Contextualización de la enseñanza de las matemáticas en las carreras tecnológicas

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RESUMEN

La enseñanza de las Matemáticas siempre ha sido un tema de gran preocupación a todos los niveles académicos y más especialmente en las carreras tecnológicas. Tanto los contenidos de las asignaturas matemáticas, como la forma de impartirlos han sido constantemente cuestionados. Actualmente los estudiantes tienen cada vez menos interés en las carreras tecnológicas y ha aumentado la tasa de abandono en estas carreras. Una de las causas principales es la falta de motivación que tienen los estudiantes hacia las asignaturas matemáticas en los primeros cursos, que viene derivada en la mayoría de los casos, de la falta de conciencia sobre la importancia de las Matemáticas para su futuro profesional. Asimismo, el conocimiento tecnológico es cada vez más especializado y fundamental para el desarrollo económico de las sociedades actuales, por lo tanto son más necesarios los especialistas en las disciplinas STEM (Science, Technology, Engineering, Mathematics) y por ello se espera un aumento en la demanda de ocupación de profesionales STEM. El hecho de que cada vez haya menos estudiantes interesados en las carreras tecnológicas y de que estos especialistas sean los más demandados, acentúa la preocupación de los educadores y profesionales implicados. El principal objetivo de esta tesis es aumentar la motivación e implicación del alumnado hacia las asignaturas matemáticas de las carreras tecnológicas a través de la contextualización de estas asignaturas en las diferentes disciplinas que componen sus carreras, de tal manera que los estudiantes tomen conciencia de la importancia de las Matemáticas para el desarrollo de su carrera profesional. De esta forma, el alumnado aumenta su interés hacia las Matemáticas, mejorando su rendimiento y disminuyendo así la tasa de fracaso escolar. Así pues, este trabajo tiene como objetivo mejorar la calidad de la enseñanza de las carreras tecnológicas, en las cuales se imparten los conocimientos necesarios para la educación en el desarrollo sostenible, y por consiguiente contribuye al cumplimiento de los ODS (Objetivos de Desarrollo Sostenible) planteados por la Unesco (Organización de las Naciones Unidas para la Educación, la Ciencia y la Cultura) en la Agenda 2030. Para conseguir estos objetivos se ha llevado a cabo la implementación de aplicaciones prácticas de las Matemáticas, relacionadas con las disciplinas que el alumnado estudiará en los próximos cursos de sus carreras. Para ello, en los cursos 2019/2020 y 2020/2021 se han impartido los seminarios “Aplicaciones de Matemáticas en Ingeniería I: Álgebra Lineal”, y “Aplicaciones de Matemáticas en la Ingeniería II: Cálculo Multivariable”, en la UPC (Universitat Politècnica de Catalunya-BarcelonaTech), basados en la enseñanza de estas asignaturas a través de la ejecución de problemas reales, donde los conceptos matemáticos que forman parte de esas asignaturas son necesarios para resolver los problemas. Para analizar el efecto que ha tenido la realización de estos seminarios en los estudiantes, al final de cada sesión se han realizado encuestas anónimas, en las que los estudiantes han valorado estas sesiones con respecto a la mejora en su motivación y al aprendizaje de las Matemáticas a través de las aplicaciones prácticas. Además, al final de cada seminario se han realizado entrevistas personales, en las cuales el alumnado ha podido expresar abiertamente su opinión sobre estas sesiones. Los resultados obtenidos confirman que la implementación de problemas matemáticos reales aumenta su motivación y mejora el aprendizaje de los conceptos matemáticos desarrollados, lo cual implica un aumento en su rendimiento y una disminución del abandono en los primeros cursos de las carreras tecnológicas, que conlleva un mayor interés por las carreras STEM, las cuales son fundamentales para el desarrollo económico de las sociedades actuales.

Palabras clave: Matemáticas, carreras tecnológicas, enseñanza, contextualización, STEM, motivación, ingeniería, aplicaciones

ABSTRACT

The teaching of Mathematics has been always a matter of great concern at all academic levels and more especially in technological degrees. Both the contents of mathematical subjects and the way they are taught, have been always questioned. Currently, students have less interest in technological degrees and the dropout rate has increased in first courses of these studies. One of the main reasons is the lack of motivation that students have towards mathematical subjects in these first academic years, which is derived in most cases, from the lack of awareness about the importance of Mathematics for their academic progress and for their future profession. Likewise, technological knowledge is becoming more specialized, as well as fundamental for economic development of present societies, thus experts in STEM (Science, Technology, Engineering, Mathematics) disciplines are more required and the occupation demand of STEM professionals is expected to increase. The fact that there are fewer students interested in technological degrees and that these specialists are the most demanded, accentuate the concern of educators and professionals involved. The main objective of this thesis is to increase students' motivation and involvement towards mathematical subjects in first academic years of technological degrees through the contextualization of these subjects in the different disciplines which compose their degrees, so that students became aware of the importance of Mathematics for the development of their degree and for their future career. In this way, students' interest towards Mathematics increase, growing their academic performance and decreasing the dropout failure rate in these degrees. So, this work has as a goal, the improvement of the quality education in technological degrees, in which the required knowledge for the education in sustainable development is taught, and therefore it contributes to the achievement of the SDG (Sustainable Development Goals) proposed by Unesco (United Nations Educational, Scientific and Cultural Organization) in the 2030 Agenda. To attain these objectives, the implementation of practical mathematical applications related to the disciplines that students will learn in next academic years, have been undertaken. To this end, in the 2019/2020 y 2020/2021 academic years, the seminars "Applications of Mathematics in Engineering I: Linear Algebra", and "Applications of Mathematics in Engineering II: Multivariable Calculus", in UPC (Universitat Politècnica de Catalunya-BarcelonaTech), based on the teaching of these subjects through the execution of real problems, where the mathematical concepts which are part of those subjects, are necessary to solve the problems. To analyze the effect that these seminars have caused on the students, anonymous surveys has been held, where students have valued these sessions regarding to their motivation improvement and to the learning of mathematical concepts through practical applications. Moreover, at the end of each seminar, personal interviews have been undertaken, in which students have been able to express straightforwardly their opinion about these sessions. The results obtained confirm that the implementation of real mathematical problems increases their motivation and improves the learning of the developed mathematical concepts, what implies a performance increase and a dropout rate decrease in first academic years of technological degrees. This entails a greater interest in STEM degrees, which are essential for economic development of present societies.

Keywords: Mathematics, technological degrees, teaching, contextualization, STEM, motivation, engineering, applications

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1. ESTADO DE LA CUESTIÓN Y OBJETIVO DE LA TESIS

1.1. Introducción

La enseñanza de las Matemáticas siempre ha sido una cuestión discutida en todos los niveles académicos, especialmente en las carreras tecnológicas, en las cuales las asignaturas matemáticas son la base para el aprendizaje de las diferentes disciplinas que constituyen estas carreras (Lusa Monforte, 1976). El valor de las Matemáticas en las carreras tecnológicas continúa siendo un problema importante y debería ser una preocupación fundamental en la elaboración de los programas de primer curso de las Ingenierías (Harris et al., 2015).

Actualmente los estudiantes universitarios cada vez tienen menos interés en las carreras tecnológicas (Joyce, 2014) y cada vez son más los casos de abandono en los primeros cursos de estas carreras (Ministerio de Universidades, 2021). Las causas de la falta de interés y de las crecientes tasas de fracaso escolar en estos estudios son principalmente debidas además de a la dificultad de las asignaturas matemáticas de los primeros cursos, a la falta de motivación del alumnado hacia estas asignaturas (Eichler & Gradwohl, 2021), ya que en la mayoría de los casos los estudiantes no ven la conexión entre lo que aprenden en Matemáticas con el resto de disciplinas tecnológicas y con los problemas ingenieriles reales (Blessner & Bolkas, 2021)(Pepin et al., 2021), debido a que en los programas de los primeros cursos de Ingeniería, las asignaturas matemáticas se enseñan como asignaturas separadas (Lepellere, 2021). Por todo ello, los estudiantes no son conscientes de la importancia y de la necesidad de las Matemáticas para la realización de sus carreras (López-Díaz & Peña, 2020), y por esta razón, las perciben más como un obstáculo para impedir la continuación de sus estudios, que como un elemento clave para su futuro.

Muchas universidades han mostrado su preocupación por estas altas tasas de fracaso observadas en las asignaturas matemáticas de los primeros cursos de las carreras tecnológicas (Sundre et al., 2012). Por ello, los educadores y profesionales implicados trabajan para identificar los factores que causan este hecho (Bradburn, 2003) así como para modificar los aspectos académicos necesarios para mejorar la calidad de la enseñanza y atraer el interés de los estudiantes a las carreras tecnológicas (Marra et al., 2012).

Por otro lado, cada vez son más requeridos los profesionales STEM (Caprile et al., 2015), ya que las sociedades actuales dependen cada vez más de su tecnología, siendo ésta un aspecto clave en el desarrollo económico de los países (Joyce, 2014). Por ello, se necesita que el conocimiento tecnológico sea más especializado y cada vez son más demandados los expertos en disciplinas STEM (Vennix et al., 2018).

Las altas tasas de abandono en las carreras STEM repercuten negativamente en el progreso de la sociedad (Stover et al., 2015). La disminución del interés de los estudiantes por las carreras STEM es un tema de preocupación en todo el mundo (Drymiotou et al., 2021). El informe de la OCDE (Organización para la Cooperación y el Desarrollo Económicos) de 2021, indica que sólo un 27,4% de estudiantes ingresaron en carreras STEM, que se considera relativamente bajo, dado que las disciplinas STEM tienen una gran importancia política, ya que los países tratan de mejorar la innovación tecnológica a través de la formación de ciudadanos con habilidades y competencias en estas disciplinas (Ministerio de Educación y Formación Profesional, 2021). Este porcentaje de estudiantes que acceden a carreras STEM es insuficiente para satisfacer la demanda de profesionales STEM, puesto que las sociedades se enfrentan a desafíos complejos, tanto en salud pública, como en el cambio climático y en el desarrollo sostenible, en los cuales se exige la presencia de este tipo de profesionales (Drymiotou et al., 2021).

Está demostrado que en la enseñanza de las Matemáticas, los estudiantes universitarios aprenden mejor los conceptos matemáticos si los aprenden en contexto, es decir, a través de aplicaciones prácticas, porque de esta forma se motivan y se involucran resolviendo problemas reales (Gasiewski et al., 2012)(Alsina, 2007)(Cárcamo et al., 2017)(Kandamby, 2018). El profesorado que incorpora aplicaciones reales consigue mejorar la comprensión y el aprendizaje del alumnado (Gasiewski et al., 2012). Por lo tanto, es imprescindible que en los primeros cursos universitarios, se muestre al alumnado que los conocimientos que están aprendiendo son esenciales para su futuro profesional, y también es necesario convencerlos de que los estudios que han elegido son fundamentales para la sociedad actual, consiguiendo de este modo, atraer, motivar, animar e implicar a los/as alumnos/as a obtener soluciones ingeniosas a problemas reales (Lopez, 2017). Además, es muy beneficioso para los estudiantes adquirir experiencias prácticas con relevancia en la vida real (Lepellere, 2021).

Asimismo, en las carreras tecnológicas se imparten los conocimientos necesarios para contribuir a la educación en el desarrollo sostenible, objetivo primordial de la Agenda 2030, establecido por la Unesco, que trabaja para asegurar que los estudiantes tengan las habilidades y conocimientos necesarios en el desarrollo sostenible y para avanzar en ciencia, tecnología e innovación en soluciones sostenibles que garanticen el futuro de la humanidad (Unesco, 2017). Por este motivo también es necesario fomentar la realización de las carreras STEM para formar a estudiantes competentes en desarrollo sostenible y contribuir de esta forma a conseguir los ODS en la Agenda 2030.

En este trabajo se estudia cómo mejorar la enseñanza de las asignaturas matemáticas a partir de la contextualización de estas asignaturas en las diferentes disciplinas que constituyen estos estudios, con el objetivo de conseguir aumentar la motivación del alumnado, disminuir el fracaso académico y atraer el interés del alumnado por estas carreras. De esta manera, se pretende conseguir un aumento del número de profesionales STEM y por tanto contribuir a la mejora de la economía de la sociedad actual, así como a impulsar la educación en el desarrollo sostenible.

1.2. Estado de la cuestión

1.2.1. La motivación del alumnado en las carreras tecnológicas

La motivación es uno de los factores más importantes en el rendimiento académico de los estudiantes universitarios de las carreras tecnológicas, en las cuales la dificultad es un elemento que dificulta la motivación, lo cual conlleva altos índices de fracaso escolar, sobre todo en los primeros cursos de estas carreras (López Fernández et al., 2014) y especialmente en las asignaturas matemáticas (Pepin et al., 2021). Además de la dificultad de estas asignaturas, la baja motivación se debe a que los estudiantes no son conscientes de la importancia de las Matemáticas, ya que éstas no están vinculadas con las asignaturas de los cursos posteriores (Czocher & Baker, 2010), ni con su futuro profesional.

Existen diferentes técnicas para motivar a los estudiantes en el aprendizaje de las Matemáticas y una de ellas es la de ayudarlos a ver las aplicaciones prácticas que tienen las Matemáticas en la vida real. Si los alumnos son conscientes de que las Matemáticas son útiles, aumentarán su interés por la asignatura (Maseda Fernández, 2011).

Varios estudios han constatado la importancia de la contextualización de las Matemáticas en las carreras tecnológicas para conseguir la motivación del alumnado. El trabajo "Matemática motivacional en el proceso de formación de ingenieros. Una perspectiva desde la Ingeniería Eléctrica" (Fernández et al., 2018), propone como método para aumentar la motivación del alumnado de estas carreras, el descubrimiento de aplicaciones, es decir, la modelización de problemas utilizando conceptos matemáticos.

En el artículo "Teaching Mathematics through concept motivation and action learning" (Abramovich et al., 2019), se propone que los estudiantes de carreras STEM alcancen las habilidades matemáticas esenciales para desarrollar las disciplinas STEM a través de aplicaciones matemáticas relacionadas con esas disciplinas. Las aplicaciones matemáticas de la vida real proporcionan una gran cantidad de estímulo según investigaciones realizadas en varias carreras tecnológicas (Abramovich et al., 2019).

Varios profesores de carreras tecnológicas han indicado que una perspectiva ingenieril puede mejorar tanto la motivación de los estudiantes como sus habilidades para aplicar sus conocimientos matemáticos en contextos ingenieriles (Varsavsky, 1995)(Lepellere, 2021). Asimismo, la contextualización de las Matemáticas puede ayudar a los estudiantes de Ingeniería a ser más conscientes de las múltiples formas de usar las Matemáticas en su carrera. Además, comprender cómo se utilizan los conceptos y el pensamiento matemáticos en aplicaciones profesionales contribuye a mejorar la motivación de los estudiantes, así como a contextualizar la enseñanza de las Matemáticas, y de este modo, determinar qué se debe enseñar a los estudiantes de estas carreras (Cardella, 2010).

1.2.2. Contextualización de las Matemáticas en las carreras tecnológicas

En este apartado se resume la situación de la contextualización de las Matemáticas en las carreras tecnológicas y se muestran algunos trabajos destacados sobre aplicaciones matemáticas realizados para mejorar el interés del alumnado por las Matemáticas.

Desde que se iniciaron las carreras de Ingeniería, se ha cuestionado qué y cómo deben enseñarse las Matemáticas a los/as ingenieros/as. Un seguimiento detallado de este debate en el marco español se recoge en (Lusa Monforte, 1976) y se resume en (Lusa Monforte, 2018). En dicho debate participaron personas tan destacadas como José Echegaray (1832-1916), matemático, ingeniero, político y dramaturgo español, divulgador de la Matemática creada en Europa y defensor del valor de las Matemáticas, no sólo por sus aplicaciones, sino como ciencia pura y abstracta (Echegaray, 1866).

A finales del siglo XIX, se genera un debate universal en torno a la educación técnica, que afecta directamente al papel de las Matemáticas en la formación de los ingenieros. Alois Riedler, profesor de la *Technische Hochschule* de Berlín, proclama un mayor entrelazamiento entre los aspectos teóricos y prácticos de las ingenierías, argumentando que en la enseñanza de las Matemáticas se prescinde de la realidad de las cosas. También el matemático inglés John Perry, se posiciona a favor de las Matemáticas prácticas y en contra las Matemáticas abstractas (Lusa Monforte, 2018).

El ingeniero de caminos Luís Gaztelu, catedrático de Cálculo Infinitesimal en la Escuela de Caminos, se inclina por las teorías de Perry y además critica que los tres grandes bloques de materias (Matemáticas puras, estudios científicos y las aplicaciones a la Ingeniería) se impartan sucesivamente y no simultáneamente, de forma que los estudiantes invierten más de tres años en conocer la verdadera finalidad de su carrera y de su profesión. En su obra "Las Matemáticas del ingeniero y su enseñanza" (1914), analiza las Matemáticas que son necesarias para el ingeniero y cómo deben enseñarse. Asimismo, el catedrático y director de la Escuela de Caminos, Vicente Machimbarrena, siguiendo las ideas de Riedler, proclama la desaparición de las Matemáticas abstractas del plan de estudios de las Ingenierías (Lusa Monforte, 2018), en contra de lo que defendía José Echegaray.

Una de las discusiones más sonadas sobre la enseñanza de las Matemáticas en la Ingeniería, fue la que mantuvieron los ingenieros industriales José Serrat Bonastre y José de Igual, partidarios del teoricismo y del practicismo, respectivamente (Lusa Monforte, 2018).

Como conclusión se constata que deben distinguirse dos aspectos fundamentales de las Matemáticas: por un lado son una herramienta, que dota a los/as ingenieros/as de un sentido práctico y resolutivo; y por otro, tienen un carácter formativo, que les confiere un pensamiento riguroso y la capacidad de abstracción, generalización y creación que necesitan los/as ingenieros/as (Lusa Monforte, 1976). Es por tanto indudable, la importancia que tienen las Matemáticas en la Ingeniería, sin embargo debe tenerse en cuenta que las Matemáticas para los/as ingenieros/as son un instrumento y no un fin, por lo tanto, el interés de los estudiantes de Ingeniería por las Matemáticas debe conseguirse a través de la conexión de los conceptos matemáticos con las disciplinas que constituyen su carrera (Lusa Monforte, 1976).

En este interés por la contextualización se destacan las aportaciones, ya en el siglo XX, de científicos tan notables como Pere Puig Adam (1900-1990), que alternó su labor como profesor de instituto y en la Escuela de Ingenieros, defendió la resolución de problemas como método para la enseñanza de las Matemáticas (Alsina, 2001) e intentó siempre contextualizar las Matemáticas tanto en sus clases como en sus libros (González Astudillo, 2008). El matemático riojano Julio Rey Pastor (1888-1963) afirmó que un matemático es quien resuelve problemas y no sólo quien conoce teorías (Español González, 1997). El ingeniero y matemático catalán Lluís Antoni Santaló (1911-2001), siempre procuró emplear aplicaciones de las Matemáticas relacionadas con la vida cotidiana y con los estudios realizados (Alsina, 2019).

También en otros países son muchos los profesionales, la mayoría de ellos ingenieros y/o matemáticos, dedicados a la educación, que han trabajado en la contextualización de las Matemáticas en las carreras tecnológicas. Destacando al mismo tiempo que muchas de las materias profesionales en la educación en Ingeniería requieren conocimientos de herramientas y conceptos matemáticos, y por consiguiente, el dominio de las Matemáticas es necesario para la comprensión más profunda de las disciplinas que componen las carreras tecnológicas (Tossavainen et al., 2019). Así, el matemático y profesor húngaro George Pólya (1887-1985) trataba de motivar a los alumnos a través de la resolución de problemas y ayudarlos a pensar por sí mismos, como plasmó en su libro de 1945 "*How to solve it*?" (Cómo plantear y resolver problemas). Afirmó que la única forma de aprender Matemáticas es resolviendo problemas (Alfaro, 2006). Hans Freudenthal (1905-1990), matemático y educador holandés, fue el fundador de la Matemática Realista, que consiste en presentar problemas en contextos de la vida diaria, de tal forma que los alumnos puedan imaginarse esas situaciones reales y a partir de ahí utilicen sus herramientas y los modelos matemáticos que les permitan resolverlos (Bressan, 2005). Y finalmente, el matemático portugués Paulo Abrantes, que aseguró que el alumnado desarrolla la competencia matemática a través de la experiencia de trabajar con problemas (Giménez et al., 2004).

Además, para esta tesis, se ha considerado importante mencionar la labor del profesor emérito de Matemáticas danés, Mogens Niss, que ya en el año 1991 definía la modelización matemática como "el arte de aplicar las Matemáticas a la vida real". Asimismo, Mogens Niss, en el danés KOM (*Competencies and the Learning of Mathematics*) Project (Niss, 2003), definió el concepto de competencia matemática como la capacidad de saber emplearlas y aplicarlas (Niss et al., 2017).

En el marco más cercano de la UPC, son numerosas las experiencias de contextualización en las últimas décadas. Por ejemplo, los trabajos del profesor de Matemáticas Joan Vicenç Gómez i Urgellés, entre los que destacan su labor por favorecer la creatividad y motivación de los estudiantes en relación a los contenidos matemáticos, poniendo a su alcance ejemplos reales con el objetivo de ilustrar la aplicabilidad de los conceptos que integran su formación, es decir, enseñar a los estudiantes a utilizar las técnicas matemáticas aprendidas en un contexto real (Gómez Urgellés, 2005), (Gómez Urgellés, 2018).

Mención especial merecen las aportaciones del catedrático de Matemáticas Claudi Alsina, firme partidario de la contextualización de las Matemáticas para conseguir la motivación de los estudiantes, tal y como refleja toda su obra (Alsina, 2007), quien además afirma que la competencia matemática consiste en tener la capacidad de describir modelos matemáticos correspondientes a problemas reales, poseer habilidades y técnicas para resolverlos y saber particularizar resultados matemáticos generales a casos concretos reales (Alsina, 2016).

En el caso más específico del Algebra Lineal se han realizado grandes trabajos y proyectos sobre contextualización, formados por aplicaciones y ejercicios prácticos de diferentes disciplinas que se resuelven con conceptos y herramientas del Álgebra Lineal. Un ejemplo de ello es el libro "Lecciones de Álgebra y Geometría", elaborado en el año 1984 por los matemáticos Enric Trillas y Claudi Alsina, en el cual presentan problemas y ejemplos sobre Arquitectura para motivar de esta forma al alumnado de esta carrera (Alsina & Trillas, 1984). Asimismo, los trabajos "*Learning Engineering to teach Mathematics*" (Ferrer et al., 2010) y "*Learning Automation to teach Mathematics*" (Ferrer et al., 2012) recopilan ejercicios prácticos sobre Electricidad y Automática, respectivamente, realizados junto a guías técnicas explicativas que sirven de ayuda tanto al alumnado, como al profesorado de Matemáticas que no esté especializado en estas disciplinas. Otro de los trabajos es "*Conservative continuous flows*", que consta de aplicaciones de varios conceptos del Álgebra Lineal en algunas de las disciplinas tecnológicas que componen los Grados de Ingeniería (Peña et al., 2018). Además, cabe destacar, el trabajo "*Applying dynamical discrete systems to teach Mathematics*", compuesto por ejemplos de sistemas dinámicos discretos que se resuelven con el uso de conceptos básicos del Álgebra Lineal (Peña, 2018). En "*Mathematical modelling in Engineering: a proposal to introduce Linear Algebra concepts*", se desarrollan ejemplos de modelización en Ingeniería que ilustran los conceptos de sistema generador y espacio generado del Álgebra Lineal (Cárcamo et al., 2017). El proyecto "Utilización del Álgebra Lineal para la clasificación por rangos" emplea conocimientos del Álgebra Lineal para estudiar, analizar y comprender el funcionamiento y la organización interna del buscador de Google (Morán, 2016). Y por último, en el trabajo "Álgebra Lineal en la educación para el desarrollo sostenible", se muestran diversos proyectos que se resuelven utilizando conceptos del Álgebra Lineal y en los que se desarrollan diversos aspectos de los ODS (García Planas et al., 2018).

Asimismo, se han realizado trabajos sobre contextualización del Cálculo en carreras tecnológicas, como el proyecto "*Using an integrated Engineering curriculum to improve freshman Calculus*", que trata de integrar la asignatura de Cálculo con las disciplinas tecnológicas de los posteriores cursos, de manera que los estudiantes aprendan aplicaciones prácticas y así mejoren su motivación y el aprendizaje de los conceptos de cálculo (Barrow & Fulling, 1998); en el trabajo "*Teach multivariable functions through applications and GeoGebra*" se relacionan los conceptos sobre funciones multivariable tanto con otras disciplinas del grado de Ingeniería Civil como con situaciones ingenieriles reales, con el propósito de que los estudiantes profundicen en el conocimiento de estos conceptos matemáticos y de estimular su creatividad (Lepellere, 2021).

Por otro lado, existen actualmente comunidades de educadores que estudian y trabajan en la contextualización de las Matemáticas, de las cuales se destacan en esta tesis algunas de las más importantes.

El ICTMA (*International Community of Teachers of Mathematical Modelling and Applications*), que desde el año 1983 organiza conferencias sobre aplicaciones y modelización matemáticas a todos los niveles educativos. La próxima conferencia tendrá lugar entre los días 24 y 27 de septiembre del presente año en la ciudad alemana de Würzburg. El ICTMA es desde el año 2003 un "*Affiliated Study Group*" de la comisión internacional ICMI (*International Commission on Mathematical Instruction*), fundada en 1908, cuyo objetivo es fomentar la reflexión, colaboración e intercambio y difusión de ideas en la enseñanza y aprendizaje de las Matemáticas a todos los niveles académicos. El programa ICMI *Studies*, que pertenece al ICMI y comenzó a mediados de los 80, contribuye a la mejora en la investigación multidisciplinar de la educación matemática.

MatRIC (*Centre for Research, Innovation and Coordination of Mathematics Teaching - Centre for Excellence in Education*), es una comunidad de aprendizaje, que pertenece a la Universidad de Agder (Noruega) y que trabaja por la enseñanza de las matemáticas en universidades y colegios universitarios noruegos. MatRIC aborda un área de prioridad nacional para la ciencia y la tecnología, establecida por el Ministerio de Educación e Investigación en el año 2010: las Matemáticas son un requisito en los marcos nacionales noruegos para la educación de ingenieros, economistas, así como para los estudios de ciencias naturales y salud, sin embargo, los resultados de los estudiantes noruegos revelan datos decepcionantes en la educación superior. MatRIC apoya la enseñanza y el aprendizaje efectivo de las Matemáticas con el objetivo de conseguir que los estudiantes estén motivados, disfruten de las Matemáticas y aprecien su relevancia; que comprendan los conceptos matemáticos de forma que puedan luego aplicarlos para resolver situaciones de problemas no rutinarios; que los/as alumnos/as posean competencias en modelización matemática y en la aplicación de las Matemáticas en su futuro profesional; que el profesorado sea competente en Matemáticas y comprenda cómo se utilizan en una variedad de situaciones comerciales, científicas, industriales y financieras.

1.3. Objetivo de la tesis

El objetivo principal de esta tesis es conseguir aumentar el interés del alumnado por las profesiones STEM, mejorando la motivación por las asignaturas matemáticas y fomentando la implicación del alumnado en las carreras tecnológicas, contribuyendo así a aumentar el rendimiento académico y a disminuir la tasa de abandono en estas carreras.

Para conseguirlo, en este trabajo se ha realizado la contextualización de las asignaturas matemáticas en las posteriores disciplinas que constituyen estos estudios, que consiste en la elaboración de diversas aplicaciones prácticas y problemas reales en los cuales son necesarios los conceptos y técnicas desarrolladas en las asignaturas Álgebra Lineal y Cálculo Multivariable para poder resolverlos. Estas aplicaciones son adecuadas al nivel académico del alumnado del primer curso de Ingeniería Industrial y se han desarrollado junto a guías técnicas donde se explican los temas tratados en cada una de las disciplinas tecnológicas. Estas guías técnicas sirven de apoyo para la comprensión, realización y aprendizaje tanto del alumnado que durante el primer curso aún no ha cursado estas disciplinas, como para el profesorado que en un futuro imparta estas asignaturas matemáticas y no sea especialista en las disciplinas tecnológicas tratadas en las aplicaciones.

Esta tesis, además tiene como objetivo ayudar a conseguir los ODS marcados en la Agenda 2030, contribuyendo a la mejora de la educación en las carreras tecnológicas y por consiguiente a garantizar una educación de calidad, en la que los estudiantes adquieran los conocimientos teóricos y prácticos para promover el desarrollo sostenible.

Asimismo, este trabajo quiere contribuir a la mejora del crecimiento económico de la sociedad actual, intentando conseguir para ello, aumentar el interés de los estudiantes en las carreras STEM, las cuales son fundamentales para el desarrollo tecnológico de los países y por tanto para su economía.

Esta tesis pretende aportar una nueva información sobre la experiencia educativa de los estudiantes de los primeros cursos de Ingeniería con respecto a las asignaturas de Álgebra Lineal y Cálculo Multivariable.

Por último, con el material elaborado en este trabajo, se intenta contribuir al desarrollo de un nuevo currículum de las asignaturas matemáticas de los primeros cursos de las carreras tecnológicas, en la cual estén integradas las disciplinas STEM.

2. METODOLOGÍA Y ESTRUCTURA DE LA TESIS

2.1. Metodología

Esta tesis se basa en el seminario “Contextualización de las Matemáticas en las carreras tecnológicas”, que se inauguró en el curso académico 2017/2018 y cuyo objetivo es la presentación de aplicaciones y ejemplos prácticos y reales en las asignaturas matemáticas de los primeros cursos de las carreras tecnológicas. En este seminario ha participado profesorado de las diversas disciplinas involucradas.

El curso 2019/2020, se llevó a cabo la implementación de estas aplicaciones prácticas y reales, dirigidas al alumnado de primer curso de grado de las carreras tecnológicas, iniciándose en la ETSEIB (*Escola Tècnica Superior d’Enginyers Industrials de Barcelona*) de la UPC, el seminario “Aplicaciones de Matemáticas en Ingeniería”, que consta de dos partes: “Aplicaciones de Matemáticas en Ingeniería I: Álgebra Lineal” y “Aplicaciones de Matemáticas en Ingeniería II: Cálculo Multivariable”, las cuales se imparten en el primer y segundo semestre del primer curso académico, respectivamente. Estos seminarios, que se han impartido durante los cursos 2019/2020 y 2020/2021, constan de diversos ejemplos y aplicaciones prácticas de los contenidos de estas asignaturas matemáticas en las diferentes disciplinas tecnológicas (Electricidad, Mecánica, Automática, etc.) de estas carreras (Ingenierías, Informática, Arquitectura, etc.).

En cada una de las sesiones de los seminarios de Álgebra Lineal y de Cálculo Multivariable que se han llevado a cabo en los cursos académicos 2019/2020 y

2020/2021, se han realizado encuestas anónimas con el objetivo de analizar los resultados que han producido estas sesiones sobre el alumnado. En estas encuestas el alumnado ha valorado los contenidos tanto matemáticos como ingenieriles, las aplicaciones tecnológicas, y ha expresado su grado de mejora de la motivación y del aprendizaje de los conceptos matemáticos a través de las aplicaciones tecnológicas. Al final de las encuestas, el alumnado tenía la posibilidad de añadir comentarios sobre las sesiones de este seminario, que han contribuido a analizar los resultados obtenidos. En el Anexo A, se incluye el formulario de estas encuestas.

Todos los estudiantes asistentes a los seminarios de Álgebra Lineal y Cálculo Multivariable impartidos en los cursos académicos 2019/2020 y 2020/2021 participaron en las encuestas y respondieron a las preguntas formuladas, valorando en una escala de 1 a 5 (1 = completamente en desacuerdo, 2 = en desacuerdo, 3 = ni de acuerdo ni en desacuerdo, 4 = de acuerdo, 5 = completamente de acuerdo).

Con el propósito de recabar más información acerca de la influencia de las sesiones en el alumnado, al final de cada seminario se han realizado entrevistas personales a cada alumno/a, en las cuales han podido expresar con detalle su opinión sobre estas sesiones. Asimismo, se ha podido conocer las aplicaciones que les han parecido más interesantes a los/as alumnos/as, así como las que les han ayudado más a comprender los conceptos matemáticos asociados y aquéllas que más les han sorprendido. Además, los estudiantes han expresado cómo ha mejorado su motivación e interés por las asignaturas matemáticas después de la realización de estos seminarios. En el anexo B, se muestran las preguntas realizadas al alumnado asistente a los seminarios.

2.1.1. Seminario “Contextualización de las Matemáticas en las carreras tecnológicas”

Este seminario consta de sesiones semanales de una hora y media y está dirigido a profesores. Se ha llevado a cabo en la UPC, está promovido por el *Vicerectorat de Política Acadèmica* de la UPC y se organiza con la colaboración del Departament de Matemàtiques, del *Departament de Física*, de la FME (*Facultat de Matemàtiques i Estadística*) y del ICE (*Institut de Ciències de l’Educació*).

Las sesiones que componen este seminario, constan de aplicaciones sobre diferentes disciplinas tecnológicas: Automática, Electricidad, Mecánica, Electrónica, etc. En cada una de estas disciplinas, se explican casos prácticos utilizando los conceptos matemáticos necesarios para resolverlos: matrices, números complejos, sistemas de ecuaciones lineales, etc. Estas sesiones han sido impartidas por profesorado tanto del departamento de Matemáticas como de departamentos de las diferentes áreas tecnológicas e ingenieriles de la universidad.

En la Tabla 1 se detallan las sesiones de que consta el seminario de "Contextualización de las Matemáticas en las carreras tecnológicas", así como el profesorado que las ha impartido y la fecha en que han tenido lugar.

Tabla 1. Seminario "Contextualización de las Matemáticas en las carreras tecnológicas"

Contextualització de les Matemàtiques en les carreres tecnològiques			
Sesión	***Título***	***Ponente***	***Fecha***
1	Conferència inaugural curs 2017/2018: "Invitació a la renovació educativa de las Matemàtiques en carreres tècniques"	**Claudi Alsina** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	10/04/2018
2	"Fluxos en xarxes"	**Josep Ferrer** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	25/04/2018
3	"Engagement amb els estudiants de primer d'Enginyeria Civil"	**M. Rosa Estela** (Departament d'Enginyeria Civil i Ambiental, Universitat Politècnica de Catalunya)	15/05/2018
4	"Les matemàtiques de Google"	**Rafael Bru** (Departament de Matemàtica Aplicada, Universitat Politècnica de València)	23/05/2018
5	"Numerical Factory: un tast numèric sobre l'ensenyament de les matemàtiques a les enginyeries"	**Antonio Susín** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	05/06/2018
6	"Com les eines matemàtiques ajuden a fabricar peces. Casos pràctics"	**Antonio Travieso** (Departament d'Enginyeria Mecànica, Universitat Politècnica de Catalunya)	03/10/2018
7	"Una proposta pera a l'ensenyament de les matemàtiques a enginyeria Informàtica"	**Joan Vicenç Gómez** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	16/10/2018
8	"Aplicacions Matemàtiques a l'Elasticitat i Resistència de Materials"	**Miquel Casafont, Miquel Ferrer i M. Magdalena Pastor** (Departament de Resistència de Materials i Estructures a l'Enginyeria, Universitat Politècnica de Catalunya)	07/11/2018
9	Sessió inaugural curs 2018/2019: "Una relación históricamente problemática: las Matemáticas en las ingenierías"	**Guillermo Lusa** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	27/11/2018
10	"Aplicacions de Realitat Virtual per a Enginyeria Biomèdica"	**Jordi Torner** (Departament d'Expressió Gràfica a l'Enginyeria, Universitat Politècnica de Catalunya)	27/02/2019
11	"Conceptes i eines matemàtiques fonamentals en tecnologia electrònica"	**Sergi Busquets** (Departament d'Enginyeria Electrònica, Universitat Politècnica de Catalunya)	21/03/2019
12	"Modelització amb sistemes d'Equacions Diferencials Ordinàries lineals"	**Rafael Ramírez** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	10/04/2019
13	"Sistemes lineals determinats per valors consecutius dels estats"	**Josep Ferrer** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	02/05/2019
14	"Conceptes i eines matemàtiques en Automàtica"	**Roberto Griñó** (Departament d'Enginyeria de Sistemes, Automàtica i Informàtica Industrial, Universitat Politècnica de Catalunya)	22/05/2019
15	"Matemàtiques animades"	**Josefina Antonijuan i Jordi Guàrdia** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	16/10/2019

16	“Probabilitats i Teoria de la Comunicació: Codificació, Caminades Aleatòries en Grafs i Algorismes”	**Josep Fàbrega** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	02/12/2020
17	“Criptografia: L’aritmètica dels nombres grans”	**Anna Rio** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	17/03/2021
18	“Les matemàtiques al servei de les actituds en enginyeria”	**Jaume Fabregat** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	04/05/2021
19	“Intel·ligència Artificial: on som, on anem i on podríem arribar?”	**Tetiana Klymchuk** (Departament de Matemàtiques, Universitat Politècnica de Catalunya)	17/11/2021

El número de profesores/as que han asistido a este seminario es de una media de 30 profesores/as por cada sesión. Los asistentes, que pertenecen tanto al departamento de Matemáticas como a otros departamentos de la universidad, han sido encuestados con el objetivo de analizar su opinión y valoración sobre el seminario. Los participantes en estas encuestas han sido un 55% de los asistentes y los resultados obtenidos han sido muy positivos, tanto en la valoración de los aspectos académicos como organizativos de este seminario.

El profesorado asistente a estas sesiones ha destacado que se han tratado aplicaciones innovadoras, ejemplos con aplicaciones en diferentes disciplinas, así como interesantes trabajos relacionados con problemas sociales.

2.1.2. Seminario “Aplicaciones de Matemáticas en Ingeniería”

Dado el éxito obtenido con el seminario dirigido al profesorado y con el objetivo de poner en práctica el material extraído de dicho seminario, durante el curso 2019/2020 se iniciaron los seminarios “Aplicaciones de Matemáticas en Ingeniería I: Álgebra Lineal” y “Aplicaciones de Matemáticas en Ingeniería II: Cálculo Multivariable”. Estos seminarios constan de sesiones de noventa minutos semanales dirigidas al alumnado de primer curso del Grado en Ingeniería en Tecnologías industriales, que se imparten durante el primer y segundo semestre, respectivamente. Se trata de sesiones voluntarias que la UPC reconoce con 1 ECTS (*European Credit Transfer and Accumulation System*) para cada semestre de asistencia al seminario.

En estas sesiones se han desarrollado aplicaciones y ejemplos prácticos de las diferentes disciplinas tecnológicas a partir de conceptos matemáticos de las asignaturas Álgebra Lineal y Cálculo Multivariable.

El objetivo de estas sesiones es aumentar la motivación e implicación del alumnado hacia las Matemáticas a través del conocimiento y desarrollo de aplicaciones prácticas relacionadas con las disciplinas tecnológicas que conforman sus estudios.

Aplicaciones de Matemáticas en Ingeniería I: Álgebra Lineal

El seminario "Aplicaciones de Matemáticas en Ingeniería I: Álgebra Lineal" se realizó durante el primer semestre del curso académico, coincidiendo con la impartición de la asignatura Álgebra Lineal del primer curso del Grado en Ingeniería en Tecnologías Industriales de Ingeniería Industrial. El temario de esta asignatura es el siguiente:

1. Estructuras algebraicas
2. Espacios vectoriales y aplicaciones lineales.
3. Reducción de aplicaciones lineales.
4. Aplicación del Álgebra Lineal a la resolución de sistemas lineales discretos.

Las sesiones que componen este seminario se indican a continuación en la Tabla 2. Estas sesiones constan de varias aplicaciones de los temas de la asignatura Álgebra Lineal.

Tabla 2. Seminario "Aplicaciones de Matemáticas en Ingeniería I: Álgebra Lineal"

Aplicacions de Matemàtiques en Enginyeria I: Àlgebra Lineal	
Sesión	***Título***
1	"Els nombres complexos en l'estudi de les oscil·lacions de preus"
2	"Els nombres complexos en l'estudi de corrents alterns"
3	"Sistemes indeterminats: variables de control"
4	"Sistemes indeterminats homogenis: les solucions són un subespai vectorial"
5	"Suma i intersecció de subespais vectorials en sistemes dinàmics discrets"
6	"Aplicacions lineals i la seva matriu"
7	"Modelització amb sistemes d'Equacions Diferencials Ordinàries lineals"
8	"Canvis de base"
9	"Valors propis, vectors propis i diagonalització a l'enginyeria"
10	"Matriu de connectivitat, índex d'accessibilitat de Gould"

En el Anexo C pueden consultarse las presentaciones que se realizaron en cada una de estas sesiones. Las cuales constan además de las aplicaciones prácticas en las diferentes disciplinas tecnológicas, de guías técnicas explicativas sobre los conceptos matemáticos asociados y sobre las disciplinas tratadas.

Aplicaciones de Matemáticas en Ingeniería II: Cálculo Multivariable

El seminario “Aplicaciones de Matemáticas en Ingeniería II: Cálculo Multivariable” se realizó durante el segundo semestre del curso académico, coincidiendo con la impartición de la asignatura Cálculo Multivariable del primer curso del grado en Ingeniería en Tecnologías Industriales. El temario de esta asignatura es el siguiente:

1. Continuidad y derivabilidad de funciones de varias variables.
2. Integración de funciones de varias variables.
3. Transformada de Laplace y series de Fourier.

Las sesiones que componen este seminario se indican a continuación en la Tabla 3. Estas sesiones constan de aplicaciones de algunos de los temas de la asignatura Cálculo Multivariable. Aunque el tema de fenómenos discontinuos no forma parte actualmente del programa de la asignatura, se ha decidido presentar algunas aplicaciones sobre dicho tema, ya que los fenómenos discontinuos son muy frecuentes en Ingeniería y están relacionados con la mayoría de las disciplinas tecnológicas, y por tanto ilustran significativamente la necesidad de las Matemáticas para la resolución de problemas ingenieriles reales.

Tabla 3. Seminario “Aplicaciones de Matemáticas en Ingeniería II: Cálculo Multivariable”

Aplicacions de Matemàtiques en Enginyeria II: Càlcul Multivariable	
Sesión	***Título***
1	“Fenòmens discontinus: histèresi, càustiques, ...”
2	“Les catàstrofes de Thom”
3	“Desenvolupaments de Taylor i de Fourier”
4	“Cadena, inversa, implícita”
5	“Cinètica inversa”
6	“Cinemàtica de mecanismes amb enllaços”
7	“Optimització”
8	“Miscel·lània”

En el Anexo D pueden consultarse las presentaciones que se explicaron en cada una de estas sesiones. Las cuales constan además de las aplicaciones prácticas en las diferentes disciplinas tecnológicas, de guías técnicas explicativas sobre los conceptos matemáticos asociados y sobre las disciplinas tratadas.

2.1.3. Análisis de la motivación e implicación del alumnado

A partir de los resultados obtenidos en las encuestas y entrevistas, se ha analizado cómo ha influido la realización de aplicaciones prácticas en el alumnado asistente al seminario.

Para hacer el estudio se ha tenido en cuenta el cuestionario SMTSL (*Students' Motivation Toward Science Learning*), en el cual se valoran seis factores de motivación, que son, la autoeficacia, las estrategias del aprendizaje activo, el valor del aprendizaje de las ciencias, el objetivo del rendimiento, el objetivo del logro y la estimulación del entorno de aprendizaje (Tuan et al., 2005). De estos factores se han seleccionado dos de ellos, ya que son los que tienen en cuenta la implementación de aplicaciones prácticas en el aprendizaje de las ciencias:

1. **El valor del aprendizaje de las ciencias**, que consiste en permitir a los estudiantes adquirir la competencia de la resolución de problemas, experimentar en actividades de investigación, estimular su propio pensamiento, y encontrar la importancia de las ciencias en la vida diaria. Si el alumnado percibe estos valores esenciales, se motivarán para aprender ciencias (Tuan et al., 2005).

2. **El objetivo de logro**, es decir, que los estudiantes sientan satisfacción al aumentar sus competencias y logros en el aprendizaje de las ciencias (Tuan et al., 2005).

Las respuestas a las preguntas planteadas en las encuestas realizadas al alumnado han permitido analizar estos dos factores, ya que el alumnado ha valorado los contenidos matemáticos e ingenieriles de las aplicaciones, así como el conocimiento de aplicaciones tecnológicas de los diferentes conceptos matemáticos. Además, los estudiantes han valorado no sólo el aumento de su motivación con el aprendizaje de los conceptos matemáticos a través de aplicaciones, sino también la mejora del aprendizaje de esos conceptos matemáticos a través de las aplicaciones tecnológicas reales.

Las respuestas del alumnado han sido muy positivas, siendo la valoración media para los dos seminarios, superior al 80% de alumnado satisfecho con los aspectos cuestionados. Por lo tanto, se considera que los dos factores de motivación seleccionados del cuestionario SMTSL, muestran resultados positivos, que permiten inferir, que la implementación de aplicaciones realizadas en los seminarios llevados a cabo para este estudio, mejora la motivación del alumnado que las ha realizado.

Además, las opiniones expresadas en estas encuestas se han reforzado con las respuestas a algunas de las preguntas realizadas en las entrevistas sobre la influencia de las sesiones en la motivación y el interés de los estudiantes hacia las Matemáticas, así como la visión sobre la importancia de estas asignaturas después de haber realizado los seminarios. Asimismo, los estudiantes han expresado cómo ha influido la realización del seminario en el desarrollo de las asignaturas Álgebra Lineal y Cálculo Multivariable, las cuales han cursado en el mismo período lectivo.

2.2. Estructura de la tesis

Esta tesis se presenta como compendio de publicaciones y consta de los siguientes apartados:

1) En el primer apartado se realiza una introducción del trabajo. Asimismo, se explica la situación actual de las carreras tecnológicas y se exponen algunos de los trabajos más importantes sobre contextualización de las Matemáticas en estas carreras realizados hasta ahora. Finalmente se indican cuáles son los objetivos de esta tesis.

2) En este punto, se detallan las fases de la metodología de trabajo llevada a cabo en la tesis y se indica cómo está estructurada la memoria.

3) En este apartado se incluyen los tres artículos que componen esta tesis.

   3.1) A partir del material desarrollado en la primera parte del seminario "Aplicaciones de Matemáticas en Ingeniería I: Álgebra Lineal", se realizó el primer artículo sobre la formación de las Matemáticas en los Grados de Ingeniería, centrado en la asignatura de primer curso de Ingeniería Industrial, Álgebra Lineal. Con el objetivo de analizar la motivación de los estudiantes y la valoración del seminario se realizaron encuestas anónimas y entrevistas personales, cuyos resultados se muestran y analizan en este artículo (Publicación 1: "*Mathematics training in Engineering degrees: an intervention from teaching staff to students*").

   3.2) A continuación, se desarrolló el segundo artículo, basado en la segunda parte del seminario, "Aplicaciones de las Matemáticas en Ingeniería II: Cálculo Multivariable", en el cual se han desarrollado problemas reales donde los conceptos de la asignatura Cálculo Multivariable de primer curso de Ingeniería Industrial son necesarios para resolverlos. Asimismo se realizaron encuestas y entrevistas para analizar el impacto de este seminario en el alumnado asistente, cuyos resultados se muestran en este segundo artículo, presentando una propuesta para la mejora del currículum de Cálculo en los Grados de Ingeniería: la introducción de aplicaciones tecnológicas (Publicación 2: "*Improving Calculus curriculum in Engineering degrees: implementation of technological applications*").

   3.3) El tercer artículo se basa en una de las aplicaciones desarrolladas en la primera parte del seminario "Aplicaciones de Matemáticas en Ingeniería I: Álgebra Lineal", la cual presenta un modelo simplificado sobre el problema de la migración de población, que tiene como objetivo mostrar al alumnado de primer curso de Ingeniería Industrial, la utilidad de los conceptos del Álgebra Lineal. La aplicación consiste en predecir el censo en el centro y en los suburbios de una población, determinar el punto de equilibrio de la población de la ciudad y hacer una interpretación sociológica de los flujos de población. Este problema práctico ha sido la base del tercer artículo de esta tesis, que estudia la mejora de la motivación y la implicación de los estudiantes en las titulaciones STEM mediante la ejecución de aplicaciones reales en asignaturas matemáticas: el problema de la migración de población (Publicación 3: "*Encouraging students' motivation and involvement in STEM degrees by the execution of real applications in mathematical subjects: the population migration problem*").

4) Por último, se realiza una discusión de los resultados obtenidos en este trabajo en relación a los objetivos planteados y se muestran cuáles son las conclusiones y los posibles futuros trabajos de investigación en esta área.

5) **Bibliografía.** Se presentan las referencias utilizadas en este trabajo.

6) **Anexos A y B.** En el anexo A se muestra el formulario utilizado para las entrevistas anónimas y en el anexo B, se incluye el documento con las preguntas realizadas en las entrevistas.

7) **Anexos C y D.** En estos anexos se incluyen las presentaciones utilizadas en los seminarios "Aplicaciones de las Matemáticas en Ingeniería". En el anexo C, las presentaciones del Seminario de Álgebra Lineal y en el anexo D, las presentaciones de Cálculo Multivariable.

3. ARTÍCULOS

3.1. Formación de las Matemáticas en los Grados de Ingeniería: una intervención del profesorado al alumnado

mathematics

Article

Mathematics Training in Engineering Degrees: An Intervention from Teaching Staff to Students

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Abstract: There has always been a great concern about the teaching of mathematics in engineering degrees. This concern has increased because students have less interest in these studies, which is mainly due to the low motivation of the students towards mathematics, and which is derived in most cases from the lack of awareness of undergraduate students about the importance of mathematics for their career. The main objective of the present work is to achieve a greater motivation for engineering students via an intervention from the teaching staff to undergraduate students. This intervention consists of teaching and learning mathematical concepts through real applications in engineering disciplines. To this end, starting in the 2017/2018 academic year, sessions addressed to the teaching staff from Universitat Politècnica de Catalunya in Spain were held. Then, based on the material extracted from these sessions, from 2019/2020 academic year the sessions "Applications of Mathematics in Engineering I: Linear Algebra" for undergraduate students were offered. With the aim of assessing these sessions, anonymous surveys have been conducted. The results of this intervention show an increase in students' engagement in linear algebra. These results encourage us to extend this experience to other mathematical subjects and basic sciences taught in engineering degrees.

Keywords: mathematics education; engineering degrees; STEM; student motivation

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1. Introduction

To improve the economy of countries, a key factor is to encourage technology. Technological production begins by encouraging and supporting students to develop professional careers in fields related to science, technology, engineering and mathematics (STEM). Therefore, STEM disciplines are considered essential for the economic development of technological societies. The critical role of integrating STEM disciplines into the promotion of students who need to equip themselves with 21st century skills has attracted much attention. Several countries have promoted STEM education as the benefits of this education for quality learning have been recognized. In addition, it has been shown that STEM education could improve the integration of students' skills and the fact that they are better prepared for their professional activity. The 21st century, as the age of information-based technology, brings new job prospects as well as upcoming jobs that demand new skills from workers. Technology is currently used in many jobs in areas such as science, business, engineering, etc.

In addition, high employment in STEM disciplines is expected [1–3]. As technological knowledge becomes increasingly specialized and economically important, more jobs are needed in STEM disciplines and this demand is expected to increase further in the coming years, as noted in [4,5]. However, in most countries, the number of students enrolling in STEM-related disciplines has decreased at secondary and tertiary levels [6]. Currently, this concern has increased because a lower interest in STEM disciplines among European university students has been detected. Therefore, the engineering education community is working to identify the factors that provoke this scenario, as indicated by [1].

One of the issues that has gained a lot of interest in academic research is the worrying levels of dropout in higher education. It was found that approximately one-third of entering college students leave their institution of higher education without obtaining a degree, especially during their first year [7]. The dropout rate increases in STEM careers, as can be observed in [8]. Several studies have focused on the importance of students' motivation and engagement [9,10], in particular for technological degrees [11]. So, to reduce the number of dropouts in the early stages of study, it is necessary to promote student engagement [12,13], which is directly related to motivation [14], student achievement [15] and academic performance [11].

Teachers have always been the most crucial element in educational reform [16,17]. Previous studies on education reform stated that teachers were the main drivers behind students' interest in STEM and their achievements [18]. Most efforts to reform undergraduate STEM education are based on a presumptive reform model related to teacher participation, based primarily on classroom innovation and the teaching–learning process. The self-efficacy and involvement of teachers in classroom teaching play a key role in the realization of integrated education in STEM [19,20]. An important aspect of mathematics education research is to address significant ways for learning and change of mathematics teachers [21–23].

Many university departments offer math courses to their first-year college students and are generally a mandatory part of their departmental programs [24]. In [25], the relationship between basic subjects and applied engineering subjects in higher engineering education curricula is evaluated. Different approaches to teaching mathematics have been considered in several works. There are numerous studies that confirm that active learning has a positive effect on increasing students' motivation and their improvement in learning, which implies enhanced performance, as indicated in [26–29]. For example, the concept of mathematical creativity and the relevance of problem-solving in teaching mathematics have been studied in [30]. In addition, key skills and qualifications expected from employees change from performing routine tasks to solving problems comprised of complex systems through construction, description, explanation, manipulation and prediction. That is, employees are expected to have problem-solving and analytical thinking skills, as well as the conceptual tools to communicate and outsource them.

Rarely do math teacher training programs include a focus on mathematical modeling or the use of models in future teachers' math courses [31,32]. The use of problem-posing in engineering degrees is a profitable tool to increase student involvement and it is known that in engineering education practical and real applications used in basic sciences encourage student engagement and motivation [14], as has been developed in previous studies [33–35]. With this methodology, students are given a problem related to a technological field, which will drive the learning process and allow students to discover what they need to learn to solve the problem. Moreover, it helps students to develop skills and competencies, such as continuous learning, autonomy, teamwork, critical thinking, communication and planning [36], which are considered very important in their profession [37]. Furthermore, theory and practice are integrated, and motivation is enhanced, which results in increased academic performance [38–41]. Another approach to math education based on action learning has been considered in [42]. Various studies have described the benefits of integrating information and communication technologies (ICT) into education [43–45]. It is hoped that the 21st century math teachers will be able to figure out how to integrate technology into all aspects of education [46,47]. Computational thinking is another essential skill to incorporate in math education [48].

In this article, we conjecture the challenge of generating an integrated STEM curriculum. In particular, the aim of this study is to present a contribution to the relationship between mathematical applications and integrated STEM education. The main objective of the present work is to achieve greater motivation for undergraduate engineering students by contextualizing basic sciences, mainly mathematics, through applications to the disciplines of technological degrees. It is expected that the material developed in this work will be introduced for future adaptation of mathematics into core subjects of engineering degrees.

Research Rationale and Research Questions

Engineering students generally do not perceive mathematics in the same way as people who want to pursue this discipline. Engineering students think differently; they want to solve engineering problems and mathematics is just one tool like any other. They need to be told what their knowledge of mathematics is for and the extent to which it is essential to their studies and their future profession. In this sense, the motivation and commitment of the students is considered a key element, making clear the relevance of these basic disciplines to the later technologies and professional exercise.

For an intervention to be more likely to be successful, it must be contextually appropriate in its disciplinary and institutional environment. In this sense, the first part of the work consists of considering the competence specificities related to engineering disciplines that focus on a type of problems of their own, as well as the type of knowledge and learning their own skills. It is very convenient that this task is done in collaboration with the teachers of mathematics and technology departments. Next, it is a question of conducting an analysis looking for a systemic understanding that goes beyond the appreciation of the individual components, to extract the mathematical concepts of the different engineering problems posed previously. This task will be carried out by the teachers of mathematics departments who will then make the extracted material available to undergraduate engineering students.

The aim of this work is to present proposals for the implementation of problems arising from the technology faced by engineering students, which will be complementary to their regular courses. These problems will be multidisciplinary and have in common the idea that mathematics is a necessary skill for solving them. Having realized the need for their knowledge of mathematical methods, students are looking forward to solving the posed problems, thus turning their attention to their mathematical education.

This paper focuses on these research questions:

- How can the mathematical curriculum of an engineering program be adapted to include technological applications?
- How do teachers value this intervention?
- How do students value this intervention?

2. Materials and Methods

The study was conducted at the Universitat Politècnica de Catalunya-BarcelonaTech (UPC) (www.upc.edu, accessed on 1 May 2021), a public university specializing in STEM. During the 2017/2018 academic year, the seminar "Contextualization of mathematics in engineering degrees" was inaugurated at UPC, supervised by one of the authors of this paper and promoted by the vice rector's Office for Academic Policy. The intervention was done in several stages, each dealing with one science subject (mathematics, physics, chemistry, etc.). In the first stage, the intervention was based on mathematics, which began in the 2017/2018 academic year and continues today.

First, these seminars consisted of lectures (an hour and a half per session) for teachers. This is the fourth academic year of this seminar for teachers, called "Contextualization of mathematics in engineering degrees", which aims to illustrate the applications of mathematics in different technological areas. Then, in accordance with the results obtained in the previous seminars for teachers, teaching is carried out for undergraduate engineering students (weekly sessions, an hour and a half each session). Previous sessions were aimed at teachers starting the 2017/2018 academic year and the 2019/2020 academic year sessions for undergraduate engineering students focused on mathematics began. Students' sessions were called "Applications of mathematics in engineering". This seminar for students aimed to bring to the classroom the material extracted from the previous seminar for teachers, in order to improve academic performance and reduce the dropout rate at the UPC, promoting students' engagement. This intervention began in the 2017/2018 academic year with teachers to enable them to implement the material extracted from these sessions later with students in the 2019/2020 academic year. Currently, the intervention is carried out in parallel with teachers and students to expand the material available.

To evaluate these interventions, anonymous surveys were conducted, both for teachers and students of each of the sessions. These questionnaires analyze the impact of the experience and collect assessments from all project members, which will be used to tailor science content to the needs and expectations of undergraduate students in upcoming academic years.

2.1. Teachers' Intervention

The teacher's intervention based on mathematics consists of sessions focused on different engineering disciplines (automation, electricity, mechanics, electronics, etc.). In each of these areas, engineering cases are presented and explained using the mathematical tools needed to solve them. To study and solve these exercises, it is necessary to apply mathematical concepts and techniques: equations of the linear system, complex numbers, matrix modeling, etc. The sessions of this seminar are taught by both teachers of the departments of basic and applied subjects engineering departments. To date, there have been eighteen sessions of math contextualization. The titles are detailed in Table 1.

Table 1. Seminar of contextualization of mathematics in engineering degrees.

Session	Title	Date
1	"Invitation to the Educative Renewal of Mathematics in Engineering degrees"	10 April 2018
2	"Network Flows"	25 April 2018
3	"Engagement with the First Course Students of Civil Engineering"	15 May 2018
4	"Mathematics of Google"	23 May 2018
5	"Numerical Factory: a Numerical Tasting about the Teaching of Mathematics in Engineering"	5 June 2018
6	"How Mathematical Tools help to Manufacture Mechanical Parts"	3 October 2018
7	"One Proposal for the Teaching of Mathematics in Computer Science"	16 October 2018
8	"Mathematical Applications in Elasticity and Resistance of Materials"	7 November 2018
9	"A Historically Problematic Relationship: Mathematics in Engineering"	27 November 2018
10	"Virtual Reality Applications for Biomedical Engineering"	27 February 2019
11	"Fundamental Mathematical Concepts and Tools in Electronic Engineering"	21 March 2019
12	"Modelling and Linear Ordinary Differential Equations Systems"	10 April 2019
13	"Determined Linear Systems for Consecutive Values of States"	2 May 2019
14	"Mathematical Concepts and Tools in Automatic"	22 May 2019
15	"Animated Mathematics"	16 October 2019
16	"Probabilities and Communication Theory: Random Walks in Graphs and Algorithms"	2 December 2020
17	"Cryptography: the Arithmetic of Large Numbers"	17 March 2021
18	"Mathematics at the Service of Engineering Attitudes"	4 May 2021

To evaluate this experience, anonymous surveys were conducted at the end of each session with the aim of analyzing teachers' opinions about the applications and practical exercises introduced.

2.2. Students' Intervention

The material from these teachers' sessions has been adapted to be useful to students. Thus, since the 2019/2020 academic year, weekly sessions have been given to undergraduate students in the first semester on "Applications of Mathematics in Engineering", based on the subject of linear algebra. This students' intervention is designed to increase the engagement and motivation of students in the early stages of their studies. These are voluntary sessions and the UPC recognizes 1 European Credit Transfer and Accumulation System (ECTS) for each semester of student attendance.

Sessions aimed at undergraduate students are organized according to the different concepts of linear algebra. The sessions "Applications of Mathematics in Engineering I: Linear Algebra" for undergraduate students consists of 10 sessions. The sessions of this seminar (Table 2) were organized following the contents of linear algebra in the first course of an engineering degree in order to show students that the concepts they are learning are useful and necessary for their degree.

Table 2. Applications of mathematics in engineering I: linear algebra.

Session	Title
1	"Complex Numbers on the Study of Price Fluctuations"
2	"Complex Numbers on the Study of Alternating Current"
3	"Indeterminate Systems: Control Variables"
4	"Mesh Flushes: a Basis of Conservative Fluxes Vector Subspace"
5	"Addition and Intersection of Vector Subspaces in Discrete Dynamical Systems"
6	"Linear Applications and Associated Matrix"
7	"Basis Changes"
8	"Eigenvalues, Eigenvectors and Diagonalization in Engineering"
9	"Modal Analysis in Discrete Dynamical Systems"
10	"Difference Equations"

To evaluate this experience, anonymous surveys were conducted at the end of each session, with the aim of analyzing students' appreciation of the applications and practical exercises introduced. In order to extract more information from the students attending the sessions "Applications of Mathematics in Engineering I: Linear Algebra", personal interviews were undertaken at the end of these sessions.

3. Results

3.1. Teachers' Results

3.1.1. Teachers' Mathematical Contents

As an example of the teachers' intervention and in order to show the seminars and the development of a session, the session "Network flows" is summarized below. The applications and linear algebra contents from this session are detailed in Table 3.

Table 3. Session 2 ("Network flows") from the seminar of contextualization of mathematics in engineering degrees.

Applications	Linear Algebra Contents
Roundabout traffic	Matrices and determinants. Equation systems.
Electrical network	Equation systems. Vectorial spaces. Vectorial subspaces. Linear applications.
Bus station	Discrete linear systems: contagious matrix, eigenvectors and eigenvalues.
Google: webs classification	Discrete linear systems: contagious matrix, eigenvectors and eigenvalues, Gould accessibility index.

With the aim of being profitable in the future and being able to be consulted and used by any member of the educational community, the sessions of the "Seminar of contextualization of mathematics in engineering degrees" have been recorded. These recordings are available in a repository of UPC (https://upcommons.upc.edu/handle/2117/118481, Catalan language, accessed on 1 May 2021).

3.1.2. Teachers' Surveys Results

The number of teachers attending the sessions "Seminar of contextualization of mathematics in engineering degrees" undertaken to date (18 sessions) is 612 (among them, around 150 different teachers), which means an average of 34 teachers per session. It is worth noting that not only the teachers from the mathematics department attended these sessions, but also those from engineering departments.

The material developed in the sessions "Seminar of contextualization of mathematics in engineering degrees" has been analyzed taking into account the results of the anonymous surveys conducted by the attending professors. Teachers' surveys assess the academic aspects of each lecture, as well as the clarity of the speaker and the general organization of the activity. The participants answered three questions valued on a 5-point scale (1 = strongly disagree, 2 = disagree, 3 = neither agree nor disagree, 4 = agree, 5 = strongly agree). In addition, there is an open field with the possibility to add a comment about the session.

Teachers' questionnaires of the sessions held until now have already been analyzed. The participants in these surveys were 337 teachers (55% of the assistants to the sessions). In Table 4 the questions and the average results are detailed.

Table 4. Teachers' surveys results.

Survey Question	Average
The assessment of academic aspects is positive	4.56
The level of satisfaction regarding the speaker is positive	4.62
General organization of the activity has been appropriate	4.56

The response of the teachers participating in these sessions has been very positive, as can be seen in Table 4. In addition, it is worth mentioning the comments of some teachers expressed in the open field of the surveys, the main themes were:

- Innovative problems.
- Examples with applications in different fields.
- Interesting works linked to social needs.

3.2. Students' Results

3.2.1. Students' Mathematical Contents

Some examples of the applications and problems addressed to students in the sessions "Applications of Mathematics in Engineering I: Linear Algebra" are summarized below. They consist of applications of linear algebra related to engineering which can be understood by undergraduate students in first-year courses.

1. Session 1 (complex numbers on the study of price fluctuations):

In dynamical systems, oscillatory modes with the following form are frequent:

$$p(k) = p_e + c\|\lambda\|^k \cos(k\varphi + \varphi_0),\ k = 0,\ 1,\ 2,\ \ldots \tag{1}$$

determined by:

$$\lambda = \|\lambda\| e^{j\varphi} \in \mathbb{C} \tag{2}$$

In this study, $p(k)$ is the merchandise price in the k-th sales season.

Price expectation for the next season from the previous season is in general:

$$\hat{p}(k) = \beta_1 p(k-1) + \beta_2 p(k-2) + \cdots \tag{3}$$

$$\beta_1 + \beta_2 + \cdots = 1 \tag{4}$$

It is demonstrated that:

$$p(k) = p_e + c\|\lambda\|^k \cos(k\varphi + \varphi_0) \tag{5}$$

where:

$$\lambda = \|\lambda\| e^{j\varphi} \in \mathbb{C} \tag{6}$$

is the "dominant root" of:

$$t^k + \frac{b}{a}\left(\beta_1 t^{k-1} + \beta_2 t^{k-2} + \cdots\right) = 0 \tag{7}$$

called "characteristic polynomial".

- Application session 1: spiderweb model:

In the spiderweb model, producers take as "price expectative" the price from the previous season:

$$\hat{p}(k) = p(k-1) \Rightarrow \beta_1 = 1,\ \beta_2 = \beta_3 = \cdots = 0 \Rightarrow \tag{8}$$

$$\Rightarrow \lambda \text{ is the dominant root of : } t + \frac{b}{a} = 0 \Rightarrow \tag{9}$$

$$\Rightarrow \lambda = -\frac{b}{a} = \frac{b}{a}e^{j\pi} \Rightarrow \begin{cases} \|\lambda\| = \frac{b}{a} \\ \text{Biannual periodicity} \end{cases} \tag{10}$$

- Application session 1: producers' reference to two previous years:

Suppose that $\frac{b}{a} = 1$, but producers refer to the two previous years:

$$\hat{p}(k) = \frac{p(k-1) + p(k-2)}{2} \Rightarrow \beta_1 = \beta_2 = \frac{1}{2}, \beta_3 = \beta_4 = \cdots = 0 \Rightarrow \tag{11}$$

$$\Rightarrow \lambda \text{ is the dominant root of : } t^2 + \frac{1}{2}(t+1) = 0 \Rightarrow \lambda = \frac{-1 \pm j\sqrt{7}}{4} \tag{12}$$

Therefore:

$$\begin{cases} \text{Triennial periodicity} \\ \text{Attenuated oscillations } \left(\text{with } \frac{\mathrm{b}}{\mathrm{a}} = 1\right) \end{cases} \tag{13}$$

Particularly, the condition $\frac{b}{a} < 1$ can be changed to $\frac{b}{a} < 2$:

$$\frac{b}{a} = 2 \Rightarrow \lambda \ \text{ is the dominant root of : } t^2 + t + 1 = 0 \Rightarrow \lambda = \frac{-1 \pm j\sqrt{3}}{2} \Rightarrow \|\lambda\| = 1 \tag{14}$$

- Application session 1: price cycle of pork meat:

In almost a century, four times/year oscillations were observed in the production of pork fat meat in the USA. It is necessary to find a model that fits into it and deduce the $\frac{b}{a}$ value to attenuate it.

It must be considered that there are two seasons of production in each year (spring and autumn) and that the raising period of fat pork is approximately one year. Therefore, k variable corresponds to semester and the "decision/production" is two of these periods (that is, $\beta_1 = 0$).

Supposing:

$$\hat{p}(k) = \frac{1}{5}(p(k-2) + p(k-3) + p(k-4) + p(k-5) + p(k-6)) \tag{15}$$

results:

$$t^6 + \frac{b}{a}\frac{1}{5}\left(t^4 + t^3 + t^2 + t + 1\right) = 0 \tag{16}$$

In fact, four times-year oscillations are obtained:

$$\lambda = e^{j\frac{\pi}{4}} \Longrightarrow \begin{cases} \lambda^6 = -j \\ \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 \cong 2.4j \end{cases} \Rightarrow \frac{b}{a} \cong \frac{5}{2.4} \cong 2.08 \tag{17}$$

Thus, it must be forced:

$$\frac{b}{a} < 2.08 \tag{18}$$

2. Session 2 (Complex numbers on the study of alternating current):

In this session, several applications of electricity in alternating current were explained, in which the use of complex numbers was necessary to solve these problems. See [38] for further information. The applications dealt in this session were:

– Analysis of alternating current circuits: an alternating current $i(t)$ must be calculated in a node, knowing the values of three alternating currents in the same node. Kirchhoff's current law is used, and currents are converted into the complex form.
– Triphasic distribution: phase/neutral voltage and phase/phase voltage must be calculated in a triphasic distribution. To solve it, voltages are converted into the complex form, and phasor representation is used in order to explain the relation between phase/neutral voltage and phase/phase voltage.
– RLC circuit: a circuit with resistance, inductance and capacitor is solved using the complex impedance.
– Resonances: the conditions in which resonance is produced in a parallel circuit must be determined.
– Annulation of reactive power: in this exercise, the capacity of a capacitor must be calculated which has to be in parallel with impedance so that the equivalent impedance is real. That means that reactive power disappears, and performance is optimized.

3. Session 3 (indeterminate systems: control variables):

The third session showed applications of indeterminate equations systems.

- Application session 3: the roundabout traffic:

One of the exercises consisted of a roundabout traffic where three double-ways converge (Figure 1). It was explained how it can be described by a linear equations system and the compatibility conditions were found and interpreted [20].

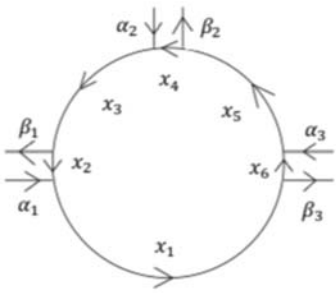

Figure 1. The roundabout traffic.

In this practical exercise it was asked to:

1. Prove that it can be described by the following linear equation system:

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ -\beta_1 \\ \alpha_2 \\ -\beta_2 \\ \alpha_3 \\ -\beta_3 \end{pmatrix}, A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (19)$$

2. Find and interpret the compatibility conditions.
3. In such case, prove that it is a 1-indeterminate system, and a solution basis of the homogeneous system is $x_1 = \cdots = x_6 = 1$.
4. How many traffic measures are needed to know $(x_1, \ldots, x_6)$?
5. Deduce that there exist solutions with $x_i \geq 0$ and that there exists a unique solution with $x_i \geq 0$ and some $x_{i_0} = 0$.
6. Interpret the solutions with $x_i > 0$.

- Application session 3: flow distribution:

Another application dealt in this session was the following flow distribution (Figure 2):

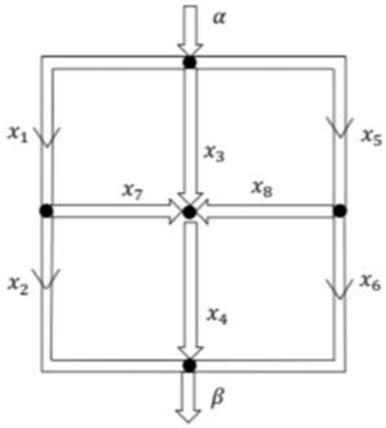

Figure 2. Flow distribution.

In this exercise it was asked:

1. To study the compatibility conditions of the system.
2. To determine how many flows must be measured to know the global circulation of the system.
3. If global circulation can be calculated measuring the flows of the four peripheric points.
4. If global circulation can be calculated measuring the flows of the four intern points.
5. To generalize the study to three branches with more the one interconnexion.

4. Session 4 (mesh fluxes: a basis of vector subspace of conservative fluxes):

In this session a simple electrical network (Figure 3) was solved in order to demonstrate that mesh fluxes are a basis of conservative fluxes. See [38] for further information.

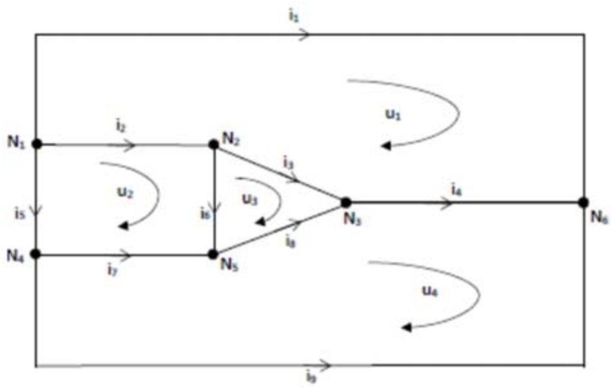

Figure 3. An electrical network.

E being the set of possible current distributions, it was demanded to find the subset $F \subset E$ verifying KCL (Kirchhoff's Current Law); that is, at each of the nodes sum of input currents must be equal to the sum of output currents.

In practice, the used currents are not the above ones indicates in the figure, but the so-called mesh currents (I_1, I_2, I_3, I_4).

To justify this use, it is asked to:

1. Prove that *E* is a vector space of dimension 9 and that *F* is a subspace of *E* of dimension 4.
2. Determinate a basis of *F* so that (I_1, I_2, I_3, I_4) are its coordinates.
3. Prove that one of Kirchhoff's equations is redundant; that is, if it is verified at 5 nodes, it must also be verified at the 6th node.

5. Session 5 (addition and intersection of vector subspaces in discrete dynamical systems):

In this session some examples about control linear systems were explained. See [49] for further information on control linear systems.

The following figure shows the diagram of a general control linear system (Figure 4):

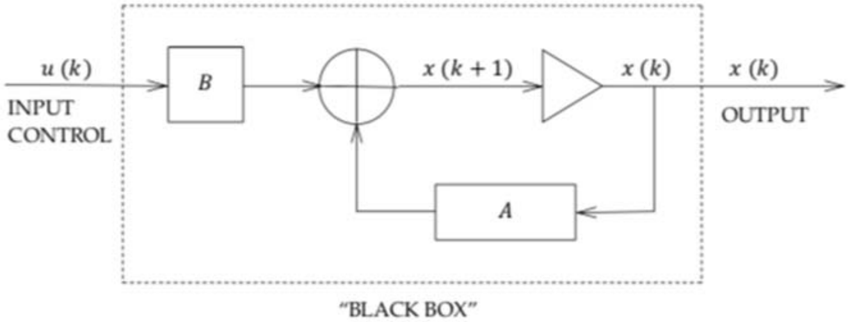

Figure 4. General control linear system.

The state of a general control linear system is:

$$x(k+1) = Ax(k) + Bu(k) \tag{20}$$

Here, different cases of control linear systems are presented.

- Application session 5: one-control case:

In the case of one control and the initial state equal to zero:

$$x(k+1) = Ax(k) + bu(k) \tag{21}$$

$$x(0) = 0 \tag{22}$$

In this case examples were proposed in which states were calculated and it was asked to find the control functions to reach a certain state.

- Application session 5: multi-control case:

In the case of multi-control and the initial state equal to zero:

$$x(k+1) = Ax(k) + Bu(k),\ B = (b_1 \dots b_m) \tag{23}$$

$$x(0) = 0 \tag{24}$$

The examples held in the multi-control case explained how to calculate the states in two conditions:

– With all of the controls, as an addition of subspaces.
– With any of the controls, as an intersection of subspaces.

- Application session 5: Kalman decomposition:

In the case of more general systems:

$$x(k+1) = Ax(k) + Bu(k) \tag{25}$$

$$y(k) = Cx(k) \tag{26}$$

Controllability subspace and observability subspace were defined.

Kalman decomposition was used to solve this case.

6. Session 6 (linear applications and associated matrix):

The applications dealt in this session were examples of linear applications and the associated matrix defined by the images of a basis.

Given a vector space E (with basis $(u_1, \dots, u_n)$), which has as image the vector space F, the following property is defined:

$$E \xrightarrow{f} F \tag{27}$$

$$u_1 \longrightarrow f(u_1) \tag{28}$$

$$u_n \longrightarrow f(u_n) \tag{29}$$

Being:

$$x = x_1u_1 + \cdots + x_nu_n \tag{30}$$

$$f(x) = x_1 f(u_1) + \cdots + x_n f(u_n) \tag{31}$$

If the basis of F is $(v_1, \ldots, v_m)$:

$$x \xrightarrow{f} f(x) \equiv y = y_1 v_1 + \cdots + y_m v_m \tag{32}$$

Therefore:

$$\begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} \xrightarrow{f} \begin{pmatrix} y_1 \\ \cdots \\ y_m \end{pmatrix} = \underbrace{\underbrace{\begin{pmatrix} & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \end{pmatrix}}_{f(u_1)\ldots f(u_n)}}_{f\ MATRIX\ in\ BASES\ \left(\begin{smallmatrix} u_1, \ldots,\ u_n \\ v_1, \ldots,\ v_m \end{smallmatrix} \right)} \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} \tag{33}$$

This property was applied in the examples hereunder.

- Application session 6: rotation of 30°:

In this example it was required to rotate a vector 30°, therefore the linear application is defined as:

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2 \tag{34}$$

It was asked to find:

$$f\begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{35}$$

The matrix in ordinary bases is calculated:

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \tag{36}$$

Therefore:

$$f\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3\frac{\sqrt{3}}{2} + 2\frac{-1}{2} \\ 3\frac{1}{2} + 2\frac{\sqrt{3}}{2} \end{pmatrix} \tag{37}$$

In general:

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2 \\ \frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 \end{pmatrix} \tag{38}$$

- Application session 6: change to italics:

This example showed how to change a letter to italics (Figure 5):

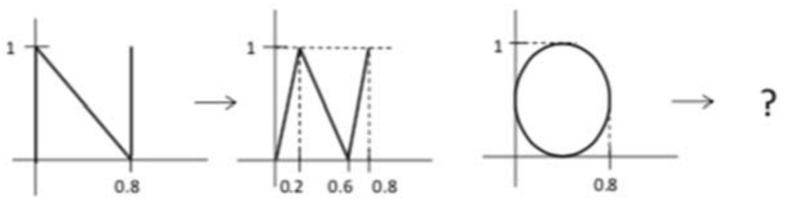

Figure 5. Change a letter to italics.

First, the matrix in ordinary bases is calculated:

$$\begin{pmatrix} 0.75 & 0.2 \\ 0 & 1 \end{pmatrix} \tag{39}$$

Therefore:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} 0.75 & 0.2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.75x_1 + 0.2x_2 \\ x_2 \end{pmatrix} \tag{40}$$

Indeed:

$$\begin{pmatrix} 0.8 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0.75 & 0.2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0.8 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 1 \end{pmatrix} \quad (41)$$

In general, fixed points are defined by:

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} 0.75 & 0.2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Longleftrightarrow \cdots \Longleftrightarrow x_1 = 0.8x_2 \quad (42)$$

7. Session 7 (Basis changes):

This session presented examples of basis changes in vectors and basis changes in linear applications. Finally, some applications in control theory were dealt. Here, one of the examples treated in the session is explained.

- Application session 7: color filters:

This is an example of basis changes in vectors.

Colors form a vector space with dimension 3. For example: yellow, green, red and blue are not linearly independent.

Different bases of three colors are used depending on if the mixed is additive (light) or subtractive (pigments), as it is going to be detailed hereunder.

The three chosen colors are called primary colors and the mixed of only two of them are called secondary colors.

Likewise, in international congress CIE (Commission Internationale de l'Éclairage) of 1931, new coordinates which depend on luminosity were stablished.

The human retina contains 6.5 million cone cells and 120 million rod cells.

The three types of cone cells respond to light of short (S cones), medium (M cones) and long (L cones) wavelengths. L cones more readily absorb red, M cones, green and S cones absorb blue.

Rod cells are sensitive to brightness and produce a black and white response.

For that reason, colors red, green and blue are used for additive mixing as primary colors.

Natural code for screens is RGB code: red (R), green (G) and blue (B). Secondary colors result as (Figure 6):

$$\begin{pmatrix} \text{R} \\ \text{G} \\ \text{B} \end{pmatrix} \quad (43)$$

G + B = CYAN (C)
R + B = MAGENTA (M)
R + G = YELLOW
R + G + B = WHITE

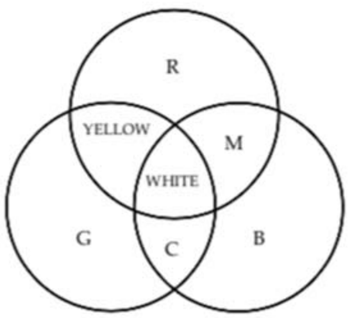

Figure 6. Colors.

But black cannot be obtained.

For subtractive mixing (printers, pigments, etc.), code CMY is used, which has as primary colors cyan, magenta and yellow:

$$\begin{pmatrix} \mathrm{C} \\ \mathrm{M} \\ \mathrm{Y} \end{pmatrix} \tag{44}$$

Secondary colors are the primary colors in the natural code:

MAGENTA + YELLOW = RED
CYAN + YELLOW = GREEN
CYAN + MAGENTA = BLUE
CYAN + MAGENTA + YELLOW = BLACK

Likewise, black is often added as a fourth pigment for saving reasons.

In additive mixing it was verified that human retina is especially sensible to brightness (black and white). For this reason, in CIE congress of 1931, the CIE code was established:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{45}$$

where x ($\equiv \mathrm{C_R}$) $\cong$ RED, y brightness and z ($\equiv \mathrm{C_B}$) $\cong$ BLUE.

A usual transformation is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.61 & 0.29 & 0.15 \\ 0.35 & 0.59 & 0.063 \\ 0.04 & 0.12 & 0.787 \end{pmatrix} \begin{pmatrix} \mathrm{R} \\ \mathrm{G} \\ \mathrm{B} \end{pmatrix} \tag{46}$$

8. Session 8 (eigenvalues, eigenvectors and diagonalization in engineering):

In this session, multiple applications of eigenvalues and eigenvectors in engineering were exposed: materials resistance, mechanics, elasticity, control, dynamics, electricity, population models, etc. Here, two of the examples are developed. More examples can be found in [50].

- Application session 8: prey/predator:

Supposing a prey (*p*) and predator (*d*) model, where respective next year populations *d*(*k*+1), *p*(*k*+1) depend linearly on present year populations *d*(*k*), *p*(*k*):

$$\begin{pmatrix} d(k+1) \\ p(k+1) \end{pmatrix} = \begin{pmatrix} 0.5 & 0.4 \\ -0.125 & 1.1 \end{pmatrix} \begin{pmatrix} d(k) \\ p(k) \end{pmatrix} \tag{47}$$

It was asked to determine the eigenvalues an eigenvectors of the matrix, which are:

$$\lambda_1 = 1;\ v_1 = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \tag{48}$$

$$\lambda_2 = 0.6; v_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{49}$$

The first one indicates a stationary distribution of 4 predators for each 5 preys, which maintains the total populations constant ($\lambda_1 = 1$).

The second one indicates another stationary distribution (4 predators for each prey), with a yearly decrease of the total population of 40% ($\lambda_2 = 0.6$).

- Application session 8: American owl:

In the study of Lamberson [51] about survival of the American owl, he experimentally obtained:

$$\begin{pmatrix} Y(k+1) \\ S(k+1) \\ A(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0.33 \\ 0.18 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{pmatrix} \begin{pmatrix} Y(k) \\ S(k) \\ A(k) \end{pmatrix} \tag{50}$$

where $Y(k)$, $S(k)$ and $A(k)$ indicate the "young" population (until 1 year old), "subadult" population (between 1 and 2 years old) and "adult" population, respectively, in the year k.

The first row of the matrix is formed by birth rate. So, the young and subadult populations do not procreate, while each adult couple has on average 2 children, each 3 years old. The coefficients 0.18 and 0.71 are the survival indices of the transition young/subadult and subadult/adult, respectively. It is clearly confirmed that the first one is critical: when the young phase finishes, they have to leave the nest, find a hunting domain, find a couple, construct a nest, etc. The coefficient 0.94 indicates that the adult population has a yearly death rate of 6%.

It was asked to find the eigenvalues of the matrix, which are:

$$\lambda_1 = 0.98;\ \lambda_2 = -0.02 \pm 0.21j \tag{51}$$

which means an annual decrease of 2%. In these conditions, the American owl converges to extinction in less than 50 years.

The extinction is avoided if and only if the dominant eigenvalue is greater than 1.

The problem is the low survival index. It was requested to verify that extinction would be avoided if the young survival index is 30% instead of 18%. In this case, the system is:

$$\begin{pmatrix} Y(k+1) \\ S(k+1) \\ A(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0.33 \\ 0.30 & 0 & 0 \\ 0 & 0.71 & 0.94 \end{pmatrix} \begin{pmatrix} Y(k) \\ S(k) \\ A(k) \end{pmatrix} \tag{52}$$

And the eigenvalues are:

$$\lambda_1 = 1.01;\ \lambda_2 = -0.03 \pm 0.26j \tag{53}$$

In these conditions, there is an annual increase of 1%. The asymptotic population distribution is given by the coordinates of the eigenvector corresponding to the dominant eigenvalue:

$$v_{DOM} \cong \begin{pmatrix} 10 \\ 3 \\ 31 \end{pmatrix} \tag{54}$$

That is, for each 10 young owls, there will be 3 subadult owls and 31 adult owls, with a growth rate of 1%.

9. Session 9 (modal analysis in discrete dynamical systems):

This session showed several exercises about dynamical discrete linear systems: bus station, Gould accessibility index and Google. See [50] for further information related to dynamical discrete systems. The application of a bus station is presented hereunder.

- Application session 9: bus station:

In this exercise four stations (A, B, C and D) were considered. The traffic is determined by the following rules:

- Stations A, B: 1/3 of buses goes to C; 1/3 of buses goes to D; 1/3 of buses remains for maintenance.
- Station C (and respectively D): 1/4 of buses goes to A; 1/4 of buses goes to B; 1/2 of buses goes to D (and respectively C).

It was asked to prove that there is asymptotic stationary distribution of the buses, and to compute it.

10. Session 10 (difference equations):

Some applications of difference equations were held in this session: Shannon information theory, queues theory and "Biking". This last application is developed here.

- Application session 10: "Biking":

It was required to organize, in 4 years, a "biking" with 400 bicycles in permanent regime, buying *b* bicycles each month.

It is known that 70% of bicycles keep in service, 25% are in the garage and reincorporate the next month, and 5% are irrecoverable.

It was asked the value of b and how many bicycles there would be in 4 years.

3.2.2. Students' Surveys and Interviews Results

So far, two editions of the sessions of "Applications of Linear Algebra in Engineering I: Linear Algebra" have been held, corresponding to the first semesters of the 2019/2020 and 2020/2021 academic years. The number of attending students to the sessions undertaken each semester was 20.

The material developed in these sessions has been analyzed considering the results of the anonymous surveys and interviews conducted to students. Students' surveys evaluate the mathematical and engineering contents and applications of each session, as well as the impact on the motivation of linear algebra. In addition, there is the possibility to add a comment, where students could express their opinion and their impression about the sessions.

Students' surveys of the sessions held until now were analyzed. The surveys were taken in the 2019/2020 and 2020/2021 academic years, when these sessions were held. The results obtained in these two academic years did not present significant differences. Thus, the answers are shown as an average of both years. The participants in these surveys have been all the attending students to the sessions. The participants have answered five questions which must be valued on a 5-point scale (1 = strongly disagree, 2 = disagree, 3 = neither agree nor disagree, 4 = agree, 5 = strongly agree). In the following figures the average of the answers to each question for all the sessions are presented.

The answers to the first question (Figure 7) show that most students, almost 85%, agree with the mathematical contents developed in the sessions.

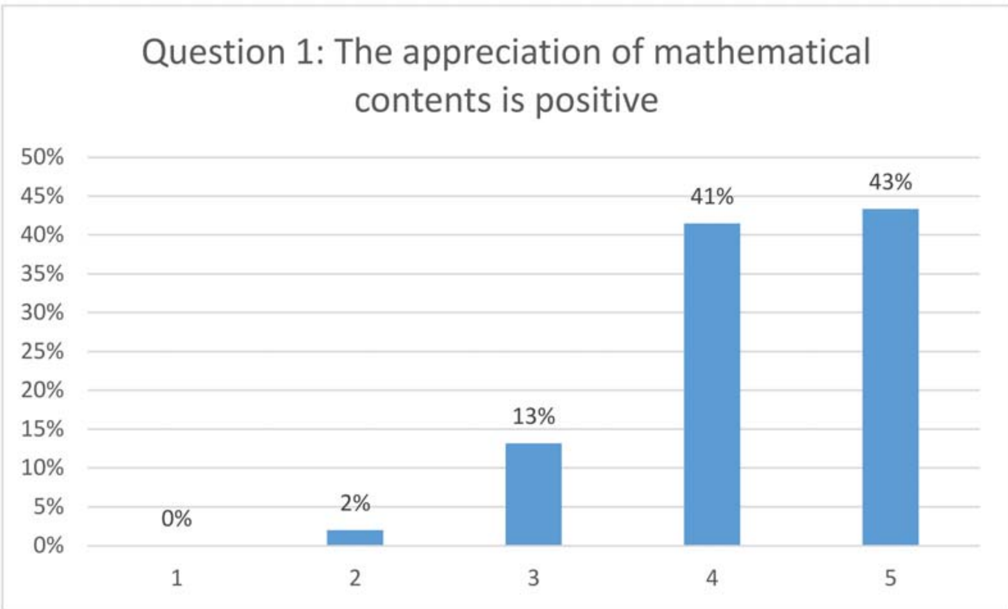

Figure 7. Answers to question 1: the appreciation of mathematical contents is positive.

In the answers to the second question (Figure 8), it can be observed that more than 80% of students agreed with the engineering contents explained in the sessions.

More than 90% of students think that the sessions "Applications of Linear Algebra in Engineering" let them know technological applications of different mathematical concepts (Figure 9).

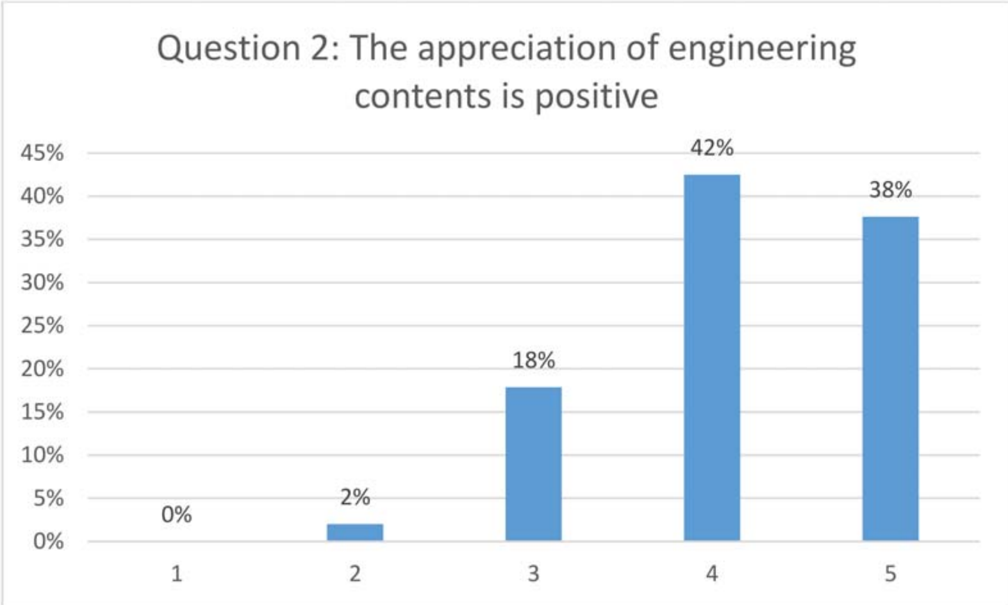

Figure 8. Answers to question 2: the appreciation of engineering contents of this session is positive.

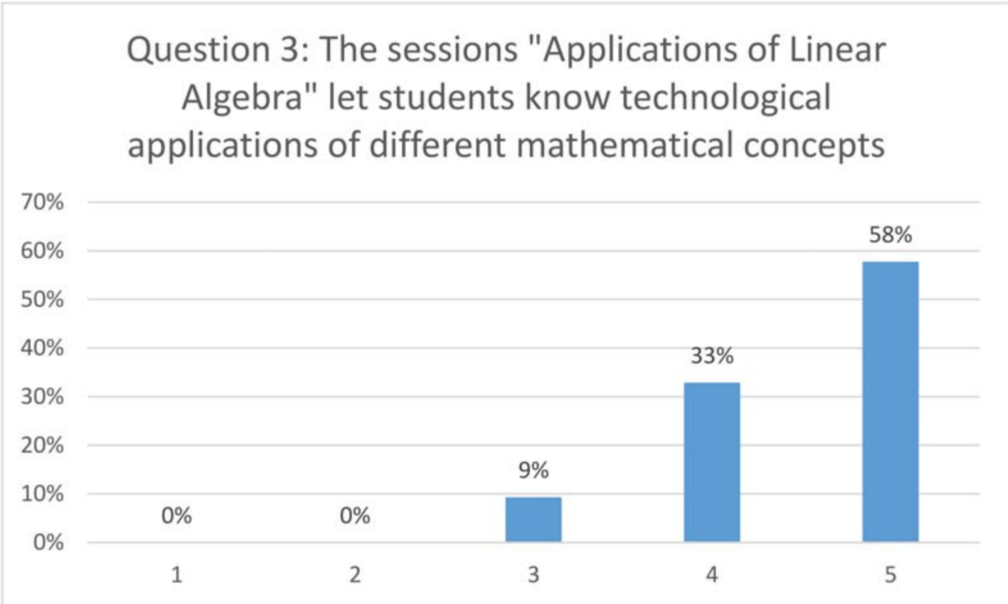

Figure 9. Answers to question 3: the sessions "Applications of Linear Algebra" let students know technological applications of different mathematical concepts.

It can be seen that 90% of students agreed that applications of mathematical concepts succeeded in increasing their motivation to study linear algebra (Figure 10).

Almost 70% of students state that the execution of practical exercises with technological applications improve the learning of mathematical concepts (Figure 11).

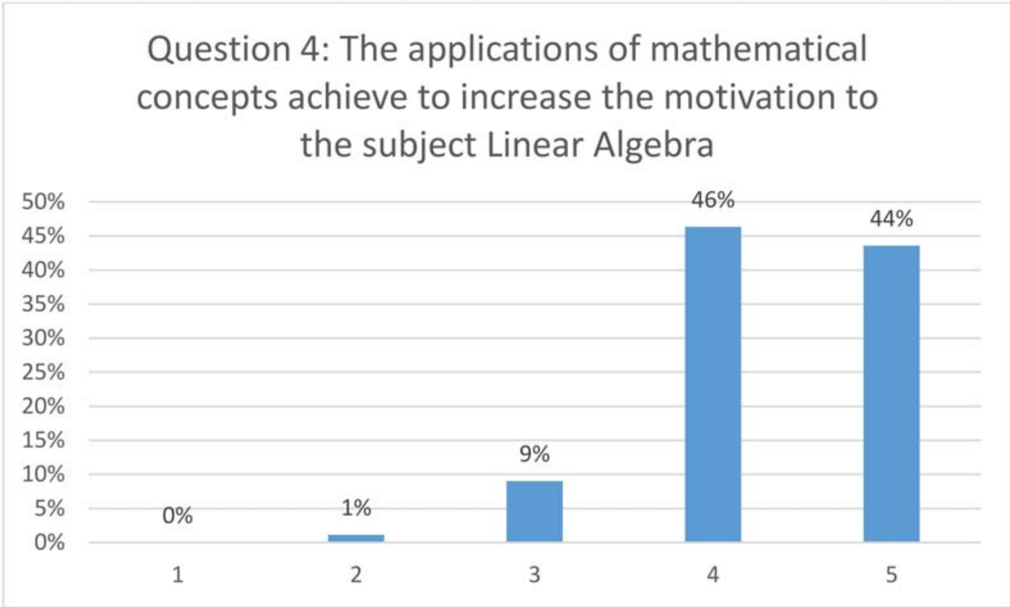

Figure 10. Answers to question 4: the applications of mathematical concepts achieve to increase the motivation to the subject linear algebra.

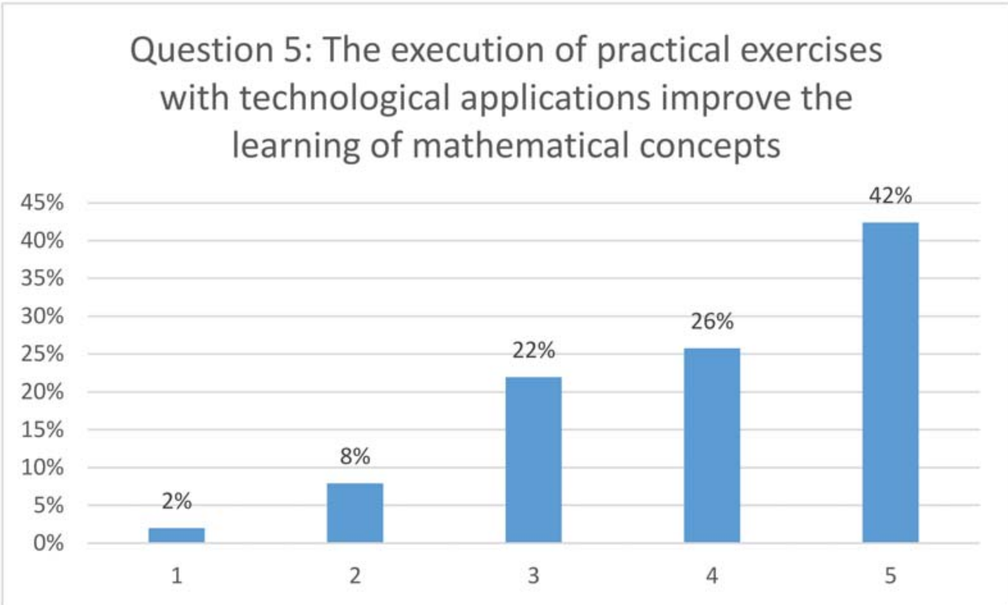

Figure 11. Answers to question 5: the execution of practical exercises with technological applications improve the learning of mathematical concepts.

The response of the attending students to these sessions in 2019/2020 and 2020/2021 academic years was very positive. It is also worth mentioning that some students' comments, expressed in the open field of anonymous surveys in both years, were along the following main themes:

- Applications let students know that mathematics is necessary.
- These sessions achieve the goal to motivate students and let them realize that linear algebra has real applications.
- Context in mathematics increases the interest and the attention of students, both in university and in secondary school.
- Applications helped students learn better linear algebra concepts.

In order to extract more information from students attending the sessions "Applications of Linear Algebra in Engineering", personal interviews were undertaken at the end of all sessions in 2019/2020 and 2020/2021 academic years. These interviews consisted of several open questions, which let students explain in detail their opinion and assessment of the sessions. The main questions set to students were:

1. What aspects do you asses more positively of these sessions?
2. What applications have been more interesting? Why?
3. How have these sessions influenced on your motivation and on your interest toward linear algebra?
4. Have these sessions helped you understand mathematical concepts of the subject linear algebra? What applications? What concepts?
5. After these sessions, do you consider that mathematics are more important and essential to the development of engineering degrees? How? Why?

The information extracted from these answers in both academic years is presented here:

- The sessions "Applications of Linear Algebra in Engineering" let students know real applications in different disciplines of engineering.
- Seeing all these applications let students know what they will be able to do in the following courses and it is very motivating.
- These applications let students realize of how important linear algebra is for engineering degrees and for their future profession.
- Interesting applications: complex numbers (electricity, economy), indeterminate compatible systems (roundabout traffic), vector subspaces (linear control systems), linear applications (change to italics, population fluxes), eigenvectors and eigenvalues (demographic control).

4. Discussion

In this work we contribute to generating an integrated STEM curriculum, presenting an intervention from the teaching staff to students about the relationship between mathematical applications and integrated STEM education. This work contributes to the connection of mathematics with technological disciplines and with technological professions, with the aim of improving the motivation and engagement of undergraduate engineering students.

In the current engineering curriculum, the first two courses consist mainly of math, science, communication and electives courses. Students take very few engineering courses in the first two years. With the intervention presented in this study basic science subjects, as mathematics, can play another role in engineering degrees and offer a wider view in STEM education [52,53]. Here, it has been presented that mathematics courses should cover examples and problems related to the main field of students enrolled degree to improve the understanding and application of these concepts, as [22] states. Teachers have always been the most crucial element in educational reform [16–18]. Teachers' intervention has been done in conjunction with the mathematics department faculty and the engineering department faculty [24,31,32]. As in [25], it is shown that the relationship between basic subjects and applied engineering subjects in higher engineering education curricula is evaluated.

Regarding the teachers' intervention, the seminar based on mathematics consists of sessions focused on electricity, automatics, mechanics, electronics, etc. For example, in the electricity area, different exercises in electrical engineering have been developed by the authors, as [38] shows. Following this structure, other engineering disciplines have been organized, as [49,50] show. Some preliminary results about the teachers' intervention were presented in [54].

Regarding the students' intervention, the main issue has been the relevance of problem-solving in the teaching of mathematics, as it is known that mathematical problem-posing provide better student's critical thinking skills effect than conventional learning [30,36]. Moreover, applications used in basic sciences subjects encourage student engagement and motivation in STEM degrees [14,33–35]. Considering the results shown in Figure 10, it can be seen that 90% of students agreed that applications of mathematical concepts managed to increase their motivation to study linear algebra. Thus, this will lead to reduced dropout, as it is related to motivation [14], student achievement [15] and academic performance [11]. It has been shown that applications let students learn mathematical concepts through practical examples increase students' motivation to study mathematics, as it was confirmed in previous studies like [29]. Moreover, students discover multiple real applications of linear algebra in engineering and other areas, what achieve to motivate them to the study of this subject, as it was developed in previous studies [14,33,34]. From the answers to the questions set to the students in the interviews made after the sessions "Applications of Linear Algebra in Engineering", it can be interpreted that most of the examples have impressed students because they did not know that linear algebra could have applications in so many different disciplines. In addition, knowing what they would be able to do in the next courses using the concepts of linear algebra was really motivating for students, as they realized how essential linear algebra was for their career and they increased their interest towards this subject.

These results confirmed that this experience allows students to get a better understanding of mathematical concepts, as it is concluded in [39], which increases students' performance in mathematical subjects of engineering degrees, as it was analyzed in previous studies like [26]. The work provide evidence that it is possible to structure teacher support so that they can make lasting pedagogical changes, rather than temporary or one-off changes as part of a specific initiative.

5. Conclusions and Future Work

The study presented here has been conducted at the Universitat Politècnica de Catalunya-BarcelonaTech (UPC), a university specialized in STEM disciplines. The work is based on an intervention starting from the teaching staff of basic science departments and engineering departments and finishing with an intervention for undergraduate students. The aim of this study is to present a contribution about the relationship between mathematical applications and integrated STEM education.

Following the successful teachers' intervention with mathematics, in the 2019/20 academic year, the seminar of contextualization for the teaching staff has been expanded to physics creating the seminar "Contextualization of basic sciences in engineering degrees at UPC". Up to now, five sessions have been held regarding physics. It is planned to continue with the other basic sciences. The work of "Contextualization of basic sciences in engineering degrees" consists of seminars which deal with the different basic sciences (mathematics, physics, chemistry, computing, statistics, etc.) in the first courses of engineering degrees.

Similarly, following the successful experience with students, new sessions to expand the applications in engineering to other mathematical subjects are planned. In particular, "Applications of Mathematics in Engineering II" based on multivariable calculus are being to be held the second semester, to complement the ones based on linear algebra held in the first semester.

It is expected that the material developed in this work will be introduced for future adaptation of basic sciences subjects in engineering degrees. This fact will lead to an increase of student engagement and a decrease in dropouts out of engineering degrees. Moreover, secondary teachers have also suggested to expand this experience to secondary education to increase the STEM vocations of the students.

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3.2. Mejora del currículum de Cálculo en los Grados de Ingeniería: implementación de aplicaciones tecnológicas

mathematics

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Article

Improving Calculus Curriculum in Engineering Degrees: Implementation of Technological Applications

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Abstract: The teaching of mathematics has always concerned all the professionals involved in engineering degrees. Curently students have less interest in these studies, what has caused an increase of this concern. The lack of awareness of students about the significance of mathematics in their careers, provoke the decrease of undergraduate students' motivation, which derives in a low interest in engineering degrees. The aim of this work is that engineering students achieve a greater motivation and involvement in first academic courses, through the implementation of real and technological applications related to their degrees in the learning of mathematical concepts. To this end, the 2019/2020 and 2020/2021 academics years, the seminar "Applications of Multivariable Calculus in Engineering" has been held in Universitat Politècnica de Catalunya-BarcelonaTech (UPC), based on the teaching of Multivariable Calculus by the execution of real problems where calculus concepts are necessary to solve them. With the aim of analyzing students' motivation and assessment of the seminar, anonymous surveys and personal interviews have been conducted. The number of attending students to the sessions in each academic year has been 16 and all of them have been participants in the surveys and interviews. The results show that students' responses were generally positive and they agree that their motivation to the subject Multivariable Calculus has increased with the use of real applications of mathematics. The execution of practical problems with engineering applications improves the acquirement of mathematical concepts, what could imply an increase of students' performance and a decrease of the dropout in the first academic courses of engineering degrees.

Keywords: calculus; engineering education; mathematics education; motivation; STEM

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1. Introduction

The economic development of countries is mainly based on technology. Thus, professionals in fields related to science, technology, engineering and mathematics (STEM) are necessary to improve the economy of countries. Technological production implies encouraging and supporting students to become technological professionals. Therefore, STEM disciplines are considered essential for the economic development of technological societies. The potential negative economic impact of undersupply is of concern due to opportunity costs and loss of competitiveness [1]. In addition, STEM education could integrate students' skills and better professional competences. The 21st century, as the age of information technologies, entails new job prospects and upcoming jobs which require new skills from professionals. Nowadays, technology is necessary in many jobs such as science, business, engineering, etc.

Moreover, high occupancy demands for STEM degrees are expected [2,3]. As technological knowledge and expertise is becoming more specialized and economically increasingly important, ever more jobs specialized in STEM disciplines are required and this demand is expected to further increase in the upcoming years, as remarked in [4,5].

However, at the present time in most countries, undergraduate students have less interest in technological degrees [1,6,7], which is mainly due to the lack of awareness of the importance of mathematical subjects in first academic courses of these studies. This lack of awareness derives, in most cases, in the decrease of students' motivation, which has as a result a low performance and a high dropout rate in the first years of these degrees. Thus, the engineering education community work to identify the causes of this situation, as indicates [2].

The worrying dropout in higher education has gained much interest in academic research. One third of undergraduate students leave university without obtaining a degree, mainly during their first academic year [8]. The dropout rate is higher in STEM careers [7].

The importance of students' motivation and engagement has been analyzed in previous studies [9,10], and in particular for technological degrees [11]. In first academic years is essential to promote student engagement [12,13], which involves the improvement of motivation [14,15], relatedness [16], student achievement [17] and academic performance [18], what imply the decrease of the dropout rate.

Practical and real applications used in mathematics subjects of engineering degrees, encourage student engagement and motivation [14], as has been studied in previous works [19–23]. A proper coordination among mathematical subjects and technological subjects of engineering degrees syllabus contribute to the decrease of dropout rates [24]. Active learning has positive results on the rise of students' motivation and on the enhancement of their learning, what entails the improvement of their performance, as it is stated in several studies [25–28]. For instance, the relationship between mathematical creativity and the relevance of problem-solving in the teaching of mathematics has been studied in [29]. Moreover, key employee expected abilities involve problem-solving and analytical thinking skills besides the competences to communicate them. The use of problem-posing in engineering degrees contributes to increase student involvement. This methodology consists of exposing a problem to students, related to technological disciplines, which will lead them to discover what they need to learn to solve this problem. Furthermore, it implies the development of essential abilities and competences for their career, as they are autonomy, continuous learning, critical thinking, planning and communications skills [30,31]. Moreover, the integration of theory and practice entails the improvement of motivation, what implies an increase of academic performance [32–35].

The purpose of this work is to generate an integrated STEM curricula, connecting mathematical applications with STEM education. The aim of the present study is to increase undergraduate engineering students' motivation by contextualization of mathematical subjects with technological applications related to the disciplines taught in the following academic courses of these degrees. The material developed in this work is expected to be introduced for a future adaptation of mathematical subjects' syllabus in engineering degrees.

Engineering students have to solve engineering problems and mathematical methodologies are the tools to solve them. They need to know the usefulness of mathematics and how essential they are for their degrees and their future careers. In this regard, the motivation and involvement of students are considered a key element, clarifying the importance of mathematics for technological subjects and for their future profession.

This study is part of the work "Applications of Mathematics in Engineering", which is formed by two seminars: "Applications of Linear Algebra in Engineering" [36] and "Applications of Multivariable Calculus in Engineering". These two seminars entail the mathematical subjects of first academic courses in technological degrees. This study focuses on the seminar "Applications of Multivariable Calculus in Engineering", whose purpose is to present real and technical applications of Multivariable Calculus related to engineering degrees with the objective of increasing students' motivation towards the learning of mathematics in first academic courses. Knowing the need of mathematical concepts to solve those technical applications, students realized of the importance of mathematics not only for the execution of their degrees but also for the development of their careers as engineers.

This article focuses on these research questions:

- How does the implementation of real and practical applications in mathematical subjects' influence on students' motivation?
- What are the benefits of including real and practical applications in mathematical subjects of first academic courses in technological degrees?

The results of the study show that for students is really motivating to know what they will be capable to do in the next courses of their degree. They also realized of how essential Multivariable Calculus is for their future profession and increased their interest towards the subject.

These results verifies that this experience lets students obtain a greater understanding of mathematical concepts, which increases students' performance in mathematical subjects of engineering degrees.

2. Materials and Methods

The study has been conducted at the Universitat Politècnica de Catalunya-BarcelonaTech (UPC), a public university specialised in STEM degrees. The work "Applications of Mathematics in Engineering" is formed by two seminars: "Applications of Linear Algebra in Engineering" [36] and "Applications of Multivariable Calculus in Engineering", which started in the 2019/2020 academic year and were undertaken in the first and the second semester, respectively. Both seminars were organised in weekly sessions of one hour and a half each session. These sessions have been held also in the 2020/2021 academic year and it is planned to repeat them during the following years.

Thus, since the 2019/2020 academic year, weekly sessions have been given to first-year students of the Industrial Engineering Bachelor's Degree from the Barcelona School of Industrial Engineering (ETSEIB) of the UPC, this degree lasts four years. Currently, the syllabus of mathematical subjects in engineering degrees do not content technological applications. Mathematical subjects focus on mathematical concepts, they are not contextualized in the technological disciplines of engineering degrees. The aim of this work is to contextualize mathematics through the connection of mathematical subjects with technological disciplines, taught in the following academic courses, and with their future technological professions. Thus, students will be able to realize of the importance of mathematics for engineering, as well as they learn engineering applications from the beginning of their degrees.

The two seminars "Applications of Linear Algebra in Engineering" and "Applications of Multivariable Calculus in Engineering" are offered in the same semesters in which the ordinary classes of the compulsory subjects Linear Algebra and Multivariable Calculus are taught, first and second semesters, respectively, so that the students who wish could complement in a parallel way and from a practical point of view the theoretical concepts introduced in the ordinary classes. The seminars have been devised with the aim of increasing students' motivation and involvement in the early stages of engineering studies. In addition to the benefits of these sessions, Universitat Politècnica de Catalunya-BarcelonaTech (UPC) recognizes with 1 European Credit Transfer and Accumulation System (ETCS) the attendance for students.

This article focuses on the seminar "Applications of Multivariable Calculus in Engineering". In each of the sessions of this seminar, applications illustrating the use of mathematical concepts related to multivariable calculus in different engineering areas are explained. The compulsory subject of Multivariable Calculus lasts one semester (14 weeks). Instead, the optional seminar presented in this work consists of 10 weeks. During the first two weeks of the semester, students are informed of the existence of this seminar in order to enable registration; and two other weeks, before the partial and final exams, no seminar sessions are given. So, this seminar consists of 10 sessions, 8 main sessions and 2 review sessions. The 8 main sessions are detailed in Table 1.

Table 1. Applications of Multivariable Calculus in Engineering.

Session	Title
1	"Discontinuous phenomena: hysteresis, caustics"
2	"Thom's catastrophes"
3	"Taylor and Fourier series"
4	"Chain, implicit and inverse theorems"
5	"Inverse kinetics"
6	"Kinematics of mechanisms with links"
7	"Optimization"
8	"Miscellany"

To evaluate the results of this study, anonymous surveys and personal interviews were conducted, with the aim of analyzing students' appreciation of the seminar "Applications of Multivariable Calculus in Engineering".

Surveys were undertaken at the end of each session and evaluate the impact of the experience on the students attending to the sessions, as regards the mathematical and engineering contents, the technological applications and the motivation towards the subject of Multivariable Calculus. These surveys consisted of five questions which must be valued on a 5-point scale (1 = Strongly disagree, 2 = Disagree, 3 = Nor agree nor disagree, 4 = Agree, 5 = Strongly agree). In addition, there is the possibility to include an opinion, where students could explain their impression about the session. The questions set in the surveys were:

Question 1: The assessment of mathematical contents is positive.

Question 2: The assessment of engineering contents is positive.

Question 3: The sessions "Applications of Multivariable Calculus in Engineering" let students know technological applications of different mathematical concepts.

Question 4: The applications of mathematical concepts achieve to increase the motivation to the subject Multivariable Calculus.

Question 5: The execution of practical exercises with technological applications improve the learning of mathematical concepts.

With the aim of extracting more opinions from the students attending the sessions "Applications of Multivariable Calculus in Engineering", personal interviews have been undertaken at the end of all the sessions in 2019/2020 and 2020/2021 academic years, which consisted of several open questions, where students could express in detail their opinion and assessment of the sessions. It should be noted that, in order to avoid bias in the answers, the person who interviewed students was not a professor but a master's student. The main questions set to students were:

1. What aspects do you assess most positively of these sessions?
2. What applications have been more interesting? Why?
3. How have these sessions influenced on your motivation and on your interest toward Multivariable Calculus?
4. Have these sessions helped you understand mathematical concepts of the subject Multivariable Calculus? What applications? What concepts?
5. After these sessions, do you consider that Mathematics are more important and essential to the development of engineering degrees? How? Why?

The influence of the implementation of the seminar "Applications of Multivariable Calculus in Engineering" on the students attending the seminar has been analyzed from the answers to the surveys and to the interviews undertaken after the sessions of this seminar.

3. Results

3.1. Students' Mathematical Contents

The 8 main sessions of the seminar "Applications of Multivariable Calculus in Engineering" consists of real and practical applications of the contents developed in the subject Multivariable Calculus, whose syllabus is:

1. Continuity and derivability of multivariable functions.
2. Integration of multivariable functions.
3. Laplace transform and Fourier series.

The first three sessions of the seminar are focused in discontinuous phenomena. Although they are not still included in mathematical subject programmes, discontinuous phenomena are very common in engineering problems and entail most of the contents of engineering subjects and contribute to illustrate the importance of these mathematical theories to solve real engineering problems. The applications developed in sessions 1, 2 and 3 include the Thom's catastrophes and Taylor and Fourier series.

The sessions 4, 5 and 6 are related to differential calculus and the basic theorems: chain rule, implicit function theorem and inverse function theorem, which are the basis of a great number of classical applications in engineering.

The session 7 is about optimization, which is the most important goal of engineering.

The last session, miscellany, deal with the use of engineering vision to solve applications in order to apply mathematical calculations, concluding that engineers must complement the use of mathematical tools with their engineering knowledge.

Some practical and real applications explained to the students in the sessions "Applications of Multivariable Calculus in Engineering" are explained below. They consist of applications of Multivariable Calculus related to engineering disciplines which can be understood and learnt by undergraduate students in the first academic courses.

3.1.1. Application 1: A Gravitational Machine

The first application is an example of discontinuous phenomena and it was explained in the session 1 (Discontinuous phenomena: hysteresis, caustics), where discontinuous phenomena were introduced highlighting how frequent they are in engineering.

In a gravitational machine appear discontinuous phenomena as it is going to be shown hereunder. A gravitational machine consists of a flat sheet limited by a parabola, leant on a horizontal plane. The most important point of this machine is that the center of gravity (CDG) is variable through the position of a magnet which can be moved on the sheet surface. Supposing the sheet mass negligible, the CDG would be the magnet position (Figure 1).

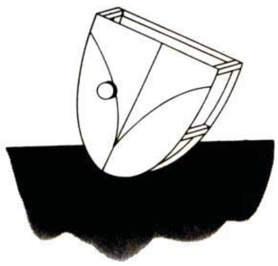

Figure 1. Centre of gravity.

When the CDG is displaced on the sheet, how will the sheet situate in a stable way?

The stability situation will happen when the CDG is placed in a minimum height, therefore the stable equilibrium point P on the parabola outline is the relative minimum of the distance between the CDG and the parabola points, that is, the orthogonal base to the parabola from the CDG, as it is shown in the following figure (Figure 2).

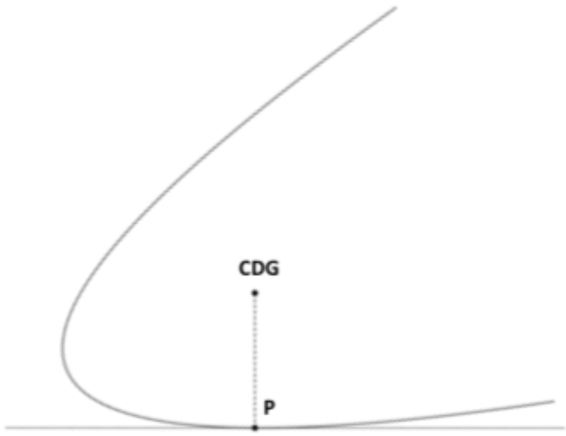

Figure 2. Position of the center of gravity.

If CDG is placed on the parabola vertical axis, the equilibrium point could be stable or instable depending on the CDG height. If the distance between the CDG and the equilibrium point is a relative minimum, there would be stable equilibrium but if this distance is a relative maximum, the equilibrium point is the parabola vertex and it would be instable equilibrium.

If we consider the parabola $\{(z,z^2),|z|\leq 2\}$ and the CDG $=(0,2)$, the distance between any point on the parabola outline and the CDG placed on the parabola axis would be:

$$E=\left(d\left(z,z^2\right),(0,2)\right)\right)^2=z^2+\left(z^2-2\right)^2=z^4-3z+4 \tag{1}$$

If we derivate and equal to zero, we obtain:

$$D\left(d\left(\left(z,z^2\right),(0,2)\right)\right)^2=4z^3-6z=0 \tag{2}$$

where $z=0$ is a relative maximum and $z=\pm\sqrt{\frac{3}{2}}$ are relative minimums.

Therefore, if CDG is = (0,2), there is instable equilibrium in the parabola vertex, $V=(0,0)$. In addition, there is stable equilibrium on the parabola outline points $P_1=\left(\sqrt{\frac{3}{2}},\frac{3}{2}\right)$, $P_2=\left(-\sqrt{\frac{3}{2}},\frac{3}{2}\right)$. These points are indicated in the following figure (Figure 3):

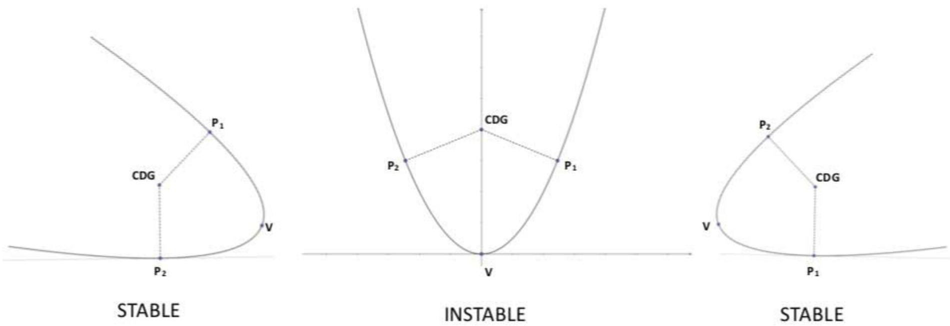

Figure 3. Stability of the center of gravity.

If the CDG height is less than $\frac{1}{2}$. there is only one relative extreme, which is the parabola vertex and, in this case, there would be 1 stable equilibrium point.

Depending on the position of de CDG, there can exist three equilibrium points or only one equilibrium point. This situation happens not only if the CDG is placed on the parabola axis but also on any point inside the parabola.

The question is in what positions of the CDG, there are three equilibrium points (two stable points and one instable point) and in what positions there is only one stable equilibrium point. To answer this question, we must analyze in what positions of the CDG there are three orthogonals, corresponding to two relative minims and one relative maxim and in what positions there is only one equilibrium point, corresponding to a relative minim. To solve it we have to make the orthogonals envelope. We must distinguish the area where there are three orthogonals and the area with only one orthogonal, the separation between these two areas is the orthogonals envelope. To obtain the expression of this envelope, we have to do the following calculations.

The orthogonal in:

$$\left(z, z^2\right) = \left\{(\beta, \alpha) : \frac{\beta - z}{-2z} = \alpha - z^2\right\} \quad (3)$$

The expression of the orthogonal is:

$$0 = \beta - z + 2z\left(\alpha - z^2\right) = \beta + (2\alpha - 1)z - 2z^3 \quad (4)$$

To eliminate z, we calculate the derivative:

$$0 = D^Z\left(\beta - z + 2z\left(\alpha - z^2\right)\right) = (2\alpha - 1) - 6z^2 \quad (5)$$

with these 2 expressions, we can obtain that:

$$\begin{gathered} z^2 = \tfrac{2\alpha-1}{6} \Rightarrow 0 = \beta + z\left((2\alpha - 1) - 2\tfrac{2\alpha-1}{6}\right) \Rightarrow 0 = \beta + z\tfrac{2}{3}(2\alpha - 1) \Rightarrow \\ \Rightarrow \beta^2 = z^2\tfrac{4}{9}(2\alpha - 1)^2 = \tfrac{2\alpha-1}{6}\tfrac{4}{9}(2\alpha - 1)^2 \end{gathered} \quad (6)$$

As a result, it can be deduced that the envelope expression is:

$$\beta^2 = \frac{16}{27}\left(\alpha - \frac{1}{2}\right)^3 \quad (7)$$

That is a cusp curve that separates the triple orthogonality area from the simple orthogonality area, as it is shown in this figure (Figure 4):

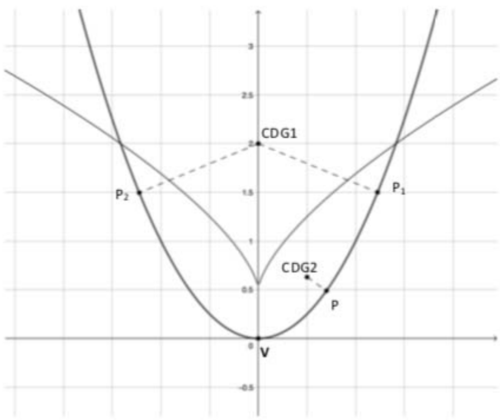

Figure 4. Cusp curve.

If the CDG is placed over the cusp, the gravitational machine will have three equilibrium points. In the figure, CDG1 is placed over the cusp and in this case the two stable equilibrium points are P1 and P2 and the instable equilibrium point is the parabola vertex V.

If the CDG is placed under the cusp, the gravitational machine has one equilibrium point. In the figure, CDG2 is under the cusp and the only one stable point is P.

The following expressions represent these conditions:

$$\text{CDG}(\beta,\alpha)\begin{cases} \beta^2 < \frac{16}{27}\left(\alpha - \frac{1}{2}\right)^3 \Rightarrow \begin{cases} \text{2 STABLE} \\ \text{1 INSTABLE} \end{cases} \\ \beta^2 > \frac{16}{27}\left(\alpha - \frac{1}{2}\right)^3 \Rightarrow \text{1 STABLE} \end{cases} \tag{8}$$

Now we are going to analyze the machine behavior when the CDG moves from a position over the envelope to a position under it. In this case the system will lose one stable equilibrium point. Therefore, it will provoke a discontinuity.

To show it, we are going to study the following figure (Figure 5).

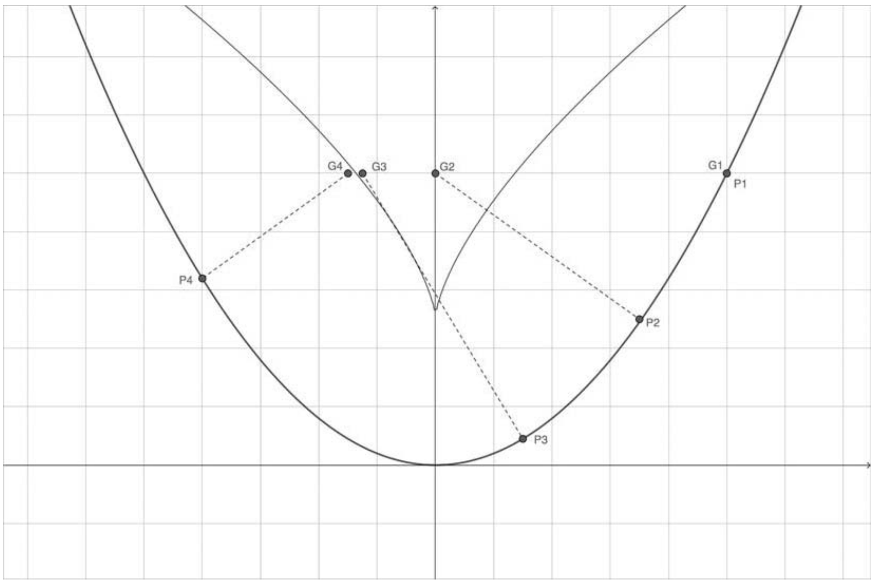

Figure 5. Machine behavior.

If CDG moves slowly among the points G_1, G_2 and G_3, the equilibrium point changes continuously among the points P_1, P_2 and P_3, respectively.

If CDG changes from G_3 to G_4, the parabola falls discontinuously, the equilibrium point jumps from P_3 to P_4. The disruption occurs when $\beta \cong -0.28$.

In the case that $\alpha = 0.4$ and the CDG changes horizontally, there will not be disruption because the CDG is always under the envelope.

3.1.2. Application 2: Euler's Arc

The second example, explained in session 3, is an application of Thom's catastrophes and of Fourier and Taylor's series.

Supposing a compressed arc (with length π) and a load m slightly off-center (ε), as it is represented in the following figure (Figure 6):

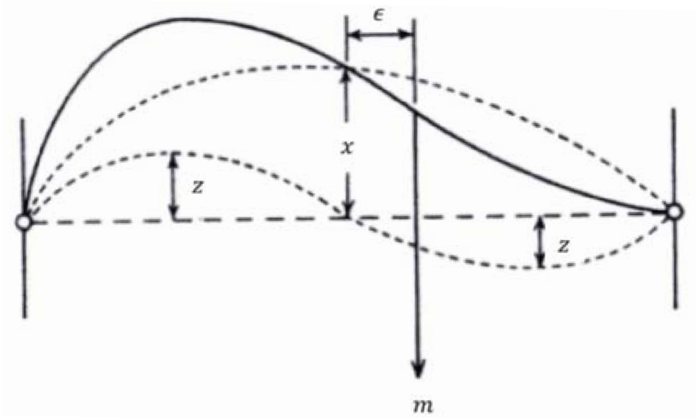

Figure 6. Compressed arc.

The beam resistance depends on two control parameters (m,ε), which must be modelled by the Thom's 2nd catastrophe.

Expressing the arc with the following function:

$$f(s), 0 \leq s \leq \pi \tag{9}$$

Fourier analysis establishes that periodic functions can be modelled by additions of harmonics of different periods. Therefore, the bean function can be expressed as an addition of sinus of different periods:

$$f(s) = \sum c_n \sin ns \tag{10}$$

In this example, supposing one or two harmonics, it is obtained that:

$$m = 0,\ \varepsilon = 0 \Rightarrow f(s) \cong r \sin s \tag{11}$$

If the elastic module is $\mu = \frac{1}{\pi}$, potential energy and elastic energy are:

$$V_P = mf\left(\frac{\pi}{2} + \varepsilon\right) \tag{12}$$

$$V_E = \frac{1}{2\pi} \int_0^\pi (f''(s))^2 \frac{1}{\left(1 + (f'(s))^2\right)^3} ds \tag{13}$$

Applying Taylor ($\varepsilon \ll$):

$$V_P = mx \cos \varepsilon + mz(-\sin 2\varepsilon) \cong mx\left(1 - \frac{\varepsilon^2}{2}\right) + mz(-2\varepsilon) \tag{14}$$

The variable x depends on the variable z because the beam distance does not change when the beam distorts:

$$d = \int_0^\pi \sqrt{1 - (f'(s))^2} ds \tag{15}$$

In both cases (considering one or two harmonics) this distance is the same:

$$\begin{aligned} d = \textstyle\int_0^\pi \sqrt{1 - (r \cos s)^2} ds \cong \int_0^\pi \sqrt{\left(1 + \frac{1}{2} r^2 \cos^2 s + \frac{-1}{8} r^4 \cos^4 s\right)}\, ds = \\ = \tfrac{\pi}{4}\left(4 - r^3 - \tfrac{3}{16} r^4 - \tfrac{5}{64} r^6\right) \end{aligned} \tag{16}$$

$$\begin{aligned} d = \textstyle\int_0^\pi \sqrt{1 - (\mathrm{x coss} + 2z \cos 2s)^2} ds \cong \ldots = \\ = \tfrac{\pi}{4}\left(4 - x^2 - 4z^2 - \tfrac{3}{16} x^4 - 3x^2 z^2 - \tfrac{5}{64} x^6\right) \end{aligned} \tag{17}$$

Equaling the two expressions below and applying Taylor approximation:

$$x \cong a_o + a_2 z^2 + a_4 z^4 \Longrightarrow a_o = r,\ a_2 = -\frac{2}{r} - \frac{3r}{4},\ a_4 = \frac{-2}{r^3} \tag{18}$$

It is obtained x as an implicit function of z:

$$x \cong r + z^2\left(-\frac{2}{r} - \frac{3r}{4}\right) + z^4 \frac{-2}{r^3} \tag{19}$$

Now we can obtain the elastic energy depending only on z:

$$V_E \cong \ldots = \text{constant} + \left(3 + \frac{13}{8} r^2\right) z^2 \tag{20}$$

The total energy is the addition of the potential and the elastic energy:

$$V = V_P + V_E \cong \text{constant} - 2m\varepsilon z + \left(\left(3 + \frac{13}{8} r^2\right) - m\left(\frac{2}{r} + \frac{3r}{4}\right)\right) z^2 - \frac{2m}{r^3} z^4 \tag{21}$$

which is the expression of the Thom's 2nd catastrophe with vertex (supposing $\varepsilon = 0$):

$$m_o = \left(3 + \frac{13}{8} r^2\right)\left(\frac{2}{r} + \frac{3r}{4}\right)^{-1} \cong \frac{3}{2} r - \frac{1}{4} r^3 \tag{22}$$

Consequently:

$$V \cong -\frac{3}{r^2} z^4 - \frac{2}{r}(m - m_o) z^2 - 2r\varepsilon z \tag{23}$$

The maxim load decreases quickly when ε increases, as it represents the Thom's cusp represented in the following figure (Figure 7):

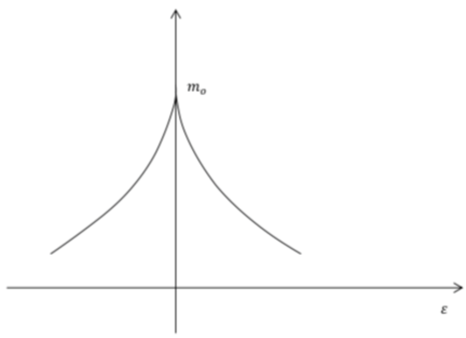

Figure 7. Thom's cusp.

3.1.3. Application 3: Crank and Connecting Rod

This exercise, explained in session 4, is an example of the implicit function theorem, which has many applications in mechanic in order to relation the different parameters operating in a mechanism.

This application is the crank/connecting rod system of explosion motors (see Figure 8), which consists of:

- one crank moving with an angle θ;
- one connecting rod whose movement depends on the crank turn;
- one piston moving horizontally on an axis.

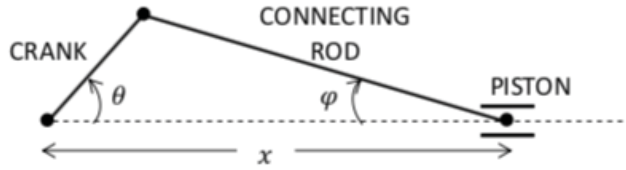

Figure 8. Crank/connecting rod system.

Supposing that the crank length r and the connecting rod length L are known, this system has three position parameters:

- x: piston distance to the crank turn center;
- θ: crank angle;
- φ: connecting rod angle.

These three parameters are related according to the following expressions:

$$\begin{cases} x = r\cos\theta + L\cos\varphi \\ r\sin\theta = L\sin\varphi \end{cases} \tag{24}$$

There are three parameters to determine the position and two equations which relate them. One of the three parameters could be expressed in function of the other two parameters and act as a control parameter determining those two parameters following the equations below.

Fixing the value of one from the three variables, we would obtain a system with two equations and two unknown factors, which would have a unique solution.

Applying the implicit function theorem:

$$(x, \theta, \varphi) \xrightarrow{f} (x - r\cos\theta - L\cos\varphi, r\sin\theta - L\sin\varphi) \tag{25}$$

Calculating the derivative matrix:

$$Df = \begin{pmatrix} 1 & r\sin\theta & L\sin\varphi \\ 0 & r\cos\theta & -L\cos\varphi \end{pmatrix} \tag{26}$$

According to this theorem, one variable acts as implicit (control variable) if the minor formed by the other columns is different to zero.

If we calculate the minor of the variable x:

$$\begin{gathered} \det\begin{pmatrix} r\sin\theta & L\sin\varphi \\ r\cos\theta & -L\cos\varphi \end{pmatrix} = -rL(\sin\theta\cos\varphi + \cos\theta\sin\varphi) = \\ = -rL\sin(\theta+\varphi) \neq 0 \text{ if}(\theta+\varphi) \neq 0, \pi \end{gathered} \tag{27}$$

Therefore, x acts as a control parameter except for the neutrals:

$$(\theta+\varphi) = 0\,, \pi \Longleftrightarrow x = \begin{matrix} L + r \\ L - r \end{matrix} \tag{28}$$

Indeed, the crank turn can be reversed in neutrals.

If we calculate the minor of the variable θ:

$$\det\begin{pmatrix} 1 & L\sin\varphi \\ 0 & -L\cos\varphi \end{pmatrix} = -L\cos\varphi \neq 0 \tag{29}$$

Therefore, the crank angle θ is a control parameter for all the values.

3.1.4. Application 4: Articulated Arm

However, in order to simplify the above computation and the further ones, a key tool is the matrix of the linear map. Let us obtain the matrix of f in ordinary basis.

This example, explained in session 5, is an application of inverse kinetics, which is, calculating input position, speed, etc., from outputs position, speed, etc.

This application explains the work of a robot articulated arm, whose scheme is represented in the following figure (Figure 9), which is composed by:

- a shoulder situated in the coordinates origin;
- an upper arm with length 5 and an angle $\theta > 0$ from the vertical;
- an elbow situated at the end of the upper arm;
- a lower arm with length 4 and angle $\varphi < \pi$ from the upper arm;
- a hand situated at the end of the lower arm, in the coordinates (x, y);
- torsion motors in the articulations (the shoulder and the elbow).

The analysis consists of a direct and an inverse kinetics study of the articulated arm.

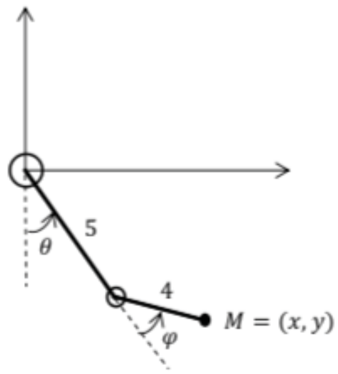

Figure 9. Articulated arm.

The direct kinetics study obtains the hand position from the shoulder and elbow angles, as it is calculated hereunder.

$$]0, \pi[\mathbb{x}]0, \pi[\xrightarrow{f} \Omega. \tag{30}$$

$$(\theta, \varphi) \rightarrow (x, y) \tag{31}$$

$$\begin{cases} x = 5\sin\theta + 4\sin(\theta + \varphi) \\ y = 5\cos\theta + 4\cos(\theta + \varphi) \end{cases} \tag{32}$$

The hand speed is calculated applying the chain rule, as it indicated hereunder:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = (Df)\begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix}, \; Df = \begin{pmatrix} 5\cos\theta + 4\cos(\theta + \varphi) & 4\cos(\theta + \varphi) \\ -5\sin\theta - 4\sin(\theta + \varphi) & -4\sin(\theta + \varphi) \end{pmatrix} \tag{33}$$

what is really interesting in robots is calculating the shoulder and the elbow rotor speeds from the hand position, that is, the inverse kinetics study. To obtain these speeds, it is necessary to apply the chain rule, the inverse function theorem and the implicit function theorem, as it is carried out in the following example.

If the output is $M = (5, 4)$, it is asked to obtain the shoulder and the elbow speeds $\dot{\theta}$ and $\dot{\varphi}$.

It is clear to see that is a functional dependence between the hand position (x, y) and the shoulder and the elbow positions (θ, φ) since there is only one possible triangle which determine the hand position from the shoulder and the elbow positions. Therefore

$$\Omega \xrightarrow{f^{-1}}]0, \pi[\mathbb{x}]0, \pi[\tag{34}$$

$$(x, y) \to (\theta, \varphi) \tag{35}$$

Applying the inverse function theorem, it is confirmed the f^{-1} derivability:

$$\begin{aligned} \det Df &= -20\cos\theta\ \sin\ (\theta+\varphi) + 20\sin\theta\ \cos\ (\theta+\varphi) = \\ &= -20\sin(\theta - (\theta+\varphi)) = -20\sin\varphi \neq 0 \end{aligned} \tag{36}$$

The relation between the shoulder and the elbow speeds, and the hand speed is:

$$\begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix} = (Df)^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \tag{37}$$

The hand position $M = (5,4)$ corresponds to the angles $= \frac{\pi}{2}$, $\varphi = \frac{\pi}{2}$.
Replacing in the expression below, it is obtained that:

$$\begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ -5 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 0 & -4 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \tag{38}$$

$$\dot{\theta} = -\frac{1}{5}\dot{y} \tag{39}$$

$$\dot{\varphi} = \frac{1}{4}\dot{x} + \frac{1}{5}\dot{y} \tag{40}$$

3.1.5. Application 5: Electrical Dispatch

This example, explained in session 7, is about the most important goal of engineering, which is the optimization of all the technological process.

The problem of the electrical dispatch deals with assigning the electrical central productions to the required power. All the distributions companies need to calculate the production of each supply central $P_1, \cdots, P_n$ to cover the instant demand P.

In each moment, it must be decided, which centrals act and with what power, considering the cost of productions of those supply centrals. The objective is reaching the minimum production cost.

The production cost of each supply electrical central is defined by the expression:

$$C_j = \alpha_j + \beta_j\, P_j + \gamma_j P_j^2, \quad 1 \leq j \leq n, \alpha_j,\ \beta_j, \gamma_j > 0 \tag{41}$$

The problem is, if we have several supply electrical centrals which have quadratic production costs and there is a certain demand P lower than the maximum, knowing the power distribution of the different centrals and the first central that must be stopped.

To illustrate the solving of this problem, we are going to use an example with only three supply electrical centrals, whose costs are hereunder indicated:

$$C_1 = 7 + P_1 + P_1^2. \tag{42}$$

$$C_2 = 4 + 2P_2 + 2P_2^2 \tag{43}$$

$$C_3 = 2 + 4P_3 + 3P_3^2 \tag{44}$$

Total power is the three powers sum:

$$P = P_1 + P_2 + P_3 \tag{45}$$

Total cost production of the three centrals is the cost productions sum:

$$\begin{aligned} C &= \left(7 + P_1 + P_1^2\right) + \left(4 + 2P_2 + 2P_2^2\right) + + \left(2 + 4(P - P_1 - P_2) + 3(P - P_1 - P_2)^2\right) = \\ &= 13 + 4P - 3P_1 - 2P_2 + P_1^2 + 2P_2^2 + 3(P - P_1 - P_2)^2 \end{aligned} \tag{46}$$

To minimize the cost, the cost partial derivatives are calculated and equaled to zero:

$$D_1C = -3 + 2\,P_1 - 6(P - P_1 - P_2) = -3 - 6P + 8P_1 + 6P_2 = 0 \tag{47}$$

$$D_2C = -2 + 4\,P_2 - 6(P - P_1 - P_2) = -2 - 6P + 6P_1 + 10P_2 = 0 \tag{48}$$

The system obtained is compatible determined, therefore has a unique solution, which is:

$$\begin{array}{l} P_1^* = \frac{1}{22}(9 + 12P) \\ P_2^* = \frac{1}{22}(-1 + 6P) \end{array} \Rightarrow P_3^* = \frac{1}{22}(-8 + 4P) \tag{49}$$

This solution is valid only if $P_1^*, P_2^*, P_3^* \geq 0$:

$$P_1^* > 0,\ \forall P \tag{50}$$

$$P_2^* > 0 \Leftrightarrow P \geq \frac{1}{6} \tag{51}$$

$$P_3^* > 0 \Leftrightarrow P \geq 2 \tag{52}$$

Therefore, the solution is valid only if $P \geq 2$.

If P decreases under $P = 2$, the solution below is not valid. From this value, P_3^* turns to be negative, what indicates that the third central must be the first central to stop.

In this case only the other two centrals act and the production cost is:

$$C = 7 + P_1 + P_1^2 + 4 + 2(P - P_1) + 2(P - P_1)^2 \tag{53}$$

The cost derivative calculation equaled to zero is:

$$P_1^* = \frac{1}{6}(1 + 4P) \tag{54}$$

$$P_2^* = \frac{1}{6}(-1 + 2P) \Leftrightarrow \frac{1}{2} \leq P \leq 2 \tag{55}$$

$$P_3^* = 0 \tag{56}$$

Therefore, if P decreases under $P = \frac{1}{2}$, P_3^* turns to be negative, what indicates that the second central must stop.

In this case, the only one central which supplies power is the first central and in this case the distribution is:

$$P_1^* = P,\ P_2^* \ = P_3^* \ = 0 \text{ si} P \leq \frac{1}{2} \tag{57}$$

3.2. Students' Surveys and Interviews Results

Up to now, two editions of the seminar "Application of Multivariable Calculus in Engineering" have been held, corresponding to the second semester of the 2019/2020 and 2020/2021 academic years. The contents explained in these sessions has been studied considering the answers to the anonymous questionnaires and to the personal interviews conducted to students.

Students' surveys of the sessions undertaken until now have been analyzed. The surveys were held in the 2019/2020 and 2020/2021 academic years, after each of the sessions. The number of attending students to the sessions has been 16 and all of them have been participants in the surveys. The results obtained in these two academic years did not have relevant differences. In the following figures the answers to each question for all the sessions in both years are presented. So, for each figure, 256 represents the total number of cases, which are the answers of 16 students in each of the 8 sessions and during two academic years.

The answers to the first question (Figure 10) show that most of the students, almost 90% of the total 256 answers of students, agree with the mathematical contents developed in the sessions.

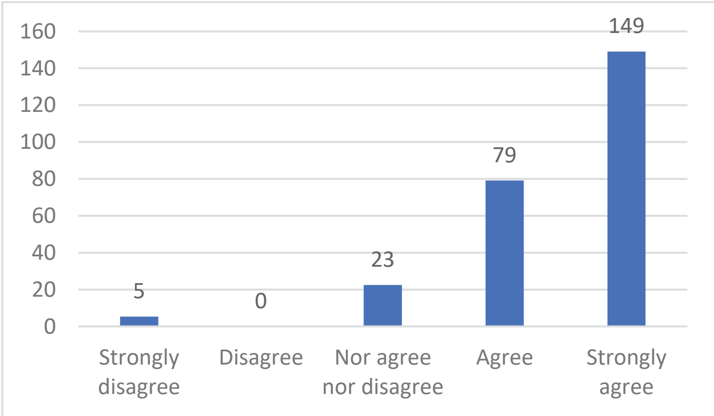

Figure 10. Answers to question 1: The assessment of mathematical contents is positive.

Likewise, in the answers to the second question (Figure 11), it can be observed that almost 90% of the total 256 answers of students agree with the engineering contents explained in the sessions.

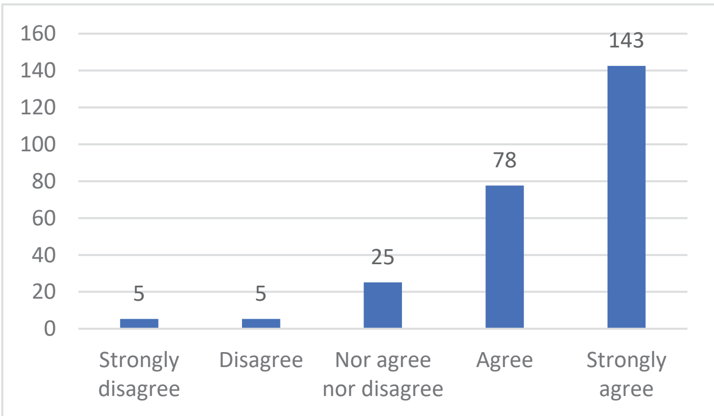

Figure 11. Answers to question 2: The assessment of engineering contents is positive.

According to the answers to question 3, almost 90% of the total 256 answers of students think that the sessions "Applications of Multivariable Calculus in Engineering" let them know technological applications of different mathematical concepts (Figure 12).

Almost 70% of the total 256 answers of students agree that applications of mathematical concepts achieve to increase their motivation to the subject Multivariable Calculus, as the answers to question 4 (Figure 13) show.

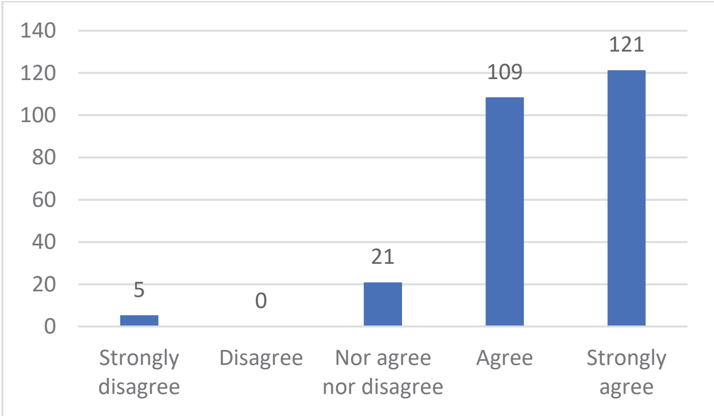

Figure 12. Answers to question 3: The sessions "Applications of Multivariable Calculus in Engineering" let students know technological applications of different mathematical concepts.

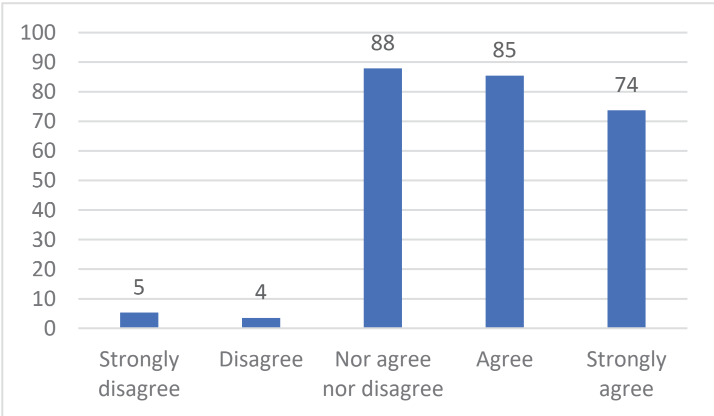

Figure 13. Answers to question 4: The applications of mathematical concepts achieve to increase the motivation to the subject Multivariable Calculus.

More than 70% of the total 256 answers of students state that the execution of practical exercises with technological applications improves the learning of mathematical concepts (Figure 14).

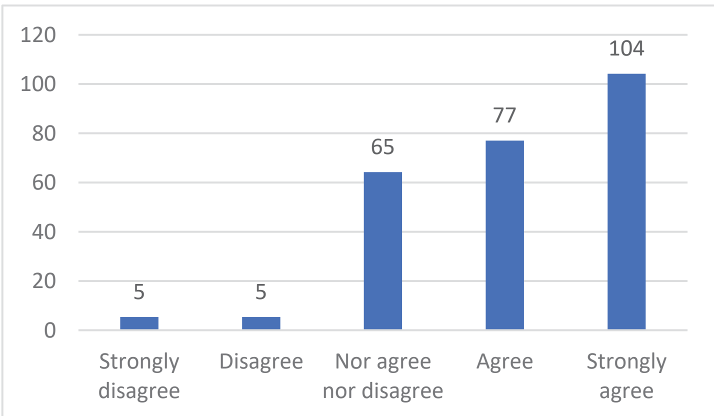

Figure 14. Answers to question 5: The execution of practical exercises with technological applications improve the learning of mathematical concepts.

The response of the attending students to these sessions in 2019/2020 and 2020/2021 academic years has been very positive. As can be observed in the above figures, the number of students who agree or strongly agree with the statements about the seminar is higher than the number of students who nor agree nor disagree, except for the question 4 (Figure 13). The reason is that some students have answered that although they assess positively the contents of the seminar, they were already motivated to the study of Calculus Multivariable before attending to the seminar.

It is also worth mentioning some students' comments expressed in the open questions asked in the anonymous surveys in both years, such as:

- These sessions let know real applications of mathematics in engineering, which gives more sense to the study of mathematics.
- Discovering that discontinuous phenomena produced in engineering processes can be modeled by mathematical theories increases the motivation towards the learning of mathematics.
- The use of mathematical concepts in technological applications, as they are implicit function theorem or Taylor and Fourier series, let students realize about the need of mathematics in engineering.
- Applications of Multivariable Calculus in mechanics and robotics increase the curiosity and the interest of students towards mathematical subjects.

The information extracted from students' answers in personal interviews in both academic years is presented hereunder:

- The real applications shown in the sessions "Applications of Multivariable Calculus in Engineering" let students realize of the usefulness of mathematics for their degree and for their future career.
- Applications studied in this seminar have been very practical and students will use them in their future profession. Learning to solve real engineering problems shows students how essential mathematical subjects are for engineers.
- Seeing how mathematics can be applied in engineering motivates to learn mathematics in order to be able to use them in the future as engineers.
- Seeing technological applications of mathematics increases the interest towards the subject.
- These applications help students understand related mathematical concepts as the implicit function theorem, the Fourier series or the calculations of maximums and minimums in functions defined in compact sets.
- Interesting applications: Zeeman machine solved with Thom's catastrophes theory and Taylor series and the crank/connecting rod system of explosion motors using the implicit function theorem.
- It has been very impressive knowing no technological applications of Thom's catastrophes, such as the analysis of dogs' behavior and sociological applications.
- Students knew that mathematics were necessary for engineering but, attending this seminar, they have discovered that mathematics are also necessary for other different disciplines.
- Mathematics are not subjects to prepare students for beginning the degree, mathematics are applications in the future work of engineers.

4. Discussion

In this work we contribute to develop an integrated STEM curriculum, introducing an implementation of mathematical applications integrated in STEM education. This study provides a connection of mathematical subjects with technological disciplines and with engineering careers, with the objective of enhancing the motivation and engagement of engineering students.

In the present engineering curriculum, the first two academic courses content very few engineering subjects, but consist of mathematics, science, communications and electives subjects. With the implementation developed in this work, mathematical subjects should cover real applications related to the main area of students enrolled degree, offering a wider view in STEM education [37,38], what would involve the improvement of the understanding and learning of mathematical concepts, as [39] states.

The main issue of this work has been the relevance of solving real and technological problems in the teaching of mathematics, since students' analytical thinking skills are enhanced with the use of mathematical problem solving [29,30,40]. In addition, the implementation of real and practical problems in basic sciences subjects promote student engagement and motivation in STEM degrees [14,19–21]. Considering the results shown in Figure 13, it can be seen that almost the 70% of the total 256 answers of students agree that applications of mathematical concepts achieve to increase their motivation to the subject of Multivariable Calculus, what will lead to reduce dropout, since it is connected to motivation [14], student achievement [17] and academic performance [11]. In the results of Figure 14, it has been shown that according to more than the 70% of the total 256 answers of students, the applications explained in the seminar, let them learn mathematical concepts trough practical examples. This fact increases their motivation to mathematics, as it is confirmed in previous studies such as [28]. In addition, as shows Figure 12, almost the 90% of the total 256 answers of students state that with this sessions they have known multiple real application of Multivariable Calculus in engineering and other disciplines, what attain to encourage and motivate them to the learning of the subject, as it was analyzed in several studies [14,19,20].

The answers to the questions taken to the students in the personal interviews after the sessions "Applications of Multivariable Calculus in Engineering", show that most of the practical problems have impressed students because they have discovered that Multivariable Calculus have applications in many different areas. Moreover, it is to highlight that for students is really motivating to know what they will be capable to do in the next academic courses, using the concepts of Multivariable Calculus. They also realized of how essential Multivariable Calculus is for their future career and increased their interest towards the subject.

The results obtained in this study support that this experience contribute to an improvement of students' learning of mathematical concepts, as it was concluded in [33], which involves the increase of students' performance in mathematical subjects of engineering degrees, as it was studied in previous works such as [25].

5. Conclusions

This study was carried out at the Universitat Politècnica de Catalunya-BarcelonaTech (UPC), a university focused on STEM fields. The work is based on the teaching of Multivariable Calculus by the execution of real and technological applications where Calculus concepts are necessary to solve them. The aim of this work is to generate and integrated STEM curriculum, presenting a contribution about the relationship among mathematical applications and STEM education. The work provides evidence that it is possible to increase students' motivation through the implementation of engineering applications in the learning of mathematics, what could imply an improvement of the learning of mathematics and therefore, an increase of students' performance and a decrease of the dropout in the first academic courses of engineering degrees. This entails a rise of interest in STEM degrees, which are essential for the economic growth of technological countries.

In view of the success of the seminar "Applications of Multivariable Calculus in Engineering", more real applications are planned to be developed. These sessions are going to be repeated in the second semester of the next academic year 2021/2022. Likewise, the seminar "Applications of Linear Algebra in Engineering" is going to be repeated in the first semester of the next academic year. These two seminars cover the most mathematical subjects of the first academic course in engineering degrees.

It is also planned to conduct surveys and interviews to the students attending the seminar of the following academic year with the aim of collecting a greater sample of surveys results and more information about students' experience in these sessions.

With the results obtained, it is expected that the contents developed in this work will be included in a future adaptation of mathematical subjects' syllabus in engineering degrees.

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3.3. Fomentar la motivación y la implicación en los Grados STEM a través de realización de aplicaciones reales en las asignaturas matemáticas: el problema de la migración de población

mathematics

Article

Encouraging Students' Motivation and Involvement in STEM Degrees by the Execution of Real Applications in Mathematical Subjects: The Population Migration Problem

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Abstract: This paper presents a simplified model of the population migration problem, addressed to first-year engineering students in order to show them the use of linear algebra tools. The study consists of predicting the census in the city centre and in the suburbs, determining the city population equilibrium point, and making a sociological interpretation of population flows. This practical problem is part of the seminar "Applications of Linear Algebra in Engineering", which is being held at the Universitat Politècnica de Catalunya-BarcelonaTech (UPC). This seminar consists in the learning of linear algebra by the implementation of real applications where mathematical tools are required to resolve them. This paper presents an application of linear algebra to the population migration problem and analyses students' appreciation through anonymous surveys and personal interviews. The surveys assessed students' motivation towards the subject of linear algebra and their learning of mathematical concepts. Personal interviews were conducted for students in order to let them express in detail their opinion about the seminar. The results confirm that the introduction of real applications in the learning of mathematics increases students' motivation and involvement, which implies an improvement in students' performance in the first courses of STEM degrees.

Keywords: engineering; linear algebra; mathematics; population migration; STEM; students' motivation

MSC: 97D30; 97D40; 97H60; 97M10; 97M50

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1. Introduction

For a long time, there have been discussions about the syllabus of mathematical subjects in first-year courses of engineering degrees regarding the contents needed for technological disciplines and how to teach them to make students aware of this necessity [1].

Currently, there is an increasing concern that derives from the alarming fact that European undergraduate students show less interest in STEM (sciences, technology, engineering, and mathematics) fields compared to other fields [2]. Therefore, STEM education professionals are working to identify the causes of this fact [3] and have determined that there are various interrelated factors, both extrinsic (science capital, learning opportunities, socio-economic status) and intrinsic (interest linked to self-efficacy issues, attitude towards science, perceived social expectations) [4].

In addition, high occupancy demands for STEM professions are foreseen [5], and it is expected to increase in the forthcoming years. Likewise, technological skills are becoming more specialized [6], and thus STEM professions are essential for technological development and economic growth of present societies [2].

The low interest in STEM disciplines is mainly due to the high failure rate in the first courses of STEM degrees [7,8], especially in mathematical subjects. This is considered one of the main difficulties for engineering students in first courses of their degrees [9], which entails a loss of student engagement [10] in the first year courses and in most cases results in the dropout of these degrees [11].

The influence of students' sense of belonging in the classroom should also be considered in improving students' motivation and involvement [12], which is especially relevant in STEM degrees [13]. The sense of class belonging is directly linked to engagement and can be as important to persist in STEM degrees as self-efficacy and intrinsic interest in these fields [7]. To settle this failure rate, it is necessary to revise mathematical subjects' syllabi and teaching methodologies as well as to encourage students' motivation and engagement [11,14,15], giving them awareness of the connections between mathematics and technological disciplines and with their future careers and illustrating real applications of the concepts and tools learnt in mathematical subjects.

Mathematics is fundamental to many professions, especially science, technology, and engineering, and it is a gateway to many scientific and technological fields [16]. Mathematics should be an integrated part of STEM disciplines, such that it constitutes specific practices related to those fields [9]. It is essential to teach mathematics using real problems in order to achieve the involvement of students, ensuring that it is significant for them; it is also essential to develop students' mathematical thinking through practical applications, rather than teaching a set of disconnected concepts and skills. If mathematics was taught as an applied subject, connections between mathematics and STEM disciplines would be reinforced, as it is shown in [16].

There is a wide consensus about the promotion of STEM degrees by the encouragement of students' motivation and involvement through the contextualization of mathematical concepts in the technological disciplines of the following courses of these degrees. It is unquestionable that applied problems used in mathematical subjects of STEM degrees encourage students' motivation and engagement [11,13,17,18], as well as the improvement in their learning [19], which improves their academic performance [20–22].

There are several investigations that support the idea that active learning has a profitable effect on the improvement in students' engagement and in their performance, as indicated in [23–25]. The use of problem-solving methodologies in mathematical subjects of STEM degrees is a profitable tool to increase students' involvement. With problem-solving, students are given a problem based on a real situation, which will lead them to learn the required mathematical concepts to develop the problem. In addition, a problem-solving methodology will facilitate students' critical thinking skills, since they will have the opportunity to consider different assumptions and discuss and make decisions about this problem. Problem-solving also lets students be more active, both individually and in groups: they can exchange knowledge, engage in peer learning, and work together [26]. Therefore, problem-solving helps students acquire skills and competences such as autonomy, continuous learning, critical thinking, teamwork, planning, organization, and communications [26], which are essential for their careers [27,28].

The objective of this study is that students understand the need for mathematical concepts to solve practical problems. With this work, it is expected that students' motivation towards the learning of mathematics in first year courses will increase, and as a result, their performance is expected to improve, as confirmed by previous studies [21,29]. At the same time, problem-solving will let students develop required skills for the execution of their degree and for their engineering careers [30].

This article focuses on the analysis of the population migration problem:

- How can migration flow problems be solved using linear algebra?
- How do students of technological degrees value the implementation of this mathematical application in the learning of linear algebra?

2. Materials and Methods

This study is part of the work "Applications of Mathematics in Engineering" [22], which emerged with the proposal of providing connections between mathematics and STEM education, through the execution of real practical problems to engineering students, with the intention of improving their engagement in first year courses of engineering degrees. The aim of this work is to analyse and solve demographic problems using mathematical tools in order to illustrate applications of linear algebra for engineering students, with the objective that they will realize of the significance of mathematics for the development of their future profession, which will entail an improvement of students' motivation in the first courses of these degrees and therefore an increase in their academic performance.

The seminar is being taught at the Industrial Engineering Bachelor's Degree from the Barcelona School of Industrial Engineering (ETSEIB) of the Universitat Politècnica de Catalunya-BarcelonaTech (UPC), a public university specialising in STEM degrees. The work "Applications of Mathematics in Engineering" comprises two seminars, "Applications of Mathematics in Engineering I: Linear Algebra" and "Applications of Mathematics in Engineering II: Multivariable Calculus", which began in the 2019/2020 academic year. Each seminar is held in one semester (the first and the second, respectively) and consists of weekly sessions one hour and a half in length. These sessions have also been given in the 2020/2021 academic year.

The two seminars, "Applications of Mathematics in Engineering I: Linear Algebra" and "Applications of Mathematics in Engineering II: Multivariable Calculus" consist of voluntary sessions, aimed at first course undergraduate students of engineering degrees. These sessions are organized according to the contents of the Linear Algebra and Multivariable Calculus syllabus, respectively, in order to show students that the concepts they are learning in ordinary mathematical subjects are essential for their degree and for their future career.

The sessions "Applications of Mathematics in Engineering I: Linear Algebra", consists of 10 sessions, which are indicated in Table 1.

Table 1. Applications of Linear Algebra in Engineering ([22]).

Session	Title
1	"Complex Numbers on the Study of Price Fluctuations"
2	"Complex Numbers on the Study of Alternating Current"
3	"Indeterminate Systems: Control Variables"
4	"Mesh Flushes: a Basis of Conservative Fluxes Vector Subspace"
5	"Addition and Intersection of Vector Subspaces in Discrete Dynamical Systems"
6	"Linear Applications and Associated Matrix"
7	"Basis Changes"
8	"Eigenvalues, Eigenvectors and Diagonalization in Engineering"
9	"Modal Analysis in Discrete Dynamical Systems"
10	"Difference Equations"

This article focuses on one of the applications studied in two of the sessions of the seminar "Applications of Linear Algebra in Engineering". In each of these two sessions of this seminar—Session 6, "Linear Applications and Associated Matrix", and Session 8—"Eigenvalues, Eigenvectors and Diagonalization in Engineering"—the population migration problem was analysed and solved illustrating the use of mathematical concepts related to linear algebra. In Session 6, the population migration problem was studied using concepts related to linear applications and associated matrix. In Session 8, eigenvalues, eigenvectors, and diagonalization are used to analyse and solve this demographic problem.

The problem presented here is a simplified model of the population migration for educational purposes. It does not consider sociological factors of migration that would give rise to more complex models, since this is not the aim of the study.

With the aim of assessing students' appreciation of the population migration problem, anonymous questionnaires and personal interviews were conducted at the end of each session. The results obtained have contributed to analysing the material developed in these sessions.

Questionnaires assess the influence of this implementation on the students attending these two sessions, in relation to the execution of demographic problems, the motivation towards the subject of linear algebra, and the learning of mathematical concepts. These surveys consisted of several statements, which were valued on a 5-point scale (1 = Strongly disagree, 2 = Disagree, 3 = Nor agree nor disagree, 4 = Agree, 5 = Strongly agree). In addition, students could add a comment expressing their opinion and explaining their impression about each session.

In order to extract further information from the students attending the sessions, personal interviews were conducted, where they could express in detail their impression and opinion about these sessions. With the aim of evading partiality in their comments, the person who interviewed students was a doctoral student instead of a teacher.

The responses to the surveys and to the interviews undertaken after the sessions were analysed to assess the influence of the implementation of the population migration problem on the students attending these sessions.

3. Results

3.1. Analysis of Population Migration Using Linear Algebra

The 10 sessions of the seminar "Applications of Mathematics in Engineering I: Linear Algebra" consisted of applied problems related to mathematical concepts developed in the subject of linear algebra, whose content is organized into four mains topics:

1. Algebraic structures (complex numbers, matrix and determinants, and systems of equations);
2. Vector spaces and linear applications (vector spaces, vector subspaces, and linear applications);
3. Reduction of linear applications (diagonalization, eigenvalues, and eigenvectors);
4. Resolution of linear discrete dynamical systems (difference equations, and linear discrete dynamical systems).

A major part of the population migration problem introduced in this article includes linear algebra, as detailed below.

This exercise is intended to be developed as the student progresses in the Linear Algebra course, essentially in four phases:

(a) Basic concepts of vector space and linear application, which are explained in the second topic of the subject, are used in the development of the population migration problem (Sections 3.1.1–3.1.3).
(b) The matrix of an application and its use, learnt at the end of the second topic of the linear algebra subject, are concepts are necessary to solve the problems in Sections 3.1.4–3.1.6.
(c) Eigenvalues, eigenvectors, diagonalization, and the calculation of matrix powers, which are the concepts studied in the third topic of the subject, are used in Sections 3.1.7 and 3.1.8.
(d) Extensions: some of the concepts developed in Sections 3.1.9–3.1.12 are explained in the fourth topic of the Linear Algebra subject.

3.1.1. Description of a Population Migration Problem

Considering a population exchange city centre and periphery, assuming, to simplify, that the total number of habitants is constant and equal to 1,000,000 inhabitants, and supposing a decreasing census in the centre of the city—1,000,000 inhabitants in 2010, 600,000 inhabitants in 2015, and 400,000 inhabitants in 2020 (provisional) [31]—the following points must be answered:

1. Inhabitants in the centre of the city in 2025.
2. Inhabitants in the centre of the city in 2030, 2035, etc.
3. Equilibrium point.
4. In general, inhabitants in 2010 + 5k.

5. Which definitive census in 2020 would be alarming?
6. Sociological interpretation.

3.1.2. A Linear Algebra Approach

The aim is to answer the above questions by means of elementary Linear Algebra tools. Therefore, the first step is to model the population migration as a linear map.

$$\text{Vector space } E = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{array}{l} x_1 = \text{ inhabitants in centre} \\ x_2 = \text{ inhabitants in periphery} \end{array} \right\} \subset \mathbb{R}^2 \quad (1)$$

$$\underbrace{E}_{\text{inhabitants in } t} \xrightarrow{f} \underbrace{E}_{\text{inhabitants in } t+5} \quad (2)$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \xrightarrow{f} \begin{pmatrix} x_1(t+5) \\ x_2(t+5) \end{pmatrix} \quad (3)$$

If x_1, x_2 are expressed in terms of millions of inhabitants, the census data show that:

$$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{2010} \xrightarrow{f} \underbrace{\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}}_{2015} \xrightarrow{f} \underbrace{\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}}_{2020 \text{ (provisional)}} \longrightarrow \dots \quad (4)$$

A key point is that f can be assumed to be linear because of two natural hypotheses: homogeneity (same behaviour in any neighbourhood of the city) and proportionality (the behaviours do not depend on the number of inhabitants).

3.1.3. Inhabitants in the Centre in 2025

The first question can be answered by means of the basic formula for a linear map f:

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \dots) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + \dots \quad (5)$$

In our case, it allows us to compute $f\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$:

$$\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \Leftrightarrow \begin{array}{c} 0.4 = \alpha + 0.6\beta \\ 0.6 = 0.4\beta \end{array} \Leftrightarrow \begin{array}{c} \alpha = -1/2 \\ \beta = 3/2 \end{array} \quad (6)$$

$$f\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \alpha f\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta f\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \Rightarrow \quad (7)$$

$$\Rightarrow 300{,}000 \text{ inhabitants in the centre in 2025} \quad (8)$$

3.1.4. The Matrix

Below, the matrix of the linear map f in ordinary basis is obtained in order to simplify the computation:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \quad (9)$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \alpha' \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + \beta' \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \alpha' \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\prime \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \Leftrightarrow \begin{array}{c} \alpha' = -3/2 \\ \beta = 5/2 \end{array} \quad (10)$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \alpha' \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + \beta' \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = -\frac{3}{2}\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} + \frac{5}{2}\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.1 \\ 0.9 \end{pmatrix} \quad (11)$$

Hence, we can conclude:

$$\text{Matrix in ordinary basis } = \begin{pmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{pmatrix} \equiv A \tag{12}$$

In particular:

$$\begin{pmatrix} x_1(t+5) \\ x_2(t+5) \end{pmatrix} = \underbrace{\begin{pmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{pmatrix}}_{A} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{13}$$

3.1.5. Inhabitants in the Centre in 2030, 2035, etc.

By means of matrix A, the two first questions can be answered in a very simple way:

$$\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{2010} \stackrel{A}{\to} \underbrace{\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}}_{2015} \stackrel{A}{\to} \underbrace{\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}}_{2020} \stackrel{A}{\to} \underbrace{\begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}}_{2025} \stackrel{A}{\to} \underbrace{\begin{pmatrix} 0.25 \\ 0.75 \end{pmatrix}}_{2030} \stackrel{A}{\to} \underbrace{\begin{pmatrix} 0.225 \\ 0.775 \end{pmatrix}}_{2035} \to \ldots \Rightarrow \tag{14}$$

$$\Rightarrow 250,000 \text{ inhabitants in the centre in 2030,} \qquad 225,000 \text{ in 2035} \ldots \tag{15}$$

3.1.6. Equilibrium Point

Moreover, again by using A, the equilibrium point, if it exists, can be computed:

$$\begin{pmatrix} x_1{}^e \\ x_2{}^e \end{pmatrix} \to (A) \begin{pmatrix} x_1{}^e \\ x_2{}^e \end{pmatrix} \Leftrightarrow \begin{matrix} x_1{}^e = 0.6x_1{}^e + 0.1x_2{}^e \\ x_2{}^e = 0.4x_1{}^e + 0.9x_2{}^e \end{matrix} \Leftrightarrow \ldots \Leftrightarrow \begin{matrix} x_1{}^e = 0.2 \\ x_2{}^e = 0.8 \end{matrix} \tag{16}$$

Therefore, the equilibrium point is 200,000 inhabitants in the city centre.

3.1.7. Inhabitants in 2010 + 5k

However, let us see that this elementary use of A is not sufficient in order to solve question 4. As the calculation is done in five-year periods:

$$\begin{cases} t \; = \; 2010 \; + \; 5k, \; k = 0, \; 1, \; 2, \; \ldots \\ \quad x(k) \equiv x(2010 + 5k) \end{cases} \tag{17}$$

$$\underbrace{x(0)}_{2010} \stackrel{A}{\to} \underbrace{x(1)}_{2015} \stackrel{A}{\to} \underbrace{x(2)}_{2020} \to \ldots \tag{18}$$

$$\begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = A \cdots A \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = A^k \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = ? \tag{19}$$

3.1.8. Eigenvalues, Eigenvectors, and Diagonalization

In order to compute A^k, more sophisticated tools are needed, namely the techniques of diagonalization by means of the eigenvalues and eigenvectors. The eigenvalues are obtained by means of the characteristic polynomial:

$$Q_A(t) = \det \begin{pmatrix} 0.6 - t & 0.1 \\ 0.4 & 0.9 - t \end{pmatrix} = t^2 - \frac{3}{2}t + \frac{1}{2} = (t-1)\left(t - \frac{1}{2}\right) \Leftrightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = \frac{1}{2} \end{cases} \tag{20}$$

The eigenvectors corresponding to these eigenvalues are:

$$\begin{aligned} v_1 &= \mathrm{Ker}(A - 1I) = \left[\begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} \right] \\ v_2 &= \mathrm{Ker}\left(A - \tfrac{1}{2}I\right) = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \end{aligned} \tag{21}$$

We conclude:

$$S^{-1}AS = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}, \text{ being } S = \begin{pmatrix} 0.2 & 1 \\ 0.8 & -1 \end{pmatrix} \tag{22}$$

Thus:

$$A^k = S\begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}^k S^{-1} = S\begin{pmatrix} 1 & 0 \\ 0 & 1/2^k \end{pmatrix} S^{-1} \tag{23}$$

Finally:

$$x(k) = S\begin{pmatrix} 1 & 0 \\ 0 & 1/2^k \end{pmatrix} S^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \ldots = \begin{pmatrix} 0.2 + 0.8\frac{1}{2^k} \\ \ldots \end{pmatrix} \tag{24}$$

The solution obtained shows the millions of inhabitants in the city centre, in the quinquennium k. If we calculate it for different values of k, we see that it coincides with the results obtained before; see Table 2.

Table 2. Number of inhabitants in the city centre in the quinquennium k.

Quinquennium	Year	Inhabitants
$k = 0$	2010	1,000,000
$k = 1$	2015	600,000
$k = 2$	2020	400,000
$k = 3$	2025	300,000
$k = 4$	2030	250,000
$k = 5$	2035	225,000
...	...	...
$k \to \infty$	$\to \infty$	200,000

As shown in Table 2, if $k \to \infty$, the population in the city center tends to an equilibrium point with 200,000 inhabitants, which confirms the study done previously.

Indeed, the general theory of dynamical systems ensures the existence of an equilibrium point because $|\lambda_1|, |\lambda_2| \leq 1$.

3.1.9. Which Definitive Census in 2020 Would Be Alarming?

In addition, Question 5 can be answered: the situation would be alarming if the centre empties, that is to say, if the equilibrium point is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This happens for a transition matrix A' such that $A'\begin{pmatrix} 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$A' = \begin{pmatrix} 0.6 & a \\ 0.4 & b \end{pmatrix} \text{ in equilibrium with } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{25}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.6 & a \\ 0.4 & b \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow \begin{matrix} a = 0 \\ b = 1 \end{matrix} \tag{26}$$

This matrix is characterised by the census in 2020:

$$A' = \begin{pmatrix} 0.6 & 0 \\ 0.4 & 1 \end{pmatrix} \Leftrightarrow A'\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.36 \\ 0.64 \end{pmatrix} \tag{27}$$

Therefore, it would be alarming if a definitive census in 2020 showed 360,000 inhabitants instead of 400,000 inhabitants.

3.1.10. Sociological Interpretation

Finally, let us see the difference between cases A and A' from a sociological point of view:

$$\begin{array}{r}A: \ x_1(k+1) = 0.6x_1(k) + 0.1x_2(k) \\ A': x_1(k+1) = 0.6x_1(k) + 0x_2(k)\end{array} \tag{28}$$

By comparing A and A', it is deduced that alarm does not depend on the percentage of inhabitants who move from centre to periphery but it does depend on the percentage of inhabitants who return from periphery to centre.

3.1.11. Linear Systems Determined by Consecutive Values of the States

We consider homogeneous linear systems of the form

$$x(k+1) = Ax(k), \ k = 0, 1 \ldots, \ A \in M_n(\mathbb{R}) \tag{29}$$

where $x(k) \in \mathbb{R}^n$ is the state variable.

Sometimes it is difficult to empirically calculate the entries of the transition matrix A, whereas it is easy to measure consecutive values of the states:

$$x(0), \ x(1), \ x(2) \ldots \tag{30}$$

Let us see how the matrix A can be obtained from these data.

As in the above example, let us consider the linear map

$$f: \ \mathbb{R}^n \rightarrow \mathbb{R}^n \ , \ x(k) \overset{f}{\rightarrow} x(k+1) \tag{31}$$

Then, the consecutive values of the states can be seen as successive images of $x(0)$.

$$x(0) \overset{f}{\rightarrow} x(1) \overset{f}{\rightarrow} \ \ldots \ \overset{f}{\rightarrow} x(n-1) \overset{f}{\rightarrow} x(n) \tag{32}$$

If $x(0), \ldots, x(n-1)$ are linearly independent, it is clear that the matrix of f in this basis is:

$$\overline{A} = \begin{pmatrix} 0 & 0 & \cdots & 0 & a_1 \\ 1 & 0 & \cdots & 0 & a_2 \\ 0 & 1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_n \end{pmatrix} \tag{33}$$

being

$$x(n) = a_1 x(0) + \ldots + a_n x(n-1) \tag{34}$$

Notice that $\overline{A}$ is a Sylvester or companion matrix, such that it is a non-derogatory matrix (that is to say, each eigenvalue has a unique eigenvector), and the coefficients of its characteristic polynomial are just the opposite of the coefficients in the last column of the Sylvester matrix (see (40)).

From $\overline{A}$, the matrix A of f in the ordinary basis can be obtained simply as

$$A = S\overline{A}S^{-1}, \ S = (x(0) \ \ldots \ x(n-1)) \tag{35}$$

where the columns of S are the coordinates of the states $x(0), \ldots, x(n-1)$. Clearly the columns of the matrix $S\overline{A}$ are the coordinates of the states $x(1), \ldots, x(n)$.

Summarizing, one has:

Proposition 1. *Let*

$$x(k+1) = Ax(k), \ k = 0, 1 \ldots, \ A \in M_n(\mathbb{R}) \tag{36}$$

be an homogeneous linear system in $\mathbb{R}^n$, where

$$x(0),\ \ldots,\ x(n-1) \text{ are linearly independent vectors} \tag{37}$$

$$x(n) = a_1 x(0) + \ldots + a_{\mathrm{n}} x(n-1) \tag{38}$$

Then:

1. *The matrix of f in ordinary basis is:*

$$A = (x(1)\ \ldots\ x(n))\ (x(0)\ \ldots\ x(n-1))^{-1} \tag{39}$$

2. *The characteristic polynomial of A is:*

$$Q_A(t) = t^n - a_n t^{n-1} - \ldots - a_2 t - a_1 \tag{40}$$

where:

$$x(n) = a_1 x(0) + \ldots + a_n x(n-1) \tag{41}$$

In particular,

$$1 \text{ is an eigenvalue of } A \ \Leftrightarrow\ a_1 + \ldots + a_n = 1 \tag{42}$$

which is a necessary condition for the existence of an equilibrium point.

3. *A is a non-derogatory matrix, that is to say, each eigenvalue has a unique eigenvector.*

In particular, there is only a Jordan block for each eigenvalue, so that

$$A \text{ diagonalizes } \Leftrightarrow\ A \text{ has n distinct eigenvalues} \tag{43}$$

If in addition 1 is a dominant eigenvalue, then its eigenvector is the equilibrium point.

3.1.12. Learning Theory to Solve Problems

We have seen that concrete problems can be solved by means of elementary linear algebra. Now, let us see that the solution is easier if further linear algebra tools are used. This fact contrasts with some topics that focus the attention on the "problems", underestimating the "theory".

Indeed, by means of the above proposition, the computations in our problem can be simplified as follows

$$\overline{A} = \begin{pmatrix} 0 & -1/2 \\ 1 & 3/2 \end{pmatrix} \tag{44}$$

$$S = \begin{pmatrix} 1 & 0.6 \\ 0 & 0.4 \end{pmatrix} \tag{45}$$

$$A = \begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix} \begin{pmatrix} 1 & -3/2 \\ 0 & 5/2 \end{pmatrix} = \begin{pmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{pmatrix} \tag{46}$$

$$Q_A(t) = t^2 - \frac{3}{2}t + \frac{1}{2} = (t-1)\left(t - \frac{1}{2}\right) \tag{47}$$

Thus, the eigenvector with an eigenvalue of 1 is the equilibrium point:

$$x^e = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix} \tag{48}$$

3.2. Students' Surveys and Interview Results

The number of students attending the seminar "Application of Linear Algebra in Engineering" undertaken in the first semesters of the 2019/2020 and 2020/2021 academic years was 20 [22], all of them answered to the surveys and interviews, and the results achieved in both academic years did not show relevant differences.

The population migration problem has been treated in two sessions of this seminar as mentioned above: in Session 6, "Linear Applications and Associated Matrix" and in Session 8, "Eigenvalues, Eigenvectors and Diagonalization in Engineering".

The answers to some of the questions held in Sessions 6 and 8, where the migration population problem was studied, are shown.

According to the surveys answers, all the students thought that the sessions "Linear Applications and Associated Matrix" and "Eigenvalues, Eigenvectors and Diagonalization in Engineering" let them know of applications of demographic problems; see Figure 1.

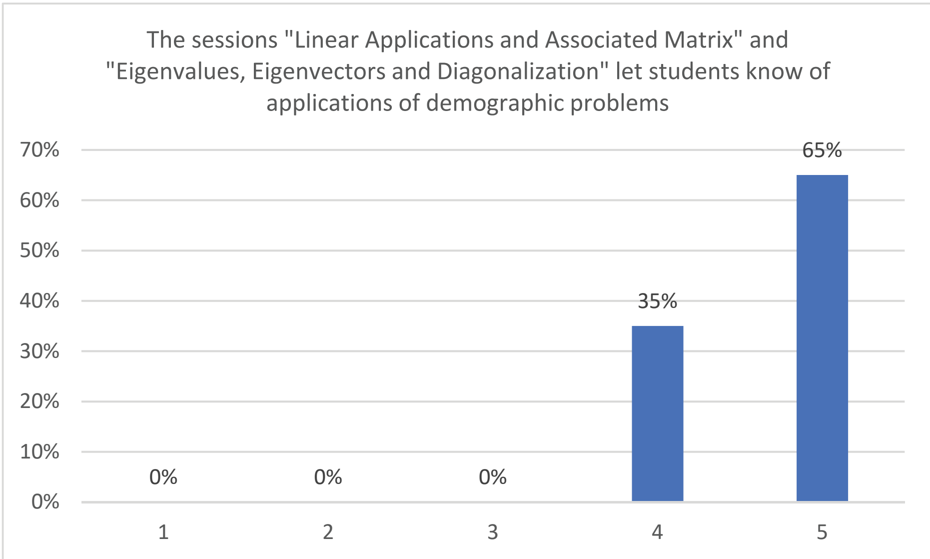

Figure 1. Answers to the question: the sessions "Linear Applications and Associated Matrix" and "Eigenvalues, eigenvectors and Diagonalization in Engineering" let students know of applications of demographic problems.

All the students agreed that the study of the migration population problem increased their motivation to understand the subject of linear algebra; see Figure 2.

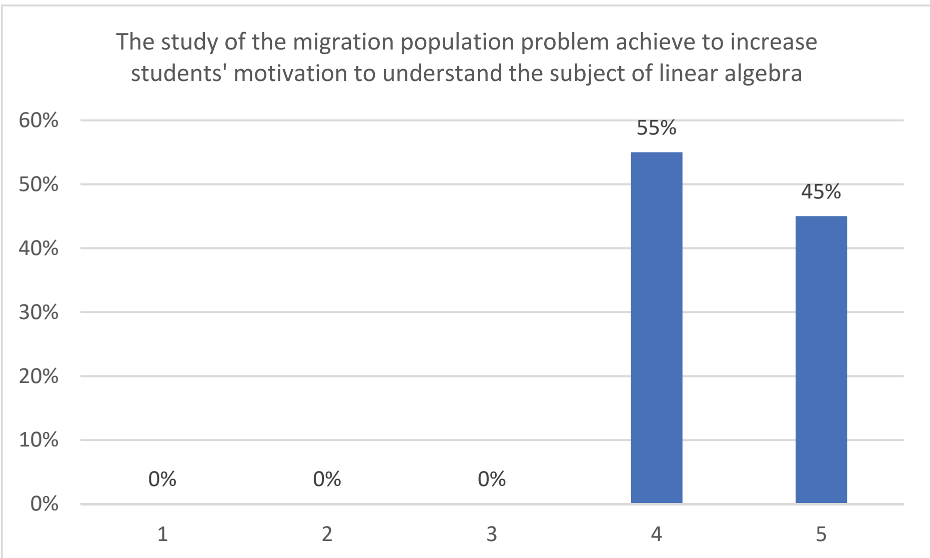

Figure 2. Answers to the question of whether the study of the migration population problem increased students' motivation to understand the subject of linear algebra.

Most of the students stated that the execution of the population migration problem improved their learning of mathematical concepts; see Figure 3.

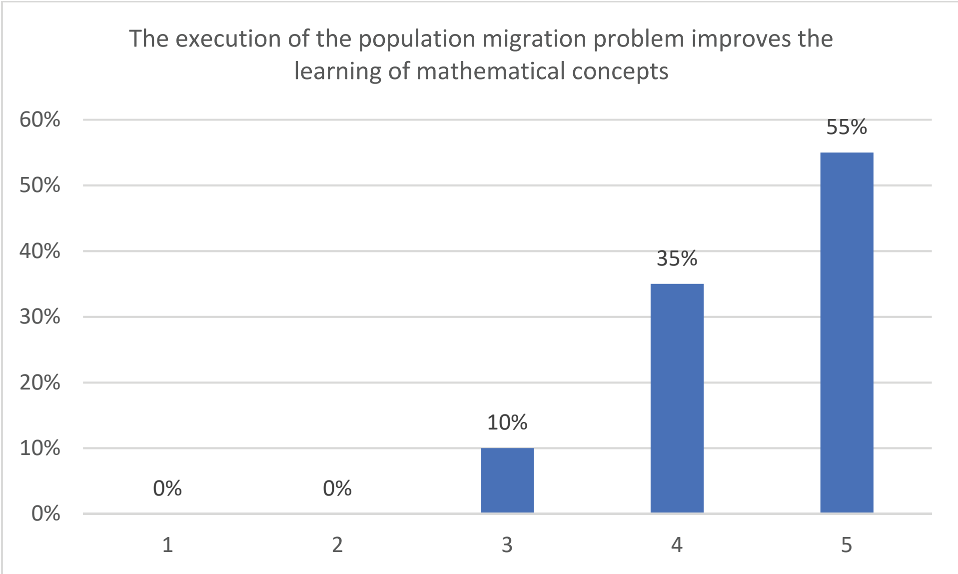

Figure 3. Answers to the question of whether the execution of the population migration problem improves the learning of mathematical concepts.

There are also noteworthy comments from students expressed in the open section of the anonymous questionnaires in the 2019/2020 and 2020/2021 academic years. The main themes were as follows:

- The population migration problem has helped students understand and learn the concepts of eigenvalues and eigenvectors.
- This application has let students realize of the importance of Linear Algebra to solve real social problems.
- Real applications, such as demographic control, motivated students to study linear algebra.

The information extracted from the students attending these two sessions was a follows:

- The sessions "Linear Applications and Associated Matrix" and "Eigenvalues, Eigenvectors and Diagonalization in Engineering" let students solve problems with real applications, such as the population migration problem.
- This application shows students what they will be capable of doing in the next courses, and it improves their motivation and involvement.
- The use of the concepts of linear algebra in the population migration problem, namely linear applications, matrices, eigenvalues, and eigenvectors, lets students know the importance of Linear Algebra for engineering degrees and for their STEM careers.
- Students classified the application of population flows and demographic control as one of the most interesting applications studied in the seminar.
- Students value positively knowing applications in sociology such as the population migration problem.

4. Discussion

Considering the students' survey and interview results, the implementation of the population migration problem in the seminar "Applications of Linear Algebra in Engineering" has been very positive, as explained below.

By analysing the results presented in Figure 2, it is shown that all the students agreed that the study of the population migration problem achieve to increase their motivation to the subject Linear Algebra, what implies the decrease of the dropout, as it is related to motivation [11], student achievement [14] and academic performance [13].

The results shown in Figure 3 confirm that 90% of the students think that the execution of the population migration problem let them learn mathematical tools through real applications. This improves their motivation in mathematical subjects, as is shown in previous studies such as [20].

Moreover, as Figure 1 shows, all the students stated that the sessions "Linear Applications and Associated Matrix" and "Eigenvalues, Eigenvectors and Diagonalization in Engineering" let students understand applications in engineering and other areas such as sociology, which motivated them to the study of mathematical subjects, as was confirmed in other works [11,17,18].

This work confirms the relevance of solving real problems in the teaching of mathematics, as shown in [26]. Problem-solving methodology improves students' critical thinking skills and provides key skills and competences for their future careers [27,28].

The responses to the questions given to the students in the personal interviews about the sessions "Linear Applications and Associated Matrix" and "Eigenvalues, Eigenvectors and Diagonalization in Engineering" show that the population migration problem amazed students because they were unaware of linear algebra's applications in demographic problems. It can also be emphasized that students' motivation and engagement increased when they realized the importance of linear algebra for their STEM degree and improved their interest towards the subject.

These findings demonstrate that the execution of the population migration problem improves the understanding of linear algebra concepts, as was shown in [19], which increases students' performance in mathematical subjects of STEM degrees, as was studied previously in [21].

5. Conclusions

This study was implemented at the Universitat Politècnica de Catalunya-BarcelonaTech (UPC). The objective of this work is to provide connections between mathematics and STEM education. The execution of the migration population problem in the seminar "Applications of Linear Algebra in Engineering" has let students know how to analyse demographic problems using concepts from linear algebra. This work confirms the possibility of illustrating in engineering degrees the concepts and basic results of linear algebra, such as the computation of the matrix by means of consecutive states, eigenvectors and eigenvalues, equilibrium points, and stability. Moreover, first-year engineering students have realized the importance of mathematics in solving real problems. Thus, students' involvement in mathematical subjects has improved, and therefore, their performance will increase. The exercises are appropriate for first-year students since they only require basic concepts of linear algebra to be solved.

In view of the success of the implementation of the population migration problem in the seminar "Applications of Linear Algebra in Engineering", the development of more real applications is planned. Furthermore, the project "Applications of Mathematics in Engineering", formed by this seminar "Applications of Mathematics in Engineering I: Linear Algebra" and the seminar "Applications of Mathematics in Engineering II: Multivariable Calculus", is programmed to be undertaken in future academic years. Likewise, surveys and interviews will be conducted among students attending the seminars, with the aim of analysing the applications that students consider more interesting and more useful for learning mathematics, as well as with the objective of collecting a greater sample of surveys results and more information about students' experience in these sessions.

This work confirms that it is possible to increase students' engagement through the introduction of STEM applications in the learning of mathematical subjects, which entails an enhancement of the learning of mathematics and, hence, an improvement in students' performance and a decline in the dropout in the early stages of engineering degrees. This involves an increasing interest in STEM degrees, which are considered essential for the technological development and economic growth of present-day society.

Since the results obtained confirm that the implementation of real applications increases students' motivation and involvement, which implies a decrease in the dropout rates of engineering degrees, the future aim of this project is to introduce the developed of applications in the seminar "Applications of Mathematics in Engineering I: Linear Algebra" in the ordinary syllabus of the Linear Algebra subject within the Industrial Engineering bachelor's degree from the Barcelona School of Industrial Engineering (ETSEIB) of the UPC. The impact of the inclusion of these applications in the linear algebra program will be assessed regarding the influence on students' motivation as well as on students' academic performance. If the results obtained are positive, there is a plan to introduce the applications developed in the seminar "Applications of Mathematics in Engineering II: Multivariable Calculus" in the subject of Calculus Multivariable within the Industrial Engineering bachelor's degree from the ETSEIB of the UPC.

In the same way, other mathematical applications linked to other studies in STEM fields could be developed and introduced into the mathematical subjects of the corresponding study plans of the Universitat Politècnica de Catalunya-BarcelonaTech (UPC), a public university specializing in STEM degrees. In this way, the connection of mathematics with the rest of the STEM disciplines would be much clearer, and students would be able to see the application of mathematics in their different careers from the first year of studies, which would contribute to an improvement in the quality of education.

These implementations will require the implication of the governing bodies of the UPC, who are acquainted with these seminaries and give support to proposals whose aims are the improvement of students' motivation, the increase in academic performance and the decrease in the dropout in the early stages of STEM degrees.

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4. DISCUSIÓN GENERAL Y CONCLUSIONES

4.1. Discusión general

En esta tesis se han desarrollado un conjunto de aplicaciones matemáticas relacionadas con las diferentes disciplinas de las carreras tecnológicas y se han analizado los efectos de la implementación de estas aplicaciones prácticas en el alumnado del primer curso del Grado en Ingeniería en Tecnologías Industriales de la ETSEIB (UPC). Este estudio ha conseguido conectar las asignaturas matemáticas con las disciplinas tecnológicas y con las carreras STEM, contribuyendo de esta forma a aumentar la motivación e implicación de los estudiantes de estas carreras. Este trabajo confirma la importancia de la resolución de problemas reales para los estudiantes de carreras tecnológicas en el aprendizaje de las Matemáticas, ya que además de mejorar su motivación hacia las asignaturas matemáticas, se fomentan habilidades mentales necesarias para ejercer estas profesiones.

4.1.1. Mejora de la motivación e implicación del alumnado

El alumnado que ha asistido a estos seminarios ha valorado positivamente los contenidos tanto matemáticos como ingenieriles desarrollados y explicados en cada una de las sesiones.

Los resultados de las encuestas realizadas al alumnado, muestran que la mayoría de los estudiantes que han asistido a estos seminarios, han aprendido mejor los conceptos matemáticos a través de las aplicaciones prácticas que han resuelto utilizando estos conceptos, como confirman estudios previos, en los que se ha analizado que los estudiantes universitarios mejoran la comprensión y el aprendizaje de las Matemáticas a través de problemas y aplicaciones reales, y gracias a ello consiguen aumentar su motivación (Gasiewski et al., 2012)(Alsina, 2007)(Cárcamo et al., 2017)(Kandamby, 2018)

Los estudiantes han destacado que han podido conocer múltiples aplicaciones tanto ingenieriles como de otras disciplinas no tecnológicas, lo cual les ha permitido darse cuenta de la utilidad de las Matemáticas para sus carreras y para otros ámbitos de la sociedad, y de esta forma los seminarios también han contribuido al aumento de su motivación hacia estas asignaturas. Tal y como ha sido analizado en otros trabajos, los/as alumnos/as se motivan cuando son conscientes de la importancia de las Matemáticas para su futuro profesional (Lopez, 2017) (Varsavsky, 1995)(Lepellere, 2021) (Fernández et al., 2018).

Considerando los resultados de las encuestas realizadas al alumnado, puede afirmarse que la realización de aplicaciones matemáticas aumenta la motivación e implicación de los estudiantes hacia estas asignaturas, tanto en Álgebra Lineal como en Cálculo Multivariable, lo cual implica un aumento del rendimiento académico y conlleva una reducción del abandono escolar, como muestran varios estudios realizados previamente (Wilson et al., 2015), (Kuh et al., 2016), (Tinto, 2016), (Gasiewski et al., 2012), (Handelsman et al., 2010).

Las respuestas del alumnado en las entrevistas llevadas a cabo después de los seminarios, demuestran que las aplicaciones desarrolladas en estos seminarios, tanto en el de Álgebra Lineal como en el de Cálculo Multivariable, han impresionado a los estudiantes, ya que desconocían que los conceptos matemáticos que forman parte de estas asignaturas, tuvieran tantas aplicaciones en áreas tan diversas. Cabe destacar, que los alumnos han expresado su aumento de motivación hacia las asignaturas matemáticas al ser conscientes de todo lo que serán capaces de hacer en los próximos cursos de sus estudios utilizando esos conceptos matemáticos. Los estudiantes han podido comprender lo esencial y necesarias que son las asignaturas matemáticas para la realización de su carrera y para su futuro profesional, lo cual ha aumentado su interés hacia las Matemáticas, tal y como confirmaban estudios anteriores (Maseda Fernández, 2011), (Abramovich et al., 2019), (Cardella, 2010).

Los resultados de este estudio confirman que la implementación de aplicaciones prácticas y reales en las asignaturas matemáticas de los primeros cursos de las carreras tecnológicas, mejora el aprendizaje de los conceptos matemáticos empleados, aumenta la implicación y motivación de los estudiantes hacia las asignaturas matemáticas, y por tanto mejora su rendimiento académico y disminuye el abandono escolar en los primeros cursos de las titulaciones STEM. Todo ello implica la mejora de la calidad de la enseñanza en las carreras STEM y un aumento del interés de los estudiantes hacia estas carreras.

La mejora de la calidad de la educación en las carreras STEM conlleva la consecución del ODS 4 (educación de calidad), y además contribuye a alcanzar otros objetivos marcados por la Unesco en la Agenda 2030 (Unesco, 2017), ya que la educación en las disciplinas STEM juega un papel clave en la consecución de dichos objetivos. La educación STEM trata de elaborar y proporcionar soluciones innovadoras para resolver problemas globales, especialmente aquellos directamente relacionados con el ODS 2 (hambre cero), el ODS 3 (buena salud y bienestar), el ODS 6 (agua limpia y saneamiento), el ODS 7 (energía asequible y limpia), el ODS 9 (industria, innovación e infraestructura), el ODS 12 (producción y consumo responsables), ODS 13 (acción por el clima), ODS 14 (vida bajo el agua) y el ODS 15 (vida de ecosistemas terrestres). Además, el ODS 8 (trabajo decente y crecimiento económico) y el ODS 11 (ciudades y comunidades sostenibles) dependen directamente del progreso de las disciplinas STEM. En el contexto de la Industria 4.0, la contribución de las carreras STEM para lograr los ODS es crucial (Ng, 2019).

Asimismo el aumento de interés por las carreras STEM supone un incremento del número de profesionales STEM, los cuales son necesarios para enfrentarse a los desafíos actuales y futuros de las países (Drymiotou et al., 2021), que dependen cada vez más de la tecnología, y que necesitan especialistas con habilidades y competencias en las disciplinas STEM para resolver estos desafíos (Ng, 2019)(Ministerio de Educación y Formación Profesional, 2021). El aumento de profesionales STEM implica la mejora en el desarrollo tecnológico, y por tanto, el crecimiento económico de los países (Joyce, 2014).

4.1.2. Desarrollo de aplicaciones matemáticas reales

Con este trabajo se han desarrollado una colección de aplicaciones prácticas de dos de las asignaturas matemáticas del primer curso de ingeniería, Álgebra Lineal y Cálculo Multivariable, las cuales además de ser adecuadas para los estudiantes de primer curso, están acompañadas de guías técnicas explicativas dirigidas tanto al alumnado como al profesorado que pueda impartirlas en el futuro y no sea especialista en las disciplinas en que las aplicaciones están contextualizadas.

Estas aplicaciones prácticas tratan problemas reales ingenieriles, tecnológicos y de otros ámbitos de la vida cotidiana, que necesitan conceptos y herramientas del Álgebra Lineal y del Cálculo Multivariable para ser resueltos. El conjunto de aplicaciones reales y prácticas desarrollado en esta tesis complementa y amplia los contenidos para la contextualización del Álgebra Lineal (Alsina & Trillas, 1984), (Ferrer et al., 2010), (Ferrer et al., 2012), (Peña et al., 2020), (Peña, 2018), (Cárcamo et al., 2017), (Morán, 2016), (García Planas et al., 2018) y del Cálculo Multivariable (Barrow & Fulling, 1998), (Lepellere, 2021) en las diferentes disciplinas de las carreras tecnológicas, y colabora con la integración y conexión de las Matemáticas con el resto de asignaturas que componen estos estudios.

Esta tesis contribuye, con la elaboración de este material, al futuro desarrollo de nuevos programas de las asignaturas matemáticas de primer curso de las carreras tecnológicas, en los cuales estén integradas las disciplinas tecnológicas que constituyen estos estudios.

4.2. Conclusiones e investigaciones futuras

El presente estudio ha sido llevado a cabo en la UPC. El trabajo ha consistido en la enseñanza de dos asignaturas matemáticas de primer curso de Ingeniería Industrial, Álgebra Lineal y Cálculo Multivariable, implementando la realización de aplicaciones prácticas y reales, donde los conceptos desarrollados en estas asignaturas son necesarios para resolverlas.

Los resultados obtenidos en este trabajo aportan nueva información sobre la experiencia educativa del alumnado de Ingeniería en las asignaturas de Álgebra Lineal y Cálculo Multivariable. Con este estudio se ha comprobado que la contextualización de las asignaturas matemáticas Álgebra Lineal y Cálculo Multivariable, ha conseguido aumentar la motivación del alumnado hacia estas asignaturas así como ha mejorado el aprendizaje de los conceptos desarrollados en estas asignaturas, lo que conlleva un aumento en el rendimiento académico y una disminución de la tasa de abandono en los primeros cursos de las carreras tecnológicas. Todo esto supone un aumento en el interés de las carreras STEM, que implica un mayor número de profesionales STEM y por consiguiente contribuye al desarrollo tecnológico y económico de la sociedad.

Por todo ello, se espera que los contenidos desarrollados en esta tesis puedan ser incluidos en futuras adaptaciones de los programas de las asignaturas Álgebra Lineal y Cálculo Multivariable, y que conlleven la mejora de la enseñanza de las carreras tecnológicas y el aumento de la tasa de éxito en estos estudios.

Asimismo, la mejora de la educación en las carreras tecnológicas, consideradas fundamentales para el aprendizaje y la educación en el desarrollo sostenible, contribuye a la consecución de los ODS propuestos por la Unesco en la Agenda 2030.

Además, se anima a realizar futuros trabajos en los cuales se extienda este estudio a otras asignaturas matemáticas, así como a otras ciencias básicas de las carreras tecnológicas.

Por último, se plantea realizar una experiencia similar en la educación secundaria, con el objetivo de inculcar al alumnado de este ciclo el interés por las carreras tecnológicas y por las profesiones STEM.

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ANEXO A. Formulario encuestas anónimas seminario “Aplicaciones de Matemáticas en Ingeniería”

Enquesta "APLICACIONS DE LES MATEMÀTIQUES A L'ENGINYERIA I"

SESSIÓ I

Iniciar sesión en Google para guardar lo que llevas hecho. Más información

*Obligatorio

VALORACIÓ ACADÈMICA GENERAL

1. La meva valoració sobre els continguts matemàtics de la sessió és positiva *

	1	2	3	4	5	
Completament en desacord	○	○	○	○	○	Completament d'acord

2. La meva valoració sobre els continguts enginyerils de la sessió és positiva *

	1	2	3	4	5	
Completament en desacord	○	○	○	○	○	Completament d'acord

VALORACIÓ DELS CONTINGUTS

3. L'assignatura permet conèixer les aplicacions tecnològiques dels diferents conceptes matemàtics *

	1	2	3	4	5	
Completament en desacord	○	○	○	○	○	Completament d'acord

4. Les aplicacions dels diferents conceptes matemàtics aconsegueixen augmentar la motivació cap a l'assignatura *

	1	2	3	4	5	
Completament en desacord	○	○	○	○	○	Completament d'acord

5. La realització d'exercicis amb aplicacions tecnològiques facilita l'aprenentatge dels conceptes matemàtics *

	1	2	3	4	5	
Completament en desacord	○	○	○	○	○	Completament d'acord

VALORACIÓ DEL PROFESSORAT

6. Les explicacions del professorat han estat clares *

	1	2	3	4	5	
Completament en desacord	○	○	○	○	○	Completament d'acord

7. El professorat ha solucionat els dubtes satisfactòriament *

	1	2	3	4	5	
Completament en desacord	○	○	○	○	○	Completament d'acord

8. El professorat ha aconseguit mantenir l'atenció i motivar als alumnes *

	1	2	3	4	5	
Completament en desacord	○	○	○	○	○	Completament d'acord

VALORACIÓ ORGANITZATIVA

9. He estat ben informat de la programació i dels objectius de les sessions *

	1	2	3	4	5	
Completament en desacord	○	○	○	○	○	Completament d'acord

10. Les infraestructures (aules, mitjans audiovisuals i tecnològics, etc.) han estat adequades *

	1	2	3	4	5	
Completament en desacord	○	○	○	○	○	Completament d'acord

OBSERVACIONS I SUGERIMENTS (opcional)

Indica els dos aspectes que t'hagin resultat més interessants:

Tu respuesta

Indica els dos aspectes que pensis que s'han de millorar:

Tu respuesta

Enviar

Borrar formulario

ANEXO B. Preguntas entrevista seminario “Aplicaciones de Matemáticas en Ingeniería”

I. VALORACIÓ DE L'ASSIGNATURA "APLICACIONS DE LES MATEMÀTIQUES A L'ENGINYERIA I":

1) Quins aspectes valores més positivament d'aquestes sessions?

2) Indica quines aplicacions t'han resultat més interessants. Per què?

3) Quines aplicacions t'han ajudat més a comprendre els conceptes matemàtics relacionats?

4) Les guies tècniques t'han ajudat a comprendre les aplicacions i els conceptes matemàtics relacionats? Què creus que es podria millorar?

5) Indica, si és el cas, les dificultats que has trobat per al seguiment de les sessions.

6) Com creus que es podrien millorar aquestes sessions?

7) Com han influït aquestes sessions en la teva motivació i en el teu interès cap a les Matemàtiques?

8) Després d'haver realitzat aquestes sessions, ha canviat la teva visió sobre la importància de les Matemàtiques per al desenvolupament del Grau d'Enginyeria? Com? Per què?

II. INFLUÈNCIA EN L'ASSIGNATURA "ÀLGEBRA LINEAL":

1) Com creus que ha influït cursar l'assignatura "Aplicacions de les Matemàtiques a l'Enginyeria I" en el desenvolupament de l'assignatura d'Àlgebra Lineal? Ha millorat el teu interès cap a l'Àlgebra Lineal? T'ha ajudat a comprendre els conceptes matemàtics de l'Àlgebra Lineal?

2) Quins han estat els teus resultats en l'assignatura Àlgebra Lineal? Creus que ha influït cursar l'assignatura "Aplicacions de les Matemàtiques a l'Enginyeria I" en els teus resultats en l'assignatura d'Àlgebra Lineal? (CONTESTAR DESPRÉS DE L'EXAMEN - Em tornaré a posar en contacte amb vosaltres després de l'examen)

ANEXO C. Sesiones “Aplicaciones de Matemáticas en Ingeniería I: Álgebra Lineal”

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

ETSEIB

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA I: ÀLGEBRA LINEAL

SESSIÓ I: ELS NOMBRES COMPLEXOS EN L'ESTUDI DE LES OSCIL·LACIONS DE PREUS

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

ELS NOMBRES COMPLEXOS EN L'ESTUDI DE LES OSCIL·LACIONS DE PREUS

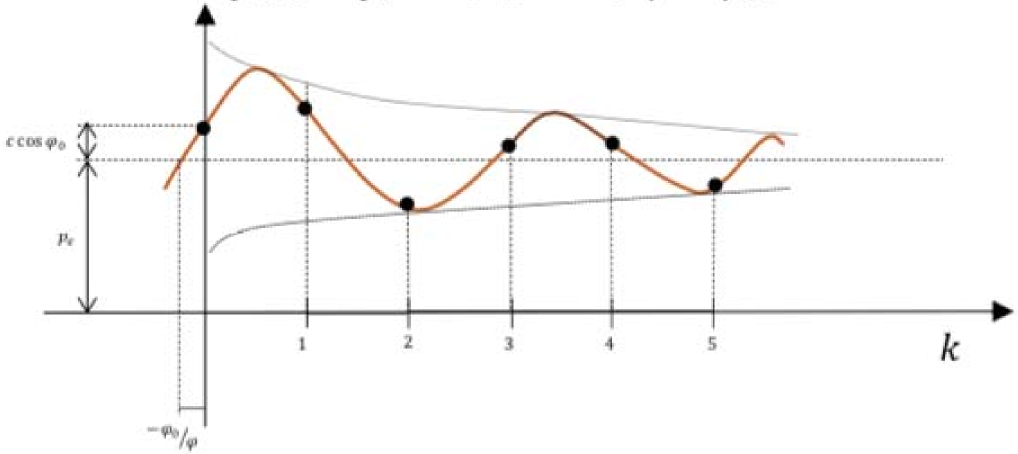
INDEX

- INTRODUCCIÓ
- GUÍA TÈCNICA ECONOMIA
- EXEMPLES

INTRODUCCIÓ

En sistemes dinàmics sovint apareixen modes oscil·latoris de la forma:

$$p(k) = p_e + c\|\lambda\|^k \cos(k\varphi + \varphi_0) \;\; ; \quad \boldsymbol{k = 0, 1, 2, \ldots}$$



Determinats per:

$\lambda = \|\lambda\| e^{j\varphi} \in \mathbb{C}$

(c, φ_0 només es relacionen amb les "condicions inicials")

(j designa en enginyeria la "unitat imaginària", que en matemàtiques s'escriu *i*)

Observeu que no és fàcil de determinar a partir dels valors empírics (punts negres). Vegem-ho quan p(k) és el preu d'una mercaderia en la k-èsima temporada de vendes.

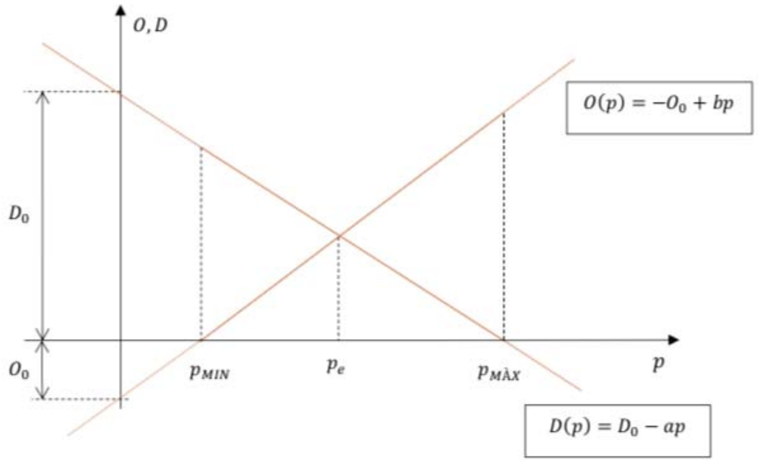
ELS NOMBRES COMPLEXOS EN L'ESTUDI DE LES OSCIL·LACIONS DE PREUS

GUIA TÈCNICA ECONOMIA

1. OFERTA/DEMANDA: PUNT D'EQUILIBRI
2. MODEL DE LA TERANYINA
3. CAS GENERAL

1. OFERTA/DEMANDA: PUNT D'EQUILIBRI

Suposem que la oferta (O) i la demanda (D) depenen del preu (p) segons les gràfiques



On:
- $a, b, O_0, D_0 > 0$
- $p_{MIN} \left(= \frac{O_0}{b}\right)$: preu mínim acceptable per als productors
- $p_{MÀX} \left(= \frac{D_0}{a}\right)$: preu màxim acceptable per als compradors

El preu d'equilibri ideal p_e seria:

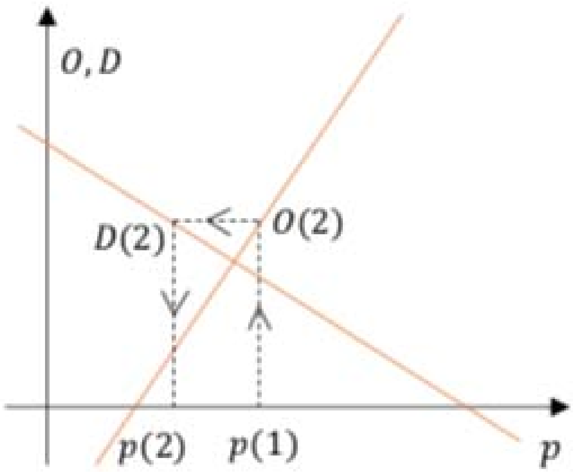
$$-O_0 + bp_e = D_0 - ap_e \Leftrightarrow \boxed{p_e = \frac{D_0 + O_0}{a+b}}$$

2. MODEL DE LA TERANYINA

Sovint el preu del mercat no és p_e.

2.1) Suposem que en la temporada $k = 1$ el preu ha estat $p(1) > p_e$ i que els productors planifiquen la temporada $k = 2$ segons aquest preu, de manera que produeixen una elevada $O(2)$.

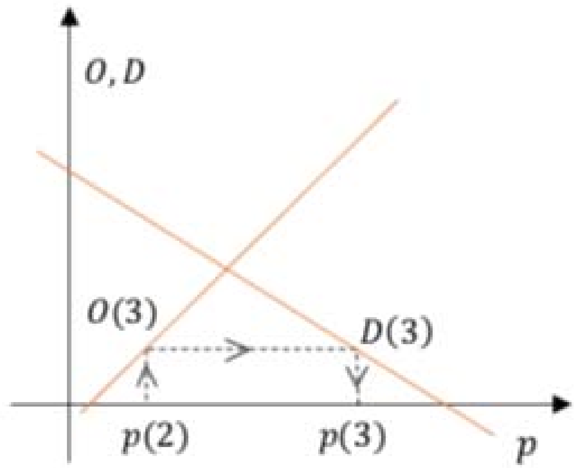
Tanmateix, els compradors absorbiran tota aquesta oferta ($\Leftrightarrow D(2) = O(2)$) només si el preu baixa a $p(2)$.



2. MODEL DE LA TERANYINA

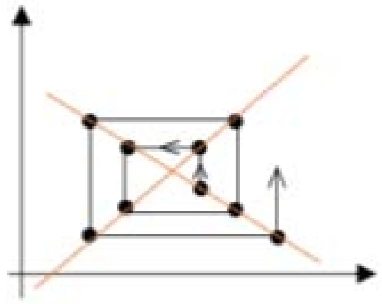
2.2) Per la següent temporada $k = 3$, la producció $O(3)$ serà determinada per aquesta expectativa $p(2)$ i per tant, serà molt més baixa que $O(2)$.

Però aquesta escassa oferta obligarà als compradors a acceptar un preu més alt $p(3)$.

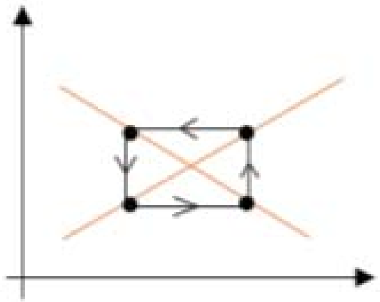


2. MODEL DE LA TERANYINA

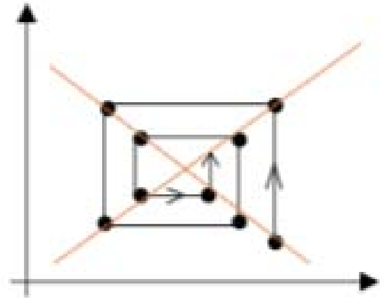
2.3) I així successivament, amb 3 possibilitats:



$b > a$



$b = a$



$b < a$

Apareixen, doncs, oscil·lacions biennals, només amb tendència a atenuar-se si $\frac{b}{a} < 1$

Cal doncs
- Disminuir b (⇔ desincentivar la producció)
- Augmentar a (⇔ estimular la compra)

3. CAS GENERAL

3.1) <u>EXPECTATIVA DE PREU</u>: $\hat{p}(k)$

La clau de volta és l'estratègia per preveure el preu de la pròxima temporada a partir dels preus de les anteriors, és a dir, com es calcula l'"expectativa de preu", $\hat{p}(k)$, a partir de $p(k-1), p(k-2)$,...

En efecte, determina el preu real $p(k)$ mitjançant la seqüència:

$$\ldots \rightarrow \quad \hat{p}(k) = ?? \rightarrow \quad O(k) = -O_0 + \; b\hat{p}(k) \quad \rightarrow \quad D(k) = O(k) \quad \rightarrow \quad p(k) = \frac{-D(k)+D_0}{a} \quad \rightarrow \quad \ldots$$

Per tant:

$$p(k) = \frac{1}{a}(D_0 + O_0 - b\hat{p}(k)) = p_e + \frac{b}{a}(p_e - \hat{p}(k))$$

3. CAS GENERAL

En el model de la teranyina, aquesta estratègia és la mes simple: els productors han actuat prenent com "expectativa de preu" el de la temporada anterior:

$$\hat{p}(2) = p(1),\ \hat{p}(3) = p(2),\ \dots,\ \hat{p}(k) = p(k-1)$$

Amb errors considerables:

$$\hat{p}(2) \gg p(2),\ \hat{p}(3) \ll p(3), \dots$$

Cal esperar un canvi d'estratègia, amb una "expectativa de preu" referenciada a un període previ més llarg, com ara:

$$\hat{p}(3) = \frac{p(2)+p(1)}{2}, \dots,\quad \hat{p}(k) = \frac{p(k-1)+p(k-2)}{2}$$

O encara més sofisticada:

$$\hat{p}(k) = \frac{3p(k-1) + 2p(k-2) + p(k-3)}{6}$$

3. CAS GENERAL

3.2) RESULTAT GENERAL

En el cas més general

$$\hat{p}(k) = \beta_1 p(k-1) + \beta_2 p(k-2) + \cdots$$
$$\beta_1 + \beta_2 + \cdots = 1$$

Es demostra que, aleshores:

$$p(k) \cong p_e + c\|\lambda\|^k \cos(k\varphi + \varphi_0)$$

on $\lambda = \|\lambda\| e^{j\varphi} \in \mathbb{C}$ és l'arrel "dominant" ($\Leftrightarrow$ de mòdul més gran) de:

$$t^k + \frac{b}{a}\left(\beta_1 t^{k-1} + \beta_2 t^{k-2} + \cdots\right) = 0$$

que s'anomena "polinomi característic".

Recordem que c, φ_0 només es relacionen amb les "condicions inicials".

3. CAS GENERAL

El comportament posterior ve determinat per $\lambda \in \mathbb{C}$:

$$\|\lambda\|: \text{estabilitat} \begin{cases} \|\lambda\| > 1 \Leftrightarrow \text{oscil·lacions amplificades} \\ \|\lambda\| = 1 \Leftrightarrow \text{oscil·lacions d'amplitud constant} \\ \|\lambda\| < 1 \Leftrightarrow \text{oscil·lacions atenuades} \end{cases}$$

φ: periodicitat ($\cong {}^{2\pi}/_{\varphi}$)

ELS NOMBRES COMPLEXOS EN L'ESTUDI DE LES OSCIL·LACIONS DE PREUS

EXEMPLES PRÀCTICS

Exemple 1: Model de la teranyina

Exemple 2: Els productors es referencien als 2 anys anteriors

Exemple 3: El cicle del preu de la carn de porc

EXEMPLE 1. MODEL DE LA TERANYINA

Model de la Teranyina:

$$\hat{p}(k) = p(k-1) \Rightarrow \beta_1 = 1, \beta_2 = \beta_3 = \cdots = 0 \Rightarrow$$

$\Rightarrow \lambda$ és l'arrel dominant de: $t + \frac{b}{a} = 0 \Rightarrow$

$$\Rightarrow \lambda = -\frac{b}{a} = \frac{b}{a} e^{j\pi} \Rightarrow \begin{cases} \|\lambda\| = {}^{b}/_{a} \\ \text{Periodicitat biennal} \end{cases}$$

EXEMPLE 2. ELS PRODUCTORS ES REFERENCIEN ALS 2 ANYS ANTERIORS

Suposem $^{b}/_{a} = 1$, però els productors es referencien als 2 anys anteriors:

$$\hat{p}(k) = \frac{p(k-1) + p(k-2)}{2} \Rightarrow \beta_1 = \beta_2 = \frac{1}{2}, \beta_3 = \beta_4 = \cdots = 0 \Rightarrow$$

$\Rightarrow \lambda$ és l'arrel dominant de: $t^2 + \frac{1}{2}(t+1) = 0 \Rightarrow \lambda = \frac{-1 \pm j\sqrt{7}}{4} \cong 0{,}7e^{\pm j110^o}$

Per tant:
- Periodicitat triennal
- Oscil·lacions atenuades (amb $^{b}/_{a} = 1$)

En particular, es pot relaxar la condició $^{b}/_{a} < 1$ fins a $^{b}/_{a} < 2$:

$\frac{b}{a} = 2 \Rightarrow \lambda$ és l'arrel dominant de: $t^2 + t + 1 = 0 \Rightarrow \lambda = \frac{-1 \pm j\sqrt{3}}{2} \Rightarrow \|\lambda\| = 1$

EXEMPLE 3. EL CICLE DEL PREU DE LA CARN DE PORC (LUENBERGER)

***El cicle del preu de la carn de porc** (**LUENBERGER**) :*

Durant gairebé un segle, varen observar-se oscil·lacions quadriennals en la producció de carn de tocino als USA. Es tracta de trobar un model de 3.2) que s'hi adeqüi per tal de deduir el valor de b/a que l'atenuaria.

Cal tenir present que hi ha dues temporades de producció cada any (primavera i tardor) i que el període de criança del tocino és aproximadament un any. Per tant, la variable k correspondrà a semestre i que la "decisió/producció" és de 2 d'aquests períodes (és a dir, $\beta_1 = 0$).

EXEMPLE 3. EL CICLE DEL PREU DE LA CARN DE PORC (LUENBERGER)

- Si suposem:

$$\hat{p}(k) = p(k-2)$$

Resulta:

$$t^2 + \frac{b}{a} = 0 \Rightarrow \lambda = \pm j\sqrt{\frac{b}{a}}$$

Per tant, periodicitat 4-semestral (⟺ biennal) que no s'ajusta a la realitat.

- Si suposem:

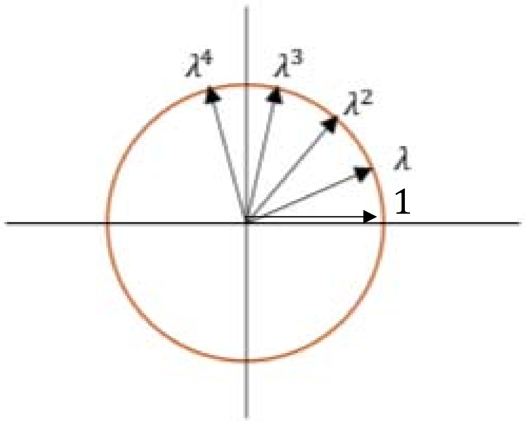
$$\hat{p}(k) = \frac{1}{5}\big(p(k-2) + p(k-3) + p(k-4) + p(k-5) + p(k-6)\big)$$

Resulta:

$$t^6 + \frac{b}{a}\frac{1}{5}(t^4 + t^3 + t^2 + t + 1) = 0$$

EXEMPLE 3. EL CICLE DEL PREU DE LA CARN DE PORC (LUENBERGER)

Busquem arrels de mòdul 1, ja que les oscil·lacions eren sostingudes.



Observem que si $\|\lambda\| = 1$, per simetria resulta

$$\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 = \alpha\lambda^2 \ , \qquad \alpha \in \mathbb{R}$$

Per tant:

$$\lambda^6 + \frac{b}{a}\frac{1}{5}\alpha\lambda^2 = 0$$

D'aquí podem deduir $arg\lambda$:

$$arg\lambda^6 = arg\lambda^2 + \pi$$

$$arg\lambda = \frac{\pi}{4}$$

EXEMPLE 3. EL CICLE DEL PREU DE LA CARN DE PORC (LUENBERGER)

Efectivament, resulten oscil·lacions quadriennals

Finalment:

$$\lambda = e^{j\frac{\pi}{4}} \Longrightarrow \left[\begin{array}{l} \lambda^6 = -j \\ \lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 \cong 2{,}4j \end{array} \right] \Rightarrow \frac{b}{a} \cong \frac{5}{2{,}4} \cong 2{,}08$$

Cal, doncs, forçar que:

$$\boxed{\frac{b}{a} < 2{,}08}$$

FINAL SESSIÓ I

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

ETSEIB

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA I: ÀLGEBRA LINEAL

SESSIÓ II: ELS NOMBRES COMPLEXOS EN L'ESTUDI DE CORRENTS ALTERNS

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

ELS NOMBRES COMPLEXOS EN L'ESTUDI DE CORRENTS ALTERNS

INDEX

ELS NOMBRES COMPLEXOS EN L'ESTUDI DE CORRENTS ALTERNS

INTRODUCCIÓ

- En corrents continus, les magnituds elèctriques (intensitat, voltatge, resistència) es representen per NOMBRES REALS.
- Per corrents alterns es representen per NOMBRES COMPLEXOS, de manera que són vàlides fórmules i tècniques anàlogues.

ELS NOMBRES COMPLEXOS EN L'ESTUDI DE CORRENTS ALTERNS

GUIA TÈCNICA ELECTRICITAT

1. ANÀLISI DE CIRCUITS DE CORRENTS CONTINUS: LLEIS DE KIRCHHOFF I LLEI DE OHM
2. CORRENT ALTERN: MAGNITUDS COSINUIDALS
3. ANÀLISI DE CIRCUITS DE CORRENT ALTERN
4. FASOR (COMPLEX) ASSOCIAT A UNA MAGNITUD COSINUIDAL
5. LLEIS DE KIRCHHOFF FASORIALS
6. LA IMPEDÀNCIA COMPLEXA

ELS NOMBRES COMPLEXOS EN L'ESTUDI DE CORRENTS ALTERNS

EXERCICIS PRÀCTICS

Exercici 1: Anàlisi de circuits de Corrent Altern

Exercici 2: Distribució trifàsica

Exercici 3: Circuit RLC

Exercici 4: Resonàncies

Exercici 5: Anul·lació de la potència reactiva

Exercici 6: Matriu d'Impedàncies

GUIA TÈCNICA I. ANÀLISI DE CIRCUITS DE CORRENTS CONTINUS: LLEIS DE KIRCHHOFF I LLEI DE OHM

MAGNITUDS ELÈCTRIQUES

- INTENSITAT: $i(t) = I \in \mathbb{R}$
- VOLTATGE: $u(t) = U \in \mathbb{R}$

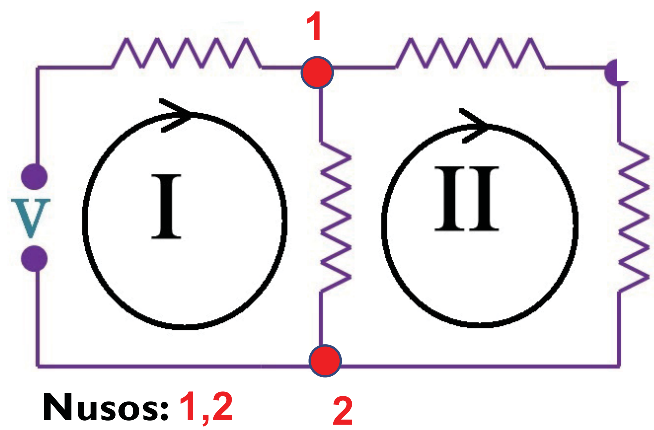
Per determinar I_k, U_k a cada branca d'un circuit:

<u>1ª Llei de Kirchhoff</u>: en cada **nus** $\sum I_k = 0$

<u>2ª Llei de Kirchhoff</u>: en cada **malla** $\sum U_k = 0$

<u>Llei de Ohm</u>: en cada **branca** $U_k = R_k \cdot I_k$,

on $R_k \in \mathbb{R}$ és la RESISTÈNCIA de la branca

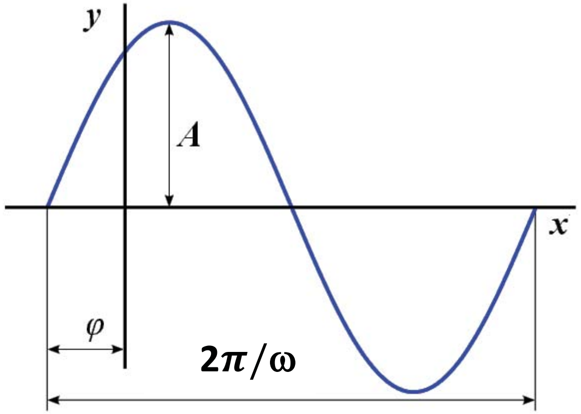


Nusos: 1,2
Malles: I, II
3 Branques

GUIA TÈCNICA 2. CORRENTS ALTERNS: MAGNITUDS COSINUIDALS

En corrent altern, la intensitat i(t) és una ona cosinusoidal (de freqüència ω, que a Europa és de 50 Hz)

$$i(t) = A cos(\omega t + \varphi) = \sqrt{2} \cdot I \cdot sin(\omega t + \varphi)$$



caracteritzada per:

- **VALOR EFICAÇ: $I = A / \sqrt{2}$**
- **FASE: φ**

I anàlogament: $u(t) = \sqrt{2} \cdot U \cdot cos(\omega t + \varphi)$

El valor eficaç és el valor del corrent continu i(t) = I amb la mateixa energia:

$$\int_0^{\frac{2\pi}{\omega}} I^2 dt = I^2 \frac{2\pi}{\omega}$$

$$\int_0^{\frac{2\pi}{\omega}} (i(t))^2 dt = A^2 \int_0^{\frac{2\pi}{\omega}} \frac{1 + cos2(\omega t + \varphi)}{2} dt = A^2 \frac{\pi}{\omega}$$

Per tant: $A = \sqrt{2} \cdot I$

Per corrents alterns, les fórmules equivalents a les lleis de Kirchhoff i a la llei de Ohm resulten de molt difícil tractament amb magnituds cosinuidals:

1ª Llei de Kirchhoff: en cada nus $\sum i_k(t) = 0$

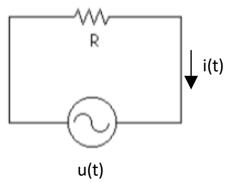
2ª Llei de Kirchhoff: en cada malla $\sum u_k(t) = 0$

Vegeu, per exemple, l'exercici 1:

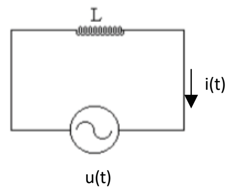
$$4 \cdot \sqrt{2} \cdot cos\omega t + 3 \cdot \sqrt{2} \cdot cos\left(\omega t + \frac{\pi}{3}\right) + 2 \cdot \sqrt{2} \cdot cos(\omega t + \frac{\pi}{4}) = \ldots ??$$

Igualment, la Llei de Ohm en cada branca es complica

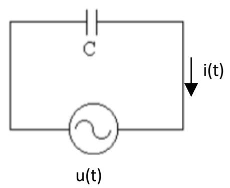
RESISTÈNCIA: $u(t) = R \cdot i\ (t)$



INDUCTÀNCIA: $u(t) = L \cdot i'(t)$



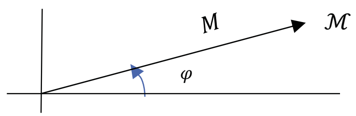
CAPACITAT: $i(t) = C \cdot u'(t)$



GUIA TÈCNICA 4. FASOR ASSOCIAT A UNA MAGNITUD COSINUIDAL

Representem la magnitud cosinuidal: $m(t) = \sqrt{2} \cdot M \cdot Cos(\omega t + \varphi)$

Mitjançant el nombre complex: $\mathcal{M} = M \cdot e^{j\varphi} \in \mathbb{C}$



que se'n diu el FASOR ASSOCIAT

(j designa en enginyeria la "unitat imaginària", que en matemàtiques s'escriu *i*)

Es comprova fàcilment que és compatible amb la SUMA i el PRODUCTE PER ESCALARS:

- FASOR ASSOCIAT a $m_1(t) + m_2(t)$: $\mathcal{M}_1 + \mathcal{M}_2$
- FASOR ASSOCIAT a $\lambda \cdot m(t)$: $\lambda \cdot \mathcal{M}$

GUIA TÈCNICA 5. LLEIS DE KIRCHHOFF FASORIALS

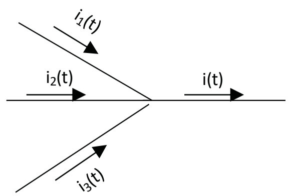
Amb aquesta representació fasorial, les lleis de Kirchhoff per CORRENTS ALTERNS són anàlogues a les de CORRENTS CONTINUS, però ara com sumes de **nombres complexos**:

1ª Llei de Kirchhoff: en cada nus $\sum \mathcal{I}_k = 0$

2ª Llei de Kirchhoff: en cada malla $\sum \mathcal{U}_k = 0$

EXERCICI 1. ANÀLISI DE CIRCUITS DE CORRENT ALTERN

Calcula la intensitat i(t) a partir de les intensitats $i_1(t)$, $i_2(t)$, $i_3(t)$:



$$\left.\begin{aligned} i_1(t) &= 4 \cdot \sqrt{2} \cdot cos\omega t \\ i_2(t) &= 3 \cdot \sqrt{2} \cdot cos\left(\omega t + \frac{\pi}{3}\right) \\ i_3(t) &= 2 \cdot \sqrt{2} \cdot cos(\omega t + \frac{\pi}{4}) \end{aligned}\right\} \; i(t) = \; i_1(t) + i_2(t) + i_3(t)$$

$$\left.\begin{aligned} \mathcal{I}_1 &= 4 \\ \mathcal{I}_2 &= 3e^{j\frac{\pi}{3}} \\ \mathcal{I}_3 &= 2e^{j\frac{\pi}{4}} \end{aligned}\right\} \; \mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \; = \left(4 + 3\,\frac{1}{2} + 2\,\frac{1}{\sqrt{2}}\right) + j\left(3\,\frac{\sqrt{3}}{2} + 2\,\frac{1}{\sqrt{2}}\right) \cong 6{,}9 + j4 \cong 8e^{j\frac{\pi}{6}}$$

$$i(t) \cong 8 \cdot \sqrt{2} \cdot cos(\omega t + \frac{\pi}{6})$$

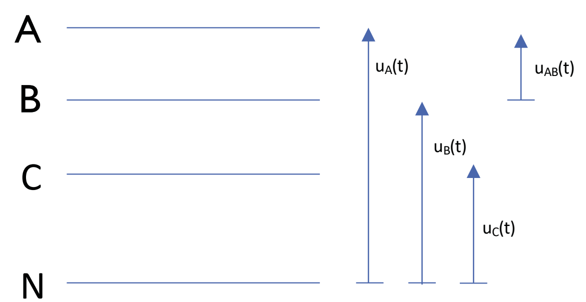
EXERCICI 2. DISTRIBUCIÓ TRIFÀSICA

Calcula la relació entre les tensions fase / neutre (u_A(t), u_B(t), …) i les tensions fase / fase (u_{AB}(t), …) en una xarxa trifàsica:

En la distribució d'electricitat s'empren:

3 FASES (A, B, C)
- MATEIX VOLTATGE
- FASES 0°, 120°, 240°

1 NEUTRE (N): TENSIÓ NUL·LA



EXERCICI 2. DISTRIBUCIÓ TRIFÀSICA

És a dir:

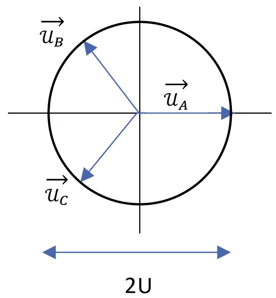
$$u_A(t) = \sqrt{2}\,U cos\omega t$$

$$u_B(t) = \sqrt{2}\,U cos\left(\omega t + \frac{2\pi}{3}\right)$$

$$u_C(t) = \sqrt{2}\,U cos(\omega t + \frac{4\pi}{3})$$

N pot obtenir-se connectant les 3 fases: $u_A(t) + u_B(t) + u_C(t) = 0$

Es pot veure fàcilment en REPRESENTACIÓ FASORIAL:



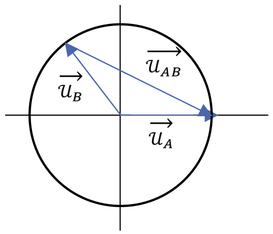
EXERCICI 2. DISTRIBUCIÓ TRIFÀSICA

Obtenim major TENSIÓ entre 2 FASES:

TENSIÓ $_{\text{FASE – FASE}}$ = $\sqrt{3}$ x TENSIÓ $_{\text{FASE – NEUTRE}}$

En efecte:

$$\mathcal{U}_{AB} = \mathcal{U}_A - \mathcal{U}_B = U - Ue^{j\frac{2\pi}{3}} = \sqrt{3}Ue^{-j\frac{\pi}{6}}$$



En particular:

TENSIÓ $_{\text{FASE – NEUTRE}}$ = 120 V $\Longrightarrow$ TENSIÓ $_{\text{FASE – FASE}}$ $\approx$ 210 V

GUIA TÈCNICA 6. LA IMPEDÀNCIA COMPLEXA

Vegem que la representació fasorial permet també adaptar la Llei de Ohm en la forma $\mathcal{U} = Z \cdot \mathcal{I}$, on ara $\mathcal{Z} \in \mathbb{C}$

Observem que:

$$m(t) = \sqrt{2}Mcos(\omega t + \varphi) \Rightarrow m'(t) = -\sqrt{2}M\omega sin(\omega t + \varphi) = \sqrt{2}M\omega cos\left(\omega t + \varphi + \frac{\pi}{2}\right)$$

Per tant, el fasor associat a $m'(t)$ és: $\mathcal{M}' = M\omega e^{j(\varphi+\frac{\pi}{2})} = \mathcal{M}\omega j$

En particular:

$$u(t) = R \cdot i(t) \Leftrightarrow \mathcal{U} = R \cdot \mathcal{I}$$
$$u(t) = L \cdot i'(t) \Leftrightarrow \mathcal{U} = L\omega j \cdot \mathcal{I}$$
$$i(t) = C \cdot u'(t) \Leftrightarrow \mathcal{I} = C\omega j \cdot \mathcal{U}$$

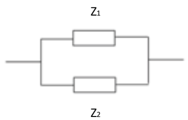
Equivalentment:

$$\mathcal{U} = \mathcal{Z} \cdot \mathcal{I} \quad \begin{cases} \mathcal{Z} = R \\ \mathcal{Z} = L\omega j \\ \mathcal{Z} = \dfrac{1}{Cj\omega} = \dfrac{-1}{C\omega}j \end{cases}$$

GUIA TÈCNICA 6. LA IMPEDÀNCIA COMPLEXA

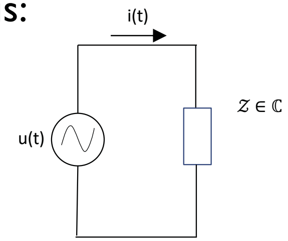
Són igualment vàlides les fórmules per CONNEXIONS EN SÈRIE i en PARAL·LEL:

$$\mathcal{Z}_s = \mathcal{Z}_1 + \mathcal{Z}_2$$



$$\frac{1}{\mathcal{Z}_p} = \frac{1}{\mathcal{Z}_1} + \frac{1}{\mathcal{Z}_2}$$

En definitiva, tot circuit es redueix a la fórmula següent, anàloga a la Llei de Ohm per a corrents continus:

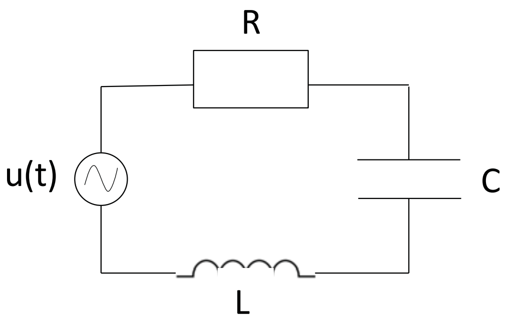


$$\mathcal{U} = \mathcal{Z} \cdot \mathcal{I}$$

$\mathcal{Z} \in \mathbb{C}$, que se'n diu la IMPEDÀNCIA COMPLEXA

EXERCICI 3. CIRCUIT RLC

Calcula la impedància del següent circuit:



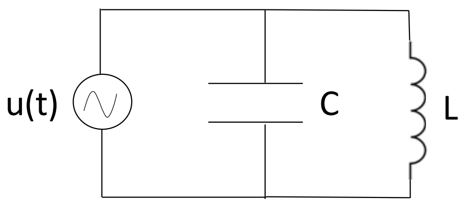
Es tracta d'una connexió en sèrie. Per tant:

$$\mathcal{U} = \mathcal{Z}_{RLC} \cdot \mathcal{I}$$

$$\mathcal{Z}_{RLC} = \mathrm{R} + \mathrm{j}(\omega L - \frac{1}{\omega C})$$

EXERCICI 4. RESONÀNCIES

Troba en quines condicions es produeix resonància ($\mathcal{Z} \rightarrow \infty$) en el següent circuit:



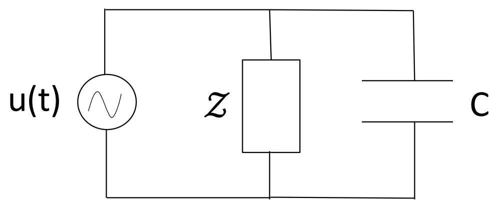
Es tracta d'una connexió en paral·lel. Per tant:

$$\mathcal{Z} = \frac{1}{\frac{1}{j\omega L} + \frac{1}{-j\omega C}} = \frac{j\omega L}{1 - \omega^2 LC}$$

$$\boxed{\mathcal{Z} \rightarrow \infty \Leftrightarrow \omega^2 LC = 1}$$

EXERCICI 5. ANUL·LACIÓ DE LA POTÈNCIA REACTIVA

Calcula la capacitat del condensador que s'ha de col·locar en paral·lel per tal que la impedància sigui un nombre real (la qual cosa anul·la la potència reactiva del circuit, i per tant optimitza el seu rendiment)



EXERCICI 6. MATRIUS D'IMPEDÀNCIES

Per simplificar la MATRIU D'IMPEDÀNCIES de les MÀQUINES TRIFÀSIQUES, s'utilitza la MATRIU:

$$F = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{pmatrix}, \alpha \neq 1, \alpha^3 = 1$$

És a dir

$$\alpha = e^{j\frac{2\pi}{3}}, \qquad \alpha^2 = e^{j\frac{4\pi}{3}}$$

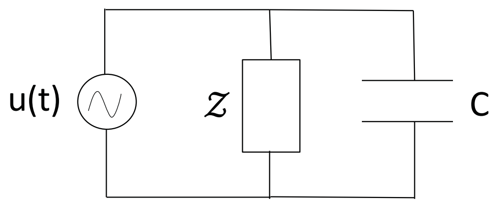
$$\alpha = e^{j\frac{4\pi}{3}}, \qquad \alpha^2 = e^{j\frac{2\pi}{3}}$$

Calculeu F^2 i F^4.

FINAL SESSIÓ II

EXERCICI 5. ANUL·LACIÓ DE LA POTÈNCIA REACTIVA

Calcula la capacitat del condensador que s'ha de col·locar en paral·lel per tal que la impedància sigui un nombre real (la qual cosa anul·la la potència reactiva del circuit, i per tant optimitza el seu rendiment)



Es tracta d'una connexió en paral·lel. Per tant:

$$\frac{1}{\mathcal{Z}_p} = \frac{1}{\mathcal{Z}} + j\omega C$$

Per minimitzar les pèrdues cal anul·lar la part imaginària de la impedància $\mathcal{Z}_p$:

$$\mathcal{Z}_p \in \mathbb{R} \Leftrightarrow Im\left(\frac{1}{\mathcal{Z}}\right) + \omega C = 0 \Leftrightarrow \boxed{C = \frac{-Im(\mathcal{Z})}{\omega \|\mathcal{Z}\|^2}}$$

EXERCICI 6. MATRIUS D'IMPEDÀNCIES

$$F^2 = \frac{1}{3}\begin{pmatrix} 3 & 1+\alpha+\alpha^2 & 1+\alpha+\alpha^2 \\ 1+\alpha+\alpha^2 & 1+\alpha^2+\alpha^4 & 1+\alpha^3+\alpha^3 \\ 1+\alpha+\alpha^2 & 1+\alpha^3+\alpha^3 & 1+\alpha^2+\alpha^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$F^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA I: ÀLGEBRA LINEAL

SESSIÓ III. SISTEMES INDETERMINATS: VARIABLES DE CONTROL

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

SISTEMES INDETERMINATS :VARIABLES DE CONTROL

INDEX

INTRODUCCIÓ

En **sistemes d'equacions compatibles indeterminats**, interessen:

<u>VARIABLES INDEPENDENTS</u> o <u>DE CONTROL</u>

- Poden prendre <u>valors arbitraris</u>
- Els seus valors <u>determinen els de la resta</u>

$\Rightarrow$

$\Rightarrow$ Parametritzen el conjunt de solucions:

{VAR. INDEP.} $\longleftrightarrow$ {SOLUCIONS}

INTRODUCCIÓ

Exemple:

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 2x_2 + x_3 + x_4 &= 0 \end{aligned}\right\}$$

- SÍ $\quad x_1, x_2 : \quad (x_1, x_2) \longrightarrow (x_1, x_2, \underbrace{-x_1 - 2x_2}_{x_3}, \underbrace{x_1}_{x_4})$
- NO $\quad x_1, x_4$
  - No valors arbitraris: ha de ser $x_1 = x_4$
  - Tot i $x_1 = x_4$, no queden determinats x_2, x_3

PROBLEMA

- Quantes VARIABLES INDEPENDENTS o DE CONTROL
- Quines VARIABLES INDEPENDENTS o DE CONTROL

SISTEMES INDETERMINATS :VARIABLES DE CONTROL

GUIA TÈCNICA
VARIABLES DE CONTROL

1. SISTEMES HOMOGENIS

2. SISTEMES COMPLETS

1. SISTEMES HOMOGENIS

$$m \updownarrow \overset{\longleftarrow\ n\ \longrightarrow}{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \updownarrow m$$

- SEMPRE COMPATIBLES ($x_1 = \cdots = x_n = 0$ n'és solució)
- $r \equiv rangA(\leq n, m) =$ nº restriccions efectives

En principi, cada equació restringeix 1 grau de llibertat però pot ser redundant amb les anteriors.

El nº de restriccions efectives no és m, sinó r

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 & \text{RESTRINGEIX} \\ 2x_2 + x_3 + x_4 = 0 & \text{RESTRINGEIX} \\ x_1 - x_4 = 0 & \text{REDUNDANT!} \end{cases}$$

I. SISTEMES HOMOGENIS

- nº VAR. IND. o de CONTROL = $n - r$ = ordre d'indeterminació = nº graus de llibertat preservats
- $(n-r)$ variables són INDEPENDENTS o de CONTROL $\Longleftrightarrow$ La resta de columnes de A tenen també rang r

<u>Exemple</u> (anterior):

$$rang\begin{pmatrix}1 & 2 & 1 & 0\\ 0 & 2 & 1 & 1\end{pmatrix} = 2 \Rightarrow \text{nº VAR. IND.} = 4 - 2 = 2$$

$$x_1, x_2 \quad \text{VAR. IND. ? : } rang\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} = 2 \Longrightarrow \text{SÍ}$$

$$x_1, x_4 \quad \text{VAR. IND. ? : } rang\begin{pmatrix}2 & 1\\ 2 & 1\end{pmatrix} = 1 \Longrightarrow \text{NO}$$

2. SISTEMES COMPLETS

$$\underset{m}{\updownarrow}\overset{\longleftarrow\ n\ \longrightarrow}{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}}_{b \neq 0} \underset{m}{\updownarrow}$$

- COMPATIBLE $\Leftrightarrow rangA = rang(A/b) = r$
- Aleshores: $\left\{ \begin{matrix} SOLUCIONS \\ del\ COMPLET \end{matrix} \right\} = \begin{matrix} una\ SOLUCIÓ \\ PARTICULAR \end{matrix} + \left\{ \begin{matrix} SOLUCIONS \\ de\ l'HOMOGENI \end{matrix} \right\}$
- Són valides la resta de consideracions anteriors

2. SISTEMES COMPLETS

<u>Exemple</u>:

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= b_1 \\ 2x_2 + x_3 + x_4 &= b_2 \\ x_1 - x_4 &= b_3 \end{aligned}\right\} \qquad A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$A \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow rangA = 2$$

$$(A|b) \sim \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & b_1 \\ 0 & 2 & 1 & 1 & b_2 \\ 0 & 0 & 0 & 0 & b_3 - b_1 + b_2 \end{array}\right) \Longrightarrow \text{COMPATIBLE} \quad \text{sii } b_3 - b_1 + b_2 = 0$$

2. SISTEMES COMPLETS

$b_3 = b_1 - b_2$

La 3ª equació és redundant (pot suprimir-se)

Ordre d'indeterminació $= 4 - 2 = 2$

x_1, x_2 són VARIABLES INDEPENDENTS o de CONTROL

SOLUCIÓ PARTICULAR: $(0,0,b_1,b_2 - b_1)$ (per exemple)

{SOLUCIONS} $= (0,0,b_1,b_2 - b_1) + \{(x_1, x_2, -x_1 - 2x_2, x_1)\} =$

$= \{(x_1, x_2, b_1 - x_1 - 2x_2, b_2 - b_1 + x_1)\}$

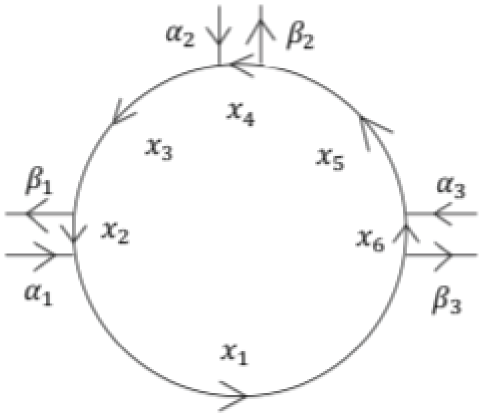
SISTEMES INDETERMINATS :VARIABLES DE CONTROL

EXERCICIS PRÀCTICS

Exercici 1: Rotonda de tràfic

Exercici 2: Distribució de cabals

EXERCICI 1. ROTONDA DE TRÀFIC



$$\begin{cases} x_2 - x_1 = -\alpha_1 \\ x_3 - x_2 = \beta_1 \\ x_4 - x_3 = -\alpha_2 \\ x_5 - x_4 = \beta_2 \\ x_6 - x_5 = -\alpha_3 \\ x_1 - x_6 = \beta_3 \end{cases}$$

$$\overbrace{\left(\begin{matrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{matrix}\right.}^{A} \left|\begin{matrix} \overbrace{-\alpha_1}^{b} \\ \beta_1 \\ -\alpha_2 \\ \beta_2 \\ -\alpha_3 \\ \beta_3 \end{matrix}\right) \sim \left(\begin{matrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right|\left.\begin{matrix} -\alpha_1 \\ \beta_1 \\ -\alpha_2 \\ \beta_2 \\ -\alpha_3 \\ \beta_3 - \alpha_1 + \beta_1 - \alpha_2 + \beta_2 - \alpha_3 \end{matrix}\right) \Longrightarrow$$

EXERCICI 1. ROTONDA DE TRÀFIC

$\Longrightarrow$

COMPATIBLE $\Leftrightarrow \alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3$ ⟶ SÍ!

Aleshores:

1) $rang A = rang(A|b) = 5 \Longrightarrow 1 - INDETERMINAT$

2) $VARIABLE\ DE\ CONTROL$: qualsevol de les 6 variables

$\{SOLUCIONS\ HOMOGENI\} = \{x_1 = x_2 = x_3 = x_4 = x_5 = x_6\}$

3) $\{SOLUCIONS\} = una\ SOLUCIÓ\ PARTICULAR\ + \{k(1,1,1,1,1,1),\ k \in \mathbb{R}\}$

|| per exemple

$(\alpha_1, 0, \beta_1, -\alpha_2+\beta_1, \beta_2 - \alpha_2+\beta_1, -\alpha_3 + \beta_2 - \alpha_2+\beta_1)$

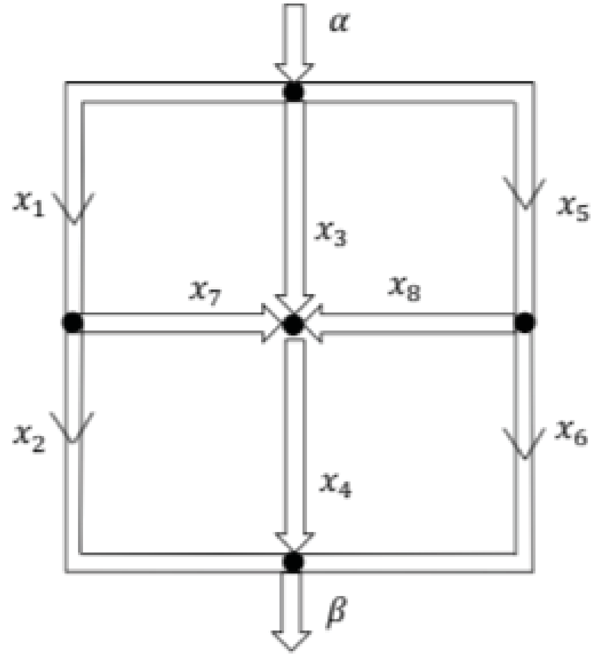
EXERCICI 1. ROTONDA DE TRÀFIC

Per tant:

- ✓ COMPATIBLE
- 3) ✓ ∃ SOLUCIONS POSITIVES! (la solució particular triada no ho és, però amb la k prou gran, sí que resulta)
- 3) ✓ Les solucions es diferencien només per "k cotxes donant voltes" (sense sortir)
- 1) ✓ Suficient 1 SENSOR per conèixer la situació a tota la rotonda
- 2) ✓ Col·locat en QUALSEVOL TRAM de la rotonda

EXERCICI 2. DISTRIBUCIÓ DE CABALS

Considerem 3 ramals interconnectats:



(1) Estudieu les condicions de compatibilitat del sistema.

(2) Determineu quants cabals cal mesurar per tal de conèixer la circulació global del sistema.

(3) En particular: podem determinar la circulació global mesurant els cabals dels quatre punts perifèrics?

(4) Ídem dels quatre trams interiors.

(5) Generalitzeu l'estudi a 3 ramals amb més d'una interconnexió com la de la figura.

FINAL SESSIÓ III

EXERCICI 2. DISTRIBUCIÓ DE CABALS

(1)

$$\begin{cases} \alpha = x_1 + x_3 + x_5 \\ x_1 = x_2 + x_7 \\ \beta = x_2 + x_4 + x_6 \\ x_3 + x_7 + x_8 = x_4 \\ x_5 = x_6 + x_8 \end{cases}$$

$$\left(\begin{array}{cccccccc|c} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \alpha \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & \beta \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha - \beta \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & \beta \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 \end{array}\right)$$

EXERCICI 2. DISTRIBUCIÓ DE CABALS

- COMPATIBLE $\Leftrightarrow \alpha = \beta$. Suposarem, doncs, $\alpha = \beta$ d'ara endavant.

(2) $r = rang A = 4$

$n^o\ VARIABLES\ de\ CONTROL = 8 - 4 = 4$

(3) $VARIABLES\ de\ CONTROL$ x_1, x_2, x_5, x_6 ? : SÍ **TRAMS PERIFÈRICS**

(4) $VARIABLES\ de\ CONTROL$ x_3, x_4, x_7, x_8 ? : NO

$VARIABLES\ de\ CONTROL$ $x_1, x_2, x_7, *$? : NO

(5) <u>Per a 3 ramals</u> (amb els trams que siguin), controlem amb els cabals dels <u>TRAMS PERIFÈRICS.</u>

<u>NO</u> és cert per a més de 3 ramals.

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APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA I: ÀLGEBRA LINEAL

SESSIÓ IV. ELS FLUXOS DE MALLA: UNA BASE DEL SUBESPAI VECTORIAL DELS FLUXOS CONSERVATIUS

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

SISTEMES INDETERMINATS HOMOGENIS: {SOLUCIONS} ÉS UN SUBESPAI VECTORIAL

INDEX

INTRODUCCIÓ

Havíem vist a l'anterior sessió les VARIABLES INDEPENDENTES o de CONTROL:

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$r \equiv rang\ A$

- QUANTES ? : $n - r$ = ordre d'indeterminació =
  = nº graus de llibertat
- QUINES ? $\Longleftrightarrow$ RANG (les altres columnes) $= r$
- Aleshores: $\{VAR.IND.\} \leftrightarrow \{SOLUCIONS\}$

INTRODUCCIÓ

Exemples:

(1)

$$x_1 + 2x_2 + x_3 = 0$$
$$2x_2 + x_3 + x_4 = 0$$
$$A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$
$$rang\ A = 2$$

QUANTES ? : $4 - 2 = 2$

x_1, x_2 ho són? SÍ ($\Leftarrow rang \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 2$)

$$\{(x_1, x_2)\} \leftrightarrow \{(x_1, x_2, \underbrace{-x_1 - 2x_2}_{x_3}, \underbrace{x_1}_{x_4})\}$$

(2) ROTONDA DE TRÀFIC

QUANTES ? : 1

QUINES ? $QUALSEVOL$

És suficient 1 sensor, en qualsevol tram

(3) DISTRIBUCIÓ CABALS

QUANTES ? : 4

QUINES ?

- PERIFÈRIQUES: SÍ
- INTERIORS: NO

SISTEMES INDETERMINATS HOMOGENIS: {SOLUCIONS} ÉS UN SUBESPAI VECTORIAL

GUIA TÈCNICA ELECTRICITAT

1. {SOLUCIONS} ÉS UN SUBESPAI VECTORIAL
2. FLUXOS CONSERVATIUS
3. ELS FLUXOS DE MALLA SÓN UNA BASE DELS FLUXOS CONSERVATIUS

SISTEMES INDETERMINATS HOMOGENIS: {SOLUCIONS} ÉS UN SUBESPAI VECTORIAL

EXERCICIS PRÀCTICS

Exercici 1: Anàlisi de circuits elèctrics

Exercici 2: Demostració enginyeril de $\boldsymbol{rang\ A = N - 1}$

GUIA TÈCNICA 1. {SOLUCIONS} ÉS UN SUBESPAI VECTORIAL

Vegem que {SOLUCIONS} és un SUBESPAI VECTORIAL, i per tant:

- Podem precisar que "nº GRAUS de LLIBERTAT" és la seva dimensió.
- Tenim moltes més possibilitats per descriure'l:

  {SOLUCIONS} = {COMBINACIONS LINEALS de QUALSEVOL BASE}

GUIA TÈCNICA 1. {SOLUCIONS} ÉS UN SUBESPAI VECTORIAL

<u>TEOREMA</u>:

$$(A)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$r \equiv rangA$

(1) {SOLUCIONS} és SUBESPAI VECTORIAL de $\mathbb{R}^n$

Se'n diu "definit per les equacions"; s'escriu $Nuc\ A$

(2) dim {SOLUCIONS} = n$-r$

$\equiv Nuc\ A$

(3) $n-r$ SOLUCIONS $X_1, \dots, X_{n-r}$:

BASE $\Leftrightarrow$ són LINEALMENT INDEPENDENTS

$\Leftrightarrow$ ho són les seves COORDENADES corresponents a var. indep.

(4) Aleshores:

$Nuc\ A \equiv$ {SOLUCIONS}=$\{ \lambda_1 X_1 + \dots + \lambda_{n-r} X_{n-r} ;\ \lambda_1, \dots, \lambda_{n-r} \in \mathbb{R} \}$

GUIA TÈCNICA I. {SOLUCIONS} ÉS UN SUBESPAI VECTORIAL

Exemple.- El (1) anterior:

- dim {SOLUCIONS} = 2

- $(1,0,-1,1)$

  $(0,1,-2,0)$

  L.I.

  BASE: $\{SOLUCIONS\} = \{\lambda_1(1,0,-1,1) + \lambda_2(0,1,-2,0)\}$, amb $x_1 = (1,0,-1,1)$ i $x_2 = (0,1,-2,0)$

- $(1,1,-3,1)$

  $(1,-1,1,1)$

  L.I.

  BASE: $\{SOLUCIONS\} = \{\lambda_1(1,1,-3,1) + \lambda_2(1,-1,1,1)\}$

GUIA TÈCNICA 2. FLUXOS CONSERVATIUS

Es tracta d'emprar BASES especialment significatives: vegem que per a FLUXOS CONSERVATIUS ho són els FLUXOS de MALLA.

Considerem:

- XARXA
  - N nusos: $N_1, N_2, \ldots$
  - B branques (orientades): $B_1, B_2, \ldots$
  - M malles: $M_1, M_2, \ldots$

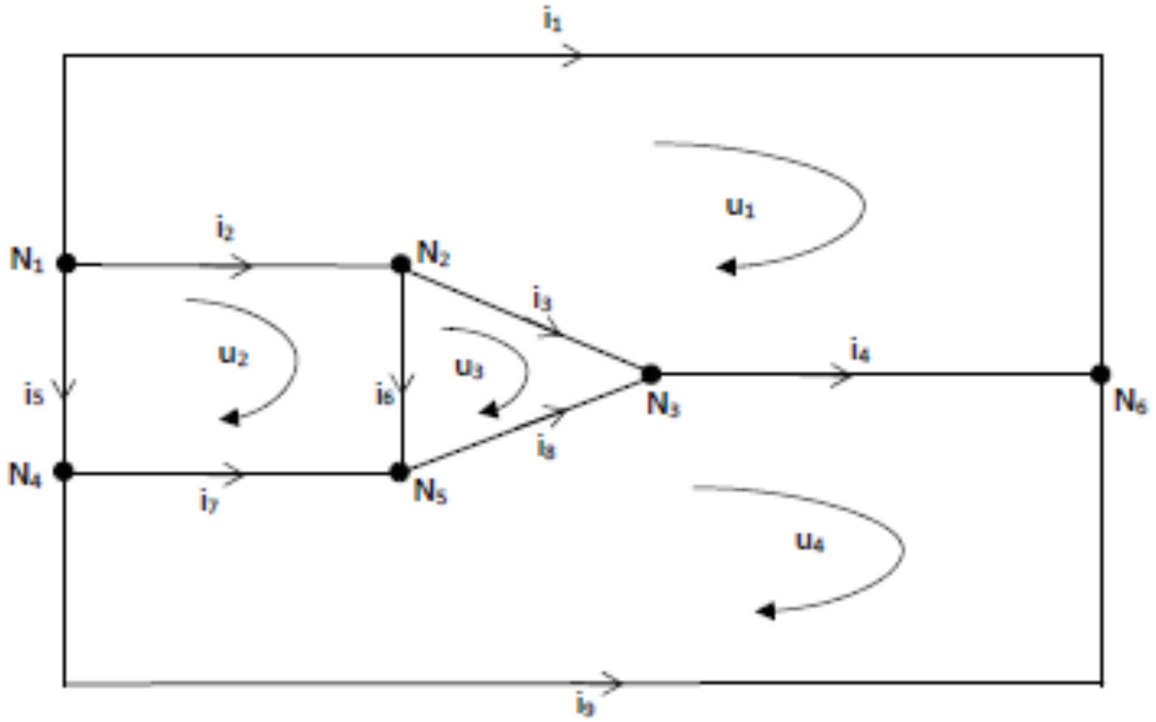
  $N - B + M = 1$ (TEOREMA DE EULER)

  En particular: $M > 1 \Rightarrow \mathrm{B} > \mathrm{N}$

  $N > 1 \Rightarrow \mathrm{B} > \mathrm{M}$

- VARIABLES: els FLUXOS en CADA BRANCA $(i_1, \ldots, i_B) \in \mathbb{R}^B$

EXEMPLE



- 9 BRANQUES
- 6 NUSOS
- 4 MALLES $(= 1 - 6 + 9)$

- FLUXOS: $(i_1, \dots, i_9) \in \mathbb{R}^9$

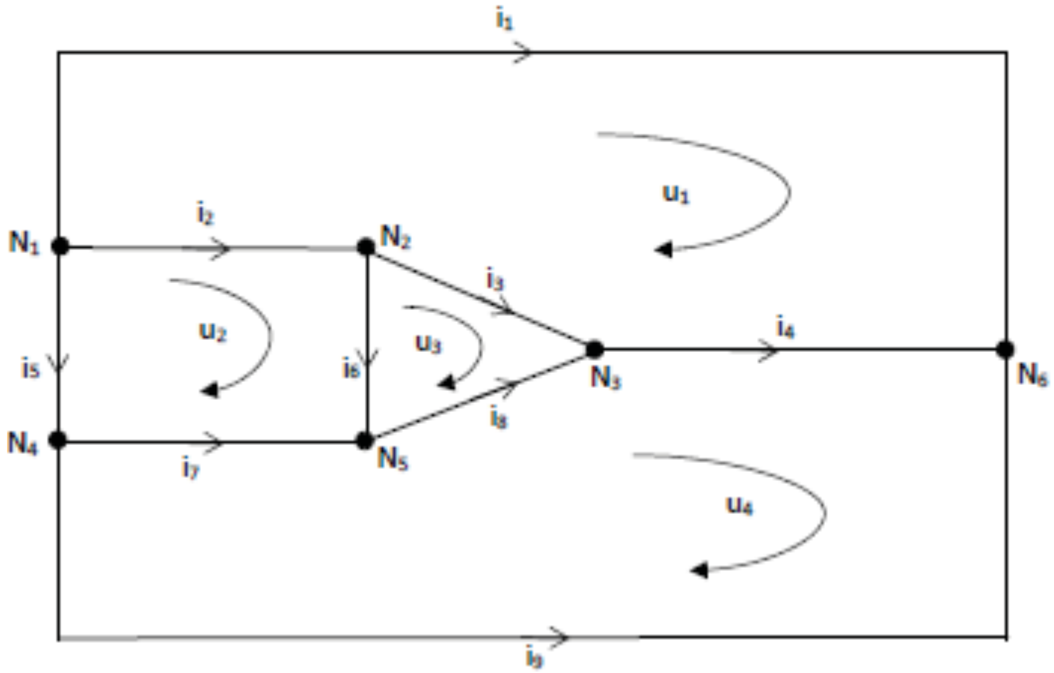
GUIA TÈCNICA 2. FLUXOS CONSERVATIUS

- FLUXOS CONSERVATIUS (fluids, corrents, ...): en cada NUS, $\sum i_k = 0$

  ($\Longleftrightarrow$ a cada NUS SURT TANT FLUX COM ENTRA)

- FLUXOS de MALLA UNITARIS: $(U_1, U_2, \dots, U_M)$
  - FLUX ±1, en sentit horari, en les branques d'una malla; 0 a les altres
  - Clarament és CONSERVATIU
- FLUXOS de MALLA: múltiples dels unitaris $(I_1 U_1, I_2 U_2, \dots)$

EXEMPLE



- FLUXOS CONSERVATIUS:

$$(N_1)\ -i_1 - i_2 - i_5 = 0$$
$$(N_2)\quad i_2 - i_3 - i_6 = 0$$
$$(N_3)\quad i_3 - i_4 + i_8 = 0$$
$$(N_4)\quad i_5 - i_7 - i_9 = 0$$
$$(N_5)\quad i_6 + i_7 - i_8 = 0$$
$$(N_6)\quad i_1 + i_4 + i_9 = 0$$

- FLUXOS de MALLA UNITARIS:

$$U_1 = (1,-1,-1,-1,0,0,0,0,0)$$
$$U_2 = (0,1,0,0,-1,1,-1,0,0)$$
$$U_3 = (0,0,1,0,0,-1,0,-1,0)$$
$$U_4 = (0,0,0,1,0,0,1,1,-1)$$

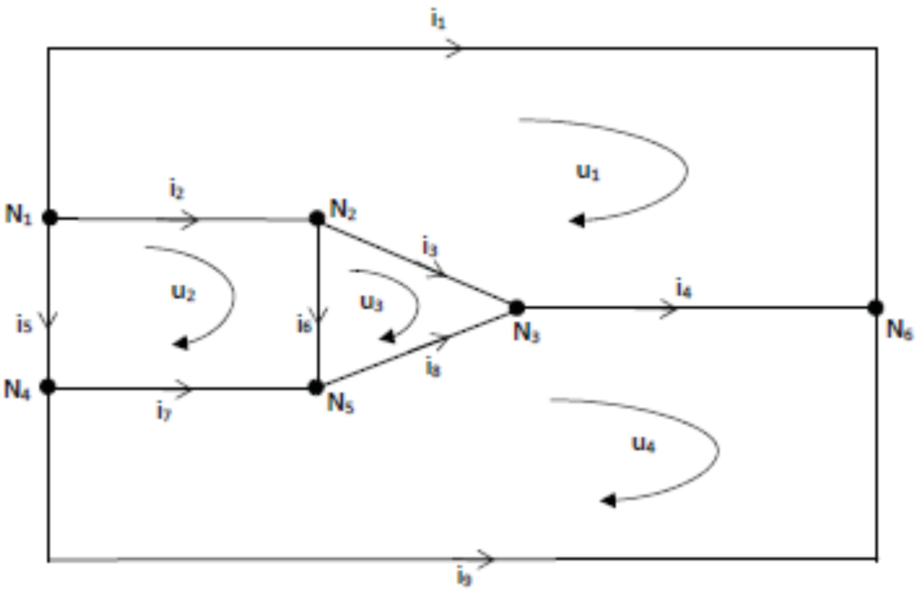
GUIA TÈCNICA 2. FLUXOS CONSERVATIUS

- <u>MATRIU d'INCIDÈNCIA COMPLETA</u>: $N \updownarrow \overset{\overset{B}{\longleftrightarrow}}{(A_T)}$
  - En cada columna ($\Leftrightarrow$ BRANCA)
  - -1 a la FILA del NUS de SORTIDA
  - +1 a la FILA del NUS d'ARRIBADA
  - 0 a les altres files

- <u>MATRIU d'INCIDÈNCIA REDUÏDA</u>: $N\text{-}1 \updownarrow \overset{\overset{B}{\longleftrightarrow}}{(A)}$
  - Traient una FILA a A_T

- <u>MATRIU de MALLES ADJACENTS</u>: $M \updownarrow \overset{\overset{B}{\longleftrightarrow}}{(F)}$
  - En cada columna ($\Leftrightarrow$ BRANCA)
  - ±1 a les FILES de MALLES adjacents
  - 0 a les altres files

EXEMPLE



$$A_T = \begin{pmatrix} -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = A$$

$$F = \begin{pmatrix} 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

GUIA TÈCNICA 3. ELS FLUXOS DE MALLA SÓN UNA BASE DELS FLUXOS CONSERVATIUS

Lema.- (1) $rang\ A_T = rang\ A$
(2) $rang\ A = N - 1$

Dem.- (1) La suma de totes les files és 0
(2) ? ? ? (Exercici 2)

Teorema (FLUXOS CONSERVATIUS).- En les condicions anteriors:

(1) $\{FLUXOS\ CONSERVATIUS\} = Nuc\ A_T = Nuc\ A = \left\{(A)\begin{pmatrix} i_1 \\ \vdots \\ i_B \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}\right\}$

(2) Formen SUBESPAI VECTORIAL de $\mathbb{R}^B$, i

$$\dim\{FLUXOS\ CONSERVATIUS\} = B - (N-1) = M$$

(3) En són BASE, els FLUXOS de MALLA UNITARIS $U_1, \dots, U_M$

(4) $U_1, \dots, U_M$ són les FILES de F

(5) $\{FLUXOS\ CONSERVATIUS\} = \left\{(F^t)\begin{pmatrix} I_1 \\ \dots \\ I_M \end{pmatrix}, I_1, \dots, I_M \in \mathbb{R}\right\}$

GUIA TÈCNICA 3. ELS FLUXOS DE MALLA SÓN UNA BASE DELS FLUXOS CONSERVATIUS

Dem.-

(1) Comprovació directa

(2) Teorema anterior + EULER

(3) $U_1, \dots, U_M$

- Són CONSERVATIUS (ja vist)
- Tants com la DIMENSIÓ (obvi)
- Són LINEALMENT INDEPENDENTS:

  Per recurrència: $\sum \lambda_k U_k = 0 \Rightarrow \lambda_k = 0$ per a les malles exteriors $\Rightarrow$

  $\Rightarrow \lambda_k = 0$ per a les següents $\Rightarrow$ ...

(4) Comprovació directa

(5) Conseqüència de (3)

EXEMPLE

- Aplicant el teorema anterior:

$$\{FLUXOS\ CONSERVATIUS\} = Nuc\ A_T = (F^t)\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = I_1(1,-1,-1,-1,0,0,0,0) +$$

$$+I_2(0,1,0,0,-1,1,-1,0,0) + I_3(0,0,1,0,0,-1,0,-1,0) + I_4(0,0,0,1,0,0,1,1,-1)$$

- També, aplicant la TEORIA GENERAL:

i_1, i_2, i_3, i_4 són variables independents $\Rightarrow$ una BASE és

$$\left.\begin{matrix} V_1 = (1,0,0,0,-1,0,0,0,-1) \\ V_2 = (0,1,0,0,-1,1,-1,0,0) \\ V_3 = (0,0,1,0,0,-1,0,-1,0) \\ V_4 = (0,0,0,1,0,0,1,1,1) \end{matrix}\right\} \Rightarrow$$

LIN. IND.

$$\Rightarrow \{FLUXOS\ CONSERVATIUS\} = \{i_1V_1 + i_2V_2 + i_3V_3 + i_4V_4;\ i_1, i_2, i_3, i_4 \in \mathbb{R}\}$$

$$= \{(i_1, i_2, i_3, i_4, \underbrace{-i_1 - i_2}_{i_5}, \underbrace{i_2 - i_3}_{i_6}, \underbrace{-i_2 + i_4}_{i_7}, \underbrace{-i_3 + i_4}_{i_8}, \underbrace{-i_1 - i_4}_{i_9})\}$$

PERÒ LA SIGNIFICACIÓ FÍSICA NO ÉS CLARA

EXERCICI 1. ANÀLISI DE CIRCUITS ELÈCTRICS

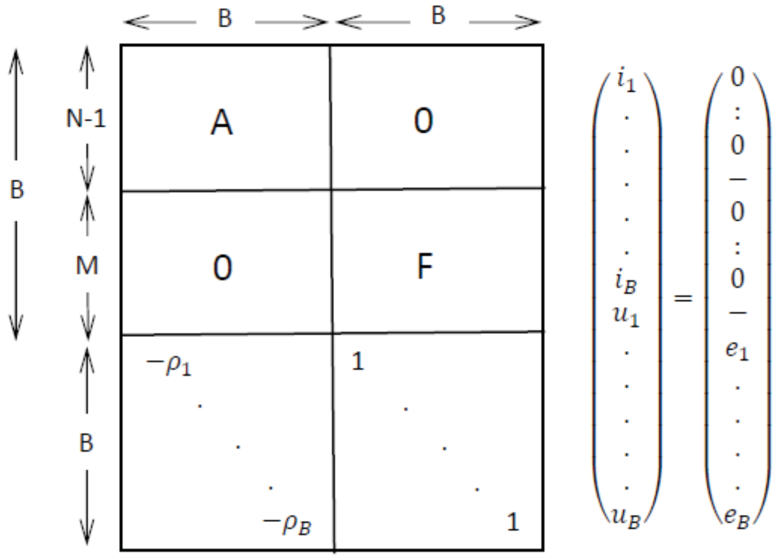
Ho farem per a corrents continus; per alterns seria anàleg, considerant els fasors complexes associats i les impedàncies

- Denotem, per la BRANCA k-èsima
  - i_k= INTENSITAT del CORRENT; u_k= CAIGUDA de POTENCIAL
  - ρ_k= RESISTÈNCIA; e_k= FORÇA ELECTROMOTRIU APORTADA
- Cal determinar:

  $i_1, \dots, i_B, u_1, \dots, u_B$ emprant
  - (KCL) en CADA NUS: $\Sigma i_k = 0$
  - (KVL) en CADA MALLA: $\Sigma u_k = 0$
  - (OHM) en CADA BRANCA: $u_k = \rho_k i_k + e_k$

EXERCICI 1. ANÀLISI DE CIRCUITS ELÈCTRICS

Cal resoldre el sistema amb 2B incògnites:



O equivalentment:

$$(A)\begin{pmatrix} i_1 \\ \vdots \\ i_B \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$(F)\begin{pmatrix} u_1 \\ \vdots \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -\rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\rho_B \end{pmatrix}\begin{pmatrix} i_1 \\ \vdots \\ i_B \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_B \end{pmatrix} = \begin{pmatrix} e_1 \\ \vdots \\ e_B \end{pmatrix}$$

EXERCICI 1. ANÀLISI DE CIRCUITS ELÈCTRICS

Pel teorema anterior, les solucions del primer sistema:

$$\begin{pmatrix} i_1 \\ \vdots \\ i_B \end{pmatrix} = (F^t)\begin{pmatrix} I_1 \\ \vdots \\ I_M \end{pmatrix}$$

Substituint:

$$(F)\left[\begin{pmatrix} e_1 \\ \vdots \\ e_B \end{pmatrix} - \begin{pmatrix} -\rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\rho_B \end{pmatrix}(F^t)\begin{pmatrix} I_1 \\ \vdots \\ I_M \end{pmatrix}\right] = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

O sigui:

$$(F)\begin{pmatrix} -\rho_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -\rho_B \end{pmatrix}(F^t)\begin{pmatrix} I_1 \\ \vdots \\ I_M \end{pmatrix} = (F)\begin{pmatrix} e_1 \\ \vdots \\ e_B \end{pmatrix}$$

que només té M incògnites $I_1, \dots, I_M$

Un cop trobades:

$$\begin{pmatrix} i_1 \\ \vdots \\ i_B \end{pmatrix} = (F^t) \begin{pmatrix} I_1 \\ \vdots \\ I_M \end{pmatrix} \qquad \begin{pmatrix} u_1 \\ \vdots \\ u_B \end{pmatrix} = \begin{pmatrix} e_1 \\ \vdots \\ e_B \end{pmatrix} + \begin{pmatrix} \rho_1 i_1 \\ \vdots \\ \rho_B i_B \end{pmatrix}$$

En l'exemple anterior:

4 INCÒGNITES $(I_1, \dots, I_4)$, en lloc de les 18 $(i_1, \dots, i_9, u_1, \dots, u_9)$

EXERCICI 2. DEMOSTRACIÓ ENGINYERIL DE *rang* $A=N$-1

Hem obtingut una demostració ENGINYERIL del LEMA anterior:

- El sistema homogeni:

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & F \\ \hline \begin{matrix} -\rho_1 & & \\ & \ddots & \\ & & -\rho_B \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{array}\right] \begin{pmatrix} i_1 \\ \vdots \\ i_B \\ u_1 \\ \vdots \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

és clarament determinat, ja que:

$e_1 = \cdots = e_B = 0 \Longrightarrow$ No hi ha aportació d'energia $\Longrightarrow i_1 = \cdots = i_B = u_1 = \cdots = u_B = 0$

EXERCICI 2. DEMOSTRACIÓ ENGINYERIL DE *rang* $A=N$-1

- Per tant:

$$rang \begin{bmatrix} A & 0 \\ 0 & F \\ \begin{matrix} -\rho_1 & & \\ & \ddots & \\ & & -\rho_B \end{matrix} & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{bmatrix} = 2B$$

- Per tant totes les files són LINEALMENT INDEPENDENTS, en particular:

$$rang\ A = N - 1; rang\ F = M$$

FINAL SESSIÓ IV

UNIVERSITAT POLITÈCNICA DE CATALUNYA
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APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA I: ÀLGEBRA LINEAL

SESSIÓ V. SUMA I INTERSECCIÓ DE SUBESPAIS VECTORIALS EN SISTEMES DINÀMICS DISCRETS

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

SUMA I INTERSECCIÓ DE SUBESPAIS VECTORIALS EN SISTEMES DINÀMICS DISCRETS

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INTRODUCCIÓ

SISTEMES DINÀMICS LINEALS DISCRETS

Exemple.-

En el quinquenni k
k= 0, 1, 2, ...

- $x_1(k)$ = habitants al centre
- $x_2(k)$ = habitants a la perifèria
- $u(k)$ = immigració al centre

Canvi quinquennal

- 40 % centre → perifèria
- 10 % perifèria → centre

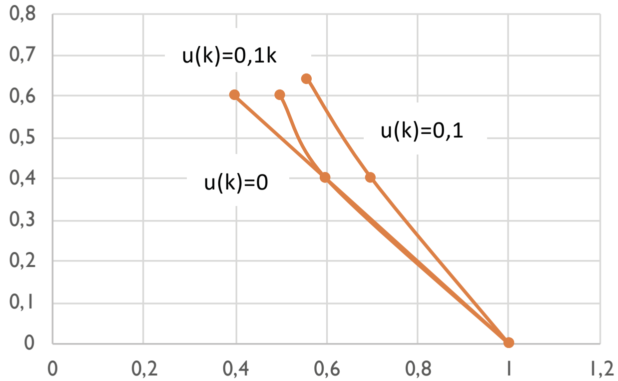
$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \underbrace{\begin{pmatrix} 0{,}6 & 0{,}1 \\ 0{,}4 & 0{,}9 \end{pmatrix}}_{A} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{b} u(k)$$

INTRODUCCIÓ

$$u(k) = 0 \Longrightarrow x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x(1) = \begin{pmatrix} 0{,}6 \\ 0{,}4 \end{pmatrix}, x(2) = \begin{pmatrix} 0{,}4 \\ 0{,}6 \end{pmatrix}, \dots$$

$$u(k) = 0{,}1 \Longrightarrow x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x(1) = \begin{pmatrix} 0{,}7 \\ 0{,}4 \end{pmatrix}, x(2) = \begin{pmatrix} 0{,}56 \\ 0{,}64 \end{pmatrix}, \dots$$

$$u(k) = 0{,}1k \Longrightarrow x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, x(1) = \begin{pmatrix} 0{,}6 \\ 0{,}4 \end{pmatrix}, x(2) = \begin{pmatrix} 0{,}5 \\ 0{,}6 \end{pmatrix}, \dots$$



INTRODUCCIÓ

PREGUNTES

CONTROL

- Quins estats són assolibles (amb algun control)?
- Amb quin u(k)?
- I si tenim també immigració a perifèria?

En general:

n ESTATS: $x(k) = \begin{pmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{pmatrix}$

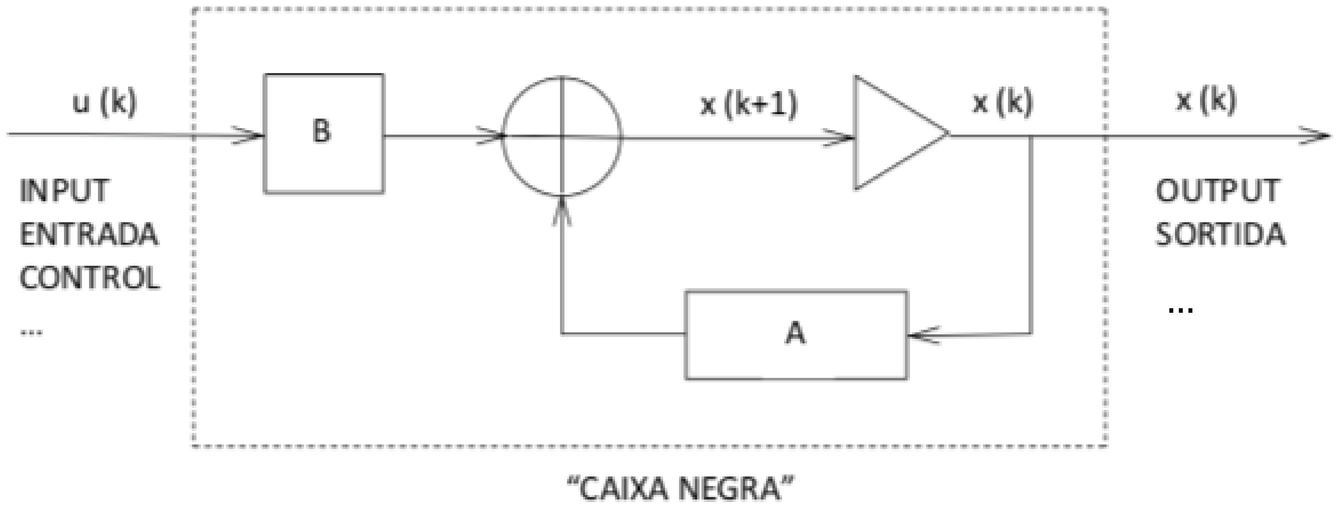
m CONTROLS: $u(k) = \begin{pmatrix} u_1(k) \\ \vdots \\ u_m(k) \end{pmatrix}$

$$\begin{pmatrix} x_1(k+1) \\ \vdots \\ x_n(k+1) \end{pmatrix} = (A) \begin{pmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{pmatrix} + (b_1 \dots b_m) \begin{pmatrix} u_1(k) \\ \vdots \\ u_m(k) \end{pmatrix}$$

B

INTRODUCCIÓ

$$x(k+1) = Ax(k) + Bu(k)$$



SUMA I INTERSECCIÓ DE SUBESPAIS VECTORIALS EN SISTEMES DINÀMICS DISCRETS

GUIA TÈCNICA

1. CAS 1 CONTROL :
SUBESPAIS DEFINITS PER GENERADORS
2. CAS MULTICONTROL :
SUMA I INTERSECCIÓ DE SUBESPAIS
3. DESCOMPOSICIÓ DE KALMAN :
BASES DE GRASSMANN

SUMA I INTERSECCIÓ DE SUBESPAIS VECTORIALS EN SISTEMES DINÀMICS DISCRETS

EXEMPLES PRÀCTICS

Exemple 1: Cas 1 control

Exemple 2: Cas 1 control

Exemple 3: Cas multicontrol

Exemple 4: Descomposició de Kalman

Exemple 5: Descomposició de Kalman

GUIA TÈCNICA I. CAS 1 CONTROL

CAS 1 CONTROL (amb x(0) = 0)

$$\left.\begin{array}{l} x(k+1) = Ax(k) + bu(k) \\ x(0) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x(1) = bu(0) \\ x(2) = Abu(0) + bu(1) \\ x(3) = A^2bu(0) + Abu(1) + Abu(0) \\ \ldots \end{array}\right.$$

Per tant:

- Estats assolibles des de l'origen en k passos amb algun control $\quad R(k) = [b, Ab, \ldots, A^{k-1}b]$
- $R(1) \subset R(2) \subset \cdots \subset R(n) = R(n+1) = R(n+2) = \cdots \equiv R$
- Estats assolibles des de l'origen, amb algun control (no importa el nº de passos) $\quad R(k) = [b, Ab, \ldots, A^{n-1}b]$

Que són SUBESPAIS VECTORIALS DEFINITS PER GENERADORS

EXEMPLE I. CAS 1 CONTROL

$$x(k+1) = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 2 & 1 & 3 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u(k)$$

<u>ESTATS ASSOLIBLES</u>:

$$R(1) = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right] = \{x_1 = x_3 = 0\}$$

$$R(2) = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right] = \{x_1 + x_3 = 0\} \supset R(1)$$

$$R = R(3) = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} \right] = R(2)$$

EXEMPLE I. CAS 1 CONTROL

FUNCIONS DE CONTROL PER ARRIBAR A $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \in R(2)$:

EN 2 PASSOS: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u(1) + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} u(0) \Leftrightarrow \begin{cases} u(0) = 1 \\ u(1) = -1 \end{cases}$

EN 3 PASSOS: $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u(2) + \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} u(1) + \begin{pmatrix} -2 \\ 2 \\ 2 \end{pmatrix} u(0) \Leftrightarrow \begin{cases} -1 = -2u(0) - u(1) \\ 0 = 2u(0) + u(1) + u(2) \\ 1 = 2u(0) + u(1) \end{cases}$

$$\left(\begin{array}{ccc|c} -2 & -1 & 0 & -1 \\ 2 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} -2 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} \text{1-INDETERMINAT} \\ u(0) \text{ VARIABLE INDEPENDENT} \\ u(1) = 1 - 2u(0) \\ u(2) = -1 \end{cases}$$

EXEMPLE I. CAS 1 CONTROL

FUNCIONS DE CONTROL PER ARRIBAR A $\begin{pmatrix} x_1 \\ x_2 \\ -x_1 \end{pmatrix} \in R(2) = R$:

EN 2 PASSOS: $u(0) = -x_1, \quad u(1) = x_2 - u(0)$

EN 3 PASSOS: $u(0)$ VAR. IND., $u(1) = -x_1 - 2u(0), \quad u(2) = x_1 + x_2$

EXEMPLE 2. CAS 1 CONTROL

$$x(k+1) = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 2 & 1 & 3 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} u(k)$$

$$R(1) = \left[\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right] = \{x_1 = 0, \qquad x_2 + x_3 = 0\}$$

$$R(2) = \left[\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \right] = \{x_2 + x_3 = 0\} \supset R(1)$$

$$R = R(3) = \left[\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \begin{pmatrix} 8 \\ -8 \\ 8 \end{pmatrix} \right] = R(2)$$

GUIA TÈCNICA 2. CAS MULTICONTROL

CAS MULTICONTROL

$$\left.\begin{array}{l} x(k+1) = Ax(k) + Bu(k) \\ x(0) = 0 \end{array}\right\} \quad B = (b_1 \dots b_m) \quad \Rightarrow \quad \left\{\begin{array}{l} x(1) = Bu(0) \\ x(2) = ABu(0) + Bu(1) \\ \dots \end{array}\right.$$

ESTATS ASSOLIBLES AMB TOTS ELS CONTROLS: SUMA DE SUBESPAIS

$$\left\{\begin{array}{l} R(k) = [B, AB, \dots, A^{k-1}B] = R(k; b_1) + \cdots + R(k; b_m) \\ R = R(n) = [B, AB, \dots, A^{n-1}B] = R(b_1) + \cdots + R(b_m) \end{array}\right.$$

ESTATS ASSOLIBLES AMB QUALSEVOL DELS CONTROLS: INTERSECCIÓ DE SUBESPAIS

$$\left\{\begin{array}{l} R(k; b_1) \cap \cdots \cap R(k; b_m) \\ R(b_1) \cap \cdots \cap R(b_m) \end{array}\right.$$

EXEMPLE 3. CAS MULTICONTROL

$$x(k+1) = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 1 & -1 \\ 2 & 1 & 3 \end{pmatrix} x(k) + \begin{pmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} u(k)$$

$$\quad b_1 \quad b_2$$

ESTATS ASSOLIBLES AMB ELS 2 CONTROLS:

$$R(1) = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right] = \{x_1 = 0\}$$

$$R(2) = \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} \right] = \mathbb{R}^3$$

EXEMPLE 3. CAS MULTICONTROL

FUNCIONS DE CONTROL:

$$\left.\begin{aligned} x_1 &= -u_1(0) + 2u_2(0) \\ x_2 &= u_1(1) - u_2(1) + u_1(0) - 2u_2(0) \\ x_3 &= u_2(1) + u_1(0) + 2u_2(0) \end{aligned}\right\}$$

$$\left(\begin{array}{cccc|c} 0 & 0 & -1 & 2 & x_1 \\ 1 & -1 & 1 & -2 & x_2 \\ 0 & 1 & 1 & 2 & x_3 \end{array}\right) \Rightarrow \cdots \Rightarrow \left\{\begin{array}{l} \text{1-INDETERMINAT} \\ u_2(0)\ \text{VARIABLE INDEPENDENT} \\ u_1(0) = -x_1 + 2u_2(0) \\ u_1(1) = 2x_1 + x_2 + x_3 - 4u_2(0) \\ u_2(1) = x_1 + x_3 - 4u_2(0) \end{array}\right.$$

ESTATS ASSOLIBLES AMB QUALSEVOL DELS 2 CONTROLS:

$$R(b_1) \cap R(b_2) = \{x_1 + x_3 = 0\} \cap \{x_2 + x_3 = 0\} = \left[\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right]$$

DESCOMPOSICIÓ DE KALMAN

Per a sistemes més generals:
$$\left.\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned}\right\}$$

SUBESPAI DE CONTROLABILITAT: $R = Im\ K,\ \ K = (B, AB, \dots, A^{n-1}B)$

SUBESPAI D'OBSERVABILITAT: $U = Nuc\ L,\ \ L = \begin{pmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{pmatrix}$

DESCOMPOSICIÓ de KALMAN = BASE de GRASSMANN adaptada a R i U

EXEMPLE 4. DESCOMPOSICIÓ DE KALMAN

$$A=\begin{pmatrix}1 & 1 & 1 & 1/2\\ 0 & -2 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 2\end{pmatrix}, B=\begin{pmatrix}1/2\\ 1\\ 0\\ 0\end{pmatrix}, C=(0\ 2\ 0\ 0)$$

$$K=\begin{pmatrix}1/2 & 3/2 & -1/2 & 7/2\\ 1 & -2 & 4 & -8\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, R=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\right]=\{x_3=x_4=0\}$$

$$L=\begin{pmatrix}0 & 2 & 0 & 0\\ 0 & -4 & 0 & 0\\ 0 & 8 & 0 & 0\\ 0 & -16 & 0 & 0\end{pmatrix}, U=\{x_2=0\}=\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right]$$

EXEMPLE 4. DESCOMPOSICIÓ DE KALMAN

$$R \cap U = \{x_2 = x_3 = x_4 = 0\} = \left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right]$$

$$R + U = \mathbb{R}^4$$

DESCOMPOSICIÓ de KALMAN: $\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\begin{pmatrix}0\\1\\0\\0\end{pmatrix};\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\begin{pmatrix}0\\0\\0\\1\end{pmatrix};\emptyset$

$R \cap U$

R

U

EXEMPLE 5. DESCOMPOSICIÓ DE KALMAN

A, C com abans; $B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

$$K = \begin{pmatrix} 0 & 3/2 & 3 & 9/2 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, R = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right] = \{x_2 = 0; x_3 - x_4 = 0\}$$

L, U com abans $\qquad R \cap U = R \qquad R + U = U$

DESCOMPOSICIÓ de KALMAN: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}; \emptyset; \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

FINAL SESSIÓ V

UNIVERSITAT POLITÈCNICA DE CATALUNYA
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APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA I: ÀLGEBRA LINEAL

SESSIÓ VI. APLICACIONS LINEALS I LA SEVA MATRIU

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

APLICACIONS LINEALS I LA SEVA MATRIU

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GUIA TÈCNICA I. CAS PARTICULAR DE DEFINIDES IMPLÍCITAMENT PER UN SISTEMA D'EQUACIONS INDETERMINAT

Exemple 1

$$\left.\begin{array}{r} x_1 + 2x_2 + x_3 = 0 \\ 2x_2 + x_3 + x_4 = 0 \\ x_1 - x_4 = 0 \end{array}\right\} A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Longrightarrow \left\{\begin{array}{l} \text{2-INDETERMINAT} \\ x_1, x_2 \text{ VARIABLES INDEPENDENTS} \end{array}\right.$$

Podem re-escriure'l:

$$\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = 0$$

Per tant:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \underbrace{-\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}}_{\text{MATRIU en BASES ORDINÀRIES}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

GUIA TÈCNICA I. CAS PARTICULAR DE DEFINIDES IMPLÍCITAMENT PER UN SISTEMA D'EQUACIONS INDETERMINAT

PROPOSICIÓ.- $(A)\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \dots \\ 0 \end{pmatrix}; r = rang\ A$ ($\Longrightarrow$ $(n-r)$-INDETERMINAT)

PIVOTANT i REORDENANT FILES i COLUMNES:

$$\underbrace{\begin{pmatrix} A_{IND} \\ - \\ 0 \end{pmatrix}}_{\text{COLUMNES de les VARIABLES INDEPENDENTS}} \begin{pmatrix} VAR \\ IND \end{pmatrix} + \underbrace{\begin{pmatrix} A_{DEP} \\ - \\ 0 \end{pmatrix}}_{\text{COLUMNES de les VARIABLES DEPENDENTS}} \begin{pmatrix} VAR \\ DEP \end{pmatrix} = (0)$$

ALESHORES: $\mathbb{R}^{n-r} \to \mathbb{R}^r$

$$\begin{pmatrix} VAR \\ IND \end{pmatrix} \to \begin{pmatrix} VAR \\ DEP \end{pmatrix} = \underbrace{-(A_{DEP})^{-1}(A_{IND})}_{\text{MATRIU en BASES ORDINÀRIES}} \begin{pmatrix} VAR \\ IND \end{pmatrix}$$

GUIA TÈCNICA 2. DEFINIDES PER LES IMATGES D'UNA BASE

PROPOSICIÓ.- $E \xrightarrow{f} F$

BASE de E: $u_1 \longrightarrow f(u_1)$, …, $u_n \longrightarrow f(u_n)$

(1) $f(x) = x_1 f(u_1) + \cdots + x_n f(u_n)$, amb $x = x_1 u_1 + \cdots + x_n u_n$

(2) BASE de F: $v_1, \dots, v_m$

$$x \longrightarrow f(x) \equiv y = y_1 v_1 + \cdots + y_m v_m$$

$$\Updownarrow$$

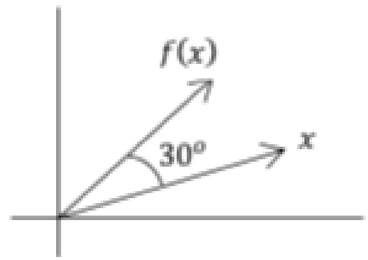
$$\begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} \xrightarrow{f} \begin{pmatrix} y_1 \\ \cdots \\ y_m \end{pmatrix} = \begin{pmatrix} \vdots & \cdots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$$

$f(u_1) \dots f(u_n)$ *en BASE* $v_1, \dots, v_m$

MATRIU de f *en BASES* $(u_1, \dots u_n)$ *i* $(v_1, \dots, v_m)$

GUIA TÈCNICA 2. DEFINIDES PER LES IMATGES D'UNA BASE

Exemple 2: ROTACIÓ de 30º: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$



- $f\binom{3}{2} = ?$

$$\binom{1}{0} \longrightarrow \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

$$\binom{0}{1} \longrightarrow \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$$

$$\Rightarrow f\binom{3}{2} = 3\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} + 2\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 3\,\dfrac{\sqrt{3}}{2} + 2\,\dfrac{-1}{2} \\ 3\,\dfrac{1}{2} + 2\,\dfrac{\sqrt{3}}{2} \end{pmatrix}$$

$$\binom{3}{2} = 3\binom{1}{0} + 2\binom{0}{1}$$

Exemple 2: ROTACIÓ de 30º: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

- MATRIU en BASES ORDINÀRIES: $\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

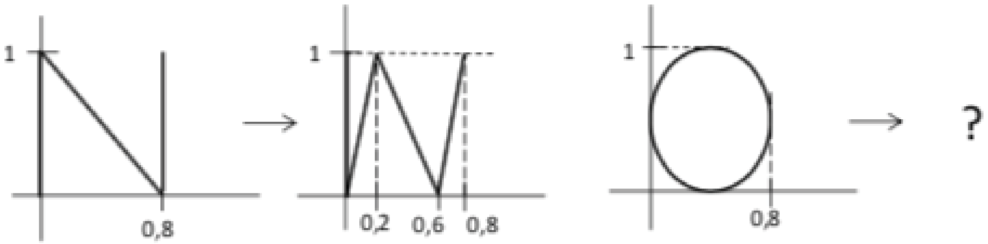
$$\underset{f\binom{1}{0}}{} \qquad \underset{f\binom{0}{1}}{}$$

$$\binom{3}{2} \longrightarrow \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \binom{3}{2} = \begin{pmatrix} 3\,\frac{\sqrt{3}}{2} + 2\,\frac{-1}{2} \\ 3\,\frac{1}{2} + 2\,\frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\binom{x_1}{x_2} \longrightarrow \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \binom{x_1}{x_2} = \begin{pmatrix} \frac{\sqrt{3}}{2}x_1 - \frac{1}{2}x_2 \\ \frac{1}{2}x_1 + \frac{\sqrt{3}}{2}x_2 \end{pmatrix}$$

GUIA TÈCNICA 2. DEFINIDES PER LES IMATGES D'UNA BASE

Exemple 3: PASSAR a CURSIVA: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$



$$\begin{pmatrix}1\\0\end{pmatrix} \longrightarrow \begin{pmatrix}0{,}75\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}0{,}2\\1\end{pmatrix} \Rightarrow$$

MATRIU en BASES ORDINÀRIES: $\begin{pmatrix}0{,}75 & 0{,}2\\0 & 1\end{pmatrix}$

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} \longrightarrow \begin{pmatrix}0{,}75 & 0{,}2\\0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0{,}75x_1 + 0{,}2x_2\\x_2\end{pmatrix}$$

GUIA TÈCNICA 2. DEFINIDES PER LES IMATGES D'UNA BASE

Exemple 3: PASSAR a CURSIVA: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

EFECTIVAMENT : $\begin{pmatrix} 0,8 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0,75 & 0,2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0,8 \\ 0 \end{pmatrix} = \begin{pmatrix} 0,6 \\ 0 \end{pmatrix}$

EN GENERAL, PUNTS FIXOS:

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 0,75 & 0,2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Leftrightarrow \cdots \Leftrightarrow x_1 = 0,8x_2$$

GUIA TÈCNICA 3. CAS PARTICULAR DE DEFINIDES PER IMATGES SUCCESSIVES

Exemple 4: CENTRE / PERIFÈRIES

SUPOSEM

CIUTAT amb 1.000.000 HABITANTS (CONSTANT, per SIMPLIFICAR)

	2010	2015	2020 (PROVISIONAL)
CENS AL CENTRE	1.000.000	600.000	400.000

PREGUNTES

(1) 2025

(2) 2030, 2035, …

(3) EQUILIBRI (SI N'HI HA)

(4) EN GENERAL: 2010 + 5k

(5) QUIN CENS DEFINITIU a 2020 SERIA ALARMANT

(6) INTERPRETACIÓ SOCIOLÒGICA

Exemple 4: CENTRE / PERIFÈRIES

(0) PLANTEJAMENT: $E = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \begin{matrix} x_1 = \text{HABITANTS al CENTRE} \\ x_2 = \text{HABITANTS a la PERIFÈRIA} \end{matrix} \right\} \simeq \mathbb{R}^2$

$$E \xrightarrow{f} F \qquad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \longrightarrow \begin{pmatrix} x_1(t+5) \\ x_2(t+5) \end{pmatrix}$$

HABITANTS en t → HABITANTS en t+5

LINEAL $\left(\Longleftrightarrow \left\{ \begin{matrix} \text{PROPORCIONALITAT} \\ \text{HOMOGENEITAT} \end{matrix} \right. \right)$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{f} \begin{pmatrix} 0{,}6 \\ 0{,}4 \end{pmatrix} \xrightarrow{f} \begin{pmatrix} 0{,}4 \\ 0{,}6 \end{pmatrix} \longrightarrow \cdots$$

2009 — 2015 — 2020 (PROV)

GUIA TÈCNICA 3. CAS PARTICULAR DE DEFINIDES PER IMATGES SUCCESSIVES

Exemple 4: <u>CENTRE / PERIFÈRIES</u>

(1) 2025? : $f\begin{pmatrix}0,4\\0,6\end{pmatrix} = \alpha f\begin{pmatrix}1\\0\end{pmatrix} + \beta f\begin{pmatrix}0,6\\0,4\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}0,6\\0,4\end{pmatrix} + \frac{3}{2}\begin{pmatrix}0,4\\0,6\end{pmatrix} \Rightarrow$ 300.000 al 2025

$$\alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0,6\\0,4\end{pmatrix} \Leftrightarrow \begin{matrix}0,4 = \alpha + 0,6\beta\\ 0,6 = \quad 0,4\beta\end{matrix} \Leftrightarrow \begin{matrix}\alpha = {-1}/{2}\\ \beta = {3}/{2}\end{matrix}$$

(1') CALCULEM LA MATRIU en BASES ORDINÀRIES:

$$\begin{pmatrix}1\\0\end{pmatrix} \longrightarrow \begin{pmatrix}0,6\\0,4\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \alpha'\begin{pmatrix}0,6\\0,4\end{pmatrix} + \beta'\begin{pmatrix}0,4\\0,6\end{pmatrix} \Leftrightarrow \begin{pmatrix}0\\1\end{pmatrix} = \alpha'\begin{pmatrix}1\\0\end{pmatrix} + \beta'\begin{pmatrix}0,6\\0,4\end{pmatrix} \Leftrightarrow \begin{matrix}\alpha' = {-3}/{2}\\ \beta' = {5}/{2}\end{matrix}$$

Exemple 4: CENTRE / PERIFÈRIES

$$\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \alpha'\begin{pmatrix}0,6\\0,4\end{pmatrix} + \beta'\begin{pmatrix}0,4\\0,6\end{pmatrix} = -\frac{3}{2}\begin{pmatrix}0,6\\0,4\end{pmatrix} + \frac{5}{2}\begin{pmatrix}0,4\\0,6\end{pmatrix} = \begin{pmatrix}0,1\\0,9\end{pmatrix} \Rightarrow$$

$\Rightarrow$ MATRIU en BASES ORDINÀRIES $= \begin{pmatrix}0,6 & 0,1\\0,4 & 0,9\end{pmatrix} \equiv A \Rightarrow \boxed{\begin{pmatrix}x_1(t+5)\\x_2(t+5)\end{pmatrix} = \underbrace{\begin{pmatrix}0,6 & 0,1\\0,4 & 0,9\end{pmatrix}}_{A}\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}}$

(2) $$\begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{A} \begin{pmatrix}0,6\\0,4\end{pmatrix} \xrightarrow{A} \begin{pmatrix}0,4\\0,6\end{pmatrix} \xrightarrow{A} \begin{pmatrix}0,3\\0,7\end{pmatrix} \xrightarrow{A} \begin{pmatrix}0,25\\0,75\end{pmatrix} \xrightarrow{A} \begin{pmatrix}0,225\\0,775\end{pmatrix} \longrightarrow \cdots$$

2010 2015 2020 2025 2030 2035

GUIA TÈCNICA 3. CAS PARTICULAR DE DEFINIDES PER IMATGES SUCCESSIVES

Exemple 4: CENTRE / PERIFÈRIES

(3) EQUILIBRI :

$$\begin{pmatrix} {x_1}^e \\ {x_2}^e \end{pmatrix} \longrightarrow (A) \begin{pmatrix} {x_1}^e \\ {x_2}^e \end{pmatrix} \Leftrightarrow \begin{matrix} {x_1}^e = 0{,}6{x_1}^e + 0{,}1{x_2}^e \\ {x_2}^e = 0{,}4{x_1}^e + 0{,}9{x_2}^e \end{matrix} \Leftrightarrow \cdots \Leftrightarrow \begin{matrix} {x_1}^e = 0{,}2 \\ {x_2}^e = 0{,}8 \end{matrix}$$

PER TANT: EQUILIBRI amb 200.000

(4) SI COMPTEM per QUINQUENNIS : $\begin{cases} t = 2010 + 5k, & k = 0, 1, 2, \ldots \\ x(k) \equiv x(2010 + 5k) \end{cases}$

$$x(0) \xrightarrow{A} x(1) \xrightarrow{A} x(2) \longrightarrow \cdots$$

2010 2015 2020

$$\begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = A \cdots A \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = A^k \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 0{,}6 & 0{,}1 \\ 0{,}4 & 0{,}9 \end{pmatrix}^k \begin{pmatrix} 1 \\ 0 \end{pmatrix} = ?$$

Exemple 4: CENTRE / PERIFÈRIES

(5) ALARMANT $\Leftrightarrow$ ES BUIDA EL CENTRE $\Leftrightarrow A' = \begin{pmatrix} 0{,}6 & a \\ 0{,}4 & b \end{pmatrix}$ AMB EQUILIBRI a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0{,}6 & a \\ 0{,}4 & b \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow \begin{matrix} a = 0 \\ b = 1 \end{matrix}$$

$$A' = \begin{pmatrix} 0{,}6 & 0 \\ 0{,}4 & 1 \end{pmatrix} \Leftrightarrow A' \begin{pmatrix} 0{,}6 \\ 0{,}4 \end{pmatrix} = \begin{pmatrix} 0{,}36 \\ 0{,}64 \end{pmatrix}$$

PER TANT: ALARMANT $\Leftrightarrow$ CENS DEFINITIU a 2020 = 360.000 en lloc de 400.000

Exemple 4: CENTRE / PERIFÈRIES

(6)

$$A: x_1(k+1) = \underbrace{0{,}6}_{*}x_1(k) + \underbrace{0{,}1}_{**}x_2(k)$$

$$A': x_1(k+1) = 0{,}6x_1(k) + \underbrace{0}_{***}x_2(k)$$

* 40% CENTRE ⟶ PERIFÈRIA

** 10% PERIFÈRIA ⟶ CENTRE

*** NO HI HA RETORN

PER TANT L'ALARMA
- NO DEPEN del % CENTRE ⟶ PERIFÈRIA
- SÍ DEPEN del RETORN PERIFÈRIA ⟶ CENTRE

FINAL SESSIÓ VI

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

ETSEIB

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA I: ÀLGEBRA LINEAL

SESSIÓ VII. CANVIS DE BASE

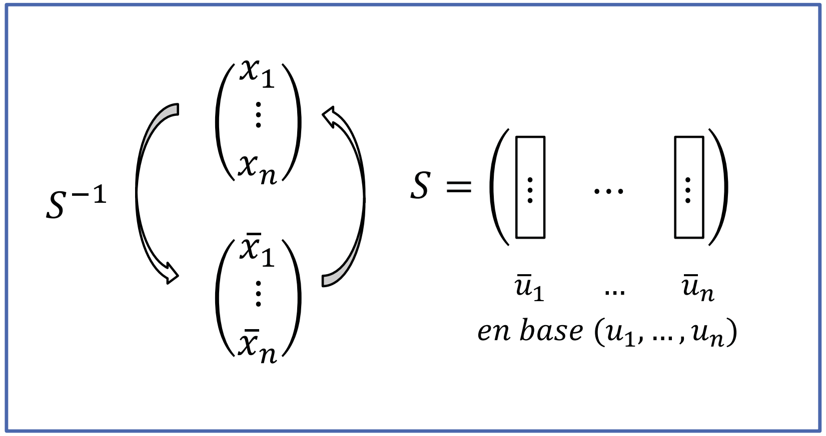
PROFESSORAT: JOSEP FERRER
MARTA PEÑA

CANVIS DE BASE

INDEX

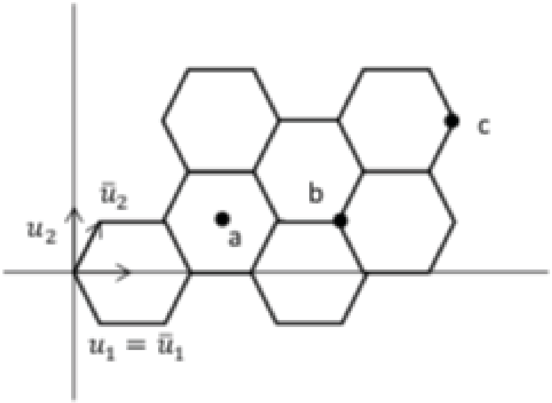
GUIA TÈCNICA I. CANVI DE BASE EN VECTORS

BASES $\begin{cases} u_1, \dots, u_n \\ \bar{u}_1, \dots, \bar{u}_n \end{cases}$ $\qquad x = \begin{cases} x_1 u_1 + \cdots + x_n u_n \\ \bar{x}_1 \bar{u}_1 + \cdots + \bar{x}_n \bar{u}_n \end{cases}$



GUIA TÈCNICA I. CANVI DE BASE EN VECTORS

Exemple 1.- RETICLES HEXAGONALS (GRAFÈ, EMPAQUETAMENT COMBUSTIBLE NUCLEAR, ...)



- COORDENADES MÉS FÀCILS en $(\bar{u}_1, \dots, \bar{u}_n)$:

	a	b	c
COORDENADES en (u_1, u_2)	?	?	*
COORDENADES en $(\bar{u}_1, \bar{u}_2)$	$\binom{2}{1}$	$\binom{4}{1}$	$\binom{5}{3}$

GUIA TÈCNICA I. CANVI DE BASE EN VECTORS

Exemple 1.- RETICLES HEXAGONALS (GRAFÈ, EMPAQUETAMENT COMBUSTIBLE NUCLEAR, ...)

- CANVIS DE BASE: $S = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}$; $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = S \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} \bar{x}_1 + \frac{1}{2}\bar{x}_2 \\ \frac{\sqrt{3}}{2}\bar{x}_2 \end{pmatrix}$

- PER EXEMPLE: $* = S \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 + 3/2 \\ 3\sqrt{3}/2 \end{pmatrix}$

$$d(x,0) = \sqrt{x_1^2 + x_2^2} = \sqrt{(\bar{x}_1 + \frac{1}{2}\bar{x}_2)^2 + (\frac{\sqrt{3}}{2}\bar{x}_2)^2} = \sqrt{\bar{x}_1^2 + \bar{x}_1\bar{x}_2 + \bar{x}_2^2}$$

Exemple 2.- COLORS: DIMENSIÓ 3

- Els COLORS formen un espai vectorial de DIMENSIÓ 3.
  Per exemple: groc, verd, vermell i blau NO són linealment independents.

- S'empren DIFERENTS BASES de 3 colors
  segons la mescla sigui ADDITIVA (llum) o SUBSTRACTIVA (pigments)
  com detallem a continuació.
  Els 3 colors triats se'n diuen PRIMARIS i
  les mescles de només 2 d'ells, SECUNDARIS

- Tanmateix, al congrés internacional CIE (1931) es van establir unes noves coordenades on intervé la LLUMINOSITAT

GUIA TÈCNICA I. CANVI DE BASE EN VECTORS

Exemple 2.- COLORS: DIMENSIÓ 3

- ULL HUMÀ
  - 6,5 milions de cons: L, ROIG; M, VERD; S, BLAU
  - 120 milions de bastons: LLUMINOSITAT BLANC/NEGRE
- Per això, per a mescles additives s'empren els ROIG, VERD i BLAU com a colors primaris

CODI NATURAL (pantalles...): $\begin{pmatrix} R \\ G \\ B \end{pmatrix}$ $\begin{matrix} \text{ROIG} \\ \text{VERD} \\ \text{BLAU} \end{matrix}$

Els secundaris resulten:

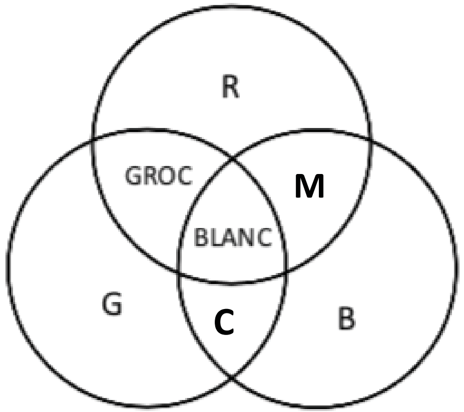
G + B = CIAN (≅ turquesa)

R + B = MAGENTA (≅ fúcsia)

R + G = GROC

També: R + G + B = BLANC

Però no es pot obtenir el NEGRE



Exemple 2.- COLORS: DIMENSIÓ 3

- Per a mescles substractives (PIGMENTS, IMPRESSORES...) s'empren els CIAN, MAGENTA i GROC com a colors primaris:

CODI CMY (impressores...): $\begin{pmatrix} C \\ M \\ Y \end{pmatrix} \begin{matrix} \text{CIAN} \\ \text{MAGENTA} \\ \text{GROC} \end{matrix}$

Ara els secundaris són els primaris del codi natural:

MAGENTA + GROC = ROIG

CIAN + GROC = VERD

CIAN + MAGENTA = BLAU

i amb tots tres:

CIAN + MAGENTA + GROC = NEGRE

Tanmateix, sovint s'afegeix el negre com a 4rt pigment per raons d'estalvi i de major "rigitud"

Exemple 2.- COLORS: DIMENSIÓ 3

- Tornant a les mescles additives (IL·LUMINACIÓ) es va constatar que

l'ull humà és especialment sensible a la BRILLANTOR (BLANC/NEGRE)

Per això el congrés internacional CIE va establir

CODI CIE (1931): $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

x $(\equiv C_R) \cong$ ROIG

y BRILLANTOR, LLUMINOSITAT

z $(\equiv C_B) \cong$ BLAU

Un canvi usual:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0{,}61 & 0{,}29 & 0{,}15 \\ 0{,}35 & 0{,}59 & 0{,}063 \\ 0{,}04 & 0{,}12 & 0{,}787 \end{pmatrix} \begin{pmatrix} R \\ G \\ B \end{pmatrix}$$

GUIA TÈCNICA 2. CANVI DE BASE EN APLICACIONS LINEALS

$$
\begin{array}{ccc}
E & \xrightarrow{\ f\ } & F \\
x & \longrightarrow & f(x) \equiv y
\end{array}
$$

$$
S \left\{ \begin{array}{l} u_1, \dots, u_n \\ \bar{u}_1, \dots, \bar{u}_n \end{array} \right. \qquad
T \left\{ \begin{array}{l} v_1, \dots, v_m \\ \bar{v}_1, \dots, \bar{v}_m \end{array} \right.
$$

$$
\begin{array}{ccccc}
 & \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} & \xrightarrow{\ A\ } & \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} & \\
S \uparrow & & & & \uparrow T \\
 & \begin{pmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_n \end{pmatrix} & \xrightarrow{\ \bar{A}\ } & \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_m \end{pmatrix} &
\end{array}
$$

$$A = T\bar{A}S^{-1}$$

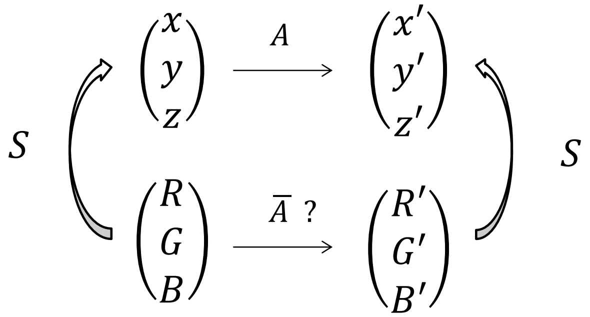
$$\bar{A} = T^{-1}AS$$

GUIA TÈCNICA 2. CANVI DE BASE EN APLICACIONS LINEALS

Exemple 3.- FILTRES DE COLORS

A en CODI CIE

? en CODI $\begin{pmatrix} R \\ G \\ B \end{pmatrix}$



$$\bar{A} = S^{-1}AS$$

GUIA TÈCNICA 2. CANVI DE BASE EN APLICACIONS LINEALS

Exemple 4.- <u>MATRIU D'APLICACIONS PER ESTATS SUCCESSIUS</u>

- Recordem: $\begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{f} \begin{pmatrix}0{,}6\\0{,}4\end{pmatrix} \xrightarrow{f} \begin{pmatrix}0{,}4\\0{,}6\end{pmatrix} \xrightarrow{f} \cdots$

Considerem BASES: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$S \left[\begin{array}{l} \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \\ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0{,}6\\0{,}4\end{pmatrix} \end{array} \right. \qquad T \left[\begin{array}{l} \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \\ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \end{array} \right.$$

$$S = \begin{pmatrix}1 & 0{,}6\\0 & 0{,}4\end{pmatrix} \qquad T = Id$$

$$A = ?$$

$$\bar{A} = \begin{pmatrix}0{,}6 & 0{,}4\\0{,}4 & 0{,}6\end{pmatrix} \Longrightarrow$$

$$\Longrightarrow A = T\bar{A}S^{-1} = \begin{pmatrix}0{,}6 & 0{,}4\\0{,}4 & 0{,}6\end{pmatrix}\begin{pmatrix}1 & 0{,}6\\0 & 0{,}4\end{pmatrix}^{-1}$$

Exemple 4.- MATRIU D'APLICACIONS PER ESTATS SUCCESSIUS

- En general:

$$x(0) \xrightarrow{f} x(1) \xrightarrow{f} \cdots \xrightarrow{f} x(n-1) \xrightarrow{f} x(n) \longrightarrow \cdots$$

Si $x(0), x(1) \dots x(n-1)$ són L.I. (genèricament cert), aleshores:

La MATRIU en BASES ORDINÀRIES és:

$$A = \begin{pmatrix} \boxed{\vdots} & \cdots & \boxed{\vdots} \\ & \ddots & \\ & \cdots & \end{pmatrix} \begin{pmatrix} \boxed{\vdots} & \cdots & \boxed{\vdots} \\ & \ddots & \\ & \cdots & \end{pmatrix}^{-1}$$

$$\quad x(1) \quad \dots \quad x(n) \qquad x(0) \quad \dots \quad x(n-1)$$

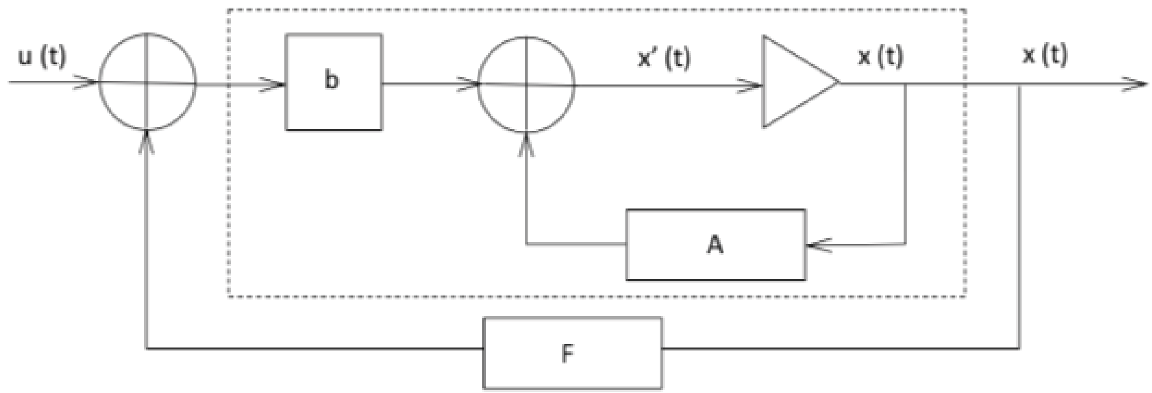
En canvi, el càlcul directe dels coeficients d'A (per enquestes, ...)

SERIA MOLT COMPLICAT!

GUIA TÈCNICA 3. APLICACIÓ A TEORIA DE CONTROL: ASSIGNACIÓ DE POLS PER REALIMENTACIÓ MITJANÇANT LA FORMA CANÒNICA DE CONTROL

PLANTEJAMENT DEL PROBLEMA

- Donat $x'(t) = Ax(t) + bu(t)$, els seus "POLS" són molt significatius (per exemple, freqüències de ressonància, ...)
- Ens proposem canviar-los mitjançant una "REALIMENTACIÓ" exterior F:



$$x'(t) = Ax(t) + b\big(u(t) + Fx(t)\big) = (A + bF)x(t) + bu(t)$$

GUIA TÈCNICA 3. APLICACIÓ A TEORIA DE CONTROL: ASSIGNACIÓ DE POLS PER REALIMENTACIÓ MITJANÇANT LA FORMA CANÒNICA DE CONTROL

PLANTEJAMENT DEL PROBLEMA

- Els "POLS" resulten ser els VAPs de A

Per tant: podem triar els VAPs de $A + bF$ amb F adient??

- Veurem que:

- (A, b) es pot transformar mitjançant un CANVI DE BASE S en la seva "FORMA CANÒNICA de CONTROL" de (A_c, b_c) amb:

$$A_c = S^{-1}AS \quad \text{una matriu de SYLVESTER (o COMPANYA)} ; \quad b_c = S^{-1}b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

- és fàcil trobar F_c de manera que:

$A_c + b_c F_c$ tingui els VAPs desitjats

- Finalment, prenent

$$F = F_c S^{-1}$$

obtenim els VAPs desitjats ja que:

$$A + bF = SA_cS^{-1} + S\, b_cF_cS^{-1} = S\,(A_c + b_c\, F_c)\, S^{-1}$$

3.1. MATRIUS DE SYLVESTER

- Se'n diuen les de la forma

$$A_C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}$$

- El seu POLINOMI CARACTERÍSTIC és

$$Q(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

per la qual cosa es diu matriu COMPANYA de $Q(z)$

- És clar que podem obtenir qualsevol altre polinomi característic

$$Q'(z) = z^n + a'_{n-1}z^{n-1} + \cdots + a'_1 z + a'_0$$

"realimentant" en la forma

$$A_c + \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix} F_c \quad \text{amb} \quad F_c = \begin{pmatrix} a_0 - a'_0 & \dots & a_{n-1} - a'_{n-1} \end{pmatrix}$$

3.2. FORMA CANÒNICA DE CONTROL

PROBLEMA.-

- Donat $x'(t) = Ax(t) + bu(t)$. Ens preguntem si amb un CANVI de BASE S podem:

$$S^{-1}AS \equiv A_C = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \boxed{\quad * \quad} \end{pmatrix} ; \quad S^{-1}b \equiv b_C = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

- Com que A_C té la FORMA de SYLVESTER (o COMPANYA), ha de ser:

$$\boxed{\quad * \quad} = -a_0 \quad -a_1 \quad \dots \quad -a_{n-1}$$

on: $Q_A(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ (POLINOMI CARACTERÍSTIC)

3.2. FORMA CANÒNICA DE CONTROL

Exemple 5:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \; Q_A(z) = z^3 - 2z + 1$$

- CANVI de BASE: A^2b, Ab, b

$$S_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\bar{A} = (S_1)^{-1}AS = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\bar{b} = (S_1)^{-1}b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

3.2. FORMA CANÒNICA DE CONTROL

Exemple 5:

- SEGON CANVI de BASE: $S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \Longrightarrow$

$$(S_2)^{-1}\bar{A}S_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}$$

$$(S_2)^{-1}\bar{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

GUIA TÈCNICA 3. APLICACIÓ A TEORIA DE CONTROL: ASSIGNACIÓ DE POLS PER REALIMENTACIÓ MITJANÇANT LA FORMA CANÒNICA DE CONTROL

3.2. FORMA CANÒNICA DE CONTROL

TEOREMA.- Suposem

$$\begin{cases} x'(t) = Ax(t) + bu(t) \\ rang\,(b, Ab, \dots, A^{n-1}b) = n \\ Q_A(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \end{cases}$$

Fent 2 CANVIS de BASE consecutius:

$$S_1 = \begin{pmatrix} \underbrace{\begin{bmatrix} \vdots \end{bmatrix}}_{A^{n-1}b} & \cdots & \underbrace{\begin{bmatrix} \vdots \end{bmatrix}}_{Ab} & \underbrace{\begin{bmatrix} \vdots \end{bmatrix}}_{b} \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a_{n-1} & \ddots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & 0 \\ a_2 & \cdots & \ddots & \ddots & 0 \\ a_1 & a_2 & \cdots & a_{n-1} & 1 \end{pmatrix}$$

3.2. FORMA CANÒNICA DE CONTROL

Resulta:

$$A_C = (S_2)^{-1}(S_1)^{-1}AS_1S_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix}$$

$$b_C = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

FORMA CANÒNICA DE CONTROL

3.3. ASSIGNACIÓ DE POLS PER REALIMENTACIÓ

Exemple 6: Donats $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Volem F per tal que: $Q_{A+bF}(z) = z^3 + a_2' z^2 + a_1' z + a_0'$

- Fem-ho primer per la seva FORMA CANÒNICA de CONTROL:

$$A_C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}, b_C = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \; F_C = (\alpha \;\; \beta \;\; \gamma)$$

3.3. ASSIGNACIÓ DE POLS PER REALIMENTACIÓ

Exemple 6:

$$A_C + b_C F_C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (\alpha \;\; \beta \;\; \gamma) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1+\alpha & 2+\beta & \gamma \end{pmatrix}$$

$$Q_{A_C+b_C F_C}(z) = z^3 + a_2' z^2 + a_1' z + a_0' \Leftrightarrow \left[\begin{matrix} -1+\alpha = -a_0' \\ 2+\beta = -a_1' \\ \gamma = -a_2' \end{matrix} \right. \Leftrightarrow \begin{matrix} \alpha = 1 - a_0' \\ \beta = -2 - a_1' \\ \gamma = -a_2' \end{matrix}$$

3.3. ASSIGNACIÓ DE POLS PER REALIMENTACIÓ

Exemple 6:

- Aleshores: $S_1 S_2 (A_C + b_C F_C)(S_2)^{-1}(S_1)^{-1} =$

$$= S_1 S_2 A_C (S_2)^{-1}(S_1)^{-1} + S_1 S_2 b_C F_C (S_2)^{-1}(S_1)^{-1} = A + bF_C (S_2)^{-1}(S_1)^{-1}$$

Prenent $\boxed{F = F_C (S_2)^{-1}(S_1)^{-1}}$ resulta:

$$Q_{A+bF}(z) = Q_{A_C + b_C F_C}(z) = z^3 + a_2' z^2 + a_1' z + a_0'$$

- En definitiva:

$$F = (1 - a_0' \quad -2 - a_1' \quad -a_2')(S_2)^{-1}(S_1)^{-1}$$

3.3. ASSIGNACIÓ DE POLS PER REALIMENTACIÓ

TEOREMA.- Suposem
$$\begin{cases} x'(t) = Ax(t) + bu(t) \\ rang\ (b, Ab, \dots, A^{n-1}b) = n \\ Q_A(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \end{cases}$$

Volem F per tal que: $Q_{A+bF}(z) = z^n + a'_{n-1}z^{n-1} + \cdots + a'_1 z + a'_0$

Siguin S_1, S_2 com al TEOREMA anterior

Aleshores: $$F = (a_0 - a'_0 \quad a_1 - a'_1 \quad \dots \quad a_{n-1} - a'_{n-1})(S_2)^{-1}(S_1)^{-1}$$

FINAL SESSIÓ VII

UNIVERSITAT POLITÈCNICA DE CATALUNYA BARCELONATECH

ETSEIB

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA I: ÀLGEBRA LINEAL

SESSIÓ VIII. VAPs, VEPs I DIAGONALITZACIÓ A L'ENGINYERIA

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

VAPs, VEPs I DIAGONALITZACIÓ A L'ENGINYERIA

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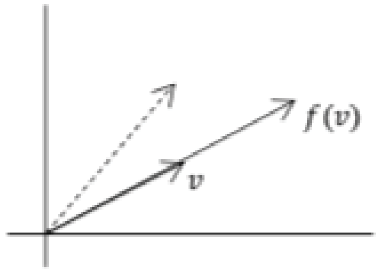
GUIA TÈCNICA I. VAPs I VEPs; DIAGONALITZACIÓ

DEF.- $\left.\begin{array}{ll} v(\neq 0) & VEP \\ \lambda & VAP \end{array}\right\} \Leftrightarrow f(v) = \lambda v \Leftrightarrow (A)(v) = \lambda(v) \Leftrightarrow$



PROP.- A DIAGONALITZA $\Leftrightarrow \exists$ BASE VEPs $v_1 \dots v_n \Leftrightarrow$

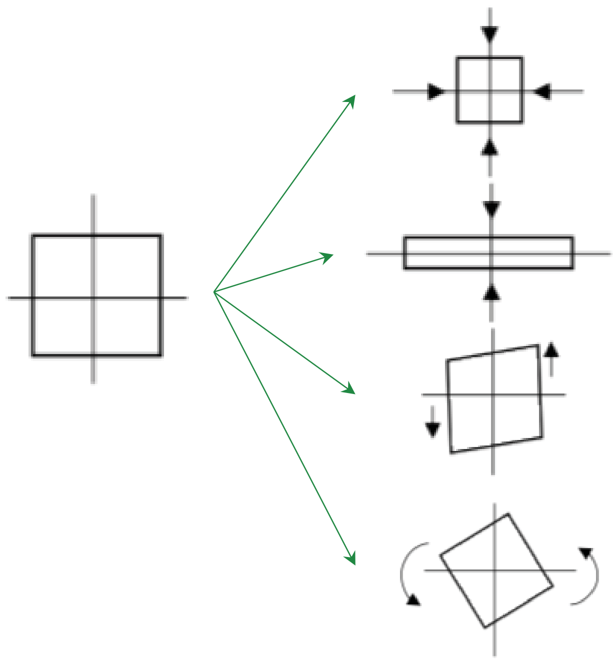
$$S^{-1}AS = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \quad \text{VAPs}$$

$$S = \begin{pmatrix} \boxed{\vdots} & \cdots & \boxed{\vdots} \end{pmatrix}, \quad v_1 \quad \dots \quad v_n \quad \text{VEPs L.I.}$$

OBS.- Es generalitza a matrius no diagonalitzables mitjançant la FORMA REDUÏDA de JORDAN

- <u>RESISTÈNCIA DE MATERIALS</u>:

  4 tipus bàsics de sol·licitacions externes, caracteritzades pels VEPs



APLICACIÓ/MATRIU	VEPs	VAPs
	TOTS	$\lambda < 1$
	$\begin{cases} e_1 \\ e_2 \end{cases}$	$\lambda_1 > 1$ $\lambda_2 < 1$
	e_2	$\lambda = 1$
	CAP	

GUIA TÈCNICA 2. VAPs I VEPs A L'ENGINYERIA

	APLICACIÓ/MATRIU	VEPs	VAPs
• MECÀNICA	Tensor d'inèrcia (Rotació → Moment angular)	Eixos principals d'inèrcia (eix de simetria axial, normal a pla de simetria, ...)	Moments principals d'inèrcia
• ELASTICITAT	Tensor deformació (Punt → Deformació)	Direccions principals de deformació	...
		‖ (LLEI DE HOOKE)	
	Tensor tensió	Direccions principals de tensió	...

GUIA TÈCNICA 2. VAPs I VEPs A L'ENGINYERIA

	APLICACIÓ/MATRIU	VEPs	VAPs
• CONTROL	Matriu de transició d'estats	...	Pols (Ressonàncies, ...)
• DINÀMICA, ELECTRICITAT ...: funcions pròpies de les EDOs fonamentals	D	CONSTANTS $e^{\lambda t}$	0 λ
	D^2	CONSTANTS $e^{\lambda t}$ $\sin \omega t, \cos \omega t$	0 λ^2 $-\omega^2$
• MODELS POBLACIONALS	Matriu de transició	Distribucions poblacionals estacionàries	Taxa de creixement
• ETC.			

Exemple 1.- PRESA / DEPREDADOR

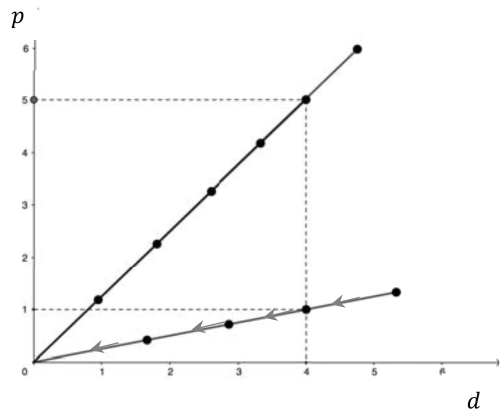
Suposem un model presa/depredador, on la població respectiva $d(k+1), p(k+1)$ de l'any vinent depèn linealment de les de l'any actual $d(k)/p(k)$:

$$\begin{pmatrix} d(k+1) \\ p(k+1) \end{pmatrix} = \begin{pmatrix} 0{,}5 & 0{,}4 \\ -0{,}125 & 1{,}1 \end{pmatrix} \begin{pmatrix} d(k) \\ p(k) \end{pmatrix}$$

VAPs VEPs

1 $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ EQUILIBRI
(per QUALSSEVOL poblacions amb PROPORCIÓ d/p = 4/5)

$0{,}6$ $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ EXTINCIÓ
(DECREIXEMENT 40% anual i PROPORCIÓ constant d/p = 4/1)



OBS.- Models més sofisticats (no lineals) donen òrbites tancades al voltant d'un únic punt d'equilibri.

PROP.-

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \Rightarrow \begin{cases} A^k = S\begin{pmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{pmatrix} S^{-1} \\ e^{At} = S\begin{pmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{pmatrix} S^{-1} \\ \sqrt{A} = \cdots \\ \sin A = \cdots \\ \text{ETC.} \end{cases}$$

GUIA TÈCNICA 4. SOLUCIONS DE SISTEMES DINÀMICS (DIAGONALITZABLES)

PROP.- (1) $x(k+1) = Ax(k)$

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \Rightarrow \boxed{x(k) = A^k x(0) = S \begin{pmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{pmatrix} S^{-1} x(0)}$$

(2) $\dot{x}(t) = Ax(t)$

$$S^{-1}AS = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} \Rightarrow x(t) = e^{At} x(0) = S \begin{pmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{pmatrix} S^{-1} x(0)$$

GUIA TÈCNICA 4. SOLUCIONS DE SISTEMES DINÀMICS (DIAGONALITZABLES)

Exemple 2.- CENTRE / PERIFÈRIES

$$\begin{pmatrix}1\\0\end{pmatrix} \xrightarrow{A} \begin{pmatrix}0{,}6\\0{,}4\end{pmatrix} \xrightarrow{A} \begin{pmatrix}0{,}4\\0{,}6\end{pmatrix} \xrightarrow{A} \dots \qquad A = \begin{pmatrix}0{,}6 & 0{,}1\\0{,}4 & 0{,}9\end{pmatrix}$$

$$S^{-1}AS = \begin{pmatrix}1 & 0\\0 & 1/2\end{pmatrix} \Rightarrow x(k) = S\begin{pmatrix}1 & 0\\0 & 1/2^k\end{pmatrix}S^{-1}\begin{pmatrix}1\\0\end{pmatrix} = \dots = \begin{pmatrix}0{,}2 + 0{,}8\,\dfrac{1}{2^k}\\ \dots\end{pmatrix}$$

$\lambda_1 \quad \lambda_2$

$$S = \begin{pmatrix}0{,}2 & 1\\0{,}8 & -1\end{pmatrix}$$

$v_1 \quad v_2$

Per tant: HAB. CENTRE QUINQUENNI k $= 200.000 + 800.000\,\dfrac{1}{2^k}$

- $k = 0$: 1.000.000
- $k = 1$: 600.000
- $k = 2$: 400.000
- $k = 3$: 300.000
- $\vdots$
- $k \to \infty$: 200.000

Que recopila el que havíem obtingut a la sessió anterior

GUIA TÈCNICA 5. ESTABILITAT

GENERALITZEM L'EXEMPLE ANTERIOR:

PROP.- (1) $x(k+1) = Ax(k) + b$
A DIAGONALITZABLE

$$x(k) \rightarrow UN\ PUNT\ D'EQUILIBRI \Leftrightarrow |VAPs| \leq 1$$

$$\Updownarrow$$

$$x_e = Ax_e + b$$

(2) $\dot{x}(t) = Ax(t) + b$
A DIAGONALITZABLE

$$x(t) \rightarrow UN\ PUNT\ D'EQUILIBRI \Leftrightarrow \mathcal{R}e(VAPs) \leq 0$$

$$\Updownarrow$$

$$0 = Ax_e + b$$

Exemple anterior.- CENTRE / PERIFÈRIES

VAPs = 1, $^1/_2 \Rightarrow$ $x(k) \rightarrow \text{Punt d'equilibri}$

Exemple 3.- MUSSOL AMERICÀ; LAMBERSON, 1992

- Es va detectar una DISMINUCIÓ de MUSSOLS a Califòrnia (icona del país):

Lamberson va dividir la població segons 3 edats

COHORTS
- JOVES, J: $\leq$ 1 ANY (al niu matern)
- SUBADULTS, S: 1 $\div$ 2 ANYS (busquen parella, terreny de caça, ...)
- ADULTS, A: $\geq$ 2 ANYS (ja aposentats)

Per a la variació d'un any per al següent va trobar:

TRANSICIÓ
$$\begin{pmatrix} J(k+1) \\ S(k+1) \\ A(k+1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0{,}33 \\ 0{,}18 & 0 & 0 \\ 0 & 0{,}71 & 0{,}94 \end{pmatrix} \begin{pmatrix} J(k) \\ S(k) \\ A(k) \end{pmatrix}$$

VAPs = $0{,}98; -0{,}02 \pm 0{,}21i \;\Rightarrow$ EXTINCIÓ! (en menys de 50 anys: 2% disminució anual)

Exemple 3.- MUSSOL AMERICÀ; LAMBERSON, 1992

- El problema és la baixa supervivència J⤳S, que només era del 18%
  Si aconseguim:

$\begin{pmatrix} \cdots & \cdots & \cdots \\ 0{,}30 & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$, aleshores VAPs $= \ 1{,}01; \ -0{,}03 \pm 0{,}26i \Rightarrow$

- 1% creixement anual
- DISTRIBUCIÓ POBLACIONAL $\cong \begin{pmatrix} 10 \\ 3 \\ 31 \end{pmatrix}$

- Calia: més bosc disponible
  O sigui: menys tales per les papereres!
  I es va fer per LLEI!!!

OBS.- La proposició anterior es pot generalitzar al cas de matrius A no diagonalitzables:

$$|VAPs| < 1 \quad \Rightarrow \quad x(k) \to UN\ PUNT\ D'EQUILIBRI$$

$$\mathcal{R}e(VAPs) < 0 \quad \Rightarrow \quad x(t) \to UN\ PUNT\ D'EQUILIBRI$$

Exemple 4.- (una ETS molt EXIGENT)

En una ETS molt exigent només aprova el 30% de l'estudiantat en cada curs del grau. La resta repeteixen curs, excepte a primer, on el 50% del total abandonen. Cada any ingressen 600 estudiants nous. Volem estudiar:

(1) Quants estudiants hi haurà a cada curs en règim estacionari.

(2) Si s'hi tendeix cap a aquest estat estacionari qualssevol que siguin les condicions inicials.

(3) L'evolució de l'estudiantat de primer curs en un canvi de pla d'estudis.

Exemple 4.- (una ETS molt EXIGENT)

Si designem per $x_1(k), x_2(k), x_3(k), x_4(k)$ el nombre d'estudiants en cada curs l'any k, el sistema que regeix el model és:

$$x(k+1) = Ax(k) + b \ , \qquad A = \begin{pmatrix} 0,2 & 0 & 0 & 0 \\ 0,3 & 0,7 & 0 & 0 \\ 0 & 0,3 & 0,7 & 0 \\ 0 & 0 & 0,3 & 0,7 \end{pmatrix}, \qquad b = \begin{pmatrix} 600 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

En règim estacionari:

$$x^e = Ax^e + b \iff \begin{matrix} x_1^e = 0,2x_1^e + 600 \\ x_i^e = 0,3x_{i-1}^e + 0,7x_i^e \ , \ i = 2,3,4 \end{matrix} \iff x_1^e = x_2^e = x_3^e = x_4^e = 750$$

Com que els VAPs són $\lambda_1 = 0,2$ (simple) i $\lambda_2 = 0,7$ (triple) i $|\lambda_1|, |\lambda_2| < 1$, resulta:

$$x(k) \to x^e$$

A primer curs la convergència és força ràpida

$$x_1(1) = 600 \ \Rightarrow \ x_1(2) = 720 \ \Rightarrow \ x_1(3) = 744 \ \Rightarrow \ x_1(4) = 749 \ldots$$

FINAL SESSIÓ VIII

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

ETSEIB

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA I: ÀLGEBRA LINEAL

SESSIÓ IX. ANÀLISI MODAL EN SISTEMES DINÀMICS DISCRETS

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

ANÀLISI MODAL EN SISTEMES DINÀMICS DISCRETS

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GUIA TÈCNICA 6. COMPORTAMENT ASSIMPTÒTIC CAP AL MODE DOMINANT

Havíem vist que x(k) tendeix cap a un punt d'equilibri si $|VAPs| \leq 1$. Precisem-ho:

Definició.- Direm (si n'hi ha) que λ és un *VAP* DOMINANT, $\lambda_{DOM}, \vec{u}$:
és simple i $|\lambda_{DOM}| > |\lambda_2| \dots |\lambda_n|$

Aleshores:

- v_{DOM} (*VEP* DOMINANT) és un seu *VEP*, amb $\|v_{DOM}\| = 1$
- MODES DOMINANTS són les solucions particulars de la forma $c\lambda_{DOM}^{k} v_{DOM}, \quad c \in \mathbb{R}$

GUIA TÈCNICA 6. COMPORTAMENT ASSIMPTÒTIC CAP AL MODE DOMINANT

Proposició.- $x(k+1) = Ax(k) + b$

A diagonalitzable

λ_{DOM} *VAP* DOMINANT

x_e un punt d'equilibri

$\Rightarrow$ $\boxed{x(k) \cong c\lambda_{DOM}^{k} v_{DOM} + x_e}$ $\quad , si \;\; k \gg \;\; i \;\; c \neq 0$

És a dir:

- v_{DOM} dona la "DISTRIBUCIÓ ESTACIONÀRIA ASSIMPTÒTICA":

$$lim \frac{x(k) - x_e}{\|x(k) - x_e\|} = \pm v_{DOM}$$

- λ_{DOM} és la "TAXA DE CREIXEMENT ASSIMPTÒTICA":

$$lim \frac{\|x(k+1) - x_e\|}{\|x(k) - x_e\|} = |\lambda_{DOM}|$$

GUIA TÈCNICA 6. COMPORTAMENT ASSIMPTÒTIC CAP AL MODE DOMINANT

En particular: $|\lambda_{DOM}| \leq 1 \Longrightarrow x(k) \longrightarrow$ UN PUNT D'EQUILIBRI

OBSERVACIONS.- (1) Es diu que x(k) TENDEIX ASSIMPTÒTICAMENT cap al seu MODE DOMINANT
(2) Veurem que $c \in \mathbb{R}^n$ depèn de les condicions inicials x(0)

Exemple anterior.- (MUSSOLS AMERICANS)

Índex de SUPERVIVÈNCIA dels JOVES

$$0{,}18 \Longrightarrow x(k) \overset{k \gg}{\cong} c \cdot 0{,}98^k \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0{,}3 \Longrightarrow x(k) \overset{k \gg}{\cong} c \cdot 1{,}01^k \begin{pmatrix} 10 \\ 3 \\ 31 \end{pmatrix}$$

GUIA TÈCNICA 6. COMPORTAMENT ASSIMPTÒTIC CAP AL MODE DOMINANT

Exemple 4.- CENTRAL D'AUTOBUSOS

Considerem una companya d'autobusos amb 4 quatre centrals A, B, C i D. Cada setmana el moviment d'autobusos és el següent:

- dels que hi ha a cadascuna de les centrals A i B, un terç se'n va cap a C, un terç cap a D i l'altre terç roman a la pròpia central
- dels que hi ha a cadascuna de les centrals C i D, se'n va un terç cap a cadascuna de les altres tres centrals

La matriu de transició seria: $\begin{pmatrix} 1/3 & 0 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix}$

Exemple 4.- CENTRAL D'AUTOBUSOS

Els *VAPs* són:

$$\lambda_1 = 1, \lambda_2 = -{}^{1}/_{3}\,(doble), \lambda_3 = {}^{1}/_{3}$$

Per tant:

$$\lambda_{DOM} = \lambda_1 = 1$$

El *VEP* corresponent és $v_{DOM} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Per tant, siguin quines siguin les distribucions inicials dels autobusos, tendeix a igualar-se el nombre en cada central.

Exemple 5.- ÍNDEX D'ACCESSIBILITAT DE GOULD

Es tracta de quantificar l'accessibilitat de cada nus d'una xarxa no orientada. En particular, quins nusos estan més ben connectats.

Els índex de Gould han estat utilitzats en diversos problemes geogràfics, com ara xarxes de transport o moviments migratoris, configurant una xarxa (o graf) on els nusos representen ciutats (o altres entitats geogràficament significatives) i els arcs les connexions entre elles. Per exemple per establir MAGATZEMS de DISTRIBUCIÓ, CENTRES OPERATIUS, ...

Exemple 5.- ÍNDEX D'ACCESSIBILITAT DE GOULD

GOULD proposa:

- MATRIU *A* d'ADJACÈNCIA: $a_{ii} = 1$

  $a_{ij} \begin{cases} 1 & \text{sii estan connectats els nusos i, j} \\ 0 & \text{altrament} \end{cases}$

- ÍNDEX d'ACCESSIBILITAT del nus i´èsim = coordenada i´èsima del *VEP* DOMINANT (normalitzat per la suma de coordenades)

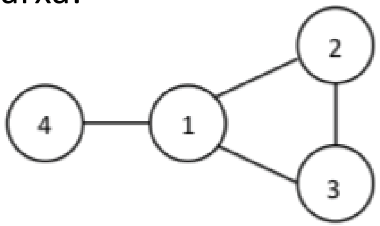
La IDEA és:

- el sistema $x(k+1) = Ax(k)$ fa que en cada transició k, A envia una "CÒPIA" dels objectes en un nus a cadascun dels ADJACENTS
- l'ÍNDEX d'ACCESSIBILITAT d'un nus serà el PERCENTATGE de "CÒPIES" que acumuli assimptòticament

Exemple 5.- ÍNDEX D'ACCESSIBILITAT DE GOULD

Per exemple, considerem la xarxa:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$



VAPs i *VEPs*
$$\begin{cases} \lambda_1 = 3'17 & v_1 = (0'61, 0'52, 0'52, 0'28) \\ \lambda_2 = 1'31 & v_2 = (0'25, -0'37, -0'37, 0'82) \\ \lambda_3 = 0 & v_3 = (0, 0'71, 0'71, 0) \\ \lambda_4 = -0'48 & v_4 = (0'75, -0'30, -0'30, -0'51) \end{cases}$$

Exemple 5.- ÍNDEX D'ACCESSIBILITAT DE GOULD

Com que λ_1 és DOMINANT, la DISTRIBUCIÓ ASSIMPTÒTICA serà:

$$^{v_1}/_{\Sigma} = (0'32, 0'27, 0'27, 0'14)$$

que seran respectivament els ÍNDEX d'ACCESSIBILITAT de GOULD de cada vèrtex.

Observem:

- el MÉS ACCESSIBLE és el nus 1 (amb índex 0'32)
- el MENYS ACCESSIBLE és el nus 4 (amb índex 0'14)
- els nusos 2 i 3 tenen la MATEIXA ACCESSIBILITAT (amb índex 0,27)

Exemple 5.- ÍNDEX D'ACCESSIBILITAT DE GOULD

De fet, la convergència asimptòtica és molt ràpida amb independència de la distribució inicial dels "objectes":

$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(3)/_{\Sigma}$
$(1,1,1,1)$	$(4,3,3,2)$	$(12,10,10,6)$	$(38,32,32,18)$	$(0'31,0'27,0'27,0'15)$
$(1,2,3,4)$	$(10,6,6,5)$	$(27,22,22,15)$	$(86,71,71,42)$	$(0'32,0'26,0'26,0'16)$
$(4,3,2,1)$	$(10,9,9,5)$	$(33,28,28,15)$	$(104,89,89,48)$	$(0'32,0'27,0'27,0'14)$

Exemple 6.- GOOGLE

- D'una manera similar a Gould, Brin and Page consideren la matriu de connexions entre pàgines web (milers de milions de files i columnes), amb certes modificacions.
- Aleshores el vector PageRank és el *VEP* DOMINANT.
- L'aparició en pantalla és en ordre decreixent de les seves coordenades.

GUIA TÈCNICA 7. ANÀLISI MODAL

Més en general, estudiem les diferents evolucions segons les condicions inicials.

PROP.- $x(k+1) = Ax(k) + b$
Suposem *VAPs* $\lambda_1, \dots, \lambda_n$
amb *VEPs* $v_1, \dots, v_n$
Suposem x^e punt d'equilibri
$(\Leftrightarrow x^e = Ax^e + b)$

$\Longrightarrow$ $x(k) = c_1\lambda_1^k v_1 + \cdots + c_n\lambda_n^k v_n + x^e$

on $c_1, \dots, c_n$ queden determinades per les condicions inicials:

$$x(0) = c_1 v_1 + \cdots + c_n v_n$$

OBSV.-
- Cada sumand $\lambda_i^k v_i$ se'n diu un MODE PROPI.
- Ja hem vist que $x(k)$ tendeix assiptòticament cap al MODE PROPI DOMINANT (corresponent al *VAP* DOMINANT).
- Les trajectòries són combinacions lineals dels MODES PROPIS, segons les condicions inicials.
- El mapa de trajectòries es diu DIAGRAMA DE FASES.

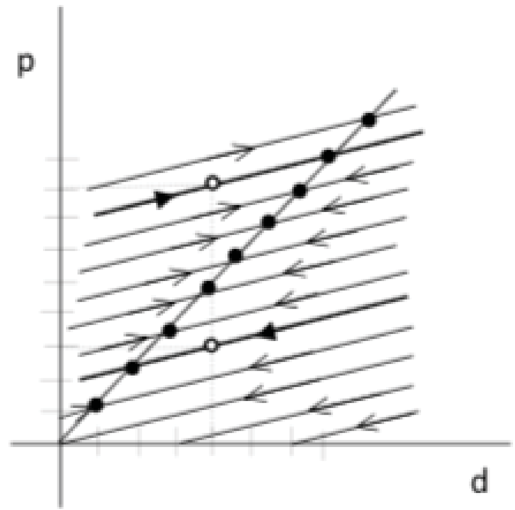
Exemple anterior.- (PRESA / DEPREDADOR)

Per al sistema:

$$\begin{pmatrix} d(k+1) \\ p(k+1) \end{pmatrix} = \begin{pmatrix} 0{,}5 & 0{,}4 \\ -0{,}125 & 1{,}1 \end{pmatrix} \begin{pmatrix} d(k) \\ p(k) \end{pmatrix}$$

La solució general serà:

$$\begin{pmatrix} d(k) \\ p(k) \end{pmatrix} = c_1 1^k \begin{pmatrix} 4 \\ 5 \end{pmatrix} + c_2 0'6^k \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$



que dona les trajectòries, dependent de les condicions inicials.

A la figura es presenta el DIAGRAMA DE FASES resultant.

Exemple anterior.- (PRESA / DEPREDADOR)

Per exemple:

$$\begin{pmatrix} d(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Leftrightarrow c_1 = c_2 = \frac{1}{2} \Leftrightarrow \begin{pmatrix} d(k) \\ p(k) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 \\ 5 \end{pmatrix} + \frac{1}{2} 0'6^k \begin{pmatrix} 4 \\ 1 \end{pmatrix} \to \begin{pmatrix} 2 \\ 2'5 \end{pmatrix}$$

$$\begin{pmatrix} d(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Leftrightarrow c_1 = \frac{7}{4}, c_2 = -\frac{11}{4} \Leftrightarrow \begin{pmatrix} d(k) \\ p(k) \end{pmatrix} = \frac{7}{4} \begin{pmatrix} 4 \\ 5 \end{pmatrix} - \frac{11}{4} 0'6^k \begin{pmatrix} 4 \\ 1 \end{pmatrix} \to \begin{pmatrix} 7 \\ 8'75 \end{pmatrix}$$

Observem que:

$d(0)/p(0) > \frac{4}{5} \Longrightarrow d(k), p(k)$ decreixen fins a l'equilibri

$d(0)/p(0) < \frac{4}{5} \Longrightarrow d(k), p(k)$ creixen fins a l'equilibri

En particular, eliminar depredadors pot fer créixer la seva població!

Exemple 7.- PRESA / DEPREDADOR

Generalitzem l'exemple anterior a diferents voracitats $\alpha \in \mathbb{R}$:

$$\begin{pmatrix} d(k+1) \\ p(k+1) \end{pmatrix} = \begin{pmatrix} 0{,}5 & 0{,}4 \\ -\alpha & 0{,}1 \end{pmatrix} \begin{pmatrix} d(k) \\ p(k) \end{pmatrix}$$

- Per a voracitats baixes $(\alpha < 0'125)$, hi ha algun *VAP* major que 1 i per tant les poblacions poden créixer en certes condicions inicials.

Per exemple per a $\alpha < 0'104$:

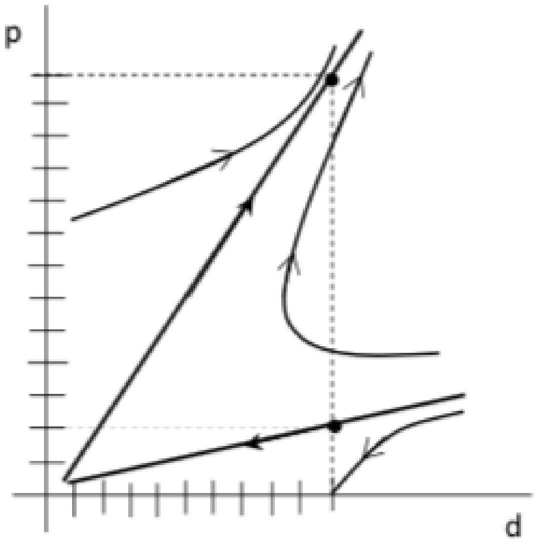
$$\begin{pmatrix} d(k) \\ p(k) \end{pmatrix} = c_1 1'02^k \begin{pmatrix} 10 \\ 13 \end{pmatrix} + c_2 0'58^k \begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

Exemple 7.- PRESA / DEPREDADOR

Si $c_1 > 0$, tendeix assimptòticament a:

- CREIXEMENT ANUAL 2%
- PROPORCIÓ $\frac{d}{p} = \frac{10}{13}$

Si $c_1 < 0$, tendeix a $p = 0$ (i per tant $d = 0$)



- Per a voracitats altes($\alpha > 0'125$), tots els *VAPs* són menors que 1 i per tant les poblacions s'extingeixen, qualssevol que siguin les condicions inicials.

FINAL SESSIÓ IX

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

ETSEIB

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA I: ÀLGEBRA LINEAL

SESSIÓ X. EQUACIONS EN DIFERÈNCIES

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

EQUACIONS EN DIFERÈNCIES

INDEX

1.1. GUIA TÈCNICA: INTRODUCCIÓ

Una EQUACIÓ en DIFERÈNCIES (EED) d'ordre n, lineal, amb coeficients constants, és una relació de recurrència:

$$y(k+n)+a_{n-1}y(k+n-1)+\cdots+a_1y(k+1)+a_0y(k)=b$$

Es diu $\begin{cases}\text{HOMOGÈNIA si } b=0\\ \text{COMPLETA si } b\neq 0\end{cases}$

Es tracta de determinar la successió SOLUCIÓ:

$$y(0), y(1), \dots, y(k), \dots$$

Segons les CONDICIONS INICIALS:

$$y(0), y(1), \dots, y(n-1)$$

Exemple.- PROGRESSIÓ ARITMÈTICA: $y(k+1)=y(k)+d \Rightarrow y(k)=y(0)+dk$

Exemple.- PROGRESSIÓ GEOMÈTRICA: $y(k+1)=\lambda y(k) \Rightarrow y(k)=\lambda^k y(0)$

1.2. GUIA TÈCNICA: CAS HOMOGENI

Diem:

- POLINOMI CARACTERÍSTIC: $Q(t) = t^n + a_{n-1}\, t^{n-1} + \cdots + a_1 t + a_0$
- VALORS CARACTERÍSTICS: les arrels de $Q(t)$

Prop.- RESOLUCIÓ del CAS HOMOGENI ($\Leftrightarrow b = 0$)

Si hi ha n valors característics diferents $\lambda_1, \dots, \lambda_n$:

- SOLUCIÓ GENERAL: $\boxed{y(k) = \alpha_1 \lambda_1^k + \cdots + \alpha_n \lambda_n^k}$
- SOLUCIONS PARTICULARS per cadascuns de $\alpha_1, \dots, \alpha_n$ determinats per les condicions inicials

$$\begin{cases} y(0) = \alpha_1 + \cdots + \alpha_n \\ \quad \cdots \\ y(n-1) = \alpha_1 \lambda_1^{n-1} + \cdots + \alpha_n \lambda_n^{n-1} \end{cases}$$

Observació.- Com veurem a les aplicacions, els coeficients $\alpha_1, \dots, \alpha_n$ poden ser determinats per altres condicions

Exemple.- SUCCESSIÓ de FIBONACCI

En la successió de Fibonacci cada terme és la suma dels dos anteriors:

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

Es considera la primera EED (vinculada a la reproducció dels conills!):

$$\begin{array}{l} y(k+2) = y(k+1) + y(k) \\ y(0) = 0, \qquad y(1) = 1 \end{array}$$

Polinomi característic: $Q(t) = t^2 - t - 1$

Valors característics: $\frac{1 \pm \sqrt{1+4}}{2}$

$\lambda_1 = \frac{1+\sqrt{5}}{2}$ (dominant)

$\lambda_2 = \frac{1-\sqrt{5}}{2}$

1.2. GUIA TÈCNICA: CAS HOMOGENI

Solució general: $y(k) = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^k + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^k$

Solució particular: $y(0) = 0 = \alpha_1 + \alpha_2$

$$y(1) = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) \Rightarrow \alpha_1 = \frac{1}{\sqrt{5}}, \ \alpha_2 = -\frac{1}{\sqrt{5}} \Rightarrow$$

$$\Rightarrow \boxed{y(k) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k}$$

Sorprèn que, per tot k, resulta un nombre natural!:

$$y(2) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1$$

$$y(3) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^3 - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^3 = 2$$

$\dots$

1.2. GUIA TÈCNICA: CAS HOMOGENI

Prop.- COMPORTAMENT ASSIMPTÒTIC per al CAS HOMOGENI

Si $|\lambda_1| > |\lambda_2|, \dots, |\lambda_n|$:

- VALOR CARACTERÍSTIC DOMINANT: λ_1
- MODE DOMINANT: $\alpha_1 \lambda_1^k$

Aleshores:

$y(k) \cong \alpha_1 \lambda_1^k$	si $k \gg, \alpha_1 \neq 0$
$lim \frac{y(k+1)}{y(k)} = \lambda_1$	si $\alpha_1 \neq 0$

Exemple.- SUCCESSIÓ de FIBONACCI

$$lim \frac{y(k+1)}{y(k)} = \lambda_1 = \frac{1+\sqrt{5}}{2}$$ (la famosa RELACIÓ ÀURIA!)

Es a dir:

$$1, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8} \dots \to \frac{1+\sqrt{5}}{2}$$

$$1, 2, 1'5, 1'666, 1'6, 1'625 \dots \to 1'618 \dots$$

1.2. GUIA TÈCNICA: CAS HOMOGENI

Exemple.- Per calcular el límit de la successió: $1, 3, \frac{5}{3}, \frac{11}{5}, \frac{21}{11}, \frac{43}{21}, \ldots$

observem que és la de QUOCIENTS CONSECUTIUS de: $1, 3, 5, 11, 21, 43 \ldots$

que correspon a l'EED $\begin{cases} y(k+2) = y(k+1) + 2y(k) \\ y(1) = 1, \quad y(2) = 3 \end{cases}$

Per tant:

POLINOMI CARACTERÍSTIC: $Q(t) = t^2 - t - 2$

VALORS CARACTERÍSTICS: $\frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$ → 2 (dominant), -1

SOLUCIÓ GENERAL: $y(k) = \alpha_1 2^k + \alpha_2 (-1)^k$

SOLUCIÓ PARTICULAR: $\left.\begin{array}{l} y(1) = 1 = 2\alpha_1 - \alpha_2 \\ y(2) = 3 = 4\alpha_1 + \alpha_2 \end{array}\right\} \Rightarrow \alpha_1 = \frac{2}{3}, \alpha_2 = \frac{1}{3} \Rightarrow \boxed{y(k) = \frac{2}{3} 2^k + \frac{1}{3} (-1)^k}$

En conclusió: $\boxed{lim \frac{y(k+1)}{y(k)} = 2}$

1.3. GUIA TÈCNICA: CAS COMPLET

Prop.- CAS COMPLET: PUNTS d'EQUILIBRI; ESTABILITAT

(1) Se'n diu PUNT d'EQUILIBRI una solució constant $y(k) = y_e$

	1 no és valor característic	1 és valor característic
$b \neq 0$	$y_e = \dfrac{b}{1 + a_{n-1} + \cdots + a_1 + a_0}$	cap
$b = 0$	$y_e = 0$	y_e qualsevol

(2) Aleshores:

- SOLUCIÓ GENERAL: $y(k) = \alpha_1 \lambda_1^k + \cdots + \alpha_n \lambda_n^k + y_e$
- SOLUCIÓ PARTICULAR: per les condicions inicials, com abans
- $|\lambda_1|, \ldots, |\lambda_n| < 1 \Longrightarrow \lim y(k) = y_e$ (punt d'equilibri ESTABLE)
- λ_1 DOMINANT $\Longrightarrow y(k) \cong \alpha_1 \lambda_1^k + y_e$, si $k \gg, \alpha_1 \neq 0$

1.3. GUIA TÈCNICA: CAS COMPLET

Exemples:

(1) $y(k+1) = y(k) + d$

- $d = 0 \Longrightarrow \; y_e = y(0)$, qualsevol
- $d \neq 0 \Longrightarrow$ cap punt d'equilibri

(2) $y(k+1) = \lambda y(k)$

- $\lambda = 1 \Longrightarrow \; y_e = y(0)$, qualsevol
- $\lambda \neq 1 \Longrightarrow \; y_e = 0$

(3) $y(k+2) = y(k+1) + y(k) + b$

- PUNT d'EQUILIBRI: $y_e = -b$ (INESTABLE)
- SOLUCIÓ GENERAL: $y(k) = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) - b$

1.3. GUIA TÈCNICA: CAS COMPLET

Observació.- Les EED són casos particulars de SISTEMES DISCRETS:

$$y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1\, y(k+1) + a_0\, y(k) = b \Leftrightarrow$$

$$\Leftrightarrow x(k+1) = Ax(k) + B$$

on: $x(k) = \begin{pmatrix} y(k) \\ y(k+1) \\ \cdots \\ y(k+n-1) \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \dots \dots \dots \dots \dots 0 \\ 0 & 0 \dots \dots \dots \dots \dots 0 \\ \dots & \dots \dots \dots \dots \dots \dots \dots \\ 0 & 0 \dots \dots \dots \dots \dots 1 \\ -a_0 & -a_1 \dots \dots \dots - a_{n-1} \end{pmatrix}$

Observem que A és la MATRIU COMPANYA de $Q(t)$. Per tant:

polinomi característic de l'EED = polinomi característic de la matriu A
valors característics de l'EED = valors propis de la matriu A
n valors característics diferents $\lambda_1, \cdots, \lambda_n \Longrightarrow A$ diagonalitza

La resolució es generalitza al cas de valors característics múltiples tenint en compte que la reduïda de Jordan de A presenta un sol bloc per cada valor propi.

2.1. APLICACIONS: TEORIA DE LA INFORMACIÓ

Exemple.- TEORIA de la INFORMACIÓ de SHANNON

Suposem un SISTEMA de TRANSMISSIÓ:

- ALFABET de 2 SÍMBOLS: PUNT GUIÓ
- Cadascun amb un TEMPS de TRANSMISSIÓ
- CORDA = conjunt de símbols consecutius
- $N_t = n^{\underline{o}}$ de cordes amb temps de transmissió t
- CAPACITAT $\equiv C = \lim_{t \to \infty} \frac{\log_2 N_t}{t}$

(1) Suposem el cas trivial:

TEMPS de TRANSMISSIÓ: 1 per al PUNT, 1 per al GUIÓ $\Rightarrow N_t = 2^t \Rightarrow \boxed{C = 1}$

2.1. APLICACIONS: TEORIA DE LA INFORMACIÓ

(2) TEMPS de TRANSMISSIÓ < 1 per al PUNT / 2 per al GUIÓ $\Rightarrow$

$N_1 = 1$ (una única corda: ·)

$N_2 = 2$ (dues possibles cordes < ·· / –)

En general: $N_t = N_{t-1} + N_{t-2}$ (FIBONACCI!) $\Rightarrow$

$$\Rightarrow N_t = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{t+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{t-1} \Rightarrow$$

Criteri de STÖLZ ↓

$$\Rightarrow C = \lim_{t\to\infty} \frac{\log_2 N_t}{t} = \lim_{t\to\infty} (\log_2 N_t - \log_2 N_{t-1}) = \lim_{t\to\infty} \log_2 \frac{N_t}{N_{t-1}} \Rightarrow$$

$$\Rightarrow \boxed{C = \log_2 \frac{1+\sqrt{5}}{2}}$$

2.2. APLICACIONS: TEORIA DE CUES

Exemple.- <u>TEORIA de CUES</u>

Suposem un petit taller { treballa sota comanda / mantenint l'ordre de recepció

- RECEPCIÓ de comandes en 1 hora
  - probabilitat p de rebre'n 1
  - probabilitat 0 de rebre'n més d'1
- COMPLETACIÓ de comandes en 1 hora
  - probabilitat q de completar-ne 1
  - probabilitat 0 de completar-ne més d'1
- $p < q \ll$

Es tracta de calcular: en mitjana, QUANTES en CUA d'ESPERA?

2.2. APLICACIONS: TEORIA DE CUES

Sigui: $y(n)$ = probabilitat de que la cua sigui n

Tenim
$$\begin{cases} y(0) = qy(1) + (1-p)y(0) \\ y(n) = qy(n+1) + (1-p-q)y(n) + py(n-1), n > 0 \\ \sum y(n) = 1 \end{cases}$$

Per tant: $y(n+1) = \frac{p+q}{q} y(n) - \frac{p}{q} y(n-1)$

$$Q(t) = t^2 - \frac{p+q}{q} t + \frac{p}{q}$$

$$\frac{\frac{p+q}{q} \pm \sqrt{\left(\frac{p+q}{q}\right)^2 - 4\frac{p}{q}}}{2} = \frac{\frac{p+q}{q} \pm \frac{p-q}{q}}{2}$$

$\lambda_1 = 1$

$\lambda_2 = \frac{p}{q} (< 1)$

$$y(n) = \alpha_1 + \alpha_2 \left(\frac{p}{q}\right)^n$$

2.2. APLICACIONS: TEORIA DE CUES

Calculem α_1, α_2: $\quad y(n) = \alpha_1 + \alpha_2 \left(\frac{p}{q}\right)^n$

- $y(1) = \frac{p}{q}\, y(0)$

$$y(2) = 2 \begin{cases} \frac{p+q}{q}\frac{p}{q} y(0) - \frac{p}{q} y(0) = \left(\frac{p}{q}\right)^2 y(0) \\ \alpha_1 + \alpha_2 \left(\frac{p}{q}\right)^2 \end{cases} \Rightarrow \begin{array}{l} \alpha_1 = 0 \\ \alpha_2 = y(0) \end{array} \Rightarrow \; y(n) = \left(\frac{p}{q}\right)^n y(0)$$

- $1 = \sum y(n) = y(0)\left(1 + \frac{p}{q} + \left(\frac{p}{q}\right)^2 + \cdots\right) = y(0)\,\frac{1}{1-\frac{p}{q}} = y(0)\,\frac{q}{q-p} \Rightarrow y(0) = \frac{q-p}{q} \Rightarrow$

$$\Rightarrow \boxed{y(n) = \left(1 - \frac{p}{q}\right)\left(\frac{p}{q}\right)^n}$$

Finalment: $\text{CUA MITJANA} = \sum_{n\geq 0} n\, y(n) = \left(1 - \frac{p}{q}\right) \sum_{n\geq 0} n \left(\frac{p}{q}\right)^n$

- $|x| < 1{:}\;\; 1 + x + x^2 + \cdots = \frac{1}{1-x} \Rightarrow 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2} \Rightarrow$

$$\Rightarrow x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2} \Rightarrow 1 + x + 2x^2 + \cdots = 1 + \frac{x}{(1-x)^2}$$

- $\sum n\, y(n) = \left(1 - \frac{p}{q}\right)\left(1 + \frac{p/q}{\left(1-p/q\right)^2}\right) = \frac{1}{1-\frac{p}{q}}\left(\left(1 - \frac{p}{q}\right)^2 + \frac{p}{q}\right) = \frac{q}{q-p}\left(1 - \frac{p}{q} + \left(\frac{p}{q}\right)^2\right) = 1 + \frac{p^2}{(q-p)q}$

- $\boxed{\text{CUA MITJANA} = 1 + \frac{p^2}{(q-p)q}} = 1 + \frac{\left(p/q\right)^2}{1-\left(p/q\right)}$

Exemple.- <u>BICING</u>

Es pretén organitzar, en 4 anys, un BICING amb 400 bicicletes en règim permanent mitjançant la compra de b BICICLETES cada mes.

Se sap que, cada mes:
- 70% es mantenen en servei
- 25% al taller i es reincorporen al mes següent
- 5% irrecuperables

Ens preguntem:
- el valor de b
- quantes bicicletes als 4 anys

- EED $\begin{cases} y(k+2) = 0'7y(k+1) + 0'25y(k) + b \\ y(0) = 20, \qquad y(1) = 34 \end{cases}$

- PUNT d'EQUILIBRI $= 400 \Longrightarrow 400 = \frac{b}{1-0'7-0'25} \Longrightarrow \boxed{b = 20}$

(De fet: en règim permanent caldrà repostar cada mes el 5% de 400)

2.3. APLICACIONS: BICING

- POLINOMI CARACTERÍSTIC: $Q(t) = t^2 - 0'7t - 0'25$
- VALORS CARACTERÍSTICS: $\frac{0'7 \pm \sqrt{0'49+1}}{2} \cong \frac{0'7 \pm 1'22}{2}$ < $0'96$ (DOMINANT), $-0'26$
- SOLUCIÓ GENERAL: $y(k) = \alpha_1 0'96^k + \alpha_2(-0'26)^k + 400$
- SOLUCIÓ PARTICULAR: $\left.\begin{array}{l} y(0) = 20 = \alpha_1 + \alpha_2 + 400 \\ y(1) = 34 = 0'96\alpha_1 - 0'26\alpha_2 + 400 \end{array}\right\} \Rightarrow \begin{array}{l} \alpha_1 \cong -381 \\ \alpha_2 \cong 1 \end{array} \Rightarrow$

$$\Rightarrow \boxed{y(k) \cong -381(0'96)^k + (-0'26)^k + 400}$$

ALS 4 ANYS: $y(48) \cong -381(0'96)^{48} + 400 \cong -381 \cdot 0'141 + 400 \Rightarrow \boxed{y(48) \cong 346}$

FINAL SESSIÓ X

ANEXO D. Sesiones “Aplicaciones de Matemáticas en Ingeniería II: Cálculo Multivariable”

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA II: CÀLCUL MULTIVARIABLE

SESSIÓ I. FENÒMENS DISCONTINUS: HISTÈRESI, CAÚSTIQUES, ...

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

FENÒMENS DISCONTINUS: HISTÈRESI, CÀUSTIQUES, …

INDEX

1. INTRODUCCIÓ

RENÉ THOM (1972) va abordar l'estudi sistemàtic de DISCONTINUITATS en els comportaments dinàmics (físics, socials, ...). Fins aleshores només s'havien estudiat casuísticament.

Exemple.- UN SALTAMARTÍ OCASIONAL

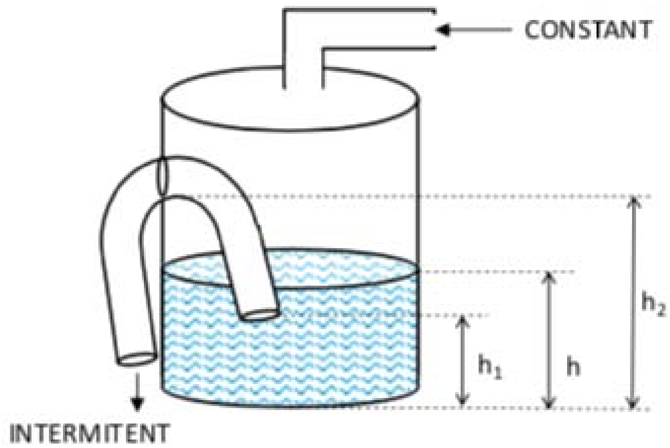
Considerem una ampolla de base un casquet esfèric.
L'ESTABILITAT depèn del NIVELL del LÍQUID: per a quins nivells actua com un saltamartí?

1. INTRODUCCIÓ

Exemple.- UNA FONT INTERMITENT

Un recipient amb:
FLUX D'ENTRADA CONSTANT (petit) ⇒ Flux de SORTIDA INTERMITENT
un sobreeixidor tubular enfonsat

- Per $h_1 < h < h_2$: s'omple SENSE RAJAR
- Quan $h = h_2$: comença a RAJAR i segueix fins que $h = h_1$
- Quan $h = h_1$: s'INTERROMP el buidatge i torna a omplir-se sense rajar



2. HISTÈRESI

Els CICLES d'HISTÈRESI són un dels casos més freqüents de fenòmens discontinus.

El cas d'HISTÈRESI MAGNÈTICA és paradigmàtic i històric:

- un CAMP MAGNÈTIC EXTERIOR variable força
- l'orientació dels DIPOLS INTERNS d'un material ferromagnètic

Les característiques CLAU:

- RESISTÈNCIA al canvi d'orientació dels dipols interns quan canvia la del camp magnètic exterior
- CANVI SOBTAT i complet de l'orientació dels dipols interns

  quan la contrària del camp extern assoleix un cert LLINDAR de disrupció

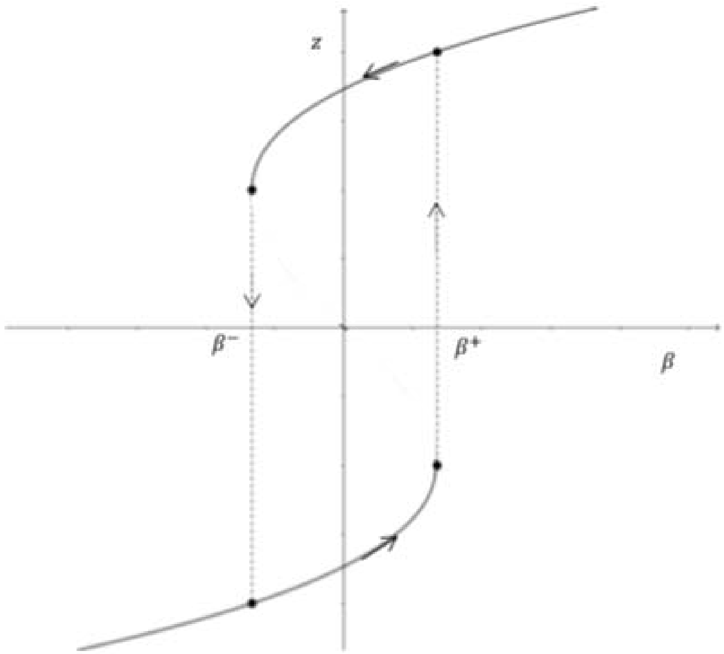
2. HISTÈRESI

Per tant, canvis ALTERNANTS en el camp exterior generen un CICLE D'HISTÈRESI com a la figura:

z : orientació dels dipols interns

β : camp magnètic extern

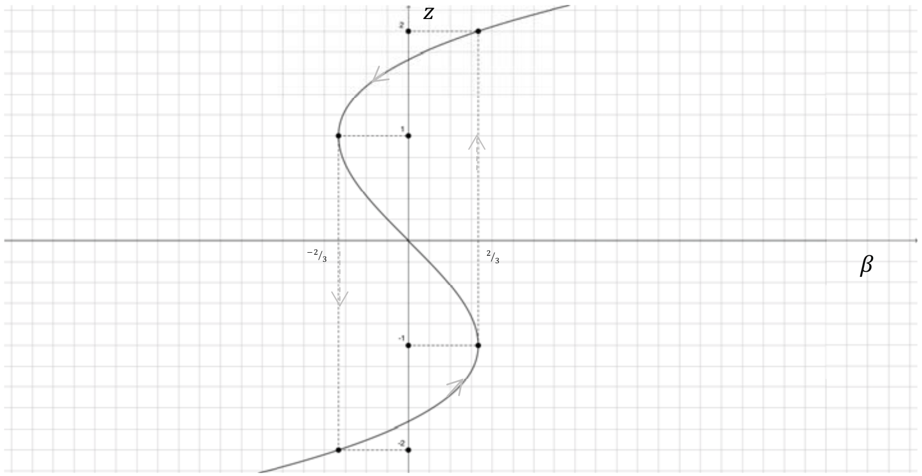
β^{+}, β^{-}: llindars de disrupció que provoquen la REORIENTACIÓ SOBTADA de la imantació interna



2. HISTÈRESI

Un MODEL MATEMÀTIC per al cicle d'histèresi: els valor possibles de z, β són les solucions de l'equació:

$$\frac{z^3}{3} - z - \beta = 0 \quad , \quad |z| \geq 1$$



Els llindars de disrupció són:

$$\beta^+ = \frac{2}{3} \qquad \beta^- = -\frac{2}{3}$$

Per exemple:

$\beta = \frac{2}{3} \Rightarrow z$ passa bruscament de -1 a 2

$\beta = -\frac{2}{3} \Rightarrow z$ passa bruscament de 1 a -2

Veurem que és GENÈRICAMENT VÀLID per a histèresis en qualsevol àmbit.
En particular només $|z| \geq 1$ perquè corresponen a punts d'equilibri estable.
Els $|z| < 1$ correspondran a punts d'equilibri inestable.

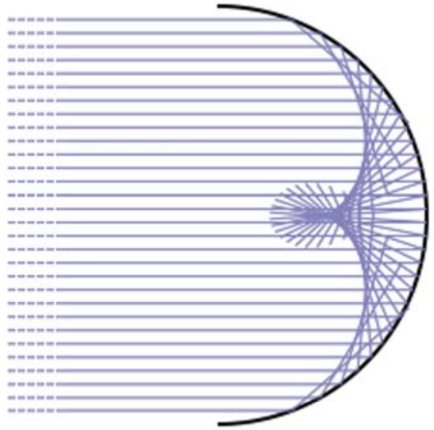
3. CÀUSTIQUES

- CÀUSTIQUES ÓPTIQUES = línies o superfície d'extraordinària lluminositat degut a la CONCENTRACIÓ de RAJOS procedents d'una reflexió.
- Típiques FORMES CUSPOIDALS: són genèriques en molt diferents àmbits, com veurem.
- Geomètricament són les ENVOLUPANTS dels rajos reflectits (= corbes tangents a tots ells).

En el cas de la figura resulta la "esferoide":

$$x = \cos\theta - \frac{1}{2}\cos\theta\cos 2\theta$$

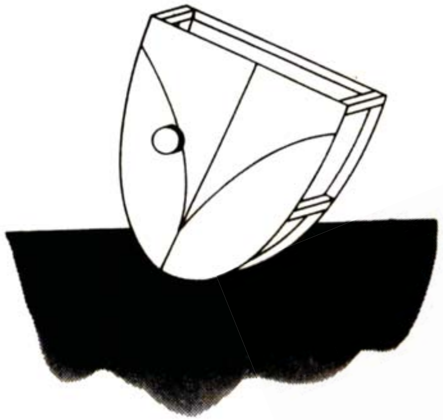
$$y = sen\,\theta - \frac{1}{2}\cos\theta\, sen 2\theta$$



4. UNA MÀQUINA GRAVITACIONAL

Ambdós fenòmens anteriors (HISTÈRESI, CAÚSTICA) apareixen en la següent màquina gravitacional:

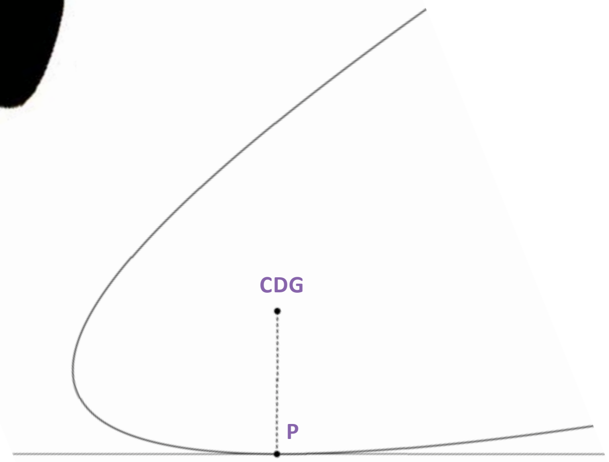
una placa limitada per una paràbola recolzada en un pla horitzontal amb centre de gravetat (CDG) variable (per exemple, un imant).



Tendirà a minimitzar l'altura del CDG:

EQUILIBRI ESTABLE sobre P ⇔

⇔ P MÍNIM RELATIU de la distància CDG/punts de la paràbola ⇒
⇒ P peu d'una ORTOGONAL a la paràbola des del CDG



Tanmateix, també:

EQUILIBRI INESTABLE sobre P'⇔

⇔ P' MÀXIM RELATIU de la distància CDG/punts de la paràbola ⇒
⇒ P' peu d'una ORTOGONAL a la paràbola des del CDG

4. UNA MÀQUINA GRAVITACIONAL

Per exemple:

paràbola $\{(z, z^2), |z| \leq 2\}$

CDG $= (0,2)$

Aleshores:

$$\left(d\left((z, z^2), (0,2)\right)\right)^2 = z^2 + (z^2 - 2)^2 = z^4 - 3z + 4$$

$$D\left(d\left((z, z^2), (0,2)\right)\right)^2 = 4z^3 - 6z = 0$$

$z = 0$, MÀXIM RELATIU

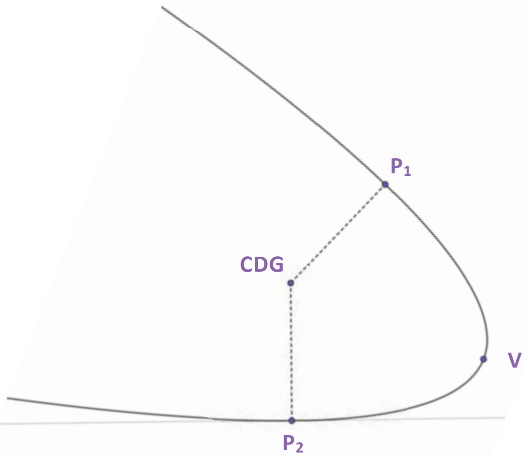
$z = \pm\sqrt{3/2}$, MÍNIMS RELATIUS

Per tant:

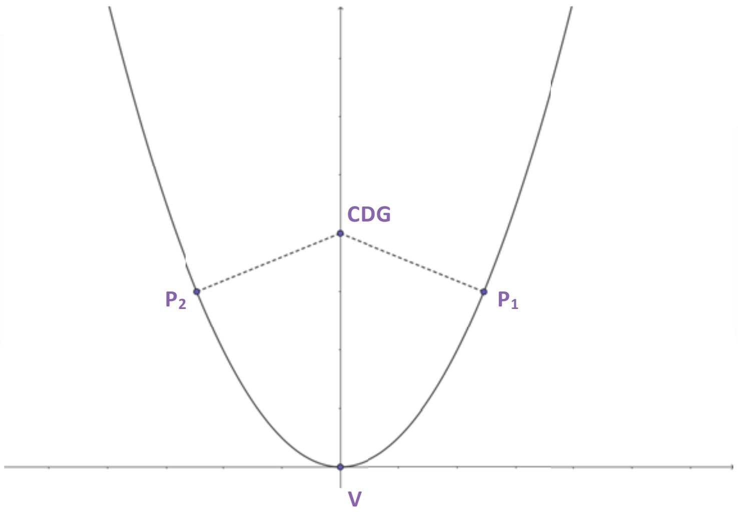
CDG $= (0,2) \Rightarrow$

- EQUILIBRI INESTABLE sobre $V = (0,0)$
- EQUILIBRI ESTABLE sobre $P_1 = \left(\sqrt{\frac{3}{2}}, \frac{3}{2}\right)$, $P_2 = \left(-\sqrt{\frac{3}{2}}, \frac{3}{2}\right)$

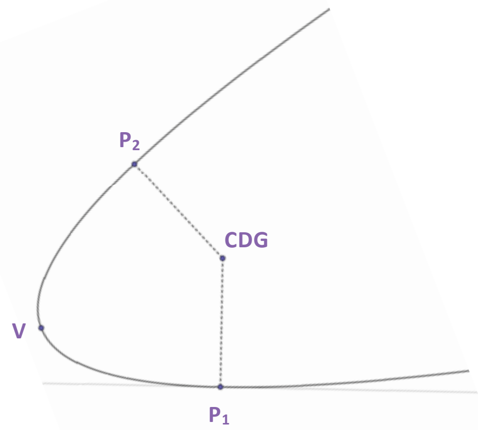
4. UNA MÀQUINA GRAVITACIONAL



ESTABLE



INESTABLE



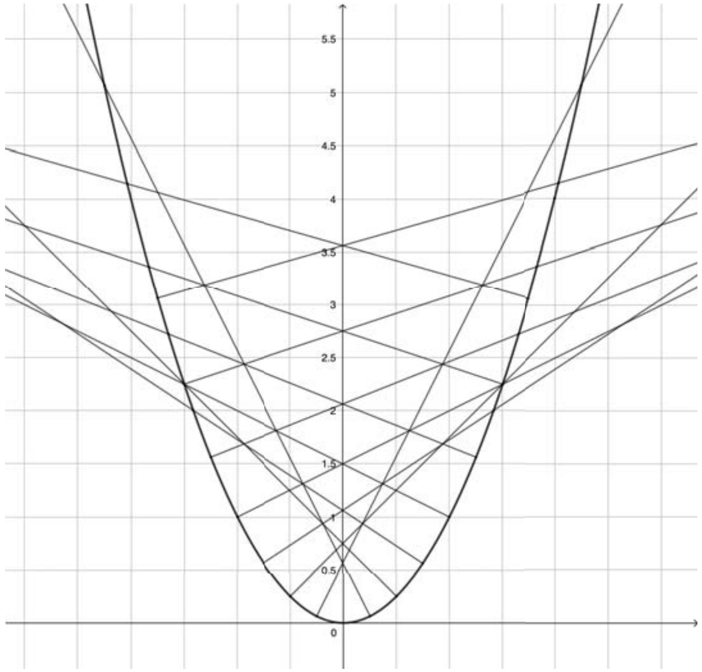
ESTABLE

4. UNA MÀQUINA GRAVITACIONAL

Tanmateix:

CDG $= (0, \alpha)$

$\alpha < \frac{1}{2}$ $\Rightarrow$ ÚNIC EXTREM RELATIU en $V = (0,0)$, MÍNIM $\Rightarrow$ ÚNIC PUNT d'EQUILIBRI en $V = (0,0)$, ESTABLE



Per tant, hi ha dues possibilitats:

- des del CDG
  - 3 ORTOGONALS a la PARÀBOLA (2 MÍNIMS, 1 MÀXIM)
  - 1 ORTOGONAL a la PARÀBOLA (MÍNIM)

- separació de les dues formes = ENVOLUPANT de les ORTOGONALS

4. UNA MÀQUINA GRAVITACIONAL

Per calcular-la:

- ortogonal en $(z, z^2) = \left\{(\beta, \alpha): \frac{\beta - z}{-2z} = \alpha - z^2\right\}$

- eliminem z:

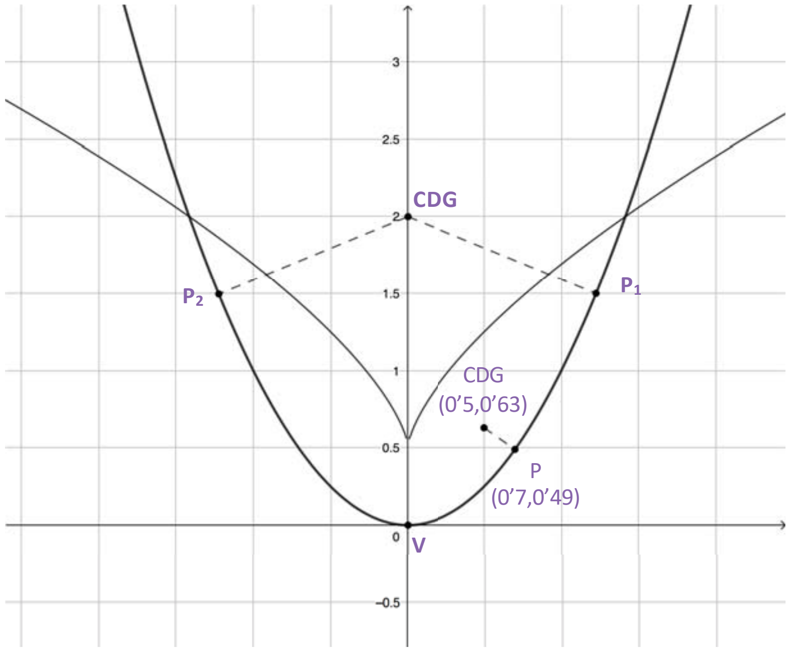
$$\left.\begin{array}{l} 0 = \beta - z + 2z(\alpha - z^2) = \beta + (2\alpha - 1)z - 2z^3 \\ 0 = D_z(\beta - z + 2z(\alpha - z^2)) = (2\alpha - 1) - 6z^2 \end{array}\right] \Rightarrow z^2 = \frac{2\alpha - 1}{6} \Rightarrow 0 = \beta + z((2\alpha - 1) - 2\,\frac{2\alpha - 1}{6}) \Rightarrow$$

$$\Rightarrow 0 = \beta + z\frac{2}{3}(2\alpha - 1) \Rightarrow \beta^2 = z^2\,\frac{4}{9}(2\alpha - 1)^2 = \frac{2\alpha - 1}{6}\frac{4}{9}(2\alpha - 1)^2 \Rightarrow \boxed{\beta^2 = \frac{16}{27}(\alpha - \frac{1}{2})^3}$$

4. UNA MÀQUINA GRAVITACIONAL

Retrobem, com a les càustiques òptiques, una CÚSPIDE que separa comportaments dinàmics ben diferents:

$$CDG(\beta,\alpha)\begin{cases}\beta^2 < \frac{16}{27}(\alpha-\frac{1}{2})^3 \Rightarrow \begin{cases}2\ EQ.ESTABLE\\1\ EQ.INESTABLE\end{cases}\\ \beta^2 > \frac{16}{27}(\alpha-\frac{1}{2})^3 \Rightarrow 1\ EQ.ESTABLE\end{cases}$$



4. UNA MÀQUINA GRAVITACIONAL

Per tant:

quan el CDG travessa la CÚSPIDE ⇒ DESAPAREIX 1 PUNT d'EQUILIBRI ESTABLE

En particular, retrobem un CICLE d'HISTÈRESI:

per exemple: $\alpha = 1, -1 \leq \beta \leq 1$

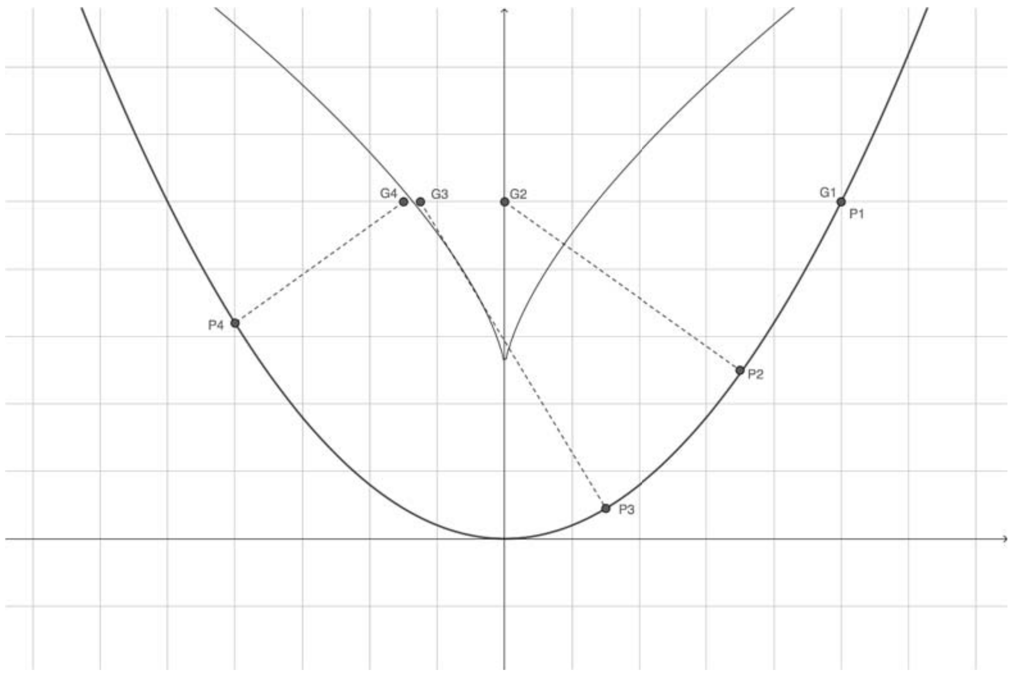
CDG

- $G_1 = (1{,}1) \Rightarrow$ EQ. ESTABLE sobre $P_1 = (1{,}1)$
- $G_2 = (0{,}1) \Rightarrow$ EQ. ESTABLE sobre $P_2 \cong (0'7{,}0'5)$
- $G_3 = (-0'25{,}1) \Rightarrow$ EQ. ESTABLE sobre $P_3 \cong (0'3{,}0'09)$
- $G_4 = (-0'3{,}1) \Rightarrow$ EQ. ESTABLE sobre $P_4 \cong (-0'8{,}0'64)$

4. UNA MÀQUINA GRAVITACIONAL

Per $\alpha = 1$ i β disminuint

- Si el CDG varia entre G_1, G_2 i G_3 el punt de suport varia CONTINUAMENT entre els punts P_1, P_2 i P_3 respectivament
- Si el CDG varia de G_3 a G_4 la paràbola tomba DISCONTINUAMENT: el punt de suport salta de P_3 a P_4
- La disrupció es produeix per $\beta \cong -0'28$



En canvi, per $\alpha = 0{,}4$

- NO hi ha disrupció ni histèresi en variar β

5. GENERICITAT D'AQUESTS EXEMPLES

Com veurem, aquests exemples representen una situació ben general:

- en fenòmens discontinus dependents de 2 paràmetres externs,
  les disrupcions apareixen sobre corbes CUSPIDALS de la forma $\beta^2 = \lambda\alpha^3$

- en particular, hi apareixen CICLES d'HISTÈRESI quan varia β
  (per valors de α superiors al del vèrtex cuspoidal)

- el MODEL MATEMÀTIC a l'apartat 2 (pàg. 7) és genèricament vàlid per aquests cicles d'histèresi

- per valors de α inferiors al del vèrtex cuspoidal
  DESAPAREIXEN les disrupcions i la histèresi variant β

FINAL SESSIÓ I

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

ETSEIB

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA II: CÀLCUL MULTIVARIABLE

SESSIÓ II. LES CATÀSTROFES DE THOM

PROFESSORAT: JOSEP FERRER
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LES CATÀSTROFES DE THOM

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1. LA TEORIA DE CATÀSTROFES (RENÉ THOM, 1972)

TEOR.- z, VARIABLE d'ESTAT (o interna)

Comportament depenent d'un POTENCIAL $V(z;\ \alpha, \beta, \ldots)$

on $\alpha, \beta, \ldots$ són PARÀMETRES de CONTROL (o variables externes)

$\Rightarrow$ es pot reduir a:

1 PARÀMETRE: $V(z;\alpha) = \frac{z^3}{3} + \alpha z$

2 PARÀMETRES: $V(z;\alpha,\beta) = \frac{z^4}{12} + \alpha\frac{z^2}{2} + \beta z$

3 PARÀMETRES: ...

4 PARÀMETRES: ...

7 CATÀSTROFES ELEMENTALS de THOM

1. LA TEORIA DE CATÀSTROFES (RENÉ THOM, 1972)

COROL.- (1) PUNTS d'EQUILIBRI $= \{(z;\ \alpha, \beta, \dots):\ D_Z V(z;\ \alpha, \beta, \dots) = 0\}$ (≡ SUPERFÍCIE d'EQUILIBRI)

(2) quan varien $\alpha, \beta, \dots$ poden DESAPAREIXER PUNTS d'EQUILIBRI ESTABLES ⟺ "CATÀSTROFE"

CONJUNT DE BIFURCACIÓ $= \{(\alpha, \beta, \dots)$: on hi ha "catàstrofe"$\}$

OBS.- Es generalitza a casos NO DEPENENTS de POTENCIAL

Es generalitza a DIVERSES VARIABLES d'ESTAT

2. CAS D'1 PARÀMETRE: PLEGAMENT

$$V(z;\alpha) = \frac{z^3}{3} + \alpha z$$

Calculem la SUPERFÍCIE d'EQUILIBRI:

$D_z V(z;\alpha) = z^2 + \alpha$

- CAP punt d'equilibri $si \ \ \alpha > 0$
- $z = \ 0 \ \ si \ \alpha = 0$
- $z = \ \pm\sqrt{-\alpha} \ \ si \ \alpha < 0$

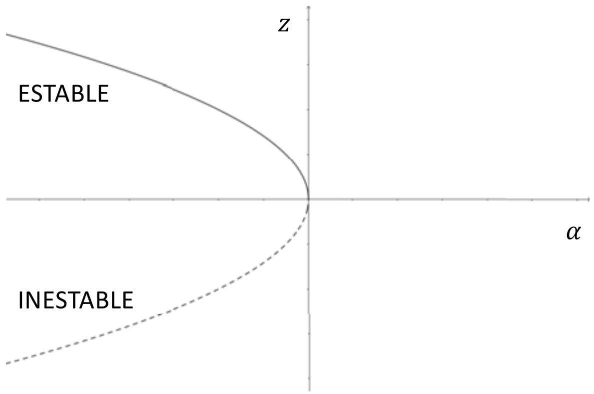
$D_z^2 V(z;\alpha) = 2z$

- $> 0 \ en \ z = \ +\sqrt{-\alpha} \ \Rightarrow$

  $\Rightarrow$ EQ. ESTABLE $en \ z = \ +\sqrt{-\alpha}\ , \alpha < 0$

- $< 0 \ en \ z = -\sqrt{-\alpha} \ \Rightarrow$

  $\Rightarrow$ EQ. INESTABLE $en \ z = -\sqrt{-\alpha}\ , \alpha < 0$

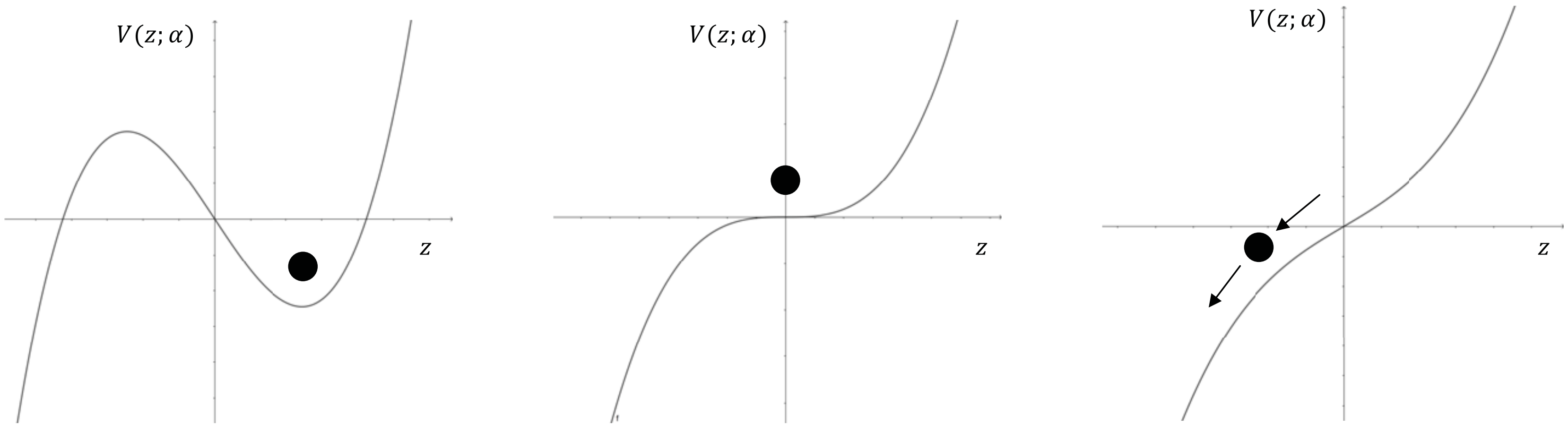


Per tant, CONJUNT de BIFURCACIÓ = $\{\alpha = 0\}$

2. CAS D'1 PARÀMETRE: PLEGAMENT

En efecte:

α CREIX $\Rightarrow$ L'EQ. ESTABLE DESAPAREIX per $\alpha = 0$



$\alpha < 0$ $\alpha = 0$ $\alpha > 0$

3. CAS DE 2 PARÀMETRES: LA CÚSPIDE DE THOM

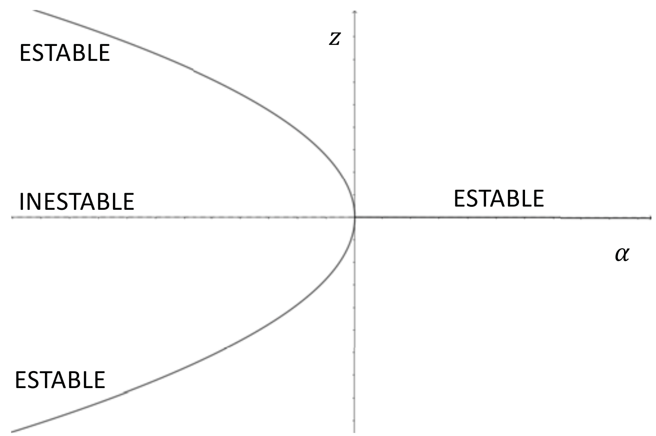
$$V(z;\alpha,\beta) = \frac{z^4}{12} + \alpha\frac{z^2}{2} + \beta z$$

Calculem la SUPERFÍCIE d'EQUILIBRI:

$$D_z V(z;\alpha,\beta) = \frac{z^3}{3} + \alpha z + \beta = 0$$

$$D_Z^2 V(z;\alpha,\beta) = z^2 + \alpha$$

Per $\beta = 0$
- $\alpha > 0 : z = 0$, ESTABLE
- $\alpha < 0$
  - $z = 0$, INESTABLE
  - $z = \pm\sqrt{-3\alpha}$, ESTABLES



Observeu que per $\beta = 0$ NO resulta el cas anterior

3. CAS DE 2 PARÀMETRES: LA CÚSPIDE DE THOM

En general, resulta:

SUPERFÍCIE d'EQUILIBRI

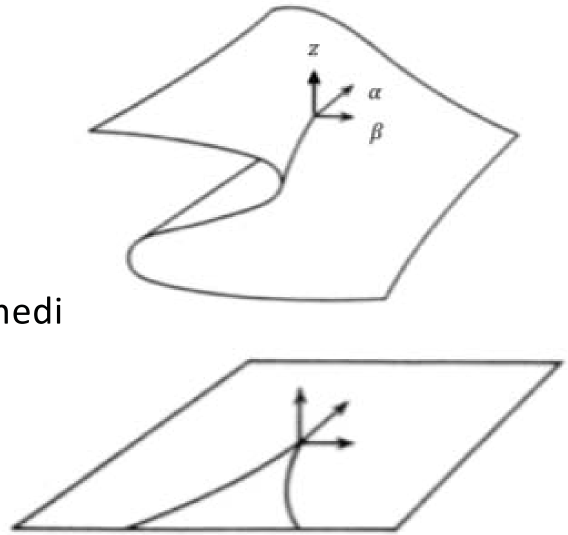
$$\frac{z^3}{3} + \alpha z + \beta = 0$$

EQUILIBRIS INESTABLES: al plec intermedi

EQUILIBRIS ESTABLES: la resta

CONJUNT de BIFURCACIÓ

$$4\beta^2 = -9\alpha^3$$



CÚSPIDE DE THOM

Observem que recuperem el que ja hem vist tallant per $\beta = 0$

3. CAS DE 2 PARÀMETRES: LA CÚSPIDE DE THOM

En efecte, la superfície d'equilibri és:

$$D_z V(z;\alpha,\beta) = \frac{z^3}{3} + \alpha z + \beta = 0$$

$$D_z^2 V(z;\alpha,\beta) = z^2 + \alpha$$

resulta

- $\alpha > 0 \Rightarrow D_z V$ CREIXENT $\Rightarrow$ 1 ÚNIC PUNT d'EQUILIBRI ESTABLE
- $\alpha < 0 \Rightarrow D_z V$
  - MÀXIM RELATIU $\beta - \frac{2}{3}\alpha\sqrt{-\alpha}$, en $z = -\sqrt{-\alpha}$
  - MÍNIM RELATIU $\beta + \frac{2}{3}\alpha\sqrt{-\alpha}$, en $z = \sqrt{-\alpha}$

  $\Rightarrow$

  $\Rightarrow D_z V$
  - 3 ZEROS si $\beta + \frac{2}{3}\alpha\sqrt{-\alpha} < 0 < \beta - \frac{2}{3}\alpha\sqrt{-\alpha}$
  - 1 ZERO, altrament

  $\Rightarrow$

  $\Rightarrow D_z V$
  - 3 ZEROS si $\frac{2}{3}\alpha\sqrt{-\alpha} < -\beta < -\frac{2}{3}\alpha\sqrt{-\alpha}$
  - 1 ZERO, altrament

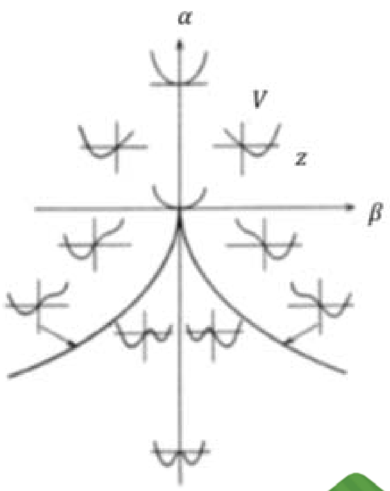
3. CAS DE 2 PARÀMETRES: LA CÚSPIDE DE THOM

En definitiva:

$4\beta^2 > -9\alpha^3 \Rightarrow$ 1 EQUILIBRI ESTABLE

$4\beta^2 < -9\alpha^3 \Rightarrow \begin{cases} 2 & \text{EQUILIBRIS ESTABLES} \\ 1 & \text{EQUILIBRI INESTABLE} \end{cases}$

CONJUNT de BIFURCACIÓ $= \{4\beta^2 = -9\alpha^3\}$ (≡ CÚSPIDE de THOM)

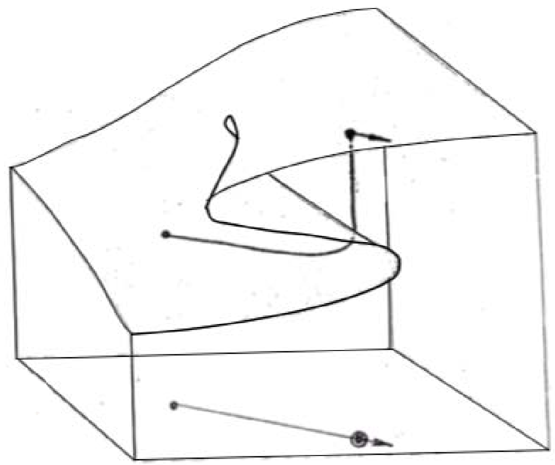


4. PROPIETATS DINÀMIQUES DE LA CÚSPIDE DE THOM

(1) Quan (α, β) varia travessant la cúspide, DESAPAREIX UN DELS PUNTS d'EQ. ESTABLES, i es produeix la DISRUPCIÓ CAP A L'ALTRE.

Exemple:

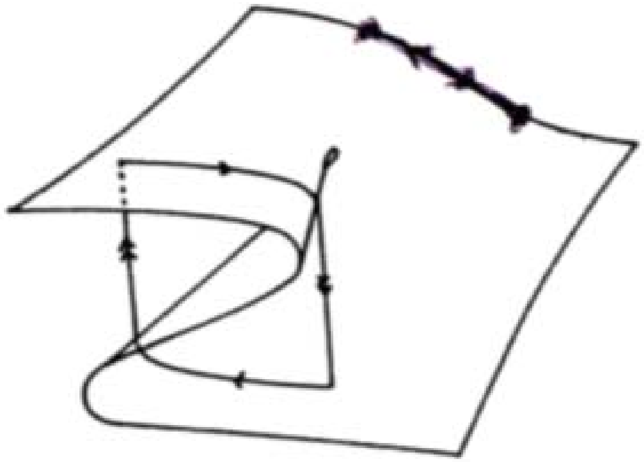
$$\begin{cases} \alpha < 0, \text{ CONSTANT} \\ \beta \quad \text{CREIXENT} \end{cases} \Rightarrow \text{DISRUPCIÓ per } \beta = \sqrt{-\frac{9}{4}\alpha^3}$$



És el cas de la màquina gravitatòria quan tomba sobtadament d'un costat a l'altre

4. PROPIETATS DINÀMIQUES DE LA CÚSPIDE DE THOM

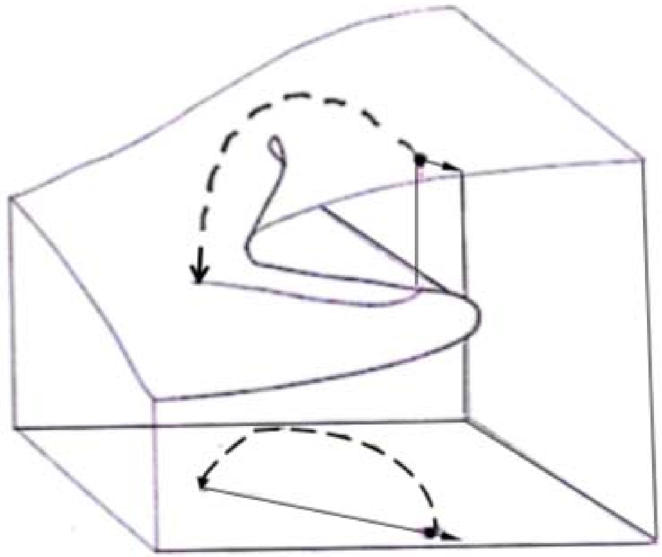
(2) Retrobem el CICLE d'HISTÈRESI ja conegut si β CREIX/DECREIX alternativament



(3) Observem que DESAPAREIX la HISTÈRESI si α CREIX fins $\alpha > 0$, encara que β continuï oscil·lant

4. PROPIETATS DINÀMIQUES DE LA CÚSPIDE DE THOM

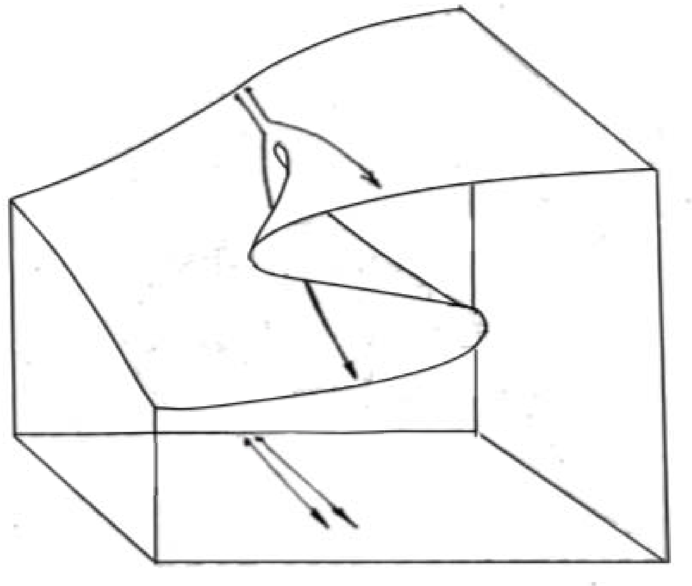
(4) Igualment mitjançant el paràmetre α,
una DISRUPCIÓ pot REVERTIR-SE DE FORMA CONTÍNUA com a la figura



4. PROPIETATS DINÀMIQUES DE LA CÚSPIDE DE THOM

(5) Un fenomen de DIVERGÈNCIA o RADICALITZACIÓ

- $\alpha > 0$, $\beta_1 \lesssim 0 \lesssim \beta_2$ $\Rightarrow$ PUNTS d'EQ. PRÒXIMS
- α DECREIX fins $\alpha < 0$, MATEIXOS β_1, β_2 $\Rightarrow$ PUNTS d'EQ. ALLUNYATS



5. APLICACIONS

La cúspide de Thom s'ha aplicat en àmbits molt diversos:

física, ciències socials, biologia, lingüística, ...

Persisteix la polèmica sobre la validesa d'algunes d'aquestes aplicacions (vegeu darrera pàgina)

En física ja vam veure la MÀQUINA GRAVITATÒRIA

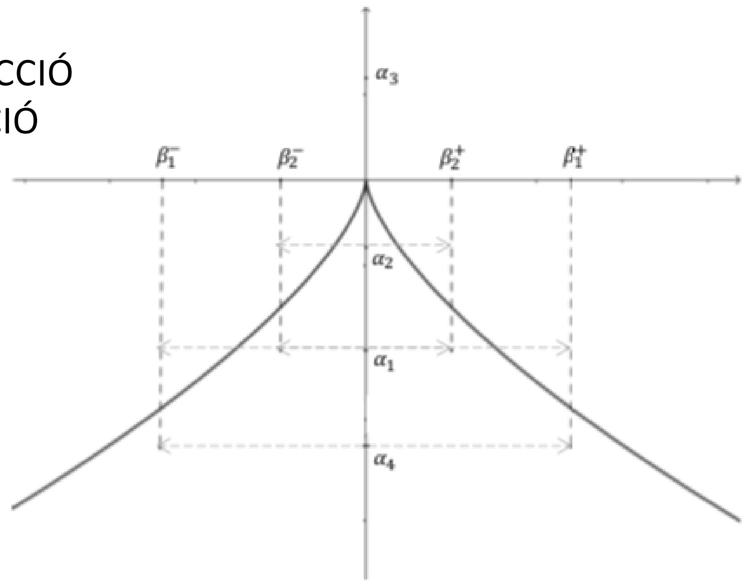
i veurem altres aplicacions en la pròxima sessió.

Vegem ara exemples en biologia.

5. APLICACIONS

Exemple .- FUNCIONAMENT DEL COR

Considerem:
- z, ELONGACIÓ MÚSCUL CARDÍAC (CONTRACCIÓ / RELAXACIÓ)
- α, PRESSIÓ SANGUÍNIA
- β, IMPULS NERVIÓS



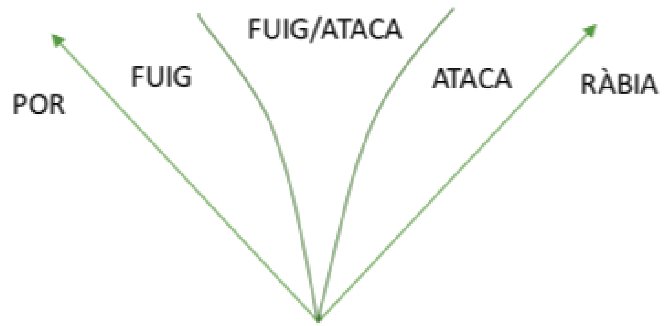
- HISTÈRESI $(\beta_1^- \leq \beta \leq \beta_1^+; \alpha = \alpha_1) \Longleftrightarrow$ BATECS DEL COR
  - APLICACIÓ IMPULS NERVIÓS $\Longrightarrow$ SÍSTOLE
  - ANUL·LACIÓ IMPULS NERVIÓS $\Longrightarrow$ DIÀSTOLE
- IMPULS NERVIÓS INSUFICIENT $(\beta_2^- \leq \beta \leq \beta_2^+; \alpha = \alpha_1)$:
  - NO HI HA BATEC (FIBRILACIÓ)
  - Es pot SUBSANAR REDUINT la PRESSIÓ SANGUÍNIA $(\alpha = \alpha_2)$
- PRESSIÓ MASSA BAIXA O MASSA ALTA $(\alpha = \alpha_3, \alpha_4) \Longrightarrow$ NO HI HA BATEC

5. APLICACIONS

Exemple.- COMPORTAMENT ATAC / FUGIDA D'UN GOS

- KONRAD LORENZ (Nobel Medicina 1973):

  2 PARÀMETRES de CONTROL: RÀBIA (⟺ BOCA), POR (⟺ ORELLES)



- ZERIK ZEEMAN: CÚSPIDE de THOM, amb RÀBIA / POR a les bisectrius

- En particular:

  FUIG / ATACA segons HISTÒRIA PRÈVIA

  El canvi és DISRUPTIU

5. APLICACIONS

Exemples.- COMPORTAMENTS SOCIALS (no hi ha unanimitat sobre la seva validesa)

- PRESONS: AMOTINAMENTS SOBTATS
- CAIGUDA de l'IMPERI ROMÀ
- COTITZACIONS en BORSA
- ELECCIONS:
  - PREDOMINI del CENTRE
  - RADICALITZACIÓ BIMODAL

FINAL SESSIÓ II

UNIVERSITAT POLITÈCNICA DE CATALUNYA
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APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA II: CÀLCUL MULTIVARIABLE

SESSIÓ III. DESENVOLUPAMENTS DE TAYLOR I DE FOURIER

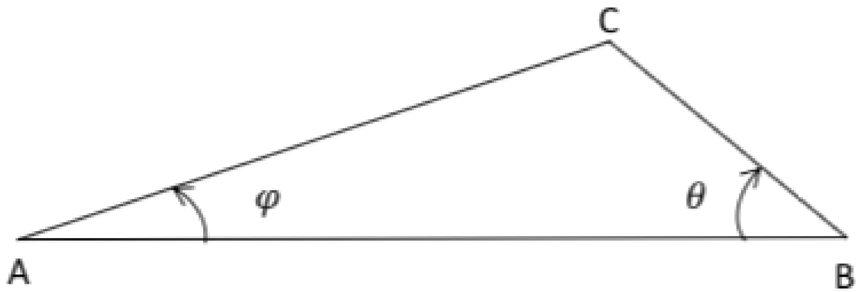
PROFESSORAT: JOSEP FERRER
MARTA PEÑA

DESENVOLUPAMENTS DE TAYLOR I DE FOURIER

INDEX

1. TRIANGULACIÓ TOPOGRÀFICA (REY PASTOR)

En una triangulació topogràfica (figura) es tracta de calcular $\overline{AC}$ i $\overline{BC}$ a partir de $\overline{AB}$, φ i θ.



Ens preguntem l'error $\Delta\overline{BC}$ si es cometen errors Δφ i $\Delta\theta$.

1. TRIANGULACIÓ TOPOGRÀFICA (REY PASTOR)

Clarament $\overline{BC} = \overline{AB}\,\frac{sen\theta}{sen(\theta+\varphi)}$

Segons la fórmula de TAYLOR:

$$\Delta\overline{BC} \cong \overline{AB}D_{\theta}\,\frac{sen\theta}{sen(\theta+\varphi)}\,\Delta\theta + \overline{AB}D_{\varphi}\,\frac{sen\theta}{sen(\theta+\varphi)}\,\Delta\varphi = \overline{AB}\,\frac{sen\varphi}{sen^2(\theta+\varphi)}\,\Delta\theta \;-\; \overline{AB}D_{\varphi}\,\frac{sen\theta cos(\theta+\varphi)}{sen^2(\theta+\varphi)}\,\Delta\varphi$$

Per exemple, si:

$$\overline{AB} = 100\text{ m},\ \ \varphi \cong 29^\circ\,, \theta \cong 55^\circ$$

resulta:

$$\boxed{\Delta\overline{BC} \cong 82'82\Delta\theta - 5'12\Delta\varphi}$$

<u>Per tant cal fer especial atenció en la medició de θ</u>

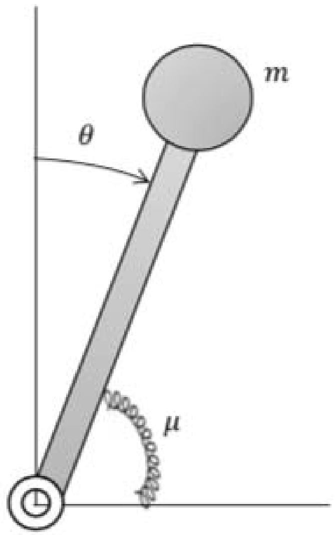
2. MODEL D'AUGUSTI

Es considera:

- una BARRA recta, rígida i lleugera (que per simplificar suposarem de llargària 1)
- una MASSA m en la part superior
- la subjecció inferior només permet girs en el pla i

  una MOLLA rotacional (de mòdul μ) intenta mantenir-la vertical
- tanmateix, per una IMPERFECCIÓ constructiva,

  la molla és en repòs per a $\theta = \theta_o$ (en lloc de $\theta = 0$)



Ens preguntem el comportament de θ quan m augmenta.

2. MODEL D'AUGUSTI

Com que hi ha 2 PARÀMETRES de CONTROL (m, θ_o)

hauria de modelitzar-se per la SEGONA CATÀSTROFE de THOM

Vegem-ho explicant TAYLOR, per θ petit.

L'ENERGIA del sistema (referida a la vertical):

$$V(\theta; \mu, \theta_o) = \left(\frac{\mu}{2}(\theta - \theta_o)^2 - \frac{\mu}{2}{\theta_o}^2\right) + (mgcos\theta - mg) \cong \frac{\mu}{2}(\theta^2 - 2\theta\theta_o) + mg\left(-\frac{\theta^2}{2} + \frac{\theta^4}{24}\right) =$$

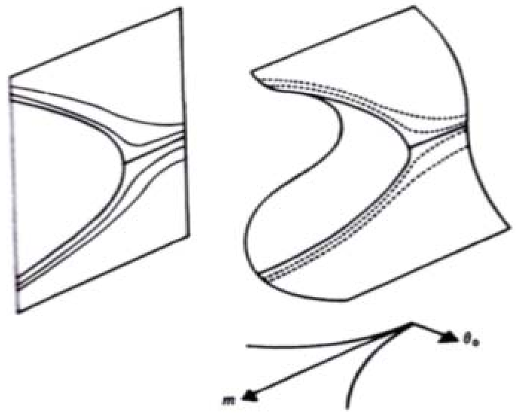
$$= \frac{mg}{24}\,\theta^4 + \left(\frac{\mu}{2} - \frac{mg}{2}\right)\theta^2 - \mu\theta_o\,\theta$$

2. MODEL D'AUGUSTI

Per tant, si AUGMENTA m (amb θ_o constant):

$\theta_o = 0 \Rightarrow$ línea gruixuda de la figura $\Leftrightarrow$
- vertical fins $m = {}^{\mu}/_{g}$
- s'inclina ràpidament per $m > {}^{\mu}/_{g}$

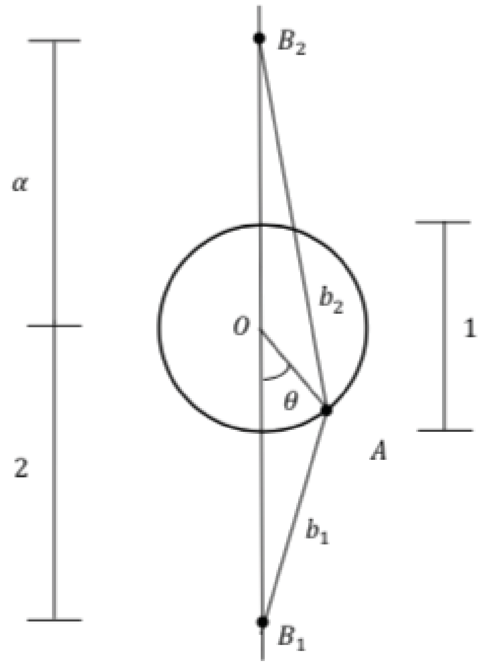
$0 < \theta_o << \Rightarrow$ líneas fines/discontinues de la figura $\Leftrightarrow$
- la inclinació creix lentament fins $m = {}^{\mu}/_{g}$
- la inclinació creix ràpidament per $m > {}^{\mu}/_{g}$



3. MÀQUINA DE ZEEMAN

Considerem:

- una RODA d'1 cm de diàmetre i centre 0 fix
- un punt A de la seva vora
- dues BANDES ELÀSTIQUES AB_1 i AB_2 de llargària en repòs 1 cm
- $B_1 = (0, -2)$
- $B_2 = (\beta, \alpha)$
- $\theta =$ angle B_1OA



Es pregunta, per a $\beta = 0$:

- valors de α per als quals $\theta = 0$ és un punt d'equilibri estable
- valors de α per als quals $\theta = 0$ és un punt d'equilibri inestable
- valor de DISRUPCIÓ de α

3. MÀQUINA DE ZEEMAN

Segons la llei de Hooke, l'ENERGIA del sistema és (en unitats adequades):

$$V(\theta;\alpha,\beta) = (b_1 - 1)^2 + (b_2 - 1)^2$$

Si $\beta = 0$, per TAYLOR (fins ordre 4):

$$V(\theta;\alpha,0) = \left(\sqrt{\left(2-\frac{1}{2}cos\theta\right)^2 + \left(\frac{1}{2}sen\theta\right)^2} - 1\right)^2 + \left(\sqrt{\left(\alpha+\frac{1}{2}cos\theta\right)^2 + \left(\frac{1}{2}sen\theta\right)^2} - 1\right)^2 \cong$$

$$\cong \left(\sqrt{\frac{9}{4}+\theta^2} - 1\right)^2 + \left(\sqrt{\left(\alpha+\frac{1}{2}\right)^2 - \frac{\alpha}{2}\theta^2} - 1\right)^2 \cong \left(\frac{3}{2}+\frac{1}{3}\theta^2 - 1\right)^2 + \left(\left(\alpha+\frac{1}{2}\right) - \frac{\alpha}{2}\frac{1}{2\alpha+1}\theta^2 - 1\right)^2 \cong$$

$$\cong \frac{1}{4} + \left(\alpha - \frac{1}{2}\right)^2 + \left(\frac{1}{3} - \frac{\alpha}{2}\frac{2\alpha-1}{2\alpha+1}\right)\theta^2$$

El punt de DISRUPCIÓ serà:

$$\frac{1}{3} - \frac{\alpha}{2}\frac{2\alpha-1}{2\alpha+1} = 0 \Leftrightarrow \boxed{\alpha = \frac{7+\sqrt{97}}{12} \cong 1{,}4}$$

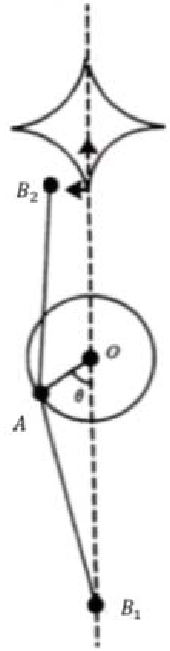
3. MÀQUINA DE ZEEMAN

Anàlogament trobaríem una SEGONA DISRUPCIÓ (DUAL) per $\alpha \cong 2'46$,

corresponent al punt d'equilibri $\theta = \pi$

Si considerem β variable com 2n PARÀMETRE de CONTROL,

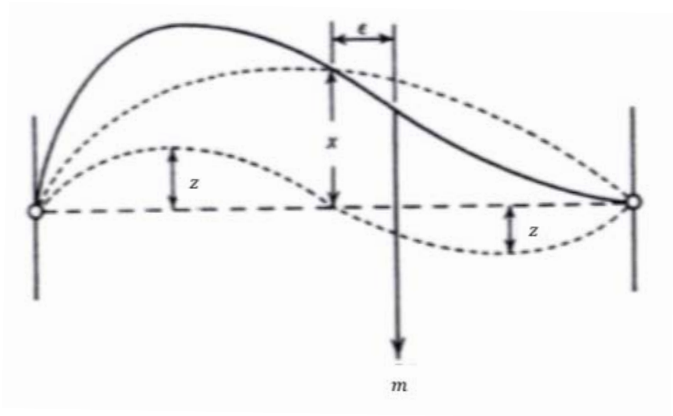
trobaríem la SEGONA CATÀSTROFE de Thom amb 4 CÚSPIDES de BIFURCACIÓ



4. ARC D'EULER

Suposem (vegeu figura):

- ARC COMPRIMIT (de llargària π, per simplificar)
- una CÀRREGA m
- lleugerament DESCENTRADA (ε)



Depenent de 2 PARÀMETRES de CONTROL (m,ε)

hauria de modelitzar-se per la SEGONA CATÀSTROFE de THOM

4. ARC D'EULER

FORMA de l'arc: $f(s), \qquad 0 \leq s \leq \pi$

Per FOURIER:

$$f(s) = \sum c_n \operatorname{sen} ns$$

Podem simplificar, suposant 1 o 2 HARMÒNICS

- $m = 0, \varepsilon = 0 \Rightarrow f(s) \cong r \operatorname{sen} s$
- $m \neq 0, 0 \neq \varepsilon \ll \Rightarrow f(s) \cong x \operatorname{sen} s + z \operatorname{sen} 2s$

4. ARC D'EULER

Suposem mòdul elàstic $\mu = {}^{1}/_{\pi}$ per simplificar:

ENERGIA POTENCIAL $\equiv V_P = mf\left(\frac{\pi}{2} + \varepsilon\right)$

ENERGIA ELÀSTICA $\equiv V_E = \frac{1}{2\pi}\int_0^{\pi}(f''(s))^2 \frac{1}{(1+(f'(s))^2)^3} ds$

Apliquem TAYLOR (per $\varepsilon \ll$):

$$V_P = mx \cos \varepsilon + mz(-\operatorname{sen} 2\varepsilon) \cong mx\left(1 - \frac{\varepsilon^2}{2}\right) + mz(-2\varepsilon)$$

4. ARC D'EULER

Les variables x, z estan relligades pel fet que la distància d entre els extrems de la biga:

$$d = \int_0^{\pi} \sqrt{1 - (f'(s))^2}\, ds$$

és la mateixa en els dos casos:

$$d = \int_0^{\pi} \sqrt{1 - (r\cos s)^2}\, ds \cong \int_0^{\pi} \sqrt{\left(1 + \frac{1}{2} r^2 \cos^2 s + \frac{-1}{8} r^4 \cos^4 s\right)}\, ds = \frac{\pi}{4}\left(4 - r^3 - \frac{3}{16} r^4 - \frac{5}{64} r^6\right)$$

$$d = \int_0^{\pi} \sqrt{1 - (x\cos s + 2z\cos 2s)^2}\, ds \cong \cdots = \frac{\pi}{4}\left(4 - x^2 - 4z^2 - \frac{3}{16} x^4 - 3x^2 z^2 - \frac{5}{64} x^6\right)$$

Igualant, x és una FUNCIÓ IMPLÍCITA de z per a la qual novament apliquem TAYLOR:

$$x \cong a_o + a_2 z^2 + a_4 z^4 \Longrightarrow \quad a_o = r, \quad a_2 = -\frac{2}{r} - \frac{3r}{4}, \quad a_4 = \frac{-2}{r^3}$$

Per tant:

$$x \cong r + z^2\left(-\frac{2}{r} - \frac{3r}{4}\right) + z^4 \frac{-2}{r^3}$$

4. ARC D'EULER

Anàlogament:

$$V_E \cong \cdots = constant + \left(3 + \frac{13}{8}r^2\right) z^2$$

Per tant:

$$V = V_P + V_E \cong constant - 2m\varepsilon z + \left(\left(3 + \frac{13}{8}r^2\right) - m\left(\frac{2}{r} + \frac{3r}{4}\right)\right) z^2 - \frac{2m}{r^3} z^4$$

que dona efectivament una CÚSPIDE de THOM amb vèrtex $(\varepsilon = 0)$:

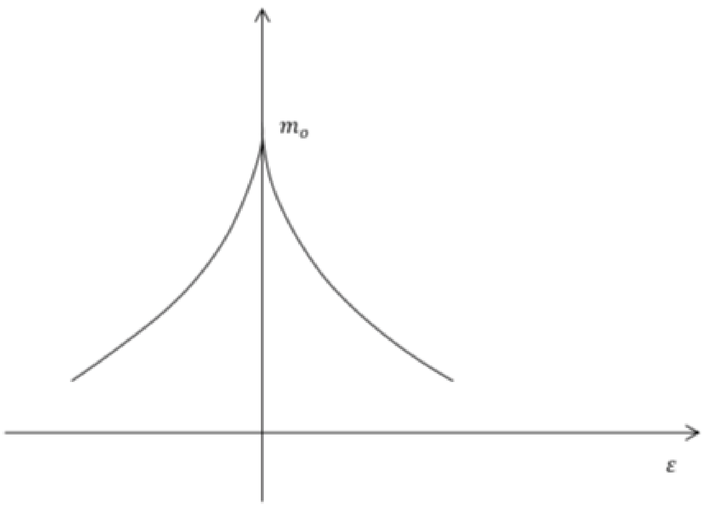
$$m_o = \left(3 + \frac{13}{8}r^2\right)\left(\frac{2}{r} + \frac{3r}{4}\right)^{-1} \cong \frac{3}{2}r - \frac{1}{4}r^3$$

En definitiva:

$$V \cong -\frac{3}{r^2}z^4 - \frac{2}{r}(m - m_o)z^2 - 2r\varepsilon z$$

4. ARC D'EULER

En particular, la CÀRREGA MÀXIMA disminueix molt ràpidament en augmentar ε



Cosa que <u>coincideix amb els resultats experimentals</u>

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APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA II: CÀLCUL MULTIVARIABLE

SESSIÓ IV. CADENA, INVERSA, IMPLÍCITA

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

CADENA, INVERSA, IMPLÍCITA

INDEX

1. TÈCNIQUES D'APROXIMACIÓ LINEAL

En tots 3 teoremes

- **TAYLOR**: APROXIMACIÓ LINEAL del PROBLEMA
- **ÀLGEBRA LINEAL**: RESOLUCIÓ d'aquest PROBLEMA LINEAL APROXIMAT
- **TEOREMA**: és l'APROXIMACIÓ LINEAL de la SOLUCIÓ!

PROBLEMA --- ? ---> SOLUCIÓ

PROBLEMA --- 1-TAYLOR ---> PROBLEMA LINEAL APROXIMAT

SOLUCIÓ --- 1-TAYLOR ---> APROXIMACIÓ LINEAL de la SOLUCIÓ

PROBLEMA LINEAL APROXIMAT --- ALG. LIN. ---> SOLUCIÓ del PROBLEMA LINEAL APROXIMAT

SOLUCIÓ del PROBLEMA LINEAL APROXIMAT // APROXIMACIÓ LINEAL de la SOLUCIÓ

TEOREMA!
- CADENA
- INVERSA
- IMPLÍCITA

<u>Observació</u>.- Recordem que l'aproximació de Taylor és vàlida LOCALMENT (en l'entorn d'un punt)

1. TÈCNIQUES D'APROXIMACIÓ LINEAL: Regla de la cadena

- REGLA de la CADENA per al problema de COMPOSICIÓ de FUNCIÓ

$$x \xrightarrow{f} y \xrightarrow{g} z \quad \overset{?}{\dashrightarrow} \quad x \xrightarrow{g \circ f} z$$

1-TAYLOR

1-TAYLOR

$$\Delta x \xrightarrow{D_x f} \Delta y \xrightarrow{D_y g} \Delta z \quad \overset{\text{ALG. LIN.}}{\dashrightarrow}$$

$$\Delta x \xrightarrow{D_y g D_x f} \Delta z \quad // \quad \Delta x \xrightarrow{D_x (g \circ f)} \Delta z$$

REGLA de la CADENA

1. TÈCNIQUES D'APROXIMACIÓ LINEAL: Teorema de la inversa

- TEOREMA de la FUNCIÓ INVERSA per al problema de <u>INVERSA de FUNCIONS</u>

$x \xrightarrow{\text{f}} y$ --- ? ---> $y \xrightarrow{f^{-1}} x$

1-TAYLOR (from $x \xrightarrow{\text{f}} y$ down to $\Delta x \xrightarrow{D_x f} \Delta y$)

1-TAYLOR (from $y \xrightarrow{f^{-1}} x$ down to $\Delta y \xrightarrow{D_y(f^{-1})} \Delta x$)

$\Delta x \xrightarrow{D_x f} \Delta y$ --- ALG. LIN. ---> $\Delta y \xrightarrow{(D_x f)^{-1}} \Delta x$

$$\Delta y \xrightarrow{D_y(f^{-1})} \Delta x$$

$$\Delta y \xrightarrow{(D_x f)^{-1}} \Delta x \quad = \quad \text{TEOREMA de la INVERSA}$$

$$Si \;\; \det \;\; D_x \mathrm{f} \neq 0$$

1. TÈCNIQUES D'APROXIMACIÓ LINEAL: Teorema de la implícita

- TEOREMA de la FUNCIÓ IMPLÍCITA per al problema de FUNCIONS DEFINIDES per SISTEMES INDETERMINATS

$f(x,y) = 0$ → ? → $y = y(x)$

$f(x,y) = 0$ → 1-TAYLOR → $D_x f \Delta x + D_y f \Delta y = 0$

$y = y(x)$ → 1-TAYLOR → $\Delta y = D_x y \Delta x$

$D_x f \Delta x + D_y f \Delta y = 0$ → ALG. LIN. → $\Delta y = -\left(D_y f\right)^{-1} D_x f \Delta x$

$$\Delta y = D_x y \Delta x$$

// TEOREMA de la IMPLÍCITA

$$\Delta y = -\left(D_y f\right)^{-1} D_x f \Delta x$$

$Si\;\det\; D_y \mathrm{f} \neq 0$

2. DERIVADES DIRECCIONALS

En particular, la FÒRMULA del GRADIENT per a derivades direccionals:

$$(x, y, z) \longrightarrow V(x, y, z)$$

$$grad\ V \equiv (D_x V,\ D_y V,\ D_z V)$$

<u>PROPOSICIÓ</u>:

- DERIVADA segons $v \epsilon \mathbb{R}^3$ } $D_v V(a) = < v, grad\ V(a) >$

- DERIVADA DIRECCIONAL MÀXIMA en $\mathbb{R}$ } DIRECCIÓ $= grad\ V(a)$; VALOR $= \|grad\ V(a)\|$

2. DERIVADES DIRECCIONALS: Aplicacions

Aplicacions:

- TRAJECTÒRIES de MÀXIMA PENDENT $\Leftrightarrow$

  $\Leftrightarrow$ tangents al gradient $\Leftrightarrow$ ORTOGONALS a les CORBES de NIVELL

- LÍNIES de FORÇA d'un CAMP POTENCIAL $\Leftrightarrow$

  $\Leftrightarrow$ tangents al gradient $\Leftrightarrow$ ORTOGONALS a les SUPERFÍCIES EQUIPOTENCIALS

- ALGORISMES d'OPTIMITZACIÓ :

  seguiment iterat del GRADIENT

3. EXEMPLES D'APLICACIÓ: Cinètica amb paràmetres de control

Vegem algunes aplicacions senzilles. En les pròximes sessions en veurem de més complexes.

Exemple 1.- VELOCITAT i ACCELERACIÓ amb PARÀMETRES de CONTROL

Suposem
- (x, y) POSICIÓ
- (α, β) PARÀMETRES de CONTROL
- $t \to (\alpha, \beta) \to (x, y)$

Per la REGLA de la CADENA:

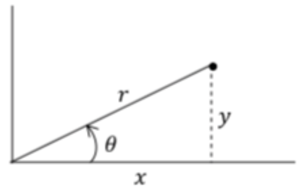
- VELOCITAT: $$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} D_\alpha x & D_\beta x \\ D_\alpha y & D_\beta y \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix}$$
- ACCELERACIÓ: $$\ddot{x} = (\dot{\alpha}, \dot{\beta}) \begin{pmatrix} D_{\alpha\alpha} x & D_{\alpha\beta} x \\ D_{\beta\alpha} x & D_{\beta\beta} x \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} + (D_\alpha x \quad D_\beta x) \begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \end{pmatrix}$$
  $$\ddot{y} = \cdots$$

3. EXEMPLES D'APLICACIÓ: Coordenades polars

Exemple 2.- COORDENADES POLARS (PLANES, CILÍNDRIQUES, ESFÈRIQUES)

Les COORDENADES POLARS PLANES

$$\begin{aligned} x &= r\cos\theta \\ y &= r\,\mathrm{sen}\,\theta \end{aligned}$$



$$(r,\theta) \xrightarrow{\ \phi\ } (x,y) \xrightarrow{\ f\ } z, \qquad F: (r,\theta) \to z$$

$$D\phi = \begin{pmatrix} \cos\theta & \mathrm{sen}\,\theta \\ -r\mathrm{sen}\,\theta & r\cos\theta \end{pmatrix} \qquad D(\phi^{-1}) = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{pmatrix}$$

$$F(r,\theta) = f(r\cos\theta\,, r\mathrm{sen}\,\theta)$$

$$(D_rF \quad D_\theta F) = (D_xf \quad D_yf)D\phi \qquad\qquad (D_xf \quad D_yf) = (D_rF \quad D_\theta F)D(\phi^{-1})$$

Múltiples aplicacions en geometria, càlcul integral, equacions diferencials:

3. EXEMPLES D'APLICACIÓ: Coordenades polars

- LEMNISCATA de BERNOUILLI: el símbol de l'infinit

$$\text{punts } (x, y) \text{ amb } \mathrm{d}\big((x, y), (0, a)\big) \cdot \mathrm{d}\big((x, y), (0, -a)\big) = a^2 \Leftrightarrow$$

$$\Leftrightarrow (x^2 + y^2)^2 = 2a^2(x^2 - y^2) \Leftrightarrow \boxed{r^2 = 2a^2 \cos 2\theta}$$

- ESPIRAL d'ARQUÍMEDES: soles dels vinils, enrotllar (una corda, una tela, ...)

$$\left.\begin{array}{l} \dot{r} = a \\ \dot{\theta} = b \end{array}\right\} \Leftrightarrow \boxed{r = \frac{a}{b}\theta}$$

- EXTREMS de $f(x, y) = xy \; en \; x^2 + y^2 = 2$:

$$\left.\begin{array}{l} f(x, y) = xy \\ x^2 + y^2 = 2 \end{array}\right\} \left.\begin{array}{l} F(r, \theta) = f(\mathrm{r}\cos\theta, \mathrm{r}\operatorname{sen}\theta) = r^2 \dfrac{\operatorname{sen} 2\theta}{2} \\ r = \sqrt{2} \end{array}\right\} \Leftrightarrow$$

$$\Leftrightarrow \theta = \begin{cases} \frac{\pi}{4}, \frac{5\pi}{4} & \text{MÀXIMS} \\ \frac{3\pi}{4}, \frac{7\pi}{4} & \text{MÍNIMS} \end{cases} \Leftrightarrow \boxed{\begin{array}{l} \text{MÀXIMS PER } (1,1), (-1,-1) \\ \text{MÍNIMS PER } (1,-1), (-1,1) \end{array}}$$

3. EXEMPLES D'APLICACIÓ : Coordenades polars

- SISTEMES DINÀMICS

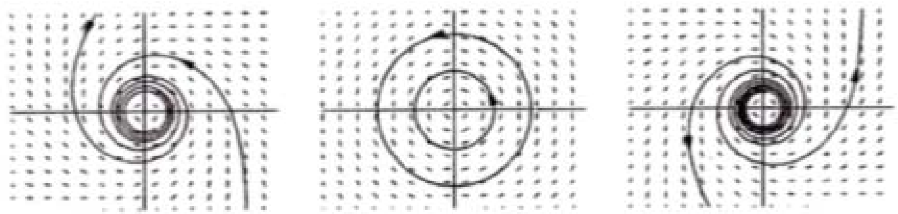
Ens preguntem les trajectòries dels sistema dinàmic

$$\dot{x} = -y + ax(x^2 + y^2)$$
$$\dot{y} = x + ay(x^2 + y^2)$$

Passant a POLARS:

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \cdots = \begin{pmatrix} ar^3 \\ 1 \end{pmatrix} \Leftrightarrow \boxed{\begin{array}{l} \dot{r} = ar^3 \\ \dot{\theta} = 1 \end{array}}$$

Ara és fàcil representar-les:



$a<0$ $a=0$ $a>0$

3. EXEMPLES D'APLICACIÓ : Coordenades polars

- EQUACIONS en DERIVADES PARCIALS

Ens preguntem les funcions $f(x, y)$ que compleixin:

$$yD_xf = xD_yf$$

Passant a POLARS:

$$(r,\theta) \xrightarrow{\phi} (x,y) \xrightarrow{f} z, \qquad F = f \circ \phi$$

$$F(r,\theta) = f(\text{rcos}\,\theta\,, \text{rsen}\,\theta)$$

$$(D_xf \quad D_yf) = (D_rF \quad D_\theta F)\begin{pmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{pmatrix}$$

Substituint:

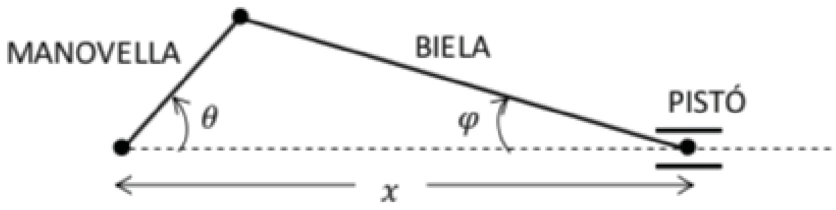
$$yD_xf = xD_yf \Leftrightarrow \cdots \Leftrightarrow D_\theta F = 0$$

Per tant, les solucions només dependran de r:

$$\boxed{f(x,y) = g(x^2 + y^2)}$$

3. EXEMPLES D'APLICACIÓ: Biela / Manovella

Exemple 3.- BIELA / MANOVELLA



r: llargària de la manovella
L: llargària de la biela

$$\begin{cases} x = r\cos\theta + L\cos\varphi \\ r \operatorname{sen}\theta = L \operatorname{sen}\varphi \end{cases}$$

Apliquem el teorema de la IMPLÍCITA amb:

- $(x,\theta,\varphi) \xrightarrow{f} (x - r\cos\theta - L\cos\varphi\,, r\operatorname{sen}\theta - \ L\operatorname{sen}\varphi)$

$$Df = \begin{pmatrix} 1 & r\operatorname{sen}\theta & L\operatorname{sen}\varphi \\ 0 & r\cos\theta & -L\cos\varphi \end{pmatrix}$$

3. EXEMPLES D'APLICACIÓ: Biela / Manovella

Exemple 3.- BIELA / MANOVELLA

- $$det\begin{pmatrix} r\,\text{sen}\,\theta & L\,\text{sen}\,\varphi \\ r\cos\theta & -L\cos\varphi \end{pmatrix} = -rL\,(\text{sen}\,\theta\cos\varphi + \cos\theta\,\text{sen}\,\varphi) =$$

$$= -rL\,\text{sen}(\theta+\varphi) \neq 0 \; si \; (\theta+\varphi) \neq 0,\pi \Rightarrow$$

$$\Rightarrow \boxed{x \text{ controla fora dels PUNTS MORTS}} \; \left(\Leftrightarrow x = \begin{matrix} L+r \\ L-r \end{matrix}\right)$$

En efecte: en els punts morts es podria invertir el gir de la manovella!

- $$det\begin{pmatrix} 1 & L\,\text{sen}\,\varphi \\ 0 & -L\cos\varphi \end{pmatrix} = -L\cos\varphi \neq 0 \quad \forall\varphi \Rightarrow \boxed{\theta \text{ controla en tots els PUNTS}}$$

3. EXEMPLES D'APLICACIÓ: Trobeu l'error

Exemple 4.- TROBEU L'ERROR !!

Preguntem els extrems relatius de la DISTÀNCIA del punt $(0,2)$ als de la PARÀBOLA $y = x^2$

Equivalentment, els extrems relatius de

$$f(x,y) = (0-x)^2 + (2-y)^2 = x^2 + 4 - 4y + y^2, \quad \text{quan } y = x^2$$

$$f\big(x, y(x)\big) = x^2 + 4 - 4x^2 + x^4 = x^4 - 3x^2 + 4;\ 4x^3 - 6x = 0 \Leftrightarrow \begin{cases} x = 0\ ;\ y = 0 & \text{(MÀXIM)} \\ x = \pm\sqrt{\frac{3}{2}}\ ;\ y = \frac{3}{2} & \text{(MÍNIMS)} \end{cases}$$

$$f(x(y), y) = y + 4 - 4y + y^2 = y^2 - 3y + 4;\ 2y - 3 = 0 \Leftrightarrow y = \frac{3}{2};\ x = \pm\sqrt{\frac{3}{2}} \quad \text{(MÍNIMS)}$$

PER QUÈ en el segon cas NO APAREIX el MÀXIM RELATIU en (0,0) ??

3. EXEMPLES D'APLICACIÓ: Trobeu l'error

Exemple 4.- TROBEU L'ERROR !!

Apliquem el TEOREMA de la IMPLÍCITA a l'equació:

$$x^2 - y = 0$$

- DEFINEIX $x \to y(x)$ en TOTS els PUNTS $\Rightarrow$ podem substituir y per x^2 en tots els punts
- DEFINEIX $y \to x(y)$ FORA de $x = 0 \Rightarrow$ podem substituir x^2 per y només si $x \neq 0$

Per tant: la segona substitució no és vàlida per a $x = 0$

de manera que no es detecta el MÀXIM en aquest punt

3. EXEMPLES D'APLICACIÓ: Acceleració d'un flux

Exemple 5.- ACCELERACIÓ D'UN FLUX

- Suposem
  - POSICIÓ (x, y)
  - VELOCITAT (u, v)
  - EQUACIONS de CARACTERITZACIÓ: $\begin{cases} xy + u + v^2 = 0 \\ x^2 + y^2 + uv = 2 \end{cases}$
- Matriu de derivades del sistema: $\begin{pmatrix} y & x & 1 & 2v \\ 2x & 2y & v & u \end{pmatrix}$
- Per a $(x, y) = (1, -1), (u, v) = (1,0)$: $\begin{pmatrix} -1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{pmatrix}$

  $det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq 0 \Rightarrow$ FUNCIÓ IMPLÍCITA $(x, y) \rightarrow (u, v)$, LOCALMENT
- Calculem l'ACCELERACIÓ $(\dot{u}, \dot{v})$ en el punt anterior:

$$\begin{rcases} y\dot{x} + x\dot{y} + \dot{u} + 2v\dot{v} = 0 \\ 2x\dot{x} + 2y\dot{y} + v\dot{u} + u\dot{v} = 0 \end{rcases} \xrightarrow[u = \dot{x} = 1,\ v = \dot{y} = 0]{x = 1,\ y = -1} \boxed{\begin{matrix} \dot{u} = 1 \\ \dot{v} = -2 \end{matrix}}$$

FINAL SESSIÓ IV

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

ETSEIB

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA II: CÀLCUL MULTIVARIABLE

SESSIÓ V. CINÈTICA INVERSA

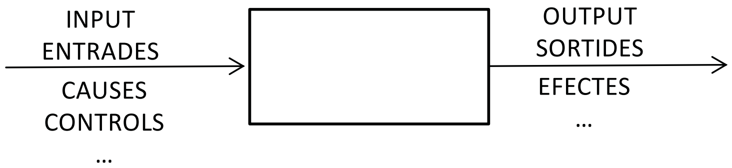
PROFESSORAT: JOSEP FERRER
MARTA PEÑA

CINÈTICA INVERSA

INDEX

1. INTRODUCCIÓ

En sistemes generals



ANÀLISI: ENTRADES ⟶ SORTIDES
- quina sortida resulta de cada entrada
- quines sortides són possibles

...

CONTROL: ENTRADES ⟵ SORTIDES
- quina entrada cal per obtenir la sortida desitjada
- calcular l’entrada mitjançant mesurar la sortida

...

En SISTEMES CINEMÀTICS: les entrades i sortides són POSICIONS, VELOCITATS, ...

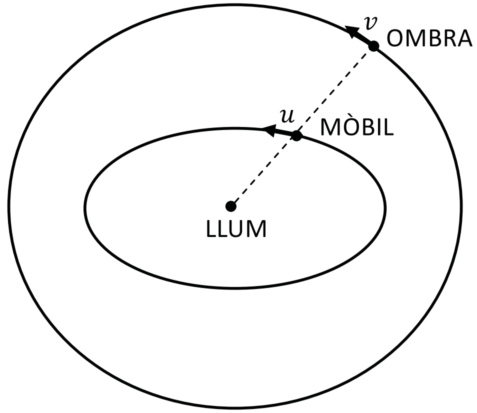
CINÈTICA DIRECTA: posicions, velocitats, ... de les entrades ⟶ posicions, velocitats, ... de les sortides

<u>CINÈTICA INVERSA</u>: “ ⟵ “

2. OMBRA D'UN MÒBIL SOBRE UNA PANTALLA

Suposem:

- PUNT LLUMINÓS en $(0,0)$
- (x, y): MÒBIL ORBITANT-LO
- (α, β): OMBRA sobre una pantalla
- CELERITAT del MÒBIL: $u \equiv \|(\dot{x}, \dot{y}\| = \sqrt{\dot{x}^2 + \dot{y}^2}$
- CELERITAT de l'OMBRA: $v \equiv \left\|(\dot{\alpha}, \dot{\beta}\right\| = \sqrt{\dot{\alpha}^2 + \dot{\beta}^2}$



CINÈTICA

- DIRECTA: mòbil $\longrightarrow$ ombra
- INVERSA: mòbil $\longleftarrow$ ombra

Volem CALCULAR u, mitjançant MESURAR v; és a dir, la cinètica inversa per a les celeritats

2. OMBRA D'UN MÒBIL SOBRE UNA PANTALLA

Ho farem per al cas particular:

ÒRBITA: $x^2 + 2y^2 = 3$

PANTALLA: $\alpha^2 + \beta^2 = 8$

QUAN el mòbil travessa la bisectriu del primer quadrant:

$$x = y = 1; \qquad \alpha = \beta = 2$$

Observem que la relació mòbil/ombra és:

$$\frac{y}{x} = \frac{\beta}{\alpha} \Leftrightarrow x\beta = y\alpha$$

2. OMBRA D'UN MÒBIL SOBRE UNA PANTALLA

- CINÈTICA DIRECTA: mòbil ⟶ ombra

$x^2 + 2y^2 = 3$ DEFINEIX IMPLÍCITAMENT: $x \longrightarrow y$

$$\left.\begin{aligned} \alpha^2 + \beta^2 &= 8 \\ x\beta &= y\alpha \end{aligned}\right\}$$ DEFINEIX IMPLÍCITAMENT: $(x, y) \longrightarrow (\alpha, \beta)$

Per a les VELOCITATS:

$$\boxed{\dot{y} = -\frac{1}{2}\frac{x}{y}\dot{x}}$$

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = -\begin{pmatrix} 2\alpha & 2\beta \\ -y & x \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \Rightarrow \boxed{\begin{aligned} \dot{\alpha} &= \frac{\beta}{\alpha x + \beta y}(\beta\dot{x} - \alpha\dot{y}) \\ \dot{\beta} &= \frac{-\alpha}{\alpha x + \beta y}(\beta\dot{x} - \alpha\dot{y}) \end{aligned}}$$

2. OMBRA D'UN MÒBIL SOBRE UNA PANTALLA

- CINÈTICA INVERSA: ombra $\longrightarrow$ mòbil

$\alpha^2 + \beta^2 = 8$ DEFINEIX IMPLÍCITAMENT: $\alpha \longrightarrow \beta$

$\left.\begin{matrix} x^2 + 2y^2 = 3 \\ x\beta = y\alpha \end{matrix}\right]$ DEFINEIX IMPLÍCITAMENT: $(\alpha, \beta) \longrightarrow (x, y)$

Per a les VELOCITATS:

$$\boxed{\dot{\beta} = -\frac{\alpha}{\beta}\dot{\alpha}}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\begin{pmatrix} 2x & 4y \\ \beta & -\alpha \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ -y & x \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} \Rightarrow \boxed{\begin{matrix} \dot{x} = \frac{-2y}{x\alpha+2y\beta}(-y\dot{\alpha} + x\dot{\beta}) \\ \dot{y} = \frac{x}{x\alpha+2y\beta}(-y\dot{\alpha} + x\dot{\beta}) \end{matrix}} \begin{matrix} = \frac{-2y}{x\alpha+2y\beta}\left(-y - \frac{\alpha x}{\beta}\right)\dot{\alpha} \\ = \frac{x}{x\alpha+2y\beta}\left(-y - \frac{\alpha x}{\beta}\right)\dot{\alpha} \end{matrix}$$

2. OMBRA D'UN MÒBIL SOBRE UNA PANTALLA

- La cinètica inversa en el punt $x = y = 1, \alpha = \beta = 2$:

$$\dot{\beta} = -\dot{\alpha}$$
$$\dot{x} = \frac{2}{3}\dot{\alpha}$$
$$\dot{y} = -\frac{1}{3}\dot{\alpha}$$

- Per a les CELERITATS en aquest punt:

$$v = \sqrt{\dot{\alpha}^2 + \dot{\beta}^2} = \sqrt{2}\,\dot{\alpha}$$

$$u = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\sqrt{5}}{3}\,\dot{\alpha}$$

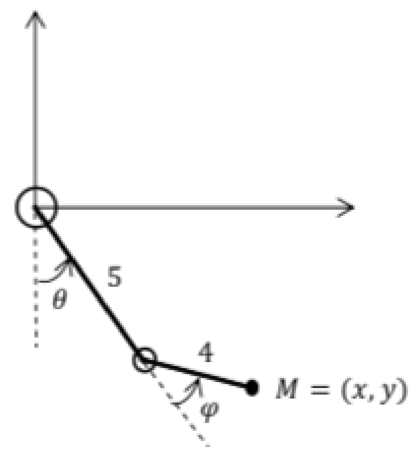
Per tant:

$$\boxed{u = \frac{\sqrt{5}}{3\sqrt{2}}\,v}$$

3. BRAÇ ARTICULAT

Suposem:

- BRAÇ ARTICULAT (figura)
- MOTORS de TORSIÓ a les articulacions
- (x, y): posició de la mà M
- (θ, φ): angles de les articulacions $(0 < \theta, \varphi < \pi)$



Es demana:

- estudiar les cinètiques directa i inversa
- en particular: suposant la mà $M = (5{,}4)$, calcular les velocitats $\dot{\theta}, \dot{\varphi}$

  per tal que M es MOGUI amb una velocitat $(\dot{x}, \dot{y})$ volguda

3. BRAÇ ARTICULAT

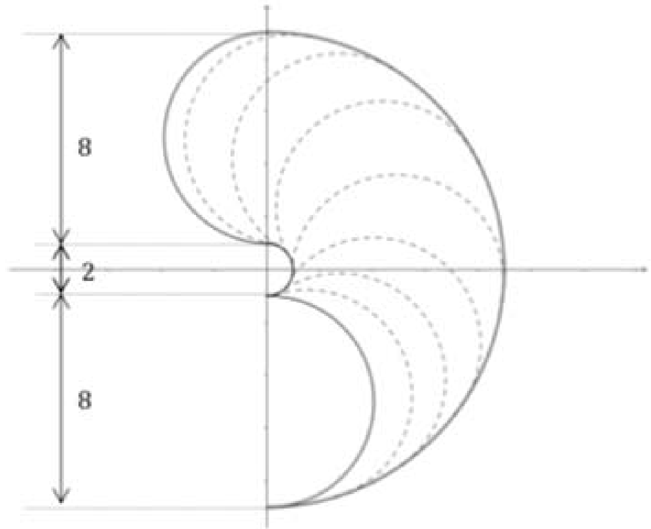
- CINÈTICA DIRECTA:

  Per a les posicions: $]0,\pi[\ \times\ ]0,\pi[\ \xrightarrow{f}\ \Omega$

$$(\theta,\varphi) \longrightarrow (x,y)$$

$$\begin{cases} x = 5\,\mathrm{sen}\,\theta + 4\,\mathrm{sen}(\theta+\varphi) \\ y = 5\cos\theta + 4\cos(\theta+\varphi) \end{cases}$$

Ω: vegem figura (la frontera és formada per semicircumferències amb diàmetre sobre l'eix vertical)



Per a les velocitats:

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = (Df)\begin{pmatrix}\dot{\theta}\\ \dot{\varphi}\end{pmatrix},\ Df = \begin{pmatrix} 5\cos\theta + 4\cos(\theta+\varphi) & 4\cos(\theta+\varphi) \\ -5\,\mathrm{sen}\,\theta - 4\,\mathrm{sen}(\theta+\varphi) & -4\,\mathrm{sen}(\theta+\varphi) \end{pmatrix}$$

3. BRAÇ ARTICULAT

- CINÈTICA INVERSA:

Per a les posicions:

Fixat $M = (x, y) \in \Omega \Longrightarrow$ només hi ha 2 triangles OCM amb $O = (0{,}0)$, $\overline{OC} = 5$, $\overline{CM} = 4$ (simètrics respecte a OM) i només 1 amb $\varphi > 0$

Per tant:

$$\begin{array}{ccc} \Omega & \xrightarrow{f^{-1}} &]0,\pi[\ \times \ ]0,\pi[\\ \downarrow & & \\ (x,y) & \longrightarrow & (\theta,\varphi) \end{array}$$

Per a la diferenciabilitat de f^{-1} :

$$detDf = -20 \cos\theta \ sen\ (\theta + \varphi) + 20\,\text{sen}\,\theta\ cos\ (\theta + \varphi) =$$
$$= -20\ sen\ (\theta - (\theta + \varphi)) = -20\ sen\ \varphi \neq 0$$

3. BRAÇ ARTICULAT

Per a les velocitats:

$$\begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix} = (Df)^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

En particular: $M = (5,4) \Leftrightarrow \theta = \frac{\pi}{2}, \varphi = \frac{\pi}{2}$

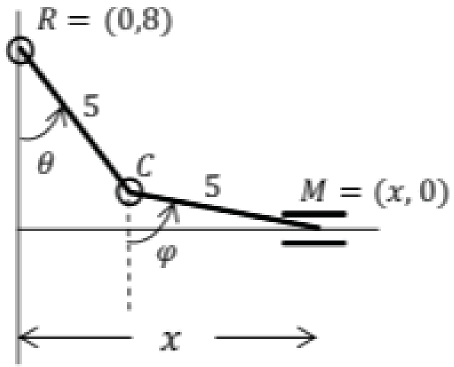
$$\begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ -5 & 0 \end{pmatrix}^{-1} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 0 & -4 \\ 5 & 4 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

$$\boxed{\begin{aligned} \dot{\theta} &= -\frac{1}{5}\dot{y} \\ \dot{\varphi} &= \frac{1}{4}\dot{x} + \frac{1}{5}\dot{y} \end{aligned}}$$

4. MECANISME XERRAC

Suposem:

- ROTOR alternant en $R = (0,8)$ (espatlla)
- fa oscil·lar l'articulació C (colze)
- i lliscar la CORREDERA M (mà)
- θ: angle del rotor
- x: abscissa de la corredissa



Es demanen les GRÀFIQUES de les:

- CINÈTICA DIRECTA: $\theta \longrightarrow x$
- CINÈTICA INVERSA: $x \longrightarrow \theta$

4. MECANISME XERRAC

- EQUACIONS del SISTEMA:

$$\begin{cases} 8 = 5\cos\theta + 5\cos\varphi \\ x = 5\,\text{sen}\,\theta + 5\,\text{sen}\,\varphi \end{cases}$$

- CINÈTICA DIRECTA:

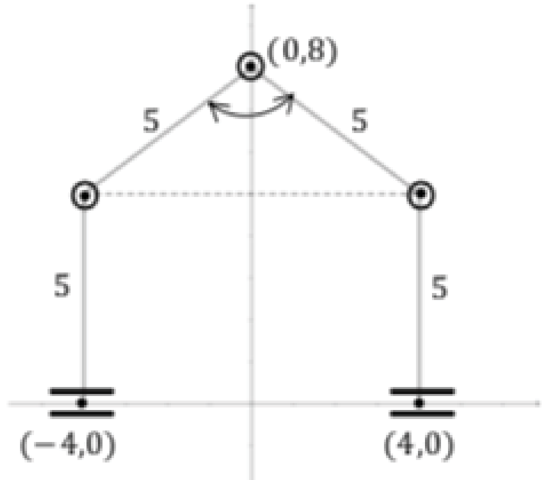
ordenada de $C \leq 5 \Rightarrow -arcos\,\frac{8-5}{5} \leq \theta \leq arcos\,\frac{8-5}{5}$

$$\det\begin{pmatrix} 5\,\text{sen}\,\varphi & 0 \\ -5\cos\varphi & 1 \end{pmatrix} = 5\,\text{sen}\,\varphi \neq 0 \quad si\ \varphi \neq 0, \pi \Rightarrow$$

$\Rightarrow$

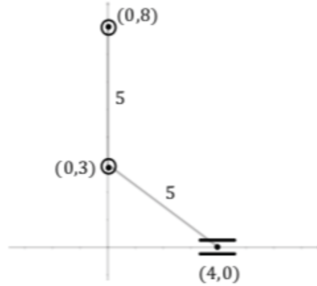
FUNCIÓ IMPLÍCITA $\theta \longrightarrow (x, \varphi)$

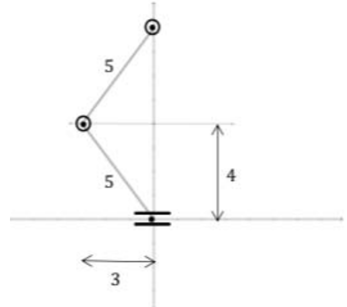
$-arcos\,\frac{3}{5} < \theta < arcos\,\frac{3}{5}$



4. MECANISME XERRAC

interseccions amb eixos $\begin{cases} x = 0 \Leftrightarrow \theta = -arcos\,\dfrac{4}{5} \\ \theta = 0 \Leftrightarrow x = 4 \end{cases}$

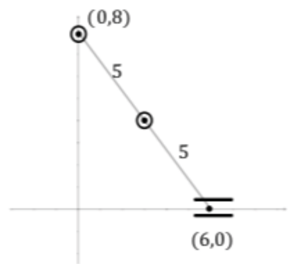




extrems: $\begin{cases} 5\,\mathrm{sen}\,\theta + 5\,\mathrm{sen}\,\varphi \quad \varphi' = 0 \\ x' - 5\cos\theta - 5\cos\varphi \quad \varphi' = 0 \end{cases}$

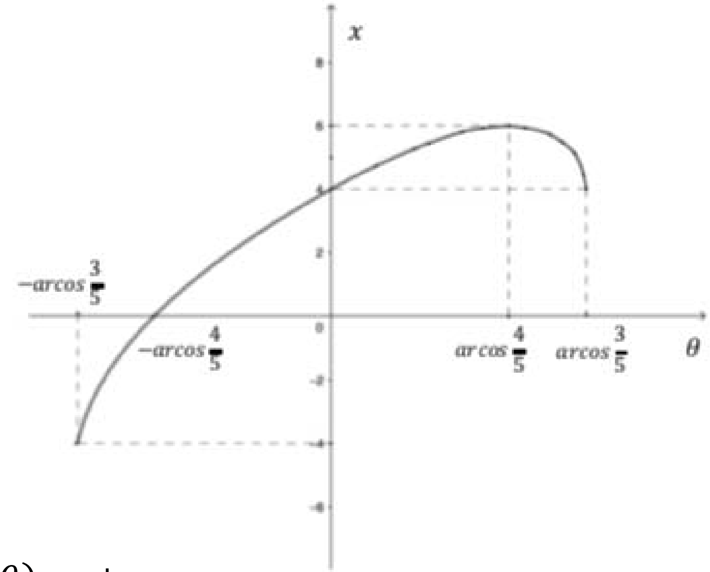
$$\begin{cases} \varphi' = -\dfrac{\mathrm{sen}\,\theta}{\cos\theta} \\ x' = 5\cos\theta + 5\cos\varphi\,\dfrac{-\mathrm{sen}\,\theta}{\mathrm{sen}\,\theta} = \dfrac{5}{\mathrm{sen}\,\varphi}\,sen(\theta - \varphi) \end{cases}$$

$$x'(\theta) = 0 \Leftrightarrow \theta = \varphi \Leftrightarrow \theta = arcos\,\frac{8}{10} \Rightarrow x = 6$$



4. MECANISME XERRAC

Per tant, la gràfica aproximada de $\theta \longrightarrow x$:



Observem que: $$\lim_{\theta \to -arcos\frac{3}{5}} x'(\theta) = +\infty$$

4. MECANISME XERRAC

- CINEMÀTICA INVERSA:

$$-6 \le x \le 6$$

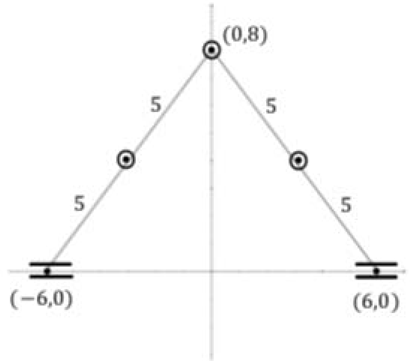
$$det\begin{pmatrix} 5sen\theta & 5sen\varphi \\ -5cos\theta & -5cos\varphi \end{pmatrix} =$$

$$= -25(sen\theta cos\varphi - cos\theta sen\varphi) =$$

$$= -25sen(\theta - \varphi) \neq 0 \; si \; \theta \neq \varphi \; (\Leftrightarrow x \neq \pm 6) \Rightarrow$$

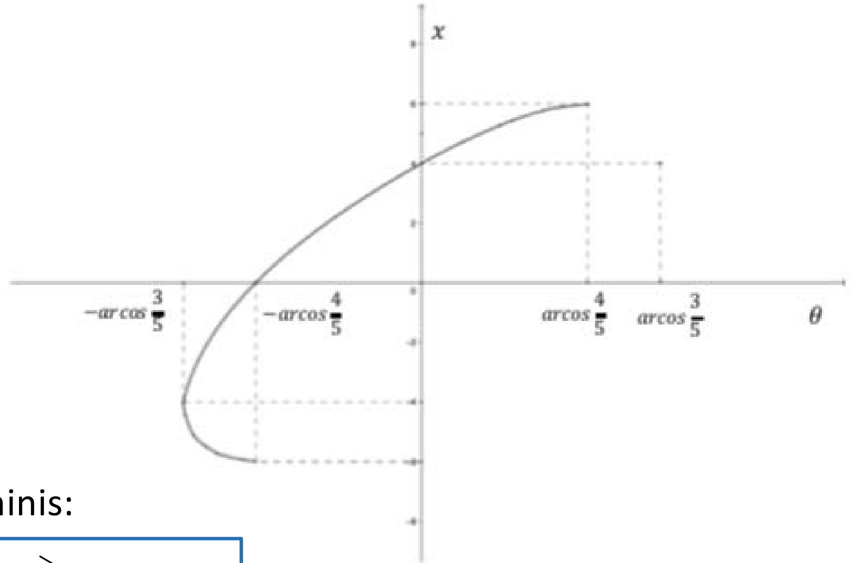
$\Rightarrow$ FUNCIÓ IMPLÍCITA $x \longrightarrow (\theta, \varphi)$

$$-6 < x < 6$$



4. MECANISME XERRAC

De forma anàloga, la gràfica aproximada $x \longrightarrow \theta, \quad -6 < x < 6$
És la simètrica de la de la figura:



- Observem que la BIJECTIVITAT es verifica si restringim els dominis:

$$\theta \rightleftarrows x$$

$$-arcos\frac{3}{5} < \theta < arcos\frac{4}{5} \qquad -4 < x < 6$$

FINAL SESSIÓ V

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA II: CÀLCUL MULTIVARIABLE

SESSIÓ VI. CINEMÀTICA DE MECANISMES AMB ENLLAÇOS

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

CINEMÀTICA DE MECANISMES AMB ENLLAÇOS

INDEX

1. INTRODUCCIÓ

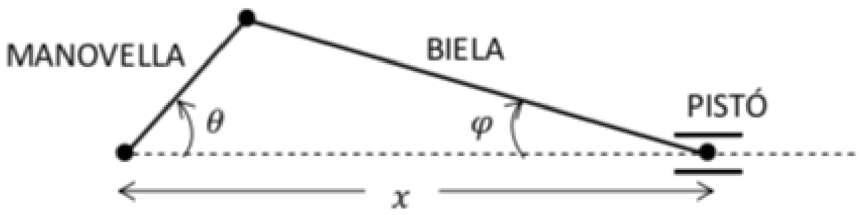
L'ANÀLISI CINEMÀTICA de MECANISMES amb ENLLAÇOS comporta:

- Quins paràmetres <u>GOVERNEN el MECANISME</u> ⟺
  - poden variar lliurament
  - determinen la resta

- Expressar en funció de velocitats i acceleracions dels paràmetres governants, les <u>VELOCITATS i ACCELERACIONS</u> de la resta

Una eina clau és el teorema de la FUNCIÓ IMPLÍCITA !!

2. BIELA / MANOVELLA



r: llargària de la manovella
L: llargària de la biela

$$\begin{cases} x = r\cos\theta + L\cos\varphi \\ r\,\mathrm{sen}\,\theta = L\,\mathrm{sen}\,\varphi \end{cases}$$

Ja vist: θ CONTROLA el mecanisme en totes les posicions.

Ara estudiem la velocitat i acceleració del pistó, suposant:

v = velocitat de gir del CIGONYAL (constant !)

En particular: per quin valor de θ s'assoleix la màxima velocitat del pistó??

2. BIELA / MANOVELLA

- $\theta \longrightarrow (x,\varphi) \Rightarrow \begin{cases} \dot{x} = x'\dot{\theta}\ ;\ \ddot{x} = x''(\dot{\theta})^2 + x'\ddot{\theta} \\ \dot{\varphi} = \varphi'\dot{\theta}\ ;\ \ddot{\varphi} = \varphi''(\dot{\theta})^2 + \varphi'\ddot{\theta} \end{cases}$

- $\dot{\theta} = v = \text{constant} \Rightarrow \begin{cases} \dot{x} = x'v;\ \ddot{x} = x''v^2 \\ \dot{\varphi} = \varphi'v;\ \ddot{\varphi} = \varphi''v^2 \end{cases}$

- $\begin{Bmatrix} x' = -r\,\text{sen}\,\theta - L\,\text{sen}\,\varphi\,\varphi' \\ r\cos\theta = L\cos\varphi\,\varphi' \end{Bmatrix} \Rightarrow \varphi' = \frac{r\cos\theta}{L\cos\varphi}\ ;\ x' = -r\,\text{sen}\,\theta - L\,\text{sen}\,\varphi\,\frac{r\cos\theta}{L\cos\varphi}$

$$\theta = 0 \Rightarrow \varphi = 0 \Rightarrow \varphi'(0) = \frac{r}{L}\ ; x'(0) = 0 \ \Rightarrow\ \boxed{\dot{x}(0) = 0}$$

$$\theta = \pi \Rightarrow \varphi = 0 \Rightarrow \varphi'(\pi) = -\frac{r}{L}\ ; x'(\pi) = 0 \ \Rightarrow\ \boxed{\dot{x}(\pi) = 0}$$

$$\theta = \frac{\pi}{2} \Rightarrow \text{sen}\,\varphi = \frac{r}{L} \Rightarrow \varphi'\left(\frac{\pi}{2}\right) = 0; x'\left(\frac{\pi}{2}\right) = -r \ \Rightarrow\ \boxed{\dot{x}\left(\frac{\pi}{2}\right) = -rv}$$

2. BIELA / MANOVELLA

- $\left\{\begin{array}{l} x'' = -r\cos\theta - L\cos\varphi\,(\varphi')^2 - L\,\mathrm{sen}\,\varphi\,\varphi'' \\ -r\,\mathrm{sen}\,\theta = -L\,\mathrm{sen}\,\varphi\,(\varphi')^2 + L\cos\varphi\,\varphi'' \end{array}\right\} \Rightarrow \varphi'' = \dfrac{-r\,\mathrm{sen}\,\theta + L\,\mathrm{sen}\,\varphi(\varphi')^2}{L\cos\varphi}$

$$x'' = -r\cos\theta - L\cos\varphi\,(\varphi')^2 - L\,\mathrm{sen}\,\varphi\,\varphi''$$

$$\theta = 0 \Rightarrow \varphi''(0) = 0\,;\ x''(0) = -r - L\left(-\frac{r}{L}\right)^2 \Rightarrow \boxed{\ddot{x}(0) = \left(-r - \frac{r^2}{L}\right)v^2} < 0$$

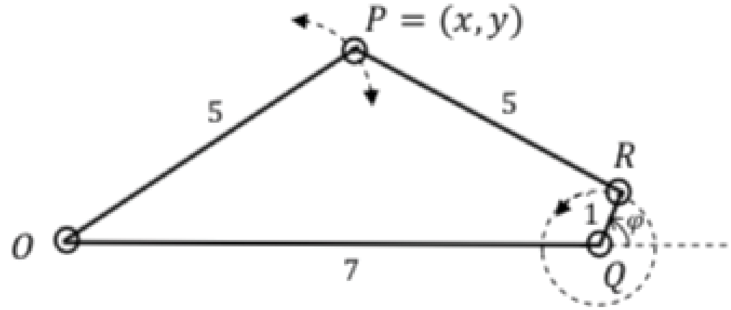
$$\theta = \pi \Rightarrow \varphi''(\pi) = 0\,;\ x''(\pi) = r - L\left(-\frac{r}{L}\right)^2 \Rightarrow \boxed{\ddot{x}(\pi) = \left(r - \frac{r^2}{L}\right)v^2} = \frac{r}{L}(L - r) > 0$$

$$\theta = \frac{\pi}{2} \Rightarrow \varphi''\left(\frac{\pi}{2}\right) = \frac{-r}{L\sqrt{1 - \frac{r^2}{L^2}}}\,;\ x''\left(\frac{\pi}{2}\right) = -L\frac{r}{L}\frac{-r}{\sqrt{L^2 - r^2}} \Rightarrow \boxed{\ddot{x}\left(\frac{\pi}{2}\right) = \frac{r^2}{\sqrt{L^2 - r^2}}v^2} > 0$$

- En particular existeix $\theta < \theta_M < \frac{\pi}{2}$, $\dot{x}(\theta_M) < 0$, $\boxed{\dot{x}(\theta_M)\text{ MÀXIMA}}$

3. QUADRILÀTER ARTICULAT

Suposem (figura)
- QUADRILÀTER ARTICULAT de costats: $5, 5, 7, 1$
- O, Q fixos
- R pot moure's per tota la circumferència: radi 1, centre 0



Pregunta: POSSIBLES POSICIONS de P?

P es mourà sobre un ARC (radi 5, centre 0): cal trobar-ne els EXTREMS

3. QUADRILÀTER ARTICULAT

- EQUACIONS DE LLIGAM:

$$\begin{cases} d(P,0) = 5 \\ d(P,R) = 5 \end{cases} \Rightarrow \begin{cases} 25 = x^2 + y^2 \\ 25 = \left(x - (7 + \cos\varphi)\right)^2 + (y - \operatorname{sen}\varphi)^2 = \end{cases}$$

$$= x^2 + 49 + cos^2\varphi - 14x - 2x\cos\varphi + 14\ \cos\varphi + y^2 + sen^2\varphi - 2y\operatorname{sen}\varphi =$$

$$= 25 + 49 + 1 - 14x - 2x\cos\varphi + 14\ \cos\varphi - 2y\operatorname{sen}\varphi$$

$$\Leftrightarrow \boxed{\begin{cases} x^2 + y^2 = 25 \\ x(7 + \cos\varphi) - 7\ \cos\varphi + y\operatorname{sen}\varphi = 25 \end{cases}}$$

3. QUADRILÀTER ARTICULAT

- Vegem que les posicions són DETERMINADES per φ:

$$det\begin{pmatrix} 2x & 2y \\ 7+\cos\varphi & \operatorname{sen}\varphi \end{pmatrix} = 0 \Leftrightarrow 7+\cos\varphi = \frac{x \operatorname{sen}\varphi}{y}$$

$$\left.\begin{array}{c} x^2+y^2=25 \\ x(7+\cos\varphi)-7\ \cos\varphi+y\operatorname{sen}\varphi=25 \\ 7+\cos\varphi=\frac{x\operatorname{sen}\varphi}{y} \end{array}\right\} \Rightarrow x^2\operatorname{sen}\varphi+y^2\operatorname{sen}\varphi=(7\cos\varphi+25)\,y \Rightarrow$$

$$\Rightarrow y=\frac{25\operatorname{sen}\varphi}{7\cos\varphi+25}\ ;\ x=(7+\cos\varphi)\,\frac{25}{7\cos\varphi+25}$$

$$\Rightarrow 25=\frac{25^2}{(7\cos\varphi+25)^2}\left((7+\cos\varphi)^2+sen^2\ \varphi\right)=$$

$$=\frac{25^2}{(7\cos\varphi+25)^2}(49+14\cos\varphi+1)=\frac{25^2\cdot 2}{7\cos\varphi+25}$$

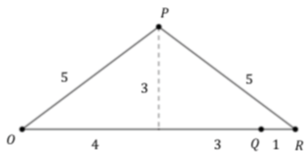
$$\Rightarrow \cos\varphi=\frac{1}{7}(50-25)>1\ \ !!$$

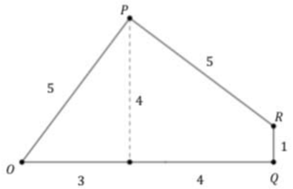
Per tant, existeix FUNCIÓ IMPLÍCITA DIFERENCIABLE: $\boxed{\varphi \longrightarrow (x,y)}$

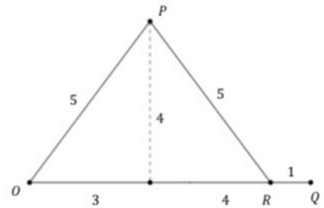
3. QUADRILÀTER ARTICULAT

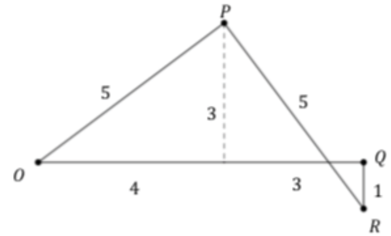
- Vegem algunes POSICIONS:

$\varphi = 0 \Rightarrow P = (4,3)$ $\varphi = \frac{\pi}{2} \Rightarrow P = (3,4)$ $\varphi = \pi \Rightarrow P = (3,4)$ $\varphi = \frac{3\pi}{2} \Rightarrow P = (4,3)$

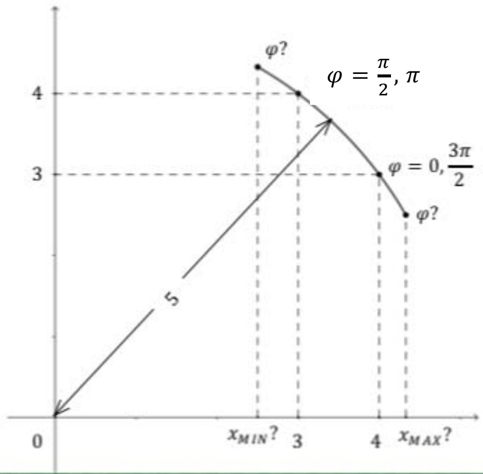








Per tant:



3. QUADRILÀTER ARTICULAT

- Per determinar els punts extrems de l'arc:

> Suposem $\dot{\varphi} = 1$
> aleshores: PUNTS EXTREMS de l'ARC $\Leftrightarrow \dot{x} = \dot{y} = 0$

Serà:

$$\dot{x} = x'\dot{\varphi} = x'; \dot{y} = y'\,\dot{\varphi} = y'$$

Per tant:

$$\dot{x} = \dot{y} = 0 \Leftrightarrow x' = y' = 0$$

derivant a les equacions de lligam:

$$\begin{cases} 2xx' + 2yy' = 0 \\ x'(7 + \cos\varphi) - x\ \operatorname{sen}\varphi + y'\operatorname{sen}\varphi + y\cos\varphi + 7\operatorname{sen}\varphi = 0 \end{cases}$$

$$x' = y' = 0 \Leftrightarrow -x\ \operatorname{sen}\varphi + y\cos\varphi + 7\operatorname{sen}\varphi = 0$$

En definitiva:

$$\text{PUNTS EXTREMS} \Leftrightarrow \begin{cases} x^2 + y^2 = 25 \\ 7x - (7 - x)\cos\varphi + y\operatorname{sen}\varphi = 25 \\ (7 - x)\operatorname{sen}\varphi + y\cos\varphi = 0 \end{cases}$$

3. QUADRILÀTER ARTICULAT

De la tercera: $\text{tg}\,\varphi = \frac{-y}{7-x} \Rightarrow \cos\varphi = \sqrt{\frac{1}{1+\frac{y^2}{(7-x)^2}}} = \frac{7-x}{\sqrt{(7-x)^2+y^2}}$; $\text{sen}\,\varphi = \frac{-y}{\sqrt{(7-x)^2+y^2}}$

Substituint a la segona:

$$7x - (7-x)\frac{7-x}{\sqrt{(7-x)^2+y^2}} + y\frac{-y}{\sqrt{(7-x)^2+y^2}} = 25 \Leftrightarrow 7x - 25 = \frac{(7-x)^2+y^2}{\sqrt{(7-x)^2+y^2}} = \sqrt{(7-x)^2+y^2}$$

$$\Leftrightarrow 49x^2 - 350x + 25^2 = 49 - 14x + x^2 + y^2$$

Emprant la primera: $0 = 49x^2 - 336x + 25^2 - 49 - 25 = 49x^2 - 336x + 551 \Leftrightarrow$

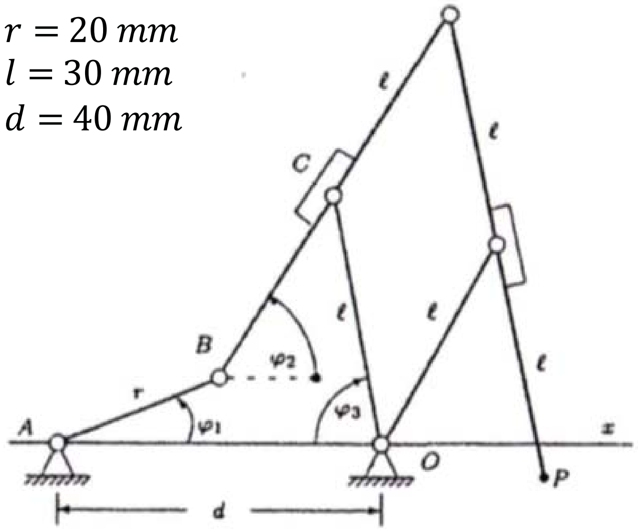
$$\Leftrightarrow 7x^2 - 48x + \frac{551}{7} = 0 \Leftrightarrow x = \frac{48 \pm \sqrt{48^2 - 4\cdot 551}}{14} = \frac{48 \pm 10}{14} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \boxed{x = {}^{29}/_{7}} \cong 4'14\,,\ \boxed{y = {}^{8\sqrt{6}}/_{7}} \cong 2'78 \\ \boxed{x = {}^{19}/_{7}} \cong 2'71\,,\ \boxed{y = {}^{12\sqrt{6}}/_{7}} \cong 4'2 \end{cases}$$

4. ANÀLISI CINEMÀTICA D'UN MECANISME AMB ENLLAÇOS*

Considerem el mecanisme amb enllaços de la figura

$r = 20\,mm$
$l = 30\,mm$
$d = 40\,mm$



Podem suposar $r < l < d < r + l$

1. Obtingueu les dues equacions d'enllaç geomètriques entre els angles φ_1, φ_2 i φ_3.
2. Estudieu si φ_1 governa el mecanisme diferenciablement (és a dir, si φ_2 , φ_3 queden determinats per φ_1 i en depenen diferenciablement).
3. Expresseu les velocitats $\dot{\varphi}_2$ i $\dot{\varphi}_3$ en funció de la $\dot{\varphi}_1$.
4. Expresseu les acceleracions $\ddot{\varphi}_2$ i $\ddot{\varphi}_3$ en funció de la $\ddot{\varphi}_1$, i de les velocitats $\dot{\varphi}_1$, $\dot{\varphi}_2$ i $\dot{\varphi}_3$.

(*) S. Cardona, D. Clos: Teoria de màquines, Ed. UPC

4. ANÀLISI CINEMÀTICA D'UN MECANISME AMB ENLLAÇOS

- EQUACIONS D'ENLLAÇ:

$$\begin{cases} r\cos\varphi_1 + l\cos\varphi_2 + l\cos\varphi_3 = d \\ r\,\mathrm{sen}\,\varphi_1 + l\,\mathrm{sen}\,\varphi_2 = l\,\mathrm{sen}\,\varphi_3 \end{cases}$$

- $det\begin{pmatrix} -l\,\mathrm{sen}\,\varphi_2 & -l\mathrm{sen}\,\varphi_3 \\ l\cos\varphi_2 & -l\cos\varphi_3 \end{pmatrix} = l^2(\mathrm{sen}\,\varphi_2\cos\varphi_3 + \cos\varphi_2\,\mathrm{sen}\,\varphi_3) = l^2\,\mathrm{sen}(\varphi_2+\varphi_3) \neq 0$

ja que
- $\varphi_2 + \varphi_3 = 0 \Leftrightarrow C = \text{punt mig de } BO$
- $\varphi_2 + \varphi_3 = \pi \Leftrightarrow B = 0$

Per tant: $\varphi_1 \longrightarrow (\varphi_2, \varphi_3)$ FUNCIÓ IMPLÍCITA DIFERENCIABLE

4. ANÀLISI CINEMÀTICA D'UN MECANISME AMB ENLLAÇOS

- DERIVANT:

$$\begin{cases} -r \operatorname{sen} \varphi_1 \dot{\varphi}_1 - l \operatorname{sen} \varphi_2 \dot{\varphi}_2 - l \operatorname{sen} \varphi_3 \dot{\varphi}_3 = 0 \\ r \cos \varphi_1 \dot{\varphi}_1 + l \cos \varphi_2 \dot{\varphi}_2 - l \cos \varphi_3 \dot{\varphi}_3 = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} \dot{\varphi}_2 \\ \dot{\varphi}_3 \end{pmatrix} = -\begin{pmatrix} -l \operatorname{sen} \varphi_2 & -l \operatorname{sen} \varphi_3 \\ l \cos \varphi_2 & -l \cos \varphi_3 \end{pmatrix}^{-1} \begin{pmatrix} -r \operatorname{sen} \varphi_1 \\ r \cos \varphi_1 \end{pmatrix} \dot{\varphi}_1 =$$

$$= -\frac{1}{l^2(\operatorname{sen} \varphi_2 \cos \varphi_3 + \cos \varphi_2 \operatorname{sen} \varphi_3)} \begin{pmatrix} -l \cos \varphi_3 & l \operatorname{sen} \varphi_3 \\ -l \cos \varphi_2 & -l \operatorname{sen} \varphi_2 \end{pmatrix} = \begin{pmatrix} -r \operatorname{sen} \varphi_1 \\ r \cos \varphi_1 \end{pmatrix} \dot{\varphi}_1 =$$

$$= -\frac{lr}{l^2 \operatorname{sen}(\varphi_2 + \varphi_3)} \begin{pmatrix} -\cos \varphi_3 \operatorname{sen} \varphi_1 + \operatorname{sen} \varphi_3 \cos \varphi_1 \\ -\cos \varphi_2 \operatorname{sen} \varphi_1 + \operatorname{sen} \varphi_2 \cos \varphi_1 \end{pmatrix} \dot{\varphi}_1 \Rightarrow$$

$$\Rightarrow \boxed{\begin{aligned} \dot{\varphi}_2 &= \frac{r}{l} \frac{sen(\varphi_3 - \varphi_1)}{sen(\varphi_2 + \varphi_3)} \dot{\varphi}_1 \\ \dot{\varphi}_3 &= \frac{r}{l} \frac{sen(\varphi_2 - \varphi_1)}{sen(\varphi_2 + \varphi_3)} \dot{\varphi}_1 \end{aligned}}$$

4. ANÀLISI CINEMÀTICA D'UN MECANISME AMB ENLLAÇOS

- DERIVANT de NOU:

$$\begin{cases} -r\cos\varphi_1\,(\dot\varphi_1)^2 - r\,\mathrm{sen}\,\varphi_1\ddot\varphi_1 - l\cos\varphi_2(\dot\varphi_2)^2 - l\,\mathrm{sen}\,\varphi_2\ddot\varphi_2 - l\cos\varphi_3(\dot\varphi_3)^2 - l\,\mathrm{sen}\,\varphi_3\ddot\varphi_3 = 0 \\ -r\,\mathrm{sen}\,\varphi_1\,(\dot\varphi_1)^2 + r\cos\varphi_1\ddot\varphi_1 - l\,\mathrm{sen}\,\varphi_2(\dot\varphi_2)^2 + l\cos\varphi_2\ddot\varphi_2 + l\,\mathrm{sen}\,\varphi_3(\dot\varphi_3)^2 - l\cos\varphi_3\ddot\varphi_3 = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{pmatrix} -r\cos\varphi_1 & -l\cos\varphi_2 & -l\cos\varphi_3 \\ -r\,\mathrm{sen}\,\varphi_1 & -l\,\mathrm{sen}\,\varphi_2 & l\,\mathrm{sen}\,\varphi_3 \end{pmatrix}\begin{pmatrix}(\dot\varphi_1)^2\\(\dot\varphi_2)^2\\(\dot\varphi_3)^2\end{pmatrix} + \begin{pmatrix}-r\,\mathrm{sen}\,\varphi_1\\ r\cos\varphi_1\end{pmatrix}\ddot\varphi_1 + \begin{pmatrix}-l\,\mathrm{sen}\,\varphi_2 & -l\,\mathrm{sen}\,\varphi_3\\ l\cos\varphi_2 & -l\cos\varphi_3\end{pmatrix}\begin{pmatrix}\ddot\varphi_2\\ \ddot\varphi_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}\ddot\varphi_2\\ \ddot\varphi_3\end{pmatrix} = -\begin{pmatrix}-l\,\mathrm{sen}\,\varphi_2 & -l\,\mathrm{sen}\,\varphi_3\\ l\cos\varphi_2 & -l\cos\varphi_3\end{pmatrix}^{-1}\left[\begin{pmatrix} -r\cos\varphi_1 & -l\cos\varphi_2 & -l\cos\varphi_3 \\ -r\,\mathrm{sen}\,\varphi_1 & -l\,\mathrm{sen}\,\varphi_2 & l\,\mathrm{sen}\,\varphi_3 \end{pmatrix}\begin{pmatrix}(\dot\varphi_1)^2\\(\dot\varphi_2)^2\\(\dot\varphi_3)^2\end{pmatrix} + \begin{pmatrix}-r\,\mathrm{sen}\,\varphi_1\\ r\cos\varphi_1\end{pmatrix}\ddot\varphi_1\right] =$$

$$= \frac{1}{l sen(\varphi_2+\varphi_3)}\begin{pmatrix}\cos\varphi_3 & -\mathrm{sen}\,\varphi_3\\ \cos\varphi_2 & \mathrm{sen}\,\varphi_2\end{pmatrix}\left[\left[-r\begin{pmatrix}\cos\varphi_1\\ \mathrm{sen}\,\varphi_1\end{pmatrix} - \frac{r^2}{l}\frac{1}{sen^2(\varphi_2+\varphi_3)}\begin{pmatrix}\cos\varphi_2 & cos\,\varphi_3\\ \mathrm{sen}\,\varphi_2 & -\mathrm{sen}\,\varphi_3\end{pmatrix}\begin{pmatrix}sen^2(\varphi_3-\varphi_1)\\ sen^2(\varphi_2-\varphi_1)\end{pmatrix}\right](\dot\varphi_1)^2 - r\begin{pmatrix}\mathrm{sen}\,\varphi_1\\ -\cos\varphi_1\end{pmatrix}\ddot\varphi_1\right] = \cdots$$

FINAL SESSIÓ VI

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA II: CÀLCUL MULTIVARIABLE

SESSIÓ VII. OPTIMITZACIÓ

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

OPTIMITZACIÓ

INDEX

1. INTRODUCCIÓ

OPTIMITZAR (costos, residus, ...) és un problema transversal de molts àmbits de l'enginyeria

Es presenten:

- diversos tipus de plantejament
- diverses tècniques de resolució

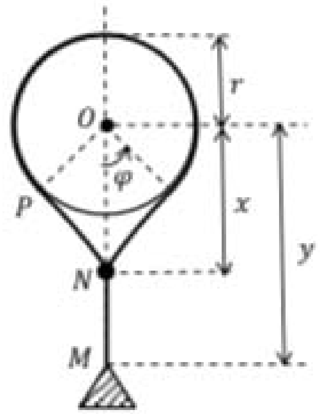
El punt de partida és el teorema que assegura que:

Tota funció contínua en un compacte té màxim i mínim (absoluts i assolibles)

Això ens estalviarà càlculs i verificacions

2. NUS CORREDÍS

Suposem (figura):
- BIGA cilíndrica de radi r i centre O
- CORDA de llargària L
- PES lligat a una de les punes M
- NUS CORREDÍS a l'altra punta N



Es pregunta:
- POSICIÓ del nus (x)
- POSICIÓ del pes (y)

Diguem:
- $P =$ punt de contacte de la corda amb la biga
- $\varphi =$ angle de OP amb la vertical

Es tracta de: **determinar φ per tal que y sigui MÀXIM**

2. NUS CORREDÍS

Trobem els zeros de $y'(\varphi)$:

$$x = \frac{r}{cos\varphi}$$

$$y = x + (L - 2r(\pi - \varphi) - 2r\, tg\varphi)$$

$$y' = \frac{rsen\varphi}{cos^2\varphi} - 2r - 2r\frac{1}{cos^2\varphi} = \frac{r}{cos^2\varphi}(sen\varphi - 2cos^2\varphi - 2) = \frac{r}{cos^2\varphi}(sen\varphi - 2sen^2\varphi)$$

$$y' = 0 \Leftrightarrow \begin{cases} sen\varphi = 0 \Leftrightarrow \varphi = 0 \\ sen\varphi = \frac{1}{2} \Leftrightarrow \varphi = \frac{\pi}{6} \end{cases}$$

Clarament el MÀXIM és per $\varphi = \frac{\pi}{6}$(també: $y''\left(\frac{\pi}{6}\right) = \cdots < 0$)

$$\varphi_{MAX} = \frac{\pi}{6};\ x_{MAX} = \frac{2r}{\sqrt{3}};\ y_{MAX} = \frac{2r}{\sqrt{3}} + L - 2r\frac{5\pi}{6} - 2r\frac{1}{\sqrt{3}} = L - \frac{5}{3}\pi r \cong L - 5'23r$$

Observem que:

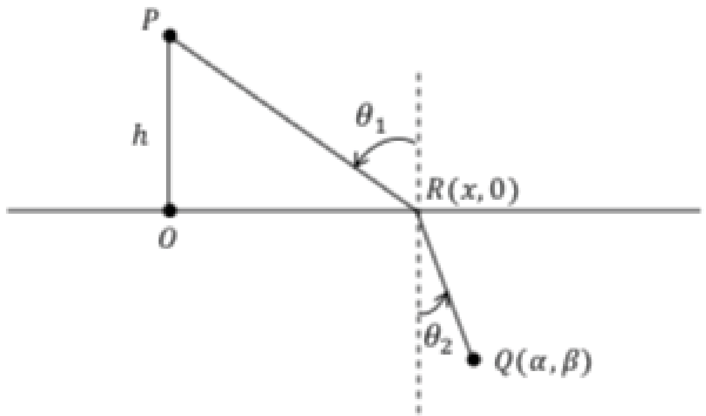
$$\varphi = 0 \Rightarrow x(0) = r; y(0) = r + L - 2\pi r = L - (2\pi - 1)r \cong L - 5'28r$$

$$\varphi = \frac{\pi}{3} \Rightarrow x\left(\frac{\pi}{3}\right) = 2r; y\left(\frac{\pi}{3}\right) = 2r + L - 2r\frac{2\pi}{3} - 2r\sqrt{3} = L + \left(1 - \frac{2\pi}{3} - \sqrt{3}\right)2r \cong L - 5'6r$$

3. LLEI DE SNELL

La llei de Snell relaciona els ANGLES d'INCIDÈNCIA i de REFRACCIÓ d'un raig de llum en travessar la interfície de separació de dos medis.

Suposem
- raig de llum sortint de $P = (0, h)$
- i arribant a $Q = (\alpha, \beta)$



Es tracta
- De MINIMITZAR el temps del trajecte en funció de les VELOCITATS c_1, c_2 de la llum en els medis respectius

3. LLEI DE SNELL

$$t = \frac{d(P,R)}{c_1} + \frac{d(R,Q)}{c_2} = \frac{\sqrt{x^2+h^2}}{c_1} + \frac{\sqrt{(\alpha-x)^2+\beta^2}}{c_2}$$

$$Dt = \frac{1}{c_1}\frac{x}{\sqrt{x^2+h^2}} + \frac{1}{c_2}\frac{-(\alpha-x)}{\sqrt{(\alpha-x)^2+\beta^2}} = 0 \Leftrightarrow \frac{1}{c_1}\frac{x}{\sqrt{x^2+h^2}} = \frac{1}{c_2}\frac{\alpha-x}{\sqrt{(\alpha-x)^2+\beta^2}} \Leftrightarrow \boxed{\frac{sen\theta_1}{c_1} = \frac{sen\theta_2}{c_2}}$$

En particular $\begin{cases} c_1 = 0'4c\,;\, c_2 = 0'2c \\ h = 1 \\ \alpha = 3\,;\, \beta = -2 \end{cases}$

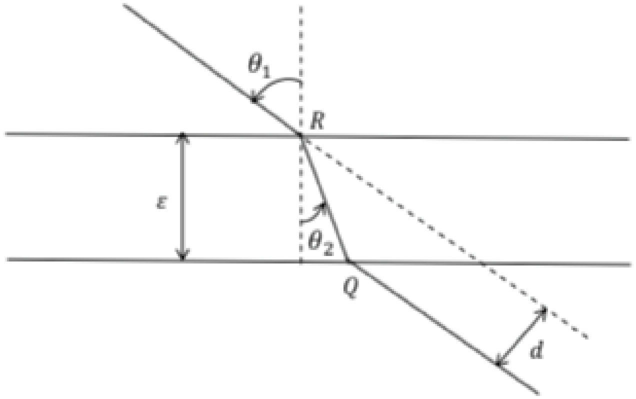
$$\frac{1}{0'4c}\frac{x}{\sqrt{x^2+1}} = \frac{1}{0'2c}\frac{3-x}{\sqrt{(3-x)^2+4}} \Leftrightarrow x\sqrt{(3-x)^2+4} = 2(3-x)\sqrt{x^2+1} \Leftrightarrow$$

$$\Leftrightarrow x^2\,(13-6x+x^2) = 4(9-6x+x^2)(x^2+1) = 36x^2+36-24x^3-24x+4x^4+4x^2 \Leftrightarrow$$

$$\Leftrightarrow x^4 - 6x^3 + 9x^2 - 8x + 12 = 0 \Rightarrow \boxed{x = 2}$$ (única arrel amb $\frac{3}{2} \leq x \leq 3$)

3. LLEI DE SNELL

Observem que si travessa una PLACA d'ESPESSOR ε el raig emergent és PARAL·LEL a l'incident a una DISTÀNCIA d:



$$d(R,Q) = \frac{d}{sen(\theta_1 - \theta_2)} = \frac{\varepsilon}{cos\theta_2} \Rightarrow$$

$$\Rightarrow \boxed{d = \varepsilon \frac{sen(\theta_1 - \theta_2)}{cos\theta_2}} = \varepsilon \left(sen\theta_1 - \frac{cos\theta_1}{cos\theta_2} sen\theta_2 \right) = \varepsilon sen\theta_1 \left(1 - \frac{c_2}{c_1} \frac{cos\theta_1}{cos\theta_2} \right)$$

4. ALGORISME DEL GRADIENT

Considerem
- la xarxa de NOMBRES ENTERS
- la funció $f(x, y) = xy(x-4)(y-5)$

Emprarem un algorisme del gradient simplificat per trobar els nusos de la xarxa en els quals la funció $f(x, y)$ assoleix un MÀXIM relatiu en el RECTANGLE de vèrtex $(0,0)$, $(4,0)$, $(0,5)$, $(4,5)$.

Observem que $f(x, y)$ n'anul·la en la frontera del rectangle.

4. ALGORISME DEL GRADIENT

$$gradf = (y(x-4)(y-5) + xy(y-5), x(x-4)(y-5) + xy(x-4)) =$$

$$= \left(y(y-5) + (2x-4), x(x-4)(2y-5)\right)$$

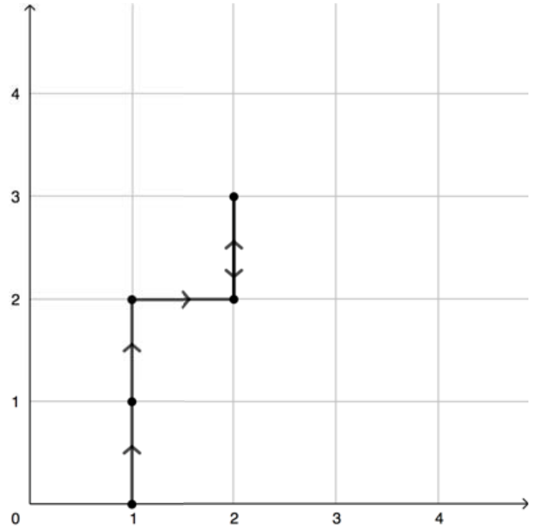
$nus(1{,}0)$: $f(1{,}0) = 0; gradf(1{,}0) = (0{,}15)$ ($\Rightarrow$ PUJAR)

$nus(1{,}1)$: $f(1{,}1) = 12; gradf(1{,}1) = (8{,}9)$ ($\Rightarrow$ PUJAR)

$nus(1{,}2)$: $f(1{,}2) = 18; gradf(1{,}2) = (12{,}3)$ ($\Rightarrow$ DRETA)

$nus(2{,}2)$: $f(2{,}2) = 24; gradf(2{,}2) = (0{,}4)$ ($\Rightarrow$ PUJAR)

$nus(2{,}3)$: $f(2{,}3) = 24; gradf(2{,}3) = (0,-4)$ ($\Rightarrow$ BAIXAR)



Per tant, assoleix el màxim 24 en els NUSSOS (2,2) i (2,3)

De fet, el màxim és 25, en el punt $\left(2, \frac{5}{2}\right)$

5. DESPATX ELÈCTRIC

Problema del “despatx elèctric”:

- ASSIGNAR les produccions $P_1, \cdots, P_n$ a les centrals elèctriques
- per tal que la POTÈNCIA TOTAL P sigui la requerida per la xarxa
- amb el MÍNIM COST de PRODUCCIÓ
- se suposa que el COST de funcionament de cada central:

$$C_j = \alpha_j + \beta_j\ P_j + \gamma_j P_j^2,\ 1 \leq j \leq n\ , \alpha_j, \beta_j, \gamma_j > 0$$

Es demostra que el problema té SOLUCIÓ ÚNICA: $P_1^*, \ldots, P_n^*$: potències assignades

Un element clau és: QUINES CENTRALS CAL ATURAR en funció del requeriment P

Aquí ho resoldrem per a 3 centrals, amb:

$$C_1 = 7 + P_1 + P_1^2$$
$$C_2 = 4 + 2P_2 + 2P_2^2$$
$$C_3 = 2 + 4P_3 + 3P_3^2$$

$$P = P_1 + P_2 + P_3$$

5. DESPATX ELÈCTRIC

Es tracta de MINIMITZAR:

$$C = (7 + P_1 + P_1^2) + (4 + 2P_2 + 2P_2^2) + (2 + 4(P - P_1 - P_2) + 3(P - P_1 - P_2)^2) =$$

$$= 13 + 4P - 3P_1 - 2P_2 + P_1^2 + 2P_2^2 + 3(P - P_1 - P_2)^2$$

$$\begin{array}{l} D_1C = -3 + 2P_1 - 6(P - P_1 - P_2) = -3 - 6P + 8P_1 + 6P_2 = 0 \\ D_2C = -2 + 4P_2 - 6(P - P_1 - P_2) = -2 - 6P + 6P_1 + 10P_2 = 0 \end{array} \Leftrightarrow$$

$$\Leftrightarrow \boxed{\begin{array}{l} P_1^* = \frac{1}{22}(9 + 12P) \\ P_2^* = \frac{1}{22}(-1 + 6P) \end{array} \Rightarrow P_3^* = \frac{1}{22}(-8 + 4P)}$$

Tanmateix: VÀLIDES $\Leftrightarrow P_1^*, P_2^*, P_3^* \geq 0 \Leftrightarrow \begin{array}{c} P \geq 1/6 \\ P \geq 2 \end{array} \Leftrightarrow \boxed{P \geq 2}$

5. DESPATX ELÈCTRIC

Si P baixa: $\boxed{P = 2 \Rightarrow \text{cal ATURAR la 3ª central}}$ ‼

Aleshores:

$$C = 7 + P_1 + P_1^2 + 4 + 2(P - P_1) + 2(P - P_1)^2$$

$$DC = 1 + 2P_1 - 2 - 4(P - P_1) = -1 - 4P + 6P_1 = 0 \Leftrightarrow$$

$$\Leftrightarrow \boxed{\begin{array}{ll} P_1^* = \frac{1}{6}(1 + 4P) & \\ P_2^* = \frac{1}{6}(-1 + 2P) & \text{si } \frac{1}{2} \leq P \leq 2 \\ P_3^* = 0 & \end{array}}$$

Si P baixa més: $\boxed{P = \frac{1}{2} \Rightarrow \text{cal ATURAR la 2ª central}}$ ‼

Aleshores: $\boxed{P_1^* = P,\ P_2^* = P_3^* = 0 \text{ si } P \leq \frac{1}{2}}$

6. UBICACIÓ

Problema
- UBICACIÓ d'una escola E
- per abastir 3 POBLES A, B, C
- atenent al NOMBRE d'ESCOLARS de cadascuna n_A, n_B, n_C

Un possible model és minimitzar una "FUNCIÓ COST":

$$f(x,y) = n_A d\big(A,(x,y)\big) + n_B d\big(B,(x,y)\big) + n_C d\big(C,(x,y)\big)$$

6. UBICACIÓ

Suposem, per exemple
$$\begin{cases} A = (0,0); n_A = 50 \\ B = (b,0); n_B = 70 \\ C = (c,d); n_C = 90 \end{cases}$$

Denotant $E = (x,y)$:

$$f(x,y) = 50\sqrt{x^2 + y^2} + 70\sqrt{(b-x)^2 + y^2} + 90\sqrt{(c-x)^2 + (d-y)^2}$$

$$\begin{cases} D_x f = 50\dfrac{x}{\sqrt{x^2+y^2}} + 70\dfrac{-(b-x)}{\sqrt{(b-x)^2+y^2}} + 90\dfrac{-(c-x)}{\sqrt{(c-x)^2+(d-y)^2}} = 0 \\ D_y f = 50\dfrac{y}{\sqrt{x^2+y^2}} + 70\dfrac{y}{\sqrt{(b-x)^2+y^2}} + 90\dfrac{-(d-y)}{\sqrt{(c-x)^2+(d-y)^2}} = 0 \end{cases}$$

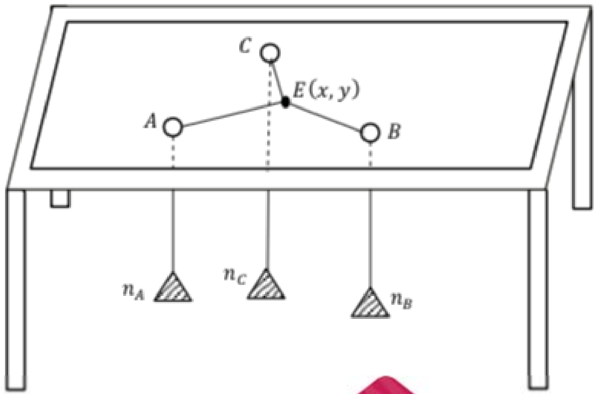
de resolució no pas fàcil

6. UBICACIÓ

Una resolució mecànica (vegeu figura)

- una TAULA amb forats en els punts A, B, C;
- 3 CORDES de llargàries L_A, L_B, L_C (qualssevol)
- LLIGADES per un dels extrems en un NUS E;
- l'altre EXTREM passant pels forats A, B, C
- i penjant-hi un PES n_A, n_B, n_C

Aleshores: UBICACIÓ ÒPTIMA = posició del nus E (quan s'estabilitza)



6. UBICACIÓ

En efecte, si l'alçària de la taula és h $(> L_A, L_B, L_C)$, l'ENERGIA POTENCIAL del sistema per a la posició $E = (x, y)$ del NUS és:

$$V(x,y) = n_A\big(h - (L_A - d(A,E))\big) + n_B\big(h - (L_B - d(B,E))\big) + n_C\big(h - (L_C - d(C,E))\big) =$$

$$= f(x,y) + h(n_A + n_B + n_C) - n_A\, L_A - n_B\, L_B - n_C\, L_C = f(x,y) + \text{constant}$$

Per tant: MÍNIM de $V(x,y)$ = MÍNIM de $f(x,y)$

FINAL SESSIÓ VII

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH

ETSEIB

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA II: CÀLCUL MULTIVARIABLE

SESSIÓ VIII. MISCEL·LÀNIA

PROFESSORAT: JOSEP FERRER
MARTA PEÑA

MISCEL·LÀNIA

INDEX

1. INTRODUCCIÓ

Es presenta una versió simplificada dels problemes dels anys 2001 i 2002 del

CONCURS MATEMÀTIC del FÒRUM E.T.S.E.I.B.

Destaquem que consideracions GEOMÈTRIQUES permeten simplificar càlculs.

En aquest sentit, acabem amb un exercici en el qual la VISIÓ SINTÈTICA supera el raonament analític.

Molt important a l'ENGINYERIA !

2. EL PENJOLL

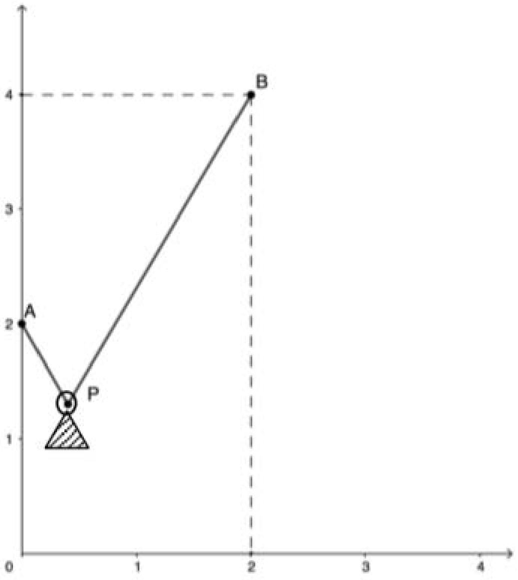
Considerem
- un PENJOLL P penja de
- una CORDA de llargària 4, amb extrems A i B:
- $A = (0,2)$ és fix
- diferents restriccions per B

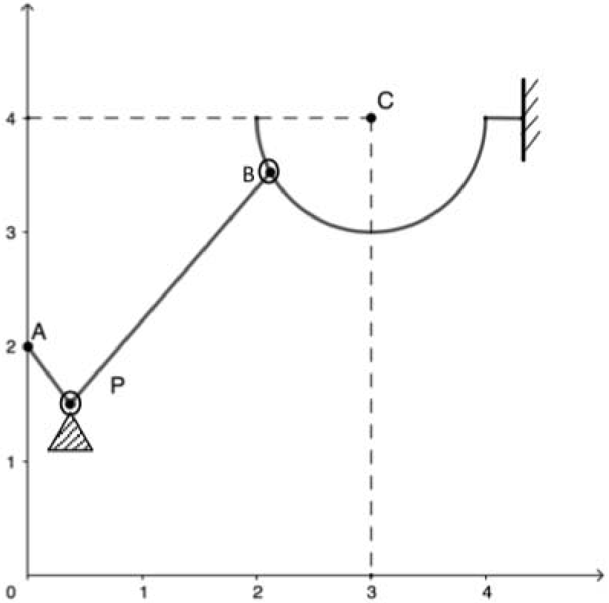
En qualsevol cas: POSICIÓ d'EQUILIBRI del PENJOLL $\Leftrightarrow$ MÍNIMA ALÇÀRIA possible

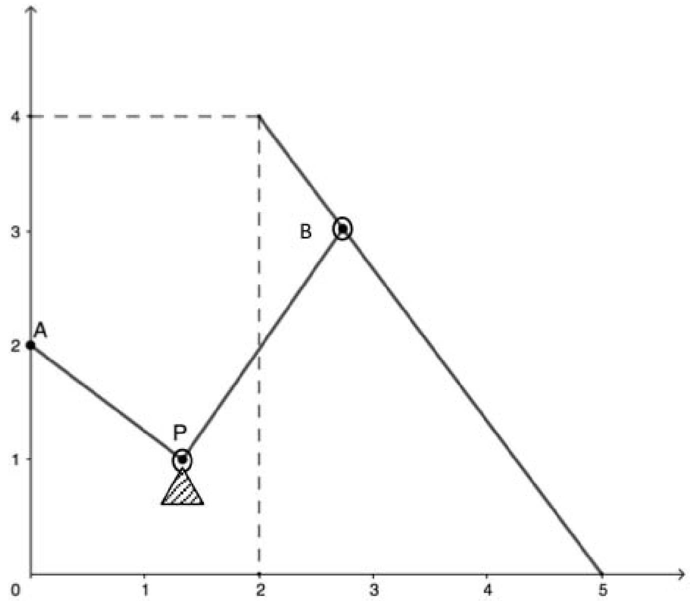
Determineu-la per als casos següents (vegeu figures):

(1) $B = (2,4)$
(2) B pot lliscar per un GANXO SEMICIRCULAR de radi 1 i centre $C = (3,4)$
(3) B pot lliscar per una BARRA rectilínia d'extrems $(2,4)$ i $(5,0)$
(4) Cas general per B lliscant per un GANXO qualsevol

2. EL PENJOLL







2. EL PENJOLL

(1) S'ha de verificar:

$$4 = d(A,P) + d(B,P)$$

Si diem $P = (x,y)$, cal MINIMITZAR $y(x)$ amb

$$4 = \sqrt{x^2 + (2-y)^2} + \sqrt{(2-x)^2 + (4-y)^2}$$

Derivant:

$$0 = \frac{2x-2(2-y)y'}{2\sqrt{x^2+(2-y)^2}} + \frac{-2(2-x)-2(4-y)y'}{\sqrt{(2-x)^2+(4-y)^2}}$$

$$\boxed{y' = 0} \Leftrightarrow \frac{x}{2\sqrt{x^2+(2-y)^2}} = \frac{2-x}{\sqrt{(2-x)^2+(4-y)^2}} \Leftrightarrow \frac{1}{\sqrt{1+\left(\frac{2-y}{x}\right)^2}} = \frac{1}{\sqrt{1+\left(\frac{4-y}{2-x}\right)^2}} \Leftrightarrow \boxed{\frac{2-y}{x} = \frac{4-y}{2-x}} \Leftrightarrow$$

$$\Leftrightarrow 4 - 2x - 2y + xy = 4x - yx \Leftrightarrow x = \frac{-y+2}{-y+3} \Leftrightarrow 2 - x = \frac{-y+4}{-y+3}$$

2. EL PENJOLL

Substituint:

$$4 = \sqrt{\left(\frac{2-y}{3-y}\right)^2 + (2-y)^2} + \sqrt{\left(\frac{4-y}{3-y}\right)^2 + (4-y)^2} = (2-y)\sqrt{\frac{1}{(3-y)^2}+1} + (4-y)\sqrt{\frac{1}{(3-y)^2}+1} \Leftrightarrow$$

$$\Leftrightarrow 4^2 = (2-y+4-y)^2\left(\frac{1}{(3-y)^2}+1\right) = \frac{(6-2y)^2}{(3-y)^2}(1+(3-y)^2) = 2^2(y^2-6y+10) \Leftrightarrow$$

$$\Leftrightarrow 0 = y^2-6y+6 \Leftrightarrow y = \frac{6\pm\sqrt{36-24}}{2} = 3\pm\sqrt{3} \begin{cases} \cancel{3+\sqrt{3}} \\ 3-\sqrt{3} \end{cases}$$

ja que clarament $y_{MIN} < 2$

Per tant: $\boxed{y_{MIN} = 3-\sqrt{3}} \cong 1'3 \; ; \; x \cong \frac{0'7}{1'7} \cong 0'41$

2. EL PENJOLL

(1 bis) Observem que la condició clau

dibujar una elipse del punto P (pág.5) con las normales como dice
[Si esto es muy complicado no hace falta que lo hagas]

$$y'(P) = 0 \Leftrightarrow \frac{2-y}{x} = \frac{4-y}{2-x}$$

pot obtenir-se per condicions geomètriques:

- P ha de pertànyer a l'EL·LIPSE de $\begin{cases} \text{FOCUS } A, B \\ \text{SEMIEIX MAJOR} = 2 \end{cases}$
- el raig AP s'ha de REFLECTIR per l'altre focus $B \Leftrightarrow$ **la NORMAL en P és bisectriu de l'angle APB**
- $y'(P) = 0 \Leftrightarrow$ TANGENT a l'el·lipse en P és HORITZONTAL

  $\Leftrightarrow$ **la NORMAL en P és VERTICAL**

Per tant:

$$y'(P) = 0 \;\Leftrightarrow\; \text{PENDENT } AP = -\text{ PENDENT } BP \;\Leftrightarrow\; \frac{2-y}{0-x} = -\frac{4-y}{2-x}$$

2. EL PENJOLL

(2) Clarament:

B en equilibri $\Rightarrow PB$ ortogonal al ganxo $\Leftrightarrow\ PB$ passa per C

Per tant:

problema (2) $\Leftrightarrow$ problema (1) amb CORDA $\begin{cases}\text{de llargària } 5\\ \text{d'extrems } A \text{ i } C\end{cases}$

Procedim com abans:

- $\frac{2-y}{x} = \frac{5-y}{3-x} \Leftrightarrow \cdots \Leftrightarrow x = \frac{6-3y}{7-2y} \Leftrightarrow 3 - x = \frac{15-3y}{7-2y}$
- $5 = \sqrt{x^2 + (2-y)^2} + \sqrt{(3-x)^2 + (5-y)^2} = \sqrt{\frac{3^2(2-y)^2}{(7-2y)^2} + (2-y)^2} + \sqrt{\frac{3^2(5-y)^2}{(7-2y)^2} + (5-y)^2} =$

$$= \frac{2-y+5-y}{7-2y}\sqrt{3^2 + (7-2y)^2} = \sqrt{9 + 49 - 28y + 4y^2} \Leftrightarrow 25 = 58 - 28y + 4y^2 \Leftrightarrow$$

$$\Leftrightarrow y = \frac{28 \pm \sqrt{28^2 - 4^2 \cdot 33}}{8} = \frac{7 \pm 4}{8} = \begin{cases} 5'5 \ (> 2) \\ 1'5 \end{cases}$$

En definitiva:

$\boxed{y_{MIN} = 1'5}$, $x \cong 0'375$

2. EL PENJOLL

(3) Ara les condicions d'equilibri són:

$$\begin{cases} 4 = d(A,P) + d(B,P) \\ PB \text{ PERPENDICULAR a la BARRA (amb } B\epsilon\text{barra)} \end{cases}$$

Dient $B = (\alpha, \beta)$, de la primera resulta (com abans)

$$4 = \sqrt{x^2 + (2-y)^2} + \sqrt{(\alpha - x)^2 + (\beta - y)^2}$$

$$y' = 0 \Leftrightarrow \frac{2-y}{x} = \frac{\beta - y}{\alpha - x} \Leftrightarrow \cdots \Leftrightarrow x = \frac{\alpha(2-y)}{\beta+2-2y} \Leftrightarrow \boxed{\alpha - x = \frac{\alpha(\beta - y)}{\beta+2-2y}}$$

$$4 = \frac{2-y}{\beta+2-2y}\sqrt{\alpha^2 + (\beta+2-2y)^2} + \frac{\beta-y}{\beta+2-2y}\sqrt{\alpha^2 + (\beta+2-2y)^2} = \sqrt{\alpha^2 + (\beta+2-2y)^2} \Leftrightarrow$$

$$\Leftrightarrow 4^2 - \alpha^2 = (\beta+2-2y)^2 \Leftrightarrow \boxed{y = \begin{matrix} \cdots \\ \frac{\beta + 2 - \sqrt{16-\alpha^2}}{2} \end{matrix}}$$

2. EL PENJOLL

Apliquem la segona condició:

$$\begin{cases} B\epsilon \text{barra} \Leftrightarrow \boxed{\beta = \frac{4}{3}(5-\alpha)} \\ 0 = <(\alpha - x, \beta - y), (-3,4)> \Leftrightarrow \boxed{3(\alpha - x) = 4(\beta - y)} \end{cases}$$

En definitiva:

$$\left.\begin{array}{c} \alpha - x = \dfrac{\alpha(\beta - y)}{\beta + 2 + 2 - 2y} \\ y = \dfrac{\beta + 2 - \sqrt{16 - \alpha^2}}{2} \\ \beta = \dfrac{4}{3}(5 - \alpha) \\ 3(\alpha - x) = 4(\beta - y) \end{array}\right\} \Rightarrow \cdots \Rightarrow \begin{array}{l} \boxed{y_{MIN} = 1} \\ x = \dfrac{4}{3} \\ \alpha = \dfrac{16}{5} \\ \beta = \dfrac{12}{5} \end{array}$$

2. EL PENJOLL

(4) Refent (3) per a un GANXO qualsevol $\beta = \varphi(\alpha)$:

$$x = \frac{\alpha(2-y)}{\beta + 2 - 2y}$$

$$y = \frac{\beta + 2 - \sqrt{16-\alpha^2}}{2}$$

$$\beta = \varphi(\alpha)$$

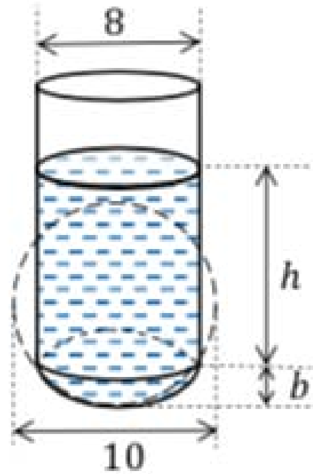
$$0 = < (\alpha - x, \beta - y), \big(1, \varphi'(\alpha)\big) >$$

de les 4 equacions resulten $y_{MIN}, x, \alpha, \beta$ com a (3)

3. L'AMPOLLA SALTAMARTÍ

Tornem al primer exemple de la sessió 13:

Considerem una AMPOLLA (vegeu figura) formada per
- una superfície CILÍNDRICA de 8 cm de diàmetre
- com a base, un CASQUET ESFÈRIC de 5 cm de radi



És clar que la POSICIÓ vertical és
- ESTABLE per $h \ll$
- INESTABLE per $h \gg$

on $h =$ nivell d'aigua a la part cilíndrica

Es tracta de determinar el NIVELL h de DISRUPCIÓ

3. L'AMPOLLA SALTAMARTÍ

Clarament:

alçària del CENTRE de CURVATURA de la base $= 5$

Per tant:

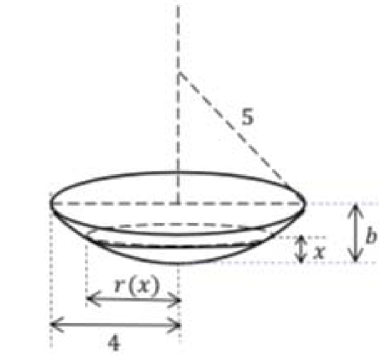
- alçària del CENTRE de GRAVETAT $\begin{cases} < 5 \Rightarrow \text{ESTABLE} \\ > 5 \Rightarrow \text{INESTABLE} \end{cases}$
- $h =$ nivell de DISRUPCIÓ $\Leftrightarrow$ alçària del CENTRE de GRAVETAT $= 5$

Per simplificar, suposem:

- densitat del líquid $= 1$
- PES de l'ampolla negligible front al del líquid

3. L'AMPOLLA SALTAMARTÍ

(1) Estudiem el CASQUET ESFÈRIC:



- $b = 5 - \sqrt{5^2 - 4^2} = 5 - 3 = 2$

- $r(x) = \sqrt{5^2 - (5 - x)^2} = \sqrt{10x - x^2}$

- MASSA $= \int_0^b \pi(r(x))^2 dx = \int_0^b \pi(10x - x^2)dx = \pi\left[\frac{10x^2}{2} - \frac{x^3}{3}\right]_0^2 = \pi\left(20 - \frac{8}{3}\right) = \frac{52\pi}{3}$

- ALÇÀRIA del c.d.g. $= \frac{1}{52\pi/_3}\int_0^b x\pi(r(x))^2 dx = \frac{3}{52}\int_0^2 (10x^2 - x^3)dx = \cdots = \frac{68}{52} \cong 1'3$

3. L'AMPOLLA SALTAMARTÍ

(2) Part CILÍNDRICA (depenent de h):

- MASSA $= \pi 4^2 h$
- ALÇÀRIA del c.d.g. $= 2 + \frac{h}{2}$

(3) En l'AMPOLLA (depenent de h):

- MASSA $= \frac{52\pi}{3} + \pi 4^2 h$
- ALÇÀRIA del c.d.g. $\equiv x_G = \frac{1}{\frac{52\pi}{3}+\pi 16h}\left(1'3\,\frac{52\pi}{3} + \left(2+\frac{h}{2}\right)\pi 16h\right)$

(4) Per tant:

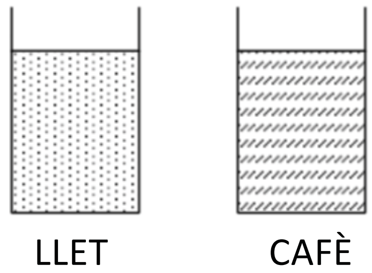
- $x_G = 5 \Leftrightarrow 5\left(\frac{52\pi}{3} + \pi 16h\right) = 1'3\,\frac{52\pi}{3} + \left(2+\frac{h}{2}\right)\pi 16h \Leftrightarrow \cdots \Leftrightarrow h^2 - 6h - 3'7\,\frac{6'5}{3} = 0 \Leftrightarrow$

$$\Leftrightarrow h \cong \frac{6 \pm \sqrt{36+32}}{2} \cong \frac{6 \pm 8'2}{2} = < \begin{matrix} 7'1 \\ -1'1 \end{matrix}$$

- h de DISRUPCIÓ $\cong 7'1$

4. CAFÈ O LLET ?

Partim de $\begin{cases} 1\ L \text{ de CAFÈ} \\ 1\ L \text{ de LLET} \end{cases}$



Operació de transferència:

- es passa una CULLERADA de LLET al cafè
- es barreja fins a homogeneïtzat i
- es retorna una CULLERADA de BARREJA a la llet

Es pregunta:

(1) Esbrineu si:

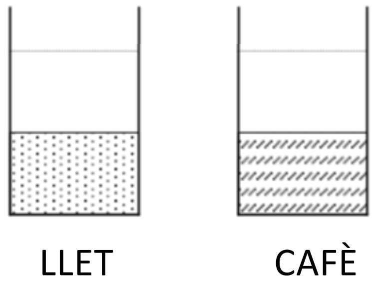
$$\text{CAFÈ a la LLET} \overset{?}{\gtrless} \text{LLET al CAFÈ}$$

(2) Ídem si es reitera la operació

4. CAFÈ O LLET ?

La resposta
- no requereix càlculs
- no depèn del nombre d'operacions, ni del grau d'homogeneïtzació, ni ...

Per qualssevol operacions de transferència:



$$\text{CAFÈ a la LLET} \overset{!!}{=} \text{LLET al CAFÈ}$$

ja que:

- en cada recipient hi ha finalment $1\ L$
- entre els dos recipients: $1\ L$ de CAFÈ, $1\ L$ de LLET

APLICACIONS DE MATEMÀTIQUES EN ENGINYERIA II: CÀLCUL MULTIVARIABLE

FINAL SESSIÓ VIII