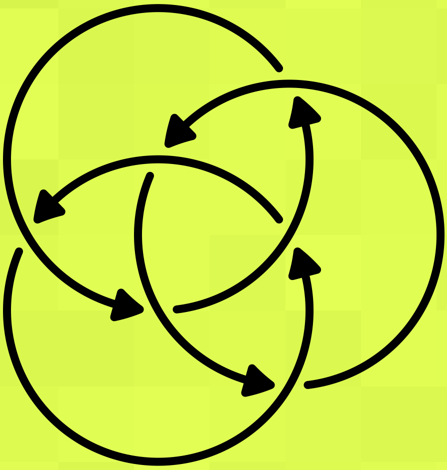

Cosmological perturbations including matter loops

A study in de Sitter

Universitat de Barcelona

Markus Benjamin Fröb

Programa de doctorat en Física

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A study in de Sitter

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PhD Dissertation

Grup de Gravitació i Cosmologia

Departament de Física Fonamental

Facultat de Física

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Barcelona, October 2013

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Typeset in 10pt Bitstream Charter, using pdf-ε-LaTeX with KOMA-Script and the packages *scrpage2*, *mathdesign*, *amsmath*, *natbib*, *tabu*, *booktabs*, *tocstyle*, *fancyvrb* and *microtype*.

Date of the PhD defense:

Defense committee members

President

Vocal

Secretary

Acknowledgements

At the end of a long journey full of heights and depths, I would like to give out my thanks to all who have accompanied and supported me.

First of all, my supervisor, Enric Verdaguer, accepting me as a PhD student after a ten-minute discussion in bumpy Spanish without knowing me, not giving up on me in the first months where it seemed that nothing worked out, and guiding me through all this time. Every time I had a question, the door of your office was open for me, and yet you gave me all the freedom and time I needed to explore this fascinating subject of physics. I couldn't have asked for a better supervisor. Gràcies, Enric.

A very special thanks also goes to my co-supervisor, Albert Roura. Even though our interaction was short, it was intense just as in high-energy physics, and I have learned a lot from your experience and thoughts on many aspects of physics in long and profound discussions. Gràcies, Albert.

Likewise, I would like to express my gratitude to Atsushi Higuchi who made my stay in York a very pleasant one. Working and discussing with you was very relaxed yet fruitful, and I hope we can continue this collaboration. ありがとうございます

To all the other people in the Gravitation and Cosmology group at the University of Barcelona, Jaume Garriga, Yuko Urakawa, Roberto Emparan, Kentaro Tanabe, Bartomeu Fiol and David Mateos, I want to say thanks as well. Thank you for lively discussions, for interesting seminars and for creating a good working athmosphere.

The exactly same thing goes for the people in the Mathematical Physics group at the University of York, Chris Fewster, Bernard Kay, Henning Bostelmann, Niall MacKay, Simon Eveson, Martina Balagovic and Eli Hawkins. Thank you.

Furthermore I like to thank Ian Morrison, whose work on Mellin-Barnes transformations in de Sitter space introduced me to this interesting subject, with whom I enjoyed stimulating discussions and who made it possible for me to come to Cambridge. We will surely meet again, I'm looking forward to it.

Let me thank also the many friends I have made in Barcelona and York during this time, people who have been with me and with whom I shared many unforgettable moments. Ivan, Tamara, for everything. Dani, Luis, for crazy nights and lots of discussions. Outi, Daniela, for the dances. Blai, Adriana, too many things to list. Marina, Carmen, best girls in town. Vanessa, Ellyn, you know why. Rafa, Manolo, nossa! Lourdes, Natalia, pel català. Aya, Sri, Karaoke anyone? Hans, Umberto, for general awesomeness.

I also must not forget all the other people around that have formed part of my experience, have helped me and had fun: my current and former office mates Martin, Nico, Nidal, María; other office's mates Alejandro, Genís, Paolo, Blai P, Paco, Albert, Aldo, Pol, Kolja; Barcelona people Núria, Pilar, Giuliana, Dani Curu, Blanca, Elsa, Tina, Oriol, Patrizia, Pau, Patricia, Savitri, Hicham, Valentino, Andrea, Carlos, Ilze; York folks John, Benjamin, James, Spiros, Andrew, Johnny, Horacio, Anna, Oyin, Jenny, Harold, Carla, Wang, Aurélien, Gus, Molly, Tara, Miral, Angela, Gabriela, Ami, Farina, Dani, and generally great guys Helvi, Wilke and Sebastian. Thank y'all.

Last but not least, a big hearty hug goes out to my parents Dirk and Katharina, my sister Elisabeth, my grandmothers Birgit and Waltraud and the rest of my family for their continued support, understanding and love, and without whom this thesis would be quite literally nonexistent. Ihr seid die Besten.

Financial support through an APIF scholarship from the Universitat de Barcelona and a FPU scholarship no. AP2010-5453 from the Spanish Ministerio de Educación, Cultura y Deporte is gratefully acknowledged, as well as partial financial support by the Research Projects MCI FPA2007-66665-C02-02, FPA2010-20807-C02-02, CPAN CSD2007-00042, within the program Consolider-Ingenio 2010, and AGAUR 2009-SGR-00168.

Contents

Conclusions

1 Introduction

Our current theoretical description of nature is divided into two categories: the Standard Model of particle physics which unifies electromagnetism, weak and strong interactions and the General Theory of Relativity which describes gravitation. Both theories have been extremely well tested at a wide range of energy and length scales, the Standard Model in high-energy collision experiments such as the Large Hadron Collider in Geneva and General Relativity in astrophysical observations in the solar system and beyond, among others.

However, General Relativity is a classical theory and must fail to provide the correct description at sufficiently high-energy scales, comparable to the Planck scale, where a putative theory of Quantum Gravity takes over. Since such energy scales are beyond our experimental reach, work on such theories is purely theoretical and can only be judged by internal consistency and simplicity. The only possible perspective on experimental evidence for a theory of quantum gravity (except maybe for analogue models of gravity [1]) comes from observations of the cosmic microwave background (CMB), which were recently vastly improved upon by the Planck satellite [2–4] and which allow us to catch a glimpse of the very early history of our universe where quantum gravity effects played a role. Remnants of interactions of that time that are still observable today therefore can give us clues about physics at those scales. According to the standard ΛCDM model of cosmology, the universe underwent a period of rapid expansion in its early history, known as inflation [5–8], and which can be modeled by a part of de Sitter spacetime up to small corrections (so-called slow-roll corrections). In this case, the driving factor is a homogeneous scalar field, the inflaton, with a large potential term which changes very slowly during the inflationary period, and can thus be modeled effectively by a constant. Furthermore, there is a real cosmological constant which is however very small, and which is responsible for the current accelerated expansion of the cosmos [4, 9]. Models which replace also this constant with some dynamical term are known under the generic term dark energy (see [10, 11] for reviews), however, to this date there are no observational indications which favor anything different from a constant. A puzzling feature of the CMB, namely its almost perfect homogeneity over regions which have never been in causal contact, can be explained by such an inflationary period with exponential expansion, i.e., faster than the speed of light. Tiny variations of the temperature distribution are thought to be sourced by quantum fluctuations of the metric through which the photons of the CMB propagate, with most of the effect coming from fluctuations which were generated during inflation [8, 12].

Nonetheless, Planck scales are not reached during the inflationary period, and in such an intermediate regime quantum gravity can be studied perturbatively in the manner of effective field theories (EFT). In these theories, we parametrize our ignorance of the true high-energy physics by some effective interactions, including all possible terms whose form is compatible with the assumed symmetries of the system under consideration. A prominent example is the Fermi theory of four-fermion interaction, where the exchange of a W boson between two fermions at energy scales well below the mass of the W boson is described by an effective vertex where four fermions interact, with a coupling constant inversely proportional to the squared mass of the W [13]. This theory gives correct predictions at all energies which are well below the scale set by the mass of the W boson. It is not renormalizable, and to calculate loop effects one would have to include operators of higher mass dimension, i.e., six- and more-fermion vertices with coupling constants which must be determined by experiment. However, these higher operators are suppressed by increasingly higher powers of the W boson mass, and at low energy scales do not make appreciable contributions. The situation is analogous in the case of perturbative quantum gravity. At large distance and low curvature (as compared to the Planck scale), we can add to the Einstein-Hilbert action for gravity higher-dimensional operators, which are higher powers of curvature tensors and their covariant derivatives. These operators are suppressed by powers of the Planck mass, and so for low-energy processes give only small corrections that can be trusted at those scales.

In this thesis, we will investigate two effects that can be studied in this scenario: the stability of de Sitter spacetime under small metric perturbations, and their correlation functions which can be related to cosmological observables.

The standard theoretical analysis of perturbations in primordial cosmology relies on linear perturbation theory, which in the quantum theory amounts to a tree level calculation. However, the effects due to loop corrections can be potentially significant if, by some novel physical process in curved backgrounds, the usual suppression of these corrections can be overcome. Already some time ago, Tsamis and Woodard proposed that radiative corrections due to loops of gravitons could lead to a screening of the cosmological constant [14–16], which then could potentially serve as a mechanism to end inflation, independently of an inflaton field. In this case, the fundamental cosmological constant would be only a few orders of magnitude smaller than the Planck scale, and the value that is observed today is the screened one. The mechanism by which this works relies on the continuous excitation of graviton modes during the inflationary period which are not diluted by the exponential expansion, which then – due to the nonlinear nature of gravitational interaction – leads to a slowing down of this expansion. While this claim is very attractive from a physical point of view, its validity and the interpretation of concrete calculations have been doubted [17, 18], and it is still an unsettled issue whether it may work at all. A similar effect had been proposed to appear for the massless, minimally coupled scalar field [19], but has since been shown to be an artifact of perturbation theory [20, 21].

In addition, because of the absence of a global time-like Killing vector in de Sitter space, there exists the possibility that even massive theories on a fixed de Sitter background may have instabilities. This idea has been analyzed by a variety of authors both at tree level [22–27] and including loop corrections [28–30]. In this context, the continuation to an Euclidean formalism (i.e., calculations on the sphere) has been proven to be very useful. The Euclidean vacuum state and correlation functions in this state, which are defined by an appropriate analytic continuation from the sphere to de Sitter space, have a number of attractive properties, which include infrared finiteness and full de Sitter invariance. For massive scalar fields, calculations done on the sphere and using the in-in formalism in de Sitter space have been proven to be fully equivalent by Higuchi, Marolf and Morrison [31], and generalizations to very light and massless fields have also been developed [20, 21]. Furthermore, the Euclidean vacuum constitutes a late-time attractor for generic initial states, meaning that correlation functions in other states approach the Euclidean ones for late times in a precisely defined sense [32–34].

These findings for interacting matter fields (for sufficiently weak coupling such that perturbation theory is applicable) extend classical results about the late-time attractor character of de Sitter spacetime both for linear perturbations and the nonlinear case [35–40], which is known as "no-hair" property similar to the case of black holes. However, considering test fields on a fixed background only gives part of the full answer, and for the complete problem one has to take the backreaction of the quantum fields on the spacetime geometry into account. Studies of this type have been done in semiclassical gravity, where the metric is still considered classical, but the quantum nature of matter is already taken into account, and similar attractor properties have been found [41]. Instead of a classical stress tensor, on the right-hand side of the Einstein equation an expectation value of a quantum stress tensor operator appears [42, 43]. However, it is important as well to also take the quantum nature of the metric into account: studying gravity as an effective field theory, this amounts to a quantization of metric perturbations around a fixed background.

In the first half of this thesis, we will thus extend these considerations to the gravitational case, studying the stability of de Sitter spacetime under small metric perturbations interacting with matter fields. Partial studies in this direction have been done. For example, flat space has been shown to be stable under linearized perturbations interacting with matter fields [44, 45], in de Sitter space a vanishing correction to classical modes has been found for the interaction of tensor modes with massless, minimally coupled free scalar fields [46], and also the stability of de Sitter spacetime for spatially isotropic perturbations was established [47, 48]. Extending these calculations, here we consider the stability under general linearized metric perturbations (of scalar, vector and tensor type) due to the interaction with conformal fields, and give arguments on why our conclusions should smoothly extend to other kinds of matter. In this context, we employ the so-called order reduction method, which in contrast to a strictly perturbative treatment provides solutions which are reliable over a long time, a feature which is obviously crucial to study stability questions. This method not only eliminates spurious solutions which lie outside the validity of the effective field theory approach that we are pursuing, but also has the advantage to generate backreaction equations which are relatively simple to solve exactly, and thus eliminating the need to use approximations beyond the ones imposed by the EFT ansatz and the consideration of linear perturbations.

The second half of the thesis concerns correlation functions of these metric perturbations. This goes one step further than the previous objective, where the equations that govern the evolution of metric perturbations are the same as the ones that apply to their expectation value. It is of course important to take fluctuations around the mean value into account. Their relative size in relation to the mean value can be used as a criterion for the validity of the mean field description [49–51], and they are related to cosmological observables such as the tensor power spectrum at the end of inflation [8, 12]. For the case of the interaction with conformal fields, we calculate explicitly the two-point function of scalar, vector and tensor perturbations, using a generalization of the flat space $\mathrm{i}\epsilon$ prescription which allows us to define a proper interacting vacuum state at past infinity. From this two-point function, we calculate the power spectrum for tensorial perturbations, which can be observed through the spectrum of the temperature fluctuations in the CMB.

However, while the gauge fixing that is usually used in cosmology (and that we use in this thesis) for the scalar–vector–tensor decomposition is complete for perturbations that fall off at spatial infinity, it is not local since one needs to specify boundary conditions. Objects constructed from correlation functions in this gauge are therefore not observables in the strict sense of the word, since we only have observational access to a finite part of the universe. Therefore, one has to search for "sufficiently local" observables which characterize the geometrical properties in regions of finite physical size. This last requirement has been crucial in the construction of so-called infrared-safe observables, which give finite results in situations that would lead to divergences without an explicit infrared cutoff [18, 52, 53]. Furthermore, this gauge fixing which is done in the conformally flat coordinate patch does not respect the symmetries of the underlying de Sitter background, and hence obscures such symmetries in the correlation functions. It is of course difficult to find a gauge-invariant and local observable, even in perturbative quantum gravity [54], but if we exclude graviton loops and restrict to a de Sitter (or other maximally symmetric) background, the linearized Riemann tensor provides such an observable. We thus calculate its two-point function, starting from the two-point function of the metric perturbations for the case where they interact with conformal fields. In order to better understand the structure of the result, it is decomposed in Weyl and Ricci tensor and scalar correlation functions. We then make use of the Bianchi identities to bypass the long calculation, showing that the Riemann tensor two-point function can be obtained directly from the two-point function of the stress tensor.

This thesis consists of four parts: Preliminaries, Metric perturbations, Riemann tensor, and Conclusions. In the first part, after fixing our conventions in chapter 2, in chapter 3 we explain how to calculate the effective action, which incorporates quantum corrections to the metric perturbations due to matter fields, and from which these perturbations can be calculated. This exposition is completely general. In chapter 4, we specialize to de Sitter space and calculate the necessary matter expectation values which appear in the effective action, paying special attention to renormalization. In the second part, chapter 5 is dedicated to the first objective, the study of the stability of de Sitter space under these perturbations, and in chapter 6 we calculate their two-point function as well as the tensor power spectrum. The third part, chapter 7, gives the two-point function of the Riemann tensor as a proper local observable, and explains a procedure by which it can be obtained directly from the two-point function of the matter stress tensor. We conclude the thesis in the last part with a discussion of the obtained results, some technical appendices and a summary in Spanish.

Some results of this thesis have already been published. The relevant publications are:

- M. B. Fröb, A. Roura, and E. Verdaguer, *One-loop gravitational wave spectrum in de Sitter spacetime*, *JCAP* **1208** (2012) 009, [`arXiv:1205.3097`]
- M. B. Fröb, D. B. Papadopoulos, A. Roura, and E. Verdaguer, *Nonperturbative semiclassical stability of de Sitter spacetime for small metric deviations*, *Phys. Rev. D* **87** (2013) 064019, [`arXiv:1301.5261`]
- M. B. Fröb and A. Higuchi, *Mode-sum construction of the two-point functions for the Stueckelberg vector fields in the Poincaré patch of de Sitter space*, `arXiv:1305.3421`
- M. B. Fröb, *Fully renormalized stress tensor correlator in flat space*, *Phys. Rev. D* **88** (2013) 045011, [`arXiv:1305.0217`]

Preliminaries

Before turning to those moral and mental aspects of the matter which present the greatest difficulties, let the enquirer begin by mastering more elementary problems.

— Sir Arthur Conan Doyle, A study in scarlet

Its form of value is simple and collective, thus general.
(Ihre Wertform ist einfach und gemeinschaftlich, daher allgemein.)

— Karl Marx, Capital

2 Notations and Conventions

In this thesis, we work with natural units such that $c = \hbar = 1$. Furthermore, we define $\kappa^2 = 16\pi G_\mathrm{N}$ with Newton's constant G_N. Our sign conventions are a mostly plus metric, $R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \ldots$ and $R_{bd} = R^a{}_{bad}$, i.e., the $+++$ convention in the list of [59]. Depending on the context, Latin indices can be understood as abtract indices or as ranging over space and time, while Greek indices always refer to spatial coordinates only. Spatial vectors are also written in boldface without indices. In the first half of the thesis, we work in n dimensions, and after renormalization switch to $n = 4$ in the second half.

We do not differentiate in notation between a function and its Fourier transform, since the arguments always make it clear which one is meant. The Fourier transform from coordinate to momentum space is defined by $f(p) = \int f(x)\,\mathrm{e}^{-\mathrm{i}px}\,\mathrm{d}^n x$, while the reverse transform changes the sign in the exponential and has an additional factor $(2\pi)^{-n}$. Since we use the Minkowski metric η_{ab} to define the scalar product px, for the time components the signs are reversed.

Many different quantum states are defined in this thesis; the state $|\mathrm{in}\rangle$ is the interacting vacuum, the state $|0\rangle$ is the free-field (Bunch-Davies) vacuum state. $\langle\rangle_\phi$ (defined by (3.14)) denotes an expectation value with respect to a matter action of fields ϕ (which can be free or interacting), $\langle\rangle_h$ (defined by (6.4)) denotes an expectation value with respect to the effective action (3.15) for metric perturbations, and $\langle\rangle$ denotes a connected correlation function, with the state in which this function is evaluated ($|\mathrm{in}\rangle$ or $|0\rangle$) being clear from the context.

The background de Sitter metric is g_{ab}, while the perturbed metric is denoted by $\tilde{g}_{ab} = g_{ab} + \kappa h_{ab}$ with the perturbation h_{ab}. Objects with a tilde refer to this perturbed metric, while objects without tilde refer to the background metric. We will denote successive perturbative orders of a perturbative expansion in h_{ab} with a superscript, e.g., $\tilde{g}^{(0)}_{ab} = g_{ab}$ and $\tilde{g}^{(1)}_{ab} = h_{ab}$.

3 The effective action

In this section, we explain how to calculate an effective action which only depends on the metric perturbations but incorporates quantum corrections due to their interaction with matter fields. The presentation is generic, and its specialization to a background de Sitter spacetime will be given in the next section.

3.1. The *in-in* formalism and the iϵ prescription

In the quantum field theoretical treatment of scattering problems one usually calculates the transition matrix element of a Hermitean operator $\hat{A}$ between two states,

$$\langle\alpha|\hat{A}|\beta\rangle\,, \tag{3.1}$$

where $|\alpha\rangle$ and $|\beta\rangle$ can be taken to be two different *in* and *out* vacuum states, with the particle content of the real states incorporated into the operator $\hat{A}$. For time-ordered matrix elements between these vacuum states, the path integral representation

$$\langle\mathrm{out}|\mathcal{T}\hat{A}|\mathrm{in}\rangle = \frac{\int A[\phi]\,\mathrm{e}^{\mathrm{i}S[\phi]}\,\mathcal{D}\phi}{\int \mathrm{e}^{\mathrm{i}S[\phi]}\,\mathcal{D}\phi}\,. \tag{3.2}$$

with the action S depending on the fields ϕ is well known.

To implement the standard flat-space choice of vacuum in the path integral (3.2), one slightly tilts the time integration contour on the complex plane to include an imaginary part

$$t \to t(1-\mathrm{i}\epsilon) \tag{3.3}$$

with $\epsilon > 0$ (see for instance section 4.2 in [60]). In the case of a time-independent Hamiltonian, which is the usual situation (e.g., in the Standard Model of particle physics), this prescription selects the asymptotic vacuum as the state of lowest energy of the full interacting theory, which includes appropriate correlations at the initial time between the different fields or even different modes of the same field. If the Hamiltonian is time-dependent, this prescription still selects an adiabatic vacuum of the theory at early times. The standard procedure is then to calculate the integral from some initial time t_0 to some final time T and to take the limits $t_0 \to -\infty$, $T \to \infty$ in a slightly imaginary direction. One may even rotate the time axis further onto the imaginary axis, which gives rise to *Euclidean* quantum field theory.

However, in a cosmological setting one is not interested in transition matrix elements but rather in true correlation functions, which are expectation values of operators. Moreover, one typically needs to impose initial conditions at early times instead of boundary conditions at both early and late times. Furthermore, as was explicitly shown in [25] for de Sitter space, in an exponentially expanding spacetime *in-out* perturbation theory has an infrared divergence because of the expanding space volume. For all these reasons we are naturally led to consider the *in-in* formalism, where one specifies initial conditions at some initial time (in certain cases, such as the exponentially expanding patch of de Sitter with spatially flat sections, one can specify these initial conditions at past infinity and define an asymptotic *in* vacuum).

We now want to derive an analogue formula to (3.2) for this case. We thus insert the identity operator as a sum over an orthonormal basis of field-configuration eigenvectors $|\alpha, T\rangle$ in the Heisenberg picture at some final time T, i.e. $\hat{\phi}(T)|\alpha, T\rangle = \alpha|\alpha, T\rangle$, to obtain

$$\langle \mathrm{in}|(\mathcal{T}^{-1}\hat{A})(\mathcal{T}\hat{B})|\mathrm{in}\rangle = \sum_\alpha \langle \mathrm{in}|\mathcal{T}^{-1}\hat{A}|\alpha, T\rangle\langle\alpha, T|\mathcal{T}\hat{B}|\mathrm{in}\rangle = \sum_\alpha \big(\langle\alpha, T|\mathcal{T}\hat{A}|\mathrm{in}\rangle\big)^* \langle\alpha, T|\hat{A}|\mathrm{in}\rangle$$
$$\sim \int A[\phi^-]B[\phi^+]\delta\big[\phi^+(T)-\alpha\big]\delta\big[\phi^-(T)-\alpha\big]\,\mathrm{e}^{\mathrm{i}S[\phi^+]}\,\mathrm{e}^{-\mathrm{i}S[\phi^-]}\,\mathcal{D}\phi^+\,\mathcal{D}\phi^-\,\mathcal{D}\alpha\,. \tag{3.4}$$

Since we have two path integrals for each degree of freedom, we need two copies of the fields which we have labeled ϕ^+ and ϕ^-. Instead of enforcing the separate equality of both fields to α and then integrating over all field configurations α at time T, in the following we can directly enforce the equality of the fields ϕ^+ and ϕ^- at that time. Rather than considering two separate fields, this suggests yet another way, namely modifying the time integration contour (which in the usual case goes from $-\infty$ to $+\infty$) to run from $-\infty$ to T and turning back to $-\infty$ [61]. In the second half of this contour, time runs backwards which provides for the minus sign in front of $S[\phi^-]$ in (3.4). Thus, the *in-in* formalism is also known as the *closed-time-path* (CTP) formalism, and one in general obtains *path-ordered* (denoted by $\mathcal{P}$) correlation functions, with fields lying on the backward contour ordered to the left of fields lying on the forward contour (see figure 3.1). This contour being implicitly understood, the formula connecting expectation values and path integrals is almost identical to the *in-out* result (3.2)

$$\langle \mathrm{in}|\mathcal{P}\hat{A}|\mathrm{in}\rangle = \frac{\int A[\phi]\,\mathrm{e}^{\mathrm{i}S[\phi]}\,\mathcal{D}\phi}{\int \mathrm{e}^{\mathrm{i}S[\phi]}\,\mathcal{D}\phi}\,. \tag{3.5}$$

The iϵ prescription can also be carried over to the CTP formalism. However, due to the complex conjugation of the integrand in the path integral for the ϕ^- fields, we need to take the complex conjugate prescription for them

$$t \to t(1+\mathrm{i}\epsilon)\,. \tag{3.6}$$

One therefore has to integrate along a contour going from t_0^+ to T, returning to t_0^- with $t_0^- = (t_0^+)^*$, and taking at the end of the calculation $t_0^+ \to -\infty(1-\mathrm{i}\epsilon)$. The dependence on T disappears in the final result as long as it is larger than all times of interest (i.e., all the times in the arguments of the correlation function one wants to calculate), as required by causality. The deformed contour is also shown in figure 3.1.

As explained before, in the case of a time-dependent Hamiltonian this prescription is suitable for selecting an adiabatic vacuum of the interacting theory as $t \to -\infty$ as asymptotic initial state. It is clear that it only can work when the behavior of the modes for free fields is dominated at past infinity by the same kind of oscillatory behavior as in Minkowski space (factors with a power-law or weaker time dependence are allowed). This is the case for the exponentially expanding patch of de Sitter spacetime with spatially flat sections, but would not be appropriate

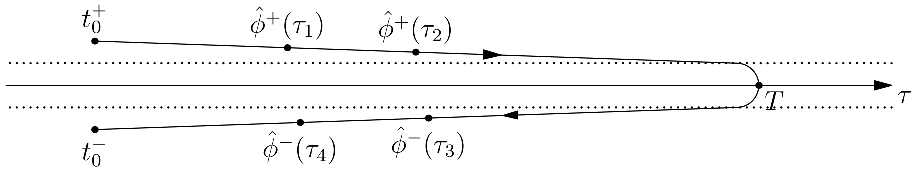

Figure 3.1.: The deformed CTP integration contour, together with 4 path-ordered fields. For any permutation of the indices we have $\mathcal{P}\hat{\phi}^{\pm}(\tau_a)\hat{\phi}^{\pm}(\tau_b)\hat{\phi}^{\pm}(\tau_c)\hat{\phi}^{\pm}(\tau_d) = \hat{\phi}^{-}(\tau_4)\hat{\phi}^{-}(\tau_3)\hat{\phi}^{+}(\tau_2)\hat{\phi}^{+}(\tau_1)$. Here, $\hat{\phi}^{+}(\tau_a)$ and $\hat{\phi}^{-}(\tau_a)$ simply represent the operator $\hat{\phi}$ evaluated at a time lying, respectively, on the upper and lower branch of the deformed complex contour for time integration, and correspond to the fields ϕ^+ and ϕ^- respectively in the path integrals.

for global de Sitter, which is instead exponentially contracting coming from past infinity. In this case, the proper definition of asymptotic initial states is much more involved [62].

For further details about the CTP formalism see for instance [61, 63–67]. In the following, we will mostly exhibit general considerations using the CTP contour, splitting it only for the calculation of explicit examples. However, switching between the two formulations is very easy: starting with the expression for the closed contour, each field obtains an extra capital Latin index A which can take the values $+$ and $-$, and every time integration becomes two integrations, with an additional minus sign for the second one which runs over the $-$ branch of the contour.

3.2. Calculating the effective action

We will now take the full metric to be composed of a background and perturbations

$$\tilde{g}_{ab} = g_{ab} + \kappa h_{ab} \,. \tag{3.7}$$

As an action for gravity, we consider the standard Einstein-Hilbert action with a cosmological constant Λ

$$S_{\mathrm{G}}[h] = \frac{1}{\kappa^2} \int (\tilde{R} - 2\Lambda)\sqrt{-\tilde{g}}\, \mathrm{d}^n x \,, \tag{3.8}$$

some matter action $S_{\mathrm{M}}[h, \phi]$ (with ϕ being a general field, not necessary scalar) and counterterms needed for renormalization which in our case are quadratic in the curvature tensors

$$S_{\mathrm{Q}}[h] = \frac{2}{3}\alpha \int \left(\tilde{R}^{abcd}\tilde{R}_{abcd} - \tilde{R}^{ab}\tilde{R}_{ab}\right)\sqrt{-\tilde{g}}\, \mathrm{d}^n x + \beta \int \tilde{R}^2 \sqrt{-\tilde{g}}\, \mathrm{d}^n x \,. \tag{3.9}$$

In the spirit of effective field theories [68–70], these are all the local and covariant terms of mass dimension four, which is the next term in a low-curvature expansion of some unknown high-energy theory (other terms of dimension four such as $\Box R$ are total derivatives and do not contribute to the bulk action). Furthermore, in four dimensions the Gauß-Bonnet theorem tells us that the integral of the Euler density

$$E_4 = R^{abcd}R_{abcd} - 4R^{ab}R_{ab} + R^2 \tag{3.10}$$

is a topological invariant and hence does not contribute to the correlation functions of h_{ab} in which we are interested. It is therefore sufficient to consider only two counterterms with coefficients α and β, and we choose the above for simplicity. Also in four dimensions, the first term in (3.9) reduces to the square of the Weyl tensor plus a multiple of the Euler density, as one can see from the expansions of appendix A.

In the following, we only want to consider quantum corrections due to loops of the matter fields appearing in S_{M}. This can be formally implemented in a natural way by considering a large N expansion for N matter fields interacting with the gravitational field [71]. One then rescales the gravitational constant $1/\kappa^2 \to N/\bar{\kappa}^2$ and takes N identical fields (this is equivalent to $S_{\mathrm{M}} \to N S_{\mathrm{M}}$) and takes the limit $N \to \infty$ with $\bar{\kappa}$ held fixed. In this limit, the path integrals in (3.5) can be evaluated by the saddle-point method, and graviton loops are suppressed by factors of $1/N$. This can be seen clearly from the expansion of the gravitational action in successive powers of the perturbation h_{ab}, and hence we do not need to perform the $1/N$-expansion explicitly but can simply truncate the expansion in h_{ab} after the quadratic term. For the matter action, this expansion can be parametrized as

$$S_{\mathrm{M}}[h,\phi] = S_{\mathrm{M}}^{(0)}[\phi] + \frac{1}{2}\kappa \int h^{ab} T_{ab}[\phi] \sqrt{-g}\,\mathrm{d}^n x + \kappa^2 \int h^{ab} h^{cd} U_{abcd}[\phi] \sqrt{-g}\,\mathrm{d}^n x\,, \tag{3.11}$$

where T^{ab} is the usual stress tensor of the matter.

Since we are not interested in correlation functions of the matter fields, we can integrate them out. Formula (3.5) can then be written in the form (we recall that the time integration runs over the CTP contour 3.1)

$$\langle \mathrm{in}|\mathcal{P}\hat{A}[h]|\mathrm{in}\rangle = \frac{\int A[h]\, \mathrm{e}^{\mathrm{i}S_{\mathrm{eff}}[h]}\, \mathcal{D}h}{\int \mathrm{e}^{\mathrm{i}S_{\mathrm{eff}}[h]}\, \mathcal{D}h} \tag{3.12}$$

with an effective action S_{eff}, which only depends on the metric perturbation h_{ab}, defined by

$$\mathrm{e}^{\mathrm{i}S_{\mathrm{eff}}[h]} = \int \mathrm{e}^{\mathrm{i}S[h,\phi]}\, \mathcal{D}\phi = \mathrm{e}^{\mathrm{i}(S_{\mathrm{G}}[h]+S_{\mathrm{Q}}[h])} \int \mathrm{e}^{\mathrm{i}S_{\mathrm{M}}[h,\phi]}\, \mathcal{D}\phi\,. \tag{3.13}$$

If we define the expectation value $\langle\rangle_\phi$ by

$$\langle \mathcal{P}A[\phi]\rangle_\phi = \frac{\int A[\phi]\, \mathrm{e}^{\mathrm{i}S_{\mathrm{M}}^{(0)}[\phi]}\, \mathcal{D}\phi}{\int \mathrm{e}^{\mathrm{i}S_{\mathrm{M}}^{(0)}[\phi]}\, \mathcal{D}\phi}\,, \tag{3.14}$$

we obtain up to terms quadratic in h_{ab}

$$\begin{aligned} S_{\mathrm{eff}} = S_{\mathrm{G}}[h] + S_{\mathrm{Q}}[h] + \frac{1}{2}\kappa \int h^{ab} \langle T_{ab}\rangle_\phi \sqrt{-g}\,\mathrm{d}^n x + \kappa^2 \int h^{ab} h^{cd} \langle U_{abcd}\rangle_\phi \sqrt{-g}\,\mathrm{d}^n x \\ + \frac{\mathrm{i}}{8}\kappa^2 \iint \left[\left\langle \mathcal{P} T_{ab}(x) T_{c'd'}(x')\right\rangle_\phi - \langle T_{ab}(x)\rangle_\phi \left\langle T_{c'd'}(x')\right\rangle_\phi \right] \\ \times h^{ab}(x) h^{c'd'}(x') \sqrt{-g}\,\mathrm{d}^n x \sqrt{-g}\,\mathrm{d}^n x'\,. \end{aligned} \tag{3.15}$$

We see that S_{eff} really deserves its name, including corrections of higher order in κ to the gravitational action due to the interaction with matter. The earliest example of such effective actions is the Euler-Heisenberg Lagrangian [72] which includes corrections to the Maxwell Lagrangian due to vacuum polarization effects. However, note that this is a different effective action from the one obtained by a Legendre transformation.

As explained before, one has to consider an appropriate initial state which in general includes correlations between the metric perturbations and the matter fields. In this light, the factorization between matter and metric perturbation degrees of freedom implied by the definition of the effective action (3.13) does not seem to be valid. However, if we employ the iϵ prescription and adopt the above factorization, the interacting *in* vacuum state that is selected by this prescription will in some sense contain the minimally necessary correlations. Furthermore, it has been shown (see [73] and references therein) that other initial states (described by an arbitrary density matrix) can be incorporated into this approach if one adds additional terms to the effective action which are supported at the initial time (i.e., are proportional to $\delta(t-t_0)$).

For the calculation of this effective action, we need the specified expectation values, a calculation that will be done in the next section. For free fields, this amounts to a one-loop calculation, while interacting fields can be treated as usually as a power series in the interaction constant. A special case are conformal theories, for which these low-order correlation functions are completely determined by conformal symmetry, even when they are strongly interacting (see, e.g. [74]). In this thesis, we concentrate on specific examples in the free-field case; the generalization to interacting theories should be straightforward.

4 Free quantum fields in de Sitter space

4.1. Facts on de Sitter space

n-dimensional de Sitter space is most easily defined by embedding it into a flat $n+1$-dimensional manifold. Denoting the Cartesian embedding coordinates of a point x by $X^A(x)$, it is the hyperboloid satisfying

$$X^A(x)X_A(x) = H^{-2}\,, \tag{4.1}$$

where H is constant. As a measure of distance between two points, we simply define

$$Z(x,x') = H^2 X^A(x)X_A(x')\,, \tag{4.2}$$

which is related to the Minkowskian distance d between two points in the embedding space as

$$d^2(x,x') = \left(X^A(x) - X^A(x')\right)\left(X_A(x) - X_A(x')\right) = 2H^{-2}\left(1 - Z(x,x')\right)\,. \tag{4.3}$$

Symmetries of de Sitter space are Lorentz transformations of the embedding Minkowski space, since they leave the embedding condition (4.1) unchanged. It is clear that $Z(x,x')$ is also invariant under such transformations, and hence will be denoted invariant distance. The Lorentz group $\mathrm{O}(n,1)$ is therefore contained in the symmetry group of dS_n. Since it is generated by $n(n+1)/2$ generators – Killing vectors –, which is the maximum for a n-dimensional space, de Sitter space is a maximally symmetric space, and its symmetry group is exactly $\mathrm{O}(n,1)$. As for all maximally symmetric spaces, the Riemann tensor can be expressed in terms of the metric as

$$R_{abcd} = k\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)\,, \tag{4.4}$$

and by choosing a specific embedding (such as (4.6)) one calculates $k = H^2$.

When the two points x and x' are connected by a geodesic, we can relate $Z(x,x')$ to the geodesic (proper) distance $\mu(x,x')$ between those points

$$Z(x,x') = \cos(H\mu(x,x'))\,, \tag{4.5}$$

where the distance is to be taken along the shortest geodesic in case there are more than one. From the relation to the Minkowskian distance of the embedding space (4.3) as well as to the de Sitter geodesic distance (4.5), we see immediately that $Z(x,x') = 1$ if the points are null-related, $\mu(x,x') = d(x,x') = 0$. When they are time-like separated, $d^2(x,x') < 0$ and $Z(x,x') > 1$, from which it follows that the geodesic distance becomes imaginary $\mu(x,x') = \mathrm{i}|\mu(x,x')|$; while for space-like separated points we have $d^2(x,x') > 0$ and consequently $Z(x,x') < 1$. In this case, there are points which cannot be connected by a geodesic, namely when $Z(x,x') < -1$.

One special symmetry of O(n,1) deserves further mention, which is coordinate inversion $X^A \to -X^A$. The image $\bar{x}$ of a point x in de Sitter space under this symmetry is called antipodal point, with $Z(x,\bar{x}') = Z(\bar{x},x') = -Z(x,x') = -Z(\bar{x},\bar{x}')$. One sees immediately that if two points x and x' cannot be connected by a geodesic, x and $\bar{x}'$ as well as $\bar{x}$ and x' can be, and in fact, this geodesic is then time-like.

Since the only object which is left invariant by all isometries of flat space is the Minkowski distance d^2, any scalar function of two points (a *biscalar*) which is invariant under all de Sitter isometries must be a function of Z since their symmetry groups are identical as shown above.

Extending in the obvious way the definition of tensor fields, one can now define *bitensors* which depend on two points and have indices referring to either one of them. In total correspondence to the scalar case, a bitensor is said to be maximally symmetric if it is invariant under all de Sitter isometries. Since parallel transport commutes with the action of isometries, covariant derivatives of $Z(x,x')$ are maximally symmetric bitensors. That they form a complete set in the sense that any maximally symmetric bitensor can be expressed as a sum of products of derivatives of Z with coefficients which are invariant, has been shown by Allen and Jacobson [75] (however, not directly for derivatives of Z but for a related set of bitensors – see table 4.1).

Additionally, the set of covariant derivatives of $Z(x,x')$ is finite. Taking the conformally flat coordinate system defined by

$$X^0 = -\frac{\boldsymbol{x}^2 - \eta^2 + H^{-2}}{2\eta}\,, \qquad X^\alpha = -\frac{x^\alpha}{H\eta}\,, \qquad X^4 = \frac{\boldsymbol{x}^2 - \eta^2 - H^{-2}}{2\eta} \tag{4.6}$$

with $-\infty < \eta < 0$, we get for the invariant distance

$$Z(x,x') = \frac{\eta^2 + \eta'^2 - (\boldsymbol{x} - \boldsymbol{x}')^2}{2\eta\eta'} = 1 - \frac{(\boldsymbol{x} - \boldsymbol{x}')^2 - (\eta - \eta')^2}{2\eta\eta'}\,. \tag{4.7}$$

The induced metric is given by

$$\mathrm{d}s^2 = \eta_{AB}\frac{\partial X^A}{\partial x^m}\frac{\partial X^B}{\partial x^n}\,\mathrm{d}x^m\,\mathrm{d}x^n = \frac{1}{H^2\eta^2}\left(-\mathrm{d}\eta^2 + \mathrm{d}\boldsymbol{x}^2\right) \tag{4.8}$$

and one can easily calculate that

$$\nabla_a\nabla_b Z = -ZH^2 g_{ab}\,, \tag{4.9}$$

so that the first derivatives of Z and the mixed second derivative already form a complete set.

In the literature an overwhelming number of different bitensor sets are used, and for the benefit of the reader we have compiled a short table (4.1) to convert between them. Of course, they have all one or the other advantage and disadvantage, but only one of them (apart from derivatives of Z) will be important in this work, namely the set (a). While n_a is defined as the normal vector to the shortest geodesic connecting the points x and x' (if it exists), its importance stems from the fact that it is normalized, $n^a n_a = 1$, and so suited for the asymptotic expansion of bitensors as $Z \to \pm\infty$. On the other hand, the parallel propagator $g_{ab'}$, which propagates vectors along the aforementioned geodesic, is essential to calculate contractions of bitensors since only indices which refer to the same point can be contracted.

	Z	$\nabla_a Z = Z_{;a}$	$\nabla_a \nabla_{b'} Z = Z_{;ab'}$
(a)	$\mu = H^{-1} \arccos Z$	$n_a = -\dfrac{Z_{;a}}{H\sqrt{1-Z^2}}$	$g_{ab'} = H^{-2}\left(Z_{;ab'} - \dfrac{Z_{;a}Z_{;b'}}{1+Z} \right)$
	$z = \frac{1}{2}(1+Z)$	-	-
(b)	$y = 2(1-Z)$	$y_{;a} = -2Z_{;a}$	$y_{;ab'} = -2Z_{;ab'}$
(c)	$\theta = \arccos Z$	$\hat{x}_a = -\dfrac{Z_{;a}}{H\sqrt{1-Z^2}}$	$I_{ab'} = H^{-2}\left(Z_{;ab'} - \dfrac{Z_{;a}Z_{;b'}}{1+Z} \right)$
	$s = 2H^{-2}(1-Z)$	-	$\mathcal{J}_{ab'} = H^{-2}\left(Z_{;ab'} + \dfrac{Z_{;a}Z_{;b'}}{1-Z} \right)$
	-	-	$\hat{I}_{ab'} = H^{-2} Z_{;ab'}$

Table 4.1.: de Sitter-invariant bitensor sets in use in the literature. Authors using those conventions are, e.g., Allen, Jacobson and Higuchi for (a), Tsamis and Woodard for (b) and Osborn and Shore for (c).

Of later use will be explicit expressions for those bitensors in the conformally flat coordinate system (4.8). They are easily calculated from equation (4.7), and we get

$$\begin{aligned}
\eta\eta' Z_{;a} &= -(\boldsymbol{x}-\boldsymbol{x}')_a + \delta^0_a \left(\eta - \eta' Z\right) \\
\eta\eta' Z_{;a'} &= (\boldsymbol{x}-\boldsymbol{x}')_a + \delta^0_a \left(\eta' - \eta Z\right) \\
\eta\eta' Z_{;ab'} &= \eta_{ab} - \eta\delta^0_b Z_{;a} - \eta'\delta^0_a Z_{;b'} - \delta^0_a\delta^0_b (Z-1) \, ,
\end{aligned} \tag{4.10}$$

with $(\boldsymbol{x}-\boldsymbol{x}')_0 = 0$ understood. Also, contractions of those bitensors will be useful that can be obtained by using the relations in table 4.1 and $g_{ab'}n^{b'} = -n_a$ as well as $n^a n_a = 1$, giving

$$\begin{aligned}
Z_{;a}Z^{;a} &= H^2(1-Z^2) \\
Z_{;ab'}Z^{;b'} &= -H^2 Z Z_{;a} \\
Z_{;ab'}Z^{;cb'} &= H^4\delta^c_a - H^2 Z_{;a}Z^{;c} \, .
\end{aligned} \tag{4.11}$$

The flat space limit can be obtained by taking the de Sitter radius H^{-1} to infinity, keeping the geodesic distance $\mu(x,x')$ between two points x and x' fixed. From equation (4.5), this gives

$$Z \to 1 - \frac{1}{2}H^2\mu^2(x,x') + \mathcal{O}\left(H^4\right) \, , \tag{4.12}$$

and by recalling that in Minkowski space we have $\mu^2(x,x') = (x-x')^2$, also covariant derivatives of Z can be easily obtained in the flat space limit.

4.2. Scalar and vector fields

In this thesis, we are interested in free scalar fields ϕ of arbitrary mass and curvature coupling and massless vector (gauge) fields A^b. However, in order to not burden us with the treatment of gauge-fixing and ghosts, we will treat the massless vector fields as the limit $m \to 0$ of massive vector fields. The corresponding actions are given by

$$S_{\mathrm{S}} = -\frac{1}{2} \int \left((\nabla^a \phi)(\nabla_a \phi) + m^2 \phi^2 + \xi R \phi^2 \right) \sqrt{-g} \, \mathrm{d}^n x \tag{4.13}$$

and

$$S_{\mathrm{V}} = -\frac{1}{4} \int \left(F^{ab} F_{ab} + 2m^2 A^b A_b \right) \sqrt{-g} \, \mathrm{d}^n x \tag{4.14}$$

with $m \geq 0$ the mass of the corresponding field and ξ a parameter controlling the coupling to the curvature scalar. Furthermore, $F_{ab} = \nabla_a A_b - \nabla_b A_a$ is the field strength tensor for the vector field. For $\xi = (n-2)/[4(n-1)]$ and vanishing mass, the scalar action is invariant under a rescaling of the metric $g_{ab} \to \mathrm{e}^{2\omega} g_{ab}$ and the scalar field $\phi \to \exp\left(\frac{2-n}{2}\omega\right)\phi$, as can be easily checked using the formulas of appendix B. The same is true for the vector action in the case of vanishing mass in four dimensions; both theories are then said to be scale invariant. Sometimes they are also denoted as "conformal theories"; nevertheless, conformal invariance demands more than just scale invariance [76, 77] and it is not yet fully determined if scale invariance always implies conformal invariance [78–80]. Furthermore, a theory that is classically conformally invariant may not stay so after quantization when quantum corrections break the conformal symmetry. This is known as *conformal anomaly*: for those theories, there exists no regularization and renormalization procedure that preserves the scale invariance of the theory. The stress tensor of classically scale invariant theories is traceless, while the trace of the renormalized stress tensor of the quantum fields does not vanish. If conformality is preserved in the quantum theory, all contributions to this trace anomaly must cancel out, such as happens, e.g., in the case of $\mathcal{N} = 4$ Super-Yang-Mills theory ([81–83]).

However, here we will only consider free theories. The equations of motion that follow from this action are

$$\left(\Box - m^2 - \xi R \right) \phi = 0 \tag{4.15}$$

for the scalar field and

$$\nabla_a F^{ab} = m^2 A^b \tag{4.16}$$

for the vector field. For the massive case, this equation was first postulated by Proca [84], while the massless case comprise just the source-free Maxwell equations. Note that because of the antisymmetry of the field-strength tensor, one can derive

$$m^2 \nabla_b A^b = 0 \,, \tag{4.17}$$

and so in the massive case the vector field is automatically transverse.

In order to calculate (path-ordered) correlation functions using (3.14), we need to know the propagator of the respective fields, which is a fundamental solution to their equations of motion. The choice of vacuum state is encoded in the addition of a homogeneous solution, and in the following we will assume that the matter fields are in the so-called Bunch-Davies vacuum state, a state which resembles the standard Minkowski vacuum for high momenta. In the case of the scalar field, the Feynman propagator $G^{\mathrm{F}}(x, x')$ has to fulfill

$$\left(\Box - m^2 - \xi R \right) G^{\mathrm{F}}(x, x') = \delta(x, x') \tag{4.18}$$

with the covariant δ distribution

$$\delta(x,x') = \frac{\delta^n(x-x')}{\sqrt{-g}}\,, \tag{4.19}$$

while for the vector field we have

$$\left(g_{ab}\,\Box - \nabla_b\nabla_a - m^2 g_{ab}\right)G^{bc'} = \delta_a^{c'}\,\delta(x,x')\,. \tag{4.20}$$

4.3. Canonical quantization

To calculate the propagator in the Bunch-Davies vacuum state, one can use canonical quantization in the conformally flat coordinate system (4.8). In this coordinate system, the scalar action (4.13) reads

$$S_\mathrm{S} = -\frac{1}{2}\int \left[-H^2\eta^2(\phi')^2 + H^2\eta^2\,(\nabla\phi)^2 + m^2\phi^2 + n(n-1)H^2\xi\phi^2\right](-H\eta)^{-n}\,\mathrm{d}^n x\,. \tag{4.21}$$

The canonical momentum is then given by

$$\pi = (-H\eta)^{2-n}\phi' \tag{4.22}$$

and the fields are postulated to fulfill the canonical equal-time commutation relations

$$[\phi(\boldsymbol{x},\eta),\pi(\boldsymbol{y},\eta)] = \mathrm{i}\delta^{n-1}(\boldsymbol{x}-\boldsymbol{y})\,, \tag{4.23}$$

with other commutators vanishing.

As usual, those relations are solved by decomposing the field ϕ in a set of modes

$$\phi(\boldsymbol{x},\eta) = \int \left(a(\boldsymbol{p})f(\boldsymbol{p},\eta) + a^\dagger(-\boldsymbol{p})f^*(-\boldsymbol{p},\eta)\right)\mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{x}}\frac{\mathrm{d}^{n-1}p}{(2\pi)^{n-1}}\,, \tag{4.24}$$

where the operators $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ fulfill the commutation relation

$$[a(\boldsymbol{p}),a^\dagger(\boldsymbol{q})] = (2\pi)^{n-1}\delta^{n-1}(\boldsymbol{p}-\boldsymbol{q}) \tag{4.25}$$

and the mode functions $f(\boldsymbol{p},\eta)$ are normalized to

$$f(\boldsymbol{p},\eta)f'^*(\boldsymbol{p},\eta) - f^*(-\boldsymbol{p},\eta)f'(-\boldsymbol{p},\eta) = \mathrm{i}(-H\eta)^{n-2}\,, \tag{4.26}$$

in order to ensure the canonical commutation relations (4.23). Since the scalar field satisfies the equation of motion (4.15), the mode functions are solutions of the same equation, which expressed in the conformally flat coordinate system reads

$$\eta^2 f''(\boldsymbol{p},\eta) - (n-2)\eta f'(\boldsymbol{p},\eta) + \eta^2\boldsymbol{p}^2 f(\boldsymbol{p},\eta) + (m^2H^{-2} + n(n-1)\xi)f(\boldsymbol{p},\eta) = 0\,. \tag{4.27}$$

For the Bunch-Davies vacuum, the correctly normalized solution to this equation is given (up to phase factors) by

$$f(\boldsymbol{p},\eta) = \frac{\sqrt{\pi}}{2}(-H\eta)^{\frac{n-2}{2}}\,\mathrm{e}^{\frac{1}{4}\mathrm{i}\pi(2\nu+1)}\sqrt{-\eta}\;\mathrm{H}_\nu^{(1)}(-|\boldsymbol{p}|\eta)\,, \tag{4.28}$$

where we defined

$$\nu = \sqrt{\frac{(n-1)^2}{4} - (m^2 H^{-2} + n(n-1)\xi)}\,. \tag{4.29}$$

In the massless, minimally coupled case we have $\nu = (n-1)/2$, while for the massless, conformally coupled case with $\xi = (n-2)/[4(n-1)]$ we obtain $\nu = 1/2$.

For high momentum $|\boldsymbol{p}| \to \infty$, these mode functions become positive-frequency modes

$$f(\boldsymbol{p},\eta) \sim (-H\eta)^{\frac{n-2}{2}} \sqrt{\frac{1}{2|\boldsymbol{p}|}}\, \mathrm{e}^{-\mathrm{i}|\boldsymbol{p}|\eta}\,, \tag{4.30}$$

so that this choice reduces to the standard Minkowski vacuum in the appropriate limit. The Bunch-Davies vacuum state $|0\rangle$ (a standard Fock vacuum) is then defined by demanding that $a(\boldsymbol{p})|0\rangle = 0$ for all $\boldsymbol{p}$. This vacuum state is also called Euclidean vacuum (since its two-point function that we calculate in the following can also be obtained by analytic continuation from the Euclidean version of de Sitter space, the sphere), and has been convincingly argued to be the only consistent state when interactions are taken into account [85–90]. The Bunch-Davies vacuum state is of Hadamard form [91, 92], and Hadamard states have a lot of nice physical properties (for example, the Hadamard form is preserved under Cauchy evolution, and vacuum states in static spacetimes are Hadamard). It has also been shown in examples that non-Hadamard states have unpleasant behaviour (such as an infinite stress tensor expectation value with usual renormalization procedures or state-dependent renormalization) [93–103]. The high-momentum expansion (4.30) becomes exact for the massless, conformally coupled case. In that case, the mode functions are naturally given by the Minkowski ones times the conformal factor $(-H\eta)^{\frac{n-2}{2}}$, which is consistent with our choice.

We can now proceed to calculate correlation functions, which by the Wick theorem for free fields can all be calculated from the two-point function

$$\begin{aligned} \langle 0|\phi(x)\phi(x')|0\rangle &= \int f(\boldsymbol{p},\eta) f^*(\boldsymbol{p},\eta')\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')} \frac{\mathrm{d}^{n-1}p}{(2\pi)^{n-1}} \\ &= \frac{\pi}{4H}(H^2\eta\eta')^{\frac{n-1}{2}} \int \mathrm{H}^{(1)}_\nu(-|\boldsymbol{p}|\eta)\, \mathrm{H}^{(2)}_\nu\left(-|\boldsymbol{p}|\eta'\right) \mathrm{e}^{\mathrm{i}\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{x}')} \frac{\mathrm{d}^{n-1}p}{(2\pi)^{n-1}} \end{aligned} \tag{4.31}$$

(using the identity $\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu}\mathrm{H}^{(1)}_\nu(x)\left[\mathrm{e}^{\mathrm{i}\frac{\pi}{2}\nu}\mathrm{H}^{(1)}_\nu(x')\right]^* = \mathrm{H}^{(1)}_\nu(x)\,\mathrm{H}^{(2)}_\nu(x')$, valid for either real or purely imaginary ν). The integral is calculated in appendix F, giving

$$\langle 0|\phi(x)\phi(x')|0\rangle = I_\nu(Z - \mathrm{i}0\,\mathrm{sgn}(\eta-\eta')) \tag{4.32}$$

with

$$\begin{aligned} I_\nu(Z) = \frac{H^{n-2}}{\Gamma\left(\frac{n}{2}\right)(4\pi)^{\frac{n}{2}}} \Gamma\left(\frac{n-1}{2}+\nu\right)\Gamma\left(\frac{n-1}{2}-\nu\right) \times \\ \times\, {}_2\mathrm{F}_1\left(\frac{n-1}{2}+\nu, \frac{n-1}{2}-\nu; \frac{n}{2}; \frac{1+Z}{2}\right) \end{aligned} \tag{4.33}$$

and including the correct prescription to go around the branch cut of the Gauß hypergeometric function for $Z \in [1,\infty)$ (see appendix D for more information on distributions which such prescriptions). It also has the correct flat space limit, which is obtained when $H \to 0$. The limit of Z was calculated before and is given by (4.12), but the calculation of the limiting form of $I_\nu(Z)$ is more involved. We can obtain it by using a Mellin-Barnes representation of the hypergeometric function (G.62)

$$ {}_2\mathrm{F}_1(a,b;c;z) = \int\limits_{\mathcal{C}} \frac{\Gamma(c)\Gamma(-s)\Gamma(c-a-b-s)\Gamma(s+a)\Gamma(s+b)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} (1-z)^s \frac{\mathrm{d}s}{2\pi\,\mathrm{i}}, \tag{4.34} $$

where the integration contour $\mathcal{C}$ runs from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$ in the way described in appendix G. $I_\nu(Z)$ can thus be written in the form

$$ I_\nu(Z) = \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}} \int\limits_{\mathcal{C}} \frac{\Gamma(-s)\Gamma\left(-\frac{n-2}{2}-s\right)\Gamma\left(s+\frac{n-1}{2}+\nu\right)\Gamma\left(s+\frac{n-1}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}-\nu\right)\Gamma\left(\frac{1}{2}+\nu\right)} \left(\frac{1-Z}{2}\right)^s \frac{\mathrm{d}s}{2\pi\,\mathrm{i}}. \tag{4.35} $$

In the Minkowski limit, we have (4.29) $\nu \sim \mathrm{i}m/H$ (the sign of ν does not matter) and

$$ \Gamma(x+\nu)\Gamma(x-\nu) \sim 2\pi \left(\frac{H}{m}\right)^{1-2x} \mathrm{e}^{-\frac{m}{H}\pi}, \tag{4.36} $$

so that equation (4.35) reduces to

$$ I_\nu(Z) \to \frac{m^{n-2}}{(4\pi)^{\frac{n}{2}}} \int\limits_{\mathcal{C}} \Gamma(-s)\Gamma\left(-\frac{n-2}{2}-s\right)\left(\frac{m^2(x-x')^2}{4}\right)^s \frac{\mathrm{d}s}{2\pi\,\mathrm{i}}. \tag{4.37} $$

This is however nothing else than the Mellin-Barnes representation of the modified Bessel function K [104], so that we have

$$ I_\nu(Z) \to \frac{m^{n-2}}{(2\pi)^{\frac{n}{2}}} \left(m\sqrt{(x-x')^2}\right)^{\frac{2-n}{2}} \mathrm{K}_{\frac{n-2}{2}}\left(m\sqrt{(x-x')^2}\right), \tag{4.38} $$

which is the correct flat-space result [105]. In the massless case, this reduces to

$$ \frac{\Gamma\left(\frac{n-2}{2}\right)}{4\pi^{\frac{n}{2}}} \left[(x-x')^2\right]^{\frac{2-n}{2}}. \tag{4.39} $$

Other two-point functions may then be calculated easily from the above result. For example, the Feynman propagator is given by

$$ \begin{aligned} G_\mathrm{F}(x,x') &= -\mathrm{i}\langle 0|\mathcal{T}\phi(x)\phi(x')|0\rangle \\ &= -\mathrm{i}\Theta(\eta-\eta')\langle 0|\phi(x)\phi(x')|0\rangle - \mathrm{i}\Theta(\eta'-\eta)\langle 0|\phi(x')\phi(x)|0\rangle \\ &= -\mathrm{i}I_\nu(Z-\mathrm{i}0), \end{aligned} \tag{4.40} $$

and one may check that it fulfills the correct equation (4.18) (which is most easily done by employing the form given in the second line together with the integral representation (4.31)).

A special case is the massless, minimally coupled scalar, with $m = \xi = 0$. In this case, we have $\nu = (n-1)/2$ and the integral in equation (4.31) does not converge for small $|\boldsymbol{p}|$. It is possible to place an infrared cutoff on this integral and obtain a closed-form expression; the infrared cutoff must then not appear in physical observables (i.e., in any physical observable it must be possible to take the limit where the cutoff tends to zero). This issue has been debated a lot in the literature [106–108], and there exist quantization procedures in closed de Sitter space where the cutoff does not appear [109]; also the choice of a different vacuum for the problematic low-momentum modes has been considered [110] (this does not change the Hadamard character of the resulting state). In the following, all observables that we consider will only involve mixed second derivatives of the propagator, for which all those procedures agree. Especially, we can take the massless limit of the massive propagator after taking these derivatives

$$\nabla_a \nabla_{b'} \left[G(x,x') \right]_{m=\xi=0} = \lim_{m,\xi\to 0} \nabla_a \nabla_{b'} G(x,x') = -\mathrm{i} \left(Z_{;ab'} I'_{\frac{n-1}{2}}(Z) + Z_{;a} Z_{;b'} I''_{\frac{n-1}{2}}(Z) \right) , \quad (4.41)$$

with the appropriate prescription for the Wightman and Feynman functions.

Another special case is the massless, conformally coupled scalar, for which we have $\nu = 1/2$ independent of the dimension. As remarked at the beginning of this section, in this case the mode functions are naturally given by the Minkowski mode functions times a power of the conformal factor, and so it should be clear that also the propagator is given by the Minkowski one times twice this power of the conformal factor. In flat space, the propagators of the massless, conformally coupled scalar and the massless, minimally coupled scalar agree and are just given by the appropriate power of the Minkowski distance $(x-x')^2$, so that in de Sitter space the conformal propagator is given by the same power of the de Sitter invariant distance $1-Z(x,x') = (x-x')^2/(2\eta\eta')$ (4.7). This follows in fact from the general expression (4.32), (4.33) by setting $\nu = 1/2$ and we obtain

$$I_{\frac{1}{2}}(Z) = \frac{H^{n-2}}{2(2\pi)^{\frac{n}{2}}} \Gamma\left(\frac{n-2}{2} \right) (1-Z)^{\frac{2-n}{2}} . \quad (4.42)$$

As said before, we will treat the massless gauge vector field as the limit of the massive one. Unlike the case of the massless, minimally coupled scalar, the mode functions are not infrared divergent in the massless limit (with a covariant gauge-fixing term present [58]), and thus for vector fields the massless limit is smooth as in flat space. We then follow the same steps as in the scalar case, expressing the massive vector action (4.14) in the conformally flat coordinate system

$$\begin{aligned} \mathcal{L} = &-\frac{1}{2} m^2 (-H\eta)^{2-n} (-A_0^2 + A^\alpha A_\alpha) \\ &- \frac{1}{2} (-H\eta)^{4-n} \left(-(A'^\alpha - \partial^\alpha A_0)(A'_\alpha - \partial_\alpha A_0) + F^{\alpha\beta} (\partial_\alpha A_\beta) \right) . \end{aligned} \quad (4.43)$$

The canonical momenta are given by

$$\pi^\alpha = (-H\eta)^{4-n} (A'^\alpha - \partial^\alpha A_0) , \qquad \pi^0 = 0 , \quad (4.44)$$

and we see that since the time component is non-dynamical, its canonical momentum vanishes. Instead the transversality condition (4.17) gives

$$A_0 = -\frac{(-H\eta)^{n-2}}{m^2} \triangle \pi . \quad (4.45)$$

We then impose the canonical commutation relations for spatial components

$$[A_\alpha(\boldsymbol{x},\eta), \pi^\beta(\boldsymbol{x}',\eta)] = \mathrm{i} \delta^\beta_\alpha \delta^{n-1}(\boldsymbol{x}-\boldsymbol{x}') . \quad (4.46)$$

The equations of motion that follow from the Lagrangian (4.43) are coupled. To decouple them, we separate the field and the canonical momentum into the transverse and longitudinal parts,

$$A_\alpha = B_\alpha + \partial_\alpha A, \qquad \pi^\alpha = \varpi^\alpha + \partial^\alpha \pi\,, \tag{4.47}$$

with $\partial^\alpha B_\alpha = \partial_\alpha \varpi^\alpha = 0$. After some rearrangements, we obtain for the transverse parts

$$\begin{aligned} B''_\alpha - \frac{n-4}{\eta} B'_\alpha - \triangle B_\alpha = -(-H\eta)^{-2} m^2 B_\alpha\,, \\ \varpi_\alpha = (-H\eta)^{4-n} B'_\alpha\,, \end{aligned} \tag{4.48}$$

where $\triangle = \partial^\alpha \partial_\alpha$. The equation for the longitudinal component is intertwined with the temporal component, but it is possible to obtain a single equation for its conjugate momentum

$$\pi'' + \frac{n-2}{\eta}\pi' - \triangle\pi = -(-H\eta)^{-2} m^2 \pi\,, \tag{4.49}$$

in terms of which the longitudinal component of the field is given by

$$A = -\frac{(-H\eta)^{n-2}}{m^2}\pi'\,. \tag{4.50}$$

After introducing the usual decomposition into modes just as in the case of the scalar field, which we don't repeat here for brevity and since it is not enlightening in any way, the two-point function of the vector field in the Bunch-Davies vacuum state reads

$$\begin{aligned} \langle 0|A_0(x)A_0(x')|0\rangle &= -\frac{1}{m^2}\left(\triangle I_\rho(\eta,\eta',\boldsymbol{x}-\boldsymbol{x}') + \partial_\eta\partial_{\eta'} I_\mu(\eta,\eta',\boldsymbol{x}-\boldsymbol{x}')\right)\,, \\ \langle 0|A_0(x)A_{\beta'}(x')|0\rangle &= -\frac{1}{m^2}(H^2\eta\eta')^{n-2}\partial_{\beta'}\partial_{\eta'}\left((H^2\eta\eta')^{2-n} I_\rho(\eta,\eta',\boldsymbol{x}-\boldsymbol{x}')\right)\,, \\ \langle 0|A_\alpha(x)A_0(x')|0\rangle &= -\frac{1}{m^2}(H^2\eta\eta')^{n-2}\partial_\alpha\partial_\eta\left((H^2\eta\eta')^{2-n} I_\rho(\eta,\eta',\boldsymbol{x}-\boldsymbol{x}')\right)\,, \\ \langle 0|A_\alpha(x)A_{\beta'}(x')|0\rangle &= (H^2\eta\eta')^{-1}\left(\eta_{\alpha\beta'} + \frac{\partial_\alpha\partial_{\beta'}}{\triangle}\right) I_\rho(\eta,\eta',\boldsymbol{x}-\boldsymbol{x}') \\ &\quad - \frac{1}{m^2}(H^2\eta\eta')^{n-2}\frac{\partial_\alpha\partial_{\beta'}}{\triangle}\partial_\eta\partial_{\eta'}\left((H^2\eta\eta')^{2-n} I_\rho(\eta,\eta',\boldsymbol{x}-\boldsymbol{x}')\right)\,, \end{aligned} \tag{4.51}$$

with the same integral I_ν (4.33), and the parameter

$$\rho = \sqrt{\frac{(n-3)^2}{4} - \frac{m^2}{H^2}}\,. \tag{4.52}$$

Inserting the explicit expression for I_ν and performing the derivatives (in the case of purely spatial components, one has to go back to the definition of I_ν in terms of a Fourier integral and integrate by parts [57]), we obtain the de Sitter-invariant result

$$\langle 0|A_a(x)A_{b'}(x')|0\rangle = \frac{1}{m^2} K_{ab'}(Z(x,x') - \mathrm{i}0\,\operatorname{sgn}(\eta-\eta'))\,, \tag{4.53}$$

where we defined

$$\begin{aligned} K_{ab'}(Z) = &\left((n-1)I'_\rho(Z) + Z I''_\rho(Z)\right) Z_{;a}Z_{;b'} \\ &+ \left(-(n-1)Z I'_\rho(Z) + (1-Z^2) I''_\rho(Z)\right) Z_{;ab'}\,. \end{aligned} \tag{4.54}$$

The tensor $K_{ab'}(Z)$ is transverse, which can readily be verified by using the relations (4.11). In this case there is a subtlety in defining the time-ordered function: since the transversality condition (4.17) holds as an operator equation, the naive generalization of the scalar definition (4.40) is not covariant and must be amended by additional local terms (i.e., $\sim \delta(x-x')$). The divergence of the Feynman propagator then does not vanish (as it does for the Wightman function), but it is also local. In order to keep the simple transversality condition, one can quantize the massive theory using a covariant gauge-fixing term $1/(2\xi)(\nabla_b A^b)^2$ and take the limit $\xi \to \infty$ after quantization. In this case, the same result (4.53) is obtained [57], and the only difference between the Feynman and the Wightman function is the appropriate prescription. We will therefore drop the corresponding prescription and only display it when it there is a difference in the treatment between the Wightman function and the time-ordered one.

Recalling the definition of the field strength tensor $F_{ab} = \nabla_a A_b - \nabla_b A_a$, from this result we can easily calculate the correlation function of the field strength. Using that I_ν satisfies

$$(1-Z^2)I''_\nu(Z) - nZI'_\nu(Z) + \left(\nu^2 - \frac{(n-1)^2}{4}\right) I_\nu(Z) = 0 \tag{4.55}$$

as a consequence of the scalar field equation of motion (4.15), the field strength tensor correlation function has a very simple expression

$$\langle 0|F_{ab}F_{c'd'}|0\rangle = 4H^{-2}\left(I'_\rho(Z) Z_{;a[c'}Z_{;d']b} - I''_\rho(Z) Z_{;[a}Z_{;b][c'}Z_{;d']}\right) . \tag{4.56}$$

In this correlation function, we may now take the massless limit, which does not exist for the two-point function of the vector field (4.53), a fact well known from Minkowski space. In the massless case, we have $\rho = (n-3)/2$ and the field strength correlation function reads explicitly

$$\begin{aligned}\langle 0|F_{ab}F_{c'd'}|0\rangle = \frac{2H^{n-4}}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma(n-1)}{\Gamma\left(\frac{n+2}{2}\right)} \Bigg[{}_2\mathrm{F}_1\left(2, n-1; \frac{n+2}{2}; \frac{1+Z}{2}\right) Z_{;a[c'}Z_{;d']b} \\ - \frac{2(n-1)}{n+2}\, {}_2\mathrm{F}_1\left(3, n; \frac{n+4}{2}; \frac{1+Z}{2}\right) Z_{;[a}Z_{;b][c'}Z_{;d']} \Bigg] \\ \to \frac{Z_{;a[c'}Z_{;d']b}}{2\pi^2(1-Z)^2} - \frac{Z_{;[a}Z_{;b][c'}Z_{;d']}}{\pi^2(1-Z)^3} \quad (n \to 4) .\end{aligned} \tag{4.57}$$

Another method to derive the propagators is by solving the equations (4.18) and (4.20) that they fulfill. One assumes a de Sitter-invariant vacuum state and looks for solutions to the homogeneous equation, i.e., outside of coincidence. Of the two possible solutions, one takes the one which is only singular for light-like related points, which means $Z = 1$, but regular for antipodal points, $Z = -1$. The overall normalization is determined by comparing with the flat-space limit, and the result is the same as obtained by canonical quantization. For the scalar propagator, this approach was first used by Candelas and Raine [111], while the vector propagator was first derived by Allen and Jacobson [75]. However, canonical quantization has an advantage in as much as it tells us that we do not need to assume the existence of a de Sitter-invariant vacuum, but that the Bunch-Davies vacuum is the appropriate one which gives invariant correlation functions.

4.4. Stress tensors

As we have seen in the expansion of the effective action (3.15), the basic interaction between matter and gravity occurs through the stress tensor (and the tensor U_{abcd}). In this section, we therefore determine the classical expressions for the stress tensor of scalar and vector fields, and their (dimensionally regularized) one- and two-point functions, as well as expressions for U_{abcd}. For the sake of brevity, we restrict ourselves to minimally coupled scalar fields (i.e., $\xi = 0$), except for the case of the massless, conformally coupled scalar (with $\xi = (n-2)/[4(n-1)]$); however, all calculations can be generalized to the case of general mass and coupling to the curvature scalar.

For the minimally coupled scalar field, the necessary expansion has the general form (3.11) with

$$\begin{aligned} T_{ab} &= (\nabla_a\phi)(\nabla_b\phi) - \frac{1}{2} g_{ab} (\nabla^s\phi)(\nabla_s\phi) - \frac{1}{2} g_{ab} m^2 \phi^2 \,, \\ U_{abcd} &= -\frac{1}{16}\left[(\nabla^s\phi)(\nabla_s\phi) + m^2\phi^2\right](g_{ab}g_{cd} - 2g_{a(c}g_{d)b}) - \frac{1}{2}\left(\nabla_{(a}\phi\right) g_{b)(c}\left(\nabla_{d)}\phi\right) \\ &\quad + \frac{1}{8} g_{ab} (\nabla_c\phi)(\nabla_d\phi) + \frac{1}{8} g_{cd} (\nabla_a\phi)(\nabla_b\phi) \,, \end{aligned} \tag{4.58}$$

while for the massless, conformally coupled scalar field we have

$$2(n-1)T_{ab} = n(\nabla_a\phi)(\nabla_b\phi) - g_{ab}(\nabla^s\phi)(\nabla_s\phi) + \frac{(n-2)^2}{4} g_{ab} H^2 \phi^2 - (n-2)\phi\nabla_a\nabla_b\phi \tag{4.59}$$

(using the equation of motion (4.15) to simplify the result). In this case, the explicit expression for U_{abcd} is not necessary. Finally, for the (massless) vector field the expansion reads

$$\begin{aligned} T_{ab} &= F_a{}^s F_{bs} - \frac{1}{4} g_{ab} F^{mn} F_{mn} \\ 8U_{abcd} &= 2F_{a(c}F_{d)b} + F_a{}^s F_{bs} g_{cd} + F_c{}^s F_{ds} g_{ab} + 4F^s{}_{(a} g_{b)(c} F_{d)s} - \frac{1}{4} F^{mn} F_{mn} \left(g_{ab}g_{cd} - 2g_{a(c}g_{d)b}\right) . \end{aligned} \tag{4.60}$$

Since the stress tensor involves the product of field operators at the same point, divergences result when the expectation value is taken. We may regulate those divergences by separating the points (i.e., take one field at a point x and another at a point x'), calculate the expectation value in n dimensions and taking the limit as $x' \to x$. The resulting expression will be finite for $n \neq 4$, but will develop poles as $n \to 4$ which later on have to be subtracted with suitable counterterms. Let us take the expectation value of the stress tensor for the massless vector field as a concrete example. In this case, we calculate

$$\begin{aligned} \langle 0|T_{ab}(x)|0\rangle &= \lim_{x'\to x}\left[g^{mn'}\langle 0|F_a{}^m(x)F_{b'n'}(x')|0\rangle - \frac{1}{4} g_{ab'} g^{mp'} g^{nq'} \langle 0|F_{mn}(x)F_{p'q'}(x')|0\rangle \right] \\ &= \lim_{x'\to x}\left[(n-2)\left(\frac{2}{1+Z} I'_{\frac{n-3}{2}}(Z) + I''_{\frac{n-3}{2}}(Z)\right) Z_{;a}Z_{;b'} \right. \\ &\quad \left. + \frac{1}{2} H^2 g_{ab'} \left(-[(n-5)(n-2) + 2(n-3)Z] I'_{\frac{n-3}{2}}(Z) + (n-3)(1-Z^2) I''_{\frac{n-3}{2}}(Z)\right)\right], \end{aligned} \tag{4.61}$$

where we used the relations between the parallel propagator and derivatives of Z and their contractions given in table 4.1. In taking the coincidence limit, which corresponds to $Z \to 1$, we can replace the hypergeometric functions by their value at 1 by Gauß's theorem

$$ {}_2\mathrm{F}_1\left(a, b; c; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \,. \tag{4.62} $$

This identity is only valid for $\mathfrak{Re}\,(c-a-b) > 0$; however, this is fulfilled in some region of the complex n plane, and we obtain therefore from equation (4.33)

$$ \begin{aligned} I_\nu(1) &= \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}} \Gamma\left(1-\frac{n}{2}\right) \frac{\Gamma\left(\frac{n-1}{2}+\nu\right)\Gamma\left(\frac{n-1}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)} \,, \\ I'_\nu(1) &= \frac{H^{n-2}}{2(4\pi)^{\frac{n}{2}}} \Gamma\left(-\frac{n}{2}\right) \frac{\Gamma\left(\frac{n+1}{2}+\nu\right)\Gamma\left(\frac{n+1}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)} \,. \end{aligned} \tag{4.63} $$

Furthermore, in this limit we have $g_{ab'} \to g_{ab}$ and $Z_{;a} \to 0$, so that the end result for the massless vector field is

$$ \langle 0|T_{ab}(x)|0\rangle = -(n-4)\frac{H^n}{4(4\pi)^{\frac{n}{2}}} \frac{\Gamma(n)}{\Gamma\left(\frac{n+2}{2}\right)} g_{ab} \to 0 \qquad (n \to 4) \,. \tag{4.64} $$

We see that the regularized expectation value actually vanishes in four dimensions. By the same procedure we calculate

$$ \begin{aligned} \langle 0|U_{abcd}(x)|0\rangle &= \frac{H^n}{8(4\pi)^{\frac{n}{2}}} \frac{\Gamma(n-2)}{n\Gamma\left(\frac{n-2}{2}\right)} \left[-(n^2-9n+16) g_{ab} g_{cd} + 2(n^2-9n+12) g_{a(c} g_{d)b} \right] \\ &\to \frac{H^4}{8(4\pi)^4} \left(g_{ab} g_{cd} - 4 g_{a(c} g_{d)b} \right) \qquad (n \to 4) \,. \end{aligned} \tag{4.65} $$

Again, this is already finite in four dimensions.

In the same manner, we obtain for the massive, minimally coupled scalar field

$$ \begin{aligned} \langle 0|T_{ab}(x)|0\rangle &= \frac{H^n}{2(4\pi)^{\frac{n}{2}}} \Gamma\left(-\frac{n}{2}\right) \frac{\Gamma\left(\frac{n+1}{2}+\nu\right)\Gamma\left(\frac{n+1}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)} g_{ab} \\ &\to \frac{m^2-2H^2}{32\pi^2(n-4)} m^2 g_{ab} + \mathcal{O}\left((n-4)^0\right) \quad (n \to 4) \,, \\ \langle 0|U_{abcd}(x)|0\rangle &= \frac{H^n}{8(4\pi)^{\frac{n}{2}}} \Gamma\left(-\frac{n}{2}\right) \frac{\Gamma\left(\frac{n-1}{2}+\nu\right)\Gamma\left(\frac{n-1}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)} \left(g_{ab} g_{cd} - 2 g_{a(c} g_{d)b} \right) \\ &\to \frac{m^2-2H^2}{128\pi^2(n-4)} m^2 \left(g_{ab} g_{cd} - 2 g_{a(c} g_{d)b} \right) + \mathcal{O}\left((n-4)^0\right) \quad (n \to 4) \,. \end{aligned} \tag{4.66} $$

Note that the limit $m \to 0$ of these expressions is not the correct result for the massless, minimally coupled scalar field, since a cancellation between terms which diverge like $\sim m^{-2}$ as $m \to 0$ and explicit factors of m^2 takes place [108, 109] (however, the limit is smooth if one takes the non-minimally coupled scalar with $\xi = -m^2/(6H^2)$ [109]). We therefore have to calculate this case separately, giving

$$
\begin{aligned}
\langle 0|T_{ab}(x)|0\rangle &= (n-2)\frac{H^n}{4(4\pi)^{\frac{n}{2}}}\frac{\Gamma(n)}{\Gamma\left(\frac{n+2}{2}\right)}g_{ab} \to \frac{3H^4}{32\pi^2}g_{ab} \qquad (n\to 4)\,,\\
\langle 0|U_{abcd}|0\rangle &= (n-4)\frac{H^n}{32(4\pi)^{\frac{n}{2}}}\frac{\Gamma(n)}{\Gamma\left(\frac{n+2}{2}\right)}\left(g_{ab}g_{cd}-2g_{a(c}g_{d)b}\right) \to 0 \qquad (n\to 4)\,.
\end{aligned}
\tag{4.67}
$$

Lastly, we need these expectation values for the case of the massless, conformally coupled scalar field. Whatever the explicit expressions are, they will all involve the regularized coincidence limit of the propagator (or its derivatives); however, for the conformal scalar these are just n-dependent powers of $1-Z$ which have vanishing coincidence limit in dimensional regularization. Therefore, in this case we just have

$$
\langle 0|T_{ab}|0\rangle = \langle 0|U_{abcd}|0\rangle = 0\,. \tag{4.68}
$$

This result might seem to be at odds with the known conformal anomaly, explained in section 4.2. However, since dimensional regularization preserves the classical equations of motion in the quantum theory, the *regularized* expectation value of the trace vanishes just like the classical one. In the maximally symmetric Bunch-Davies vacuum state that we have chosen, the expectation value of the stress tensor must be proportional to the de Sitter metric and hence is completely determined by its trace, so it vanishes. The trace anomaly results from the inability to choose a regularization and renormalization procedure that preserves conformal symmetries in general (see [112] and references therein); however this does not exclude the possibility that it works for a specific background. In fact, since $\langle T_{ab}\rangle$ must be proportional to g_{ab} in a maximally symmetric vacuum state it can always be forced to vanish by choosing the cosmological constant appropriately, as we will see later on.

These expectation values generally diverge in four dimensions, and we have to renormalize them. We therefore expand the gravitational action (3.8, 3.9) in terms of the perturbation h_{ab} using appendix A so that the effective action (3.15) reads

$$
S_{\text{eff}}[h] = \kappa S^{(1)}_{\text{eff}}[h] + \kappa^2 S^{(2)}_{\text{eff}}[h] \tag{4.69}
$$

with

$$
\begin{aligned}
2S^{(1)}_{\text{eff}}[h] &= \frac{1}{\kappa^2}\left((n-1)(n-2)H^2-2\Lambda\right)\int h\sqrt{-g}\,\mathrm{d}^n x + \int h^{ab}\langle T_{ab}\rangle_\phi\sqrt{-g}\,\mathrm{d}^n x\\
&\quad + (n-1)(n-4)\left(-\frac{2}{3}(n-3)\alpha + n(n-1)\beta\right)H^4\int h\sqrt{-g}\,\mathrm{d}^n x\,,
\end{aligned}
\tag{4.70}
$$

the part quadratic in the perturbation $S^{(2)}_{\text{eff}}[h]$ is too long to be given here. Since the renormalization of the parameters α and β is determined in the next section, to renormalize the expectation value of the stress tensor we only have Λ and $1/\kappa^2$ at our disposal. In fact, we may completely null the linear part of the effective action $S^{(1)}_{\text{eff}}[h]$ by choosing

$$
\frac{2\Lambda}{\kappa^2} = \frac{(n-1)(n-2)H^2}{\kappa^2} + \frac{g^{ab}}{n}\langle T_{ab}\rangle_\phi + \frac{(n-1)(n-4)}{3}\left(3n(n-1)\beta - 2(n-3)\alpha\right)H^4\,. \tag{4.71}
$$

Note that it is essential for this procedure to work that $\langle T_{ab} \rangle_\phi$ is proportional to the de Sitter metric. However, if one considers a different Hadamard state, the divergent part will still be the same [91, 92], but there will be an additional finite part which may have a different form. Also, since the background value of the Ricci scalar is given by $(n-1)(n-2)H^2$, this equation actually gives a definition of the renormalization of the cosmological constant as well as of Newton's constant. Of the examples calculated so far, the only divergent stress tensor expectation value is the one of the massive, minimally coupled scalar field (4.66). We see that the divergent part naturally separates into two, one proportional to m^4 and one proportional to $m^2 H^2$, and by choosing

$$\begin{aligned} \frac{\Lambda}{\kappa^2} &\to \frac{\Lambda}{\kappa^2} + \frac{m^4}{64\pi^2(n-4)} \,, \\ \frac{1}{\kappa^2} &\to \frac{1}{\kappa^2} + \frac{m^2}{96\pi^2(n-4)} \,, \end{aligned} \tag{4.72}$$

we can absorb all the divergences. This agrees with the well-known calculation by t'Hooft and Veltman [113] as well as other previous calculations in Minkowski space [114]. Note also that we did not introduce a renormalization scale at this point; the Hubble constant H arises naturally as the scale where the theory is renormalized, in the sense that the finite part of the stress tensor after the above renormalization includes a term $\ln(m/H)$. It is of course possible to introduce a different renormalization scale μ; in this case we are left with a scale-dependent cosmological constant $\Lambda(\mu)$ and Newton's constant in the form $1/\kappa^2(\mu)$ as well as a renormalized stress tensor expectation value $\langle T_{ab} \rangle^{\text{ren}}(\mu)$.

Of all the cases we consider, the expectation value of U_{abcd} is only divergent for the massive, minimally coupled scalar field (4.66), and we see that the divergent part agrees with the divergent part of the stress tensor expectation value (apart from the different tensor structure). By expanding the part of the effective action which is quadratic in the perturbation, $S^{(2)}_{\text{eff}}[h]$ (which is conveniently done using the tensor algebra package xAct, see appendix H, using the expansions from appendix A), we see that these divergences are absorbed as well by the renormalization of the cosmological constant and Newton's constant.

4.5. Stress tensor two-point functions

The last correlation function that we need is the (connected) two-point function of the stress tensor

$$\left\langle T_{ab}(x) T_{c'd'}(x') \right\rangle = \langle 0 | T_{ab}(x) T_{c'd'}(x') | 0 \rangle - \langle 0 | T_{ab}(x) | 0 \rangle \langle 0 | T_{c'd'}(x') | 0 \rangle \,. \tag{4.73}$$

In the connected correlation function, the divergences that arise from taking fields at the same spacetime point cancel out, and so it is already finite as long as x and x' are not lightlike related. However, in the null separation limit $(x-x')^2 \to 0$ the correlation functions still diverge after smearing with test functions; they are not well-defined distributions in four dimensions. While the regularized two-point function can be quite easily calculated [115], the separation of these remaining divergences is in general a hard problem. For the massless, conformally coupled scalar field this correlation function was renormalized by Campos and Verdaguer [67, 116] by transforming to flat space, a method that is applicable for any conformal theory. In the following, I will present a general technique whose only drawback is that it has so far only been formulated for flat space since a generalization to de Sitter space has some technical difficulties. Afterwards, I explain a different method which works in de Sitter space on the example of the massless, minimally coupled scalar field.

However, first we need to calculate the regularized two-point function in the various cases. Using the same point-splitting prescription as in the last section, schematically we have to calculate correlation functions of the type

$$\lim_{y,y'\to x,x'}\left[\langle 0|\phi(x)\phi(y)\phi(x')\phi(y')|0\rangle - \langle 0|\phi(x)\phi(y)|0\rangle\langle 0|\phi(x')\phi(y')|0\rangle\right] . \tag{4.74}$$

At the free-field level that we are considering, the first correlation function factorizes per Wick's theorem, and by noting that it does not matter if we take the limit $y' \to x'$ or $x' \to y'$, this reduces to

$$2\langle 0|\phi(x)\phi(x')|0\rangle\langle 0|\phi(x)\phi(x')|0\rangle . \tag{4.75}$$

From this explicit expression, the assertion made above is clear: as long as x and x' are not null separated, this expression is already finite. As an example, let us take again the massless vector field, where we have

$$\begin{aligned}\left\langle T_{ab}(x)T_{c'd'}(x')\right\rangle = &\langle 0|F_a{}^s(x)F_{c'}{}^{t'}(x')|0\rangle\langle 0|F_{bs}(x)F_{d't'}(x')|0\rangle \\ &+ \langle 0|F_a{}^s(x)F_{d't'}(x')|0\rangle\langle 0|F_{bs}(x)F_{c'}{}^{t'}(x')|0\rangle \\ &- \frac{1}{2}g_{ab}\langle 0|F^{mn}(x)F_{c'}{}^{t'}(x')|0\rangle\langle 0|F_{mn}(x)F_{d't'}(x')|0\rangle \\ &- \frac{1}{2}g_{c'd'}\langle 0|F_a{}^s(x)F^{p'q'}(x')|0\rangle\langle 0|F_{bs}(x)F_{p'q'}(x')|0\rangle \\ &+ \frac{1}{8}g_{ab}g_{c'd'}\langle 0|F^{mn}(x)F^{p'q'}(x')|0\rangle\langle 0|F_{mn}(x)F_{p'q'}(x')|0\rangle .\end{aligned} \tag{4.76}$$

We can now insert the explicit expression (4.57) to obtain

$$\left\langle T_{ab}(x)T_{c'd'}(x')\right\rangle = \sum_{k=1}^{5} {}^{(k)}\mathcal{T}_{abc'd'}(Z(x,x'))\mathcal{T}^{(k)}(Z(x,x') - \mathrm{i}0\ \mathrm{sgn}(\eta-\eta')) \tag{4.77}$$

with the bitensor set

$$\begin{aligned} {}^{(1)}\mathcal{T}_{abc'd'}(Z) &= g_{ab}g_{c'd'} \\ {}^{(2)}\mathcal{T}_{abc'd'}(Z) &= H^{-2}(Z_{;a}Z_{;b}g_{c'd'} + g_{ab}Z_{;c'}Z_{;d'}) \\ {}^{(3)}\mathcal{T}_{abc'd'}(Z) &= H^{-4}Z_{;a}Z_{;b}Z_{;c'}Z_{;d'} \\ {}^{(4)}\mathcal{T}_{abc'd'}(Z) &= 4H^{-4}Z_{;(a}Z_{;b)(c'}Z_{;d')} \\ {}^{(5)}\mathcal{T}_{abc'd'}(Z) &= 2H^{-4}Z_{;a(c'}Z_{;d')b} \end{aligned} \tag{4.78}$$

and the coefficients

$$
\begin{aligned}
\mathcal{T}^{(1)}(Z) &= \frac{5-n}{2} H^4 (1-Z^2) \Big[4Z I''_{\frac{n-3}{2}}(Z) I'_{\frac{n-3}{2}}(Z) - (1-Z^2) I''^2_{\frac{n-3}{2}}(Z) \Big] \\
&\qquad + H^4 (n^2 - 11n + 26 - 2(5-n)Z^2) I'^2_{\frac{n-3}{2}}(Z) \to -\frac{H^8}{16\pi^4 (1-Z)^4} \quad (n \to 4) \\
\mathcal{T}^{(2)}(Z) &= (n-4) H^4 \Big[4 I'^2_{\frac{n-3}{2}}(Z) + 4Z I''_{\frac{n-3}{2}}(Z) I'_{\frac{n-3}{2}}(Z) + (1-Z^2) I''^2_{\frac{n-3}{2}}(Z) \Big] \to 0 \quad (n \to 4) \\
\mathcal{T}^{(3)}(Z) &= 2H^4 \Big[4 I'^2_{\frac{n-3}{2}}(Z) + 4Z I''_{\frac{n-3}{2}}(Z) I'_{\frac{n-3}{2}}(Z) + (n-3+Z^2) I''^2_{\frac{n-3}{2}}(Z) \Big] \to \frac{H^8}{4\pi^4 (1-Z)^6} \quad (n \to 4) \\
\mathcal{T}^{(4)}(Z) &= H^4 \Big[-4Z I'^2_{\frac{n-3}{2}}(Z) + 2(n-2-2Z^2) I''_{\frac{n-3}{2}}(Z) I'_{\frac{n-3}{2}}(Z) + Z(1-Z^2) I''^2_{\frac{n-3}{2}}(Z) \Big] \\
&\qquad \to \frac{H^8}{8\pi^4 (1-Z)^5} \quad (n \to 4) \\
\mathcal{T}^{(5)}(Z) &= H^4 \Big[4(n-3+Z^2) I'^2_{\frac{n-3}{2}}(Z) - 4Z(1-Z^2) I''_{\frac{n-3}{2}}(Z) I'_{\frac{n-3}{2}}(Z) + (1-Z^2)^2 I''^2_{\frac{n-3}{2}}(Z) \Big] \\
&\qquad \to \frac{H^8}{8\pi^4 (1-Z)^4} \quad (n \to 4) \,.
\end{aligned}
\tag{4.79}
$$

For the tedious tensor algebra, it is advisable to use the tensor manipulation package xAct, as detailed in appendix H.

For the time-ordered two-point function, the formula (4.77) holds but with the Feynman prescription instead of the Wightman one, i.e.,

$$
\left\langle \mathcal{T} T_{ab}(x) T_{c'd'}(x') \right\rangle = \sum_{k=1}^{5} {}^{(k)}\mathcal{T}_{abc'd'}(Z(x,x')) \mathcal{T}^{(k)}(Z(x,x') - \mathrm{i}0) \tag{4.80}
$$

with the same coefficients $\mathcal{T}^{(k)}$ from equation (4.79). In this case, one can see from the explicit expressions that they are too singular to be integrable in four dimensions, and hence do not give a well-defined distribution. One might wonder if for the time-ordered two-point function additional local terms $\sim \delta(x,x')$ are obtained when the derivatives act on the Feynman propagator of the respective field, like in equation (4.18). However, since a priori the time-ordered stress tensor correlation function is not well defined, there is no reason to keep those local terms, and we will see later on that our choice leads to the correct well-known counterterms needed for renormalization.

The two-point function of the stress tensor for the massive, minimally coupled scalar field has the same general expression (4.77), but with the different coefficients

$$
\begin{aligned}
\mathcal{T}^{(1)}(Z) &= \frac{1}{2} H^4 (n-5+(n-1)^2 Z^2) I'^2_\nu(Z) + m^4 I^2_\nu(Z) \\
&\qquad + m^2 H^2 [(n-1) Z I'_\nu(Z) I_\nu(Z) + (1-Z^2) I'^2_\nu(Z)] \,, \\
\mathcal{T}^{(2)}(Z) &= H^4 I'^2_\nu(Z) - H^4 (n-2) Z I'_\nu(Z) I''_\nu(Z) - m^2 H^2 \left(I_\nu(Z) I''_\nu(Z) + I'^2_\nu(Z) \right) \,, \\
\mathcal{T}^{(3)}(Z) &= 2H^4 I''^2_\nu(Z) \,, \\
\mathcal{T}^{(4)}(Z) &= H^4 I'_\nu(Z) I''_\nu(Z) \,, \\
\mathcal{T}^{(5)}(Z) &= H^4 I'^2_\nu(Z) \,.
\end{aligned}
\tag{4.81}
$$

As in the case of the stress tensor expectation value, the expression for the massless, minimally coupled scalar field cannot be obtained by taking the limit $m \to 0$ of this result, since there is a cancellation between explicit powers of m^2 and the divergence $1/m^2$ in I_ν [115]. The correct result for the massless, minimally coupled case is given by

$$\begin{aligned}
\mathcal{T}^{(1)}(Z) &= \frac{1}{2}H^4(n-1)(1+(n-1)Z^2)I'^2_{\frac{n-1}{2}}(Z) - 2H^4 I'^2_{\frac{n-1}{2}}(Z) + \frac{H^{2n}}{2(4\pi)^n}\frac{\Gamma^2(n)}{\Gamma^2\left(\frac{n}{2}\right)} \\
&\quad + (n-1)\frac{H^{n+2}}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} Z I'_{\frac{n-1}{2}}(Z) \to \frac{H^8}{128\pi^4}\frac{(5-Z)(1+Z)}{(1-Z)^4} \quad (n \to 4) \\
\mathcal{T}^{(2)}(Z) &= H^4 I'^2_{\frac{n-1}{2}}(Z) - H^4(n-2)Z I'_{\frac{n-1}{2}}(Z) I''_{\frac{n-1}{2}}(Z) - \frac{H^{n+2}}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} I''_{\frac{n-1}{2}}(Z) \\
&\to -\frac{H^8}{64\pi^4}\frac{5-Z}{(1-Z)^5} \quad (n \to 4) \\
\mathcal{T}^{(3)}(Z) &= 2H^4 I''^2_{\frac{n-1}{2}}(Z) \to \frac{H^8}{32\pi^4}\frac{(3-Z)^2}{(1-Z)^6} \quad (n \to 4) \\
\mathcal{T}^{(4)}(Z) &= H^4 I'_{\frac{n-1}{2}}(Z) I''_{\frac{n-1}{2}}(Z) \to \frac{H^8}{64\pi^4}\frac{(2-Z)(3-Z)}{(1-Z)^5} \quad (n \to 4) \\
\mathcal{T}^{(5)}(Z) &= H^4 I'^2_{\frac{n-1}{2}}(Z) \to \frac{H^8}{64\pi^4}\frac{(2-Z)^2}{(1-Z)^4} \quad (n \to 4) ,
\end{aligned} \tag{4.82}$$

and we have simplified the expressions using the relation

$$(1-Z^2) I''_{\frac{n-1}{2}}(Z) - nZ I'_{\frac{n-1}{2}}(Z) = \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)} , \tag{4.83}$$

which in the massless case replaces (4.55).

Lastly, for the conformally coupled scalar field we obtain

$$\begin{aligned}
\mathcal{T}^{(1)}(Z) &= -\frac{H^{2n}}{4(n-1)(2\pi)^n}\Gamma^2\left(\frac{n}{2}\right)(1-Z)^{-n} \\
\mathcal{T}^{(2)}(Z) &= 0 \\
\mathcal{T}^{(3)}(Z) &= \frac{H^{2n} n}{4(n-1)(2\pi)^n}\Gamma^2\left(\frac{n}{2}\right)(1-Z)^{-n-2} \\
\mathcal{T}^{(4)}(Z) &= \frac{H^{2n} n}{8(n-1)(2\pi)^n}\Gamma^2\left(\frac{n}{2}\right)(1-Z)^{-n-1} \\
\mathcal{T}^{(5)}(Z) &= \frac{H^{2n} n}{8(n-1)(2\pi)^n}\Gamma^2\left(\frac{n}{2}\right)(1-Z)^{-n} .
\end{aligned} \tag{4.84}$$

This expression is especially simple, consisting of single powers of $1-Z$. In fact, up to an overall constant the two-point function of the stress tensor is fixed by conformal invariance for any conformal theory, and has been calculated for a variety of cases by Osborn and collaborators [117, 118]. Since the massless vector field is conformally invariant in four dimensions, its stress tensor two-point function agrees with the above for $n = 4$, up to an overall numerical factor 12.

A check on these results is provided by the conservation of the stress tensor $\nabla_a T^{ab} = 0$, which also applies to its correlation functions (in the case of time-ordered functions, there are additional local terms $\sim \delta(x, x')$). By demanding that $\nabla^a \langle T_{ab}(x) T_{c'd'}(x') \rangle = 0$ and using the decomposition (4.77), we obtain

$$
\begin{aligned}
\mathcal{T}^{(1)\prime}(Z) &= -(n+4-(n+2)^2 Z^2)\mathcal{T}^{(4)}(Z) - (2n+5)(1-Z^2)Z\mathcal{T}^{(4)\prime}(Z) + (1-Z^2)^2 \mathcal{T}^{(4)\prime\prime}(Z) \\
&\quad + (n+1)^2 Z\mathcal{T}^{(5)}(Z) - ((n+2)-(2n+3)Z^2)\mathcal{T}^{(5)\prime}(Z) - Z(1-Z^2)\mathcal{T}^{(5)\prime\prime}(Z) , \\
\mathcal{T}^{(2)}(Z) &= (n+2)Z\mathcal{T}^{(4)}(Z) - (1-Z^2)\mathcal{T}^{(4)\prime}(Z) + (n+1)\mathcal{T}^{(5)}(Z) + Z\mathcal{T}^{(5)\prime}(Z) , \\
(1-Z^2)\mathcal{T}^{(3)\prime}(Z) - (n+3)Z\mathcal{T}^{(3)}(Z) &= (n+2)\mathcal{T}^{(4)}(Z) - (n+2)Z\mathcal{T}^{(4)\prime}(Z) \\
&\quad + (1-Z^2)\mathcal{T}^{(4)\prime\prime}(Z) - (n+2)\mathcal{T}^{(5)\prime}(Z) - Z\mathcal{T}^{(5)\prime\prime}(Z) .
\end{aligned}
\tag{4.85}
$$

The relations are fulfilled in each case, as one can check using the equality (4.55), and we see that there are only two independent functions of Z (and two integration constants) which determine the complete stress tensor two-point function.

4.6. Renormalizing stress tensor correlation functions

As can be seen from the explicit expressions in the last section, the (time-ordered) two-point function of the stress tensor is too singular to be a well-defined distribution in four dimensions, and needs to be renormalized. The leading singularity is of the same strength in all the cases, being proportional to $(1-Z)^{-n}$ which is more singular than the leading singularity in the product of two propagators that is only proportional to $(1-Z)^{2-n}$. This is of course due to the fact that we took derivatives to arrive at the final expressions, and to reduce the strength of the singularity we have to extract those derivatives again, in the form of some fourth-order differential operator acting on a kernel. Unfortunately, this procedure is in general hard to implement, but there is a method that works at least in flat space [58].

Let us take the massive, minimally coupled scalar field as an example. The flat space expressions for the stress tensor two-point function can be obtained by taking $H \to 0$. The limit of Z is given by (4.12), and the limit of $I_\nu(Z)$ was calculated in (4.38). Taking the limit of the expressions (4.78) and (4.81), the stress tensor two-point function then reads

$$
\begin{aligned}
\langle T_{ab}(x) T_{c'd'}(0) \rangle &= 8\eta_{a(c'}\eta_{d')b} I_\nu'^2(x^2) + 32 x_{(a}\eta_{;b)(c'} x_{d')} I_\nu'(x^2) I_\nu''(x^2) + 32 x_a x_b x_{c'} x_{d'} I_\nu''^2(x^2) \\
&\quad + 4(x_a x_b \eta_{c'd'} + \eta_{ab} x_{c'} x_{d'}) \Big[2(n-2) I_\nu'(x^2) I_\nu''(x^2) - m^2 \left(I_\nu(x^2) I_\nu''(x^2) + I_\nu'^2(x^2) \right) \Big] \\
&\quad + \eta_{ab}\eta_{c'd'} \Big[2(n^2-n-4) I_\nu'^2(x^2) + 2m^2 [-(n-1) I_\nu'(x^2) I_\nu(x^2) + 2 I_\nu'^2(x^2)] + m^4 I_\nu^2(x^2) \Big] ,
\end{aligned}
\tag{4.86}
$$

where I_ν now denotes its flat-space limit (4.38), and we have set $x' = 0$, which can be done without loss of generality because of the Poincaré invariance of the correlation function. In this expression, the Wightman prescription $x \to x + \mathrm{i}0 \operatorname{sgn} t$ is implicitly understood; the time-ordered correlation function has the same form but with the prescription $x \to x + \mathrm{i}0$. Each function $I_\nu(x^2)$ has a Mellin-Barnes representation (4.37), but it is also possible to obtain a single Mellin-Barnes representation for the whole stress tensor correlation function, which reads

$$\langle T_{ab}(x)T_{c'd'}(0)\rangle = \int\limits_{C} (m^2)^z (x^2)^{z-n} \frac{1}{4^{z+2}\pi^n} \frac{\Gamma(n-z)\Gamma^2\left(\frac{n}{2}-z\right)\Gamma(-z)}{\Gamma(n+2-2z)} T_{abc'd'}(z) \frac{\mathrm{d}z}{2\pi\,\mathrm{i}} \tag{4.87}$$

with

$$\begin{aligned} T_{abc'd'}(z) = 2\big[&\eta_{ab}\eta_{c'd'}(n^2-n-4-2z(2n-1)+4z^2)+4\eta_{a(c'}\eta_{d')b}\big](n+1-2z)(n-2z) \\ &+16\frac{x_{(a}\eta_{b)(c'}x_{d')}}{x^2}(n-z)(n-2z)(n+1-2z) \\ &-4\frac{\eta_{ab}x_{c'}x_{d'}+x_a x_b\eta_{c'd'}}{x^2}(n-z)(n-2z)(n-2-2z)(n+1-2z) \\ &+8\frac{x_a x_b x_{c'} x_{d'}}{(x^2)^2}(n+1-z)(n-z)(n-2z)^2 \end{aligned} \tag{4.88}$$

(see appendix G for details). Although this result looks complicated, it is quite simple and has a very nice property: the integrand is conserved, independently of the value of z

$$\partial^a\left[(x^2)^{z-n}T_{abc'd'}(z)\right] = 0\,. \tag{4.89}$$

Since this is the most important property of a stress tensor, one may expect that renormalization works at the level of the integrand, which is a simple power of x^2.

As said before, we have to extract differential operators before we can renormalize, and the exact form of those operators can be determined by demanding automatic conservation, independently of the kernel on which they are acting. Furthermore, we have seen in the previous section that there are only two independent functions determining the stress tensor two-point function. We therefore consider the operators

$$S_{ab} = \partial_a\partial_b - \eta_{ab}\,\Box\,, \tag{4.90}$$

which are transverse $\partial^a S_{ab} = 0$ when acting on any tensor $T_{m\cdots n}$. An ansatz for the stress tensor two-point function which respects the symmetries is then given by

$$\langle T_{ab}(x)T_{c'd'}(0)\rangle = 2(S_{a(c'}S_{d')b} - S_{ab}S_{c'd'})f(x^2) + S_{ab}S_{c'd'}\,g(x^2)\,, \tag{4.91}$$

where the functions f and g (the kernels) are unconstrained. To determine them, we make a Mellin-Barnes ansatz similar to the one of the stress tensor correlation function (4.87)

$$f(x^2) = \int\limits_{C} (m^2)^z (x^2)^{z+2-n} \frac{1}{4^{z+2}\pi^n} \frac{\Gamma(n-z)\Gamma^2\left(\frac{n}{2}-z\right)\Gamma(-z)}{\Gamma(n+2-2z)} F(z) \frac{\mathrm{d}z}{2\pi\,\mathrm{i}} \tag{4.92}$$

and analogously for $g(x^2)$. By comparing our ansatz (4.91) with the result (4.87), we obtain five relations for $F(z)$ and $G(z)$ which are solved by

$$\begin{aligned} F(z) &= \frac{2z-n}{2(z+2-n)(z+1-n)(2z+2-n)} \\ G(z) &= \frac{(n-2z)^2}{2(z+2-n)(z+1-n)}\,, \end{aligned} \tag{4.93}$$

so that we obtain

$$\langle T_{ab}(x) T_{c'd'}(0) \rangle = (S_{a(c'} S_{d')b} - S_{ab} S_{c'd'}) \int\limits_{\mathcal{C}} (m^2)^z (x^2)^{z+2-n} \times \frac{\Gamma(n-2-z)\Gamma\left(\frac{n}{2}-1-z\right)\Gamma\left(\frac{n}{2}+1-z\right)\Gamma(-z)}{2^{2z+4}\pi^n \Gamma(n+2-2z)} \frac{\mathrm{d}z}{2\pi\mathrm{i}} + S_{ab} S_{cd} \int\limits_{\mathcal{C}} (m^2)^z (x^2)^{z+2-n} \frac{\Gamma(n-2-z)\Gamma^2\left(\frac{n}{2}+1-z\right)\Gamma(-z)}{2^{2z+3}\pi^n \Gamma(n+2-2z)} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \,. \tag{4.94}$$

The contour runs left of $\mathfrak{Re}\, z = 0$, where the first pole is encountered, and we could in principle close it to the right to obtain a closed-form expression for the stress tensor two-point function in terms of higher hypergeometric functions. However, the residue of this first pole is (in four dimensions) proportional to $(x^2)^{-2}$ which is still not a well-defined distribution, while the residues of all other poles give rise to well-defined distributions (with the corresponding Wightman or Feynman prescription). We therefore lift the contour over this pole (see figure 4.1)

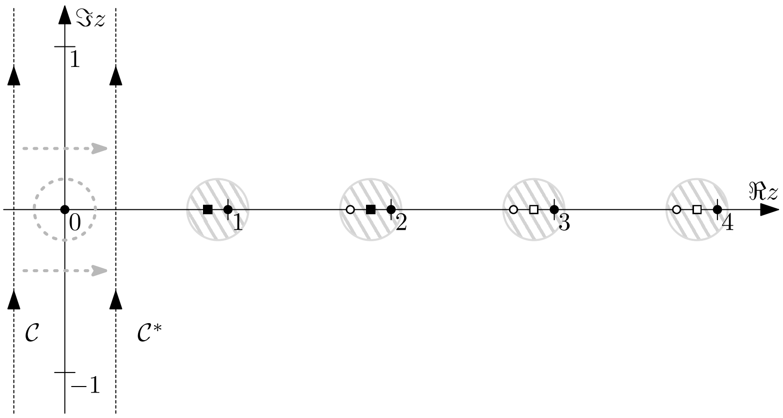

Figure 4.1.: Poles in the first Mellin-Barnes integral appearing in the stress tensor two-point function. There are simple poles at $z = k$ (black dots), at $z = n-2+k$ (white dots) and at $z = \frac{n}{2} - 1 + k$ (black squares) and double poles at $z = \frac{n}{2} + 1 + k$ (white squares), shown here for $n = 3.75$. The original contour $\mathcal{C}$ runs left of all poles. We lift it over the pole at $z = 0$ to obtain the contour $\mathcal{C}^*$ and the residue of this pole (shown in gray). In the remaining integral over $\mathcal{C}^*$, we can take the limit $n \to 4$, and the poles flow together (shaded circles).

to extract the problematic term and take the limit $n \to 4$ in the remaining integral, obtaining

$$\langle T_{ab}(x) T_{c'd'}(0) \rangle = \frac{\Gamma(n-2)\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}+1\right)}{16(n-2)\pi^n \Gamma(n+2)} \left(2 S_{a(c'} S_{d')b} + (n^2-2n-2) S_{ab} S_{c'd'}\right) (x^2)^{2-n} + (S_{a(c'} S_{d')b} - S_{ab} S_{c'd'}) \int\limits_{\mathcal{C}^*} (m^2)^z (x^2)^{z-2} \frac{\Gamma(-z)\Gamma(1-z)\Gamma(2-z)\Gamma(3-z)}{2^{2z+4}\pi^4 \Gamma(6-2z)} \frac{\mathrm{d}z}{2\pi\mathrm{i}} + S_{ab} S_{cd} \int\limits_{\mathcal{C}^*} (m^2)^z (x^2)^{z-2} \frac{\Gamma(-z)\Gamma(2-z)\Gamma^2(3-z)}{2^{2z+3}\pi^4 \Gamma(6-2z)} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \,. \tag{4.95}$$

The term $(x^2)^{2-n}$ can now be renormalized. For this, we first exhibit an explicit factor of $(n-4)^{-1}$ by extracting a d'Alembertian operator,

$$(x^2)^{2-n} = \frac{1}{2(n-3)(n-4)} \Box (x^2)^{3-n} \,. \tag{4.96}$$

We then add and subtract $\mu^{n-4}(x^2)^{\frac{2-n}{2}}$ with an arbitrary mass scale μ (the renormalization scale) which is necessary to have a dimensionally correct result, and partially take the limit $n \to 4$ to get

$$(x^2)^{2-n} = -\frac{1}{4} \Box \left(\frac{\ln(\mu^2 x^2)}{x^2} \right) + \frac{\mu^{n-4}}{2(n-3)(n-4)} \Box (x^2)^{\frac{2-n}{2}} \,. \tag{4.97}$$

The first term is a well-defined distribution in four dimensions, while the second one is proportional to the massless scalar propagator (4.39). Since the Wightman function G^+ fulfills the homogeneous Klein-Gordon equation $\Box G^+(x,x') = 0$, the second term vanishes in this case. For the Feynman propagator G^{F}, however, we get a delta distribution (4.18). Taking the explicit form of the Feynman propagator (4.39), for the Feynman prescription the second term thus reduces to

$$\pi^2 \left(\frac{2}{n-4} - 2 + \gamma + \ln(\pi \mu^2) \right) \mathrm{i} \delta^n(x - x') \,. \tag{4.98}$$

Since the original term $(x^2)^{2-n}$ is well defined whenever one smears it with a test function that has support only off the diagonal $x^2 = 0$, this formula effects the extension to a distribution that is also well defined for test functions which have support on the diagonal, and the procedure is known as dimensional regularization and renormalization in coordinate space [119–121].

Putting everything together, the two-point function of the stress tensor can be decomposed into a regular and a singular part, viz.

$$\begin{aligned}
\langle \mathcal{T} T_{ab}(x) T_{c'd'}(0) \rangle &= \mathrm{i} \langle T_{ab} T_{c'd'} \rangle^{\mathrm{sing}}(x) + \langle T_{ab} T_{c'd'} \rangle^{\mathrm{ren}}(x^2 + \mathrm{i}0) \\
\langle T_{ab}(x) T_{c'd'}(0) \rangle &= \langle T_{ab} T_{c'd'} \rangle^{\mathrm{ren}}(x^2 + \mathrm{i}0 \operatorname{sgn} t) \\
\langle T_{c'd'}(0) T_{ab}(x) \rangle &= \langle T_{ab} T_{c'd'} \rangle^{\mathrm{ren}}(x^2 - \mathrm{i}0 \operatorname{sgn} t) \\
\left\langle \mathcal{T}^{-1} T_{ab}(x) T_{c'd'}(0) \right\rangle &= -\mathrm{i} \langle T_{ab} T_{c'd'} \rangle^{\mathrm{sing}}(x) + \langle T_{ab} T_{c'd'} \rangle^{\mathrm{ren}}(x^2 - \mathrm{i}0) \,.
\end{aligned} \tag{4.99}$$

In this decomposition, the regular part is given by (4.95) with the power $(x^2)^{2-n}$ replaced by the first term in equation (4.97). To simplify this expression further, we extract also from the Mellin-Barnes integral a d'Alembertian operator

$$(x^2)^{z-2} = \frac{1}{4(z-1)z} \Box (x^2)^{z-1} \tag{4.100}$$

and shift the contour back over $z = 0$, adding the corresponding residue (which is now a well defined distribution). This gives

$$\begin{aligned}
\langle T_{ab} T_{c'd'} \rangle^{\mathrm{ren}}(x^2) &= (S_{a(c'} S_{d')b} - S_{ab} S_{c'd'}) \Box \int_C (m^2)^z (x^2)^{z-1} \frac{\Gamma^2(-z)\Gamma(1-z)}{2048 \pi^{\frac{7}{2}} \Gamma\left(\frac{7}{2} - z\right)} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \\
&\quad + S_{ab} S_{cd} \Box \int_C (m^2)^z (x^2)^{z-1} \frac{\Gamma^2(-z)\Gamma(3-z)}{1024 \pi^{\frac{7}{2}} \Gamma\left(\frac{7}{2} - z\right)} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \\
&\quad - \frac{1}{1920 \pi^4} \ln\left(\frac{\mu}{m}\right) (S_{a(c'} S_{d')b} + 3 S_{ab} S_{c'd'}) \Box \left(\frac{1}{x^2} \right) .
\end{aligned} \tag{4.101}$$

The singular part, which only appears for the time-ordered and anti-time-ordered functions, contains everything proportional to $\delta^n(x-x')$ and reads

$$\langle T_{ab} T_{c'd'} \rangle^{\text{sing}}(x) = \frac{1}{960\pi^2} \left(\frac{2}{n-4} + \gamma + \ln\left(\frac{\mu^2}{4\pi}\right)\right) (S_{a(c'} S_{d')b} + 3 S_{ab} S_{c'd'})\, \delta^n(x)\,. \tag{4.102}$$

It can be shown [58] that the regular part coincides with earlier results [114, 122, 123]. Note that the massless limit is finite for the regular part thanks to the last line in (4.101), which cancels the potential logarithmic divergence as $m \to 0$.

In the integral

$$\iint h^{ab}(x) \big\langle T_{ab}(x) T_{c'd'}(x') \big\rangle h^{c'd'}(x')\, \mathrm{d}^n x\, \mathrm{d}^n x'\,, \tag{4.103}$$

we may now integrate these differential operators by parts and simplify the resulting expressions using the curvature tensor expansions from appendix A, specialized to flat space. This results in

$$\iint h^{ab}(x) \big[S_{ab} S_{c'd'} f((x-x')^2) \big] h^{c'd'}(x')\, \mathrm{d}^n x\, \mathrm{d}^n x' = \iint \tilde{R}^{(1)}(x) f((x-x')^2) \tilde{R}^{(1)}(x')\, \mathrm{d}^n x\, \mathrm{d}^n x' \tag{4.104}$$

and

$$\begin{aligned} &\iint h^{ab}(x) \big[S_{a(c'} S_{d')b} f((x-x')^2) \big] h^{c'd'}(x')\, \mathrm{d}^n x\, \mathrm{d}^n x' \\ &\quad = \eta^{am'} \eta^{bn'} \eta^{cp'} \eta^{dq'} \iint \tilde{R}^{(1)}{}_{abcd}(x) f((x-x')^2) \tilde{R}^{(1)}_{m'n'p'q'}(x')\, \mathrm{d}^n x\, \mathrm{d}^n x'\,. \end{aligned} \tag{4.105}$$

That is, the integral (4.103) with the appropriate stress tensor two-point function (4.99) can be written in a manifestly gauge-invariant form involving the linearized curvature tensors, making explicit a theorem by Wald [124]. Using further integration by parts, we can bring the integral involving the singular part $\langle T_{ab} T_{c'd'} \rangle^{\text{sing}}(x)$ into the form

$$\frac{1}{1440\pi^2} \left(\frac{2}{n-4} + \gamma + \ln\left(\frac{\mu^2}{4\pi}\right)\right) \int \left(2\tilde{R}^{(1)}{}_{abcd} \tilde{R}^{(1)abcd} - 2\tilde{R}^{(1)}{}_{ab} \tilde{R}^{(1)ab} + 5\tilde{R}^{(1)2} \right) \mathrm{d}^n x\,, \tag{4.106}$$

where the divergences can be cancelled by choosing the bare coefficients α and β in the effective action (3.15, 3.9) appropriately,

$$\begin{aligned} \alpha &= \alpha(\mu) - \frac{1}{3840\pi^2} \left(\frac{2}{n-4} + \gamma + \ln\left(\frac{\mu^2}{4\pi}\right)\right) \\ \beta &= \beta(\mu) - \frac{1}{2304\pi^2} \left(\frac{2}{n-4} + \gamma + \ln\left(\frac{\mu^2}{4\pi}\right)\right)\,. \end{aligned} \tag{4.107}$$

These counterterms coincide also with well-known results [113].

In de Sitter space, the above construction should in principle work in exactly the same way. The practical implementation fails however in two parts: first, it seems very difficult to obtain a single Mellin-Barnes representation for the stress tensor two-point function à la (4.87) due to a lack of a generalization of Barnes' lemma (G.55) for a higher number of Γ functions in the integrand (in flat space, the propagator representation (4.37) only involves two Γ functions, while in de Sitter we need four (4.35)). Second, it is not clear how the generalization of the differential operators (4.90) and therefore the ansatz (4.91) can be done; while in flat space S_{ab} can be defined alternatively by $R = \kappa S_{ab} h^{ab} + \mathcal{O}\big(\kappa^2\big)$ (see appendix A) and the term $S_{ab}S_{c'd'}$ has a straightforward generalization, the mixed term $S_{a(c'}S_{d')b}$ is not as easy to generalize – because covariant derivatives do not commute, it fails to be transverse. As a replacement, a differential operator obtained by perturbatively expanding the Weyl tensor has been proposed [46, 117, 118], but it is not yet clear if this works in all cases. Nevertheless, we can apply the general idea in specific cases.

Since the massless, conformally coupled scalar field can be treated by flat-space methods after a conformal transformation, we will explain the method for the case of the massless, minimally coupled scalar field. In this case, we first extract the most singular terms from the stress tensor two-point function (4.82), namely those which would not be well defined in four dimensions. For the other terms, we may already take the limit $n \to 4$, and it results

$$\begin{aligned}
\mathcal{T}^{(1)}(Z) &= \frac{H^{2n}}{4(2\pi)^n}\Gamma^2\left(\frac{n}{2}\right)\Bigg[\frac{1}{2}(n^2-n-4)(1-Z)^{-n} + \frac{1}{4}(n^3-5n^2+4n-4)(1-Z)^{-n+1} \\
&\qquad + \frac{1}{16}(n^4-8n^3+19n^2-28n+8)(1-Z)^{-n+2} + 8\frac{n-1}{n(n-2)}(1-Z)^{-\frac{n}{2}}\Bigg] \\
\mathcal{T}^{(2)}(Z) &= \frac{H^{2n}}{4(2\pi)^n}\Gamma^2\left(\frac{n}{2}\right)\Bigg[-\frac{1}{2}n(n-2)(1-Z)^{-n-1} - \frac{1}{4}(n^3-5n^2+6n-4)(1-Z)^{-n} \\
&\qquad - \frac{1}{16}(n^3-7n^2+12n-12)n(1-Z)^{-n+1} - 2\frac{n-1}{n-2}(1-Z)^{-\frac{n}{2}-1}\Bigg] \\
\mathcal{T}^{(3)}(Z) &= \frac{H^{2n}}{4(2\pi)^n}\Gamma^2\left(\frac{n+2}{2}\right)\Bigg[2(1-Z)^{-n-2} + (n-2)(1-Z)^{-n-1} + \frac{1}{4}(n^2-3n-2)(1-Z)^{-n}\Bigg] \\
\mathcal{T}^{(4)}(Z) &= \frac{H^{2n}}{4(2\pi)^n}\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+2}{2}\right)\Bigg[(1-Z)^{-n-1} + \frac{1}{2}(n-1)(1-Z)^{-n} \\
&\qquad + \frac{1}{8}(n+1)(n-2)(1-Z)^{-n+1} - 2\frac{n-1}{n(n-2)}(1-Z)^{-\frac{n}{2}-1}\Bigg] \\
\mathcal{T}^{(5)}(Z) &= \frac{H^{2n}}{4(2\pi)^n}\Gamma^2\left(\frac{n}{2}\right)\Bigg[(1-Z)^{-n} + \frac{n}{2}(1-Z)^{-n+1} + \frac{1}{8}(n+1)n(1-Z)^{-n+2} \\
&\qquad - 4\frac{n-1}{n(n-2)}(1-Z)^{-\frac{n}{2}}\Bigg].
\end{aligned} \tag{4.108}$$

In flat space, we have seen that the integral involving the stress tensor two-point function in the effective action can be written in a manifestly gauge-invariant form involving the linearized Riemann tensor, and so it seems that also in curved spacetime we should be able to write the integral

$$\iint h^{ab}(x)\big\langle T_{ab}(x)T_{c'd'}(x')\big\rangle h^{c'd'}(x')\sqrt{-g}\,\mathrm{d}^n x'\sqrt{-g}\,\mathrm{d}^n x \tag{4.109}$$

in the form

$$\iint \tilde{R}^{(1)ab}{}_{cd}(x)K_{ab}{}^{cd}{}_{m'n'}{}^{p'q'}(x,x')\tilde{R}^{(1)m'n'}{}_{p'q'}(x')\sqrt{-g}\,\mathrm{d}^n x'\sqrt{-g}\,\mathrm{d}^n x \tag{4.110}$$

with a kernel K that needs to be determined. Furthermore, since the stress tensor correlation function is a maximally symmetric bitensor, it is reasonable to suppose that also K is such a bitensor. A complete set of maximally symmetric bitensors that, when antisymmetrized in the index pairs ab, cd, $m'n'$ and $p'q'$, respects the symmetries of the Riemann tensor is given by

$$
\begin{aligned}
{}^{(1)}\mathcal{R}_{abcdm'n'p'q'}(Z) &= g_{ac}g_{bd}g_{m'p'}g_{n'q'} \\
{}^{(2)}\mathcal{R}_{abcdm'n'p'q'}(Z) &= H^{-2}g_{ac}g_{m'p'}\left(g_{bd}Z_{;n'}Z_{;q'} + Z_{;b}Z_{;d}g_{n'q'}\right) \\
{}^{(3)}\mathcal{R}_{abcdm'n'p'q'}(Z) &= H^{-4}g_{ac}g_{m'p'}Z_{;b}Z_{;d}Z_{;n'}Z_{;q'} \\
{}^{(4)}\mathcal{R}_{abcdm'n'p'q'}(Z) &= 4H^{-4}g_{ac}g_{m'p'}Z_{;(b}Z_{;d)(n'}Z_{;q')} \\
{}^{(5)}\mathcal{R}_{abcdm'n'p'q'}(Z) &= 2H^{-4}g_{ac}g_{m'p'}Z_{;b(n'}Z_{;q')d} \\
{}^{(6)}\mathcal{R}_{abcdm'n'p'q'}(Z) &= 2H^{-6}\left(g_{ac}Z_{;m'}Z_{;p'} + Z_{;a}Z_{;c}g_{m'p'}\right)Z_{;b(n'}Z_{;q')d} \\
{}^{(7)}\mathcal{R}_{abcdm'n'p'q'}(Z) &= 2H^{-8}Z_{;a}Z_{;c}Z_{;m'}Z_{;p'}Z_{;b(n'}Z_{;q')d} \\
{}^{(8)}\mathcal{R}_{abcdm'n'p'q'}(Z) &= 8H^{-8}Z_{;(a}Z_{;c)(m'}Z_{;p')}Z_{;b(n'}Z_{;q')d} \\
{}^{(9)}\mathcal{R}_{abcdm'n'p'q'}(Z) &= 4H^{-8}Z_{;a(m'}Z_{;p')c}Z_{;b(n'}Z_{;q')d}\,.
\end{aligned}
\tag{4.111}
$$

Using appendix A, the linearized Riemann tensor in de Sitter spacetime reads

$$
\tilde{R}^{(1)ab}{}_{cd} = -2\left(H^2\delta^{[a}_{[c} + \nabla^{[a}\nabla_{[c}\right)h^{b]}_{d]}\,, \tag{4.112}
$$

and involves two derivatives. If each derivative acts on a coefficient function which multiplies the bitensors (4.111), it lowers the powers of $(1-Z)$ that appear by 4, and since the most divergent power appearing in the stress tensor two-point function (4.108) is $(1-Z)^{-n-2}$ our unknown coefficient functions should involve $(1-Z)^{2-n}$ and lesser powers. We make therefore the ansatz

$$
\begin{aligned}
K_{abcdm'n'p'q'}(Z) = \frac{H^{2n-4}}{8(2\pi)^n}\sum_{k=1}^{9}{}^{(k)}\mathcal{R}_{abcdm'n'p'q'}(Z) \\
\times\left(\alpha_k(1-Z)^{2-n} + \beta_k(1-Z)^{3-n} + 2\gamma_k\frac{(1-Z)^{4-n}-(1-Z)^{2-\frac{n}{2}}}{4-n} + \delta_k\right)
\end{aligned}
\tag{4.113}
$$

with unknown coefficients α_k, β_k, γ_k and δ_k. Integrating by parts in equation (4.110), we can then determine the coefficients by comparison with the result (4.108), and the outcome is given in table 4.2. Since there are nine bitensors in K (4.113) but only five bitensors in the stress tensor two-point function (4.77), there is a large freedom in choosing those coefficients (namely, terms which upon integration by parts vanish identically), and we have taken a choice where as many coefficients as possible are zero.

The only power of $1-Z$ that needs to be renormalized is $(1-Z)^{2-n}$. We proceed as in flat space, extracting first a d'Alembertian operator

$$
(1-Z)^{2-n} = \frac{1}{H^2(n-3)(n-4)}\left(\Box - 2H^2(n-3)\right)(1-Z)^{3-n}\,, \tag{4.114}
$$

and then adding and subtracting $(1-Z)^{\frac{2-n}{2}}$ to obtain

$$
(1-Z)^{2-n} = \left(1-\frac{1}{2H^2}\Box\right)\frac{\ln(1-Z)}{1-Z} + \frac{1}{H^2(n-3)(n-4)}\left(\Box-2H^2(n-3)\right)(1-Z)^{\frac{2-n}{2}}\,, \tag{4.115}
$$

k	1	2	3	4	5	6–8	9
α_k	$\frac{8}{15} - \frac{29}{450}(n-4)$	0	0	0	$-\frac{19}{15} + \frac{8}{225}(n-4)$	0	$\frac{1}{5}$
β_k	$\frac{44}{25}$	$\frac{227}{50}$	$-\frac{48}{5}$	$\frac{299}{50}$	$-\frac{74}{25}$	0	$\frac{1}{5}$
γ_k	3	$-\frac{9}{2}$	0	0	-3	0	0
δ_k	$-\frac{589}{150}$	0	0	$-\frac{12}{5}$	$\frac{387}{75}$	0	0

Table 4.2.: Coefficients of powers of $(1-Z)$ in the kernel K (4.113). Since there is considerable freedom in choosing them, we have taken a choice where as many coefficients as possible vanish. Note that all coefficients may actually be calculated as exact functions of n, but we only give the (sufficient) expansion to order $(n-4)^1$ (for α_1, α_5 and α_9) or $(n-4)^0$ (all other coefficients).

As in flat space, the second term gives a δ distribution only for the Feynman prescription, as one can see from the equation of motion (4.15) for the massless, conformally coupled scalar field

$$\left(\Box - \frac{n(n-2)}{4} H^2 \right) G^{\mathrm{F}}(x,x') = \delta(x,x') \tag{4.116}$$

together with the explicit form for its propagator (4.42). In total, we have

$$\begin{aligned}(1-Z)^{2-n} &= \left(1 - \frac{1}{2H^2} \Box \right) \frac{\ln(1-Z)}{1-Z} - \frac{1}{2(1-Z)} \\ &\quad + \frac{4\pi^2}{H^4} \left(\frac{2}{n-4} - 2 + \gamma - 2\ln H + \ln 2 + \ln \pi \right) \mathrm{i} \frac{\delta^n(x-x')}{\sqrt{-g}} \,,\end{aligned} \tag{4.117}$$

where the local term $\sim \delta^n(x-x')$ is absent for the Wightman prescription. Note that as for the stress tensor expectation value, in this case there was no need to explicitly introduce a renormalization scale μ for dimensional reasons since Z is dimensionless; the Hubble constant H naturally arises as the scale where the theory is renormalized. Of course, one can easily introduce a different renormalization scale μ.

As in flat space (4.99), we can split the regularized expression into a renormalized, finite term and a singular part. The singular part of the integral (4.110) is given by

$$\begin{aligned}&\mathrm{i} \frac{1}{32\pi^2} \left(\frac{2}{n-4} - 2 + \gamma + \ln\left(\frac{H^2}{2\pi} \right) \right) \iint \tilde{R}^{(1)ab}{}_{cd}(x) \tilde{R}^{(1)m'n'}{}_{p'q'}(x') \\ &\quad \times \left[\sum_{k=1,5,9} \alpha_k {}^{(k)}\mathcal{R}_{ab}{}^{cd}{}_{m'n'}{}^{p'q'}(Z) \right] \frac{\delta^n(x-x')}{\sqrt{-g}} \sqrt{-g} \, \mathrm{d}^n x' \sqrt{-g} \, \mathrm{d}^n x \,.\end{aligned} \tag{4.118}$$

We now insert the bitensors ${}^{(k)}\mathcal{R}$ from equation (4.111) and the coefficients α_k from table 4.2, note that in the coincidence limit we have $Z_{;ab'} \to H^2 g_{ab}$ and obtain after some algebraic manipulations (taking advantage of the symmetries of the Riemann tensor)

$$\begin{aligned}
&\mathrm{i}\frac{1}{480\pi^2}\left(\frac{2}{n-4}-2+\gamma+\ln\left(\frac{H^2}{2\pi}\right)\right)\int\left[9\tilde{R}^{abcd}\tilde{R}_{abcd}+\left(-38+\frac{16}{15}(n-4)\right)\tilde{R}^{ab}\tilde{R}_{ab}\right.\\
&\quad\left.+\left(8-\frac{29}{30}(n-4)\right)\tilde{R}^2-\frac{96}{5}(n-4)H^2(\tilde{R}-6H^2)\right]\sqrt{-g}\,\mathrm{d}^n x\,.
\end{aligned}\tag{4.119}$$

This is not yet in the form of the counterterms in S_{Q} (3.9). To bring it into the appropriate form, we first use the fact that we can replace $\sqrt{-g}$ by its perturbed value since our original expression (4.110) was already of second order in the perturbation. Then we use the identity

$$\begin{aligned}
&\int\left(\tilde{R}^{abcd}\tilde{R}_{abcd}-4\tilde{R}^{ab}\tilde{R}_{ab}+\tilde{R}^2\right)\sqrt{-\tilde{g}}\,\mathrm{d}^n x\\
&\quad-2H^2(n-4)(n-3)\int\left(\tilde{R}-\frac{1}{2}(n-2)(n-1)H^2\right)\sqrt{-\tilde{g}}\,\mathrm{d}^n x=\text{const.}+\mathcal{O}\big(\kappa^3\big)\,,
\end{aligned}\tag{4.120}$$

which in four dimensions reduces to the Gauß-Bonnet identity (3.10) and obtain

$$\begin{aligned}
&-\mathrm{i}\frac{1}{1440\pi^2}\left(\frac{2}{n-4}-2+\gamma+\ln\left(\frac{H^2}{2\pi}\right)\right)\int\left[2\left(1-\frac{8}{15}(n-4)\right)\left(\tilde{R}^{abcd}\tilde{R}_{abcd}-\tilde{R}^{ab}\tilde{R}_{ab}\right)\right.\\
&\quad\left.+5\left(1+\frac{11}{30}(n-4)\right)\tilde{R}^2-\frac{2}{5}(n-4)H^2\tilde{R}-\frac{858}{5}(n-4)H^4\right]\sqrt{-\tilde{g}}\,\mathrm{d}^n x\,.
\end{aligned}\tag{4.121}$$

The counterterms we need to cancel these divergences are exactly the same as in flat space (4.107), but here we take the choice of an additional finite renormalization to also remove the finite parts. The terms proportional to $\tilde{R}$ and the constant term do not receive additional divergent contributions, so that the renormalization of Newton's constant and the cosmological constant is still determined by (4.72); however, we can add to those relations the finite renormalizations needed to get rid of the above finite parts. To switch from the renormalization at the scale H to a different scale μ, we just add $\ln(\mu/H)$ to the regular part and subtract it from the singular part. The renormalized kernel K^{ren} is then given by

$$\begin{aligned}
K^{\text{ren}}_{abcdm'n'p'q'}(Z,\mu)=\frac{H^4}{8(2\pi)^4}\sum_{k=1}^{9}{}^{(k)}\mathcal{R}_{abcdm'n'p'q'}(Z)&\left[\alpha_k\left(1-\frac{1}{2H^2}\square\right)\frac{\ln\big((1-Z)\mu^2/H^2\big)}{1-Z}\right.\\
&\left.+\left(\beta_k-\frac{1}{2}\alpha_k\right)(1-Z)^{-1}+\gamma_k\ln(1-Z)+\delta_k\right],
\end{aligned}\tag{4.122}$$

with the coefficients from table 4.2.

This method works in practice for stress tensor correlation functions which have a simple form in four dimensions, so that the number of terms in an expansion around the most singular terms as in (4.108) is manageable. However, there is no reason to suspect it does not work in principle for all stress tensor two-point functions.

With the calculation of these expectation values, the effective action (3.15) is completely determined, and the renormalized effective action reads

$$
\begin{aligned}
S_{\text{eff}}[h] = {} & \frac{1}{\kappa^2(\mu)} \int (\tilde{R} - 2\Lambda(\mu)) \sqrt{-\tilde{g}}\, \mathrm{d}^4 x + \frac{2}{3}\alpha(\mu) \int \left(\tilde{R}^{abcd}\tilde{R}_{abcd} - \tilde{R}^{ab}\tilde{R}_{ab} \right) \sqrt{-\tilde{g}}\, \mathrm{d}^4 x \\
& + \beta(\mu) \int \tilde{R}^2 \sqrt{-\tilde{g}}\, \mathrm{d}^4 x + \frac{1}{2}\kappa(\mu) \int h^{ab} \langle T_{ab} \rangle^{\text{ren}}(\mu) \sqrt{-g}\, \mathrm{d}^4 x \\
& + \kappa^2(\mu) \int h^{ab} h^{cd} \langle U_{abcd} \rangle^{\text{ren}}(\mu) \sqrt{-g}\, \mathrm{d}^4 x \\
& + \frac{\mathrm{i}}{8}\kappa^2(\mu) \iint \tilde{R}^{(1)ab}{}_{cd}(x) K^{\text{ren}}{}_{ab}{}^{cd}{}_{m'n'}{}^{p'q'}(Z(x,x'),\mu) \tilde{R}^{(1)m'n'}{}_{p'q'}(x') \sqrt{-g}\, \mathrm{d}^4 x \sqrt{-g}\, \mathrm{d}^4 x' \,.
\end{aligned}
\tag{4.123}
$$

Note that all time integrations in this effective action run over the CTP contour 3.1; alternatively one can split the contour. Taking the last term as an example, this reads with the proper prescriptions for the kernel K^{ren}

$$
\begin{aligned}
& + \iint \tilde{R}^{(1)+ab}{}_{cd}(x) K^{\text{ren}}{}_{ab}{}^{cd}{}_{m'n'}{}^{p'q'}(Z - \mathrm{i}0, \mu) \tilde{R}^{(1)+m'n'}{}_{p'q'}(x') \sqrt{-g}\, \mathrm{d}^n x \sqrt{-g}\, \mathrm{d}^n x' \\
& - \iint \tilde{R}^{(1)-ab}{}_{cd}(x) K^{\text{ren}}{}_{ab}{}^{cd}{}_{m'n'}{}^{p'q'}(Z - \mathrm{i}0 \operatorname{sgn}(\eta - \eta'), \mu) \tilde{R}^{(1)+m'n'}{}_{p'q'}(x') \sqrt{-g}\, \mathrm{d}^n x \sqrt{-g}\, \mathrm{d}^n x' \\
& - \iint \tilde{R}^{(1)+ab}{}_{cd}(x) K^{\text{ren}}{}_{ab}{}^{cd}{}_{m'n'}{}^{p'q'}(Z + \mathrm{i}0 \operatorname{sgn}(\eta - \eta'), \mu) \tilde{R}^{(1)-m'n'}{}_{p'q'}(x') \sqrt{-g}\, \mathrm{d}^n x \sqrt{-g}\, \mathrm{d}^n x' \\
& + \iint \tilde{R}^{(1)-ab}{}_{cd}(x) K^{\text{ren}}{}_{ab}{}^{cd}{}_{m'n'}{}^{p'q'}(Z + \mathrm{i}0, \mu) \tilde{R}^{(1)-m'n'}{}_{p'q'}(x') \sqrt{-g}\, \mathrm{d}^n x \sqrt{-g}\, \mathrm{d}^n x' \,,
\end{aligned}
\tag{4.124}
$$

where now all time integrations extend over $-\infty < \eta < 0$.

Metric perturbations

He had hardly started, however, before he realized the difficulty which faced him. In his eagerness he had wandered far past the ravines which were known to him, and it was no easy matter to pick out the path which he had taken.

— Sir Arthur Conan Doyle, A study in scarlet

At first sight, a good seems to be a self-evident, trivial thing. Its analysis yields that it is a very intricate thing, full of metaphysical pernicketinesses and theological quirks.
(Eine Ware scheint auf den ersten Blick ein selbstverständliches, triviales Ding. Ihre Analyse ergibt, daß sie ein sehr vertracktes Ding ist, voll metaphysischer Spitzfindigkeit und theologischer Mucken.)

— Karl Marx, Capital

5 Effective field equations and the stability of de Sitter space

5.1. The semiclassical Einstein equation

The semiclassical Einstein equation which describes the backreaction of the matter fields on the background can be obtained by varying the renormalized effective action (4.123) with respect to the perturbation h_{ab} and setting it to zero afterwards. We first define the tensors A_{ab} and B_{ab} by the variations

$$\begin{aligned} A_{ab} &= \frac{\delta}{\sqrt{-g}\,\delta g^{ab}} \int C^{abcd} C_{abcd} \sqrt{-g}\, \mathrm{d}^4 x = -4\nabla^{(m}\nabla^{n)} C_{manb} - 2R^{mn} C_{manb} \\ &= -4R^{mn}R_{manb} + \frac{4}{3} R R_{ab} + g_{ab} R^{mn} R_{mn} - \frac{1}{3} g_{ab} R^2 - 2\,\Box R_{ab} + \frac{2}{3}\nabla_a \nabla_b R + \frac{1}{3} g_{ab}\,\Box R\,, \\ B_{ab} &= \frac{\delta}{\sqrt{-g}\,\delta g^{ab}} \int R^2 \sqrt{-g}\, \mathrm{d}^4 x = \frac{1}{2} g_{ab} R^2 - 2RR_{ab} + 2\nabla_a \nabla_b R - 2 g_{ab}\,\Box R\,. \end{aligned} \tag{5.1}$$

Using the fact that the first term in equation (3.9) reduces to the square of the Weyl tensor in four dimensions plus a multiple of the Euler density (3.10) which vanishes under variation, the semiclassical Einstein equations read

$$\frac{1}{\kappa^2(\mu)} G_{ab} + \frac{\Lambda(\mu)}{\kappa^2(\mu)} g_{ab} = \alpha(\mu) A_{ab} + \beta(\mu) B_{ab} + \frac{1}{2} \langle T_{ab} \rangle^{\text{ren}}(\mu)\,. \tag{5.2}$$

Since the Bunch-Davies vacuum which we chose as the state for the scalar fields gives rise to a de Sitter-invariant stress tensor expectation value as calculated in the last section, the background is still a de Sitter space, but with a quantum-corrected cosmological constant. On this de Sitter background, the tensors A_{ab} and B_{ab} vanish, and the semiclassical Einstein equation then gives the relation between the cosmological constant Λ and the inverse de Sitter radius H

$$\Lambda(\mu) = 3H^2 + \frac{1}{8} \kappa^2(\mu) g^{ab} \langle T_{ab} \rangle^{\text{ren}}(\mu)\,. \tag{5.3}$$

This relation only depends on the matter field content of the theory and is independent of the unknown parameters $\alpha(\mu)$ and $\beta(\mu)$, so that it is an unambiguous prediction of low-energy effective quantum gravity.

5.2. Equations for metric perturbations

The equations for the perturbations h_{ab} can be obtained by the same variation. When one splits the CTP contour, it is necessary to take a variational derivative with respect to the + fields only, and set afterwards $h^+_{ab} = h^-_{ab} = h_{ab}$. In the other case, it is still useful to split the CTP contour after variation to exhibit clearly the causal structure of the resulting equations. Of course, in both cases the same result is obtained, and the only difference to an *in-out* treatment is that we have to take the difference between the kernels K^{ren} (4.122) with the Feynman and the negative Wightman prescription, instead of only the Feynman one. After taking into account the semiclassical background (5.3), these equations read

$$\begin{aligned}
&\frac{1}{\kappa^2(\mu)} \tilde{G}^{(1)}{}^b_a(x) = \alpha(\mu)\tilde{A}^{(1)}{}^b_a(x) + \beta(\mu)\tilde{B}^{(1)}{}^b_a(x) + 2h^{cd} g^{bs} \langle U_{ascd} \rangle^{\text{ren}}(x,\mu) \\
&- \frac{\mathrm{i}}{2} \int \left[H^2 \delta^c_d + \nabla_d \nabla^c \right] \left(K^{\text{ren}}(Z(x,x') - \mathrm{i}0, \mu) - K^{\text{ren}}(Z(x,x') + \mathrm{i}0 \operatorname{sgn}(\eta - \eta')) \right)_{[ac]}{}^{[bd]}{}_{m'n'}{}^{p'q'} \\
&\qquad \times \tilde{R}^{(1)m'n'}{}_{p'q'}(x') \sqrt{-g}\, \mathrm{d}^4 x' \,,
\end{aligned} \tag{5.4}$$

with the time integration ranging over $-\infty < \eta < 0$. They are strictly causal, since the integral is constrained to the past lightcone of the point x. If x' and x are timelike related, the difference between the kernels K^{ren} vanishes if x' is to the future of x, and if x' and x are spacelike related the prescription is not necessary to define a distribution as one can see from the explicit form (4.122), so that the integrand also vanishes. Furthermore, these equations are of higher than second order, and so would need more initial conditions when taken at face value. In this case, they would admit so-called *runaway* solutions which grow very quickly, on a timescale of order $1/\kappa^2$. These solutions are therefore outside of the validity of the effective field theory approach, where corrections to the lowest-order solutions must be small, and it is now generally accepted that early results based on such solutions [125–128] must be regarded as unphysical. In order to eliminate these unphysical solutions, there are several approaches. One could, for instance, simply calculate all solutions and then simply disregard all that show this unstable behaviour [129]. However, solving the full equations is hard and in many cases simply impossible. Another possibility would be a perturbative development of the solution $h_{ab} = h^{(0)}_{ab} + \kappa^2 h^{(1)}_{ab}$. The lowest order equation is then solved for $h^{(0)}_{ab}$, and the result is treated as a source term in the equation for $h^{(1)}_{ab}$. However, in many situations the real expansion parameter is not the small parameter (in this case κ^2), but this quantity multiplied by some characteristic scale of the problem such as the total time. The perturbative expansion then ceases to be valid when this product grows, so that it may fail to capture the correct long-time behaviour due to the appearance of these so-called *secular* terms. A method with overcomes this limitation is the *order reduction* method, which transforms the original higher-order equation into one that is valid up to the same order in the perturbative parameter, but which does not contain higher-derivative terms. Solutions of the order-reduced equation are then taken as exact (i.e., without a further expansion in the perturbative parameter). They agree locally with the perturbative solutions constructed around any point in time, but provide a long-time interpolation between all of them, so that long-time phenomena can be studied. This is especially important when corrections are locally small, but can build up over time and give rise to large accumulated effects. Examples of such situations are given by the circular motion of an electric charge in a uniform magnetic field, where the continuous emission of electromagnetic radiation slowly decreases its energy and thus the radius of its orbit, or an evaporating black hole which slowly decreases its mass because of the continuous emission of Hawking radiation. Order reduction has been applied to problems of this type [130–133], including calculations in semiclassical gravity [43, 134]. One should take into account that sometimes order reduction cannot be applied in a straightforward way [43], however, it is perfectly viable in our case.

The lowest order term of equation (5.4) reads

$$\tilde{G}^{(1)b}{}_a(x) = \mathcal{O}\big(\kappa^2\big) \,, \tag{5.5}$$

and we may substitute this equation on the right-hand side of (5.4) to obtain an equivalent equation which is valid to the same order in κ^2. The explicit expressions (5.1) show that the tensors $\tilde{A}^{(1)}_{ab}$ and $\tilde{B}^{(1)}_{ab}$ all involve the Ricci tensor or scalar, and so are of higher order in κ^2. Moreover, all terms in the kernel K^{ren} which involve a metric, the terms ${}^{(k)}\mathcal{R}$ for $k = 1, \dots, 5$ (4.111), contract the linearized Riemann tensor in the integral to a linearized Ricci tensor which thus also is of higher order in κ^2. Since for the massless, minimally coupled scalar for which we calculated the kernel K^{ren} explicitly in the last section only the term involving ${}^{(9)}\mathcal{R}$ is non-vanishing, let us focus on this term (for the others, a similar analysis can be done). Let us further assume for a moment that the coefficient in the renormalized kernel K^{ren} were a simple function $f(Z)$ as opposed to a genuine distribution. In this case, we can insert the explicit expression from equation (4.111) into the integral and perform the covariant derivatives to obtain (using the contractions for covariant derivatives of Z (4.11) and omitting overall constant factors)

$$\int Z_{;bn'} Z^{;dq'} \left(Z_{;m'} Z^{;p'} g'(Z) - H^2 \delta^{p'}_{m'} Z f'(Z) \right) \tilde{R}^{(1)m'n'}{}_{p'q'}(x') \sqrt{-g} \, \mathrm{d}^4 x' \tag{5.6}$$

with

$$g(Z) = \int \left(15 f(Z) + 9 Z f'(Z) + Z^2 f''(Z) \right) \mathrm{d}Z \,. \tag{5.7}$$

We now write $Z_{;m'} g'(Z) = \nabla_{m'} g(Z)$ and integrate by parts, which leads to

$$\begin{aligned} & H^2 \int Z_{;bn'} Z^{;dq'} \delta^{p'}_{m'} Z [g(Z) - f'(Z)] \tilde{R}^{(1)m'n'}{}_{p'q'}(x') \sqrt{-g} \, \mathrm{d}^4 x' \\ & - H^2 \int Z_{;bn'} Z^{;d} Z^{;q'} \delta^{p'}_{m'} g(Z) \tilde{R}^{(1)m'n'}{}_{p'q'}(x') \sqrt{-g} \, \mathrm{d}^4 x' \\ & - \int Z_{;bn'} Z^{;dq'} Z^{;p'} g(Z) \nabla_{m'} \tilde{R}^{(1)m'n'}{}_{p'q'}(x') \sqrt{-g} \, \mathrm{d}^4 x' \,. \end{aligned} \tag{5.8}$$

The terms involving $\delta^{p'}_{m'}$ contract the linearized Riemann tensor into a linearized Ricci tensor, which is of higher order in κ^2 and can be neglected. For the last term, we use the second Bianchi identity

$$\tilde{\nabla}_{[s} \tilde{R}_{ab]}{}^{cd} = 0 \tag{5.9}$$

can be contracted to give

$$\tilde{\nabla}_m \tilde{R}^{mn}{}_{pq} = 2 \tilde{\nabla}_{[p} \tilde{R}_{q]}{}^n \,. \tag{5.10}$$

To linear order in the perturbation h_{ab}, the covariant derivative can be taken to be the background derivative ∇, and since the background Riemann tensor (4.4) has vanishing covariant derivative, we conclude that the identity is applicable to the last term in the integral (5.8). Therefore, also this term is of higher order when we use order reduction.

However, in our case the kernel K^{ren} is a genuine distribution, and this procedure is not directly applicable. What is shows instead is that non-local terms in K^{ren} do not contribute in equation (5.4) when we use order reduction, but there are local terms $\sim \delta^4(x-x')$ which remain. The correct extraction of those local terms is difficult and can be best performed by calculating the spatial Fourier transform of the kernel K^{ren}.

The same order reduction procedure can of course be applied to the case of other fields, for which we did not calculate the kernel K^{ren}. However, it should be given by the same general form, i.e., a sum of the nine bitensors (4.111) with coefficients which are distributions of Z, and so also only the local terms in this coefficients contribute.

5.2.1. The massless, conformally coupled scalar

Since the calculation of the effective action for the massless, conformally coupled scalar field by Campos and Verdaguer [67, 116] was done using a conformal transformation to flat space, it is easy to perform a spatial Fourier transform on the corresponding kernels. In this case, it turns out that also for the metric perturbations it is easier to work in the conformally related almost flat spacetime with metric $\hat{g}_{ab}$ defined by

$$\tilde{g}_{ab} = g_{ab} + \kappa h_{ab} = \frac{1}{(-H\eta)^2}\hat{g}_{ab} = \frac{1}{(-H\eta)^2}\left(\eta_{ab} + \kappa\hat{h}_{ab}\right), \tag{5.11}$$

and all tensors with a hat refer to this metric. It is important to keep in mind that these tensors get their indices moves with the hatted metric $\hat{g}_{ab}$, and so the "hatted version" of tensor fields depends on their index position. For the curvature tensors, the corresponding transformed tensors are given in appendix B. For N fields, the equations for the metric perturbations then read

$$\begin{aligned}\tilde{G}_{ab} + \Lambda\tilde{g}_{ab} &= \alpha(\mu)A_{ab} + \left(\beta - \frac{N}{960\pi^2}\right)\kappa^2\tilde{B}_{ab} \\ &+ \frac{N}{5760\pi^2}\kappa^2\left[-\tilde{R}_{am}\tilde{R}^m_b + \frac{2}{3}\tilde{R}\tilde{R}_{ab} + \frac{1}{2}\tilde{g}_{ab}\tilde{R}^{mn}\tilde{R}_{mn} - \frac{1}{4}\tilde{g}_{ab}\tilde{R}^2 - 2\tilde{R}^{mn}\tilde{C}_{ambn}\right] \\ &+ \frac{N}{1920\pi^2}\kappa^2H^2\eta^2\left[-4\nabla^m\nabla^n\left(\hat{C}_{ambn}\ln(-H\eta)\right) + \int H(x-x',\mu)\hat{A}_{ab}(x')\,\mathrm{d}^4x'\right].\end{aligned} \tag{5.12}$$

These equations also encompass the semiclassical Einstein equations (5.2). Furthermore, in the calculation a choice $\bar{\mu}$ of the renormalization scale was made such that the coefficient in front of the squared Weyl tensor in (4.123) vanishes, $\alpha(\bar{\mu}) = 0$, but we have restored it here for clarity. A further peculiarity of conformal fields is the fact that they do not renormalize either the cosmological or Newton's constant since the expectation values of T_{ab} and U_{abcd} (4.68) vanish, and consequently Λ and κ^2 do not depend on the renormalization scale μ. Moreover, also the coefficient β does not receive any renormalization at all. Finally, the kernel $H(x-x',\mu)$, which in this case is the nonlocal part of K^{ren}, is given by

$$H(x,\mu) = -\frac{1}{2}\int\left[\ln\left|\frac{p^2}{\mu^2}\right| - \mathrm{i}\pi\Theta(-p^2)\operatorname{sgn}p^0\right]\mathrm{e}^{\mathrm{i}px}\frac{\mathrm{d}^4p}{(2\pi)^4}, \tag{5.13}$$

and by employing the Fourier transforms of appendix D, we can write it in the form

$$H(x,\mu) = -\frac{1}{2}\int\cos\left[|\boldsymbol{p}|(\eta-\eta')\right]\mathcal{P}'\frac{\Theta(\eta-\eta')}{\eta-\eta'}\,\mathrm{e}^{\mathrm{i}\boldsymbol{px}}\frac{\mathrm{d}^3p}{(2\pi)^3}. \tag{5.14}$$

Setting the perturbation h_{ab} to zero, we obtain the semiclassical Einstein equation (5.2) for this case, which in accordance with the general formula (5.3) determines the relation between Λ and the de Sitter radius H^{-1},

$$\Lambda = 3H^2 \left(1 - \frac{N}{5760\pi^2} \kappa^2 H^2 \right) . \tag{5.15}$$

For non-vanishing perturbations, these equations are invariant under the gauge transformations induced by local diffeomorphisms of the perturbed de Sitter space, namely

$$h_{ab} \to h_{ab} + 2\nabla_{(a}\xi_{b)} \tag{5.16}$$

with an arbitrary vector ξ_a. In terms of the rescaled perturbation $\hat{h}_{ab}$ and $\hat{\xi}_a = (-H\eta)^2 \xi_a$, this reads

$$\hat{h}_{ab} \to \hat{h}_{ab} + 2\partial_{(a}\hat{\xi}_{b)} + \frac{2}{\eta}\eta_{ab}\hat{\xi}_0 . \tag{5.17}$$

We may use this freedom to fix the gauge and set some components of the perturbation to zero. For this, we first decompose both $\hat{h}_{ab}$ and $\hat{\xi}_a$ into tensorial, vectorial and scalar parts with respect to spatial transformations and rotations [133, 135]. The spatial part $\hat{h}_{\mu\nu}$ decomposes as

$$\hat{h}_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} + 2\partial_{(\mu}w^{\mathrm{T}}_{\nu)} + \partial_\mu\partial_\nu\sigma + \tau\delta_{\mu\nu} , \tag{5.18}$$

where $\delta^{\mu\nu}\partial_\mu h^{\mathrm{TT}}_{\nu\rho} = 0 = \delta^{\mu\nu}h^{\mathrm{TT}}_{\mu\nu}$ and $\delta^{\mu\nu}\partial_\mu w^{\mathrm{T}}_\nu = 0$. The temporal components can be similarly decomposed as

$$\hat{h}_{0\mu} = v^{\mathrm{T}}_\mu + \partial_\mu\psi , \qquad \hat{h}_{00} = \phi , \tag{5.19}$$

where $\delta^{\mu\nu}\partial_\mu v^{\mathrm{T}}_\nu = 0$. In total, we have four scalars ϕ, ψ, σ, τ, two transverse vectors w^{T}_μ and v^{T}_μ (with two independent components each) and a transverse traceless tensor $h^{\mathrm{TT}}_{\mu\nu}$ (with two independent components as well). Decomposing also the spatial part of the vector field $\hat{\xi}_\mu$ as

$$\hat{\xi}_\mu = \xi^{\mathrm{T}}_\mu + \partial_\mu\xi , \tag{5.20}$$

where $\delta^{\mu\nu}\partial_\mu\xi^{\mathrm{T}}_\nu = 0$, we obtain the behavior of the various components under a gauge transformation,

$$\begin{aligned} h^{\mathrm{TT}}_{\mu\nu} &\to h^{\mathrm{TT}}_{\mu\nu} & w^{\mathrm{T}}_\mu &\to w^{\mathrm{T}}_\mu + \xi^{\mathrm{T}}_\mu \\ \sigma &\to \sigma + 2\xi & \tau &\to \tau + \frac{2}{\eta}\xi_0 \\ v^{\mathrm{T}}_\mu &\to v^{\mathrm{T}}_\mu + \xi'^{\mathrm{T}}_\mu & \psi &\to \psi + \xi' + \xi_0 \\ \phi &\to \phi + 2\xi'_0 - \frac{2}{\eta}\xi_0 . \end{aligned} \tag{5.21}$$

Choosing ξ^{T}_μ, ξ and ξ_0 appropriately, we can set w^{T}_μ, σ and τ to zero, so that the perturbation of the spatial metric is entirely given by the tensorial component

$$h_{\mu\nu} = h^{\mathrm{TT}}_{\mu\nu} . \tag{5.22}$$

This is the *transverse traceless* gauge, also known as *spatially flat* gauge when focusing on the scalar perturbations. If we restrict ourselves to metric perturbations that fall off at spatial infinity, the gauge is completely fixed by this choice. On the other hand, if we also include perturbations which tend to a constant or grow at infinity, there is some residual gauge freedom left.

On one hand, there are transformations which leave $h_{\mu\nu}$ invariant. One possibility are those generated by ξ^{T}_μ which are functions only of the conformal time η, and which do not change the spatial part of the perturbation because w^{T}_μ only enters by a spatial derivative. However, they change v^{T}_μ and thus temporal components of $\hat{h}_{ab}$. A second possibility involves transformations generated by $\xi = f(\eta)\boldsymbol{r}^2$, $\xi_0 = -2\eta f(\eta)$ which change σ and τ in such a way as to leave $\hat{h}_{\mu\nu}$ invariant; but ϕ and ψ are changed by this transformation. On the other hand, the transformations generated by $\xi^{\mathrm{T}}_\mu = E_{\mu\nu}x^\nu$ with a constant traceless matrix $E_{\mu\nu}$ induce changes of $h_{\mu\nu}$, but leave it transverse and traceless. These residual gauge transformations will play a role in the next section to show that certain solutions are pure gauge.

The full form of the equations (5.12) with the metric perturbations decomposed and gauge-fixed is long and complicated. For tensor perturbations, these equations coincide with ones derived by Starobinsky [56, 126]. Nevertheless, after applying order reduction as explained above most terms are of higher order and do not contribute. Introducing the parameter

$$\varsigma = \frac{N}{960\pi^2}\kappa^2 H^2\,, \tag{5.23}$$

the order-reduced equations read

$$h''^{\mathrm{TT}}_{\mu\nu} - \frac{2}{\eta}(1-\varsigma)h'^{\mathrm{TT}}_{\mu\nu} - (1-2\varsigma)\,\triangle\, h^{\mathrm{TT}}_{\mu\nu} = \mathcal{O}\big(\kappa^4\big)\,, \tag{5.24a}$$

$$\triangle v^{\mathrm{T}}_\mu = \mathcal{O}\big(\kappa^4\big)\,, \tag{5.24b}$$

$$\triangle \phi = \mathcal{O}\big(\kappa^4\big)\,, \tag{5.24c}$$

$$\triangle\psi + \frac{3}{2\eta}\left(1+\frac{4}{9}\varsigma\right)\phi = \mathcal{O}\big(\kappa^4\big)\,. \tag{5.24d}$$

As explained in section 5.2, after order reduction these equations do not contain any nonlocal terms, and are independent of the unknown parameter $\alpha(\mu)$ and $\beta(\mu)$, depending only on the matter content (in this case N massless, conformally coupled scalars) through the parameter ς.

5.2.2. Stability of de Sitter space

The solutions of equation (5.24b) for the components of the vector perturbation v^{T}_μ are arbitrary functions of time, which can be eliminated by a gauge transformation. Indeed, by using the residual gauge freedom described in the last section and choosing ξ^{T}_μ as an appropriate function of time only, we can set $v^{\mathrm{T}}_\mu = 0$.

For ϕ, the solution of equation (5.24c) is also an arbitrary function of time. The solution for ψ is then given by

$$\psi = f(\eta) - \frac{1}{4\eta}\left(1+\frac{4}{9}\nu\right)\phi(\eta)\boldsymbol{r}^2\,. \tag{5.25}$$

Since ψ enters into the perturbation $\hat{h}_{ab}$ only through a spatial derivative according to equation (5.19), the arbitrary function $f(\eta)$ does not change the perturbation $\hat{h}_{ab}$ and we can set it to zero. If we want to start with bounded initial perturbations, we must exclude solutions which are unbounded and have to take $\phi = 0$. On the other hand, if we had not excluded such unbounded solutions, we need to take advantage of the residual gauge transformations explained in the previous section. We would first have to choose ξ_0 appropriately to make ϕ vanish, and would need to take a similar unbounded function $\xi = -1/(2\eta)\xi_0 \boldsymbol{r}^2$ so that the combination $\partial_\mu\partial_\nu\sigma + \tau\delta_{\mu\nu}$ which enters into the decomposition of the perturbation (5.18) still vanishes. The solution for ψ is then an arbitrary function of time which we can set to zero as above.

Both vector and scalar parts are therefore pure gauge when order reduction is employed and can be completely eliminated by a residual gauge transformation of the kind mentioned at the end of the last section. This conclusion was also reached by Anderson et al. [129], who investigated scalar perturbations without using order reduction (called perturbations of the first kind in their work) and found only solutions which lie outside the range of validity of the semiclassical theory. However, their intermediate expressions are not directly comparable to the ones presented here since they introduce gauge-invariant variables which only have simple forms in a gauge quite different from the above.

For the tensor perturbations we first take the Fourier transform with respect to the spatial coordinates, which gives

$$h^{\mathrm{TT}}_{\mu\nu}(\eta, \boldsymbol{x}) = \sum_{s=\pm} \int e^s_{\mu\nu}(\boldsymbol{p}) g_s(\eta, \boldsymbol{p})\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{x}} \frac{\mathrm{d}^3 p}{(2\pi)^3}\,, \tag{5.26}$$

where $e^\pm_{\mu\nu}(\boldsymbol{p})$ are a pair of transverse and traceless tensors corresponding to two different polarizations. Equation (5.24a) then becomes

$$g''_\pm(\eta, \boldsymbol{p}) - \frac{2}{\eta}(1-\varsigma) g'_\pm(\eta, \boldsymbol{p}) + (1-2\varsigma)\boldsymbol{p}^2 g_\pm(\eta, \boldsymbol{p}) = \mathcal{O}\big(\kappa^4\big)\,. \tag{5.27}$$

Setting $\omega^2 = (1-2\varsigma)\boldsymbol{p}^2$ and $g_\pm = (-\omega\eta)^{\frac{3}{2}-\varsigma} f_\pm(-\omega\eta)$, this reduces to a Bessel equation for $f_\pm$, whose general solution is

$$g_\pm(\eta, \boldsymbol{p}) = (-\omega\eta)^{\frac{3}{2}-\varsigma}\Big[C^\pm_1 \mathrm{J}_{\frac{3}{2}-\varsigma}(-\omega\eta) + C^\pm_2 \mathrm{Y}_{\frac{3}{2}-\varsigma}(-\omega\eta)\Big]\,, \tag{5.28}$$

where $C^\pm_1$ and $C^\pm_2$ are integration constants which may depend on ω. The solution for $|\boldsymbol{p}| = 0$ can then be obtained by choosing $C^\pm_i$ appropriately as functions of ω and taking the limit $\omega \to 0$, resulting in

$$h^{\mathrm{TT}}_{\mu\nu}(\eta, \boldsymbol{x}) = C^1_{\mu\nu}(-\eta)^{3-2\varsigma} + C^2_{\mu\nu} \tag{5.29}$$

with two constant and traceless tensors $C^i_{\mu\nu}$. The second term can be eliminated by a residual gauge transformation with $\xi^{\mathrm{T}}_\mu = C^2_{\mu\nu} x^\nu/2$, as explained in the previous section.

To discuss the importance of those solutions, we first need to consider an appropriate observable. While the tensor perturbation $h^{\mathrm{TT}}_{\mu\nu}(\eta, \boldsymbol{x})$ is a gauge-invariant object (as can be seen from equation (5.21)) with a well-defined meaning, namely the amplitude of free gravitons, it is not local and can therefore not be completely measured by an observer which only has access to a region of finite physical site. Given a generic perturbation, the extraction of the tensorial component needs the specification of boundary conditions at spatial infinity, but these are beyond the reach of a realistic observer. This question and its implications have been recently discussed by Tanaka and Urakawa [136].

An observable which characterizes the geometry in a fixed physical region, i.e., locally, is the (linearized) Riemann tensor. In terms of the conformally rescaled metric perturbation $\hat{h}_{ab}$, it is given from (4.112) by

$$\tilde{R}^{(1)ab}{}_{cd} = 2H^2 \delta^a_{[c} \delta^b_{d]} \hat{h}_{00} + 2H^2 \eta^2 \eta^{m[a} \eta^{b]n} \partial_n \partial_{[c} \hat{h}_{d]m} + 2H^2 \eta \delta^m_{[c} \delta^{[b}_{d]} \eta^{a]n} \left(2\partial_{(m} \hat{h}_{n)0} - \hat{h}'_{mn} \right) . \tag{5.30}$$

For tensor perturbations, we perform a Fourier transform with respect to the spatial coordinates to obtain

$$\tilde{R}^{(1)ab}{}_{cd} = 2H^2 \int \left(S_+^{[ab]}{}_{[cd]} + S_-^{[ab]}{}_{[cd]} \right) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{x}} \frac{\mathrm{d}^3 p}{(2\pi)^3} , \tag{5.31}$$

where

$$\begin{aligned} S_{\pm}^{0\mu}{}_{0\rho} &= -\eta^{\mu\alpha} e^{\pm}_{\alpha\rho} \eta g'_{\pm} , \\ S_{\pm}^{\mu\nu}{}_{0\rho} &= \mathrm{i}\eta \eta^{\mu\alpha} p_\alpha S_{\pm}^{0\nu}{}_{0\rho} , \\ S_{\pm}^{\mu\nu}{}_{\rho\sigma} &= \eta^{\mu\alpha} p_\alpha p_\rho (e^{\pm})^\nu_\sigma \eta^2 g_{\pm} + \delta^\mu_\rho S_{\pm}^{0\nu}{}_{0\sigma} . \end{aligned} \tag{5.32}$$

Hence, we can see that all the Riemann components can be written in terms of $g_{\pm}$ and $g'_{\pm}$. Since everything that will be said is entirely equivalent for both polarizations, in the remainder of this section we will omit the subindices $\pm$ labeling the two transverse polarizations associated with each momentum $\boldsymbol{p}$.

Let us now calculate the late-time limit $\eta \to 0$ of the above expressions. From the known behaviour of the Bessel functions [104], the solution for $g(\eta)$ from equation (5.28) tends to

$$\begin{aligned} g &\to -\frac{C_2}{\pi} \Gamma\left(\frac{3}{2} - \varsigma \right) 2^{\frac{3}{2}-\varsigma} = \text{const.} \\ g' &\to \omega \frac{C_2}{\pi} \Gamma\left(\frac{1}{2} - \varsigma \right) 2^{\frac{1}{2}-\varsigma} = \text{const.} \end{aligned} \tag{5.33}$$

The quantum correction to the Riemann tensor (5.31) vanishes as $\mathcal{O}(\eta)$ at late times; thus, the constant limit of g as $\eta \to 0$ is pure gauge just like the second term in (5.29). De Sitter spacetime is therefore stable against any linear perturbation in the semiclassical limit, since (as shown before) the vector and scalar perturbations are gauge-equivalent to zero. This extends classical results known as "no-hair" theorems [35–40], which are obtained for $\varsigma = 0$; in fact the only effect of the quantum corrections is to alter the order of the Bessel functions appearing in the solution (5.28) so that they fall off a little slower than the uncorrected ones.

So far, we have not paid much attention to initial conditions, integrating freely by parts in equations such as (5.8). Because in the CTP formalism the equations are causal, as we have seen above, partial integration cannot create surface terms at future timelike or spatial infinity, but there may be surface terms at the initial surface where the quantum state was prepared. Furthermore, as explained in section 3.1, the interacting quantum state differs from the vacuum state of the free theory, and appropriate cross correlations must be taken into account, so-called initial state corrections [122, 137]. In the Poincaré patch of de Sitter spacetime, the iϵ prescription explained in section 3.1 selects the right asymptotic initial state at past infinity (as demonstrated in section 6 where two-point functions of h_{ab} are calculated), and in this case integration by parts does not generate extra correction terms. However, if we want to consider an initial state at a finite time η_0, we have to take these corrections into account. Furthermore, one may consider a state which is not vacuum at the initial time, but contains some excitations.

All these effects can be subsumed by an additional stress tensor δT_{ab} on the right-hand side of equation (5.4) (or for the conformal case, equation (5.12)). In this way, one can of course also incorporate an additional classical stress tensor provided it is not too large such that the linear approximation breaks down (this has been analyzed for the classical theory in [138, 139]). Especially, it has been shown in [47, 48] that a suitable stress tensor for a wide class of initial states can be generated by evolving an asymptotic Bunch-Davies vacuum state from $-\infty$ to the initial time η_0 in a given, nondynamical perturbed geometry which is asymptotically of the de Sitter form as $\eta \to -\infty$ and matches smoothly to the dynamical metric $\tilde{g}_{ab}$ at η_0. The corresponding stress tensor is then given by

$$\delta T_{ab}(\eta, \boldsymbol{p}) = \frac{N\alpha}{960\pi^2}(-H\eta)^2 \int_{-\infty}^{\eta_0} \hat{A}_{ab}(\eta', \boldsymbol{p}) H(\eta - \eta', \boldsymbol{p}, \bar{\mu}) \, \mathrm{d}\eta' \,, \tag{5.34}$$

i.e., the nonlocal part of (5.12) integrated over the preparation period. Note that while the kernel H depends on η, the integration only extends until η_0 because afterwards $\hat{A}_{ab}$ vanishes when we use order reduction as explained earlier. The metric perturbations during this initial period can be fairly arbitrary, subject only to the requirement that they decay quickly enough as $\eta \to -\infty$ so that the above integral converges, small enough so that the linear approximation is valid for all times and matching smoothly enough at η_0 (up to the fourth derivative, which is the maximum that occurs in A_{ab} (5.1) – this is the condition for the initial state to be of fourth adiabatic order which is a standard requirement for finite stress tensor expectation values [140]). More extensive studies on the conditions needed to obtain a well-defined, finite initial state have also been done [141–143]. As long as $\eta > \eta_0$, the integral is finite under these conditions, and especially has a well-defined limit as $\eta \to 0$. Therefore, these contributions fall off at least as fast as η^2 when $\eta \to 0$.

After order reduction the generalization of equations (5.24) reads

$$h''^{\mathrm{TT}}_{\mu\nu} - \frac{2}{\eta}(1-\varsigma) h'^{\mathrm{TT}}_{\mu\nu} - (1-2\varsigma) \triangle h^{\mathrm{TT}}_{\mu\nu} = \kappa^2 \delta T^{\mathrm{TT}}_{\mu\nu} + \mathcal{O}\big(\kappa^4\big) \,, \tag{5.35a}$$

$$\triangle v^{\mathrm{T}}_{\mu} = -\kappa^2 \delta T^{\mathrm{T}}_{0\mu} + \mathcal{O}\big(\kappa^4\big) \,, \tag{5.35b}$$

$$\triangle \phi = -\kappa^2 \left(\delta T_{00} - \frac{1}{2}\eta \delta T'_{00} + \frac{1}{2}\eta^{ab} \delta T_{ab} \right) + \mathcal{O}\big(\kappa^4\big) \,, \tag{5.35c}$$

$$\triangle \psi + \frac{3}{2\eta}\left(1 + \frac{4}{9}\varsigma\right)\phi = \frac{1}{4}\eta \kappa^2 \delta T_{00} + \mathcal{O}\big(\kappa^4\big) \,, \tag{5.35d}$$

where we introduced a decomposition of this stress tensor exactly analogous to the case of the metric perturbations (5.18, 5.19), taking into account its covariant conservation. In the case of vector and scalar perturbations, the elliptic equation gets transformed into a algebraic equation in spatial Fourier space, and the fall-off of those components can be directly inferred from the fall-off of the corresponding stress tensor components. For the above explained construction, the stress tensor components decay at least as fast as η^2, and therefore also v^{T}_{μ} and ϕ decay at least as fast, while (due to the extra factor $1/\eta$ in the corresponding equation) we can only assert that ψ decays at least like η. For tensor perturbations, we first extract the two functions contained in $\delta T^{\mathrm{TT}}_{\mu\nu}$ by a Fourier transformation similar to (5.26),

$$\delta T^{\mathrm{TT}}_{\mu\nu}(\eta, \boldsymbol{x}) = \sum_{s=\pm} \int e^s_{\mu\nu}(\boldsymbol{p}) J_s(\eta, \boldsymbol{p}) \, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{x}} \frac{\mathrm{d}^3 p}{(2\pi)^3} \,. \tag{5.36}$$

The solution of the equation for the tensor modes $g_\pm(\eta, \boldsymbol{p})$ is then given by the homogeneous solution (5.28) plus the inhomogeneous term

$$
\begin{aligned}
g_\pm^{\text{inh}}(\eta, \boldsymbol{p}) &= \int_{\eta_0}^{0} G_{\text{ret}}(\eta, \eta') J_\pm(\eta', \boldsymbol{p}) \, \mathrm{d}\eta' \\
&= (-\omega\eta)^{\frac{3}{2}-\varsigma} \mathrm{Y}_{\frac{3}{2}-\varsigma}(-\omega\eta) \frac{\pi}{2\omega} \int_{\eta_0}^{\eta} (-\omega\eta')^{-\frac{1}{2}+\varsigma} \mathrm{J}_{\frac{3}{2}-\varsigma}\left(-\omega\eta'\right) J_\pm(\eta') \, \mathrm{d}\eta' \\
&\quad - (-\omega\eta)^{\frac{3}{2}-\varsigma} \mathrm{J}_{\frac{3}{2}-\varsigma}(-\omega\eta) \frac{\pi}{2\omega} \int_{\eta_0}^{\eta} (-\omega\eta')^{-\frac{1}{2}+\varsigma} \mathrm{Y}_{\frac{3}{2}-\varsigma}\left(-\omega\eta'\right) J_\pm(\eta') \, \mathrm{d}\eta' \,,
\end{aligned} \tag{5.37}
$$

with the retarded propagator G_{ret} pertaining to the tensor mode equation (5.27), whose explicit expression in terms of the homogeneous solutions (using the standard Wronski formula) has been inserted in the second line. Since the stress tensor components $J_\pm(\eta)$ fall off as fast as or faster than η^2 for the initial states described above, we can use the behaviour of the Bessel functions near zero [104] to conclude that the integrals vanish as $\eta \to 0$. Hence, the inhomogeneous contribution is subdominant to the homogeneous contribution which tends to a constant in this limit (5.33).

Specializing equation (5.30) to scalar and vector perturbations using the decomposition (5.18, 5.19) we obtain

$$
\begin{aligned}
\tilde{R}^{(1)0\mu}{}_{0\rho} &= -\mathrm{i}H^2\eta\eta^{\mu\alpha}p_{(\alpha}\left(\eta v'^{\mathrm{T}}_{\rho)} - v^{\mathrm{T}}_{\rho)}\right) + \frac{1}{2}H^2\left(2\delta^\mu_\rho\phi - \delta^\mu_\rho\eta\phi' - \eta^2\eta^{\mu\alpha}p_\alpha p_\rho\phi\right) \\
&\qquad - H^2\eta\eta^{\mu\alpha}p_\alpha p_\rho\left(\psi - \eta\psi'\right) , \\
\tilde{R}^{(1)0\mu}{}_{\rho\sigma} &= H^2\eta^2\eta^{\mu\alpha}p_\alpha p_{[\rho}v^{\mathrm{T}}_{\sigma]} + \mathrm{i}H^2\eta\delta^\mu_{[\rho}p_{\sigma]}\phi \,, \\
\tilde{R}^{(1)\mu\nu}{}_{0\rho} &= -H^2\eta^2 p_\rho\eta^{\mu\alpha}\eta^{\nu\beta}p_{[\alpha}v^{\mathrm{T}}_{\beta]} - \mathrm{i}H^2\eta\delta^{[\mu}_\rho\eta^{\nu]\alpha}p_\alpha\phi \,, \\
\tilde{R}^{(1)\mu\nu}{}_{\rho\sigma} &= 2H^2\delta^{[\mu}_{[\rho}\left(\mathrm{i}\eta p_{\sigma]}\eta^{\nu]\alpha}v^{\mathrm{T}}_\alpha + \mathrm{i}\eta v^{\mathrm{T}}_{\sigma]}\eta^{\nu]\alpha}p_\alpha + \delta^{\nu]}_{\sigma]}\phi + 2\eta p_{\sigma]}\eta^{\nu]\alpha}p_\alpha\psi\right) .
\end{aligned} \tag{5.38}
$$

We can see that the Riemann tensor components fall off at least as fast as the corresponding perturbations (and in many cases faster, with additional explicit factors of η). Summarizing, it is clear that the above conclusion of the semiclassical late-time stability of de Sitter space as $\eta \to 0$ is not changed by the inclusion of initial state corrections.

What can change in this result if we take into account the contribution of other fields? Assuming that the effective action can always be brought into the form (4.123), we have seen that nonlocal terms do not contribute when we employ order reduction. The local terms must then take a form similar to (5.24), with different numerical coefficients. Since the renormalized kernel K^{ren} is a distribution depending on Z and thus de Sitter-invariant, the local terms are invariant under the simultaneous rescaling

$$
\boldsymbol{x}, \eta, \eta' \to \alpha\boldsymbol{x}, \alpha\eta, \alpha\eta' \tag{5.39}
$$

for constant α. However, the most general second-order equation compatible with this rescaling has the form

$$
h''^{\mathrm{TT}}_{\mu\nu} - \frac{2}{\eta}(1+\varsigma_1)h'^{\mathrm{TT}}_{\mu\nu} - (1+\varsigma_2)\,\triangle\, h^{\mathrm{TT}}_{\mu\nu} + \frac{3\varsigma_3}{\eta^2}h^{\mathrm{TT}}_{\mu\nu} = \mathcal{O}\left(\kappa^4\right) , \tag{5.40}
$$

and similar equations for v^{T}_μ, ψ and ϕ and where the ς_k are small parameters proportional to κ^2. If the equations for the vectorial and scalar components are still elliptic, for appropriate boundary conditions they can be gauge transformed to zero as before. However, no time derivatives of these components can occur in the order-reduced equations. If we examine the explicit expression of the linearized Riemann tensor (5.38), we see that time derivatives of these components only occur in the term $\tilde{R}^{(1)0\mu}{}_{0\rho} = \tilde{R}^{(1)\mu}{}_{\rho} - \tilde{R}^{(1)\alpha\mu}{}_{\alpha\rho}$. Since $\tilde{R}^{(1)\mu}{}_{\rho}$ is of higher order when using order reduction, and $\tilde{R}^{(1)\alpha\mu}{}_{\alpha\rho}$ does not contain time derivatives of the vectorial and scalar components, time derivatives will always be of higher order and the order-reduced equations for those components will be elliptic.

For the tensorial components, we again may take a Fourier transform as in equation (5.26), and the solution of equation (5.40) for the modes $g_\pm(\eta, \boldsymbol{p})$ reads

$$g_\pm(\eta, \boldsymbol{p}) = (-\omega\eta)^{\frac{3}{2}+\varsigma_1} \left[C_1^\pm \, \mathrm{J}_{\frac{3}{2}+\varsigma_1-\varsigma_3}(-\omega\eta) + C_2^\pm \, \mathrm{Y}_{\frac{3}{2}+\varsigma_1-\varsigma_3}(-\omega\eta) \right] \tag{5.41}$$

with $\omega^2 = (1+\varsigma_2)\boldsymbol{p}^2$ instead of (5.28). At late times $\eta \to 0$, instead of tending to a constant as in equation (5.33), $g_\pm$ and its derivative behave asymptotically $\sim (-\omega\eta)^{\varsigma_3}$. However, from the expression of the linearized Riemann tensor in terms of the tensorial components (5.32), the components of the Riemann tensor decay like $\sim (-\omega\eta)^{1+\varsigma_3}$ and thus vanish at late times just as before. Therefore, our conclusions are unchanged also including loop corrections due to other matter fields.

6 The two-point function of the metric perturbations

For the calculation of the two-point function, we will use a perturbative approach. Following formula (3.12), the path-ordered two-point function is given by

$$\langle \mathrm{in}|\mathcal{P} h_{ab}(x) h_{c'd'}(x')|\mathrm{in}\rangle = \frac{\int h_{ab}(x) h_{c'd'}(x')\, \mathrm{e}^{\mathrm{i}S_{\mathrm{eff}}[h]}\, \mathcal{D}h}{\int \mathrm{e}^{\mathrm{i}S_{\mathrm{eff}}[h]}\, \mathcal{D}h}\,, \tag{6.1}$$

where the time integration extends over the CTP contour (see figure 3.1). Since the background spacetime g_{ab} fulfills the semiclassical Einstein equation (5.2), the effective action S_{eff} (4.123) consists only of terms which are quadratic in the perturbation h_{ab}. Furthermore, we can decompose this quadratic part into powers of κ^2

$$S_{\mathrm{eff}}[h] = S_{\mathrm{eff},0}[h] + \kappa^2 S_{\mathrm{eff},2}[h]\,. \tag{6.2}$$

Using the formulas from appendix A, this decomposition explicitly reads

$$\begin{aligned}
S_{\mathrm{eff},0}[h] &= \int \left(\frac{1}{2} h \tilde{R}^{(1)} + \tilde{R}^{(2)} + \frac{3}{4} H^2 \left(h^2 - 2 h_{mn} h^{mn} \right) \right) \sqrt{-g}\, \mathrm{d}^4 x\,, \\
S_{\mathrm{eff},2}[h] &= -\frac{1}{32} \int g^{ab} \langle T_{ab} \rangle^{\mathrm{ren}}(\mu) \left(h^2 - 2 h_{mn} h^{mn} \right) \sqrt{-g}\, \mathrm{d}^4 x \\
&\quad + \int h^{ab} h^{cd} \langle U_{abcd} \rangle^{\mathrm{ren}}(\mu) \sqrt{-g}\, \mathrm{d}^4 x + \alpha(\mu) \int \tilde{C}^{(1)abcd} \tilde{C}^{(1)}_{abcd} \sqrt{-\tilde{g}}\, \mathrm{d}^4 x \\
&\quad + \beta(\mu) \int \left(\tilde{R}^{(1)2} + 24 H^2 \tilde{R}^{(2)} + 12 H^2 h \tilde{R}^{(1)} + 18 H^4 \left(h^2 - 2 h_{mn} h^{mn} \right) \right) \sqrt{-g}\, \mathrm{d}^4 x \\
&\quad + \frac{\mathrm{i}}{8} \iint \tilde{R}^{(1)ab}{}_{cd}(x) K^{\mathrm{ren}}{}_{ab}{}^{cd}{}_{m'n'}{}^{p'q'}(Z(x,x'),\mu) \tilde{R}^{(1)m'n'}{}_{p'q'}(x') \sqrt{-g}\, \mathrm{d}^4 x \sqrt{-g}\, \mathrm{d}^4 x' \\
&= \frac{1}{2} \iint h^{ab}(x) V_{abc'd'}(x,x') h^{c'd'}(x') \sqrt{-g}\, \mathrm{d}^4 x \sqrt{-g}\, \mathrm{d}^4 x'\,,
\end{aligned} \tag{6.3}$$

where we defined the kernel $V_{abc'd'}(x,x')$ in the last line. By partial integration, the effective action can always be brought in such a form, and single integrals over x can be expressed as double integrals using a covariant Dirac δ distribution $\delta(x,x') = \delta^4(x-x')/\sqrt{-g}$ – we will refer to those parts of V as local parts. If we split the time integration contour in $+$ and $-$ parts and introduce capital indices $A,B = \pm$ for the kernel V, its non-local part which involves K^{ren} is given with the proper prescriptions by equation (4.124), while the local parts always come with a Kronecker δ_{AB}.

If we define the expectation value $\langle\rangle_h$ by

$$\langle \mathcal{P}A[h]\rangle_h = \frac{\int A[h]\,\mathrm{e}^{\mathrm{i}S_{\text{eff},0}[h]}\,\mathcal{D}h}{\int \mathrm{e}^{\mathrm{i}S_{\text{eff},0}[h]}\,\mathcal{D}h}\,, \tag{6.4}$$

the propagator for the perturbations is given by

$$G_{abc'd'}(x,x') = -\mathrm{i}\big\langle \mathcal{P}h_{ab}(x)h_{c'd'}(x')\big\rangle_h\,. \tag{6.5}$$

Because of gauge invariance, it is well known that this expression, and in fact all expectation values calculated using (6.4) are actually ill-defined. There are two possibilities to resolve this issue, working in an exact gauge or adding gauge-fixing terms to the action. To work in an exact gauge, one imposes certain conditions on the perturbation as in the last section (5.21) which are then also fulfilled by the propagator; this is also the way we choose in the next section. Standard textbook lore tells us that adding an additional gauge-fixing term to the action can be thought of as imposing an exact gauge depending on a parameter, and then integrating over this parameter with some weighting function, so that these gauges are also known as average gauges – depending on the weighting function, the propagator fulfills the imposed condition in some average sense. However, for the calculation of the gauge-invariant Riemann tensor two-point function which is done in section 7.1, it does not matter which possibility is used. Expanding the effective action in (6.1), we thus obtain a perturbative expansion for the two-point function of the metric perturbations

$$\begin{aligned}\langle \mathrm{in}|\mathcal{P}h_{ab}(x)h_{c'd'}(x')|\mathrm{in}\rangle &= \big\langle \mathcal{P}h_{ab}(x)h_{c'd'}(x')\big\rangle_h \\ &+ \mathrm{i}\kappa^2\Big(\big\langle \mathcal{P}h_{ab}(x)h_{c'd'}(x')S_{\text{eff},2}[h]\big\rangle_h - \big\langle \mathcal{P}h_{ab}(x)h_{c'd'}(x')\big\rangle_h\big\langle \mathcal{P}S_{\text{eff},2}[h]\big\rangle_h\Big) + \mathcal{O}\big(\kappa^4\big)\,.\end{aligned} \tag{6.6}$$

Using Wick's theorem, the expectation values $\langle\rangle_h$ can be evaluated by contracting fields in all possible ways and replacing two-point correlation functions by the propagator (6.5). This leads to our final expression

$$\begin{aligned}\langle \mathrm{in}|\mathcal{P}h_{ab}(x)h_{c'd'}(x')|\mathrm{in}\rangle &= \mathrm{i}G_{abc'd'}(x,x') \\ &- \mathrm{i}\kappa^2 \iint G_{abmn}(x,y)V^{mnp'q'}(y,y')G_{c'd'p'q'}(x',y')\sqrt{-g}\,\mathrm{d}^4y\,\sqrt{-g}\,\mathrm{d}^4y' + \mathcal{O}\big(\kappa^4\big)\,.\end{aligned} \tag{6.7}$$

The propagator $G_{abc'd'}$ satisfies also an equation similar to (4.18) for the scalar field. In fact, since the equation of motion for linearized metric perturbations is (5.4), which to lowest order in κ^2 just reads $\tilde{G}^{(1)a}{}_b = 0$, the analogue of equation (4.18) is given by

$$E^{abmn}\big\langle \mathcal{T}h_{mn}(x)h_{c'd'}(x')\big\rangle_h = \mathrm{i}\delta^a_{(c'}\delta^b_{d')}\delta(x,x')\,, \tag{6.8}$$

with the differential operator E_{abcd} defined by the expansion of the linearized Einstein tensor

$$\tilde{G}^{(1)a}{}_b = E^a{}_b{}^{mn}h_{mn}\,. \tag{6.9}$$

Of course, the Wightman function of the metric perturbations satisfies the homogeneous version of (6.8). The two-point function of the Einstein tensor can then be calculated by applying this operator to both arguments of the metric perturbation two-point function (6.7). Let us take its Wightman function, for which we obtain

$$\begin{aligned}\langle \mathrm{in}|\tilde{G}^{(1)a}{}_b(x)\tilde{G}^{(1)c'}{}_{d'}(x')|\mathrm{in}\rangle &= E^a{}_b{}^{mn}E^{c'}{}_{d'}{}^{p'q'}\langle \mathrm{in}|h^-_{mn}(x)h^+_{p'q'}(x')|\mathrm{in}\rangle \\ &= -\mathrm{i}\kappa^2 E^a{}_b{}^{mn}E^{c'}{}_{d'}{}^{p'q'}V^{mnp'q'}_{-+}(x,x')\,.\end{aligned} \tag{6.10}$$

All other terms of equation (6.7) involve at least one Wightman function of the metric perturbations, and hence vanish under the action of the differential operator E. Also, there is a cancellation between two minus signs, one that comes from the splitting of the CTP contour as explained in section 3.1, and one because the anti-time-ordered or Dyson propagator G^{--} which surfaces in this splitting satisfies equation (6.8) with a minus sign on the right-hand side. Since all local terms in the kernel V are of the $++$ or $--$ type, only the non-local term contributes to V^{+-}. However, this term is nothing but the renormalized stress tensor correlation function which we rewrote in terms of the kernel K^{ren} in section 4.6 – but since the Wightman function does not need to be renormalized, we simply have

$$\left\langle \tilde{G}^{(1)a}{}_{b}(x)\tilde{G}^{(1)c'}{}_{d'}(x')\right\rangle = \frac{\kappa^2}{4}\left\langle T^{a}{}_{b}(x)T^{c'}{}_{d'}(x')\right\rangle . \tag{6.11}$$

In this last equation, since the expectation value of $\tilde{G}^{(1)}$ vanishes, we directly passed to the connected correlation function. Perturbatively, the correlation functions of the linearized Einstein tensor (and by appropriate index contractions and subtractions) the linearized Ricci tensor and scalar are therefore directly related to the stress tensor two-point function. For the time-ordered function of the Einstein tensor, in addition to a change in the prescription $Z \to Z - \mathrm{i}0$ there are additional local terms $\sim \delta(x,x')$ coming from V^{++}.

6.1. The massless, conformally coupled scalar

An explicit calculation of the metric perturbations using equation (6.7) has been done including loops of massless, conformally coupled scalars. As in the case of effective field equations, it is advantageous to consider a conformally rescaled perturbation $\hat{h}_{ab}$, defined by equation (5.11). We also consider the same exact gauge as in that section, where the spatial components are purely tensorial (5.22), while scalar and vector perturbations reside in temporal components. Furthermore, it is easier to work in spatial Fourier space.

The two-point function (6.5) for the tensorial components in this gauge has been calculated by Ford and Parker [106], and the result is

$$\left\langle \hat{h}^{ab}(\eta,\boldsymbol{p})\hat{h}^{cd}(\eta',\boldsymbol{q})\right\rangle_h^{\mathrm{TT}} = (2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{q})f(\eta,\eta',|\boldsymbol{p}|)P^{abcd}(\boldsymbol{p})\,, \tag{6.12}$$

where

$$f(\eta,\eta',|\boldsymbol{p}|) = \frac{H^2}{2|\boldsymbol{p}|^3}(|\boldsymbol{p}|\eta - \mathrm{i})(|\boldsymbol{p}|\eta' + \mathrm{i})\,\mathrm{e}^{-\mathrm{i}|\boldsymbol{p}|(\eta-\eta')} = f^*(\eta',\eta,|\boldsymbol{p}|)\,, \tag{6.13}$$

and

$$P^{abcd}(\boldsymbol{p}) = P^{ad}P^{bc} + P^{ac}P^{bd} - P^{ab}P^{cd} \tag{6.14}$$

is the polarization tensor which can be written in terms of the projection tensor P^{ab} defined by

$$\begin{aligned} P^{\mu\nu} &= \eta^{\mu\nu} - \frac{p^\mu p^\nu}{\boldsymbol{p}^2}\,, \\ P^{00} &= P^{0\mu} = P^{\mu 0} = 0\,. \end{aligned} \tag{6.15}$$

The projection tensor P^{ab} satisfies

$$P^{ab}\eta_{ab} = 2\,, \qquad P^{ab}P^{cd}\eta_{bc} = P^{ad}\,. \tag{6.16}$$

The Feynman propagator as well as other propagators we need can then be easily obtained from this result, namely

$$G^{AB}_{abcd}(\eta,\eta',\boldsymbol{p}) = G^{AB}(\eta,\eta',\boldsymbol{p}) P_{abcd}(\boldsymbol{p}) \,, \tag{6.17}$$

where the components of G^{AB} are

$$\begin{aligned}
G^{0++}(\eta,\eta',\boldsymbol{p}) &= -\mathrm{i}\big[\Theta(\eta-\eta')f(\eta,\eta',|\boldsymbol{p}|) + \Theta(\eta'-\eta)f^*(\eta,\eta',|\boldsymbol{p}|)\big] \\
G^{0+-}(\eta,\eta',\boldsymbol{p}) &= -\mathrm{i}f^*(\eta,\eta',|\boldsymbol{p}|) \\
G^{0-+}(\eta,\eta',\boldsymbol{p}) &= -\mathrm{i}f(\eta,\eta',|\boldsymbol{p}|) \\
G^{0--}(\eta,\eta',\boldsymbol{p}) &= -\mathrm{i}\big[\Theta(\eta-\eta')f^*(\eta,\eta',|\boldsymbol{p}|) + \Theta(\eta'-\eta)f(\eta,\eta',|\boldsymbol{p}|)\big] \,.
\end{aligned} \tag{6.18}$$

One can easily check that the Feynman propagator solves the suitably restricted and conformally transformed version of (6.8), which after stripping off the polarization tensor factor reads

$$\left(\partial_\eta^2 - \frac{2}{\eta}\partial_\eta + \boldsymbol{p}^2\right) G^{0++}(\eta,\eta',\boldsymbol{p}) = -H^2\eta^2\delta(\eta-\eta') \,, \tag{6.19}$$

and which is the same equation as the one satisfied by the massless, minimally coupled scalar (4.18), expressed in the conformally flat coordinate system (4.8). Hence, for small $|\boldsymbol{p}|$, the function $f(\eta,\eta',\boldsymbol{p})$ (6.13) is too singular to have a well-defined Fourier transform just as for the massless, minimally coupled scalar field, and an infrared cutoff must be introduced. However, if one considers appropriate observables such as the stress tensor for the massless, minimally coupled scalar or the linearized Riemann tensor for the metric perturbation, it is possible to take this cutoff to zero, and so we will not concern ourselves with the intensely waged discussion in the literature about its potential effects.

Since the scalar and vectorial components are not dynamical, the equation that they satisfy is purely elliptic, or algebraic in spatial Fourier space. For the vectorial part, the appropriate restriction of equation (6.8) is given by

$$\boldsymbol{p}^2 G^{++}_{abcd}(\eta,\eta',\boldsymbol{p}) = H^2\eta^2\delta(\eta-\eta')\delta^0_{(a}P_{b)(c}\delta^0_{d)} \,, \tag{6.20}$$

and this algebraic equation is trivially solved for the Feynman propagator. For the Wightman function, the right-hand side vanishes, and so the Wightman function must be taken to vanish, too. While this may sound surprising, it is just a consequence of choosing the exact transverse traceless gauge (5.21, 5.22) as opposed to an average gauge where (usually) all metric components are dynamical, and there is no problem in the calculation. Similarly, the Feynman propagator for the scalar components is given by

$$G^{++}_{abcd}(\eta,\eta',\boldsymbol{p}) = \delta^0_{(a}\left[\mathrm{i}\eta\left(p_{b)}\delta^0_{(c} - \delta^0_{b)}p_{(c}\right) + 3\frac{p_{b)}p_{(c}}{\boldsymbol{p}^2}\right]\delta^0_{d)}\frac{H^2\eta^2}{\boldsymbol{p}^2}\delta(\eta-\eta') \,, \tag{6.21}$$

with $p_0 = 0$ understood, while the scalar Wightman function vanishes.

For this case, the effective action has been calculated in [67, 116]. They actually obtained a result for general FLRW geometries, but we can specialize it to de Sitter spacetimes. The local part of $S_{\text{eff},2}[h]$ is then given by the terms quadratic in the perturbation h_{ab} of

$$\begin{aligned}
&-\frac{N}{1920\pi^2}\int \hat{C}^{abcd}\hat{C}_{abcd}\ln(-H\eta)\sqrt{-\hat{g}}\,\mathrm{d}^4x - \frac{N}{34560\pi^2}\int \tilde{R}^2\sqrt{-\tilde{g}}\,\mathrm{d}^4x \\
&+\frac{N}{5760\pi^2}\int \left[\eta^{-2}\hat{G}^{00} + 2\hat{g}^{00}H^2\left(2(-H\eta)^{-1}\hat{\nabla}^m\hat{\nabla}_m(-H\eta)^{-1} - \hat{g}^{00}H^2(-H\eta)^{-4}\right)\right]\sqrt{-\hat{g}}\,\mathrm{d}^4x \,,
\end{aligned} \tag{6.22}$$

together with the terms multiplied by $\alpha(\mu)$ and $\beta(\mu)$ from equation (6.3), while the nonlocal part has the form

$$
\begin{aligned}
&\frac{N}{3840\pi^2} \iint \hat{C}^{+}_{abmn}(x) \hat{C}^{+abmn}(y) \left(L(x-y,\mu) + \mathrm{i} N(x-y) \right) \mathrm{d}^4 x \, \mathrm{d}^4 y \\
&+ \frac{N}{3840\pi^2} \iint \hat{C}^{+}_{abmn}(x) \hat{C}^{-abmn}(y) \left(D(x-y) - \mathrm{i} N(x-y) \right) \mathrm{d}^4 x \, \mathrm{d}^4 y \\
&- \frac{N}{3840\pi^2} \iint \hat{C}^{-}_{abmn}(x) \hat{C}^{+abmn}(y) \left(D(x-y) - \mathrm{i} N(x-y) \right) \mathrm{d}^4 x \, \mathrm{d}^4 y \\
&- \frac{N}{3840\pi^2} \iint \hat{C}^{-}_{abmn}(x) \hat{C}^{-abmn}(y) \left(L(x-y,\mu) - \mathrm{i} N(x-y) \right) \mathrm{d}^4 x \, \mathrm{d}^4 y \,.
\end{aligned} \tag{6.23}
$$

Of course, the concrete form of these local and non-local terms is a bit different from the general ansatz (4.123) owing to the different method that was used to calculate them, but the overall structure is the same – the non-local part consists of a kernel sandwiched between two curvature tensors. In this case, the kernels are given by

$$
\begin{aligned}
L(x,\mu) &= -\int \ln \left| \frac{p^2}{\mu^2} \right| \mathrm{e}^{\mathrm{i}px} \frac{\mathrm{d}^4 p}{(2\pi)^4} = L(-x,\mu) \,, \\
N(x) &= \pi \int \Theta(-p^2) \, \mathrm{e}^{\mathrm{i}px} \frac{\mathrm{d}^4 p}{(2\pi)^4} = N(-x) \,, \\
D(x) &= \mathrm{i}\pi \int \Theta(-p^2) \, \operatorname{sgn} p^0 \, \mathrm{e}^{\mathrm{i}px} \frac{\mathrm{d}^4 p}{(2\pi)^4} = -D(-x) \,,
\end{aligned} \tag{6.24}
$$

and with the help of appendix D we can calculate their mixed form

$$
\begin{aligned}
L(\eta-\eta',\boldsymbol{p}) &= \cos\left[|\boldsymbol{p}|(\eta-\eta') \right] \mathcal{P}' \frac{1}{|\eta-\eta'|} \,, \\
D(\eta-\eta',\boldsymbol{p}) &= \cos\left[|\boldsymbol{p}|(\eta-\eta') \right] \mathcal{P} \frac{1}{\eta-\eta'} \,, \\
N(\eta-\eta',\boldsymbol{p}) &= -\frac{\sin\left[|\boldsymbol{p}|(\eta-\eta') \right]}{\eta-\eta'} + \pi \delta(\eta-\eta') \,.
\end{aligned} \tag{6.25}
$$

Note that the kernel H given in equation (5.13) can be written as $H(x,\mu) = [L(x,\mu) + D(x)]/2$, in line with the general result (5.4) that the kernel in the equations for the metric perturbations is the difference between the $++$ and $+-$ parts (and taking into account an extra minus sign in (6.23) from the decomposition of the contour). Note also that all kernels are real. Since the integrands of L and N in (6.24) are even under the reflection $p \to -p$, only the cosine term of the exponential contributes, while for D which is odd under this reflection only the sine term of the exponential contributes.

The well-known advantage of considering a spatial Fourier transform is that the spatial integrals in the two-point function (6.7) reduce to multiplications in Fourier space, so that only the time integrals remain. Of course, for the vectorial and scalar parts where the propagator is proportional to a δ distribution (6.20, 6.21), those integrals are trivial. For the tensorial part, let us explain the calculation on the example of a part of the first term of the effective action in equation (6.22), which reads

$$-\frac{N}{1920\pi^2}\int \hat{C}^{abcd}\hat{C}_{abcd}\ln H\sqrt{-\hat{g}}\,\mathrm{d}^4x\,. \tag{6.26}$$

For the Weyl tensor of the almost flat metric $\hat{g}_{ab}$ (5.11), we calculate using appendix A and the definition of the tensorial part (5.18) in the transverse traceless gauge

$$\begin{aligned}\hat{C}_{abcd} &= \kappa\left(2\delta^m_{[c}\delta^p_{d]}\delta^q_{[a}\delta^n_{b]} - \delta^p_{[c}\eta_{d][a}\delta^q_{b]}\eta^{mn}\right)\partial_m\partial_n h^{\mathrm{TT}}_{pq} \\ &= \kappa T_{abcd}{}^{mnpq}\partial_m\partial_n h^{\mathrm{TT}}_{pq}\,,\end{aligned} \tag{6.27}$$

where we defined the tensor T in the second equality. The contribution of this part to the two-point function (6.7) is therefore given by

$$\mathrm{i}\frac{N}{960\pi^2}\kappa^4 T_{efgh}{}^{rsmn}T^{efghklpq}\int(\partial_r\partial_s G_{abmn}(x,y))(\partial_k\partial_l G_{pqc'd'}(y,x'))\ln H\,\mathrm{d}^4y \tag{6.28}$$

with the time integration ranging over the CTP contour and the derivatives referring to the point y (note the extra factor 2 which comes from the definition of the interaction kernel V (6.3)). We now split the contour, so that for the Wightman function (−+) we have to calculate schematically

$$+\int_{t_0^+}^{T} G^{-+}(\eta,\tau)G^{++}(\tau,\eta')\,\mathrm{d}\tau - \int_{t_0^-}^{T} G^{--}(\eta,\tau)G^{-+}(\tau,\eta')\,\mathrm{d}\tau\,, \tag{6.29}$$

i.e., the external indices are fixed while we sum over the index corresponding to the integration variable, and the − branch obtains an extra minus sign. As explained in section 3.1, the integration goes from an initial time $t_0^\pm$ depending on the branch of the contour until a final time T, such that we have $t_0^\pm < \eta, \eta' < T$. After Fourier transforming the spatial parts, inserting the explicit expressions (6.17) and performing some judicious integrations by parts, we obtain

$$\begin{aligned}
&+\mathrm{i}\frac{N}{960\pi^2}\kappa^4 P_{abcd}\Big[\big[(\partial_\tau^2-\boldsymbol{p}^2)G^{-+}(\eta,\tau,|\boldsymbol{p}|)\big]\big[\partial_\tau G^{++}(\tau,\eta',|\boldsymbol{p}|)\big] \\
&\qquad\qquad -\big[(\partial_\tau^2+3\boldsymbol{p}^2)\partial_\tau G^{-+}(\eta,\tau,|\boldsymbol{p}|)\big]G^{++}(\tau,\eta',|\boldsymbol{p}|)\Big]_{t_0^+}^{T}\ln H \\
&-\mathrm{i}\frac{N}{960\pi^2}\kappa^4 P_{abcd}\Big[\big[\partial_\tau G^{--}(\tau,\eta,|\boldsymbol{p}|)\big]\big[(\partial_\tau^2-\boldsymbol{p}^2)G^{-+}(\tau,\eta',|\boldsymbol{p}|)\big] \\
&\qquad\qquad -G^{--}(\tau,\eta,|\boldsymbol{p}|)\big[(\partial_\tau^2+3\boldsymbol{p}^2)\partial_\tau G^{-+}(\tau,\eta',|\boldsymbol{p}|)\big]\Big]_{t_0^-}^{T}\ln H \\
&+\mathrm{i}\frac{N}{960\pi^2}\kappa^4 P_{abcd}\int_{t_0^+}^{T}\big[(\partial_\tau^2+\boldsymbol{p}^2)^2G^{-+}(\eta,\tau,|\boldsymbol{p}|)\big]G^{++}(\tau,\eta',|\boldsymbol{p}|)\ln H\,\mathrm{d}\tau \\
&-\mathrm{i}\frac{N}{960\pi^2}\kappa^4 P_{abcd}\int_{t_0^-}^{T}G^{--}(\tau,\eta,|\boldsymbol{p}|)\big[(\partial_\tau^2+\boldsymbol{p}^2)^2G^{-+}(\tau,\eta',|\boldsymbol{p}|)\big]\ln H\,\mathrm{d}\tau\,.
\end{aligned} \tag{6.30}$$

From the formulas (6.18) and the definition of the function f (6.13), one easily obtains that

$$(\partial_\tau^2 + \boldsymbol{p}^2)^2 G^{-+}(\eta, \tau, |\boldsymbol{p}|) = (\partial_\tau^2 + \boldsymbol{p}^2)^2 G^{-+}(\tau, \eta', |\boldsymbol{p}|) = 0\,, \tag{6.31}$$

and therefore the two integrals vanish. The two boundary terms can also be evaluated explicitly, and the above contribution reduces to

$$\begin{aligned}
&-\frac{N}{480\pi^2}\kappa^4 H^2 P_{abcd} f(\eta, \eta', |\boldsymbol{p}|) \ln H \\
&-\mathrm{i}\frac{N}{960\pi^2}\kappa^4 P_{abcd}\frac{H^4}{2|\boldsymbol{p}|^3}\Big[(|\boldsymbol{p}|\eta - \mathrm{i})(|\boldsymbol{p}|\eta' - \mathrm{i})(-1 + (1+\mathrm{i})|\boldsymbol{p}|t_0^+)(\mathrm{i} + (1+\mathrm{i})|\boldsymbol{p}|t_0^+)\,\mathrm{e}^{\mathrm{i}|\boldsymbol{p}|(2t_0^+ - \eta - \eta')} \\
&\qquad + (|\boldsymbol{p}|\eta + \mathrm{i})(|\boldsymbol{p}|\eta' + \mathrm{i})(1 + (1+\mathrm{i})|\boldsymbol{p}|t_0^-)(-\mathrm{i} + (1+\mathrm{i})|\boldsymbol{p}|t_0^-)\,\mathrm{e}^{-\mathrm{i}|\boldsymbol{p}|(2t_0^- - \eta - \eta')}\Big]\ln H\,.
\end{aligned} \tag{6.32}$$

If we now set $t_0^\pm = t_0(1 \mp \mathrm{i}\epsilon)$ (3.3, 3.6) and then take the limit $t_0 \to -\infty$, the second line vanishes and we are left with the first line as the final result for this term. It is independent of the arbitrary final time T, since contributions from the $+$ and the $-$ branch of the CTP contour cancel out for all times larger than both η and η'; this fact could be made manifest by rewriting the Feynman and Dyson propagators in (6.29) in terms of Wightman functions and retarded propagators. Furthermore, it can be seen very nicely how the $\mathrm{i}\epsilon$ prescription for the deformation of the CTP contour leads to a good definition of the interacting vacuum. The exponential factors $\exp(\mathrm{i}|\boldsymbol{p}|t_0\epsilon)$ (and in general $\exp(\mathrm{i}|\boldsymbol{p}|t\epsilon)$ before the integration over τ) serve to switch the interaction on for early times, so that the free vacuum can adiabatically evolve into an interacting state with proper correlations.

The other terms can be calculated similarly. For the non-local terms in the effective action, which involve two integrations, it is advisable to suitably combine the kernels L, D and N (6.25) into exponentials. This allows to take the limit $t_0^\pm \to -\infty(1 \mp \mathrm{i}\epsilon)$ already after the first (inner) integration and simplifies the calculation of the remaining integral a lot, as can already be seen above (6.32). The final result for the Wightman function is given by

$$\begin{aligned}
\langle \mathrm{in}|\hat{h}_{ab}(\eta, \boldsymbol{p})\hat{h}_{cd}(\eta', \boldsymbol{q})|\mathrm{in}\rangle = (2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{q})P_{abcd}\Big[&f(\eta, \eta', \boldsymbol{p})\big(1 + \kappa^2 H^2 \lambda\big) \\
&+ \frac{N}{1920\pi^2}\kappa^2 H^4\Big(I_1(\eta, \eta', \boldsymbol{p}) - I_2(\eta, \eta', \boldsymbol{p}) - I_2^*(\eta', \eta, \boldsymbol{p})\Big) \\
&+ \frac{N}{1920\pi^2}\kappa^2 H^4\Big(I_3(\eta, \eta', \boldsymbol{p}) - I_4(\eta, \eta', \boldsymbol{p}) + I_5(\eta, \eta', \boldsymbol{p})\Big)\Big] + \mathcal{O}\big(\kappa^4\big)\,,
\end{aligned} \tag{6.33}$$

where

$$\begin{aligned}
I_1(\eta, \eta', \boldsymbol{p}) &= 2|\boldsymbol{p}|^{-1}\eta\eta'\,\mathrm{e}^{-\mathrm{i}|\boldsymbol{p}|(\eta-\eta')} \\
I_2(\eta, \eta', \boldsymbol{p}) &= |\boldsymbol{p}|^{-3}\,\mathrm{e}^{\mathrm{i}|\boldsymbol{p}|(\eta+\eta')}(|\boldsymbol{p}|\eta + \mathrm{i})\big(|\boldsymbol{p}|\eta' + \mathrm{i}\big)\left[\mathrm{Ein}(-2\mathrm{i}|\boldsymbol{p}|\eta) + \ln(2\mathrm{i}|\boldsymbol{p}|\eta) + \gamma\right] \\
&\quad + |\boldsymbol{p}|^{-3}\,\mathrm{e}^{-\mathrm{i}|\boldsymbol{p}|(\eta-\eta')}(|\boldsymbol{p}|\eta - \mathrm{i})\big(|\boldsymbol{p}|\eta' + \mathrm{i}\big)\ln(-2|\boldsymbol{p}|\eta) \\
I_3(\eta, \eta', \boldsymbol{p}) &= |\boldsymbol{p}|^{-3}\,\mathrm{e}^{-\mathrm{i}|\boldsymbol{p}|(\eta-\eta')}(|\boldsymbol{p}|\eta - \mathrm{i})\big(|\boldsymbol{p}|\eta' + \mathrm{i}\big)\big[\ln[2\mathrm{i}|\boldsymbol{p}|(\eta - \eta')] + \gamma\big] \\
I_4(\eta, \eta', \boldsymbol{p}) &= |\boldsymbol{p}|^{-3}\,\mathrm{e}^{\mathrm{i}|\boldsymbol{p}|(\eta-\eta')}(|\boldsymbol{p}|\eta + \mathrm{i})\big(|\boldsymbol{p}|\eta' - \mathrm{i}\big)\big[\mathrm{Ein}[-2\mathrm{i}|\boldsymbol{p}|(\eta - \eta')] + \ln[2\mathrm{i}|\boldsymbol{p}|(\eta - \eta')] + \gamma\big] \\
I_5(\eta, \eta', \boldsymbol{p}) &= \eta^2(\eta')^2\big[N(\eta - \eta', \boldsymbol{p}) - \mathrm{i}D(\eta - \eta', \boldsymbol{p})\big]\,.
\end{aligned} \tag{6.34}$$

and

$$\lambda = 6\alpha(\mu) - 24\beta + \frac{N}{480\pi^2} \ln\left(\frac{\mu}{H}\right) - \frac{N}{960\pi^2} \,. \tag{6.35}$$

The definition of the function $\operatorname{Ein}(x)$ can be found in appendix C. Note that λ is independent of the renormalization scale μ, since the dependence of $\alpha(\mu)$ on μ cancels the explicit logarithm, and as said before β does not get renormalized for conformal fields.

For the scalar and vectorial parts, we obtain

$$\begin{aligned}\langle \mathrm{in}|\hat{h}_{ab}(\eta,\boldsymbol{p})\hat{h}_{cd}(\eta',\boldsymbol{q})|\mathrm{in}\rangle &= -\frac{N}{5760\pi^2}\kappa^2 H^4\Bigg[9\delta^0_{(a}p_{b)}\delta^0_{(c}p_{d)}\frac{\eta^2(\eta')^2}{(\boldsymbol{p}^2)^2}\partial_\eta^2 - \delta^0_{(a}p_{b)}\delta^0_{(c}p_{d)}\eta^3(\eta')^3 \\ &+ 3\delta^0_{(a}p_{b)}\delta^0_{(c}p_{d)}\frac{\eta^2(\eta')^2}{\boldsymbol{p}^2}(\eta-\eta')\partial_\eta - 3\,\mathrm{i}\delta^0_a\delta^0_b\delta^0_{(c}p_{d)}\frac{\eta^3(\eta')^2}{\boldsymbol{p}^2}\partial_\eta^2 + 3\,\mathrm{i}\delta^0_{(a}p_{b)}\delta^0_c\delta^0_d\frac{\eta^2(\eta')^3}{\boldsymbol{p}^2}\partial_\eta^2 \\ &+ \mathrm{i}\delta^0_{(a}\left(\delta^0_{b)}p_{(c} + p_{b)}\delta^0_{(c}\right)\delta^0_{d)}\eta^3(\eta')^3\partial_\eta + \delta^0_a\delta^0_b\delta^0_c\delta^0_d\eta^3(\eta')^3\partial_\eta^2\Bigg]\big(N(\eta-\eta',\boldsymbol{p}) - \mathrm{i}D(\eta-\eta',\boldsymbol{p})\big)\end{aligned} \tag{6.36}$$

and

$$\begin{aligned}\langle \mathrm{in}|\hat{h}_{ab}(\eta,\boldsymbol{p})\hat{h}_{cd}(\eta',\boldsymbol{q})|\mathrm{in}\rangle &= -(2\pi)^3\delta^3(\boldsymbol{p}+\boldsymbol{q})\delta^0_{(a}P_{b)(c}\delta^0_{d)}\frac{N}{480\pi^2}\kappa^2H^4\eta^2(\eta')^2 \\ &\times\frac{1}{\boldsymbol{p}^2}\left(\partial_\eta^2 + \boldsymbol{p}^2\right)\big(N(\eta-\eta',\boldsymbol{p}) - \mathrm{i}D(\eta-\eta',\boldsymbol{p})\big)\,.\end{aligned} \tag{6.37}$$

We should stress that this is the exact one-loop correlation function for metric perturbations including loops of massless, conformally coupled scalars to order κ^2, and does not resort to approximations such as leading order in $\boldsymbol{p}^2$ or equal times. Furthermore, in addition to the manifest invariance under spatial rotations and translations it depends (apart from a global factor $|\boldsymbol{p}|^{-3}$) only on the physical momentum $\tilde{p} = -H\eta|\boldsymbol{p}|$, and is therefore invariant under the simultaneous rescaling

$$\boldsymbol{p} \to \alpha^{-1}\boldsymbol{p}\,, \qquad \eta,\eta' \to \alpha\eta,\alpha\eta' \tag{6.38}$$

(the analogue of (5.39)), which is a necessary, but not sufficient conditions for the de Sitter invariance of gauge-invariant observables obtainable from this two-point function.

Another important point concerns the dependence on initial conditions found in a similar calculation by Hsiang et. al. [144]. In our case, the iϵ prescription selects the correct adiabatic interacting vacuum state at past infinity, and no dependence on the initial time t_0 is left, which is crucial for the result to be invariant under the rescaling (6.38). In contrast, Hsiang et. al. obtained a result which diverges linearly as $t_0 \to -\infty$, and derived from this divergence limits on the maximum duration of inflation, where the initial state is prepared at a large but finite time t_0. However, this divergence must be interpreted as an artifact of their calculation method. As one goes to earlier and earlier times, the Poincaré patch of de Sitter space shrinks and the strength of interactions grow (as can be seen, e.g., from the first term in (6.22) which grows logarithmically). However, in a regime of strong interaction perturbation theory, which is based on the decomposition in free theory and an interaction that is supposed to be absent at the initial time, fails. On therefore has to resort to either modifying the interaction to decay adiabatically, something which is provided by the iϵ prescription, or to nonperturbative methods, as has been done in sections 5.2.1 and 5.2.2 to study the long-time stability of de Sitter space (note that nonperturbative here refers to the time evolution, and not to the intrinsically perturbative expansion of the metric $\tilde{g}_{ab}$ in powers of h_{ab}).

6.2. Cosmological Observables

As an example of an observable that is measured in cosmology, in this section we calculate the tensor power spectrum. Through observations of the cosmic microwave background, we can infer the spectrum of metric perturbations at the end of inflation and compare it with models. This spectrum was first calculated in the context of inflation about thirty years ago [145–149], with a scale-invariant result, and its high agreement with observations makes it a crucial feature that must be reproduced by all models. Here, we concern ourselves with quantum corrections to this spectrum induced by loops of massless, conformally coupled scalars (or other conformal matter), for which the two-point function of the metric perturbations was calculated in the last section.

The power spectrum δ^2 is defined, up to a factor and a Fourier transform, by the equal-time limit of the contracted two-point function for tensorial perturbations [8, 12],

$$\begin{aligned}\delta^2(|\boldsymbol{p}|,\eta) &= \frac{\kappa^2}{4(2\pi)^3}|\boldsymbol{p}|^3\eta^{ac}\eta^{bd}\int\langle\text{in}|\hat{h}_{ab}(\eta,\boldsymbol{x})\hat{h}_{cd}(\eta,\boldsymbol{0})|\text{in}\rangle\,\mathrm{e}^{-\mathrm{i}\boldsymbol{p}\boldsymbol{x}}\,\mathrm{d}^3x \\ &= \frac{\kappa^2}{32\pi^3}|\boldsymbol{p}|^3\eta^{ac}\eta^{bd}\int\langle\text{in}|\hat{h}_{ab}(\eta,\boldsymbol{p})\hat{h}_{cd}(\eta,\boldsymbol{q})|\text{in}\rangle\frac{\mathrm{d}^3q}{(2\pi)^3}\,.\end{aligned} \tag{6.39}$$

However, in our case there is problem with this definition, since the kernels N and D (6.25) are genuine distributions with singular support, for which the equal-time limit does not make sense. Therefore, strictly speaking, the power spectrum is ill-defined at one-loop level. Since it is an ultraviolet effect, it will also appear for other Hadamard states, and in other curved spacetimes, and one has to find a way to reconcile it with observations.

It is clear that physical observations can not be made with infinite precision, and so truly observable quantities will always involve an integration over the time arguments instead of a strict equal-time limit. Especially, the power spectrum is observed through the interaction of CMB photons with the metric perturbations from the time of last scattering until today, and it is those photons which interacted at different times which are measured. We then may model the whole interaction and measurement process by the convolution of the two-point function of the metric perturbations with a measurement function, for which we take a Gaussian of small width σ,

$$g_\sigma(\eta_0,\eta) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\eta-\eta_0)^2}{2\sigma^2}\right)\,. \tag{6.40}$$

The correct generalization of the naive definition (6.39) is then given by

$$\delta^2(|\boldsymbol{p}|,\eta,\sigma) = \frac{\kappa^2}{32\pi^3}|\boldsymbol{p}|^3\eta^{ac}\eta^{bd}\iiint g_\sigma(\eta,\tau)g_\sigma(\eta,\tau')\langle\text{in}|\hat{h}_{ab}(\tau,\boldsymbol{p})\hat{h}_{cd}(\tau',\boldsymbol{q})|\text{in}\rangle\frac{\mathrm{d}^3q}{(2\pi)^3}\,\mathrm{d}\tau\,\mathrm{d}\tau' \tag{6.41}$$

in the limit where σ tends to zero. For the regular terms in the correlation function (6.33), this limit exists and is simple given by the equal-time limit $\eta' = \eta$ of those terms, while for the kernels D and N we calculate

$$\iint g_\sigma(\eta,\tau)g_\sigma(\eta,\tau')\big(N(\tau-\tau',\boldsymbol{p}) - \mathrm{i}D(\tau-\tau',\boldsymbol{p})\big)\,\mathrm{d}\tau\,\mathrm{d}\tau' = \frac{\sqrt{\pi}}{\sigma} - |\boldsymbol{p}| + \mathcal{O}(\sigma)\,. \tag{6.42}$$

Since D is odd under the exchange of τ and τ' (6.25), it is only N that gives a contribution here. From its Fourier transform (6.24), we see that it is positive definite, so that smearing with an arbitrary test function must produce a positive result. This is the case for the result above, but we also see that it is necessary to take the singular distribution into account – if we would have neglected the δ distribution in N, the result would have been just the second term $-|\boldsymbol{p}|$ which is negative. This provides also the resolution to the question raised by some studies which found negative power spectra neglecting such singular terms [150, 151].

We can now calculate the modified power spectrum for tensorial perturbations. Recalling the definition (6.14) of the projection tensor, we obtain

$$\begin{aligned}\delta^2(|\boldsymbol{p}|,\eta,\sigma) = \frac{\kappa^2 H^2}{16\pi^3}\left(1+\boldsymbol{p}^2\eta^2\right)\left(1+\kappa^2 H^2\lambda\right) + \frac{N}{960\pi^2}\frac{\kappa^4 H^4}{16\pi^3}\Bigg[2|\boldsymbol{p}|^2\eta^2 - |\boldsymbol{p}|^4\eta^4 + |\boldsymbol{p}|^4\eta^4\frac{\sqrt{\pi}}{|\boldsymbol{p}|\sigma} \\ - 2\left(1+\boldsymbol{p}^2\eta^2\right)\ln(-2|\boldsymbol{p}|\eta) - 2\Re\left[\mathrm{e}^{2\,\mathrm{i}|\boldsymbol{p}|\eta}\left(|\boldsymbol{p}|\eta+\mathrm{i}\right)^2\left(\operatorname{Ein}(-2\,\mathrm{i}|\boldsymbol{p}|\eta)+\mathrm{i}\frac{\pi}{2}\right)\right] \\ + 2\left[\cos(2|\boldsymbol{p}|\eta)\left(1-\boldsymbol{p}^2\eta^2\right)+2|\boldsymbol{p}|\eta\sin(2|\boldsymbol{p}|\eta)\right]\left(\ln(-2|\boldsymbol{p}|\eta)+\gamma\right)\Bigg] + \mathcal{O}\left(\kappa^6\right).\end{aligned} \tag{6.43}$$

This expression is of course not very illuminative, but we can study its behaviour in two important limits. Since the power spectrum only depends on the physical momentum $\tilde{p} = -H\eta|\boldsymbol{p}|$, it is time-independent in the physical background de Sitter spacetime. For high momenta $\tilde{p} \gg H$, the power spectrum should be independent of the curvature of spacetime, since in the chosen Bunch-Davies vacuum the corresponding modes go over into Minkowski modes (4.30), as explained in section 4.3. In this *sub-horizon* limit, we obtain

$$\delta^2(|\boldsymbol{p}|,\eta,\sigma) = \frac{\kappa^2}{16\pi^3}\tilde{p}^2\left(1-\frac{N}{960\pi^2}\kappa^2\tilde{p}^2\right) + \mathcal{O}\left(\frac{H}{\tilde{p}}\right). \tag{6.44}$$

This shows that we have a power correction to the standard Minkowski spectrum [152], and in fact the curvature of spacetime (here represented by the Hubble constant H) does not show up. Furthermore, one can see nicely the limits of the effective field theory approach – for momenta which are comparable with the Planck scale where the effective field theory description ceases to be valid, we obtain a negative power spectrum, which is however not sensible as explained above. On the other hand, for the so-called *super-horizon* modes which have $\tilde{p} \ll H$, we get

$$\delta^2(|\boldsymbol{p}|,\eta,\sigma) = \frac{\kappa^2 H^2}{16\pi^3}\left(1+\kappa^2 H^2\lambda + \kappa^2 H^2\frac{N}{480\pi^2}\gamma\right) + \mathcal{O}\left(\frac{\tilde{p}}{H}\right), \tag{6.45}$$

a small constant shift of the scale-invariant power spectrum which is found at tree level. In contrast to previous studies of the one-loop corrections to tensor [153] and scalar [154, 155] perturbations, there is no logarithmic dependence on the comoving momentum $|\boldsymbol{p}|$, in line with more recent results on loops corrections for scalar perturbations [156]. In this last study also the origin of this discrepancy was identified – the additional contribution from the spacetime volume measure $\sqrt{-g}$ in dimensional regularization was not properly taken into account.

Regrettably, these corrections are too small to be observable. We could only obtain an observable effect if the Hubble constant H is of the order of the inverse Planck length, but in this regime can effective field theory approach that we have used breaks down, and the prediction could not be trusted anymore.

Riemann tensor

So startling would his results appear to the uninitiated that until they learned the processes by which he had arrived at them, they might well consider him as a necromancer.

— Sir Arthur Conan Doyle, A study in scarlet

Research has to acquire the matter in detail, to analyze its different evolutionary stages and to track down their inner consistence. The real movement can be portrayed appropriately only after this work is accomplished.
(Die Forschung hat den Stoff sich im Detail anzueignen, seine verschiednen Entwicklungsformen zu analysieren und deren innres Band aufzuspüren. Erst nachdem diese Arbeit vollbracht, kann die wirkliche Bewegung entsprechend dargestellt werden.)

— Karl Marx, Capital

7 The Riemann tensor two-point function

In this section, we calculate the two-point function of the Riemann tensor. As explained in the introduction, the Riemann tensor is a gauge-invariant observable which furthermore is local, i.e., it does not depend on boundary conditions at infinity. Classically, it measures the non-commutativity of parallel transport along any curve, and thus completely determines the geometry. Therefore, one can expect that also in the quantum theory its correlation functions give complete information about the (quantum) geometry. Since in de Sitter space it has a non-vanishing background value, we will consider the connected correlation function

$$\left\langle \tilde{R}^{ab}{}_{cd}(x)\tilde{R}^{m'n'}{}_{p'q'}(x')\right\rangle = \langle \text{in}|\tilde{R}^{ab}{}_{cd}(x)\tilde{R}^{m'n'}{}_{p'q'}(x')|\text{in}\rangle - \langle \text{in}|\tilde{R}^{ab}{}_{cd}(x)|\text{in}\rangle\langle \text{in}|\tilde{R}^{m'n'}{}_{p'q'}(x')|\text{in}\rangle\,, \tag{7.1}$$

where this background contribution is subtracted. Furthermore, this has the advantage that we do only need to consider its expansion in the perturbation h_{ab} to linear order, since the quadratic terms in the expansion at x, which would give a contribution when combined with the background value of the Riemann tensor at the point x', cancel out. The above is thus equivalent to

$$\left\langle \tilde{R}^{ab}{}_{cd}(x)\tilde{R}^{m'n'}{}_{p'q'}(x')\right\rangle = \langle \text{in}|\tilde{R}^{(1)ab}{}_{cd}(x)\tilde{R}^{(1)m'n'}{}_{p'q'}(x')|\text{in}\rangle\,, \tag{7.2}$$

with the linearized Riemann tensor given by (4.112), and gauge invariance under the transformation $h_{ab} \to h_{ab} + \nabla_{(a}\xi_{b)}$ is evident from the explicit expression of the linearized Riemann tensor. Note that it is only with the index position given that gauge invariance holds, because then its Lie derivative in the background spacetime with respect to an arbitrary vector field w^a, which gives the gauge transformation by diffeomorphisms defined by w^a, vanishes,

$$\mathcal{L}_w R^{ab}{}_{cd} = \mathcal{L}_w\left(2H^2\delta^a_{[c}\delta^b_{d]}\right) = 0\,. \tag{7.3}$$

In the next two sections we explain two different methods to arrive at this correlation function, first by explicit calculation from the two-point function of the metric perturbations, calculated with corrections due to massless, conformally coupled scalars in section 6.1, and second by using Bianchi identities which relate it to the stress tensor two-point functions calculated in section 4.5.

7.1. Using the metric two-point function

To calculate the Riemann tensor two-point correlation function, we apply the differential operator implied in equation (5.30) to each argument of the two-point function of the metric perturbations given in equations (6.33, 6.36, 6.37). After applying those differential operators, we have to invert the Fourier transform and replace the resulting tensor structure, which will consist of linear combinations of η_{ab}, δ^0_a and $(x-x')^a$ (with $(x-x')^0 = 0$), by maximally symmetric bitensors. The coefficient functions, which will be functions of η, η' and $\boldsymbol{x}-\boldsymbol{x}'$, also have to be replaced by the de Sitter-invariant biscalar $Z(x,x')$. The replacements can be deduced from the expression of $Z(x,x')$ in our spatially flat coordinate system (4.7) and its derivatives (4.10). Explicitly, we have

$$\begin{aligned}
\eta_{ab} &\to H^2\eta^2 g_{ab}\,, & (x-x')_a &\to -\eta\eta' Z_{;a} + \delta^0_a\left(\eta-\eta' Z\right)\,,\\
\eta_{a'b'} &\to H^2(\eta')^2 g_{a'b'}\,, & (x-x')_{a'} &\to \eta\eta' Z_{;a'} - \delta^0_{a'}\left(\eta'-\eta Z\right)\,,\\
\eta_{ab'} &\to \eta\eta' Z_{;ab'} + \eta\delta^0_{b'}Z_{;a} + \eta'\delta^0_a Z_{;b'} + \delta^0_a\delta^0_{b'}(Z-1)\,.
\end{aligned}$$

Technical details for the calculation of the Fourier transformation are given in appendix E.

For the two-point function (6.33, 6.37, 6.36) which includes loops of N massless, conformally coupled scalars, it turns out after a long calculation (conveniently done with the tensor algebra package xAct, see appendix H) that all terms which are not de Sitter-invariant cancel. Finally we are left with a de Sitter-invariant result which can be expressed as

$$\begin{aligned}
&\left\langle \tilde{R}^{ab}{}_{cd}(x)\tilde{R}^{m'n'}{}_{p'q'}(x')\right\rangle = \\
&\quad \frac{4\kappa^2H^6}{\pi^2}\sum_{k=1}^{9} {}^{(k)}\mathcal{R}^{[ab]}{}_{[cd]}{}^{[m'n']}{}_{[p'q']}(Z(x,x'))\bigg[\frac{N}{1920\pi^2}\kappa^2H^2\mathcal{R}^{(1,k)}(Z(x,x')-\mathrm{i}0\,\mathrm{sgn}(\eta-\eta')) \\
&\qquad + \left(1+\kappa^2H^2\lambda\right)\mathcal{R}^{(0,k)}(Z(x,x')-\mathrm{i}0\,\mathrm{sgn}(\eta-\eta'))\bigg]\,,
\end{aligned} \tag{7.4}$$

with λ given by equation (6.35). Here, ${}^{(k)}\mathcal{R}$ are the bitensors constructed from the metric and covariant derivatives of Z given by equation (4.111), which (when antisymmetrized in each pair of indices as indicated in equation (7.4)) have the appropriate symmetries, and the coefficients $\mathcal{R}^{(0,k)}(Z)$ and $\mathcal{R}^{(1,k)}(Z)$ are de Sitter-invariant functions of the biscalar Z. Since they are rather long expressions, and furthermore can be inferred from the decomposition which we detail in the following, it is not necessary to give their explicit expressions here. However, an important feature of those coefficients is that they are well-defined distributions which are only singular as $Z\to 1$, i.e., when x and x' are on the light cone, which is also the case for the two-point function of the matter field ϕ. This can also be seen from the decomposition which follows.

In obtaining this result, de Sitter invariance has not been assumed, but deduced from the the two-point metric correlations computed in the Poincaré patch, which are not only gauge-dependent but also not manifestly de Sitter-invariant. The only de Sitter-invariant input is the selection of a de Sitter-invariant asymptotic initial vacuum state for metric fluctuations as well as matter fields, the (interacting generalization of the) Bunch-Davies vacuum. Furthermore, this result must also be seen as a nontrivial check of the two-point correlation functions of the metric perturbations given by (6.33, 6.37, 6.36).

Of course we can also express the result using other bitensor sets from table 4.1, especially the one given by the normal vectors n_a, $n_{a'}$ and the parallel propagator $g_{ab'}$. The corresponding complete set of maximally symmetric bitensors, which respect the symmetries of the Riemann tensor when antisymmetrized properly is given by

$$\begin{aligned}
{}^{(1)}\mathcal{N}_{abcdm'n'p'q'}(Z) &= g_{ac}g_{bd}g_{m'p'}g_{n'q'} \\
{}^{(2)}\mathcal{N}_{abcdm'n'p'q'}(Z) &= g_{ac}g_{m'p'}\left(g_{bd}n_{n'}n_{q'} + n_b n_d g_{n'q'}\right) \\
{}^{(3)}\mathcal{N}_{abcdm'n'p'q'}(Z) &= g_{ac}g_{m'p'}n_b n_d n_{n'} n_{q'} \\
{}^{(4)}\mathcal{N}_{abcdm'n'p'q'}(Z) &= 4g_{ac}g_{m'p'}n_{(b}g_{d)(n'}n_{q')} \\
{}^{(5)}\mathcal{N}_{abcdm'n'p'q'}(Z) &= 2g_{ac}g_{m'p'}g_{b(n'}g_{q')d} \\
{}^{(6)}\mathcal{N}_{abcdm'n'p'q'}(Z) &= 2\left(g_{ac}n_{m'}n_{p'} + n_a n_c g_{m'p'}\right)g_{b(n'}g_{q')d} \\
{}^{(7)}\mathcal{N}_{abcdm'n'p'q'}(Z) &= 2n_a n_c n_{m'} n_{p'} g_{b(n'}g_{q')d} \\
{}^{(8)}\mathcal{N}_{abcdm'n'p'q'}(Z) &= 8n_{(a}g_{c)(m'}n_{p')}g_{b(n'}g_{q')d} \\
{}^{(9)}\mathcal{N}_{abcdm'n'p'q'}(Z) &= 4g_{a(m'}g_{p')c}g_{b(n'}g_{q')d}\,.
\end{aligned} \tag{7.5}$$

By using the relations from table 4.1, we can easily express the Riemann tensor two-point function (7.4) in the basis furnished by these bitensors. Again, the expressions coefficients are long and can be inferred from the following decomposition, so that we do not give them here.

In order to better analyze the structure of this correlation function, we decompose the Riemann tensor two-point function into the two-point function of the Weyl tensor, the two-point function of the Ricci tensor and scalar and mixed correlation functions. Using equation (A.9), one thus obtains

$$\begin{aligned}
\left\langle \tilde{R}^{ab}{}_{cd}(x)\tilde{R}^{m'n'}{}_{p'q'}(x')\right\rangle = &\left\langle \tilde{C}^{ab}{}_{cd}(x)\tilde{C}^{m'n'}{}_{p'q'}(x')\right\rangle \\
&+ 2\delta^{[a}_{[c}\left\langle \tilde{R}^{b]}{}_{d]}(x)\tilde{C}^{m'n'}{}_{p'q'}(x')\right\rangle + 2\left\langle \tilde{C}^{ab}{}_{cd}(x)\tilde{R}^{[m'}{}_{[p'}(x')\right\rangle\delta^{n']}_{q']} \\
&- \frac{1}{3}\delta^{[a}_{[c}\delta^{b]}_{d]}\left\langle \tilde{R}(x)\tilde{C}^{m'n'}{}_{p'q'}(x')\right\rangle - \frac{1}{3}\left\langle \tilde{C}^{ab}{}_{cd}(x)\tilde{R}(x')\right\rangle\delta^{[m'}_{[p'}\delta^{n']}_{q']} \\
&+ 4\delta^{[a}_{[c}\left\langle \tilde{R}^{b]}{}_{d]}(x)\tilde{R}^{[m'}{}_{[p'}(x')\right\rangle\delta^{n']}_{q']} + \frac{1}{9}\delta^{[a}_{[c}\delta^{b]}_{d]}\left\langle \tilde{R}(x)\tilde{R}(x')\right\rangle\delta^{[m'}_{[p'}\delta^{n']}_{q']} \\
&- \frac{2}{3}\delta^{[a}_{[c}\delta^{b]}_{d]}\left\langle \tilde{R}(x)\tilde{R}^{[m'}{}_{[p'}(x')\right\rangle\delta^{n']}_{q']} - \frac{2}{3}\delta^{[a}_{[c}\left\langle \tilde{R}^{b]}{}_{d]}(x)\tilde{R}(x')\right\rangle\delta^{[m'}_{[p'}\delta^{n']}_{q']}\,.
\end{aligned} \tag{7.6}$$

The correlation functions on the right-hand side are obtained from the correlation function of the Riemann tensor by using the explicit expression (inferable from equation (A.9))

$$C^{ab}{}_{cd} = \left(\delta^a_k\delta^b_l\delta^s_c\delta^t_d + 2\delta^s_k\delta^{[a}_l\delta^{b]}_{[c}\delta^t_{d]} + \frac{1}{3}\delta^{[a}_{[c}\delta^{b]}_{d]}\delta^s_k\delta^t_l\right)R^{kl}{}_{st} \tag{7.7}$$

for the two-point functions that involve the Weyl tensor, while for two-point functions which involve the Ricci tensor or scalar one can simply contract indices appropriately. From the previous decomposition it also follows that the scalar two-point correlation functions defined by

$$\begin{aligned}
\left\langle \tilde{R}ie^2(x,x')\right\rangle &= g_{am'}g_{bn'}g^{cp'}g^{dq'}\left\langle \tilde{R}^{ab}{}_{cd}(x)\tilde{R}^{m'n'}{}_{p'q'}(x')\right\rangle, \\
\left\langle \tilde{C}^2(x,x')\right\rangle &= g_{am'}g_{bn'}g^{cp'}g^{dq'}\left\langle \tilde{C}^{ab}{}_{cd}(x)\tilde{C}^{m'n'}{}_{p'q'}(x')\right\rangle, \\
\left\langle \tilde{R}ic^2(x,x')\right\rangle &= g_{bn'}g^{dq'}\left\langle \tilde{R}^b{}_d(x)\tilde{R}^{n'}{}_{q'}(x')\right\rangle,
\end{aligned} \tag{7.8}$$

are related by

$$\left\langle \tilde{R}ie^2(x,x') \right\rangle = \left\langle \tilde{C}^2(x,x') \right\rangle + 2\left\langle \tilde{R}ic^2(x,x') \right\rangle - \frac{1}{3}\left\langle \tilde{R}(x)\tilde{R}(x') \right\rangle . \tag{7.9}$$

As a peculiarity of the conformal case which already occurs in Minkowski spacetime [114], all correlation functions which include the Ricci scalar vanish, so that in the following we only need to calculate the correlation functions involving the Weyl and Ricci tensors.

7.1.1. The two-point Weyl correlation function

Since the Weyl tensor is traceless, the corresponding coefficients for the bitensor set (4.111) are not independent because those basic bitensors are not traceless. In fact, there are only three combinations of invariant bitensors which have vanishing trace on any contraction. These are

$$\begin{aligned}
{}^{(1)}\mathcal{C}_{[ab][cd][m'n'][p'q']}(Z) &= \left[-2Z\,{}^{(1)}\mathcal{R} - 3\,{}^{(4)}\mathcal{R} + 6Z\,{}^{(5)}\mathcal{R} + 2\,{}^{(8)}\mathcal{R} - 2Z\,{}^{(9)}\mathcal{R} \right]_{[ab][cd][m'n'][p'q']}(Z) \,, \\
{}^{(2)}\mathcal{C}_{[ab][cd][m'n'][p'q']}(Z) &= \Big[-(5-Z^2)\,{}^{(1)}\mathcal{R} + 6\,{}^{(2)}\mathcal{R} + 12\,{}^{(5)}\mathcal{R} - 6\,{}^{(6)}\mathcal{R} - 6\,{}^{(7)}\mathcal{R} \\
&\qquad + 2Z\,{}^{(8)}\mathcal{R} - (3+Z^2)\,{}^{(9)}\mathcal{R} \Big]_{[ab][cd][m'n'][p'q']}(Z) \,, \\
{}^{(3)}\mathcal{C}_{[ab][cd][m'n'][p'q']}(Z) &= \left[(1-Z^2)\left[{}^{(1)}\mathcal{R} - 3\,{}^{(5)}\mathcal{R} + {}^{(9)}\mathcal{R} \right] - 6\,{}^{(3)}\mathcal{R} + 6\,{}^{(7)}\mathcal{R} \right]_{[ab][cd][m'n'][p'q']}(Z) \,.
\end{aligned} \tag{7.10}$$

By using equation (7.7) and rearranging, we therefore obtain the (connected) two-point correlation function of the Weyl tensor linearized over the de Sitter background as

$$\begin{aligned}
\left\langle \tilde{C}^{ab}{}_{cd}(x)\tilde{C}^{m'n'}{}_{p'q'}(x') \right\rangle &= \frac{4\kappa^2 H^6}{\pi^2} \sum_{k=1}^{3} {}^{(k)}\mathcal{C}^{[ab]}{}_{[cd]}{}^{[m'n']}{}_{[p'q']}(Z) \\
&\quad \times \Bigg[\frac{N}{11520\pi^2}\kappa^2 H^2 \mathcal{C}^{(1,k)}(Z - \mathrm{i}0\,\mathrm{sgn}(\eta-\eta')) \\
&\qquad + \left(1+\kappa^2 H^2 \lambda\right) \mathcal{C}^{(0,k)}(Z - \mathrm{i}0\,\mathrm{sgn}(\eta-\eta')) \Bigg] ,
\end{aligned} \tag{7.11}$$

where the coefficients $\mathcal{C}^{(0,k)}(Z)$ and $\mathcal{C}^{(1,k)}(Z)$ are given explicitly by

$$\begin{aligned}
\mathcal{C}^{(0,1)}(Z) &= \frac{1}{4}(2-Z)(1-Z)^{-4} \,, \\
\mathcal{C}^{(0,2)}(Z) &= \frac{1}{8}(1-Z)^{-4} \,, \\
\mathcal{C}^{(0,3)}(Z) &= \frac{1}{4}(3-Z)(1-Z)^{-5} \,, \\
\mathcal{C}^{(1,1)}(Z) &= 12(1+4Z^2-Z^4)(1+Z)^{-4}(1-Z)^{-4} \ln\left[\frac{1}{2}(1-Z)\right] \\
&\qquad + (21+28Z+52Z^2-28Z^3-25Z^4)(1+Z)^{-3}(1-Z)^{-5} \,, \\
\mathcal{C}^{(1,2)}(Z) &= 12Z(1+Z^2)(1+Z)^{-4}(1-Z)^{-4} \ln\left[\frac{1}{2}(1-Z)\right] \\
&\qquad + (8+35Z+28Z^2+25Z^3)(1+Z)^{-3}(1-Z)^{-5} \,,
\end{aligned} \tag{7.12}$$

$$\mathcal{C}^{(1,3)}(Z) = 12Z(7+10Z^2-Z^4)(1+Z)^{-5}(1-Z)^{-5}\ln\left[\frac{1}{2}(1-Z)\right] \\ +(30+177Z+142Z^2+184Z^3-28Z^4-25Z^5)(1+Z)^{-4}(1-Z)^{-6}\,.$$

Even though it seems that those coefficient functions are also singular for $Z \to -1$ (which would correspond to antipodal points), this is only apparent. Developing the logarithm around $Z=-1$ as

$$\ln\left[\frac{1}{2}(1-Z)\right] = -\sum_{k=1}^{\infty}\frac{(1+Z)^k}{2^k\,k}\,, \tag{7.13}$$

we get

$$\mathcal{C}^{(1,1)}(Z) = \frac{11}{128}+\mathcal{O}(Z+1)\,,\quad \mathcal{C}^{(1,2)}(Z) = \frac{23}{768}+\mathcal{O}(Z+1)\,,\quad \mathcal{C}^{(1,3)}(Z) = \frac{19}{240}+\mathcal{O}(Z+1)\,, \tag{7.14}$$

which are perfectly regular. This means that the Weyl tensor two-point function is only singular when the two points x and x' of the two-point correlation function are on the light cone, $Z(x,x')=1$.

Of course, we may also express the basic bitensor set (7.10) for the two-point Weyl correlation function in terms of the complete set n_a, $n_{a'}$ and $g_{ab'}$ using the relations from table 4.1. One easily calculates

$$\begin{aligned}
{}^{(1)}\mathcal{C}_{abcdm'n'p'q'}(Z) &= -2Z^{(1)}\mathcal{D}_{abcdm'n'p'q'}+(1-Z)^{2(2)}\mathcal{D}_{abcdm'n'p'q'}\,,\\
{}^{(2)}\mathcal{C}_{abcdm'n'p'q'}(Z) &= -(3+Z^2)^{(1)}\mathcal{D}_{abcdm'n'p'q'}-(3-Z)(1-Z)^{(2)}\mathcal{D}_{abcdm'n'p'q'}\\
&\quad +(1-Z^2)^{(3)}\mathcal{D}_{abcdm'n'p'q'}\,,\\
{}^{(3)}\mathcal{C}_{abcdm'n'p'q'}(Z) &= (1-Z^2)^{(1)}\mathcal{D}_{abcdm'n'p'q'}+(1-Z)^2(1+Z)^{(2)}\mathcal{D}_{abcdm'n'p'q'}\,,
\end{aligned} \tag{7.15}$$

where the (traceless) tensors ${}^{(k)}\mathcal{D}$ are defined by

$$\begin{aligned}
{}^{(1)}\mathcal{D}_{abcdm'n'p'q'} &= \left[{}^{(1)}\mathcal{N}-3^{(5)}\mathcal{N}+{}^{(9)}\mathcal{N}\right]_{abcdm'n'p'q'}\\
{}^{(2)}\mathcal{D}_{abcdm'n'p'q'} &= \left[-12^{(3)}\mathcal{N}-3^{(4)}\mathcal{N}+12^{(7)}\mathcal{N}+2^{(8)}\mathcal{N}\right]_{abcdm'n'p'q'}\\
{}^{(3)}\mathcal{D}_{abcdm'n'p'q'} &= \left[-2^{(1)}\mathcal{N}+6^{(2)}\mathcal{N}-12^{(3)}\mathcal{N}+3^{(4)}\mathcal{N}+3^{(5)}\mathcal{N}-6^{(6)}\mathcal{N}+12^{(7)}\mathcal{N}\right]_{abcdm'n'p'q'}\,,
\end{aligned} \tag{7.16}$$

with the basic bitensor set ${}^{(k)}\mathcal{N}$ introduced in equation (7.5). The two-point Weyl correlation function (7.44) then reads

$$\left\langle \tilde{C}^{ab}{}_{cd}(x)\tilde{C}^{m'n'}{}_{p'q'}(x')\right\rangle = \frac{4\kappa^2H^6}{\pi^2}\sum_{k=1}^{3}{}^{(k)}\mathcal{D}^{[ab]}{}_{[cd]}{}^{[m'n']}{}_{[p'q']} \\ \times\left[\left(1+\kappa^2H^2\lambda\right)\mathcal{D}^{(0,k)}(Z-\mathrm{i}0\,\mathrm{sgn}(\eta-\eta'))+\frac{N}{11520\pi^2}\kappa^2H^2\mathcal{D}^{(1,k)}(Z-\mathrm{i}0\,\mathrm{sgn}(\eta-\eta'))\right], \tag{7.17}$$

where the coefficient functions $\mathcal{D}^{(0,k)}$ and $\mathcal{D}^{(1,k)}$ are given by

$$
\begin{aligned}
\mathcal{D}^{(0,1)}(Z) &= \frac{1}{8}(3-Z)(1-Z)^{-3} \\
\mathcal{D}^{(0,2)}(Z) &= \frac{1}{8}(7-Z)(1-Z)^{-3} \\
\mathcal{D}^{(0,3)}(Z) &= \frac{1}{8}(1+Z)(1-Z)^{-3} \\
\mathcal{D}^{(1,1)}(Z) &= 24Z(1+Z)^{-3}(1-Z)^{-3}\ln\left[\frac{1}{2}(1-Z)\right] + 6(1+5Z)(1+Z)^{-2}(1-Z)^{-4} \\
\mathcal{D}^{(1,2)}(Z) &= 12(1+2Z+3Z^2)(1+Z)^{-3}(1-Z)^{-3}\ln\left[\frac{1}{2}(1-Z)\right] \\
&\qquad + 3(9+20Z+19Z^2)(1+Z)^{-2}(1-Z)^{-4} \\
\mathcal{D}^{(1,3)}(Z) &= 12Z(1+Z^2)(1+Z)^{-3}(1-Z)^{-3}\ln\left[\frac{1}{2}(1-Z)\right] \\
&\qquad + (8+35Z+28Z^2+25Z^3)(1+Z)^{-2}(1-Z)^{-4} \,.
\end{aligned}
\tag{7.18}
$$

Using either equation (7.11) and the expressions (4.11) for contractions of derivatives of Z, or equation (7.17) together with the relations $g_{ab'}n^{b'} = -n_a$ and $n^a n_a = 1$, we now may compute the contracted Weyl two-point function defined in equation (7.8). It is given by

$$
\begin{aligned}
\left\langle \tilde{C}^2(x,x') \right\rangle = &-\frac{N\kappa^4 H^8}{16\pi^4}(1+Z)^{-3}\ln\left[\frac{1}{2}(1-Z+\mathrm{i}0\,\mathrm{sgn}(\eta-\eta'))\right] \\
&-\frac{N\kappa^4 H^8}{192\pi^4}(7-35Z+29Z^2-25Z^3)(1+Z)^{-2}(1-Z+\mathrm{i}0\,\mathrm{sgn}(\eta-\eta'))^{-4} \,.
\end{aligned}
\tag{7.19}
$$

Again, the singularity for $Z \to -1$ is only apparent, as can be seen by developing the logarithm around $Z = -1$ using the expansion (7.13).

7.1.2. The two-point Ricci and the Ricci-Weyl correlation function

Suitably contracting indices in (7.4) and using the expressions (4.11) for contractions of derivatives of Z, we get for the two-point Ricci correlation function

$$
\begin{aligned}
\left\langle \tilde{R}^b{}_d(x)\tilde{R}^{n'}{}_{q'}(x') \right\rangle = \frac{N\kappa^4 H^4}{768\pi^4}\Big[&-H^4\delta^b_d\delta^{n'}_{q'}(1-Z)^2 + 4Z^{;b}Z_{;d}Z^{;n'}Z_{;q'} \\
&+ 2\left(Z^{;b}Z^{;n'}_{;d}Z_{;q'} + Z_{;d}Z^{;bn'}Z_{;q'} + Z^{;b}Z_{;dq'}Z^{;n'} + Z_{;d}Z^{;b}_{;q'}Z^{;n'}\right)(1-Z) \\
&+ 2\left(Z^{;bn'}Z_{;dq'} + Z^{;b}_{;q'}Z^{;n'}_{;d}\right)(1-Z)^2\Big](1-Z+\mathrm{i}0\,\mathrm{sgn}(\eta-\eta'))^{-6} \,.
\end{aligned}
\tag{7.20}
$$

As explained before, for the conformal case correlation functions involving the Ricci scalar vanish, which can be checked easily with this explicit form. A check on this result is provided by comparing it with the four-dimensional limit of the stress tensor two-point function for massless, conformally coupled scalars, which has the form (4.77) with the coefficients (4.84). We obtain

$$
\left\langle \tilde{R}^b{}_d(x)\tilde{R}^{n'}{}_{q'}(x') \right\rangle = N\frac{\kappa^4}{4}\left\langle T^b{}_d(x)T^{n'}{}_{q'}(x') \right\rangle \,,
\tag{7.21}
$$

which coincides with the previously derived result (6.11) if we take into account that correlation functions of the Ricci scalar vanish and that we consider N conformal fields.

Using equation (7.7) we then obtain for the Ricci-Weyl correlation function

$$\begin{aligned}\left\langle \tilde{R}^b{}_d(x)\tilde{C}^{m'n'}{}_{p'q'}(x')\right\rangle = \frac{N\kappa^4 H^4}{960\pi^4}\Big[&2H^4\delta^b_d\delta^{[m'}_{[p'}\delta^{n']}_{q']}(1-Z^2) - 2H^2 Z^{;b}Z_{;d}\delta^{[m'}_{[p'}\delta^{n']}_{q']} \\ &-6H^2\left(\delta^b_d - H^{-2}Z^{;b}Z_{;d}\right)\delta^{[m'}_{[p'}Z^{;n']}Z_{;q']} \\ &-3\delta^{[m'}_{[p'}\left(Z_{;q']}Z^{;n']b}Z_{;d} + Z^{;n']}Z^{;b}_{;q']}Z_{;d} + Z_{;q']}Z^{;n']}_{;d}Z^{;b} + Z^{;n']}Z_{;q']d}Z^{;b}\right)Z \\ &-3\delta^{[m'}_{[p'}\left(Z^{;b}_{;q']}Z^{;n']}_{;d} + Z^{;n']b}Z_{;q']d}\right)(1-Z^2) \\ &+6H^{-2}Z^{;[m'}Z_{;[p'}\left(Z^{;b}_{;q']}Z^{;n']}_{;d} + Z^{;n']b}Z_{;q']d}\right)\Big](1-Z+\mathrm{i}0\,\mathrm{sgn}(\eta-\eta'))^{-5}\,. \end{aligned} \tag{7.22}$$

For the Weyl-Ricci correlation function one also has

$$\left\langle \tilde{C}^{ab}{}_{cd}(x)\tilde{R}^{n'}{}_{q'}(x')\right\rangle = \left\langle \tilde{R}^{n}{}_{q}(x)\tilde{C}^{a'b'}{}_{c'd'}(x')\right\rangle, \tag{7.23}$$

which (for a de Sitter-invariant result) can be seen as a consequence of $Z(x,x') = Z(x',x)$. Note that there are no tree level contributions to the correlation functions involving the Ricci tensor. This can be explained by the fact that we consider the conformal scalar field in its vacuum state, where the de Sitter-invariant expectation value of the stress tensor operator is (with the appropriate index position) proportional to δ^a_b, which is not affected by metric perturbations, just like the cosmological constant term. Therefore, connected correlation functions involving the Ricci tensor first receive contributions at one-loop order.

As explained before, any further contractions vanish, i.e., the two-point function of the Ricci scalar with any other curvature tensor is zero. Lastly, for the contracted two-point Ricci correlation function defined in equation (7.8) we obtain

$$\left\langle \tilde{R}ic^2(x,x')\right\rangle = \frac{N\kappa^4 H^8}{64\pi^4}(1-Z+\mathrm{i}0\,\mathrm{sgn}(\eta-\eta'))^{-4}\,. \tag{7.24}$$

In the bitensor basis furnished by the normal vectors n_a and $n_{a'}$ and the parallel propagator $g_{ab'}$, the above correlation functions (7.20) and (7.22) read

$$\begin{aligned}\left\langle \tilde{R}^b{}_d(x)\tilde{R}^{n'}{}_{q'}(x')\right\rangle = \frac{N\kappa^4 H^8}{768\pi^4}\Big[&-\delta^b_d\delta^{n'}_{q'} + 16n^b n_d n^{n'}n_{q'} + 2\left(g^{bn'}g_{dq'} + g^b_{q'}g^{n'}_d\right) \\ &+4\left(n^b g^{n'}_d n_{q'} + n_d g^{bn'}n_{q'} + n^b g_{dq'}n^{n'} + n_d g^b_{q'}n^{n'}\right)\Big]\times \\ &\times(1-Z+\mathrm{i}0\,\mathrm{sgn}(\eta-\eta'))^{-4}\end{aligned} \tag{7.25}$$

and

$$\left\langle \tilde{R}^b{}_d(x)\tilde{C}^{m'n'}{}_{p'q'}(x')\right\rangle = \frac{N\kappa^4 H^8}{960\pi^4}\Big[2\left(\delta^b_d - n^b n_d\right)\delta^{[m'}_{[p'}\delta^{n']}_{q']} - 6\delta^b_d\delta^{[m'}_{[p'}n^{n']}n_{q']}$$
$$- 6\delta^{[m'}_{[p'}\left(n_{q']}g^{n']b}n_d + n^{n']}g^b_{q']}n_d + n_{q']}g^{n']}_d n^b + n^{n']}g_{q']d}n^b\right)Z$$
$$- 3\left(\delta^{[m'}_{[p'} - 2n^{[m'}n_{[p'}\right)\left(g^b_{q']}g^{n']}_d + g^{n']b}g_{q']d}\right)\Big]\times$$
$$\times (1+Z)(1-Z+\mathrm{i}0\,\mathrm{sgn}(\eta-\eta'))^{-4}\,, \tag{7.26}$$

respectively. A further check on these results is provided by verifying that they satisfy the Bianchi identities, which will be explained in the next section. Moreover, the physical interpretation of these results will be given in the section afterwards.

7.2. Using Bianchi identities

Independent of the concrete form of the metric, the Riemann tensor obeys certain identities arising from its definition, the Bianchi identities. In the case that we are considering, these identities can actually be used to fully determine the form of its two-point function, up to two constants of integration. For all the tensor algebra in this section, we have used the tensor manipulation package xAct (see appendix H).

The identity we use is the second Bianchi identity (5.9). By contracting indices and taking other symmetries of the Riemann tensor into account, from this identity we can derive

$$\tilde{\nabla}_a\tilde{R}^{ab} = \frac{1}{2}\tilde{\nabla}^b\tilde{R}, \tag{7.27}$$

$$\tilde{\nabla}_{[s}\tilde{C}^{ab}{}_{cd]} = \left(\delta^a_{[s}\delta^p_c\delta^l_{d]}\delta^b_k - \delta^b_{[s}\delta^p_c\delta^l_{d]}\delta^a_k + \frac{1}{3}\delta^{[a}_{[s}\delta^{b]}_c\delta^p_{d]}\delta^l_k\right)\tilde{\nabla}_p\tilde{R}^k{}_l \tag{7.28}$$

and

$$\tilde{\nabla}_a\tilde{C}^{ab}{}_{cd} = \tilde{\nabla}_{[c}\left(\tilde{R}^b{}_{d]} - \frac{1}{6}\delta^b_{d]}\tilde{R}\right). \tag{7.29}$$

We now want to apply these identities to correlation functions of the curvature tensors. For the linearized Riemann tensor, the Bianchi identities are operator identities since there is no problem with operator-ordering. If we only consider the connected two-point function which is bilinear in the perturbation h_{ab}, we can take (to this order) the derivative $\tilde{\nabla}$ to be the background derivative ∇. Furthermore, if we concentrate on Wightman functions, no additional local terms due to the time-ordering can arise if we take the derivative outside the expectation value. All in all, the Bianchi identities apply therefore without any additional contact terms also to the two-point function of the Riemann tensor.

To use them to constrain the form of the connected Riemann tensor two-point function, we decompose it into the two-point function of the Weyl tensor, the two-point functions of the Ricci tensor and scalar and mixed correlation functions (7.6), as was already done in the last section. It has been shown in section 6 that the Einstein tensor correlation function is simply related to the two-point function of the stress tensor (6.11),

$$\left\langle \tilde{G}^a{}_b(x)\tilde{G}^{c'}{}_{d'}(x)\right\rangle = \frac{\kappa^4}{4}\left\langle T^a{}_b(x)T^{c'}{}_{d'}(x)\right\rangle = \frac{\kappa^2}{4}\sum_{k=1}^{5}{}^{(k)}\mathcal{T}^a{}_b{}^{c'}{}_{d'}\mathcal{T}^{(k)}(Z(x,x')), \tag{7.30}$$

with the decomposition of the stress tensor correlation function (4.77). Using the definition of the Einstein tensor

$$\tilde{G}^a{}_b = \tilde{R}^a{}_b - \frac{1}{2}\tilde{R}\delta^a_b, \tag{7.31}$$

we therefore obtain

$$\begin{aligned}\left\langle \tilde{R}^a{}_b(x)\tilde{R}^{m'}{}_{n'}(x')\right\rangle &= \left(\delta^a_c\delta^d_b - \frac{1}{2}\delta^a_b\delta^d_c\right)\left(\delta^{m'}_{p'}\delta^{q'}_{n'} - \frac{1}{2}\delta^{m'}_{n'}\delta^{q'}_{p'}\right)\left\langle \tilde{G}^c{}_d(x)\tilde{G}^{p'}{}_{q'}(x')\right\rangle \\ &= \frac{\kappa^4}{4}\sum_{k=1}^{5}\left({}^{(k)}\mathcal{T}^{a\,m'}_{b\,n'} - \frac{1}{2}\delta^{m'}_{n'}{}^{(k)}\mathcal{T}^{as'}_{bs'} - \frac{1}{2}\delta^a_b{}^{(k)}\mathcal{T}^{cm'}_{cn'} + \frac{1}{4}\delta^a_b\delta^{m'}_{n'}{}^{(k)}\mathcal{T}^{cs'}_{cs'}\right)\mathcal{T}^{(k)}(Z).\end{aligned} \tag{7.32}$$

Under the assumption that the stress tensor correlation function is de Sitter-invariant, the Einstein tensor correlation function and therefore the two-point Ricci correlation function are also de Sitter-invariant. The same applies to the two-point correlation function of the Ricci scalar and the correlation functions between Ricci scalar and Ricci tensor, which can be obtained by simply contracting the two-point Ricci tensor correlation function (7.32) in an appropriate way. In the following, we will only be interested in the Wightman function but omit the necessary prescription $Z \to Z - \mathrm{i}0\,\mathrm{sgn}(\eta-\eta')$, in order not to overburden the already long formulas.

7.2.1. The Weyl-Ricci correlation function

Applying the identity (7.29) to the Weyl-Ricci correlation function, we get

$$\nabla_a\left\langle \tilde{C}^{ab}{}_{cd}\tilde{R}^{m'}{}_{n'}(x')\right\rangle = \left(\delta^b_k\delta^l_{[d} - \frac{1}{6}\delta^l_k\delta^b_{[d}\right)\nabla_{c]}\left\langle \tilde{R}^k{}_l\tilde{R}^{m'}{}_{n'}(x')\right\rangle. \tag{7.33}$$

Considering that we only calculate the two-point correlation functions to lowest order in the metric perturbation h_{mn}, the (perturbed) derivative $\tilde{\nabla}$ contributes only with its background value ∇ and can be taken outside of the expectation value as explained before.

What is the most general form that the Weyl-Ricci correlation function can have? For quantization using the CTP formalism in de Sitter space, one can choose the Poincaré patch and manifestly preserve spatial rotations and translations in the conformally flat coordinate system (4.8). The tensor structure of any correlation function may then contain the spatial vector $\boldsymbol{x}-\boldsymbol{x}'$, temporal coefficients δ^a_0 and the Minkowski metric, which using equation (4.10) can be alternatively expressed using covariant derivatives of Z and δ^a_0. Furthermore, instead of δ^0_a we can use the comoving velocity

$$u_a = (-H\eta)^{-1}\delta^0_a \tag{7.34}$$

which is normalized to $u_a u^a = -1$ and whose covariant derivative is given by

$$\nabla_a u_b = -H(u_a u_b + g_{ab}). \tag{7.35}$$

The contraction of u_a with a covariant derivative of Z is given by

$$u^s Z_{;s} = H\left(\frac{\eta}{\eta'} - Z\right), \tag{7.36}$$

which is easily calculated using the definition of u_a and equation (4.10).

From these objects, we construct the most general six-index bitensor with arbitrary coefficient functions. Imposing the symmetries of the Ricci and Weyl tensor and the vanishing trace of the Weyl tensor on any index contraction, only some of these terms survive, and the most general form of the Weyl-Ricci correlation function under the above assumptions is given by

$$\left\langle \tilde{C}^{ab}{}_{cd}(x)\tilde{R}^{m'}{}_{n'}(x')\right\rangle = \sum_{k=1}^{3} {}^{(k)}F^{[ab]}{}_{[cd]}{}^{m'}{}_{n'}\, f_k(Z,\eta,\eta'). \tag{7.37}$$

In this expression, the f_k are arbitrary functions of Z, η and η' while the tensor factors ${}^{(k)}F$ are given by

$$\begin{aligned} {}^{(1)}F_{abcdm'n'} = {} & g_{ac}g_{bd}g_{m'n'}(1-Z^2) - H^{-2}g_{ac}g_{bd}Z_{;m'}Z_{;n'} - 3H^{-2}g_{ac}Z_{;b}Z_{;d}g_{m'n'} \\ & + 3H^{-4}g_{ac}Z_{;b}Z_{;d}Z_{;m'}Z_{;n'} - 6H^{-4}g_{ac}Z_{;(b}Z_{;d)(m'}Z_{;n')}Z \\ & - 3H^{-4}g_{ac}Z_{;b(m'}Z_{;n')d}(1-Z^2) + 6H^{-6}Z_{;a}Z_{;c}Z_{;b(m'}Z_{;n')d}\,, \end{aligned} \tag{7.38}$$

$$\begin{aligned} {}^{(2)}F_{abcdm'n'} = {} & g_{ac}g_{bd}g_{m'n'} + 3g_{ac}u_bu_dg_{m'n'} + g_{ac}g_{bd}u_{m'}u_{n'}\frac{\eta^2}{\eta'^2} - 2H^{-1}g_{ac}g_{bd}u_{(m'}Z_{;n')}\frac{\eta}{\eta'} \\ & - 3H^{-2}g_{ac}u_bu_dZ_{;m'}Z_{;n'} + 6H^{-3}g_{ac}u_{(b}Z_{;d)(m'}Z_{;n')} - 6H^{-2}g_{ac}u_{(b}Z_{;d)(m'}u_{n')}\frac{\eta}{\eta'} \\ & - 3H^{-4}g_{ac}Z_{;b(m'}Z_{;n')d} - 6H^{-4}u_bu_dZ_{;a(m'}Z_{;n')c} \end{aligned} \tag{7.39}$$

and

$$\begin{aligned} {}^{(3)}F_{abcdm'n'} = {} & -3g_{ac}u_bu_dg_{m'n'}\left(\frac{\eta}{\eta'} - Z\right) - g_{ac}g_{bd}u_{m'}u_{n'}\left(\frac{\eta}{\eta'} - Z\right)\frac{\eta^2}{\eta'^2} - 3H^{-1}g_{bd}u_{(a}Z_{;c)}g_{m'n'} \\ & + H^{-1}g_{ac}g_{bd}u_{(m'}Z_{;n')}\left(2\frac{\eta}{\eta'} - Z\right)\frac{\eta}{\eta'} - H^{-2}g_{ac}g_{bd}Z_{;m'}Z_{;n}\frac{\eta}{\eta'} \\ & + 6H^{-2}g_{ac}u_{(b}Z_{;d)(m'}u_{n')}\left(\frac{\eta}{\eta'} - Z\right)\frac{\eta}{\eta'} + 3H^{-2}g_{ac}u_bu_dZ_{;m'}Z_{;n'}\left(\frac{\eta}{\eta'} - Z\right) \\ & + 3H^{-3}g_{ac}Z_{;(b}Z_{;d)(m'}u_{n')}\frac{\eta}{\eta'} + 3H^{-3}g_{ac}u_{(b}Z_{;d)}Z_{;m'}Z_{;n'} \\ & - 3H^{-3}g_{ac}u_{(b}Z_{;d)(m'}Z_{;n')}\left(2\frac{\eta}{\eta'} - Z\right) - 3H^{-4}g_{ac}Z_{;(b}Z_{;d)(m'}Z_{;n')} \\ & + 6H^{-4}u_bu_dZ_{;a(m'}Z_{;n')c}\left(\frac{\eta}{\eta'} - Z\right) + 6H^{-5}u_{(b}Z_{;d)}Z_{;a(m'}Z_{;n')c}\,. \end{aligned} \tag{7.40}$$

To constrain the functions f_k, we now use equation (7.33) on the general form (7.37). Since the two-point Ricci tensor correlation function is de Sitter-invariant, all terms in the Weyl-Ricci correlation function which (after taking the covariant derivative) contain u_a must vanish. Calculating this explicitly, we see that it is only possible if $f_2(Z,\eta,\eta') = f_3(Z,\eta,\eta') = 0$ and $\partial_\eta f_1(Z,\eta,\eta') = 0$, so that the tensor structure is de Sitter invariant and f_1 does not depend explicitly on η. Calculating now the right-hand side of equation (7.33) using the expression (7.32) for the two-point Ricci correlation function, we obtain

$$f_1(Z,\eta,\eta') = \frac{\kappa^4}{15}\left(Z\mathcal{T}^{(4)}(Z) + (1-Z^2)\mathcal{T}^{(4)\prime}(Z) - (1-Z^2)\mathcal{T}^{(3)}(Z) - Z\mathcal{T}^{(5)\prime}(Z)\right) . \tag{7.41}$$

That is, the Weyl-Ricci correlation function is de Sitter-invariant and can be given in the explicit form

$$\begin{aligned}\left\langle \tilde{C}^{ab}{}_{cd}(x)\tilde{R}^{m'}{}_{n'}(x')\right\rangle = \frac{\kappa^4}{30H^4}\Big[&2H^2\delta^a_{[c}\delta^b_{d]}\left(H^2\delta^{m'}_{n'}(1-Z^2) - Z^{;m'}Z_{;n'}\right) \\ &-6\delta^{[a}_{[c}Z^{;b]}Z_{;d]}\left(H^2\delta^{m'}_{n'} - Z^{;m'}Z_{;n'}\right) \\ &-3\delta^{[a}_{[c}\left(Z^{;b]}Z^{;m'}_{;d]}Z_{;n'} + Z^{;b]}Z_{;d]n'}Z^{;m'} + Z_{;d]}Z^{;b]m'}Z_{;n'} + Z_{;d]}Z^{;b]}_{;n'}Z^{;m'}\right)Z \\ &-3\left((1-Z^2)\delta^{[a}_{[c} - 2H^{-2}Z^{;[a}Z_{;[c}\right)\left(Z^{;b]m'}Z_{;d]n'} + Z^{;b]}_{;n'}Z^{;m'}_{;d]}\right)\Big] \\ \times\Big(-(1-Z^2)&\left(\mathcal{T}^{(3)}(Z) - \mathcal{T}^{(4)\prime}(Z)\right) + Z\left(\mathcal{T}^{(4)}(Z) - \mathcal{T}^{(5)\prime}(Z)\right)\Big) .\end{aligned} \tag{7.42}$$

One easily checks that applying the full Bianchi identity (7.28) to this correlation function does not give any new information.

For the Ricci-Weyl correlation function the same reasoning applies, and it results

$$\left\langle \tilde{R}^m{}_n(x)\tilde{C}^{a'b'}{}_{c'd'}(x')\right\rangle = \left\langle \tilde{C}^{ab}{}_{cd}(x)\tilde{R}^{m'}{}_{n'}(x')\right\rangle . \tag{7.43}$$

The correlation function of the Weyl tensor and the Ricci scalar, obtained by contracting the Weyl-Ricci correlation function with $\delta^{n'}_{m'}$, vanishes identically, so that it only remains to calculate the two-point correlation function of the Weyl tensor, which we do in the next subsection.

7.2.2. The two-point Weyl correlation function

In contrast to the Weyl-Ricci correlation function, all terms involving u_a drop out when one imposes the symmetries of the Weyl tensor in the most general form for the two-point Weyl tensor correlation function. It is given by

$$\left\langle \tilde{C}^{ab}{}_{cd}(x)\tilde{C}^{m'n'}{}_{p'q'}(x')\right\rangle = \sum_{k=1}^{3} {}^{(k)}\mathcal{C}^{[ab]}{}_{[cd]}{}^{[m'n']}{}_{[p'q']}\mathcal{C}^{(k)}(Z,\eta,\eta') , \tag{7.44}$$

where $\mathcal{C}^{(k)}(Z,\eta,\eta')$ are arbitrary functions, and where the bitensors ${}^{(k)}\mathcal{C}_{abcdm'n'p'q'}$ are given by equation (7.10).

For the two-point Weyl tensor correlation function, it is necessary to apply the uncontracted Bianchi identity (7.28) to maximally constrain the functions $\mathcal{C}^{(k)}(Z,\eta,\eta')$. Again, to lowest order in the metric perturbations we can take the derivative outside of the expectation value and obtain

$$\nabla_{[s}\left\langle \tilde{C}^{ab}{}_{cd]}(x)\tilde{C}^{m'n'}{}_{p'q'}(x')\right\rangle = \left(\delta^a_{[s}\delta^p_c\delta^l_{d]}\delta^b_k - \delta^b_{[s}\delta^p_c\delta^l_{d]}\delta^a_k + \frac{1}{3}\delta^{[a}_{[s}\delta^{b]}_c\delta^p_{d]}\delta^l_k\right) \times \nabla_p\left\langle \tilde{R}^k{}_l(x)\tilde{C}^{m'n'}{}_{p'q'}(x')\right\rangle . \tag{7.45}$$

The Ricci-Weyl correlation function is given by equation (7.43), and is de Sitter-invariant. Therefore, after taking the covariant derivative in the ansatz (7.44) all terms which contain u^a must vanish, which is only possible if the functions $\mathcal{C}^{(k)}(Z,\eta,\eta')$ do not depend explicitly on η. Applying the Bianchi identity (7.28) to the Weyl tensor at x', we see that they also cannot depend explicitly on η', so that the two-point Weyl tensor correlation function must be de Sitter-invariant. Calculating explicitly both sides of equation (7.45) using the Ricci-Weyl correlation function given by (7.43), we obtain

$$\begin{aligned}\mathcal{C}^{(1)}(Z) &= -6Z\mathcal{C}^{(2)}(Z) + (1-Z^2)\mathcal{C}^{(2)\prime}(Z) \\ &\quad + \frac{\kappa^4}{15}\left[2Z(1-Z^2)\left(\mathcal{T}^{(3)}(Z)-\mathcal{T}^{(4)\prime}(Z)\right) + (5-2Z^2)\left(\mathcal{T}^{(4)}(Z)-\mathcal{T}^{(5)\prime}(Z)\right)\right]\end{aligned} \tag{7.46}$$

and

$$\begin{aligned}\mathcal{C}^{(3)}(Z) &= 6\mathcal{C}^{(2)}(Z) + Z\mathcal{C}^{(2)\prime}(Z) \\ &\quad + \frac{\kappa^4}{15}\left[(3+2Z^2)\left(\mathcal{T}^{(3)}(Z)-\mathcal{T}^{(4)\prime}(Z)\right) + 2Z\left(\mathcal{T}^{(4)}(Z)-\mathcal{T}^{(5)\prime}(Z)\right)\right] ,\end{aligned} \tag{7.47}$$

while $\mathcal{C}^{(2)}(Z)$ is the solution of the second-order ordinary differential equation

$$(1-Z^2)\mathcal{C}^{(2)\prime\prime}(Z) - 10Z\mathcal{C}^{(2)\prime}(Z) - 20\mathcal{C}^{(2)}(Z) = \frac{\kappa^4}{15}S(Z) , \tag{7.48}$$

where

$$S(Z) = (7-2Z^2)\mathcal{T}^{(3)}(Z) - 2Z\mathcal{T}^{(4)}(Z) - 2(6-Z^2)\mathcal{T}^{(4)\prime}(Z) + 2Z\mathcal{T}^{(5)\prime}(Z) + 5\mathcal{T}^{(5)\prime\prime}(Z) . \tag{7.49}$$

This equation can be solved easily by the method of variation of parameters. A complete set of solutions of the homogeneous equation is found to be $c_1(Z) = (1+Z)^{-4}$ and $c_2(Z) = Z(1+Z^2)(1-Z^2)^{-4}$, from which the method of variation of parameters gives

$$\mathcal{C}^{(2)}(Z) = \frac{\kappa^4}{15(1+Z)^4}\left[\frac{Z(1+Z^2)}{(1-Z)^4}\int (1-Z)^4 S(Z)\,\mathrm{d}Z - \int Z(1+Z^2)S(Z)\,\mathrm{d}Z\right] . \tag{7.50}$$

That is, also the two-point Weyl tensor correlation function is de Sitter-invariant and, up to two integration constants, completely determined by the stress tensor correlation function.

In the rest of this subsection we will argue that those integration constants can (and should) always be chosen such as to make the limit $Z \to -1$ of the two-point Weyl tensor correlation function (which corresponds to antipodal points) finite, if it is finite for the stress tensor correlation function.

Assuming that the stress tensor correlation function is only singular as $Z \to 1$, equation (7.49) shows that $S(Z)$ and its derivatives are finite at $Z = -1$. We can then fix the limits of the integrals in the solution (7.50) to be -1 and Z and exhibit explicitly the integration constants C_i, which make the contribution

$$\frac{\kappa^4}{15(1+Z)^4}\left(\frac{Z(1+Z^2)}{(1-Z)^4}C_1 - C_2\right) . \tag{7.51}$$

Expanding this solution around $Z=-1$, we thus obtain

$$\mathcal{C}^{(2)}(Z) \sim -\frac{C_1+8C_2}{2(1+Z)^4} + \frac{C_1}{32} + \frac{5C_1+32S(-1)}{80}(1+Z) + \mathcal{O}\big((1+Z)^2\big) , \tag{7.52}$$

so that the choice $C_1=-8C_2$ makes the limit $Z\to -1$ finite. This is also what one expects on physical grounds: the CTP formalism manifestly preserves causality in the sense that all integrals extend only over the past light cone. For a vacuum state which is only singular for points x and x' which are on the light cone, such integrations should clearly not result in a singular behavior at antipodal points, which are never in causal contact. One therefore has to make the above choice of the integration constants, and we are certain that an explicit calculation will conform this choice for any particular case.

The coefficient $\mathcal{C}^{(2)}(Z)$ is thus given by

$$\begin{aligned}\mathcal{C}^{(2)}(Z) = \frac{\kappa^4}{15(1+Z)^4}&\left[\frac{Z(1+Z^2)}{(1-Z)^4}\int_{-1}^{Z}(1-Z)^4 S(Z)\,\mathrm{d}Z - \int_{-1}^{Z} Z(1+Z^2)S(Z)\,\mathrm{d}Z\right] \\ &+ C(1-Z)^{-4} ,\end{aligned} \tag{7.53}$$

where the remaining ambiguity is the free choice of the constant C. Inserting only the term with this constant into equations (7.46) and (7.47), the undetermined part of the two-point Weyl correlation function is an arbitrary multiple of

$$\begin{aligned}&2\,{}^{(1)}\mathcal{C}^{[ab]}{}_{[cd]}{}^{[m'n']}{}_{[p'q']}(2-Z)(1-Z)^{-4} + {}^{(2)}\mathcal{C}^{[ab]}{}_{[cd]}{}^{[m'n']}{}_{[p'q']}(1-Z)^{-4} \\ &\quad + 2\,{}^{(3)}\mathcal{C}^{[ab]}{}_{[cd]}{}^{[m'n']}{}_{[p'q']}(3-Z)(1-Z)^{-5} ,\end{aligned} \tag{7.54}$$

which is proportional to the two-point Weyl correlation function for free gravitons [157, 158]. This is a physically sensible result: the whole two-point Riemann correlation function is determined by the matter fields in form of the stress tensor correlation function, except for a part which corresponds to free gravitons. The strength of this part (i.e., the overall coefficient) will of course in general receive quantum corrections. This also gives a very intuitive physical picture: gravitons propagating through spacetime are hindered in their motion by the interaction with matter fields, i.e., the matter fluctuations around the otherwise empty vacuum induce a kind of refractive index (see also [159, 160]). Of course, it has to be seen if this simple and intuitive result still holds when one takes interaction between gravitons into account.

7.2.3. Examples

In this section we will use the formulas derived in previous sections to calculate the two-point curvature tensor correlation functions for the fields studied in section 4.2, the massless vector field, the massless, conformally coupled scalar, the massive, minimally coupled scalar and the massless, minimally coupled scalar.

Since the Wightman function does not need to be renormalized, we can take the four-dimensional limit of the stress tensor two-point functions calculated in section 4.5. Let us start with the massless vector field, for which the stress tensor correlation function is given by the general form (4.77), with the coefficients (4.79). The source term $S(Z)$ (7.49) reads in this case

$$S(Z) = \frac{3H^8}{4\pi^4} \frac{9+Z}{(1-Z)^6} , \tag{7.55}$$

and we determine $\mathcal{C}^{(2)}$ using equation (7.53) to be

$$\begin{aligned} \mathcal{C}^{(2)}(Z) = {} & \frac{\kappa^4 H^8}{240\pi^4} \frac{25Z^3 + 28Z^2 + 35Z + 8}{(1-Z)^5(1+Z)^3} + \frac{\kappa^4 H^8}{20\pi^4} \frac{Z(1+Z^2)}{(1-Z^2)^4} \ln\left(\frac{1-Z}{2}\right) \\ & + C(1-Z)^{-4} , \end{aligned} \tag{7.56}$$

adjusting the constant C to shorten the resulting expression. From this expression, we now calculate $\mathcal{C}^{(1)}$ (7.46) and $\mathcal{C}^{(3)}$ (7.47), for which it results

$$\begin{aligned} \mathcal{C}^{(1)}(Z) = {} & \frac{\kappa^4 H^8}{240\pi^4} \frac{21 + 28Z + 52Z^2 - 28Z^3 - 25Z^4}{(1-Z)^5(1+Z)^3} + \frac{\kappa^4 H^8}{20\pi^4} \frac{1 + 4Z^2 - Z^4}{(1-Z^2)^4} \ln\left(\frac{1-Z}{2}\right) \\ & + 2C \frac{2-Z}{(1-Z)^4} \\ \mathcal{C}^{(3)}(Z) = {} & \frac{\kappa^4 H^8}{240\pi^4} \frac{30 + 177Z + 78Z^2 + 184Z^3 - 28Z^4 - 25Z^5}{(1-Z)^6(1+Z)^4} \\ & + \frac{\kappa^4 H^8}{20\pi^4} \frac{Z(7 + 10Z^2 - Z^4)}{(1-Z^2)^5} \ln\left(\frac{1-Z}{2}\right) + 2C \frac{3-Z}{(1-Z)^5} . \end{aligned} \tag{7.57}$$

For the combination of stress tensor components appearing in the Weyl-Ricci correlation function (7.42), we get

$$-(1-Z^2)\left(\mathcal{T}^{(3)}(Z) - \mathcal{T}^{(4)\prime}(Z)\right) + Z\left(\mathcal{T}^{(4)}(Z) - \mathcal{T}^{(5)\prime}(Z)\right) = \frac{3H^8}{8\pi^4(1-Z)^5} , \tag{7.58}$$

and with this result all curvature tensor correlation functions are determined: the Einstein tensor two-point function is related by (6.11) to the stress tensor two-point functions, and one can calculate easily that all correlation functions involving the Ricci scalar vanish just as for the massless, conformally coupled scalar calculated in section 7.1. The Weyl-Ricci correlator is given by the general formula (7.42) with the coefficient given by (7.58), and the Weyl tensor two-point function has the form (7.44) with the coefficients from (7.56, 7.57).

In fact, by comparing with the result for the massless, conformally coupled scalar (7.22, 7.11), we see that the result for loops of massless vectors is exactly 12 times the result for loops of massless, conformally coupled scalars (up to the undetermined constant C). This result should not come as a surprise: the main ingredient in the effective action (3.15) is the stress tensor two-point function, and we have seen in section 4.5 that in four dimensions massless vectors and massless, conformally coupled scalars give rise to the same stress tensor correlation function, up to this factor of 12. The differences between the two (such as non-equivalence in n dimensions and different stress tensor expectation values) can all be subsumed during the renormalization procedure in the renormalized couplings, which then may result in a different choice for the constant C.

For the massless, conformally coupled scalar the procedure gives the same result up to the mentioned factor 12, and this then coincides exactly with the previously obtained result (7.22, 7.11) if we identify the constant C with

$$C = \frac{\kappa^2 H^6}{2\pi^2} \left(1 + \kappa^2 H^2 \lambda\right) . \tag{7.59}$$

Since everything else in the (Wightman) curvature tensor correlation functions is completely determined by the Wightman stress tensor two-point function, which does not get renormalized, the renormalized couplings $\alpha(\mu)$ and $\beta(\mu)$ can only appear in this constant C (in this case through λ (6.35)).

For the free massless, minimally coupled scalar field, the stress tensor two-point function is given by the same general decomposition (4.77) with the coefficients from equation (4.82). The two-point Ricci tensor correlation function can be calculated using equation (7.32), and reads

$$\left\langle \tilde{R}^a{}_b(x) \tilde{R}^{m'}{}_{n'}(x') \right\rangle = \frac{\kappa^4}{4} \sum_{k=3}^{5} {}^{(k)}\mathcal{T}^a{}_b{}^{m'}{}_{n'} \; {}^{(k)}\mathcal{T}(Z) . \tag{7.60}$$

The correlation function between the Ricci tensor and the Ricci scalar is obtained by contracting indices, and we obtain

$$\left\langle \tilde{R}^a{}_b(x) \tilde{R}(x') \right\rangle = \frac{\kappa^4 H^8}{128\pi^4} \left[\delta^a_b (2-Z)^2 (1-Z)^{-4} + H^{-2} Z^{;a} Z_{;b} (5-Z)(1-Z)^{-5} \right] , \tag{7.61}$$

as well as

$$\left\langle \tilde{R}(x) \tilde{R}^{m'}{}_{n'}(x') \right\rangle = \left\langle \tilde{R}^m{}_n(x) \tilde{R}(x') \right\rangle \tag{7.62}$$

and

$$\left\langle \tilde{R}(x) \tilde{R}(x') \right\rangle = \frac{3\kappa^4 H^8}{128\pi^4} (7 - 4Z + Z^2)(1-Z)^{-4} . \tag{7.63}$$

The Weyl-Ricci correlation function is calculated in equation (7.42), which gives for the massless, minimally coupled case

$$\begin{aligned} \left\langle \tilde{C}^{ab}{}_{cd}(x) \tilde{R}^{m'}{}_{n'}(x') \right\rangle = \frac{\kappa^4 H^4}{1920\pi^4} \Big[& 2H^2 \delta^a_{[c} \delta^b_{d]} \left(H^2 \delta^{m'}_{n'} (1-Z^2) - Z^{;m'} Z_{;n'} \right) \\ & - 3\delta^{[a}_{[c} \left(Z^{;b]} Z^{;m'}_{;d]} Z_{;n'} + Z^{;b]} Z_{;d]n'} Z^{;m'} + Z_{;d]} Z^{;b]m'} Z_{;n'} + Z_{;d]} Z^{;b]}_{;n'} Z^{;m'} \right) Z \\ & - 3 \left((1-Z^2) \delta^{[a}_{[c} - 2H^{-2} Z^{;[a} Z_{;[c} \right) \left(Z^{;b]m'} Z_{;d]n'} + Z^{;b]}_{;n'} Z^{;m'}_{;d]} \right) \\ & - 6\delta^{[a}_{[c} Z^{;b]} Z_{;d]} \left(H^2 \delta^{m'}_{n'} - Z^{;m'} Z_{;n'} \right) \Big] (7-5Z)(1-Z)^{-5} . \end{aligned} \tag{7.64}$$

The two-point Weyl correlation function has the general form (7.44) with the coefficients given in equations (7.46), (7.47) and (7.53). For our case, we calculate

$$S(Z) = \frac{H^8}{256\pi^4} \frac{19 - 9Z}{(1-Z)^6} \tag{7.65}$$

and thus

$$
\begin{aligned}
{}^{(2)}\mathcal{C}(Z) &= \frac{\kappa^4 H^8}{1920\pi^4}(27+20Z+72Z^2-60Z^3+5Z^4)(1+Z)^{-3}(1-Z)^{-5} \\
&\quad - \frac{3\kappa^4 H^8}{80\pi^4} Z(1+Z^2)(1-Z^2)^{-4}\ln\left(\frac{1-Z}{2}\right) + C(1-Z)^{-4}\,, \\
{}^{(1)}\mathcal{C}(Z) &= -\frac{\kappa^4 H^8}{960\pi^4}(7+76Z-96Z^2+4Z^3+25Z^4)(1+Z)^{-3}(1-Z)^{-5} \\
&\quad - \frac{3\kappa^4 H^8}{80\pi^4}(1+4Z^2-Z^4)(1-Z^2)^{-4}\ln\left(\frac{1-Z}{2}\right) + 2C\frac{2-Z}{(1-Z)^4}\,, \\
{}^{(3)}\mathcal{C}(Z) &= \frac{\kappa^4 H^8}{960\pi^4}(60+19Z+274Z^2-252Z^3+34Z^4+25Z^5)(1-Z)^{-6}(1+Z)^{-4} \\
&\quad - \frac{3\kappa^4 H^8}{80\pi^4} Z(7+10Z^2-Z^4)(1-Z^2)^{-5}\ln\left(\frac{1-Z}{2}\right) + 2C\frac{3-Z}{(1-Z)^5}\,.
\end{aligned}
\tag{7.66}
$$

Again, we used the freedom to redefine the arbitrary constant C to make the expressions simpler. The overall structure is strikingly similar to the massless, conformally coupled scalar; especially, the behaviors as $Z \to 1$ and $Z \to \pm\infty$ are completely equal, except for overall numerical factors.

For the massive, minimally coupled scalar the analogous determination of the curvature tensor correlation functions fails due to a technical problem – the components of the stress tensor two-point function are products of hypergeometric functions, and there is no closed-form expression for the antiderivative of such a product multiplied by a power of Z which is needed for the coefficient $\mathcal{C}^{(2)}$ (7.53).

However, we can check if important assumptions we made in the last section are satisfied. For the massive, minimally coupled scalar field, we can expand the source term $S(Z)$ (7.49) around $Z = 1$ and $Z = -1$ using well-known formulas for the hypergeometric function [104], and after some algebra it results

$$
\begin{aligned}
S(Z) &= \frac{m^4}{H^4}(2H^2-m^2)^2(4H^2+m^2)\frac{14H^2+5m^2}{73728\pi^2\cos^2(\pi\nu)} + \mathcal{O}(1+Z)\,, \\
S(Z) &= \frac{5H^8}{8\pi^4(1-Z)^6} + \mathcal{O}\big((1-Z)^{-5}\big)\,,
\end{aligned}
\tag{7.67}
$$

that is, it is finite as $Z \to -1$ and has the same leading singular behaviour as $Z \to -1$ as for the massless vector (7.55) or the massless, minimally coupled scalar (7.65). The behaviour of the curvature tensor correlation functions should therefore be similar to these cases.

7.3. Interpretation

We now want to interpret these results. From the explicit expressions for the coefficients appearing in the Weyl tensor two-point function (7.12), (7.56, 7.57) or (7.66), we see that the quantum corrections are more singular near the lightcone (as $Z \to 1$) than the tree-level coefficients. In the opposite case, for $Z \to \pm\infty$, we see that the quantum corrections decay faster, i.e., in this limit the tree-level result dominates. In total, we can say that the quantum corrections are concentrated near the light cone. However, the bitensors consisting of covariant derivatives of Z are not really appropriate to study these limits, since they are not normalized and grow or shrink with Z. Since all calculated correlation functions show the same behaviour up to numerical coefficients, let us discuss the two-point functions including corrections due to massless, conformally coupled scalars which are already given in the normalized basis consisting of the parallel propagator $g_{ab'}$ and normal vectors n_a, $n_{b'}$ in equations (7.17, 7.18) for the Weyl tensor and in equations (7.26, 7.25) for correlation functions involving the Ricci tensor.

For large spacelike separations, we can set the times η and η' of the two points x and x' equal by a de Sitter isometry. The proper physical distance $d(x,x')$ between those points is then given by $d^2 = (-H\eta)^{-2}(\boldsymbol{x}-\boldsymbol{x}')^2$. From the explicit expression of Z given in (4.7), this distance can be written as $d^2 = 2H^{-2}(1-Z)$ which is equal to the Minkowski distance in the embedding space between those points (4.3). Therefore, large spacelike separations $(\boldsymbol{x}-\boldsymbol{x}')^2 \to \infty$, which correspond to superhorizon scales $d \gg H^{-1}$, yield $Z \to -\infty$, and we see that the components of the Weyl tensor two-point function which decay more slowly go like

$$\sim \kappa^2 H^6 |Z|^{-2} + \kappa^4 H^8 |Z|^{-2} + \kappa^4 H^8 |Z|^{-3} \ln|Z| + \dots\,, \tag{7.68}$$

which in terms of the proper physical distance d reads

$$\sim \kappa^2 H^2 d^{-4} + \kappa^4 H^4 d^{-4} + \kappa^4 H^2 d^{-6} \ln(Hd) + \dots\,. \tag{7.69}$$

For the short-distance limit when the two points are spacelike separated, i.e., at subhorizon scales $d \ll H^{-1}$, one should recover the flat space result. In fact, using the previous expression connecting d and Z for points at equal time, one can see that $d \to 0$ implies $Z \to 1$, and that the logarithmic terms in (7.18) are subdominant in this limit:

$$\kappa^4 H^8 (1-Z)^{-3} \ln\left[\frac{1}{2}(1-Z)\right] \sim \kappa^4 H^2 d^{-6} \ln\left(\frac{1}{2}H^2 d^2\right) \ll \kappa^4 d^{-8} \sim \kappa^4 H^8 (1-Z)^{-4}\,. \tag{7.70}$$

Thus the dominant terms do not contain H, as one would expect, and we obtain

$$\sim \kappa^2 d^{-6} + \kappa^4 d^{-8} + \dots\,. \tag{7.71}$$

We see that at large distances at superhorizon scales the two-point Weyl correlation function decays more slowly than in flat space, a behavior that was also found for the stress tensor correlation function for massive, minimally coupled scalar fields [115]. Note that the short-distance limit $d \ll H^{-1}$ is different from the light cone limit $Z \to 1$, in which the two points must be near the light cone, but may otherwise be separated by an arbitrarily large distance along it, and which contains additional terms proportional to H.

For large timelike separations, we can achieve $\boldsymbol{x} = \boldsymbol{x}'$ by a de Sitter isometry. The expression of Z in spatially flat coordinates (4.7) then shows that $|\eta - \eta'| \to \infty$ corresponds to $Z \to \infty$, and we get the result (7.68). The proper time elapsed along a geodesic connecting the two points is given by $\tau^2 = -\mu^2(x,x')$, where the relation between the geodesic distance μ and the biscalar Z is given by equation (4.5). In the limit $Z \to \infty$ it follows that $\tau = H^{-1} \ln Z$, and therefore the components of the two-point Weyl correlation function which decay more slowly go like

$$\sim \kappa^2 H^6 \,\mathrm{e}^{-2H\tau} + \kappa^4 H^8 \,\mathrm{e}^{-2H\tau} + \kappa^4 H^9 \tau \,\mathrm{e}^{-3H\tau} + \dots\,, \tag{7.72}$$

i.e., they decay exponentially in proper time separation. This exponential decay is typical for massive theories in flat spacetime [161], with correlation functions decaying like $\exp(-m\tau)$ for large separations. In contrast, correlation functions for massless fields only decay like a power of the proper time separation and thus much more slowly. We thus see that the background curvature acts like an effective mass for the graviton field, which otherwise is massless. This can also be seen from the Minkowski limit given in equation (7.71) for spacelike separations, but which has the same form with d replaced by τ for timelike separations.

For the Ricci-Weyl correlation function (7.26) and the Ricci-Ricci correlation function (7.25), a similar analysis can be done. There are no tree-level contributions to these correlation functions, since the background spacetime is vacuum. As for the Weyl tensor two-point functions, both of these correlation functions are only singular as $Z \to 1$. For large time- or spacelike separations where $Z \to \pm\infty$, the components that decay more slowly go like $\sim \kappa^4 H^8 Z^{-4}$ for the Ricci tensor two-point function, and like $\sim \kappa^4 H^8 Z^{-2}$ for the Ricci-Weyl correlation function, i.e., also exponentially in proper time separation. We see that the Ricci-Weyl correlation function decays as slowly as the Weyl tensor two-point function, while the Ricci tensor two-point function vanishes faster. Since for massless, minimally coupled scalars the results are the same up to numerical coefficients, the same fall-off behaviour is obtained, and the same conclusions apply.

In all cases, these results contribute to our understanding of quantum field theory in de Sitter space. Especially, they shed light on possible generalizations of flat-space theorems for quantum field theory to curved spacetimes, such as the cluster decomposition theorem. This states that connected correlation functions decay when their arguments are widely separated, in the manner described above, and we see that this is also the case for the two-point function of the Riemann tensor, including loops of massless, minimally coupled or conformally coupled scalars. This stands in strong opposition to results obtained by Tsamis, Woodard and collaborators which grow with increasing separation, usually like logarithms of the scale factor, which corresponds to polynomial growth in proper time, and therefore would oppose probable generalizations of the cluster decomposition theorem. However, their results are obtained for different cases which include internal gravitons and graviton loops, and it is possible that these effects are gauge-dependent. If one could identify a proper gauge-independent, local observable for those cases, it is still possible that its correlation functions also would fall off.

Conclusions

"You sum up the difficulties of the situation succinctly and well," he said. "There is much that is still obscure, though I have quite made up my mind on the main facts."

— Sir Arthur Conan Doyle, A study in scarlet

A double result has thus arisen.
(Ein doppeltes Resultat hat sich also ergeben.)

— Karl Marx, Capital

8 Discussion

In this thesis, we have studied the influence of matter loop corrections to metric perturbations. First, the the effective action for the metric perturbations which incorporates these corrections was derived for some specific cases. In the case of a flat space background, it was possible to obtain the general result for minimally coupled scalar fields of arbitrary mass, and the generalization of the method that was used to all other matter fields is straightforward. For the case of a background de Sitter space, apart from the existing calculation for massless, conformally coupled scalar fields which already had been done, we calculated the effective action for a massless, minimally coupled scalar field. While the generalization of the technique used in flat space is not possible because of a technical issue, we extended the general idea to a curved space background and were able to apply it successfully to this last case. In principle, there is no reason why this extension should not work also for the general case of arbitrary mass, but in practice a lot of effort would be needed for stress tensor two-point functions which do not have a simple form in four dimensions.

With the calculation of the effective action, it became possible to study not only the semiclassical Einstein equation, which in this case gives a small correction to the relation between the cosmological constant and the Hubble constant (or inverse de Sitter radius), but also the equations satisfied by small metric perturbations. These equations were solved using the nonperturbative method of order reduction, both for a initial Bunch-Davies vacuum state (resp. its interacting generalization) as well as for a wide class of generic initial states. In both cases, the changes induced in the linearized Riemann tensor, which is a gauge-invariant local observable, are small and vanish at late times. These results constitute an extension of the classical and semiclassical "no-hair" theorems for de Sitter space to the quantum case, at least for the interaction studied. Such theorems assure that at sufficiently large times, any perturbation dies away in the sense that for any region of fixed physical size observations with fixed precision cannot detect them anymore. In the case studied, we have seen that while tensorial perturbations tend to a constant at late times, i.e., their wavelength tends to infinity, these are only detectable if an observer has access to the whole universe. If one considers local observables such as the Riemann tensor, this constant does not contribute, and so the conclusions of the "no-hair" theorems are unchanged. Furthermore, for the tensorial perturbations which are the dynamical part of generic metric perturbations, the corrections due to an initial non-vacuum state fall off faster than the perturbation itself. This supports existing calculations about the late-time attraction behaviour not only of de Sitter space, but also of perturbations starting from a de Sitter-invariant state. While these results were derived for the interaction of gravitons with a massless, conformally coupled scalars, the de Sitter invariance of the non-local part of the effective action allowed us to constrain the form of the order-reduced equations of motion in the general case, such that these conclusions remain unchanged.

After these findings, we calculated the two-point function of scalar, vectorial and tensorial metric perturbations including loops of massless, conformally coupled scalars. In contrast to earlier results that concentrated on leading-order effects and the equal-time limit, we obtained the full one-loop two-point function for separate times. The result is invariant under the subset of de Sitter isometries that is preserved by our gauge choice, most importantly under the simultaneous constant rescaling of time and space coordinates. From this correlation function, we could calculate a cosmological observable, the power spectrum for tensorial perturbations, which is the equal-time limit of the Fourier transform of the contracted two-point function. However, since the one-loop result for the Fourier transform is a genuine distribution (as opposed to a regular function which arises at tree level), the equal-time limit is divergent. In order to obtain a finite result, we smear the distribution with a test function, taken to be a Gaussian of very small width, modeling in this way the measurement process. For the regular part of the two-point function, this reduces to the standard definition when the width of the Gaussian goes to zero. The obtained power spectrum depends only on physical momentum, and is thus time-independent in the physical spacetime. This is in contrast to very early results, which found a logarithmic dependence on comoving momentum, but in line with more recent studies who argued (for scalar perturbations) that such a dependence is erroneous, identifying the cause of the discrepancy. Unfortunately for observations, these corrections are then much too small to be observable. Their size would only be appreciable if the Hubble constant had a magnitude comparable to the Planck scale, where the effective field theory ceases to be valid.

However, due to the specific gauge fixing used observables calculated from this metric two-point functions are not local and (through the gauge fixing) implicitly depend on boundary conditions at spatial infinity. We thus focused our attention on the two-point function of the linearized Riemann tensor, which is a gauge-invariant and local observable when we restrict to matter loops only. Since classically the Riemann tensor encodes all information about the local geometry of spacetime, one may expect that its correlation functions also give full information in the quantum case. Its two-point function may be calculated from the two-point function of the metric perturbations, and results being de Sitter-invariant. While the gauge fixing for the metric perturbations was not invariant, the gauge-invariant Riemann tensor makes the de Sitter symmetries that entered into the calculation, namely of the stress tensor correlation functions and the asymptotic in vacuum state, manifest, and shows that for the interaction of metric perturbations with loops of conformal matter there is no physical breaking of this invariance. The result illustrates that the quantum corrections are concentrated near the light cone, i.e., they are more singular there than the tree level result. Furthermore, the two-point function decays exponentially for large separations between the two points, showing that the background curvature acts as an effective mass for the graviton.

It is naturally interesting to see if these calculations can be generalized to the interaction with other kinds of matter. In the last part of the thesis, we thus exploited the Bianchi identities to show that the Riemann tensor two-point function is always de Sitter-invariant if the stress tensor two-point function is. We gave explicit formulas to calculate its coefficients from the coefficients of the stress tensor correlation function, and the result for the Riemann two-point function is fully determined up to an integration constant. This constant can naturally be interpreted as the strength of free gravitons which are propagating through the background spacetime, and depends on the unknown coefficients which multiply terms in the gravitational action which are quadratic in the curvature tensors.

What do these outcomes tell us? From a cosmological point of view, we may safely say that loop corrections are irrelevant (at least for the considered cases), which maybe sounds disappointing. This conclusion also does not change if we include matter self-interactions, as long as the stress tensor correlation functions stay de Sitter-invariant. Of course, the inclusion of graviton loops can change this picture tremendously if the breaking of de Sitter invariance, which seems to be a robust feature in calculations that have been done for individual cases, is indeed physical and not only a gauge artifact. In order to reach a conclusion in this matter, it is first necessary to construct an appropriate gauge-invariant and local observable which can then play the same role as the linearized Riemann tensor does in this thesis: separate physical from gauge effects. However, from a mathematical point of view the results obtained in this thesis are extremely satisfying. In Minkowski space many quantization procedures, regularization methods and gauge fixing terms exist that keep manifest the Poincaré symmetry of the background. Considering quantum fields on a fixed curved background (i.e., not considering effects that radically change this background such as the formation of black holes), one naturally expects that there is a way to preserve the background symmetries, and we see that this is indeed the case. While there is a manifestly de Sitter-invariant graviton propagator, derived from the Euclidean version of de Sitter space, the sphere, its validity has been questioned and it is rewarding to see that the choice of propagator does not matter for appropriate observables. Furthermore, these findings are a further step on the way to establish equivalence between calculations done in de Sitter and its Euclidean version, a connection which in Minkowski space has been cast into a theorem.

Appendix

A. Metric expansion

We decompose the full metric $\tilde{g}_{ab}$ into a background g_{ab} and perturbations h_{ab},

$$\tilde{g}_{ab} = g_{ab} + \kappa h_{ab} \,. \tag{A.1}$$

All indices are moved with the background metric g_{ab}, i.e. we see h_{ab} as a tensor field on the manifold. To calculate the inverse metric, we use the most general form and determine the coefficients from $\tilde{g}_{ab}\tilde{g}^{bc} = \delta^c_a$, which gives

$$\tilde{g}^{ab} = g^{ab} - \kappa h^{ab} + \kappa^2 h^{am} h^b_m + \mathcal{O}\big(\kappa^3\big) \,. \tag{A.2}$$

For the determinant we use

$$\begin{aligned}
\sqrt{-\tilde{g}} = \sqrt{-\det(\tilde{g}_{ab})} &= -\exp\left[\frac{1}{2}\,\mathrm{tr}\ln(\tilde{g}_{ab})\right] \\
&= -\exp\left[\frac{1}{2}\,\mathrm{tr}\ln(g_{ab}) + \frac{1}{2}\,\mathrm{tr}\left(\kappa h_{ab} - \frac{1}{2}\kappa^2 h_{am} h^m_b\right)\right] \\
&= \sqrt{-g}\,\exp\left(\frac{1}{2}\kappa h - \frac{1}{4}\kappa^2 h_{mn} h^{mn}\right) \\
&= \sqrt{-g}\left(1 + \frac{1}{2}\kappa h + \frac{1}{8}\kappa^2 h^2 - \frac{1}{4}\kappa^2 h_{mn} h^{mn}\right) + \mathcal{O}\big(\kappa^3\big) \,.
\end{aligned} \tag{A.3}$$

For the Christoffel symbols we get after a short direct calculation

$$\begin{aligned}
\tilde{\Gamma}^a_{bc} &= \Gamma^a_{bc} + \frac{1}{2}\kappa S^a{}_{bc} - \frac{1}{2}\kappa^2 h^{am} S_{mbc} + \mathcal{O}\big(\kappa^3\big) \\
S^a{}_{bc} &= \nabla_b h^a_c + \nabla_c h^a_b - \nabla^a h_{bc} \\
\nabla_c h_{ab} &= S_{(ab)c} \,,
\end{aligned} \tag{A.4}$$

where ∇ denotes the derivative with respect to the background metric g_{ab}.

The curvature tensors can equally well be calculated in a straightforward way, which gives

$$\begin{aligned}
\tilde{R}^a{}_{bcd} &= R^a_{bcd} + \kappa \tilde{R}^{(1)a}{}_{bcd} + \kappa^2 \tilde{R}^{(2)a}{}_{bcd} + \mathcal{O}\big(\kappa^3\big) \,, \\
\tilde{R}_{bd} &= R_{bd} + \kappa \tilde{R}^{(1)}_{bd} + \kappa^2 \tilde{R}^{(2)}_{bd} + \mathcal{O}\big(\kappa^3\big) \,, \\
\tilde{R} &= R + \kappa \tilde{R}^{(1)} + \kappa^2 \tilde{R}^{(2)} + \mathcal{O}\big(\kappa^3\big)
\end{aligned} \tag{A.5}$$

with

$$\begin{aligned}
\tilde{R}^{(1)a}{}_{bcd} &= \nabla_{[c} S^a{}_{d]b} \,, \\
\tilde{R}^{(2)a}{}_{bcd} &= -\nabla_{[c}\left(S^s{}_{d]b} h^a_s\right) + \frac{1}{2} S^a{}_{s[c} S^s{}_{d]b} \,,
\end{aligned} \tag{A.6}$$

$$
\begin{aligned}
\tilde{R}^{(1)}{}_{bd} &= \frac{1}{2}\left(\nabla_c \nabla_b h^c_d + \nabla_c \nabla_d h^c_b - \Box h_{bd} - \nabla_b \nabla_d h\right) , \\
\tilde{R}^{(2)}{}_{bd} &= \frac{1}{2} h^{mn}\left(\nabla_m \nabla_n h_{bd} + \nabla_d \nabla_b h_{mn} - 2\nabla_m \nabla_{(b} h_{d)n}\right) \\
&\quad + \frac{1}{4}\left(\nabla^m h - 2\nabla_n h^{mn}\right)\left(2\nabla_{(b} h_{d)m} - \nabla_m h_{bd}\right) \\
&\quad + \frac{1}{4}\left(\nabla_b h_{mn}\right)\left(\nabla_d h^{mn}\right) + \left(\nabla_{[m} h_{n]b}\right)\left(\nabla^{[m} h^{n]}_d\right)
\end{aligned}
\tag{A.7}
$$

and

$$
\begin{aligned}
\tilde{R}^{(1)} &= -h^{bd} R_{bd} + \nabla_m \nabla_n h^{mn} - \Box h , \\
\tilde{R}^{(2)} &= h^{mn}\left(\nabla_m \nabla_n h + \Box h_{mn} - 2\nabla_m \nabla^s h_{sn}\right) + h^{mn} h^{pq} R_{mpnq} \\
&\quad - \frac{1}{4}\left(2\nabla_n h^{mn} - \nabla^m h\right)\left(2\nabla^s h_{sm} - \nabla_m h\right) \\
&\quad + \frac{1}{4}\left(\nabla_s h_{mn}\right)\left(\nabla^s h^{mn}\right)\left(\nabla_{[m} h_{n]s}\right)\left(\nabla^{[m} h^{n]s}\right) ,
\end{aligned}
\tag{A.8}
$$

where $\Box = \nabla^s \nabla_s$.

The Weyl tensor can be obtained from the Riemann tensor by subtracting all traces; in four dimensions it is given by

$$
\tilde{C}^{ab}{}_{cd} = \tilde{R}^{ab}{}_{cd} - 2\tilde{R}^{[a}_{[c}\delta^{b]}_{d]} + \frac{1}{3}\tilde{R}\delta^{[a}_{[c}\delta^{b]}_{d]} .
\tag{A.9}
$$

For its square we obtain

$$
\tilde{C}^{abcd}\tilde{C}_{abcd} = \tilde{R}^{abcd}\tilde{R}_{abcd} - 2\tilde{R}^{ab}\tilde{R}_{ab} + \frac{1}{3}\tilde{R}^2 .
\tag{A.10}
$$

B. Conformal transformations

Under the conformal transformation

$$
\tilde{g}_{ab} = \mathrm{e}^{2\omega}\hat{g}_{ab} ,
\tag{B.11}
$$

the transformed Christoffel symbols are given by

$$
\tilde{\Gamma}^a_{bc} = \hat{\Gamma}^a_{bc} + \left(\delta^a_b \delta^m_c + \delta^a_c \delta^m_b - \hat{g}_{bc}\hat{g}^{am}\right)\partial_m \omega ,
\tag{B.12}
$$

and the curvature tensors follow as

$$
\begin{aligned}
\tilde{R}^a{}_{bcd} &= \hat{R}^a_{bcd} + 4\hat{g}^{ak}\delta^m_{[c}\hat{g}_{d][k}\left[\hat{\nabla}_{b]}\hat{\nabla}_m\omega - (\hat{\nabla}_{b]}\omega)(\hat{\nabla}_m\omega)\right] - 2\delta^a_{[c}\hat{g}_{d]b}(\hat{\nabla}^m\omega)(\hat{\nabla}_m\omega) \\
\tilde{R}_{bd} &= \hat{R}_{bd} - 2\left[\hat{\nabla}_b\hat{\nabla}_d\omega - (\hat{\nabla}_b\omega)(\hat{\nabla}_d\omega) + \hat{g}_{bd}(\hat{\nabla}^m\omega)(\hat{\nabla}_m\omega)\right] - \hat{g}_{bd}\hat{\nabla}^m\hat{\nabla}_m\omega \\
\mathrm{e}^{2\omega}\tilde{R} &= \hat{R} - 6\hat{\nabla}^m\hat{\nabla}_m\omega - 6(\hat{\nabla}^m\omega)(\hat{\nabla}_m\omega) ,
\end{aligned}
\tag{B.13}
$$

where the covariant derivative $\hat{\nabla}$ refers to the metric $\hat{g}_{ab}$.

C. Special functions

We define the entire function Ein(z) by

$$\operatorname{Ein}(z) = \int_0^z \frac{e^t - 1}{t}\, dt = \sum_{k=1}^{\infty} \frac{z^k}{k\, k!}\,. \tag{C.14}$$

Its asymptotic expansion at infinity ($r \to \infty$) is given by

$$\operatorname{Ein}(\alpha r + \beta) \sim -\gamma - \ln(-(\alpha r + \beta)) + e^{\alpha r + \beta}\left[\frac{1}{\alpha r} + \frac{1-\beta}{\alpha^2 r^2} + \mathcal{O}\big(r^{-3}\big)\right], \tag{C.15}$$

where γ is the Euler-Mascheroni constant.

The following useful integrals involving Ein(z) can be easily obtained by partial integration:

$$\begin{aligned}
\int \frac{e^{at}}{bt+c}\, dt &= \frac{1}{b}\, e^{-\frac{ac}{b}}\left[\operatorname{Ein}\left[\frac{a}{b}(bt+c)\right] + \ln(bt+c)\right] \\
\int e^{at} \ln(bt+c)\, dt &= \frac{1}{a}\left[\left(e^{at} - e^{-\frac{ac}{b}}\right)\ln(bt+c) - e^{-\frac{ac}{b}} \operatorname{Ein}\left[\frac{a}{b}(bt+c)\right]\right] \\
\int e^{at} \operatorname{Ein}(bt+c)\, dt &= \frac{1}{a}\, e^{at} \operatorname{Ein}(bt+c) + \frac{1}{a}\, e^{-\frac{ac}{b}}\left[\operatorname{Ein}\left(\frac{a}{b}(bt+c)\right) - \operatorname{Ein}\left(\frac{a+b}{b}(bt+c)\right)\right].
\end{aligned} \tag{C.16}$$

These integrals depend continuously on the parameters a and c, so that for instance the integral of Ein($bt + c$) can be calculated by taking the limit $a \to 0$ on the right side of the last integral.

D. Distributions and their Fourier transforms

In this section, we give details on the distributions appearing in sections 5.2.1 and 6.1, and their Fourier transforms. General information about the theory of distributions can be found in [162, 163].

In general, one can define distributions as the limit of regular functions. For example, the principal value distribution $\mathcal{P}\frac{1}{t}$ can be defined as the limit as $\epsilon \to 0$ of

$$\frac{\Theta(t-\epsilon) + \Theta(-t-\epsilon)}{t}\,, \tag{D.17}$$

but this limit can only be taken after smearing with a test function, i.e.,

$$\int \mathcal{P}\frac{1}{t} f(t)\, dt = \lim_{\epsilon \to 0}\left[\int \frac{\Theta(t-\epsilon) + \Theta(-t-\epsilon)}{t} f(t)\, dt\right], \tag{D.18}$$

where $f(t)$ is such an appropriate (fast decaying) test function. We will denote this limit by replacing ϵ by 0 in (D.17), so that

$$\mathcal{P}\frac{1}{t} = \frac{\Theta(t-0) + \Theta(-t-0)}{t}\,. \tag{D.19}$$

Its Fourier transform can be calculated to give

$$\begin{aligned}\int \mathcal{P}\frac{1}{t}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t &= \lim_{\epsilon\to 0}\left[\left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\right)\frac{\mathrm{e}^{\mathrm{i}\omega t}}{t}\,\mathrm{d}t\right] = \lim_{\epsilon\to 0}\left[\lim_{r\to\infty}\int_{-r}^{-\epsilon}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{t}\,\mathrm{d}t + \lim_{r\to\infty}\int_{\epsilon}^{r}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{t}\,\mathrm{d}t\right]\\ &= \lim_{\epsilon\to 0}\left[\lim_{r\to\infty}\left[\mathrm{Ein}\,(\mathrm{i}\omega t)+\ln|t|\right]_{-r}^{-\epsilon} + \lim_{r\to\infty}\left[\mathrm{Ein}\,(\mathrm{i}\omega t)+\ln|t|\right]_{\epsilon}^{r}\right]\\ &= \lim_{\epsilon\to 0}\left[\mathrm{Ein}\,(-\mathrm{i}\omega\epsilon)+\ln(\mathrm{i}\omega)-\ln(-\mathrm{i}\omega)-\mathrm{Ein}\,(\mathrm{i}\omega\epsilon)\right]\\ &= \mathrm{i}\arg(\mathrm{i}\omega)-\mathrm{i}\arg(-\mathrm{i}\omega)=\mathrm{i}\pi\,\mathrm{sgn}\,\omega\,,\end{aligned} \tag{D.20}$$

where we used the expansion (C.15) for the Ein function. Alternatively, one can employ the Sokhotsky formula

$$\frac{1}{t-\mathrm{i}0} = \mathcal{P}\frac{1}{t} + \mathrm{i}\pi\delta(t)\,, \tag{D.21}$$

which gives the same result.

Another distribution of interest is $\mathcal{P}'\frac{\Theta(t)}{t}$, which is defined as

$$\mathcal{P}'\frac{\Theta(t)}{t} = \left[\frac{\Theta(t-0)}{t} + \delta(t)(\ln(\mu 0)+\gamma)\right], \tag{D.22}$$

where $\mu > 0$ is an arbitrary reference energy scale to make the definition dimensionally correct, and we use the symbol $\mathcal{P}'$ to distinguish it from the previous $\mathcal{P}$ which did not include a δ distribution. Its Fourier transform can be calculated by proceeding as in the previous case, and is given by

$$\begin{aligned}\int \mathcal{P}'\frac{\Theta(t)}{t}\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}t &= \lim_{\epsilon\to 0}\left[\int_{\epsilon}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{t}\,\mathrm{d}t + \ln(\mu\epsilon)+\gamma\right]\\ &= \lim_{\epsilon\to 0}\lim_{r\to\infty}\left[\left[-\gamma-\ln(-\mathrm{i}\omega)-\ln r\right]+\ln r-\mathrm{Ein}\,(\mathrm{i}\omega\epsilon)-\ln\epsilon+\ln\epsilon+\gamma+\ln\mu\right]\\ &= \ln\left(\frac{\mu}{|\omega|}\right)+\frac{\mathrm{i}\pi}{2}\,\mathrm{sgn}\,\omega\,.\end{aligned} \tag{D.23}$$

Furthermore, combining these two we can define

$$\mathcal{P}'\frac{1}{|t|} = 2\mathcal{P}'\frac{\Theta(t)}{t} - \mathcal{P}\frac{1}{t}\,. \tag{D.24}$$

The Fourier transform of the kernels L, D and N (6.24) can then be calculated using

$$\int \ln\left|p^2\right|\mathrm{e}^{-\mathrm{i}p^0 t}\frac{\mathrm{d}p^0}{2\pi} = \int\left[\ln\left|p^0+|\boldsymbol{p}|\right|+\ln\left|p^0-|\boldsymbol{p}|\right|\right]\mathrm{e}^{-\mathrm{i}p^0 t}\frac{\mathrm{d}p^0}{2\pi} = -\cos(|\boldsymbol{p}|t)\,\mathcal{P}'\frac{1}{|t|}+\delta(t)\ln\mu^2 \tag{D.25}$$

with the scale μ from equation (D.22), which can be identified with the renormalization scale,

$$\int \Theta(-p^2)\,\mathrm{e}^{-\mathrm{i}p^0 t}\frac{\mathrm{d}p^0}{2\pi} = \int\left[1-\Theta(\boldsymbol{p}^2-(p^0)^2)\right]\mathrm{e}^{-\mathrm{i}p^0 t}\frac{\mathrm{d}p^0}{2\pi} = \delta(t)-\frac{\sin(|\boldsymbol{p}|t)}{\pi t} \tag{D.26}$$

and

$$\int \Theta(-p^2)\Theta(p^0)\,\mathrm{e}^{-\mathrm{i}p^0 t}\frac{\mathrm{d}p^0}{2\pi} = \lim_{\epsilon\to 0}\int \left[1-\Theta(\boldsymbol{p}^2-(p^0)^2)\right]\Theta(p^0)\,\mathrm{e}^{-\mathrm{i}p^0 t-\epsilon p^0}\frac{\mathrm{d}p^0}{2\pi} \\ = \frac{1}{2}\delta(t) - \frac{\mathrm{i}}{2\pi}\,\mathrm{e}^{-\mathrm{i}|\boldsymbol{p}|t}\mathcal{P}\frac{1}{t}\,, \tag{D.27}$$

where in the last step equation (D.21) was used.

E. Fourier transforms for Riemann tensor two-point functions

For the calculation of the Riemann tensor correlation function (7.4) we need Fourier transformations of the type

$$f_{i_1\cdots}(x,x') = \int \tilde{f}_{i_1\cdots}(\boldsymbol{p},\eta,\eta')\,\mathrm{e}^{\mathrm{i}\boldsymbol{p}\boldsymbol{r}}\frac{\mathrm{d}^3p}{(2\pi)^3} = \mathcal{F}(\boldsymbol{p},\boldsymbol{r})\left[\tilde{f}_{i_1\cdots}(\boldsymbol{p},\eta,\eta')\right], \tag{E.28}$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$. To compute this integrals we first strip the expressions of any tensorial factors: η_{ab} and δ^0_a can simply be pulled out, while

$$\mathcal{F}(\boldsymbol{p},\boldsymbol{r})\left[p_a\tilde{f}_{i_1\cdots}(\boldsymbol{p},\eta,\eta')\right] = -\mathrm{i}\left(\partial_a - \delta^0_a\partial_\eta\right)\mathcal{F}(\boldsymbol{p},\boldsymbol{r})\left[\tilde{f}_{i_1\cdots}(\boldsymbol{p},\eta,\eta')\right]. \tag{E.29}$$

The Fourier transformation of the remaining scalar factor must then be understood in the sense of distributions, i.e. we calculate

$$f(x,x') = \operatorname*{d-lim}_{\epsilon\to 0}\left[\frac{1}{2\pi^2|\boldsymbol{r}|}\int_0^\infty \mathrm{e}^{-\epsilon|\boldsymbol{p}|}\tilde{f}(|\boldsymbol{p}|,\eta,\eta')\sin(|\boldsymbol{p}||\boldsymbol{r}|)\,|\boldsymbol{p}|\,\mathrm{d}|\boldsymbol{p}|\right], \tag{E.30}$$

where we take into account that because of rotational symmetry those scalar factors only depend on the absolute value of $\boldsymbol{p}$.

As a technical detail, we note that for the Fourier transformation of e.g. $p_a p_b|\boldsymbol{p}|^{-3}$ we have to introduce a lower bound ξ for the integral in equation (E.30). After derivation such as indicated in equation (E.29), all dependence on ξ disappears. This is only a technical detail: while the Fourier transformation of $p_a p_b|\boldsymbol{p}|^{-3}$ is well defined (and is given by the result of this process), the Fourier transformation of $|\boldsymbol{p}|^{-3}$ alone is not.

Using the integrals for the Ein function given in appendix C and using partial integration to reduce any powers of $|\boldsymbol{p}|$, most Fourier transformations can be readily calculated in this way. Additionally, we need the indefinite integral

$$\int \ln(ap)\frac{1}{p}\,\mathrm{d}p = \frac{1}{2}\ln^2(ap)\,, \tag{E.31}$$

and the definite integral

$$I = \int_0^\infty \mathrm{e}^{-\epsilon p}\,\mathrm{e}^{\mathrm{i}ap}\,\mathrm{Ein}(\mathrm{i}bp)\frac{1}{p}\,\mathrm{d}p \tag{E.32}$$

for real parameters a and b. To compute this integral, since it is absolutely convergent, we may insert the series expansion of the Ein function to get

$$I = \sum_{k=1}^{\infty} \frac{(\mathrm{i}b)^k}{k\,k!} \int_0^\infty \mathrm{e}^{-\epsilon p}\,\mathrm{e}^{\mathrm{i}ap} p^{k-1}\,\mathrm{d}p = \sum_{k=1}^{\infty} \frac{1}{k^2}\left(-\frac{b}{a+\mathrm{i}\epsilon}\right)^k = \mathrm{Li}_2\left(-\frac{b}{a+\mathrm{i}\epsilon}\right). \tag{E.33}$$

Here, $\mathrm{Li}_2(z)$ is the dilogarithm function. However, this definite integral only occurs together with p_a, so that its derivative

$$\mathrm{Li}_2'(z) = -\frac{\ln(1-z)}{z}\,, \tag{E.34}$$

which can also be directly inferred from the series expansion, is the only expression we need for our computation.

F. Evaluation of the integral $I_\nu(Z)$

In this section, we explicitly calculate the integral $I_\nu(Z)$ in n dimensions, which is essentially the scalar two-point function. In n dimensions, we obtain for the angular integration

$$\begin{aligned} \int f(|\boldsymbol{p}|)\,\mathrm{e}^{\mathrm{i}\boldsymbol{pr}} \frac{\mathrm{d}^{n-1}p}{(2\pi)^{n-1}} &= \frac{\Omega_{n-3}}{(2\pi)^{n-1}} \int_0^\infty \int_0^\pi f(q)\,\mathrm{e}^{\mathrm{i}qr\cos\theta} \sin^{n-3}\theta\, q^{n-2}\,\mathrm{d}\theta\,\mathrm{d}q \\ &= \frac{r^{\frac{3-n}{2}}}{(2\pi)^{\frac{n-1}{2}}} \int_0^\infty f(q)\,\mathrm{J}_{\frac{n-3}{2}}(qr)\, q^{\frac{n-1}{2}}\,\mathrm{d}q\,, \end{aligned} \tag{F.35}$$

where we set $r = |\boldsymbol{r}|$ and defined $\Omega_D = 2\pi^{D+\frac{1}{2}}/\Gamma\left(\frac{D}{2}+1\right)$, the area of the unit D-sphere. Hence, the integral defined by (4.31) reads

$$\begin{aligned} I_\nu(\eta,\eta',\boldsymbol{r}) &= \frac{\pi}{4H}(H^2\eta\eta')^{\frac{n-1}{2}} \int \mathrm{H}^{(1)}_\nu(-|\boldsymbol{p}|\eta)\,\mathrm{H}^{(2)}_\nu\left(-|\boldsymbol{p}|\eta'\right)\mathrm{e}^{\mathrm{i}\boldsymbol{pr}} \frac{\mathrm{d}^{n-1}p}{(2\pi)^{n-1}} \\ &= \frac{1}{H2^{\frac{n+3}{2}}\pi^{\frac{n-3}{2}}}(H^2\eta\eta')^{\frac{n-1}{2}} r^{\frac{3-n}{2}} \int_0^\infty \mathrm{H}^{(1)}_\nu(-q\eta)\,\mathrm{H}^{(2)}_\nu\left(-q\eta'\right)\mathrm{J}_{\frac{n-3}{2}}(qr)\, q^{\frac{n-1}{2}}\,\mathrm{d}q\,. \end{aligned}$$

This integral can be evaluated by expressing the Hankel functions by Bessel functions and using a result due to Bailey [164]. This result of Bailey's in the general case is

$$\begin{aligned} &\int_0^\infty \mathrm{J}_\mu(ax)\,\mathrm{J}_\nu(bx)\,\mathrm{J}_\rho(cx)\,x^{\lambda-1}\,\mathrm{d}x \\ &= \frac{2^{\lambda-1}}{\pi c^{\lambda+\mu+\nu}} a^\mu b^\nu \sin\left(\frac{\pi}{2}(-\rho+\lambda+\mu+\nu)\right) \frac{\Gamma\left(\frac{\rho+\lambda+\mu+\nu}{2}\right)\Gamma\left(\frac{-\rho+\lambda+\mu+\nu}{2}\right)}{\Gamma(1+\mu)\Gamma(1+\nu)} \\ &\quad\times \mathrm{F}_4\left(\frac{\rho+\lambda+\mu+\nu}{2}, \frac{-\rho+\lambda+\mu+\nu}{2}; 1+\mu, 1+\nu; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right), \end{aligned} \tag{F.36}$$

where F_4 is the fourth Appell hypergeometric function defined by

$$F_4(a,b;c,d;x,y) = \sum_{k,l=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n m!n!} x^m y^n \,. \tag{F.37}$$

The integral is convergent if $a,b,c>0$ with $c>a+b$, together with the conditions $\mathfrak{Re}(\lambda)<\frac{5}{2}$ and $\mathfrak{Re}(\rho+\lambda+\mu+\nu)>0$. In the case where $\lambda=\rho+2$ and $\mu=\pm\nu$, which is all we need for evaluating $I_\nu(Z)$, equation (F.36) simplifies as

$$\int_0^\infty J_\nu(ax) J_{-\nu}(bx) J_\rho(cx)\, x^{\rho+1}\, dx = 0 \tag{F.38}$$

$$\begin{aligned}\int_0^\infty J_\nu(ax) J_\nu(bx) J_\rho(cx)\, x^{\rho+1}\, dx &= \frac{2^{\rho+1}}{\pi c^{\rho+2+2\nu}} a^\nu b^\nu \sin(\pi(1+\nu)) \frac{\Gamma(\rho+1+\nu)}{\Gamma(1+\nu)} \\ &\quad \times F_4\left(\rho+1+\nu, 1+\nu; 1+\nu, 1+\nu; \frac{a^2}{c^2}, \frac{b^2}{c^2}\right) .\end{aligned} \tag{F.39}$$

Now we combine the formula

$$F_4\left(\alpha,\beta;\beta,\beta;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right) = (1-x)^\alpha(1-y)^\alpha\, {}_2F_1(\alpha,1+\alpha-\beta;\beta;xy) \tag{F.40}$$

due to Bailey and the transformation formula for the Gauß hypergeometric function

$${}_2F_1\left(\alpha,1+\alpha-\beta;\beta;z^2\right) = (1-z)^{-2\alpha}\, {}_2F_1\left(\alpha,\beta-\frac{1}{2};2\beta-1;-\frac{4z}{(1-z)^2}\right) \tag{F.41}$$

to find

$$\begin{aligned}&F_4\left(\alpha,\beta;\beta,\beta;-\frac{x}{(1-x)(1-y)},-\frac{y}{(1-x)(1-y)}\right) \\ &\quad = (1-x)^\alpha(1-y)^\alpha(1-\sqrt{xy})^{-2\alpha}\, {}_2F_1\left(\alpha,\beta-\frac{1}{2};2\beta-1;-\frac{4\sqrt{xy}}{(1-\sqrt{xy})^2}\right) .\end{aligned} \tag{F.42}$$

Introducing

$$u^2 = -\frac{x}{(1-x)(1-y)}\,, \qquad v^2 = -\frac{y}{(1-x)(1-y)} \tag{F.43}$$

with the (relative) sign defined by

$$uv = \frac{\sqrt{xy}}{(1-x)(1-y)}\,, \tag{F.44}$$

one obtains after some algebra

$$-\frac{4\sqrt{xy}}{(1-\sqrt{xy})^2} = \frac{4uv}{(u+v)^2-1}\,, \qquad \frac{(1-x)(1-y)}{(1-\sqrt{xy})^2} = \frac{1}{1-(u+v)^2}\,. \tag{F.45}$$

By substituting these formulae in (F.42) we obtain

$$\mathrm{F}_4\left(\alpha,\beta;\beta,\beta;u^2,v^2\right)=\left(1-(u+v)^2\right)^{-\alpha}{}_2\mathrm{F}_1\left(\alpha,\beta-\frac{1}{2};2\beta-1;\frac{4uv}{(u+v)^2-1}\right). \tag{F.46}$$

By using this formula in (F.39) we find

$$\begin{aligned}\int_0^\infty \mathrm{J}_\nu(ax)\mathrm{J}_\nu(bx)\mathrm{J}_\rho(cx)x^{\rho+1}\,\mathrm{d}x &= -\frac{2^{\rho+1}}{\pi}\sin(\pi\nu)\frac{\Gamma(\rho+1+\nu)}{\Gamma(1+\nu)}\\ &\times c^\rho a^\nu b^\nu\left(c^2-(a+b)^2\right)^{-(\rho+1+\nu)}{}_2\mathrm{F}_1\left(\rho+1+\nu,\frac{1}{2}+\nu;1+2\nu;\frac{4ab}{(a+b)^2-c^2}\right).\end{aligned} \tag{F.47}$$

We write the Hankel functions in (F.36) in terms of Bessel functions

$$\mathrm{H}_\nu^{(1)}(z)=\frac{\mathrm{J}_{-\nu}(z)-\mathrm{e}^{-\mathrm{i}\pi\nu}\mathrm{J}_\nu(z)}{\mathrm{i}\sin(\pi\nu)},\quad \mathrm{H}_\nu^{(2)}(z)=\frac{\mathrm{e}^{\mathrm{i}\pi\nu}\mathrm{J}_\nu(z)-\mathrm{J}_{-\nu}(z)}{\mathrm{i}\sin(\pi\nu)}, \tag{F.48}$$

use (F.38) and (F.47) and apply a standard hypergeometric transformation which sends the argument to its inverse. We also use the duplication and reflection formulas for the Γ function

$$\Gamma(\pm 2\nu)=\frac{\Gamma(\pm\nu)\Gamma\left(\frac{1}{2}\pm\nu\right)}{2^{1\mp 2\nu}\sqrt{\pi}},\quad \Gamma(\pm\nu)\Gamma(1\mp\nu)=\pm\frac{\pi}{\sin(\pi\nu)}. \tag{F.49}$$

Thus, we obtain

$$\begin{aligned}\int_0^\infty \mathrm{H}_\nu^{(1)}(ax)\mathrm{H}_\nu^{(2)}(bx)\mathrm{J}_\rho(cx)x^{\rho+1}\,\mathrm{d}x &= \frac{1}{\pi^{\frac{3}{2}}ab}\left(\frac{c}{2ab}\right)^\rho\frac{\Gamma(\rho+1+\nu)\Gamma(\rho+1-\nu)}{\Gamma\left(\rho+\frac{3}{2}\right)}\\ &\times {}_2\mathrm{F}_1\left(\rho+1+\nu,\rho+1-\nu;\rho+\frac{3}{2};\frac{(a+b)^2-c^2}{4ab}\right).\end{aligned} \tag{F.50}$$

This equation was derived under the condition $c > a+b$, and all transformations we have applied were valid under this condition, as well as $\mathfrak{Re}(\rho) < \frac{1}{2}$. (We also need the condition $-1+|\mathfrak{Re}(\nu)| < \mathfrak{Re}(\rho)$, but it is always satisfied in the cases we are interested in except for the case $m^2=0$ for $\partial_a I_\mu(Z)$, which can be found by taking the limit $m^2\to 0$ of the result for positive mass.) However, now we can use analytic continuation to affirm that this formula is actually valid for all a, b and c. To this end we recall that the asymptotic expansion of the Hankel functions at large argument is given by

$$\mathrm{H}_\nu^{(1)}(ax)\sim\sqrt{\frac{2}{\pi ax}}\,\mathrm{e}^{\mathrm{i}[ax-(2\nu+1)\pi/4]},\quad \mathrm{H}_\nu^{(2)}(bx)\sim\sqrt{\frac{2}{\pi bx}}\,\mathrm{e}^{-\mathrm{i}[bx-(2\nu+1)\pi/4]}. \tag{F.51}$$

Hence the integral is convergent if we let $a\to a(1+\mathrm{i}\epsilon)$ and $b\to b(1-\mathrm{i}\epsilon)$ with $\epsilon>0$ for all positive values of a, b and c and for all ρ satisfying $-1+|\mathfrak{Re}(\nu)|<\mathfrak{Re}(\rho)$. This observation determines how the hypergeometric function in (F.50) should be continued to $(a+b)^2-c^2>4ab>0$. Thus, the argument of the hypergeometric function is given, in the limit $\epsilon\to 0$, by

$$\frac{(a+b)^2-c^2}{4ab}+\mathrm{i}\epsilon\,\mathrm{sgn}(a-b). \tag{F.52}$$

Hence, by letting $a = -\eta(1 + i\epsilon)$, $b = -\eta'(1 - i\epsilon)$ and $c = |\boldsymbol{r}|$ in (F.50) we find indeed (4.33), i.e.

$$I_\nu(\eta, \eta', \boldsymbol{r}) = \frac{H^{n-2}}{(4\pi)^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n-1}{2} + \nu\right)\Gamma\left(\frac{n-1}{2} - \nu\right)}{\Gamma\left(\frac{n}{2}\right)} \times {}_2F_1\left(\frac{n-1}{2} + \nu, \frac{n-1}{2} - \nu; \frac{n}{2}; \frac{1+Z}{2} - i\epsilon \operatorname{sgn}(\eta - \eta')\right) \tag{F.53}$$

with Z defined by (4.7).

G. Mellin-Barnes integrals

Integrals of Mellin-Barnes type have found applications in a variety of contexts. For theoretical physics, this includes the calculation of Feynman diagrams in momentum space [120], the AdS/CFT correspondence [165–168] and also physics in de Sitter space [32–34, 169, 170]. Mellin-Barnes integrals are contour integrals in the complex plane which involve products of Γ functions, such as

$$\int_{\mathcal{C}} \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z)x^z \frac{dz}{2\pi i}. \tag{G.54}$$

Here, the integration path $\mathcal{C}$ runs from $-i\infty$ to $+i\infty$ and is deformed in such a way that poles of the Γ functions of the type $\Gamma(x+s)$, called left poles, are to the left and poles of the type $\Gamma(x-s)$, called right poles, to the right of this contour. For the example integral (G.54) with $a = \frac{1}{4}$, $b = \frac{1}{2}(1+i)$, $c = \frac{3}{16}(-2+i)$ and $d = 0$, the path is shown in figure G.1. It is

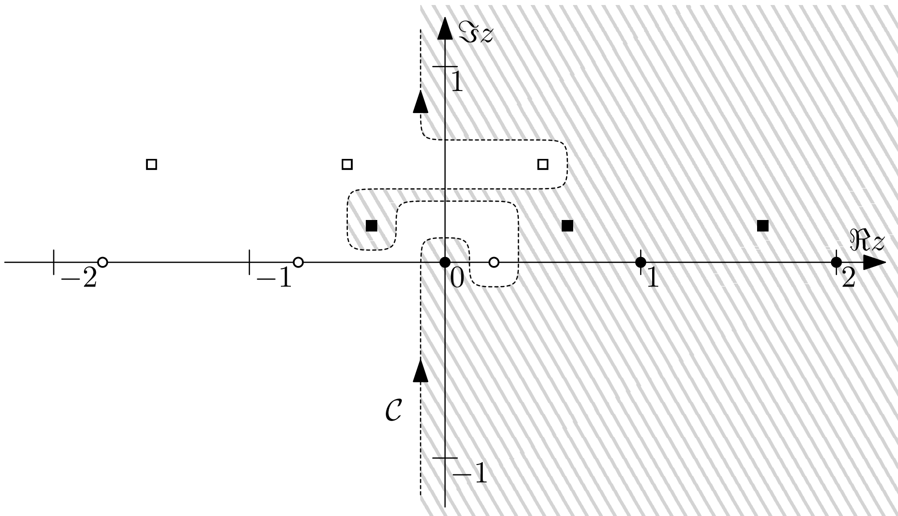

Figure G.1.: The integration path $\mathcal{C}$ for the example Mellin-Barnes integral. It separates the complex plane in a left (unshaded) and a right (shaded) half such that the left poles (shown in white) all lie in the left half and the right poles (shown in black) all lie in the right half.

always possible to find such a contour as long as no difference between the a's is an integer, so that no left pole coincides with any right pole. These integrals are well defined because the Γ functions decay exponentially in imaginary directions, and therefore are also suited to numerical evaluation. If the integrand also decays in a real direction $z \to \pm\infty$, they can be evaluated by contour integration, summing the resulting residue series.

An important result is Barnes' Lemma [171, 172]

$$\int_{\mathcal{C}} \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(d-z)\frac{\mathrm{d}z}{2\pi\,\mathrm{i}} = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}\,, \tag{G.55}$$

and another case which is often needed is

$$\int_{\mathcal{C}} \frac{\Gamma(\alpha+z)\Gamma(-z)}{\Gamma(\alpha)} x^z \frac{\mathrm{d}z}{2\pi\,\mathrm{i}} = \frac{1}{(1+x)^\alpha}\,, \tag{G.56}$$

where the contour can be closed to the left if $|x| > 1$ and to the right if $|x| < 1$.

This process can of course also be done in reverse. Let us take the series definition of the Gauß hypergeometric function

$${}_2\mathrm{F}_1\,(a,b;c;x) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+k)} \frac{x^k}{k!}\,. \tag{G.57}$$

The terms in this sum are the residues of

$$\frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+z)}\Gamma(-z)(-x)^z \tag{G.58}$$

for $z = 0, 1, \ldots$, and one therefore obtains directly the Mellin-Barnes representation of the hypergeometric function

$${}_2\mathrm{F}_1\,(a,b;c;x) = \int_{\mathcal{C}} \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+z)}\Gamma(-z)(-x)^z \frac{\mathrm{d}z}{2\pi\,\mathrm{i}}\,. \tag{G.59}$$

Take now the formula (G.56) in the form

$$\Gamma(-s)(1-x)^s = \int_{\mathcal{C}} \Gamma(z-s)\Gamma(-z)(-x)^z \frac{\mathrm{d}z}{2\pi\,\mathrm{i}}\,, \tag{G.60}$$

multiply by $\Gamma(c-a-b-s)\Gamma(a+s)\Gamma(b+s)$ on both sides and integrate over s using the appropriate Mellin-Barnes contour $\mathcal{C}$. In the resulting double integral on the right-hand side, we can switch the order of integration because both integrals are absolutely convergent. It thus follows

$$\begin{aligned} &\int_{\mathcal{C}} \Gamma(-s)\Gamma(c-a-b-s)\Gamma(a+s)\Gamma(b+s)(1-x)^s \frac{\mathrm{d}s}{2\pi\,\mathrm{i}} \\ &= \int_{\mathcal{C}}\int_{\mathcal{C}} \Gamma(c-a-b-s)\Gamma(a+s)\Gamma(b+s)\Gamma(z-s)\frac{\mathrm{d}s}{2\pi\,\mathrm{i}}\Gamma(-z)(-x)^z \frac{\mathrm{d}z}{2\pi\,\mathrm{i}} \\ &= \int_{\mathcal{C}} \frac{\Gamma(c-a)\Gamma(c-b)\Gamma(a+z)\Gamma(b+z)}{\Gamma(c+z)}\Gamma(-z)(-x)^z \frac{\mathrm{d}z}{2\pi\,\mathrm{i}} \end{aligned} \tag{G.61}$$

where in the last line we applied Barnes' Lemma (G.55). Comparing with (G.59), we therefore obtain another Mellin-Barnes representation of the Gauß hypergeometric function,

$$ {}_2\mathrm{F}_1\left(a,b;c;x\right) = \int\limits_{C} \frac{\Gamma(a+z)\Gamma(b+z)\Gamma(c-a-b-z)\Gamma(-z)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} (1-x)^z \frac{\mathrm{d}z}{2\pi\,\mathrm{i}} \,. \tag{G.62} $$

A further application of Mellin-Barnes integrals concerns the product of two flat-space propagators (4.37). These have the schematic form

$$ G(x^2) \sim \int\limits_{C} \Gamma(-s)\Gamma\left(-\frac{n-2}{2}-s\right)\left(\frac{m^2x^2}{4}\right)^s \frac{\mathrm{d}s}{2\pi\,\mathrm{i}} \,, \tag{G.63} $$

where we stripped unimportant prefactors. For the product of two such propagators, we multiply and shift the inner integration variable to obtain

$$ G^2(x^2) \sim \int\limits_{C}\int\limits_{C} \Gamma(-s)\Gamma(-t+s)\Gamma\left(-\frac{n-2}{2}-s\right)\Gamma\left(-\frac{n-2}{2}-t+s\right)\left(\frac{m^2x^2}{4}\right)^t \frac{\mathrm{d}t}{2\pi\,\mathrm{i}}\frac{\mathrm{d}s}{2\pi\,\mathrm{i}} \,. \tag{G.64} $$

We can then exchange the integration order, again because everything converges absolutely, and by using Barnes' Lemma (G.55) it follows

$$ G^2(x^2) \sim \int\limits_{C} \frac{\Gamma^2\left(-t-\frac{n-2}{2}\right)\Gamma(-t)\Gamma(-t-n+2)}{\Gamma(-2t-n+2)}\left(\frac{m^2x^2}{4}\right)^t \frac{\mathrm{d}t}{2\pi\,\mathrm{i}} \,. \tag{G.65} $$

Similarly, one derives Mellin-Barnes formulas for the product of derivatives of such propagators. The generalization of these formulas to de Sitter space is however not so simple, mainly because the Mellin-Barnes representation of the de Sitter propagator (4.35) contains four Γ functions (instead of two as in flat space), and there is no analogue of Barnes' Lemma (G.55) involving eight Γ functions.

H. Calculating with xAct

In this section, we explain an example calculation with the open source tensor manipulation package xAct [173] for Mathematica, which was used extensively in the calculations presented in this thesis. Commands to be entered are shown in black, while the corresponding output is given in grey. Comments are always at the end of the line. and shown in normal font. Since the output sometimes can be lengthy, we only give a description of it, prefixed by -.

First one needs to load the appropriate packages.

```
<< xAct`xTensor`;                         This is the main xAct package.
-- Copyright messages --
<< xAct`TexAct`;                          This package provides nice TEX formatting.
-- Copyright messages --
```

The next step is to define a manifold `M` of dimension `nn`, and give a set of indices that we want to use later.

```
DefManifold[M, nn, a, b, c, d, k, l, m, n, p, q, r, s, t, v, w, x, y, z];
** DefManifold: Defining manifold M.
** DefVBundle: Defining vbundle TangentM.
DefConstantSymbol[nn];
** DefConstantSymbol: Defining constant symbol nn.
```

Avoid derivatives like ∇_ann.

The definition of the manifold is completed with the definition of a metric gg, a covariant derivative CD and tensors that we need in the following.

```
DefMetric[-1, gg[-a, -b], CD, ";", "∇"];
-- Metric, covariant derivative and curvature tensors are defined --
DefTensor[Z0[], M];
DefTensor[Z1n[a], M];
DefTensor[Z1s[a], M];
DefTensor[Z2[a,b], M];
DefTensor[ggn[a,b], M, Symmetric[a,b]];
DefTensor[ggs[a,b], M, Symmetric[a,b]];
DefTensor[un[a], M];
DefTensor[us[a], M];
DefTensor[eta[], M];
DefTensor[etas[], M];
DefConstantSymbol[H];
** DefTensor: Defining tensor Z0[].
-- Definitions for Z1n, Z1s, Z2, gg1, gg2 and H --
PrintAs[Z0]^="Z";
```

This is Z (4.5). Its covariant derivative $\nabla_a Z$. Its covariant derivative $\nabla_{a'} Z$. The mixed derivative $\nabla_a \nabla_{b'} Z$. The metric at the point x. The metric at the point x'. The comoving velocity u_a at x (7.34). The comoving velocity $u_{a'}$ at x'. Nicer output. The same can be done for all other objects.

Next, we define two index sets, one from which we can take indices which refer to the point x and one for the point x'.

```
nindices={a,b,c,d,v,w,x,y,z};
sindices={k,l,m,n,p,q,r,s};
```

The distinction between these two sets is necessary in order to properly define the rules for covariant derivatives of Z, u_a (7.35), η and η'. Covariant derivatives are written in the form CD[a]@T[], which corresponds to $\nabla^a T$, and contravariant indices have to be prefixed with a minus sign.

```
mkZ1nrule[e_]:=Flatten[{Map[(CD[#]@Z0[]->Z1n[#])&,e],
                        Map[(CD[-#]@Z0[]->Z1n[-#])&,e]}]
mkZ1srule[e_]:=Flatten[{Map[(CD[#]@Z0[]->Z1s[#])&,e],
                        Map[(CD[-#]@Z0[]->Z1s[-#])&,e]}]
mkggZ1nrule[e_]:=Flatten[{Map[(CD[#]@Z1n[i_]:>-Z0[] H^2 ggn[#,i])&,e],
                          Map[(CD[-#]@Z1n[i_]:>-Z0[] H^2 ggn[-#,i])&,e]}]
mkggZ1srule[e_]:=Flatten[{Map[(CD[#]@Z1s[i_]:>-Z0[] H^2 ggs[#,i])&,e],
                          Map[(CD[-#]@Z1s[i_]:>-Z0[] H^2 ggs[-#,i])&,e]}]
mkZ2Z1nrule[e_]:=Flatten[{Map[(CD[#]@Z1n[i_]:>Z2[i,#])&,e],
                          Map[(CD[-#]@Z1n[i_]:>Z2[i,-#])&,e]}]
mkZ2Z1srule[e_]:=Flatten[{Map[(CD[#]@Z1s[i_]:>Z2[#,i])&,e],
                          Map[(CD[-#]@Z1s[i_]:>Z2[-#,i])&,e]}]
mkZ1sZ2rule[e_]:=Flatten[{Map[(CD[#]@Z2[i_,j_]:>-Z1s[j] H^2 ggn[#,i])&,e],
                         Map[(CD[-#]@Z2[i_,j_]:>-Z1s[j] H^2 ggn[-#,i])&,e]}]
mkZ1nZ2rule[e_]:=Flatten[{Map[(CD[#]@Z2[i_,j_]:>-Z1n[i] H^2 ggs[#,j])&,e],
                         Map[(CD[-#]@Z2[i_, j_]:>-Z1n[i] H^2 ggs[-#,j])&,e]}]
mkunnrule[e_]:=Flatten[{Map[(CD[#]@un[i_]:>-H (un[#]un[i]+ggn[#,i]))&,e],
                        Map[(CD[-#]@un[i_]:>-H (un[-#]un[i]+ggn[-#,i]))&,e]}]
mkussrule[e_]:=Flatten[{Map[(CD[#]@us[i_]:>-H (us[#]us[i]+ggs[#,i]))&,e],
                        Map[(CD[-#]@us[i_]:>-H (us[-#]us[i]+ggs[-#,i]))&,e]}]
mkunsrule[e_]:=Flatten[{Map[(CD[#]@un[i_]:>0)&,e],
                        Map[(CD[-#]@un[i_]:>0)&,e]}]
mkusnrule[e_]:=Flatten[{Map[(CD[#]@us[i_]:>0)&,e],
```

Derivatives with respect to the point x' vanish, as one can see from the

```
                        Map[(CD[-#]@us[i_]:>0)&,e]}]
```

explicit form of u_a (7.34)

```
mketanrule[e_]:=Flatten[{Map[(CD[#]@eta[]:>-H eta[] un[#])&,e],
                         Map[(CD[-#]@eta[]:>-H eta[] un[-#])&,e]}]
mketasrule[e_]:=Flatten[{Map[(CD[#]@etas[]:>-H etas[] us[#])&,e],
                         Map[(CD[-#]@etas[]:>-H etas[] us[-#])&,e]}]
mketansrule[e_]:=Flatten[{Map[(CD[#]@eta[]:>0)&,e],
                          Map[(CD[-#]@eta[]:>0)&,e]}]
mketasnrule[e_]:=Flatten[{Map[(CD[#]@etas[]:>0)&,e],
                          Map[(CD[-#]@etas[]:>0)&,e]}]
```

These rules are then defined for each index from the two sets, together with rules for the contraction of covariant derivatives of Z (4.11) and u_a (7.36).

```
ConvertZrules=Flatten[{makeZ1nrule[nindices], makeZ1srule[sindices],
  makeggZ1nrule[nindices], makeggZ1srule[sindices], makeZ2Z1nrule[sindices],
  makeZ2Z1srule[nindices], makeZ1sZ2rule[nindices], makeZ1nZ2rule[sindices],
  Z1n[i_]Z1n[-i_]:>H^2(1-Z0[]^2), Z1s[i_]Z1s[-i_]:>H^2(1-Z0[]^2),
  Z2[i_,j_]Z1n[-i_]:>-H^2 Z0[]Z1s[j], Z2[-i_,j_]Z1n[i_]:>-H^2 Z0[]Z1s[j],
  Z2[i_,j_]Z1s[-j_]:>-H^2 Z0[]Z1n[i], Z2[i_,-j_]Z1s[j_]:>-H^2 Z0[]Z1n[i],
  Z2[i_,j_]Z2[-i_,k_]:>H^4 ggs[j,k]-H^2 Z1s[j]Z1s[k],
  Z2[i_,j_]Z2[k_,-j_]:>H^4 ggn[i,k]-H^2 Z1n[i]Z1n[k],
  u1n[a_]Z1n[-a_]:>H(eta[]/etas[]-Z0[]), u1n[-a_]Z1n[a_]:>H(eta[]/etas[]-Z0[]),
  u1s[a_]Z1s[-a_]:>H(etas[]/eta[]-Z0[]), u1n[-s_]Z1n[s_]:>H(etas[]/eta[]-Z0[]),
  u1n[a_]Z2[-a_,i_]:>H(eta[]/etas[] H us[i]-Z1s[i]),
  u1n[-a_]Z2[a_,i_]:>H(eta[]/etas[] H us[i]-Z1s[i]),
  u1s[a_]Z2[i_,-a_]:>H(etas[]/eta[] H un[i]-Z1n[i]),
  u1s[-a_]Z2[i_,a_]:>H(etas[]/eta[] H un[i]-Z1n[i]),
  CD[_]@ggn[__]:>0, CD[_]@ggs[__]:>0, ggn[i_,-i_]:>nn, ggn[-i_,i_]:>nn,
  ggs[i_,-i_]:>nn, ggs[-i_,i_]:>nn, ggn[a_,b_]Z1n[-a_]:>Z1n[b],
  ggn[a_,b_]Z1n[-b_]:>Z1n[a], ggn[-a_,b_]Z1n[a_]:>Z1n[b],
  ggn[a_,-b_]Z1n[b_]:>Z1n[a], ggs[a_,b_]Z1s[-a_]:>Z1s[b],
  ggs[a_,b_]Z1s[-b_]:>Z1s[a], ggs[-a_,b_]Z1s[a_]:>Z1s[b],
  ggs[a_,-b_]Z1s[b_]:>Z1s[a], ggn[a_,b_]Z2[-a_,i_]:>Z2[b,i],
  ggn[a_,b_]Z2[-b_,i_]:>Z2[a,i], ggn[-a_,b_]Z2[a_,i_]:>Z2[b,i],
  ggn[a_,-b_]Z2[b_,i_]:>Z2[a,i], ggs[a_,b_]Z2[i_,-a_]:>Z2[i,b],
  ggs[a_,b_]Z2[i_,-b_]:>Z2[i,a], ggs[-a_,b_]Z2[i_,a_]:>Z2[i,b],
  ggs[a_,-b_]Z2[i_,b_]:>Z2[i,a], ggn[i_,a_]ggn[-a_,j_]:>ggn[i,j],
  ggn[i_,-a_]ggn[a_,j_]:>ggn[i,j], ggn[i_,a_]ggn[j_,-a_]:>ggn[i,j],
  ggn[a_,i_]ggn[-a_,j_]:>ggn[i,j], ggs[i_,a_]ggs[-a_,j_]:>ggs[i,j],
  ggs[i_,-a_]ggs[a_,j_]:>ggs[i,j], ggs[i_,a_]ggs[j_,-a_]:>ggs[i,j],
  ggs[a_,i_]ggs[-a_,j_]:>ggs[i,j], ggn[a_,b_]un[-a_]:>un[b],
  ggn[a_,b_]un[-b_]:>un[a], ggn[-a_,b_]un[a_]:>un[b],
  ggn[a_,-b_]un[b_]:>un[a], ggs[a_,b_]us[-a_]:>us[b],
  ggs[a_,b_]us[-b_]:>us[a], ggs[-a_,b_]us[a_]:>us[b],
  ggs[a_,-b_]us[b_]:>us[a]}];
```

Finally we define a function which applies these rules to an expression, along with some other contractions and expansions.

```
ConvertZ[e_]:=ToCanonical[ContractMetric[Expand[e],OverDerivatives->True]
                          //.ConvertZrules]
```

Another very useful function groups expressions by their tensorial structure, as in (4.77), (4.113) or (7.4). To define this function, we need a helper function which checks if a given term has indices

```
HasIndices[f_]:=Length[IndicesOf[Free][f]]>0 || Length[IndicesOf[Dummy][f]]>0
```

as well as one that can expand an expression that was previously `Collect[]`ed according to a specific criterion.

```
ExpandCollect[expr_, crit_]:=If[Head[expr] === Times,
 If[Length[Select[expr, ((Head[#] === Plus) && crit[#])&, 1]]>0,
   Module[{pos = Position[expr, _?(((Head[#] === Plus) && crit[#]) &), {1},
           Heads -> False][[1, 1]]},
     Map[ExpandCollect[Delete[expr, pos] #, crit] &, expr[[pos]]]],
   expr],
 If[Head[expr] === Plus,
   If[crit[expr], Map[ExpandCollect[#, crit] &, expr], expr],
  expr]];
```

This function works in the following way: on each subexpression that is a product, each factor is checked if it fulfills the specified criterion, and sorted into ones that do and ones that do not. On each subexpression that is a sum, the procedure is applied to each summand. The grouping function can then be defined by

```
TCollect[f_]:=ExpandCollect[Collect[f, {gg1[__], gg2[__], Z1n[_], Z1s[_],
                            Z2[__], u1n[_], u1s[_], H}, Factor], HasIndices]
```

After defining these functions, we are ready to tackle specific calculations. For example, it is easy to check the tracelessness of the bitensors (7.10) appearing in the Weyl-Weyl correlation functions. We first define the Riemann bitensor set (4.111)

```
R1 = Antisymmetrize[Antisymmetrize[Antisymmetrize[Antisymmetrize[
      gg1[a,c]gg1[b,d]gg2[m,p]gg2[n,q],{a,b}],{c,d}],{m,n}],{p,q}];
```

and so on, and then define

```
C1 = - 2 Z0[]R1 - 3R4 + 6 Z0[]R5 + 2R8 - 2 Z0[]R9;
C2 = -(5-Z0[]^2) R1 + 6R2 + 12R5 - 6R6 - 6R7 + 2 Z0[]R8 - (3+Z0[]^2)R9;
C3 = (1-Z0[]^2) (R1 - 3R5 + R9) - 6R3 + 6R7;
```

Since these expressions are valid in four dimensions, but we defined our manifold with dimension nn, we have to set nn to 4 after the contraction:

```
ConvertZ[gg1[-a,-b] C1 /.{nn->4}]
0
ConvertZ[gg1[-a,-b] C2 /.{nn->4}]
0
ConvertZ[gg1[-a,-b] C3 /.{nn->4}]
0
```

Similarly is is quite easy to derive the stress tensor correlation functions from section 4.5, for example the massless vector propagator (4.76). The two-point function of the field strength tensor is given by (4.56). First we define a scalar function for $I_\rho(Z)$

```
DefScalarFunction[Irho]
** DefScalarFunction: Defining scalar function Irho.
```

and then we define the field strength tensor two-point function

```
FF[a_,b_,c_,d_]:=2 H^(-2) (Z2[a,c]Z2[b,d]-Z2[a,d]Z2[b,c]) Irho'[Z0[]]
        - H^(-2) (Z1n[a]Z2[b,c]Z1s[d]-Z1n[b]Z2[a,c]Z1s[d]
                 -Z1n[a]Z2[b,d]Z1s[c]+Z1n[b]Z2[a,d]Z1s[c]) Irho''[Z0[]]
```

The stress tensor two-point function can be directly copied from equation (4.76), taking care to use dummy indices from the proper set depending on whether they refer to the point x or x'.

```
TT = FF[a,v,m,r] FF[b,-v,n,-r] + FF[a,v,n,r] FF[b,-v,m,-r]
     - 1/2 gg1[a,b] FF[v,w,m,r] FF[-v,-w,n,-r]
     - 1/2 gg2[m,n] FF[a,v,r,s] FF[b,-v,-r,-s]
     + 1/8 gg1[a,b] gg2[m,n] FF[v,w,r,s] FF[-v,-w,-r,-s];
TCollect[ConvertZ[ConvertZ[TT]]]
g[a,b]g[m,n]H^4 (5-nn)( 2(1-Z0[]^2)Z0[] Irho''[Z0[]] Irho'[Z0[]] - ...) + ...
```

Of course, for other calculations such as the determination of the Riemann tensor two-point function (7.4) from the correlation function of the metric perturbations (6.33, 6.37, 6.36), more tensors have to be defined and rules for them must be declared.

Resumen en castellano

Nuestra descripción teórica actual de la naturaleza está dividida en dos categorías: el Modelo Estándar de la Física de Partículas que unifica electromagnetismo y las interacciones débiles y fuertes, y la Teoría de Relatividad General que describe la gravitación. Ambas teorías han sido bien probadas en un rango amplio de escalas de energía y longitudes: el Modelo Estándar en experimentos de colisión de altas energías como en el Large Hadron Collider en Ginebra y la Relatividad General en observaciones astrofísicas en el sistema solar y fuera de él, entre otras.

No obstante, la Relatividad General es una teoría clásica y falla en dar la descripción correcta en escalas de energía suficientemente altas, comparables con la escala de Planck, donde una teoría de Gravedad Cuántica asume el mando. Debido a que estas escalas de energía están fuera de nuestro alcance experimental, el trabajo con teorías de este estilo es puramente teórico y solo se puede juzgar por consistencia interna y simplicidad. La única perspectiva posible de evidencia experimental para una teoría de gravedad cuántica (quizás con la excepción de modelos análogos de gravedad) viene de observaciones del fondo cósmico de microondas (CMB). Estas observaciones se han mejorado inmensamente por el satélite Planck hace poco y nos han permitido echar un vistazo a los primeros momentos de la historia de nuestro universo cuando efectos cuánticos de gravedad jugaban un papel relevante. Vestigios de interacciones de aquel tiempo que todavía son observables hoy en día nos pueden dar una pista sobre la física a estas escalas. Según el modelo estándar ΛCDM de cosmología, en su infancia el universo sufrió un periodo de expansión rápida, conocida como inflación, el cual se puede modelar por una parte del espacio-tiempo de Sitter hasta pequeñas correcciones (denominadas correcciones de slow-roll). En este caso, el factor propulsor es un campo escalar homogéneo, el inflatón, con un término potencial grande que cambia muy despacio durante el periodo inflacionario, y entonces se puede modelar efectivamente por una constante. Además, existe una constante cosmológica real que, sin embargo, es muy pequeña y que es responsable de la expansión acelerada actual del cosmos. Modelos que también reemplazan esta constante con algún término dinámico están resumidos bajo el nombre genérico de energía oscura; no obstante, hasta ahora no hay ninguna indicación observacional que favorezca algo diferente de una constante. Una característica sorprendente del CMB, su homogeneidad casi perfecta por encima de regiones que nunca se encontraron en contacto causal, puede ser explicada por tal periodo inflacionario con expansión exponencial, o sea, más rápido que la velocidad de la luz. Se piensa que la fuente de pequeñas variaciones de la distribución de temperatura son fluctuaciones cuánticas de la métrica en la cual los fotones del CMB se propagan, con la mayoría del efecto proveniente de fluctuaciones que fueron generadas a lo largo de la inflación.

Sin embargo, no se llega a escalas de Planck durante el periodo inflacionario, y en tal régimen intermediario se puede estudiar la gravedad cuántica perturbativamente utilizando teorías de campos efectivas (EFT). En estas teorías, parametrizamos nuestra ignorancia de la física real de altas energías con unas interacciones efectivas, incluyendo todos los términos posibles cuya forma es compatible con las simetrías del sistema bajo consideración que fueron asumidas. Un ejemplo prominente es la teoría de Fermi de la interacción de cuatro fermiones, en la cual el intercambio de un bosón W entre dos fermiones a escalas de energía bien por debajo de la masa del bosón W está descrito por un vértice efectivo con interacción de cuatro fermiones, con una constante de acoplo inversamente proporcional al cuadrado de la masa del W. Esta teoría hace predicciones correctas para todas las energías inferiores a la escala dada por la masa del bosón W. No es renormalizable, y para calcular efectos de lazos se tendrían que incluir operadores de dimensión más alta, o sea, vértices de seis fermiones y más, con constantes de acoplo que se tienen que determinar por el experimento. No obstante, estos operadores están suprimidos por potencias más altas de la masa del bosón W, y a escalas bajas de energía no aportan correcciones apreciables. La situación es análoga en el caso de gravedad cuántica perturbativa. A distancias largas y curvatura baja (comparadas con la escala de Planck), podemos añadir a la acción de gravedad de Einstein-Hilbert operadores con dimensiones más altas, que son potencias más altas de los tensores de curvatura y sus derivadas covariantes. Estos operadores están suprimidos por potencias de la masa de Planck, y para procesos de energía baja solo dan correcciones pequeñas confiables.

En esta tesis investigamos dos efectos que se pueden estudiar en este escenario: la estabilidad del espacio-tiempo de Sitter bajo perturbaciones débiles de la métrica, y sus funciones de correlación que se pueden relacionar con observables cosmológicos.

El análisis teórico estándar de perturbaciones en cosmología primordial se basa en la teoría lineal de perturbaciones, que en la teoría cuántica es equivalente a un cálculo a nivel árbol. Sin embargo, los efectos debidos a correcciones de lazos podrían ser significativos si la supresión habitual de estas correcciones está superada por algún proceso físico nuevo en espacios curvos. Ya hace tiempo, Tsamis y Woodard propusieron que correcciones radiativas debidas a lazos de gravitones podrían llevar a un apantallamiento de la constante cosmológica, lo que entonces serviría como mecanismo para acabar con el periodo inflacionario independientemente de la existencia de un inflatón. En este caso, la constante cosmológica fundamental tendría un valor que solo sería pocos ordenes de magnitud más pequeña que la escala de Planck, y el valor que observamos hoy es el valor apantallado. El mecanismo por el cual funciona se basa en la excitación continua de modos del gravitón en el periodo inflacionario que no están diluidos por la expansión exponencial, lo que entonces – por la naturaleza no lineal de la interacción gravitatoria – conduce a la frenada de la expansión. Aunque esta afirmación es muy atractiva desde un punto de vista físico, su validez y la interpretación de cálculos concretos se han puesto en duda, y todavía sigue siendo un problema irresuelto determinar si realmente funciona.

Además, por la ausencia de un vector de Killing global de tipo tiempo en el espacio-tiempo de Sitter, existe la posibilidad de que incluso teorías masivas en un fondo fijo de Sitter puedan ser inestables. Esta idea fue analizada por varios autores tanto al nivel árbol como incluyendo correcciones de lazos. En este contexto, la continuación a un formalismo euclídeo (o sea, cálculos en la esfera) se ha mostrado muy útil. El estado del vacío euclídeo y funciones de correlación en este estado, definidas por una continuación analítica apropiada de la esfera al espacio-tiempo de Sitter, tienen algunas propiedades muy atractivas que incluyen finitud infrarroja e invariancia completa de Sitter. Para campos escalares masivos, Higuchi, Marolf y Morrison demostraron que cálculos hechos en la esfera y usando el formalismo in-in en el espacio-tiempo de Sitter son completamente equivalentes, y también se estudiaron generalizaciones para campos muy ligeros y sin masa. A parte de eso, el vacío euclídeo es un atractor en tiempos tardíos de estados iniciales genéricos, es decir, funciones de correlación en otros estados se acercan a las funciones euclídeas en el futuro lejano en un sentido precisamente definido.

Estos hallazgos para campos de materia interactuantes (para un acoplo suficientemente débil tal que se puede aplicar teoría de perturbaciones) extienden resultados clásicos sobre el carácter atractor en el futuro lejano del espacio-tiempo de Sitter, tanto para perturbaciones lineales como para el caso non lineal, conocido bajo el nombre de propiedad "sin pelo" como en el caso de agujeros negros. Sin embargo, considerando campos de prueba en un fondo fijo solo da una parte de la respuesta completa, y para el problema entero se tiene que tener en cuenta la retroacción de los campos cuánticos a la geometría del espacio-tiempo. Estudios de este estilo se hicieron en gravedad semiclásica, donde la métrica todavía se considera clásicamente pero se tiene en cuenta la naturaleza cuántica de la materia, y se encontraron propiedades similares del tipo atractor. En vez de un tensor de energía-momento clásico aparece un valor esperado de un operador cuántico correspondiente al tensor de energía-momento en la parte derecha de las ecuaciones de Einstein. No obstante, también es importante tener en cuenta la naturaleza cuántica de la métrica: estudiando gravedad como una teoría efectiva, esto equivale a una cuantización de perturbaciones de la métrica alrededor de un fondo fijo.

En la primera mitad de esta tesis, extendimos estas consideraciones al caso gravitatorio, estudiando la estabilidad del espacio-tiempo de Sitter bajo perturbaciones débiles de la métrica que interactúan con campos de materia. Estudios parciales en esta dirección ya existen. Por ejemplo, se ha demostrado que el espacio-tiempo plano es estable bajo perturbaciones linealizadas en interacción con campos de materia, se ha encontrado una corrección nula a los modos clásicos en el espacio-tiempo de Sitter para la interacción de modos tensoriales con campos escalares libres sin masa y mínimamente acoplados, y también se ha establecido la estabilidad del espacio-tiempo de Sitter para perturbaciones isótropas en el espacio. Extendiendo estos cálculos, consideramos primero las ecuaciones semiclásicas de Einstein, que en este caso dan una pequeña corrección a la relación entre la constante cosmológica y la constante de Hubble (o el radio inverso de Sitter). Derivamos entonces las ecuaciones que satisfacen las perturbaciones linealizadas generales de la métrica (de los tipos escalares, vectoriales y tensoriales) debidas a la interacción con campos conformes, y consideramos la estabilidad del espacio-tiempo de Sitter bajo estas perturbaciones. En este contexto, empleamos el método de reducción del orden, que en contraste a un tratamiento estrictamente perturbativo produce soluciones que son fiables por un tiempo extendido, una cualidad que obviamente es crucial para estudiar cuestiones de estabilidad. Este método no solo desvanece soluciones casuales que están fuera de la validez de la teoría efectiva que usamos, sino que también tiene la ventaja de generar ecuaciones de retroacción que se pueden resolver bastante fácilmente, y así elimina la necesidad de usar aproximaciones más allá de las que están impuestas por el uso de la EFT y la consideración de perturbaciones lineales.

Resolvimos las ecuaciones tanto para un estado inicial de vacío (la generalización del vacío de Bunch-Davies que incluye interacciones) como para una clase amplia de estados iniciales generales. En ambos casos, los cambios inducidos en el tensor de Riemann linealizado, que es un observable invariante gauge y local, son pequeños y desaparecen en el futuro infinito. Estos resultados extienden los teoremas clásicos y semiclásicos llamados "sin pelo" del espacio-tiempo de Sitter al caso cuántico, por lo menos para los casos que se estudiaron. Teoremas de este tipo aseguran que para tiempos suficientemente grandes, cualquier perturbación amaina, en el sentido de que para cualquier región de tamaño físico fijo, observaciones con precisión limitada no pueden detectar la perturbación. En el caso estudiado, vimos que aunque las perturbaciones tensoriales tienden a una constante para tiempos grandes, o sea, su longitud de onda tiende a infinito, solo son detectables si un observador tiene acceso a todo el universo. Si uno considera observables locales como el tensor de Riemann, esta constante no contribuye, y las conclusiones del teorema “sin pelo” son inalteradas. Además, para las perturbaciones tensoriales que son la parte dinámica de perturbaciones generales de la métrica, las correcciones debidas a un estado inicial diferente del vacío decaen más rápidamente que la misma perturbación. Esto da soporte a cálculos existentes sobre el carácter atractor del futuro lejano no solo del espacio-tiempo de Sitter, sino también de perturbaciones que empiezan su evolución en un estado invariante de Sitter. Aunque estos resultados fueron derivados para la interacción de gravitones con campos escalares sin masa y conformemente acoplados, la invariancia de Sitter de la parte no local de la acción efectiva nos permitió restringir la forma de las ecuaciones de movimiento en el caso general, tal que estas conclusiones permanecen inalteradas.

Para llegar a ese objetivo, primero derivamos la acción efectiva para las perturbaciones de la métrica que incluye las correcciones debidas a lazos de materia para algunos casos. Para un espacio de fondo plano, fue posible de obtener el resultado general para campos escalares mínimamente acoplados y con masa arbitraria, y la generalización del método usado a otros campos de materia es fácil. Para el caso de un fondo de Sitter, aparte del cálculo existente para campos escalares sin masa y conformemente acoplados que ya estaba hecho, calculamos la acción efectiva para un campo escalar sin masa y mínimamente acoplado. Mientras la generalización de la técnica usada en el espacio plano fue imposible por un problema técnico, extendimos la idea general a un fondo de espacio curvo y fuimos capaz de aplicarla con éxito a este último caso. En principio no hay ninguna razón por la cual esta extensión no funcione en el caso general de masa arbitraria, pero en la práctica se necesitaría mucho esfuerzo para funciones de dos puntos del tensor de energía-momento que no tienen una forma fácil en cuatro dimensiones.

La segunda mitad de tesis trata funciones de correlación de estas perturbaciones de la métrica. Esto va un paso adelante del objetivo anterior, donde las ecuaciones que dirigen la evolución de las perturbaciones de la métrica son las mismas que las que se aplican a su valor esperado. Claramente es importante tener en cuenta las fluctuaciones alrededor del valor medio. Su tamaño relativo en relación al valor medio se puede usar como criterio para la validez de la descripción del campo medio, y están relacionadas a observables cosmológicos como el espectro de potencia tensorial al final del periodo inflacionario. Para el caso de la interacción con campos conformes, calculamos explícitamente la función de dos puntos de perturbaciones escalares, vectoriales y tensoriales, usando una generalización de la prescripción $\mathrm{i}\epsilon$ del espacio-tiempo plano que nos permite definir un estado adecuado de vacío con interacción en el pasado infinito. Contrario a resultados anteriores que se concentraban en efectos de primer orden y el límite de tiempos iguales, obtuvimos la función de dos puntos entera para tiempos distintos. El resultado es invariante bajo el subconjunto de isometrías de Sitter que está preservado por nuestro selección de gauge, lo más importante de estas siendo el rescalado constante simultaneo de coordenadas temporales y espaciales. De esta función de dos puntos, calculamos un observable cosmológico, el espectro de potencia para las perturbaciones tensoriales que se puede observar mediante el espectro de las fluctuaciones de temperatura en el CMB. Este espectro es el límite de tiempos iguales de la transformación de Fourier de la función contraída de dos puntos. No obstante, este límite de tiempos iguales es divergente porque el resultado a un lazo para la transformación de Fourier es una distribución verdadera (en oposición a una función regular que es el resultado a nivel árbol). Para obtener un resultado finito, integramos la distribución con una función de prueba que es una función gaussiana de ancho pequeño, modelando de tal manera el proceso de medir. Para la parte regular de la función de dos puntos, esto se reduce a la definición estándar cuando el ancho de la función gaussiana se hace cero. El espectro obtenido solo depende del momento físico, y por tanto es independiente del tiempo en el espacio-tiempo físico. Este hecho está en contraste con resultados anteriores, que encontraron una dependencia logarítmica en el momento comóvil, pero concuerda con estudios más recientes que argumentan (para perturbaciones escalares) que una dependencia de este estilo es errónea, identificando también la causa de la discrepancia. Desafortunadamente, estas correcciones son demasiado pequeñas para ser medidas. Su tamaño solo sería apreciable si la constante de Hubble tuviera una magnitud comparable a la escala de Planck, donde la teoría efectiva que usamos deja de ser válida.

Sin embargo, mientras que la fijación del gauge que se usa habitualmente en cosmología (y que usamos en esta tesis) para la descomposición escalar–vectorial–tensorial es completa para perturbaciones que decaen en el infinito espacial, no es local dado que se necesitan especificar condiciones del contorno. Objetos que están construidos a partir de funciones de correlación en este gauge entonces no son observables en el sentido estricto, ya que solo tenemos acceso observacional a una parte finita del universo. Por lo tanto, uno tiene que buscar observables "suficientemente locales" que caractericen las propiedades geométricas en regiones de tamaño físico finito. Este último requerimiento ha sido importante en la construcción de observables llamados de "seguro infrarrojo", que dan resultados finitos en situaciones que en el caso contrario llevan a divergencias sin un corte en el infrarrojo. Además, esta fijación del gauge que se hace en el sistema de coordenadas conformemente plano no respeta las simetrías del fondo de Sitter, y por eso oculta tales simetrías en las funciones de correlación. Evidentemente es difícil encontrar un observable invariante gauge y local, incluso en gravedad cuántica perturbativa, pero si excluimos lazos de gravitones y nos restringimos a un fondo de Sitter (u otros fondos máximamente simétricos), el tensor de Riemann linealizado facilita un observable de este tipo. En la teoría clásica, este tensor contiene toda la información sobre la geometría local del espacio-tiempo, y se puede esperar que sus funciones de correlación también den la información completa en el caso cuántico.

Calculamos entonces su función de dos puntos, empezando desde la función de dos puntos de las perturbaciones de la métrica en el caso de interacción con campos conformes. Para entender mejor la estructura del resultado, este se descompone en las funciones de correlación de los tensores de Weyl y Ricci y del escalar de Ricci. A pesar de que la fijación de gauge para las perturbaciones de la métrica no era completamente invariante de Sitter, el resultado para la función de dos puntos del tensor de Riemann sí que lo es. Esto muestra que el tensor de Riemann divide los efectos físicos de los efectos gauge, y exhibe manifiestamente las simetrías de Sitter que entraban en el cálculo, la invariancia de Sitter de las funciones de correlación del tensor de energía-momento y del estado asintótico del vacío con interacciones, mostrando que para la interacción de perturbaciones de la métrica con lazos de materia conforme no hay rotura física de esta invariancia. El resultado ilustra que las correcciones cuánticas están concentradas cerca del cono de la luz, o sea, que allí son más singulares que el resultado a nivel árbol. Además, la función de dos puntos decae exponencialmente para separaciones largas entre los dos puntos, mostrando que la curvatura del fondo actúa como una masa efectiva para el gravitón.

Naturalmente es interesante saber si estos cálculos se pueden generalizar a la interacción con otros tipos de materia. En la última parte de la tesis explotamos entonces las identidades de Bianchi para demostrar que la función de dos puntos del tensor de Riemann siempre es invariante de Sitter si lo es para el tensor de energía-momento. Dados los coeficientes para la función de dos puntos del tensor de energía-momento, dimos fórmulas explícitas para calcular los coeficientes de la función de correlación del tensor de Riemann, y vimos que el resultado está completamente determinado salvo una constante de integración. Esta constante naturalmente se puede interpretar como potencia de gravitones libres que se propagan en el espacio-tiempo de fondo, y depende de los coeficientes desconocidos que multiplican a los términos en la acción gravitatoria que involucran cuadrados de tensores de curvatura.

¿Qué nos dicen estos resultados? Desde un punto de vista cosmológico, podemos decir con bastante certeza que correcciones de lazos son irrelevantes (para los casos considerados), lo cual podría sonar decepcionante. Esta conclusión tampoco cambia si incluimos auto-interacciones de los campos de materia, en cuanto las funciones de correlación del tensor de energía-momento siguen siendo invariantes de Sitter. Evidentemente la inclusión de lazos de gravitones puede cambiar enormemente este imagen si la rotura de la invariancia de Sitter, que parece ser una característica robusta en cálculos que se hacían para casos individuales, es de verdad física y no solo un artefacto gauge. Para llegar a una conclusión en este asunto, primero es necesario construir un observable adecuado, invariante gauge y local que entonces puede jugar el mismo papel que el tensor de Riemann linealizado en esta tesis: separando efectos físicos de efectos gauge. Sin embargo, desde un punto de vista matemático los resultados obtenidos en esta tesis son extremadamente satisfactorios. En el espacio-tiempo de Minkowski existen muchos métodos de cuantización y regularización y términos para fijar el gauge que preservan manifiestamente la simetría Poincaré del fondo. Considerando campos cuánticos en un fondo curvo fijo (o sea, no considerando efectos que radicalmente cambian este fondo como la formación de agujeros negros), uno naturalmente espera que haya una manera de mantener las simetrías del fondo, y vimos que en efecto es el caso. Si bien existe un propagador de gravitones que es invariante de Sitter, derivado de la versión euclídea del espacio-tiempo de Sitter, la esfera, su validez se ha puesto en cuestión y es gratificante ver que la elección del propagador no influye el resultado para observables adecuados. Además, estos hallazgos son un paso más en el camino para establecer la equivalencia de cálculos hechos en de Sitter y su versión euclídea, una conexión que en el espacio plano se ha podido poner en forma de teorema.

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