Chapter 5

Deficiency

5.1 Introduction

In this chapter, we will study "how close" is a graph to be super magic. Of course it is possible to think about many different ways to attack this question. However we will follow the line introduced by Kotzig and Rosa in [28], who presented the following definition. The magic deficiency of a graph G, denoted by $\mu(G)$, is defined to be the minimum number of isolated vertices that we have to union G with, so that the resulting graph is magic. Of course if G is magic, we have that $\mu(G) = 0$. The first question that appears in relation to this definition is: for any graph G, does there exists always a finite number of isolated vertices n, so that the graph $G \cup nK_1$ is a magic graph? Kotzig and Rosa [28] provided an affirmative answer to this question, proving that for any graph G, $\mu(G) = n$ for some n in $\mathbb{Z}$.

Motivated by this work Figueroa, et al. [16] defined the concept of super magic deficiency of a graph G as follows. Let G be a graph and let

$$M(G) = \{n \geq 0 \mid G \cup nK_1 \text{ is super magic}\}.$$

Then the super magic deficiency of G, denoted by $\mu_s(G)$, is defined to be

$$\mu_s(G) = \begin{cases} \min M(G), & \text{if } M(G) \neq \emptyset; \\ +\infty, & \text{if } M(G) = \emptyset. \end{cases}$$

Of course, if G is super magic, we have that $\mu_s(G) = 0$. Also it is a direct consequence of the definitions that $\mu(G) \leq \mu_s(G)$, and in fact, as we will see in this chapter, there are graphs which are neither magic nor super magic, and for which the magic and super magic deficiency are equal. One of the main differences between the magic and super magic deficiencies is that while the magic deficiency is always finite, the super magic deficiency is not necessarily finite, and as we will see, graphs with infinite super magic deficiency are not hard to find. One of the main goals in this chapter will be to try to find conditions that guarantee infinite super magic deficiency. Also, we will study the super magic deficiency of the complete bipartite graphs, and motivated by the Ringel-Lladó conjecture [34] that all trees are super magic [], we will prove that the super magic deficiency of such graphs is always finite by proving that the super magic deficiency of forests is always finite. Finally, we will compute the exact super magic deficiency of some forests and 2-regular graphs, as for instance the graph nK_2, and the cycle C_n.

5.2 Graphs Whose Super Magic Deficiency is Infinity

As we already mentioned in the introduction, one of the main differences between magic and super magic deficiencies is that while the magic deficiency is always finite, the super magic deficiency is not necessarily finite. In fact, it is not hard to find graphs with infinite super magic deficiency. In this section we will provide examples of different types of graphs with infinite super magic deficiency. In order to do this, we will first state the following two results introduced first by Figueroa et al. in [].

Lemma 5.1. *Let G be a graph of size q having the property that, for all sets V_1 and V_2 such that $V_1 \cup V_2 = V(G)$, $V_1 \cap V_2 = \emptyset$ and*

$$|\{uv \in E(G) \mid u \in V_1 \text{ and } v \in V_2\}|$$

is neither $\lfloor q/2 \rfloor$ nor $\lceil q/2 \rceil$. Then, $\mu_s(G) = +\infty$.

The proof of this lemma is basically identical to the proof of Lemma 2.8, and hence it will be omitted.

Although Lemma 5.1 is theoretically interesting it is not always easy to use. The next corollary gives an easier to use, although less powerful condition. The proof of the corollary will be omitted since it is similar to the proof of Theorem 2.9.

Lemma 5.2. *Let G be a graph of size q such that $\deg v$ is even for all $v \in V(G)$, and $q \equiv 2 \pmod 4$. Then $\mu_s(G) = +\infty$.*

It is worthwhile to remark that there are graphs satisfying the hypothesis of Lemma 5.1, but not those of the previous theorem. For example, take the graph $G \cong K_{12}$. Then, $|E(K_{12})| \neq |E(K_{n,12-n})|$ for $n = 1, 2, \ldots, 11$ since the resulting quadratic equation $n^2 - 12n + 33 = 0$ has no integer solutions.

Another type of graphs that have infinite magic deficiency are those with a "large" *clique*, where the clique of a graph G is the largest complete subgraph of G. In order to formalize the previous assertion, we will borrow some ideas from what today is known as additive number theory.

Kotzig defined in 1972 [26] (although this concept was first introduced by Sidom, and it's sometimes refered to as a weak Sidom set, in 1949 []), a set $x = \{x_1 < x_2 < \cdots < x_n\} \subset \mathbb{Z}$ to be a well-spread set (WS set for short), if the sums $x_i + x_j$ for $i < j$ are all different. Furthermore, Kotzig in the same paper also defined the smallest span of pairwise sums of cardinality n, denoted by $\rho^*(n)$ to be

$$\rho^*(n) = \min\{x_n + x_{n-1} - x_2 - x_1 + 1 \mid \{x_1 < x_2 < \cdots < x_n\} \text{ is a WS-set.}\}$$

The next lemma provided also by Kotzig [26] states the values of $\rho^*(n)$ for $n = 4, 5, 6, 7$, and 8, and provides a lower bound for $n \geq 9$.

Lemma 5.3. *The smallest span of pairwise sums of cardinality n, $\rho^*(n)$, satisfies $\rho^*(4) = 6$, $\rho^*(5) = 4$, $\rho^*(6) = 19$, $\rho^*(7) = 30$, $\rho^*(8) = 43$, and $\rho^*(n) \geq n^2 - 5n + 14$ for $n \geq 9$.*

With all this information in mind, we are now ready to state and prove the following theorem by Figueroa et al. [16].

Theorem 5.4. *Let G be a graph that contains the complete subgraph K_n. If $|E(G)| < \rho^*(n)$ then $\mu_s(G) = +\infty$.*

Proof.

We will use an indirect argument in order to prove the theorem. Suppose that there exists a graph G containing the complete subgraph $H \cong K_n$ with $|E(G)| < \rho^*(n)$ and such that $\mu_s(G) = m$ where $m \in \mathbb{N} \cup \{0\}$.

Now assume that f is a super magic labeling of $G \cup mK_1$ and let

$$S = \{f(u) + f(v) \mid uv \in E(G)\}$$

Then S is a set of $|E(G)|$ consecutive integers and hence

$$\{f(v) \mid uv \in E(G)\} = \{x_1 < x_2 < \cdots < x_n\}$$

is a WS-set. Thus $|E(G)| = \max S - \min S + 1 \geq x_n + x_{n-1} - x_2 - x_1 + 1 \geq \rho^*(n) > |E(G)|$, and therefore the desired contradiction has been reached. □

As an immediate corollary of the previous theorem, we can compute the super magic deficiency of the complete graph K_n as follows [16].

Theorem 5.5. *The super magic deficiency of the complete graph K_n satisfies the following formula.*

$$\mu_s(K_n) = \begin{cases} 0, & \text{if } n = 1, 2, 3; \\ 1, & \text{if } n = 4; \\ +\infty, & \text{if } n \geq 5. \end{cases}$$

Proof.
The graphs K_1, K_2 and K_3 are trivially super magic and thus

$$\mu_s(K_1) = \mu_s(K_2) = \mu_s(K_3) = 0.$$

Also, K_4 is certainly not super magic; however $K_4 \cup K_1$ is super magic as one simply needs to label the isolated vertex with 2, and the rest of the vertices with the remaining labels. Thus, $\mu_s(K_4) = 1$. Finally, by Theorem 5.4 we have that $\mu_s(K_n) = +\infty$ for $n \geq 5$. □

5.3 The Super Magic Deficiency of Complete Bipartite Graphs

Before beginning this section, we point out that all results in it were first proved by Figueroa et al. in [16].

Our first result in this section provides an upper bound for the super magic deficiency of $K_{m,n}$ for all m and n, implying that $\mu_s(K_{m,n}) < +\infty$.

Theorem 5.6. *The super magic deficiency of the complete bipartite graph satisfies that $\mu_s(K_{m,n}) \leq (m-1)(n-1)$ for all possible positive integer values of m and n.*

Proof.
Let V_1 and V_2 be the partite sets of $K_{m,n}$, and let G be isomorphic to $K_{m,n} \cup (m-1)(n-1)K_1$. Then it suffices to present a super magic labeling of G.

Thus, we consider the vertex labeling $f : V(G) \to \{1, 2, \ldots, mn+1\}$ such that $f(V_1) = \{1, 2, \ldots, m\}$ and $f(V_2) = \{m+1, 2m+1, \ldots, nm+1\}$, and the remaining labels from 1 to $nm+1$ are placed arbitrarily on the isolated vertices of G, which extends to a super magic labeling of G. □

For the next proof, we will need the following notation. Let $A \subseteq \mathbb{R}$ and $b \in \mathbb{R}$ then $A + b = \{a + b \mid a \in A\}$.

Theorem 5.7. *The super magic deficiency of the graph $K_{2,n}$ is exactly equal to $(n-1)$ for every positive integer n.*

Proof.

First, notice that, by a previous theorem $\mu_s(K_{m,2}) \leq m-1$, so we assume $\mu_s(K_{m,2}) = n$ and let $G \cong K_{m,2} \cup nK_1$ such that

$$V(G) = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_m\} \cup \{w_1, w_2, \ldots, w_n\}$$

and

$$E(G) = \{uv \mid i = 1, 2 \text{ and } j = 1, 2, \ldots, m\}.$$

Now, let

$$f : V(G) \cup E(G) \to \{1, 2, \ldots, 3m + n + 2\}$$

be a super magic labeling of G such that $f(u_1) < f(u_2)$ and $f(v_i) < f(v_j)$ if and only if $i < j$. Also let $\alpha = f(u_2) - f(u_1) - 1$ and $\beta_i = f(v_i) - f(v_{i-1}) - 1$ for $i = 2, 3, \ldots, m$, and define $\varepsilon = \max\{i \mid \beta_i = 0 \text{ for all } j \leq i\}$. First, we show that $\alpha = \varepsilon - 1$.

If $\varepsilon = m$, then $f(u_1) + f(v_m) + 1 = f(u_2) + f(v_1)$. Thus, $f(v_m) - f(v_1) = \alpha$; however, $f(v_m) = f(v_1) + m - 1$, so $\alpha = \varepsilon - 1$. Hence, we may assume that $\varepsilon < m$.

Next consider

$$\begin{aligned} S &= \{f(x) + f(y) \mid xy \in E(G)\} \\ &= \{f(u_i) + f(v_j) \mid i = 1, 2 \text{ and } j = 1, 2, \ldots, m\}, \end{aligned}$$

which is a set of $2m$ consecutive integers by Lemma 1. Now, by subtracting $f(u_1)$ from each element of S, we obtain

$$R = \{f(v_1), f(v_2), \ldots, f(v_m)\} \cup \{f(v_1), f(v_2), \ldots, f(v_m)\} + (\alpha + 1),$$

which is also a set of $2m$ consecutive integers.

Since $\varepsilon < m$, it follows that $\beta_{\varepsilon+1} = f(v_{\varepsilon+1}) - f(v_\varepsilon) - 1 > 0$ is well defined.

Then $f(v_\varepsilon) + \beta_{\varepsilon+1} + 1 = f(v_{\varepsilon+1}) - 1$. This implies that

$$\{1, 2, \ldots, \beta_{\varepsilon+1}\} + f(v_\varepsilon) \subset \{f(v_1), f(v_2), \ldots, f(v_m)\}$$

and thus,

$$\{1, 2, \ldots, \beta_{\varepsilon+1}\} + f(v_\varepsilon) \subset \{f(v_1), f(v_2), \ldots, f(v_m)\} + (\alpha + 1).$$

Assume now that $f(v_1) + \alpha + 1 < f(v_\varepsilon) + 1$, then $f(v_1) + \alpha + 1 \leq f(v_\varepsilon)$ and hence $f(v_1) + \alpha + 1 \in \{f(v_1), f(v_2), \ldots, f(v_\varepsilon)\}$, which is a contradiction. Thus, $f(v_1) + \alpha + 1 \geq f(v_\varepsilon) + 1$ which implies that $f(v_1) + \alpha + 1 = f(v_\varepsilon) + 1$ since $f(v_\varepsilon) + 1 \in \{f(v_1), f(v_2), \ldots, f(v_m)\} + (\alpha + 1)$.

Hence, since $f(v_\varepsilon) = f(v_1) + \varepsilon - 1$, it follows that $\alpha = \varepsilon - 1$.

Next, we want to show that $m = \varepsilon\delta$ for some integer δ,

$$\beta_i = \begin{cases} \varepsilon, & \text{if } i \equiv 1 \pmod{\varepsilon}; \\ 0, & \text{otherwise}; \end{cases} \tag{5.1}$$

for $i = 2, 3, \ldots, m$, and that if we order the elements of R, we get the sequence $\overrightarrow{R} = (r_k)_{k=1}^{2m}$, where

$$r_{2i\varepsilon+j} = f(v_{i\varepsilon+j}) \text{ and } r_{(2i+1)\varepsilon+j} = f(v_{i\varepsilon+j}) + \varepsilon \tag{5.2}$$

if $1 \le i \le \delta - 1$ and $1 \le j \le \varepsilon$. To do this, we proceed by induction.

First, notice that if $m = \varepsilon$, we are done, so we assume that $\varepsilon < m$.

Next, by assumption $\beta_i = 0$ and $r_i = f(v_i)$ for $i = 1, 2, \ldots, \varepsilon$. Now, notice that since $\{1, 2, \ldots, \beta_{\varepsilon+1}\} + f(v_\varepsilon)$ is a set of consecutive numbers in R starting with $f(v_1) + \varepsilon$, it follows that $\beta_{\varepsilon+1} \le \varepsilon$.

Note that since $\alpha = \varepsilon - 1$, it follows that $f(u_2) = f(u_1) + \varepsilon$. Also, $f(v_\varepsilon) + 1 \le f(v_{\varepsilon+1})$, however, $f(v_1) + \varepsilon = f(v_\varepsilon) + 1$ and hence $f(v_1) + \varepsilon < f(v_{\varepsilon+1})$. Now, assume that $f(v_\varepsilon) + \varepsilon \ge f(v_{\varepsilon+1})$ then since $f(v_1) + 1, f(v_2) + \varepsilon, \ldots, f(v_\varepsilon) + \varepsilon$ are consecutive, there exists $1 \le k \le \varepsilon$ such that $f(v_k) + \varepsilon = f(v_{\varepsilon+1})$. Thus,

$$f(v_k) + f(u_2) = f(v_k) + f(u_1) + \varepsilon = f(v_{\varepsilon+1}) + f(u_1),$$

in other words, we have two distinct edges to which f assigns the same label, which is a contradiction. Therefore, $f(v_\varepsilon) + \varepsilon < f(v_{\varepsilon+1})$ and since $f(v_\varepsilon) + \beta_{\varepsilon+1} + 1 = f(v_{\varepsilon+1})$ we have $\beta_{\varepsilon+1} \ge \varepsilon$. Thus, we conclude that $\beta_{\varepsilon+1} = \varepsilon$ and $r_{\varepsilon+k} = f(v_k) + \varepsilon$ for $k = 1, 2, \ldots, \varepsilon$ and $r_{2\varepsilon+1} = f(v_{\varepsilon+1})$.

Now assume that (5.1) holds for $\beta_1, \beta_2, \ldots \beta_{i\varepsilon+1}$ and consequently, (5.2) also holds for $r_1, r_2, \ldots r_{2i\varepsilon+1}$ for a fixed i with $1 \le i \le (m/\varepsilon) - 1$. Then we wish to prove that (5.1) holds for

$$\beta_{i\varepsilon+2,}\beta_{i\varepsilon+3}, \ldots, \beta_{(i+1)\varepsilon+1}$$

and (5.2) holds for ,

$$r_{2i\varepsilon+2,}r_{2i\varepsilon+3}, \ldots, r_{2(i+1)\varepsilon+1.}$$

In order to do this, we first show that

$$\beta_{i\varepsilon+2} = \beta_{i\varepsilon+3} = \cdots = \beta_{(i+1)\varepsilon} = 0; \tag{5.3}$$

so assume, the contrary, that there exist

$$j = \min\left\{k \mid \beta_k > 0 \text{ and } i\varepsilon + 2 \le k \le (i+1)\varepsilon\right\}.$$

Then $\beta_j = f(v_j) - f(v_{j-1}) - 1$ so $f(v_{j-1}) + \beta_j = f(v_j) - 1$, which implies that $\{1, 2, \ldots, \beta_j\} + f(v_{j-1})$ is a set of β_j consecutive elements of R wedged between $f(v_{j-1})$ and $f(v_j)$. Thus, since we have assigned $\{f(v_1), f(v_2), \ldots, f(v_{i\varepsilon})\} + \varepsilon$ already to terms in the sequence $\overrightarrow{R}$, it follows that

$$\{1, 2, \ldots, \beta_j\} + f(v_{j-1}) \subset \{f(v_{i\varepsilon+1}), f(v_{i\varepsilon+2}), \ldots, f(v_m)\} + \varepsilon.$$

Then

$$f(v_{i\varepsilon+1}) + \varepsilon \leq f(v_{j-1}) + 1 = f(v_{i\varepsilon+1}) + j + i\varepsilon - 1.$$

Thus, $(i+1)\varepsilon \leq j - 1$, so $(i+1)\varepsilon < j$, which is a contradiction, since $i\varepsilon + 2 \leq j \leq (i+1)\varepsilon$. Therefore $f(v_{i\varepsilon+1}) + \varepsilon > f(v_{j-1}) + 1$, which is also a contradiction, and we have that (5.3) holds, which implies that $r_{2i\varepsilon+j} = f(v_{i\varepsilon+j})$ for $j = 2, 3, \ldots, \varepsilon$.

Second, we show that $\beta_{(i+1)\varepsilon+1} = \varepsilon$. Notice first that

$$\left\{1, 2, \ldots, \beta_{(i+1)\varepsilon+1}\right\} + f(v_{(i+1)\varepsilon}) \subset \{f(v_{i\varepsilon+1}), f(v_{i\varepsilon+2}), \ldots, f(v_m)\} + \varepsilon,$$

assume that $f(v_{i\varepsilon+1}) + \varepsilon < f(v_{(i+1)\varepsilon}) + 1$, so $f(v_{i\varepsilon+1}) + \varepsilon \leq f(v_{(i+1)\varepsilon}) + 1$ and hence

$$f(v_{i\varepsilon+1}) + \varepsilon \in \left\{f(v_{i\varepsilon+1}), f(v_{i\varepsilon+2}), \ldots, f(v_{(i+1)\varepsilon})\right\}$$

which is a contradiction. Thus, $f(v_{i\varepsilon+1}) + \varepsilon \geq f(v_{(i+1)\varepsilon}) + 1$, which implies that $f(v_{i\varepsilon+1}) + \varepsilon = f(v_{(i+1)\varepsilon}) + 1$. Therefore, $\beta_{(i+1)\varepsilon+1} \leq \varepsilon$.

Notice, that $f(v_{(i+1)\varepsilon}) + 1 \leq f(v_{(i+1)\varepsilon+1})$. Also since

$$\left\{f(v_{i\varepsilon+1}), f(v_{i\varepsilon+2}), \ldots, f(v_{(i+1)\varepsilon})\right\}$$

are consecutive, $f(v_{(i+1)\varepsilon}) + 1 = f(v_{i\varepsilon+1}) + \varepsilon$, thus $f(v_{i\varepsilon+1}) + \varepsilon \leq f(v_{(i+1)\varepsilon+1})$.

Now, assume

$$f(v_{(i+1)\varepsilon}) + \varepsilon \geq f(v_{(i+1)\varepsilon+1}),$$

then since

$$\left\{f(v_{i\varepsilon+1}), f(v_{i\varepsilon+2}), \ldots, f(v_{(i+1)\varepsilon})\right\} + \varepsilon$$

are consecutive, there exists $1 \leq k \leq \varepsilon$ such that $f(v_{i\varepsilon+k}) + \varepsilon = f(v_{(i+1)\varepsilon+1})$, which implies that

$$f(v_{i\varepsilon+k}) + f(u_2) = f(v_{i\varepsilon+k}) + f(u_1) + \varepsilon = f(v_{(i+1)\varepsilon+1}) + f(u_1),$$

in other words, f assigns to two distinct edges the same label, which is a contradiction. Hence, $f(v_{(i+1)\varepsilon+1}) > f(v_{(i+1)\varepsilon}) + \varepsilon$, so $\beta_{(i+1)\varepsilon+1} \geq \varepsilon$. Therefore, we conclude that $\beta_{(i+1)\varepsilon+1} = \varepsilon$ and $r_{(2i+1)\varepsilon+j} = f(v_{i\varepsilon+j}) + \varepsilon$. This concludes the inductive part of the proof.

Finally, the above facts yield that $f(u_2) = f(u_1) + \varepsilon, f(v_{i\varepsilon+j}) = f(v_1) + 2i\varepsilon + j - 1$ for $1 \leq j \leq \varepsilon$ and $1 \leq i \leq (m/\varepsilon) - 1$ and $\sum_{i=2}^{m} \beta = m - \varepsilon$, which we use to conclude that $\mu_s(K_{m,2}) \geq m - 1$. We do this by computing the minimum number of elements that would have to be added to the set $\{f(u_1), f(u_2)\} \cup \{f(v_1), f(v_2), \ldots, f(v_m)\}$ to obtain a set of consecutive integers starting at 1. For this, we need two cases.
Case 1: Let $f(u_1) < f(u_2) < f(v_1)$. We would need to add in this case at least

$$(f(u_1) - 1) + \alpha + (f(v_1) - f(u_2) - 1) + \sum_{i=2}^{m} \beta = m + f(v_1) - 2 \geq m - 1$$

since $f(u_2) = f(u_1) + \varepsilon$ and $(f(v_1) \geq 3$.
Case 2: Let $f(v_m) < f(u_1) < f(u_2)$. we would need to add in this case at least

$$(f(v_1) - 1) + \sum_{i=2}^{m} \beta_i + (f(u_1) - (f(v_m) - 1) + \alpha = f(u_1) - m + \varepsilon - 2 \geq m - 1$$

since $f(v_m) = f(v_1) + 2m - \varepsilon - 1$ and $f(u_1) \geq 2m + \varepsilon + 1$.

Notice that no other cases are possible since $f(u_2) - f(u_1) = \varepsilon - 1$. Therefore, we arrive to the desired result. □

A computer search of small cases together with Theorems 7 and 8 lead us to conjecture that in fact $\mu_s(K_{m,n}) = (m - 1)(n - 1)$.

5.4 The Deficiency of Forests

In [7], Enomoto et al. conjectured that all trees are super magic, so a very natural question is wether one can compute, or at least bound, $\mu_s(T)$ for any tree T. Figueroa et al. [16] computed a possible bound, which is presented in the next theorem.

Theorem 5.8. *Let F be a forest, then $\mu_s(F) < +\infty$.*

Proof.

Without loss of generality, assume that in this proof, forests have no isolated vertices. Notice that any forest F can be thought of as having its vertices to be ordered triples that satisfy the following properties. Let $v \in V(F)$, then we call $v \cdot \mathbf{i}$, $v \cdot \mathbf{j}$ and $v \cdot \mathbf{k}$, respectively, the component, label and position of v ("·" denotes the inner product of two vectors and $\mathbf{i} = (1, 0, 0)^T$, $\mathbf{j} = (0, 1, 0)^T$ and $\mathbf{k} = (0, 0, 1)^T$). Then the component of

v ranges from 1 to $k(F)$, the number of components of F. Also, the set $R = \{(x,0,0)^T \mid 1 \leq x \leq k(F)\}$ is a subset of $V(F)$. We let $r(v) = (v \cdot \mathbf{i}, 0, 0)^T$ be called the root of the component that v lies in; thus the level of v is $v \cdot \mathbf{j} = d(v, r(v))$, which is the distance from v to $r(v)$. Now, let $v \in V(F) - R$, then let $f(v)$, which we call the father of v be the vertex adjacent to v in the $(v, r(v))$-path in F. Then if $u, v \in V(F)$ have the same component and level and the position of the father of u is less than or equal to the position of the father of v ($u \cdot \mathbf{i} = v \cdot \mathbf{i}$, $u \cdot \mathbf{j} = v \cdot \mathbf{j}$ and $f(u) \cdot \mathbf{k} \leq f(v) \cdot \mathbf{k}$), then the position of u is less than or equal to the position of v ($u \cdot \mathbf{k} \leq v \cdot \mathbf{k}$). Finally, for $x, y \in \mathbb{N}$, let $l(x,y) = \left|\{(x,y,z)^T \in V(F) \mid z \in \mathbb{N}\}\right|$ the number of vertices in component x and level y. Then $\{z \mid (x,y,z)^T \in V(F)\} = \{1, 2, \ldots, l(x,y)\}$. That is given a fixed component x and a fixed level y of F, the vertices within them both, have consecutive positions which range from 1 to $l(x,y)$. Now, if $e \in E(F)$ and $e = uv$, where $u = (x,y,z)^T$ and $v = (x, y+1, \delta)^T$, then we denote e by an ordered triple in brackets as follows: $e = [x, y+1, \delta]$. We will call x, $y+1$, δ the component, level and position of e, respectively. Let y^* denote the maximum level of the component $x-1$. We will call an edge even or odd depending on the parity of the level of the edge. Also, the set of even edges and odd edges will be denoted by E_e and E_o, respectively. Define the function $\varepsilon : E(F) \to \{-1, 1\}$, such that $\varepsilon(e) = -1$ if $e \in E_o$ and 1 if $e \in E_e$.

Next, define the function $g : E(F) \to \mathbb{Z}$ recursively as follows. Let $e = [1, y, \delta]$, then

$$g(e) = \begin{cases} y - 1 - \varepsilon(e)\delta, \text{ for all } \delta \in \{1, 2, \ldots, l(1,y)\}, & \text{if } y \in \{1,2\}; \\ g([1, y-2, l(1, y-2)]) - \varepsilon(e)\delta, & \text{if } y \notin \{1,2\}. \end{cases}$$

Let α and β be the minimum and maximum labels, respectively, among all the labels of the edges of component $x-1$. Let $e = [x, y, \delta]$ with $x \neq 1$, then if $g([x-1, y^*, l(x-1, y)]) > 0$ we have that

$$g(e) = \begin{cases} \alpha - \delta, & \text{if } y = 1; \\ \beta + \delta, & \text{if } y = 2; \\ g([x, y-2, l(x, y-2)]) + \varepsilon(e)\delta, & \text{if } y \notin \{1,2\}. \end{cases}$$

If $g([x-1, y^*, l(x-1, y)]) \leq 0$, then we have that

$$g(e) = \begin{cases} \beta + \delta, & \text{if } y = 1; \\ \alpha - \delta, & \text{if } y = 2; \\ g([x, y-2, l(x, y-2)]) - \varepsilon(e)\delta, & \text{if } y \notin \{1,2\}. \end{cases}$$

Next, define the function $h : V(F) \to \mathbb{Z}$ recursively as follows. First, let $h\left((1,0,0)^T\right) = 0$. Second, if $v = (x,y,z)^T \in V(F) - R$ and $[x,y,z] =$

$\{(x, y-1, \delta)^T, (x, y, z)^T\}$, then $h(v) = g([x, y, z]) - h\left((x, y-1, \delta)^T\right)$. Now, for $v = (x, 0, 0)^T$ with, $x \neq 1$, we let $\gamma = h\left((x-1, y^*, l(x-1, y^*))^T\right)$, then $h(v) = \gamma - 1$ if $\gamma \leq 0$ and $h(v) = \gamma + 1$ if $\gamma > 0$.

Finally, let $\lambda : V(F) \to \mathbb{N}$ be such that $\lambda(v) = h(v) + m$, where $m = \min\{h(v) + 1 \mid v \in V(F)\}$. Then for all $u, v \in V(F)$, we have that $\lambda(u) \geq 1$ and $\lambda(u) = \lambda(v)$ if and only if $u = v$. Also, $\{\lambda(u) + \lambda(v) \mid uv \in E(F)\}$ is a set of $|E(F)|$ consecutive integers. Therefore, $\mu_s(F) < +\infty$. □

5.5 Some Particular Families of Forests and 2-Regular Graphs

Unless stated otherwise, all the results on this section where first introduced by Figueroa et al. in [15] and [16]

We will calculate the super magic deficiency of some particular families of forests and 2-regular graphs. We begin this section calculating the magic and super magic deficiencies of n disjoint copies of the graph K_2.

Theorem 5.9. *The magic and super magic deficiencies of nK_2 $(n \geq 1)$ are given by the formula*

$$\mu_s(nK_2) = \mu(nK_2) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

Proof.

Note first that Kotzig and Rosa [6] showed that nK_2 is magic if and only if n is an odd integer. Hence, $\mu(nK_2) = 0$ if n is odd and $\mu(nK_2) \geq 1$ when n is even. Actually, the magic labeling that they provide in their proof for nK_2 when n is odd, is in fact, a super magic labeling, which implies that $\mu_s(nK_2) = \mu(nK_2) = 0$ for n odd. Thus, assume without loss of generality, that n is even.

Consider the graph $G = nK_2 \cup K_1$ such that $V(G) = \{y_i, z_i \mid 1 \leq i \leq n\} \cup \{x\}$ and $E(G) = \{y_i, z_i \mid 1 \leq i \leq n\}$. Then we construct the vertex labeling $g : V(nK_2 \cup K_1) \to \{1, 2, \ldots, 2n+1\}$ as follows,

$$g(w) = \begin{cases} \frac{3n+2}{2}, & \text{if } w = x; \\ i, & \text{if } w = y_i \text{ and } 1 \leq i \leq n; \\ \frac{3n+2}{2}, & \text{if } w = z_i \text{ and } 1 \leq i \leq \frac{n}{2}; \\ \frac{n}{2} + i, & \text{if } w = z_i \text{ and } \frac{n}{2} + 1 \leq i \leq n. \end{cases}$$

Then g is the canonical form of a magic labeling of G. Hence, $\mu_s(nK_2) \leq 1$. Therefore, we conclude that $\mu(nK_2) = \mu_s(nK_2) = 1$ when n is even. □

This family of graphs is an example that shows that the magic and super magic deficiencies of a graph that is not magic may be equal.

Our next immediate goal is to characterize the super magic deficiency of the forest formed by the disjoint union of a path and a star. In order to do this we will present the following lemmas.

Lemma 5.10. *For every positive integer n, we have that*

$$\mu_s(P_2 \cup K_{1,n}) = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof.

First, note that the forest $P_2 \cup K_{1,n} \cong K_{1,1} \cup K_{1,n}$ is shown to be super magic in [13] if and only if n is even. Consequently, it is sufficient to verify that $\mu_s(P_2 \cup K_{1,n}) \leq 1$ when n is odd.

Now assume that n is odd, and define the forest $F \cong P_2 \cup K_{1,n} \cup K_1$ with

$$V(F) = \{u_1, u_2\} \cup \{v_1 \mid 1 \leq i \leq n\} \cup \{v, w\}$$

and

$$E(F) = \{u_1, u_2\} \cup \{uv_1 \mid 1 \leq i \leq n\}.$$

The vertex labeling $f : V(F) \to \{1, 2, \ldots, n+4\}$ such that

$$f(x) = \begin{cases} 1, & \text{if } x = u_1; \\ n+4, & \text{if } x = u_2; \\ \frac{n+5}{2} & \text{if } x = v; \\ i+1, & \text{if } x = v_i \text{ and } 1 \leq i \leq \frac{n+1}{2}; \\ i+2, & \text{if } x = v_i \text{ and } \frac{n+3}{2} \leq i \leq n; \\ n+3, & \text{if } x = w. \end{cases}$$

is the canonical form of a super magic labeling of F with valence $(5n+19)/2$.

Therefore, $\mu_s(P_2 \cup K_{1,n}) \leq 1$ when n is odd, and this completes the proof.

□

Theorem 5.11. *For every integer $n \geq 2$,*

$$\mu_s(P_3 \cup K_{1,n}) = \begin{cases} 0, & \text{if } n \equiv 0 \pmod 3; \\ 1, & \text{if } n \equiv 1, 2 \pmod 3. \end{cases}$$

Proof.

First, note that it is proven in [13] that the forest $P_3 \cup K_{1,n} \cup K_1$ with

$$V(F) = \{u, u_1, u_2, v, w\} \cup \{v_i \mid 1 \leq i \leq n\}$$

and

$$E(F) = \{uu_1, uu_2\} \cup \{uv_i \mid 1 \le i \le n\}$$

is super magic. Now consider two cases.

Case 1: Assume that n is a positive integer such that $n \equiv 1$ or $5 \pmod 6$. Then the vertex labeling $f : V(F) \to \{1, 2, \ldots, n+5\}$ such that

$$f(x) = \begin{cases} 1, & \text{if } x = u; \\ n+i+3, & \text{if } x = u_i \text{ and } 1 \le i \le 2; \\ \frac{n+5}{2}, & \text{if } x = v; \\ i+1, & \text{if } x = v_i \text{ and } 1 \le i \le \frac{n+1}{2}; \\ i+3, & \text{if } x = v_i \text{ and } \frac{n+3}{2} \le i \le n; \\ \frac{n+7}{2}, & \text{if } x = w. \end{cases}$$

extends to a super magic labeling of F with valence $(5n+23)/2$. Thus, we conclude that $\mu_s(P_3 \cup K_{1,n}) \le 1$ in this case.

Case 2: Assume that $n(\ge 2)$ is an integer such that $n \equiv 2$ or $4 \pmod 6$. Then the vertex labeling $f : V(F) \to \{1, 2, \ldots, n+5\}$ such that

$$f(x) = \begin{cases} 1, & \text{if } x = u; \\ n+i+3, & \text{if } x = u_i \text{ and } 1 \le i \le 2; \\ \frac{n+6}{2} & \text{if } x = v; \\ i+1, & \text{if } x = v_i \text{ and } 1 \le i \le \frac{n}{2}; \\ i+3, & \text{if } x = v_i \text{ and } \frac{n}{2} \le i \le n; \\ \frac{n+4}{2}, & \text{if } x = w. \end{cases}$$

extends to a super magic labeling of F with valence $(5n+24)/2$. Thus, we conclude that $\mu_s(P_3 \cup K_{1,n}) \le 1$ in this case.

Therefore, the proof is completed. □

Now, the previous two theorems, together with the facts that the graph $P_1 \cup K_{1,n}$ is super magic and that for $n \ge 4$, $P_n \cup K_{1,n}$ is super magic, this last one proved in Chapter 2 of this thesis, we obtain the following result.

Theorem 5.12. *For every positive integer n, the super magic deficiency of the forest $P_m \cup K_{1,n}$ is given by*

$$\mu_s(P_m \cup K_{1,n}) = \begin{cases} 1, & \text{if } m = 2 \text{ and } n \text{ is odd,} \\ & \text{or } m = 3 \text{ and } n \not\equiv 0 \pmod 3 \\ 0, & \text{otherwise.} \end{cases}$$

At this point, we will study the super magic deficiency of some types of two regular graphs.

First of all we recall the following lemma already proved in Chapter 2.

Lemma 5.13. *The graph mC_n is super magic if and only if $m \geq 1$ and $n \geq 3$ are both odd.*

With this result in mind, we are ready to state and prove the following three results concerning the super magic deficiency of C_n, $2C_n$ and $3C_n$.

Theorem 5.14. *The super magic deficiency of the cycle C_n is given by*

$$\mu_s(C_n) = \begin{cases} 0, & \text{if } n \equiv 1 \text{ or } 3 \pmod 4; \\ 1, & \text{if } n \equiv 0 \pmod 4; \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof.
Let n be odd, then C_n is super magic implying that $\mu_s(C_n) = 0$. For $n \equiv 0 \pmod 4$, C_n is not super magic. That is, $\mu_s(C_n) \geq 1$.
For the other inequality, define the graph $G \cong C_n \cup K_1$ having

$$V(G) = \left\{x_i, y_i \mid 1 \leq i \leq \frac{n}{2}\right\} \cup \{z\}$$

and

$$E(G) = \left\{x_i, y_i \mid 1 \leq i \leq \frac{n}{2}\right\} \cup \left\{x_{i-1}y_i \mid 2 \leq i \leq \frac{n}{2}\right\} \cup \left\{x_{\frac{n}{2}}y_1\right\},$$

where $n \equiv 0 \pmod 4$. The following vertex labeling f extends to a super magic labeling of G, where

$$f(v) = \begin{cases} i, & \text{if } v = x_i \text{ and } 1 \leq i \leq \frac{n}{2}; \\ i + \frac{n}{2}, & \text{if } v = y_i \text{ and } 1 \leq i \leq \frac{n}{4}; \\ i + \frac{n}{2} + 1, & \text{if } v = y_i \text{ and } \frac{n}{4} + 1 \leq i \leq \frac{n}{2}; \\ \frac{3n}{2} + 1, & \text{if } v = z. \end{cases}$$

Thus, $\mu_s(C_n) \leq 1$, which leads us to conclude that $\mu_s(C_n) = 1$.
Finally the remaining case immediately follows from Lemma 5.2. □

Theorem 5.15. *For every positive integer n, the super magic deficiency of the 2-regular graph $2C_n$ is given by*

$$\mu_s(2C_n) = \begin{cases} 1, & \text{if } n \text{ is even}; \\ +\infty, & \text{if } n \text{ is odd}. \end{cases}$$

Proof.
First, suppose that n is odd. Thus, by Lemma 5.2, we obtain that $\mu_s(2C_n) = +\infty$. Next, assume that n is even, then the 2-regular graph $2C_n$ is not super magic by Lemma 5.13. Hence, $\mu_s(2C_n) \geq 1$.

For the reverse inequality, define the graph $G \cong 2C_n \cup K_1$ with

$$V(G) = \{x_i, y_i \mid 1 \le i \le n\} \cup \{z\}$$

and

$$\begin{aligned} E(G) &= \left\{x_{\frac{n}{2}}, y_1, x_n y_{\frac{n}{2}} + 1\right\} \cup \{x_i, y_i \mid 1 \le i \le n\} \\ &\cup \left\{x_i y_{i+1} \mid 1 \le i \le \frac{n}{2} - 1\right\} \cup \{x_i y_{i+1} \mid 1 \le i \le n-1\}. \end{aligned}$$

Next, two cases will be considered for the vertex labeling

$$f : V(G) \to \{1, 2, \ldots, 2n+1\}.$$

Case 1: For $n = 4k$, where k is a positive integer, let

$$f(w) = \begin{cases} 4k+i+1, & \text{if } w = x_i \text{ and } 1 \le i \le k; \\ 8k+i+2, & \text{if } w = x_i \text{ and } k+1 \le i \le 2k; \\ i-k+1, & \text{if } w = x_i \text{ and } 2k+1 \le i \le 3k-1, \text{ where } k \ge 2; \\ i+1, & \text{if } w = x_i \text{ and } 3k \le i \le 4k; \\ i, & \text{if } w = y_i \text{ and } 1 \le i \le k; \\ 4k-i+2, & \text{if } w = y_i \text{ and } k+2 \le i \le 2k, \text{ where } k \ge 2; \\ 8k-i+2, & \text{if } w = y_i \text{ and } 2k+1 \le i \le 3k; \\ 4k+1+1, & \text{if } w = y_i \text{ and } 3k+1 \le i \le 4k; \\ 2k+1, & \text{if } w = z. \end{cases}$$

Case 2: For $n = 4k+2$, where k is a positive integer, let

$$f(w) = \begin{cases} 6k-i+5, & \text{if } w = x_i \text{ and } 1 \le i \le k-1, \text{ where } k \ge 2; \\ 6k+i+4, & \text{if } w = x_i \text{ and } k \le i \le 2k+1; \\ i-2k-1, & \text{if } w = x_i \text{ and } 2k+2 \le i \le 3k+2; \\ 6k-i+5, & \text{if } w = x_i \text{ and } 3k+3 \le i \le 4k+2; \\ 2k-i+2, & \text{if } w = y_i \text{ and } 1 \le i \le k; \\ 2k+i+2, & \text{if } w = y_i \text{ and } k+1 \le i \le 3k+3; \\ 10k-i+7, & \text{if } w = y_i \text{ and } 3k+4 \le i \le 4k+2; \\ 2k+2, & \text{if } w = z. \end{cases}$$

Then f extends to a super magic labeling of G with valence $5n+4$, which implies that $\mu_s(2C_n) = 1$ when n is even and completes the proof. □

Theorem 5.16. *For every positive integer n, the super magic deficiency of the 2-regular graph $3C_n$ is given by*

$$\mu_s(3C_n) = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ 1, & \text{if } n \equiv 0 \pmod 4; \\ +\infty, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof.

First, let n be odd, then the 2-regular graph $3C_n$ is super magic by Lemma 5.13, implying that $\mu_s(3C_n) = 0$. For $n \equiv 0 \pmod 4$, the 2-regular graph $3C_n$ is not super magic by Lemma 5.13, that is, $\mu_s(3C_n) \geq 1$.

For the other inequality, assume that $n = 4k$, where $K \geq 2$, and define the graph $G \cong 3C_n \cup K_1$ with

$$V(G) = \{x_i, y_i \mid 1 \leq i \leq 6k\} \cup \{z\}$$

and

$$\begin{aligned} E(G) &= \{x_i, y_i \mid 1 \leq i \leq 6k\} \cup \{x_i y_{i+1} \mid 1 \leq i \leq 2k-1\} \\ &\cup \{x_{2k}, y_1\} \cup \{x_i y_{i+1} \mid 2k+1 \leq i \leq 4k-1\} \\ &\cup \{x_{4k}, y_{2k+1}\} \cup \{x_i y_{i+1} \mid 4k+1 \leq i \leq 6k-1\} \cup \{x_{6k}, y_{4k+1}\}. \end{aligned}$$

We now consider the vertex labeling $f : V(G) \to \{1, 2, \ldots, 3n+1\}$ such that

$$f(w) = \begin{cases} i, & \text{if } w = x_i \text{ and } 1 \leq i \leq 6k; \\ 8k-1, & \text{if } w = y_1; \\ 6k+i-1, & \text{if } w = y_i \text{ and } 2 \leq i \leq k; \\ 9k, & \text{if } w = y_{k+1}; \\ 6k+i-2, & \text{if } w = y_i \text{ and } k+2 \leq i \leq 2k; \\ 6k+i, & \text{if } w = y_i \text{ and } 2k+1 \leq i \leq 3k-1; \\ 6k+i+3, & \text{if } w = y_i \text{ and } 3k \leq i \leq 4k-1; \\ 8k, & \text{if } w = y_{4k}; \\ 6k+i+2, & \text{if } w = y_i \text{ and } 4k+1 \leq i \leq 5k; \\ 9k+2, & \text{if } w = x_{5k+1}; \\ 6k+i+1, & \text{if } w = y_i \text{ and } 5k+2 \leq i \leq 6k; \\ 9k+1, & \text{if } w = z. \end{cases}$$

Then f is the canonical form of a super magic labeling g with valence $(15n)/2+3$. Also, it is possible to verify by exhaustive search that $3C_4 \cup K_1$ is super magic or not. Thus, we conclude that $\mu_s(3C_n) = 1$ when $n \equiv 0 \pmod 4$.

Finally, the remaining case immediately follows from Lemma 5.2. □