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}$$

$$+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{\nu_2=m_2}^{\infty} |\lambda^{\tau}_{2^{m_1-1},2^{\nu_2}} - \lambda^{\tau}_{2^{m_1-1},2^{\nu_2-1}}| \Big[\sum_{n_1=m_1}^{\infty} \sum_{n_2=\nu_2}^{\infty} \Delta^2_{n_1+1,n_2+1} \Big]^{\frac{\tau}{2}} \Big\}^{\frac{p}{\tau}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}$$

$$+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{\nu_1=m_1}^{\infty} \sum_{\nu_2=m_2}^{\infty} |\lambda^{\tau}_{2^{\nu_1},2^{\nu_2}} - \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2}} - \lambda^{\tau}_{2^{\nu_1},2^{\nu_2-1}} + \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2-1}}|$$

$$\times \Big[\sum_{n_1=\nu_1}^{\infty} \sum_{n_2=\nu_2}^{\infty} \Delta^2_{n_1+1,n_2+1} \Big]^{\frac{\tau}{2}} \Big\}^{\frac{p}{\tau}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} \Big)^{\frac{1}{\tau}}$$

$$= \Big(\lambda^{\tau}_{2^{m_1-1},2^{m_2-1}} \Big(\int_0^{2\pi}\int_0^{2\pi} \Big[\sum_{n_1=m_1}^{\infty} \sum_{n_2=m_2}^{\infty} \Delta^2_{n_1+1,n_2+1} \Big]^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}$$

$$+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \Big\{ \sum_{\nu_1=m_1}^{\infty} \Big[\sum_{n_1=\nu_1}^{\infty} \sum_{n_2=m_2}^{\infty}$$

$$|\lambda^{\tau}_{2^{\nu_1},2^{m_2-1}} - \lambda^{\tau}_{2^{\nu_1-1},2^{m_2-1}}|^{\frac{2}{\tau}} \Delta^2_{n_1+1,n_2+1} \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}$$

$$+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \Big\{ \sum_{\nu_2=m_2}^{\infty} \Big[\sum_{n_2=\nu_2}^{\infty} \sum_{n_1=m_1}^{\infty}$$

$$|\lambda^{\tau}_{2^{m_1-1},2^{\nu_2}} - \lambda^{\tau}_{2^{m_1-1},2^{\nu_2-1}}|^{\frac{2}{\tau}} \Delta^2_{n_1+1,n_2+1} \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{\tau}{p}}$$

$$+ \Big(\int_0^{2\pi}\int_0^{2\pi} \Big\{ \sum_{\nu_2=m_2}^{\infty} \Big\{ \Big\{ \sum_{\nu_1=m_1}^{\infty} \Big[\sum_{n_1=\nu_1}^{\infty} \sum_{n_2=\nu_2}^{\infty} |\lambda^{\tau}_{2^{\nu_1},2^{\nu_2}} - \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2}}$$

$$- \lambda^{\tau}_{2^{\nu_1},2^{\nu_2-1}} + \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2-1}}|^{\frac{2}{\tau}} \Delta^2_{n_1+1,n_2+1} \Big]^{\frac{\tau}{2}} \Big\}^{\frac{2}{\tau}} \Big\}^{\frac{\tau}{2}} \Big\}^{\frac{p}{\tau}} dx_1 dx_2 \Big)^{\frac{\tau}{p}} \Big)^{\frac{1}{\tau}}.$$

Further, using Minkowski's inequality and Lemma 1.5.6 for $\tau \geqslant 2$, we obtain

$$J_4 \lesssim \Big(\lambda^{\tau}_{2^{m_1-1},2^{m_2-1}}\Big(\int\limits_0^{2\pi}\int\limits_0^{2\pi}\Big[\sum_{n_1=m_1}^{\infty}\sum_{n_2=m_2}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int\limits_0^{2\pi}\int\limits_0^{2\pi}\Big\{\sum_{n_1=m_1}^{\infty}\Big\{\sum_{\nu_1=m_1}^{n_1}\Big[\sum_{n_2=m_2}^{\infty}$$

$$|\lambda^{\tau}_{2^{\nu_1},2^{m_2-1}}-\lambda^{\tau}_{2^{\nu_1-1},2^{m_2-1}}|^{\frac{2}{\tau}}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{\tau}{2}}\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int\limits_0^{2\pi}\int\limits_0^{2\pi}\Big\{\sum_{n_2=m_2}^{\infty}\Big\{\sum_{\nu_2=m_2}^{n_2}\Big[\sum_{n_1=m_1}^{\infty}$$

$$|\lambda^{\tau}_{2^{m_1-1},2^{\nu_2}}-\lambda^{\tau}_{2^{m_1-1},2^{\nu_2-1}}|^{\frac{2}{\tau}}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{\tau}{2}}\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int\limits_0^{2\pi}\int\limits_0^{2\pi}\Big\{\sum_{\nu_2=m_2}^{\infty}\Big\{\sum_{n_1=m_1}^{\infty}\Big\{\sum_{\nu_1=m_1}^{n_1}\Big[\sum_{n_2=\nu_2}^{\infty}|\lambda^{\tau}_{2^{\nu_1},2^{\nu_2}}-\lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2}}$$

$$-\lambda^{\tau}_{2^{\nu_1},2^{\nu_2-1}}+\lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2-1}}|^{\frac{2}{\tau}}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{\tau}{2}}\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{\tau}{2}}\Big\}^{\frac{p}{\tau}}dx_1dx_2\Big)^{\frac{\tau}{p}}\Big)^{\frac{1}{\tau}}$$

$$\lesssim \Big(\lambda^{\tau}_{2^{m_1-1},2^{m_2-1}}\Big(\int\limits_0^{2\pi}\int\limits_0^{2\pi}\Big[\sum_{n_1=m_1}^{\infty}\sum_{n_2=m_2}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int\limits_0^{2\pi}\int\limits_0^{2\pi}\Big\{\sum_{n_1=m_1}^{\infty}\sum_{n_2=m_2}^{\infty}\Delta^2_{n_1+1,n_2+1}$$

$$\Big\{\sum_{\nu_1=m_1}^{n_1}\Big[|\lambda^{\tau}_{2^{\nu_1},2^{m_2-1}}-\lambda^{\tau}_{2^{\nu_1-1},2^{m_2-1}}|^{\frac{2}{\tau}}\Big]^{\frac{\tau}{2}}\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int\limits_0^{2\pi}\int\limits_0^{2\pi}\Big\{\sum_{n_2=m_2}^{\infty}\sum_{n_1=m_1}^{\infty}$$

$$\Big\{\sum_{\nu_2=m_2}^{n_2}\Big[|\lambda^{\tau}_{2^{m_1-1},2^{\nu_2}}-\lambda^{\tau}_{2^{m_1-1},2^{\nu_2-1}}|^{\frac{2}{\tau}}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{\tau}{2}}\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int\limits_0^{2\pi}\int\limits_0^{2\pi}\Big\{\sum_{\nu_2=m_2}^{\infty}\Big\{\sum_{n_1=m_1}^{\infty}\sum_{n_2=\nu_2}^{\infty}\Big\{\sum_{\nu_1=m_1}^{n_1}\Big[|\lambda^{\tau}_{2^{\nu_1},2^{\nu_2}}-\lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2}}$$

$$- \lambda^{\tau}_{2^{\nu_1},2^{\nu_2-1}} + \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2-1}}|^{\frac{2}{\tau}}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{\tau}{2}}\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{\tau}{2}}\Big\}^{\frac{p}{\tau}}dx_1dx_2\Big)^{\frac{\tau}{p}}\Big)^{\frac{1}{\tau}}$$

$$\lesssim \Big(\lambda^{\tau}_{2^{m_1-1},2^{m_2-1}}\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=m_1}^{\infty}\sum_{n_2=m_2}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_1=m_1}^{\infty}\sum_{n_2=m_2}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big\{\sum_{\nu_1=m_1}^{n_1}|\lambda^{\tau}_{2^{\nu_1},2^{m_2-1}} - \lambda^{\tau}_{2^{\nu_1-1},2^{m_2-1}}|\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=m_2}^{\infty}\sum_{n_1=m_1}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big\{\sum_{\nu_2=m_2}^{n_2}|\lambda^{\tau}_{2^{m_1-1},2^{\nu_2}} - \lambda^{\tau}_{2^{m_1-1},2^{\nu_2-1}}|\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=m_2}^{\infty}\Big\{\sum_{\nu_2=m_2}^{n_2}\Big\{\sum_{n_1=m_1}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big\{\sum_{\nu_1=m_1}^{n_1}|\lambda^{\tau}_{2^{\nu_1},2^{\nu_2}} - \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2}}$$

$$- \lambda^{\tau}_{2^{\nu_1},2^{\nu_2-1}} + \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2-1}}|\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{\tau}{2}}\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}\Big)^{\frac{1}{\tau}}.$$

Applying Lemma 1.5.6 and additional conditions (3.4.10), we get

$$J_4 \lesssim \Big(\lambda^{\tau}_{2^{m_1-1},2^{m_2-1}}\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=m_1}^{\infty}\sum_{n_2=m_2}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_1=m_1}^{\infty}\sum_{n_2=m_2}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big\{\sum_{\nu_1=m_1}^{n_1}|\lambda^{\tau}_{2^{\nu_1},2^{m_2-1}} - \lambda^{\tau}_{2^{\nu_1-1},2^{m_2-1}}|\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=m_2}^{\infty}\sum_{n_1=m_1}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big\{\sum_{\nu_2=m_2}^{n_2}|\lambda^{\tau}_{2^{m_1-1},2^{\nu_2}} - \lambda^{\tau}_{2^{m_1-1},2^{\nu_2-1}}|\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=m_2}^{\infty}\sum_{n_1=m_1}^{\infty}\Delta^2_{n_1+1,n_2+1}\Big\{\sum_{\nu_2=m_2}^{n_2}\Big\{\Big\{\sum_{\nu_1=m_1}^{n_1}|\lambda^{\tau}_{2^{\nu_1},2^{\nu_2}} - \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2}}$$

$$- \lambda^{\tau}_{2^{\nu_1},2^{\nu_2-1}} + \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2-1}}|\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{\tau}{2}}\Big\}^{\frac{2}{\tau}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}\Big)^{\frac{1}{\tau}}$$

$$\lesssim \Big(\lambda^{\tau}_{2^{m_1-1},2^{m_2-1}}\Big(\int_0^{2\pi}\int_0^{2\pi}\Big[\sum_{n_1=m_1+1}^{\infty}\sum_{n_2=m_2+1}^{\infty}\Delta^2_{n_1,n_2}\Big]^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_1=m_1+1}^{\infty}\sum_{n_2=m_2+1}^{\infty}\Delta^2_{n_1,n_2}\lambda^2_{2^{n_1-1},2^{m_2-1}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=m_2+1}^{\infty}\sum_{n_1=m_1+1}^{\infty}\Delta^2_{n_1,n_2}\lambda^2_{2^{m_1-1},2^{n_2-1}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}$$

$$+\Big(\int_0^{2\pi}\int_0^{2\pi}\Big\{\sum_{n_2=m_2+1}^{\infty}\sum_{n_1=m_1+1}^{\infty}\Delta^2_{n_1,n_2}\lambda^2_{2^{n_1-1},2^{n_2-1}}\Big\}^{\frac{p}{2}}dx_1dx_2\Big)^{\frac{\tau}{p}}\Big)^{\frac{1}{\tau}}.$$

It is easy to see that, for any $n_1 \geqslant m_1$ and $n_2 \geqslant m_2$,

$$\lambda^\tau_{2^{m_1-1},2^{m_2-1}} = \sum_{\nu_2=m_2}^{n_2-1}(\lambda^\tau_{2^{m_1-1},2^{\nu_2-1}} - \lambda^\tau_{2^{m_1-1},2^{\nu_2}}) + \lambda^\tau_{2^{m_1-1},2^{n_2}}$$

$$\leqslant \sum_{\nu_2=m_2}^{n_2-1}|\lambda^\tau_{2^{m_1-1},2^{\nu_2-1}} - \lambda^\tau_{2^{m_1-1},2^{\nu_2}}| + \lambda^\tau_{2^{m_1-1},2^{n_2-1}}$$

$$\leqslant \sum_{\nu_2=m_2}^{n_2-1}|\lambda^\tau_{2^{m_1-1},2^{\nu_2-1}} - \lambda^\tau_{2^{m_1-1},2^{\nu_2}}| + \sum_{\nu_1=m_1}^{n_1-1}|\lambda^\tau_{2^{\nu_1-1},2^{n_2-1}} - \lambda^\tau_{2^{\nu_1},2^{n_2-1}}| + \lambda^\tau_{2^{n_1-1},2^{n_2-1}}$$

$$= \sum_{\nu_1=m_1}^{n_1-1}|\lambda^\tau_{2^{\nu_1-1},2^{n_2-1}} - \lambda^\tau_{2^{\nu_1},2^{n_2-1}}| + \lambda^\tau_{2^{n_1-1},2^{n_2-1}}$$

$$+ \sum_{\nu_2=m_2}^{n_2-1}|\sum_{\nu_1=m_1}^{n_1-1}\Big(\big(\lambda^\tau_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^\tau_{2^{\nu_1-1},2^{\nu_2}}\big) - \big(\lambda^\tau_{2^{\nu_1},2^{\nu_2-1}} - \lambda^\tau_{2^{\nu_1},2^{\nu_2}}\big)\Big)$$

$$+ \big(\lambda^\tau_{2^{n_1-1},2^{\nu_2-1}} - \lambda^\tau_{2^{n_1-1},2^{\nu_2}}\big)|$$

$$\leqslant \sum_{\nu_1=m_1}^{n_1-1}|\lambda^\tau_{2^{\nu_1-1},2^{n_2-1}} - \lambda^\tau_{2^{\nu_1},2^{n_2-1}}| + \lambda^\tau_{2^{n_1-1},2^{n_2-1}}$$

$$+ \sum_{\nu_2=m_2}^{n_2-1}|\sum_{\nu_1=m_1}^{n_1-1}\Big|\big(\lambda^\tau_{2^{\nu_1-1},2^{\nu_2-1}} - \lambda^\tau_{2^{\nu_1-1},2^{\nu_2}}\big) - \big(\lambda^\tau_{2^{\nu_1},2^{\nu_2-1}} - \lambda^\tau_{2^{\nu_1},2^{\nu_2}}\big)\Big|$$

$$+ \sum_{\nu_2=m_2}^{n_2-1}\Big|\lambda^\tau_{2^{n_1-1},2^{\nu_2-1}} - \lambda^\tau_{2^{n_1-1},2^{\nu_2}}\Big| \lesssim \lambda^\tau_{2^{n_1-1},2^{n_2-1}}.$$

Using this estimate, we derive that

$$J_4 \lesssim \Big(\int_0^{2\pi} \int_0^{2\pi} \Big\{ \sum_{n_2=m_2+1}^{\infty} \sum_{n_1=m_1+1}^{\infty} \Delta^2_{n_1,n_2} \lambda^2_{2^{n_1-1},2^{n_2-1}} \Big\}^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{1}{p}}$$

$$= \Big(\int_0^{2\pi} \int_0^{2\pi} \Big[\sum_{k_1=m_1}^{\infty} \sum_{k_2=m_2}^{\infty} \lambda^2_{2^{k_1},2^{k_2}} \Delta^2_{k_1+1,k_2+1} \Big]^{\frac{p}{2}} dx_1 dx_2 \Big)^{\frac{1}{p}}$$

$$\lesssim Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p.$$

Thus,

$$\Bigg\{ \lambda^{\tau}_{2^{m_1-1},2^{m_2-1}} Y^{\tau}_{2^{m_1}-1,2^{m_2}-1}(f)_p + \sum_{\nu_1=m_1}^{\infty} |\lambda^{\tau}_{2^{\nu_1},2^{m_2-1}} - \lambda^{\tau}_{2^{\nu_1-1},2^{m_2-1}}| Y^{\tau}_{2^{\nu_1}-1,2^{m_2}-1}(f)_p$$

$$+ \sum_{\nu_2=m_2}^{\infty} |\lambda^{\tau}_{2^{m_1-1},2^{\nu_2}} - \lambda^{\tau}_{2^{m_1-1},2^{\nu_2-1}}| Y^{\tau}_{2^{m_1}-1,2^{\nu_2}-1}(f)_p$$

$$+ \sum_{\nu_1=m_1}^{\infty} \sum_{\nu_2=m_2}^{\infty} |\lambda^{\tau}_{2^{\nu_1},2^{\nu_2}} - \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2}} - \lambda^{\tau}_{2^{\nu_1},2^{\nu_2-1}} + \lambda^{\tau}_{2^{\nu_1-1},2^{\nu_2-1}}| Y^{\tau}_{2^{\nu_1}-1,2^{\nu_2}-1}(f)_p \Bigg\}^{\frac{1}{\tau}}$$

$$\lesssim Y_{2^{m_1}-1,2^{m_2}-1}(f^{(\lambda,\beta_1,\beta_2)})_p,$$

completing the proof. □

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