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A qualitative and quantitative study of some planar differential equations

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Certifico que aquesta memòria ha estat realitzada per Johanna Denise García Saldaña sota la meva supervisió i que constitueix la seva Tesi per a aspirar al grau de Doctor en Matemàtiques per la Universitat Autònoma de Barcelona.

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Contents

Introduction

Differential equations are a fundamental tool in the description of physical phenomena through mathematical models that cover almost any area of human knowledge. The theoretical development of the Theory of Ordinary Differential Equations began in 1675 with Isaac Newton (1642–1727) and Gottfried W. von Leibniz (1646–1716).

The main concern of mathematicians during the 17th and 18th century was focused primarily on integration of differential equations by means of elementary functions. Due to the works of several great mathematicians, all known elementary methods for solving first order differential equations had been found practically by the end of the 17th century. Many differential equations of second order were derived, in the beginning of the 18th century, as models for problems in classical Mechanics. Also other phenomena led to differential equations of third order.

In the middle of the 19th century, Joseph Liouville (1809-1882) showed the impossibility of expressing the general solution of certain differential equations by a combination of elementary functions or Liouville functions. Hence, a new approach to the study of differential equations should be developed.

The qualitative theory of differential equations was born at the end of the 19th century with the works of Henri Poincaré (1854–1912) [89] and Aleksandr M. Lyapunov (1857–1918) [70]. Its aim consists in obtaining the local and global behavior of the solutions without having them explicitly. The main goal of the qualitative theory is the topological description of properties and configurations of solutions of differential systems in the whole space.

When restricted to the planar case, Poincaré showed in the second part of [89], that differential equations can have *limit cycles*: isolated periodic solutions in the set of all periodic solutions, see also [90]. They strongly attracted Poincaré's attention and he developed several tools for their study, like the Poincaré's map, the Annular Region Theorem, the method of small parameters, etc. In addition, he noticed the close relationship between the study of limit cycles and the global behavior of the solutions of differential equations. Thus, the limit cycles became one of the most interesting objects to be considered in the study of any differential equation. In 1900 David Hilbert stated the research on limit cycles in planar polynomial differential equations as one of the most important problems for the 20th century, see [60] and the interesting surveys [62, 65]. Afterwards, Steve Smale proposed the same problem, but restricted to polynomial Liénard differential equations, as one of the challenging problems for the 21st century [94].

In 1901, Ivar Bendixson (1861–1935) gave a rigorous proof of the Annular Region Theorem and extended it to the well-known Poincaré–Bendixson Theorem on the limit sets of trajectories of differential equations in a bounded region [11]. Moreover, he first applied Green's formula to establish the relationship between the existence of a closed trajectory and the divergence of a planar vector field. This result was improved in 1933 by Henri Dulac (1870-1955). The Bendixson–Dulac Theorem has been generalized in the plane, see [50], and extended in several directions: for proving non-existence of periodic orbits in higher dimensions [37, 66], or to control the number of isolated periodic solutions of some non-autonomous Abel differential equations, see for instance [6, 18].

In the 1930s, van der Pol and A. A. Andronov showed that the closed orbit in the phase plane of a self-sustained oscillation occurring in a vacuum tube circuit is a limit cycle as considered by Poincaré. After this observation, the existence, non-existence, uniqueness and other properties of limit cycles have been studied extensively by mathematicians and physicists. Moreover, many mathematical models from physics, engineering, chemistry, biology, economics, etc., were displayed as plane autonomous systems with limit cycles.

According to the previous discussion we can distinguish two different, important and interlinked directions in the research on differential equations: quantitative and qualitative theories. While the objective of the first one is to find a solution either in closed form or else by some process of approximation, the main goal of the second one is to provide a qualitative description of the behavior of the solutions.

Essentially, we can distinguish three broad categories of techniques for analyzing the quantitative and qualitative properties of nonlinear differential systems: asymptotic techniques (for instance the averaging method), heuristic techniques (Galerkin methods, the harmonic balance method, etc.), and analytical mathematical results about differential equations. In this thesis we will concentrate on the last two categories, carrying out a thorough study of the qualitative and quantitative properties of the solutions of certain classes of planar ordinary differential equations.

This thesis is divided into two parts. The first one is concerned with both the application and the theoretical basis of the Harmonic Balance Method (HBM). The second part is devoted to the analytical analysis of bifurcation diagrams, phase portraits, and limit cycles of some planar polynomial differential systems. Indeed, the results of the second part of this work, address in the issues proposed by Coppel in his well-known paper [29], taking into account the results of [35]. Recall that Coppel says: "Ideally one might hope to characterize the phase portraits of quadratic systems by means of algebraic inequalities on the coefficients". However, in [35] the authors proved that there are bifurcation curves in quadratic systems which are neither algebraic nor analytic. Therefore, not only for quadratic, but for polynomial systems there is no hope of finding analytic or algebraic expressions of the bifurcation curves. Similarly, for one-parameter families of polynomial vector fields it will not be possible to obtain all the bifurcation values algebraically. Hence to obtain rational upper and lower bounds of the non-algebraic bifurcation values is a natural and interesting question.

Each chapter of this work is almost self-contained and has a detailed introduction. For this reason, we will only give a brief description of the ideas, methods, and tools that we use and develop in our research, as well as a summary of the main results that we have obtained.

The Harmonic Balance Method has been widely used to study periodic solutions of nonlinear systems. We know that each periodic solution has a convergent Fourier series representation. The so-called N-th order Harmonic Balance Method (HBM) consists of approximating the periodic solutions of a nonlinear differential equation by using truncated Fourier series of order N. This procedure allows to transform an autonomous system of nonlinear ordinary differential equations into a system of nonlinear algebraic equations whose unknowns are the coefficients of the truncated Fourier series and the frequency of the sought periodic orbit. In most of the applications this method is used to approach isolated periodic solutions, see for instance [61, 74, 78, 76, 75]. Since the HBM also provides an approximation of the angular frequency of the desired periodic solution, it can be also used to get its period. Thus, the HBM can be applied to systems of differential equations having a continuum of periodic orbits in order to obtain approximations to their period function. This approach is also used for instance in [9, 10, 79]. To determine the local or global behavior of this period function is an interesting problem in the qualitative theory of differential equations either as a theoretical question or due to its appearance in many situations of applied science.

In Chapter 1 we consider several families of potential non-isochronous systems. The main goal is to show that some properties of their associated period functions, like their global monotonicity or their local behavior near the critical point or infinity, are captured by the HBM. To this end, we use analytical tools to obtain the local or global expression of the period function, we compute an approximation to this function by applying the HBM and we see that they have similar Taylor series.

For instance, in Section 1.7 we consider the family of polynomial potential systems

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + k\,x^3 + x^5, \quad k \in \mathbb{R}, \end{cases} \tag{1}$$

which for some values of k has a global center, and whose associated period function T has at most one oscillation [71, Thm. 1.1 (b)]. Here $T(A)$ denotes the (minimal) period of the periodic orbits of system (1) passing through the point $(A, 0)$. Two of the main results of this chapter are the following, which show that the first approximation of the period function obtained by applying the first order HBM captures and reproduces quite well the actual behavior of $T(A)$.

Theorem 1.5. *Consider system (1) with $k \in (-2, \infty)$. Let T be the period function associated to the origin, which is a global center. Then:*

(i) The function T is monotonous decreasing for $k \geq 0$.

(ii) The function T starts increasing, until a maximum (a critical period) and then decreases towards zero, for $k \in (-2, 0)$.

(iii) At the origin

$$T(A) = 2\pi - \frac{3}{4}k\pi A^2 + \frac{57k^2 - 80}{128}\pi A^4 + O(A^6),$$

and at infinity

$$T(A) \sim \frac{2B(\frac{1}{6}, \frac{1}{2})}{\sqrt{3}} \frac{1}{A^2} \approx \frac{8.4131}{A^2}.$$

Proposition 1.6. *Let $T_1(A)$ be the approximation of the period function $T(A)$ of system (1) obtained by applying the first-order HBM to the same system. Then:*

$$T_1(A) = \frac{8\pi}{\sqrt{16 + 12kA^2 + 10A^4}}.$$

In particular,

(i) The function $T_1(A)$ is decreasing for $k \geq 0$.

(ii) The function $T_1(A)$ starts increasing, has a maximum and then decreases towards zero, for $k \in (-2, 0)$.

(iii) At the origin

$$T_1(A) = 2\pi - \frac{3}{4}k\pi A^2 + \frac{54k^2 - 80}{128}\pi A^4 + O(A^6),$$

and at infinity

$$T_1(A) \sim \frac{4\pi\sqrt{10}}{5}\frac{1}{A^2} \approx \frac{7.9477}{A^2}.$$

In order to continue with the exploration about the validity of the HBM for approaching the period function, in Chapter 2 we perform an exhaustive study of the differential equation $x\ddot{x}+1=0$, already considered in [78] and proposed in [2] as a model for certain phenomena in plasma physics. We prove that it has a continuum of continuous weak periodic solutions and we compute their periods.

In practice, the N-th order HBM is often used with $N=1,2$ because of the difficulty to solve the nonlinear algebraic systems obtained with this approach. However, by using the Gröbner basis method we can apply the HBM until sixth order to approach the periods of the weak periodic solutions of $x\ddot{x}+1=0$, and to illustrate how the sharpness of the method increases with the order.

We have not yet been able to give a theoretical basis of the applicability of the HBM to approximate the period functions. Aiming towards a complete understanding of the validity of this approach we have restricted our attention to a simpler case. More precisely, in Chapter 3 we provide a result for one-dimensional non-autonomous ordinary differential equations which implies that the HBM can be used to prove the existence of limit cycles of planar differential systems as well as for determining its localization and hyperbolicity. Our main result, which is strongly based in the pioneering works of Stokes [97] and Urabe [99] on the subject, provides a theoretical basis for the above question. To state it we introduce some concepts.

Let $\bar{x}(t)$ be a real 2π-periodic $\mathcal{C}^1$-function; we will say that $\bar{x}(t)$ is *noncritical* with respect to the differential equation $x'=X(x,t)$ if

$$\int_0^{2\pi} \frac{\partial}{\partial x} X(\bar{x}(t),t)\,dt \neq 0.$$

Observe that if $\bar{x}(t)$ is a periodic solution of $x'=X(x,t)$ then the concept of noncritical is equivalent to the one of being *hyperbolic*; see [68].

As we will see in Lemma 3.3, if $\bar{x}(t)$ is noncritical w.r.t. $x'=X(x,t)$, the linear periodic system

$$y' = \frac{\partial}{\partial x} X(\bar{x}(t),t)\,y + b(t)$$

has a unique periodic solution $y_b(t)$ for each smooth 2π-periodic function $b(t)$. Moreover, once X and $\bar{x}$ are fixed, there exists a constant M, which will be called a *deformation constant associated to $\bar{x}$ and X*, such that

$$||y_b||_\infty \leq M||b||_2.$$

Here, as usual, for a continuous 2π-periodic function f,

$$||f||_2 = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} f^2(t)dt}, \quad ||f||_\infty = \max_{x\in\mathbb{R}} |f(x)| \quad \text{and} \quad ||f||_2 \le ||f||_\infty .$$

Finally, consider

$$s(t) := \bar{x}'(t) - X(\bar{x}(t), t).$$

We say that $\bar{x}(t)$ is *an approximate solution of* $x' = X(x,t)$ *with accuracy* $S = ||s||_2$.

Theorem 3.1. *Let $\bar{x}(t)$ be a 2π-periodic $\mathcal{C}^1$ -function such that*

- *it is noncritical w.r.t. $x' = X(x,t)$ and has M as a deformation constant,*
- *it has accuracy S w.r.t. $x' = X(x,t)$.*

Given $I := [\min_{\{t\in\mathbb{R}\}} \bar{x}(t) - 2MS, \max_{\{t\in\mathbb{R}\}} \bar{x}(t) + 2MS] \subset \Omega$, let $K < \infty$ be a constant such that

$$\max_{(x,t)\in I\times[0,2\pi]} \left| \frac{\partial^2}{\partial x^2} X(x,t) \right| \le K.$$

Therefore, if

$$2M^2KS < 1,$$

there exists a 2π-periodic solution $x^*(t)$ of $x' = X(x,t)$ satisfying

$$||x^* - \bar{x}||_\infty \le 2MS,$$

and it is the unique periodic solution of the equation entirely contained in this strip. If in addition

$$\left| \int_0^{2\pi} \frac{\partial}{\partial x} X(\bar{x}(t), t)\, dt \right| > \frac{2\pi}{M},$$

then the periodic orbit $x^*(t)$ is hyperbolic, and its stability is given by the sign of this integral.

The above result is applied to two examples of planar rigid systems to localize and give an approximated expression of their corresponding limit cycles.

In the last two chapters of this thesis we focus on the study of one-parameter families of planar differential equations, in order to give as much information as possible about their bifurcation diagrams. The main difference between these families is that one of them is a semi-complete family of rotated vector field (SCFRVF) and the other one is not.

As is well-known, if a one-parameter family of differential systems is a SCFRVF, then there are many results that allow to control the possible bifurcations; see

[33, 87, 83]. One of the most useful ones is the so-called *non-intersection property.* It asserts that if γ_1 and γ_2 are limit cycles corresponding to systems with different values of the parameter, then $\gamma_1 \cap \gamma_2 = \emptyset$. As a consequence, the study of one-parameter bifurcation diagrams is much more simple in this case.

More specifically, in Chapter 4 we study the number of limit cycles and the bifurcation diagram in the Poincaré sphere of the one-parameter family of rotated vector fields of degree five

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + (b^2 - x^2)(y + y^3), \quad b \in \mathbb{R}^+ \cup \{0\}, \end{cases} \tag{2}$$

which is reminiscent of the celebrated van der Pol system. In previous papers [59, 102] it was proved that the family can have limit cycles if and only if $b \in (0, b^*)$ and $b^* \in (0, \sqrt[6]{9\pi^2/16}) \approx (0, 1.33)$. Moreover it was shown that when they exist they are unique and hyperbolic. By using numerical methods it is not difficult to get that $b^* \approx 0.80629$. Nevertheless, as far as we know there are no analytical tools to obtain the value b^*. This is the main goal of this chapter. By using a rational Dulac function we provide an interval of length 27/1000 where b^* lies. To our knowledge the tools used to determine this interval are new and are based on the construction of algebraic curves without contact by the flow of the differential equation. These curves are obtained using analytic information about the separatrices of the infinite critical points of the vector field. Moreover, during our study we have also realized that there is a bifurcation value not obtained in the previous studies. Our main result is:

Theorem 4.1. *Consider system* (2)*. There exist two positive numbers* $\hat{b}$ *and* b^* *such that:*

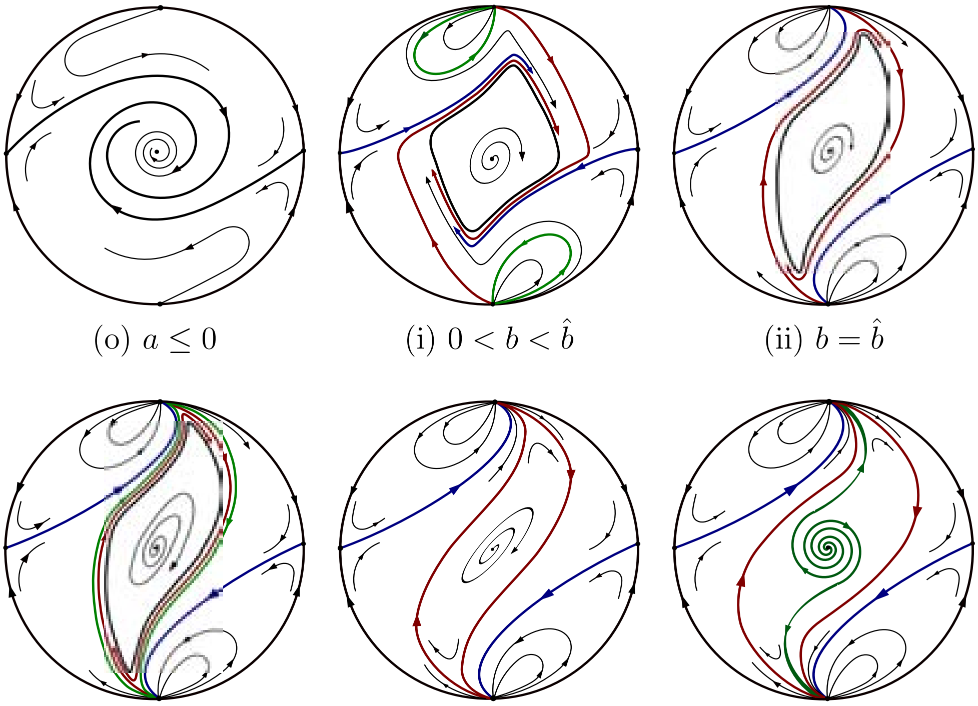
(*a*) *It has a limit cycle if and only if* $0 < b < b^*$*. Moreover, when it exists, it is unique, hyperbolic and stable.*

(*b*) *The only bifurcation values of the system are 0,* $\hat{b}$ *and* b^**. In consequence there are exactly six different global phase portraits on the Poincaré disc, which are the ones displayed in Figure 1.*

(*c*) *It holds that* $0.79 < \hat{b} < b^* < 0.817$.

The phase portraits missing in [102] are (ii) and (iii) of Figure 1.

In order to prove that the Bendixson–Dulac Theorem applies, we develop a method for studying whether one-parameter families of polynomials in two variables do not vanish. That method is based on the computation of the so-called double discriminant ($\triangle^2$) and on the concept of *uniformly isolated points*, see Appendix II of Chapter 4 for more details. Our result reads as follows.



(o) $a \leq 0$ (i) $0 < b < \hat{b}$ (ii) $b = \hat{b}$

(iii) $\hat{b} < b < b^*$ (iv) $b = b^*$ (v) $b > b^*$

Figure 1: Phase portraits of system (2).

Proposition 4.25. *Let $F_b(x,y)$ be a family of real polynomials depending also polynomially on a real parameter b and let $\Omega \subset \mathbb{R}^2$ be an open connected subset having a boundary $\partial\Omega$ formed by finitely many algebraic curves. Suppose that there exists an open interval $I \subset \mathbb{R}$ such that:*

(i) For some $b_0 \in I$, $F_{b_0}(x,y) > 0$ on Ω.

(ii) For all $b \in I$, $\triangle^2(F_b) \neq 0$.

(iii) For all $b \in I$, all points of $F_b = 0$ at infinity which are also in Ω do not depend on b and are uniformly isolated.

(iv) For all $b \in I$, $\{F_b = 0\} \cap \partial\Omega = \emptyset$.

Then for all $b \in I$, $F_b(x,y) > 0$ on Ω.

In Chapter 5 we consider the one-parameter family of planar quintic systems,

$$\begin{cases} \dot{x} = y^3 - x^3, \\ \dot{y} = -x + my^5, \qquad m \in \mathbb{R}, \end{cases} \tag{3}$$

proposed by A. Bacciotti, during a conference about the stability of analytic dynamical systems held in Florence in 1985. Two years later, Galeotti and Gori in [40] published an extensive study of this family that we complete in this work. A main difficulty in considering such a family is that it is not a semi-complete family of rotated vector fields.

Theorem 5.1. *Consider the family of systems* (3).

(i) It has neither periodic orbits, nor polycycles, when $m \in (-\infty, 0.547] \cup [0.6, \infty)$. *Otherwise, it has at most one periodic orbit or one polycycle, but can not coexist. Moreover, when the limit cycle exists, it is hyperbolic and unstable.*

(ii) For $m > 0$, *their phase portraits on the Poincaré disc, are given in Figure 2.*

(iii) Let $\mathcal{M}$ *be the set of values of* m *for which it has a heteroclinic polycycle. Then* $\mathcal{M}$ *is finite, non-empty and it is contained in* $(0.547, 0.6)$. *Moreover, the system corresponding to* $m \in \mathcal{M}$ *has no limit cycles and its phase portrait is given by Figure 2* (b).

Our simulations show that (a), (b) and (c) of Figure 2 occur when $m \in (0, m^*)$, $m = m^*$ and $m > m^*$, respectively, for some $m^* \in (0.547, 0.6)$, which numerically we have found to be $m^* \approx 0.560115$. We have not been able to prove the existence of this special value m^*, because of our system is not a SCFRVF and this fact hinders the obtention of the full bifurcation diagram. From our analysis, we know the existence of finitely many values m_j^*, $j = 1, \ldots, k$, where $k \geq 1$, satisfying $0.547 < m_1^* < m_2^* < \cdots m_k^* < 0.6$, such that phase portrait (b) only occurs for these values. Moreover, for $m \in (0.547, m_1^*)$, phase portrait (a) holds, for $m \in (m_k^*, 0.6)$ phase portrait (c) holds, and for each one of the remaining $k-1$ intervals, the phase portrait does not vary on each interval and is either (a) or (c).

Furthermore, we answer an open question left in [40] about the change of stability of the origin for an extension of system (3). In short, we prove

Theorem 5.3. *Consider the system*

$$\begin{cases} \dot{x} = y^3 - x^{2k+1}, \\ \dot{y} = -x + my^{2s+1}, \qquad m \in \mathbb{R} \quad \textit{and} \quad k, s \in \mathbb{N}^+. \end{cases}$$

(i) When $s < 2k$, *the origin is an attractor for* $m \leq 0$ *and a repeller for* $m > 0$.

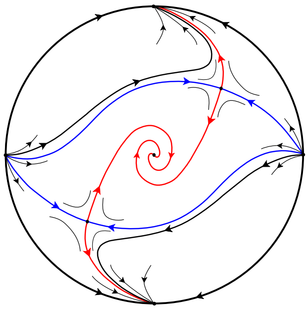
(ii) When $s > 2k$, *the origin is always an attractor.*

(iii) When $s = 2k$, *the origin is an attractor for*

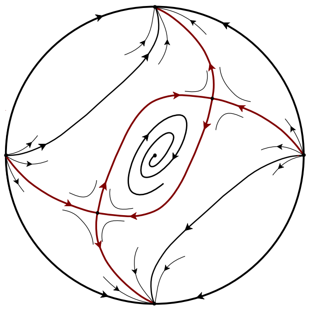
$$m < \frac{(2k+1)!!}{(4k+1)!!!!}$$

and a repeller when the reverse inequality holds. Moreover, when $k = 1$ and $m = 3/5$ the origin is a repeller and for $m \lesssim 3/5$ the system has at least one limit cycle near the origin.

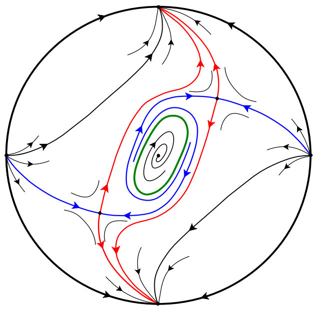
In [40], the authors gave the stability of the origin when $s \neq 2k$ and ask whether it is true or not that the change of stability of the origin when $s = 2k$ is at the value $m = (2k+1)/(4k+1)$. We can see that their guess was only correct for $k = 1$.



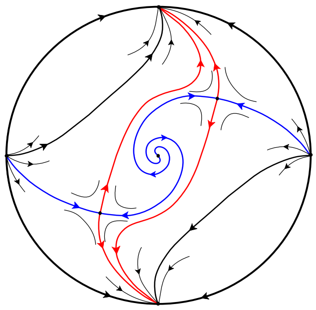
(a) When $m \in (0, 0.547]$, or when $m \in (0.547, 0.6)$ and neither the polycycle nor the limit cycle exist.



(b) When $m \in (0.547, 0.6)$ and the polycycle exists.



(c) When $m \in (0.547, 0.6)$ and the limit cycle exists.



(d) For $m \in [0.6, \infty)$

Figure 2: Phase portraits of system (3).

From our point of view, to introduce tools for studying one-parameter families that are not SCFRVF is a challenge for the differential equations community. Further topics considered and developed in last two chapters are the construction of Dulac functions for applying a generalization of the Bendixson–Dulac Theorem in order to prove existence, non-existence, uniqueness and hyperbolicity of the limit cycles, and the determination of explicit lower bounds of the basin of attraction of attracting critical points. We believe that the tools that we have introduced in this part of the work can be applied to other families of polynomial vector fields to provide an analytic control of the bifurcation values for them.

The results of Chapter 1 and 2 have been submitted for publication see [42] and [43], respectively. The results of Chapter 3 are already published in “Journal of Differential Equations” [41]. These three papers have been written in collaboration with Armengol Gasull.

The results of Chapter 4 have been accepted for publication in “Discrete and Continuous Dynamical Systems” [44] and the ones of Chapter 5 have already been published in “Journal of Mathematical Analysis and Applications” [45]. The results of both chapters have been obtained in joint works with Armengol Gasull and Hector Giacomini.

Part I

The Harmonic Balance Method (HBM)

Chapter 1

The period function of potential systems and the HBM

1.1 Introduction and main results

Given a planar differential system having a continuum of periodic orbits, its period function is defined as the function that associates to each periodic orbit its period. To determine the global behavior of this period function is an interesting problem in the qualitative theory of differential equations either as a theoretical question or due to its appearance in many situations. For instance, the period function is present in mathematical models in physics or ecology, see [28, 91, 101] and the references therein; in the study of some bifurcations [23, pp. 369-370]; or to know the number of solutions of some associated boundary value problems, see [19, 20].

In particular, there are several works giving criteria for determining the monotonicity of the period function associated with some systems, see [19, 39, 54, 93, 105] and the references therein. Results about non monotonous period functions have also recently appeared, see for instance [46, 55, 71].

The so-called N-th order Harmonic Balance Method (HBM) consists on approximating the periodic solutions of a nonlinear differential equation by using truncated Fourier series of order N. It is mainly applied with practical purposes, although in many cases there is no a theoretical justification. In most of the applications this method is used to approach isolated periodic solutions, see for instance [41, 61, 74, 78, 76, 75]. Since the HBM also provides an approximation of the angular frequency of the searched periodic solution, it can be also used to get its period.

Hence, applying the HBM to systems of differential equations having a continuum of periodic orbits we can obtain approximations of the corresponding period functions. The main goal of this chapter is to illustrate this last assertion through the study of several concrete planar systems. This approach is also used for instance in [9, 10, 79]. A main difference among these papers and our work is that we also carry out a detailed analytic study of the involved period functions.

More specifically, in this chapter we will consider several families of planar potential systems, $\ddot{x} = f(x)$, having continua of periodic orbits. We will study analytically their corresponding period functions and we will see that the approximations of the period functions obtained using the N-th order HBM, for $N = 1$, keep the essential properties of the actual period functions: local behavior near the critical point and infinity, monotonicity, oscillations,... For the case of the Duffing oscillator we also consider $N = 2$ and 3. In particular, the method that we introduce using resultants gives an analytic way to deal with the 3rd order HBM, answering question (iv) in [79, p. 180].

First, we focus in the following two families of potential differential systems:

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + x^{2m-1}, \qquad m \in \mathbb{N} \quad \text{and} \quad m \geq 2, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\frac{x}{(x^2+k^2)^m}, \qquad k \in \mathbb{R} \setminus \{0\}, \ m \in [1, \infty). \end{cases} \tag{1.2}$$

Each system of these families has a continuum of periodic orbits around the origin. Thus, we can talk about its period function T which associates to each periodic orbit passing through $(x, y) = (A, 0)$ its period $T(A)$. In addition, we will denote by $T_N(A)$ the approximation to $T(A)$ by using N-th order HBM; see Section 1.3 for the precise definition of $T_N(A)$.

System (1.1) is an extension of the Duffing-harmonic oscillator which corresponds to the case $m = 2$. The case $m = 2$ has been studied by many authors, see [61, 67, 76, 80]. The exact period function of this particular system is given as an elliptic function and so it is easier to obtain analytic properties of T. Our analytic study is valid for all integers $m \geq 2$.

System (1.2) with $m = 1$ and by taking the limit $k \to 0$ is equivalent to the second order differential equation $x\ddot{x} + 1 = 0$, which is studied in [78] as a model of plasma physics. Thus, system (1.2) can be seen as an extension of the singular second order differential equation $x\ddot{x} + 1 = 0$. In the next chapter we explore the relationship between the periodic solutions of (1.2) with $m = 1$ and their corresponding periods with the solutions of the limiting case $x\ddot{x} + 1 = 0$.

We have chosen these two families due to their simplicity and because, as we will see, their corresponding period functions are monotonous, being the first one decreasing and the second one increasing.

For the first family (1.1), in addition to the monotonicity of T, we perform a more detailed study of some properties of T. More precisely, we give the behavior of

T near to the origin and at infinity and we compare them with the results obtained by using the HBM.

Theorem 1.1. *System* (1.1) *has a global center at the origin and its period function T is decreasing. Moreover, at $A = 0$,*

$$T(A) = 2\pi\left(1 - \frac{(2m-1)!!}{(2m)!!}A^{2m-2} + S(m)\,A^{4m-4} + O(A^{6m-6})\right), \tag{1.3}$$

where $S(m) = \frac{(2m-1)(4m-1)!!}{m(4m)!!} - \frac{(m-1)(2m-1)!!}{m(2m)!!}$; and

$$T(A) \sim B\left(\frac{1}{2m}, \frac{1}{2}\right)\frac{2}{\sqrt{m}\,A^{m-1}}, \quad A \to \infty, \tag{1.4}$$

where $B(\cdot,\cdot)$ is the Beta function.

Proposition 1.2. *By applying the first-order HBM to system* (1.1) *we get the decreasing function*

$$T_1(A) = \frac{2^m\pi}{\sqrt{\frac{(2m-1)!}{(m-1)!m!}A^{2m-2} + 2^{2m-2}}}. \tag{1.5}$$

Moreover, at $A = 0$,

$$T_1(A) = 2\pi\left(1 - \frac{(2m-1)!!}{(2m)!!}A^{2m-2} + \frac{3}{2}\left(\frac{(2m-1)!!}{(2m)!!}\right)^2 A^{4m-4} + O(A^{6m-6})\right),$$

and

$$T_1(A) \sim \frac{2^m\pi}{\sqrt{\frac{(2m-1)!}{(m-1)!m!}}\,A^{m-1}}, \quad A \to \infty. \tag{1.6}$$

By Theorem 1.1, we know that the period function T of system (1.1) is decreasing. Proposition 1.2 asserts that this property is already present in its first order approximation obtained with the HBM. Additionally, we can see that the first and second terms of the Taylor series at $A = 0$ of $T(A)$ and $T_1(A)$ coincide, while the third one is different. Furthermore, from (1.4) and (1.6) it follows that $T(A)$ and $T_1(A)$ have similar behaviors at infinity.

In the case of the Duffing-harmonic oscillator ($m = 2$ in (1.1)) we will apply the N-th order HBM, $N = 2, 3$, for computing the approximations $T_N(A)$ of the period function $T(A)$, see Section 1.6. We prove that $T(A) - T_N(A) = O(A^{2N+4})$ at $A = 0$. We believe that similar results hold for (1.1) with $m > 2$, nevertheless, we do not study this question in this work. We also will see that the approximations $T_N(A)$, $N = 1, 2, 3$, at infinity become sharper by increasing N.

For the family (1.2) we have similar results. We only will deal with the global behaviors of T and T_1 skipping the study of these functions near zero and infinity.

Theorem 1.3. *System* (1.2) *has a center at the origin and its period function T is increasing. Moreover, the center is global for $m = 1$ and non-global otherwise.*

Proposition 1.4. *By applying the first-order HBM to system* (1.2) *we obtain the increasing function*

$$T_1(A) = 2\pi\sqrt{\sum_{j=0}^{m}\left(\frac{1}{2}\right)^{2j}\binom{m}{j}\binom{2j+1}{j}k^{2(m-j)}A^{2j}}. \tag{1.7}$$

Note that again, as in system (1.1), with the first-order HBM we obtain that $T_1(A)$ and $T(A)$ have the same monotonicity behavior.

In Section 1.7 we consider the family of polynomial potential systems

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + k\,x^3 + x^5, \quad k \in \mathbb{R}, \end{cases} \tag{1.8}$$

which for some values of k has a global center. In [71, Thm. 1.1 (b)] it is proved that the period function associated to the global center at the origin has at most one oscillation. Joining this result with a similar study that the one made for system (1.1) at the origin and at infinity, we obtain:

Theorem 1.5. *Consider system* (1.8) *with $k \in (-2, \infty)$. Let T be the period function associated to the origin, which is a global center. Then:*

(*i*) *The function T is monotonous decreasing for $k \geq 0$.*

(*ii*) *The function T starts increasing, until a maximum (a critical period) and then decreases towards zero, for $k \in (-2, 0)$.*

(*iii*) *At the origin*

$$T(A) = 2\pi - \frac{3}{4}k\pi A^2 + \frac{57k^2 - 80}{128}\pi A^4 + O(A^6),$$

and at infinity

$$T(A) \sim \frac{2B(\frac{1}{6}, \frac{1}{2})}{\sqrt{3}}\frac{1}{A^2} \approx \frac{8.4131}{A^2}.$$

We prove:

Proposition 1.6. *By applying the first-order HBM to the family* (1.8) *we get:*

$$T_1(A) = \frac{8\pi}{\sqrt{16 + 12kA^2 + 10A^4}}.$$

In particular,

(*i*) *The function* $T_1(A)$ *is decreasing for* $k \geq 0$.

(*ii*) *The function* $T_1(A)$ *starts increasing, has a maximum and then decreases towards zero, for* $k \in (-2, 0)$.

(*iii*) *At the origin*

$$T_1(A) = 2\pi - \frac{3}{4}k\pi A^2 + \frac{54k^2 - 80}{128}\pi A^4 + O(A^6),$$

and at infinity

$$T_1(A) \sim \frac{4\pi\sqrt{10}}{5}\frac{1}{A^2} \approx \frac{7.9477}{A^2}.$$

Once more, we can see that the function $T_1(A)$ obtained by applying the first order HBM captures and reproduces quite well the actual behavior of $T(A)$.

Remark 1.7. *In fact, the shape of the function* $T_1(A)$ *for* $k \in (-2, 0)$ *does not vary until* $k = -2\sqrt{10}/3 \approx -2.107$. *For* $k \leq -2\sqrt{10}/3$, *it is no more defined for all* $A \in \mathbb{R}$. *Somehow, this phenomenon reflects the fact that for* $k \leq -2$ *the center is not global. Notice, that for* $k < -2$, *system* (1.8) *has three centers.*

Motivated by all our results, in Section 1.8 we study the relationship between the Taylor series of $T(A)$ and $T_N(A)$ with $N = 1, 2$ at $A = 0$ for an arbitrary smooth potential.

When the system has a center and its period function is constant, then the center is called *isochronous.* The problem about the existence and characterization of isochronous center has also been extensively studied, see [24, 25, 26, 58, 72]. To end this introduction we want to comment that we have not succeeded in applying the HBM to detect isochronous potentials. We have unfold in 1-parameter families one of the simplest potential isochronous systems, the one given by a rational potential function, see [14]. Our attempts to use the low order HBM to detect the value of the parameter that corresponds to the isochronous case have not succeed.

The chapter is organized as follows. In Section 1.2 we give some preliminary results which include a known result for studying the monotonicity of the period function. Next, we describe the N-th order HBM. In Section 1.4 we prove our analytical results about the monotonicity of the period function of systems (1.1) and (1.2) and their local behavior at the origin and at infinity, see Theorems 1.1 and 1.3. In Section 1.5 we prove Propositions 1.2 and 1.4, both dealing with the HBM. In Section 1.6 we focus on the study of the Duffing-harmonic oscillator and we also apply the 2-th order and 3-rd order HBM. Section 1.7 deals with the family of planar polynomial potential systems having a non-monotonous period function. Finally, Section 1.8 studies the local behavior near zero of $T(A)$ and $T_N(A)$ with $N = 1, 2$, of an arbitrary smooth potential system.

1.2 Preliminary results

This section is divided in two parts. The first one is devoted to recall some definitions, as well as, to give the framework for the study of the period function of (1.1) and (1.2) from an analytical point of view. In the second one we will give the description of the N-order Harmonic Balance Method, which we will apply in our second analysis of the period function.

Definitions and some analytical tools

The systems studied in this chapter are all potential systems,

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = F'(x), \end{cases} \tag{1.9}$$

with associated Hamiltonian function $H(x,y) = y^2/2 + F(x)$, where $F : \Omega \subset \mathbb{R} \to \mathbb{R}$ is a real smooth function, $F(0) = 0$ and $0 \in \Omega$, an open real interval.

Let p_0 be a singular point of (1.9). It is said that p_0 is a *center* if there exists an open neighborhood U of p_0 such that each solution $\gamma(t)$ of (1.9) with $\gamma(0) \in U - \{p_0\}$ defines a periodic orbit γ surrounding p_0. The largest neighborhood $\mathcal{P}$ with this property is called the *period annulus* of p_0. If $\Omega = \mathbb{R}$ and $\mathcal{P} = \mathbb{R}^2$, then p_0 is called a *global center*.

The following result characterizes systems (1.9) having global centers.

Lemma 1.8. *If $F(x)$ has a minimum at* 0*, then system* (1.9) *has a center at the origin. Moreover, the center is global if and only if $F'(x) \neq 0$ for all $x \neq 0$ and $F(x)$ tends to infinity when $|x|$ does.*

Suppose that (1.9) has a center with period annulus $\mathcal{P}$. For each periodic orbit $\gamma \in \mathcal{P}$ we define $T(\gamma)$ to be the period of γ. Thus, the map

$$T : \mathcal{P} \to \mathbb{R}_+, \qquad \gamma \mapsto T(\gamma),$$

is called the *period function* associated with $\mathcal{P}$. It is said that the map T is *monotone increasing* (respectively *monotone decreasing*) if for each couple of periodic orbits γ_0 and γ_1 in $\mathcal{P}$, with γ_0 in the interior of bounded region surrounded by γ_1, it holds that $T(\gamma_1) - T(\gamma_0) > 0$ (respectively < 0). When T is constant, then the center is called *isochronous center*.

If we fix a transversal section Σ to $\mathcal{P}$ and we take a parametrization $\sigma(A)$ of Σ with $A \in (0, A^*) \subset \mathbb{R}_+$, then we can denote by γ_A the periodic orbit passing through $\sigma(A)$ and by $T(A)$ its period. That is, we have the map $T : (0, A^*) \to \mathbb{R}_+$, $A \mapsto T(A)$. When T is not monotonous then either it is constant or it has local maxima or minima. The isolated zeros of $T'(A)$ are called *critical periods*. It is not difficult to prove that the number of critical periods does not depend neither of Σ nor of its parametrization.

Next, we will recall two results about some properties of the period function T which we will apply in our study of the families (1.1), (1.2) and (1.8). The first result is an adapted version to system (1.9), of statement 3 of [39, Prop. 10] and gives a criterion about the monotonicity of T. The second one is an adapted version of [25, Thm. C], which will allow us to describe the behavior of T at infinity.

Proposition 1.9. *Suppose that system* (1.9) *has a center at the origin. Let T be the period function associated to the period annulus of the center. Then*

(*i*) *If $F'(x)^2 - 2F(x)F''(x) \geq 0$ (not identically 0) on Ω, then T is increasing.*

(*ii*) *If $F'(x)^2 - 2F(x)F''(x) \leq 0$ (not identically 0) on Ω, then T is decreasing.*

To state the second result, we need some previous constructions and definitions.

Let $\gamma_h(t) = (x_h(t), y_h(t))$ be a periodic orbit of (1.9) contained in $\mathcal{P}$ corresponding to the level set $\{H = h\}$. This orbit crosses the axis $y = 0$ at the points determined by $F(x_h(t)) = h$. Since F has a minimum at $x = 0$, near the origin the above equation has two solutions, one of them on $x > 0$ which will be denoted by $F_+^{-1}(h)$ and the other one on $x < 0$ which will be denoted by $F_-^{-1}(h)$. We note that this property remains for all $h \in (0, h^*) := H(\mathcal{P}) \setminus \{0\}$. For each $h > 0$ we define the function

$$l_F(h) = F_+^{-1}(h) - F_-^{-1}(h) \tag{1.10}$$

which gives the length of the projection to the x-axis of γ_h. See Figure 1.1.

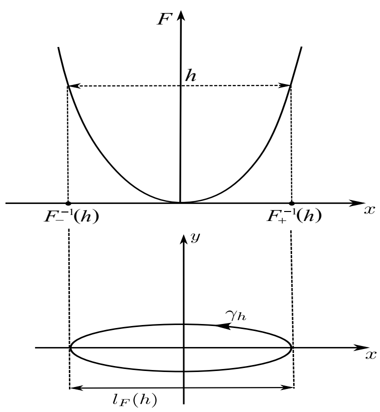


Figure 1.1: Definition of l_F for system (1.9).

Definition 1.10. *Given two real numbers a and M, it is said that a continuous function $g(x)$ has Mx^a as* **dominant term** *of its asymptotic expansion at $x = x_0 \in \mathbb{R} \cup \{\infty\}$ if*

$$\lim_{x \to x_0} \frac{g(x) - Mx^a}{x^a} = 0.$$

This property is denoted by $g(x) \sim Mx^a$ at $x = x_0$.

Theorem 1.11. *Assume that (1.9) has a global center at the origin. Let $l_F(h)$ be as in (1.10) and suppose that $l_F'(h) \sim Mh^a$, at $h = \infty$ with $a > -1$ and $M > 0$. Then, the period function of (1.9) satisfies $T(h) \sim Ch^{a+1/2}$ at $h = \infty$, where $C = \sqrt{2}MB(a+1, 3/2)$, and $B(\cdot,\cdot)$ is the Beta function.*

Next lemma computes the function $l_F(h)$ for system (1.1).

Lemma 1.12. *The function $l_F(h)$ associated to (1.1) satisfies that $l_F(h) \sim 2(2mh)^{\frac{1}{2m}}$ at $h = \infty$.*

Proof. We start studying the algebraic curve $\mathcal{C} := \{p(x,h) = F(x) - h = 0\}$ at infinity. For that, we consider the homogenization

$$P(X, H, Z) = X^{2m} + mX^2Z^{2m-2} - 2mHZ^{2m-1}, \tag{1.11}$$

of $p(x,h)$ in the real projective plane $\mathbb{RP}^2$. From (1.11) it follows that $[0 : 1 : 0]$ is the unique point at infinity of $\mathcal{C}$, and $\mathcal{C}$ in the chart that contains such point is given by the set of zeros of the polynomial

$$\tilde{p}(\tilde{x}, \tilde{z}) := P(\tilde{x}, 1, \tilde{z}) = \tilde{x}^{2m} + m\tilde{x}^2z^{2m-2} - 2m\tilde{z}^{2m-1}.$$

For studying $\mathcal{C}$ at infinity we will obtain a parametrization of it close to the point $[0 : 1 : 0]$. As usual, we will use the Newton polygon associated to $\tilde{p}$. The *carrier* of $\tilde{p}$ is $\operatorname{carr}(\tilde{p}) = \{(2m, 0), (2, 2m-2), (0, 2m-1)\}$, whence the Newton polygon is the straight line joining $(2m, 0)$ and $(0, 2m-1)$ whose equation is

$$\tilde{z} = -\left(\frac{2m-1}{2m}\right)\tilde{x} + 2m - 1.$$

In $\tilde{p}(\tilde{x}, \tilde{z})$ we replace $\tilde{z} = z_0t^{2m}$ and $\tilde{x} = x_0t^{2m-1}$, then

$$\tilde{p}(x_0t^{2m-1}, z_0t^{2m}) = mx_0^2z_0^{2(m-1)}t^{2(2m^2-1)} + (x_0^{2m} - 2mz_0^{2m-1})t^{2m(2m-1)}.$$

For x_0 fixed we consider

$$\begin{aligned} \phi(z_0, t) &= (x_0^{2m} - 2mz_0^{2m-1})t^{2m(2m-1)} + mx_0^2z_0^{2(m-1)}t^{2(2m^2-1)} \\ &= t^{2m(2m-1)}\tilde{\phi}(z_0, t), \end{aligned}$$

where $\tilde{\phi}(z_0, t) = x_0^{2m} - 2mz_0^{2m-1} + mx_0^2 z_0^{2(m-1)} t^{2(m-1)}$. It is clear that $\tilde{\phi}(z_0^*, 0) = 0$ for z_0^* solution of $x_0^{2m} - 2mz_0^{2m-1} = 0$. Moreover

$$\frac{\partial \tilde{\phi}}{\partial z_0}(z_0^*, 0) = -2m(2m-1)(z_0^*)^{2(m-1)} \neq 0.$$

From the implicit function theorem there exists a function $z_0(t) : (\mathbb{R}, 0) \to \mathbb{R}$ such that $z_0(0) = z_0^*$ and $\phi(z_0(t), t) = 0$ for $t \in (\mathbb{R}, 0)$. Since $\phi(z_0(t), t)$ is an analytic function, $z_0(t)$ also is it. Hence we can write $z_0(t) = c_0 + c_1 t + c_2 t^2 + \ldots$, moreover as $z_0(0) = z_0^*$ then

$$z_0(t) = z_0^* + O(t).$$

From $x_0^{2m} - 2mz_0^{2m-1} = 0$ and the above equation it follows that

$$x_0(t) = \pm((2m)^{\frac{1}{2m}} (z_0^*)^{\frac{2m-1}{2m}} + O(t^{\frac{2m-1}{2m}})).$$

Then the parametrization of $\mathcal{C}$ is $t \mapsto (\tilde{x}(t), \tilde{z}(t))$ where

$$\tilde{x}(t) = t^{2m-1} x_0(t) = (2m)^{\frac{1}{2m}} (z_0^*)^{\frac{2m-1}{2m}} t^{2m-1} + O(t^{\frac{4m^2-1}{2m}}) \tag{1.12}$$

$$\tilde{z}(t) = t^{2m} z_0(t) = z_0^* t^{2m} + O(t^{2m+1}). \tag{1.13}$$

Recall that the relation between $(\tilde{x}, \tilde{z})$ and (x, h) is given by $\tilde{x} = x/h$ and $\tilde{z} = 1/h$. From (1.13) it follows that $1/h \sim z_0^* t^{2m}$ at $h = \infty$. Using this behavior and (1.12) we get $x \sim \pm(2mh)^{\frac{1}{2m}}$ at $h = \infty$. Hence $F_\pm^{-1}(h) \sim (2mh)^{\frac{1}{2m}}$ and from (1.10) it follows that $l_F(h) \sim 2(2mh)^{\frac{1}{2m}}$. □

1.3 Description of the HBM

In this section we recall the N-th order HBM adapted to our setting. Consider the second order differential equation

$$\ddot{x} = f(x, \alpha), \quad \alpha \in \mathbb{R}.$$

Suppose that it has a T-periodic solution $x(t)$ such that $x(0) = A$ and $\dot{x}(0) = 0$. This T-periodic function $x(t)$ satisfies the functional equation

$$\mathcal{F} := \mathcal{F}(x(t), \ddot{x}(t), \alpha) = \ddot{x}(t) - f(x(t), \alpha) = 0.$$

On the other hand, $x(t)$ has the Fourier series:

$$x(t) = \frac{\tilde{a}_0}{2} + \sum_{k=1}^{\infty} \left(\tilde{a}_k \cos(k\omega t) + \tilde{b}_k \sin(k\omega t) \right),$$

where $\omega := 2\pi/T$ is the angular frequency of $x(t)$ and the coefficients $\tilde{a}_k$ and $\tilde{b}_k$ are the so-called Fourier coefficients, which are defined as

$$\tilde{a}_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega t)\,dt \quad \text{and} \quad \tilde{b}_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega t)\,dt \quad \text{for } k \geq 0.$$

Although we not write explicitly, $\tilde{a}_k$, $\tilde{b}_k$, and ω depend on α and A, that is, $\tilde{a}_k := \tilde{a}_k(\alpha, A)$, $\tilde{b}_k := \tilde{b}_k(\alpha, A)$, and $\omega := \omega(\alpha, A)$. Hence it is natural to try to approximate the periodic solutions of the functional equation $\mathcal{F} = 0$ by using truncated Fourier series of order N, *i.e.* trigonometric polynomials of degree N.

The N-th order HBM consists of the following four steps.

1. Consider a trigonometric polynomial

$$x_N(t) = \frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k\cos(k\omega_N t) + b_k\sin(k\omega_N t)\right). \tag{1.14}$$

2. Compute the T-periodic function $\mathcal{F}_N := \mathcal{F}(x_N(t), \ddot{x}_N(t))$, which has also an associated Fourier series, that is,

$$\mathcal{F}_N = \frac{\mathcal{A}_0}{2} + \sum_{k=1}^{\infty} \left(\mathcal{A}_k\cos(k\omega_N t) + \mathcal{B}_k\sin(k\omega_N t)\right),$$

where $\mathcal{A}_k = \mathcal{A}_k(\mathbf{a}, \mathbf{b}, \omega)$ and $\mathcal{B}_k = \mathcal{B}_k(\mathbf{a}, \mathbf{b}, \omega)$, $k \geq 0$, with $\mathbf{a} = (a_0, a_1, \ldots, a_N)$ and $\mathbf{b} = (b_1, \ldots, b_N)$.

3. Find values $\mathbf{a}$, $\mathbf{b}$, and ω such that

$$\mathcal{A}_k(\mathbf{a}, \mathbf{b}, \omega) = 0 \quad \text{and} \quad \mathcal{B}_k(\mathbf{a}, \mathbf{b}, \omega) = 0 \quad \text{for} \quad 0 \leq k \leq N. \tag{1.15}$$

4. Then the expression (1.14), with the values of $\mathbf{a}$, $\mathbf{b}$, and ω obtained in point 3, provides candidates to be approximations of the actual periodic solutions of the initial differential equation. In particular the values $2\pi/\omega$ give approximations of the periods of the corresponding periodic orbits.

We end this short explanation about HBM with several comments:

(a) The above set of equations (1.15) is a system of polynomial equations which usually is very difficult to solve. For this reason in many works, see for instance [78, 79] and the references therein, only small values of N are considered. We also remark that in general the coefficients of $x_N(t)$ and $x_{N+1}(t)$ do not coincide at all. Hence, going from order N to order $N+1$ in the method, implies to compute again all the coefficients of the Fourier polynomial.

(b) The equations $\mathcal{A}_k(\mathbf{a}, \mathbf{b}, \omega) = 0$ and $\mathcal{B}_k(\mathbf{a}, \mathbf{b}, \omega) = 0$ for $0 \leq k \leq N$ are equivalent to

$$\frac{2}{T}\int_0^T \mathcal{F}_N\cos(k\omega t)\,dt = 0 \quad \text{and} \quad \frac{2}{T}\int_0^T \mathcal{F}_N\sin(k\omega t)\,dt = 0 \quad \text{for} \quad 0 \leq k \leq N.$$

(c) The linear combination, $a_k \cos(k\omega t) + b_k \sin(k\omega t)$, of the harmonics of order k, with $k = 0, 1, \dots N$, can be expressed as

$$a_k \cos(k\omega t) + b_k \sin(k\omega t) = \frac{\bar{c}_k\, e^{ik\omega t} + c_k\, e^{-ik\omega t}}{2},$$

where $c_k = a_k + ib_k$, and $\bar{c}_k$ is the complex conjugated of c_k. Therefore, we can use the HBM with the last notation, because the truncated Fourier series can be written as

$$x_N(t) = \sum_{k=1}^{N} \frac{\bar{c}_k\, e^{ik\omega_N t} + c_k\, e^{-ik\omega_N t}}{2}, \quad k = 1, \dots, N. \tag{1.16}$$

(d) In general, although in many concrete applications HBM seems to give quite accurate results, it is not proved that the found Fourier polynomials are approximations of the actual periodic solutions of differential equation.

1.4 The period function from the analytical point of view

In this section we prove our main results concerning the period function of systems (1.1) and (1.2). For proving Theorem 1.1 we will apply Lemma 1.8 and Proposition 1.9 to determine the existence of a global center of (1.1) and the monotonicity of its period function. To find the Taylor series of T at the origin we will use an old idea, due to Cherkas([15]), which consists in transforming (1.1) into an Abel equation. Finally, in the last part of the proof, that corresponds to the behavior at infinity of T, we will use Theorem 1.11 and Lemma 1.12. Theorem 1.3 follows using similar tools.

Proof of Theorem 1.1. System (1.1) is of the form (1.9) with $F(x) = x^2/2 + x^{2m}/2m$. Clearly, by Lemma 1.8, the origin $(0, 0)$ is a global center. Moreover, the set $\{(A, 0) \in \mathbb{R}^2 \,|\, A > 0\}$ is a transversal section to $\mathcal{P}$. Thus, T can be expressed as function depending on the parameter A.

Some easy computations give that

$$F'(x)^2 - 2F(x)F''(x) = -(\tfrac{m-1}{m})((2m-1) + x^{2m-2})x^{2m} \leq 0.$$

Therefore, Proposition 1.9.(ii) implies that the period function $T(A)$ associated to $\mathcal{P}$ is decreasing for all m.

For obtaining the Taylor series of T at $A = 0$ we will consider system (1.1) in polar coordinates and initial condition $(A, 0)$, that is,

$$\begin{cases} \dot{R} = \sin(\theta)\cos^{2m-1}(\theta)R^{2m-1} \\ \dot{\theta} = 1 + \cos^{2m}(\theta)R^{2m-2}, \end{cases} \quad R(0) = A, \quad \theta(0) = 0, \tag{1.17}$$

which is equivalent to the differential equation

$$\frac{dR}{d\theta} = \frac{\sin(\theta)\cos^{2m-1}(\theta)R^{2m-1}}{1+\cos^{2m}(\theta)R^{2m-2}}, \quad R(0) = A.$$

By applying the Cherkas transformation [15]: $r = r(R;\theta) = \frac{R^{2m-2}}{1+\cos^{2m}(\theta)R^{2m-2}}$ to the previous equation, we obtain the Abel differential equation

$$\frac{dr}{d\theta} = P(\theta)r^3 + Q(\theta)r^2, \quad r(A;0) = \frac{A^{2m-2}}{1+A^{2m-2}}, \tag{1.18}$$

where $P(\theta) = (2-2m)\sin(\theta)\cos^{4m-1}(\theta)$ and $Q(\theta) = 2(2m-1)\sin(\theta)\cos^{2m-1}(\theta)$. Near the solution $r = 0$, the solutions of this Abel equation can be written as the power series

$$r(A;0) = \frac{A^{2m-2}}{1+A^{2m-2}} + \sum_{i=2}^{\infty} u_i(\theta)\left(\frac{A^{2m-2}}{1+A^{2m-2}}\right)^i \tag{1.19}$$

for some functions $u_i(\theta)$ such that $u_i(0) = 0$ which can be computed solving recursively linear differential equations obtained by replacing (1.19) in (1.18). For instance,

$$u_2(\theta) = \int_0^\theta Q(\psi)d\psi \qquad \text{and} \qquad u_3(\theta) = \int_0^\theta (P(\psi) + 2Q(\psi)u_2(\psi))d\psi.$$

From the expression of $\dot{\theta}$ in (1.17) and using variables (r,θ) again, we obtain

$$\begin{aligned} T(A) &= \int_0^{2\pi} \frac{d\theta}{1+\cos^{2m}(\theta)R^{2m-2}} = \int_0^{2\pi} (1-\cos^{2m}(\theta)\, r)d\theta = \\ &= 2\pi - \int_0^{2\pi} \cos^{2m}(\theta)\left(\frac{A^{2m-2}}{1+A^{2m-2}} + \sum_{i=2}^{\infty} u_i(\theta)\left(\frac{A^{2m-2}}{1+A^{2m-2}}\right)^i\right)d\theta. \end{aligned}$$

Then, we have

$$T(A) = 2\pi - \sum_{k\geq 1} \mathcal{S}_k \left(\frac{A^{2m-2}}{1+A^{2m-2}}\right)^k,$$

with

$$\mathcal{S}_1 = \int_0^{2\pi} \cos^{2m}(\theta)d\theta \quad \text{and} \quad \mathcal{S}_k = \int_0^{2\pi} \cos^{2m}(\theta)u_k(\theta)d\theta \quad \text{for} \quad k \geq 2.$$

It is easy to see that for $|A| < 1$,

$$\frac{A^{2m-2}}{1+A^{2m-2}} = A^{2m-2} - A^{4m-4} + O(A^{6m-6}), \left(\frac{A^{2m-2}}{1+A^{2m-2}}\right)^2 = A^{4m-4} + O(A^{6m-6}).$$

Thus,

$$T(A) = 2\pi - \mathcal{S}_1 A^{2m-2} - (\mathcal{S}_2 - \mathcal{S}_1)A^{4m-4} - O(A^{6m-6}).$$

Easy computations show that

$$\mathcal{S}_1 = 2\pi\frac{(2m-1)!!}{(2m)!!}, \qquad \mathcal{S}_2 = 2\pi\left(\frac{2m-1}{m}\right)\left(\frac{(2m-1)!!}{(2m)!!} - \frac{(4m-1)!!}{(4m)!!}\right),$$

where, given $n \in \mathbb{N}^+$, $n!!$ is defined recurrently as $n!! = n \times (n-2)!!$ with 1!!=1 and 2!!=2. Hence, introducing $S(m) = \mathcal{S}_2 - \mathcal{S}_1$ we obtain (1.3), as we wanted to prove.

Finally, for studying the behavior of T at infinity we will apply Theorem 1.11. By Lemma 1.12, we have that at $h = \infty$, $l_F(h) \sim 2(2mh)^{\frac{1}{2m}}$. Then

$$l_F'(h) \sim \frac{(2m)^{\frac{1}{2m}}}{m} h^{-\frac{2m-1}{2m}}.$$

If we denote by $\widetilde{T}(h)$ the period function of (1.1) in terms of h, then, from Theorem 1.11, it follows that $\widetilde{T}(h)$ at $h = \infty$ satisfies

$$\widetilde{T}(h) \sim B\left(\frac{1}{2m}, \frac{1}{2}\right) 2^{\frac{m+1}{2m}} m^{-\frac{2m-1}{2m}} h^{-\frac{m-1}{2m}}. \tag{1.20}$$

Now, using that $h = A^2/2 + A^{2m}/2m$, we get

$$h^{-\frac{m-1}{2m}} = \left(\frac{A^2}{2} + \frac{A^{2m}}{2m}\right)^{-\frac{m-1}{2m}} = A^{-(m-1)}\left(\frac{1}{2A^{2m-2}} + \frac{1}{2m}\right)^{-\frac{m-1}{2m}}$$

and we have

$$\lim_{A\to\infty} \frac{A^{-(m-1)}\left(\frac{1}{2A^{2m-2}} + \frac{1}{2m}\right)^{-\frac{m-1}{2m}} - (2m)^{\frac{m-1}{2m}} A^{-(m-1)}}{A^{-(m-1)}} = 0.$$

Hence $T(A) = \widetilde{T}(A^2/2 + A^{2m}/2m)$, and from previous equation and (1.20) we obtain

$$T(A) \sim B\left(\frac{1}{2m}, \frac{1}{2}\right) 2^{\frac{m+1}{2m}} m^{-\frac{2m-1}{2m}} (2m)^{\frac{m-1}{2m}} A^{-(m-1)},$$

which after a simplification reduces to (1.4). □

Proof of Theorem 1.3. By using the transformation $u = x/k$, $v = yk^{m-1}$, and the rescaling of time $\tau = -t/k^m$, system (1.2) becomes

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = \dfrac{x}{(x^2+1)^m}, \qquad m \in [1, \infty), \end{cases} \tag{1.21}$$

where we have reverted to the original notation (x,y) and t.

The associated Hamiltonian function to (1.21) is $H(x,y)=\frac{y^2}{2}+F(x)$ with

$$F(x)=\begin{cases} -\frac{1}{2}\ln(x^2+1), & \text{if } m=1,\\ -\frac{1}{2(m-1)(x^2+1)^{m-1}}+\frac{1}{2(m-1)}, & \text{if } m>1.\end{cases}$$

It is clear that for all m the function F is smooth at the origin and has a non-degenerate minimum. Thus, from Lemma 1.8, system (1.21) has a center at the origin with some period annulus $\mathcal{P}$.

From a straightforward computation we get

$$F'(x)^2-2F(x)F''(x)=\begin{cases} \frac{x^2+(x^2-1)\ln(x^2+1)}{(x^2+1)^2}, & \text{if } m=1,\\ \frac{1-mx^2+((2m-1)x^2-1)(x^2+1)^{m-1}}{(m-1)(x^2+1)^{2m}}, & \text{if } m>1.\end{cases}$$

To prove that the period function T associated to $\mathcal{P}$ is increasing we will apply Proposition 1.9.(i). Hence we need only to show that $F'(x)^2-2F(x)F''(x)\geq 0$. For $m=1$ it is clear. For $m>1$ the denominator of $F'(x)^2-2F(x)F''(x)$ is positive, then remains to prove that its numerator is positive.

By taking $w=x^2+1$, the numerator of $F'(x)^2-2F(x)F''(x)$ with $m>1$ is $(2m-1)w^m-2mw^{m-1}-mw+m+1$ or equivalently,

$$(w-1)^2\left((2m-1)w^{m-2}+(2m-2)w^{m-3}+\ldots+m+1\right),$$

which is clearly positive.

To finish the proof, we will discuss about the globality of the center. For $m=1$ the $(0,0)$ is a global minimum of H. Thus, (1.21) and therefore (1.2) have a global center at the origin. For $m>1$ the level curve

$$\mathcal{C}_{\frac{1}{2m-2}}=\left\{\frac{1}{2(m-1)(x^2+1)^{m-1}}-\frac{y^2}{2}=0\right\}$$

has two disjoin components. Indeed, it is formed by the graphics of the functions

$$y=\pm\frac{1}{\sqrt{(m-1)(x^2+1)^{m-1}}},$$

which are well-defined for all $x\in\mathbb{R}$ because $m>1$. This implies that the center at the origin of (1.21) is bounded by $\mathcal{C}_{\frac{1}{2m-2}}$ and therefore it is not global. The same happens with (1.2). □

1.5 The period function from the HBM point of view

In this section we prove Propositions 1.2 and 1.4.

Proof of Proposition 1.2. System (1.1) is equivalent to the second order differential equation $\ddot{x} + x + x^{2m-1} = 0$ with initial conditions $x(0) = A$, $\dot{x}(0) = 0$. For applying the HBM we consider the functional equation

$$\mathcal{F}(x(t), \ddot{x}(t)) = \ddot{x}(t) + x(t) + x(t)^{2m-1} = 0. \tag{1.22}$$

By symmetry, for applying the 1st order HBM we can look for a solution of the form $x(t) = a_1 \cos(\omega_1 t)$. We substitute it in (1.22). By using that

$$\cos^{2m-1}(\omega_1 t) = \frac{1}{2^{2m-2}} \sum_{k=0}^{m-1} \binom{2m-1}{k} \cos((2m-2k-1)\omega_1 t)$$

and reordering terms we have that the vanishing of the coefficient of $\cos(\omega_1 t)$ in $\mathcal{F}_1(x(t), \ddot{x}(t))$ implies

$$2^{2m-2}(\omega_1^2 - 1) - \frac{(2m-1)!}{(m-1)!\, m!} a_1^{2m-2} = 0.$$

From the initial conditions we have $a_1 = A$, whence

$$\omega_1 = \frac{1}{2^{m-1}} \sqrt{\frac{(2m-1)!}{(m-1)!m!} A^{2m-2} + 2^{2m-2}}.$$

Therefore, the first approximation $T_1(A)$ to $T(A)$ of system (1.1) is

$$T_1(A) = \frac{2\pi}{\frac{1}{2^{m-1}} \sqrt{\frac{(2m-1)!}{(m-1)!m!} A^{2m-2} + 2^{2m-2}}}. \tag{1.23}$$

Easy computations shows that the Taylor series of T_1 at $A = 0$ is

$$T_1(A) = 2\pi \left(1 - \frac{(2m)!}{(m!)^2\, 2^{2m}} A^{2m-2} + \left(\frac{(2m)!}{(m!)^2\, 2^{2m}} \right)^2 A^{4m-4} + O(A^{6m-6}) \right).$$

By using the identities $(2m)!/(2^m\, m!) = (2m-1)!!$ and $2^m\, m! = (2m)!!$ we have the expression of the statement.

For studying the behavior at infinity we can write $T_1(A)$ as

$$T_1(A) = 2^m \pi A^{-m+1} \left(\frac{(2m-1)!}{(m-1)!\, m!} + \frac{2^{2m-2}}{A^{2m-2}} \right)^{-1/2},$$

thus,

$$\lim_{A\to\infty} \frac{2^m\pi A^{-m+1}\left(\frac{(2m-1)!}{(m-1)!m!}+\frac{2^{2m-2}}{A^{2m-2}}\right)^{-1/2}-2^m\pi A^{-m+1}\left(\frac{(2m-1)!}{(m-1)!m!}\right)^{-1/2}}{A^{-m+1}}=0.$$

Hence $T_1(A)$ at infinity satisfies (1.6). □

Proof of Proposition 1.4. System (1.2) is equivalent to the second order differential equation

$$(x^2+k^2)^m\ddot{x}+x=0, \tag{1.24}$$

with initial conditions $x(0)=A$, $\dot{x}(0)=0$. For simplicity in the computations, we consider the complex form, given in (1.16), of the first-order HBM

$$x_1(t)=\frac{1}{2}\left(\bar{c}e^{i\omega_1 t}+ce^{-i\omega_1 t}\right), \tag{1.25}$$

where $c=a+bi$. By using the binomial expression

$$(x^2+k^2)^m=\sum_{j=0}^{m}\binom{m}{j}x^{2j}k^{2(m-j)},$$

and by replacing (1.25) in (1.24), after some computations we get

$$-\omega_1^2\sum_{j=0}^{m}\binom{m}{j}\left(\frac{1}{2}\right)^{2j+1}k^{2(m-j)}\sum_{l=0}^{2j+1}\binom{2j+1}{l}(c^l\bar{c}^{2j-l+1})(e^{i\omega_1 t})^{2j-2l+1}$$
$$+\frac{(\bar{c}e^{i\omega_1 t}+ce^{-i\omega_1 t})}{2}=0.$$

We are concerned only with the first-order harmonics, i.e. $j=l$ or $l=j+1$ in the above equation

$$-\frac{1}{2}\left(\omega_1^2\sum_{j=0}^{m}\left(\frac{1}{2}\right)^{2j}\binom{m}{j}\binom{2j+1}{j}k^{2(m-j)}(c\bar{c})^j-1\right)\bar{c}e^{i\omega_1 t}$$
$$-\frac{1}{2}\left(\omega_1^2\sum_{j=0}^{m}\left(\frac{1}{2}\right)^{2j}\binom{m}{j}\binom{2j+1}{j+1}k^{2(m-j)}(c\bar{c})^j-1\right)ce^{-i\omega_1 t}+HOH=0.$$

Since $\binom{2j+1}{j}=\binom{2j+1}{j+1}$, the previous equation can be written as

$$-\left(\omega_1^2\sum_{j=0}^{m}\left(\frac{1}{2}\right)^{2j}\binom{m}{j}\binom{2j+1}{j}k^{2(m-j)}(c\bar{c})^j-1\right)\left(\frac{\bar{c}e^{i\omega_1 t}+ce^{-i\omega_1 t}}{2}\right)=0,$$

whence

$$\omega_1 = \frac{1}{\sqrt{\sum_{j=0}^{m} \left(\frac{1}{2}\right)^{2j} \binom{m}{j} \binom{2j+1}{j} k^{2(m-j)} (c\bar{c})^j}}.$$

By the initial conditions we have $a_1 = A$ and $b_1 = 0$ then $c\bar{c} = A^2$. Therefore, the approximation $T_1(A)$ of $T(A)$ associated to system (1.2) is

$$T_1(A) = 2\pi \sqrt{\sum_{j=0}^{m} \left(\frac{1}{2}\right)^{2j} \binom{m}{j} \binom{2j+1}{j} k^{2(m-j)} A^{2j}}.$$

□

1.6 The Duffing-harmonic oscillator

This section is devoted to the study of the Duffing-harmonic oscillator. We compare the approximations $T_N(A)$, $N = 1, 2, 3$, given by the N-th order HBM with the exact period function $T(A)$ of the system

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x + x^3, \end{cases} \tag{1.26}$$

with initial conditions $x(0) = A$, $y(0) = 0$, both near the origin and at infinity. Our results extend those of [79], where only the cases $N = 1, 2$ are studied and where the analytic comparaison is restricted to a neighborhood of the origin.

Remark 1.13. *Some papers (for instance [38, 104]) consider the Duffing-harmonic oscillator $\dot{x} = -y$, $\dot{y} = x + \epsilon x^3$, $\epsilon \neq 0$, however, it is not difficult to see that by applying the transformation $x = \epsilon^{-1/2} u$, $y = \epsilon^{-1/2} v$, this system becomes* (1.26).

As in [79], we compute the period function $T(A)$ of (1.26) via elliptic functions. Let us remember the **K** complete elliptic integral of the first kind see [1, pp. 590]

$$\mathbf{K}(k) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k\,z^2)}},$$

whose Taylor expansion at $k = 0$, for $|k| < 1$ is

$$\mathbf{K}(k) = \frac{1}{2}\pi \left[1 + \left(\frac{1}{2}\right)^2 k + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^2 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^3 + \cdots \right]. \tag{1.27}$$

Lemma 1.14. *The period function $T(A)$ associated the system* (1.26) *is given by*

$$T(A) = \frac{4}{\sqrt{1+\frac{1}{2}A^2}}\mathbf{K}\left(\frac{-A^2}{2+A^2}\right). \tag{1.28}$$

Moreover, its Taylor series at $A = 0$ is

$$T(A) = 2\pi - \frac{3}{4}\pi A^2 + \frac{57}{128}\pi A^4 - \frac{315}{1024}\pi A^6 + \frac{30345}{131072}\pi A^8 + O(A^{10}), \tag{1.29}$$

and its behavior at infinity is

$$T(A) \sim B\left(\frac{1}{4}, \frac{1}{2}\right)\frac{\sqrt{2}}{A} \approx \frac{7.4163}{A}. \tag{1.30}$$

Proposition 1.15. *Let $T_N(A), N = 1,2,3,$ be the approximations of the period function for system* (1.26) *obtained applying the N-th order HBM. Then:*

(*i*) *The first approximation is*

$$T_1(A) = \frac{4\pi}{\sqrt{3A^2+4}}.$$

Its Taylor series at $A = 0$ is

$$T_1(A) = 2\pi - \frac{3}{4}\pi A^2 + \frac{27}{64}\pi A^4 + O(A^6), \tag{1.31}$$

and its behavior at infinity

$$T_1(A) \sim \frac{4\pi}{\sqrt{3}A} \approx \frac{7.2551}{A}. \tag{1.32}$$

(*ii*) *The second approximation is*

$$T_2(A) = \frac{2\pi}{\omega_2(A)},$$

where $\omega_2(A) := \omega_2$ is the real positive solution to the equation

$$1058\,\omega_2^6 - 3(219A^2+322)\omega_2^4 - \frac{9}{4}(21A^4+80A^2+40)\omega_2^2 - \frac{27}{64}A^2(7A^2+8)^2 - 2 = 0.$$

Moreover, its Taylor series at $A = 0$ is

$$T_2(A) = 2\pi - \frac{3}{4}\pi A^2 + \frac{57}{128}\pi A^4 - \frac{633}{2048}\pi A^6 + O(A^8), \tag{1.33}$$

and its behavior at infinity is given by

$$T_2(A) \sim \frac{\bar{\Delta}}{A} \approx \frac{7.4018}{A}, \tag{1.34}$$

where

$$\bar{\Delta} = \frac{92\sqrt{2}\pi\Delta}{\sqrt{1033992 + 876\Delta + \Delta^2}}, \quad \Delta = (1763014086 + 71386434\sqrt{393})^{1/3}.$$

(iii) The third approximation is given implicitly as one of the branches of an algebraic curve $h(A^2, T^2) = 0$ that has degree 11 with respect to A^2 and T^2 and total degree 44. In particular, at $A = 0$,

$$T_3(A) = 2\pi - \frac{3}{4}\pi A^2 + \frac{57}{128}\pi A^4 - \frac{315}{1024}\pi A^6 + \frac{30339}{131072}\pi A^8 + O(A^{10}),$$

and at infinity,

$$T_2(A) \sim \frac{\delta}{A} \approx \frac{7.4156}{A}, \tag{1.35}$$

where δ is the positive real root of an even polynomial of degree 22.

Notice that by Lemma 1.14 and Proposition 1.15 it holds that

$$T(A) - T_N(A) = O(A^{2N+4}), \quad N = 1, 2, 3,$$

result that evidences that, at least locally and for these values of N, the N-th order HBM improves when N increases. Moreover, the dominant terms at $A = \infty$ of $T_N(A)$ also improve when N increases.

In Figure 1.2 it is shown the absolute error between the exact period function $T(A)$ and first and second approximation by using HBM.

Proof of Lemma 1.14. The Hamiltonian function associated to system (1.26) is $H(x, y) = y^2/2 + x^2/2 + x^4/4$. The energy level is $H(x, y) = A^2/2 + A^4/4 := h$. Then, the period function is

$$T(A) = 4\int_0^A \frac{dx}{\sqrt{2h - x^2 - \frac{1}{2}x^4}}.$$

Making the change of variable $z = x/A$, we can write the above expression as

$$T(A) = \frac{4}{\sqrt{1 + \frac{1}{2}A^2}} \int_0^1 \frac{dz}{\sqrt{(1 - z^2)\left(1 + \frac{A^2}{2+A^2}z^2\right)}},$$

which gives (1.28) as we wanted to prove. By using (1.27) and (1.28), straightforward computations yield to (1.29). The behavior at infinity of $T(A)$ is a direct consequence of Theorem 1.1 with $m = 2$.

□

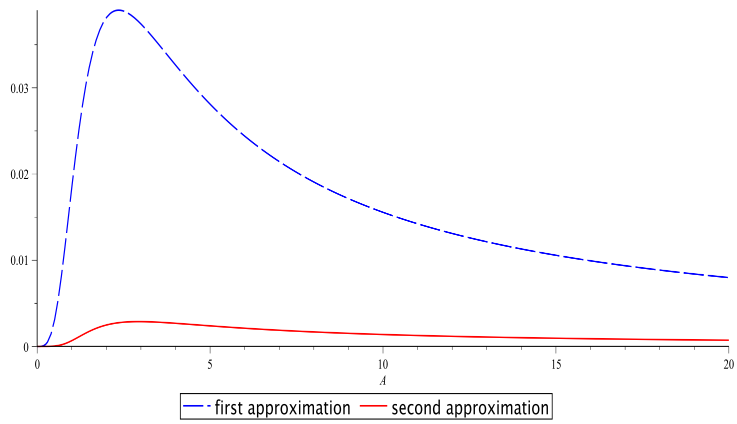


Figure 1.2:

Proof of Proposition 1.15. Notice that result (i) corresponds to the particular case $m = 2$ in Proposition 1.2. In this case, expression (1.23) gives

$$T_1(A) = \frac{4\pi}{\sqrt{3A^2+4}}.$$

Straightforward computations show that its Taylor series at $A = 0$ is (1.31). Moreover, writing $T_1(A)$ as

$$T_1(A) = \frac{4\pi}{A\sqrt{3+\frac{4}{A^2}}},$$

it is clear that $\frac{4\pi}{\sqrt{3}A}$ is its dominant term at infinity.

(ii) System (1.26) is equivalent to the second order differential equation

$$\ddot{x} + x + x^3 = 0, \tag{1.36}$$

with initial conditions $x(0) = A$, $\dot{x}(0) = 0$.

For applying second-order of the HBM we look for a solution of (1.36) of the form $x_2(t) = a_1\cos(\omega_2 t) + a_3\cos(3\omega_2 t)$. The vanishing of the coefficients of $\cos(\omega_2 t)$ and $\cos(3\omega_2 t)$ in the Fourier series of $\mathcal{F}_2$ provides the nonlinear system

$$\begin{aligned} -4\omega_2^2 + 3a_1^2 + 3a_1a_3 + 6a_3^2 + 4 &= 0,\\ -9a_3\omega_2^2 + \frac{1}{4}a_1^3 + \frac{3}{2}a_1^2a_3 + a_3 + \frac{3}{4}a_3^3 &= 0. \end{aligned}$$

From the initial conditions we have $a_1 = A - a_3$. Hence the above system becomes

$$\begin{aligned} -4\omega_2^2 + 6a_3^2 - 3a_3A + 3A^2 + 4 &= 0,\\ -9\omega_2^2a_3 + 2a_3^3 - \frac{9}{4}a_3^2A + \frac{3}{4}a_3A^2 + a_3 + \frac{1}{4}A^3 &= 0. \end{aligned}$$

Doing the resultant of these equations with respect to a_3, we obtain the polynomial

$$1058\omega_2^6 - 3(219A^2 + 322)\omega_2^4 - \frac{9}{4}(21A^4 + 80A^2 + 40)\omega_2^2 - \frac{1323}{64}A^6 - \frac{189}{4}A^4 - 27A^2 - 2.$$

Thus, ω_2 is the unique real positive root of the above polynomial, that is,

$$\omega_2 = \frac{\sqrt{2}}{92}\left(\frac{2166784 + 3272256\,A^2 + 1033992\,A^4 + (1288 + 876A^2)\,R^{1/3} + R^{2/3}}{R^{1/3}}\right)^{1/2}$$

where

$$\begin{aligned} R &= 3189506048 + 7956430848\,A^2 + 6507324864\,A^4 + 1763014086\,A^6 \\ &\quad + 3174\,(320 + 357A^2)\,A\,S, \\ S &= (4521984 + 9925632\,A^2 + 6899904\,A^4 + 1559817\,A^6)^{1/2}. \end{aligned}$$

Therefore, the second approximation $T_2(A)$ to $T(A)$ of (1.26) is $T_2(A) = 2\pi/\omega_2$, and it is not difficult to see that its Taylor series at $A = 0$ is (1.33).

For studying the behavior of T_2 at infinity we rewrite ω_2 as

$$\omega_2 = \frac{\sqrt{2}\,A}{92\,\bar{R}^{1/6}}\left(\frac{2166784}{A^4} + \frac{3272256}{A^2} + 1033992 + \left(\frac{1288}{A^4} + \frac{876}{A^2}\right)\bar{R}^{1/3} + \bar{R}^{2/3}\right)^{1/2}$$

where

$$\begin{aligned} \bar{R} &= \frac{3189506048}{A^6} + \frac{7956430848}{A^4} + \frac{6507324864}{A^2} + 1763014086 \\ &\quad + \frac{1015680}{A^5}\bar{S} + 1133118\bar{S}, \\ \bar{S} &= \left(\frac{4521984}{A^6} + \frac{9925632}{A^4} + \frac{6899904}{A^2} + 1559817\right)^{1/2}. \end{aligned}$$

From the previous expressions we have

$$\lim_{A\to\infty} \bar{S} = 63\sqrt{393}, \qquad \lim_{A\to\infty} \bar{R} = 1763014086 + 71386434\sqrt{393}.$$

Thus,

$$\lim_{A\to\infty} \frac{2\pi A}{\omega_2} = \frac{92\sqrt{2}\pi\Delta}{\sqrt{1033992 + 876\Delta + \Delta^2}},$$

where $\Delta = (1763014086 + 71386434\sqrt{393})^{1/3}$.

Hence,

$$\lim_{A\to\infty} \frac{T_2(A) - \bar{\Delta}A^{-1}}{A^{-1}} = 0,$$

where

$$\bar{\Delta} = \frac{92\sqrt{2}\pi\Delta}{\sqrt{1033992 + 876\Delta + \Delta^2}}.$$

Therefore, we have proved (ii).
(iii) When $N = 3$ we look for a solution of (1.36) of the form $x_2(t) = a_1 \cos(\omega_3 t) + a_3 \cos(3\omega_3 t) + a_5 \cos(5\omega_3 t)$. Using the initial conditions we get that $a_1 = A - a_3 - a_5$. Afterwards, imposing that the first three significative harmonics vanish, we obtain the system of three equations:

$$\begin{aligned}
P =& A - {\omega_3}^2 A + \frac{3}{4}A^3 + \left({\omega_3}^2 - \frac{3}{2}A^2 - 1\right) a_3 + \left({\omega_3}^2 - \frac{9}{4}A^2 - 1\right) a_5 + \frac{9}{2}a_3 a_5 A \\
&+ \frac{9}{4}A{a_3}^2 + \frac{15}{4}{a_5}^2 A - \frac{9}{4}{a_5}^3 - 3{a_3}^2 a_5 - \frac{9}{2}a_3 {a_5}^2 - \frac{3}{2}{a_3}^3 = 0,\\
Q =& \frac{1}{4}A^3 + \left(1 + \frac{3}{4}A^2 - 9{\omega_3}^2\right) a_3 - \frac{3}{2}a_3 a_5 A - \frac{3}{4}{a_5}^2 A - \frac{9}{4}A{a_3}^2 \\
&+ \frac{3}{2}{a_3}^2 a_5 + \frac{9}{4}a_3 {a_5}^2 + 2{a_3}^3 + \frac{1}{2}{a_5}^3 = 0,\\
R =& \frac{3}{4}A^2 a_3 + \left(-25{\omega_3}^2 + \frac{3}{2}A^2 + 1\right) a_5 - 3{a_5}^2 A - \frac{9}{2}a_3 a_5 A - \frac{3}{4}A{a_3}^2 \\
&+ \frac{15}{4}{a_3}^2 a_5 + \frac{9}{4}{a_5}^3 + \frac{15}{4}a_3 {a_5}^2 = 0.
\end{aligned}$$

Since all the equations are polynomial, the searching of its solutions can be done by using successive resultants, see for instance [98]. We compute the following polynomials

$$PQ := \frac{\operatorname{Res}(P, Q, a_3)}{A - a_5}, \quad QR := \operatorname{Res}(Q, R, a_3),$$

and finally

$$PQR := \frac{\operatorname{Res}(PQ, QR, a_5)}{3A^2 + 4 - 36\omega_3^2}.$$

This last expression is a polynomial with rational coefficients that only depends on A and ω_3 and has total degree 70. Fortunately, it factorizes as $PQR(A, \omega_3) = f(A, \omega_3)g(A, \omega_3)$, with factors of respective degrees 22 and 48. Although both factors could give solutions of our system we continue our study only with the factor f. It is clear that if we consider the following numerator

$$h(A^2, T^2) := \operatorname{Num}\left(f\left(A, \frac{2\pi}{T}\right)\right),$$

we have an algebraic curve $h(A^2, T^2) = 0$ that gives a restriction that has to be satisfied in order to have a solution of our initial system. This function is precisely the one that appears in the statement of the proposition.

Once we have this explicit algebraic curve it is not difficult to obtain the other results of the statement. So, to obtain the local behavior near the origin we consider $T_3(A) = \sum_k^m t_{2k} A^{2k}$ and we impose that $h(A^2, (T_3(A))^2) \equiv 0$, obtaining easily the first values t_{2k}. Similarly, for A big enough, we impose that $T_3(A) \sim \delta/A$ obtaining the value of δ. □

1.7 Non-monotonous period function

In this section we study the family of systems (1.8) whose period function has a critical period (a maximum of the period function) and we show that the HBM also captures this behavior.

It is not difficult to establish the existence of values of $k \gtrsim -2$ for which the period function is not monotonous. It holds that, for all k,

$$\lim_{A \to 0} T(A) = 2\pi \quad \text{and} \quad \lim_{A \to \infty} T(A) = 0. \tag{1.37}$$

We remark that when $k \leq -2$ the center is no more global but there is also a neighborhood of infinity full of periodic orbits. When $k = -2$, the system has also the critical points $(\pm 1, 0)$ and all the orbits of the potential system are closed, except the heteroclinic ones joining these two points. Hence, for $k = -2$ and from the continuity of the flow of (1.8) with respect to initial conditions, it follows that the periodic orbits close to these heteroclinic orbits have periods arbitrarily high; thus, the period of nearby periodic orbits, for $k > -2$ with $k + 2$ small enough, is also arbitrarily high due to the continuity of the flow of (1.8) with respect to parameters. Therefore, from this property and (1.37) it follows that $T(A)$ is not monotonous.

The proof that $T(A)$ has only one maximum is much more difficult and indeed was the main objective of [71]. In that paper the authors proved this fact showing first that $T(h)$, where h is the energy level of the Hamiltonian associated with (1.8), satisfies a Picard-Fuchs equation. As a consequence, the function $x(h) = T'(h)/T(h)$ satisfies a Riccati equation. Finally, they study the flow of this equation for showing that $x(h)$ vanish at most at a single point.

Proof of Proposition 1.6. For applying first-order HBM we write the family (1.8) as the second order differential equation

$$\ddot{x} + x + kx^3 + x^5 = 0.$$

We look for a solution of the form $x_1(t) = a_1\cos(\omega_1 t)$. The vanishing of the coefficient of $\cos(\omega_1 t)$ in $\mathcal{F}_1$, and the initial conditions $x_1(0) = A > 0$, $\dot{x}_1(0) = 0$, provides the algebraic equation

$$16 + 12kA^2 + 10A^4 - 16\omega_1^2 = 0.$$

Solving for ω_1 we obtain

$$\omega_1(A) = \frac{1}{4}\sqrt{16 + 12kA^2 + 10A^4}.$$

Then, the first approximation $T_1(A)$ to $T(A)$ is

$$T_1(A) = \frac{8\pi}{\sqrt{16 + 12kA^2 + 10A^4}},$$

which is well defined for all $A \in \mathbb{R}$ only for $k \in (-2\sqrt{10}/3, \infty)$. It is clear that if $k \geq 0$, then $T_1(A)$ is decreasing, which proves (i). Moreover

$$T_1'(A) = \frac{-16\pi\, A\,(3\,k + 5\,A^2)}{(8 + 6\,kA^2 + 5\,A^4)\,\sqrt{16 + 12\,kA^2 + 10\,A^4}}.$$

Hence, $T_1(A)$ has a non-zero critical point only when $k \in (-2\sqrt{10}/3, 0)$, and it is $A = \sqrt{-3k}/\sqrt{5}$. Moreover, it is easy to see that such critical point is a maximum.

The proof of items (ii) and (iii) is straightforward. □

1.8 General potential system

In this section we consider the smooth potential system

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + \displaystyle\sum_{i=2}^{\infty} k_i x^i. \end{cases} \tag{1.38}$$

Since its Hamiltonian function has a non degenerated minimum at the origin, it has a period annulus surrounding the origin. Thus, we have a period function $T(A)$ associated to this period annulus. The behavior near the origin of $T(A)$ is given in the following result.

Proposition 1.16. *The period function $T(A)$ of the system* (1.38) *at $A = 0$ is*

$$\begin{aligned} T(A) =& 2\,\pi + \left(\frac{5}{6}\,{k_2}^2 - \frac{3}{4}\,k_3\right)\pi A^2 + \left(\frac{5}{9}\,{k_2}^3 - \frac{1}{2}\,k_2\,k_3\right)\pi A^3 \\ &+ \left(\frac{385}{288}k_2^4 - \frac{275}{96}k_2^2k_3 + \frac{7}{4}k_2k_4 + \frac{57}{128}k_3^2 - \frac{5}{8}k_5\right)\pi A^4 + O\left(A^5\right). \end{aligned}$$

The proof of this proposition follows by using standard methods in the local study of the period function [22, 53].

By applying the HBM to the next family of potential systems

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x + \sum_{i=2}^{M} k_i x^i, \end{cases} \tag{1.39}$$

for $M = 3, 4, 5, 6, 7$, we obtain the corresponding $T_{1,M}(A)$ which satisfy

$$T_{1,M}(A) = 2\,\pi + \left({k_2}^2 - \frac{3}{4}\, k_3 \right) \pi A^2 + O_M \left(A^3 \right).$$

As can be seen, the quadratic terms do not depend on M. These first terms only coincide with the corresponding ones of $T(A)$ when $k_2 = 0$. Notice that this is the situation in Propositions 1.6 and 1.15.

To get a more accurate approach of $T(A)$ we have applied the second order HBM to (1.39) with $M = 3$ obtaining

$$T_2(A) = 2\,\pi + \left(\frac{5}{6}\, {k_2}^2 - \frac{3}{4}\, k_3 \right) \pi A^2 + O \left(A^3 \right),$$

result that coincides with the actual value of $T(A)$.

1.9 Conclusions

Studying several examples of potential systems we have seen that the approximations $T_N(A)$ calculated using the N-th order HBM keep some of the properties (analytic and qualitative) of the actual period function $T(A)$. Moreover, this matching seems to improve when N increases.

We believe that obtaining general results to strengthen the above relationship is a challenging question.

Chapter 2

Weak periodic solutions of $x\ddot{x}+1=0$ and the HBM

2.1 Introduction and main results

The nonlinear differential equation

$$x\ddot{x}+1=0, \tag{2.1}$$

appears in the modeling of certain phenomena in plasma physics [2]. In [78], Mickens calculates the period of its periodic orbits and also uses the N-th order HBM, for $N=1,2$, to obtain approximations of these periodic solutions and of their corresponding periods. Strictly speaking, it can be easily seen that neither equation (2.1), nor its associated system

$$\begin{cases} \dot{x}=y, \\ \dot{y}=-\frac{1}{x}, \end{cases} \tag{2.2}$$

which is singular at $x=0$, have periodic solutions. Our first result gives two different interpretations of Mickens' computation of the period. The first one in terms of weak (or generalized) solutions. In this work a weak solution will be a function satisfying the differential equation (2.1) on an open and dense set, but being of class $\mathcal{C}^0$ at some isolated points. The second one, as the limit, when k tends to zero, of the period of actual periodic solutions of the extended planar differential system

$$\begin{cases} \dot{x}=y, \\ \dot{y}=-\frac{x}{x^2+k^2}, \end{cases} \tag{2.3}$$

which, for $k\neq 0$, has a global center at the origin, as proved in Chapter 1.

Theorem 2.1. *(i) For the initial conditions* $x(0) = A \neq 0$, $\dot{x}(0) = 0$, *the differential equation* (2.1) *has a weak* $\mathcal{C}^0$*-periodic solution with period* $T(A) = 2\sqrt{2\pi}A$.

(ii) Let $T(A;k)$ *be the period of the periodic orbit of system* (2.3) *with initial conditions* $x(0) = A$, $y(0) = 0$. *Then*

$$T(A;k) = 4\,A \int_0^1 \frac{ds}{\sqrt{\ln\left(\frac{A^2+k^2}{A^2s^2+k^2}\right)}}$$

and

$$\lim_{k\to 0} T(A;k) = 4A \int_0^1 \frac{1}{\sqrt{-2\ln s}} ds = 2\sqrt{2\pi}A = T(A).$$

Recall that the N-th order HBM consists in approximating the solutions of differential equations by truncated Fourier series with N harmonics and an unknown frequency; In [79, p. 180] the author asks for techniques for dealing analytically with the N-th order HBM, for $N \geq 3$. In Chapter 1 it is shown how resultants can be used when $N = 3$. Here we utilize a more powerful tool, the computation of Gröbner basis ([31, Ch. 5]), for going further in the obtention of approximations of the function $T(A)$ introduced in Theorem 2.1.

Notice that equation (2.1) is equivalent to the family of differential equations

$$x^{m+1}\ddot{x} + x^m = 0, \tag{2.4}$$

for any $m \in \mathbb{N} \cup \{0\}$. Hence it is natural to approach the period function,

$$T(A) = 2\sqrt{2\pi}A \approx 5.0132A,$$

by the periods of the trigonometric polynomials obtained applying the N-th order HBM to (2.4). Next theorem gives our results for $N \leq 6$. Here $[a]$ denotes the integer part of a.

Theorem 2.2. *Let* $\mathcal{T}_N(A;m)$ *be the period of the truncated Fourier series obtained applying the* N*-th order HBM to equation* (2.4). *It holds:*

(i) For all $m \in \mathbb{N} \cup \{0\}$,

$$\mathcal{T}_1(A;m) = 2\pi \sqrt{\frac{2[\frac{m+1}{2}]+1}{2[\frac{m+1}{2}]+2}}\,A. \tag{2.5}$$

(ii) For $m = 0$,

$$\mathcal{T}_1(A;0) = \sqrt{2}\,\pi A \approx 4.4428A, \qquad \mathcal{T}_2(A;0) = (\sqrt{218}/9)\,\pi A \approx 5.1539A,$$
$$\mathcal{T}_3(A;0) = \frac{13810534\,\pi A}{3\sqrt{5494790257313+115642506449\sqrt{715}}} \approx 4.9353A, \quad \mathcal{T}_4(A;0) \approx 5.0455A,$$
$$\mathcal{T}_5(A;0) \approx 4.9841A, \qquad \mathcal{T}_6(A;0) \approx 5.0260A,$$

(iii) For $m = 1$,

$$\mathcal{T}_1(A;1) = \sqrt{3}\,\pi A \approx 5.4414A, \qquad \mathcal{T}_2(A;1) \approx 5.2733A,$$
$$\mathcal{T}_3(A;1) \approx 5.1476A, \qquad \mathcal{T}_4(A;1) \approx 5.1186A.$$

(iv) For $m = 2$,

$$\mathcal{T}_1(A;2) = \sqrt{3}\,\pi A \approx 5.4414A, \qquad \mathcal{T}_2(A;2) \approx 5.2724A,$$
$$\mathcal{T}_3(A;2) \approx 5.1417A.$$

Moreover, the approximate values appearing above are roots of given polynomials with integer coefficients. Whereby the Sturm sequences approach can be used to get them with any desirable precision.

Notice that the values $\mathcal{T}_1(A;m)$, for $m \in \{0,1,2\}$ given in items (ii), (iii) and (iv), respectively, are already computed in item (i). We explicit this only to clarify the reading.

$e_N(m)$	$m = 0$	$m = 1$	$m = 2$
$N = 1$	11.38%	8.54%	8.54%
$N = 2$	2.80%	5.19%	5.17%
$N = 3$	1.55%	2.68%	2.56%
$N = 4$	0.64%	2.10%	–
$N = 5$	0.58%	–	–
$N = 6$	0.25%	–	–

Table 2.1: Percentage of relative errors $e_N(m)$.

Observe that the comparison of (2.5) with the value $T(A)$ given in Theorem 2.1 shows that when $N = 1$ the best approximations of $T(A)$ happen when $m \in \{1,2\}$. For this reason we have applied the HBM for $N \leq 6$ and $m \leq 2$ to elucidate which of the approaches is better. In the Table 2.1 we will compare the percentage of the relative errors

$$e_N(m) = 100\left|\frac{\mathcal{T}_N(A;m)-T(A)}{T(A)}\right|.$$

The best approximation that we have found corresponds to $\mathcal{T}_6(A;0)$. Our computers have had problems to get the Gröbner basis needed to fill the gaps of the table.

The chapter is organized as follows. First, we prove Theorem 2.1. In Section 2.2 we describe the N-th order HBM in the framework of symmetric equations. Finally, in Section 2.3 we use this method to demonstrate Theorem 2.2.

Proof of Theorem 2.1.

Proof. (*i*) We start proving that the solution of (2.1) with initial conditions $x(0) = A$, $\dot{x}(0)=0$ and for $t \in \left(-\frac{\sqrt{2\pi}}{2}A, \frac{\sqrt{2\pi}}{2}A\right)$ is

$$x(t) = \phi_0(t) := Ae^{-\left(\operatorname{erf}^{-1}\left(\frac{2t}{\sqrt{2\pi}A}\right)\right)^2}, \tag{2.6}$$

where erf^{-1} is the inverse of the error function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}\,ds.$$

Notice that $\lim_{t\to\pm\frac{\sqrt{2\pi}}{2}}\phi_0(t) = 0$ and $\lim_{t\to\pm\frac{\sqrt{2\pi}}{2}}\phi_0'(t) = \mp\infty$. To obtain (2.6), observe that from system (2.2) we arrive at the simple differential equation

$$\frac{dx}{dy} = -xy,$$

which has separable variables and can be solved by integration. The particular solution that passes by the point $(x,y)=(A,0)$ is

$$x = Ae^{-y^2/2}. \tag{2.7}$$

Combining (2.2) and (2.7) we obtain

$$\frac{dy}{dt} = -\frac{e^{y^2/2}}{A},$$

again a separable equation. It has the solution

$$y(t) = -\sqrt{2}\operatorname{erf}^{-1}\left(\frac{2t}{\sqrt{2\pi}A}\right), \tag{2.8}$$

which is well defined for $t \in \left(-\frac{\sqrt{2\pi}}{2}A, \frac{\sqrt{2\pi}}{2}A\right)$ since $\mathrm{erf}^{-1}(\cdot)$ is defined in $(-1,1)$. Finally, by replacing $y(t)$ in (2.7) we obtain (2.6), as we wanted to prove.

By using $x(t)$ and $y(t)$ given by (2.6) and (2.8), respectively, or using (2.7), we can draw the phase portrait of (2.2) which, as we can see in Figure 2.1.(b), is symmetric with respect to both axes. Notice that its orbits do not cross the y-axis, which is a singular locus for the associated vector field. Moreover, the solutions of (2.1) are not periodic (see Figure 2.1.(a)), and the transit time of $x(t)$ from $x = A$ to $x = 0$ is $\sqrt{2\pi}\, A/2$.

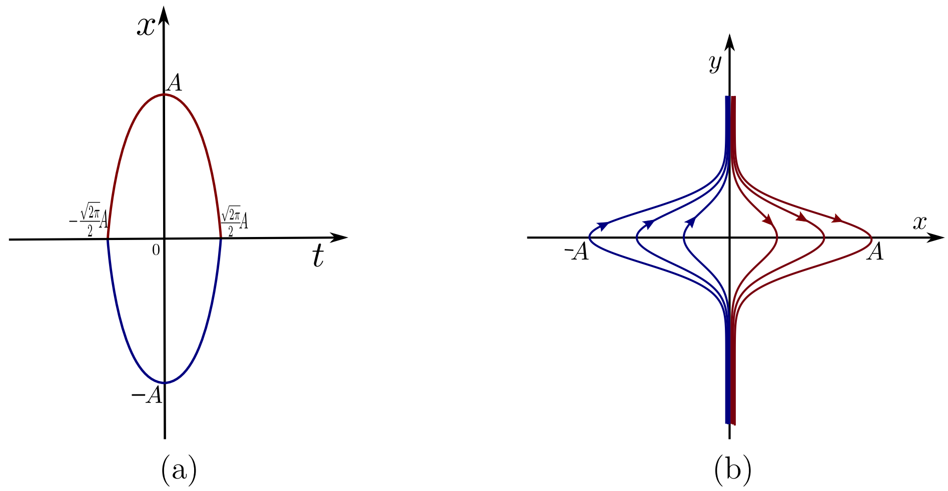


Figure 2.1: (a) Two solutions of equation (2.1). (b) Phase-portrait of system (2.2).

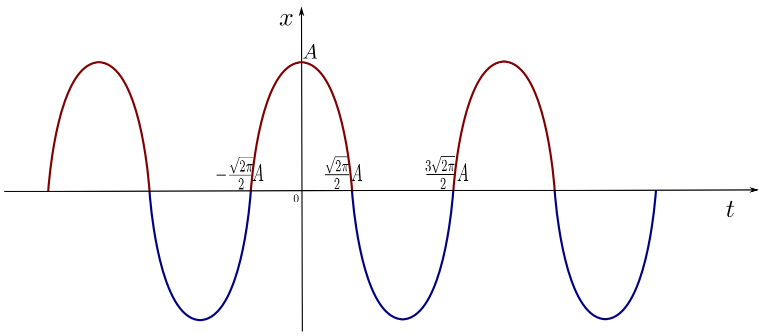


Figure 2.2: A weak $\mathcal{C}^0$-periodic solution of (2.1).

From (2.6) we introduce the $\mathcal{C}^0$-function, defined on the whole $\mathbb{R}$, as

$$\phi(t) = \begin{cases} (-1)^n \phi_0(t - n\sqrt{2\pi}), & \text{for} \quad t \in \left(\frac{2n-1}{2}\sqrt{2\pi}, \frac{2n+1}{2}\sqrt{2\pi}\right), \quad n \in \mathbb{Z}, \\ 0 & \text{for} \quad t = \frac{2n+1}{2}\sqrt{2\pi}, \qquad n \in \mathbb{Z}, \end{cases}$$

see Figure 2.2. It is a $\mathcal{C}^0$-periodic function of period $T(A)=2\sqrt{2\pi}A$ and $x=\phi(t)$ satisfies (2.1), for all $t\in\mathbb{R}\setminus\cup_{n\in\mathbb{Z}}\{\frac{2n+1}{2}\sqrt{2\pi}\}$. Hence (i) of the theorem follows.

Notice that directly from (2.1), it is easy to see that this equation can not have $\mathcal{C}^2$-solutions such that $x(t^*)=0$ for some $t^*\in\mathbb{R}$, because this would imply that $\lim_{t\to t^*}\ddot{x}(t)=\infty$.

(ii) System (2.3) is Hamiltonian with Hamiltonian function

$$H(x,y)=\frac{y^2}{2}+\frac{\ln(x^2+k^2)}{2}.$$

Since $\ln(x^2+k^2)$ has a global minimum at 0 and $\ln(x^2+k^2)$ tends to infinity when $|x|$ does, system (2.3) has a global center at the origin. In Figure (2.3) we can see its phase portrait for some values of k. This figure also illustrates how the periodic orbits of (2.3) approach to the solutions of system (2.2). Its period function is

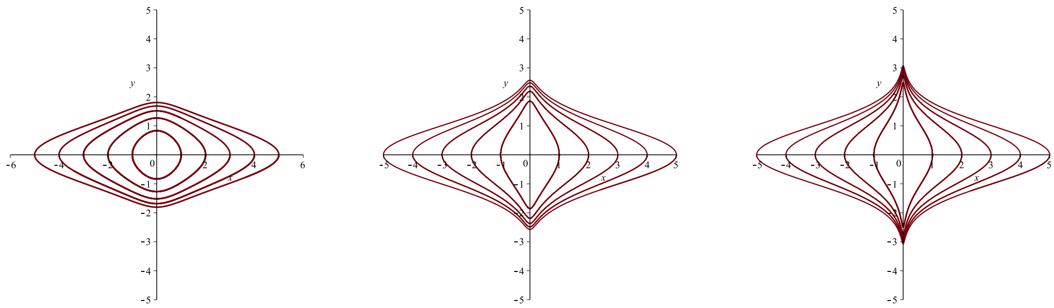


Figure 2.3: Phase portraits of (2.3) for different values of k.

$$T(A;k)=2\int_{-A}^{A}\frac{dx}{y(x)}=2\int_{-A}^{A}\frac{dx}{\sqrt{2h-\ln\left(x^2+k^2\right)}},$$

where $h=\ln(A^2+k^2)/2$ is the energy level of the orbit passing through the point $(A,0)$. Therefore,

$$T(A;k)=2\int_{-A}^{A}\frac{dx}{\sqrt{\ln\left(\frac{A^2+k^2}{x^2+k^2}\right)}}=4\,A\int_{0}^{1}\frac{ds}{\sqrt{\ln\left(\frac{A^2+k^2}{A^2s^2+k^2}\right)}},$$

where we have used the change of variable $s=x/A$ and the symmetry with respect to x. Then,

$$\lim_{k\to 0}T(A;k)=\lim_{k\to 0}\int_{0}^{1}\frac{4\,A\,ds}{\sqrt{\ln\left(\frac{A^2+k^2}{A^2s^2+k^2}\right)}}.$$

If we prove that

$$\lim_{k\to 0}\int_0^1 \frac{4\,A\,ds}{\sqrt{\ln\left(\frac{A^2+k^2}{A^2s^2+k^2}\right)}} = \int_0^1 \lim_{k\to 0}\frac{4\,A\,ds}{\sqrt{\ln\left(\frac{A^2+k^2}{A^2s^2+k^2}\right)}}, \tag{2.9}$$

then

$$\lim_{k\to 0} T(A;k) = 4\,A\int_0^1 \frac{ds}{\sqrt{-2\ln(s)}} = 2\sqrt{2\pi}A = T(A)$$

and the theorem will follow. Therefore, for completing the proof, it only remains to show that (2.9) holds. For proving that, take any sequence $1/z_n$, with z_n tending monotonically to infinity, and consider the functions $f_n(s) = \left(\ln\left(\frac{A^2z_n^2+1}{A^2z_n^2s^2+1}\right)\right)^{-1/2}$. We have that the sequence $\{f_n(s)\}_{n\in\mathbb{N}}$ is formed by measurable and positive functions defined on the interval $(0,1)$. It is not difficult to prove that it is a decreasing sequence. In particular, $f_n(s) < f_1(s)$ for all $n > 1$. Therefore, if we show that $f_1(s)$ is integrable, then we can apply the Lebesgue's dominated convergence theorem ([92]) and (2.9) will follow. To prove that $\int_0^1 f_1(s) < \infty$ note that for s close to 1,

$$f_1(s) = \left(\ln\left(\frac{A^2z_1^2+1}{A^2z_1^2s^2+1}\right)\right)^{-1/2} \sim \left(\frac{2A^2z_1^2(1-s)}{A^2z_1^2+1}\right)^{-1/2}.$$

Since this last expression is integrable the result follows by the comparison test for improper integrals. □

2.2 The HBM for symmetric equations

Although in Chapter 1 we did a description of the HBM, we want to repeat it but in the context of certain second order symmetric differential equations. Since, in this case, there will be simplifications in the computations.

Consider the second order differential equations

$$\mathcal{F} := \mathcal{F}(x(t),\ddot{x}(t)) = 0, \tag{2.10}$$

with $\mathcal{F}(-u,-v) = \mathcal{F}(u,v)$. Notice that if $x(t)$ is a solution of (2.10) then $x(-t)$ also is a solution.

Suppose that equation (2.10) has a T-periodic solution $x(t)$ with initial conditions $x(0) = A$, $\dot{x}(0) = 0$ and period $T = T(A)$. If $x(t)$ satisfies $x(t) = x(-t)$ it is clear that its Fourier series has no sinus terms and writes as

$$\sum_{k=1}^{\infty} a_k\cos(k\omega t), \quad \text{with} \quad \sum_{k=1}^{\infty} a_k = A \quad \text{and} \quad \omega = \frac{2\pi}{T}.$$

As we have seen in previous section, the weak periodic solutions of equation $x(t)\ddot{x}(t)+1=0$ that we want to approach satisfy the above property. Moreover $x(T/4)=0$ and $\dot{x}(T/4)$ does not exist. In any case, if we are searching smooth approximations to this $x(t)$, they should also satisfy $\dot{x}(t)\ddot{x}(t)+x(t)\dddot{x}(t)=0$, and hence $\dot{x}(T/4)=0$. For this reason, in this chapter we will search Fourier series in cosines, not having the even terms $\cos(2j\omega t)$, $j\in\mathbb{N}\cup\{0\}$, which do not satisfy this property. This type of a priori simplifications are similar to the ones introduced in [77] for other problems.

Hence, in our setting, the HBM of order N follows the next five steps:

1. Consider a trigonometric polynomial

$$x_{N}(t)=\sum_{j=1}^{N}a_{2j-1}\cos((2j-1)\omega_N t)\quad\text{with}\quad\sum_{j=1}^{N}a_{2j-1}=A. \tag{2.11}$$

2. Compute the $2\pi/\omega_N$-periodic function $\mathcal{F}_N:=\mathcal{F}(x_{N}(t),\ddot{x}_{N}(t))$, which has also an associated Fourier series,

$$\mathcal{F}_N(t)=\sum_{j\geq 0}\mathcal{A}_j\cos(j\,\omega_N t),$$

where $\mathcal{A}_j=\mathcal{A}_j(\mathbf{a},\omega_N,A)$ $j\geq 0$, with $\mathbf{a}=(a_1,a_3,\ldots,a_{2N-1})$.

3. Find all values $\mathbf{a}$ and ω_N such that

$$\mathcal{A}_j(\mathbf{a},\omega_N,A)=0\qquad\text{for}\quad 1\leq j\leq j_N, \tag{2.12}$$

where j_N is the value such that (2.12) consists exactly of N non trivial equations. Notice also that each equation $\mathcal{A}_j(\mathbf{a},\omega_N,A)=0$ is equivalent to

$$\int_0^{2\pi/\omega_N}\cos(j\omega_N t)\mathcal{F}_N(t)\,dt=0. \tag{2.13}$$

4. Then the expression (2.11), with the values of $\mathbf{a}=\mathbf{a}(A)$ and $\omega_N=\omega_N(A)$ obtained in point 3, provide candidates to be approximations of the actual periodic solutions of the initial differential equation. In particular, the functions $\mathcal{T}_N=\mathcal{T}_N(A)=2\pi/\omega_N$ give approximations of the periods of the corresponding periodic orbits (as can be seen in Chapter 1).

5. Choose, as final approximation, the one associated to the solution that minimizes the *accuracy* of the solution given by the norm

$$||\mathcal{F}_N(t)||_2=\sqrt{\frac{1}{\mathcal{T}_N}\int_0^{\mathcal{T}_N}\mathcal{F}_N^2(t)\,dt},$$

see also the Section 3.1 of the Chapter 3. We remark again some points about the HBM, some of which were already mentioned in Section 1.3 of Chapter 1.

Remark 2.3. *(i) Going from order N to order $N+1$ in the method, implies to compute again all the coefficients of the Fourier polynomial, because in general the common Fourier coefficients of $x_{N}(t)$ and $x_{N+1}(t)$ do not coincide.*

(ii) The above set of equations (2.12) *is a system of polynomial equations which usually is not easy to solve. For this reason in many works, see for instance [78, 79] and the references therein, only the values of $N=1,2$ are considered. In Chapter 1, by using the method of successive resultants, we do the computations for $N=3$. In this chapter, for solving system* (2.12) *for $N\geq 3$ we use the Gröbner basis approach ([31]). In general this method is faster that using successive resultants and moreover it does not give spurious solutions.*

(iii) As far as we know, the test proposed in point 5 to select the best approach is not commonly used. We propose it following the definition of accuracy of an approximated solution used in Chapter 3 and inspired in the classical works [97, 99].

2.3 Application of the HBM

We start proving a lemma that will allow to reduce our computations to the case $A=1$.

Lemma 2.4. *Let $\mathcal{T}_N(A;m)$ be the period of the truncated Fourier series obtained applying the N-th order HBM to equation* (2.4). *Then there exists a constant $C_N(m)$ such that $\mathcal{T}_N(A;m)=C_N(m)A$.*

Proof. Consider $\mathcal{F}_N=x_N^{m+1}\ddot{x}_N+x_N^m=0$, with x_N given in (2.11). We have to solve the set of $N+1$ non-trivial equations

$$\int_0^{2\pi/\omega_N}\cos(j\omega_N t)\mathcal{F}_N(t)\,dt=0\qquad 1\leq j\leq j_N,\qquad \sum_{j=1}^{N}a_{2j-1}=A, \tag{2.14}$$

with $N+1$ unknowns $a_1,a_3,\ldots,a_{2N-1}$ and ω_N and $A\neq 0$. The lemma clearly follows if we prove next assertion: $\tilde{a}_1,\tilde{a}_3,\ldots,\tilde{a}_{2N-1}$ and $\tilde{\omega}_N$ is a solution of (2.14) with $A=1$ if and only if $A\tilde{a}_1,A\tilde{a}_3,\ldots,A\tilde{a}_{2N-1}$ and $\tilde{\omega}_N/A$ is a solution of (2.14). This equivalence is a consequence of the fact that the change of variables $s=At$ writes the integral equation in (2.14) as

$$\frac{1}{A}\int_0^{2\pi A/\omega_N}\cos\Big(j\frac{\omega_N}{A}s\Big)\mathcal{F}_N\Big(\frac{s}{A}\Big)\,ds=0$$

and from the structure of the right hand side equation of (2.14). Hence, $\mathcal{T}_N(A;m)=\mathcal{T}_N(1;m)A=:C_N(m)A$, as we wanted to prove. □

Proof of Theorem 2.2. Due to the above lemma, in the application of the N-th order HBM, we can restrict our attention to the case $A=1$.

(i) Following section 2.2, we consider $x_1(t)=\cos(\omega_1 t)$ as the first approximation to the actual solution of the functional equation $\mathcal{F}(x(t),\ddot{x}(t))=x^{m+1}\ddot{x}+x^m=0$. Then

$$\mathcal{F}_1(t)=-\omega_1^2\cos^{m+2}(\omega_1 t)+\cos^m(\omega_1 t).$$

When $m=2k$ the above expression writes as

$$\mathcal{F}_1(t)=-\omega_1^2\cos^{2k+2}(\omega_1 t)+\cos^{2k}(\omega_1 t)=0. \tag{2.15}$$

Using (2.13) for $j=0$ we get

$$\int_0^{2\pi/\omega_1}\mathcal{F}_1(t)\,dt=-\omega_1^2 I_{2k+2}+I_{2k}=0, \tag{2.16}$$

where $I_{2\ell}=\int_0^{2\pi/\omega_1}\cos^{2\ell}(\omega_1 t)dt$. By using integration by parts we prove that $(2k+2)I_{2k+2}=(2k+1)I_{2k}$. Combining this equality and (2.16) we obtain that

$$\omega_1=\sqrt{\frac{2k+2}{2k+1}},$$

or equivalently,

$$\mathcal{T}_1(A;m)=2\pi A\sqrt{\frac{2k+1}{2k+2}},$$

that in terms of m coincides with (2.5). The case m odd follows similarly. The only difference is that instead of condition (2.16) to find $\mathcal{T}_1(A;m)$ we have to impose that

$$\int_0^{2\pi/\omega_1}\cos(\omega_1 t)\mathcal{F}_1(t)\,dt=0,$$

because $\int_0^{2\pi/\omega_1}\mathcal{F}_1(t)\,dt\equiv 0$.

(ii) Case $m=0$. Consider the functional equation $\mathcal{F}(x(t),\ddot{x}(t))=x(t)\ddot{x}(t)+1=0$. When $N=2$, we take as approximation $x_2(t)=a_1\cos(\omega_2 t)+a_3\cos(3\omega_2 t)$. The vanishing of the coefficients of 1 and $\cos(2\omega_2 t)$ in the Fourier series of $\mathcal{F}_2$ provides the nonlinear system

$$\begin{aligned}1-\frac{1}{2}(a_1^2+9a_3^2)\omega_2^2&=0,\\ a_1+10a_3&=0,\\ a_1+a_3-1&=0.\end{aligned}$$

By solving it and applying point 5 in the HBM we get $\omega_2 = 18/\sqrt{218}$. Therefore,

$$\mathcal{T}_2(A;0) = \frac{\sqrt{218}}{9}\pi A \approx 5.1539A,$$

as we wanted to prove.

For the third-order HBM we use as approximate solution $x_3(t) = a_1\cos(\omega_3 t) + a_3\cos(3\omega_3 t) + a_5\cos(5\omega_3 t)$. Imposing that the coefficients of 1, $\cos(2\omega_3 t)$, and $\cos(4\omega_3 t)$ in $\mathcal{F}_3$ vanish we arrive at the system

$$\begin{aligned}
P &= 2 - \left(a_1^2 + 9a_3^2 + 25a_5^2\right)\omega_3^2 = 0,\\
Q &= a_1^2 + 10a_1a_3 + 34a_3a_5 = 0,\\
R &= 5a_3 + 13a_5 = 0,\\
S &= a_1 + a_3 + a_5 - 1 = 0.
\end{aligned}$$

Since all the equations are polynomial, the searching of its solutions can be done by using the Gröbner basis approach, see [31]. Recall that the idea of this method consists in finding a new systems of generators, say $G_1, G_2, \dots, G_\ell$, of the ideal of $\mathbb{R}[a_1, a_3, a_5, \omega_3]$ generated by P, Q, R and S. Hence, solving $P = Q = R = S = 0$ is equivalent to solve $G_i = 0$, $i = 1, \dots, \ell$. In general, choosing the lexicographic ordering in the Gröbner basis approach, we get that the polynomials of the equivalent system have triangular structure with respect to the variables and it can be easily solved.

Now, by computing the Gröbner basis of $\{P, Q, R, S\}$ with respect to the lexicographic ordering $[a_1, a_3, a_5, \omega_3]$ we obtain a new basis with four polynomials ($\ell = 4$), being one of them,

$$G_1(\omega_3) = 1553685075\omega_3^8 - 3692301106\omega_3^6 + 2143547654\omega_3^4 - 402413472\omega_3^2 + 20301192.$$

Solving $G_1(\omega_3) = 0$ and using again point 5 of our approach to HBM we get that the solution that gives the better approximation is

$$\omega_3 = \frac{3\sqrt{5494790257313 + 115642506449\sqrt{715}}}{6905267}.$$

Hence the expression $\mathcal{T}_3(A;0) = 2\pi A/\omega_3$ of the statement follows.

When $N = 4$ we consider $x_4(t) = a_1\cos(\omega_4 t) + a_3\cos(3\omega_4 t) + a_5\cos(5\omega_4 t) + a_7\cos(7\omega_4 t)$, and we arrive at the system

$$\begin{aligned}
P &= 2 - \left(a_1^2 + 9a_3^2 + 25a_5^2 + 49a_7^2\right)\omega_4^2 = 0,\\
Q &= a_1^2 + 10a_1a_3 + 34a_3a_5 + 74a_5a_7 = 0,\\
R &= 5a_1a_3 + 13a_1a_5 + 29a_3a_7 = 0,\\
S &= 9a_3^2 + 50a_1a_7 + 26a_1a_5 = 0,\\
U &= a_1 + a_3 + a_5 + a_7 - 1 = 0.
\end{aligned}$$

The Gröbner basis of $\{P, Q, R, S, U\}$ with respect to the lexicographic ordering $[a_1, a_3, a_5, a_7, \omega_4]$ is a new basis with five polynomials, being one of them an even polynomial in ω_4 of degree 16 with integers coefficients. Solving it we obtain that the best approximation is $\omega_4 \approx 1.2453$, which gives $\mathcal{T}_4(A; 0) \approx 5.0455\, A$.

For $N = 5$ and $N = 6$ we have done similar computations. In the case $N = 5$ one of the generators of the Gröbner basis is an even polynomial in ω_5 with integers coefficients and degree 32. When $N = 6$ the same happens but with a polynomial of degree 64 in ω_6. Solving the corresponding polynomials we get that $\omega_5 \approx 1.2606$ and $\omega_6 \approx 1.2501$, and consequently, $\mathcal{T}_5(A; 0) \approx 4.9843\, A$, and $\mathcal{T}_6(A; 0) \approx 5.0260\, A$.

(iii) Case $m = 1$. We apply the HBM to $\mathcal{F}(x(t), \ddot{x}(t)) = x^2(t)\ddot{x}(t) + x(t) = 0$. When $N = 2$, doing similar computations that in item (ii), we arrive at

$$\begin{aligned}
P &= 4 - \left(3a_1^2 + 11a_1a_3 + 38a_3^2\right) \omega_2^2 = 0,\\
Q &= 4a_3 - \left(a_1^3 + 22a_1^2a_3 + 27a_3^3\right) \omega_2^2 = 0,\\
R &= a_1 + a_3 - 1 = 0.
\end{aligned}$$

Again, by searching the Gröbner basis of $\{P, Q, R\}$ with respect to the lexicographic ordering $[a_1, a_3, \omega_2]$ we obtain a new basis with three polynomials, being one of them

$$G_1(\omega_2) = 7635411\omega_2^8 - 14625556\omega_2^6 + 5833600\omega_2^4 - 661376\omega_2^2 + 13824.$$

Notice that the equation $G(\omega_2) = 0$ can be algebraically solved. Nevertheless, for the sake of shortness, we do not give the exact roots. Following again step 5 of our approach we get that the best solution is $\omega_2 \approx 1.1915$, or equivalently that $\mathcal{T}_2(A; 1) \approx 5.2733A$.

The HBM when $N = 3$ produces the system

$$\begin{aligned}
P &= 4a_1 - \left(3a_1^3 + 11a_1^2a_3 + 38a_1a_3^2 + 70a_1a_3a_5 + 102a_1a_5^2 + 43a_3^2a_5\right) \omega_3^2 = 0,\\
Q &= 4a_3 - \left(a_1^3 + 22a_1^2a_3 + 27a_1^2a_5 + 70a_1a_3a_5 + 27a_3^3\right) \omega_3^2 = 0,\\
R &= 4a_5 - \left(11a_1^2a_3 + 54a_1^2a_5 + 19a_1a_3^2 + 86a_3^2a_5 + 75a_5^3\right) \omega_3^2 = 0,\\
S &= a_1 + a_3 + a_5 - 1 = 0.
\end{aligned}$$

Computing the Gröbner basis of $\{P, Q, R, S\}$ with respect to the lexicographic ordering $[a_1, a_3, a_5, \omega_3]$ we get that one of the polynomials of the new basis is an even polynomial in ω_3 of degree 26 with integer coefficients. By solving it we obtain that the best approximation is $\omega_3 \approx 1.2206$, which produces the value $\mathcal{T}_3(A; 1)$ of the statement.

When $N = 4$ we arrive at five polynomial equations, that we omit. Once more, using the Gröbner basis approach we obtain a polynomial condition in ω_4 of degree 80. Finally, $\omega_4 \approx 1.2275$ and $\mathcal{T}_4(1; A) \approx 5.1186$.

(iv) When $m = 2$ we have to approach the solutions of $\mathcal{F}(x(t), \ddot{x}(t)) = x^3(t)\ddot{x}(t) + x^2(t) = 0$. We do not give the details of the proof because we get our results by using exactly the same type of computations.

□

Remark 2.5. *For each N and m our computations also provide a trigonometric polynomial that approaches the continuous weak periodic solution $\phi(t)$ given in the proof of Theorem 2.1.*

Chapter 3

A theoretical basis for the HBM

3.1 Introduction and main results

Consider the real non-autonomous differential equation

$$x' = X(x, t), \tag{3.1}$$

where the prime denotes the derivative with respect to t, $X : \Omega \times [0, 2\pi] \to \mathbb{R}$ is a $\mathcal{C}^2$-function, 2π-periodic in t, and $\Omega \subset \mathbb{R}$ is a given open interval.

There are several methods for finding approximations to the periodic solutions of (3.1). For instance, the Harmonic Balance Method (HBM), recalled in section 3.3, or simply the numerical approximations of the solutions of the differential equations. In any case, from all the methods we can get a truncated Fourier series, namely a trigonometric polynomial, that "approximates" an actual periodic solution of the equation. The aim of this chapter is to recover some old results of Stokes and Urabe that allow the use of these approximations to prove that near them there are actual periodic solutions and also provide explicit bounds, in the infinity norm, of the distance between both functions. To the best of our knowledge these results are rarely used in the papers dealing with the HBM.

When the methods are applied to concrete examples one has to manage the coefficients of the truncated Fourier series that are rational numbers which renders the subsequent computations more difficult. See the examples of Section 3.5. At this point we introduce in this setting a classical tool that as far as we know has never been used in this type of problems: we approximate all the coefficients of the truncated Fourier series by suitable convergents of their respective expansions in continuous fractions. This is done in such a way that by using these new coefficients we obtain a new approximate solution that is essentially at the same distance to the actual solution as the starting approximation. With this method we obtain trigonometric polynomials with nice rational coefficients that approximate the periodic solutions.

Before stating our main result, and following [97, 99], we introduce some concepts. Let $\bar{x}(t)$ be a 2π-periodic $\mathcal{C}^1$-function; we will say that $\bar{x}(t)$ is *noncritical* with respect to (3.1) if

$$\int_0^{2\pi} \frac{\partial}{\partial x} X(\bar{x}(t), t)\, dt \neq 0. \tag{3.2}$$

Notice that if $\bar{x}(t)$ is a periodic solution of (3.1) then the concept of noncritical is equivalent to the one of being *hyperbolic*; see [68].

As we will see in Lemma 3.3, if $\bar{x}(t)$ is noncritical w.r.t. equation (3.1), the linear periodic system

$$y' = \frac{\partial}{\partial x} X(\bar{x}(t), t)\, y + b(t)$$

has a unique periodic solution $y_b(t)$ for each smooth 2π-periodic function $b(t)$. Moreover, once X and $\bar{x}$ are fixed, there exists a constant M such that

$$||y_b||_\infty \leq M||b||_2, \tag{3.3}$$

where as usual, for a continuous 2π-periodic function f,

$$||f||_2 = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} f^2(t) dt}, \quad ||f||_\infty = \max_{x\in\mathbb{R}} |f(x)| \quad \text{and} \quad ||f||_2 \leq ||f||_\infty.$$

Any constant satisfying (3.3) will be called a *deformation constant associated to* $\bar{x}$ *and* X. Finally, consider

$$s(t) := \bar{x}'(t) - X(\bar{x}(t), t). \tag{3.4}$$

We will say that $\bar{x}(t)$ is *an approximate solution of* (3.1) *with accuracy* $S = ||s||_2$. For simplicity, if $\tilde{S} > S$, we also will say that $\bar{x}(t)$ has accuracy $\tilde{S}$. Notice that actual periodic solutions of (3.1) have accuracy 0; in this sense, the function $s(t)$ measures how far is $\bar{x}(t)$ from being an actual periodic solution of (3.1).

The next theorem improves some of the results of Stokes [97] and Urabe [99] in the one-dimensional setting. More concretely, in those papers they prove the existence and uniqueness of the periodic orbit when $4M^2KS < 1$. We present a similar proof with the small improvement $2M^2KS < 1$. Moreover our result gives, under an additional condition, the hyperbolicity of the periodic orbit.

Theorem 3.1. *Let* $\bar{x}(t)$ *be a* 2π*-periodic* $\mathcal{C}^1$*–function such that*

- *it is noncritical w.r.t. equation* (3.1) *and has* M *as a deformation constant,*

- *it has accuracy* S *w.r.t. equation* (3.1).

Given $I := [\min_{\{t\in\mathbb{R}\}} \bar{x}(t) - 2MS, \max_{\{t\in\mathbb{R}\}} \bar{x}(t) + 2MS] \subset \Omega$, *let* $K < \infty$ *be a constant such that*

$$\max_{(x,t)\in I\times[0,2\pi]} \left| \frac{\partial^2}{\partial x^2} X(x,t) \right| \leq K.$$

Therefore, if

$$2M^2KS < 1,$$

there exists a 2π*-periodic solution* $x^*(t)$ *of* (3.1) *satisfying*

$$||x^* - \bar{x}||_\infty \leq 2MS,$$

and it is the unique periodic solution of the equation entirely contained in this strip. If in addition

$$\left| \int_0^{2\pi} \frac{\partial}{\partial x} X(\bar{x}(t), t)\, dt \right| > \frac{2\pi}{M},$$

then the periodic orbit $x^*(t)$ *is hyperbolic, and its stability is given by the sign of this integral.*

Once some approximate solution is guessed, for applying Theorem 3.1 we need to compute the three constants appearing in its statement. In general, K and S can be easily obtained. Recall for instance that $||s||_2$, when s is a trigonometric polynomial, can be computed from Parseval's theorem. On the other hand, M is much more difficult to estimate. In Lemma 3.5 we give a result useful for computing it in concrete cases, that is different from the approach used in [97, 99, 100].

Assuming that a non-autonomous differential equation has a hyperbolic periodic orbit, the results of [99] also guarantee that, if given a suitable trigonometric polynomial $\bar{r}(t)$ of a sufficiently high degree, we can apply the first part of Theorem 3.1. Intuitively, while the value of the accuracy S goes to zero when we increase the degree of the trigonometric polynomial, the values M and K remain bounded. Thus, at some moment, it holds that $2M^2KS < 1$.

In Section 3.5 we apply Theorem 3.1 to study and localize the limit cycles of two planar polynomial autonomous systems. The first one is considered in Subsection 3.5.1 and is a simple example for which the exact limit cycle is already known. We do our study step by step to illustrate how the method suggested by Theorem 3.1 works in a concrete example. In particular we obtain an approximation $\bar{x}(t)$ of the periodic orbit by using a combination between the HBM until order 10 and a suitable choice of the convergents obtained from the theory of continuous fractions applied to the approach obtained by the HBM.

The second case corresponds to the system

$$\begin{cases} \dot{x} = -y + \frac{x}{10}(1 - x - 10x^2), \\ \dot{y} = \, x + \frac{y}{10}(1 - x - 10x^2). \end{cases}$$

In polar coordinates it is written as $\dot{r} = r/10 - \cos(\theta)r^2/10 - \cos^2(\theta)r^3$, $\dot{\theta} = 1$, or equivalently,

$$r' = \frac{dr}{dt} = \frac{1}{10}\, r - \frac{1}{10}\cos(t)\, r^2 - \cos^2(t)\, r^3, \tag{3.5}$$

and it has a unique positive periodic orbit; see also [56]. Notice that we have renamed θ as t. We prove the following:

Proposition 3.2. *Consider the periodic function*

$$\bar{r}(t) = \frac{4}{9} - \frac{1}{693}\cos(t) - \frac{1}{51}\sin(t) - \frac{1}{653}\cos(2t) - \frac{1}{45}\sin(2t) - \frac{1}{780}\cos(3t).$$

The differential equation (3.5) *then has a periodic solution* $r^*(t)$ *such that*

$$||\bar{r} - r^*||_\infty \leq 0.042,$$

which is hyperbolic and stable, and it is the only periodic solution of (3.5) *contained in this strip.*

As we will see, in this example we will find computational difficulties to obtain the third approximation given by the HBM. Therefore we will get it first by numerically approaching the periodic solution, then by numerically computing the first terms of its Fourier series and finally by using the continuous fractions approach to simplify the values appearing in our computations. We also will see that the same approach works for other concrete rigid systems. Similar examples for second-order differential equations have also been studied in [100].

3.2 Preliminary results

This section contains some technical lemmas that are useful for proving Theorem 3.1 and for obtaining in concrete examples the constants appearing in its statement. We also include a very short overview of the HBM adapted to the setting of one-dimensional 2π-periodic non-autonomous differential equations.

As usual, given $A \subset \mathbb{R}$, $\mathbf{1}_A : \mathbb{R} \to \mathbb{R}$ denotes the *characteristic function of* A: the function takes the value 1 when $x \in A$, and the value is 0 otherwise.

Lemma 3.3. *Let* $a(t)$ *and* $b(t)$ *be continuous real* 2π*-periodic functions. Consider the non-autonomous linear ordinary differential equation*

$$x' = a(t)x + b(t). \tag{3.6}$$

If $A(2\pi) \neq 0$*, where* $A(t) := \int_0^t a(s)ds$*, then for each* $b(t)$ *the equation* (3.6) *has a unique* 2π*-periodic solution* $x_b(t) := \int_0^{2\pi} H(t,s)b(s)ds$*, where the kernel* $H(t,s)$ *is given by the piecewise function*

$$H(t,s) = \frac{e^{A(t)}}{1 - e^{A(2\pi)}} \left[e^{-A(s)}\mathbf{1}_{[0,t]}(s) + e^{A(2\pi)-A(s)}\mathbf{1}_{[t,2\pi]}(s) \right]. \tag{3.7}$$

Moreover $||x_b||_\infty \le 2\pi \max_{t\in[0,2\pi]} ||H(t,\cdot)||_2 \, ||b||_2$.

Proof. Since (3.6) is linear, its general solution is

$$x(t) = e^{A(t)} \left(x_0 + \int_0^t b(s)e^{-A(s)}ds \right). \tag{3.8}$$

If we impose that the solution is 2π-periodic, i.e. $x(0) = x(2\pi)$, we get

$$x_0 = \frac{e^{A(2\pi)}}{1-e^{A(2\pi)}} \int_0^{2\pi} b(s)e^{-A(s)}ds, \tag{3.9}$$

then (3.8) becomes

$$\begin{aligned} x_b(t) &= \frac{e^{A(t)}}{1-e^{A(2\pi)}} \left[e^{A(2\pi)} \int_0^{2\pi} b(s)e^{-A(s)}ds + (1-e^{A(2\pi)}) \int_0^t b(s)e^{-A(s)}ds \right] \\ &= \frac{e^{A(t)}}{1-e^{A(2\pi)}} \left[e^{A(2\pi)} \int_t^{2\pi} b(s)e^{-A(s)}ds + \int_0^t b(s)e^{-A(s)}ds \right] \\ &= \int_0^{2\pi} H(t,s)b(s)ds. \end{aligned}$$

Therefore, the first assertion follows. On the other hand, by the Cauchy-Schwarz inequality,

$$|x_b(t)| \le \sqrt{\int_0^{2\pi} H^2(t,s)ds} \sqrt{\int_0^{2\pi} b^2(s)ds}.$$

Therefore,

$$||x_b||_\infty \le 2\pi \max_{t\in[0,2\pi]} ||H(t,\cdot)||_2 \, ||b||_2.$$

This completes the proof. □

Corollary 3.4. *A deformation constant M associated with $\bar{x}$ and X is*

$$M := 2\pi \max_{t\in[0,2\pi]} ||H(t,\cdot)||_2,$$

where H is given in (3.7) *with* $A(t) = \int_0^t \frac{\partial}{\partial x} X(\bar{x}(t),t)\,dt$.

Now we prove a technical result that will allow us to compute in practice deformation constants. In fact we will find an upper bound of M that will avoid the integration step needed in the computation of the norm $||\cdot||_2$. First, we introduce some notations.

Given a function $A : [0,2\pi] \to \mathbb{R}$, a positive number ℓ, and a partition $t_i = ih$ with $i = 0,1,\dots,N$, of the interval $[0,2\pi]$, where $h = 2\pi/N$, we consider the function $L : [0, 2\pi] \to \mathbb{R}$ given by the continuous linear piecewise function joining the points $(t_i, A(t_i) - \ell)$. Notice that $L(t) = \sum_{i=0}^{N-1} L_i(t)\mathbf{1}_{I_i}$, where $I_i = [t_i, t_{i+1}]$ and

$$L_i(t) = \frac{A(t_{i+1}) - A(t_i)}{h}(t - t_i) + f(t_i) := -\frac{1}{2}(\alpha_i t + \beta_i).$$

We will say that L is *an adequate lower bound of* A if it holds that $L(t) < A(t)$ for all $t \in [0, 2\pi]$. Clearly, smooth functions always have adequate functions that approach them.

For each $m = 0, 1, \ldots, N$ we define the function

$$\Psi_m(t) := \sum_{i=0}^{m-1} J_i + \lambda^2 \sum_{i=m-1}^{N-1} J_i + (1 - \lambda^2)\frac{e^{\beta_m}}{\alpha_m}\left(e^{\alpha_m t} - e^{\alpha_m t_m}\right), \tag{3.10}$$

where $\lambda = e^{A(2\pi)}$, and

$$J_i := \int_{t_i}^{t_{i+1}} e^{-2L(s)} ds = \int_{t_i}^{t_{i+1}} e^{-2L_i(s)} ds = \frac{e^{\beta_i}}{\alpha_i}\left(e^{\alpha_i t_{i+1}} - e^{\alpha_i t_i}\right).$$

Lemma 3.5. *Let L be an adequate lower bound of A, where A is the function given in Lemma 3.3. Consider the functions $\Psi_m(t)$, with $m = 0, 1, \ldots, N - 1$. Therefore, also following the notation introduced in that Lemma, it holds that $||x_b||_\infty \le N\,||b||_2$, where*

$$N = \frac{\sqrt{2\pi}}{|1 - \lambda|} \max_{t \in [0, 2\pi]} e^{A(t)} \sqrt{\sum_{m=0}^{N-1} \Psi_m(t)\mathbf{1}_{I_m}(t)}.$$

Proof. Recall that from Lemma 3.3, $||x_b||_\infty \le M\,||b||_2$, where

$$M := 2\pi \max_{t \in [0, 2\pi]} ||H(t, \cdot)||_2.$$

Thus, we will find an upper bound of M. Since

$$H(t, s) = \frac{e^{A(t)}}{1 - e^{A(2\pi)}}\left[e^{-A(s)}\mathbf{1}_{[0,t]}(s) + e^{A(2\pi) - A(s)}\mathbf{1}_{[t,2\pi]}(s)\right],$$

it holds that

$$||H(t, \cdot)||_2 = \frac{1}{\sqrt{2\pi}}\frac{e^{A(t)}}{|1 - \lambda|}\sqrt{G(t)}$$

where

$$G(t) := \int_0^t e^{-2A(s)} ds + \lambda^2 \int_t^{2\pi} e^{-2A(s)} ds < \int_0^t e^{-2L(s)} ds + \lambda^2 \int_t^{2\pi} e^{-2L(s)} ds,$$

because $L(t) < A(t)$ for all $t \in [0, 2\pi]$.

Assume that $t \in I_m$. Thus,

$$\int_0^t e^{-2L(s)}ds = \sum_{i=0}^{m-1} J_i + \int_{t_m}^t e^{-2L_m(s)}ds,$$
$$\int_t^{2\pi} e^{-2L(s)}ds = \sum_{i=m}^{N-1} J_i + \int_t^{t_{m+1}} e^{-2L_m(s)}ds = \sum_{i=m-1}^{N-1} J_i - \int_{t_m}^t e^{-2L_m(s)}ds.$$

Therefore, for $t \in I_m$,

$$G(t) < \sum_{i=0}^{m-1} J_i + \lambda^2 \sum_{i=m-1}^{N-1} J_i + (1-\lambda^2)\int_{t_m}^t e^{\alpha_m s+\beta_m}ds = \Psi_m(t).$$

As a consequence, for $t \in [0, 2\pi]$,

$$G(t) < \sum_{m=0}^{N-1} \Psi_m(t)\mathbf{1}_{I_m}(t),$$

and the result follows. □

Remark 3.6. *Notice that the above lemma provides a way for computing a deformation constant where there is no need of computing integrals. This will be very useful in concrete application, where the primitive of* $e^{-2A(t)}$ *is not computable, and so Corollary 3.4 is difficult to apply for obtaining* M.

In the next result, which introduces the constant K appearing in Theorem 3.1, D° denotes the topological interior of D.

Lemma 3.7. *Consider* X *as in* (3.1). *Let* D *be a closed interval, and let* $\bar{x}(t)$ *be a* 2π*-periodic* $\mathcal{C}^1$*-function such that* $\{\bar{x}(t) : t \in \mathbb{R}\} \subset D^\circ$. *Define*

$$R(z,t) := X(\bar{x}(t)+z,t) - X(\bar{x}(t),t) - \frac{\partial}{\partial x}X(\bar{x}(t),t)z \tag{3.11}$$

for all z *such that* $\{\bar{x}(t)+z : t \in \mathbb{R}\} \subset D$. *Then*

(*i*) $|R(z,t)| \le \frac{K}{2}|z|^2$,

(*ii*) $|R(z,t) - R(\bar{z},t)| \le K \max(|z|,|\bar{z}|)\,|z-\bar{z}|$,

where

$$K := \max_{(x,t)\in D\times[0,2\pi]} \left|\frac{\partial^2}{\partial x^2}X(x,t)\right|.$$

Proof. (i). By using Taylor's formula, for each t it holds that

$$X(\bar{x}(t)+z,t) = X(\bar{x}(t),t) + \frac{\partial}{\partial x}X(\bar{x}(t),t)z + \frac{1}{2}\frac{\partial^2}{\partial x^2}X(\xi(t),t)z^2$$

for some $\xi(t) \in \langle \bar{x}(t), \bar{x}(t)+z \rangle$. Therefore

$$|R(z,t)| = \left|\frac{1}{2}\frac{\partial^2}{\partial x^2}X(\xi(t),t)\right| |z|^2 \leq \frac{K}{2}|z|^2,$$

as we wanted to prove.
(ii). From Rolle's theorem, for each fixed t it follows that there exists $\eta(t) \in \langle z, \bar{z} \rangle$ such that

$$|R(z,t) - R(\bar{z},t)| \leq \left|\frac{\partial}{\partial z}R(\eta(t),t)\right| |z-\bar{z}|.$$

Applying again this theorem, but now to $\frac{\partial}{\partial z}R$, and by noticing that $\frac{\partial}{\partial z}R(z,t)\big|_{z=0} = 0$, we obtain

$$\left|\frac{\partial}{\partial z}R(\eta(t),t)\right| \leq \left|\frac{\partial^2}{\partial z^2}R(\omega(t),t)\right| |\eta(t)| = \left|\frac{\partial^2}{\partial x^2}X(\omega(t),t)\right| |\eta(t)| \leq K|\eta(t)|,$$

where $\omega(t) \in \langle 0, \eta(t) \rangle$. Note also that

$$|\eta(t)| \leq \max(|z|, |\bar{z}|).$$

Hence, the result follows combining the three inequalities. □

3.3 The HBM for non-autonomous equations

Although in Section 1.3 of Chapter 1 we defined the HBM, in this section we want to recall it adapted to the setting of one-dimensional 2π-periodic non-autonomous differential equations. The main difference between the non-autonomous case treated here and the autonomous one is that in the second situation the periods of the searched periodic orbits, or equivalently their frequencies, are also treated as unknowns (see Chapter 1 and 2).

So, we are interested in finding periodic solutions of the 2π-periodic differential equation (3.1), or equivalently, periodic functions which satisfy the following functional equation

$$\mathcal{F}(x(t)) := x'(t) - X(x(t),t) = 0. \tag{3.12}$$

Since any smooth 2π-periodic function $x(t)$ can be written as its Fourier series

$$x(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(mt) + b_m \sin(mt)\right),$$

where

$$a_m = \frac{1}{\pi}\int_0^{2\pi} x(t)\cos(mt)\,dt, \quad \text{and} \quad b_m = \frac{1}{\pi}\int_0^{2\pi} x(t)\sin(mt)\,dt,$$

for all $m \geq 0$. Then, it is natural to try to approximate the periodic solutions of the functional equation (3.12) by using truncated Fourier series, *i.e.* trigonometric polynomials.

Now, let us describe the HBM of order N. Consider a trigonometric polynomial

$$y_{N}(t) = \frac{r_0}{2} + \sum_{m=1}^{N}(r_m\cos(mt) + s_m\sin(mt))$$

with unknowns $r_m = r_m(N), s_m = s_m(N)$ for all $m \leq N$. Compute then the 2π-periodic function $\mathcal{F}(y_{N}(t))$. It has also an associated Fourier series

$$\mathcal{F}(y_{N}(t)) = \frac{\mathcal{A}_0}{2} + \sum_{m=1}^{\infty}(\mathcal{A}_m\cos(mt) + \mathcal{B}_m\sin(mt))\,,$$

where $\mathcal{A}_m = \mathcal{A}_m(\mathbf{r},\mathbf{s})$ and $\mathcal{B}_m = \mathcal{B}_m(\mathbf{r},\mathbf{s})$, $m \geq 0$, with $\mathbf{r} = (r_0, r_1, \ldots, r_{N})$ and $\mathbf{s} = (s_1, \ldots, s_{N})$. The HBM consists of finding values $\mathbf{r}$ and $\mathbf{s}$ such that

$$\mathcal{A}_m(\mathbf{r},\mathbf{s}) = 0 \quad \text{and} \quad \mathcal{B}_m(\mathbf{r},\mathbf{s}) = 0 \quad \text{for} \quad 0 \leq m \leq N. \tag{3.13}$$

As we have seen in previous chapters, the above set of equations is usually a very difficult nonlinear system of equations, and for this reason in various works including [78] and the references therein, only small values of N are considered. We also remark that in general the coefficients of $y_{N}(t)$ and $y_{N+1}(t)$ do not coincide at all.

The hope of the method is that the trigonometric polynomials found using this approach are “near” actual periodic solutions of the differential equation (3.1). In any case, as far as we know, the HBM for small N is only a heuristic method that sometimes works quite well.

3.4 Proof of the main Theorem

Proof of Theorem 3.1. As a first step we prove the following result: consider the nonlinear differential equation

$$z' = X(z + \bar{x}(t), t) - X(\bar{x}(t), t) - s(t), \tag{3.14}$$

where $s(t)$ is given in (3.4). A 2π-periodic function $z(t)$ is then a solution of (3.14) if and only if $z(t) + \bar{x}(t)$ is a 2π-periodic solution of (3.1).

This is a consequence of the following equalities:

$$(z(t)+\bar{x}(t))' = [X(z(t)+\bar{x}(t),t) - X(\bar{x}(t),t) - s(t)] + [X(\bar{x}(t),t) + s(t)]$$
$$= X(z(t)+\bar{x}(t),t).$$

By using the function

$$R(z,t) = X(z+\bar{x}(t),t) - X(\bar{x}(t),t) - \frac{\partial}{\partial x}X(\bar{x}(t),t)z,$$

introduced in Lemma 3.7, equation (3.14) can be written as

$$z' = \frac{\partial}{\partial x}X(\bar{x}(t),t)z + R(z,t) - s(t). \tag{3.15}$$

Let $\mathcal{P}$ be the space of 2π-periodic $\mathcal{C}^0$-functions. To prove the first part of the theorem it suffices to see that equation (3.15) has a unique $\mathcal{C}^1$, 2π-periodic solution $z^*(t)$, which belongs to the set

$$\mathcal{N} = \{z \in \mathcal{P} : ||z||_\infty \leq 2MS\}.$$

To prove this last assertion, we will construct a contractive map $T : \mathcal{N} \to \mathcal{N}$. Because $\mathcal{N}$ is a complete space with the $||\cdot||_\infty$ norm, its fixed point will be a continuous function in $\mathcal{N}$ that will satisfy an integral equation, equivalent to (3.15). Finally we will see that this fixed point is in fact a $\mathcal{C}^1$ function, and it satisfies equation (3.15).

Let us define T. If $z \in \mathcal{N}$, then $T(z)$ is defined as the unique 2π-periodic solution of the linear differential equation

$$y' = \frac{\partial}{\partial x}X(\bar{x}(t),t)y + R(z(t),t) - s(t).$$

Notice that this map is well-defined, by Lemma 3.3, because $\bar{x}(t)$ is noncritical w.r.t. equation (3.1). Thus, z_1 satisfies

$$z_1' = \frac{\partial}{\partial x}X(\bar{x}(t),t)z_1 + R(z(t),t) - s(t).$$

Let us prove that T maps $\mathcal{N}$ into $\mathcal{N}$ and that it is a contraction. By Lemmas 3.3 and 3.7 and the hypotheses of the theorem

$$||T(z)||_\infty = ||z_1||_\infty \leq M||R(z(\cdot),\cdot) - s(\cdot)||_2 \leq M\left(||R(z(\cdot),\cdot)||_2 + S\right),$$
$$\leq M(||R(z(\cdot),\cdot)||_\infty + S) \leq M(\frac{K}{2}||z||_\infty^2 + S),$$
$$\leq M(2KM^2S^2 + S) < 2MS,$$

where we have used in the last inequality that $2M^2KS < 1$.

To show that T is a contraction on $\mathcal{N}$, take $z, \bar{z} \in \mathcal{N}$ and denote by $z_1 = T(z)$, $\bar{z}_1 = T(\bar{z})$. Then

$$z_1' = \frac{\partial}{\partial x} X(\bar{x}(t), t) z_1 + R(z(t), t) - s(t),$$
$$\bar{z}_1' = \frac{\partial}{\partial x} X(\bar{x}(t), t) \bar{z}_1 + R(\bar{z}(t), t) - s(t).$$

Therefore,

$$(z_1 - \bar{z}_1)' = \frac{\partial}{\partial x} X(\bar{x}(t), t)(z_1 - \bar{z}_1) + R(z(t), t) - R(\bar{z}(t), t).$$

Again by Lemmas 3.3 and 3.7 and the hypotheses of the theorem,

$$\begin{aligned} ||T(z) - T(\bar{z})||_\infty =& ||z_1 - \bar{z}_1||_\infty \leq M||R(z(\cdot), \cdot) - R(\bar{z}(\cdot), \cdot)||_\infty \\ \leq& MK \max(||z||_\infty, ||\bar{z}||_\infty)||z - \bar{z}||_\infty \leq 2M^2KS||z - \bar{z}||_\infty, \end{aligned}$$

as we wanted to prove, because recall that $2M^2KS < 1$.

Thus, the sequence of functions $\{z_n(t)\}$ defined as

$$z_{n+1}'(t) = \frac{\partial}{\partial x} X(\bar{x}(t), t) z_{n+1}(t) + R(z_n(t), t) - s(t),$$

with any $z_0(t) \in \mathcal{N}$, and $z_{n+1}(t)$ chosen to be periodic, converges uniformly to some function $x^*(t) \in \mathcal{N}$. In fact we also have that

$$z_{n+1}(t) = z_{n+1}(0) + \int_0^t \left(\frac{\partial}{\partial x} X(\bar{x}(w), w) z_{n+1}(w) + R(z_n(w), w) - s(w) \right) dw.$$

Therefore,

$$x^*(t) = x^*(0) + \int_0^t \left(\frac{\partial}{\partial x} X(\bar{x}(w), w) x^*(w) + R(x^*(w), w) - s(w) \right) dw.$$

We know that $x^*(t)$ is a continuous function, but from the above expression we obtain that it is indeed of class $\mathcal{C}^1$. Therefore $x^*(t)$ is a periodic solution of (3.15) and is the only one in $\mathcal{N}$, as we wanted to see.

To prove the hyperbolicity of $x^*(t)$, it suffices to show that

$$\int_0^{2\pi} \frac{\partial}{\partial x} X(x^*(t), t) dt \neq 0,$$

and study its sign; see [68]. We have that, fixed t,

$$\frac{\partial}{\partial x} X(x^*(t), t) = \frac{\partial}{\partial x} X(\bar{x}(t), t) + \frac{\partial^2}{\partial x^2} X(\xi(t), t)(x^*(t) - \bar{x}(t)),$$

for some $\xi(t) \in \langle x^*(t), \bar{x}(t)\rangle$. Therefore, since we have already proved that $|x^*(t) - \bar{x}(t)| < 2MS$,

$$\left| \frac{\partial}{\partial x} X(\bar{x}(t), t) - \frac{\partial}{\partial x} X(x^*(t), t) \right| \leq 2KMS.$$

Then

$$\left| \int_0^{2\pi} \frac{\partial}{\partial x} X(\bar{x}(t), t) dt - \int_0^{2\pi} \frac{\partial}{\partial x} X(x^*(t), t) dt \right| \leq 4\pi KMS < \frac{2\pi}{M},$$

and the result follows because by hypothesis the first integral is, in absolute value, bigger that $2\pi/M$. □

3.5 Applications to some planar rigid systems

In this section we apply Theorem 3.1 for proving the existence and uniqueness of hyperbolic limit cycles, in a suitable region, of some planar rigid cubic systems which after some transformations can be converted into differential equation of the form (3.1).

Concretely, we consider the family of planar differential systems

$$\left\{ \begin{array}{rl} \dot{x} = & -y + x(a + bx + cy + dx^2 + exy + fy^2), \\ \dot{y} = & x + y(a + bx + cy + dx^2 + exy + fy^2), \end{array} \right. \tag{3.16}$$

where $a, b, c, d, e, f \in \mathbb{R}$ and $d^2 + e^2 + f^2 \neq 0$. In polar coordinates it writes as

$$\dot{r} = ar + (b\cos(\theta) + c\sin(\theta))r^2 + (d\cos^2(\theta) + e\sin(\theta)\cos(\theta) + f\sin^2(\theta))r^3, \quad \dot{\theta} = 1,$$

or equivalently,

$$r' = \frac{dr}{dt} = ar + (b\cos(t) + c\sin(t))r^2 + (d\cos^2(t) + e\sin(t)\cos(t) + f\sin^2(t))r^3, \tag{3.17}$$

where we have renamed θ as t.

We study two cases. In the first case, we consider a concrete integrable case. Although we know explicitly the limit cycle in that system, we first use the HBM to approximate it and then Theorem 3.1 to prove, in an alternative way, its existence. In the second case, we found numerically an approximation of the limit cycle and from this approximation we propose a truncated Fourier series as a simpler approximation. Finally, Theorem 3.1 is used again to prove the existence and localize the limit cycle.

3.5.1 An integrable case

By fixing $a = -1$, $d = 3$, $e = 2$ and $f = 1$ in (3.17) we obtain a Bernoulli equation that can be solved explicitly. Then we have

$$r' = -r + (\cos(2t) + \sin(2t) + 2)r^3 := X(r,t). \tag{3.18}$$

Its solutions are $r(t) \equiv 0$ and

$$r(t) = \pm\frac{1}{\sqrt{2 + \cos(2t) + ke^{2t}}}.$$

Therefore its unique positive periodic solution, which corresponds to the only limit cycle of (3.16) for the given values of the parameters, is given by the ellipse

$$r^*(t) = \frac{1}{\sqrt{2 + \cos(2t)}}. \tag{3.19}$$

Moreover since

$$\int_0^{2\pi} \frac{\partial}{\partial r} X(r^*(t), t)\, dt = 4\pi > 0,$$

it is hyperbolic and unstable, see [68]. Its Fourier series is

$$r^*(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_{2k}\cos(2kt), \tag{3.20}$$

where

$$\begin{aligned}
a_0 &= \tfrac{4K}{\sqrt{3}\pi} \approx 1.491498374, \quad a_0/2 \approx 0.745749187,\\
a_2 &= \tfrac{12E-8K}{\sqrt{3}\pi} \approx -0.2016837219,\\
a_4 &= \tfrac{-32E+20K}{\sqrt{3}\pi} \approx 0.04065713288,\\
a_6 &= \tfrac{476E-296K}{\sqrt{3}\pi} \approx -0.009092598292,\\
a_8 &= \tfrac{-10624E+6604K}{\sqrt{3}\pi} \approx 0.002133790322,\\
a_{10} &= \tfrac{105548E-65608K}{\sqrt{3}\pi} \approx -0.0005148662408,
\end{aligned}$$

being $K = K(\sqrt{6}/3)$ and $E = E(\sqrt{6}/3)$ the complete elliptic integrals of the first and second kind respectively, see [12].

Let us forget that we know the exact solution and its full Fourier series to illustrate how to use the HBM and Theorem 3.1 for equation (3.18) to obtain an approach to the actual periodic solution (3.19).

Following the HBM, consider the equation

$$\mathcal{F}(r(t)) = r'(t) + r(t) - (\cos(2t) + \sin(2t) + 2)r^3(t) = 0, \tag{3.21}$$

which is clearly equivalent to (3.18).

Searching for a solution of the form $r(t) = r_0$ and imposing that the first harmonic of $\mathcal{F}(r(t))$ vanishes we get that $r_0 + 2r_0^3 = 0$. The only positive solution of the equation is $r_0 = \sqrt{2}/2 \approx 0.7071$ and this is the first order solution given by HBM.

Motivated by the symmetries of (3.18) for applying the second order HBM we search for an approximation of the form

$$r(t) = r_0 + r_2 \cos(2t).$$

The vanishing of the coefficients of 1 and $\cos(2t)$ in the Fourier series of $\mathcal{F}(r(t))$ give the nonlinear system:

$$g(r_0, r_2) := r_0 - 2r_0^3 - \frac{3}{2}r_2 r_0^2 - 3r_2^2 r_0 - \frac{3}{8}r_2^3 = 0,$$
$$h(r_0, r_2) := r_2 - r_0^3 - 6r_2 r_0^2 - \frac{9}{4}r_2^2 r_0 - \frac{3}{2}r_2^3 = 0.$$

Doing the resultants $\mathrm{Res}(g, h, r_0)$, $\mathrm{Res}(g, h, r_2)$ we obtain that the solutions of the above system are also solutions of

$$219720r_0^8 - 18852r_0^6 + 4269r_0^4 - 328r_0^2 + 8 = 0,$$
$$49437r_2^8 - 70956r_2^6 + 30708r_2^4 - 4288r_2^2 + 128 = 0.$$

One of its solutions is $r_0 \approx 0.7440456581 =: \tilde{r}_0$, $r_2 \approx -0.2013905597 =: \tilde{r}_2$.

To know the accuracy of the periodic function $\tilde{r}(t) = \tilde{r}_0 + \tilde{r}_2 \cos(2t)$ as a solution of (3.18) we compute

$$\widetilde{S} = ||\tilde{r}'(t) + \tilde{r}(t) - (2 + \sin(2t) + \cos(2t))\tilde{r}(t)^3||_2 \approx 0.1361$$

Since it is enough for our purposes we can consider simpler rational approximations of $\tilde{r}_0$ and $\tilde{r}_1$, but keeping a similar accuracy. For finding these rational approximations, we search them doing the continuous fraction expansion of these values. For instance

$$\tilde{r}_0 = [0, 1, 2, 1, 9, 1, 21, 17, 3, 11]$$

giving the convergents 1, 2/3, 3/4, 29/39, 32/43,.... Similarly $\tilde{r}_2$ gives 1/4, 1/5, 28/139, 29/144,.... At this point we have the following new candidate to be an approximation of the periodic solution

$$\bar{r}(t) = \frac{3}{4} - \frac{1}{5}\cos(2t).$$

Its accuracy w.r.t. equation (3.18) is

$$S = ||\bar{r}'(t) + \bar{r}(t) - (2 + \sin(2t) + \cos(2t))\bar{r}(t)^3||_2 = \frac{\sqrt{50069}}{1600} \approx 0.1398 < 0.14,$$

and so, quite similar to the one of $\tilde{r}(t)$.

Therefore $\tilde{r}(t)$ and $\bar{r}(t)$ are solutions of (3.18) with similar accuracy so we keep $\bar{r}(t)$ as the second order approximation given by this modification of the HBM. For this $\bar{r}(t)$ we already know that its accuracy is $S = 0.14$.

We need to know the value of M given in Theorem 3.1. With this aim we will apply Lemma 3.5. We consider in that lemma a function $L(t)$ formed by 13 straight lines and $\ell = 1/9$. Then we get that we can take $M = 2.3$. Therefore, since $2MS = 0.644$ and $0.55 = \frac{11}{20} \le \bar{r}(t) \le \frac{19}{20} = 0.95$.

We have that $I = [-0.094, 1.594]$ in Theorem 3.1. Moreover

$$\left|\frac{\partial^2}{\partial r^2}X(r,t)\right| \le 6|2 + \sin(2t) + \cos(2t)||r| \le (12 + 6\sqrt{2})|r| \le \frac{41}{2}|r|$$

Thus taking $K = \frac{41}{2}(1.594) \approx 32.68$ we get that $2M^2KS \approx 48.4 > 1$ and we can not apply Theorem 3.1.

Doing similar computations with the successive approaches given by the HBM we obtain

$$\begin{aligned} \bar{r}(t) &= \tfrac{3}{4} - \tfrac{1}{5}\cos(2t) + \tfrac{1}{25}\cos(4t), \\ \bar{r}(t) &= \tfrac{3}{4} - \tfrac{1}{5}\cos(2t) + \tfrac{1}{25}\cos(4t) - \tfrac{1}{110}\cos(6t). \end{aligned}$$

It is worth to comment that the above two functions are periodic functions that approximate to solution of (3.18) with accuracies 0.045 and 0.018, respectively, while the solutions obtained solving approximately the nonlinear systems with ten significative digits have similar accuracies, namely 0.043 and 0.013, respectively. For none of both approaches Theorem 3.1 applies. Let us see that the next order HBM works for this example.

If we do all the computations we obtain the candidate to be solution

$$\tilde{r}(t) = \sum_{k=0}^{4} r_{2k}\cos(2kt),$$

with

$$\begin{aligned} &r_0 = 0.7457489122, \quad &&r_2 = -0.2016836610, \quad &&r_4 = 0.04065712547, \\ &r_6 = -\,0.009092599917, \quad &&r_8 = 0.002133823488. && \end{aligned}$$

Computing the accuracy of $\tilde{r}(t)$ we obtain that it is 0.0039. If we take the approximation, using some convergents of r_{2k},

$$\bar{r}(t) = \frac{3}{4} - \frac{1}{5}\cos(2t) + \frac{1}{25}\cos(4t) - \frac{1}{110}\cos(6t) + \frac{1}{468}\cos(8t)$$

it has accuracy 0.0125. This means that we have lost significative digits and we need to take convergents of r_{2k} that have at least 3 significative digits. For instance some convergents of r_0 are 1, 2/3, 3/4, 41/55, 44/59,... and we choose 44/59. Finally we consider

$$\bar{r}(t) = \frac{44}{59} - \frac{24}{119}\cos(2t) + \frac{2}{49}\cos(4t) - \frac{1}{110}\cos(6t) + \frac{1}{468}\cos(8t). \tag{3.22}$$

The accuracy of $\bar{r}$ is 0.00394 quite similar to the one of $\tilde{r}(t)$. So we take $S = 0.004$. Let us see that Theorem 3.1 applies if we take this approximate periodic solution.

In this case, by applying Lemma 3.5, using the piecewise linear function L formed by 10 pieces and $\ell = 1/10$, we obtain that we can take $M = 2.4$.

Since it can be seen that $0.5 \le \bar{r}(t) \le 1$ and $2MS = 0.0192$ we can take in Theorem 3.1 the interval $I := [0.4808, 1.0192]$.

Then

$$\max_{I\times[0,2\pi]} \left| \frac{\partial^2}{\partial r^2} X(r,t) \right| \le \frac{41}{2}(1.02) = 20.91 =: K.$$

Finally, $2M^2KS \approx 0.96 < 1$ and Theorem 3.1 applies.

Finally, it is easy to see that

$$\int_0^{2\pi} \frac{\partial}{\partial r} X(\bar{r}(t), t)dt > 12.5,$$

which is bigger than $2\pi/M \approx 2.6$. Therefore the hyperbolicity of the periodic orbit given by Theorem 3.1 follows. In short we have proved,

Proposition 3.8. *Consider the periodic function $\bar{r}(t)$ given in* (3.22)*. Then there is a periodic solution $r^*(t)$ of* (3.18)*, such that*

$$||\bar{r} - r^*||_\infty \le 0.0192,$$

which is hyperbolic and unstable and it is the only periodic solution of (3.18) *in this strip.*

Remark 3.9. *Using the known analytic expression of $r^*(t)$ it can be seen that indeed*

$$||\bar{r} - r^*||_\infty \le 0.0007.$$

Notice that by using a high enough HBM we have obtained a proof of the existence of a hyperbolic periodic orbit and an effective approximation $\bar{r}(t)$ without integrating the differential equation.

3.5.2 A biparametric family

Take $d = 1$, $e = 0$, $f = 0$ in the family of rigid cubic systems (3.16), that is,

$$\begin{cases} \dot{x} &= -y - x(a + bx + x^2), \\ \dot{y} &= x - y(a + bx + x^2), \end{cases} \tag{3.23}$$

already considered in [56]. In that paper it is proved that (3.23) has at most one limit cycle, and when it exists, it is hyperbolic. We study some concrete cases.

With our point of view we will find an explicit approximation of the limit cycle; see Proposition 3.2. In order to simplify the computations we first consider the case $a = -b = 1/10$, which in polar coordinates is written as (3.5):

$$r' = \frac{dr}{dt} = \frac{1}{10}\,r - \frac{1}{10}\cos(t)\,r^2 - \cos^2(t)\,r^3 := X(r,t).$$

We want to find an approximation of the periodic solution of (3.5), which we will use for applying Theorem 3.1.

First attempt: the HBM. According to section 3.3, we consider the equation

$$\mathcal{F}(r(t)) = r'(t) - \frac{1}{10}\,r + \frac{1}{10}\cos(t)\,r^2 + \cos^2(t)\,r^3 = 0, \tag{3.24}$$

which is clearly equivalent to (3.5).

Searching for a solution of the form $r(t) = r_0$ and imposing that the first harmonic of

$$\frac{1}{2}r_0^3 - \frac{1}{10}r_0 + \frac{1}{10}\cos(t)r_0^2 + \frac{1}{2}\cos(2t)r_0^3$$

vanishes, we obtain

$$\frac{1}{2}r_0\left(r_0^2 - \frac{1}{5}\right) = 0.$$

Hence $r_0 = \sqrt{5}/5 \approx 0.4472135954$ is the first order solution given by the HBM. We obtain that the positive approximate solution is $r = \sqrt{5}/5$. For applying the second-order HBM we search for an approximation of the form

$$r(t) = r_0 + r_1\cos(t) + s_1\sin(t).$$

The vanishing of the coefficients of 1, $\cos(t)$ and $\sin(t)$ in $\mathcal{F}(r(t))$, provides the nonlinear system

$$\begin{aligned}
f(r_0,r_1,s_1) &:= \frac{1}{2}r_0^2 + \frac{9}{8}r_1^2 - \frac{1}{10} + \frac{3}{8}s_1^2 + \frac{1}{10}r_1 = 0,\\
g(r_0,r_1,s_1) &:= \frac{9}{4}r_0^2r_1 - \frac{5}{8}r_1^3 + \frac{3}{8}r_1s_1^2 + \frac{1}{10}r_0^2 + \frac{3}{40}r_1^2 + \frac{1}{40}s_1^2 - \frac{1}{10}r_1 + s_1 = 0,\\
h(r_0,r_1,s_1) &:= \frac{3}{4}r_0^2s_1 + \frac{3}{8}r_1^2s_1 + \frac{1}{8}s_1^3 + \frac{1}{20}r_1s_1 - \frac{1}{10}s_1 - r_1 = 0.
\end{aligned}$$

Doing the resultants $\mathrm{Res}(f, g, r_0)$ and $\mathrm{Res}(f, h, r_0)$ we obtain respectively

$$1775\, {r_1}^3 + 525\, r_1 {s_1}^2 + 240\, {r_1}^2 + 20\, {s_1}^2 - 132\, r_1 - 400\, s_1 - 8 = 0,$$
$$105\, {r_1}^2 s_1 + 35\, {s_1}^3 + 8\, r_1 s_1 + 80\, r_1 - 4\, s_1 = 0.$$

Repeating the resultant between these last two equations with respect to r_1 we have

$$1715000000\, {s_1}^9 + 3697050000\, {s_1}^7 - 3528000000\, {s_1}^6 - 45425216000\, {s_1}^5 - 988160000\, {s_1}^4 - 55802705600\, {s_1}^3 + 26417920000\, {s_1}^2 + 370408960000\, s_1 + 7270400000 = 0.$$

The approximate real solution of this equation is $\tilde{s}_1 = -0.0196567414$, and then we have the respective approximate solutions $\tilde{r}_0 = 0.4471066159$, $\tilde{r}_1 = -0.0009814101$.

For our purposes we can consider simpler rational approximations of $\tilde{r}_0$, $\tilde{r}_1$ and $\tilde{s}_1$, with maintaining a similar accuracy. For finding these rational approximations, we seek them by performing the continued fraction expansion of these values. For instance,

$$\tilde{r}_0 = [0, 2, 4, 4, 2, 2, 2, 2, 4, 2, 1, 1],$$

giving the convergents 0, 1/2, 4/9, 17/38, 38/85,... Similarly $\tilde{r}_1$ gives -1, 0, $-1/1018$, $-1/1019$,... and $\tilde{s}_1$ gives -1, 0, $-1/50$, $-1/51$,... At this point we have the following new candidate to be an approximation of the periodic solution

$$\bar{r}(t) = \frac{1}{2} - \frac{1}{1018}\cos(t) - \frac{1}{50}\sin(t).$$

Its accuracy w.r.t. equation (3.5) is

$$S = ||\bar{r}'(t) - \frac{1}{10}\bar{r}(t) + \frac{1}{10}\cos(t)\bar{r}(t)^2 + \cos^2(t)\,\bar{r}(t)^{\,3}||_2 \approx 0.046.$$

Doing all the computations needed to apply Theorem 3.1 we get that we are not under its hypotheses. Therefore we need to continue with the third-order HBM.

Performing the third-order approach we obtain five algebraic polynomial equations that we omit for the sake of simplicity. Unfortunately, neither using the resultant method as in the previous case, nor using the more sophisticated tool of Gröbner basis, our computers are able to obtain an approximate solution to start our theoretical analysis.

A numerical approach. First, we search for a numerical solution of (3.5) by using the Taylor series method. From this approximation we compute, again numerically, its first Fourier terms obtaining

$$\tilde{r}(t) = \sum_{k=0}^{3} r_k \cos(kt) + s_k \sin(kt),$$

where

$$
\begin{aligned}
&r_0 = 0.4483561517, && r_1 = -0.0024133439, && s_1 = -0.0193837572,\\
&r_2 = -\,0.0037463296, && s_2 = -0.0220176517,\\
&r_3 = -\,0.0012390886, && s_3 = 0.0003784656.
\end{aligned}
$$

The accuracy of $\tilde{r}(t)$ is 0.00289. If we take a new simpler approximation, using again some convergents of r_k and s_k, we obtain

$$\bar{r}(t) = \frac{4}{9} - \frac{1}{693}\cos(t) - \frac{1}{51}\sin(t) - \frac{1}{653}\cos(2t) - \frac{1}{45}\sin(2t) - \frac{1}{780}\cos(3t), \tag{3.25}$$

with accuracy 0.00298, quite similar to the one of $\tilde{r}(t)$. Note that (3.25) is precisely the approximation of the periodic solution of (3.5) stated in Proposition 3.2.

Proof of Proposition 3.2. We already know that the accuracy of $\bar{r}(t)$ is $S := 0.003$. To apply Theorem 3.1 we will compute M and K.

First we calculate $A(t) = \int_0^t \frac{\partial}{\partial r} X(\bar{r}(t), t)$.

$$
\begin{aligned}
A(t) = {} & \tfrac{2891685439}{72733752000} - \tfrac{3478883508132995559}{1778094556332494400}t - \tfrac{561179}{36756720}\cos(t) - \tfrac{685338551}{8000712720}\sin(t)\\
& - \tfrac{757058717}{48004276320}\cos(2t) - \tfrac{40221206418131}{273447836421760}\sin(2t) - \tfrac{2923231}{576974475}\cos(3t)\\
& + \tfrac{37724429}{36003207240}\sin(3t) - \tfrac{353400139}{96008552640}\cos(4t) + \tfrac{176710017086539999}{42674269351979865600}\sin(4t)\\
& + \tfrac{5358811}{300026727000}\cos(5t) + \tfrac{4708003}{20001781800}\sin(5t) + \tfrac{1537}{207810720}\cos(6t)\\
& + \tfrac{43551971479}{1438264594166400}\sin(6t) + \tfrac{1}{327600}\cos(7t) - \tfrac{1}{4753840}\sin(7t) - \tfrac{1}{12979200}\sin(8t).
\end{aligned}
$$

Now, by using Lemma 3.5, we find a deformation constant M. In this case we use as a lower bound for A the piecewise function L formed by 7 straight lines and $\ell = 1/18$. We obtain that we can take $M = 7$. Therefore $2MS \approx 0.042$. Since it can be seen that $0.4 \le \bar{r}(t) \le 0.47$ we can consider the interval $I = [0.358, 0.512]$ in Theorem 3.1. In addition,

$$\max_{I\times[0,2\pi]} \left| \frac{\partial^2}{\partial r^2} X(r,t) \right| \le \frac{1}{5} + 6||\bar{r}||_\infty = \frac{1}{5} + 6(0.512) = 3.272 =: K$$

Finally, $2M^2KS \approx 0.962 < 1$, and the first part of Theorem 3.1 applies. Hence equation (3.5) has a periodic solution $r^*(t)$ satisfying

$$||\bar{r} - r^*||_\infty \le 0.042, \tag{3.26}$$

which is the only one in this strip.

Moreover,

$$\left|\int_0^{2\pi} \frac{\partial}{\partial r} X(\bar{r}(t), t) dt\right| > 1.2.$$

Since $2\pi/M \approx 0.9$, the hyperbolicity of $r^*(t)$ follows by applying the second part of the theorem. □

Notice that the example of the system (3.23) that we have studied is $a = \lambda$ and $b = -\lambda$ with $\lambda = 1/10$. With the same techniques we see that the same function $\bar{r}(t)$ given in the statement of Proposition 3.2 is an approximation of the unique periodic orbit of the system when $|\lambda - 1/10| < 1/500$, which also satisfies (3.26).

Part II

Bifurcation diagram of planar vector fields

Chapter 4

A family of rotated vector fields

4.1 Introduction and main results

Consider the one-parameter family of quintic differential systems

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + (a - x^2)(y + y^3), \quad a \in \mathbb{R}. \end{cases} \tag{4.1}$$

Notice that without the term y^3, (4.1) coincides with the famous van der Pol system. This family was studied in [102] and the authors concluded that it has only two bifurcation values, 0 and a^*, and exactly four different global phase portraits on the Poincaré disc. Moreover, they concluded that there exists $a^* \in (0, \sqrt[3]{9\pi^2/16}) \approx (0, 1.77)$, such that the system has limit cycles only when $0 < a < a^*$ and then if the limit cycle exists, is unique and hyperbolic. Later, it was pointed out in [59] that the proof of the uniqueness of the limit cycle had a gap and a new proof was presented.

System (4.1) has no periodic orbits when $a \leq 0$ because in this case the function $x^2 + y^2$ is a global Lyapunov function. Thus, from now on, we restrict our attention to the case $a > 0$ and for convenience we write $a = b^2$, with $b > 0$. That is, we consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + (b^2 - x^2)(y + y^3), \quad b \in \mathbb{R}^+ \cup \{0\}. \end{cases} \tag{4.2}$$

Therefore the above family has limit cycles if and only if $b \in (0, b^*)$ with $b^* = \sqrt{a^*}$ and $b^* \in (0, \sqrt[6]{9\pi^2/16}) \approx (0, 1.33)$. Following [102] we also know that the value $b = 0$ corresponds to a Hopf bifurcation and the value b^* to the disappearance of the limit cycle in an unbounded polycycle. By using numerical methods it is not difficult to approach the value b^*. Nevertheless, as far as we know there are no analytical tools to obtain the value b^*. This is the main goal of this chapter.

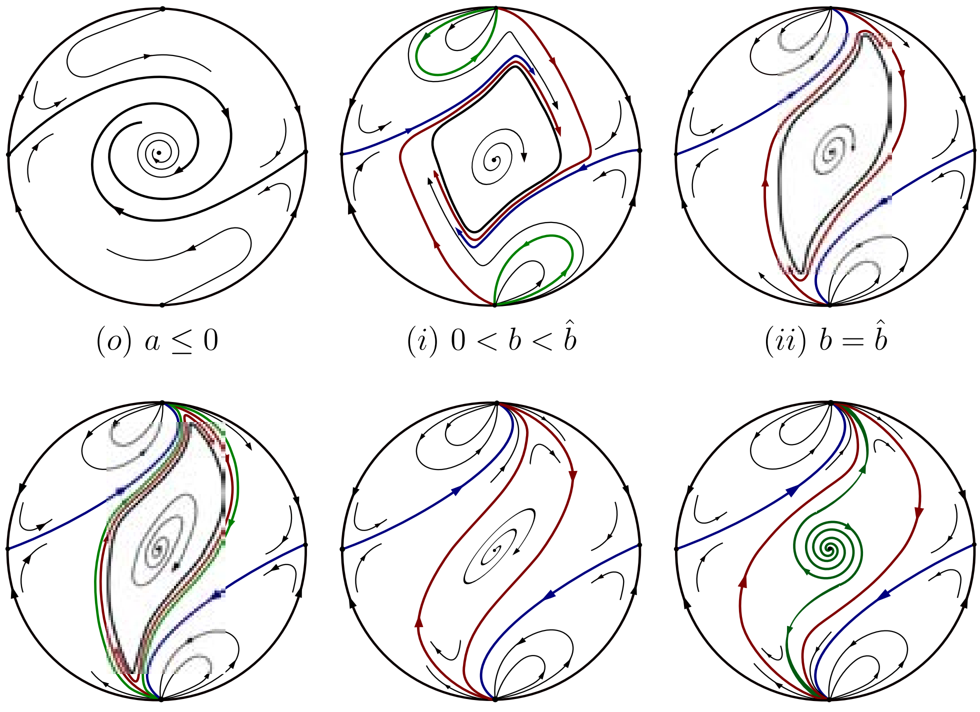
We have succeed in finding an interval of length 0.027 containing b^* and during our study we have also realized that there was a bifurcation value that has not been observed in the previous studies. Our main result is:

Theorem 4.1. *Consider system* (4.2). *There exist two positive numbers* $\hat{b}$ *and* b^* *such that:*

(a) *It has a limit cycle if and only if* $0 < b < b^*$. *Moreover, when it exists, it is unique, hyperbolic and stable.*

(b) *The only bifurcation values of the system are 0,* $\hat{b}$ *and* b^*. *In consequence there are exactly six different global phase portraits on the Poincaré disc, which are the ones showed in Figure 4.1.*

(c) *It holds that* $0.79 < \hat{b} < b^* < 0.817$.



(o) $a \leq 0$ (i) $0 < b < \hat{b}$ (ii) $b = \hat{b}$

(iii) $\hat{b} < b < b^*$ (iv) $b = b^*$ (v) $b > b^*$

Figure 4.1: Phase portraits of systems (4.1) and (4.2). When $a \geq 0$, then $b = \sqrt{a}$.

The phase portraits missing in [102] are (*ii*) and (*iii*) of Figure 4.1.

The key steps in our proof of Theorem 4.1 are the following:

- Give analytic asymptotic expansions of the separatrices of the critical points at infinity, see Section 4.2.

- Use these expansions to construct explicit piecewise rational curves, and prove that they are without contact for the flow given by (4.2). These curves allow to control the global relative positions of the separatrices of the infinite critical points, see Section 4.4.

- Provide an alternative proof of the uniqueness and hyperbolicity of the limit cycle, which is based in the construction of an explicit rational Dulac function, see Section 4.3.

By solving numerically the differential equations we can approach the bifurcation values given in the theorem, see Remark 4.7. We have obtained that $\hat{b} \approx 0.8058459066$, $b^* \approx 0.8062901027$ and then $b^* - \hat{b} \approx 0.000444$. As we have said the main goal of this chapter is to get an analytic approach to the more relevant value b^*, because it corresponds to the disappearance of the limit cycle.

Although all our efforts have been focused on system (4.2), the tools that we introduce in this work can be applied to other families of polynomial vector fields and they can provide an analytic control of the bifurcation values for these families. In Section 4.5 we give more details about the applicability of our approach. As we will see, our approach is not totally algorithmic and following it we do not know how to improve the interval presented in Theorem 4.1 for the values $\hat{b}$ and b^*.

One of the main computational difficulties that we have found has been to prove that certain polynomials in x, y and b, with high degree, do not vanish on some given regions. To treat this question, in Appendix II we propose a general method that uses the so-called double discriminant and that we believe that can be useful in other settings, see for instance [3, 88]. In our context this discriminant turns out to be a huge polynomial in b^2 with rational coefficients. In particular we need to control, on a given interval with rational extremes, how many reals roots has a polynomial of degree 965, with enormous rational coefficients. Although Sturm algorithm theoretically works, in practical our computers can not deal with this problem using it. Fortunately we can utilize a kind of bisection procedure based on the Descartes rule ([63]) to overcome this difficulty, see Appendix I.

4.2 Structure at infinity and relative positions of the separatrices

As usual, for studying the behavior of the solutions at infinity of system (4.2) we use the Poincaré compactification. That is, we will use the transformations $(x, y) = (1/z, u/z)$ and $(x, y) = (v/z, 1/z)$, with a suitable change of time to transform system (4.2) into two new polynomial systems, one in the (u, z)-plane and another one in the (v, z)-plane respectively (see [7] for details). Then, for understanding the behavior of the solutions of (4.2) near infinity we will study the structure of the critical points of the transformed systems which are localized on the line $z = 0$. Recall that these points are the *critical points at infinity* of system (4.2) and their separatrices play a key role for knowing the bifurcation diagram of the system. In fact, it follows from the works of Markus [73] and Newmann [82] that it suffices to know the behavior of these separatrices, the type of finite critical points and the number and type of periodic orbits to know the phase portraits of the system. We obtain the following result:

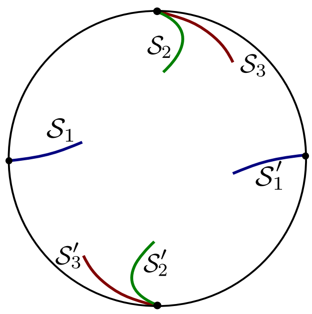


Figure 4.2: Separatrices at infinity for system (4.2).

Theorem 4.2. *System* (4.2) *has six separatrices at infinity, which we denote by* $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_1', \mathcal{S}_2'$ *and* $\mathcal{S}_3'$*, see Figure 4.2. Moreover:*

(*i*) *Each* $\mathcal{S}_k'$ *is the image of* $\mathcal{S}_k$ *under the transformation* $(x, y) \to (-x, -y)$.

(*ii*) *The separatrices* $\mathcal{S}_2$ *and* $\mathcal{S}_3$ *near infinity are contained in the curve* $\{y - \phi(x) = 0\}$ *where* $\phi(x) = \tilde{\phi}(x - b)/(x - b)^2$, $\tilde{\phi}(u)$ *is an analytic function at the origin that satisfies*

$$\tilde{\phi}(u) = \frac{1}{b} - \frac{1}{3b^2}u + \frac{1}{9b^3}u^2 - \frac{359}{27b^4}u^3 + O(u^4). \tag{4.3}$$

In particular, $\mathcal{S}_2$ *corresponds to* $x \lesssim b$ *and* $\mathcal{S}_3$ *to* $x \gtrsim b$.

(*iii*) *The separatrix* $\mathcal{S}_1$ *near infinity is contained in the curve* $\{y - \varphi(x) = 0\}$ *where* $\varphi(x) = \tilde{\varphi}(1/x)$ *and* $\tilde{\varphi}$ *is an analytic function at the origin that satisfies*

$$\tilde{\varphi}(u) = -u - (b^2 - 1)u^3 - (b^4 - 3b^2 + 2)u^5 + O\left(u^7\right). \tag{4.4}$$

Remark 4.3. *In the statements* (*ii*) *and* (*iii*) *of Theorem 4.2 the Taylor expansions of the functions* $\tilde{\phi}$ *and* $\tilde{\varphi}$ *can be obtained up to any given order. In fact, in Section 4.4 we will use the approximation of* $\tilde{\phi}$ *until order 16.*

As a consequence of the above theorem we have the following result:

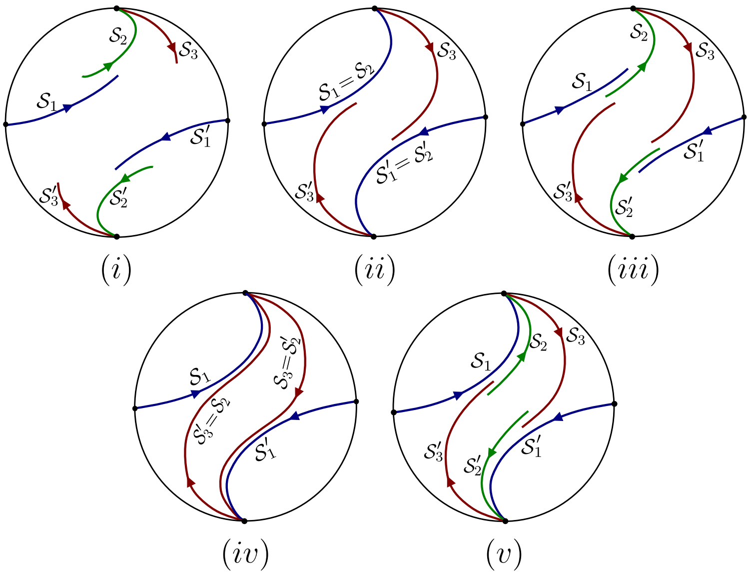


Figure 4.3: Relative position of the separatrices of system (4.2).

Corollary 4.4. *All the possible relative positions of the separatrices of system* (4.2) *in the Poincaré disc are given in Figure 4.3.*

To prove the above theorem we need some preliminary lemmas.

Lemma 4.5. *By using the transformation* $(x, y) = (1/z, u/z)$ *and the change of time* $dt/d\tau = 1/z^4$ *system* (4.2) *is transformed into the system*

$$\begin{cases} u' = -(1+u^2)z^4 - u(1 - b^2z^2)(u^2 + z^2), \\ z' = -uz^5, \end{cases} \tag{4.5}$$

where the prime denotes the derivative respect to τ. *The origin is the unique critical point of* (4.5) *and it is a saddle. Moreover, the stable manifold is the* u*-axis, the* *unstable manifold, $\mathcal{S}_1$, is locally contained in the curve $\{u - \psi(z) = 0\}$, where $\psi(z)$ is an analytic function at the origin that satisfies*

$$\psi(z) = -z^2 - (b^2 - 1)z^4 - (b^4 - 3b^2 + 2)z^6 + O(z^8), \tag{4.6}$$

see Figure 4.4.

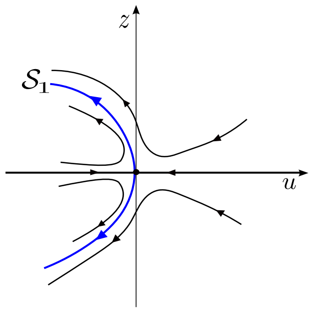


Figure 4.4: Phase portrait of system (4.5).

Proof. From the expression of (4.5) it is clear that the origin is its unique critical point. For determining its structure we will use the directional blow-up since the linear part of the system at this point vanishes identically.

The u-directional blow-up is given by the transformation $u = u$, $q = z/u$; and by using the change of time $dt/d\tau = u^2$, system (4.5) becomes

$$\begin{cases} \dot{u} = -u - (1 - b^2u^2)uq^2 - (1 - b^2u)u^2q^4 - u^4q^4, \\ \dot{q} = q + (1 - b^2u^2)q^3 + (1 - b^2u)uq^5. \end{cases} \tag{4.7}$$

This system has a unique critical point at origin and it is a saddle with eigenvalues ± 1.

The z-directional blow-up is given by the transformation $r = u/z$, $z = z$. Doing the change of time $dt/d\tau = -z^2$, system (4.5) becomes

$$\begin{cases} \dot{r} = z + (1 - b^2z^2)(r + r^3), \\ \dot{z} = rz^4. \end{cases} \tag{4.8}$$

This system has a unique critical point at the origin which is semi-hyperbolic. We will use the results of [7, Theorem 65] to determine its type. By applying the linear change of variables $r = -\xi + \eta$, $z = \xi$ system (4.8) is transformed into

$$\begin{cases} \dot{\xi} = (\eta - \xi)\xi^4, \\ \dot{\eta} = \eta - N(\xi, \eta), \end{cases}$$

where $N(\xi,\eta) = (\eta-\xi)(b^2\xi^2-\xi^4)+(\eta-\xi)^3(b^2\xi^2-1)$. It is easy to see that if $\eta = n(\xi)$ is the solution of $\eta - N(\xi,\eta) = 0$ passing for the origin, then $n(\xi) = -(b^2-1)\xi^3-(b^4-3b^2+2)\xi^5+O(\xi^7)$. Thus $(n(\xi)-\xi)\xi^4 = -\xi^5+O(\xi^7)$. Therefore from [7, Theorem 65] we know that the origin is a semi-hyperbolic saddle. Moreover, its stable manifold is the η-axis and its unstable manifold is given by

$$\eta = -(b^2-1)\xi^3-(b^4-3b^2+2)\xi^5+O(\xi^7).$$

In the plane (r,z) the local expression of this manifold is

$$r = -z-(b^2-1)z^3-(b^4-3b^2+2)z^5+O(z^7).$$

Finally, in the (u,z)-plane the unstable manifold is contained in the curve (4.6) and from the analysis of the phase portraits of systems (4.7) and (4.8) we obtain that the local phase portrait of system (4.5) is the one given in Figure 4.4. □

Lemma 4.6. *By using the transformation $(x,y)=(v/z,1/z)$ and the change of time $dt/d\tau = 1/z^4$ system (4.2) is transformed into the system*

$$\begin{cases} v' = v(1+z^2)(v^2-b^2z^2)+(1+v^2)z^4, \\ z' = z(1+z^2)(v^2-b^2z^2)+vz^5, \end{cases} \tag{4.9}$$

where the prime denotes the derivative respect to τ. System (4.9) has a unique critical point at the origin and its local phase portrait is the one showed in Figure 4.5. Moreover, the separatrices $\mathcal{S}_2$ and $\mathcal{S}_3$ are locally contained in the curve $\{v-g(U)=0\}$ where $U=z/v-1/b$ and $g(U)$ is an analytic function at the origin that satisfies

$$g(U) = b^6U^2-\frac{10}{3}b^7U^3+\frac{22}{3}b^8U^4+O\left(U^5\right). \tag{4.10}$$

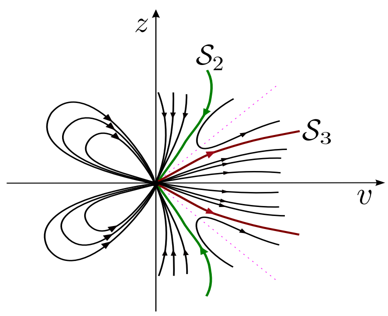


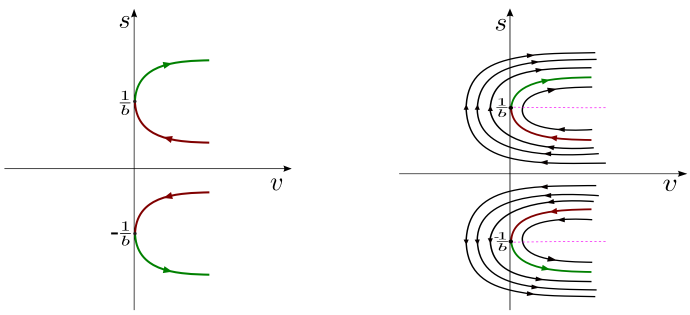
Figure 4.5: Topological local phase portrait of system (4.9). All the solutions are tangent to the v-axis but for aesthetical reasons this fact is not showed in the figure.

Proof. From the expression of system (4.9) it is clear that the origin is its unique critical point. As in Lemma 4.5 we will use the directional blow-up technique to determine its structure since the linear part of the system at this point is identically zero.

It is well-known, see [7], that since at the origin $z'v - v'z = -z^5 + O(z^6)$, all the solution, arriving or leaving the origin have to be tangent to $z = 0$. So it suffices to consider the v-directional blow-up given by the transformation $v = v$, $s = z/v$. Performing it, together with the change of time $dt/d\tau = -v^3$, system (4.9) is transformed into

$$\begin{cases} \dot{v} = -(1+v^2s^2)(1-b^2s^2) - vs^4(1+v^2), \\ \dot{s} = s^5. \end{cases} \tag{4.11}$$

This system has no critical points. However, by studying the vector field on the s-axis we will obtain relevant information for knowing the phase portrait of system (4.9). If $s = 0$ then $\dot{v} = -1$ and $\dot{s} = 0$, that is, the v axis is invariant. If $v = 0$ then $\dot{v} = -1 + b^2s^2$ and $\dot{s} = s^5$, this implies that $\dot{v} = 0$ if $s = \pm 1/b$. In addition, a simple computation shows that $\ddot{v} > 0$ at the points $(0, \pm 1/b)$. Therefore the solutions through these points are as it is showen in Figure 4.6.(a), and by the continuity of solutions with respect to initial conditions, we have that the phase portrait of system (4.11), close to these points, is as it is showed in Figure 4.6.(b).



(a) (b)

Figure 4.6: Local phase portrait of system (4.11).

Then by using the transformation $(v, z) = (v, sv)$ and the phase portrait showed in Figure 4.6.(b) we can obtain the phase portrait of system (4.9). Recall that the mapping swaps the second and the third quadrants in the v-directional blow-up. In addition, taking into account the change of time $dt/d\tau = -v^3$ it follows that the vector field in the first and fourth quadrant of the plane (v, z) has the opposite direction to the showed in the (v, s)-plane. Therefore the local phase portrait of (4.9) is the showed in Figure 4.5.

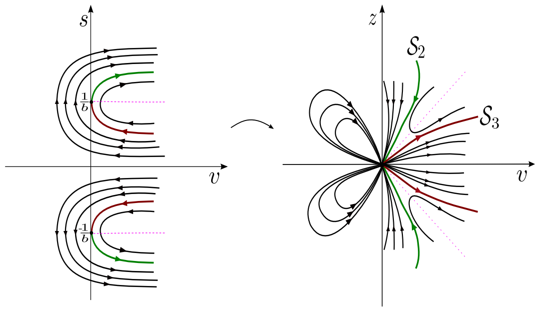


Figure 4.7: Transformation between system (4.11) and system (4.9).

To show that the separatrices $\mathcal{S}_2$ and $\mathcal{S}_3$ are contained in the curve (4.10) we proceed as follows. First, we will obtain the curve that contains the solution through the point $(0, 1/b)$ in the plane (v, s). Second, by using the transformation $(v, z) = (v, sv)$ we will obtain the corresponding curve in the (v, z)-plane and we will show that such curve is exactly the curve given by (4.10).

Since $\dot{s}$ is positive in $(0, \infty)$, the solution through the point $(0, 1/b)$ (respectively $(0, -1/b)$) is contained in the curve $\{v - g(s) = 0\}$ (respectively $\{v - \tilde{g}(s) = 0\}$), where $g(s)$ (respectively $\tilde{g}(s)$) is an analytical function defined in an open neighborhood of the point, moreover it is clear that $g(1/b) = 0$ and $g'(1/b) = 0$. Consider the Taylor series of $g(s)$ around $(1/b)$:

$$g(s) = \sum_{i=2}^{\infty} \frac{g^{(i)}\left(\frac{1}{b}\right)}{i!} \left(s - \frac{1}{b}\right)^i . \tag{4.12}$$

Since the curve $\{v - g(s) = 0\}$ is invariant then $\langle \nabla(v - g(s)), \widetilde{X} \rangle = 0$ at all the points of $\{v - g(s) = 0\}$, where $\widetilde{X}$ is the vector field associated to system (4.11). Thus, we have a function, $\langle \nabla(v - g(s)), \widetilde{X} \rangle$, for which all its coefficients have to be zero. From this observation we obtain linear recurrent equations in the coefficients, $g^{(i)}(1/b)$ of $g(s)$. Simple computations show that the first 3 terms of the Taylor series of $g(s)$ are:

$$b^6 \left(s - \tfrac{1}{b}\right)^2 - \tfrac{10}{3} b^7 \left(s - \tfrac{1}{b}\right)^3 + \tfrac{22}{3} b^8 \left(s - \tfrac{1}{b}\right)^4 .$$

Thus, in the plane (v, z), the curve corresponding to $\{v - g(s) = 0\}$ is

$$\left\{ v - b^6 \left(\tfrac{z}{v} - \tfrac{1}{b}\right)^2 + \tfrac{10}{3} b^7 \left(\tfrac{z}{v} - \tfrac{1}{b}\right)^3 - \tfrac{22}{3} b^8 \left(\tfrac{z}{v} - \tfrac{1}{b}\right)^4 + O\left(\left(\tfrac{z}{v} - \tfrac{1}{b}\right)^5\right) = 0 \right\}.$$

Finally, if $U = z/v - 1/b$, we obtain (4.10). $\square$

Remark 4.7. *The proof of the above lemma gives a natural way for finding a numerical approximation of the value b^*. Notice that in the coordinates (v, s) the* *point* $(0, 1/b)$ *corresponds to both separatrices* $\mathcal{S}_2$ *and* $\mathcal{S}_3$*. Since it is a regular point we can start our numerical method (we use a Taylor method) without initial errors and then follow the flow of the system, both forward and backward for given fixed times, say* $t^+ > 0$ *and* $t^- < 0$*. We arrive to the points* $(v^\pm, s^\pm)$ *with* $s^\pm \neq 0$ *for* $t = t^\pm$*, respectively. These two points have associated two different points* $(x^\pm, y^\pm)$ *in the plane* (x, y)*, because of the transformation* $(v, s) = (x/y, 1/x)$*. Now, we integrate numerically the system (4.2) with initial conditions* $(x^\pm, y^\pm)$ *to continue obtaining approximations of the separatrices* $\mathcal{S}_2$ *and* $\mathcal{S}_3$*, respectively. The next step is to compare the points of intersection* $\tilde{x}^+ = \tilde{x}^+(b) < 0$ *and* $\tilde{x}^- = \tilde{x}^-(b) > 0$ *of these approximations with the* x*-axis.*

We consider the function $b \to \Pi(b) := x^+(b) + \tilde{x}^-(b)$ *and we use the bisection method to find one approximate zero of* Π*. Note that if* $\Pi(\bar{b}) = 0$ *then* $\mathcal{S}_2' = \mathcal{S}_3$ *and by the symmetry of the system* $\mathcal{S}_3' = \mathcal{S}_2$*, and therefore* $b^* = \bar{b}$*. Taking* $b_0 = 0.8062901027$*,* $t^+ = 0.05$ *and* $t^- = -0.5$ *we obtain that* $\tilde{x}^+(b_0) + \tilde{x}^-(b_0) \approx -4.58036036 \times 10^{-11}$ *and so* $b^* \approx b_0$*.*

Following a similar procedure, but now using Lemma 4.5 to have an initial condition almost on $\mathcal{S}_1$*, we get that* $\hat{b} \approx 0.8058459066$*.*

Proof of Theorem 4.2. (*i*) The result follows because system (4.2) is invariant by the transformation $(x, y) \to (-x, -y)$.

(*ii*). From (4.10) and by using the change of variables $(v, z) = (x/y, 1/y)$ we obtain that the separatrices $\mathcal{S}_2$ and $\mathcal{S}_3$ are contained in the curve

$$\left\{ \frac{x}{y} - b^6 \left(\frac{1}{x} - \frac{1}{b} \right)^2 + \frac{10}{3} b^7 \left(\frac{1}{x} - \frac{1}{b} \right)^3 - \frac{22}{3} b^8 \left(\frac{1}{x} - \frac{1}{b} \right)^4 + O\left(\left(\frac{1}{x} - \frac{1}{b} \right)^5 \right) = 0 \right\},$$

or equivalently

$$\left\{ y - \overline{\phi}(x) = 0 \right\}, \tag{4.13}$$

where

$$\overline{\phi}(x) = \frac{x}{b^6 \left(\frac{1}{x} - \frac{1}{b} \right)^2 - \frac{10}{3} b^7 \left(\frac{1}{x} - \frac{1}{b} \right)^3 + \frac{22}{3} b^8 \left(\frac{1}{x} - \frac{1}{b} \right)^4 + O\left(\left(\frac{1}{x} - \frac{1}{b} \right)^5 \right)}.$$

We can write the function $\overline{\phi}(x)$ as

$$\overline{\phi}(x) = \left(\frac{1}{(x-b)^2} \right) \overline{\phi}_1(x), \tag{4.14}$$

where

$$\overline{\phi}_1(x) = \frac{b^2 x^3}{b^6 + \frac{10}{3} b^7 \left(\frac{x-b}{bx} \right) + \frac{22}{3} b^8 \left(\frac{x-b}{bx} \right)^2 + O\left(\left(\frac{x-b}{bx} \right)^3 \right)}.$$

The function $\overline{\phi}_1(x)$ is analytical at $x = b$ and it is not difficult to see that it has the following Taylor expansion

$$\overline{\phi}_1(x) = \frac{1}{b} - \frac{(x-b)}{3b^2} + \frac{(x-b)^2}{9b^3} - \frac{359(x-b)^3}{27b^4} + O((x-b)^4).$$

Then (4.14) can be written as

$$\overline{\phi}(x) = \frac{1}{b(x-b)^2} - \frac{1}{3b^2(x-b)} + \frac{1}{9b^3} - \frac{359}{27b^4}(x-b) + O((x-b)^2).$$

Hence from (4.13) and taking $\overline{\phi}(x) = \tilde{\phi}(x-b)/(x-b)^2$ we complete the proof.

The proof of *(iii)* follows by applying the previous ideas, considering the expression given by (4.6) and the change of variables $(u, z) = (y/x, 1/x)$. □

Proof of Theorem 4.1

We start proving a preliminary result that is a consequence of some general properties of semi-complete family of rotated vector fields with respect one parameter, SCFRVF for short, see [33, 83].

Proposition 4.8. *Consider system* (4.2) *and assume that for* $b = \bar{b} > 0$ *it has no limit cycles. Then there exists* $0 < b^* \leq \bar{b}$ *such that the system has limit cycles if and only if* $b \in (0, b^*)$*. Moreover, for* $b = b^*$ *its phase portrait is like (iv) in Theorem 4.1 and when* $b > b^*$ *it is like (v) in Theorem 4.1.*

Proof. It is easy to see that the system has a limit cycle for $b \gtrsim 0$, which appears from the origin through an Andronov-Hopf bifurcation.

If we denote by $X_b(x,y) = (P_b(x,y), Q_b(x,y))$ the vector field associated to (4.2) then

$$\frac{\partial}{\partial b^2} \arctan\left(\frac{Q_b(x,y)}{P_b(x,y)}\right) = \frac{P_b(x,y)\frac{\partial Q_b(x,y)}{\partial b^2} - Q_b(x,y)\frac{\partial P_b(x,y)}{\partial b^2}}{P_b^2(x,y) + Q_b^2(x,y)}$$
$$= \frac{y^2(1+y^2)}{P_b^2(x,y) + Q_b^2(x,y)} \geq 0.$$

This means that system (4.2) is a SCFRVF with respect to the parameter b^2.

We will recall two properties of SCFRVF. The first one is the so called *non-intersection property.* It asserts that if γ_1 and γ_2 are limit cycles corresponding to different values of b, then $\gamma_1 \cap \gamma_2 = \emptyset$. Informally, we like to call this property *Atila's property,*[1] because it implies that, if for some value of b a limit cycle passes through

[1]Recall that it was said about Atila, King of the Huns, that "the grass never grew on the spot where his horse had trod".

a region of the phase plane, this region becomes forbidden for the periodic orbits that the system could have for any other value of the parameter.

The second one is called *planar termination principle*: [85, 84] if varying the parameter we follow with continuity a limit cycle generated from a critical point $\mathbf{p}$, we get that the union of all the limit cycles covers a 1-connected open set $\mathcal{U}$, whose boundaries are $\mathbf{p}$ and a cycle of separatrices of X_b. The corners of this cycle of separatrices are finite or infinite critical points of X_b. Since in our case X_b only has the origin as a finite critical point we get that $\mathcal{U}$ has to be unbounded. Notice that in this definition, when a limit cycle goes to a semistable limit cycle then we continue the other limit cycle that has collided with it. This limit cycle has to exist, again by the properties of SCFRVF.

If for some value of $b = \bar{b} > 0$ the system has no limit cycle it means that the limit cycle starting at the origin for $b = 0$, has disappeared for some b^*, $0 < b^* \leq \bar{b}$ covering the whole set $\mathcal{U}$. Since $\mathcal{U}$ fills from the origin until infinity, from the non intersection property, the limit cycle cannot either exist for $b \geq b^*$, as we wanted to prove.

The origin is a repeller for $b > 0$, hence from the Poincaré–Bendixson Theorem and Corollary 4.4 we conclude that the phase portraits (i), (ii) and (iii) in Figure 4.1 have at least one limit cycle. Then, the phase portraits for $b \geq b^*$ have to be like (iv) or (v) in the same figure. Since the phase portrait (iv) is the only one having a cycle of separatrices it corresponds to $b = b^*$. Again by the properties of SCFRVF, the phase portrait (iv) does not appear again for $b > b^*$. Hence, for $b > b^*$ the phase portrait has to be like (v) and the proposition follows. □

Remark 4.9. *In Lemma 4.12 we will give a simple proof that when $b = 1$ system* (4.2) *has no limit cycles, based on the fact that for this value of the parameter it has the hyperbola $xy + 1 = 0$ invariant by the flow. From the above proposition it follows that $b^* < 1$. This result already improves the upper bound of b^*, given in [102], $\sqrt[6]{9\pi^2/16} \approx 1.33$. Theorem 4.1 improves again this upper bound, but as we will see, the proof is much more involved.*

Proof of Theorem 4.1. Recall that for $a \leq 0$ the function $V(x, y) = x^2 + y^2$ is a global Lyapunov function for system (4.1) and therefore the origin is global asymptotically stable. Then it is easy to see that its phase portrait is like (o) in Figure 4.1. To prove the theorem we list some of the key points that we will use and that will be proved in the forthcoming sections:

($\mathbf{R}_1$) System (4.2) has at most one limit cycle for $b \in (0, 0.817]$ and when it exists it is hyperbolic and attractor, see Section 4.3.

($\mathbf{R}_2$) System (4.2) has an odd number of limit cycles, with multiplicities taken into account, when $b \leq 0.79$ and the configuration of its separatrices is like (i) in Figure 4.3, see Proposition 4.14 in Section 4.4.

($\mathbf{R}_3$) System (4.2) has an even number of limit cycles, with multiplicities taken into account, when $b = 0.817$ and the configuration of its separatrices is like (v) in Figure 4.3, see again Proposition 4.14 in Section 4.4.

The theorem for $b \geq b^*$ is a consequence of Proposition 4.8. Notice that again by this proposition and ($\mathbf{R}_3$), $b^* < 0.817$. Hence, the limit cycles can exist only when $b \in (0, b^*) \subset (0, 0.817]$ and by ($\mathbf{R}_1$), when they exist, there is only one and it is hyperbolic and attractor.

As a consequence of ($\mathbf{R}_2$) and the uniqueness and hyperbolicity of the limit cycle we have that the phase portrait for $b \leq 0.79$ is like (i) in Figure 4.1.

To study the phase portraits for the remaining values of b, that is $b \in (0.79, b^*)$, first notice that all of them have exactly one limit cycle, which is hyperbolic and stable. So it only remains to know the behavior of the infinite separatrices. We denote by $x_2(b)$ and $x_3'(b)$ the points of intersection of the separatrices $\mathcal{S}_2$ and $\mathcal{S}_3'$ of system (4.2) with the x-axis (when they exist), see also the forthcoming Figure 4.13. Notice that for $b > b^*$, $x_3'(b) < x_2(b) < 0$ and $x_3'(b^*) = x_2(b^*) < 0$. The properties of the SCFRVF imply that $x_2(b)$ is monotonically increasing and that $x_3'(b)$ is monotonically decreasing. Hence, for $b \lesssim b^*$ the phase portrait of the system is like (iii) in Figure 4.1. Since we already know that for $b = 0.79$ the phase portrait is like (i), it should exists at least one value, say $b = \hat{b}$, with phase portrait (ii). Since for SCFRVF the solution for a given value of b, say $b = \bar{b}$, becomes a curve without contact for the system when $b \neq \bar{b}$, we have that the phase portraits corresponding to heteroclinic orbits, that is (ii) and (iv) of Figure 4.1, only appear for a single value of b (in this case $\hat{b}$ and b^*, respectively). Therefore, the theorem follows. □

4.3 Uniqueness of limit cycles

In this section we will prove the uniqueness of the limit cycle of system (4.2) when $b \leq 0.817$. The idea of the proof is to find a suitable rational Dulac function for applying the following generalization of Bendixson–Dulac criterion.

Proposition 4.10. *Consider the C^1-differential system*

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y), \end{cases} \tag{4.15}$$

and let $\mathcal{U} \subset \mathbb{R}^2$ be an open region with boundary formed by finitely many algebraic curves. Assume that:

(I) There exists a rational function $V(x, y)$ such that

$$M := \frac{\partial V}{\partial x} P + \frac{\partial V}{\partial y} Q - V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \tag{4.16}$$

does not change sign on $\mathcal{U}$. *Moreover,* M *only vanishes on points, or curves that are not invariant by the flow of* (4.15).

(II) All the connected components of $\mathcal{U} \setminus \{V = 0\}$, *except perhaps one, say* $\widetilde{\mathcal{U}}$, *are simple connected. The component* $\widetilde{\mathcal{U}}$, *if exists, is 1-connected.*

Then the system has at most one limit cycle in $\mathcal{U}$ *and when it exists is hyperbolic and it is contained in* $\widetilde{\mathcal{U}}$. *Moreover its stability is given by the sign of* $-VM$ *on* $\widetilde{\mathcal{U}}$.

The above statement is a simplified version of the one given in [49] adapted to our interests. Similar results can be seen in [16, 47, 68, 103].

Remark 4.11. *Looking at the proof of Proposition 4.10 we also know that:*

(i) The Dulac function used in the proof is $1/V$.

(ii) In the region $\mathcal{U}$, *the curve* $\{V(x, y) = 0\}$ *is without contact for the flow of* (4.15). *In particular, by the Poincaré–Bendixson Theorem, the ovals of the set* $\{V(x, y) = 0\}$ *must surround some of the critical points of the vector field.*

To give an idea of how we have found the function V that we will use in our proof we will first study the van der Pol system and then the uniqueness in our system when $b \leq 0.615$. Although we will not use these two results, we believe that to start studying them helps to a better understanding of our approach.

4.3.1 The van der Pol system

Consider the van der Pol system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + (b^2 - x^2)y. \end{cases} \tag{4.17}$$

Due to the expression of the above family of differential equations, in order to apply Proposition 4.10, it is natural to start considering functions of the form

$$V(x, y) = f_2 y^2 + f_1(x)y + f_0(x).$$

For this type of functions, the corresponding M is a polynomial of degree 2 in y, with coefficients being functions of x. In particular the coefficient of y^2 is

$$f_1'(x) + f_2(b^2 - x^2).$$

Taking $f_1(x) = (x^2 - 3b^2)f_2 x/3$ we get that it vanishes. Next, fixing $f_2 = 6$, and imposing to the coefficient of y to be zero we obtain that $f_0(x) = 6x^2 + c$, for any constant c. Finally, taking $c = b^2(3b^2 - 4)$, we arrive to

$$V_b(x,y) = 6y^2 + 2(x^2 - 3b^2)xy + 6x^2 + b^2(3b^2 - 4). \tag{4.18}$$

From (4.16) of Proposition 4.10, the corresponding M, which only depends on x, is

$$M_b(x,y) = 4x^4 + b^2(3b^2 - 4)(x^2 - b^2).$$

It is easy to see that for $b \in (0, 2/\sqrt{3}) \approx (0, 1.15)$, $M_b(x,y) > 0$. Notice that $V_b(x,y) = 0$ is quadratic in y and so is not difficult to see that it has at most one oval, see Figure 4.8 for $b = 1$. Then we can apply Proposition 4.10 to prove the uniqueness and hyperbolicity of the limit cycle for these values of b.

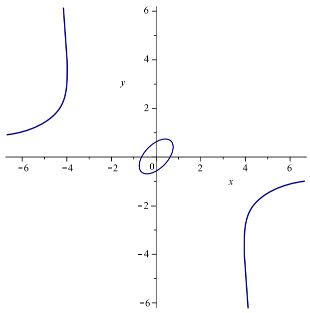


Figure 4.8: The algebraic curve $V_b(x,y) = 0$ with $b = 1$.

We remark that taking a more suitable polynomial Dulac function, it is possible to prove the uniqueness of the limit cycle for all values of b, see [21, p. 105]. We have only included this explanation as a first step towards the construction of a suitable rational Dulac function for our system (4.2).

System (4.2) with $b \leq 651/1000$

By making some modifications to the function V_b given by (4.18), we get an appropriate function for system (4.2). Consider

$$\begin{aligned} V_b(x,y) =& \big(2x^3 + 6b^2(1-b^2)x\big)y^3 + 6(1-b^2)y^2 + 2(x^2 - 3b^2)xy \\ &+ 6(1-b^2)x^2 + b^2(3b^2 - 4). \end{aligned}$$

Computing the double discriminant $\triangle^2(V_b)$ of the function V_b, introduced in Appendix II, we get that

$$\triangle^2(V_b) = b^2(3b^2 - 4)(b^2 - 1)^{15}(P_{19}(b^2))^2,$$

where P_{19} is a polynomial of degree 19. By using for instance the Sturm method, we prove that the smallest positive root of $\triangle^2(V_b)$ is greater than 0.85. Therefore, by Proposition 4.20 we know that for $b \in (0, 0.85]$ the algebraic curve $V_b(x,y) = 0$ has no singular points and therefore the set $\{V_b(x,y) = 0\} \subset \mathbb{R}^2$ is a finite disjoint union of ovals and smooth curves diffeomorphic to open intervals.

By applying Proposition 4.10 to system (4.2) with $V = V_b$, we get that

$$\begin{aligned} M_b(x,y) = \; & 6(2-3b^2)x^4y^2 - 12b^2(2-b^2)x^3y^3 + 6(2-b^2)x^2y^4 \\ & +2(2-3b^2)x^4 - 3b^2(14-15b^2)x^2y^2 + 12b^4(2-b^2)xy^3 \\ & -b^2(4-9b^2)x^2 + 3b^4(2-3b^2)y^2 + b^4(4-3b^2). \end{aligned} \tag{4.19}$$

In Subsection 4.5.4 of Appendix II we prove that M_b does not vanish on $\mathbb{R}^2$ for $b \in (0, 0.651]$. Then by Remark 4.11 all the ovals of $\{V_b(x,y) = 0\}$ must surround the origin, which is the unique critical point of the system. Since the straight line $x = 0$ has at most two points on the algebraic curve $V_b(x,y) = 0$, it can have at most one closed oval surrounding the origin. Then by Proposition 4.10 it follows the uniqueness, stability and hyperbolicity of the limit cycle of system (4.2) for these values of the parameter b.

System (4.2) with $b \leq 817/1000$

The hyperbola $xy + 1 = 0$ will play an important role in the study of this case. We first prove a preliminary result.

Lemma 4.12. *Consider system* (4.2)*.*

(I) For $b \neq 1$ the hyperbola $xy+1 = 0$ is without contact for its flow. In particular its periodic orbits never cut it.

(II) For $b = 1$ the hyperbola $xy + 1 = 0$ is invariant for its flow and the system has not periodic orbits.

Proof. Define $F(x,y) = xy + 1$ and set $X = (P,Q) := (y, -x + (b^2 - x^2)(y + y^3))$. Simple computations give that for $x \neq 0$,

$$(F_xP + F_yQ)\big|_{y=-1/x} = \frac{1+x^2}{x^2}\left(1-b^2\right).$$

Therefore (I) follows and we have also proved that when $b = 1$, the hyperbola is invariant by the flow.

(II) When $b = 1$,

$$F_xP + F_yQ = KF, \tag{4.20}$$

where $K = K(x,y) = y^2 - x^2 - xy(xy - 1)$ is the so-called *cofactor* of the invariant curve $F = 0$.

Let us prove that the system has no limit cycle. Recall that the origin is a repeller. Therefore if we prove that any periodic orbit Γ of the system is also repeller we will have proved that there is no limit cycle.

This will follow if we show that

$$\int_0^T \operatorname{div}(X)(\gamma(t))\,dt > 0, \tag{4.21}$$

where $\gamma(t) := (x(t), y(t))$ is the time parametrization of Γ and $T = T(\Gamma)$ its period.

To prove (4.21) notice that the divergence of X can be written as $\operatorname{div}(X) = 3K + 2x^2 + 1 - 3xy$. Then,

$$\int_0^T \operatorname{div}(X)(\gamma(t))\,dt = 3\int_0^T K(x(t), y(t))dt + \int_0^T (2x^2(t) + 1)dt - 3\int_0^T x(t)y(t)dt.$$

Observe that from (4.20) we have that

$$\begin{aligned} \int_0^T K(x(t), y(t))\,dt &= \int_0^T \frac{F_x(x(t), y(t))\dot{x} + F_y(x(t), y(t))\dot{y}}{F(x(t), y(t))} dt \\ &= \int_0^T \frac{d}{dt} \ln|F(x(t), y(t))| dt = \ln|F(x(t), y(t))|\Big|_0^T = 0 \end{aligned}$$

and that

$$\int_0^T x(t)y(t)dt = \int_0^T x(t)\dot{x}(t)dt = \frac{x^2(t)}{2}\Big|_0^T = 0.$$

Therefore,

$$\int_0^T \operatorname{div}(X)(\gamma(t))\,dt = \int_0^T (2x^2(t) + 1)dt > 0,$$

as we wanted to see. □

Theorem 4.13. *System (4.2) for $b \in (0, 0.817]$ has at most one limit cycle. Moreover when it exists it is hyperbolic and attractor.*

Proof. Based on the function V_b used in the Subsection 4.3.1 we consider the function $V_b(x, y) = \widehat{V}_b(x, y)/(5 + 6b^{18}x^2)$, where

$$\begin{aligned} \widehat{V}_b(x, y) = \;& \tfrac{1}{2}\, b^{18}x^6 + \tfrac{1}{2}\, b^{18}x^4y^2 + \left(1 + \tfrac{1}{2}\, b^{12}\right) x^3y^3 + \left(1 + \tfrac{3}{2}\, b^2\right) x^3y \\ & - \left(\tfrac{3}{5}\, b^{10} + \tfrac{5}{3}\, b^{14} + 2\, b^{16}\right) x^2y^2 + \left(3\, b^2 - 3\, b^4 + \tfrac{21}{10}\, b^6\right) xy^3 \\ & + (3 - 3\, b^2 + 2\, b^4)\, x^2 - b^2 \left(3 - \tfrac{1}{10}\, b^4\right) xy \\ & + (3 - 3\, b^2 + 2\, b^4)\, y^2 + \tfrac{3}{2}\, b^4 - 2\, b^2. \end{aligned} \tag{4.22}$$

We have added the non-vanishing denominator to increase a little bit the range of values for which Proposition 4.10 works. Indeed, it can be seen that the above function, but without the denominator, is good for showing that the system has at most one limit cycle for $b \leq 0.811$.

To study the algebraic curve $\widehat{V}_b(x,y)=0$ we proceed like in the previous subsection. The double discriminant introduced in Appendix II is

$$\triangle^2(\widehat{V}_b) = b^{182}(3b^2-4)(4b^{36}+27b^{24}+108b^{12}+108)(P_{152}(b^2))^2,$$

where P_{152} is a polynomial of degree 152. It can be seen that the smallest positive root of $\triangle^2(\widehat{V}_b)$ is greater than 0.88. Therefore by Proposition 4.20 we know that for $b \in (0, 0.88]$ this algebraic curve has no singular points. Hence the set $\{V_b(x,y) = 0\} \subset \mathbb{R}^2$ is a finite disjoint union of ovals and smooth curves diffeomorphic to open intervals.

The function that we have to study in order to apply Proposition 4.10 is

$$M_b(x,y) = \frac{N_b(x,y)}{30(6b^{18}x^2+5)^2}, \tag{4.23}$$

where $N_b(x,y)$ is given in (4.34) of Subsection 4.5.4. The denominator of M_b is positive for all $(x,y) \in \mathbb{R}^2$. By Lemma 4.12 we know that the limit cycles of the system must lay in the open region $\Omega = \mathbb{R}^2 \cap \{xy+1>0\}$. In Subsection 4.5.4 of Appendix II we will prove that N_b does not change sign on the region Ω and if it vanishes it is only at some isolated points.

Notice also that the set $\{\widehat{V}_b(x,y)=0\}$ cuts the y-axis at most in two points, therefore by the previous results and arguing as in Subsection 4.3.1, we know that it has at most one oval and that when it exists it must surround the origin.

Therefore we are under the hypotheses of Proposition 4.10, taking $\mathcal{U} = \Omega$, and the uniqueness and hyperbolicity of the limit cycle follows. □

4.4 About the existence of limit cycles

This section is devoted to find the relative position of the separatrices of the infinite critical points when $b \leq 0.79$ and when $b = 0.817$. The main tool will be the construction of algebraic curves that are without contact by the flow of system (4.2). These curves are essentially obtained by using the functions $\phi_i(x) := \tilde{\phi}_i(x-b)/(x-b)^2$ and $\varphi_i(x) := \tilde{\varphi}_i(1/x)$ where $\tilde{\phi}_i$ and $\tilde{\varphi}_i$ are the approximations of order i of the separatrices of the infinite critical points, given in the expressions (4.3) and (4.4) of Theorem 4.2, respectively. That is, we use algebraic approximations of $\mathcal{S}_i$ and $\mathcal{S}'_i$, for $i=1,2,3$.

As usual for knowing when a vector field X is without contact with a curve of the form $y=\psi(x)$ we have to control the sign of

$$N_\psi(x) := \langle \nabla(y-\psi(x)), X\rangle\big|_{y=\psi(x)}.$$

In this section we will repeatedly compute this function when $\psi(x)$ is either $\varphi_i(x)$, $\phi_i(x)$ or modifications of these functions.

We prove the following result.

Proposition 4.14. *Consider system* (4.2). *Then:*

(I) For $b \leq 79/100$ *the configuration of its separatrices is like (i) in Figure 4.3. Moreover it has an odd number of limit cycles, taking into account their multiplicities.*

(II) For $b = 817/1000$ *the configuration of its separatrices is like (v) in Figure 4.3. Moreover it has an even number of limit cycles, taking into account their multiplicities.*

Proof. (I) Consider the two functions

$$\varphi_1(x) = -\frac{1}{x}, \qquad \text{and} \qquad \varphi_2(x) = -\frac{1}{x} - \frac{(b^2-1)}{x^3},$$

which are the corresponding expressions in the plane (x, y) of the first and second approximation of the separatrix $\mathcal{S}_1$.

If $b < 1$ then $(\varphi_1 - \varphi_2)(x) = (b^2-1)/x^3 > 0$ for $x < 0$. This implies that the separatrix $\mathcal{S}_1$ in the (x, y)-plane and close to $-\infty$ is below the graphic of $\varphi_1(x)$. Moreover

$$N_{\varphi_1}(x) = -\frac{(x^2+1)(b^2-1)}{x^3} < 0 \qquad \text{for } x < 0.$$

This inequality implies that the separatrix $\mathcal{S}_1$ in the plane (x, y) cannot intersect the graphic of $\varphi_1(x)$ for $x < 0$, see Figure 4.9.

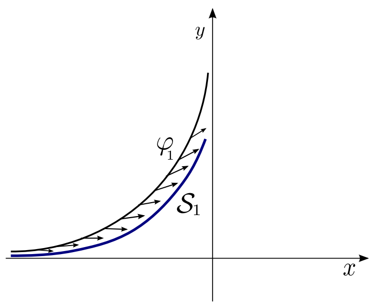


Figure 4.9: Behavior of $\mathcal{S}_1$ for $b < 1$.

Now, we consider the third approximation to the separatrices $\mathcal{S}_2$ and $\mathcal{S}_3$, that is we consider the first three terms in (4.3). It is given by the graph of the function

$$\phi_3(x) = \frac{(x^2 - 5bx + 13b^2)}{9b^3(x-b)^2}.$$

Let us prove that when $b \in (0, \sqrt{2/3})$, the graphs of $\varphi_1(x)$ and $\phi_3(x)$ intersect at a unique point, (x_0, y_0) with $x_0 < 0$ and $y_0 > 0$. For this is sufficient to show that the function $(\varphi_1 - \phi_3)(x)$ has a unique zero at some $x_0 < 0$.

It is clear that $\lim_{x\to 0^-}(\varphi_1 - \phi_3)(x) = +\infty$ and we have that $(\varphi_1 - \phi_3)(-2b) = (3b^2 - 2)/6b^3$, then for $b < \sqrt{2/3}$, $(\varphi_1 - \phi_3)(-2b) < 0$ hence $(\varphi_1 - \phi_3)(x)$ has a zero at a point x_0 with $-2b < x_0 < 0$. Moreover this zero is unique because the numerator of $(\varphi_1 - \phi_3)(x)$ is a monotonous function.

It also holds that $\nabla(y - \phi_3(x)) = (-\phi_3'(x), 1)$ where $\phi_3'(x) = (7b - x)/(3b^2(b-x)^3)$ is a positive function for $x < 0$, and a simply computation shows that

$$\begin{aligned} N_{\phi_3}(x) &= \tfrac{-1}{729b^9(b-x)^2}\Big((81b^6+1)x^4 + (729b^8 - 405b^6 - 11)bx^3 \\ &\quad -9(162b^8 - 108b^6 - 7)b^2x^2 + (729b^8 + 405b^6 - 178)b^3x \\ &\quad -13(81b^6 - 20)b^4\Big). \end{aligned}$$

To control the sign of N_{ϕ_3} we compute the discriminant of its numerator with respect to x. It gives $\mathrm{dis}(N_{\phi_3}(x), x) = b^{12}P_{22}(b^2)$, where P_{22} is a polynomial of degree 22 with integer coefficients.

By using the Sturm method we obtain that $P_{22}(b^2)$ has exactly four real zeros. By Bolzano Theorem the positive ones belong to the intervals $(0.7904, 0.7905)$ and $(2.6, 2.7)$.

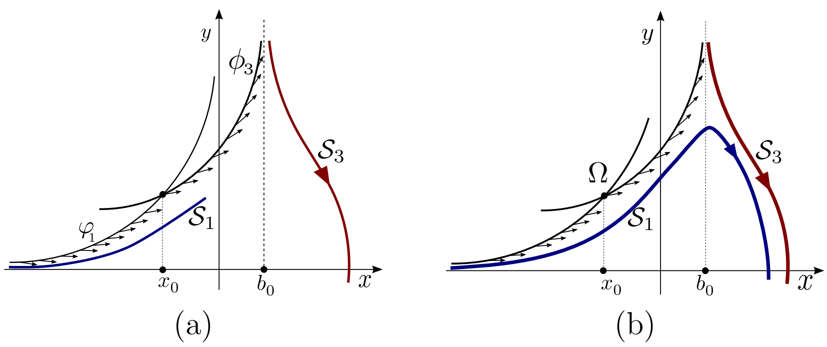


Figure 4.10: Behavior of $\mathcal{S}_1$ and $\mathcal{S}_3$ for $b \le 0.79$

If we fix $b_0 \le 79/100$ then $b_0 < \sqrt{2/3}$ and moreover according to previous paragraph the graphics of $\varphi_1(x)$ and $\phi_3(x)$ intersect at a unique point (x_0, y_0) with $x_0 < 0$ and $y_0 > 0$. Furthermore, $\frac{\partial N_{\phi_3}}{\partial b}(b_0) > 0$ in (x_0, b_0) and $N_{\phi_3} < 0$ in (x_0, b) for all $b \in (0, b_0]$. Therefore the vector field associated to (4.2) on these curves is the one showed in Figure 4.10.(a).

From Figure 4.10.(a) it is clear that the separatrix $\mathcal{S}_1$ cannot intersect the set $\Omega = \{(x, \varphi_1(x)) | -\infty < x \le x_0\} \cup \{(x, \phi_3(x)) | x_0 \le x < b_0\}$. Moreover, since the separatrix $\mathcal{S}_2$ forms an hyperbolic sector together with $\mathcal{S}_3$ we obtain that $\mathcal{S}_1$ cannot be asymptotic to the line $x = b_0$. Hence we must have the situation showed in Figure 4.10.(b). We know that the origin is a source and from the symmetry of system (4.2) we conclude that for $b \leq 0.79$ the system has an odd number of limit cycles (taking into account multiplicities) and the phase portrait is the one showed in Figure 4.11.

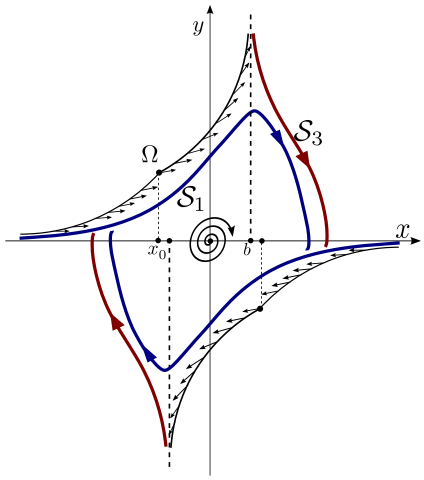


Figure 4.11: For $0 < b \leq 0.79$, system (4.2) has at least one limit cycle and phase portrait (i) of Figure 4.1.

(II) We start proving the result when $b = b_0 := 89/100$ because the method that we use is the same that for studying the case $b = 817/1000$, but the computations are easier. Recall that we want to prove that the configuration of separatrices is like (v) in Figure 4.3. That the number of limit cycles must be even (taking into account multiplicities) is then a simply consequence of the Poincaré–Bendixson Theorem, because the origin is a source.

We consider the approximation of eight order to $\mathcal{S}_2$ and $\mathcal{S}_3$ given by the graph of the function $\phi_8(x)$.

By using again the Sturm method it is easy to see that $N_{\phi_8}(x) < 0$ for $x \in (b_0, x_0)$, where $x_0 = 1.924$ is a left approximation to the root of the function $\phi_8(x)$, and $N_{\phi_8}(x) > 0$ for $x \in (x_1, b_0)$, where $x_1 = -2.022$ is a right approximation to the root of the function $N_{\phi_8}(x)$. That is, we have the situation shown in Figure 4.12.(a). Now, we consider the function $\hat{\phi}_8(x) = \phi_8(x) - 1/(9b^3)$, is clear that $(\phi_8 - \hat{\phi}_8)(x) > 0$. We have $N_{\hat{\phi}_8}(x) > 0$ for $x \in (b_0, x_2)$ where $x_2 = 1.6467$ is a left approximation to the root of the function $\hat{\phi}_8(x)$, moreover the line $x = x_2$ is transversal to the vector field for $y > 0$, thus the separatrix $\mathcal{S}_3$ intersects the x-axis at a point $\bar{x}$ of the interval (x_2, x_0), see again Figure 4.12.(a).

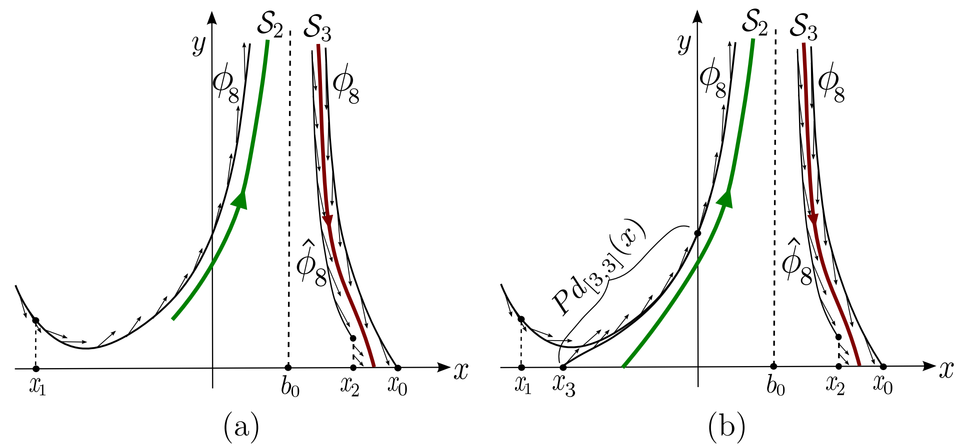


Figure 4.12: Behavior of $\mathcal{S}_2$ and $\mathcal{S}_3$ for $b \in \{0.817, 0.89\}$.

At this point, the idea is to show that $\mathcal{S}_2$ intersects the x-axis at a point $\hat{x}$, with $-x_2 < \hat{x} < 0$. For proving this, we utilize the Padé approximants method, see [8].

Recall that given a function $f(x)$, its Padé approximant $\mathrm{Pd}_{[n,m]}(f)(x, x_0)$ of order (n, m) at a point x_0, or simply $\mathrm{Pd}_{[n,m]}(f)(x)$ when $x_0 = 0$, is a rational function of the form $F_n(x)/G_m(x)$, where F_n and G_m are polynomials of degrees n and m, respectively, and such that

$$\left| f(x) - \frac{F_n(x)}{G_m(x)} \right| = O\left((x - x_0)^{n+m+1}\right).$$

Consider the Padé approximant $\mathrm{Pd}_{[3,3]}(\phi_8)$. It satisfies that $\mathrm{Pd}_{[3,3]}(\phi_8)(0) = \phi_8(0)$ and by the Sturm method it can be seen that there exists $x_3 < 0$ such that $\mathrm{Pd}_{[3,3]}(\phi_8)(x_3) = 0$, $\mathrm{Pd}_{[3,3]}(\phi_8)$ is positive and increasing on the interval $(x_3, 0)$ and a left approximation to x_3 is -1.595. Moreover, it is easy to see that $N_{\mathrm{Pd}_{[3,3]}(\phi_8)}(x) > 0$ for $x \in (x_3, 0)$. Therefore $\mathcal{S}_2$ cannot intersect neither the graph of $y = \mathrm{Pd}_{[3,3]}(\phi_8)(x)$ in $(x_3, 0)$ nor the graph of $\phi_8(x)$ in $[0, b_0)$. Hence, $\mathcal{S}_2$ intersects the x-axis in a point $\hat{x}$ contained in the interval $(x_3, 0)$. This implies that $-x_2 < \hat{x} < 0$ as we wanted to see, because $-x_2 < x_3$. Hence, the behavior of the separatrices is like Figure 4.12.(b). See also Figure 4.13.

When $b_0 = 817/1000$ we follow the same ideas. For this case we consider the functions $\phi_{16}(x)$ and $\hat{\phi}_{16}(x) = \phi_{16}(x) - 1/(9b^3)$. Recall that the graphic of $\phi_{16}(x)$ is the sixteenth order approximation to $\mathcal{S}_2$ and $\mathcal{S}_3$. It is not difficult to prove that $N_{\hat{\phi}_{16}} > 0$ on the interval (b_0, x_2), with $x_2 = 1.6421$ and since the line $x = x_2$ is transversal to X for $y > 0$, $\mathcal{S}_3$ intersects the x-axis at a point $\bar{x} > x_2$. Also we have that $N_{\phi_{16}} > 0$ on the interval $(-3/100, b_0)$ and using the Padé approximant $\mathrm{Pd}_{[5,1]}(\phi_{16})(x, -3/100)$ we obtain that $\mathcal{S}_2$ intersect to the x-axis in a point $\hat{x} \in (x_3, 0)$

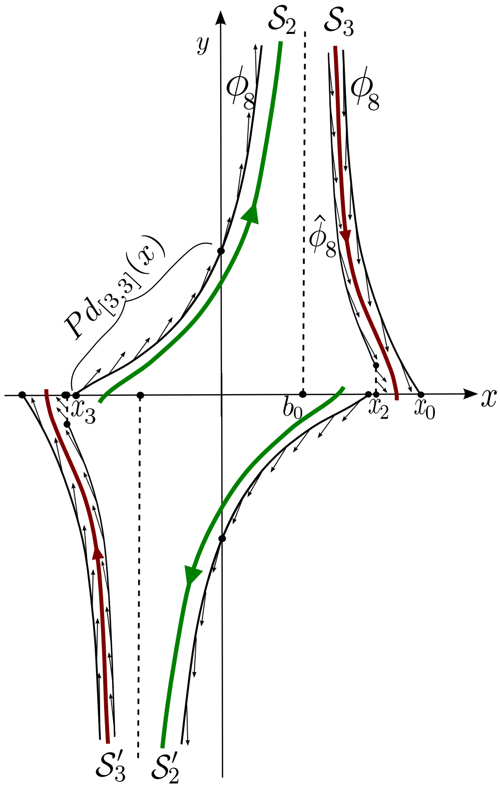


Figure 4.13: Behavior of $\mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_2'$ and $\mathcal{S}_3'$ for $b \in \{0.817, 0.89\}$.

with $x_3 > -1.638$. This implies that $-x_2 < \hat{x} < 0$ as in the case $b = 0.89$. Hence, we have the same situation that in Figure 4.13. □

Remark 4.15. *As it is shown in the proof of Theorem 4.1, the values* 0.79 *and* 0.817*, obtained in the previous proposition, provide a lower and an upper bound for* b^**. We have tried to shrink the interval where* b^* *lies using higher order approximations of the separatrices, but we have not been able to diminish its size.*

4.5 Applicability of the techniques to other families

In this section we explain how the approach introduced in this chapter for obtaining analytic estimations of the values of the bifurcation parameter for the concrete quintic family (4.2) can be adapted to be used in many other one-parameter families of planar polynomial vector fields $\dot{\mathbf{x}} = X_b(\mathbf{x})$.

Recall that the key points for obtaining the bifurcation diagrams for families of planar polynomial differential equations are the knowledge of the global behavior of the separatrices of their critical points and the control of the number of limit cycles that these equations can have.

The first step of our approach consists on searching algebraic explicit approximations of the separatrices of the finite and infinite critical points. The existence of these approximations, even for degenerate singularities, is guaranteed by the well-known fact that all the singularities of analytic vector fields can be desingularized after a finite number of blow-ups, see [4, 34]. Recall that hyperbolic and semi-hyperbolic points have analytic separatrices. Therefore, undoing the chains of desingularizing blow-ups of their truncated Taylor series at the critical points, we obtain rational curves that approximate the actual separatrices. Other examples of this approach appear in [51, 52].

The second step consists on joining pieces of these approximated separatrices and then prove that they are curves without contact by the flows of the vector fields. Notice that the fact that the approximated separatrices are without contact for the flow of the vector field in a sufficiently extended neighborhood of the critical point is not assured by the algorithm of construction of these curves. This fundamental property must be proved. The validity of this property is a bonus of the method that we have verified in the study of several families of planar vector fields. Owing to the difficult algebraic calculations that are necessary to prove the transversality property it is not possible to give simple conditions on the vector field in order to assure its validity in a sufficiently extended neighborhood of the critical point. By the moment, given a family of planar vector fields, we are not able to know in advance if our method will be successful for the given family. The different directions of these flows on the given curves and the stability of the critical points allow to distinguish the cases where the number of limit cycles, taking into account their multiplicities, is either even or odd.

Afterwards, the exact number of limit cycles is studied by using a convenient version of the generalized Bendixson-Dulac criterion. The most difficult part is to figure out the type of Dulac function, B, that works for the concrete family of differential equations. For instance, for systems of the form

$$\begin{cases} \dot{x} = f_0(x) + f_1(x)y, \\ \dot{y} = g_0(x) + g_1(x)y + g_2(x)y^2, \end{cases} \tag{4.24}$$

it is often useful to write $B(x, y) = (\sum_{k=0}^{p} c_k(x)y^k)^r(\sum_{k=0}^{q} d_k(x)y^k)^{-s}$ with p, q, r, s non-negative integer numbers, and then try to find suitable functions c_k and d_k. This is the case in this work and also in [48], [17] and other papers of the same authors. It is worth to comment here that system (4.24) contains many interesting planar differential equations: quadratic systems, Liénard and Kukles systems, predator-prey models, etc.

Notice that the last two steps of our approach imply the control of the sign, on a given region, of a polynomial function that depends on x, y and the parameter b. In general, this is not an easy task. In Appendix II we give some results on this setting that can also be useful in other problems, see for instance [3, 88]. As we will see, these results reduce all the question to control the zeros of several one-variable polynomials with rational coefficients. A priori we can not control the highest degree among all the polynomials that we will have to study (recall for our family this degree has been 965). Fortunately the powerful tools given by the Sturm algorithm and a combination of Descartes and Bisection algorithms, see Appendix I, allow in many cases to deal with the problem.

Appendix I: The Descartes method

Given a real polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$ and a real interval $I = (\alpha, \beta)$ such that $P(\alpha)P(\beta) \neq 0$, there are two well-known methods for knowing the number of real roots of P in I: the Descartes rule and the Sturm method.

Theoretically, when all the $a_i \in \mathbb{Q}$ and $\alpha, \beta \in \mathbb{Q}$, the Sturm approach solves completely the problem. If all the roots of P are simple it is possible to associate to it a sequence of $n+1$ polynomials, the so-called *Sturm sequence*, and knowing the signs of this sequence evaluated at α and β we obtain the exact number of real roots in the interval. If P has multiple roots it suffices to start with $P/(\gcd(P, P'))$, see [96, Sec. 5.6].

Nevertheless when the rational numbers have big numerators and denominators and n is also big, the computers have not enough capacity to perform the computations to get the Sturm sequence. On the other hand the Descartes rule is not so powerful but a careful use, in the spirit of bisection method, can many times solve the problem.

To recall the Descartes rule we need to introduce some notation. Given an ordered list of real numbers $[b_0, b_1, \ldots, b_{n-1}, b_n]$ we will say that it has C changes of sign if the following holds: denote by $[c_0, c_1, \ldots, c_{m-1}, c_m]$, $m \leq n$ the new list obtained from the previous one after removing the zeros and without changing the order of the remaining terms. Consider the m non-zero numbers $\delta_i := c_i c_{i+1}$, $i = 0, \ldots, m-1$. Then C is the number of negative δ_i.

Theorem 4.16 (Descartes rule)**.** *Let C be the number of changes of sign of the list of ordered numbers*

$$[a_0, a_1, a_2, \ldots, a_{n-1}, a_n].$$

Then the number of positive zeros of the polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$, counted with their multiplicities, is $C - 2k$, for some $k \in \mathbb{N} \cup \{0\}$.

Corollary 4.17. *With the notations of Theorem 4.16 if $C = 0$ then $P(x)$ has not positive roots and if $C = 1$ it has exactly one simple positive root.*

In order to apply Descartes rule to arbitrary open intervals we introduce the following definition:

Definition 4.18. *Given a real polynomial $P(x)$ and a real interval (α, β) we construct a new polynomial*

$$N_\alpha^\beta(P)(x) := (x+1)^{\deg P} P\left(\frac{\beta x + \alpha}{x+1}\right).$$

We will call $N_\alpha^\beta(P)$, the normalized version of P with respect to (α, β). Notice that the number of real roots of $P(x)$ in the interval (α, β) is equal to the number of real roots of $N_\alpha^\beta(P)(x)$ in $(0, \infty)$.

The method suggested in [63] consists in writing $(\alpha, \beta) = \bigcup_{i=1}^k (\alpha_i, \alpha_{i+1})$, with $\alpha = \alpha_1 < \alpha_2 < \cdots < \alpha_k < \alpha_{k+1} = \beta$ in such a way that on each (α_i, α_{i+1}) it is possible to apply Corollary 4.17 to the normalized version of the polynomial. Although there is no systematic way of searching a suitable decomposition, we will see that a careful use of these type of ideas has been good enough to study the number and localization of the roots for a huge polynomial of degree 965, see Subsection 4.5.4 in Appendix II.

Appendix II: Polynomials in two variables

The main result of this appendix is a new method for controlling the sign of families of polynomials with two variables. As a starting point we prove a simple result for one-parameter families of polynomials in one variable.

Let $G_b(x)$ be a one-parametric family of polynomials. As usual, we write $\triangle_x(P)$ to denote the discriminant of a polynomial $P(x) = a_n x^n + \cdots + a_1 x + a_0$, that is,

$$\triangle_x(P) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_n} \operatorname{Res}(P(x), P'(x)),$$

where $\operatorname{Res}(P, P')$ is the resultant of P and P'.

Lemma 4.19. *Let*

$$G_b(x) = g_n(b)x^n + g_{n-1}(b)x^{n-1} + \cdots + g_1(b)x + g_0(b),$$

be a family of real polynomials depending also polynomially on a real parameter b and set $\Omega = \mathbb{R}$. Suppose that there exists an open interval $I \subset \mathbb{R}$ such that:

(i) There is some $b_0 \in I$, such that $G_{b_0}(x) > 0$ on Ω.

(ii) For all $b \in I$, $\triangle_x(G_b) \neq 0$.

(iii) For all $b \in I$, $g_n(b) \neq 0$.

Then for all $b \in I$, $G_b(x) > 0$ on Ω.

Moreover, if $\Omega = \Omega_b = (c(b), \infty)$ for some smooth function $c(b)$, the same result holds changing Ω by this new Ω_b if we add the additional hypothesis

(iv) For all $b \in I$, $G_b(c(b)) \neq 0$.

Proof. The key point of the proof is that the roots (real and complex) of G_b depend continuously of b, because $g_n(b) \neq 0$. Notice that hypotheses (iii) and (iv) prevent that moving b some root enters in Ω either from infinity or from the boundary of Ω, respectively. On the other hand if moving b some real roots appear from $\mathbb{C}$, they do appear through a double real root that is detected by the vanishing of $\triangle_x(G_b)$. Since by item (ii), $\triangle_x(G_b) \neq 0$ no real root appears in this way. Hence, for all $b \in I$, the number of real roots of any G_b is the same. Since by item (i) for $b = b_0$, $G_{b_0} > 0$ on Ω, the same holds for all $b \in I$. □

To state the corresponding result for families of polynomials with two variables inspired in the above lemma, see Proposition 4.25, we need to prove some results about the iterated discriminants (to replace hypothesis (ii) of the lemma) and to recall how to study the infinity of planar curves (to replace hypothesis (iii)).

4.5.1 The double discriminant

Let $F(x, y)$ be a complex polynomial on $\mathbb{C}^2$. We write F as

$$F(x, y) = a_n y^n + a_{n-1} y^{n-1} + a_{n-2} y^{n-2} + \ldots + a_1 y + a_0, \qquad (4.25)$$

where $a_i = a_i(x) \in \mathbb{C}[x]$. Then

$$\triangle_y(F) = (-1)^{\frac{n(n-1)}{2}} \frac{1}{a_n} \operatorname{Res}(F, \partial F/\partial y),$$

and this resultant can be computed as the determinant of the Sylvester matrix of dimension $(2n-1)\times(2n-1)$, see [31],

$$S=\begin{pmatrix} a_n & 0 & 0 & 0 & na_n & 0 & 0 & 0\\ a_{n-1} & a_n & 0 & 0 & (n-1)a_{n-1} & na_n & 0 & 0\\ a_{n-2} & a_{n-1} & \ddots & 0 & (n-2)a_{n-2} & (n-1)a_{n-1} & \ddots & 0\\ \vdots & & \ddots & a_n & \vdots & & \ddots & na_n\\ & \vdots & & a_{n-1} & & \vdots & & (n-1)a_{n-1}\\ a_0 & & & & a_1 & & & \\ 0 & a_0 & & \vdots & 0 & a_1 & & \vdots\\ 0 & 0 & \ddots & & 0 & 0 & \ddots & \\ 0 & 0 & 0 & a_0 & 0 & 0 & 0 & a_1 \end{pmatrix}.$$

We will write $\triangle^2_{y,x}(F)=\triangle_x(\triangle_y(F))$. Analogously we can compute $\triangle^2_{x,y}(F)$. This so-called double discriminant plays a special role in the characterization of singular curves of $\{F(x,y)=0\}$ and it is also used in applications, see for instance [3, 64, 88]. In particular we prove the following result.

Proposition 4.20. *Let $F(x,y)$ be a complex polynomial on $\mathbb{C}^2$. If $\{F(x,y)=0\}\subset\mathbb{C}^2$ has a singular point, that is, if there exists a point $(x_0,y_0)\in\mathbb{C}^2$ such that $F(x_0,y_0)=\partial F(x_0,y_0)/\partial x=\partial F(x_0,y_0)/\partial y=0$, then $\triangle^2_{y,x}(F)=\triangle^2_{x,y}(F)=0$.*

Proof. We write $F(x,y)$ in the form (4.25). Without lost of generality we assume that $(x_0,y_0)=(0,0)$. Then from the assumptions it follows that $a_0(0)=a_0'(0)=0$ and $a_1(0)=0$, that is, $a_0(x)=x^2\hat{a}_0(x)$ and $a_1(x)=x\hat{a}_1(x)$, with both $\hat{a}_i$ also polynomials.

By using the Sylvester matrix S defined above, we have that

$$\det S=(-1)^n a_0\det(S(2n-1\mid n-1))+a_1\det(S(2n-1\mid 2n-1)),\qquad(4.26)$$

where $S(i\mid j)$ means the matrix obtained from S by removing the i-th row and the j-th column.

Notice that the elements of the last row of $S(2n-1\mid 2n-1)$ are only $0,a_0$ and a_1. Therefore, developing the determinant of this matrix from this row we get that $\det(S(2n-1\mid 2n-1))=xQ(x)$, for some polynomial $Q(x)$.

Hence, by using (4.26), we get that $\det S=x^2P(x)$ with $P(x)$ another polynomial. This implies that $\triangle_y(F)$ has a double zero at $x=0$ and hence $\triangle^2_{y,x}(F)=0$.

Analogously we can prove that $\triangle_x(F)$ has a double zero at $y=0$ and hence $\triangle^2_{x,y}(F)=0$. □

Corollary 4.21. *Consider a one-parameter family of polynomials $F_b(x,y)$, depending also polynomially on b. The values of b such that the algebraic curve $F_b(x,y)=0$ has some singular point in $\mathbb{C}^2$ have to be zeros of the polynomial*

$$\triangle^2(F_b):=\gcd\left(\triangle^2_{x,y}(F_b),\triangle^2_{y,x}(F_b)\right).$$

By simplicity we will also call the polynomial $\triangle^2(F_b)$, *double discriminant of the family* $F_b(x,y)$. As far as we know the above necessary condition for detecting algebraic curves with singular points is new.

Remark 4.22. *(i) Notice that if in Corollary 4.21, instead of imposing that for* $b \in I$, $\triangle^2(F_b) \neq 0$, *it suffices to check only that either* $\triangle^2_{x,y}(F_b) \neq 0$ *or* $\triangle^2_{y,x}(F_b) \neq 0$.

(ii) The converse of the Proposition 4.20 is not true. For instance if we consider the polynomial $F(x,y) = x^3y^3 + x + 1$ *then* $\triangle^2_{y,x}(F) = \triangle^2_{x,y}(F) = 0$, *however* $F_x(x,y) = 3x^2y^3 + 1$ *and* $F_y(x,y) = 3x^3y^2$ *hence* $\{F(x,y) = 0\}$ *does not have singular points.*

(iii) Sometimes $\triangle^2_{y,x}(F) \neq \triangle^2_{x,y}(F)$. *For instance this is the case when* $F = y^2 + x^3 + bx^2 + bx$ *because*

$$\triangle^2_{x,y}(F) = -110592b^9(b-4)(b-3)^6 \quad \text{and} \quad \triangle^2_{y,x}(F) = 256b^3(b-4).$$

Notice that $\triangle^2(F) = b^3(b-4)$.

4.5.2 Algebraic curves at infinity

Let

$$F(x,y) = F^0(x,y) + F^1(x,y) + \cdots + F^n(x,y)$$

be a polynomial on $\mathbb{R}^2$ of degree n. We denote by

$$\widetilde{F}(x,y,z) = z^n F^0(x,y) + z^{n-1}F^1(x,y) + \cdots + F^n(x,y)$$

its homogenization in $\mathbb{RP}^2$.

For studying $\widetilde{F}(x,y,z)$ in $\mathbb{RP}^2$ we can use its expressions in the three canonical charts of $\mathbb{RP}^2$, $\{[x:y:1]\}$, $\{[x:1:z]\}$, and $\{[1:y:z]\}$, which can be identified with the real planes $\{(x,y)\}$,$\{(x,z)\}$, and $\{(y,z)\}$ respectively. Of course the expression in the chart $\{[x:y:1]\}$, that is, in the (x,y)-plane is precisely $F(x,y)$.

We denote by $\widetilde{F}_1(x,z)$ and $\widetilde{F}_2(y,z)$ the expressions of the function $\widetilde{F}$ in the planes $\{(x,z)\}$ and $\{(y,z)\}$, respectively. Therefore $\widetilde{F}_1(x,z) = \widetilde{F}(x,1,z)$ and $\widetilde{F}_2(y,z) = \widetilde{F}(1,y,z)$.

Let $[x^*:y^*:z^*] \in \mathbb{RP}^2$ be a point of $\{\widetilde{F} = 0\}$. If $z^* \neq 0$, then $[x^*:y^*:z^*]$ corresponds to a point in $\mathbb{R}^2$, otherwise it is said that $[x^*:y^*:0]$ is a point of F at infinity. Notice that the points at infinity of F correspond to the points $[x^*:y^*:0]$ where $(x^*,y^*) \neq (0,0)$ is a solution of the homogeneous part of degree n of F,

$$\mathcal{H}_n(F(x,y)) = F^n(x,y),$$

that is $F^n(x^*,y^*) = 0$. Equivalently, these are the zeros of $\widetilde{F}_1(x,0)$ and $\widetilde{F}_2(y,0)$. In other words, $[x^*:y^*:0]$ is a point at infinity of F if and only if x^*/y^* is a zero of $\widetilde{F}_1(x,0) = F^n(x,1)$ or y^*/x^* is a zero of $\widetilde{F}_2(y,0) = F^n(1,y)$.

Let $\Omega \subset \mathbb{R}^2$ be an unbounded open subset with boundary $\partial\Omega$ formed by finitely many algebraic curves. It is clear that this subset can be extended to $\mathbb{RP}^2$. We will call the adherence of this extension $\bar{\Omega}$. When a point at infinity of F is also in $\bar{\Omega}$, for short we will say that is a point at infinite which is also in Ω.

4.5.3 Isolated points of families of algebraic curves

To state our main result we need explicit conditions to check when a point of a real algebraic curve $G(x, y) = 0$ is isolated. Recall that it is said that a point $\mathbf{p} \in \mathbb{R}^2$ on the curve is *isolated* if there exists an open neighborhood $\mathcal{U}$ of $\mathbf{p}$, such that

$$\mathcal{U} \cap \{(x, y) \in \mathbb{R}^2 : G(x, y) = 0\} = \{\mathbf{p}\}.$$

Clearly isolated points are singular points of the curve. Next result provides an useful criterion to deal with this question.

Lemma 4.23. *Let $G(x, y)$ be a real polynomial. Assume that $(0, 0) \in \{G(x, y) = 0\}$ and that there are natural numbers p, q and m, with $\gcd(p, q) = 1$, and a polynomial G^0 satisfying $G^0(\varepsilon^p X, \varepsilon^q Y) = \varepsilon^m G^0(X, Y)$, and such that for all $\varepsilon > 0$,*

$$G(\varepsilon^p X, \varepsilon^q Y) = \varepsilon^m G^0(X, Y) + \varepsilon^{m+1} G^1(X, Y, \varepsilon),$$

for some polynomial function G^1. If the only real solution of $G^0(X, Y) = 0$ is $(X, Y) = (0, 0)$, then the origin is an isolated point of $G(x, y) = 0$.

Proof. Assume without loss of generality that $G^0 \geq 0$. We start proving that $K := \{(x, y) \in \mathbb{R}^2 : G^0(x, y) = 1\}$ is a compact set. Clearly it is closed, so it suffices to prove that it is bounded. Since G^0 is a quasi-homogeneous polynomial we know that there exists a natural number m_0 such that $m = m_0 pq$ and $G^0(x, y) = P_{m_0}(x^q, y^p)$, where P_{m_0} is a real homogeneous polynomial of degree m_0. The fact that the only real solution of the equation $G^0(x, y) = 0$ is $x = y = 0$ implies that P_{m_0} has not linear factors when we decompose it as a product of real irreducible factors. Hence m_0 is even and $P_{m_0}(x, y) = \prod_{i=1}^{m_0/2}(A_i x^2 + B_i xy + C_i y^2)$, with $B_i^2 - 4A_iC_i < 0$. As a consequence,

$$G^0(x, y) = \prod_{i=1}^{m_0/2} (A_i x^{2q} + B_i x^q y^p + C_i y^{2p}), \quad \text{with} \quad B_i^2 - 4A_iC_i < 0. \tag{4.27}$$

Assume, to arrive to a contradiction, that K is unbounded. Therefore it should exist a sequence $\{(x_n, y_n)\}$, tending to infinity, and such that $G^0(x_n, y_n) = 1$. But this is impossible because the conditions $B_i^2 - 4A_iC_i < 0$, $i = 1, \ldots, m_0/2$, imply that all the terms $A_i x_n^{2q} + B_i x_n^q y_n^p + C_i y_n^{2p}$ in (4.27) go to infinity. So K is compact.

Let us prove that $(0,0)$ is an isolated point of $\{(x,y) \in \mathbb{R}^2 : G(x,y) = 0\}$. Assume, to arrive to a contradiction, that it is not. Therefore there exists a sequence of points $\{(x_n, y_n)\}$, tending to 0 and such that $G(x_n, y_n) = 0$ for all $n \in \mathbb{N}$. Consider $G^0(x_n, y_n) =: (g_n)^m > 0$. It is clear that $\lim_{n\to\infty}(g_n)^m = 0$. Write $(x_n, y_n) = ((g_n)^p u_n, (g_n)^q v_n)$. Notice that

$$(g_n)^m = G^0(x_n, y_n) = G^0(g_n^p u_n, g_n^q v_n) = (g_n)^m G^0(u_n, v_n).$$

Then $G^0(u_n, v_n) = 1$ and $(u_n, v_n) \in K$, for all $n \in \mathbb{N}$. Therefore, taking a subsequence if necessary, we can assume that

$$\lim_{n\to\infty} (u_n, v_n) = (u^*, v^*) \in K. \tag{4.28}$$

We have that $0 = G(x_n, y_n) = (g_n)^m + (g_n)^{m+1} G^1(u_n, v_n, g_n)$. Dividing by $(g_n)^m$ we obtain that $0 = 1 + g_n G^1(u_n, v_n, g_n)$, and passing to the limit we get that $1 = 0$ which gives the desired contradiction.

Notice that to prove that $\lim_{n\to\infty} g_n G^1(u_n, v_n, g_n) = 0$ we need to know that the sequence $\{(u_n, v_n)\}$ remains bounded and this fact is a consequence of (4.28). □

We remark that the suitable values p, q and m and the function G^0 appearing in the statement of Lemma 4.23 are usually found by using the Newton diagram associated to G.

We also need to introduce a new related concept for families of curves. Consider a one-parameter family of algebraic curves $G_b(x,y) = 0$, $b \in I$, also depending polynomially of b. Let $(x_0, y_0) \in \mathbb{R}^2$ be an isolated point of $G_b(x,y) = 0$ for all $b \in I$, we will say that (x_0, y_0) is *uniformly isolated* for the family $G_b(x,y) = 0$, $b \in I$ if for each $b \in I$ there exist neighborhoods $\mathcal{V} \subset I$ and $\mathcal{W} \subset \mathbb{R}^2$, of b and (x_0, y_0) respectively, such that for all $b \in \mathcal{V}$,

$$\{(x,y) \in \mathbb{R}^2 : G_b(x,y) = 0\} \cap \mathcal{W} = \{(x_0, y_0)\}. \tag{4.29}$$

Next example shows a one-parameter family of curves that has the origin isolated for all $b \in \mathbb{R}$ but it is not uniformly isolated for $b \in I$, with $0 \in I$,

$$G_b(x,y) = (x^2 + y^2)(x^2 + y^2 - b^2)(x - 1). \tag{4.30}$$

It is clear that the origin is an isolated point of $\{G_b(x,y) = 0\}$ for all $b \in \mathbb{R}$, but there is no open neighborhood $\mathcal{W}$ of $(0,0)$, such that (4.29) holds for any b in a neighborhood of $b = 0$.

Next result is a version of Lemma 4.23 for one-parameter families. In its proof we will use some periodic functions introduced by Lyapunov in his study of the stability of degenerate critical points, see [69]. Let us recall them.

Let $u(\varphi) = \mathrm{Cs}(\varphi)$ and $v(\varphi) = \mathrm{Sn}(\varphi)$ be the solutions of the Cauchy problem:

$$u' = -v^{2p-1},\ v' = u^{2q-1}, \quad u(0) = \sqrt[2q]{1/p} \quad \text{and} \quad v(0) = 0,$$

where the prime denotes the derivative with respect to φ.

Then $x = \mathrm{Cs}(\varphi)$ and $y = \mathrm{Sn}(\varphi)$ parameterize the algebraic curve $px^{2q}+qy^{2p} = 1$, that is $p\,\mathrm{Cs}^{2q}(\varphi) + q\,\mathrm{Sn}^{2p}(\varphi) = 1$, and both functions are smooth $T_{p,q}$-periodic functions, where

$$T = T_{p,q} = 2p^{-1/2q}q^{-1/2p}\frac{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{1}{2q}\right)}{\Gamma\left(\frac{1}{2p}+\frac{1}{2q}\right)},$$

and Γ denotes the Gamma function.

Proposition 4.24. *Let $G_b(x,y)$ be a family of real polynomials which also depends polynomially on b. Assume that $(0,0) \in \{G_b(x,y) = 0\}$ and that there are natural numbers p, q and m, with $\gcd(p,q) = 1$, and a polynomial G_b^0 satisfying $G_b^0(\varepsilon^p X, \varepsilon^q Y) = \varepsilon^m G_b^0(X,Y)$, and such that for all $\varepsilon > 0$,*

$$G_b(\varepsilon^p X, \varepsilon^q Y) = \varepsilon^m G_b^0(X,Y) + \varepsilon^{m+1}G_b^1(X,Y,\varepsilon),$$

for some polynomial function G_b^1. If for all $b \in I \subset \mathbb{R}$, the only real solution of $G_b^0(X,Y) = 0$ is $(X,Y) = (0,0)$, then the origin is an uniformly isolated point of $G_b(x,y) = 0$ for all $b \in I$.

Proof. Assume without loss of generality that $G_b^0 \geq 0$. Let us write the function $G_b(x,y)$ using the so-called generalized polar coordinates,

$$x = \rho^p\,\mathrm{Cs}(\varphi),\ y = \rho^q\,\mathrm{Sn}(\varphi), \quad \text{for} \quad \rho \in \mathbb{R}^+.$$

Then

$$\begin{aligned} G_b(x,y) &= G_b(\rho^p\,\mathrm{Cs}(\varphi), \rho^q\,\mathrm{Sn}(\varphi)) \\ &= \rho^m G_b^0(\mathrm{Cs}(\varphi),\mathrm{Sn}(\varphi)) + \rho^{m+1}G_b^1(\mathrm{Cs}(\varphi),\mathrm{Sn}(\varphi),\rho). \end{aligned} \tag{4.31}$$

Using the same notation that in the proof of Lemma 4.23, with the obvious modifications, we know from (4.27) that

$$G_b^0(\mathrm{Cs}(\varphi),\mathrm{Sn}(\varphi)) = \prod_{i=1}^{m_0/2}(A_i(b)\,\mathrm{Cs}^{2q}(\varphi) + B_i(b)\,\mathrm{Cs}^q(\varphi)\,\mathrm{Sn}^p(\varphi) + C_i(b)\,\mathrm{Sn}^{2p}(\varphi)),$$

with all $B_i^2(b) - 4A_i(b)C_i(b) < 0$. Therefore, it is not difficult to prove that there exists two positive continuous functions, $L(b)$ and $U(b)$ such that

$$0 < L(b) \leq G_b^0(\mathrm{Cs}(\varphi),\mathrm{Sn}(\varphi)) \leq U(b),$$

due to the periodicity of the Lyapunov functions and the discriminant conditions. Dividing the expression (4.31) by ρ^m we obtain that the points of $\{G_b(x,y) = (0,0)\} \setminus \{(0,0)\}$ are given by

$$G_b^0(\mathrm{Cs}(\varphi), \mathrm{Sn}(\varphi)) + \rho\, G_b^1(\mathrm{Cs}(\varphi), \mathrm{Sn}(\varphi), \rho) = 0. \tag{4.32}$$

Fix a compact neighborhood of b, say $\mathcal{V} \subset I$. Set $L = \min_{x\in\mathcal{V}} L(b)$. Then there exists $\delta > 0$ such that for any $||(x,y)|| \leq \delta$ and any $b \in \mathcal{V}$,

$$|\rho\, G_b^1(\mathrm{Cs}(\varphi), \mathrm{Sn}(\varphi), \rho)| < L/2.$$

Therefore (4.32) never holds in this region and

$$\{(x,y) \in \mathbb{R}^2 \,:\, G_b(x,y) = 0\} \cap \{(x,y) \in \mathbb{R}^2 \,:\, ||(x,y)|| < \delta\} = \{(0,0)\},$$

for all $b \in \mathcal{V}$, as we wanted to prove. □

Notice that, the fact that for all $b \in \mathbb{R}$, the origin of (4.30) is isolated simply follows plotting the zero level set of G_b. Alternatively, we can apply Lemma 4.23 with $p = 1, q = 1$ and $m = 2$ to prove that the origin is isolated when $b \neq 0$ and with $p = q = 1$ and $m = 4$ when $b = 0$. In any case, Proposition 4.24 can not be used.

4.5.4 A method for controlling the sign

Proposition 4.25. *Let $F_b(x,y)$ be a family of real polynomials depending also polynomially on a real parameter b and let $\Omega \subset \mathbb{R}^2$ be an open connected subset having a boundary $\partial\Omega$ formed by finitely many algebraic curves. Suppose that there exists an open interval $I \subset \mathbb{R}$ such that:*

- *(i) For some $b_0 \in I$, $F_{b_0}(x,y) > 0$ on Ω.*
- *(ii) For all $b \in I$, $\triangle^2(F_b) \neq 0$.*
- *(iii) For all $b \in I$, all points of $F_b = 0$ at infinity which are also in Ω do not depend on b and are uniformly isolated.*
- *(iv) For all $b \in I$, $\{F_b = 0\} \cap \partial\Omega = \emptyset$.*

Then for all $b \in I$, $F_b(x,y) > 0$ on Ω.

Proof. Write $I = J \cup J^c$, where

$$J := \{b \in I : \{F_b(x,y) = 0\} \cap \Omega = \emptyset\} \ \text{ and } \ J^c := \{b \in I : \{F_b(x,y) = 0\} \cap \Omega \neq \emptyset\}.$$

We will prove that both J and J^c are open sets. Since I is connected and from hypothesis (i), $J \neq \emptyset$ because $b_0 \in J$, it will follow that $I = J$, as we wanted to prove.

We will first prove that J^c is open. We take $\bar{b} \in J^c$. If $F_{\bar{b}}(x,y)$ does not change sign on Ω, then either $F_{\bar{b}}(x,y) \geq 0$ or $F_{\bar{b}}(x,y) \leq 0$ in Ω. Then, in both cases $F_{\bar{b}}(x,y)$ has a singular point and this implies that $\triangle^2(F_{\bar{b}}) = 0$ which contradicts hypothesis (ii). Therefore $F_{\bar{b}}(x,y)$ changes of sign in Ω. For continuity this property remains for all b close enough to $\bar{b}$. Hence there exists a neighborhood $U_{\bar{b}}$ of $\bar{b}$ contained in J^c, that is J^c is open, as we wanted to prove.

Next, we will show that J is open as well. Assume that this is false. Then there exists a sequence $\{b_n\}$ of J^c such that $b_n \to \bar{b}$ with $\bar{b} \in J$. Since $\{F_{b_n}(x,y) = 0\} \cap \Omega \neq \emptyset$, for each n there exists a point $(x_n, y_n) \in \Omega$ such that $F_{b_n}(x_n, y_n) = 0$. Consider now the sequence $\{(x_n, y_n)\}$. It is is either bounded or unbounded.

In the first case, we can assume that $\lim_{n\to\infty}(x_n, y_n) = (\bar{x}, \bar{y}) \in \Omega \cup \partial\Omega$. If $(\bar{x}, \bar{y}) \in \Omega$, then $F_{\bar{b}}(\bar{x}, \bar{y}) = 0$ which is impossible because $\bar{b} \in J$. If $(\bar{x}, \bar{y}) \in \partial\Omega$ then $F_{\bar{b}}(\bar{x}, \bar{y}) = 0$ which is impossible owing to hypothesis (iv).

In the second case, a subsequence of $\{(x_n, y_n)\}$ converges to a point at infinity of $\mathbb{RP}^2$, that is we can assume that $\lim_{n\to\infty}[x_n : y_n : 1] = [\bar{x} : \bar{y} : 0]$. Therefore $\lim_{n\to\infty} \widetilde{F}_{b_n}(x_n, y_n, 1) = \widetilde{F}_{\bar{b}}(\bar{x}, \bar{y}, 0) = 0$, where recall that $\widetilde{F}$ denotes the homogenization of F in $\mathbb{RP}^2$. This last equality is in contradiction with hypothesis (iii), because note that for all n, $\widetilde{F}_{b_n}(x_n, y_n, 1) = 0$ and these facts imply that $F_{\bar{b}}$ would have a non uniformly isolated singularity in Ω at infinity.

Hence J is open as we wanted to prove and the proposition follows. □

Control of the sign of (4.19)

In this subsection we will prove by using Proposition 4.25, that for $b \in (0, 0.6512)$, the function M_b given in (4.19) is positive on $\Omega = \mathbb{R}^2$.

To check hypothesis (i), we prove that $M_{1/2} > 0$ for all $\mathbb{R}^2$. For this value,

$$M_{1/2} = \tfrac{15}{2}\, x^4y^2 - \tfrac{21}{4}\, x^3y^3 + \tfrac{21}{2}\, x^2y^4 - \tfrac{123}{16}\, x^2y^2 + \tfrac{21}{16}\, xy^3 + \tfrac{5}{2}\, x^4 - \tfrac{7}{16}\, x^2 + \tfrac{15}{64}\, y^2 + \tfrac{13}{64}.$$

We think $M_{1/2}$ as a polynomial in x and y as a parameter and we apply Lemma 4.19. If $y = 0$ then $M_{1/2}$ reduces to the polynomial $(5/2)x^4 - (7/16)x^2 + 13/64$ which is positive on $\mathbb{R}$. Now, we compute $\triangle_x(M_{1/2})$ and we obtain a polynomial in the variable y of degree 20. By using the Sturm method it is easy to see that it does not have real roots. Moreover, the coefficient of x^4 is $5(3y^2 + 1)/2 > 0$. Therefore, $M_{1/2} > 0$ on $\mathbb{R}^2$, as we wanted to see.

To check hypothesis (ii) we compute the double discriminant of M_b and we obtain that $\triangle^2_{x,y}(M_b)$ is a polynomial in b of degree 1028, of the following form

$$\begin{aligned}\triangle^2_{x,y}(M_b) =& b^{320}(b^2-2)^{40}(3b^2-2)^5(3b^2-4)(2b^6-4b^4-3b^2+2)\times\\ &\times(b^6-2b^4-3b^2+2)(P_2(b^2))^8(P_6(b^2))^4(P_{32}(b^2))^2(P_{33}(b^2))^6,\end{aligned}$$

where P_i are polynomials of degree i with rational coefficients. By using the Sturm method we localize the real roots of each factor of $\triangle^2_{x,y}(M_b)$ and we obtain that in the interval $(0, 0.6512)$ none of them has real roots. In fact $P_{32}(b^2)$ has a root in $(0.6513, 0.6514)$ and that is the reason for which we can not increase more the value of b. Therefore $\triangle^2_{x,y}(M_b) \neq 0$ for all $b \in (0, 0.6512)$.

Finally we have to check hypothesis (iii). Notice that in this case $\partial\Omega = \emptyset$ and so (iv) follows directly.

The zeros at infinity are given by the directions

$$\mathcal{H}_6(M_b) = 6x^2y^2\big((2-3b^2)x^2 - 2b^2(2-b^2)xy + (2-b^2)y^2\big) = 0.$$

For $|b| < 0.7275$ it has only the non-trivial solutions $x = 0$ and $y = 0$. The homogenization of M_b is

$$\begin{aligned}\widetilde{M}_b =\ & 6(2-3b^2)x^4y^2 - 12b^2(2-b^2)x^3y^3 + 6(2-b^2)x^2y^4\\ &+2(2-3b^2)x^4z^2 - 3b^2(14-15b^2)x^2y^2z^2 + 12b^4(2-b^2)xy^3z^2\\ &-b^2(4-9b^2)x^2z^4 + 3b^4(2-3b^2)y^2z^4 + b^4(4-3b^2)z^6,\end{aligned} \tag{4.33}$$

and hypothesis (iii) is equivalent to prove that $(0,0)$ is an uniformly isolated singularity for $\widetilde{M}^1_b(x,z) = \widetilde{M}_b(x,1,z)$ and that $(0,0)$ is also an uniformly isolated singularity for $\widetilde{M}^2_b(y,z) = \widetilde{M}_b(1,y,z)$.

First we prove this result for $\widetilde{M}^1_b(x,z)$. From (4.33),

$$\begin{aligned}\widetilde{M}^1_b(x,z) =\ & 6(2-3b^2)x^4 - 12b^2(2-b^2)x^3 + 6(2-b^2)x^2 + 2(2-3b^2)x^4z^2\\ &-3b^2(14-15b^2)x^2z^2 + 12b^4(2-b^2)xz^2 - b^2(4-9b^2)x^2z^4\\ &+3b^4(2-3b^2)z^4 + b^4(4-3b^2)z^6.\end{aligned}$$

Hence,

$$\widetilde{M}^1_b(\varepsilon^2X, \varepsilon Z) = \Big(6(2-b^2)X^2 + 12b^4(2-b^2)XZ^2 + 3b^4(2-3b^2)Z^4\Big)\varepsilon^4 + O(\varepsilon^5).$$

The discriminant with respect to X of the homogeneous polynomial $T(X,W) := 6(2-b^2)X^2 + 12b^4(2-b^2)XW + 3b^4(2-3b^2)W^2$, where $W = Z^2$, is

$$\triangle_X(T) = 72W^2b^4(b^2-2)(2b^6-4b^4-3b^2+2).$$

Since its smallest positive root is greater than 0.673 it holds for $b \in (0, 673)$ that $T(X,W) = 0$ if and only if $(X,W) = (0,0)$. Therefore by Proposition 4.24 the point $(0,0)$ is an uniformly isolated point of the curve $\widetilde{M}^1_b(x,z) = 0$, for these values of b.

On the other hand, since

$$\begin{aligned}\widetilde{M}_b^2(y,z) = \; & 6(2-b^2)y^4 - 12b^2(2-b^2)y^3 + 6(2-3b^2)y^2 + 2(2-3b^2)z^2 \\ & -3b^2(14-15b^2)y^2z^2 + 12b^4(2-b^2)y^3z^2 - b^2(4-9b^2)z^4 \\ & +3b^4(2-3b^2)y^2z^4 + b^4(4-3b^2)z^6,\end{aligned}$$

we have that

$$\widetilde{M}_b^2(\varepsilon Y, \varepsilon Z) = 2(2-3b^2)\Big(3Y^2 + Z^2\Big)\varepsilon^2 + O(\varepsilon^3),$$

and the result follows for $b \in (0, \sqrt{2/3}) \approx (0, 0.816)$, by applying again the same proposition.

So, we have shown that for $b \in (0, 0.6512)$ all the hypotheses of the Proposition 4.25 hold. Therefore we have proved that for $b \in (0, 0.651]$, $M_b(x,y) > 0$ for all $(x,y) \in \mathbb{R}^2$.

Control of the sign of (4.23)

The numerator of the function M_b given in (4.23) is a polynomial of the following form

$$N_b(x,y) = f_0(x,b) + f_1(x,b)y + f_2(x,b)y^2 + f_3(x,b)y^3 + f_4(x,b)y^4, \qquad (4.34)$$

where

$$\begin{aligned} f_0(x,b) \; = \; & 90b^{36}x^{10} - 15b^{18}(6b^{20}-5)x^8 + 15b^{18}(24b^4 - 59b^2 + 24)x^6 \\ & -(378b^{24} - 810b^{22} + 360b^{20} - 300b^4 + 675b^2 - 300)x^4 \\ & -15b^2(18b^{22} - 24b^{20} + 21b^4 - 45b^2 + 20)x^2 - 75b^4(-4+3b^2), \end{aligned}$$

$$\begin{aligned} f_1(x,b) \; = \; & 180b^{36}x^7 + 12b^{18}(60b^{16} + 50b^{14} + 18b^{10} + 25)x^5 - 20b^{10}(36b^{12} \\ & -54b^{10} + 54b^8 - 30b^6 - 25b^4 - 9)x^3 - 180b^{20}(3b^2-4)x, \end{aligned}$$

$$\begin{aligned} f_2(x,b) \; = \; & 270b^{36}x^{10} - 45b^{18}(6b^{20} + 2b^{18} - 5)x^8 + 3b^{18}(30b^{20} + 120b^{16} \\ & +100b^{14} - 90b^{12} + 36b^{10} + 360b^4 - 615b^2 + 335)x^6 - (360b^{36} \\ & +300b^{34} + 108b^{30} + 2214b^{24} - 3690b^{22} + 3435b^{20} + 360b^{18} \\ & -300b^{16} - 250b^{14} + 225b^{12} - 90b^{10} - 900b^4 + 1350b^2 - 900)x^4 \\ & -b^2(468b^{22} - 540b^{20} - 1080b^{18} + 300b^{16} + 250b^{14} + 90b^{10} \\ & +1845b^4 - 3075b^2 + 2475)x^2 - 90b^4(4b^2-5), \end{aligned}$$

$$\begin{aligned} f_3(x,b) \; = \; & -180b^{20}(b^{10}-3)x^7 + 30b^2(6b^{34} + 6b^{30} - 24b^{22} + 18b^{20} - 72b^{18} \\ & -5b^{10} + 15)x^5 + 30b^2(24b^{24} - 36b^{22} + 72b^{20} + 10b^{16} + 5b^{12} \\ & -20b^4 + 15b^2 - 60)x^3 - 20b^4(36b^{18} - 54b^{16} + 54b^{14} + 30b^{12} \\ & +25b^{10} + 9b^6 - 30b^4 + 45b^2 - 90)x, \end{aligned}$$

$$\begin{aligned} f_4(x,b) &= 90b^{36}x^8 - 3b^{18}(30b^{20} + 120b^{16} + 100b^{14} + 36b^{10} - 25)x^6 \\ &\quad +b^{10}(360b^{26} + 300b^{24} + 198b^{20} + 360b^{12} - 615b^{10} + 720b^8 \\ &\quad -300b^6 - 250b^4 - 90)x^4 + (-738b^{24} + 1080b^{22} - 1080b^{20} \\ &\quad +300b^{18} + 250b^{16} + 315b^{12} + 300b^4 - 450b^2 + 900)x^2 + 15b^6. \end{aligned}$$

We will prove that $N_b \geq 0$ on $\Omega := \{(x, y) : xy + 1 > 0\}$ for all $b \in (0, 0.817]$ and if it vanishes this only happens at some isolated points. We will use again Proposition 4.25. Notice that $\partial\Omega = \{(x, y) : xy + 1 = 0\}$.

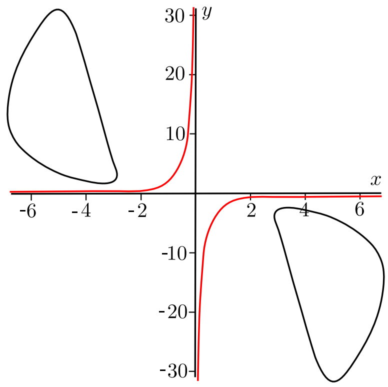


Figure 4.14: Curves $N_b = 0$ and $xy + 1 = 0$ with $b = 0.817$.

It is not difficult to verify that $\{N_b(x, y) = 0\} \cap \{xy + 1 = 0\} = \emptyset$ for $b \in (0, 0.8171)$, see Figure 4.14. It suffices to see that for these values of b, and $x \neq 0$, the one variable function $N_b(x, 1/x)$, never vanishes. We skip the details. Therefore hypothesis (iv) is satisfied.

For proving that hypothesis (ii) of Proposition 4.25 holds we compute the double discriminant $\triangle^2_{y,x}(N_b)$. It is an even polynomial in b, of degree 21852, of the following form

$$b^{7566}(3b^2 - 4)(159b^4 - 380b^2 + 225)^2(P_{71}(b^2))^2(P_{386}(b^2))^4(P_{587}(b^2))^6(P_{965}(b^2))^2, \tag{4.35}$$

where P_i are polynomials of degree i with rational coefficients. By using the Sturm method it is easy to see that its first 4 factors do not have real roots in $(0, 0.8171)$. We replace $b^2 = t$ in the next three polynomials to reduce their degrees and we obtain $\mathcal{P}_1(t) := P_{386}(t)$, $\mathcal{P}_2(t) := P_{587}(t)$, and $\mathcal{P}_3(t) := P_{965}(t)$. It suffices to study their number of real roots in $(0, 0.6678]$, because $0.6678 > (0.8171)^2$. Our computers have no enough capacity to get their Sturm sequences. Therefore we will use the Descartes approach as it is explained in Appendix I.

We consider first the polynomial $\mathcal{P}_1(t)$. Its normalized version $N_0^{0.68}(\mathcal{P}_1)$ has all their coefficients positive. Therefore $\mathcal{P}_1(t)$ has no real roots in $(0, 0.68)$ as we wanted to see.

Applying the Descartes rule to the normalized versions of $\mathcal{P}_2(t)$, $N_0^{0.561}(\mathcal{P}_2)$, $N_{0.561}^{0.811}(\mathcal{P}_2)$ and $N_{0.562}^{0.812}(\mathcal{P}_2)$, we obtain that the number of zeros in the intervals $(0, 0.561)$, $(0.561, 0.811)$ and $(0.562, 0.812)$ is 0, 1 and 0 respectively. That is, there is only one root of $\mathcal{P}_2(t)$ in $(0, 0.812)$, it is simple and it belongs to $(0.561, 0.562)$. Refining this interval with Bolzano Theorem we prove that the root is in the interval $(0.5617, 0.5618)$.

Finally, to study $\mathcal{P}_3(t)$ we consider $N_0^{11/20}(\mathcal{P}_3)$, $N_{11/20}^{7/12}(\mathcal{P}_3)$ and $N_{7/12}^{52/75}(\mathcal{P}_3)$. By Descartes rule we obtain that the number of zeros of $\mathcal{P}_3$ in the corresponding intervals is 0, 1 and 1 or 3, respectively. By Bolzano Theorem we can localize more precisely these zeros and prove that in the last interval there are exactly 3 zeros. So we have proved that the polynomial $\mathcal{P}_3$ has exactly 4 zeros in the interval $(0, 52/75) \approx (0, 0.693)$, and each one of them is contained in one of the following intervals

$$(0.5614, 0.5615)\,,\ (0.6678, 0.6679)\,,\ (0.6690, 0.6700)\,,\ (0.6870, 0.6880)\,.$$

In brief, for $t \in (0, 0.6678]$ the double discriminant $\triangle_{y,x}(N_b)$ only vanishes at two points $t = t_1$ and $t = t_2$ with $t_1 \in (0.5614, 0.5615)$ and $t_2 \in (0.5617, 0.5618)$. Therefore we are under the hypothesis (ii) of Proposition 4.25 for b belonging to each of the intervals $(0, b_1)$, (b_1, b_2) and $(b_2, 0.8171)$, where

$$b_1 := \sqrt{t_1} \approx 0.749301, \quad b_2 := \sqrt{t_2} \approx 0.749478.$$

To ensure that on each interval we are under the hypotheses (i) of the proposition we prove that N_b does not vanish on Ω for one value of b in each of the above three intervals. We take

$$\frac{1}{2} \in (0, b_1), \quad \frac{7494}{10000} \in (b_1, b_2), \quad \text{and} \quad \frac{3}{4} \in (b_2, 0.8171).$$

We study with detail the case $b = 1/2$. The other two cases can be treated similarly and we skip the details. So we have to study on Ω the sign of the function

$$\begin{aligned} N_{1/2} \;=\; & \tfrac{135}{34359738368}x^{10}y^2 + \tfrac{45}{34359738368}x^8y^4 + \tfrac{45}{34359738368}x^{10} + \tfrac{117964485}{137438953472}x^8y^2 \\ & + \tfrac{138195}{268435456}x^7y^3 + \tfrac{39253779}{137438953472}x^6y^4 + \tfrac{39321555}{137438953472}x^8 + \tfrac{45}{17179869184}x^7y \\ & + \tfrac{320504301}{137438953472}x^6y^2 + \tfrac{1932072223485}{17179869184}x^5y^3 - \tfrac{906074381}{8589934592}x^4y^4 + \tfrac{645}{1048576}x^6 \\ & + \tfrac{1229859}{1073741824}x^5y + \tfrac{5315442024413}{8589934592}x^4y^2 - \tfrac{1808748465}{4194304}x^3y^3 + \tfrac{6763995071}{8388608}x^2y^4 \\ & + \tfrac{1258289751}{8388608}x^4 + \tfrac{55625}{262144}x^3y - \tfrac{1910154937}{4194304}x^2y^2 + \tfrac{26361865}{262144}xy^3 + \tfrac{15}{64}y^4 \\ & - \tfrac{316538295}{8388608}x^2 + \tfrac{585}{1048576}xy + \tfrac{45}{2}y^2 + \tfrac{975}{64}. \end{aligned}$$

We consider $N_{1/2}$ as a polynomial in x with coefficients in $\mathbb{R}[y]$ and we apply Lemma 4.19 with $\Omega_y = (-1/y, \infty)$ when $y > 0$ and $\Omega_0 = (-\infty, \infty)$. Notice that for the symmetry of the function there is no need to study the zone $y < 0$ because $N_{1/2}(-x,-y) = N_{1/2}(x,y)$. We introduce the following notation $S_y(x) := N_{1/2}(x,y)$. We prove the following facts:

(i) If we write $S_y(x) = \sum_{i=1}^{10} s_i(y)x^i$, then $s_{10}(y) = k(1+3y^2)$ for some $k \in \mathbb{Q}^+$. Therefore $s_{10}(y) > 0$ for all $y \in \mathbb{R}$.

(ii) If $y = 0$ then $S_0(x)$ is an even polynomial of degree 10 and it is easy to see that $S_0(x) > 0$ over $\mathbb{R}$.

(iii) We already know that $\{S_y(x) = 0\} \cap \partial\Omega = \emptyset$.

(iv) Some computations give that

$$\triangle_x(S_y) = P_{35}(y^2),$$

where P_{35} is a polynomial of degree 35. Moreover, using once more the Sturm method, we get that $P_{35}(y^2)$ has only two positive roots $0 < y_1 < y_2$, with $y_1 \approx 0.588423$ and $y_2 \approx 6065.2946$. From this result it is easy to prove that:

(a) If $y \in [0, y_1) \cup (y_2, \infty)$, then $S_y(x) > 0$.

(b) If $y \in (y_1, y_2)$, then $S_y(x)$ has only two real roots, say $x_1(y) < x_2(y)$, and none of them belongs to the interval $(-1/y, \infty)$. So $S_y(x) > 0$ on $(-1/y, \infty)$.

(c) If $y \in \{y_1, y_2\}$, then $S_y(x)$ has only a real root, $x_1(y)$, which is a double root and $x_1(y) \notin (-1/y, \infty)$. So, again $S_y(x) > 0$ on $(-1/y, \infty)$.

Thus, by Lemma 4.19, the function $N_{1/2}$ is positive on $(x,y) \in \Omega$, as we wanted to see. In fact, its level curves are like the ones showed in Figure 4.14. The straight lines $y = y_1$ and $y = y_2$ correspond to the lower and upper tangents to the oval contained in the second quadrant.

To be under all the hypotheses of Proposition 4.25 it only remains to study the function $\widetilde{N}_b$ at infinity. We denote by $\widetilde{N}_b(x,y,z)$ its homogenization in $\mathbb{RP}^2$ and by $\widetilde{N}_b^1(x,z)$ and $\widetilde{N}_b^2(y,z)$ the expressions of the function $\widetilde{N}_b$ in the planes $\{(x,z)\}$ and $\{(y,z)\}$, respectively. Since $\mathcal{H}_{12}(N_b) = 90b^{36}x^8y^2(3x^2+y^2)$, the only non-trivial solutions of $\mathcal{H}_{12}(N_b) = 0$ are $x = 0$ and $y = 0$. Hence these directions give rise to two points of N_b at infinity which are also on the region Ω. They correspond to the points $(0,0)$ of the algebraic curves $\widetilde{N}_b^1(x,z) = 0$ and $\widetilde{N}_b^2(y,z) = 0$. We have to prove that both points are uniformly isolated.

Similarly that in the previous subsection, we write

$$\begin{aligned}\widetilde{N}_b^1(\varepsilon X,\varepsilon Z) = \Big(&90b^{36}X^8 - 3b^{18}(30b^{20}+120b^{16}+100b^{14}+36b^{10}-25)X^6Z^2\\ &+ b^{10}(360b^{26}+300b^{24}+198b^{20}+360b^{12}-615b^{10}+720b^8-300b^6-250b^4-90)X^4Z^4\\ &+(-738b^{24}+1080b^{22}-1080b^{20}+300b^{18}+250b^{16}+315b^{12}+300b^4-450b^2+900)X^2Z^6\\ &+15b^6Z^8\Big)\varepsilon^8+O(\varepsilon^9)\end{aligned}$$

and

$$\widetilde{N}_b^2(\varepsilon Y,\varepsilon Z) = 90b^{36}(3Y^2+Z^2)\varepsilon^2+O(\varepsilon^3).$$

By Proposition 4.24, for the second algebraic curve it is clear that $(0,0)$ is an isolated point for all $b>0$.

For studying the first one we denote by $R(X,Z)$ the homogenous polynomial accompanying ε^8 and we obtain that

$$\triangle_X(R(X,Z)) = Z^{56}b^{150}(P_{71}(b^2))^2,$$

for some polynomial P_{71} of degree 71 and integer coefficients. Since the smallest positive root of this polynomial is greater that 0.92 we can easily prove that for $b<0.92$, $R(X,Z)=0$ if and only if $X=Z=0$. Therefore we can use again Proposition 4.24 and prove that $(0,0)$ is an uniformly isolated point of the curve for these values of b.

So, if we write

$$(0,0.8171) = (0,b_1)\cup\{b_1\}\cup(b_1,b_2)\cup\{b_2\}\cup(b_2,0.8171),$$

we can apply Proposition 4.25 to each one of the open intervals to prove that for $b\in(0,0.817]\setminus\{b_1,b_2\}$ it holds that $N_b(x,y)>0$ for all (x,y) in Ω. By continuity, for the two values $b\in\{b_1,b_2\}$, we obtain that $N_b(x,y)\geq 0$. Since $\triangle_y(N_b)\not\equiv 0$ either it is always positive or it vanishes only at some isolated points, as we wanted to prove.

It can be seen that for $b \gtrsim \hat{b} \approx 0.81722$, $N_b(x,y)$ changes sign on Ω because there appears one oval in the set $\{N_b(x,y)=0\}$. The value $\hat{b}^2\approx 0.6678492$ corresponds to the root of $\mathcal{P}_3$ in the interval $(0.6678, 0.6679)$ that has appeared in the proof as a root of the double discriminant.

Chapter 5

A family of non rotated vector fields

5.1 Introduction and main results

A. Bacciotti, during a conference about the stability of analytic dynamical systems held in Florence in 1985, proposed to study the stability of the origin of the following quintic system

$$\begin{cases} \dot{x} = y^3 - x^3, \\ \dot{y} = -x + my^5, \qquad m \in \mathbb{R}. \end{cases} \tag{5.1}$$

Two years later, Galeotti and Gori in [40] published an extensive study of (5.1). They proved that system (5.1) has no limit cycles when $m \in (-\infty, 0.36] \cup [0.6, \infty)$, otherwise, it has at most one. Their proofs are mainly based on the study of the stability of the limit cycles which is controlled by the sign of its characteristic exponent, together with a transformation of the system using a special type of adapted polar coordinates. Their proof of the uniqueness of the limit cycle does not cover its hyperbolicity.

In this chapter we refine the above results. To guess which is the actual bifurcation diagram we first did a numerical study, obtaining the following results. It seems that there exists a value $m^* > 0$ such that:

(i) System (5.1) has no limit cycles if $m \in (-\infty, m^*] \cup [0.6, \infty)$. Moreover, for $m = m^*$ it has a heteroclinic polycycle formed by the separatrices of the two saddle points located at $(\pm m^{-1/4}, \pm m^{-1/4})$.

(ii) For $m \in (m^*, 0.6)$ the system has exactly one unstable limit cycle.

(iii) The value m^* is approximately 0.560115.

Recall that a polycycle is a simple, closed curve, formed by several solutions of the system, which admits a Poincaré return map. The claims (i) and (ii) above coincide with the results described in [40]. Concerning the location of the value m^* however, our computations differ from the results proposed in [40] where it is claimed that m^* is between 0.58 and 0.59.

The first aim of this work is to obtain analytic results that confirm, as accurate as possible, the above claims. To clarify the phase portraits of the system, we will study them on the Poincaré disc, see [7, 95].

For $m \leq 0$, system (5.1) has no periodic orbits because $x^2/2 + y^4/4$ is a global Lyapunov function. Therefore, the origin is a global attractor. In particular, its phase portrait is trivial. Therefore, we will concentrate on the case $m > 0$. In this case, the system has three critical points, $(\pm m^{-1/4}, \pm m^{-1/4})$ and $(0, 0)$. The first two points are saddles and the third one is a monodromic nilpotent singularity. Its stability can be determined using the tools introduced in [5, 81], see Subsection 5.2 and Theorem 5.3 below. We prove:

Theorem 5.1. *Consider system* (5.1).

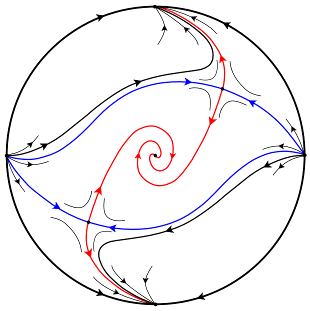
(i) It has neither periodic orbits, nor polycycles, when $m \in (-\infty, 0.547] \cup [0.6, \infty)$. *Otherwise, it has at most one periodic orbit or one polycycle, but can not coexist. Moreover, when the limit cycle exists, it is hyperbolic and unstable.*

(ii) For $m > 0$, *their phase portraits on the Poincaré disc, are given in Figure 5.1.*

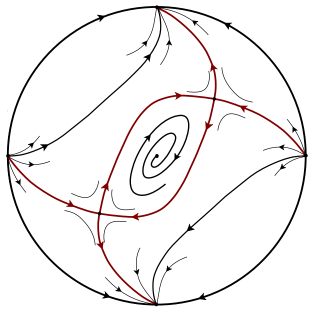
(iii) Let $\mathcal{M}$ *be the set of values of* m *for which it has a heteroclinic polycycle. Then* $\mathcal{M}$ *is finite, non-empty and it is contained in* $(0.547, 0.6)$. *Moreover, the system corresponding to* $m \in \mathcal{M}$ *has no limit cycles and its phase portrait is given by Figure 5.1* (b).

Our simulations show that (a), (b) and (c) of Figure 5.1 occur when $m \in (0, m^*)$, $m = m^*$ and $m > m^*$, respectively, for some $m^* \in (0.547, 0.6)$, that numerically we have found to be $m^* \approx 0.560115$. We have not been able to prove the existence of this special value m^* rigorously, because our system is not a semi-complete family of rotated vector fields (SCFRVF) and this fact hinders the obtention of the full bifurcation diagram; see the discussion in Subsection 5.3.1 and Example 5.20. This is precisely the reason why we have decided to push forward the study of system (5.1). Our approach can be useful to understand other interesting polynomial systems of differential equations that have been considered previously; see for instance [13, 32].

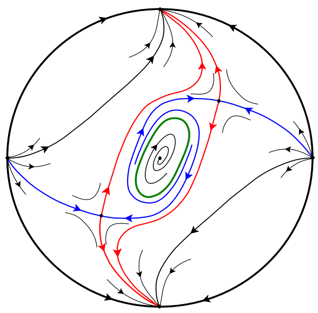
From our analysis, we know the existence of finitely many values m_j^*, $j = 1, \ldots, k$, where $k \geq 1$, satisfying $0.547 < m_1^* < m_2^* < \cdots m_k^* < 0.6$, such that phase portrait (b) only occurs for these values. Moreover, for $m \in (0.547, m_1^*)$, phase portrait (a) holds, for $m \in (m_k^*, 0.6)$ phase portrait (c) holds, and for each one of the remaining $k - 1$ intervals, the phase portrait does not vary on each interval and is either (a) or (c).



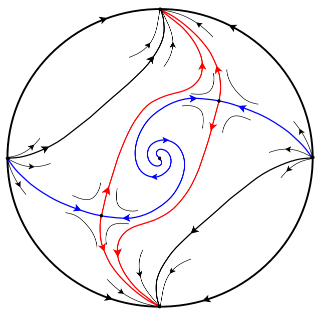
(a) When $m \in (0, 0.547]$, or when $m \in (0.547, 0.6)$ and neither the polycycle nor the limit cycle exist.



(b) When $m \in (0.547, 0.6)$ and the polycycle exists.



(c) When $m \in (0.547, 0.6)$ and the limit cycle exists.



(d) For $m \in [0.6, \infty)$

Figure 5.1: Phase portraits of system (5.1).

As a byproduct of our approach we can also give explicit algebraic restrictions on the initial conditions which ensure that the corresponding solutions tend to the origin.

Recall that when a critical point, $\mathbf{p} \in \mathbb{R}^n$, of a differential system is an attractor we can define its basin of attraction as

$$\mathcal{W}^s_{\mathbf{p}} = \{\mathbf{x} \in \mathbb{R}^n \, : \, \lim_{t \to +\infty} \varphi(t, \mathbf{x}) = \mathbf{p}\},$$

where φ denotes the solution of the differential system such that $\varphi(0, \mathbf{x}) = \mathbf{x}$. A very interesting question, mainly motivated by Control Theory problems, consists in obtaining testable conditions for ensuring that some initial condition is in $\mathcal{W}^s_{\mathbf{p}}$. Usually these conditions are obtained using suitable Lyapunov functions. In the proof of the following result however, we use a different approach based on the construction of Dulac functions.

Proposition 5.2. *Let* $\mathcal{W}_{\mathbf{0}}^s$ *be the basin of attraction of the origin of system* (5.1)*. Consider* $V_m(x,y) = g_{0,m}(y) + g_{1,m}(y)x + g_{2,m}(y)x^2$*, with*

$$\begin{aligned} g_{2,m}(y) =& \frac{1}{89100}(3-10m)(3+35m)y^{12} - \frac{1}{6300}(75-125m)^{2/3}(3-13m)y^8 \\ &+ \frac{1}{90}(3-10m)y^6 - \frac{1}{25}(75-125m)^{2/3}y^2 + 1, \end{aligned}$$

$g_{1,m}(y) = g'_{2,m}(y)$ *and* $g_{0,m}(y) = g''_{2,m}(y)/2 - my^5 g'_{2,m}(y)/2 + 5my^4 g_{2,m}(y)/3$*. Then, for* $m \in (0.5, 0.6)$*,* $\mathcal{U}_m \subset \mathcal{W}_{\mathbf{0}}^s$*, where* $\mathcal{U}_m$ *is the bounded connected component of* $\{(x,y) \in \mathbb{R}^2 : V_m(x,y) \leq 0\}$ *that contains the origin and whose boundary is the oval of* $V_m(x,y) = 0$*, see Figure 5.2.*

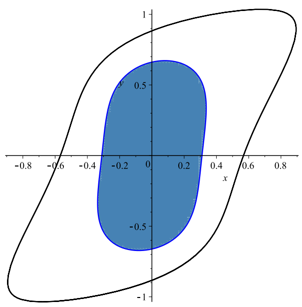


Figure 5.2: The limit cycle of system (5.1) and the set $\mathcal{U}_m$, introduced in Proposition 5.2, when $m = 0.57$.

As we will see, the proof of the above proposition is a straightforward consequence of Proposition 5.17. Using the same tools, it can be shown that the same result also holds for smaller values of m. In any case, notice that this proposition covers all the values of m for which the system has limit cycles.

While studying the stability of the origin of system (5.1) we realized that, using the same tools, we could solve an open question left in [40]. Our third result studies the stability of the origin of the following generalization of system (5.1):

$$\begin{cases} \dot{x} = y^3 - x^{2k+1}, \\ \dot{y} = -x + my^{2s+1}, \qquad m \in \mathbb{R} \quad \text{and} \quad k, s \in \mathbb{N}^+. \end{cases} \tag{5.2}$$

In [40], the authors gave the stability of the origin when $s \neq 2k$ and ask whether it is true or not that the change of stability of the origin when $s = 2k$ is at the value $m = (2k+1)/(4k+1)$. We will prove that their guess was not correct for $k > 1$. The new result shows that when $s = 2k$, the stability changes at

$$m = \frac{(2k+1)!!}{(4k+1)!!!!}, \tag{5.3}$$

where, given $n \in \mathbb{N}^+$, $n!!$ and $n!!!!$ are defined recurrently, as follows,

$$n!! = n \times (n-2)!!, \quad n!!!! = n \times (n-4)!!!!,$$

with $1!! = 1, 2!! = 2$ and $j!!!! = j$ for $1 \leq j \leq 4$. Notice that when $k = 1$, the right-hand side of (5.3) and $(2k+1)/(4k+1)$ coincide and give $m = 3/5$, which is one of the values appearing in Theorem 5.1.

Theorem 5.3. *Consider system* (5.2)*.*

(i) When $s < 2k$, the origin is an attractor for $m \leq 0$ and a repeller for $m > 0$.

(ii) When $s > 2k$, the origin is always an attractor.

(iii) When $s = 2k$, the origin is an attractor for $m < (2k+1)!!/(4k+1)!!!!$ and a repeller when the reverse inequality holds. Moreover, when $k = 1$ and $m = 3/5$ the origin is a repeller and for $m \lesssim 3/5$ system (5.1) *has at least one limit cycle near the origin.*

The method used to study the stability of the origin of (5.2), when $s = 2k$ and $k = 1$, also works for deducing its stability in the case not covered by the above theorem: $s = 2k$, $k > 1$ and m as in (5.3). Nevertheless, the computations are tedious and we have decided not to perform them.

The chapter is structured as follows. In Section 5.2 we prove Theorem 5.3. In Section 5.3 we recall some preliminary results. We start with a discussion on the differences between being or not, a SCFRVF. Then, Subsection 5.3.2 is devoted to studying the singularities of system (5.1) at infinity and their phase portraits on the Poincaré disc. Afterwards, we present some Bendixson–Dulac type results that we will use to prove non-existence or uniqueness of periodic orbits or polycycles. Finally, we introduce a result for controlling the number of roots of 1-parameter families of polynomials and we show that our system can be reduced to an Abel differential equation.

In Section 5.4 we prove the non-existence results for $m \in (-\infty, 0.36] \cup [0.6, \infty)$. Our proof is different from that of [40] and it is mainly based on the use of Dulac functions.

In Section 5.5 we prove that there exists at most one periodic orbit when $m \in (1/2, 0.6)$. Our approach also shows the hyperbolicity of the orbit and again uses a Bendixson–Dulac type results. This section also includes the proof of Proposition 5.2.

Section 5.6 is devoted to enlarging the region where we can assure the non-existence of periodic orbits and polycycles, proving this for $m \in (0.36, 0.547]$. The proof uses once more a suitable Dulac function in a part of the interval and the

Poincaré–Bendixon theorem, together with the hyperbolicity of the limit cycle, whenever it exists, for the remaining values of m.

Section 5.7 deals with the existence of polycycles for the system. Finally, in Section 5.7, we combine all of the above results to prove Theorem 5.1.

5.2 Stability of the origin

Notice that the origin of (5.1) and (5.2) are nilpotent critical points and there are several tools for studying its local stability, see for instance [5, 57, 81]. We will follow the approach of [5, 57], based on the polar coordinates introduced by Lyapunov in [69], to study the stability of degenerate critical points.

Let $u(\theta) = \mathrm{Cs}(\theta)$ and $v(\theta) = \mathrm{Sn}(\theta)$ be the solutions of the Cauchy problem:

$$\dot{u} = -v^{2p-1},\ \dot{v} = u^{2q-1}, \quad u(0) = \sqrt[2q]{1/p} \quad \text{and} \quad v(0) = 0,$$

where the prime denotes the derivative with respect to θ.

The Lyapunov *generalized polar coordinates* are $x = r^p\,\mathrm{Cs}(\theta)$ and $y = r^q\,\mathrm{Sn}(\theta)$. They parameterize the algebraic curves $px^{2q} + qy^{2p} = r^{2pq}$, that correspond to the level sets of the above (p,q)-quasi-homogeneous Hamiltonian system. In particular, $p\,\mathrm{Cs}^{2q}(\theta) + q\,\mathrm{Sn}^{2p}(\theta) = 1$, and both functions are smooth $T_{p,q}$-periodic functions, where

$$T = T_{p,q} = 2p^{-1/2q}q^{-1/2p}\frac{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{1}{2q}\right)}{\Gamma\left(\frac{1}{2p}+\frac{1}{2q}\right)},$$

and Γ denotes the Gamma function. The general expression of a differential system in these coordinates is:

$$\dot{r} = \frac{x^{2q-1}\dot{x} + y^{2p-1}\dot{y}}{r^{2pq-1}}, \quad \dot{\theta} = \frac{px\dot{y} - qy\dot{x}}{r^{p+q}}. \tag{5.4}$$

In the nilpotent monodromic case, the component $\dot{\theta}$ does not vanish in a punctured neighborhood of the critical point. Hence, system (5.4) can be written in a neighborhood of $r = 0$ as

$$\frac{dr}{d\theta} = \sum_{i=1}^{\infty} R_i(\theta) r^i, \tag{5.5}$$

where $R_i(\theta)$, $i \geq 1$ are T-periodic functions. The solution of (5.5) that passes through $r = \rho$ when $\theta = 0$ can be written as the power series

$$r(\theta, \rho) = \rho + \sum_{i=2}^{\infty} u_i(\theta)\rho^i, \quad \text{with} \quad u_i(0) = 0, \tag{5.6}$$

and the functions u_i can be computed solving recursive linear differential equations obtained by plugging (5.6) into (5.5). It is well known that the stability of the origin is given by the first nonvanishing generalized Lyapunov constant $V_k := u_k(T)$.

To effectively compute some integrals of the above generalized trigonometric functions we will use the following result, see [57].

Lemma 5.4. *Let* Sn *and* Cs *be the (1,q)-trigonometrical functions and let T be their period. Then, for $i, j \in \mathbb{N}$,*

(i) $\int_0^T \operatorname{Sn}^i(\theta)\operatorname{Cs}^j(\theta)\,d\theta = 0$ *when either i or j are odd.*

(ii) $\int_0^T \operatorname{Sn}^i(\theta)\operatorname{Cs}^j(\theta)\,d\theta = \dfrac{2\Gamma\left(\frac{i+1}{2}\right)\Gamma\left(\frac{j+1}{2q}\right)}{q^{\frac{i+1}{2}}\Gamma\left(\frac{i+1}{2}+\frac{j+1}{2q}\right)}$ *when i and j are both even.*

(iii) For $q=2$, $\int_0^\theta \operatorname{Cs}^8(\psi)\,d\psi = \dfrac{6\operatorname{Sn}(\theta)\operatorname{Cs}^5(\theta)+10\operatorname{Sn}(\theta)\operatorname{Cs}(\theta)+5\theta}{21}.$

(iv) For $q=2$, $\int_0^\theta \operatorname{Sn}^4(\psi)\,d\psi = \dfrac{-\operatorname{Sn}^3(\theta)\operatorname{Cs}(\theta)-\operatorname{Sn}(\theta)\operatorname{Cs}(\theta)+\theta}{7}.$

Proof of Theorem 5.3. By using the transformation $(x, y) \to (y, x)$, system (5.2) becomes

$$\begin{cases} \dot{x} = -y + mx^{2s+1}, \\ \dot{y} = x^3 - y^{2k+1}. \end{cases} \tag{5.7}$$

We use (5.4), with $p = 1$ and $q = 2$, to transform it into

$$\begin{cases} \dot{r} = m\operatorname{Cs}^{2s+4}(\theta)r^{2s+1} - \operatorname{Sn}^{2k+2}(\theta)r^{4k+1}, \\ \dot{\theta} = r - \operatorname{Cs}(\theta)\operatorname{Sn}^{2k+1}(\theta)r^{4k} - 2m\operatorname{Cs}^{2s+1}(\theta)\operatorname{Sn}(\theta)r^{2s}, \end{cases}$$

or equivalently,

$$\frac{dr}{d\theta} = \frac{m\operatorname{Cs}^{2s+4}(\theta)r^{2s} - \operatorname{Sn}^{2k+2}(\theta)r^{4k}}{1 - \operatorname{Cs}(\theta)\operatorname{Sn}^{2k+1}(\theta)r^{4k-1} - 2m\operatorname{Cs}^{2s+1}(\theta)\operatorname{Sn}(\theta)r^{2s-1}}. \tag{5.8}$$

Depending on the parameters s and k, the Taylor series of the right-hand side of the above equation gives rise to three different situations at the origin.

(i) When $s < 2k$, then (5.8) becomes

$$\frac{dr}{d\theta} = m\operatorname{Cs}^{2s+4}(\theta)r^{2s} + O(r^{4k}).$$

Therefore, using the method explained above and Lemma 5.4, we get that its first Lyapunov constant is

$$V_{2s} = m\int_0^T \operatorname{Cs}^{2s+4}(\theta)\,d\theta = \frac{m\sqrt{2\pi}\,\Gamma\left(\frac{2s+5}{4}\right)}{\Gamma\left(\frac{2s+7}{4}\right)}. \tag{5.9}$$

Then $m = 0$ is the bifurcation value, and the origin of (5.2) changes its stability from attractor to repeller as m goes from negative values to positive values. The case $m = 0$ follows using the Lyapunov function $x^4/4 + y^2/2$.

(*ii*) Suppose $s > 2k$, then the Taylor expansion of (5.8) at $r = 0$ is

$$\frac{dr}{d\theta} = -\operatorname{Sn}^{2k+2}(\theta) r^{4k} + O(r^{2s}).$$

By using the same method, we obtain that the first Lyapunov constant is

$$V_{4k} = \int_0^T -\operatorname{Sn}^{2k+2}(\theta)\, d\theta = -\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{2k+3}{2}\right)}{2^{\frac{2k+1}{2}}\Gamma\left(\frac{4k+7}{4}\right)} < 0, \tag{5.10}$$

and the stability of the origin of (5.2) is independent of m and it is an attractor for all m.

(*iii*) Finally, when $s = 2k$ we have

$$\frac{dr}{d\theta} = \left(m \operatorname{Cs}^{4k+4}(\theta) - \operatorname{Sn}^{2k+2}(\theta)\right) r^{4k} + O(r^{8k-1}). \tag{5.11}$$

Hence the first non-vanishing generalized Lyapunov constant is given by

$$V_{4k} = \int_0^T \left(m \operatorname{Cs}^{4k+4}(\theta) - \operatorname{Sn}^{2k+2}(\theta)\right) d\theta.$$

Using (5.9) with $s = 2k$ and (5.10), after some simplifying calculations, we obtain that

$$V_{4k} = \frac{2\pi^{3/2}\left(m(4k+1)!!!! - (2k+1)!!\right)}{\left(\Gamma\left(\frac{3}{4}\right)\right)^2 (4k+3)!!!!}.$$

Therefore the origin of (5.2) is an attractor for $m < (2k+1)!!/(4k+1)!!!!$ and a repeller for $m > (2k+1)!!/(4k+1)!!!!$, as we wanted to prove.

In the particular case $s = 2k$ and $k = 1$, which corresponds to system (5.1), and when $m = 3/5$ we have that $V_4 = 0$. To continue the proof we compute the next non-zero Lyapunov constant. For $s = 2$ and $k = 1$, equation (5.8) writes as

$$\frac{dr}{d\theta} = R_4(\theta) r^4 + R_7(\theta) r^7 + R_{10}(\theta) r^{10} + O(r^{13}),$$

with $R_4(\theta) = m \operatorname{Cs}^8(\theta) - \operatorname{Sn}^4(\theta)$,

$$R_7(\theta) = 2m^2 \operatorname{Cs}^{13}(\theta)\operatorname{Sn}(\theta) + m \operatorname{Cs}^9(\theta)\operatorname{Sn}^3(\theta) - 2m \operatorname{Cs}^5(\theta)\operatorname{Sn}^5(\theta) - \operatorname{Cs}(\theta)\operatorname{Sn}^7(\theta)$$

and

$$\begin{aligned} R_{10}(\theta) =& 4m^3 \operatorname{Cs}^{18}(\theta)\operatorname{Sn}^2(\theta) + 4m^2 \operatorname{Cs}^{14}(\theta)\operatorname{Sn}^4(\theta) + m(1-4m)\operatorname{Cs}^{10}(\theta)\operatorname{Sn}^6(\theta) \\ & - 4m \operatorname{Cs}^6(\theta)\operatorname{Sn}^8(\theta) - \operatorname{Cs}^2(\theta)\operatorname{Sn}^{10}(\theta), \end{aligned}$$

with $m = 3/5$. Following the procedure explained at the beginning of this section we obtain that $u_2 = u_3 = 0$,

$$\begin{aligned} u_4(\theta) &= \int_0^\theta R_4(\psi)\, d\psi, \quad u_5 = u_6 = 0, \\ u_7(\theta) &= \int_0^\theta \big(R_7(\psi) + 4R_4(\psi)u_4(\psi)\big)\, d\psi, \quad u_8 = u_9 = 0, \\ u_{10}(\theta) &= \int_0^\theta \Big(R_{10}(\psi) + 7R_7(\psi)u_4(\psi) + 4R_4(\psi)u_7(\psi) + 6R_4(\psi)u_4^2(\psi)\Big)\, d\psi. \end{aligned}$$

Using Lemma 5.4 and some straightforward computations we get that $V_1 = \cdots = V_9 = 0$. Finally, it suffices to compute

$$V_{10} = \int_0^T \Big(R_{10}(\theta) + 7R_7(\theta)u_4(\theta) + 4R_4(\theta)u_7(\theta)\Big)\, d\theta,$$

because $\dfrac{du_4^3(\theta)}{d\theta} = 3R_4(\theta)u_4^2(\theta)$. Performing integration by parts and using the expression of u_7' we arrive at

$$V_{10} = \int_0^T \Big(R_{10}(\theta) + 3u_4(\theta)u_7'(\theta)\Big)\, d\theta = \int_0^T \Big(R_{10}(\theta) + 3u_4(\theta)R_7(\theta)\Big)\, d\theta. \quad (5.12)$$

Notice that, applying (iii) and (iv) of Lemma 5.4, we find that

$$u_4(\theta) = \int_0^\theta \Big(\frac{3}{5}\operatorname{Cs}^8(\psi) - \operatorname{Sn}^4(\psi)\Big)\, d\psi = \frac{6\operatorname{Sn}(\theta)\operatorname{Cs}^5(\theta) + 15\operatorname{Sn}(\theta)\operatorname{Cs}(\theta) + 5\operatorname{Sn}^3(\theta)\operatorname{Cs}(\theta)}{35}.$$

Plugging this expression into (5.12), using several times (i) and (ii) of Lemma 5.4 and the properties of the Γ function, we arrive at

$$V_{10} = \frac{128}{1625}\frac{\left(\Gamma\left(\frac{3}{4}\right)\right)^2}{\sqrt{\pi}} > 0.$$

Hence the origin is unstable for $m = 3/5$. As a consequence, we obtain that at $m = 3/5$ the system has a Hopf-like bifurcation. Therefore the system has at least one limit cycle near the origin for $m \lesssim 3/5$. □

5.3 Preliminary results

This section is a miscellaneous one and it is divided into several short subsections containing either some tools that we will use to prove Theorems 5.1 and 5.2 or some preliminary results.

5.3.1 Rotated vector fields *vs* non rotated vector fields

As is widely known, if we have a 1-parameter family of differential systems is a SCFRVF, then there are many results that allow to control the possible bifurcations; see [33, 87, 83]. One of the most useful ones is the so-called *non-intersection property.* It asserts that if γ_1 and γ_2 are limit cycles corresponding to systems with different values of m, then $\gamma_1 \cap \gamma_2 = \emptyset$. As a consequence the study of 1-parameter bifurcation diagrams is much more simple in this case.

For instance, consider a 1-parameter SCFRVF satisfying the following property: **(P)** *For each* $m \in (m_0, m_1)$*, the system has at most one limit cycle, which we denote by* γ_m*. Here, if for some* m *the corresponding system has no limit cycles then* $\gamma_m = \emptyset$*. Moreover, assume that* $\cup_{m\in(m_0,m_1)}\gamma_m$ *covers a region of the plane that all the periodic orbits of the system have to pass.*

Under this assumption, *for* $m \in \mathbb{R} \setminus (m_0, m_1)$ *the system has no periodic orbits.*

The above property has very important practical consequences if we want to determine the values m_0 and m_1, that constitute, in many cases, the most difficult ones to be obtained to complete the bifurcation diagram. Usually, one of the values, say m_0 corresponds to a Hopf-like bifurcation, and it is obtained by some local analysis. Then, for instance, if for some value, say $\widetilde{m} > m_0$, the system has no limit cycles then $m_1 < \widetilde{m}$. The same idea can also be applied to obtain lower bounds of m_1. These facts simplify a lot the obtention of analytic bounds for the value m_1, because it suffices to deal with concrete systems, with fixed values of m. This approach has been successfully applied in many works; see for instance Chapter 4 and [51, 86, 83, 102].

On the other hand, if for a general family of vector fields we have that the same property **(P)** given above holds, we can say nothing of what happens for $m \in \mathbb{R} \setminus (m_0, m_1)$. For this reason, when we study system (5.1), we can not ensure the existence of a unique value of m for which the phase portrait looks like in Figure 5.1 (b); see also Example 5.20. We remark that system (5.1) is not a SCFRVF with respect to m, and moreover we have not been able to transform it into one.

From our point of view, to introduce tools for studying 1-parameter families that are not SCFRVF is a challenge for the differential equations community.

5.3.2 Global phase portraits

We will draw the phase portraits of system (5.1) on the Poincaré disc, [7, 95]. Recall that, from the works of Markus [73] and Newmann [82], to characterize a phase portrait it suffices to determine the type of critical points (finite and at infinity), the configuration of their separatrices, and the number and character of their periodic orbits.

We start by studying the critical points at infinity of the Poincaré compactification of the system. That is, we will use the transformations $(x, y) = (1/z, u/z)$ and $(x, y) = (v/z, 1/z)$, with a suitable change of time to transform system (5.1) into two new polynomial systems, one in the (u, z)-plane and another one in the (v, z)-plane; see [7] for the details. Then, to understand the behavior of the solutions of (5.1) near infinity it suffices to study the type of critical points of the transformed systems which are localized on the line $z = 0$. These points are precisely the so-called critical points at infinity of system (5.1).

Lemma 5.5. *By using the transformation* $(x, y) = (v/z, 1/z)$ *and the change of time* $dt/d\tau = 1/z^4$ *system* (5.1) *is transformed into the system*

$$\begin{cases} v' = -mv + (1 - v^3)z^2 + v^2 z^4, \\ z' = -mz + vz^5, \end{cases} \tag{5.13}$$

where the prime denotes the derivative with respect to τ*. The origin is the unique critical point of* (5.13) *on* $z = 0$ *and it is an attracting node.*

The proof of the above result is straightforward.

Lemma 5.6. *By using the transformation* $(x, y) = (1/z, u/z)$ *and the change of time* $dt/d\tau = 1/z^4$ *system* (5.1) *is transformed into the system*

$$\begin{cases} u' = (u - z^2)z^2 + u^4(mu - z^2), \\ z' = (1 - u^3)z^3, \end{cases} \tag{5.14}$$

where the prime denotes the derivative with respect to τ*. The origin is the unique critical point of* (5.14) *on* $z = 0$ *and it is a repeller.*

Proof. It is clear that the origin of system (5.14) is its unique critical point on $z = 0$. To determine its nature we will use the directional blow-up, since the linear part of the system at this point vanishes identically; see again [7].

We apply the z-directional blow-up given by the transformation $r = u/z$, $z = z$. Together with the change of time $dt/d\tau = z^3$, system (5.14) is transformed into

$$\begin{cases} \dot{r} = -1 + mzr^5, \\ \dot{z} = 1 - z^3 r^3. \end{cases} \tag{5.15}$$

System (5.15) has no critical points on $z = 0$. Then by using the transformation $(u, z) = (rz, z)$ we can obtain the phase portrait of system (5.15). Recall that the mapping swaps the third and fourth quadrants in the z-directional blow-up. In addition, taking into account the change of time $dt/d\tau = z^3$, it follows that the vector field in the third and fourth quadrant of the plane (u, z) points in the opposite direction compared to the one obtained in the (r, z)-plane.

Next, we need to perform the u-directional blow-up to know the phase portrait in that direction. After that, collecting the information about the blow-ups in both directions, we will have the phase portrait of system (5.14).

The u-directional blow-up is given by the transformation $u = u$, $q = z/u$, and with the change of time $dt/d\tau = u^3$, system (5.14) is transformed into

$$\begin{cases} \dot{u} = -q^2(uq^2 - 1) - u^2(uq^2 - m), \\ \dot{q} = q^5 - muq. \end{cases} \tag{5.16}$$

On $q = 0$, the origin is the unique critical point of the system, and since the linear part of the system at this point vanishes identically we have to use again some directional blow-ups.

Since the lower degree term of $\dot{q}u - \dot{u}q$ is $-q(2mu^2 + q^2)$, and it only vanishes on the direction $q = 0$, to study the origin of system (5.16) it suffices to consider the u-directional blow-up. It is given by the transformation $u = u$, $s = q/u$. Doing the change of time $dt/d\tau = u$, system (5.16) becomes

$$\begin{cases} \dot{u} = -us^2(u^3s^2 - 1) - u(u^3s^2 - m), \\ \dot{s} = s^3(u^3 - 1) + 2s(u^3s^4 - m). \end{cases} \tag{5.17}$$

For $s = 0$, system (5.17) has a unique critical point at the origin. The linearization matrix at the origin has eigenvalues m and $-2m$. Thus the origin of system (5.17) is a saddle.

Then, by using the transformation $(u, q) = (u, su)$, we can obtain the phase portrait of system (5.16). Recall that the mapping swaps the second and the third quadrants in the u-directional blow-up. In addition, taking into account the change of time $dt/d\tau = u$ it follows that the vector field in the second and third quadrants of the plane (u, q) points in the opposite direction compared to the one in the (u, s)-plane. Once we have the phase portrait in the (u, q)-plane, we apply the transformation $(u, z) = (u, qu)$.

By considering the properties of the blow-up technique and from the analysis of all the intermediate phase portraits we obtain that the origin of system (5.14) is a repeller. □

Recall that the finite critical points are hyperbolic saddles at $(\pm m^{-1/4}, \pm m^{-1/4})$ and a monodromic nilpotent singularity $(0, 0)$, whose stability is given in Theorem 5.3. Finally, notice that the vector field is symmetric with respect to the origin. By adding to these properties all the information concerning the infinite critical points and using the existence and uniqueness results on the number of limit cycles and polycycles given in Theorem 5.1, we obtain the global phase portraits of system (5.1) given in Figure 5.1.

5.3.3 Some Bendixson–Dulac type criteria

The next statement is a Bendixson–Dulac type result, that mixes the Bendixson–Dulac Test given in the classical book [7, Thm. 31] and the one given in [49, Prop. 2.2]. It is adapted to serve our interests. Similar results appear in [16, 47, 68, 103].

Proposition 5.7 (Bendixson–Dulac Criterion). *Let $X = (P, Q)$ be the vector field associated to the C^1-differential system*

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y), \end{cases} \tag{5.18}$$

and let $\mathcal{U} \subset \mathbb{R}^2$ be an open region which has its boundary formed by finitely many algebraic curves. Assume that there exist a rational function $V(x, y)$ and $k \in \mathbb{R}^+$ such that

$$M = M_{\{V,k\}}(x, y) = \langle \nabla V, X\rangle - kV \operatorname{div}(X) \tag{5.19}$$

does not change sign in $\mathcal{U}$ and M only vanishes on points or curves that are not invariant by the flow of X. Then:

(I) If all the connected components of $\mathcal{U} \setminus \{V = 0\}$ are simply connected then the system has neither periodic orbits nor polycycles.

(II) If all the connected components of $\mathcal{U} \setminus \{V = 0\}$ are simply connected, except one, say $\widetilde{\mathcal{U}}$, that is 1-connected, then, either the system has neither periodic orbits nor polycycles or it has at most one of them in $\mathcal{U}$. Moreover, when it has a limit cycle, it is hyperbolic, it is contained in $\widetilde{\mathcal{U}}$, and its stability is given by the sign of $-VM$ on $\widetilde{\mathcal{U}}$.

Proof. Consider the Dulac function $g(x, y) = |V(x, y)|^{-1/k}$. Then

$$\begin{aligned} \operatorname{div}(gX) &= \frac{\partial g}{\partial x}P + \frac{\partial g}{\partial y}Q + g(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}) = \langle \nabla g, X\rangle + g \operatorname{div}(X) \\ &= -\frac{1}{k}\operatorname{sgn}(V)|V|^{-\frac{k+1}{k}}\left(\langle \nabla V, X\rangle - kV \operatorname{div}(X)\right) \\ &= -\frac{1}{k}\operatorname{sgn}(V)|V|^{-\frac{k+1}{k}}M_{\{V,k\}} = -\frac{1}{k}\operatorname{sgn}(V)|V|^{-\frac{k+1}{k}}M. \end{aligned}$$

By the hypotheses, $M|_{\{V=0\}} = \langle \nabla V, X\rangle|_{\{V=0\}}$ does not change sign in $\mathcal{U}$ and there is no solution contained in $\{M = 0\}$. Therefore, neither the periodic orbits nor the polycycles of the vector field in $\mathcal{U}$ can intersect $\{V = 0\}$.

For proving (I) we follow the proof of the Bendixson–Dulac Criterion given in [7, Thm. 31]. Assume, to arrive a contradiction, that the system has a simple closed curve Γ which is the union of trajectories of the vector field. Let $C \subset \mathcal{U}$ be the bounded region with boundary Γ. Then, by Stokes Theorem, we have that

$$\iint_C \operatorname{div}(gX) = \int_\Gamma \langle gX, \mathbf{n} \rangle,$$

where Γ is oriented in a suitable way. Note that the right-hand side in the above equality is zero because gX is tangent to the curve Γ and the left one is non-zero by our hypothesis. This fact leads to the desired contradiction.

In case (II), applying a similar argument to the region bounded by two possible simple closed curves formed by trajectories of the vector field, we arrive again at a contradiction.

To end the proof, let us show the hyperbolicity of the possible limit cycle Γ. Write $\Gamma = \{\gamma(t) := (x(t), y(t)), t \in [0, T]\} \subset \widetilde{U}$, where T is its period, and its characteristic exponent $h(\Gamma) = \int_0^T \operatorname{div}\left(X(\gamma(t))\right)\, dt$. We need to prove that $h(\Gamma) \neq 0$ and that its sign coincides with the sign of $-VM$ on $\widetilde{U}$. We know that

$$\frac{M}{V} = \frac{\langle \nabla V, X \rangle}{V} - k \operatorname{div}(X).$$

Remember that $\Gamma \cap \{V = 0\} = \emptyset$. Evaluating this last equality on γ and integrating between 0 and T we obtain that

$$\begin{aligned} \int_0^T \frac{M}{V}(\gamma(t))\, dt &= \int_0^T \frac{\langle \nabla V, X \rangle}{V}(\gamma(t))\, dt - k \int_0^T \operatorname{div}(X)(\gamma(t))\, dt \\ &= \ln |V(\gamma(t))| \Big|_{t=0}^{t=T} - k\, h(\Gamma) = -k\, h(\Gamma). \end{aligned} \tag{5.20}$$

Therefore, the result follows. □

Next result is a straightforward consequence of the above proposition. It states that when we construct a suitable Dulac function, the same method provides an effective estimation of the basin of attraction of the attracting critical points.

Corollary 5.8. *Assume that we are under the hypotheses of the above theorem and moreover that $\{V(x, y) = 0\}$ has an oval such that this set and the bounded region surrounded by it, say $\mathcal{W}$, are contained in $\mathcal{U}$. If the differential system has only a critical point $\mathbf{p}$ in $\mathcal{W}$ which is an attractor, then $\mathcal{W}$ is contained in the basin of attraction of $\mathbf{p}$.*

Observe that when we are under the hypotheses of the above corollary, but we already know that the system has a limit cycle in $\mathcal{U}$ and that $\mathcal{U}$ is simply connected, then, unless the set $\{V(x, y) = 0\}$ reduces to a single point, there is no need to assume that $\{V(x,y)=0\}$ has an oval. The existence of the oval is already guaranteed by the method itself.

Sometimes the hypothesis that M does not change sign can be replaced for another one, which we explain in the following remark.

Remark 5.9. *Assume that in Proposition 5.7 we can not ensure that the function M, given in (5.19), maintains its sign on the whole domain $\mathcal{U}$. Then, this hypothesis can be exchanged for another one. Define $\{M=0\}^*$ to be the subset of $\{M=0\}$ formed by curves that separate the regions $\{M>0\}$ and $\{M<0\}$. Thus, the new hypothesis is that the set $\{M=0\}^*$ is without contact by the flow of X. Hence, the conclusions (I) and (II) of Proposition 5.7 are still hold, if we replace the assumption for the connected components of $\mathcal{U}\setminus\{V=0\}$ by the assumption for $\mathcal{U}\setminus\big(\{V=0\}\cup\{M=0\}^*\big)$. We will use this idea in the proof of Proposition 5.18.*

5.3.4 Zeros of 1-parameter families of polynomials

As usual, for a polynomial $P(x)=a_nx^n+\cdots+a_1x+a_0$, we write $\triangle_x(P)$ to denote its discriminant, that is,

$$\triangle_x(P)=(-1)^{\frac{n(n-1)}{2}}\frac{1}{a_n}\operatorname{Res}(P(x),P'(x)),$$

where $\operatorname{Res}(P,P',x)$ is the resultant of P and P' with respect to x; see [31].

By using the same techniques as in Chapter 4 Lemma 4.19, it is not difficult to prove the following result, which will be used in several parts of this chapter.

Lemma 5.10. *Let $G_m(x)=g_n(m)x^n+g_{n-1}(m)x^{n-1}+\cdots+g_1(m)x+g_0(m)$ be a family of real polynomials depending continuously on a real parameter m and set $\Lambda_m=(c(m),d(m))$ for some continuous functions $c(m)$ and $d(m)$. Suppose that there exists an interval $I\subset\mathbb{R}$ such that:*

(i) For some $m_0\in I$, G_{m_0} has exactly r zeros in Λ_{m_0} and all of them are simple.

(ii) For all $m\in I$, $G_m(c(m))\cdot G_m(d(m))\neq 0$.

(iii) For all $m\in I$, $\triangle_x(G_m)\neq 0$.

Then for all $m\in I$, $G_m(x)$ has also exactly r zeros in Λ_m and all of them are simple.

The idea of the proof consists in looking at the roots of G as continuous functions of m. The hypothesis (*ii*) prevents that real roots of G_m pass through the boundary of Λ_m when m varies. The hypothesis (*iii*) forbids the appearance of some multiple root of G_m when m varies.

Notice that the above result transforms the control of the zeros of a function depending on two variables, x and m, into three problems of only one variable, the one of item (i) with the variable x and the two remainder ones with the variable m. If the dependence on m is also polynomial, and the polynomial has rational coefficients, then these three simpler questions can be solved by applying the well-known Sturm method. As we will see in the proof of Proposition 5.17, this approach can also extended when the polynomial has some irrational coefficients.

5.3.5 Transformation into an Abel equation

System (5.1) can be seen as the sum of two quasi-homogeneous vector fields, see [27]. It is known that in many cases these systems can be transformed into Abel equations.

Proposition 5.11. *The periodic orbits of system* (5.1) *correspond to positive T-periodic solutions of the Abel equation*

$$\frac{d\rho}{d\theta} = \alpha(\theta)\rho^3 + \beta(\theta)\rho^2, \tag{5.21}$$

where

$$\alpha(\theta) = 3\,\mathrm{Cs}(\theta)\,\mathrm{Sn}(\theta)\left(2m\,\mathrm{Cs}^4(\theta) + \mathrm{Sn}^2(\theta)\right)\left(m\,\mathrm{Cs}^8(\theta) - \mathrm{Sn}^4(\theta)\right)$$

and

$$\beta(\theta) = 5m\,\mathrm{Cs}^8(\theta) - 4\,\mathrm{Sn}^4(\theta) + (3-10m)\,\mathrm{Cs}^4(\theta)\,\mathrm{Sn}^2(\theta),$$

being Sn *and* Cs *the functions introduced in Section 5.2 and T their period.*

Proof. The result follows by applying the Cherkas transformation

$$\rho = \frac{r^3}{1 - r^3\,\mathrm{Sn}(\theta)\,\mathrm{Cs}(\theta)\left(\mathrm{Sn}^2(\theta) + 2m\,\mathrm{Cs}^4(\theta)\right)},$$

to the expression of system (5.1) in the quasi-homogeneous polar coordinates introduced in Section 5.2. It is used that the periodic orbits of the system do not intersect the curve $\dot{\theta} = 0$, and therefore the above transformation is well defined over them, see [27]. □

Using the above expression it is not difficult to reproduce the proof of the existence of the Hopf-like bifurcation given in Subsection 5.2. Unfortunately, although expression (5.21) looks quite simple, the results about the number of limit cycles of Abel equations that we are aware of are not applicable to (5.21).

5.4 First results about non-existence of limit cycles and polycycles

In this section we prove the non-existence results of periodic orbits already given in [40] and extend them to the non-existence of polycycles. Our proof is different and based on the Bendixson–Dulac theorem and other classical tools. We study separately each interval.

Proposition 5.12. *For* $m \in (0, 9/25]$*, system* (5.1) *has neither periodic orbits nor polycycles.*

Proof. Recall that for $m \in (0, 9/25]$ the origin is an attractor. Therefore if we prove that any periodic orbit Γ of the system is also an attractor we will have proved that the system has not periodic orbits. In order to prove the stability of the limit cycle we need to compute $\int_0^T \operatorname{div}(X(\gamma(t)))\, dt$, where $\gamma(t) := (x(t), y(t))$ is the time parametrization of Γ, and $T = T(\Gamma)$ its period.

From equation (5.19), for any function V such that $\{V(x,y) = 0\} \cap \Gamma = \emptyset$, we have

$$\operatorname{div}(X) = \frac{M_{\{V,k\}} - \langle \nabla V, X\rangle}{-kV}.$$

Hence,

$$\int_0^T \operatorname{div}(X(\gamma(t)))\, dt = -\int_0^T \frac{M_{\{V,k\}}(\gamma(t))}{kV(\gamma(t))} dt,$$

where we have followed similar computations to those in (5.20). Then the stability of Γ is given by the sign of $-MV$. If we show that for $m \in (0, 9/25]$ there exist a non-negative V and $k \in \mathbb{R}^+$, such that its corresponding M is non-negative, then we will have proven that the limit cycle is hyperbolic and an attractor.

By considering $V(x,y) = 2x^2 + y^4$ and $k = 2/3$ equation (5.19) becomes

$$M_{\{V,\frac{2}{3}\}} = \frac{2}{3}\left((3-10m)x^2 + my^4\right) y^4,$$

which clearly is non-negative on $\mathbb{R}^2$ for $m \in (0, 3/10]$.

If we use the same $V(x,y)$ as in previous case, but $k = K(m) = 8(11m + R)/(10m+3)^2$, with $R = \sqrt{m(1-4m)(25m-9)}$, then we have

$$M_{\{V,K(m)\}} = \left(\frac{2}{3+10m}\left(\frac{(m+R)(11m+R)}{m}\right)^{1/2} x^2 + \frac{2(3-10m)}{3+10m}\left(\frac{m(11m+R)}{(m+R)}\right)^{1/2} y^4\right)^2.$$

Hence, $M_{\{V,K(m)\}}$ is non-negative on $\mathbb{R}^2$ for $m \in (1/4, 9/25]$. Therefore system (5.1) has no limit cycles for $m \in (0, 9/25]$ as we wanted to show.

To prove the non-existence of polycycles for $m \in (0, 9/25)$ we use a different approach. Following [95], we can associate to each polycycle Γ, with k hyperbolic saddles at its corners, the number $\rho(\Gamma) = \prod_{i=1}^{k} b_i/a_i$, where $-a_i < 0 < b_i$, $i = 1, \ldots k$, are the eigenvalues at the saddles. Then, Γ is stable (respectively, unstable) if $\rho(\Gamma) < 1$ (respectively, $\rho(\Gamma) > 1$). In our case

$$\rho(\Gamma) = \frac{\left(5\sqrt{m} - 3 + \sqrt{25m + 18\sqrt{m} + 9}\right)^4}{48^2 m}.$$

Then, easy computations show that the polycycle is an attractor if $m < 9/25$ and a repeller if $m > 9/25$. Assume, to arrive to a contradiction, that for $m < 9/25$ the polycycle exists. Then both, the polycycle and the origin, would be attractors. Applying the Poincaré–Bendixson Theorem we could ensure that the system would have at least one periodic orbit between them. This result is in contradiction with the first part of the proof, where the non-existence of periodic orbits is established.

It only remains to show that for $m = 9/25$ the polycycle does not exist either. To prove this fact we could study the stability of the polycycle showing that if it exists it would be an attractor, arriving again at a contradiction. Nevertheless it is easier to apply Proposition 5.7 with the V and $k = K(9/25)$ used to prove the non-existence of periodic orbits. Indeed, this latter approach, taking the corresponding V and k, could also be used for all values of $m \in (0, 9/25]$, but we have preferred to include a proof based on the study of the stability of the limit cycle and the polycycle. □

Lemma 5.13. *Let X be the vector field associated to system* (5.1)*.*

(i) If we take $k = 1/3$ and $V_1(x, y) = g_0(y) + g_1(y)x$ where $g_0(y) = g_1'(y)$ and $g_1(y)$ a solution of the second order linear ordinary differential equation

$$-g_1''(y) + my^5 g_1'(y) - \frac{5}{3}my^4 g_1(y) = 0, \tag{5.22}$$

then (5.19) *reduces to the function*

$$M_1 := M_{\{V_1, \frac{1}{3}\}}(x, y) = \frac{1}{3}\, y^3 \left(3my^2 g_1''(y) - 5myg_1'(y) + 3\, g_1(y)\right). \tag{5.23}$$

(ii) If we take $k = 2/3$ and $V_2(x, y) = g_0(y) + g_1(y)x + g_2(y)x^2$, with

$$\begin{aligned} g_1(y) &= g_2'(y), \\ g_0(y) &= (1/2)g_2''(y) - (1/2)my^5 g_2'(y) + (5/3)my^4 g_2(y), \end{aligned} \tag{5.24}$$

then (5.19) *becomes*

$$
\begin{aligned}
M_2 :=& M_{\{V_2,\frac{2}{3}\}}(x,y) \\
=& \Big(-\frac{1}{2}g_2'''(y) + \frac{3}{2}my^5 g_2''(y) - \frac{5}{2}my^4 g_2'(y) + \frac{2}{3}(3-10m)y^3 g_2(y)\Big)x \\
&+ \frac{1}{18}y^3\Big(9my^2 g_2'''(y) - m(30+9my^6)yg_2''(y) + 3(6+5m^2y^6)g_2'(y) \\
&\qquad + 20m^2y^5 g_2(y)\Big). \qquad (5.25)
\end{aligned}
$$

Proof. (*i*). If $V_1(x,y) = g_0(y) + g_1(y)\,x$ and $k = 1/3$, then

$$
\begin{aligned}
M_1 =& \langle \nabla V_1, X\rangle - \frac{1}{3}\operatorname{div}(X)V_1 \\
=& \left(g_0(y) - g_1'(y)\right)x^2 + \Big(- g_0'(y) + my^5 g_1'(y) - \frac{5}{3}my^4 g_1(y)\Big)x \\
&+ \frac{1}{3}y^3\left(3mg_0'(y)y^2 - 5mg_0(y)y + 3g_1(y)\right).
\end{aligned}
$$

By choosing $g_0(y) = g_1'(y)$ the coefficient of x^2 in M_1 vanishes, and we obtain

$$
M_1 = \left(-g_1''(y) + my^5 g_1'(y) - \frac{5}{3}my^4 g_1(y)\right)x + \frac{1}{3}y^3\left(3mg_1''(y)y^2 - 5myg_1'(y) + 3g_1(y)\right).
$$

Finally, if $g_1(y)$ is a solution of (5.22) we get (5.23).

(*ii*). If $k = 2/3$ and $V_2(x,y) = g_0(y) + g_1(y)x + g_2(y)x^2$, then

$$
\begin{aligned}
M_2 =& \langle \nabla V_2, X\rangle - \frac{2}{3}\operatorname{div}(X)V_2 \\
=& (g_1(y) - g_2'(y))x^3 + \Big(my^5 g_2'(y) - \frac{10}{3}my^4 g_2(y) - g_1'(y) + 2g_0(y)\Big)x^2 + \Big(2y^3 g_2(y) \\
&+ my^5 g_1'(y) - \frac{10}{3}my^4 g_1(y) - g_0'(y)\Big)x + \frac{1}{3}y^3\big(3g_1(y) + 3my^2 g_0'(y) - 10myg_0(y)\big).
\end{aligned}
$$

By choosing $g_1(y) = g_2'(y)$ and $g_0(y) = (1/2)g_2''(y) - (1/2)my^5 g_2'(y) + (5/3)my^4 g_2(y)$ the coefficients of x^2 and x^3 in M_2 vanish. Then we have (5.25). □

Remark 5.14. *Notice that if $g_2(y)$ is a solution of the linear ordinary differential equation*

$$
-\frac{1}{2}g_2'''(y) + \frac{3}{2}my^5 g_2''(y) - \frac{5}{2}my^4 g_2'(y) + \frac{2}{3}(3-10m)y^3 g_2(y) = 0, \qquad (5.26)
$$

then (5.19) *reduces to a function depending only on the variable y.*

Proposition 5.15. *For $m \in [3/5, \infty)$, system* (5.1) *has neither periodic orbits nor polycycles.*

Proof. We want to apply Proposition 5.7, taking $k = 1/3$ and $V_1(x, y) = g_0(y) + g_1(y)x$, with g_0 and g_1 as in (i) of Lemma 5.13. Applying the transformation $z = my^6/6$, equation (5.22) becomes

$$zg_1''(z) + \left(\frac{5}{6} - z\right) g_1'(z) + \frac{5}{18} g_1(z) = 0,$$

which is a Kummer equation, see [1, pp. 504]. A particular solution of this equation is

$$g_1(z) = z^{1/6} \sum_{j=0}^{\infty} \frac{(-\frac{1}{9})_j}{(\frac{7}{6})_j} \frac{z^j}{j!},$$

where $(a)_j := a(a+1)(a+2)\cdots(a+j-1)$ and $(a)_0 = 1$. Therefore we consider

$$g_1(y) = \left(\frac{m}{6}\right)^{1/6} y \sum_{j=0}^{\infty} \frac{(-\frac{1}{9})_j}{(\frac{7}{6})_j} \left(\frac{m}{6}\right)^j \frac{y^{6j}}{j!},$$

which is convergent on the whole of $\mathbb{R}$ and satisfies (5.22). Its derivatives are

$$g_1'(y) = \left(\frac{m}{6}\right)^{1/6} \sum_{j=0}^{\infty} \frac{(-\frac{1}{9})_j}{(\frac{7}{6})_j} \left(\frac{m}{6}\right)^j (6j+1) \frac{y^{6j}}{j!},$$

$$g_1''(y) = \left(\frac{m}{6}\right)^{1/6} \sum_{j=0}^{\infty} \frac{(-\frac{1}{9})_j}{(\frac{7}{6})_j} \left(\frac{m}{6}\right)^j 6j(6j+1) \frac{y^{6j-1}}{j!}.$$

Replacing the above functions in (5.23) we obtain

$$\begin{aligned} M_1 = &\left(\frac{3-5m}{3}\right)\left(\frac{m}{6}\right)^{1/6} y^4 \\ &+ \left(\frac{1}{3}\right)\left(\frac{m}{6}\right)^{1/6} \sum_{j=1}^{\infty} \frac{(-\frac{1}{9})_j}{(\frac{7}{6})_j} \left(\frac{m}{6}\right)^j \left(\frac{1}{j!}\right) \big(m(6j+1)(18j-5)+3\big) y^{6j+4}. \end{aligned}$$

Since $(-\frac{1}{9})_j$ is negative for all j, it follows that $M_1 \le 0$ for $m \ge 3/5$, and vanishes only on $y = 0$. Therefore the result follows by applying Proposition 5.7. $\square$

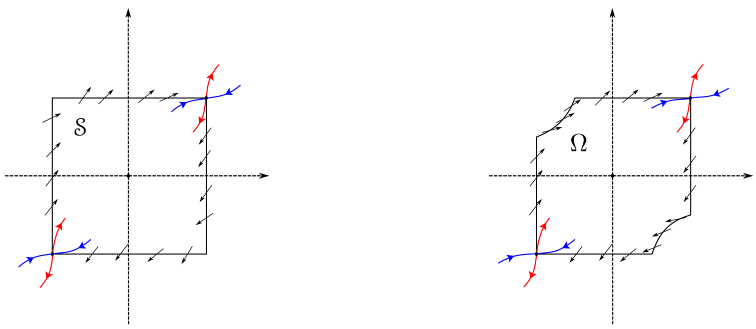
5.5 The uniqueness and hyperbolicity of the limit cycle

In this section we prove that for $m \in (1/2, 3/5)$, system (5.1) has at most one limit cycle or one polycycle and the two of them never coexist. Moreover, we show that when the limit cycle exists, it is hyperbolic. The uniqueness of the limit cycle was already proved in [40]. Our approach is different and, like in the previous section, it is based on the construction of a suitable Dulac function. This section ends with the proof of Proposition 5.2.

Lemma 5.16. *Let $\mathcal{S}$ be the open set bounded by the lines $x = \pm m^{-1/4}$ and $y = \pm m^{-1/4}$ and let Ω be the connected component containing the origin and bounded by the above four straight lines and the hyperbola $xy + 1 = 0$, see Figure 5.3. Then, for $m \in (0, 1)$, the following holds:*

(i) The vector field X associated to system (5.1) is transversal to the boundary $\partial\mathcal{S}$ of the square $\mathcal{S}$ except at the two saddle critical points of system (5.1).

(ii) If system (5.1) has a periodic orbit or a polycycle, it must be contained in $\Omega \subset \mathcal{S}$.



(a) (b)

Figure 5.3: Regions Ω and $\mathcal{S}$.

Proof. (i). Consider the function $f(x, y) = x - m^{-1/4}$. It is not difficult to see that $\langle \nabla f, X \rangle$ restricted to $x - m^{-1/4} = 0$ has the expression $y^3 - m^{-1/4}$ which is negative for $y \in (-m^{-1/4}, m^{-1/4})$. Analogously, we can see that the direction of X along $\partial\mathcal{S}$ is as showed in Figure 5.3 (a).

(ii). It is well known that the sum of the indices of all the singularities surrounded by a periodic orbit, or a polycycle is one. Recall that the indices of the saddle points are -1 and the index of a monodromic point is $+1$. Hence, if a periodic orbit or a polycycle Γ exist they must surround only the origin. Moreover, by statement (i), Γ cannot intersect $\partial\mathcal{S}$. Finally, a simple computation shows that $\langle\nabla(xy+1), X\rangle$ restricted to $xy+1=0$ is $(1-m)/x^4$, which implies that X is transversal to $xy+1=0$. Hence X is transversal to $\partial\Omega$ and the lemma follows. □

Proposition 5.17. *For $m \in [1/2, 3/5)$, system (5.1) has at most one limit cycle and one polycycle and both never coexist. Moreover, when the limit cycle exists it is hyperbolic and a repeller.*

Proof. Following statement (ii) of Lemma 5.13 we take $k = 2/3$ and a function $V_2(x,y) = g_0(y) + g_1(y)x + g_2(y)x^2$ adequate to apply Proposition 5.7 for proving the uniqueness of the limit cycles or polycycles for system (5.1).

We will take $g_2(y)$ as a truncated Taylor series at the origin of a suitable solution of (5.26) such that the curve $\{V_2 = 0\}$ has an oval surrounding the origin, and that M_2 does not change sign in Ω. These two properties will imply the result.

The general solution of (5.26) is the linear combination of generalized hypergeometric functions

$$g_2(y)=C_0\sum_{j=0}^{\infty}\frac{(\phi^+(m))_j\,(\phi^-(m))_j}{\left(\frac{2}{3}\right)_j\left(\frac{5}{6}\right)_j}\left(\frac{m}{2}\right)^j\frac{y^{6j}}{j!}+C_1y\sum_{j=0}^{\infty}\frac{(\varphi^+(m))_j\,(\varphi^-(m))_j}{\left(\frac{5}{6}\right)_j\left(\frac{7}{6}\right)_j}\left(\frac{m}{2}\right)^j\frac{y^{6j}}{j!}$$
$$+\,C_2y^2\sum_{j=0}^{\infty}\frac{(\psi^+(m))_j\,(\psi^-(m))_j}{\left(\frac{7}{6}\right)_j\left(\frac{4}{3}\right)_j}\left(\frac{m}{2}\right)^j\frac{y^{6j}}{j!},\tag{5.27}$$

where $\phi^{\pm}(m) = \pm A(m) - 2/9$, $\varphi^{\pm}(m) = \pm A(m) - 1/18$, $\psi^{\pm}(m) = \pm A(m) + 1/9$, with $A(m) = \sqrt{(14m-3)/m}/9$.

We look for an even solution, so we take $C_1 = 0$. As we will consider $C_0 \neq 0$, it is not restrictive to choose $C_0 = 1$. Finally, the constant $C_2 = -(3/5-m)^{2/3}$ is fixed after some previous simulations and taking into account that we already know that at $m = 3/5$ there is a Hopf-like bifurcation.

Once we have fixed the above constants, we calculate the Taylor polynomial of degree 12 of g_2 at $y = 0$, $\mathcal{T}_{12}(g_2)$, obtaining

$$\mathcal{T}_{12}(g_2(y)) =\frac{1}{89100}(3-10m)(3+35m)y^{12}-\frac{1}{6300}(75-125m)^{2/3}(3-13m)y^8$$
$$+\frac{1}{90}(3-10m)y^6-\frac{1}{25}(75-125m)^{2/3}y^2+1.\tag{5.28}$$

So, in *(ii)* Lemma 5.13, we fix g_2 as $\mathcal{T}_{12}(g_2(y))$. Then the corresponding g_0 and g_1 are given by (5.24). Thus, M_2 is of the form $M_2 = (\phi(y)x + \psi(y))y^4$ where

$$\phi(y) =\frac{1}{9450}\Big(\frac{7}{99}\,(3-10\,m)\,(242\,m+3)\,(35\,m+3)\,y^{11}$$
$$+\,(75-125\,m)^{2/3}\,(86\,m+3)\,(13\,m-3)\,y^7\Big),$$

$$\begin{aligned}\psi(y) = &-\frac{247}{400950}\, m^2\,(3-10\,m)\,(35\,m+3)\,y^{16}\\ &-\frac{13}{4050}\, m^2\,(75-125\,m)^{2/3}\,(13\,m-3)\,y^{12}\\ &+\frac{1}{7425}\,(3-10\,m)\left(550\,m^2+145\,m+3\right)y^{10}\\ &+\frac{2}{4725}\,(75-125\,m)^{2/3}\left(196\,m^2-45\,m-9\right)y^{6}\\ &+\frac{1}{15}\,(3-5\,m)\,y^4-\frac{2}{75}\,(75-125\,m)^{2/3}\,(3-5\,m)\,.\end{aligned}$$

The proposition follows if we prove that M_2 does not change sign on the region Ω. In fact, it is sufficient to prove that $M := M_2/y^4$ does not change sign on Ω.

The idea is to show that $\{M = 0\}$ does not intersect Ω. Since M is linear in the variable x, $\{M = 0\}$ cannot have ovals inside Ω. If $\{M = 0\}$ has a component in Ω, this component would have to cross $\partial\Omega$ by continuity of the function. Then, it suffices to see that $\{M = 0\}$ does not intersect $\partial\Omega$. Moreover, as M satisfies $M(x, y) = M(-x, -y)$, it is sufficient to study M on half of $\partial\Omega$. To deal only with polynomials we introduce the new variables $n = \sqrt[4]{m}$ and $s = (75 - 125m)^{2/3}$. Notice that $s^3 = (75 - 125n^4)^2$.

We split the half of the boundary of Ω in four pieces:

- The segment $\gamma_1 = \{(x, 1/n) \,:\, -n < x < 1/n\}$,
- The segment $\gamma_2 = \{(1/n, y) \,:\, -n < y < 1/n\}$,
- The piece of hyperbola $\gamma_3 = \{(x, -1/x) \,:\, n < x < 1/n\}$,
- The corners $\gamma_4 = \{(1/n, 1/n), (1/n, -n), (n, -1/n)\}$

and we have to prove that $\{M = 0\} \cap \gamma_i = \emptyset$ for each $i = 1, 2, 3, 4$.

These facts can be seen proving that for $n \in I := [\sqrt[4]{1/2}, \sqrt[4]{3/5}\,)$,

- $Q_1(x, n, s) := M(x, 1/n) \neq 0$, for $x \in (-n, 1/n)$.
- $Q_2(y, n, s) := M(1/n, y) \neq 0$, for $y \in (-n, 1/n)$.
- $Q_3(x, n, s) := M(x, -1/x) \neq 0$, for $x \in (n, 1/n)$.
- $M(1/n, 1/n) \cdot M(1/n, -n) \cdot M(n, -1/n) \neq 0$.

Lemma 5.10, with $r = 0$, is a convenient tool to prove the first three items. The proof of the last item is a straightforward consequence of the Sturm method.

We will give the details of the proof that $Q_2(y, n, s) \neq 0$, which is the most elaborate case. The remaining two cases follow similarly.

Writing $Q(y,n,s) := 2806650\,n\,Q_2(y,n,s)$ we get that

$$\begin{aligned}Q(y,n,s) = {} & 1729\,n^9(35n^4+3)(10n^4-3)y^{16} - 9009\,n^9 s(13n^4-3)y^{12}\\ & - 21(10n^4-3)(242n^4+3)(35n^4+3)y^{11}\\ & - 378n(10n^4-3)(550n^8+145n^4+3)y^{10} + 297s(13n^4-3)(86n^4+3)y^7\\ & + 1188ns(196n^8-45n^4-9)y^6 - 187110n(5n^4-3)y^4 + 74844ns(5n^4-3).\end{aligned}$$

Looking at Lemma 5.10 with $r=0$, it suffices to prove the following three facts:

(i) When $n = \sqrt[4]{1/2} \in I$, $Q(y,n,s) \neq 0$ for $y \in (-n, 1/n)$.

(ii) For $n \in I$, $\triangle_y Q(y,n,s) \neq 0$.

(iii) For $n \in I$, $Q(-n,n,s) \cdot Q(1/n,n,s) \neq 0$.

Since the polynomial has no rational coefficients the proof of item (i) requires some special tricks. Notice that when $n = \sqrt[4]{1/2}$ then $s = 5\sqrt[3]{10}/2$. Hence,

$$\begin{aligned}R(y) := Q\Big(y, \frac{1}{\sqrt[4]{2}}, \frac{5}{2}\sqrt[3]{10}\Big) = {} & \frac{70889}{8}\sqrt[4]{8}\,y^{16} - \frac{315315}{32}\sqrt[4]{8}\sqrt[3]{10}\,y^{12} - 106764y^{11}\\ & - 80514\sqrt[4]{8}\,y^{10} + \frac{239085}{2}\sqrt[3]{10}\,y^7 + \frac{51975}{2}\sqrt[4]{8}\sqrt[3]{10}\,y^6\\ & + \frac{93555}{2}\sqrt[4]{8}\,y^4 - \frac{93555}{2}\sqrt[4]{8}\sqrt[3]{10}.\end{aligned}$$

We will prove that the above polynomial has no real roots in $[-1, 12/10] \supset (-n, 1/n)$. The Sturm method gives polynomials with huge coefficients and our computers have problems to deal with them. We use a different approach. We know, that

$$\underline{n} := \frac{3002}{1785} < \sqrt[4]{8} < \frac{37}{22} =: \overline{n}, \quad \underline{s} := \frac{28}{13} < \sqrt[3]{10} < \frac{265}{123} =: \overline{s},$$

where these four rational approximations are obtained computing the continuous fraction expansion of both irrational numbers. If we construct the polynomial, with rational coefficients,

$$\begin{aligned}R^+(y) = {} & \frac{70889}{8}\overline{n}\,y^{16} - \frac{315315}{32}\underline{n}\,\underline{s}\,y^{12} - 106764y^{11} - 80514\underline{n}\,y^{10}\\ & + \frac{239085}{2}\overline{s}\,y^7 + \frac{51975}{2}\overline{n}\,\overline{s}\,y^6 + \frac{93555}{2}\overline{n}\,y^4 - \frac{93555}{2}\underline{n}\,\underline{s},\end{aligned}$$

it is clear that for $y \geq 0$, $R(y) < R^+(y)$. In fact,

$$
\begin{aligned}
R^+(y) =& \frac{2622893}{176} y^{16} - \frac{2427117}{68} y^{12} - 106764 y^{11} - \frac{11509668}{85} y^{10} \\
&+ \frac{21119175}{82} y^7 + \frac{15442875}{164} y^6 + \frac{314685}{4} y^4 - \frac{37446948}{221}
\end{aligned}
$$

and, now, using the Sturm method it is quite easy to prove that $R^+(y) < 0$ for $y \in [0, 12/10]$. Hence, in this interval, $R(y) < R^+(y) < 0$, as we wanted to prove.

To study the values of $y < 0$ we construct a similar upper bound,

$$
\begin{aligned}
R^-(y) =& \frac{70889}{8} \overline{n}\, y^{16} - \frac{315315}{32} \underline{n}\, \underline{s}\, y^{12} - 106764 y^{11} - 80514 \underline{n}\, y^{10} \\
&+ \frac{239085}{2} \underline{s}\, y^7 + \frac{51975}{2} \overline{n}\, \overline{s}\, y^6 + \frac{93555}{2} \overline{n}\, y^4 - \frac{93555}{2} \underline{n}\, \underline{s},
\end{aligned}
$$

and applying the same method the result follows.

To prove (ii) we compute

$$\triangle_y Q(y, n, s) = n^{42} s^3 (5n^4 - 3)^5 (35n^4 + 3)^3 (10n^4 - 3)^3 P_{258}(n, s),$$

where $P_{258}(n, s)$ is a polynomial in n and s of degree 258. Clearly, the roots of the first five factors of the above discriminant are not relevant for our problem because the corresponding n is not in I. To study whether $P_{258}(n, s)$ vanishes or not we compute

$$\mathrm{Res}(P_{258}(n, s), (75 - 125n^4)^2 - s^3, s) = (5n^4 - 3)^{24} P_{390}(n^2),$$

where $P_{390}(n^2)$ is a polynomial of degree 390 in n^2. Applying again the Sturm method we get that $P_{390}(n^2)$ has no significant roots for our study. Finally, the numerator of $Q(-n, n, s) \cdot Q(1/n, n, s)$ is a polynomial in n and s of degree 49. Using the same trick as above we prove item (iii). In this case the polynomial we have to deal with has degree 152 in n.

Therefore $\{M = 0\} \cap \partial\Omega = \emptyset$ and as a consequence $\{M = 0\} \cap \Omega = \emptyset$.

Finally, it is not difficult to see, because V is quadratic in x, that the set $\{V(x, y) = 0\}$ has exactly one oval surrounding the origin. Hence, the proposition follows. □

Proof of Proposition 5.2. Notice that the function V used in the proof of Proposition 5.17 coincides with the function $V(x, y, m)$ of the statement of the proposition. Taking $k = 2/3$ we are also under the hypotheses of Corollary 5.8. Therefore the set $\mathcal{U}_m$ is contained in $\mathcal{W}^s_{\mathbf{0}}$, as we wanted to prove. □

We remak that following similar ideas as in the above proof we can construct bigger sets contained in $\mathcal{W}^s_{\mathbf{0}}$. For a given m, let us denote by $\mathcal{T}_\ell(g_2(x; C_2))$ the Taylor polynomial of degree ℓ at $x = 0$, of the function (5.27) with $C_0 = 1$, $C_1 = 0$. Then for each $\ell \in \mathbb{N}$ and $C_2 \in \mathbb{R}$ we can take this function as a new seed g_2 for constructing the corresponding V as in (ii) of Lemma 5.13. Then checking that the oval contained in $\{V = 0\}$ is crossed inwards by the flow of the system, the result follows for the function V constructed with these ℓ and C_2.

5.6 Other results about non-existence of limit cycles and polycycles

This section contains new non-existence results for system (5.1). We split the interval into the subintervals $(9/25, 1/2)$ and $[1/2, 0.547]$. Recall that our numerical study shows that the system has no limit cycles for $m < 0.56011\ldots$ As m becomes closer to this bifurcation value the proof of non-existence of periodic orbits and polycycles becomes harder.

Proposition 5.18. *For $m \in (9/25, 1/2)$, system (5.1) has neither limit cycles nor polycycles.*

Proof. We would like to apply Proposition 5.7. To this end we will follow similar steps to the ones in the proof of Proposition 5.17, but with a function V such that the set $\{V = 0\}$ has no oval in Ω. Recall that Ω is the domain introduced in Lemma 5.16, where the limit cycles and the polycycles must lie. We take $V = V_2(x, y) = g_0(y) + g_1(y)x + g_2(y)x^2$ with $g_1(y) = g_2'(y)$, $g_0 = (1/2)g_2''(y) - (1/2)my^5 g_2'(y) + (5/3)my^4 g_2(y)$. Now we consider $g_2(y) = a_0 + a_2y^2 + a_4y^4 + a_6y^6 + a_8y^8$, with coefficients to be determined. From statement (ii) of Lemma 5.13 it follows that the corresponding M_2 is a polynomial function in x of the form $M_2 = \phi(y)x + \psi(y)$ where $\phi(y)$ and $\psi(y)$ are polynomials in the variable y whose coefficients depend on a_{2j}, $j = 0, 1, \ldots, 4$. In order to simplify the computations, we change the parameter m by n^4 to transform V into a polynomial in the variables x, y, and n. Since $m \in (9/25, 1/2)$ we can restrict our study to $n \in (0.77, 0.844)$.

We consider the values of a_4, a_6 and a_8 such that $\phi(y)$ has a zero at $y = 0$ of multiplicity nine, we choose the value of a_2 by imposing that M_2 vanishes at the two saddle points of the system and, finally, we use the freedom of changing $g_2(y)$ by $\lambda g_2(y)$, for any $0 \neq \lambda \in \mathbb{R}$, to remove all the denominators. We obtain that

$$\begin{aligned}
g_2(y) =& 270(9 + 51n^2 + 213n^4 + 535n^6 + 924n^8 + 756n^{10}) \\
& - 756n^2(9 + 42n^2 + 105n^4 + 130n^6)y^2 \\
& + 3(3 - 10n^4)(9 + 51n^2 + 213n^4 + 535n^6 + 924n^8 + 756n^{10})y^6 \\
& - 3n^2(3 - 13n^4)(9 + 42n^2 + 105n^4 + 130n^6)y^8.
\end{aligned}$$

The corresponding M_2 is of the form

$$M_2(x,y) = \frac{2}{3}y^4\left(\phi(y)x + \psi(y)\right) =: \frac{2}{3}y^4\, M(x,y), \tag{5.29}$$

where

$$\begin{aligned}
\phi(y) =& 3(3-10n^4)(3+35n^4)(9+51n^2+213n^4+535n^6+924n^8+756n^{10})y^5 \\
&- 3n^2(3-13n^4)(3+86n^4)(9+42n^2+105n^4+130n^6)y^7, \\
\psi(y) =& -756n^2(3-5n^4)(9+42n^2+105n^4+130n^6) \\
&+ 27(3-5n^4)(9+51n^2+213n^4+535n^6+924n^8+756n^{10})y^4 \\
&- 12n^2(9+42n^2+105n^4+130n^6)(9+45n^4-196n^8)y^6 \\
&- 40n^8(3-10n^4)(9+51n^2+213n^4+535n^6+924n^8+756n^{10})y^{10} \\
&+ 91n^{10}(3-13n^4)(9+42n^2+105n^4+130n^6)y^{12}.
\end{aligned}$$

Recall that the main hypothesis in Proposition 5.7 is that M does not change sign on Ω. As we will see, this happens only for $n \in J := (0.77, \widetilde{n}]$ where $\widetilde{n} \approx 0.8045592$ will be precisely defined afterwards. When $n \in K := (\widetilde{n}, 0.844)$ the result will be a consequence of the variation of Proposition 5.7 described in Remark 5.9.

For $n \in J$, proceeding similarly to proof of Proposition 5.17, we divide the half of the boundary of Ω in five pieces:

- The segment $\gamma_1 = \{(x, 1/n) \,:\, -n < x < 1/n\}$,
- The segment $\gamma_2 = \{(1/n, y) \,:\, -n < y < 1/n\}$,
- The piece of hyperbola $\gamma_3 = \{(x, -1/x) \,:\, n < x < 1/n\}$,
- The corners $\gamma_4 = \{(1/n, -n), (n, -1/n)\}$,
- The corner $\gamma_5 = \{(1/n, 1/n)\}$

and we will prove that $\{M = 0\} \cap \gamma_i = \emptyset$ for each $i = 1,2,3,4$ and that although $(1/n, 1/n) \in \partial\Omega$, the set $\{M = 0\}$ does not enter in Ω. From these results we will have proved that M does not change sign on Ω and, as a consequence, the proposition will follows for $n \in J$.

To prove the fifth assertion it suffices to study the function M in a neighborhood of the point $(1/n, 1/n) \in \partial\Omega$. By the construction of M, it holds that $M(1/n, 1/n) = 0$. By computing the partial derivatives of M at this point we obtain the tangent vector of the curve at $(1/n, 1/n)$. Then, it is easy to see that when $n \in J$, in a punctured neighborhood $\mathcal{W}$ of $(1/n, 1/n)$, it holds that $\mathcal{W} \cap \{M = 0\} \cap \Omega = \emptyset$. In fact, $\widetilde{n} \in \partial J$ is a solution of the equation

$$\operatorname{num}\Big(\frac{\partial M(x,y)}{\partial x}\Big|_{(x,y)=(1/n,1/n)}\Big) = 0,$$

where num(·) denotes the numerator of the rational function. Moreover,

$$M\Big(x, \frac{1}{n}\Big) = -\frac{9(nx-1)}{n^4}\Big(88200n^{16} + 107800n^{14} - 4930n^{12} - 37380n^{10} - 15855n^8 - 2736n^6 + 576n^4 + 108n^2 - 27\Big) \qquad (5.30)$$

and $\widetilde{n}$ is also the positive root of the polynomial in n appearing in the right-hand side of the above formula. Notice that when $n = \widetilde{n}$, the straight line $\{y = 1/\widetilde{n}\}$ is a subset of $\{M = 0\}$. This fact is the reason for which this approach only works for $n \in J = (0.77, \widetilde{n}]$.

Let us prove the remaining four assertions. As in the proof of Proposition 5.17, they follow by showing that when $n \in J$,

- $R_1(x, n) := \text{num}(M(x, 1/n)) \neq 0$, for $x \in (-n, 1/n)$.
- $R_2(y, n) := \text{num}(M(1/n, y)) \neq 0$, for $y \in (-n, 1/n)$,
- $R_3(x, n) := \text{num}(M(x, -1/x)) \neq 0$, for $x \in (n, 1/n)$.
- $M(1/n, -n) \cdot M(n, -1/n) \neq 0$.

That R_1 has no zeros in J, is a straightforward consequence of (5.30).

To study R_2 and R_3 we will use Lemma 5.16. We start computing the discriminants,

$$S_2(n) = \triangle_y(R_2(y, n)), \quad S_3(n) = \triangle_x(R_3(x, n)),$$

and analyze whether they vanish on J or not. Using the Sturm method we get that on J, S_2 vanishes only at one value $n_2 \approx 0.8040188$ and S_3 also vanishes only at one value $n_3 \approx 0.8045576$. The root n_2 of S_2 forces us to split the study of $R_2(y, n)$ in the three subcases: $n \in (0.77, n_2)$, $n = n_2$ and $n \in (n_2, \widetilde{n}]$. Doing the same type of computations and reasoning as in the previous section we can prove all the above assertions when $n \neq n_2$. The case $n = n_2$ follows by continuity arguments, because in this situation R_2 has a real multiple root but it is not in $(-n_2, 1/n_2)$. The study of R_3 is similar to the one of R_2 and we omit it. We also get that R_3 does not vanish on $(n, 1/n)$ either.

The fact that $M(1/n, -n)\cdot M(n, -1/n) \neq 0$ for $n \neq \widetilde{n}$ is once more a consequence of the Sturm method.

Therefore, when $n \in J$, we are under the hypotheses of Proposition 5.7, and we will know that the system has no limit cycles once we have proved that the set $\{V = 0\}$ has no ovals. We defer the proof of this fact until we have considered the case $n \in K = (\widetilde{n}, 0.844)$.

When $n \in K$, we know that $\{M = 0\} \cap \Omega \neq \emptyset$ and we are no more under the hypotheses of Proposition 5.7. Let us see that we can apply the ideas of Remark 5.9.

To this end we have to prove that $\{M_2 = 0\}^* \cap \Omega$ is without contact for the flow of X. Note that $\{M_2 = 0\}^* = \{M = 0\}^*$.

We need to show that $\dot{M} = \langle \nabla M, X \rangle$ does not vanish on $\{M = 0\}^* \cap \Omega$. We study the common points of $\{M = 0\}$ and $\{\dot{M} = 0\}$ and prove that they are not in Ω. First, we compute

$$\dot{M}(x, y) = \langle \nabla M(x, y), X(x, y) \rangle =: y^3 N(x, y),$$

and we remove the factor y^3. We do not care about the points on $\{y = 0\}$ because

$$M(x, 0) = 756n^2 \left(5n^4 - 3\right) \left(9 + 42n^2 + 105n^4 + 130n^6\right) \neq 0,$$

for $n \in (0, 0.88]$.

The resultant $\operatorname{Res}(M, N, x)$ factorizes as

$$\operatorname{Res}(M, N, x) = y^2(n^2y^2 - 1)(P_{n,2}(y))(P_{n,34}(y)),$$

where $P_{n,2}(y)$ and $P_{n,34}(y)$ are polynomials in the variable y with respective degrees 2 and 34 and whose coefficients are polynomial functions with rational coefficients in the variable n.

Clearly, $(n^2y^2 - 1)$ does not vanish on $-1/n < y < 1/n$. By using once more Lemma 5.10 it is not difficult to prove that $P_{n,2}(y)$ does not vanish either on $-1/n < y < 1/n$, for $n \in (\widetilde{n}, 0.844)$. Hence we will focus on the factor $P_{n,34}(y)$.

We will use again Lemma 5.10. By using the Sturm method we get that the polynomial $\triangle_y(P_{n,34}(y))$ has no zeros in the interval K. In fact one zero is $\widetilde{n} \in \partial K$ and another one is $n^* \approx 0.8445 \notin J$ and this is the reason for which we can only prove the result until $n = 0.844 < n^*$. By using Sturm method, it can be shown that $P_{n,34}(-1/n) \cdot P_{n,34}(1/n) \neq 0$ for all $n \in K$ and, for instance, for $n = n_0 = 83/100 \in K$, the polynomial $P_{n_0,34}(y)$ has exactly two (simple) zeros in $-1/n_0 < y < 1/n_0$. Then, Lemma 5.10 with $r = 2$, implies that $P_{n,34}(y)$ has exactly two (simple) zeros in $-1/n < y < 1/n$, for all $n \in K$. We call them $y = y_i(n), i = 1, 2$ and they are continuous functions of n. Therefore, we need to prove that the corresponding points in $\{M = 0\} \cap \{N = 0\}$ are outside of Ω.

Notice that due to the expression of M, given in (5.29), the points in $\{M = 0\}$ are on the curve $\Gamma = \{\big(-\frac{\psi(y)}{\phi(y)}, y\big) : y \in \mathbb{R} \setminus \{0\}\}$. Moreover it can be easily seen that $\phi(y) \neq 0$ on the region that we are considering. Therefore the points in $\{M = 0\} \cap \{N = 0\}$ are given by the two continuous curves

$$\gamma_i := \Big\{ \Big(-\frac{\psi(y_i(n))}{\phi(y_i(n))}, y_i(n) \Big) \, : \, n \in K \Big\}, \quad i = 1, 2.$$

For a fixed $n \in K$ it is not difficult to prove that the points in $\gamma_i, i = 1, 2$ are outside of Ω. If for some $n \in K$ there was a point inside Ω, by continuity it would be at least one point in one of the pieces of boundary of Ω formed by the straight line $\{x-1/n=0\}$ and the hyperbola $\{xy+1=0\}$. To prove that such a point does not exist we compute the following two resultants

$$\begin{aligned}\operatorname{Res}\Big(\operatorname{num}\Big(-\frac{\psi(y)}{\phi(y)}-\frac{1}{n}\Big),P_{n,34}(y),y\Big)&=P_{1250}(n),\\ \operatorname{Res}\Big(\operatorname{num}\Big(-y\frac{\psi(y)}{\phi(y)}+1\Big),P_{n,34}(y),y\Big)&=P_{1260}(n),\end{aligned}$$

where $P_\ell(n)$ are given polynomials with rational coefficients and degree ℓ. Both polynomials factorize in several factors and, using once more the Sturm method, we can easily prove that they do not vanish on K. Hence, $\{M=0\}\cap\{N=0\}\cap\Omega=\emptyset$ which implies that $\{M=0\}\cap\Omega$ is without contact by the flow of X, as we wanted to prove.

Since M is linear in the variable x, $\{M=0\}$ cannot have ovals. Therefore, by Remark 5.9, to end the proof we need to show that the set $\{V=0\}$ has no ovals either in Ω. We claim that the set $\{V=0\}\cap\Omega$ is without contact by the flow of the system. If this happens and $\{V=0\}$ had an oval then it would be without contact. Then by the Poincaré–Bendixson Theorem it should surround the origin. However, by considering the straight line passing through the origin $y=9x/10$ it is easy to prove, by using again Lemma 5.10, that the function $V(x,9x/10)$ does not vanish on the interval $-1/n<x<1/n$ for all $n\in(0.77,0.844)$. Thus, $\{V=0\}\cap\{y-9x/10=0\}=\emptyset$. Hence, V has no ovals inside Ω as we wanted to see and the proposition follows by using all the above results and the reasoning explained in Remark 5.9.

To prove the above claim, it suffices to see that $\{M=0\}\cap\{V=0\}\cap\Omega=\emptyset$. This is because precisely, $M\big|_{\{V=0\}}=\dot V$.

Recall that when $n\in J=(0.77,\widetilde n]$ then $\{M=0\}\cap\Omega=\emptyset$ and so the result follows.

Let us consider the case $n\in K=(\widetilde n,0.844)$. To study if $\{V=0\}$ and $\{M=0\}$ intersect, we compute the resultant of M and V with respect to x. We have

$$\operatorname{Res}(V,M,x)=(n^2y^2-1)P_{n,30}(y),$$

where $P_{n,30}(y)$ is a polynomial of degree 30 and whose coefficients are polynomial functions in the variable n with rational coefficients. We want to prove that $\operatorname{Res}(V,M,x)$ does not vanish on the interval $-1/n<y<1/n$ for $n\in K$. It suffices to study $P_{n,30}(y)$. We will use once more Lemma 5.10.

The polynomial $P_{n,30}(-1/n)\cdot P_{n,30}(1/n)$ has no real roots when $n\in K$. Moreover hypothesis (i) of Lemma 5.10 holds with $r=0$ (no real roots) by considering for instance $n_0=82/100$. To see that condition (iii) of the lemma holds, we compute $\triangle_y(P_{n,30}(y))$. It is a polynomial of degree 2728 in the variable n which factorizes in several factors, the largest one being of degree 594. From this decomposition we can prove that $\triangle_y(P_{n,30}(y))$ has no zeros for $n \in K$. Therefore, by Lemma 5.10 we conclude that $P_{n,30}(y)$ does not vanish on the whole interval $-1/n < y < 1/n$ for $n \in K$, and the claim follows. □

Proposition 5.19. *For $m \in [0.5, 0.547]$, system (5.1) has neither limit cycles nor polycycles.*

Proof. We will construct a positive invariant region $\mathcal{R}$ having the two saddle points in its boundary. As we will see, the proposition follows once we have constructed this region, simply by using the uniqueness and hyperbolicity of the limit cycle, whenever it exits. We remark that in this proof we will not use the Bendixson–Dulac theorem.

Assume that such a positive invariant region $\mathcal{R}$ exits. By the Index theory, if the system had a limit cycle, it should surround only the origin. By Proposition 5.17 we already know that for $n \in [0.5, 0.6) \supset L := [0.5, 0.547]$, the limit cycle would be unique, hyperbolic and repeller. By the Bendixson–Poincaré Theorem the above facts force the existence of another limit cycle and so a contradiction. It is straightforward that the existence of this positive invariant region is not compatible with the existence of a polycycle connecting both saddle points.

To construct $\mathcal{R}$ we consider a function $V_2(x, y) = g_0(y) + g_1(y)x + g_2(y)x^2$, with g_0 and g_1 as in (5.24) and g_2 an even polynomial function of degree 12 of the form

$$g_2(y) = 1 + \sum_{k=1}^{6} a_{2k} y^{2k},$$

to be determined. By statement (ii) of Lemma 5.13, the function M_2, given in (5.25), associated to this V_2 and $k = 2/3$ is of the form $M_2 = \phi(y)x + \psi(y)$, where $\phi(y)$ and $\psi(y)$ are polynomials in the variable y whose coefficients depend on the unknowns a_{2k} with $k = 1 \ldots 6$.

We fix a_4 and a_6 in such a way that $\phi(y)$ has a zero at $y = 0$ of multiplicity nine; we get the value of a_8 by imposing that V_2 vanishes at the two saddle points; the values of a_2 and a_{10} are chosen so that the curve $V_2 = 0$ is tangent to both separatrices at the saddle points of the system. Finally, after experimenting with several values for a_{12} and m, so that the region with boundary $\{V_2 = 0\}$ is positively invariant, we fix $a_{12} = -157(10m - 3)(35m + 3)/44550000$.

The region $\mathcal{R}$ will be the bounded connected component of $\mathbb{R}^2 \setminus \{V_2 = 0\}$ containing the origin, see Figure 5.4 (a).

We need to prove that the curve $\{V_2 = 0\} \cap \mathcal{S}$ (see Figure 5.4 (b)) is such that the vector field X points inwards on all its points. We introduce the new parameter $m = n^2$ and we compute $\dot{V}_2 = \langle \nabla V_2, X \rangle$ and

$$\operatorname{Res}(V_2, \dot{V}_2, x) = \frac{y^8(ny^2 - 1)^4 (P_{n,12}(y))^3 P_{n,36}(y)}{n^{28}(120n^3 + 113n^2 - 3)^6}, \tag{5.31}$$

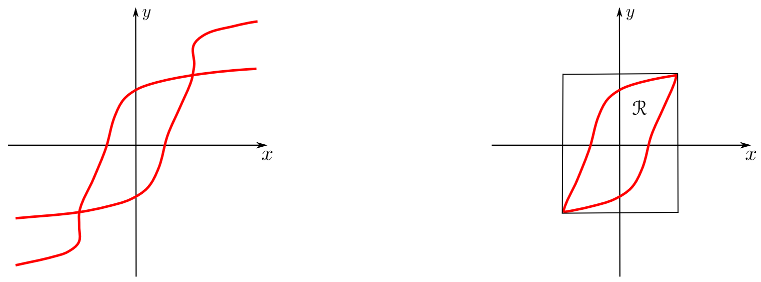


Figure 5.4: Positively invariant region $\mathcal{R}$ with boundary $\{V_2 = 0\}$.

where $P_{n,12}(y)$ and $P_{n,36}(y)$ are polynomials of degree 12 and 36, respectively, and whose coefficients are polynomial functions in the variable n.

Notice that since $m \in [0.5, 0.547]$ then $n \in T := [0.707, 0.7396]$. Since the denominator of (5.31) is positive, we only need to study its numerator.

Using once more Lemma 5.10 and the same tools as in the previous sections we prove that $P_{n,12}(y) \cdot P_{n,36}(y)$ is positive for all $y \in (-1/n, 1/n)$ and $n \in T$. We omit the details.

Hence, we have proved that the numerator of $\mathrm{Res}(V_2, \dot{V}_2, x)$ is non-negative and it only vanishes on $y = 0$ and $y = \pm n^{-1/2}$. Therefore the sets $\{V_2 = 0\}$ and $\{\dot{V}_2 = 0\}$ only can intersect on $\{y = 0\}$. Indeed, the sets $\{V_2 = 0\} \cap \mathcal{S} \cap \{y = 0\}$ and $\{\dot{V}_2 = 0\} \cap \mathcal{S} \cap \{y = 0\}$ coincide and have two points $(\pm\widehat{x}(n), 0)$ for each $n \in T$. Studying the local Taylor expansions of $V_2(x, y)$ and $\dot{V}_2(x, y)$ at these points we get that the respective curves $V_2(x, y) = 0$ and $\dot{V}_2(x, y) = 0$ have a fourth order contact point on them and, as a consequence, $\dot{V}_2$ does not change sign on $\{V_2 = 0\} \cap \mathcal{S}$, as we wanted to prove. That, on $\{V_2 = 0\}$, the vector field X points in, is a simple verification. Hence the proof follows. □

5.7 Existence of polycycles

This section is devoted to prove that the phase portrait (b) in Figure 5.1 can only appear for finitely many values of m. Notice that this phase portrait is the only one representing a polycycle. As we have already explained, the main difficulty is that we are dealing with a family that is not a SCFRVF. To see that the control of the existence of polycycles for general polynomial 1-parameter families can be a non easy task, we present a simple family for which a polycycle appears at least for two values of the parameter.

Example 5.20. *For $m = 0$ and $m = 1$, the planar systems*

$$\begin{cases} \dot{x} = -2y + (3m-4)x + (4-2m)x^3 + xy^2 - x^5 = P_m(x,y), \\ \dot{y} = (4-m)x + xy^2 - 2mx^3 - x^5 = Q_m(x,y), \qquad m \in \mathbb{R}. \end{cases} \tag{5.32}$$

have a heteroclinic polycycle connecting the saddle points located at $(\pm\sqrt{2-m}, 0)$.

Proof. The above family has been cooked to have explicit algebraic polycycles. Consider the family of algebraic curves $H_m(x,y) = y^2 - (x^2 + m - 2)^2 = 0$ and compute

$$W_m(x,y) = \langle \nabla H_m(x,y), (P_m(x,y), Q_m(x,y)) \rangle.$$

Doing the resultant with respect to x of W_m and H_m we obtain

$$\mathrm{Res}(W_m(x,y), H_m(x,y), x) = m^4(1-m)^4 y^4 R(y,m),$$

where R is a polynomial of degree 4 in both variables, m and y. This implies that for $m = 0$ and $m = 1$ the algebraic curve $H_m(x,y) = 0$ is invariant by the flow of (5.32). These sets coincide with the invariant manifolds of the saddle points $(\pm\sqrt{2-m}, 0)$ and contain the corresponding heteroclinic polycycles. □

We have simulated the phase portraits of (5.32) for several values of m and it seems that no polycycles appear for other values of m. In any case, the example shows the differences between SCFRVF, for which as we have discussed in Subsection 5.3.1, the polycycle usually appears for a single value of the parameter, and families that are not SCFRVF.

Let us continue the study of system (5.1). We denote by $\mathbf{p}_m^{\pm} = (\pm m^{-1/4}, \pm m^{-1/4})$ the two saddle points of the system.

Proposition 5.21. *Let $(0, y^s(m))$ be the first cut of the stable manifold of $\mathbf{p}_m^+$ with the Oy^+-axis. Similarly, let $(0, y^u(m))$ be the first cut of the unstable manifold of $\mathbf{p}_m^-$ with the same axis, see Figure 5.5 (a). Then the function $\delta(m) := y^s(m) - y^u(m)$ is an analytic function.*

Proof. This result is a consequence of the tools introduced in [86]. We only give the key points of that proof.

Fix a value $\widehat{m}$ for which $\delta(m)$ is defined. Simply because the Oy^+ is transversal for the flow, the function δ is well defined in a neighborhood of $\widehat{m}$. It is clear that it suffices to prove that $y^s(m)$ is analytic at $m = \widehat{m}$, because the $y^u(m)$ can be studied similarly. To prove this fact we will write the map $y^s(m)$ as the composition of two analytic maps.

Consider a vertical straight line $\mathcal{L} := \{(x,y) \,:\, x = \widehat{m}^{-1/4} - \varepsilon\}$, for $\varepsilon > 0$ small enough. Denote by $(\widehat{m}^{-1/4} - \varepsilon, y_1(m))$ the first cutting point of the stable manifold

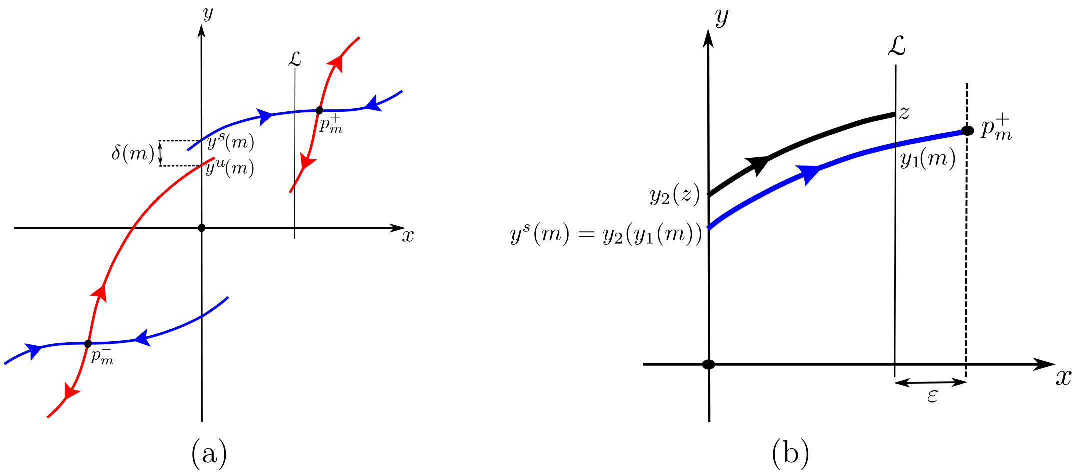


Figure 5.5: Definition of the maps $\delta(m)$ and $y^s(m)$ in Proposition 5.21.

of $\mathbf{p}_m^+$ with this line. Because $\mathcal{L}$ is close enough to the saddle point it can be seen that the local stable manifold cuts this line transversally. Moreover, the tools given in [86] prove that $y_1(m)$ is analytic at $m = \widehat{m}$, because of the hyperbolicity of the saddle point. Next, consider the orbit starting on $\mathcal{L}$ with y-coordinate $y_1(\widehat{m})$. In backward time, this orbit cuts also transversally the Oy^+-axis at the point with y-coordinate $y^s(\widehat{m})$ and needs a finite time to arrive to this point see Figure 5.5 (b). Because of the transversality to both lines, and the finiteness of the time needed for going from one to the other, it is clear that the map $y_2(z)$ induced by the flow of the system between $\mathcal{L}$ and the Oy^+-axis is analytic at $z = y_1(\widehat{m})$. Since $y^s(m) = y_2(y_1(m))$, the result follows. □

Proof of (iii) of Theorem 5.1. Notice that each value of m that is a zero of the map $\delta(m)$, introduced in Proposition 5.21, corresponds to a system (5.1) with a polycycle, i.e. $\mathcal{M} = \{m \in (0.547, 0.6) : \delta(m) = 0\}$. From Proposition 5.19 we know that $\delta(0.547) > 0$ and from Proposition 5.15 that $\delta(0.6) < 0$. Hence the set $\mathcal{M}$ is non-empty. Finally, because of the non-accumulation property of the zeros of analytic functions, the finiteness of $\mathcal{M}$ follows. □

Proof of Theorem 5.1

The proof of Theorem 5.1 simply consists in combining the corresponding results proved in the chapter. More concretely:

- The non-existence of limit cycles and polycycles when $m \in (-\infty, 0.547] \cup [3/5, \infty)$ is given in the following results:

  - For $m \in (-\infty, 0]$, trivially in the introduction.
  - For $m \in (0, 9/25]$ in Proposition 5.12,
  - For $m \in (9/25, 1/2)$ in Proposition 5.18,
  - For $m \in [1/2, 0.547]$ in Proposition 5.19,
  - For $m \in [3/5, \infty)$ in Proposition 5.15.

- The existence of at most one limit cycle and one polycycle when $m \in [1/2, 3/5)$, the fact that they never coexist, and the hyperbolicity and instability of the limit cycle, in Proposition 5.17.

- The phase portraits of the system in the Poincaré disc and the study of the origin, in Subsection 5.3.2 and Section 5.2, respectively.

- The proof of the existence of the phase portrait (b) in Figure 5.1, only for finitely many values of m, in Section 5.7.

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