Chapter 6

Results using additive number theory

6.1 Magic Graphs and Sidon Sets

6.1.1 Introduction

The results found in this section appear in [30].In the 1996 Conference of Kalamazoo, after a talk by G. Ringel, P. Erdős asked: "How large a clique in a magic graph can be?".

Throughout the paper $G = (V, E)$ denotes a finite graph without loops nor multiple edges.

In this chapter we address the Erdős' question about the maximum size of a clique in a connected magic graph. An upper bound had been given already by Kotzig and Rosa [28], where they proved that if $G = (V, E)$ is a magic graph containing a complete graph of order $n > 8$ then

$$|V| + |E| \geq n^2 - 5n + 14.$$

Their result was improved by Enomoto et al. [8] to

$$|V| + |E| \geq 2n^2 + O(n^{3/2}), \tag{6.1}$$

by using the known bounds for the size of a Sidon set. Recall that a set A of integers is said to be a Sidon set if the sums $a_i + a_j$, where a_i, a_j are two (non necessarily different) elements of A, are pairwise distinct. Erdős and Turán [10] showed that a Sidon set A contained in $\{1, 2, \ldots, N\}$ has cardinality

$$|A| \leq N^{1/2} + O(n^{1/4})$$

and they asked if the bound can be improved to $|A| \leq N^{1/2} + C$ for some constant C, but this question is still unanswered.

The upper bound for the size of a Sidon set in $[1, N]$ gives essentially an upper bound for the size of a clique in a magic graph. The reason is that, if f is a magic labeling of $G = (V, E)$ and the subgraph induced by $X \subset V$ is a complete graph, then all sums of *distinct* elements in $f(X)$ are different. Kotzig [26] called a set $A \subset \mathbb{Z}$ a *well spread sequence* if all sums of distinct elements in A are pairwise different. He showed that, if $A \subset [1, N]$, then $N \geq 4 + \binom{n-1}{2}$. Ruzsa [36] calls such a set a *weak Sidon* set, and gives a very nice short proof of the following inequality

$$|A| \leq N^{1/2} + 4N^{1/4} + 11. \tag{6.2}$$

Since $f(X)$ is a weak Sidon set then $|X|$ is bounded above by (6.2). Enomoto et al. [8] give also a lower bound for the size of the largest possible clique in a connected magic graph. Actually they use a more restricted kind of labelings. A magic labeling is said to be *super magic* if $f(V) = [1, |V|]$, that is, f assigns the smallest labels to vertices and the largest ones to edges. By using the construction of Singer [37] for dense difference sets they show that, for any graph H with n vertices and m edges, there is a connected super magic graph G which contains H as an induced subgraph such that $|V(G)| \leq 2m + 2n^2 + o(n^2)$. In particular, there are super magic graphs such that

$$|V(G)| \leq 3n^2 + o(n^2) \text{ and } K_n \subset G. \tag{6.3}$$

Moreover, if G is a super magic graph which contains a clique, then (6.1) becomes

$$|V(G)| \geq n^2 - O(n^{3/2}). \tag{6.4}$$

We prove the following result.

Theorem 6.1. *Let $A \subset [1, N]$ be a weak Sidon set of order n. There is a connected supermagic graph G of order $2N$ which contains a clique of order $n + 1$.*

Let $f(n)$ denote the size of the largest Sidon set in $[0, n]$ and $f_w(n)$ be the size of the largest weak Sidon set in $[0, n]$, so that $f'(n) \geq f(n)$. By (6.2), it is clear that $lim_{n\to\infty} n^{-1/2} f_w(n) \geq 1$. By the Erdős -Chowla Theorem,

$$lim_{n\to\infty} n^{-1/2} f(n) = 1,$$

so that the same is true for $f_w(n)$. Therefore, we can find weak Sidon sets of order n contained in $[1, f_w(n))]$ with $f_w(n) = n^2 + o(n^2)$. Hence, Theorem 6.1 implies that there is a connected graph G of order

$$|V(G)| = 2n^2 + o(n^2) \tag{6.5}$$

containing a clique of order n, improving on (6.3). In Section 3 we get the following improvement on the constant in 6.5.

Theorem 6.2. *For every $\epsilon > 0$ there is n_0 such that, for each $n > n_0$ there is a connected magic graph G of order*

$$|V(G)| \leq (\frac{1}{4} + \epsilon)n^2,$$

which contains a clique of order n.

Let $f(n)$ be the the smallest order of a connected magic graph G containing a clique of order n. We conjecture the following.

Conjecture 6.3. $f(n) = n^2 + o(n^2)$.

6.1.2 Magic graphs containing a large clique

Let $A = \{a_1, \ldots, a_n\} \subset [1, N]$ be a weak Sidon set, that is, the sums $a_i + a_j, i < j$ are pairwise distinct. Note that $A + x$ is also a weak Sidon set for any integer x, so we may assume that $1 \in A$. We also assume $N \in A$. We denote by $S = A \wedge A$ the set of sums of distinct elements in A. We have $S \subset [3, 2N - 1]$ and $|S| = \binom{n}{2}$.

Lemma 6.4. *Let $\{1, N\} \subset A \subset [1, N]$ be a weak Sidon set and $A_1 = A \setminus \{1\}$. Let S_1 be the set of sums of distinct elements in A_1. If*

$$|\{x, x + N - 1\} \cap S_1| \leq 1, \; x \in \mathbb{N}, \tag{6.6}$$

then there is a supermagic connected graph G of order N such that $K_n \subset G$.

Proof. Define the graph G with vertex set $[1, N]$ and set of edges $E_1 \cup E_2 \cup E_3$ with

$$E_1 = \{a_i a_j, i \neq j, a_i, a_j \in A\}, \; E_2 = \{1j, \; 2 \leq j \leq N - 1, 1 + j \notin S\},$$

$$E_3 = \{jN, 2 \leq j \leq N - 1; j + N \notin S\}.$$

Then the vertices in A clearly induce a clique of order n in G and the map $f(i) = i, 1 \leq i \leq N$, $f(ij) = 3N - (i + j)$ is a supermagic labeling of the graph. Finally, by the condition on S, for each vertex $x \notin A$, either $x + 1 \notin S$ or $x + N \notin S$, so that x is connected to at least one of the two vertices 1 and N, which both belong to the same clique. Hence G is connected. □

Note that if (6.6) is not satisfied for a weak Sidon set $A \subset [1, N]]$ then clearly there is $x \in S_1 \cap [1, N]$, so that there is an element $a \in A_1 \cap [1, N/2]$.

Lemma 6.5. *Let $A = \{1 = a_1 < a_2 < \cdots < a_n = N\}$ be a weak Sidon set. Then $B = \{1\} \cup (A + a_n) \subset [1, 2N]$ is a weak Sidon set with $n + 1$ elements containing $\{1, 2N - 1\}$ such that*

$$|\{x, x + 2N - 1\} \cap (B_1 \wedge B_1)| \geq 1, x \in \mathbb{N}, \tag{6.7}$$

where $B_1 = B \setminus \{1\}$.

Proof. $B_1 = A + a_n$ is clearly a weak Sidon set with set with $B_1 \wedge B_1 \subset [2N + 5, 4N - 2]$. Since $1 + B_1 \subset [1, 2N]$, B is also a weak Sidon set. Since the smallest element b_1 in B_1 satisfies $b_1 = a_1 + a_n > (a_{n-1} + a_n)/2 = b_n/2$, then (6.7) follows by the remark preceding this lemma. □

We are now ready for the proof of Theorem 6.1.

Proof of Theorem 6.1 By Lemma 6.5, there is a weak Sidon set $B \subset [1, 2N]$ satisfying condition (6.7). Then the result follows from Lemma 6.4. □

6.1.3 Embeding a clique in a large magic graph

In this Section we show that, for N large enough, there is a magic graph G of order $|V(G)| \leq cn^2 + o(n^2)$ for any constant $c > 5/4$. We first need the following lemmas.

Lemma 6.6. *Let $G = (V, E)$ be a graph of order N and $f : V \to [1, N]$ a bijection such that the edge sums $f(x) + f(y)$, $xy \in E$ are pairwise different. Then there is a super magic graph G' of order N which contains G as a spanning subgraph.*

Proof. Denote the vertices of G by $x_1, \ldots, x_N$ such that $f(x_i) = i$, $1 \leq i \leq N$. Let $B = \{f(x_i) + f(x_j),\ x_i x_j \in E\} = \{b_1 < \cdots < b_m\}$, where $m = |E|$, be the edge sumset. Let $Y = [b_1, b_m] \setminus B$ and set $Y_1 = Y \cap [b_1, N + 1]$, $Y_2 = Y \cap [N + 2, b_m]$. Consider the graph $G' = (V, E \cup E')$ where

$$E' = \{x_1 x_i,\ i + 1 \in Y_1\} \cup \{x_i, x_N,\ i + N \in Y_2\}.$$

It is easily checked that E' is well defined and that $\{i + j,\ x_i x_j \in E \cup E'\} = [b_1, b_m]$. Define f on the set of edges of G' as $f(x_i x_j) = k - i - j$, where $k = 3N$. Then, f is a super magic labeling of G'. □

Lemma 6.7. *Let $G = (V, E)$ be a connected graph graph and $f : V \to [1, N]$ an injective map such that the edge sums $f(x) + f(y)$, $xy \in E$ are pairwise different. Let $S = \{f(x)) + f(y),\ xy \in E(G)\}$ denote the edge sumset of f. If there is an increasing map $g : ([1, N] \setminus f(V)) \to ([3, 2N - 1] \setminus S$, such that*

(i) *$i < g(i) \leq N + i$ for all $i \in [1, N] \setminus f(V)$, and*

(ii) $g(i) \neq 2i$, *for all* $i \in [1, N] \setminus f(V)$,

then there is a supermagic connected graph G' *of order* N *which contains* G *a s a subgraph.*

Proof. Consider the graph G'_1 with vertex set $[1, N]$ and set of edges $E_1 \cup E_2$ where

$$E_1 = \{ij : i, j \in f(V) \text{ and } f^{-1}(i)f^{-1}(j) \in E(G)\},$$

and

$$E_2 = \{ij : i \in [1, N] \setminus f(V) \text{ and } j = g(i) - i\}.$$

Graph G'_1 clearly contains G as a subgraph. By the conditions on g, set E_2 is well defined, contains no loops and it is disjoint from E_1. Let us show that G'_1 is connected.

Suppose on the contrary that G'_1 is not connected. Denote by $A = f(V)$, $X = [1, N] \setminus A$ and $Y = [3, 2N - 1] \setminus S$. Since the subgraph of G'_1 induced by the vertices in A contains (an isomorphic copy of) G as a spanning subgraph, there is a connected component of G'_1 containing only vertices in X and edges in E_2. Let $X' \subset X$ be the vertex set of such a component. Give an orientation to each edge xy in the induced subgraph $G'_1[X']$ as (x, y) if and only if $y = g(x) - x$. In the resulting digraph, every vertex has out–degree 1, so that we have a directed cycle C'. Let $z_1, z_2, \ldots z_l$ denote the vertices of C' such that (z_i, z_{i+1}) is an arc of the directed cycle for each $i = 1, \ldots, l$, the subscripts taken modulo l, that is, $g(z_i) = z_i + z_{i+1}$. Since g is an injective function, we have $l > 2$. We may assume that $z_1 = \min\{z_1, \ldots, z_l\}$ and set $z_j = \max\{z_1, z_2, \ldots, z_l\}$. If $j = l$ then $g(z_{l-1}) = z_{l-1} + z_l > z_1 + z_l = g(z_l)$ contradicting the assumption that g is an increasing function. Suppose that $j < l$. We claim that $z_2 > z_l$. If $j > 2$, we have $z_j > z_l$ and $z_{j-1} > z_1$ which imply $g(z_{j-1}) = z_{j-1} + z_j > z_1 + z_l = g(z_l)$. Since g is an increasing function, we have $z_{j-1} > z_l$. By iterating the argument if necessary we eventually get $z_2 > z_l$. But then, $g(z_l) = z_l + z_1 < z_2 + z_1 = g(z_1)$, contradicting the minimality of z_1. These contradictions show that G'_1 must be a connected graph.

Note that the identity map $\iota : V(G'_1) \to [1, N]$ has all edge sums pairwise distinct. This is so for pair of edges from E_1 by the hypothesis on G and in all other cases by the definition of E_2, whose edge sums are in the complement of S, and by the injectivity of g. Therefore, G'_1 satisfies the conditions of Lemma 6.6and there is a magic graph G' containing G. □

Proof of Theorem 2. Let $\epsilon > 0$ be given and set $c = \sqrt{1 + 4\epsilon}$. From Ruzsa's bound (6.2), there is N_0 such that, for all $N > N_0$ and every weak

Sidon set $A \subset [1, N]$, we have $|A| \leq cN^{1/2}$. Therefore the set of sums satisfies

$$|S| = |A \wedge A| < |A|^2/2 \leq \frac{c^2}{2}N). \tag{6.8}$$

Let G be a complete graph of order $n = |A|$ with vertex set $V(G) = A$. We will show that there is a connected magic graph G' of order $N' = (\frac{5}{4}+\epsilon)N$ containing G as a subgraph by using Lemma 6.7.

For a set $U \subset \mathbb{N}$ and integers $x < y$ we denote by $U(x, y) = |U \cap [x, y]|$.

Note that $-A+(N+1)$ is also a weak Sidon set contained in $[1, N]$. Since one of A and $-A + (N + 1)$ has at least half of the sums in $[1, N]$, we may assume that $S(N + 1, 2N - 1) \leq S(3, N)$.

Let $N' = (1 + \frac{c^2}{4})N$ and set $X = [1, N'] \setminus A$ and $Y = [3, 2N - 1] \setminus S$. We then have

$$Y(N, 2N - 1) \geq N - S(N + 1, 2N - 1) \geq (1 - \frac{c^2}{4})N = (\frac{3}{4} - \epsilon)N. \tag{6.9}$$

Let us define $g : X \to Y$ as follows. For $x \in X((\frac{3}{4} - \epsilon)N, (\frac{5}{4} + \epsilon)N - 1)$ we have $N' + x \in Y(2N, 2N' - 1)$, so we define $g(x) = N' + x$. Now, from (6.9), we have as many elements in $Y(N, 2N - 1)$ as in $[1, (\frac{3}{4} - \epsilon)N]$. Therefore we may define an increasing map from $X \cap [1, (\frac{3}{4} - \epsilon)N]$ to $Y(N, 2N - 1)$ satisfying $x \leq g(x) \leq N' + x)$ for all $x \in X$. More precisely, if $X = \{x_1 < x_2 < \cdots < x_k\}$, and we denote by $X_i = \{x_{i+1}, \ldots, x_k\}$ de final segment of length $k - i$ of X, then

$$g(x_i) = \max\{y \in Y \setminus g(X_i) : y \leq N + x_i\}. \tag{6.10}$$

It can be easily checked that g satisfies properties (i) and (ii) in Lemma 6.7. Therefore, there is a graph G' of order $N' = (\frac{5}{4} + \epsilon)N$ containing G, a clique of order n. □

Let $f(n)$ denote the size of the largest Sidon set in $[0, n]$ and $f_w(n)$ be the size of the largest weak Sidon set in $[0, n]$, so that $f'(n) \geq f(n)$. By (6.2), it is clear that $lim_{n\to\infty} n^{-1/2} f_w(n) \geq 1$. By the Erdős -Chowla Theorem, $lim_{n\to\infty} n^{-1/2} f(n) = 1$, so that the same is true for $f_w(n)$. Therefore, we can find weak Sidon sets of order n contained in $[1, f_w(n))]$ with $f_w(n) = n^2 + o(n^2)$. Hence we have the following Corollary:

Corollary 6.8. *There is a connected graph G of order*

$$|V(G)| = 2n^2 + o(n^2)$$

containing a clique of order n.

6.2 Magic trees containing a given forest

Below we give the procedure that provides a proof of Theorem ??. We show an example which illustrates this procedure at the end of the paper.

Given a forest F, let $T_0 = (V_0, E_0)$ be any tree of order n containing F. In order to extend T_0 to a magic tree, we will introduce the following notation for the vertices of T_0. Let r be any vertex of V_0, which will represent the root of T_0. Partition the vertices of V_0 into levels

$$V_0^i = \{x \in V_0 : d(r, x) = i)\},\ i \geq 0,$$

where $d(r, x)$ denotes the distance in T_0 between x and r.

We define a labeling $f_0 : V_0 \to [1, n]$ recursively on the levels of the tree T_0 rooted at r. Set $f_0(r) = 1$. Suppose that f_0 has been defined in level $V_i,\ i \geq 0$. Take the vertex with smallest label in V_i whose neighbours in V_{i+1} have not been yet labelled, and label them with the smallest labels not yet used. In this way we define and injective map and the labels in a given level are consecutive.

Let $S = \{f_0(u) + f_0(v) : uv \in E_0\}$ denote the edge sumset of f_0. By the definition of f_0 the sums of S are pairwise different, and $|S| = |E_0| = n - 1$. If the elements of S are consecutive then, by Lemma x, T_0 is already a super magic tree.

Suppose that the elements of S are not consecutive numbers and let

$$\bar{S} = [\min S, \max S] \setminus S.$$

We have $\min S = 3$ and $\max S \leq 2n - 1$, so that

$$h = |\bar{S}| = \max S - n - 1 \leq n - 2.$$

In what follows we proceed to extend the tree T_0 in order to fill the gaps in S. This is done in at most three steps.

Let $\bar{S}_0 = \{s_1 < s_2 < \cdots < s_k\}$ be a maximal subset of $\bar{S}$ such that, for every $s_i \in \bar{S}_0$, there is $v_i \in V_0$ with $f_0(v_i) = s_i + i - 1$.

Let $X = \{x_1, \ldots, x_k\}$ be a set of k additional points and construct a new tree $T_1 = (V_1, E_1)$ with vertex set $V_1 = V_0 \cup X$ and $E_1 = E_0 \cup \{v_1x_1, \ldots, v_kx_k\}$. Consider the labeling $f_1 : V_1 \longrightarrow [-k + 1, n]$ defined by

$$f_1(v) = \begin{cases} f_0(v), & v \in V_0 \\ 1 - i, & v = x_i \in X \end{cases}$$

The edge sumset of f_1 is $S_1 = S \cup \bar{S}_0 \subset [3, \max S]$. If $S_1 = [3, \max S]$ then $f_1' = f_1 + k$ is a vertex labeling that extends to a supermagic labeling of T_1 and we are done.

Suppose that S_1 is a proper subset of $[3, \max S]$.

Let $Y = \{y_i, n+1 \le i \le \max \bar{S} + |\bar{S}| - 1\}$ be a set of additional points. Let $ww' \in E(T_1)$ be the edge with largest edge sum, $f(w) + f(w') = \max S$, where $f(w) < f(w') = n$. We extend T_1 to the tree $T_2 = (V_2, E_2)$ where $V_2 = V_1 \cup Y$ and

$$E_2 = E_1 \cup \{wy_i : y_i \in Y\}.$$

Take $f_2 : V_2 \longrightarrow [-k+1, n+|Y|]$ defined by

$$f_2(v) = \begin{cases} f_1(v), & v \in V_1 \\ i, & v = y_i \in Y \end{cases}$$

The edge sumset of f_2 is

$$S_2 = S \cup \bar{S}_0 \cup (f(w) + [n+1, \max \bar{S} + |\bar{S}| - 1]) \subset [3, f(w) + \max \bar{S} + |\bar{S}| - 1].$$

By the choice of w, the union above is disjoint and

$$\bar{S}_2 = [3, f(w) + \max \bar{S} + |\bar{S}| - 1] \setminus S_2 = \bar{S} \setminus \bar{S}_0 = \{s_{k+1} < \cdots < s_h\}.$$

Note that, for each $i \in \{k, \ldots, h-1\}$, we have $s_{i+1} + i \in [n+1, n+|Y|]$. Let $Y' = \{y_{j_i} : j_i = s_{i+1} + i,\ k \le i \le h-1\} \subset Y$ and consider a set of new points $Z = \{z_i :\ k \le i \le h-1\}$. Let $T_3 = (V_3, E_3)$ with $V_3 = V_2 \cup Z$ and $E_3 = E_2 \cup \{z_k y_{j_k}, \ldots, z_{h-1} y_{j_{h-1}}\}$. Take $f_3 : V_3 \longrightarrow [-h+1, n+|Y|]$ defined by

$$f_3(v) = \begin{cases} f_2(v), & v \in V_2 \\ -i, & v = z_i \in Z \end{cases}$$

Now the edgesum of f_3 is $S_3 = [3, f(w) + \max \bar{S} + |\bar{S}| - 1]$. Therefore, the edgesum of $g = f_3 + h$ is a set of consecutive integers. By Lemma x, g extends to a supermagic labeling of T_3, and the order of this supermagic tree, which contains T_0, is

$$N = |V_3| = n + h + |Y| = \max \bar{S} + 2h - 1 \le 4n - 7.$$

□

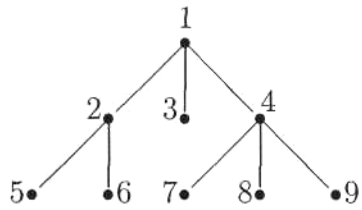

Figure 6.1: Labeling of T_0.