UNIVERSITAT POLITÈCNICA DE CATALUNYA

THE EQUATIONS OF REES ALGEBRAS OF IDEALS OF ALMOST-LINEAR TYPE

by

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Certainly the best times were when I was alone with the mathematics, free of ambition and of pretence, and indifferent to the world.

Robert P. Langlands

Mathematical retrospections

Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine.

Michael F. Atiyah

From Barry Mazur's foreword to the 2005 reprint of

Tobias Dantzig's *Number: The Language of Science.*

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Contents

Abstract

The Rees algebra $\mathbf{R}(I) = R[It] \subset R[t]$ of an ideal $I \subset R$ of a Noetherian local ring R plays a major role in commutative algebra and in algebraic geometry, since $\mathrm{Proj}(\mathbf{R}(I))$ is the blowup of the affine scheme $\mathrm{Spec}(R)$ along the closed subscheme $\mathrm{Spec}(R/I)$.

So far, the problem of describing the equations of Rees algebras of ideals, as well as other related algebras, has shown to be relevant in order to further understand these major algebraic objects. The equations of $\mathbf{R}(I)$ arise as the elements in the kernel of a symmetric presentation $\varphi: V \to \mathbf{R}(I)$. While this kernel may differ from one presentation to another, the degrees of a minimal generating set of homogeneous elements are known to be independent of φ. The top degree among such generating sets, known as the *relation type* and denoted by $\mathrm{rt}(I)$, is a coarse measurement of the complexity of the underlying Rees algebra which is nonetheless a useful numerical invariant. The ideals I such that $\mathrm{rt}(I) = 1$, known as ideals of *linear type*, have been intensely studied so far.

In this dissertation, we tackle the problem of describing the equations of $\mathbf{R}(I)$ for $I = (J, y)$, with $y \notin J$ and J being of linear type, i.e., for ideals of linear type up to one minimal generator. Throughout, such ideals will be referred to as ideals of *almost-linear type*.

The main results of this work stem from two different approaches towards the problem.

In Theorem A, we give a full description of the equations of Rees algebras of ideals of the form $I = (J, y)$, with $J = (x_1, \ldots, x_s)$ satisfying an homological vanishing condition. Theorem A permits us to recover and extend well-known results about families of ideals fulfilling the almost-linear type condition due to Vasconelos, Huckaba, Trung, Heinzer and Kim, among others.

Let $\alpha: \mathbf{S}(I) \to \mathbf{R}(I)$ be the canonical morphism from the symmetric algebra of I to the Rees algebra of I. In Theorem B, we prove that the injectivity of a single component of

$\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ propagates downwards, provided I is of almost-linear type. In particular, this result gives a partial answer to a question posed by Tchernev.

Finally, packs of examples are introduced, which illustrate the scope and applications of each of the results presented. The author also gives a collection of computations and examples which motivate ongoing and future research.

Notations

R: commutative ring with unit $1 \neq 0$, most often Noetherian and local.

k: field.

I, J: finitely generated ideals of R, with $J \subset I$.

$J_i = (x_1, \ldots, x_i)$.

M: R-module.

$(M_1 :_R M_2) = \{a \in R \mid aM_2 \subseteq M_1\}$.

$(J :_R y^\infty) = \sqcup_{n \geq 1}(J :_R y^n)$: saturation.

$M_1 \otimes_R M_2$: tensor product of R-modules.

$\mathfrak{p}$, $\mathfrak{q}$: prime ideals.

$\mathfrak{m}$: maximal (or maximal homogeneous) ideal.

$VV_J(I)_n = (J \cap I^n)/JI^{n-1}$: n-th Valabrega-Valla module of I with respect to J.

$\mathrm{Ass}(R/I)$: associated prime ideals of R/I.

$\mathrm{Min}(R/I)$: minimal prime ideals of R/I.

$\mathfrak{N}(R)$: nilradical of R.

$U = \oplus_{n \geq 0} U_n$: standard R-algebra.

$U_+ = \oplus_{n > 0} U_n$: irrelevant ideal of U.

V, W: polynomial rings over R.

φ, ψ: polynomial presentations.

$Q = \oplus_{n \geq 0} Q_n$: homogeneous ideal.

$Q\langle r \rangle$: subideal of Q generated by its homogeneous elements of degree $\leq r$.

∂: differential operator of a complex of R-modules.

$H_i(z\,;R)$: i-th Koszul homology.

$Z_i(z\,;R)$: i-th Koszul cycles.

$B_i(z\,;R)$: i-th Koszul boundaries.

$\mathbf{S}(M) = \oplus_{n\geq 0}\mathbf{S}_n(M)$: symmetric algebra of M.

$\mathbf{R}(I) = \oplus_{n\geq 0}I^n$: Rees algebra of I.

$\mathbf{G}(I) = \oplus_{n\geq 0}I^n/I^{n+1}$: associated graded ring of I.

$\mathbf{F}(I) = \oplus_{n\geq 0}I^n/\mathfrak{m}I^n$: fiber cone of I with respect to $\mathfrak{m}$.

α_I, β_I, γ_I (or simply denoted α, β and γ) are the canonical blowing-up morphisms:

$\alpha_I : \mathbf{S}(I) \to \mathbf{R}(I)$;

$\beta_I : \mathbf{S}(I/I^2) \to \mathbf{G}(I)$;

$\gamma_I : \mathbf{S}(I/\mathfrak{m}I) \to \mathbf{F}(I)$.

$E(I)_n = \ker\alpha_n/\mathbf{S}_1(I)\cdot\ker\alpha_{n-1}$: n-th module of effective relations.

X^σ, x^σ: monomials in multi-index notation.

$\mathrm{supp}(\sigma)$: support of an integer vector.

$\dim(R)$: Krull dimension.

$\lambda(M)$: length of an R-module M.

$\mu(I)$: minimal number of generators of I.

$\mathrm{ht}(I)$: height of I.

$\mathrm{pd}_R(M)$: projective dimension of an R-module M.

$\mathrm{rt}(I)$: relation type of I.

$\mathrm{r}_J(I)$: reduction number with respect to J.

$\ell(I)$: analytic spread of I.

$\mu(I) - \mathrm{ht}(I)$: deviation of I.

$\mathrm{ad}(I) = \ell(I) - \mathrm{ht}(I)$: analytic deviation of I.

$\mathrm{sd}(I) = \mu(I) - \ell(I)$: second analytic deviation of I.

Summary of results

For the sake of clarity, let us first introduce the basic assumptions and state the main results presented along this work.

The equations of Rees algebras of ideals of almost-linear type

Recall that the reduction number of I with respect to J, $J \subseteq I$, is the least integer $r \geq 0$ such that $I^{r+1} = JI^r$, denoted by $\mathrm{r}_J(I)$. Set $V = R[X_1, \ldots, X_s, Y]$ and let $\varphi : V \to \mathbf{R}(I)$ be the polynomial presentation of $\mathbf{R}(I)$, the Rees algebra of I, sending X_i to $x_i t$ and Y to yt. Let $Q = \oplus_{n \geq 1} Q_n$ be the kernel of φ. Given an integer $m \geq 0$, set $Q\langle m \rangle \subset Q$ the ideal generated by the homogeneous elements of Q of degree at most m in $X_1, \ldots, X_s, Y$. The relation type of I, denoted by $\mathrm{rt}(I)$, is the least integer $N \geq 1$ such that $Q = Q\langle N \rangle$. Let $\mathbf{G}(I) = \oplus_{n \geq 0} I^n / I^{n+1}$ be the associated graded ring of I. If $z \in R \backslash \{0\}$, we will denote by z^* the initial form of z in $\mathbf{G}(I)$, i.e., $z^* = z + I^{n+1}$, where $z \in I^n \backslash I^{n+1}$.

Theorem A. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. Let $x_1, \ldots, x_s, y$ be a minimal generating set of I, where $J = (x_1, \ldots, x_s)$ is a reduction of I with reduction number $r = \mathrm{r}_J(I)$. Assume that $x_1, \ldots, x_s$ verify the following condition for all $n \geq 2$:*

$$((x_1, \ldots, x_{i-1})I^{n-1} : x_i) \cap I^{n-1} = (x_1, \ldots, x_{i-1})I^{n-2}, \text{ for all } i = 1, \ldots, s. \qquad (\mathcal{T}_n)$$

Then, for each $n \geq 2$, the map sending $F \in Q_n$ to $F(0, \ldots, 0, 1) \in (JI^{n-1} : y^n)$ induces an isomorphism of R-modules

$$\left[\frac{Q}{Q\langle n-1 \rangle} \right]_n \cong \frac{(JI^{n-1} : y^n)}{(JI^{n-2} : y^{n-1})}.$$

In particular, $\mathrm{rt}(I) = \mathrm{r}_J(I) + 1$ and there is a form $Y^{r+1} - \sum X_i F_i \in Q_{r+1}$, with $F_i \in V_r$, such that $Q = (Y^{r+1} - \sum X_i F_i) + Q\langle r \rangle$. Moreover, if $x_1, \ldots, x_s$ is an R-sequence and $x_1^*, \ldots, x_{s-1}^*$ is a $\mathbf{G}(I)$-sequence, then $x_1, \ldots, x_s$ verify condition $(\mathcal{T}_n)$ for all $n \geq 2$.

Theorem A permits to recover, even extend, a set of well known results regarding classes of almost-linear type ideals due to Vasconelos [Vasconcelos91, Theorem 2.3.3], Trung [Trung87, Theorem 1.2, Proposition 5.1], Huckaba [Huckaba89, Theorems 1.4, 1.5] and Heinzer-Kim [HK03, Theorem 5.6].

Examples

Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $a, b \in R$ be an R-sequence. As an application of Theorem A, we give a full description of the equations of $\mathbf{R}(I)$, where I is any of the following ideals:

(*i*) $I = (a^p, b^p, ab^{p-1})$ with $p \geq 2$ an integer (see Example 2.6.1);

(*ii*) $I = (a^p, b^p, a^2 b^{p-2})$ with $p \geq 2$ an odd integer (see Example 2.6.5).

These examples may be considered folklore and they are easily computable, although this computation may be expensive, for a given $p \geq 2$. However, our approach is new.

The injectivity of the canonical blowing-up morphism

Let $\alpha_I : \mathbf{S}(I) \to \mathbf{R}(I)$ be the canonical morphism from the symmetric algebra of I to the Rees algebra of I. We write $\alpha_{I,p}$ the p-th graded component of α_I. Fixed $p \geq 2$, Tchernev asked in [Tchernev07] whether $\alpha_{I,p}$ being an isomorphism implies that $\alpha_{I,n}$ is an isomorphism for each $2 \leq n \leq p$. Observe that, as regards the following result, we do not need the Noetherian hypothesis.

Theorem B. *Let R be a commutative ring. Let $I = (x_1, \ldots, x_s, y)$ be an ideal of R and let $p \geq 2$ be an integer. Suppose that the ideal $J = (x_1, \ldots, x_s)$ verifies that $\alpha_{J,n} : \mathbf{S}_n(J) \to J^n$ is an isomorphism for all $2 \leq n \leq p$. Then the following conditions are equivalent:*

(*i*) $\alpha_{I,p} : \mathbf{S}_p(I) \to I^p$ *is an isomorphism;*

(*ii*) $\alpha_{I,n} : \mathbf{S}_n(I) \to I^n$ *is an isomorphism for each $2 \leq n \leq p$.*

Remark that the ideals of almost-linear type fulfil the hypotheses of Theorem B, hence the equivalence of (*i*) and (*ii*) holds true for such ideals.

Example

Kühl [Kühl82, Example 1.4] gave an example of a finitely generated ideal I with $\alpha_{I,n}$ being an isomorphism for $n \geq 3$, while $\ker \alpha_2 \neq 0$. In particular, $\widetilde{\alpha}_I : \mathrm{Proj}(\mathbf{R}(I)) \to \mathrm{Proj}(\mathbf{S}(I))$ is an isomorphism of schemes, but α_I is not an isomorphism of R-algebras. The following example shows, for a given integer $p \geq 2$, an ideal I such that $\alpha_{I,n}$ is an isomorphism for $n \geq p + 1$, whereas $\alpha_{I,p}$ is not.

Example 3.9.1. Let k be a field and let $p \geq 2$. Let $S = k[U_0, \ldots, U_p, X, Y]$ be a polynomial ring and let Q be the ideal of S defined as $Q = Q_1 + (U_0 X^p)$, where

$$Q_1 = (U_0 Y, U_0 X - U_1 Y, U_1 X - U_2 Y, \ldots, U_{p-1} X - U_p Y, U_p X).$$

Let R be the factor ring $S/Q = k[u_0, \ldots, u_p, x, y]$ and consider the ideal $I = (x, y) \subset R$. Then, $\alpha_{I,n}$ is an isomorphism for all $n \geq p + 1$, whereas $\alpha_{I,p}$ is not.

Chapter 1

Introduction

1.1   Rees algebras and blowing-up

So far, several graded rings associated with a (Noetherian local) ring R and an ideal I have been given much attention within the bounds of commutative algebra and algebraic geometry: the Rees algebra $\mathbf{R}(I) = R[It] = \oplus_{n \geq 0} I^n$, the associated graded ring $\mathbf{G}(I) = \oplus_{n \geq 0} I^n / I^{n+1}$, the fiber cone $\mathbf{F}(I) = \oplus_{n \geq 0} I^n / \mathfrak{m} I^n$, the extended Rees algebra $R[It, t^{-1}]$, the symbolic Rees algebra $\mathbf{R}_s(I) = \oplus_{n \geq 0} I^{(n)}$, as well as approximations to these rings such as the symmetric algebras $\mathbf{S}_R(I)$ and $\mathbf{S}_{R/I}(I/I^2)$. All these graded rings are best known as blowing-up algebras. Today, the Rees algebra plays a central role among them, although originally more attention was focused on the associated graded ring and the extended Rees algebra.

Rees algebras were originally introduced by David Rees to study several problems in commutative algebra, thus yielding, according to [HH93], to three most remarkable results that have become classical over time: the construction of a counterexample to Zariski's generalisation of Hilbert's 14-th problem [Rees58], the Artin-Rees lemma [Rees56] and a celebrated criterion for a local ring to be analytically unramified [Rees61]. Rees algebras of ideals in Noetherian rings capture in another Noetherian ring a great deal of information about how the ideal I sits in the ring R and how its powers change. The study of Rees algebras is important to several fields of study and open problems in algebra as well as in geometry.

As it is claimed in [HH93, Introduction] ever since the introduction of the Hilbert function and the discovery of its key numerical invariants, the importance of understanding how the ideals behave as we raise them to powers is widely acknowledged. The study of Cohen-Macaulayness, normality or torsion-freeness of blowing-up algebras has played a major role so far, and such topics have been often studied in terms of numerical invariants.

From an algebraic geometry perspective, given the affine scheme $X = \mathrm{Spec}(R)$ and a closed subscheme $Z = \mathrm{Spec}(R/I) \subset X$, the morphism of schemes $Y = \mathrm{Proj}(\mathbf{R}(I)) \to \mathrm{Spec}(R) = X$ is the construction known as *blowing-up of X along Z*. The procedure of blowing-up is a standard, explicit way of modifying a variety (see [HartshorneAG, I.4, II.7]). In the review [Abramovich11] the blowing-up is presented as a surgery operation in which a subvariety $Z \subset X$ is removed and replaced by the sets of normal directions of Z in X. When X and Y are smooth, Z is replaced by $\mathbb{P}(N_{Z/X})$ the projectivization of the normal bundle. In particular, if Z is a point, Z is replaced by $\mathbb{P}^{\dim X - 1}$.

Satisfactory answers to open questions related to the birrational geometry of varieties have been given using techniques relying on blowing-up. As a first example, recall that any birational map between algebraic surfaces factors as a sequence of blowing-ups modulo isomorphism [Beauville96, Chapter II] (see also [HartshorneAG, Remark V.5.6.1] for generalisations in higher dimensions). Blowing-up plays also a central role in the problem of $r$esolution of singularities, which can be stated as follows: whether given a variety X there exists a smooth variety Y birational to X, such a birational map $\pi : Y \to X$ being known as a resolution of singularities itself. In analysing nonsingular varieties in characteristic zero one can invoke, under mild conditions, the tools and techniques of complex manifold theory. In characteristic zero a resolution of singularities case can be fully achieved as a finite composition of blowing-ups. While the existence of a resolution of singularities is not known in arbitrary characteristic, it has already been shown for curves, surfaces and 3-folds (see, for instance, [Hauser10] and [Kollar07]).

A subject recently introduced in the context of computer geometric modelling are the methods of implicitization by moving curves and surfaces (see e.g. [CSCh98]). Such methods essentially provide with a description of the image of a rational map between projective spaces in terms of implicit equations. Such equations can be obtained by means of the computation

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of an elimination ideal, which turns out to be an ideal of equations of a Rees algebra. For a taste of this approach see, for instance, [Cox08], [CHW08] and [CD'A10], as well as the references therein.

Among the many sources of the work presented in this dissertation, we want to highlight the classical work of Herzog, Simis and Vasconcelos in [HSV81]. This paramount piece of work, as well as the many references therein, keep being a fundamental source of valuable insights for the algebraic study of blowing-up.

1.2   Scope of the present work

We can split the scope of our work into a list of main achievements:

(i) Establish conditions leading to procedures to explicitely describe the equations of blowing-up algebras of ideals of almost-linear type. The criteria displayed extend a set of well-known results about the equations of blowing-up algebras due to Vasconcelos, Trung, Huckaba, Heinzer and Kim.

(ii) Establish conditions under which the injectivity of a single component of the canonical morphism $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ propagates downwards the graded structure. These criteria lead to sufficient conditions for an ideal to be of linear type and provides a partial positive answer to a question posed by Tchernev.

(iii) Give examples and applications of the results presented that may help to better understand the extent of the hypotheses considered.

(iv) Present additional work in progress, including some preliminary results and ongoing experimentation suggesting future research pathways.

We will make use of elementary linear and commutative algebra as well as standard results on (graded) Koszul homology. Among the specific tools used, the *module of effective relations* of a standard algebra will play a major role. Although not so widespread across the literature, the module of effective relations is a very natural invariant to be considered in this context: it encodes useful information about the fresh generators of the equations of standard algebras and it can be defined canonically. This module can be seen as the first graded Koszul homology of a sequence of elements of a standard R-algebra. Playing with

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homological properties, we are able to establish the obstructions for such homology to vanish in terms of colon ideals.

Special attention will be paid to the degrees of the equations of blowing-up algebras: these are coarse numerical invariants. Low cost methods to know in advance what such degrees are for a given ideal, would be of great help for designing more efficient algorithms for computing these equations.

The conditions to be considered in the present work will involve most often hypotheses on the generating sets of ideals. We tackle the problem of describing the equations of $\mathbf{R}(I)$ for $I = (J, y)$, with $y \notin J$ and J being of linear type, i.e., for ideals of linear type up to one minimal generator. Throughout, such ideals will be referred to as ideals of *almost-linear type*. Remark that taking an ideal I of the form (J, y) and assuming additional hypotheses on the ideal J and the relation between J and the element y, is a natural and common approach in the context (see, for instance, [Costa85], [HMV89, Theorem 4.7], [HSV81, Proposition 3.9], [Valla80, Theorem 2.3 and Proposition 2.5]).

The numerical invariants of ideals which we will make use of explicitely are the deviation $\mu(I) - \mathrm{ht}(I)$, the analytic spread $\ell(I)$, the analytic deviation $\mathrm{ad}(I) = \ell(I) - \mathrm{ht}(I)$ and the second analytic deviation $\mathrm{sd}(I) = \mu(I) - \ell(I)$. The techniques we will rely upon and the subsequent conditions on the generators of $I = (J, y)$ and J, as well as their relationship, will often imply small upper bounds on such invariants. Equimultiple ideals, i.e., ideals with analytic deviation $= 0$, are among the classes of ideals we will come across. Such ideals include, for instance, $\mathfrak{m}$-primary ideals in a Noetherian local ring $(R, \mathfrak{m})$.

1.3   Preliminary definitions and results

For the sake of self containment we devote a condensed section to summarise a set of well-known definitions and results that will support the proofs and discussions along the core exposition in Chapters 2, 3 and 4. Throughout, R will denote a commutative ring with unit $1 \neq 0$, referred to as a commutative ring, for short.

1.3.1 Blowing-up algebras

By a standard R-algebra we mean a $\mathbb{N}$-graded commutative algebra over R of the form $U = \oplus_{n \geq 0} U_n$, with $U_0 = R$ and such that $U = R[U_1]$, with U_1 finitely generated as an R-module. We will denote by $z_1, \ldots, z_s$ a minimal generating set of U_1 as an R-module. The irrelevant ideal of U, denoted by U_+, is the graded ideal generated by the elements of positive degree: $U_+ = (U_1) = \oplus_{n > 0} U_n$.

Let $I = (x_1, \ldots, x_s)$ be an ideal of R. Among the several standard algebras associated to I, those known as blowing-up algebras will play a paramount role: the Rees algebra of I is the R-algebra $\mathbf{R}(I) = \oplus_{n \geq 0} I^n = R[It] \subset R[t]$; the associated graded ring of I is the R/I-algebra $\mathbf{G}(I) = \oplus_{n \geq 0} I^n/I^{n+1} = \mathbf{R}(I)/I\mathbf{R}(I)$; the fiber cone of I with respect to a maximal ideal $\mathfrak{m}$ is the $R/\mathfrak{m}$-algebra $\mathbf{F}(I) = \oplus_{n \geq 0} I^n/\mathfrak{m}I^n = \mathbf{R}(I)/\mathfrak{m}\mathbf{R}(I) = \mathbf{G}(I)/\mathfrak{m}\mathbf{G}(I)$.

The symmetric algebra of an R-module M is the standard R-algebra defined as $\mathbf{S}_R(M) = \oplus_{n \geq 0} \mathbf{S}_n(M) = T(M)/\mathfrak{S}$, where $T(M) = \oplus_{n \geq 0} M^{\otimes n}$ stands for the tensor algebra of M and $\mathfrak{S}$ stands for the bilateral ideal of $T(M)$ generated by all the elements of the form $x \otimes y - y \otimes x$ for $x, y \in M$. Since $\mathfrak{S}$ is generated by homogeneous elements of degree 2, $\mathbf{S}_0(M) = R$ and $\mathbf{S}_1(M) = M$ and there is a canonical injection $\iota : M \to \mathbf{S}(M)$. For an account of basic properties of symmetric algebras, see [Bourbaki, Chapter III, Section 6].

If U is an R-standard algebra, we can canonically define a surjective morphism of standard R-algebras $\alpha : \mathbf{S}(U_1) \to U$ induced by the inclusion $U_1 \hookrightarrow U$ in degree one. When $U = \mathbf{R}(I)$ we will say that $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ is the canonical blowing-up morphism. We define $\beta = \alpha \otimes \mathbf{1}_{R/I} : \mathbf{S}(I/I^2) \to \mathbf{G}(I)$ and $\gamma = \alpha \otimes \mathbf{1}_{R/\mathfrak{m}} : \mathbf{S}(I/\mathfrak{m}I) \to \mathbf{F}(I)$.

1.3.2 Sequential conditions

In the sequel we will consider different assumptions on sequences of elements in R. For a given sequence of elements $x_1, \ldots, x_s \in R$, we will consistently denote $J_i = (x_1, \ldots, x_i)$ and set $J_0 = (0)$. Let us recall some well-known definitions and results that will be useful for our aims.

Definition 1.3.1. *A sequence of elements $x_1, \ldots, x_s \in R$ is said to be an R-sequence (or regular sequence in R) if the two following conditions hold:*

(*i*) $((x_1, \ldots, x_i) : x_{i+1}) = (x_1, \ldots, x_i)$, *for all* $0 \le i \le s - 1$;

(*ii*) $(x_1, \ldots, x_s) \ne R$.

1.3.2. If I is an ideal of R generated by an R-sequence $x_1, \ldots, x_s$, then $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ is an isomorphism. Moreover, $\mathbf{S}(I) \cong \mathbf{R}(I) \cong R[X_1, \ldots, X_s]/(x_i X_j - x_j X_i \mid 1 \le i < j \le s)$. See [Micali64, Chapitre Premier, Théorème 1 and Lemme 2].

For many purposes, the elements of an R-sequence behave just like the variables in a polynomial ring over a field. This is exactly the case for a Noetherian local ring containing a field (see [Hartshorne66]).

Recall that in a Noetherian local ring, R-sequences can be detected by means of the Koszul homology.

1.3.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $x_1, \ldots, x_s$ be a sequence of elements in R. Then $x_1, \ldots, x_s$ is an R-sequence if and only if $H_i(x_1, \ldots, x_s; R) = 0$ for all $i \ge 1$. In particular, the Koszul complex gives rise to a free resolution of $R/(x_1, \ldots, x_s)$ of length s.

We will say that a sequence $x_1, \ldots, x_s \in R$ is a permutable R-sequence if $x_{\sigma(1)}, \ldots, x_{\sigma(s)}$ is an R-sequence, for every permutation $\sigma \in \mathfrak{S}_s$. If an R-sequence is contained in the Jacobson radical then it is permutable (in particular, in a local ring, every R-sequence is permutable). Monomials in permutable R-sequences behave just like monomials in indeterminates of a polynomial ring, in the sense that the basic arithmetic rules for monomial ideals hold true in a way that is made precise in the following statements (see [KS03]).

1.3.4. Let $\underline{x} = x_1, \ldots, x_s$ be a permutable R-sequence. Let $x^a = x_1^{a_1} \cdots x_s^{a_s}$ and $x^b = x_1^{b_1} \cdots x_s^{b_s}$ be monomials in $\underline{x}$. Then we can define the greatest common divisor (gcd) and the least common multiple (lcm) of x^a and x^b as follows:

$\gcd(x^a, x^b) = x^c$, where $c_i = \min\{a_i, b_i\}$ for all $1 \le i \le s$;

$\mathrm{lcm}(x^a, x^b) = x^d$, where $d_i = \max\{a_i, b_i\}$ for all $1 \le i \le s$.

1.3.5. Let $\underline{x}$ be a permutable R-sequence. Let $\mathfrak{a} = (m_1, \ldots, m_r)$, $\mathfrak{b} = (n_1, \ldots, n_t)$ be ideals generated by monomials in $\underline{x}$ (referred to as monomial ideals in $\underline{x}$). Let m be a monomial in $\underline{x}$. Then the following basic computational properties follow:

a) $\mathfrak{a} : m = \sum_{j=1}^{r} (\operatorname{lcm}(m_j, m)/m)R;$

b) $\mathfrak{a} \cap mR = \sum_{j=1}^{r} \operatorname{lcm}(m_j, m)R;$

c) $\mathfrak{a} \cap \mathfrak{b} = \sum_{j=1}^{r} \sum_{i=1}^{t} \operatorname{lcm}(m_j, n_i)R;$

d) $\mathfrak{a} : \mathfrak{b} = \cap_{i=1}^{t} \sum_{j=1}^{r} (\operatorname{lcm}(m_j, n_i)/n_i)R$

Moreover, $\mathfrak{a}$ is minimally generated by a unique generating set formed by monomials in $\underline{x}$.

If $x \in R \backslash \{0\}$, the leading form of x in $\mathbf{G}(I) = \oplus_{n \geq 0} I^n/I^{n+1}$, denoted by x^*, is the image of x in I^n/I^{n+1}, where $x \in I^n \backslash I^{n+1}$. The following result was proved in [VV78, Theorem 2.3] and will be used throughout:

1.3.6. Let R be a Noetherian ring, I an ideal of R and let $x_1, \ldots, x_s$ be an R-sequence. Then the leading forms $x_1^*, \ldots, x_s^*$ form a $\mathbf{G}(I)$-sequence if and only if, for all $1 \leq i \leq s$ and all $m \geq 1$, it is verified that

$$(x_1, \ldots, x_i) \cap I^m = \sum_{j=1}^{i} x_j I^{m-\nu(x_j)}, \qquad (VV)$$

where $\nu(x_j)$ stands for the degree of the initial form x_j^* in $\mathbf{G}(I)$. Further, if R is Noetherian local and $x_1, \ldots, x_s$ are part of a minimal generating set of I, then $\nu(x_i) = 1$ for all $1 \leq i \leq s$. Thus, setting $J_i = (x_1, \ldots, x_i)$, the condition (VV) has the form

$$J_i \cap I^m = J_i I^{m-1}.$$

Definition 1.3.7. *If $J \subset I$ are ideals of R, we will define the n-th Valabrega-Valla module of I with respect to J, denoted by $VV_J(I)_n$, as the quotient $J \cap I^n/JI^{n-1}$.*

We introduce a notion that generalises regular sequences, the notion of d-sequence, which was introduced by Huneke and has shown to be useful in a wealth of scenarios.

Definition 1.3.8. *A sequence $x_1, \ldots, x_s \in R$ is said to be a d-sequence if the following two conditions are fulfilled:*

(i) $x_1, \ldots, x_s$ *minimally generate* $(x_1, \ldots, x_s)$

(ii) $((x_1, \ldots, x_i) : x_{i+1}x_k) = ((x_1, \ldots, x_i) : x_k)$ *for all* $0 \leq i \leq s-1$ *and* $k \geq i+1$.

It is readily seen that condition (ii) in Definition 1.3.8 is equivalent to

(ii^*) $((x_1, \ldots, x_i) : x_{i+1}) \cap I = (x_1, \ldots, x_i)$ for $0 \le i \le s - 1$, where $I = (x_1, \ldots, x_s)$.

1.3.9. If I is an ideal of R generated by a d-sequence, then $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ is an isomorphism (see the works of [Huneke80, Theorem 3.1] and [Valla80, Theorem 3.15]).

1.3.10. If $\mathrm{ht}(I) = n$, then I contains a d-sequence of both length and height $= n$ (see [HSV81, Proposition 6.2]). But an ideal I may also contain d-sequences of length $> \mathrm{ht}(I)$: for instance, if $\mathfrak{p}$ is a prime ideal of R with $\mathrm{ht}(\mathfrak{p}) = n$ such that $\mathrm{grade}(\mathfrak{p}, R) = n$, $R_{\mathfrak{p}}$ is a regular local ring and $\mathfrak{p}$ has $n + 1$ generators, then $\mathfrak{p}$ is generated by a d-sequence (see [SH06, Exercise 5.24]). In any case, if $\dim R = r$ then every d-sequence has at most $r + 1$ elements (see [HSV81, 6.1.3]).

Within a polynomial ring over a field, there is a manageable criterion to decide whether a given sequence consisting of monomials is a d-sequence (see [Tang04]).

1.3.11. Let $R = k[X_1, \ldots, X_s]$ be a polynomial ring over a field k and let $f_1, \ldots, f_s \in R$ be a sequence consisting of monomials. Then $f_1, \ldots, f_s$ is a d-sequence if and only the following three conditions hold:

1. $f_1, \ldots, f_s$ is a minimal generating set of $I = (f_1, \ldots, f_s)$;
2. $\gcd(f_i, f_j) \mid f_k$, for all $1 \le i < j < k \le s$;
3. $\gcd(f_i, f_j) = \gcd(f_i, f_j^2)$, for all $1 \le i < j \le s$.

1.3.3 Reduction of ideals

Definition 1.3.12. *The ideal J, $J \subset I$, is said to be a reduction of I if there is an integer $r \ge 0$ such that $I^{r+1} = JI^r$. The reduction number of I with respect to J is the least such integer r, denoted by $\mathrm{r}_J(I)$. It is said that J is a minimal reduction of I if it is a reduction and it is minimal among all reductions of I with respect to inclusion, i.e., no ideal strictly contained in J is a reduction of I.*

1.3.13. Remark that if J is a reduction of I then $\mathrm{Min}(R/I) = \mathrm{Min}(R/J)$, thus $\sqrt{I} = \sqrt{J}$ and $\mathrm{ht}(I) = \mathrm{ht}(J)$.

Reductions of ideals can be detected by means of extension properties of the corresponding blowing-up algebras.

1.3.14. Let $(R, \mathfrak{m})$ be a Noetherian local ring with residue field $k = R/\mathfrak{m}$. Let $J \subset I$ be ideals in R. Then J is a reduction of I if and only if any of the following equivalent conditions hold:

(i) $\mathbf{R}(I)$ is a finitely generated $\mathbf{R}(J)$-module, i.e., the extension $\mathbf{R}(J) \subset \mathbf{R}(I)$ is module-finite;

(ii) If $k[J + \mathfrak{m}I/\mathfrak{m}I]$ stands for the subalgebra of $\mathbf{F}(I)$ generated over k by $J + \mathfrak{m}I/\mathfrak{m}I$, the extension $k[J + \mathfrak{m}I/\mathfrak{m}I] \subset \mathbf{F}(I)$ is module-finite.

See [SH06, Theorem 8.2.1 and Proposition 8.2.4].

Definition 1.3.15. *The analytic spread of I, denoted by $\ell(I)$, is defined as the Krull dimension of $\mathbf{F}(I)$.*

It immediately follows that the degree of the Hilbert polynomial of $\mathbf{F}(I)$ is $\ell(I) - 1$, thus $\ell(I)$ can be interpreted as the rate of growth of $\lambda(I^n/\mathfrak{m}I^n) = \mu(I^n)$ as n increases.

1.3.16. Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field k. Let J be a minimal reduction of I. Then,

(i) $\mu(J) = \ell(I)$;

(ii) any minimal generating set of J extends to a minimal generating set of I.

(iii) $\mathbf{F}(J) \cong \mathbf{S}(J/\mathfrak{m}J)$.

Let $(R, \mathfrak{m})$ be a Noetherian local ring. Throughout, we will raise almost-linear type assumptions, where $I = (J, y)$, with $y \notin J$ and $J \subset I$ will be a reduction generated either by an R-sequence or by a d-sequence. Notice that if J is generated by an R-sequence of length s, then $s \leq \operatorname{grade}(I) \leq \operatorname{ht}(I) \leq \ell(I) \leq \mu(J) = s$, whence $\operatorname{ad}(I) = \ell(I) - \operatorname{ht}(I) = 0$ and $\operatorname{sd}(I) = \mu(I) - \ell(I) = 1$. On the other hand, if I is an $\mathfrak{m}$-primary ideal with J minimally generated by a d-sequence $x_1, \ldots, x_s$ of length s, we have $s \leq \dim R + 1$ by 1.3.10. Then, since I is $\mathfrak{m}$-primary, we have $s \leq \dim R + 1 = \operatorname{ht}(I) + 1 = \ell(I) + 1 \leq \mu(J) + 1 = s + 1$. Consequently, $s - 1 \leq \operatorname{ht}(I) = \ell(I) \leq s$.

1.3.4   The equations of blowing-up algebras

Relation type

Let R be a Noetherian ring and let $U = R[z_1, \ldots, z_s]$ be a standard R-algebra. Let $V = R[T_1, \ldots, T_s]$ be a polynomial ring with variables $T_1, \ldots, T_s$ and let $\varphi : V \to U$ be the polynomial presentation of U sending T_i to z_i. Let $Q = \oplus_{n \geq 1} Q_n$ be the kernel of φ, whose elements will be referred to as the equations of U. Let $Q\langle n \rangle$ be the ideal generated by the homogeneous equations of U of degree at most n. Notice that there is an ascending chain $Q\langle 1 \rangle \subset Q\langle 2 \rangle \subset \ldots$ that stabilises at $Q\langle n \rangle$, for some integer $n \geq 1$.

1.3.17. Remark that the following isomorphisms of R-algebras hold: $V/Q\langle 1 \rangle \cong \mathbf{S}_R(U_1)$ and $V/Q \cong U$, where $\mathbf{S}_R(U_1)$ is the symmetric algebra of U_1.

Definition 1.3.18. *The relation type of U, denoted by* $\mathrm{rt}(U)$, *is the least integer $N \geq 1$, such that $Q\langle N \rangle = Q$, i.e., the maximum degree in a minimal generating set of homogeneous elements of Q.*

The case where $U = \mathbf{R}(I)$ will be of special importance for our aims. We will denote $\mathrm{rt}(\mathbf{R}(I))$ more simply as $\mathrm{rt}(I)$, the relation type of I.

1.3.19. While Q clearly depends on the polynomial presentation $\varphi : V \to U$, the relation type does not depend on the presentation of U. Moreover, even the degrees of a minimal generating set of homogeneous elements of Q remain invariant with respect to the presentation (see Remark 2.2.2).

In what follows, let $U = \mathbf{R}(I)$, $\varphi : V \to \mathbf{R}(I)$ be the polynomial presentation of $\mathbf{R}(I)$ sending T_i to $x_i t$ and $Q = \ker \varphi$.

1.3.20. The ideal Q can be computed by means of elimination theory. Let $I = (x_1, \ldots, x_s)$, $V = R[T_1, \ldots, T_s]$ and let T be an indeterminate over V. Then,

(i) $Q = (T_1 - x_1 T, \ldots, T_s - x_s T)V[T] \cap V$;

(ii) if x_1 is a non-zero-divisor of R,

$$Q = ((x_2 T_1 - x_1 T_2, \ldots, x_s T_1 - x_1 T_s) :_V x_1^\infty)$$
$$= (x_2 T_1 - x_1 T_2, \ldots, x_s T_1 - x_1 T_s, x_1 T - 1)V[T] \cap V.$$

In the context of polynomial rings these computations can be done via Gröbner basis techniques (see e.g. [Fröberg97], [CLO07]). Recall that if $L \subset k[x_1, \ldots, x_s, y_1, \ldots, y_r]$ is an ideal and $\leq_e$ is a monomial ordering such that $\mathbf{x}^\sigma \leq_e y_i$ for any $\sigma \in \mathbb{N}^s$ and $1 \leq i \leq r$, and $G = \{f_1, \ldots, f_k\}$ is a Gröbner basis of L with respect to $\leq_e$, then $G \cap k[x_1, \ldots, x_s]$ is a Gröbner basis of $L \cap k[x_1, \ldots, x_s]$ (see [Fröberg97, Proposition 8, p. 95]). A monomial ordering $\leq_e$ fulfilling the conditions above is said to be an *elimination ordering*.

The expected complexity of computing Q is high. The underlying rationale is that the theoretical complexity of computing Gröbner bases is generically exponential and doubly-exponential in the worst case (see [Lazard83] and [MM82]), as a function of the number of variables (see also [BW93, Appendix] for a further account on Gröbner bases and complexity issues). On the other hand, the cost of the computations in 1.3.20 heavily relies on the number of generators of I (see [BSV88]).

The following definition canonically encloses the fact that $Q = Q\langle 1 \rangle$, i.e., $\mathrm{rt}(I) = 1$ (see comments in 1.3.27).

Definition 1.3.21. *An ideal I of R is said to be of linear type if $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ is an isomorphism.*

1.3.22. Let I be an ideal of linear type and let $\mathfrak{p} \supset I$ a prime ideal containing I. The isomorphism $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ induces an isomorphism $\beta_\mathfrak{p} : \mathbf{S}(I_\mathfrak{p}/I_\mathfrak{p}^2) \to \mathbf{G}(I_\mathfrak{p})$, hence the Krull dimensions of $\mathbf{S}(I_\mathfrak{p}/I_\mathfrak{p}^2)$ and $\mathbf{G}(I_\mathfrak{p})$ coincide and are equal to $\dim R_\mathfrak{p}$ (see [SH06, Proposition 5.1.6]). By the surjectivity of $\mathbf{S}(I_\mathfrak{p}/I_\mathfrak{p}^2) \to \mathbf{S}(I_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}I_\mathfrak{p})$ it follows that $\mu(I_\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p})$ (see also [HSV81, Proposition 2.4]).

1.3.23. Let I be an ideal of linear type. Then, for each prime ideal $\mathfrak{p} \supset I$, $\ell(I_\mathfrak{p}) = \mu(I_\mathfrak{p})$. The converse, however, is not true in general. Consider $R = k[X, Y]/(X^2Y, XY^2) = k[x, y]$ and let $\mathfrak{p} = (y) \subset R$. Then $\mathfrak{p}$ is prime with $\ell(\mathfrak{p}R_\mathfrak{p}) = \mu(\mathfrak{p}R_\mathfrak{p}) = 0$ and $\ell(\mathfrak{p}_\mathfrak{m}) = \mu(\mathfrak{p}_\mathfrak{m}) = 1$, but $\ker \alpha_n \neq 0$ for all $n \geq 2$.

Definition 1.3.24. *An ideal I is said to be syzygetic if $Q\langle 1 \rangle = Q\langle 2 \rangle$, i.e., $\alpha_2 : \mathbf{S}_2(I) \to I^2$ is an isomorphism.*

1.3.25. A syzygetic ideal need not be an ideal of linear type. For instance, any prime ideal of height 2 in a 3-dimensional regular local ring is syzygetic (see [HSV81, Proposition 2.7

and subsequent Remark]). In the formal power series ring $k[[X, Y, Z]]$, there exist prime ideals $\mathfrak{p}$ of height 2 with arbitrarily large minimal number of generators (primes of Moh, see [Moh74]). Consequently, such prime ideals are syzygetic but they do not fulfil the inequality $\mu(\mathfrak{p}) = \mu(\mathfrak{p}_\mathfrak{m}) \leq \mathrm{ht}(\mathfrak{m}) = 3$ (see claim 1.3.22 above), thus they are not of linear type.

Effective relations

Let R be a Noetherian ring, let U be a standard R-algebra and let $\alpha : \mathbf{S}(U_1) \to U$ be the canonical morphism.

Definition 1.3.26. *If $n \geq 2$ is an integer, define the n-th module of effective relations of U as $E(U)_n = \ker \alpha_n / \mathbf{S}_1(U_1) \cdot \ker \alpha_{n-1}$. If I is an ideal of R and $U = \mathbf{R}(I)$, we set $E(I)_n = E(\mathbf{R}(I))_n$.*

1.3.27. The module $E(U)_n$ can also be described by means of a polynomial presentation $\varphi : V \to U$. If $Q = \ker \varphi$, then $E(U)_n \cong (Q/Q\langle n - 1 \rangle)_n = Q_n/V_1 Q_{n-1}$, for $n \geq 2$ (see [Planas98]). In particular, taking $U = \mathbf{R}(I)$, one can see that I is of linear type if and only if $E(U)_n = 0$ for all $n \geq 2$. Therefore, I is of linear type if and only if $Q = Q\langle 1 \rangle$, i.e., $\mathrm{rt}(I) = 1$ (see Definition 1.3.21).

The information encoded by the module of effective relations accounts for the fresh generators of $\ker \alpha$ or Q. The non-zero elements of $E(U) = \oplus_{n \geq 2} E(U)_n$ correspond to the equations of U of degree ≥ 2 which are not combination of the equations in lower degree. Consequently, if R is Noetherian, $E(U)_n = 0$ for n large enough.

1.3.28. Let R be a Noetherian ring, let $U = R[z_1, \dots, z_s]$ be a standard R-algebra and let $\underline{z} = z_1, \dots, z_s$. Then:

  (i) The module of effective relations can be expressed in terms of the graded Koszul homology: $E(U)_n \cong H_1(\underline{z}; U)_n$, for $n \geq 2$.
  (ii) The relation type $\mathrm{rt}(U)$ can be given in terms of the module of effective relations: $\mathrm{rt}(U) = \min\{s \geq 1 \mid E(U)_k = 0, \text{ for each } k \geq s + 1\}$.

1.3.29. Let R be a Noetherian ring, I an ideal of R. The following properties hold:

(*i*) There is an exact sequence $E(I)_{n+1} \to E(I)_n \to E(\mathbf{G}(I))_n \to 0$, for all $n \geq 2$;

(*ii*) As a consequence of (*i*), $\mathrm{rt}(I) = \mathrm{rt}(\mathbf{G}(I)) = r$ and $E(I)_r \cong E(\mathbf{G}(I))_r$.

See [Valla80, Theorem 1.3] for the case of $\mathrm{rt}(I) = 1$; see [Planas98, Section 3] for the general case using effective relations; and see [HKU05, Discussion 2.2] for the same result but using extended Rees and symmetric algebras. One of the advantages of using effective relations is that one is able to deduce at once that the top degree equations of $\mathbf{R}(I)$ are in correspondence with the top degree equations of $\mathbf{G}(I)$.

Geometric conditions close to linear type

We shall introduce two notions which are close to the linear type condition that will play a role in our study: geometric linear type (or g-linear type) and projectively of linear type (or p-linear type). These notions come from considering geometric properties linked to the canonical epimorphism $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$.

In Hermann-Moonen-Villamayor [HMV89], the authors consider the condition of g-linear type, which is weaker than that of linear type in the following sense: while linear type imply that the normal cone and the normal bundle of a closed subscheme of a given scheme coincide, the condition g-linear type refers only to the reduced structures of the normal cone and normal bundle.

Definition 1.3.30. *The ideal I is said to be of geometric linear type (g-linear type, for short) if the morphism induced by α, $\alpha^* : \mathrm{Spec}(\mathbf{R}(I)) \to \mathrm{Spec}(\mathbf{S}(I))$, is a homeomorphism of topological spaces.*

It is readily seen that since $\mathbf{R}(I) \cong \mathbf{S}(I)/\ker \alpha$, if I is of g-linear type, then α^* induces a homeomorphism $V(\ker \alpha) \to \mathrm{Spec}(\mathbf{S}(I))$. Consequently, the minimal primes in $V(\ker \alpha)$ correspond to the minimal primes in $\mathbf{S}(I)$, thus $\ker \alpha \subset \bigcap_{\mathfrak{p} \in \mathrm{Min}(\mathbf{S}(I))} \mathfrak{p}$, i.e., $\ker \alpha$ is nilpotent.

1.3.31. The ideal I is of g-linear type if and only if the following equivalent conditions hold (see [HMV89, Proposition 1.1]):

(*i*) $\ker \alpha$ is nilpotent;

(*ii*) $\ker \beta$ is nilpotent (see [HSV81, Theorem 3.2]);

(iii) $\mu(I_{\mathfrak{p}}) = \ell(I_{\mathfrak{p}})$, for all $\mathfrak{p} \in \mathrm{Spec}(R)$;

Clearly, if I is of linear type, then it is also of g-linear type, whereas the converse is not true in general. One of the advantages of introducing the notion of g-linear type is that this property can be checked by numerical conditions, which provides a general strategy of proving linear type by first proving g-linear type and then looking for additional conditions which assure that linear type and g-linear type coincide [HMV89].

Definition 1.3.32. *An ideal I is said to be projectively of linear type (of p-linear type, for short) if the morphism $\widetilde{\alpha} : \mathrm{Proj}(\mathbf{R}(I)) \to \mathrm{Proj}(\mathbf{S}(I))$ is an isomorphism of schemes, where $\widetilde{\alpha}$ is the morphism induced by α.*

Recall that for a standard R-algebra U, $\mathrm{Proj}(U) = \{\mathfrak{q} \in \mathrm{Spec}(U) \mid \mathfrak{q}$ homogeneous, $U_+ \not\subset \mathfrak{q}\}$ and, if α is surjective, $\widetilde{\alpha} : \mathrm{Proj}(V) \to \mathrm{Proj}(U)$ is a morphism of schemes given locally by the morphisms of affine schemes $\alpha_{u,0}^* : \mathrm{Spec}((V_{\alpha(u)})_0) \to \mathrm{Spec}((U_u)_0)$, with $u \in U_+$. It is clear that an ideal of linear type is of p-linear type.

1.3.33. Let $\alpha : U \to V$ be a surjective morphism of standard R-algebras. Then, one can show that $\widetilde{\alpha} : \mathrm{Proj}(V) \to \mathrm{Proj}(U)$ is an isomorphism of schemes if and only any of the following equivalent statements hold:

(i) there is an integer $r \geq 0$ such that $(U_+)^r \ker \alpha = 0$;

(ii) $\ker \alpha_n = 0$ for n sufficiently large.

Consequently, if I is of p-linear type, it is of g-linear type.

It is known that linear type, p-linear type and g-linear type do not coincide in general. Kühl in [Kühl82] was the first to highlight an example of an ideal of p-linear type, but not of linear type.

Example 1.3.34. (see [Kühl82, Example 1.4]). Let $A = k[U_0, U_1, U_2]$ and put

$$R = A[X, Y]/Q = A[x, y], \text{ where } Q = (U_0 X, U_1 Y, U_0 Y^2, U_1 X^2, U_1 X - U_2 Y, U_0 Y + U_2 X).$$

Then, the ideal $I = (x, y)$ of R is not of linear type, i.e., $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ is not an isomorphism; however, I is of p-linear type, i.e., $\mathrm{Proj}(\alpha) : \mathrm{Proj}(\mathbf{R}(I)) \to \mathrm{Proj}(\mathbf{S}(I))$ is an isomorphism of schemes.

One can see that if I is principal, then I is of p-linear type if and only if I is of linear type. However, a similar claim is not true when comparing g-linear type and p-linear type. Later, in Example 4.3.3, we will give a principal ideal which is of g-linear type, but not of p-linear type.

Chapter 2

The equations of Rees algebras of

ideals of almost-linear type

2.1   Introduction

The aim of this chapter is to explicitly describe the equations of Rees algebras of classes of ideals of almost-linear type. Let us first recall the basic definitions in order to properly introduce the main result of this chapter (see also Section 1.3).

Recall that the reduction number of I with respect to J, $J \subset I$, denoted by $\mathrm{r}_J(I)$, is the least integer $r \geq 0$ such that $I^{r+1} = JI^r$. Set $V = R[X_1, \ldots, X_s, Y]$ and let $\varphi : V \to \mathbf{R}(I)$ be the polynomial presentation of $\mathbf{R}(I)$, the Rees algebra of I, sending X_i to $x_i t$ and Y to yt. Let $Q = \oplus_{n \geq 1} Q_n$ be the kernel of φ. Given an integer $m \geq 0$, set $Q\langle m \rangle \subset Q$ the ideal generated by the homogeneous elements of Q of degree at most m in $X_1, \ldots, X_s, Y$. The relation type of I, denoted by $\mathrm{rt}(I)$, is the least integer $N \geq 1$ such that $Q = Q\langle N \rangle$. Let $\mathbf{G}(I) = \oplus_{n \geq 0} I^n / I^{n+1}$ be the associated graded ring of I. If $z \in R \backslash \{0\}$, we will denote by z^* the initial form of z in $\mathbf{G}(I)$, i.e., $z^* = z + I^{n+1}$, where $z \in I^n \backslash I^{n+1}$.

Theorem A. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. Let $x_1, \ldots, x_s, y$ be a minimal generating set of I, where $J = (x_1, \ldots, x_s)$ is a reduction of I with reduction number $r = \mathrm{r}_J(I)$. Assume that $x_1, \ldots, x_s$ verify the following condition for all $n \geq 2$:*

$$((x_1, \ldots, x_{i-1})I^{n-1} : x_i) \cap I^{n-1} = (x_1, \ldots, x_{i-1})I^{n-2}, \text{ for all } i = 1, \ldots, s. \qquad (\mathcal{T}_n)$$

Then, for each $n \geq 2$, the map sending $F \in Q_n$ to $F(0, \ldots, 0, 1) \in (JI^{n-1} : y^n)$ induces an isomorphism of R-modules

$$\left[\frac{Q}{Q\langle n-1 \rangle} \right]_n \cong \frac{(JI^{n-1} : y^n)}{(JI^{n-2} : y^{n-1})}.$$

In particular, $\mathrm{rt}(I) = \mathrm{r}_J(I) + 1$ and there is a form $Y^{r+1} - \sum X_i F_i \in Q_{r+1}$, with $F_i \in V_r$, such that $Q = (Y^{r+1} - \sum X_i F_i) + Q\langle r \rangle$. Moreover, if $x_1, \ldots, x_s$ is an R-sequence and $x_1^*, \ldots, x_{s-1}^*$ is a $\mathbf{G}(I)$-sequence, then $x_1, \ldots, x_s$ verify the condition $(\mathcal{T}_n)$ for all $n \geq 2$.

Roughly speaking, the theorem says how to obtain a minimal generating set of the equations of $\mathbf{R}(I)$. For the equations of degree 1, pick a minimal generating set of the first syzygies of I, viewed as elements of Q_1. For the equations of higher degree $= n$, with $2 \leq n \leq r + 1$, take representatives of the inverse images of a minimal generating set of $(JI^{n-1} : y^n)/(JI^{n-2} : y^{n-1})$ (see Remark 2.2.2 and Example 2.6.1).

Thus far, the study of the equations of $\mathbf{R}(I)$ has produced a vast literature. Some of this work is focused on ideals having small deviations as well as on the interplay between the reduction number and the relation type (see, just to mention some references, [Huckaba87], [Huckaba89], [HH92], [HH93], [Trung87], [Trung98], [Vasconcelos91], [Vasconcelos93]). The particular hypotheses and interests in this chapter owe much to the works of Cortadellas and Zarzuela in [CZ97], Heinzer and Kim in [HK03], Huckaba in [Huckaba87] and [Huckaba89], Trung in [Trung87] and [Trung98], and Vasconcelos in [Vasconcelos91]. In fact, Theorem A sprouted as an attempt to understand [Vasconcelos91, Theorem 2.3.3]. That is the reason for considering ideals I of the form $I = (J, y)$, with $y \notin J$ and J being a reduction of I generated by a sufficiently good sequence that makes J an ideal of linear type. The reader may also consult [HSV08] for a recent account on the equations of $\mathbf{R}(I)$ with similar assumptions.

Let R be a Noetherian local ring. Even for ideals of the principal class, i.e., $\mu(I) - \mathrm{ht}(I) = 0$, the equations of $\mathbf{R}(I)$ may be difficult to describe. Remarkably, any ideal I of the principal class is generated by an R-sequence $x_1, \ldots, x_s$, provided that I is prime (see [Davis67]) or R is Cohen-Macaulay. In both cases the equations are generated by the Koszul relations $x_i X_j - x_j X_i$, with $1 \leq i < j \leq s$ (see 1.3.2). In particular, I is of linear type. However, if I is not prime and R is not Cohen-Macaulay, this is no longer true. For instance, consider the ideals generated by systems of parameters. Huneke asked

in [Huneke96] whether there is a uniform bound for the relation type of these ideals in a complete local equidimensional Noetherian ring R. The full answer to this question was given in [Wang97], [Lai95] and [AGH06]. Concretely, in [AGH06, Example 2.1], it was shown that if the non-Cohen-Macaulay locus of R has dimension ≥ 2, there exist families of parametric ideals of R with unbounded relation type. This gives an idea of the complexity of the structure of the equations of $\mathbf{R}(I)$.

The case $I = (J, y)$, with $y \notin J$, $J = (x_1, \ldots, x_s)$ a reduction of I, $x_1, \ldots, x_s$ is an R-sequence and $x_1^*, \ldots, x_{s-1}^*$ a $\mathbf{G}(I)$-sequence, considered in Theorem A, verifies $\mu(I) - \mathrm{ht}(I) = 1$. In general, if $\mu(I) - \mathrm{ht}(I) = 1$ and R is Cohen-Macaulay, I is of linear type if and only if I is locally of the principal class at all minimal primes of I (see [HMV89, Theorem 4.8]). However, if I is not locally of the principal class at its minimal primes, then the relation type may be arbitrarily large. For instance, take $R = k[[x_1, x_2]]$, the power series ring in two variables over a field k and let $I = (x_1^p, x_2^p, x_1 x_2^{p-1})$, with $p \geq 2$. Then, I is (x_1, x_2)-primary and verifies $\mu(I) - \mathrm{ht}(I) = 1$, hence it is not locally of the principal class at its minimal primes. One can check that $\mathrm{rt}(I) \geq p$. In fact, I fulfils the hypotheses of Theorem A, $J = (x_1^p, x_2^p)$ being a reduction of I. Thus, the containment $(x_1 x_2^{p-1})^p \in JI^{p-1}$, induces the only equation of degree $= p$ in a minimal generating set of equations of $\mathbf{R}(I)$ and $\mathrm{rt}(I) = p$ (see Example 2.6.1).

As highlighted before, this study has its origins in the following result of Vasconcelos in [Vasconcelos91, Theorem 2.3.3]: let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and let I be an $\mathfrak{m}$-primary ideal of R. Let $x_1, \ldots, x_d, y$ be a minimal generating set of I, where $J = (x_1, \ldots, x_d)$ is a reduction of R with reduction number $\mathrm{r}_J(I) = 1$. Then there is a form $Y^2 - \sum X_i F_i \in Q_2$, with $F_i \in V_1$, such that $Q = (Y^2 - \sum X_i F_i) + Q\langle 1 \rangle$. In particular, $\mathrm{rt}(I) = 2$. Remark that the hypotheses imply that $x_1, \ldots, x_d$ is an R-sequence and that the initial forms $x_1^*, \ldots, x_d^*$ are a $\mathbf{G}(I)$-sequence (see the result of Valabrega and Valla in [VV78, Proposition 3.1]). By Theorem A, it is enough to suppose that $x_1^*, \ldots, x_{d-1}^*$ is a $\mathbf{G}(I)$-sequence. Moreover, one can consider any reduction number (see Corollary 2.5.3).

As a corollary of Theorem A, we recover a result of Heinzer and Kim in [HK03, Theorem 5.6], where they prove that the equations of the fiber cone of I, $\mathbf{F}(I) = \mathbf{R}(I) \otimes R/\mathfrak{m} = \oplus_{n \geq 1} I^n / \mathfrak{m} I^n$, are generated by a unique equation of degree $\mathrm{r}_J(I) + 1$.

2.2 Effective relations of a standard algebra

By a standard R-algebra U we mean a graded R-algebra $U = \oplus_{n\geq 0} U_n$, with $U_0 = R$, $U = R[U_1]$ and U_1 minimally generated by $\underline{z} = z_1, \ldots, z_s \in U_1$ as an R-module. For instance, the R-algebra $\mathbf{R}(I) = \oplus_{n\geq 0} I^n$, the R/I-algebra $\mathbf{G}(I) = \oplus_{n\geq 0} I^n/I^{n+1}$ and the $R/\mathfrak{m}$-algebra $\mathbf{F}(I) = \oplus_{n\geq 0} I^n/\mathfrak{m}I^n$, are standard algebras.

Let $V = R[T_1, \ldots, T_s]$ be a polynomial ring with variables $T_1, \ldots, T_s$, let $\varphi : V \to U$ be the induced presentation of U sending T_i to z_i and let $Q = \oplus_{n\geq 1} Q_n$ be the kernel of φ, the elements of which will be referred to as the equations of U. Let $\mathbf{S}(U_1)$ be the symmetric algebra of U_1 and let $\alpha : \mathbf{S}(U_1) \to U$ be the canonical morphism induced by the identity in degree one. Given $n \geq 2$, the module of effective n-relations of U is defined to be $E(U)_n = \ker(\alpha_n)/U_1 \ker(\alpha_{n-1})$. One can prove that, for any $n \geq 2$, $(Q/Q\langle n-1 \rangle)_n \cong E(U)_n$ (see [Planas98, Definition 2.2]). In particular, the relation type of U is the least integer $N \geq 1$, such that $E(U)_n = 0$ for all $n \geq N+1$.

This description of $(Q/Q\langle n-1 \rangle)_n$ as $E(U)_n = \ker(\alpha_n)/U_1 \ker(\alpha_{n-1})$ has the advantage of being canonical. On the other hand, it is the bridge to think about $(Q/Q\langle n-1 \rangle)_n$ as a Koszul homology module (see [Planas98, Corollary 2.7]). More precisely, for $n \geq 2$, we have the isomorphisms $(Q/Q\langle n-1 \rangle)_n \cong E(U)_n \cong H_1(\underline{z}; U)_n$, where $H_1(\underline{z}; U)_n$ denotes the first homology module of the complex:

$$\ldots \to \wedge_2(R^s) \otimes U_{n-2} \xrightarrow{\partial_{2,n-2}} \wedge_1(R^s) \otimes U_{n-1} \xrightarrow{\partial_{1,n-1}} U_n \to 0,$$

where the Koszul differentials are defined as follows: if $e_1, \ldots, e_s$ stands for the canonical basis of R^s, $u \in U_{n-2}$ and $v \in U_{n-1}$, then

$$\partial_{2,n-2}((e_i \wedge e_j) \otimes u) = e_j \otimes (z_i \cdot u) - e_i \otimes (z_j \cdot u) \quad \text{and} \quad \partial_{1,n-1}(e_i \otimes v) = z_i \cdot v.$$

As usual, $Z_1(\underline{z}; U)$ and $B_1(\underline{z}; U)$ will stand for the graded modules of 1-cycles and 1-boundaries, respectively, of the Koszul complex of $\underline{z}$. Observe that $B_1(\underline{z}; U)_1 = 0$.

Remark 2.2.1. For each integer $n \geq 2$, there is a well-defined isomorphism

$$\tau_n : (Q/Q\langle n-1 \rangle)_n \xrightarrow{\cong} H_1(\underline{z}; U)_n$$

sending the class of $F \in Q_n$ modulo $Q\langle n-1\rangle$ to the homology class of $(F_1(z), \ldots, F_s(z)) \in Z_1(\underline{z}; U)_n \subset U_{n-1} \oplus \overset{s)}{\ldots} \oplus U_{n-1}$, where $F_1, \ldots, F_s$ is any set of s elements in V_{n-1} satisfying $F = T_1 F_1 + \ldots + T_s F_s$.

As regards the case $n = 1$, notice that $Q\langle 0\rangle = 0$ and $B_1(\underline{z}; U)_1 = 0$, thus a similar construction leads to an isomorphism $\tau_1 : Q_1 \overset{\cong}{\to} Z_1(\underline{z}; U)_1$.

Proof: Take $f = \varphi : V \to U$ and $g = \mathbf{1}_V$ in [Planas98, Theorem 2.4].

Remark 2.2.2. Let U be a standard R-algebra and let $Q = \oplus_{n \geq 1} Q_n$ be the equations of U. Suppose that $Q = Q\langle N\rangle$ for some $N \geq 1$. Set

$$\mathcal{C} = \{F_{1,1}, \ldots, F_{1,s_1}, \ldots, F_{N,1}, \ldots, F_{N,s_N}\},$$

with $F_{i,j} \in Q_i$. Then $\mathcal{C}$ is a minimal generating set of Q if and only if, for each $n = 1, \ldots, N$, the classes of $F_{n,1}, \ldots, F_{n,s_n}$ modulo $Q\langle n-1\rangle$ are a minimal generating set of $(Q/Q\langle n-1\rangle)_n$.

2.3 Effective relations through Koszul homology

Let us set a general framework for the study of the injectivity of blowing-up morphisms by first setting some properties of the module of effective relations that we will later specialise and use. We will exploit the fact that there is an isomorphism between the n-th module of effective relations $E(U)_n$ and the n-th graded component $H_1(\underline{z}; U)_n$ (see Section 1.3.4).

Let us first examine a straightforward application of the well-known long exact sequence of the (graded) Koszul homology, which will be key for the subsequent arguments and which we will be able to specialise at low cost.

Lemma 2.3.1. *Let R be a Noetherian ring and let $U = \oplus_{n \geq 0} U_n$ be a standard R-algebra. Let $\underline{z} = z_1, \ldots, z_s$ and $\underline{z}' = \underline{z}, z_{s+1}$. Let the sequence $\underline{z}'$ be a minimal generating set of U_1 as an R-module. Then, for each integer $n \geq 2$, there is a short exact sequence*

$$0 \to \frac{H_1(\underline{z}; U)_n}{z_{s+1} H_1(\underline{z}; U)_{n-1}} \longrightarrow H_1(\underline{z}'; U)_n \overset{\sigma_n}{\longrightarrow} \frac{((\sum_{i=1}^s z_i U_{n-1}) :_R z_{s+1}^n)}{((\sum_{i=1}^s z_i U_{n-2}) :_R z_{s+1}^{n-1})} \to 0.$$

Moreover, σ_n sends the class of a cycle $(w_1, \ldots, w_s, w_{s+1})$, $w_i \in U_{n-1}$, to the class of $a \in ((\sum_{i=1}^s z_i U_{n-1}) :_R z_{s+1}^n)$, where $w_{s+1} = a z_{s+1}^{n-1} + b$ for some $b \in \sum_{i=1}^s z_i U_{n-2}$.

Proof: Consider the graded long exact sequence of Koszul homology (see e.g. [BH98, Corollary 1.6.13]):

$$H_1(\underline{z};U)_{n-1} \xrightarrow{(\rho_1)_n} H_1(\underline{z};U)_n \longrightarrow H_1(\underline{z}';U)_n \to H_0(\underline{z};U)_{n-1} \xrightarrow{(\rho_0)_n} H_0(\underline{z};U)_n,$$

where $(\rho_i)_n$ is just the multiplication by $\pm z_{s+1}$. Then we get the following short exact sequence:

$$0 \to \operatorname{coker}(\rho_1)_n \to H_1(\underline{z}';U)_n \xrightarrow{\widetilde{\sigma}_n} \ker(\rho_0)_n \to 0.$$

Clearly, $\operatorname{coker}(\rho_1)_n = H_1(\underline{z};U)_n / z_{s+1} H_1(\underline{z};U)_{n-1}$. On the other hand,

$$H_0(\underline{z};U)_{n-1} = U_{n-1} / \sum_{i=1}^{s} z_i U_{n-2}.$$

Thus $\ker(\rho_0)_n = (U_{n-1} \cap ((\sum_i^s z_i U_{n-1}) :_U z_{s+1})) / \sum_i^s z_i U_{n-2}$. One can check that $\widetilde{\sigma}_n$ maps the homology class of a cycle $(w_1, \ldots, w_s, w_{s+1}) \in \oplus_{i=1}^{s+1} U_{n-1}$ to the class of $w_{s+1} \in U_{n-1} \cap ((\sum_{i=1}^{s} z_i U_{n-1}) :_U z_{s+1})$ modulo $\sum_{i=1}^{s} z_i U_{n-2}$. Finally, consider the mapping θ_n,

$$\ker(\rho_0)_n = \frac{U_{n-1} \cap ((\sum_{i=1}^{s} z_i U_{n-1}) :_U z_{s+1})}{\sum_{i=1}^{s} z_i U_{n-2}} \xrightarrow{\theta_n} \frac{((\sum_{i=1}^{s} z_i U_{n-1}) :_R z_{s+1}^n)}{((\sum_{i=1}^{s} z_i U_{n-2}) :_R z_{s+1}^{n-1})},$$

defined as follows: for each $w \in U_{n-1} \cap ((\sum_{i=1}^{s} z_i U_{n-1}) :_U z_{s+1})$, since $U_{n-1} = z_{s+1}^{n-1} R + \sum_{i=1}^{s} z_i U_{n-2}$, take $a \in R$ and $b \in \sum_{i=1}^{s} z_i U_{n-2}$ such that $w = a z_{s+1}^{n-1} + b$. Clearly $a \in ((\sum_{i=1}^{s} z_i U_{n-1}) :_R z_{s+1}^n)$. Let $\overline{w}$ be the class of w modulo $\sum_{i=1}^{s} z_i U_{n-2}$ and let $\overline{a}$ the class of a modulo $((\sum_{i=1}^{s} z_i U_{n-2}) :_R z_{s+1}^{n-1})$. Set $\theta_n(\overline{w}) = \overline{a}$: an elementary computation shows that θ_n is a well-defined isomorphism. Finally, set $\sigma_n = \theta_n \circ \widetilde{\sigma}_n$. $\qquad\square$

Lemma 2.3.1 can be profitably specialised for either $\mathbf{R}(I)$, $\mathbf{G}(I)$ or $\mathbf{F}(I)$, thus characterising the vanishing of their modules of effective relations in terms of a mixed condition on colon ideals and Koszul homology. In the case where $\mu(I) = 2$, the vanishing of the Koszul homology itself admits a full description in terms of colon ideals that we make explicit in the following lemma.

Lemma 2.3.2. *Let R be a Noetherian ring and let $U = \oplus_{n \geq 0} U_n$ be a standard R-algebra. Suppose that z_1, z_2 is a minimal generating set of U_1 as an R-module. Then, for each integer $n \geq 2$, there is a short exact sequence*

$$0 \to \frac{(0 :_U z_1) \cap U_{n-1}}{z_2((0 :_U z_1) \cap U_{n-2})} \longrightarrow H_1(z_1, z_2;U)_n \xrightarrow{\sigma_n} \frac{(z_1 U_{n-1} :_R z_2^n)}{(z_1 U_{n-2} :_R z_2^{n-1})} \to 0.$$

Proof: Apply Lemma 2.3.1, observing that $H_1(z_1 ; U)_n \cong (0 :_U z_1) \cap U_{n-1}$. $\qquad\square$

Given ideals $I, L \subset R$, we will denote the standard algebra $\mathbf{R}(I) \otimes_R R/L = \oplus_{n \geq 0} I^n/LI^n$ by $\mathbf{R}_L(I)$. Let $\underline{z}$ denote the sequence of classes modulo I/LI of the elements of a sequence $\underline{x}$ in I. If $L = 0$, then $\underline{z}$ will stand for the sequence $\underline{x}t \subset It$.

Proposition 2.3.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let $I = (x_1, \ldots, x_s, y)$, $J = (x_1, \ldots, x_s) \subset I$ and L be ideals in R. Let $\underline{x} = x_1, \ldots, x_s$ and assume that $\underline{x}, y$ minimally generate I. Let $\underline{z}$ and $\underline{z}' = \underline{z}, z_{s+1}$ be the sequence of classes modulo I/LI of the elements of $\underline{x}$ and $\underline{x}, y$, respectively. Then the following short exact sequence holds:*

$$0 \to \frac{H_1(\underline{z} ; \mathbf{R}_L(I))}{z_{s+1} H_1(\underline{z} ; \mathbf{R}_L(I))} \to H_1(\underline{z}' ; \mathbf{R}_L(I))_n \xrightarrow{\sigma_n} \frac{(JI^{n-1} + LI^n :_R y^n)}{(JI^{n-2} + LI^{n-1} :_R y^{n-1})} \to 0.$$

In particular, if $I = (x, y)$, the preceding short exact sequence has the following form:

$$0 \to \frac{(LI^n :_R x) \cap I^{n-1}}{y((LI^{n-1} :_R x) \cap I^{n-2}) + LI^{n-1}} \to H_1(\underline{z}' ; \mathbf{R}_L(I))_n \xrightarrow{\sigma_n} \frac{(xI^{n-1} + LI^n :_R y^n)}{(xI^{n-2} + LI^{n-1} :_R y^{n-1})} \to 0.$$

Proof: One just needs to check that the expressions in the short exact sequences are isomorphic to the ones that appear in Lemma 2.3.1 and Lemma 2.3.2.

In order to prove the second short exact sequence, let $z_1, z_2 \in I/LI$ be the classes of x, y, respectively. Let $\epsilon_{n-1} : (LI^n : x) \cap I^{n-1} \to (0 :_{R_L(I)} z_1) \cap (I^{n-1}/LI^{n-1})$ be the homomorphism taking classes modulo LI^{n-1}. Then we have the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & LI^{n-2} & \longrightarrow & (LI^{n-1} :_R x) \cap I^{n-2} & \xrightarrow{\epsilon_{n-2}} & (0 :_{R_L(I)} z_1) \cap \frac{I^{n-2}}{LI^{n-2}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z_2} & & \\
0 & \longrightarrow & LI^{n-1} & \longrightarrow & (LI^n :_R x) \cap I^{n-1} & \xrightarrow{\epsilon_{n-1}} & (0 :_{R_L(I)} z_1) \cap \frac{I^{n-1}}{LI^{n-1}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & LI^{n-1}/yLI^{n-2} & \longrightarrow & \frac{(LI^n :_R x) \cap I^{n-1}}{y((LI^{n-1} :_R x) \cap I^{n-2})} & \longrightarrow & \frac{(0 :_{R_L(I)} z_1) \cap I^{n-1}/LI^{n-1}}{z_2((0 :_{R_L(I)} z_1) \cap I^{n-2}/LI^{n-2})} & \longrightarrow & 0.
\end{array}
$$

The exactness of the bottom row and Lemma 2.3.2 lead to the conclusion. $\qquad\square$

Although the next lemma is a consequence of Lemma 2.3.1 and Lemma 2.3.3, we will give an explicit proof by carrying over the ideas of Lemma 2.3.1. We keep focusing on $H_1(x_1 t, \ldots, x_s t, yt ; \mathbf{R}(I))_n$, which by Remark 2.2.1 is isomorphic to the module of effective relations of $\mathbf{R}(I)$, $E(I)_n \cong (Q/Q\langle n-1 \rangle)_n$ (see also Definition 1.3.26).

Lemma 2.3.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. Let $x_1, \ldots, x_s, y$ be a minimal generating set of I. Set $J = (x_1, \ldots, x_s)$. Then, for each integer $n \geq 2$, there is a short exact sequence*

$$0 \to \frac{H_1(x_1 t, \ldots, x_s t\,;\mathbf{R}(I))_n}{yt H_1(x_1 t, \ldots, x_s t\,;\mathbf{R}(I))_{n-1}} \longrightarrow H_1(x_1 t, \ldots, x_s t, yt\,;\mathbf{R}(I))_n \xrightarrow{\sigma_n} \frac{(JI^{n-1} :_R y^n)}{(JI^{n-2} :_R y^{n-1})} \to 0.$$

Moreover, σ_n sends the class of a cycle $(w_1 t^{n-1}, \ldots, w_s t^{n-1}, w_{s+1} t^{n-1})$, $w_i \in I^{n-1}$, to the class of $a \in (JI^{n-1} :_R y^n)$, where $w_{s+1} = ay^{n-1} + b$ for some $b \in JI^{n-2}$.

Proof: Take $\underline{z} = x_1 t, \ldots, x_s t$ and $\underline{z}' = x_1 t, \ldots, x_s t, yt$ in $It \subset \mathbf{R}(I) = R[It] = \oplus_{n \geq 0} I^n$. Consider the induced graded long exact sequence of Koszul homology:

$$H_1(\underline{z}\,;\mathbf{R}(I))_{n-1} \xrightarrow{(\rho_1)_n} H_1(\underline{z}\,;\mathbf{R}(I))_n \longrightarrow H_1(\underline{z}'\,;\mathbf{R}(I))_n \to H_0(\underline{z}\,;\mathbf{R}(I))_{n-1} \xrightarrow{(\rho_0)_n} H_0(\underline{z}\,;\mathbf{R}(I))_n,$$

where $(\rho_i)_n$ is just the multiplication by $\pm yt$. We get the following short exact sequence:

$$0 \to \operatorname{coker}(\rho_1)_n \to H_1(\underline{z}'\,;\mathbf{R}(I))_n \xrightarrow{\widetilde{\sigma}_n} \ker(\rho_0)_n \to 0.$$

Clearly $\operatorname{coker}(\rho_1)_n = H_1(\underline{z}\,;\mathbf{R}(I))_n / yt H_1(\underline{z}\,;\mathbf{R}(I))_{n-1}$. On the other hand,

$$H_0(\underline{z}\,;\mathbf{R}(I))_{n-1} = I^{n-1}/JI^{n-2}.$$

Thus $\ker(\rho_0)_n = I^{n-1} \cap (JI^{n-1} :_R y)/JI^{n-2}$. One can check that $\widetilde{\sigma}_n$ maps the homology class of a cycle $(w_1 t^{n-1}, \ldots, w_s t^{n-1}, w_{s+1} t^{n-1}) \in \oplus_{i=1}^s I^{n-1} t^{n-1}$ to the class of $w_{s+1} \in I^{n-1} \cap (JI^{n-1} :_R y)$ modulo JI^{n-2}. Finally, consider the mapping θ_n,

$$\ker(\rho_0)_n = \frac{I^{n-1} \cap (JI^{n-1} :_R y)}{JI^{n-2}} \xrightarrow{\theta_n} \frac{(JI^{n-1} :_R y^n)}{(JI^{n-2} :_R y^{n-1})},$$

defined as follows: for each $w \in I^{n-1} \cap (JI^{n-1} :_R y)$, since $I^{n-1} = y^{n-1} R + JI^{n-2}$, take $a \in R$ and $b \in JI^{n-2}$ such that $w = ay^{n-1} + b$. Clearly $a \in (JI^{n-1} :_R y^n)$. Let $\overline{w}$ be the class of w modulo JI^{n-2} and let $\overline{a}$ the class of a modulo $(JI^{n-2} :_R y^{n-1})$. Set $\theta_n(\overline{w}) = \overline{a}$. An elementary computation shows that θ_n is a well-defined isomorphism. Take $\sigma_n = \theta_n \circ \widetilde{\sigma}_n$. $\square$

Although the Lemma 2.3.4 has a version for $n = 1$, we are more interested in an analogue version obtained from considering $x_1, \ldots, x_s$ and $x_1, \ldots, x_s, y$ as sequences of elements of degree zero in $I \subset R$.

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Remark 2.3.5. With the assumptions and notations of Lemma 2.3.4, there is an exact sequence

$$0 \to \frac{H_1(x_1, \ldots, x_s; R)}{y H_1(x_1, \ldots, x_s; R)} \to H_1(x_1, \ldots, x_s, y; R) \xrightarrow{\tilde{\sigma}_1} \frac{(J :_R y)}{J} \to 0,$$

where $\tilde{\sigma}_1$ sends the homology class of a cycle $(w_1, \ldots, w_s, w_{s+1}) \in R^{s+1}$ to the class of $w_{s+1} \in (J :_R y)$. In particular, if $x_1, \ldots, x_s$ is an R-sequence, $H_1(x_1, \ldots, x_s; R) = 0$ and $\tilde{\sigma}_1$ is an isomorphism.

2.4 Vanishing of the Koszul homology

If $x_1, \ldots, x_s$ a sequence of elements belonging to an ideal I, the next lemma characterises the vanishing of the n-th graded components $H_1(x_1 t, \ldots, x_i t; \mathbf{R}(I))_n$ for all $1 \le i \le s$, in terms of sequential conditions in the $x_1, \ldots, x_s$.

Lemma 2.4.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. Let $x_1, \ldots, x_s$ be a sequence of elements belonging to I and let $n \ge 2$ be an integer. Then, the following conditions are equivalent:*

(i) $H_1(x_1 t, \ldots, x_i t; \mathbf{R}(I))_n = 0$, for all $i = 1, \ldots, s$;

(ii) $I^{n-1} t^{n-1} \cap ((x_1 t, \ldots, x_{i-1} t) :_{R(I)} x_i t) = I^{n-1} t^{n-1} \cap (x_1 t, \ldots, x_{i-1} t)$, for all $i = 1, \ldots, s$;

(iii) $((x_1, \ldots, x_{i-1}) I^{n-1} :_R x_i) \cap I^{n-1} = (x_1, \ldots, x_{i-1}) I^{n-2}$, for all $i = 1, \ldots, s$.

Proof: Let n be fixed and let us prove the equivalence by induction on s. If $s = 1$ then $H_1(x_1 t; \mathbf{R}(I))_n = (0 :_R x_1) \cap I^{n-1}$ vanishes if and only if $(0 :_R x_1) \cap I^{n-1} = 0$, hence the claim follows.

By induction, assume the lemma holds for $s - 1 > 0$. Let us denote $J_0 = 0$ and $J_i = (x_1, \ldots, x_i)$. Take $1 \le i \le s$ and set $\underline{z} = x_1 t, \ldots, x_{i-1} t$ and $\underline{z}' = x_1 t, \ldots, x_{i-1} t, x_i t$. By the induction hypothesis and using the graded long exact sequence of Koszul homology,

$$0 \to H_1(\underline{z}'; \mathbf{R}(I))_n \to H_0(\underline{z}; \mathbf{R}(I))_{n-1} \xrightarrow{(\rho_0)_n} H_0(\underline{z}; \mathbf{R}(I))_n$$

is an exact sequence for all $1 \le i \le s$. Therefore,

$$H_1(x_1 t, \ldots, x_i t; \mathbf{R}(I))_n \cong \ker(\rho_0)_n = \frac{I^{n-1} t^{n-1} \cap ((x_1 t, \ldots, x_{i-1} t) :_{R(I)} x_i t)}{I^{n-1} t^{n-1} \cap (x_1 t, \ldots, x_{i-1} t)}$$

$$\cong \frac{(J_{i-1} I^{n-1} :_R x_i) \cap I^{n-1}}{J_{i-1} I^{n-2}},$$

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whence the claim follows for all $1 \leq i \leq s$. □

Recall that the sequence $x_1, \ldots, x_s$ is a d-sequence if it minimally generates $(x_1, \ldots, x_s)$ and if $(J_i :_R x_{i+1}x_j) = (J_i :_R x_j)$ for all $0 \leq i \leq s-1$ and all $j \geq i+1$ (where $J_0 = 0$, $J_i = (x_1, \ldots, x_i)$ and $J_s = J$, as before). Recall that this second condition is equivalent to $(J_i :_R x_{i+1}) \cap J = J_i$ for all $0 \leq i \leq s-1$ (see 1.3.8 and subsequent remarks). Clearly, any R-sequence is a d-sequence.

Remark 2.4.2. If the equivalent conditions in Lemma 2.4.1 hold for all $n \geq 2$, then $x_1 t, \ldots, x_s t$ is a d-sequence of $\mathbf{R}(I)$. Furthermore, $((x_1 t, \ldots, x_{i-1} t) :_{R(I)} x_i t) \cap \mathbf{R}(I)_+ = (x_1 t, \ldots, x_{i-1} t)$ for all $i = 1, \ldots, s$.

However, being a d-sequence in $\mathbf{R}(I)$ does not imply that $H_1(x_1 t, \ldots, x_i t; \mathbf{R}(I))_n = 0$ for all $i = 1, \ldots, s$ and $n \geq 2$. Consider the ring $R = k[X, Y]_{(X,Y)}/(XY, Y^2)_{(X,Y)}$ with k a field and let $I = (x, y)$ and $J = (x)$. Then xt is a d-sequence in $\mathbf{R}(I)$, while $(0 : x) \cap (x, y) = (y) \neq 0$.

Proof: Let $x_1, \ldots, x_s$ and I as in Lemma 2.4.1. Let $\widetilde{J} = (x_1 t, \ldots, x_s t) \subset \mathbf{R}(I)_+$. The condition $I^{n-1} t^{n-1} \cap ((x_1 t, \ldots, x_{i-1}) :_{R(I)} x_i t) = I^{n-1} t^{n-1} \cap (x_1 t, \ldots, x_{i-1})$, for all $i = 1, \ldots, s$ and all $n \geq 2$, is equivalent to $((x_1 t, \ldots, x_{i-1} t) :_{R(I)} x_i t) \cap \mathbf{R}(I)_+ = (x_1 t, \ldots, x_{i-1} t)$ for all $i = 1, \ldots, s$, which implies $((x_1 t, \ldots, x_{i-1} t) :_{R(I)} x_i t) \cap \widetilde{J} = (x_1 t, \ldots, x_{i-1} t)$ for all $i = 1, \ldots, s$, i.e., $x_1 t, \ldots, x_s t$ is a d-sequence. However, the converse is not true, even if J is a reduction of I, as it is shown in Example 2.6.3. Compare this with [HSV81, Lemma 12.7]. □

In the next lemma we will prove that, for $n \geq 2$, then $H_1(x_1 t, \ldots, x_s t; \mathbf{R}(I))_n = 0$, provided that $x_1, \ldots, x_s$ an R-sequence such that $x_1^*, \ldots, x_{s-1}^*$ is a $\mathbf{G}(I)$-sequence. Notice that we are not saying that the whole $H_1(x_1 t, \ldots, x_s t; \mathbf{R}(I))$ vanishes. In fact, if $s > 1$, $x_1(x_2 t) \in x_1 t \cdot \mathbf{R}(I)$ whereas $x_1 \notin x_1 t \cdot \mathbf{R}(I)$, hence $x_1 t, \ldots, x_s t$ is not an $\mathbf{R}(I)$-sequence.

Before presenting the lemma, recall the result of Valabrega and Valla in [VV78, Corollary 2.7] which characterizes being a $\mathbf{G}(I)$-sequence. Let $J \subset I$ be two ideals of R. Let $x_1, \ldots, x_s$ denote a minimal generating set of J and keeping the notations in the proof of Lemma 2.4.1, write $J_0 = 0$ and $J_i = (x_1, \ldots, x_i)$, for $i = 1, \ldots, s$. In particular, the initial forms $x_1^*, \ldots, x_s^*$ in $\mathbf{G}(I)$ are in I/I^2. Then, $x_1^*, \ldots, x_s^*$ is a $\mathbf{G}(I)$-sequence if and only if $x_1, \ldots, x_s$ is an R-sequence and the Valabrega-Valla modules $VV_{J_i}(I)_n = J_i \cap I^n / J_i I^{n-1}$ vanish for all $i = 1, \ldots, s$ and all $n \geq 1$.

Lemma 2.4.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. Let $x_1, \ldots, x_s$ be a minimal generating set of J, where $J = (x_1, \ldots, x_s)$ is a reduction of I with reduction number $r = \mathrm{r}_J(I)$. Assume that $x_1, \ldots, x_s$ is an R-sequence and that $x_1^*, \ldots, x_{s-1}^*$ is a $\mathbf{G}(I)$-sequence. Then, for all $n \geq 2$ and all $i = 1, \ldots, s$,*

$$H_1(x_1 t, \ldots, x_i t \,;\, \mathbf{R}(I))_n = 0.$$

Proof: We will check that condition (iii) of Lemma 2.4.1 is fulfilled for all $n \geq 2$. Using the aforementioned result of Valabrega and Valla, the hypotheses of the Lemma imply

$$VV_{J_i}(I)_n = J_i \cap I^n / J_i I^{n-1} = 0$$

for all $i = 1, \ldots, s-1$ and all $n \geq 2$. Since $x_1, \ldots, x_s$ is an R-sequence, then $(J_{i-1} :_R x_i) = J_{i-1}$ for all $i = 1, \ldots, s$. Therefore

$$(J_{i-1} :_R x_i) \cap I^{n-1} = J_{i-1} \cap I^{n-1} = J_{i-1} I^{n-2},$$

for all $i = 1, \ldots, s$ and all $n \geq 2$. This identity implies

$$(J_{i-1} I^{n-1} :_R x_i) \cap I^{n-1} = J_{i-1} I^{n-2}, \tag{2.1}$$

for all $i = 1, \ldots, s$ and $n \geq 2$. But these are precisely the conditions (iii) in Lemma 2.4.1, whence the claim. $\square$

One can state a different version of Lemma 2.4.3 by considering weaker hypotheses and hence a weaker thesis, which will in turn lead to a slightly different version of the main result of this chapter (see Remark 2.5.1).

Remark 2.4.4. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. Let $x_1, \ldots, x_s$ be a minimal generating set of J, where $J = (x_1, \ldots, x_s)$ is a reduction of I with reduction number $r = \mathrm{r}_J(I)$. Assume that

(a) $x_1, \ldots, x_s$ is a d-sequence;

(b) $VV_{J_i}(I)_{r+1} = (x_1, \ldots, x_i) \cap I^{r+1} / (x_1, \ldots, x_i) I^r = 0$ for all $i = 1, \ldots, s-1$.

Then $H_1(x_1 t, \ldots, x_i t \,;\, \mathbf{R}(I))_n = 0$, for all $n \geq r + 2$ and all $i = 1, \ldots, s$.

Suppose that, in addition,

(c) $x_1, \ldots, x_s$ is an R-sequence;

(d) $VV_{J_i}(I)_r = (x_1, \ldots, x_i) \cap I^r/(x_1, \ldots, x_i)I^{r-1} = 0$ for all $i = 1, \ldots, s-1$.

Then $H_1(x_1 t, \ldots, x_i t; \mathbf{R}(I))_{r+1} = 0$, for all $i = 1, \ldots, s$.

Proof: Using (a) and (b), since $I^{r+1} = JI^r \subset J$ we get

$$(J_{i-1} :_R x_i) \cap I^{r+1} = (J_{i-1} :_R x_i) \cap J \cap I^{r+1} = J_{i-1} \cap I^{r+1} = J_{i-1}I^r,$$

for $i = 1, \ldots, s$. Therefore, for all $i = 1, \ldots, s$,

$$(J_{i-1} :_R x_i) \cap I^{r+1} = J_{i-1}I^r.$$

Using the result of Trung in [Trung98, Proposition 4.7(i)], one has $(J_{i-1} :_R x_i) \cap I^n = J_{i-1}I^{n-1}$, for all $n \geq r+1$ and all $i = 1, \ldots, s$. This identities imply that

$$(J_{i-1}I^{n-1} :_R x_i) \cap I^{n-1} = J_{i-1}I^{n-2},$$

for all $n \geq r+2$ and all $i = 1, \ldots, s$. Therefore, by Lemma 2.4.1, the claim holds. As for the second claim, (c) implies that $(J_{i-1} :_R x_i) \cap I^r = J_{i-1} \cap I^r$ for all $i = 1, \ldots, s$ (observe that that (a) is not sufficient) and using (d) we get $(J_{i-1} :_R x_i) \cap I^r = J_{i-1}I^{r-1}$, hence $(J_{i-1}I^r :_R x_i) \cap I^r = J_{i-1}I^{r-1}$ for all $i = 1, \ldots, s$, thus Lemma 2.4.1 shows that $H_1(x_1 t, \ldots, x_i t; \mathbf{R}(I))_n = 0$ for $i = 1, \ldots, s$ and $n \leq r+1$. $\qquad \square$

Remark 2.4.5. Observe that if $J \subset I$ is a reduction with small reduction number $r_J(I)$, then the graded homology vanishes with milder hypotheses than those in Lemma 2.4.3. Let $(x_1, \ldots, x_s) = J \subset I$ be a reduction with $r_J(I) = 1$. Suppose that:

(a') $x_1, \ldots, x_s$ is a d-sequence;

(b') $((x_1, \ldots, x_{i-1}) :_R x_i) \cap I = (x_1, \ldots, x_{i-1})$, for all $i = 1, \ldots, s$;

(c') $(x_1, \ldots, x_{i-1}) \cap I^2 = (x_1, \ldots, x_{i-1})I$, for all $i = 1, \ldots, s-1$.

Then, $H_1(x_1 t, \ldots, x_s t; \mathbf{R}(I))_n = 0$, for $n \geq 2$.

Proof: As in the proof of Lemma 2.4.3 and Remark 2.4.4, the goal is to reach the identities in Lemma 2.4.1 which are equivalent to the vanishing of the graded homology. From (b') we are able to prove $H_1(x_1 t, \ldots, x_s t; \mathbf{R}(I))_2 = 0$. On the other hand, (a'), (c') and [Trung98, Proposition 4.7 (i)] lead to $H_1(x_1 t, \ldots, x_s t; \mathbf{R}(I))_n = 0$, for $n \geq 3$. $\qquad \square$

Remark 2.4.6. Let $x_1, \ldots, x_s$ be a d-sequence contained in I and $J = (x_1, \ldots, x_s)$. We know that J is of linear type (see 1.3.9), hence $H_1(x_1 t, \ldots, x_s t; \mathbf{R}(J))_n = 0$ for all $n \geq 2$, i.e., the complex

$$\mathbf{R}(J)(-2)^{\binom{s}{2}} \xrightarrow{\partial_2} \mathbf{R}(J)(-1)^s \xrightarrow{\partial_1} \mathbf{R}(J)$$

is exact in degree ≥ 2. Tensoring this complex by $\mathbf{R}(I)$ and computing $\ker \partial_1 \otimes \mathbf{1}_{R(I)} / \operatorname{Im} \partial_2 \otimes \mathbf{1}_{R(I)}$ would yield $H_1(x_1 t, \ldots, x_s t; \mathbf{R}(I))$, but notice that this does not necessarily yield $H_1(x_1 t, \ldots, x_s t; \mathbf{R}(I))_n = 0$ for all $n \geq 2$ (see Example 2.6.3). For instance,

$$H_1(x_1 t, \ldots, x_s t; \mathbf{R}(I))_2$$

is the homology of the complex

$$R^{\binom{s}{2}} \xrightarrow{\widetilde{\partial_2}} J^{\oplus s} \oplus I^{\oplus s} \xrightarrow{\widetilde{\partial_1}} J^2 \oplus JI$$

where $\widetilde{\partial_2}(a_0) = (\partial_2(a_0), 0)$ and $\widetilde{\partial_1}(a_1, b_0) = (\partial_1(a_1), \partial_1(b_0))$.

2.5 Theorem A

We have now all the ingredients to prove the main result of the chapter. As in the rest of the chapter, set $V = R[X_1, \ldots, X_s, Y]$ and let Q be the kernel of the polynomial presentation $\varphi : V \to \mathbf{R}(I)$ sending X_i to $x_i t$ and Y to yt.

Theorem A. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. Let $x_1, \ldots, x_s, y$ be a minimal generating set of I, where $J = (x_1, \ldots, x_s)$ is a reduction of I with reduction number $r = \mathrm{r}_J(I)$. Assume that $x_1, \ldots, x_s$ verify the following condition for all $n \geq 2$:*

$$((x_1, \ldots, x_{i-1}) I^{n-1} : x_i) \cap I^{n-1} = (x_1, \ldots, x_{i-1}) I^{n-2}, \text{ for all } i = 1, \ldots, s. \qquad (\mathcal{T}_n)$$

Then, for each $n \geq 2$, the map sending $F \in Q_n$ to $F(0, \ldots, 0, 1) \in (JI^{n-1} :_R y^n)$ induces an isomorphism of R-modules

$$\left[\frac{Q}{Q\langle n-1 \rangle} \right]_n \cong \frac{(JI^{n-1} :_R y^n)}{(JI^{n-2} :_R y^{n-1})}.$$

In particular, $\mathrm{rt}(I) = \mathrm{r}_J(I) + 1$ and there is a form $Y^{r+1} - \sum X_i F_i \in Q_{r+1}$, with $F_i \in V_r$, such that $Q = (Y^{r+1} - \sum X_i F_i) + Q\langle r \rangle$. Moreover, if $x_1, \ldots, x_s$ is an R-sequence and $x_1^*, \ldots, x_{s-1}^*$ is a $\mathbf{G}(I)$-sequence, then $x_1, \ldots, x_s$ verify condition $(\mathcal{T}_n)$ for all $n \geq 2$.

Proof: By Lemma 2.3.4, with $\underline{z} = x_1 t, \ldots, x_s t$ and $\underline{z}' = x_1 t, \ldots, x_s t, yt$,

$$0 \to \frac{H_1(\underline{z}; \mathbf{R}(I))_n}{yt H_1(\underline{z}; \mathbf{R}(I))_{n-1}} \to H_1(\underline{z}'; \mathbf{R}(I))_n \xrightarrow{\sigma_n} \frac{(JI^{n-1} :_R y^n)}{(JI^{n-2} :_R y^{n-1})} \to 0$$

is an exact sequence for all $n \geq 2$. By Lemma 2.4.1, $H_1(\underline{z}; \mathbf{R}(I))_n = 0$ for all $n \geq 2$. Therefore, $H_1(\underline{z}'; \mathbf{R}(I))_n \xrightarrow{\sigma_n} (JI^{n-1} :_R y^n)/(JI^{n-2} :_R y^{n-1})$, for $n \geq 2$. By Remark 2.2.1, we conclude:

$$\left[\frac{Q}{Q\langle n-1\rangle}\right]_n \xrightarrow{\tau_n} H_1(\underline{z}'; \mathbf{R}(I))_n \xrightarrow{\sigma_n} \frac{(JI^{n-1} :_R y^n)}{(JI^{n-2} :_R y^{n-1})},$$

for all $n \geq 2$. Given $F \in Q_n$, write $F = \sum_{i=1}^{s} X_i F_i + YG$, with $F_i, G \in V_{n-1}$. Then the morphism τ_n sends the class of F to the homology class of $(F_1(\underline{z}'), \ldots, F_s(\underline{z}'), G(\underline{z}'))$. But $G(\underline{z}') = G(x_1, \ldots, x_s, y)t^{n-1}$ and $G(x_1, \ldots, x_s, y) = G(0, \ldots, 0, 1)y^{n-1} + b$, for some $b \in JI^{n-2}$. By Lemma 2.3.4, σ_n sends the homology class of $(F_1(\underline{z}'), \ldots, F_s(\underline{z}'), G(\underline{z}'))$ to the class of $G(0, \ldots, 0, 1)$ modulo $(JI^{n-2} :_R y^{n-1})$ and notice that $G(0, \ldots, 0, 1) = F(0, \ldots, 0, 1)$.

Since J is a reduction of I with reduction number $r_J(I) = r$, we know that $(JI^{n-1} :_R y^n) = R$ for all $n \geq r+1$. Therefore $(Q/Q\langle n-1\rangle)_n = 0$ for all $n > r+1$ and $(Q/Q\langle r\rangle)_{r+1} \cong R/(JI^{r-1} :_R y^r) \neq 0$, since $y^r \notin JI^{r-1}$. Therefore $\mathrm{rt}(I) = r+1$. Notice that the containment $y^{r+1} \in JI^r$ induces an equation of the form $Y^{r+1} - \sum_i X_i F_i$, with $F_i \in V_r$, which is sent by $\sigma_n \circ \tau_n$ to the class of 1 in $R/(JI^{r-1} :_R y^r)$.

Finally, the last claim is nothing but the application of Lemma 2.4.3, together with Lemma 2.4.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

With weaker hypotheses we get the next version of Theorem A (see Remark 2.4.4).

Remark 2.5.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. Let $x_1, \ldots, x_s, y$ be a minimal generating set of I, where $J = (x_1, \ldots, x_s)$ is a reduction of I with reduction number $r = r_J(I)$. Assume that the following two conditions hold:

(a) $x_1, \ldots, x_s$ is a d-sequence;

(b) $VV_{J_i}(I)_{r+1} = (x_1, \ldots, x_i) \cap I^{r+1} / (x_1, \ldots, x_i)I^r = 0$ for all $i = 1, \ldots, s-1$.

Then $\mathrm{rt}(I) \leq r_J(I) + 1$ and $(Q/Q\langle r\rangle)_{r+1}$ contains the non-zero class of an equation $Y^{r+1} - \sum X_i F_i$ induced by $y^{r+1} \in JI^r$.

Suppose, in addition, that the following conditions hold:

(c) $x_1, \ldots, x_s$ is an R-sequence;

(d) $VV_{J_i}(I)_r = (x_1, \ldots, x_i) \cap I^r / (x_1, \ldots, x_i) I^{r-1} = 0$ for all $i = 1, \ldots, s-1$.

Then $\mathrm{rt}(I) = \mathrm{r}_J(I) + 1$ and there is a form $Y^{r+1} - \sum X_i F_i \in Q_{r+1}$ induced by $y^{r+1} \in JI^r$, with $F_i \in V_r$, such that $Q = (Y^{r+1} - \sum X_i F_i) + Q\langle r \rangle$.

Proof: It follows from the proof of Theorem A, but using Remark 2.4.4. $\qquad \square$

Discussion 2.5.2. The hypotheses of Remark 2.5.1 are connected with the works of Huckaba in [Huckaba89, Theorem 1.4] and Trung in [Trung98, Theorem 6.4] (see also [CZ97, Theorem 3.2], [Huckaba87], [GP08, Theorem 5.3] and [Trung87]). In [Huckaba89, Theorems 1.4, 1.5], Huckaba proved that if I is an ideal with $\ell(I) = \mathrm{ht}(I) + 1 \geq 2$ and such that any minimal reduction J of I can be generated by a d-sequence $x_1, \ldots, x_s$ with $x_1^*, \ldots, x_{s-1}^*$ being a $\mathbf{G}(I)$-sequence ($s = \ell(I)$), then $\mathrm{rt}(I) \leq \mathrm{r}_J(I) + 1$. If in addition $\mu(I) = \ell(I) + 1$, then the equality $\mathrm{rt}(I) = \mathrm{r}_J(I) + 1$ holds. In particular, $r = \mathrm{r}_J(I)$ is independent of J. In fact, Trung improved this last result in [Trung98, Theorem 6.4] by showing that r coincides with the Castelnuovo-Mumford regularity of $\mathbf{R}(I)$. To prove $\mathrm{rt}(I) \leq \mathrm{r}_J(I) + 1$, Huckaba showed that the equality $I^{r+1} = JI^r$ induces an equation of $\mathbf{R}(I)$ of maximum degree. In Theorem A and with a different approach, we have fulfilled the description of the entire ideal of equations of $\mathbf{R}(I)$.

As a corollary of Theorem A, one can prove, following an alternative path, the result of Vasconcelos in [Vasconcelos91, Theorem 2.3.3]. The technique used here allows us to prove it for any reduction number, not necessarily equal to 1.

Corollary 2.5.3. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and let I be an $\mathfrak{m}$-primary ideal of R. Let $x_1, \ldots, x_d, y$ be a minimal generating set of I, with $J = (x_1, \ldots, x_d) \subset I$ being a reduction of I, with reduction number $\mathrm{r}_J(I) = r$. Suppose that $x_1^*, \ldots, x_{d-1}^*$ is a $\mathbf{G}(I)$-sequence. Then there is a form $Y^{r+1} - \sum X_i F_i \in Q_{r+1}$, induced by $y^{r+1} \in JI^r$, with $F_i \in V_r$, such that $Q = (Y^{r+1} - \sum X_i F_i) + Q\langle r \rangle$. In particular, $\mathrm{rt}(I) = r+1$.*

Proof: Since R is Cohen-Macaulay and I is $\mathfrak{m}$-primary, $x_1, \ldots, x_d$ is an R-sequence and the results follows from Theorem A. $\qquad \square$

The case $r_J(I) = 1$ has attracted much attention in the literature. In fact, this was the case covered in [Vasconcelos91, Theorem 2.3.3]. Using Remark 2.4.5, we are able to state an analogue of Theorem A with $r_J(I) = 1$, which permits us to relax other hypotheses.

Proposition 2.5.4. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let I be an ideal of R. Let $x_1, \ldots, x_s, y$ be a minimal generating set of I, where $J = (x_1, \ldots, x_s) \subset I$ is a reduction of I with reduction number $r_J(I) = 1$. Suppose that*

(a') $x_1, \ldots, x_s$ is a d-sequence;

(b') $((x_1, \ldots, x_{i-1}) :_R x_i) \cap I = (x_1, \ldots, x_{i-1})$, for all $i = 1, \ldots, s$;

(c') $(x_1, \ldots, x_{i-1}) \cap I^2 = (x_1, \ldots, x_{i-1})I$, for all $i = 1, \ldots, s-1$.

Then, $\mathrm{rt}(I) = 2$ and we have the following isomorhisms of R-modules:

$$\ker \alpha_2 \cong \left[\frac{Q}{Q\langle 1 \rangle} \right]_2 \cong \frac{(JI :_R y^2)}{(J :_R y)} = \frac{R}{(J :_R y)}.$$

In particular, there is a form $Y^2 - \sum X_i F_i \in Q_2$, with $F_i \in V_1$, such that

$$Q = (Y^2 - \sum X_i F_i) + Q\langle 1 \rangle.$$

Proof: It follows from the proof of the Theorem A, using Remark 2.4.5. □

Another corollary of Theorem A is the following result due to Heinzer and Kim in [HK03, Theorem 5.6]. Recall that for an ideal L of R and any standard R-algebra U, it is verified that $\mathrm{rt}(U \otimes R/L) \leq \mathrm{rt}(U)$ (see e.g. [Planas98, Example 3.2]), hence $\mathrm{rt}(\mathbf{F}(I)) \leq \mathrm{rt}(\mathbf{G}(I)) \leq \mathrm{rt}(I)$. In fact, for any $n \geq 2$, there is an exact sequence $E(I)_{n+1} \to E(I)_n \to E(\mathbf{G}(I))_n \to 0$. In particular, $\mathrm{rt}(I) = \mathrm{rt}(\mathbf{G}(I))$ and $E(I)_N \cong E(\mathbf{G}(I))_N$, with $N = \mathrm{rt}(I)$ (see [Planas98, Proposition 3.3]; see also [HKU05, p. 268]). Within our settings, however, $\mathrm{rt}(\mathbf{F}(I))$ is also equal to $\mathrm{rt}(I)$, as proved in the next corollary. Notice that this behaviour is by no means a general fact (see the Conjecture of Valla in [HMV89, § 2] and a counterexample in [SUV93, Example 4.4]; see also [HKU05, p. 268 and Corollary 2.6]).

Corollary 2.5.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field $k = R/\mathfrak{m}$ and let I be an ideal of R. Let $x_1, \ldots, x_s, y$ be a minimal generating set of I, where $J = (x_1, \ldots, x_s) \subset I$ is a reduction of I with reduction number $r = r_J(I)$. Assume that $x_1, \ldots, x_s$ is an R-sequence and that $x_1^*, \ldots, x_{s-1}^*$ is a $\mathbf{G}(I)$-sequence. Then there is a form $Y^{r+1} -$*

$\sum X_i F_i$, with $F_i \in k[X_1, \ldots, X_s, Y]$ *forms of degree* r *and* $\mathbf{F}(I) \cong k[X_1, \ldots, X_s, Y]/(Y^{r+1} - \sum X_i F_i)$. *In particular,* $\mathrm{rt}(\mathbf{F}(I)) = \mathrm{r}_J(I) + 1 = \mathrm{rt}(I)$.

Proof: By Theorem A, $\mathrm{rt}(\mathbf{F}(I)) \leq \mathrm{rt}(I) = r + 1$. By [HK03, Lemma 5.2], $E(\mathbf{F}(I))_n = 0$ for all $2 \leq n \leq r$ and $E(\mathbf{F}(I))_{r+1} \neq 0$. Thus $\mathbf{F}(I)$ has only equations of degree $r + 1$ and $\mathrm{rt}(\mathbf{F}(I)) = r + 1$. By Theorem A, $E(I)_{r+1}$ is cyclic and generated by the equation of $\mathbf{R}(I)$ induced by the containment $y^{r+1} \in JI^r$. Therefore the same happens with $E(\mathbf{F}(I))_{r+1}$ (see [GP08, Proposition 3.2] or [HKU05, p. 268]). $\qquad \square$

Observe that the constraints on I in Corollary 2.5.5 automatically hold for I an ideal with $\mu(I) = \ell(I) + 1$ and $\mathrm{grade}(\mathbf{G}(I)_+) \geq \ell(I) - 1$ (see [BH98, Proposition 1.5.12]).

Remark 2.5.6. An immediate consequence of Corollary 2.5.5 is that $\mathbf{F}(I)$ is Cohen-Macaulay. In fact, recall that [HK03, Proposition 5.4] already prove that, under the conditions of Corollary 2.5.5, $\mathbf{F}(I)$ is a hypersurface if and only if $\mathbf{F}(I)$ is Cohen-Macaulay.

2.6 Examples and applications

Our purpose in this section is to use Theorem A in order to deduce minimal generating sets of the equations of $\mathbf{R}(I)$. If J is a reduction of $I = (J, y)$, the ascending chain of colon ideals $\{(JI^{n-1} : y^n)\}_{n \geq 1}$, which need not be rigid but reaches R, can be calculated in any computer algebra system, giving an alternative strategy to find out the equations of $\mathbf{R}(I)$.

The ideal I in the next example, for a specific $p \geq 1$, is often used as a paradigm of an ideal of relation type $= p$. As said above, Theorem A will be crucial to our purposes.

Example 2.6.1. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let a_1, a_2 an R-sequence and $p \geq 2$. Set $x_1 = a_1^p$, $x_2 = a_2^p$ and $y = a_1 a_2^{p-1}$. Let I be the ideal generated by x_1, x_2, y. Set $V = R[X_1, X_2, Y]$ and let $\varphi : V \to \mathbf{R}(I)$ be the presentation of $\mathbf{R}(I)$ sending X_i to $x_i t$ and Y to yt. Then a minimal generating set of the ideal $Q = \ker(\varphi)$ is given by:

(i) two linear forms $F_1(X_1, X_2; Y) = a_1^{p-1} Y - a_2^{p-1} X_1$ and $G_1(X_1, X_2, Y) = a_2 Y - a_1 X_2$;

(ii) for each $2 \leq n \leq p$, a unique equation $F_n(X_1, X_2, Y) = a_1^{p-n} Y^n - a_2^{p-n} X_1 X_2^{n-1}$.

Proof: We start by proving that I fulfils the hypotheses of Theorem A. Clearly x_1, x_2 is an R-sequence and $J = (x_1, x_2)$ is a reduction of I since $I^p = JI^{p-1}$. By [KS03, Corollary 3],

a monomial m on a_1, a_2 belongs to an ideal generated by monomials $m_1, \ldots, m_r$ on a_1, a_2 if and only if m is a multiple of some m_i. It follows that $y^{p-1} \notin JI^{p-2}$ and $I^{p-1} \not\subset JI^{p-2}$. Thus $r_J(I) = p - 1$.

Claim 1: $\mathrm{grade}(\mathbf{G}(I)_+) \geq 1$ and x_1^* is $\mathbf{G}(I)$-regular.

Proof of Claim 1: By [VV78, Corollary 2.7], it suffices to prove

$$VV_{(x_1)}(I)_n = x_1 R \cap I^n / x_1 I^{n-1} = 0$$

for all $n \geq 1$. Fix $n \geq 1$. By [KS03, Proposition 1],

$$x_1 R \cap I^n = (L_{i,j,k} \mid i, j, k \text{ positive integers such that } i + j + k = n),$$

where $L_{i,j,k} = \mathrm{lcm}(a_1^p, a_1^{ip} a_2^{jp} (a_1 a_2^{p-1})^k)$. Let us prove that $L_{i,j,k}$ is in $x_1 I^{n-1}$.

Indeed, if $i \geq 1$, then $L_{i,j,k} = a_1^{ip} a_2^{jp} (a_1 a_2^{p-1})^k = a_1^p [a_1^{(i-1)p} a_2^{jp} (a_1 a_2^{p-1})^k] \in x_1 I^{n-1}$ and we have finished. Hence we can suppose $i = 0$ and $j + k = n$. If $k = 0$, then $j = n$ and $L_{0,j,0} = a_1^p a_2^{jp} \in x_1 I^{n-1}$. Suppose $0 < k \leq p$. Then $L_{0,j,k} = a_1^p a_2^{jp+k(p-1)} = a_1^p (a_2^p)^{j+k-1} a_2^{p-k} \in x_1 I^{n-1}$. Finally, if $k > p$, then $L_{0,j,k} = a_2^{jp} (a_1 a_2^{p-1})^k = a_1^p [a_1^{k-p} a_2^{jp+k(p-1)}] = a_1^p [a_1^{k-p} a_2^{(k-p)(p-1)} a_2^{jp+p(p-1)}] = a_1^p (a_2^p)^{j+p-1} (a_1 a_2^{p-1})^{k-p} \in x_1 I^{n-1}$.

Note that for $p > 2$, then x_2^* is not a $\mathbf{G}(I)$-sequence because

$$(a_1^{p+2} a_2^{p-2}) x_2 = a_1^p (a_1 a_2^{p-1})^2 = x_1 y^2 \in I^3,$$

where $(a_1^{p+2} a_2^{p-2}) \in I \setminus I^2$.

Therefore, we can apply Theorem A and deduce that, for all $n \geq 2$,

$$\left[\frac{Q}{Q\langle n-1 \rangle} \right]_n \cong \frac{JI^{n-1} : y^n}{JI^{n-2} : y^{n-1}}.$$

In particular, since $(JI^{p-2} : y^{p-1}) \subsetneq (JI^{p-1} : y^p) = R$, then $\mathrm{rt}(I) = r_J(I) + 1$.

Claim 2: $(JI^{n-1} : y^n) = (a_1^{p-n}, a_2)$ for $2 \leq n \leq p - 1$.

Proof of Claim 2: First note that, for all $2 \leq n \leq p - 1$, $a_2 \in (JI^{n-1} : y^n)$ since $a_2 y^n = a_1^n a_2^{n(p-1)+1} = a_1 a_2^p (a_1^{n-1} a_2^{(n-1)(p-1)}) = a_1 a_2^p (a_1 a_2^{p-1})^{n-1} \in JI^{n-1}$. Since $(JI^{n-1} : y^n)$ is generated by monomials on a_1, a_2, and $a_2 \in (JI^{n-1} : y^n)$, there is only one possible remaining generator: a power of a_1. Since $a_1^{p-n} y^n = a_1^{p-n} a_1^n a_2^{n(p-1)} = a_1^p a_2^{n(p-1)} = a_2^{p-n} a_1^p (a_2^p)^{n-1} \in JI^{n-1}$, then $a_1^{p-n} \in (JI^{n-1} : y^n)$. However, and using again [KS03, Corollary 3], one has $a_1^{p-n-1} \notin (JI^{n-1} : y^n)$.

Hence $(Q/Q\langle p-1\rangle)_p \cong (JI^{p-1}:y^p)/(JI^{p-2}:y^{p-1}) = R/(a_1, a_2)$ and, for $2 \le n \le p-1$,

$$\left[\frac{Q}{Q\langle n-1\rangle}\right]_n \cong \frac{JI^{n-1}:y^n}{JI^{n-2}:y^{n-1}} = \frac{(a_1^{p-n}, a_2)}{(a_1^{p-n+1}, a_2)} \cong \frac{(a_1^{p-n})}{(a_1^{p-n+1}, a_1^{p-n}a_2)}.$$

In other words, for each $2 \le n \le p$, $(Q/Q\langle n-1\rangle)_n$ is generated by a single element that corresponds to the class of a_1^{p-n} (1 if $n = p$). To find this element, consider the identity $a_1^{p-n}y^n = a_2^{p-n}a_1^p(a_2^p)^{n-1}$, which induces the equation $F_n(X_1, X_2, Y) = a_1^{p-n}Y^n - a_2^{p-n}X_1X_2^{n-1} \in Q_n$. Since the isomorphism of Theorem A sends the class of F_n to the class of $F_n(0,0,1) = a_1^{p-n}$, we are done. By Remark 2.2.2, F_n is in a minimal generating set of Q, for $2 \le n \le p$.

To finish, let us find the equations of degree one. Although this is trivial, we sketch the proof here to show the similarity with the greater degrees. As before, one shows that $J = (a_1^{p-1}, a_2)$. Using Remark 2.3.5,

$$H_1(x_1, x_2, y; R) \overset{\tilde{\sigma}_1}{\cong} \frac{(J:y)}{J} = \frac{(a_1^{p-1}, a_2)}{(a_1^p, a_2^p)}.$$

Identify Q_1 with $Z_1(x_1, x_2, y; R)$ and $B_1 = \langle x_1Y - yX_1, x_2Y - yX_2, x_1X_2 - x_2X_1 \rangle$ with $B_1(x_1, x_2, y; R)$. Then Q_1/B_1 is minimally generated by the classes of the two equations corresponding to the classes of a_1^{p-1} and a_2. The identities $a_1^{p-1}y = a_2^{p-1}a_1^p$ and $a_2y = a_1a_2^p$ induce the desired equations $F_1(X_1, X_2, Y) = a_1^{p-1}Y - a_2^{p-1}X_1$ and $G_1(X_1, X_2, Y) = a_2Y - a_1X_2 \in Q_1$, since by $\tilde{\sigma}_1$ their classes are sent to the classes of $F_1(0,0,1) = a_1^{p-1}$ and $G_1(0,0,1) = a_2$. Clearly, $B_1 \subset \langle F_1, G_1 \rangle$ and F_1, G_1 are a minimal generating set of Q_1. $\square$

The following example gives the equations of the Rees algebra of an $\mathfrak{m}$-primary ideal of almost-linear type having a reduction generated by a d-sequence. We will use Proposition 2.5.4.

Example 2.6.2. Let $S = k[[X_1, X_2, U_1, U_2]]$ be the power series ring over k with indeterminates X_1, X_2, U_1, U_2, and let $R = k[[X_1, X_2, U_1, U_2]]/L = k[[x_1, x_2, u_1, u_2]]$, where $L = (X_1, X_2) \cap (U_1, U_2) = (X_1U_1, X_1U_2, X_2U_1, X_2U_2)$. Set $J = (x_1 + u_1, x_2 + u_2)$ and $I = (J, x_1 + x_2)$. Let $\varphi : R[T_1, T_2, Y] \to \mathbf{R}(I)$ be the polynomial presentation that sends T_i into $x_i + u_i$ and Y into $x_1 + x_2$ and let $Q = \ker \varphi$. Then J is a reduction of I, with $\mathrm{r}_J(I) = 1$, $\mathrm{rt}(I) = 2$ and $\ker \alpha_{I,2} \cong (Q/Q\langle 1\rangle)_2 \cong R/(J : (x_1 + x_2)) = k$ is cyclic, generated by $Y^2 - Y(T_1 + T_2)$.

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Proof: The ring R is Noetherian local, 2-dimensional and Buchsbaum (in fact, the ideal $(X_1, X_2) \cap (U_1, U_2)$ is said to be of type $(1, 2)$, see [SV78, Theorem 3 and Definition 2, p. 742]), consequently every system of parameters is a d-sequence. Since $x_1 + u_1, x_2 + u_2$ is a system of parameters of R, it is a d-sequence. It is also readily verified that $(x_1 + u_1 : x_2 + u_2) \cap I = (x_1 + u_1)$ and $(x_1 + x_2) \cap I^2 = (x_1 + x_2)I$. By Proposition 2.5.4, the claim follows. $\qquad\square$

The next example shows that the hypotheses (a) and (b) in Remark 2.5.1 alone are not sufficient to ensure that there is only one equation of $\mathbf{R}(I)$ of maximum degree.

Example 2.6.3. Let k be a field and $R = k[X, Y]_{(X,Y)}/(XY, Y^2)_{(X,Y)}$. Set x and y the classes of X and Y in R. Let $\mathfrak{m} = (x, y)$ be the maximal ideal of R. Then, $\text{rt}(\mathfrak{m}) = 2$ and there are two quadratic equations in a minimal generating set of the equations of $\mathbf{R}(\mathfrak{m})$.

Proof: Set $J = (x)$. Since $y^2 = 0 \in J\mathfrak{m}$ and $y \notin J$, then $\mathfrak{m}^2 = J\mathfrak{m}$ and J is a reduction of $\mathfrak{m}$ with reduction number 1. Moreover, since $(0 : x) = (0 : x^2)$, x is a d-sequence. By Remark 2.5.1, $\text{rt}(\mathfrak{m}) \leq r_J(\mathfrak{m}) + 1 = 2$. Set $V = k[S, T]$ and let $\psi : V \to \mathbf{G}(\mathfrak{m})$ be the presentation of $\mathbf{G}(\mathfrak{m})$ sending S to $x + \mathfrak{m}^2$ and T to $y + \mathfrak{m}^2$. For $n \geq 2$, $\mathfrak{m}^n = (x^n)$. Thus $\mathfrak{m}^2/\mathfrak{m}^3$ is a k-vector space of dimension 1. Therefore $\ker(\psi_2) \subset V_2$ must have dimension 2. In fact, $\ker(\psi_2) = \langle ST, T^2 \rangle$. Since $\ker(\psi_1) = 0$, then $E(\mathbf{G}(\mathfrak{m}))_2$ is minimally generated by two elements. We finish by using that $E(\mathfrak{m})_2 \cong E(\mathbf{G}(\mathfrak{m}))_2$ (see 1.3.29 and [Planas98, Proposition 3.3]). Observe that in this case, J and I do not fulfil the condition (b') of Proposition 2.5.4: in fact, $(0 : x) \cap \mathfrak{m} = (y) \neq 0$. $\qquad\square$

As it has already been pointed out in Proposition 2.5.4, we can relax the hypotheses in Theorem A if the reduction number is small. The next example illustrates this by computing the equations for a non-equimultiple ideal I with $\text{ad}(I) = 1$.

Example 2.6.4. Let $(R, \mathfrak{m})$ a Noetherian regular local with infinite residue field k and dimension 4, with u, v, z, t a regular system of parameters. Set $I = (u, z) \cap (v, t)$. Then $I = (J, ut)$ where $J = (uv, vz - ut, zt)$ is a minimal reduction of I with reduction number $r_J(I) = 1$ and such that $uv, vz - ut, zt$ is a d-sequence. Let $\varphi : R[X_1, X_2, X_3, Y] \to \mathbf{R}(I)$ be the polynomial presentation relative to the minimal generating set $uv, vz - ut, zt, ut$ of I and let $Q = \ker \varphi$ be the ideal of equations of $\mathbf{R}(I)$. Then $\text{rt}(I) = 2$ and Q has only one minimal

equation of degree 2: $Y^2 + X_2 Y - X_1 X_3$. Notice that $\mathrm{grade}(I) = 2$, $\mathrm{ht}(I) = 2$, $\ell(I) = 3$ and $\mu(I) = 4$.

Proof: Notice that I is generated by uv, vz, zt, ut. We know that $\mathbf{F}(I) \cong k[uv, vz, zt, ut]$ (see [Vasconcelos05, Example 1-90, p.50]). Displaying a Noether normalisation, one can readily check that $uv, vz - ut, zt$ generate a minimal reduction J of I (see [AM69, Exercise 16, p.69]). It is readily seen that $uv, vz - ut, zt$ is a d-sequence but not an R-sequence, $uv, vz - ut, zt$ is a minimal generating set of J and $JI = I^2$, since $(ut)^2 = (uv)(zt) - (vz - ut)(ut) \in JI$, hence $\mathrm{r}_J(I) = 1$. In order to call to Proposition 2.5.4 one may check the following identities:

(b') $(0 : uv) \cap I = 0$ (for $i = 1$),

$(uv : vz - ut) \cap I = (uv)$ (for $i = 2$),

$((uv, vz - ut) : zt) \cap I = (uv, vz - ut)$ (for $i = 3$);

(c') $(uv) \cap I^2 = (uv)I$ (for $i = 2$),

$(uv, vz - ut) \cap I^2 = (uv, vz - ut)I$ (for $i = 3$).

Consequently, $Q = Q\langle 2 \rangle$, $\mathrm{rt}(I) = 2$ and $(Q/Q\langle 1 \rangle)_2 \cong (JI : (ut)^2)/(J : ut) \cong R/\mathfrak{m}$ is cyclic with generator $Y^2 + X_2 Y - X_1 X_3$. $\qquad\square$

Example 2.6.5. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let a, b an R-sequence, let $p \geq 5$ be an odd integer, and set $x_1 = a^p$, $x_2 = b^p$ and $y = a^2 b^{p-2}$. Let $J = (x_1, x_2)$ and $I = (x_1, x_2, y)$, $V = R[X_1, X_2, Y]$ the polynomial ring in three indeterminates over R and let $\varphi : V \to \mathbf{R}(I)$ be the presentation of $\mathbf{R}(I)$ sending X_i to $x_i t$ and Y to yt.

(i) The following equalities hold:

$$(JI^n : y^{n+1}) = \begin{cases} (a^{p-2n-2}, b^2) & 1 \leq n \leq \frac{p-3}{2} \\ (a, b) & \frac{p-1}{2} \leq n \leq p - 2 \\ R & n \geq p - 1 \end{cases}$$

In particular, J is a reduction of I with $\mathrm{r}_J(I) = p - 1$;

(ii) The following equalities hold: $(x_1 I^{n-1} : x_2) \cap I^{n-1}/x_1 I^{n-2} = 0$, for all $n \geq 2$; equivalently, $H_1(x_1 t, x_2 t ; \mathbf{R}(I))_n = 0$, for all $n \geq 2$;

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(iii) It is verified that $(Q/Q\langle n-1\rangle)_n \cong (JI^{n-1} : y^n)/(JI^{n-2} : y^{n-1})$, for all $n \geq 2$. In particular $\mathrm{rt}(I) = p$. Moreover,

$$Q = \{S_{1,2}, S_{1,p-2}, F_2, \ldots, F_{(p+1)/2}, F_p\}$$

is a minimal generating set of Q with

$$F_n(X_1, X_2, Y) = \begin{cases} a^{p-2n}Y^n - b^{p-2n}X_1X_2^{n-1} & 2 \leq n \leq \frac{p-1}{2} \\ bY^n - aX_1X_2^{n-1} & n = \frac{p+1}{2} \\ Y^p - X_1^2X_2^{p-2} & n = p \end{cases}$$

and $S_{1,2}(X_1, X_2, Y) = a^2X_2 - b^2Y$ and $S_{1,p-2}(X_1, X_2, Y) = a^{p-2}Y - b^{p-2}X_1$.

(iv) The initial forms x_1^*, x_2^* and y^* are zero-divisors in $\mathbf{G}(I)$.

Proof: In order to show (i), notice first that since $y^p = x_1^2x_2^{p-2} \in J^p \subset JI^{p-1}$, we have $(JI^{p-1} : y^p) = R$ and $\mathrm{r}_J(I) \leq p - 1$. Let us show that $y^{n+1} \notin JI^n$ for $n \leq p - 2$: assuming the opposite, then we can write $y^{n+1} = x_1^r x_2^s y^t$ with r, s, t positive integer such that $r + s + t = n + 1$. If $t \geq 1$ we can simplify the expressions, hence we can assume without loss of generality that $y^{n+1} = x_1^r x_2^s$ with $r + s = n + 1$ and $n \leq p - 2$. Then $a^{2(n+1)}b^{(p-2)(n+1)} = a^{rp}b^{sp}$, thus $2(n+1) = rp$. But observe that $n + 1 \leq p - 1 < p$, so r has to be 1, leading to $p = 2(n+1)$, a contradiction with the assumption that p is odd. Consequently, $\mathrm{r}_J(I) = p - 1$.

We claim that if $(p-1)/2 \leq n \leq p - 2$, then $(JI^n : y^{n+1}) = (a, b)$. Playing with the exponents we get

$$ay^{n+1} = a^p a^{2n-p+3} b^{(2n-p+3)(p-2)/2} b^{(p-1)(p-2)/2} = bx_1 x_2^{(p-3)/2} y^{n-(p-3)/2} \in JI^n,$$

with $2n - p + 3$ being positive by our assumptions on n. On the other hand,

$$by^{n+1} = aa^p a^{2n-p+1} b^{(2n-p+1)(p-2)/2} b^{p(p-1)/2} = ax_1 x_2^{(p-1)/2} y^{n-(p-1)/2} \in JI^n,$$

with $2n - p + 1$ being positive by our assumptions on n.

Observe that if $n \leq (p-3)/2$ then $b^2 \in (JI^n : y^{n+1})$, since $b^2 y^{n+1} = a^2 b^p a^{2n} b^{n(p-2)} = a^2 x_2 y^n \in JI^n$. However, in this case $b \notin (JI^n : y^{n+1})$. If $by^{n+1} \in JI^n$ we can assume without loss of generality that by^{n+1} is of the form $ax_1^r x_2^s$ with r, s positive integers such

that $r + s = n + 1$, but then $2(n + 1) = rp + 1$ and since $2(n + 1) \leq p - 2$ we arrive to a contradiction.

Likewise, if $n \leq (p-3)/2$ then $a^{p-2n-2} \in (JI^n : y^{n+1})$, since $a^{p-2n-2}y^{n+1} = a^p b^{pn} b^{p-2n-2} = b^{p-2n-2} x_1 x_2^n \in JI^n$. However, if k is a positive integer such that $k \leq p - 2n - 3$, we have $a^k \notin (JI^n : y^{n+1})$. Assuming the opposite, $a^k y^{n+1} \in (JI^n : y^{n+1})$, without loss of generality $a^k y^{n+1} = b^k x_1^r x_2^s$ with r, s positive integers such that $r + s = n + 1$. But then it follows that $2(n+1) + k = rp$, a contradiction, since $2(n+1) + k \leq p - 1$.

To finish with this description of the generators of $(JI^n : y^{n+1})$, let us show that for $n \leq (p-3)/2$ and $k \leq p - 2n - 3$, $a^k b \notin (JI^n : y^{n+1})$. If $a^k b y^{n+1} \in JI^n$ then we can assume without loss of generality that $a^k b a^{2(n+1)} b^{(p-2)(n+1)} = m x_1^r x_2^s$ with r, s positive integers such that $r + s = n + 1$ and m a monomial in a, b with $\deg(m) = k + 1$. If m is multiple of b then, simplifying b's, we get $a^k \in (JI^n : y^{n+1})$, which is in contradiction with the preceding paragraph. Thus $m = a^{k+1}$, but then, simplifying a's, we would deduce that $b \in (JI^n : y^{n+1})$, which is a contradicition with the claims above. Therefore, when $n \leq (p - 3)/2$ the ideal $(JI^n : y^{n+1})$ is the irreducible monomial ideal generated by a^{p-2n-2} and b^2.

Let us show (ii). We will show the equality $(x_1 I^n : x_2) \cap I^n = x_1 I^{n-1}$ for all $n \geq 1$, then apply Lemma 2.4.1. For that purpose we will make use of Remark 1.3.5. Concretely, we will show that any monomial generator of $(x_1 I^n : x_2) \cap I^n$ belongs to $x_1 I^{n-1}$. Each monomial generator of $(x_1 I^n : x_2) \cap I^n$ is of the form

$$G_{\sigma,\tau} = \mathrm{lcm}\left(M_\sigma, x_1^u x_2^v y^w\right),$$

where $M_\sigma = \mathrm{lcm}(x_1^{r+1} x_2^s y^t, x_2)/x_2$, with $r + s + t = n$, $u + v + w = n$; σ and τ denote the integer vectors $(r, s, t) \in \mathbb{N}^3$ and $(u, v, w) \in \mathbb{N}^3$, respectively. Let us show that $G_{\sigma,\tau} \in x_1 I^{n-1}$

Case: $s \geq 1$.
$M_\sigma = x_1^{r+1} x_2^{s-1} y^t$, then $G_{\sigma,\tau}$ is a multiple of M_σ which can be written as $x_1 \cdot (x_1^r x_2^{s-1} y^t) \in x_1 I^{n-1}$.

Case: $s = 0$.
$M_\sigma = \mathrm{lcm}(x_1^{r+1} y^t, x_2)/x_2 = \mathrm{lcm}(a^{p(r+1)+2t} b^{(p-2)t}, b^p)/b^p$. At this point it is convenient to distinguish a couple of subcases: $t = 0, 1$ and $t \geq 2$.

Sub-case: $t = 0, 1$.

If $t = 0$, $M_\sigma = a^{p(n+1)}$, then $G_{\sigma,\tau}$ is a multiple of $a^{p(n+1)} = x_1^{n+1}) \in x_1 I^n \subset x_1 I^{n-1}$. If $t = 1$, $M_\sigma = a^{pn+2}$, then $G_{\sigma,\tau} = \mathrm{lcm}(a^{pn+2}, a^{pu+2w}b^{pv+(p-2)w})$. Since $pn + 2 > pu + 2w$ for all $(u, v, w) \in \mathbb{N}^3$ such that $u + v + w = n$, $G_{\sigma,\tau} = a^{pn+2}b^{pv+(p-2)w} = a^p a^{p(n-1)+2}b^{pv+(p-2)w} \in x_1 I^{n-1}$.

Sub-case: $t \geq 2$.

In this case, since $p \geq 5$, the inequality $(p-2)t > p$ holds, then

$$M_\sigma = \mathrm{lcm}(a^{p(r+1)+2t}b^{(p-2)t}, b^p)/b^p = a^{p(r+1)+2t}b^{(p-2)t-p}.$$

Therefore, we have to check

$$G_{\sigma,\tau} = \mathrm{lcm}\left(a^{p(r+1)+2t}b^{(p-2)t-p}, x_1^u x_2^v y^w\right) \in x_1 I^{n-1}.$$

Since $G_{\sigma,\tau}$ is a multiple of $x_1^u x_2^v y^w$, if $u \geq 1$, then $G_{\sigma,\tau}$ will be multiple of $x_1 \cdot x_1^{u-1} x_2^v y^w \in x_1 I^{n-1}$. So it remains to consider the case where $u = 0$. In this case,

$$
\begin{aligned}
G_{\sigma,\tau} &= \mathrm{lcm}\left(a^{p(r+1)+2t}b^{(p-2)t-p}, a^{2w}b^{pv+(p-2)w}\right) \\
&= \mathrm{lcm}\left(a^{p(n-t+1)+2t}b^{(p-2)t-p}, a^{2w}b^{p(n-w)+(p-2)w}\right)
\end{aligned}
$$

We claim that $2w \leq p(n-t+1) + 2t$: the least value that $p(n-t+1) + 2t$ can take is $p + 2n$, whereas the greatest value that $2w$ can take is $2n$; then $2w \leq p(n-t+1) + 2t$, then $p(n-w) + (p-2)w = pn - 2w \geq pn - p(n-t+1) - 2t = t(p-2) - p > 0$. Hence $G_{\sigma,\tau} = a^{p(n-t+1)+2t}b^{p(n-w)+(p-2)w} = a^p \cdot a^{p(n-t)+2t}b^{p(n-w)+(p-2)w}$. If $w < t$ then $G_{\sigma,\tau}$ is a multiple of $a^p \cdot a^{p(n-t)+2t}b^{p(n-t)+(p-2)t} = x_1 x_1^{n-t} x_2^{n-t} y^t \in x_1 I^{2n-t} \subset x_1 I^n \subset x_1 I^{n-1}$. If $w \geq t$ then $G_{\sigma,\tau}$ is a multiple of $a^{p(n-w+1)+2w}b^{p(n-w)+(p-2)w} = a^p \cdot a^{p(n-w)+2w}b^{p(n-w)+(p-2)w} = x_1 x_1^{n-w} x_2^{n-w} y^w \in x_1 I^{2n-w} \subset x_1 I^n \subset x_1 I^{n-1}$.

To prove (*iii*) simply use (*ii*) and Theorem A.

In order to prove (*iv*) it suffices to check that both x_1^* and x_2^* are zero-divisors in $\mathbf{G}(I)$. It will be sufficient to find elements $\omega_1 \in I^{n_1}\backslash I^{n_1+1}$ and $\omega_2 \in I^{n_2}\backslash I^{n_2+1}$ such that $x_1\omega_1 \in I^{n_1+2}$ and $x_2\omega_2 \in I^{n_2+2}$. Notice that the exponents n_1, n_2 may depend on p, as the ideal I does. Let $n_1(p) = (p-3)/2$, $n_2 = 0$ and take $\omega_1 = ab^{(n_1(p)+2)\cdot(p-2)} \in I^{n_1(p)}\backslash I^{n_1(p)+1}$ and $\omega_2 = a^4 b^{p-4} \in R \notin I$. Then $x_1\omega_1 = a^{p+1}b^{(n_1(p)+2)(p-2)}$ and since $p + 1 = 2(n_1(p) + 2)$ we

get $x_1\omega_1 = (a^2b^{p-2})^{n_1(p)+2} = y^{n_1(p)+2} \in I^{n_1(p)+2}$ and, on the other hand, $x_2\omega_2 = a^4b^{2p-4} = (a^2b^{p-2})^2 = y^2 \in I^2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The next example shows the non-surprising fact that, although $(Q/Q\langle n-1\rangle)_n$ are isomorphic to quotients of the form $(JI^{n-1} : y^n)/(JI^{n-2} : y^{n-1})$, they need not be cyclic in degrees $< r_J(I) + 1$. The proof in full detail of Example 2.6.6 would be similar to that of Example 2.6.5.

Example 2.6.6. Let $(R, \mathfrak{m})$ be a Noetherian local ring. Let a, b, c an R-sequence, let $p \geq 4$ be an even integer and set $x_1 = a^p$, $x_2 = b^p$, $x_3 = c^p$ and $y = abc^{p-2}$. Let $J = (x_1, x_2, x_3)$ and $I = (x_1, x_2, x_3, y)$, $V = R[X_1, X_2, X_3, Y]$ the polynomial ring in four indeterminates over R and let $\varphi : V \to \mathbf{R}(I)$ be the presentation of $\mathbf{R}(I)$ sending X_i to $x_i t$ and Y to yt. Then,

(i) The ideal J is a reduction of I with $r_J(I) = p - 1$, concretely,

$$
(JI^n : y^{n+1}) = \begin{cases} (a^{p-n-1}, b^{p-n-1}, c^2) & 0 \leq n < \frac{p}{2} \\ (a^{p/2}, b^{p/2}, c^2, a^{p-n-1}b^{p-n-1}) & \frac{p}{2} \leq n < p-1 \\ R & n \geq p-1 \end{cases}
$$

(ii) The initial forms x_1^*, x_2^* are a $\mathbf{G}(I)$-sequence, in particular $\mathrm{grade}(\mathbf{G}(I)_+) \geq 2$; since x_1, x_2, x_3 are an R-sequence, by Lemma 2.4.3, we get $H_1(x_1 t, x_2 t, x_3 t \, ; \mathbf{R}(I))_n = 0$, for all $n \geq 2$;

(iii) It is verified that $(Q/Q\langle n-1\rangle)_n \cong (JI^{n-1} : y^n)/(JI^{n-2} : y^{n-1})$, for all $n \geq 2$. In particular, $\mathrm{rt}(I) = p$. Moreover, in terms of generators, the graded components of the modules $(Q/Q\langle n-1\rangle)_n$ are given by

$$
(Q/Q\langle n-1\rangle)_n = \begin{cases} \langle a^{p-n}Y^n - b^n c^{p-2n}X_1 X_3^{n-1}, \\ \quad b^{p-n}Y^n - a^n c^{p-2n}X_2 X_3^{n-1}\rangle & 2 \leq n \leq \frac{p}{2}, \\ \langle a^{n-2}b^{n-2}Y^n - c^{2n-4}X_1 X_2 X_3^{n-2}\rangle & n = \frac{p}{2}+1, \\ \langle a^{p-n}b^{p-n}Y^n - c^{2(p-n)}X_1 X_2 X_3^{n-2}\rangle & \frac{p}{2}+1 < n < p, \\ \langle Y^n - X_1 X_2 X_3^{n-2}\rangle & n = p. \end{cases}
$$

Identifying Q_1 with $Z_1(x_1, x_2, x_3, y \, ; R)$ and

$$B_1 = \langle x_1 Y - yX_1, x_2 Y - yX_2, x_3 Y - yX_3, x_2 X_3 - x_3 X_2, x_1 X_3 - x_3 X_1, x_1 X_2 - x_2 X_1\rangle$$

with $B_1(x_1, x_2, x_3, y; R)$, then

$$H_1(x_1, x_2, x_3, y; R) \cong \frac{Q_1}{B_1} \cong \frac{(J:y)}{J} \cong \frac{(a^{p-1}, b^{p-1}, c^2)}{(a^p, b^p, c^p)}$$

is minimally generated by the classes of the equations corresponding to the classes of a^{p-1}, b^{p-1} and c^2:

$$\frac{Q_1}{B_1} \cong \langle a^{p-1}Y - bc^{p-2}X_1, b^{p-1}Y - ac^{p-2}X_2, c^2Y - abX_3 \rangle.$$

Proof: Assertion (i) follows by applying the same strategy as in the proof of Example 2.6.5. In order to prove (ii) recall Remark 1.3.6: since x_1, x_2 is an R-sequence, it suffices to check that $(x_1) \cap I^n = (x_1)I^{n-1}$ and $(x_1, x_2) \cap I^n = (x_1, x_2)I^{n-1}$, for all $n \geq 2$. In order to prove (iii) just combine (i), (ii), Theorem A and Remark 2.3.5. $\qquad\square$

The next example shows that if $\mathrm{sd}(I) = \mu(I) - \ell(I) \neq 1$, there might be several equations of $\mathbf{R}(I)$ of top degree.

Example 2.6.7. Let $(R, \mathfrak{m})$ be a two dimensional regular local ring and let $p \geq 2$ be an integer. Then $\mathrm{rt}(\mathfrak{m}^p) = 2$ and there are $\binom{p}{2}$ quadratic equations in a minimal generating set of equations of $\mathbf{R}(\mathfrak{m}^p)$.

Proof: Let x, y be a regular system of parameters of R, $V = R[X, Y]$ and let $\varphi : V \to \mathbf{R}(\mathfrak{m})$ be the presentation of $\mathbf{R}(\mathfrak{m})$ sending X to xt and Y to yt. Since x, y is an R-sequence, $\ker(\varphi) = \langle xY - yX \rangle$. Set $V(p) = R[X^p, X^{p-1}Y, \ldots, XY^{p-1}, Y^p]$ and $\varphi(p) : V(p) \to \mathbf{R}(\mathfrak{m}^p)$ the p-th Veronese transform of φ. Note that $\ker(\varphi(p)_n) = \ker(\varphi_{pn}) = \langle xY - yX \rangle V_{pn-1}$.

Set $W = R[T_0, T_1, \ldots, T_p]$ and let $\psi : W \to V(p)$ be the polynomial presentation of $V(p)$ sending T_i to $X^{p-i}Y^i$. It is known (see e.g. [JK94, Proposition 2.5]) that the kernel of ψ is the determinantal ideal generated by the 2×2 minors of the matrix $\mathcal{M}$, where $\mathcal{M}$ is

$$\begin{pmatrix} T_0 & T_1 & \ldots & T_{p-1} \\ T_1 & T_2 & \ldots & T_p \end{pmatrix}.$$

In particular, $\ker(\psi) = \ker(\psi)\langle 2 \rangle$.

Consider $\Phi = \varphi(p) \circ \psi : R[T_0, T_1, \ldots, T_p] \to \mathbf{R}(\mathfrak{m}^p)$, the polynomial presentation of $\mathbf{R}(\mathfrak{m}^p)$ sending T_i to $x^i y^{p-i}$ and let $Q = \ker(\Phi)$ be the ideal of equations of $\mathbf{R}(\mathfrak{m}^p)$. Let us see that $Q_n = \ker(\psi_n) + W_{n-1}Q_1$, for all $n \geq 2$. Indeed, given $F \in Q_n$, since $\ker(\varphi(p)_n) =$

$\langle xY - yX \rangle V_{pn-1}$, one can find $F_i \in W_{n-1}$ and $G_i \in Q_1$ such that $\psi_n(F) = \psi_n(\sum F_i G_i)$. Therefore, $(F - \sum F_i G_i) \in \ker(\psi_n)$ and $F \in \ker(\psi_n) + W_{n-1} Q_1$.

Since $\ker(\psi) = \ker(\psi)\langle 2 \rangle$, then $Q_n = \ker(\psi_n) + W_{n-1} Q_1 = W_{n-2} \ker(\psi_2) + W_{n-1} Q_1 \subset W_{n-2} Q_2 \subset Q_n$, for all $n \geq 2$. Therefore $Q = Q\langle 2 \rangle$, $\mathrm{rt}(\mathfrak{m}^p) = 2$ and $\mu(Q_2/W_1 Q_1) \leq \binom{p}{2}$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \ker \psi_2 & \longrightarrow & W_2 & \xrightarrow{\psi_2} & V(p)_2 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \ker \psi_2 \otimes k & \longrightarrow & W_2 \otimes k & \xrightarrow{\psi_2 \otimes 1_k} & V(p)_2 \otimes k & \longrightarrow & 0
\end{array}
$$

Let us prove that $Q_2/W_1 Q_1$ is minimally generated by $\binom{p}{2}$ elements, which are precisely the classes of the 2×2 minors of $\mathcal{M}$. Since $Q_1 \subset \mathfrak{m} W_1$, then $W_1 Q_1 \subset \mathfrak{m} W_2$. Setting $L = \ker(\psi_2)$, $M = W_2$ and $N = V(p)_2$, we have

$$
\frac{Q_2}{W_1 Q_1} \otimes k \cong \frac{Q_2}{\mathfrak{m} Q_2 + W_1 Q_1} = \frac{\ker(\psi_2) + W_1 Q_1}{\mathfrak{m} \ker(\psi_2) + W_1 Q_1} \cong \frac{L}{\mathfrak{m} L + (L \cap W_1 Q_1)}.
$$

On the other hand, there is a natural epimorhism

$$
\frac{L}{\mathfrak{m} L + (L \cap W_1 Q_1)} \rightarrow \frac{L}{L \cap \mathfrak{m} M},
$$

where $L/(L \cap \mathfrak{m} M) \cong (L + \mathfrak{m} M)/\mathfrak{m} M = \ker(\psi_2 \otimes 1_k)$. Hence,

$$
\mu \left(\frac{Q_2}{W_1 Q_1} \right) \geq \dim \frac{L}{(L \cap \mathfrak{m} M)} = \dim \ker(\psi_2 \otimes 1_k),
$$

which clearly is $\binom{p}{2}$. Therefore $\mu((Q/Q\langle 1 \rangle)_2) = \binom{p}{2}$ and, by Remark 2.2.2, there are $\binom{p}{2}$ equations of degree 2 in a minimal generating set of equations of $\mathbf{R}(\mathfrak{m}^p)$.

Note that $J = (x_1^p, x_2^p)$ is a reduction of $\mathfrak{m}^p$ with reduction number $r_J(\mathfrak{m}^p) = 1$, that x_1^p, x_2^p is an R-sequence and that $(x_1^p)^*$ is a $\mathbf{G}(\mathfrak{m}^p)$-sequence. $\qquad \square$

Chapter 3

The injectivity of the canonical blowing-up morphisms

3.1 Introduction

Let R be a commutative ring and let $I = (x_1, ..., x_s, y)$ be an ideal of R. Let $\alpha_I : \mathbf{S}(I) \to \mathbf{R}(I)$ be the canonical morphism from the symmetric algebra of I to the Rees algebra of I. We will write $\alpha_{I,p}$ to denote the p-th graded component of α_I. The aim of this chapter is to show that if I is an ideal of almost-linear type, the injectivity of a single graded component $\alpha_{I,p}$ implies the injectivity of the lower graded components.

In [Kühl82, Example 1.4] (see Example 1.3.34), Kühl gave an example of a finitely generated ideal I with $\alpha_{I,n}$ being an isomorphism for n sufficiently large, in particular, $\widetilde{\alpha}_I : \mathrm{Proj}(\mathbf{R}(I)) \to \mathrm{Proj}(\mathbf{S}(I))$ is an isomorphism of schemes (see Remark 1.3.33), but such that α_I is not an isomorphism of R-algebras. On the other hand, [Tchernev07] asked whether $\alpha_{I,p}$ being an isomorphism implies that $\alpha_{I,n}$ is an isomorphism for each $2 \leq n \leq p$.

The main purpose of this chapter is to prove the following result.

Theorem B. *Let R be a commutative ring, let $I = (x_1, \ldots, x_s, y)$ be an ideal of R and let $n \geq 2$ be an integer. Suppose that the ideal $J = (x_1, \ldots, x_s)$ verifies that $\alpha_{J,n} : \mathbf{S}_n(J) \to J^n$ is an isomorphism for all $2 \leq n \leq p$. Then the following conditions are equivalent:*

(i) $\alpha_{I,p} : \mathbf{S}_p(I) \to I^p$ is an isomorphism;

(ii) $\alpha_{I,n} : \mathbf{S}_n(I) \to I^n$ is an isomorphism for each $2 \leq n \leq p$.

Notice that the ideals of almost-linear type fulfil the hypotheses of Theorem B, hence the equivalence of (i) and (ii) holds true for such ideals.

For each integer $p \geq 2$, we also display an example of an ideal I such that $\alpha_{I,n}$ is an isomorphism for all $n \geq p+1$, whereas $\alpha_{I,p}$ is not.

Example 3.9.1. Let k be a field and let $p \geq 2$. Let $S = k[U_0, \ldots, U_p, X, Y]$ be a polynomial ring and let Q be the ideal of S defined as $Q = Q_1 + (U_0 X^p)$, where

$$Q_1 = (U_0 Y, U_0 X - U_1 Y, U_1 X - U_2 Y, \ldots, U_{p-1} X - U_p Y, U_p X).$$

Let R be the factor ring $S/Q = k[u_0, \ldots, u_p, x, y]$ and consider the ideal $I = (x, y) \subset R$. Then $\alpha_{I,n}$ is an isomorphism for all $n \geq p+1$, whereas $\alpha_{I,p}$ is not.

As in Section 1.3.1, we denote by β_I and $\gamma_{I,\mathfrak{m}}$ the blowing-up morphisms $\alpha \otimes \mathbf{1}_{R/I}$: $\mathbf{S}(I/I^2) \to \mathbf{G}(I)$ and $\alpha \otimes \mathbf{1}_{R/\mathfrak{m}} : \mathbf{S}(I/\mathfrak{m}I) \to \mathbf{F}(I)$, respectively. In order to slim the notations, we will write α, β and γ when the ideals involved are clear from the context. Let us begin with some basic remarks about the influence of the injectivity of a single graded component of the canonical blowing-up morphisms on the rest of the graded components.

Notice first that in the case of $\gamma_{I,\mathfrak{m}} : \mathbf{S}(I/\mathfrak{m}I) \to \mathbf{F}_{\mathfrak{m}}(I)$ the vanishing of a single component of $\ker \gamma_{I,\mathfrak{m}}$ imposes a tight condition to the lower graded components, for if $\gamma_{I,\mathfrak{m},p+1}$ is an isomorphism, then $\gamma_{I,\mathfrak{m},p}$ is also an isomorphism. In fact, if there is a non-zero element $a \in \ker(\gamma_{I,\mathfrak{m},p})$, since every non-zero element in $\mathbf{S}_1(I/\mathfrak{m}I)$ is a non-zerodivisor of $\mathbf{S}(I/\mathfrak{m}I)$, we can pick a non-zero element ω in $\mathbf{S}_1(I/\mathfrak{m}I)$, such that $\omega \cdot a \in \ker(\gamma_{I,\mathfrak{m},p+1})$ is non-zero.

Considering this propagation of the injectivity of the graded components of $\gamma_{I,\mathfrak{m}}$ towards the lower components, it is natural to ask for the same property for the morphisms α_I and β_I. In the case of α_I it turns out to not hold in general. Indeed, in [Kühl82, Example 1.4], Kühl constructed an ideal I with $\alpha_{I,p}$ being an isomorphism for $p \geq 3$ while $\alpha_{I,2}$ was not (see Example 1.3.34). However, under certain settings the injectivity of the graded components of α propagates downwards, as Tchernev proved in [Tchernev07]: if $\alpha_{I,p}$ is an isomorphism, then $\alpha_{I,n}$ is an isomorphism for $2 \leq n \leq p$ when either $\mathrm{pd}_R(I) \leq 1$, or I is perfect Gorenstein of grade 3, or I is a maximal ideal (see [Tchernev07, Section 5]). It is worth noting that

conditions of this form (α_n being an isomorphism for all $2 \leq n \leq p$) have already been considered across the commutative algebra landscape (see, for instance, [CV99, Introduction, page 754], [Costa83, Theorem 3] or [Vasconcelos05, Definition 1.9 and Exercise 5.84]).

Theorem B provides a new class of ideals I for which this pattern of the injectivity of α_I holds. Concretely, if $I = (J, y)$ with $\alpha_{J,n} : \mathbf{S}_n(J) \to J^n$ an isomorphism for each $2 \leq n \leq p$, then $\alpha_{I,p} : \mathbf{S}_p(I) \to I^p$ being an isomorphism implies that $\alpha_{I,n} : \mathbf{S}_n(I) \to I^n$ is an isomorphism for each $2 \leq n \leq p$. In particular, if J is an ideal of linear type, i.e., if I is an ideal of almost-linear type, the hypothesis is fulfilled. Remark that writing an ideal I as $I = (J, y)$ and assuming hypotheses over J and the relation between J and y, is a common approach in this context (see e.g. [Costa85], [HMV89, Theorem 4.7], [HSV81, Proposition 3.9], [Valla80, Theorem 2.3 and Proposition 2.5]). On the other hand, the study of the morphisms α, β and γ has attracted a great deal of attention so far (see, for instance, [Costa83], [Costa85], [HMV89], [HSV81], [Huneke80], [Huneke86], [JMacL06], [Keel93], [Kühl82], [Micali64], [Tchernev07], [Valla80], [Vasconcelos94] and [Vasconcelos05]).

In Example 3.9.1 we will give a family of ideals $\{I_p\}_{p \geq 2}$ such that $\alpha_{I_p,n}$ is an isomorphism for each $n \geq p + 1$, whereas $\alpha_{I_p,p}$ is not. Furthermore, $\beta_{I_p,n}$ will be an isomorphism if and only if $n \neq p$. More explicitly, for each positive integer p we construct a $k[X, Y]$-algebra $\mathcal{A}_p$ and a 2-generated ideal $I_p \subset \mathcal{A}_p$ such that $\beta_{I_p,n}$ is an isomorphism if and only if $n \neq p$. Furthermore, the example admits a "glueing", in the sense that it is possible to construct, for any finite subset $S \subset \mathbb{N}$, an ideal I_S of a $k[X, Y]$-algebra $\mathcal{A}_S$ such that $\beta_{I_S,n}$ is an isomorphism if and only if $n \notin S$, by tensor-multiplying the collection of above defined $k[X, Y]$-algebras $\mathcal{A}_i$, for all $i \in S$. Beyond extending the example given by Kühl (focused on the case $p = 2$), our example illustrates the fact that the morphism β may have only one single non-injective component.

Many of the results presented in this chapter can also be found in [MP08]: F. Muiños, F. Planas-Vilanova. *On the injectivity of blowing-up ring morphisms.* J. Algebra **320** (2008), no. 8, 3365–3380.

3.2 Injectivity does not propagate upwards

In general, one cannot expect the injectivity of a single component of α, β or γ to propagate upwards, i.e., under mild conditions, having $\ker \alpha_n = 0$ (same for β and γ) does not imply $\ker \alpha_p = 0$ for $p > n$.

Let $\mathfrak{p}$ be a syzygetic prime ideal with $\mu(\mathfrak{p}) = r > 3$. Take for instance $R = k[[X, Y, Z]]$ and $\mathfrak{p}$ any ideal belonging to the family defined in [Moh74], which turn out to be syzygetic (i.e., $\ker \alpha_2 = 0$). The analytic spread $\ell(\mathfrak{p})$ satisfies $\ell(\mathfrak{p}) \leq \dim(R) = 3 < r$. For n large enough, $\dim_k(\mathfrak{p}^n/\mathfrak{mp}^n)$ is a polynomial in n of degree $\ell(\mathfrak{p}) - 1$, the Hilbert polynomial of $\mathbf{F}_\mathfrak{m}(I)$, whilst $\dim_k(\mathbf{S}_n(\mathfrak{p}/\mathfrak{mp})) = \binom{n+r-1}{r-1}$ is a polynomial in n of degree $r - 1$. Therefore, for n large enough, $\gamma_{\mathfrak{p},\mathfrak{m},n} : \mathbf{S}_n(\mathfrak{p}/\mathfrak{mp}) \to \mathfrak{p}^n/\mathfrak{mp}^n$ is not an isomorphism, and consequently, neither $\alpha_{\mathfrak{p},n}$ nor $\beta_{\mathfrak{p},n}$ are isomorphisms, whereas $\alpha_{\mathfrak{p},2}$, $\beta_{\mathfrak{p},2}$ and $\gamma_{\mathfrak{p},\mathfrak{m},2}$ are isomorphisms.

Remark 3.2.1. Recall that an ideal is projectively of linear type (of p-linear type, for short) if and only if $\ker \alpha_{I,n} = 0$ for n sufficiently large (see claim 1.3.33). Then if I is of p-linear type it is a necessary condition that $\ker \gamma_n$ vanish for large n, then it follows that $\ell(I) = \mu(I)$, i.e., the second analytic deviation of I must be equal to zero. However, the converse is not true unless we consider some extra conditions. For instance, consider the classical example $I = (X^3, X^2Y, Y^2Z) \subset k[[X, Y, Z]]$ (see [HMV89, p.16]): $\mu(I) = \ell(I) = 3$, while the class of $ZT_2^2 - XT_1T_3 \in \ker \alpha_{I,2}$ induces a non-zero element of $\ker \alpha_{I,n}$, for each $n \geq 2$ (e.g., multiply repeatedly by T_2).

The proof of Theorem B is based on the characterization of the vanishing of the effective relations in terms of the vanishing of the Koszul homology, which is in turn translated into a set of conditions in terms of colon ideals (see previous Chapter 2 and [Planas98]).

Sections 3.3 and 3.4 are devoted to recall and refine some basic results concerning $\ker \alpha$ and the module of effective relations. In Sections 3.5 and 3.7 we prove preliminary propositions which will be assembled in Section 3.8, where the main result is shown.

3.3 The presentation of $\ker \alpha$

The following lemma provides us with a presentation of the kernel of α.

Lemma 3.3.1. *Let R be a commutative ring and let $I = (x_1, \ldots, x_s)$ be a finitely generated ideal. Let $V = R[X_1, \ldots, X_s]$ be a polynomial ring and let $\varphi : V \to \mathbf{R}(I)$ be the induced surjective graded morphism sending X_i to $x_i t$. Let $\alpha_I : \mathbf{S}(I) \to \mathbf{R}(I)$ be the canonical morphism. Then, for $n \geq 2$,*

$$\ker(\alpha_{I,n}) \cong \ker(\varphi_n)/V_{n-1}\ker(\varphi_1).$$

Proof: Let $S(\varphi_1) : V \to \mathbf{S}(I)$ be the induced surjective graded morphism sending X_i to x_i in degree one. We have the following commutative diagram:

$$
\begin{array}{ccc}
& V & \\
{\scriptstyle S(\varphi_1)} \downarrow & & \searrow {\scriptstyle \varphi} \\
\mathbf{S}(I) & \xrightarrow{\ \alpha\ } & \mathbf{R}(I).
\end{array}
$$

The resulting isomorphism is a consequence of relating the kernels of the commutative diagram in the short exact sequence $0 \to \ker S(\varphi_1) \to \ker \varphi \to \ker \alpha \to 0$. $\qquad\square$

Example 3.3.2. Let R be a commutative ring and let $I = (x)$ be a principal ideal. Let $\alpha_I : \mathbf{S}(I) \to \mathbf{R}(I)$ be the canonical morphism. Then $\ker(\alpha_{I,n}) \cong (0 : x^n)/(0 : x)$, for all $n \geq 2$. In particular, one has the injective morphisms $\ker(\alpha_{I,2}) \hookrightarrow \ker(\alpha_{I,3}) \hookrightarrow \ldots \hookrightarrow \ker(\alpha_{I,p})$. Consequently, if $\alpha_{I,p}$ is an isomorphism for some $p \geq 2$, then α_I is an isomorphism.

Proof: Let $V = R[T]$ be the polynomial ring in one variable and let $\varphi : V \to \mathbf{R}(I)$ be the induced surjective graded morphism sending T to xt. Observe that $\ker(\varphi_n) = (0 : x^n)T^n$ and $V_{n-1}\ker(\varphi_1) = RT^{n-1}(0 : x)T = (0 : x)T^n$. Now the assertion follows by Lemma 3.3.1. The last conclusion is a consequence of the rigidity of $\{(0 : x^n)\}_{n \geq 1}$. $\qquad\square$

3.4 Obstructions to injectivity: general settings

Let R be a commutative ring and let $I = (x_1, \ldots, x_s)$ be a finitely generated ideal. Let $\alpha_I : \mathbf{S}(I) \to \mathbf{R}(I)$ be the canonical morphism. For $n \geq 2$, the *module of effective n-relations* of I is defined as $E(I)_n = \ker(\alpha_n)/\mathbf{S}_1(I)\cdot\ker(\alpha_{n-1})$. It is not difficult to interpret the effective relations as the fresh relations of $\ker \alpha$, i.e., those relations which are not combination of the relations of lower degree. Recall that for the purpose of this work, as we did in Chapter 2,

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it will be useful to describe $E(I)_n$ in terms of a graded Koszul homology (see [Planas98]):

$$E(I)_n \cong H_1(x_1 t, \dots, x_s t; \mathbf{R}(I))_n.$$

Remark 3.4.1. Let $V = R[X_1, \dots, X_s]$ be a polynomial ring over R, let $\varphi : V \to \mathbf{R}(I)$ be the polynomial presentation sending X_i to $x_i t$ and let $Q = \ker \varphi$. By Lemma 3.3.1 and using $Q\langle 1 \rangle_n = V_{n-1} \ker(\varphi_1) \subset V_1 \ker(\varphi_{n-1}) = Q\langle n-1 \rangle_n$ for $n \geq 2$, it follows that $E(I)_n$ can be calculated as

$$E(I)_n \cong \frac{\ker(\varphi_n)}{V_1 \ker(\varphi_{n-1})} \cong \left[\frac{Q}{Q\langle n-1 \rangle} \right]_n.$$

The following straightforward remark will be useful throughout:

Remark 3.4.2. Let R be a commutative ring and let $I = (x_1, \dots, x_s)$ be a finitely generated ideal. Let $\alpha_I : \mathbf{S}(I) \to \mathbf{R}(I)$ be the canonical morphism and let $p \geq 2$ be an integer. Then, the following two conditions are equivalent:

(i) $\alpha_{I,n} : \mathbf{S}_n(I) \to I^n$ is an isomorphism for each $2 \leq n \leq p$;

(ii) $E(I)_n = 0$ for every $2 \leq n \leq p$.

In the particular cases of principal ideals and two-generated ideals, the next two lemmas refine [Planas98, Example 3.1] and [Planas98, Proposition 4.5], respectively. These two lemmas clearly follow from Proposition 2.3.3, although here we provide alternative proofs.

Lemma 3.4.3. *Let R be a commutative ring and let $I = (x)$ be a principal ideal. For $n \geq 2$, there is an isomorphism $E(I)_n \cong (0 :_R x^n)/(0 :_R x^{n-1})$.*

Proof: Let $V = R[T]$ and $\varphi : V \to \mathbf{R}(I)$ be the surjective graded morphism sending T to xt. Note that $\ker(\varphi_n) = (0 :_R x^n)T^n$ and $V_1 \ker(\varphi_{n-1}) = (0 :_R x^{n-1})T^n$. Since $E(I)_n \cong \ker(\varphi_n)/V_1 \ker(\varphi_{n-1})$, the assertion follows. $\qquad \square$

Lemma 3.4.4. *Let R be a commutative ring and let $I = (x, y)$ be a two-generated ideal. For every $n \geq 2$, there exists a short exact sequence of R-modules*

$$0 \to \frac{(0 :_R x) \cap I^{n-1}}{y((0 :_R x) \cap I^{n-2})} \to E(I)_n \to \frac{(xI^{n-1} :_R y^n)}{(xI^{n-2} :_R y^{n-1})} \to 0.$$

Proof: Let $V = R[X, Y]$ be the polynomial ring in two variables and let $\varphi : V \to \mathbf{R}(I)$ be the surjective graded morphism mapping X to xt and Y to yt, so that $E(I)_n \cong \ker(\varphi_n)/V_1 \ker(\varphi_{n-1})$. Let

$$\frac{(0 :_R x) \cap I^{n-1}}{y((0 :_R x) \cap I^{n-2})} \xrightarrow{\ f\ } E(I)_n$$

be defined in the following way: for $a \in (0 :_R x) \cap I^{n-1}$, writing $a = F(x, y)$ with $F(X, Y) \in V_{n-1}$, f maps the class of a to the class of $X F(X, Y)$ in $\ker(\varphi_n)/V_1 \ker(\varphi_{n-1})$. Let

$$E(I)_n \xrightarrow{\ g\ } \frac{(xI^{n-1} :_R y^n)}{(xI^{n-2} :_R y^{n-1})}$$

be defined as follows: for $G(X, Y) \in \ker(\varphi_n)$, g maps the class of G to the class of $G(0, 1)$. It is easily checked that f and g are well-defined morphisms and determine a short exact sequence of R-modules. $\qquad\square$

Discussion 3.4.5. Notice that the two previous lemmas could have been proved using Proposition 2.3.3. Provided $\mu(I) = 1$ or 2, the homological conditions can be effortless expressed in terms of colon ideals. However, in general, we are not able to produce a strightforward conversion into explicit ideal theoretic expressions. Recall that in general we have a short exact sequence as in Lemma 2.3.4:

$$0 \to \frac{H_1(x_1 t, \dots, x_s t ; \mathbf{R}(I))_n}{yt H_1(x_1 t, \dots, x_s t ; \mathbf{R}(I))_{n-1}} \longrightarrow E(I)_n \xrightarrow{\ \sigma_n\ } \frac{(JI^{n-1} :_R y^n)}{(JI^{n-2} :_R y^{n-1})} \to 0,$$

where $J = (x_1, \dots, x_s)$ and $I = (J, y)$.

The following result (see [Planas98, Corollary 4.8]), will be fundamental as regards our approach to the proof of Theorem B. In order to state it properly, let us introduce the following notation: we will denote by $I_{i_1,\dots,i_r}$ the ideal generated by the x_j with $j \notin \{i_1, \dots, i_r\}$.

Proposition 3.4.6. *Let* $I = (x_1, \dots, x_s)$ *be an ideal generated by* $s \geq 3$ *elements and let* $p \geq 2$. *Suppose that* $E(I_s)_p = 0$. *Then* $E(I)_p = 0$ *if and only if the following two conditions hold:*

$$(a) \quad \frac{(I_i I^{p-1} :_R x_i^p)}{(I_i I^{p-2} :_R x_i^{p-1})} = 0, \text{ for each } 1 \leq i \leq s;$$

$$(b) \quad \frac{((\sum_{1 \leq i < j \leq s-1} x_i x_j I_s^{p-2}) :_R x_s) \cap I^{p-1}}{\sum_{i=1}^{s-1} x_i((I_{i,s} I_s^{p-2} :_R x_s) \cap I^{p-2})} = 0.$$

When $s \geq 3$, the vanishing of $E(I)_p$ can be characterized by means of three ideal theoretic explicit conditions. However, under the assumption that $E(I_s)_p = 0$, one of the conditions is satisfied trivially and only (a) and (b) in Proposition 3.4.6 outlive (see [Planas98, Theorem 4.7 and Corollary 4.8]).

Now consider $I = (x_1, ..., x_s)$ and suppose that $J = I_s = (x_1, \ldots, x_{s-1})$ is such that $\alpha_{J,n} : \mathbf{S}_n(J) \to J^n$ is an isomorphism for $2 \leq n \leq p$, as in the hypothesis of Theorem B. By Remark 3.4.2, $E(J)_n = 0$ for every $2 \leq n \leq p$. Therefore, and again by Remark 3.4.2, to prove the implication $(i) \Rightarrow (ii)$ in Theorem B, it will be enough to fulfil obstructions (a) and (b) in Proposition 3.4.6 for every $2 \leq n \leq p$. Each of these conditions will be studied separately in the next sections.

3.5 Vanishing of the first obstruction

The purpose of this section is to study the vanishing of the condition (a) in Proposition 3.4.6. The following proposition makes use of an explicit construction.

Proposition 3.5.1. *Let R be a commutative ring, let $I = (x_1, ..., x_s)$ be a finitely generated ideal and let $p \geq 2$ be an integer. Let $\alpha_I : \mathbf{S}(I) \to \mathbf{R}(I)$ be the canonical morphism. Then, for each $1 \leq i \leq s$, there exists a surjective morphism*

$$g_{p,i} : \ker(\alpha_{I,p}) \twoheadrightarrow \frac{(I_i I^{p-1} :_R x_i^p)}{(I_i :_R x_i)}.$$

In particular, if $\alpha_{I,p}$ is an isomorphism, then

$$\frac{(I_i I^{n-1} :_R x_i^n)}{(I_i I^{n-2} :_R x_i^{n-1})} = 0$$

for each $1 \leq i \leq s$ and for each $2 \leq n \leq p$.

At this point let us introduce some useful notations. For $\sigma = (\sigma_1, ..., \sigma_s) \in \mathbb{N}^s$ set $x^\sigma = x_1^{\sigma_1}...x_s^{\sigma_s}$ and $X^\sigma = X_1^{\sigma_1}...X_s^{\sigma_s}$. Define the *support of $\sigma \in \mathbb{N}^s$* as the subset $\mathrm{supp}(\sigma) \subset \{1, \ldots, s\}$ such that $i \in \mathrm{supp}(\sigma)$ if and only if $\sigma_i \neq 0$. Let $|\sigma|$ denote $\sum_{i=1}^s \sigma_i$. We prove now Propostion 3.5.1.

Proof: Let $V = R[X_1, \ldots, X_s] = R[\mathbf{X}]$ be the polynomial ring in s indeterminates over R and let $\varphi : V \to \mathbf{R}(I)$ be the induced surjective graded morphism sending X_i to $x_i t$. Using

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Lemma 3.3.1, present $\ker(\alpha_{I,p})$ as the quotient $\ker(\varphi_p)/V_{p-1}\ker(\varphi_1)$. Let $g_{p,i} : \ker(\alpha_{I,p}) \to$ $(I_i I^{p-1} :_R x_i^p)/(I_i :_R x_i)$ be defined as follows: for $H(\mathbf{X}) \in \ker(\varphi_p)$, $g_{p,i}$ maps the class of $H(\mathbf{X})$ to the class of $H(e_i)$ in $(I_i I^{p-1} :_R x_i^p)/(I_i :_R x_i)$, where e_i stands for the i-th vector of the natural basis of R^s. To see that $g_{p,i}$ is well-defined, take $H(\mathbf{X}) \in V_{p-1}\ker(\varphi_1)$. Since any element of V_{p-1} can be written as a linear combination of the natural basis given by the monomials of degree $p-1$, one can write $H(\mathbf{X}) = \sum_{|\sigma|=p-1}(a_{\sigma,1}X_1 + \ldots + a_{\sigma,s}X_s)X^\sigma$, where $a_{\sigma,1}x_1 + \ldots + a_{\sigma,s}x_s = 0$, $a_{\sigma,j} \in R$. Then $H(e_i) = a_{(p-1)e_i,i}$, which belongs to $(I_i :_R x_i)$. It is readily seen that $g_{p,i}$ is surjective. Finally, since $(I_i :_R x_i) \subset (I_i I :_R x_i^2) \subset \ldots \subset (I_i I^{p-1} :_R x_i^p)$, if $\alpha_{I,p}$ is an isomorphism, then $(I_i I^{n-1} :_R x_i^n)/(I_i I^{n-2} :_R x_i^{n-1}) = 0$ for each $2 \le n \le p$. $\square$

3.6   Theorem B with stronger hypotheses

In this section we state and prove a version of Theorem B that works under more restrictive assumptions than the version that we will prove later. The special version uses key results from Chapter 2 and can be proved with little effort, whereas the more general version will be more demanding in terms of preparatory results.

Theorem 3.6.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, let $I = (x_1, \ldots, x_s, y)$ be an ideal of R. Suppose that $x_1, \ldots, x_s$ verify the following condition, for all $n \ge 2$:*

$$((x_1, \ldots, x_{i-1})I^{n-1} :_R x_i) \cap I^{n-1} = (x_1, \ldots, x_{i-1})I^{n-2}, \text{ for all } i = 1, \ldots, s. \qquad (\mathcal{T}_n)$$

Then the following conditions are equivalent:

(i) $\alpha_{I,p} : \mathbf{S}_p(I) \to I^p$ *is an isomorphism;*

(ii) $\alpha_{I,n} : \mathbf{S}_n(I) \to I^n$ *is an isomorphism, for each $2 \le n \le p$;*

Proof: Let $J = (x_1, \ldots, x_s)$. Recall from Lemma 2.4.1 that the conditions in $(\mathcal{T}_n)$ hold if and only if $H_1(x_1 t, \ldots, x_i t; \mathbf{R}(I))_n = 0$, for all $i = 1, \ldots, s$. Then using the exact sequence in Discussion 3.4.5 we get the isomorphisms:

$$E(I)_n \cong \frac{(JI^{n-1} :_R I^n)}{(JI^{n-2} :_R I^{n-1})}, \text{ for all } 2 \le n \le p.$$

Since $\ker \alpha_{I,p} = 0$, from Proposition 3.5.1 we know that $E(I)_n = 0$, for all $2 \leq n \leq p$. Consider $\ker \alpha_{I,n}$, with $2 \leq n \leq p$, then using the definition of $E(I)_k$,

$$E(I)_k = \ker \alpha_{I,k}/\mathbf{S}_1(I) \cdot \ker \alpha_{k-1},$$

we get

$$\ker \alpha_{I,n} = \mathbf{S}_1(I) \cdot \ker \alpha_{I,n-1} = \ldots = (\mathbf{S}_1(I))^{n-2} \cdot \ker \alpha_{I,2}.$$

But $\ker \alpha_{I,2} = E(I)_2 = 0$, thus $\ker \alpha_{I,n} = 0$. $\qquad\square$

Corollary 3.6.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, let $I = (x_1, \ldots, x_s, y)$ be an ideal of R with $J = (x_1, \ldots, x_s)$ a reduction of I, $x_1, \ldots, x_s$ an R-sequence and $x_1^*, \ldots, x_{s-1}^*$ a $\mathbf{G}(I)$-sequence. Then the following conditions are equivalent:*

  (i) $\alpha_{I,p} : \mathbf{S}_p(I) \to I^p$ is an isomorphism;

  (ii) $\alpha_{I,n} : \mathbf{S}_n(I) \to I^n$ is an isomorphism, for each $2 \leq n \leq p$;

Proof: The result follows directly from Lemma 2.4.3 and Theorem 3.6.1. $\qquad\square$

3.7 Vanishing of the second obstruction

Let $I = (x_1, \ldots, x_s)$ be a finitely generated ideal of R and let $\alpha_I : \mathbf{S}(I) \to \mathbf{R}(I)$ be the canonical morphism. Using Lemma 3.3.1 and its terminology, for each $p \geq 2$, present $\ker(\alpha_{I,p})$ as $\ker(\varphi_p)/V_{p-1}\ker(\varphi_1)$. If $1 \leq n \leq p - 1$, since $V_{p-1}\ker(\varphi_1) \subset V_{p-n}\ker(\varphi_n)$, one has a natural surjective morphims:

$$\ker(\alpha_{I,p}) \cong \frac{\ker(\varphi_p)}{V_{p-1}\ker(\varphi_1)} \overset{\pi_{p,n}}{\twoheadrightarrow} \frac{\ker(\varphi_p)}{V_{p-n}\ker(\varphi_n)}. \qquad (\Pi_{p,n})$$

Set $J = (x_1, \ldots, x_{s-1})$ and suppose that $\alpha_{J,n} : \mathbf{S}_n(J) \to J^n$ is an isomorphism for each $2 \leq n \leq p$. In this section we will construct morphisms $f_{p,n}$, with

$$\frac{((\sum_{1 \leq i < j \leq s-1} x_i x_j I_s^{n-1}) :_R x_s) \cap I^n}{\sum_{i=1}^{s-1} x_i((I_{i,s} I_s^{n-1} :_R x_s) \cap I^{n-1})} \overset{f_{p,n}}{\longrightarrow} \frac{\ker \varphi_p}{V_{p-n} \ker \varphi_n},$$

from the modules which appear in condition (b) of Proposition 3.4.6 to $\ker(\varphi_p)/V_{p-n}\ker(\varphi_n)$ (see Propositions 3.7.1 and 3.7.2). Thus we will have the diagram:

$$\ker \alpha_{I,p}$$
$$\downarrow {\scriptstyle \pi_{p,n}}$$
$$((\textstyle\sum_{1 \leq i < j \leq s-1} x_i x_j I_s^{n-1}) :_R x_s) \cap I^n / \sum_{i=1}^{s-1} x_i((I_{i,s} I_s^{n-1} :_R x_s) \cap I^{n-1}) \overset{f_{p,n}}{\longrightarrow} \ker \varphi_p/V_{p-n} \ker \varphi_n$$

We will show that, for $p \geq 2$ and $1 \leq n \leq p-1$, $f_{p,n}$ is injective provided that I is two-generated (see Proposition 3.7.1) or, in the general case, if $\alpha_{I,p}$ is an isomorphism (see Proposition 3.7.2 and the proof of Proposition 3.7.3). In particular, if $\alpha_{I,p}$ is an isomorphism, the modules of the bottom row in the diagram above are zero and the equations (b) in Proposition 3.4.6 hold.

Two-generated case

For the sake of a better comprehension, first we treat the particular case of I being a two-generated ideal. Remark that here, unlike in the general case, we do not need the hypothesis "$\alpha_{J,n}$ isomorphic" to ensure that the morphisms $f_{p,n}$ are well-defined.

Proposition 3.7.1. *Let R be a commutative ring, let $I = (x,y)$ be a two-generated ideal and let $p \geq 2$ be an integer. Let $V = R[X,Y]$ be the polynomial ring in two variables and let $\varphi : V \to \mathbf{R}(I)$ be the induced surjective graded morphism sending X to xt and Y to yt. Then, for each $1 \leq n \leq p-1$, there exists a well-defined morphism*

$$f_{p,n} : \frac{(0:y) \cap I^n}{x((0:y) \cap I^{n-1})} \longrightarrow \frac{\ker(\varphi_p)}{V_{p-n} \ker(\varphi_n)}.$$

Moreover, if $(0:x) = (0:x^2)$, then $f_{p,n}$ is injective.

Proof: Given $a \in (0:y) \cap I^n$, $a = A(x,y)$ with $A(X,Y) \in V_n$, let $f_{p,n}$ be the map which sends the class of a to the class of $Y^{p-n}A(X,Y)$ in $\ker(\varphi_p)/V_{p-n} \ker(\varphi_n)$.

Well-defined. The definition of $f_{p,n}$ is independent of the choice of $A(X,Y)$, since if $a = B(x,y)$ for another $B(X,Y) \in V_n$, then $Y^{p-n}(A(X,Y) - B(X,Y))$ is in $V_{p-n} \ker(\varphi_n)$. In addition, if $a \in x((0:y) \cap I^{n-1})$, then $a = xK(x,y)$ where $K(X,Y) \in V_{n-1}$ and $yK(x,y) = 0$. Thus the image of the class of a is the class of $Y^{p-n}XK(X,Y) = Y^{p-n-1}XYK(X,Y) \in V_{p-n} \ker(\varphi_n)$.

Injectivity of $f_{p,n}$. Let $a = \sum_{j=0}^{n} \lambda_j x^j y^{n-j}$ with $ay = 0$. Suppose that $f_{p,n}$ maps the class of a to the zero class. Then we can write

$$Y^{p-n}\left(\sum_{j=0}^{n} \lambda_j X^j Y^{n-j}\right) = \sum_{r=0}^{p-n} G_r(X,Y) X^r Y^{p-n-r} \in V_{p-n} \ker \varphi_n,$$

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where $G_r(X, Y) \in \ker(\varphi_n)$. Set $G_r(X, Y) = \sum_{t=0}^{n} u_{r,t} X^t Y^{n-t}$. Equating the coefficients

$$\lambda_j = \sum_{r+t=j} u_{r,t} \text{ for } 1 \le j \le n,$$

where $0 \le r \le p - n$ and $0 \le t \le n$. Therefore,

$$a = \sum_{j=0}^{n} \left(\sum_{r+t=j} u_{r,t} \right) x^j y^{n-j} = u_{0,0} y^n + \sum_{j=1}^{n} \left(\sum_{r+t=j} u_{r,t} \right) x^j y^{n-j}.$$

Since $G_0(x, y) = 0$, $u_{0,0} y^n = -\sum_{j=1}^{n} u_{0,j} x^j y^{n-j}$. Then

$$a = \sum_{j=1}^{n} \left(\left(\sum_{r+t=j} u_{r,t} \right) - u_{0,j} \right) x^j y^{n-j} =$$

$$= x \sum_{j=1}^{n} \left(\left(\sum_{r+t=j} u_{r,t} \right) - u_{0,j} \right) x^{j-1} y^{n-j} = xz,$$

with $z = \sum_{j=1}^{n} \left(\left(\sum_{r+t=j} u_{r,t} \right) - u_{0,j} \right) x^{j-1} y^{n-j} \in I^{n-1}$. It remains to show that $z \in (0 : y)$. Note that when $n = 1$, $z = u_{1,0}$. Since $G_1(x, y) = 0$ and $ay = 0$, then $u_{1,0} y = -u_{1,1} x$ and $0 = ay = xzy = u_{1,0} xy = -u_{1,1} x^2$. Thus $u_{1,1} \in (0 : x^2)$, which by hypothesis is equal to $(0 : x)$. Therefore $u_{1,0} y = -u_{1,1} x = 0$ and we conclude that $a = xz = xu_{1,0} \in x(0 : y)$, proving the injectivity of $f_{p,1}$. From now on let $n \ge 2$. Then,

$$zy = \sum_{j=1}^{n} \left(\left(\sum_{r+t=j} u_{r,t} \right) - u_{0,j} \right) x^{j-1} y^{n-j+1}$$

$$= u_{1,0} y^n + \sum_{j=2}^{n} \left(\left(\sum_{r+t=j} u_{r,t} \right) - u_{0,j} \right) x^{j-1} y^{n-j+1}.$$

Since $G_1(x, y) = 0$, $u_{1,0} y^n = -\sum_{j=1}^{n} u_{1,j} x^j y^{n-j}$. Then,

$$zy = \sum_{j=2}^{n} \left(\left(\sum_{r+t=j} u_{r,t} \right) - u_{0,j} \right) x^{j-1} y^{n-j+1} - \sum_{j=1}^{n} u_{1,j} x^j y^{n-j}$$

$$= x \left[\sum_{j=2}^{n} \left(\left(\sum_{r+t=j} u_{r,t} \right) - u_{0,j} \right) x^{j-2} y^{n-j+1} - \sum_{j=1}^{n} u_{1,j} x^{j-1} y^{n-j} \right].$$

Setting $w = \sum_{j=2}^{n} \left(\left(\sum_{r+t=j} u_{r,t} \right) - u_{0,j} \right) x^{j-2} y^{n-j+1} - \sum_{j=1}^{n} u_{1,j} x^{j-1} y^{n-j}$, we have $zy = xw$. Since $0 = ay = xzy = x^2 w$, $w \in (0 : x^2)$ which, by hypothesis, is equal to $(0 : x)$. Thus $zy = xw = 0$ and $a = xz \in x((0 : y) \cap I^{n-1})$, hence $f_{p,n}$ is injective. $\qquad \square$

General case

Now let $I = (x_1, \ldots, x_s)$, $s \geq 3$, and set $J = (x_1, \ldots, x_{s-1})$. For $p \geq 2$ and $1 \leq n \leq p-1$, we are going to construct the morphisms $f_{p,n}$ under the assumption that $\alpha_{J,n} : \mathbf{S}_n(J) \to J^n$ is an isomorphisms for $2 \leq n \leq p$. For convenience, we say that a form $A(X_1, \ldots, X_s) \in V_n$ *represents* an element $a \in I^n$ if $A(x_1, \ldots, x_s) = a$.

Proposition 3.7.2. *Let R be a commutative ring, let $I = (x_1, \ldots, x_s)$ with $s \geq 3$, and let $p \geq 2$ be an integer. Let $V = R[X_1, \ldots, X_s]$ be a polynomial ring and let $\varphi : V \to \mathbf{R}(I)$ be the induced surjective graded morphism sending X_i to $x_i t$. Suppose that $J = (x_1, \ldots, x_{s-1})$ verifies that $\alpha_{J,n} : \mathbf{S}_n(J) \to J^n$ is an isomorphism for each $2 \leq n \leq p$. Then, for each $1 \leq n \leq p-1$, there exists a well-defined morphism*

$$f_{p,n} : \frac{\left(\left(\sum_{1 \leq i < j \leq s-1} x_i x_j I_s^{n-1}\right) : x_s\right) \cap I^n}{\sum_{i=1}^{s-1} x_i\left(\left(I_{i,s}I_r^{n-1} : x_s\right) \cap I^{n-1}\right)} \longrightarrow \frac{\ker \varphi_p}{V_{p-n} \ker \varphi_n}.$$

Moreover $f_{p,1}$ is injective.

Proof: Pick an element

$$a \in \left(\left(\sum_{1 \leq i < j \leq s-1} x_i x_j I_s^{n-1}\right) : x_s\right) \cap I^n.$$

Thus, $x_s a = \sum_{1 \leq i < j \leq s-1} x_i x_j a_{ij}$ with $a_{ij} \in I_s^{n-1}$. Let $A(\mathbf{X}) \in V_n$ be a form representing a and let $A_{ij}(\mathbf{X}') \in V_{n-1}$ be a form representing a_{ij}, where $\mathbf{X}' = \{X_1, \ldots, X_{s-1}\}$. Let $f_{p,n}$ be the map sending the class of a to the class of

$$X_s^{p-n-1}\left(X_s A(\mathbf{X}) - \sum_{1 \leq i < j \leq s-1} X_i X_j A_{ij}(\mathbf{X}')\right)$$

in $\ker(\varphi_p)/V_{p-n} \ker \varphi_n$.

Well-defined: Set $W = R[X_1, \ldots, X_{s-1}]$ and define $\psi : W \to \mathbf{R}(J)$ sending X_i to $x_i t$. In particular, the following diagram is commutative:

$$
\begin{array}{ccc}
W & \xrightarrow{\psi} & \mathbf{R}(J) \\
\downarrow & & \downarrow \\
V & \xrightarrow{\varphi} & \mathbf{R}(I).
\end{array}
$$

In order to check that $f_{p,n}$ is well-defined, we first prove that the image of the class of a does not depend on the choices of $A(\mathbf{X})$ and $A_{ij}(\mathbf{X'})$. Let $A(\mathbf{X}), B(\mathbf{X}) \in V_n$ be two forms representing a. Suppose that for $a_{ij}, b_{ij} \in I_s^{n-1}$

$$x_s a = \sum_{1 \le i < j \le s-1} x_i x_j a_{ij} = \sum_{1 \le i < j \le s-1} x_i x_j b_{ij}.$$

Let $A_{ij}(\mathbf{X'})$, $B_{ij}(\mathbf{X'}) \in W_{n-1}$ be two forms representing a_{ij} and b_{ij}, respectively. Then the class of

$$X_s^{p-n-1}\left(X_s\Big(A(\mathbf{X}) - B(\mathbf{X})\Big) - \sum_{1 \le i < j \le s-1} X_i X_j\Big(A_{ij}(\mathbf{X'}) - B_{ij}(\mathbf{X'})\Big)\right)$$

in $\ker \varphi_p / V_{p-n} \ker \varphi_n$ is the zero class, since $A(\mathbf{X}) - B(\mathbf{X}) \in \ker(\varphi_n)$ and

$$\sum_{1 \le i < j \le s-1} X_i X_j\big(A_{ij}(\mathbf{X'}) - B_{ij}(\mathbf{X'})\big) \in \ker(\psi_{n+1}),$$

which by hypothesis is equal to $W_1 \ker(\psi_n) \subset V_1 \ker(\varphi_n)$.

Assume now that $a \in \sum_{i=1}^{s-1} x_i\big((I_{i,s}I_s^{n-1} : x_s) \cap I^{n-1}\big)$. Write $a = \sum_{i=1}^{s-1} x_i a_i$, where $a_i \in I^{n-1}$, $x_s a_i = \sum_{j=1, j \ne i}^{s-1} x_j c_{ij}$ and $c_{ij} \in I_s^{n-1}$. Hence if $A_i(\mathbf{X}) \in V_{n-1}$ is a form representing a_i, then $\sum_{i=1}^{s-1} X_i A_i(\mathbf{X})$ is a form representing a. Note that $x_s a = \sum_{i=1}^{s-1} x_i \sum_{j=1, j \ne i}^{s-1} x_j c_{ij}$ which can be written as $\sum_{1 \le i < j \le s-1} x_i x_j d_{ij}$, where $d_{ij} = c_{ij} + c_{ji}$ for $1 \le i < j \le s-1$. Thus $f_{p,n}$ maps the class of a to the class of

$$X_s^{p-n-1}\Big(X_s \sum_{i=1}^{s-1} X_i A_i(\mathbf{X}) - \sum_{1 \le i < j \le r-1} X_i X_j D_{ij}(\mathbf{X'})\Big) =$$

$$= X_s^{p-n-1}\Big(X_s \sum_{i=1}^{s-1} X_i A_i(\mathbf{X}) - \sum_{i=1}^{s-1} X_i \sum_{\substack{j=1 \\ j \ne i}}^{s-1} X_j C_{ij}(\mathbf{X'})\Big) =$$

$$= X_s^{p-n-1} \sum_{i=1}^{s-1} X_i\Big(X_s A_i(\mathbf{X}) - \sum_{\substack{j=1 \\ j \ne i}}^{s-1} X_j C_{ij}(\mathbf{X'})\Big),$$

where $C_{ij}(\mathbf{X'}) \in W_{n-1}$ is a form representing c_{ij} and $D_{ij}(\mathbf{X'}) = C_{ij}(\mathbf{X'}) + C_{ji}(\mathbf{X'})$. Since $X_s A_i(\mathbf{X}) - \sum_{j=1, j \ne i}^{s-1} X_j C_{ij}(\mathbf{X'}) \in \ker(\varphi_n)$, then $f_{p,n}$ maps the class of a to the zero class.

Injectivity of $f_{p,1}$. Let $a \in \Big(\big(\sum_{1 \le i < j \le s-1} x_i x_j R\big) : x_s\Big) \cap I$. Then $a = \sum_{i=1}^{s} \lambda_i x_i$, with $\lambda_i \in R$, and $x_s a = \sum_{1 \le i < j \le s-1} x_i x_j a_{ij}$, $a_{ij} \in R$. Suppose that $f_{p,1}$ maps the class of a to the zero class. Then we can write

$$X_s^{p-2}\Big(X_s \sum_{i=1}^{s} \lambda_i X_i - \sum_{1 \le i < j \le s-1} X_i X_j a_{ij}\Big) = \sum_{|\sigma|=p-1} (u_{\sigma,1}X_1 + \ldots + u_{\sigma,s}X_s)X^\sigma, \qquad (3.1)$$

with $u_{\sigma,1}x_1 + \ldots + u_{\sigma,s}x_s = 0$, $u_{\sigma,j} \in R$, where recall that if $\sigma = (\sigma_1, \ldots, \sigma_s) \in \mathbb{N}^s$, $x^\sigma = x_1^{\sigma_1} \ldots x_s^{\sigma_s}$ and $X^\sigma = X_1^{\sigma_1} \ldots X_s^{\sigma_s}$. For convenience, denote

$$u_{k,t} = \begin{cases} u_{e_k + (p-2)e_s, t} & \text{for } 1 \leq k \leq s-1, \\[2mm] u_{(p-1)e_s, t} & \text{for } k = s. \end{cases}$$

where $\{e_1, \ldots, e_s\}$ stands for the canonical basis of $\mathbb{N}^s$. With these notations and from equation (3.1), we get the following identities:

$$\begin{cases} \lambda_i = u_{s,i} + u_{i,s} & \text{for } 1 \leq i \leq s-1, \\[2mm] \lambda_s = u_{s,s}. \end{cases}$$

Using the fact that $u_{s,1}x_1 + \ldots + u_{s,s}x_s = 0$, we get

$$\begin{aligned} a &= \sum_{i=1}^{s} \lambda_i x_i = \sum_{i=1}^{s-1} (u_{s,i} + u_{i,s})x_i + u_{s,s}x_s = \\ &= \sum_{i=1}^{s-1} (u_{s,i} + u_{i,s})x_i - \sum_{i=1}^{s-1} u_{s,i}x_i = \sum_{i=1}^{s-1} u_{i,s}x_i. \end{aligned}$$

Since $u_{i,1}x_1 + \ldots + u_{i,s}x_s = 0$ for $1 \leq i \leq s-1$, then $u_{i,s}x_s = -\sum_{j=1}^{s-1} u_{i,j}x_j$ and

$$ax_s = \sum_{i=1}^{s-1} u_{i,s}x_s x_i = -\sum_{i,j=1}^{s-1} u_{i,j}x_i x_j.$$

But $ax_s = \sum_{1 \leq i < j \leq s-1} x_i x_j a_{ij}$. Hence

$$R(\mathbf{X}) = \sum_{i,j=1}^{s-1} u_{i,j}X_i X_j + \sum_{1 \leq i < j \leq s-1} X_i X_j a_{ij} \tag{3.2}$$

belongs to $\ker(\psi_2)$. Since $\alpha_{J,2}$ is an isomorphism, $R(\mathbf{X})$ can be written as

$$R(\mathbf{X}) = \sum_{i=1}^{s-1} (w_{i,1}X_1 + \ldots + w_{i,s-1}X_{s-1})X_i \tag{3.3}$$

with $w_{i,1}x_1 + \ldots + w_{i,s-1}x_{s-1} = 0$, $w_{i,j} \in R$. In particular $w_{i,i} \in (I_{i,s} : x_i)$. Equating the two expressions of $R(\mathbf{X})$ in (3.2) and (3.3) we get the identities $u_{i,i} = w_{i,i} \in (I_{i,s} : x_i)$ for each $1 \leq i \leq s-1$. Then, for $1 \leq i \leq s-1$, $u_{i,s}x_s = -\sum_{j=1}^{s-1} u_{i,j}x_j = -\sum_{j=1, j\neq i}^{s-1} u_{i,j}x_j - u_{i,i}x_i \in I_{i,s}$ and $u_{i,s} \in (I_{i,s} : x_s)$. Therefore $a = \sum_{i=1}^{s-1} u_{i,s}x_i \in \sum_{i=1}^{s-1} x_i(I_{i,s} : x_s)$, proving the injectivity of $f_{p,1}$. $\qquad\square$

The next proposition proves the vanishing of equations (b) in Proposition 3.4.6 under our hypotheses.

Proposition 3.7.3. *Let R be a commutative ring, let $I = (x_1, \ldots, x_s)$ with $s \geq 3$ and let $p \geq 2$ be an integer. Suppose that $J = (x_1, \ldots, x_{s-1})$ verifies that $\alpha_{J,n} : \mathbf{S}_n(J) \to J^n$ is an isomorphism for each $2 \leq n \leq p$. If $\alpha_{I,p}$ is an isomorphism, then*

$$\frac{((\sum_{1 \leq i < j \leq s-1} x_i x_j I_s^{n-1}) : x_s) \cap I^n}{\sum_{i=1}^{s-1} x_i((I_{i,s} I_s^{n-1} : x_s) \cap I^{n-1})} = 0 \quad \text{for } 1 \leq n \leq p-1.$$

Proof: Let $V = R[X_1, \ldots, X_s]$ be the polynomial ring in s variables and let $\varphi : V \to \mathbf{R}(I)$ be the induced surjective graded morphism sending X_i to $x_i t$. If $\alpha_{I,p}$ is an isomorphism, using the natural surjective morphism in $(\Pi_{p,n})$, one deduces that

$$\frac{\ker(\varphi_p)}{V_{p-n} \ker(\varphi_n)} = 0 \quad \text{for each } 1 \leq n \leq p-1.$$

For $1 \leq n \leq p-1$, consider the morphisms $f_{p,n}$ defined in Proposition 3.7.2:

$$f_{p,n} : \frac{((\sum_{1 \leq i < j \leq s-1} x_i x_j I_s^{n-1}) : x_s) \cap I^n}{\sum_{i=1}^{s-1} x_i((I_{i,s} I_s^{n-1} : x_s) \cap I^{n-1})} \to \frac{\ker(\varphi_p)}{V_{p-n} \ker \varphi_n} = 0.$$

It suffices to show that $f_{p,n}$ is injective for $1 \leq n \leq p-1$. This will be done by induction on n, $1 \leq n \leq p-1$. By Proposition 3.7.2, $f_{p,1}$ is injective. Suppose now that $p \geq 3$, fix n with $2 \leq n \leq p-1$ and assume that $f_{p,q}$ are injective for $1 \leq q \leq n-1$. In particular,

$$\frac{((\sum_{1 \leq i < j \leq s-1} x_i x_j I_s^{q-1}) : x_s) \cap I^q}{\sum_{i=1}^{s-1} x_i((I_{i,s} I_s^{q-1} : x_s) \cap I^{q-1})} = 0 \quad \text{for } 1 \leq q \leq n-1.$$

Since $\alpha_{J,n} : \mathbf{S}_n(J) \to J^n$ is an isomorphism for each $2 \leq n \leq p$, by Remark 3.4.2, $E(J)_t = 0$ for $2 \leq t \leq p$. Since by hypothesis $\ker(\alpha_{I,p}) = 0$, Proposition 3.5.1 implies that

$$\frac{(I_i I^{t-1} : x_i^t)}{(I_i I^{t-2} : x_i^{t-1})} = 0 \quad \text{for } 1 \leq i \leq s \text{ and for } 2 \leq t \leq p.$$

Using Proposition 3.4.6, we get $E(I)_2 = 0, \ldots, E(I)_n = 0$. In particular, by Remark 3.4.2, $\alpha_{I,n}$ is an isomorphism and $\ker(\varphi_n) = V_{n-1} \ker(\varphi_1)$.

Let us prove that $f_{p,n}$ is injective. Take

$$a = \sum_{|\sigma|=n} \lambda_\sigma x^\sigma \in \left(\left(\sum_{1 \leq i < j \leq s-1} x_i x_j I_s^{n-1}\right) : x_s\right) \cap I^n.$$

Suppose that $f_{p,n}$ maps the class of a to the zero class. We will show that a can be written as $a = \sum_{i=1}^{s-1} x_i a_i$ where $a_i \in (I_{i,s} I_s^{n-1} : x_s) \cap I^{n-1}$ for $1 \leq i \leq s-1$.

By hypothesis, $x_s a$ can be written as

$$x_s a = \sum_{1 \le i < j \le s-1} x_i x_j a_{ij}, \quad \text{with } a_{ij} \in I_s^{n-1}. \tag{3.4}$$

Let $A_{ij}(\mathbf{X}')$ be a form of degree $n-1$ in the variables $\mathbf{X}' = \{X_1, \ldots, X_{s-1}\}$ representing a_{ij}. As in the proof of Proposition 3.7.2, set $W = R[X_1, \ldots, X_{s-1}]$ and $\psi : W \to \mathbf{R}(J)$ sending X_i to $x_i t$. Since $f_{p,s}$ maps the class of a to the zero class in $\ker(\varphi_p)/V_{p-n}\ker(\varphi_n)$ and $\ker(\varphi_n) = V_{n-1}\ker(\varphi_1)$, we can write

$$X_s^{p-n-1}\Big(X_s \sum_{|\sigma|=n} \lambda_\sigma X^\sigma - \sum_{1 \le i < j \le s-1} X_i X_j A_{ij}(\mathbf{X}') \Big) =$$

$$\sum_{|\omega|=p-1} (u_{\omega,1} X_1 + \ldots + u_{\omega,s} X_s) X^\omega \tag{3.5}$$

where $u_{\omega,1} x_1 + \ldots + u_{\omega,s} x_s = 0$, $u_{\omega,i} \in R$ and $\omega = (\omega_1, \ldots, \omega_s) \in \mathbb{N}^s$. The following convention will be useful: if $\omega \in \mathbb{N}^s$ with $|\omega| = p-1$ is of the form $\omega = \sigma + (p-n-1)e_s$, for a certain $\sigma \in \mathbb{N}^s$ with $|\sigma| = n$, we will denote $u_{\sigma+(p-n-1)e_s, i}$ by $v_{\sigma,i}$, for $1 \le i \le s$. In particular,

$$v_{\sigma,1} x_1 + \ldots + v_{\sigma,s} x_s = 0 \quad \text{for} \quad \sigma \in \mathbb{N}^s, \ |\sigma| = n. \tag{3.6}$$

Comparing the two polynomial expressions in equation (3.5), we observe that the monomial term $\lambda_\sigma X^{\sigma+(p-n)e_s}$ of the left hand part is equal to the sum of the following monomial terms of the right hand part: $u_{\omega,i} X^{\omega+e_i}$, for $1 \le i \le s$, where $\sigma + (p-n)e_s = \omega + e_i$. If $1 \le i \le s-1$ and $i \notin \mathrm{supp}(\sigma)$, there is no ω such that $\sigma + (p-n)e_s = \omega + e_i$. If $1 \le i \le s-1$ but $i \in \mathrm{supp}(\sigma)$ then $\omega = \sigma - e_i + e_s + (p-n-1)e_s$. If $i = s$, $\omega = \sigma + (p-n-1)e_s$. Therefore,

$$\lambda_\sigma = \sum_{\substack{1 \le i \le s-1 \\ i \in \mathrm{supp}(\sigma)}} v_{\sigma-e_i+e_s, \, i} + v_{\sigma, \, s}.$$

Then,

$$a = \sum_{|\sigma|=n} \lambda_\sigma x^\sigma = \sum_{|\sigma|=n} \Big(\sum_{\substack{1 \le i \le s-1 \\ i \in \mathrm{supp}(\sigma)}} v_{\sigma-e_i+e_s, \, i} \Big) x^\sigma + \sum_{|\sigma|=n} v_{\sigma, \, s} x^\sigma =$$

$$= \sum_{i=1}^{s-1} \sum_{\substack{|\sigma|=n \\ \sigma_i \ne 0}} v_{\sigma-e_i+e_s, \, i} \, x^\sigma + \sum_{|\sigma|=n} v_{\sigma, \, s} \, x^\sigma.$$

For a fixed $1 \le i \le s-1$, if $\sigma \in \mathbb{N}^s$ is such that $|\sigma| = n$ and $\sigma_i \ne 0$, then θ defined as $\theta = \sigma - e_i + e_s$ has $|\theta| = n$, $\theta_s \ne 0$ and $\sigma = \theta + e_i - e_s$. Conversely, if $\theta \in \mathbb{N}^s$ is such that

$|\theta| = n$ and $\theta_s \neq 0$, then σ defined as $\sigma = \theta + e_i - e_s$ has $|\sigma| = n$, $\sigma_i \neq 0$ and $\theta = \sigma - e_i + e_s$. Therefore,

$$a = \sum_{i=1}^{s-1} \sum_{\substack{|\theta|=n \\ \theta_s \neq 0}} v_{\theta,i} x^{\theta+e_i-e_s} + \sum_{|\theta|=n} v_{\theta,s} x^{\theta} =$$

$$= \sum_{\substack{|\theta|=n \\ \theta_s \neq 0}} \sum_{i=1}^{s-1} v_{\theta,i} x^{\theta+e_i-e_s} + \sum_{\substack{|\theta|=n \\ \theta_s \neq 0}} v_{\theta,s} x^{\theta} + \sum_{\substack{|\theta|=n \\ \theta_s = 0}} v_{\theta,s} x^{\theta} =$$

$$= \sum_{\substack{|\theta|=n \\ \theta_s \neq 0}} \left(\sum_{i=1}^{s-1} v_{\theta,i} x^{\theta+e_i-e_s} + v_{\theta,s} x^{\theta} \right) + \sum_{\substack{|\theta|=n \\ \theta_s = 0}} v_{\theta,s} x^{\theta}.$$

Using equation (3.6), for a $\theta \in \mathbb{N}^s$, $|\theta| = n$, then $v_{\theta,1} x_1 + \ldots + v_{\theta,s} x_s = 0$. If $\theta_s \neq 0$, then $v_{\theta,s} x^{\theta} = v_{\theta,s} x_s x^{\theta-e_s} = -\sum_{i=1}^{s-1} v_{\theta,i} x_i x^{\theta-e_s} = -\sum_{i=1}^{s-1} v_{\theta,i} x^{\theta+e_i-e_s}$. Therefore,

$$a = \sum_{\substack{|\theta|=n \\ \theta_s = 0}} v_{\theta,s} x^{\theta}. \tag{3.7}$$

From now on let $\mathcal{U} = \{\theta \in \mathbb{N}^s \mid |\theta| = n \text{ and } \theta_s = 0\}$. Then, by equation (3.6) again,

$$x_s a = \sum_{\theta \in \mathcal{U}} v_{\theta,s} x_s x^{\theta} = \sum_{\theta \in \mathcal{U}} (-v_{\theta,1} x_1 - \ldots - v_{\theta,s-1} x_{s-1}) x^{\theta}.$$

Using equation (3.4) and the subsequent definitions, it is clear that the form

$$G(\mathbf{X}') = \sum_{\theta \in \mathcal{U}} (v_{\theta,1} X_1 + \ldots + v_{\theta,s-1} X_{s-1}) X^{\theta} + \sum_{1 \leq i < j \leq s-1} X_i X_j A_{ij}(\mathbf{X}'), \tag{3.8}$$

belongs to $\ker(\psi_{n+1})$. Since $\alpha_{J,2}, \ldots, \alpha_{J,p}$ are isomorphisms, then $\ker(\psi_{n+1}) = V_n \ker(\psi_1)$ and $G(\mathbf{X}')$ can be written as

$$G(\mathbf{X}') = \sum_{\theta \in \mathcal{U}} (w_{\theta,1} X_1 + \ldots + w_{\theta,s-1} X_{s-1}) X^{\theta} \tag{3.9}$$

with $w_{\theta,1} x_1 + \ldots + w_{\theta,s-1} x_{s-1} = 0$, $w_{\theta,j} \in R$. Comparing the two polynomial expressions in (3.8) and (3.9), we observe that the term $v_{ne_i,i} X^{(n+1)e_i}$ is equal to $w_{ne_i,i} X^{(n+1)e_i}$, for $1 \leq i \leq s-1$. In particular, $v_{ne_i,i} = w_{ne_i,i} \in (I_{i,s} : x_i)$ for each $1 \leq i \leq s-1$.

Let $A(\mathbf{X}') = \sum_{\theta \in \mathcal{U}} v_{\theta,s} X^{\theta} \in W_n$, which by equation (3.7) represents a.

Set $\mathcal{U}_1 = \{\theta \in \mathcal{U} \mid \theta_1 \neq 0\}$ and $\mathcal{U}_i = \{\theta \in \mathcal{U} \mid \theta_j = 0, 1 \leq j \leq i-1 \text{ and } \theta_i \neq 0\}$ for each $2 \leq i \leq s-1$. Clearly, the $\mathcal{U}_i$ define a partition of $\mathcal{U}$.

Let $A_i(\mathbf{X}') = \sum_{\theta \in \mathcal{U}_i} v_{\theta,s} X^{\theta - e_i} \in W_{n-1}$. Then we have $A(\mathbf{X}') = X_1 A_1(\mathbf{X}') + \ldots + X_{s-1} A_{s-1}(\mathbf{X}')$. Evaluating at $x_1, \ldots, x_{s-1}$, one obtains the equality $a = A(x_1, \ldots, x_{s-1}) = \sum_{i=1}^{s-1} x_i a_i$, where $a_i = A_i(x_i, \ldots, x_{s-1}) = \sum_{\theta \in \mathcal{U}_i} v_{\theta,s} x^{\theta - e_i} \in I^{n-1}$. We claim that $a_i \in (I_{i,s} I_s^{n-1} : x_s) \cap I^{n-1}$ for each $1 \le i \le s - 1$. By equation (3.6),

$$x_s a_i = \sum_{\theta \in \mathcal{U}_i} v_{\theta,s} x_s x^{\theta - e_i} = \sum_{\theta \in \mathcal{U}_i} (-v_{\theta,1} x_1 - \ldots - v_{\theta,s-1} x_{s-1}) x^{\theta - e_i}.$$

Fix $\theta \in \mathcal{U}_i$. If $\mathrm{supp}(\theta)$ contains $j \ne i$, then $v_{\theta,i} x_i x^{\theta - e_i} = v_{\theta,i} x_i x_j x^{\theta - e_j - e_i} = v_{\theta,i} x_j x^{\theta - e_j} \in I_{i,s} I_s^{n-1}$; if $\mathrm{supp}(\theta) = \{i\}$, then $\theta = n e_i$ and $v_{\theta,i} x_i x^{\theta - e_i} = v_{ne_i,i} x_i^n \in (I_{i,s} : x_i) x_i^n \subset I_{i,s} I_s^{n-1}$. So $a = \sum_{i=1}^{s-1} x_i a_i$, where $a_i \in (I_{i,s} I_s^{n-1} : x_s) \cap I^{n-1}$, proving the injectivity of $f_{p,n}$ as desired. $\square$

3.8 Theorem B

We have now all the ingredients to prove the main result of the chapter.

Theorem B. *Let R be a commutative ring, let $I = (x_1, \ldots, x_s)$ be an ideal of R and let $p \ge 2$ be an integer. Suppose that $J = (x_1, \ldots, x_{s-1})$ verifies that $\alpha_{J,n} : \mathbf{S}_n(J) \to J^n$ is an isomorphism for each $2 \le n \le p$. Then the following conditions are equivalent:*

(i) *$\alpha_{I,p} : \mathbf{S}_p(I) \to I^p$ is an isomorphism;*

(ii) *$\alpha_{I,n} : \mathbf{S}_n(I) \to I^n$ is an isomorphism for each $2 \le n \le p$.*

Proof: Suppose (i) and assume $s \ge 3$ (add repeated generators if necessary). By Proposition 3.5.1, $(I_i I^{n-1} : x_i^n)/(I_i I^{n-2} : x_i^{n-1}) = 0$ for each $1 \le i \le s$ and $2 \le n \le p$. By Proposition 3.7.3,

$$\frac{\left(\left(\sum_{1 \le i < j \le s-1} x_i x_j I_s^{n-1} \right) : x_s \right) \cap I^n}{\sum_{i=1}^{s-1} x_i \left((I_{i,s} I_s^{n-1} : x_s) \cap I^{n-1} \right)} = 0 \text{ for each } 1 \le n \le p - 1.$$

By Remark 3.4.2, $E(J)_n = 0$ for every $2 \le n \le p$. Using Proposition 3.4.6, we have $E(I)_2 = 0, \ldots, E(I)_p = 0$ and (ii) follows by Remark 3.4.2. $\square$

Corollary 3.8.1. *Let R be a commutative ring, let $J \subset R$ be a finitely generated ideal of linear type and let $I = (J, y)$, i.e., I is an ideal of almost-linear type. Then the following conditions are equivalent:*

(i) $\alpha_{I,p}: \mathbf{S}_p(I) \to I^p$ *is an isomorphism;*

(ii) $\alpha_{I,n}: \mathbf{S}_n(I) \to I^n$ *is an isomorphism for each* $2 \le n \le p$.

In particular, this equivalence holds if either

1) *R is reduced and I is two-generated, or*

2) *R is an integrally closed domain and I is three-generated.*

Proof: If $I = (x, y)$ is a two generated ideal in a reduced ring, $J = (x)$ is of linear type. If $I = (x, y, z)$ is a three-generated ideal in an integrally closed domain, the ideal $J = (x, y)$ is of linear type by [Costa83, Theorem 3]. $\square$

Corollary 3.8.2. *Let $\{x_1, \ldots, x_s\}$ be a sequence of elements of a commutative ring R and let $J_i = (x_1, \ldots, x_i)$. Let $p \ge 2$ be an integer. Suppose that $\alpha_p: \mathbf{S}_p(J_i) \to J_i^p$ is an isomorphism for each $1 \le i \le s$. Then $\alpha_n: \mathbf{S}_n(J_i) \to J_i^n$ is an isomorphism for each $1 \le i \le s$ and for each $2 \le n \le p$.*

Proof: Proceed by induction on $s \ge 1$. Since $\alpha_{J_1,p}$ is an isomorphism, Example 3.3.2 implies that $\alpha_{J_1,n}$ is an isomorphism for each $n \ge 2$. Let $s \ge 2$ and suppose that $\alpha_{J_t,n}$ is an isomorphism for each $2 \le n \le p$ and each $2 \le t \le s-1$. Since by hypothesis $\alpha_{J_s,n}$ is an isomorphism, Theorem B implies that $\alpha_{J_s,n}$ is an isomorphism for each $2 \le n \le p$. $\square$

Remark 3.8.3. To summarize, let us gather the results of [Tchernev07] with those presented above. These are the situations in which the injectivity of the graded components of the canonical morphism α propagates downwards: *(i)* I such that $\mathbf{S}(I)$ contains a regular element in degree one; *(ii)* $I = \mathfrak{m}$ maximal ideal (see [Tchernev07, Theorem 5.5]); *(iii)* I such that $\mathrm{pd}(I) \le 1$ (see [Tchernev07, Theorem 5.1]); *(iv)* I perfect Gorenstein of grade 3 (see [Tchernev07, Theorem 5.3]); *(v)* $I = (J, y)$ with J is of linear type, i.e., I almost-linear type (see Corollary 3.8.1), in particular, either when $I = (x, y)$ and R reduced or when $I = (x, y, z)$ and R integrally closed domain.

Remark 3.8.4. Kühl proved that in a Noetherian ring, linear type and projectively of linear type are equivalent conditions for an ideal I provided that I is generated by a proper sequence (see [Kühl82, Corollary 2.4]). We have achieved new cases in which this equivalence holds: in fact, by Corollary 3.8.1, if I is an ideal of the form $I = (J, y)$ with J of linear type,

then I is of linear type if and only if I is projectively of linear type. We can state this claim together with other cases considered so far in the following corollary.

Corollary 3.8.5. *Let R be a Noetherian ring and I an ideal of R. Consider any of the situations considered in Remark 3.8.3:*

(a) $\mathbf{S}_1(I)$ *contains a non-zero-divisor of $\mathbf{S}(I)$;*

(b) $I = (J, y)$ *with J an ideal of linear type;*

(c) $\mathrm{pd}(I) \leq 1$;

(d) I *is perfect Gorenstein of grade 3;*

(e) $I = \mathfrak{m}$ *is a maximal ideal.*

In either of the situations $(a) - (e)$, I is of linear type if and only if I is of p-linear type, i.e., the canonical morphism $\alpha_I : \mathbf{S}(I) \to \mathbf{R}(I)$ is an isomorphism if and only if the induced morphism $\widetilde{\alpha}_I : \mathrm{Proj}(\mathbf{R}(I)) \to \mathrm{Proj}(\mathbf{S}(I))$ is an isomorphism of schemes.

Proof: The claim is a straightforward consequence of Remark 1.3.33 and Remark 3.8.3 above. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

3.9   Examples and applications

The following example shows that if the hypothesis of $\alpha_{J,n}$ being an isomorphism for each $2 \leq n \leq p$ is not fulfilled, then the conclusion in Theorem B may fail. Concretely, for each $p \geq 2$, we construct a two-generated ideal $I = (x, y)$, with neither (x) nor (y) of linear type, such that $\alpha_{I,n}$ is an isomorphism for each $n \geq p + 1$, whereas $\alpha_{I,p}$ is not (see also [Kühl82, Example 1.4]). Experimentation with Singular [Singular] was useful in the process of generating this as well as other examples.

Example 3.9.1. *Let k be a field and let $p \geq 2$. Let $S = k[U_0, \ldots, U_p, X, Y]$ be a polynomial ring and let Q be the ideal of S defined as $Q = Q_1 + (U_0 X^p)$, where*

$$Q_1 = (U_0 Y, U_0 X - U_1 Y, U_1 X - U_2 Y, \ldots, U_{p-1} X - U_p Y, U_p X).$$

Let R be the factor ring $S/Q = k[u_0, \ldots, u_p, x, y]$ and consider the ideal $I = (x, y) \subset R$. Then $\alpha_{I,n}$ is an isomorphism for all $n \geq p + 1$, whereas $\alpha_{I,p}$ is not. Moreover, $\beta_{I,n}$ is an isomorphism for all $n \neq p$, whereas $\beta_{I,p}$ is not.

Proof: Let $A = k[U_0, \ldots, U_p]$ and $S = A[X, Y]$. We can endow S with a grading by letting $\deg(X) = 1$, $\deg(Y) = 1$ and $\deg(w) = 0$ for each $w \in A$. Then $R = S/Q = A[x, y]$ has a standard graded A-algebra structure, with irrelevant ideal given by $I = R_+ = (x, y)$. Note that $\mathbf{G}(I) = \oplus_{p \geq 0} I^p / I^{p+1} \cong R$ and $\mathbf{S}(R/I) I/I^2 \cong \mathbf{S}_A R_1$. Consider the free presentation of R_1

$$0 \to \widetilde{Q}_1 \to Ae_x \oplus Ae_y \xrightarrow{f} R_1 \to 0,$$

where f sends e_x to x, e_y to y and $\widetilde{Q}_1$ is the submodule of $Ae_x \oplus Ae_y$ generated by $U_0 e_y, U_0 e_x - U_1 e_y, \ldots, U_{p-1} e_x - U_p e_y, U_p e_x$. Applying the symmetric functor, it gives rise to the top row in the following commutative diagram of exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Q_1 & \longrightarrow & A[X, Y] & \longrightarrow & \mathbf{S}_A(R_1) & \longrightarrow & 0 \\
& & \downarrow & & \| & & \downarrow {\scriptstyle \beta_I} & & \\
0 & \longrightarrow & Q & \longrightarrow & A[X, Y] & \longrightarrow & R & \longrightarrow & 0.
\end{array}
$$

Therefore, $\ker \beta_I \cong Q/Q_1$, which is generated by the class of $U_0 X^p$ in $A[X, Y]/Q_1$.

Let us see that $U_0 X^p \notin Q_1$. Indeed, if $U_0 X^p \in Q_1$ then $U_0 X^p = F_0 U_0 Y + F_1(U_0 X - U_1 Y) + \ldots + F_p(U_{p-1}X - U_p Y) + F_{p+1} U_p X$, for some $F_i \in S = k[U_0, \ldots, U_p, X, Y]$. We can assume that the F_i are homogeneous polynomials in X, Y of degree $p - 1$. Since $(F_0 Y - X^p + F_1 X)U_0 + (F_2 X - F_1 Y)U_1 + \ldots + (F_{p+1}X - F_p Y)U_p = 0$, we get $F_{i+1}X = F_i Y$ for each $1 \leq i \leq p$. In particular $F_1 Y^p = F_2 X Y^{p-1} = \ldots = F_{p+1} X^p$ and hence $F_1 \in X^p S$. But since $\deg(F_1) = p - 1$, F_1 must be zero, and hence $F_2 = \ldots = F_{p+1} = 0$. Thus $F_0 Y = X^p$, a contradiction.

Then $\ker(\beta_{I,p}) \neq 0$, hence $\alpha_{I,p}$ is not an isomorphism. Clearly $\ker(\beta_{I,n}) = 0$ for each $2 \leq n \leq p - 1$. On the other hand, from the polynomial equalities given by

$$
\begin{aligned}
U_0 X^{p+1} &= X^p(U_0 X - U_1 Y) + X^{p-1}Y(U_1 X - U_2 Y) + \ldots + \\
& \quad\; XY^{p-1}(U_{p-1}X - U_p Y) + Y^p(U_p X) \in Q_1, \\
U_0 X^p Y &= X^p(U_0 Y) \in Q_1,
\end{aligned}
$$

we can conclude that $\ker(\beta_{I,n}) = 0$ also for $n \geq p + 1$. Hence $\ker(\beta_{I,n}) = 0$ for each $n \neq p$.

Consider the downgrading homomorphism $\lambda : \mathbf{S}_{n+1}(I) \to \mathbf{S}_n(I)$, which maps the homogeneous element $a_1 \cdot a_2 \cdot \ldots \cdot a_{n+1}$ to $a_1(a_2 \cdot \ldots \cdot a_{n+1})$. It is well-known that $\ker(\beta_{I,n}) \cong$

$\ker(\alpha_{I,n})/\lambda(\ker(\alpha_{I,n+1}))$ (see [HSV81, Section 3]). In our case, it implies that $\ker(\alpha_{I,n}) = \lambda^t(\ker(\alpha_{I,n+t}))$ for all $n \geq p+1$ and $t \geq 1$.

Let us see now that $\ker(\alpha_{I,n}) = 0$ for all $n \geq p+1$. Since R is noetherian, $\ker(\alpha_I)$ is a finitely generated ideal of $\mathbf{S}(I)$. Therefore, there exists an integer $r \geq 1$ such that $\ker(\alpha_{I,r+t}) = \mathbf{S}_1(I)^t \cdot \ker(\alpha_{I,r})$ for all $t \geq 1$ (recall that the smallest of such integers $r \geq 1$ is $\mathrm{rt}(I)$, the relation type of I; see e.g. [Planas98] or [Vasconcelos05]). If $n \geq \max\{p+1, r\}$, then

$$\ker(\alpha_{I,n}) = \lambda^n(\ker(\alpha_{I,2n})) = \lambda^n(\mathbf{S}_1(I)^n \cdot \ker(\alpha_{I,n})) = I^n\ker(\alpha_{I,n}).$$

But $I^n\ker(\alpha_{I,n}) = 0$ since if $w \in \ker(\alpha_{I,n})$, then $I^n w \subset \alpha(w)\mathbf{S}_n(I) = 0$. On the other hand, if $p+1 \leq n < r$, observe that $\ker(\alpha_{I,n}) = \lambda^r(\ker(\alpha_{I,n+r}))$ and $n+r \geq \max\{p+1, r\}$. Hence $\ker(\alpha_{I,n}) = 0$ by the previous case. $\qquad\square$

Remark 3.9.2. Alternatively, one can use the Artin-Rees Lemma to finish the proof of the former example. Indeed, since $\ker(\beta_{I,n}) = 0$ for each $n \geq p+1$, we have $\ker(\alpha_{I,n}) = \lambda^s(\ker(\alpha_{I,n+s})) \subset I^s\mathbf{S}_n(I) \cap \ker(\alpha_{I,n})$ for each $s \geq 1$ and each $n \geq p+1$. Let s_n be the Artin-Rees number relative to the pair $\ker(\alpha_n) \subset \mathbf{S}_n(I)$ and the ideal I. Then the Artin-Rees Lemma asserts that $I^s\mathbf{S}_n(I) \cap \ker(\alpha_{I,n}) = I^{s-s_n}(I^{s_n}\mathbf{S}_n(I) \cap \ker(\alpha_{I,n})) \subset I^{s-s_n}\ker(\alpha_{I,n})$ for each $s \geq s_n$. Taking s large enough, we see that $\ker(\alpha_{I,n}) \subset I^n\ker\alpha_{I,n}$, which is zero by the argument above (see also [Kühl82, Corollary 1.3]). Therefore $\ker(\alpha_{I,n}) = 0$ for each $n \geq p+1$.

Remark 3.9.3. Let $n_1, \ldots, n_r \geq 2$ be integers. Then one can construct a Noetherian ring A and an ideal I of A such that $\beta_{I,n}$ is an isomorphism if and only if $n \neq n_1, \ldots, n_r$. This follows using a construction similar to Example 3.9.1, taking as many sets of variables, with the corresponding relations, as there are integers $n_1, \ldots, n_r$. See Example 4.2.2 for the details.

Remark 3.9.4. Remark that there are ideals I verifying the hypotheses of Theorem B and such that $\mathbf{S}(I)$ has no regular elements in degree one. Take, e.g., $R = k[X, Y, Z]/(XZ, YZ) = k[x, y, z]$ and $I = (x, y) \subset R$.

Remark 3.9.5. Graph ideals provide us with valuable examples of ideals verifying that $\alpha_{J,2}, \ldots, \alpha_{J,n}$ are isomorphisms whereas $\alpha_{J,n+1}$ is not for $n \geq 1$. Let $R = k[X_1, \ldots, X_{2n+2}]$ be

the polynomial ring in the variables $X_1, \ldots, X_{2n+2}$ over a field k. Let J be the graph ideal $(X_1 X_2, X_2 X_3, \ldots, X_{2n+1} X_{2n+2}, X_{2n+2} X_1)$ corresponding to a $(2n+2)$-cycle. It is well-known that $\alpha_{J,2}, \ldots, \alpha_{J,n}$ are isomorphisms whereas $\alpha_{J,n+1}$ is not. Remark that if $I = (J, X_1 X_3)$, one can prove that $\alpha_{I,n} : \mathbf{S}_n(I) \to I^n$ is an isomorphism (see [Villarreal95]). In particular, by Theorem B, also $\alpha_{I,2}, \ldots, \alpha_{I,n}$ are isomorphisms.

Chapter 4

Further examples and future work

We present a set of tools behind many constructions and key ideas used in the previous chapters. Many results, examples and remarks are outcomes of these techniques.

The equations of the Rees algebra of an ideal modulo a regular element have been an object of study since the influential works of [HSV81, Proposition 3.3] and [Valla80] on ideals of linear type. Here we tackle the question whether the equations of the Rees algebra of a given ideal remain invariant when taking the ideal modulo a regular element. We prove Proposition 4.1.2, result that generalises [HSV81, Proposition 3.3] and [Valla80, Theorem 2.1], relying on the module of effective relations and we are able to deduce at once yet another proof of a classical result recorded in the claim 1.3.9, namely, that an ideal generated by a d-sequence is of linear type.

When studying the equations of the Rees algebras of ideals in terms of its generating sets and reductions, it is very useful to rely on examples where the generators, basic ideal operations, computation of reduction and equations of blowing-up algebras are intuitive enough to allow simple and flexible examples and to avoid burdensome experimentation.

Efforts have been devoted to construct families of ideals with prescribed equations. We have explored the obstructions arising in the construction of examples of ideals with a predefined pattern of equations. We have proved a method for constructing ideals with predefined top degree equations, as well as we provide a general version of Example 3.9.1.

Finally, a tentative procedure for expanding sets of equations of Rees algebras is presented. It is a variant of a well-known determinantal procedure leading to the notion of

expected equations. While the procedure presented has been used by the author as a way to expand sets of non-trivial equations of Rees algebras, it is by no means well understood why it works and why it shoud work under more general settings: it remains an appealing object of future study. Surprisingly enough, in the examples examined, if we try to expand a generating set of the first syzygies, the procedure leads, with few adjustements, to full generating sets of equations.

4.1 The equations of Rees algebras of ideals modulo a regular element

The interplay between the equations of the Rees algebra of an ideal I of R, the Koszul homology of generating sets of I and conditions on these generating sets, yield results that allow for setting bounds on the relation type of I.

First of all, let us recall a result from [Planas00]:

Lemma 4.1.1 (see [Planas00])**.** *Let R be a Noetherian ring, I an ideal of R and $y \in I$. Then, for each integer $p \geq 2$, there is an exact sequence of R-modules*

$$E(I)_p \xrightarrow{\sigma_p} E(I/(y))_p \to \frac{(y) \cap I^p}{I((y) \cap I^{p-1})} \to 0.$$

Proof: Use [Planas00] with $N = (y) \subseteq M = R$. $\qquad\qquad\square$

The next result generalises [HSV81, Proposition 3.3] and [Valla80, Theorem 2.1].

Proposition 4.1.2. *Let R be a Noetherian ring and let I be an ideal of R. Let $y \in I \backslash I^2$ such that $(0 : y) \cap I = 0$ and such that the Valabrega-Valla module*

$$VV_{(y)}(I)_{p-1} = (y) \cap I^{p-1}/yI^{p-2}$$

is equal to zero, for some $p \geq 2$. Then, there is an exact sequence of R-modules

$$0 \to E(I)_p \xrightarrow{\sigma_p} E(I/(y))_p \to VV_{(y)}(I)_p \to 0.$$

In particular, if y^* is $\mathbf{G}(I)$-regular, then $E(I)_p \cong E(I/(y))_p$ for all $p \geq 2$ and $\mathrm{rt}(I) = \mathrm{rt}(I/(y))$.

Proof: Let $I = (x_1, \ldots, x_s)$ and let $R' = R/(y)$, $W = R[T_1, \ldots, T_s]$ and $W' = R'[T_1, \ldots, T_s]$. Let φ be the polynomial presentation from W to $\mathbf{R}(I)$ sending T_i to $x_i t$ and let ψ be the induced polynomial presentation from W' to $\mathbf{R}(I/(y))$, respectively, and let π_S and π_R the natural projections, as in the following diagram,

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Q & \longrightarrow & W & \xrightarrow{\;\varphi\;} & \mathbf{R}(I) & \longrightarrow & 0 \\
& & \big\downarrow & & \pi_S\big\downarrow & & \pi_R\big\downarrow & & \\
0 & \longrightarrow & Q' & \longrightarrow & W' & \xrightarrow{\;\psi\;} & \mathbf{R}(I/(y)) & \longrightarrow & 0.
\end{array}
$$

It is readily seen that, for each $p \geq 2$, π_S induces an homomorphism

$$\widetilde{\pi}_p : (Q/Q\langle p - 1 \rangle)_p \to (Q'/Q'\langle p - 1 \rangle)_p$$

such that the following is a commutative diagram:

$$
\begin{array}{ccc}
(Q/Q\langle p-1 \rangle)_p & \xrightarrow{\;\widetilde{\pi}_p\;} & (Q'/Q'\langle p-1 \rangle)_p \\
\cong\big\downarrow & & \cong\big\downarrow \\
E(I)_p & \xrightarrow{\;\sigma_p\;} & E(I/(y))_p
\end{array}
$$

In fact, $\widetilde{\pi}_p$ sends the class of a given form $F \in Q_p$ to the class $\overline{F}$ of F modulo (y) in $(Q'/Q'\langle p - 1 \rangle)_p$. If $F \in Q\langle p - 1 \rangle \cap W_p$, then $F = \sum_{i=1}^s T_i F_i$, with $F_i \in Q\langle p - 1 \rangle \cap W_{p-1}$ and it follows $F' \in Q'\langle p - 1 \rangle \cap W'_p$, hence $\widetilde{\pi}_p$ is well-defined.

We will show that $\widetilde{\pi}_p$ is injective, hence we will conclude that σ_p is injective.

Take $F \in Q_p$ a representative of an element in the kernel of σ_p. Then $\overline{F} \in W'$ can be written as $\overline{F} = \sum_{i=1}^s T_i \overline{G}_i$, where $\overline{G}_i \in Q'_{p-1}$, thus $F = \sum_{i=1}^s T_i G_i + yK$ with $K \in W_p$, $G_i \in W_{p-1}$, verifying

$$G_i(x_1, \ldots, x_s) \in (y) \cap (x_1, \ldots, x_s)^{p-1} = (y) \cap I^{p-1} = yI^{p-2}. \tag{4.1}$$

Observe that we can assume without loss of generality that $F = \sum_{i=1}^s T_i G_i$ with the G_i satisfying the last condition 4.1, since K can be written as $\sum_{i=1}^s T_i K_i$, thus we get $F = \sum_{i=1}^s T_i(G_i + yK_i)$ and $\overline{G_i + yK_i} = \overline{G}_i$.

So let us assume that $F = \sum_{i=1}^s T_i G_i$ with the G_i fulfilling condition 4.1. Let $y = \sum_{j=1}^s \lambda_j x_j$. For each $1 \leq i \leq s$, there exist $H_i \in W$, $\deg(H_i) = p - 2$ such that $G_i -$

$(\sum_{j=1}^{s} \lambda_j T_j) H_i \in Q_{p-1}$. Therefore we have:

$$F = \sum_{i=1}^{s} T_i(G_i - (\sum_{j=1}^{s} \lambda_j T_j) H_i) + \sum_{i=1}^{s} (\sum_{j=1}^{s} \lambda_j T_j) T_i H_i$$

$$= \sum_{i=1}^{s} T_i(G_i - (\sum_{j=1}^{s} \lambda_j T_j) H_i) + (\sum_{j=1}^{s} \lambda_j T_j) \sum_{i=1}^{s} T_i H_i.$$

Consequently, $y(\sum_{i=1}^{s} x_i H_i(x_1, \ldots, x_s)) = 0$. Since $I \cap (0 : y) = 0$, then

$$\sum_{i=1}^{s} x_i H_i(x_1, \ldots, x_s) = 0,$$

thus $F \in Q\langle p-1 \rangle$, showing that the homomorphism σ_p is injective. The remaining claims follow from a straightforward application of Lemma 4.1.1. □

Contrary to what one may guess in the light of some well-known results, such as [HSV81, Proposition 3.3] and [Valla80, Theorem 2.1], the inequality $\text{rt}(I) \leq \text{rt}(I/(y))$ does not hold in general, even if y is a regular element. In the following example, we observe that if $y \in I$ is an R-regular element, but y^* is not $\mathbf{G}(I)$-regular, the preceding inequality may not hold.

Example 4.1.3. Let k be a field, let $R = k[t^3, t^4, t^5]$ and let $I = (t^3, t^4)$. Then $\text{rt}(I) = 3 > \text{rt}(I/(t^3)) = 2$.

Proof: For convenience we will let $x = t^3$ and $y = t^4$. The effective relations of the ideals $I \subset R$ and $I/(x) \subset R/(x)$ can be computed using either 2.3.3 or Lemmas 3.4.3 and 3.4.4. In fact, since R is a domain we have $E(I)_n \cong (xI^{n-1} :_R y^n)/(xI^{n-2} :_R y^{n-1})$ and $E(I/(x)) \cong (x :_R y^n)/(x :_R y^{n-1})$. Straightforward computations lead to the result. Observe that x^* is not a regular element in $\mathbf{G}(I)$, since $(I^2 :_R x) = (t^3, t^4, t^5)$ but $t^5 \notin I$. □

As a consequence of the previous results and considering the elementary properties of d-sequences (see [Huneke86] and [HSV81]), we are able to recover the well-known fact that ideals generated by d-sequences are of linear type (see [Huneke80], [Valla80] and Section 1.3.9). Let us split the statement and proof of this result stating first a key lemma.

Lemma 4.1.4 (see [HSV81]). *Let $J \subset I$ be ideals of a Noetherian ring R. If I/J is an ideal of linear type in R/J, then $J \cap I^n = JI^{n-1}$, for all $n \geq 0$.*

Proposition 4.1.5. *Let R be a Noetherian ring and I an ideal of R. If I is generated by a d-sequence, then I is of linear type, i.e., $\text{rt}(I) = 1$.*

Proof: Let $x_1, \ldots, x_s$ be a d-sequence and I the ideal it generates. Let us proceed by induction on s. If $s = 1$ then being a d-sequence is equivalent to $(0 : x_1) \cap (x_1) = 0$. Then $E(I)_p \cong (0 : x_1^p)/(0 : x_1^{p-1}) = 0$, for all $p \geq 2$ and the result follows.

Suppose that the result holds for d-sequences of length $s-1$. If $x_1, \ldots, x_s$ is a d-sequence in R, then $x_2', \ldots, x_s'$ is also a d-sequence in $R' = R/(x_1)$ (see [Huneke86, Definition 1.1 and Remarks]). By the induction hypothesis, $I/(x_1)$ is of linear type. From [Huneke86, Proposition 2.1] we know that $(0 : x_1) \cap I = 0$. Using the previous lemma it follows that $(x_1) \cap I^n = (x_1)I^{n-1}$, for all $n \geq 1$ (see also [Huneke86, Theorem 2.1]). Using Proposition 4.1.2 we get $E(I)_p \cong E(I/(x_1))_p$, for all $p \geq 2$. Since $I/(x_1)$ is of linear type for all $p \geq 2$ we have $E(I/(x_1)) = 0$ thus $E(I)_p = 0$ for all $p \geq 2$ and the claim follows. $\qquad\square$

4.2   Rees algebras with prescribed equations

Two methods are presented to obtain ideals with a predefined set of equations. The first method yields two-generated irrelevant ideals of standard algebras with prescribed relation type and top degree equations. The second method stretches the arguments presented in Example 3.9.1 to show ideals with prescribed non-vanishing patterns of the graded components of $\ker \beta_I$.

4.2.1   Prescribed relation type and top degree equations

Let $A = k[U_1, \ldots, U_t]$ where k is a field and $U_1, \ldots, U_s$ are indeterminates over k. Let $X_1, \ldots, X_s$ be indeterminates over A and endow the polynomial ring $A[X_1, \ldots, X_s]$ with a grading by setting $\deg(X_i) = 1$ for all $i = 1, \ldots, s$ and $\deg(w) = 0$ for each $w \in A$. Let L be an homogeneous ideal of $A[X_1, \ldots, X_s]$, generated by elements of positive degree. Set $R = A[X_1, \ldots, X_s]/L = \oplus_{n \geq 0} R_n = A[x_1, \ldots, x_s]$, where x_i stands for the class of X_i. The ring R has a standard graded A-algebra structure, with irrelevant ideal given by $I = R_+ = (x_1, \ldots, x_s)$.

Notice that $\mathbf{G}(I) = \oplus_{n \geq 0} I^n/I^{n+1}$ is isomorphic to R as A-algebras and $\mathbf{S}_{R/I}(I/I^2) \cong \mathbf{S}_A(R_1)$. Let $\varphi : A^s \to R_1$ be a free presentation of R_1, sending each basis element e_i to X_i.

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Then there is a short exact sequence

$$0 \to \widetilde{L}_1 \to Ae_1 \oplus \ldots \oplus Ae_s \xrightarrow{f} R_1 \to 0.$$

Notice that there is an isomorphism of A-modules $\widetilde{L}_1 \cong L_1$ induced by the map $A^s \to A[X_1, \ldots, X_s]_1$ sending e_i to X_i. Applying the symmetric functor to the short exact sequence (see [Bourbaki, Algebra, Ch. 3, Section 6.2]), one gets the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L\langle 1 \rangle & \longrightarrow & A[X_1, \ldots, X_s] & \longrightarrow & \mathbf{S}_A(R_1) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle \alpha} & & \\
0 & \longrightarrow & L & \longrightarrow & A[X_1, \ldots, X_s] & \xrightarrow{\varphi} & R & \longrightarrow & 0.
\end{array}
$$

By the claims of the preceding paragraph and according to the notations adopted throughout, the canonical morphism α in the diagram above is no other than β_I. An immediate application of the snake lemma yields $\ker(\beta_I) \cong L/L\langle 1 \rangle$.

According to the claim 1.3.29, $\mathrm{rt}(I) = \mathrm{rt}(\mathbf{G}(I))$ (set r this value) and $E(I)_r \cong E(\mathbf{G}(I))_r$. On the other hand, using the isomorphism $\ker \beta_I \cong L/L\langle 1 \rangle$, we get

$$E(\mathbf{G}(I))_r = \ker \beta_{I,r}/\mathbf{S}_1(I/I^2) \cdot \ker \beta_{I,r-1} \cong (L/L\langle r-1 \rangle)_r.$$

4.2.2   Non-zero $\ker \beta$ at prescribed degrees

In Section 3.9 we constructed a ring R_p and an ideal $I_p \subset R_p$ such that $\ker \beta$ is non-zero only at a given degree p. Moreover, $\ker \beta_{I,p}$ is cyclic.

As it is asserted in Remark 3.9.3, we will prove that given a finite set $S = \{p_1, \ldots, p_r\}$ of integers ≥ 2, there exists a Noetherian ring $\mathcal{A}_S$ and an ideal $I_S \subset \mathcal{A}_S$ such that $\beta_{I_S,p}$ is an isomorphism if and only if $p \notin S$.

The idea consists in loading a new set of indeterminates and equations in these indeterminates for each new fresh equation we want to add to the kernel of β_{I_S}. If the set S is finite, this will be nothing but a tensor product of algebras. On the other hand, considering S infinite, would lead us to a non-Noetherian scenario that will not be tackled here.

Example 4.2.1. Let $U_{p,i}$ be indeterminates for integers $p \geq 2$ and $0 \leq i \leq p$. As in Example 3.9.1, define Q_p as the ideal of $A_p = k[X,Y][U_{p,0}, \ldots, U_{p,p}]$ given by

$$Q_p = (U_{p,0}Y, U_{p,0}X - U_{p,1}Y, U_{p,1}X - U_{p,2}Y, \ldots, U_{p,p-1}X - U_{p,p}Y, U_{p,p}X, U_{p,0}X^p).$$

Notice that A_p/Q_p are $k[X,Y]$-modules. Given $S = \{p_1, \ldots, p_r\}$ a finite set of integers satisfying $2 \leq p_1 < p_2 < \ldots < p_r$, define the ring $\mathcal{A}_S$ as

$$\mathcal{A}_S = (A_{p_1}/Q_{p_1}) \otimes_{k[X,Y]} (A_{p_2}/Q_{p_2}) \otimes_{k[X,Y]} \cdots \otimes_{k[X,Y]} (A_{p_r}/Q_{p_r}).$$

Then the ideal $I = (x,y) \subset \mathcal{A}_S$ verifies that

$$\ker \beta_{I,n} = \begin{cases} \langle U_{n,0} T_1^n \rangle & n \in S \\ 0 & n \notin S \end{cases}$$

In particular, I is of p-linear type and $\mathrm{rt}(I) = p_r$, thus not of linear type.

Proof: In order to not too much overload the notation, we will prove the claim for a set $S = \{p, q\}$ with $2 \leq p < q$, then the argument can easily be generalised to finite sets of integers. Let

$$Q_p = (U_0 Y, U_0 X - U_1 Y, U_1 X - U_2 Y, \ldots, U_{p-1} X - U_p Y, U_p X, U_0 X^p)$$
$$Q_q = (V_0 Y, V_0 X - V_1 Y, V_1 X - V_2 Y, \ldots, V_{q-1} X - V_q Y, V_q X, V_0 X^q),$$

and let us set

$$\mathcal{A}_S = (A_p/Q_p) \otimes_{k[X,Y]} (A_q/Q_q) = \frac{k[X,Y,U_0,\ldots,U_p]}{Q_p} \otimes_{k[X,Y]} \frac{k[X,Y,V_0,\ldots,V_q]}{Q_q}$$
$$\cong \frac{k[X,Y,U_0,\ldots,U_p,V_0,\ldots,V_q]}{(Q_p,Q_q)} = k[x,y,u_0,\ldots,u_p,v_0,\ldots,v_q],$$

$I = (x,y) \subset \mathcal{A}_S$. Let $\beta_I : \mathbf{S}(I/I^2) \to \mathbf{G}(I)$ be the canonical epimorphism onto de associated graded ring of I and consider $\psi : W = \mathcal{A}_S[T_1, T_2] \to \mathbf{G}(I)$, the polynomial presentation of $\mathbf{G}(I)$ sending T_1 to x^* and T_2 to y^*. We know that $\ker \beta_I \cong \ker \psi / W_+ \ker \psi_1$. Using this correspondence, by the previous discussion in Section 4.2.1, we known that $\ker \beta_I$ is generated by the classes of $u_0 T_1^p$ and $v_0 T_1^q$ modulo the ideal generated by linear forms

$$\ker \psi \langle 1 \rangle = W_+ \ker \psi_1 = (u_0 T_2, u_0 T_1 - u_1 T_2, u_1 T_1 - u_2 T_2, \ldots, u_{p-1} T_1 - u_p T_2, u_p T_1,$$
$$v_0 T_2, v_0 T_1 - v_1 T_2, v_1 T_1 - v_2 T_2, \ldots, v_{q-1} T_1 - v_q T_2, v_q T_1).$$

Repeating the same arguments as in the proof of Example 3.9.1, it is clear that any combination $\mu u_0 T_1^p + \nu v_0 T_1^q$ with $\deg_{T_1,T_2}(\mu) = \deg_{T_1,T_2}(\nu) \geq 1$ belongs to $\ker \psi \langle 1 \rangle$, consequently vanishes in $\ker \beta_I$. Moreover, $\mathrm{rt}(I) = q$ and from $\ker \beta_{I,n} = 0$ for all $n \geq q+1$ one deduces $\ker \alpha_{I,n} = 0$ for all $n \geq q+1$, thus I is of p-linear type. $\qquad\square$

4.3 g-linear type but not p-linear type

Recall from Section 1.3.4 that we have the following chain of implications:

$$\text{linear type} \Rightarrow p\text{-linear type} \Rightarrow g\text{-linear type.}$$

While the converse implications do not hold in general, we can see that the first one is valid for principal ideals, that is, the notions of linear type and p-linear type coincide for principal ideals.

Lemma 4.3.1. *Let R be a Noetherian ring. Let I be a principal ideal. If I is of p-linear type, then I is of linear type. Furthermore, any principal ideal I with $\ker \alpha_{I,n} = 0$ for some $n \geq 2$ is of linear type.*

Proof: Let $I = (x)$ be a principal ideal of R. By Example 3.3.2, $\ker \alpha_{I,n} \cong (0 : x^n)/(0 : x)$. By Remark 1.3.33, $\ker \alpha_{I,n} = 0$ for n large enough. It follows that $(0 : x^2) = (0 : x)$. $\qquad\square$

Remark 4.3.2. Recall that we have already seen examples of ideals of p-type with arbitrarily large relation type: recall the previous examples 3.9.1 and 4.2.1.

In the next example it is proved that g-linear type does not imply p-linear type, even for principal ideals.

Example 4.3.3. Let k be a field and $R = k[X,Y]/(X^2Y, XY^2) = k[x,y]$. Then $I = (y)$ is an ideal of g-linear type but not p-linear type.

Proof: We will show that $\ker \alpha$ is nilpotent whereas $\ker \alpha_n$ does not vanish for all $n \geq 2$. Let $\varphi : R[T] \to \mathbf{R}(I)$ sending T to y, $Q = \ker \varphi$. Then $Q/Q_1 \cong \ker \alpha$ is generated by the class of xT^2. In fact, $\ker \alpha_n \neq 0$ for all $n \geq 2$, since the class of xT^n is not zero in $\ker \alpha_n$. On the other hand, the class of $(xT^n)^2 = x^2T^{2n} = (x^2T) \cdot T^{2n-1}$ in $\ker \alpha_n$ is the zero class, since $x^2T \in Q_1$.

4.4 Computing the equations of Rees algebras by means of a determinantal closure

In this section we develop some examples that illustrate a tentative determinantal procedure to expand sets of equations of the Rees algebra of an ideal I. This procedure arises as a

natural extension of a determinantal method that computes the maximal minors of the Jacobian dual of a free presentation of I. A quick overlook of the basic settings and description of the procedure is presented (see also e.g. [SUV93], [Morey96] and [Vasconcelos91]). Then a brief explanation of the generalisation of the procedure is given.

The aim is to understand whether the equations of $\mathbf{R}(I)$ may arise as an ideal of minors of a matrix $B(\psi)$ which is iteratively obtained, starting from the first syzygies of I and leading to a sort of determinantal closure. Such a procedure has its roots in the computational insights already stressed in the literature relating the equations of Rees algebras and the first syzygies (see for instance [CHW08], [Cox08] and [CD'A10]; see also 1.3.20).

Let us begin with an introductory example:

Example 4.4.1. The following is a particular case of the family considered in Example 2.6.1. Let $I = (s^3, t^3, st^2) \subset R = k[s, t]$. Let $\varphi : R[T_1, T_2, T_3] \to \mathbf{R}(I)$ be the polynomial presentation sending T_1 to s^3, T_2 to t^3 and T_3 to st^2, and let $Q = \ker \varphi$. We get the following Hilbert-Burch presentation

$$
R(-4) \oplus R(-5) \xrightarrow{\begin{bmatrix} 0 & t^2 \\ s & 0 \\ -t & -s^2 \end{bmatrix}} R(-3)^3 \xrightarrow{\begin{bmatrix} s^3 & t^3 & st^2 \end{bmatrix}} I \longrightarrow 0.
$$

Let the linear forms $S_1, S_2 \in R[T_1, T_2, T_3]$ be given by the following identity:

$$
[S_1, S_2] = [T_1, T_2, T_3] \cdot \begin{bmatrix} 0 & t^2 \\ s & 0 \\ -t & -s^2 \end{bmatrix}.
$$

In fact, S_1, S_2 is a minimal generating set of the module of first syzygies of I using the obvious correspondence, i.e., $Q\langle 1 \rangle = (S_1, S_2)$. Observe that we have another matrix identity to describe S_1, S_2:

$$
[S_1, S_2] = [s, t] \cdot \begin{bmatrix} T_2 & -sT_3 \\ -T_3 & tT_1 \end{bmatrix}.
$$

The determinant $tT_1T_2 - sT_3^2$ of the matrix of the right hand part is an element of Q which is not in $Q\langle 1 \rangle$. Now we iterate the process by adding a column corresponding to the new element:

$$
[S_1, S_2, tT_1T_2 - sT_3^2] = [s, t] \cdot \begin{bmatrix} T_2 & -sT_3 & -T_3^2 \\ -T_3 & tT_1 & T_1T_2 \end{bmatrix}.
$$

Among the (2×2)-minors of the matrix of the right hand part, all of them belonging to Q, $T_1 T_2^2 - T_3^3$ is the only one that is not contained in the ideal $(S_1, S_2, t T_1 T_2 - s T_3^2)$. Since $T_1 T_2^2 - T_3^3 \notin (s,t) R[T_1, T_2, T_3]$, we can not play the same game again and stop.

So far we have obtained four elements of Q: S_1, S_2 of degree 1 and two more homogeneous elements of Q of degrees 2 and 3: $t T_1 T_2 - s T_3^2, T_1 T_2^2 - T_3^3$. By Example 2.6.1, we already know that they are enough to generate Q. □

For the rest of the section, consider the following framework: let R be the polynomial ring $k[Z_1, \ldots, Z_r]$ over a field k; let $I = (x_1, \ldots, x_s)$ be an ideal of R with R-free presentation:

$$R^m \xrightarrow{\psi} R^s \xrightarrow{\widetilde{\varphi}_1} I \to 0;$$

let $\varphi : R[T_1, \ldots, T_s] \to \mathbf{R}(I)$ be the polynomial presentation of $\mathbf{R}(I)$ sending T_i to $x_i t$, and let $Q = \ker \varphi$.

If M stands for the matrix of ψ, $Q\langle 1 \rangle$ is generated by the linear forms $S_1, \ldots, S_m$ defined as $[S_1, \ldots, S_m] = [T_1, \ldots, T_s] \cdot M$. Suppose that the ideal L generated by the entries of M is included in $(Z_1, \ldots, Z_r)$. Suppose also that $m = \mu(Q\langle 1 \rangle) \geq r$. Let $B(\psi)$ be an $(r \times m)$-matrix with entries in $R[T_1, \ldots, T_s]$ satisfying the equation

$$[S_1, \ldots, S_m] = [Z_1, \ldots, Z_r] \cdot B(\psi).$$

Remark 4.4.2. If Δ is an $(r \times r)$-minor of $B(\psi)$, then $\Delta \in Q$. Consequently, the ideal $I_r(B(\psi))$ generated by the $(r \times r)$-minors of M is contained in Q.

Proof: Using Cramer's rule we get $\Delta \cdot Z_i \in (S_1, \ldots, S_m) = Q\langle 1 \rangle \subset Q$, for $i = 1, \ldots, r$. Within our assumptions $\mathbf{R}(I)$ is a domain, thus Q is a prime ideal and we have $\Delta \in Q$ since $Z_i \notin Q$. □

Unless the entries of M are linear in $Z_1, \ldots, Z_r$, there will be several choices for $B(\psi)$. By Remark 4.4.2, it is clear that $(S_1, \ldots, S_m) + I_r(B(\psi)) \subseteq Q$. When the equality holds, the ideal I is said to have the *expected equations*. For a taste on such classes of ideals see for instance [Vasconcelos91].

Example 4.4.3. The following is a particular case of the family considered in Example 2.6.5.
Let $I = (s^5, t^5, s^2t^3) \subset R = k[s,t]$. We get the following Hilbert-Burch presentation:

$$R(-7) \oplus R(-8) \xrightarrow{\begin{bmatrix} 0 & t^3 \\ s^2 & 0 \\ -t^2 & -s^3 \end{bmatrix}} R(-5)^3 \xrightarrow{\begin{bmatrix} s^5 & t^5 & s^2t^3 \end{bmatrix}} I \to 0.$$

Let $S_1, S_2 \in R[T_1, T_2, T_3]$ be the linear forms given by the following identity:

$$[S_1, S_2] = [T_1, T_2, T_3] \cdot \begin{bmatrix} 0 & t^3 \\ s^2 & 0 \\ -t^2 & -s^3 \end{bmatrix}.$$

In fact, S_1, S_2 is a minimal generating set of $Q\langle 1 \rangle$. Observe that we have another matrix identity to describe S_1, S_2 in terms of a row vector $[s, t]$ and a matrix of the form $B(\psi)$:

$$[S_1, S_2] = [s, t] \cdot \begin{bmatrix} sT_2 & -s^2T_3 \\ -tT_3 & t^2T_1 \end{bmatrix}.$$

Consider the determinant $st^2T_1T_2 - s^2tT_3^2$ of the matrix of the right hand part. Decompose it into its prime components and clear the factors not belonging to Q. We get a new element $tT_1T_2 - sT_3^2$ of Q not belonging to $Q\langle 1 \rangle$. Since $tT_1T_2 - sT_3^2 \in (s,t)R[T_1, T_2, T_3]$, we add a new column:

$$[S_1, S_2, tT_1T_2 - sT_3^2] = [s, t] \cdot \begin{bmatrix} sT_2 & -s^2T_3 & -T_3^2 \\ -tT_3 & t^2T_1 & T_1T_2 \end{bmatrix},$$

thus, after clearing the (2×2)-minors of B_2 that already belong to $P_1 = (S_1, S_2, tT_1T_2 - sT_3^2)$ and the irreducible factors not belonging to Q, we get $sT_1T_2^2 - tT_3^3$. Since $sT_1T_2^2 - tT_3^3 \in (s,t)R[T_1, T_2, T_3]$, we add a new column:

$$[S_1, S_2, tT_1T_2 - sT_3^2, sT_1T_2^2 - tT_3^3] = [s, t] \cdot \begin{bmatrix} sT_2 & -s^2T_3 & -T_3^2 & T_1T_2^2 \\ -tT_3 & t^2T_1 & T_1T_2 & -T_3^3 \end{bmatrix},$$

and again after clearing the (2×2)-minors of B_3 that already belong to $P_2 = (S_1, S_2, tT_1T_2 - sT_3^2, sT_1T_2^2 - tT_3^3)$, we get the irreducible form $T_3^5 - T_1^2T_2^3$. Since $T_3^5 - T_1^2T_2^3 \notin (s,t)R[T_1, T_2, T_3]$ the procedure halts. Let $P_3 = (s^2T_2 - t^2T_3, tT_1T_2 - sT_3^2, sT_1T_2^2 - tT_3^3, T_3^5 - T_1^2T_2^3) \subset Q$: by Example 2.6.5 we know that $P_3 = Q$. $\qquad \square$

The next algorithm encodes a sequence of instructions corresponding to the steps completed in the previous discussion and examples. It is a tentative procedure to expand a generating set of $Q\langle 1\rangle$ into a generating set of Q. The reader will observe that the same process could be used to expand any finite set of elements in Q and could lead to the definition of a sort of determinantal closure for subideals of Q.

Algorithm 4.4.4. In what follows, L and K will be lists of forms in $R[T_1, \ldots, T_s]$, P will stand for an ideal in $R[T_1, \ldots, T_s]$ and B will denote a matrix of forms in $R[T_1, \ldots, T_s]$.

Compute $S = \{S_1, \ldots, S_k\}$ a minimal generating set of Q_1;

Set $L \leftarrow S$;

Let $P = (L)$, the ideal generated by the elements of the list L;

Since all the forms of the list L belong to $(Z_1, \ldots, Z_r)R[T_1, \ldots, T_s]$, there is an $(r \times k)$-matrix B_1 such that $[L] = [Z_1, \ldots, Z_r] \cdot B_1$;

Set $B \leftarrow B_1$;

while $K \neq \emptyset$ **do**

    Compute the $(r \times r)$-minors of B and put them into a list K;

    Set $K \leftarrow K \backslash \{m \in K \mid m \in P\}$, i.e., discard all the elements of that belong to P;

    for all $f \in K$ **do**

        Compute all the irreducible components $\{f_i\}$ of f in $k[Z_1, \ldots, Z_r][T_1, \ldots, T_s]$;

        Set $K \leftarrow (K \backslash f) \cup \{f_i \mid f_i \in Q\}$;

    end for

    Set $L \leftarrow L \cup K$;

    Set $P = (L)$ the ideal generated by the elements of P;

    Set $K \leftarrow K \backslash \{f \in K \mid f \notin (Z_1, \ldots, Z_r)R[T_1, \ldots, T_s]\}$;

    Since all the forms of the list K belong to $(Z_1, \ldots, Z_r)R[T_1, \ldots, T_s]$, there is a matrix B_K with r rows such that $[K] = [Z_1, \ldots, Z_r] \cdot B_K$;

    Set $B \leftarrow [B \mid B_K]$;

end while

return P. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We proceed to run Algorithm 4.4.4 in one more example, respecting the notations used in the pseudo-code.

Example 4.4.5. Let $I = (u, z) \cap (v, t) = (uv, vz, zt, ut) \subset k[u, v, z, t]$ be the ideal of the graph cycle C_4 (see also Example 2.6.4). In this case we have the following a free presentation:

$$R(-3)^4 \xrightarrow{\begin{bmatrix} z & t & 0 & 0 \\ -u & 0 & t & 0 \\ 0 & 0 & -v & u \\ 0 & -v & 0 & -z \end{bmatrix}} R(-2)^4 \xrightarrow{\begin{bmatrix} uv & vz & zt & ut \end{bmatrix}} I \rightarrow 0.$$

Let $P_0 = Q\langle 1 \rangle = (S_1, S_2, S_3, S_4)$, where

$$[S_1, S_2, S_3, S_4] = [T_1, T_2, T_3, T_4] \begin{bmatrix} z & t & 0 & 0 \\ -u & 0 & t & 0 \\ 0 & 0 & -v & u \\ 0 & -v & 0 & -z \end{bmatrix}.$$

Writing $[S_1, S_2, S_3, S_4]$ as $[u, v, z, t] \cdot B_1$, we have

$$B_1 = \begin{bmatrix} -T_2 & 0 & 0 & T_3 \\ 0 & -T_4 & -T_3 & 0 \\ T_1 & 0 & 0 & -T_4 \\ 0 & T_1 & T_2 & 0 \end{bmatrix}.$$

The only (4×4)-determinant of B_1 is $(T_1 T_4 - T_2 T_3)^2 \notin P_0$ and its only irreducible factor $T_1 T_4 - T_2 T_3$ belongs to Q, then $K_1 = \{T_1 T_4 - T_2 T_3\}$ and we set $P_1 = P_0 + (T_1 T_4 - T_2 T_3)$. Since $T_1 T_4 - T_2 T_3 \notin (u, v, z, t) R[T_1, T_2, T_3, T_4]$ the procedure halts and returns

$$P_1 = (zT_1 - uT_2, tT_1 - vT_4, tT_2 - vT_3, uT_3 - zT_4, T_1 T_4 - T_2 T_3).$$

It is readily seen from the results in [Villarreal95] that P_1 is the ideal of equations of $\mathbf{R}(I)$, i.e., $P_1 = Q$. $\qquad \square$

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