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Hopf fibration reduction of a quartic model. Applications to rotational and orbital dynamics.

Reducción de tipo Hopf de un modelo cuártico. Aplicaciones en dinámica rotacional y orbital.

D. Francisco Crespo Cutillas

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A mis padres,
que todo me han dado.

A Jorge y Elisa,
por su amor y paciencia.

To those who study the progress of exact science, the common spinning top is a symbol of the labours and the perplexities of men.

JAMES CLERK MAXWELL.

The diversity of the phenomena of nature is so great, and the treasures hidden in the heavens so rich, precisely in order that the human mind shall never be lacking in fresh nourishment.

JOHANNES KEPLER.

Contents

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Resumen

Esta tesis aborda los sistemas más conocidos de la mecánica clásica de forma unificada; consideramos para ello una familia polinómica de hamiltonianos homogéneos de cuarto grado como modelo básico. Nuestro objetivo principal es desarrollar un marco de trabajo común para el estudio de perturbaciones, dicha tarea se realiza desde un punto de vista geométrico. Para llevar a cabo nuestro proyecto, hemos estructurado esta memoria en tres partes: preliminares, el marco de trabajo para el tratamiento unificado y, por último, las aplicaciones.

Parte I. Preliminares en Mecánica Clásica y Geometría: En esta primera parte recogemos algunas herramientas que serán usadas a lo largo de nuestro estudio. En el primer capítulo fijamos la notación y se presentan algunos resultados básicos que pueden ser encontrados en [Abraham and Marsden, 1985, Marsden and Ratiu, 1999, Meyer et al., 2009]. En el segundo estudiamos el Sistema Extendido de Euler (SEE), como un problema de valor inicial paramétrico. Las distintas elecciones de parámetros en dichos sistemas nos conducen a diferentes estructuras Lie-Poisson, todas ellas admiten una generalización dentro de una estructura Poisson de seis dimensiones. La descripción de la estratificación simpléctica inducida por estas estructuras Lie-Poisson viene determinada por el uso de las integrales primeras, en el caso de sistemas SEE, estas vienen dadas por cilindros hiperbólicos y elípticos, aunque otras cuádricas pueden ser también usadas. A lo largo de este capítulo llevamos a cabo un estudio cualitativo de las soluciones generales del sistema, donde las doce funciones elípticas de Jacobi se muestran de manera unificada como dichas soluciones. Este enfoque permite derivar las principales propiedades de las funciones elípticas. En concreto, las conocidas relaciones cuadráticas entre funciones elípticas y la transformación de Jacobi para el módulo elíptico se obtienen de nuestro análisis.

Parte II. Reducción de tipo Hopf de un modelo cuártico : En el tercer capítulo estudiamos una generalización de la fibración de Hopf clásica. El uso de dicha fibración en los problemas de la mecánica clásica no es novedoso, ver [Cushman and Bates, 1997] y las referencias allí incluidas. La generalización de la que hacemos uso en esta tesis

se lleva a cabo por medio de un incremento de la dimensión de las variedades involucradas, otras generalizaciones alternativas han sido analizadas en la literatura [Gluck et al., 1986]; en la nuestra seguiremos la misma metodología que en la fibración de Hopf clásica, pero el cuerpo complejo será reemplazado por cuaternios. En el cuarto capítulo usamos los componentes de la representación cuaterniónica de la aplicación de Hopf para proponer una familia de Hamiltonianos multiparamétrica en $T^*\mathbb{R}^4$; estos Hamiltoinanos vienen dados por un polinomio cuártico homogéneo con seis parámetros, definiendo una familia integrable de sistemas Hamiltonianos [Ferrer and Crespo, 2015]. La característica clave de este modelo es su estructura Hamiltoniana-Poisson anidada, la cual aparece reflejada como dos sistemas de Euler extendidos cuando nos pasamos a las ecuaciones reducidas [Ortega and Ratiu, 2004]. Este hecho es completamente explotado en el proceso de integración, donde encontramos dos 1-DOF subsistemas y una cuadratura que liga a ambos. La solución genérica es cuasi-periódica y viene expresada por medio de funciones elípticas de Jacobi basadas en dos periodos que en general son distintos. Para una elección apropiada de los parámetros y considerando una regularización de la variable independiente, cuando sea necesario, algunos modelos destacados de la mecánica clásica tales como el sistema de Kepler, el flujo geodésico, el oscilador isotrópico de cuatro dimensiones y el sólido rígido libre aparecen como casos particulares. El análisis del modelo cuártico se lleva a cabo a través de una doble reducción. Por un lado, el sistema es geométricamente reducido y aunque no se proporcionan nuevos teoremas en teoría de reducción ni estudiamos la acción de nuevos grupos en los problemas de la mecánica, este modelo es un ejemplo detallado de reducción singular [Ortega and Ratiu, 2004], en la cual la correspondiente reconstrucción es también proporcionada. Por otro lado, la reducción simpléctica llevada a cabo a través del uso de nuevas coordenadas canónicas es analizada. En concreto, usando variables de Andoyer Proyectivas [Ferrer, 2010], que generalizan los ángulos de Andoyer al caso de cuatro dimensiones, dos simetrías del modelo cuártico aparecen como momentos conjugados de dos ángulos. Por lo tanto, estas variables representan la reducción toroidal asociada a la doble simetría del sistema paramétrico. En concreto, se muestra la relación entre la reducción geométrica y simpléctica y se proporciona la formulación explícita para todos los cambios de variables que son usados.

Parte III. Aplicaciones a la dinámica Roto-Orbital: Esta parte está dedicada al estudio de la dinámica de actitud y el movimiento orbital de modelos que aproximan un asteroide o un satélite con una triaxialidad genérica, bajo los efectos de una perturbación gravitacional. Este problema, denominado problema completo de los dos cuerpos, es un sistema dinámico Hamiltoniano no integrable, que todavía queda lejos de ser resuelto y requiere el uso de teorías de perturbaciones para su análisis. Dentro del contexto de Poincaré [Poincaré, 1899] y el refinamiento de Arnold [Arnold et al., 1993] página 185 en la aplicación de teoría KAM, una teoría de perturbación debería ser desarrollada a partir un orden cero integrable y no degenerado. Tradicionalmente, este papel ha sido encarnado por el sólido rígido libre y el sistema de kepler, los cuales son degenerados (superintegrable [Fassò, 2005]). Nosotros exploraremos otros candidatos para el orden cero, los intermediarios (see [Deprit, 1981] and the references therein).

La idea de los intermediarios consiste en definir un sistema integrable simplificado del problema en cuestión, donde el trabajo de Hill en el movimiento de la luna [Wilson, 2010] es, quizás, el mejor ejemplo conocido. En el quinto capítulo recordamos el concepto de intermediario, presentamos cinco modelos y establecemos una metodología común para su estudio. Es en este contexto donde el marco desarrollado para el modelo polinómico cuártico es completamente explotado. El sistema simplificado incluye parte del potencial donde el acoplamiento roto-orbital esta presente. Esto es, el potencial perturbado se separa en la siguiente forma, $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1$ y el nuevo orden cero está dado por

$$\mathcal{H}_0 = T_O + T_R + \mathcal{P}_0, \tag{1}$$

de tal manera, que el sistema definido por $\mathcal{H}_0$ es integrable. Los capítulos seis y siete aprovechan el marco de trabajo desarrollado en el estudio de dos intermediarios definidos en el capítulo anterior. Se asume que estos intermediarios tienen orbitas circulares y elípticas respectivamente. Algunos resultados parciales ya han sido enviados para su publicación [Ferrer et al., 2014, Molero et al., 2014], en ocasiones nos referimos a este material para más detalle.

En el capítulo seis estudiamos equilibrios relativos y bifurcaciones del intermediario circular. Este modelo de intermediario define un flujo Poisson sobre espacio multiparamétrico: tres parámetros físicos (momentos principales de inercia) y tres parámetros especiales, llamados: el módulo del vector momento angular (M), su tercera componente (ω_9) y la expresión del movimiento orbital medio (n). En el caso de un cuerpo de rotación lenta, identificamos condiciones bajo las cuales aparecen bifurcaciones de las trayectorias inestables clásicas, siendo dichos escenarios de gran interés en relación a la estabilización y control. Nuestro estudio esta basado en el uso de los invariantes definiendo el espacio reducido $\mathcal{S}^2 \times \mathcal{S}^2$ y la aplicación energía-momento. En este estudio aparecen curvas de bifurcaciones a lo largo de las cuales el sistema se muestra degenerado, dichas curvas están asociadas con el cambio de estabilidad de los equilibrios clásicos en el segundo espacio reducido (sistema de Euler). Por otro lado, también se pone de manifiesto y se estudia en detalle el papel jugado por la triaxialidad del cuerpo.

En el último capítulo la perturbacin contiene al radio y como consecuencia las órbitas obtenidas serán de tipo roseta. Este modelo se asocia a dos tipos de aplicaciones, asteroides y satélites, es decir, en nuestra última aplicación consideramos órbitas elípticas en general; también analizamos las condiciones para que este modelo admita circulares. Este escenario no lleva a considerar órbitas medias en vez de bajas como ocurría en el capítulo anterior. Un ejemplo conocido son las órbitas tipo Molniya, que son candidatas a usar este tipo de intermediario como orden cero en el desarrollo de una teorá de perturbación. El objetivo de este estudio es encontrar un modelo suficientemente simplificado para ser considerado un orden cero, pero que incorpore parte del efecto perturbativo gravitatorio.

Conclusión

En esta tesis se aborda una generalización del sistema clásico de Euler, la solución general conecta con las doce funciones elípticas de Jacobi. Usando esta generalizacion y la fibración tipo Hopf cuaterniónica, se define y estudia en detalle una familia polinómica paramétrica de Hamiltonianos. Sobre dicha familia se llevan a cabo reducciones de tipo geométrico y simpléctico y se muestra que algunos modelos de la mecánica clásica están incluidos para ciertas elecciones de los parámetros. En este sentido, la familia propuesta proporciona un marco de trabajo común para abordar estos modelos clásicos.

En las aplicaciones nos centramos en la modelización de problemas roto-orbitales. El modelo completo requiere el desarrollo de teorías perturbativas para obtener soluciones aproximadas. En este trabajo consideramos algunos candidatos para el orden cero; en la literatura se les conoce como los *intermediarios* y en las aplicaciones se estudian dos de ellos. En particular, presentamos un detallado análisis para el caso en el que el satélite presenta rotación lenta. En este escenario se presentan tipos muy distintos de dinámicas. Para cada misión concreta, el valor de los modelos dependerá de las comparaciones con experimentos numéricos.

Para el caso de radio no constante aparecen un buen número de técnicas de la mecánica clásica a investigar. En este sentido, el capítulo siete es un primer paso que requiere más investigación. Como ejemplo valga la comparación de nuestro enfoque con la eliminación de la paralaje como punto de partida. Un segundo aspecto es la producción de las correspondientes variables de ángulo-acción para los distintos intermediarios. Algunos trabajos parciales ya han sido realizados en este sentido, por ejemplo las variables ángulo-acción en cuatro dimensiones de tipo Delaunay. Estas variables presentan una alternativa al esquema de las teorás perturbativas basadas en las normalizaciones con restricciones; esta es una de nuestras líneas actuales de investigación.

Introduction

This thesis addresses some of the very well known systems in classical mechanics in a uniform manner, considering a homogeneous quartic Hamiltonian as the basic model. Our main target is to develop a common framework to deal with perturbations. Furthermore, we carry out this task from a geometric point of view. As such, the structure of this memoir comprises three parts; preliminaries, the unifying framework and the applications.

Part I. Preliminaries on Classical Mechanics and Geometry: In the first part of this memoir we gather some tools that will be used along our study. The first Chapter sets notation and presents some basic results that can be found in several basic references [Abraham and Marsden, 1985, Marsden and Ratiu, 1999, Meyer et al., 2009]. In the second Chapter we study the extended Euler systems (EES) as an initial value problem with parameters. Particular realizations of EES lead to several Lie-Poisson structures. We consider a six dimensional Poisson structure that fits all of them together. The symplectic stratification of this Lie-Poisson structure uses the first integrals which are elliptic and hyperbolic cylinders, although other quadrics may be used as well. A qualitative study of the solutions is carried out and the twelve Jacobi elliptic functions are shown in a unified way as the solutions of the EES. As a consequence of this setting, the Jacobi's transformation for the elliptic modulus is obtained.

Part II. Hopf Reduction on a Quartic Polynomial Model: In the third Chapter we study a four dimensional generalization of the classical Hopf fibration, which is performed by means of an increase of the manifold's dimension [Gluck et al., 1986]. The application of that fibration in the context of some problems of classical mechanics is not new, see [Cushman and Bates, 1997] and the references therein. We follow the same methodology as in the classical Hopf fibration, but instead of complex numbers the generalization of the classic Hopf map is defined in terms of quaternions. The fourth Chapter uses the components of the quaternionic Hopf map to propose a parametric Hamiltonian function in $\mathbb{T}^*\mathbb{R}^4$ which is an homogeneous quartic polynomial with six parameters, defining an integrable family of Hamiltonian systems [Ferrer and Crespo, 2015]. The key feature of the model is its nested Hamiltonian-Poisson structure, which appears as two extended Euler systems in the reduced equations [Ortega and Ratiu, 2004]. This is fully exploited in the process of integration, where we find two 1-DOF subsystems and a quadrature involving both of them. The generic solution is quasi-periodic, expressed by means of Jacobi elliptic functions and integrals, based on two periods. For suitable choices of the parameters, adding an appropriate regularization when needed, some remarkable classical models such as the Kepler, geodesic flow, 4-D isotropic oscillator and free rigid body systems appear as particular cases. The analysis of the quartic model is performed through a twofold reduction. On the one hand, the system is geometrically reduced and, although we do not provide new theorems in reduction theory or study interesting new groups in mechanics, this model is a detailed example of singular reduction [Ortega and Ratiu, 2004], in which the corresponding Poisson reconstruction is also provided. On the other hand, symplectic reduction by picking new canonical coordinates is examined. More precisely, using Projective Andoyer variables [Ferrer, 2010] that generalize the Andoyer angles to the four dimensional case, the two symmetries of the quartic model show up in the conjugate momenta of the angles. Thus, this variables perform the toral reduction associated to the double symmetry of the parametric system. Moreover, we show the relation between the geometric reduction and the reduction carried out by the Projective Andoyer variables. We provide detailed and explicit formulae for all the changes of variables that are used and therefore we set up the system for the further study of perturbations.

Part III. Applications to Roto-Orbital Dynamics: This part is devoted to the study of the attitude dynamics and the orbital motion of models approximating a generic triaxial spacecraft under gravity-gradient torque perturbation. The full problem is a *non-integrable* Hamiltonian dynamical system, which still remains far from being sorted out and requires the use of perturbation theories for its analysis. Within the context of Poincaré [Poincaré, 1899] and the refinement of Arnold [Arnold et al., 1993] page 185 in the application of KAM theory, a perturbation theory should be developed upon an *integrable and non-degenerate* zero order. Traditionally, this role is embodied by the free rigid body and the Kepler system, but here, rather than the classical ones, which are degenerated (superintegrable [Fassò, 2005]), we also study another candidate for the zero order, the *intermediaries* (see [Deprit, 1981] and the references therein). The idea of the intermediary is to define a simplified integrable system of the problem at stake, where the work of Hill on the Moon motion [Wilson, 2010] is, perhaps, the best known example. In the fifth Chapter we recall the concept of intermediary, present five of them and we set a common methodology. For this purpose, the framework developed for the quartic polynomial model is fully exploited. The simplified system includes part of the full potential where the roto-orbital coupling is present. That is, the perturbing potential is split $\mathcal{P} = \mathcal{P}_0 + \mathcal{P}_1$ and the new zero order is given by

$$\mathcal{H}_0 = T_O + T_R + \mathcal{P}_0, \tag{2}$$

in such a way, that the system defined by $\mathcal{H}_0$ is integrable. Sixth and seventh Chapters take advantage of the previous framework considering two intermediary models defined in the previous chapter. Those intermediaries are assumed to be in circular and elliptic orbits respectively. Partial results have been already sent for possible publication [Ferrer et al., 2014, Molero et al., 2014] and we refer to those preprints for more details.

We study relative equilibria and bifurcations of the circular intermediary in chapter six. This intermediary model defines a Poisson flow over a large parameter space: three physical parameters (principal moments of inertia) and three distinguished parameters, namely: the modulus of the angular momentum vector (M), its third component in a space rotating frame (ω_9), and the orbital mean motion (n), which are integrals of the model. In the case of slow rotational motion we identify conditions under which different bifurcations of the classical unstable trajectories occur, being those scenarios of great interest in relation to stabilization and control purposes. Our study is based on the use of the invariants defining the $\mathbb{S}^2_M \times \mathbb{S}^2_M$ reduced space and the associated energy-momentum mapping. We find bifurcation curves along which the system shows degeneracy, connected with the change of stability of the classical unstable equilibria of the second reduced space (Euler system). The role played by the triaxiality is also shown.

The final chapter examines a body moving in a rosette-like orbit. More precisely we are thinking about two types of applications, namely to artificial satellites or asteroids around a planet. In other words, we consider perturbed elliptic orbits in general; we also investigate conditions for which this model admits the circular ones. This scenario leads to medium orbits rather than to the low type of orbits studied in the preceding chapter. A very well-known example of these are the Molniya orbits, which are candidates to use the elliptic gravity-gradient as a zero order in the context of perturbation theory. The intention of this study is to analyze a model simply enough to be considered as an alternative zero order, but incorporating partially the effects of the gravity torque perturbation.

Conclusions

The main conclusion of this Memoire may be summarized as follows. A generalized study of the classical Euler system is presented, connecting its solutions with the twelve Jacobi elliptic functions. Using that and the quaternionic Hopf fibration a quartic homogeneous polynomial parametric family is proposed and studied in detail. Geometric and symplectic reductions are performed in the family. It is shown that, for suitable choices of parameters, several classical mechanical systems arise as family realizations and we provide a common framework to study them.

In the application we focus on modeling problems in the roto-orbital dynamics. The full model is a non-integrable problem which requires the development of perturbation theories in order to obtain approximate solutions. Several candidates for the zero order term, on which the whole theory relies, are considered in this context. They are known as *intermediaries* in the literature and we explore two of them. We analyze the role played by the integrals and the relation with the physical parameters involved. In particular, we present a fairly complete analysis of the case when the satellite has slow rotation, which presents several rather different types of dynamics. For each mission, the relative value of each model will finally depend on numerical experiments.

When the radius is not constant, there is a number of techniques of classical mechanics to be considered and the last chapter is just a preliminary step to do more research. As an example we mention the comparison of our approach with the elimination of the parallax as the starting point. A second aspect could be the production of the corresponding action-angle variables for the different intermediaries, partial work has been done. In particular, results have been obtained where 4-D Delaunay action-angle variables are produced. They represent an alternative for perturbation theories to the scheme of the constraint normalization proposed by others; this is one of our present lines of research.

Part I

Preliminaries on Classical Mechanics and Geometry

Chapter 1

Poisson and Symplectic Geometry

Along this preliminary chapter a minimal theoretical frame is given, with this aim, some basic concepts, facts and notation is set. Emphasis is laid upon the concept of reduction, which is exploited in the second part of this memoir. The proofs of the following well known results are referred to [Abraham and Marsden, 1985, Arnold and Givental, 1973, Berndt, 2001, Marsden and Ratiu, 1999], they provide the reader with plenty of details.

Later on, in the applications, we focus on examples coming from the classical celestial mechanics. Those systems are usually expressed in Hamiltonian formulation. Next we recall the classic definition of a Hamiltonian system, which afterwards is revisited under the more general theory of symplectic and Poisson manifolds.

Definition 1.1 (Hamiltonian system)**.** *A system of ordinary first order differential equations on the open set $E \subset \mathbb{R}^{2n}$ is said to be a Hamiltonian system if may be expressed in the following way*

$$\begin{aligned} \frac{dq_i}{dt} &= \frac{\partial \mathcal{H}(q,p)}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial \mathcal{H}(q,p)}{\partial q_i}, \quad i = 1, \cdots, n, \end{aligned} \tag{1.1}$$

where $\mathcal{H}(q,p)$ is a differentiable real valued function called the Hamiltonian function, E is the phase space and n are the degrees of freedom. This system may also be defined as follows

$$X_{\mathcal{H}}(q,p) = J\, d\mathcal{H}(q,p), \tag{1.2}$$

where$X_{\mathcal{H}}$ is the Hamiltonian vector field, J is the symplectic matrix

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},$$

0_n *and* I_n *are the* $n \times n$ *zero and identity matrices respectively.*

1.1 Geometric Hamiltonian formulation

Very often the phase space of a mechanical system lies on a symplectic or Poisson manifold, for example, for the case of the free rigid body we have $T^*SO(3)$. Thus, the Hamiltonian formulation in terms of the definition (1.1) is not applicable. Therefore, the concept of Hamiltonian system needs to be extended to those kind of manifold. We define the fundamental objects to study dynamics on manifolds and focus on the particular type of symplectic manifolds.

We start by recalling some basic concepts in differential geometry. For a complete treatment see [Abraham and Marsden, 1985, Abraham et al., 1988, Hirsch, 1994]. A differentiable or smooth manifold M is the generalization of a surface in $\mathbb{R}^3$. It may be conceived as an union of pieces of $\mathbb{R}^n$ which fix together in a nice way, that is to say, the manifold is an object without borders and the union of those pieces is smooth.

Definition 1.2 (Manifold)**.** *A differentiable manifold or smooth manifold of dimension* n *is a set* M *and a family of bijective mappings* $x_p : U_p \to x_p(U) \subseteq M$ *of open sets* $U_p \subseteq \mathbb{R}^n$ *into* M *such that*

(i) $\bigcup_p x_p(U_p) = M$.

(ii) For any pair p, q *with* $x_p(U_p) \cap x_q(U_q) \neq \emptyset$*, the sets* $x_p^{-1}(U_p)$ *and* $x_q^{-1}(U_q)$ *are open sets in* $\mathbb{R}^n$ *and the mappings* $x_q^{-1} \circ x_p$ *and* $x_p^{-1} \circ x_q$ *are differentiable.*

(iii) The family $\{(U_p, x_p)\}$ *is maximal relative to the conditions* (i) *and* (ii).

The pair (U_p, x_p) *is called a chart, parametrization or system of coordinates of* M *at* p, $x_p(U_p)$ *is a coordinate neighborhood. A family* $\{(U_p, x_p)\}$ *satisfying conditions* (i) *and* (ii) *is said to be an atlas. When the condition* (iii) *of maximality is added, we say that this family* $\{(U_p, x_p)\}$ *is a differentiable structure on* M.

Two charts (U_p, x_p) *and* (U'_p, x'_p) *are called compatible when the transformation functions* $x_p^{-1} \circ x'_p$ *and* $x_p^{'-1} \circ x_p$ *are differentiable on* $x_p^{'-1}(x_p(U_p) \cap x'_p(U'_p))$ *and* $x_p^{-1}(x_p(U_p) \cap x'_p(U'_p))$ *respectively.*

In $\mathbb{R}^n$, given two k-dimensional surfaces, not necessarily with the same dimension, there exists the concept of a differentiable map between them. Next we recall the corresponding concept in the general case of differentiable manifolds.

Definition 1.3. *A mapping $f : M \to N$ between manifolds is said to be $\mathcal{C}^k$ if for each $x \in M$ and one (equivalently: any) chart (V_q, y_q) on N with $f(p) \in V_q$ there is a chart (U_p, x_p) on M with $p \in U_p$, $f(U_p) \subseteq V_q$, and $y_q \circ f \circ x_p^{-1}$ is $\mathcal{C}^k$. We will denote by $\mathcal{C}^k(M, N)$ the space of all $\mathcal{C}^k$-mappings from M to N.*

A $\mathcal{C}^k$-mapping $f : M \to N$ is called a $\mathcal{C}^k$-diffeomorphism if $f^{-1} : N \to M$ exists and is also $\mathcal{C}^k$. Two manifolds are called diffeomorphic if there exists a diffeomorphism between them. This is the basic equivalence relation of differential topology. From differential topology, see [Hirsch, 1994], we know that if there is a $\mathcal{C}^1$-diffeomorphism between M and N, then there is also a $\mathcal{C}^\infty$-diffeomorphism.

Definition 1.4 (Immersions and Submersions)**.** *Let M^m and N^n be two smooth manifolds of dimension m and n respectively. A differentiable mapping $f : M \to N$ is said to be a submersion if $df_p : T_pM \to T_{f(p)}N$ is surjective for all $p \in M$. If $df_p : T_pM \to T_{f(p)}N$ is injective for all $p \in M$, then f is said to be an immersion. If, in addition, f is a homeomorphism onto its image $f(M) \subset N$ endowed with the induced topology, we say that f is an embedding. If $M \subset N$ and the inclusion $i : M \to N$ is an embedding, we say that M is a submanifold of N.*

Definition 1.5 (Fibered Manifolds and Sections)**.** *A triple (M, p, N), where $p : M \to N$ is a surjective submersion, is called a fibered manifold. The manifold M is called the total space and N is called the base. A fibered manifold admits local sections: For each $x \in M$ there is an open neighborhood U of $p(x)$ in N and a smooth mapping $s : U \to M$ with $p \circ s = Id_U$ and $s(p(x)) = x$.*

A dynamical system, also called a flow, is a description of how a state develops into another state over the course of time. Technically, it is a smooth action of $\mathbb{R}$ or $\mathbb{Z}$ on another object (usually a manifold). The system is called a continuous dynamical system if $\mathbb{R}$ acts, and a discrete dynamical system when $\mathbb{Z}$ acts.. Flows and vector fields are closely related, a dynamical system is defined by means of a vector fields.

Definition 1.6 (Vector Fields and Dynamical Systems)**.** *A vector field X on a manifold M is a smooth section of the tangent bundle; so $X : M \to TM$ is smooth and $\pi_M \circ X = Id_M$. A local vector field is a smooth section which is defined on an open subset only. We denote the set of all vector fields by $\mathfrak{X}(M)$. With pointwise addition and scalar multiplication $\mathfrak{X}(M)$ becomes a vector space.*

Let M be a manifold and $X \in \mathfrak{X}(M)$. A dynamical system *is given by the flow F_t*

uniquely determined by the differential equation

$$\frac{d}{dt}F_t(m) = X(F_t(m)). \tag{1.3}$$

Let $\psi(t) : M \to M$, where $t \in A \subseteq \mathbb{R}$, be a solution of (1.3). It gives rise to an action of A on the manifold M

$$\psi : A \times M \to M.$$

The parametrized curve $\psi_m(t) := \psi(t, m)$, where $t \in \mathbb{R}$ and $m \in M$ is fixed, is the *orbit* through m, and the oriented but unparameterized curve $\psi_m(t)$ is called an *trajectory*. When $A = \mathbb{R}$ it is said that the flow is *complete* and the family $\{\psi(t)\}_{t\in\mathbb{R}}$ determines a group of diffeomorphisms of M. An *equilibrium point* (*rest point, critical point, stationary point*) is a $m \in M$ such that $X(F_t(m)) = 0$. It gives rise to an *equilibrium solution* or just an *equilibrium*, that is $\psi_t(m) = m$. A periodic orbit of period τ is a solution defined as follows

$$\psi_{t+\tau}(m) = \psi_t(m), \quad \forall \tau \in \mathbb{R}. \tag{1.4}$$

1.1.1 Symplectic manifolds. Hamilton equations

Locally, a manifold may be thought of as a real vector space $\mathbb{R}^n$ and usually, this vector space has some additional structure. If this additional structure is a scalar product, we are lead to a Riemannian manifold; if this additional structure is a symplectic form, we obtain a symplectic manifold. The material for this part is taken form [Abraham et al., 1988, Berndt, 2001, Marsden and Ratiu, 1999].

The generalization of the Euclidean spaces to manifolds is endowed with the concept of differential forms. These objects are the key to obtain the basic operations of vector calculus, *div, grad* and *curl* and the integral theorems of *Green, Gauss* and *Stokes* in arbitrary manifolds.

Definition 1.7 (Tensor)**.** *A (r, s)-tensor A on the manifold M is a $C^\infty(M)$-multilineal map*

$$A : \mathfrak{X}^*(M)^s \times \mathfrak{X}(M)^r \to C^\infty(M). \tag{1.5}$$

The set of all (r, s)-tensors is denoted by $\mathcal{R}^r_s(M)$. When $r = 0$ we say that A is a covariant tensor of order s. For the case $s = 0$, A is said to be a contravariant vector of order r. A multilinear map is said to be skew-symmetric (or alternating) when it changes sign whenever two of its arguments are interchanged.

Some remarkable examples are the following:

The map $E : \mathfrak{X}^*(M) \times \mathfrak{X}(M) \to C^\infty(M), (\theta, X) \to E(\theta, X)$ given by $E(\theta, X) = \theta(X)$ is a $(1,1)$-tensor.

A vector field $X \in \mathfrak{X}(M)$ may be considered as a $(1,0)$-tensor by means of the following map

$$X : \mathfrak{X}^*(M) \to C^\infty(M),\ \theta \to X(\theta) = \theta(X).$$

Definition 1.8 (Differential k-Form). *A differential k-Form on a manifold M is a skew-symmetric covariant tensor of order k.*

Let f be a $C^\infty(M)$ function, the differential df is a $(0,1)$-tensor or a 1-form given by

$$df : \mathfrak{X}(M) \to C^\infty(M),\ X \to df(X),$$

the function f itself may be considered as a $(0,0)$-tensor or a 0-form.

Given $X \in \mathfrak{X}(M)$ and ω a k+1-form we define $i_X\omega$ as the k-form given by

$$i_X\omega(X_1, \ldots, X_k) = \omega(X, X_1, \ldots, X_k).$$

We call $i_X\omega$ the inner product of X and ω.

Definition 1.9 (Differential Symplectic Form). *A symplectic form is a non-degenerate skew-symmetric bilinear form. The non-degeneracy means that for all $v \in V$ such that $v \neq 0$, there exists $w \in V$ such that $\omega(v, w) \neq 0$.*

Definition 1.10 (Pull-Back and Push-Forward). *Let $f : M \to N$ be a $\mathcal{C}^\infty$ map between manifolds and let α be a k-form on N. Define the pull-back $f^*\alpha$ of α by f to be the k-form on M given by*

$$f^*\alpha : \mathfrak{X}(M)^k \to C^\infty(M),\ (v_1, \ldots, v_k) \to f^*\alpha(v_1, \ldots, v_k) = \alpha(df(v_1), \ldots, df(v_k)).$$

If f is a diffeomorphism, the push-forward f_* is defined by $f_* = (f^{-1})^*$.

The pull-back or push-forward operations f^* and f_* when applied to a 0-form, *i.e.* a function g, are reduced to the composition operation. More precisely, if $g \in \mathcal{C}^\infty(N)$, then $f^*g = g \circ f$ and if $g \in \mathcal{C}^\infty(M)$ we have that $f_*g = f \circ g$.

Definition 1.11 (Symplectic Manifold). *The pair (M, ω) is called a symplectic manifold, where M is a smooth manifold of dimension n and ω is a symplectic form.*

The nondegeneracy of a differential 2-form ω means that the corresponding homomorphism between the tangent bundle and its cotangent bundle

$$\omega^\flat : TM \to T^*M$$

which associates to each vector X the covector $-i_X\omega$, is an isomorphism. Observe that the restriction to each $p \in M$ makes the tangent space T_pM become a symplectic vector space, which is even dimensional. Therefore, M is always an even dimensional manifold.

A remarkable example of a symplectic manifold is the case of the cotangent bundle. Let Q be a n-dimensional smooth manifold and T^*Q the cotangent bundle associated, with cotangent coordinate chart given by $(q_1, ..., q_n, p_1, ..., p_n)$. Thus the 2-form defined as follows is a symplectic form

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i = dq \wedge dp,$$

and (T^*Q, ω) is a symplectic manifold.

Definition 1.12 (Hamiltonian dynamical system). *Let (M, ω) be a symplectic manifold and $\mathcal{H} : M \to \mathbb{R}$ a $\mathcal{C}^k$ function. The triad (M, ω, H) is a Hamiltonian dynamical system. Note that the nondegeneracy of the symplectic form ω associates naturally to each Hamiltonian system a vector field $X_{\mathcal{H}}$ defined by $\omega(X_{\mathcal{H}}, Y) = d\mathrm{H}(Y)$, which is called the Hamiltonian vector field and* H *is the Hamiltonian function.*

Next we are lead to the definition of a symplectomorphism, which is indispensable to study the equivalence between symplectic structures.

Definition 1.13 (Symplectomorphism). *A symplectomorphism, canonical map or symplectic transformation between the symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is a C^∞ mapping $F : M_1 \to M_2$ such that*

$$F^*\omega_2 = \omega_1.$$

Remark 1.1. *Since a symplectomorphism is volume preserving, its Jacobian determinant is equal to one, thus by the inverse function theorem F is a diffeomorphism.*

The classification of symplectic manifolds up to symplectomorphisms is solved locally. The Darboux Theorem states that the dimension is the only local invariant.

It also shows that the definitions 1.1 and 1.12 are equivalent in the open domain U.

Theorem 1.1 (Darboux's). *To every point (q,p) of a $2n$-dimensional symplectic manifold (M,ω), there correspond an open neighborhood U of (q,p) and a smooth map $F: U \to \mathbb{R}^{2n}$ satisfying*

$$F^*\omega_0 = \omega|_U,$$

where ω_0 is the standard symplectic form on $\mathbb{R}^{2n}$.

An equivalent formulation of the Darboux's Theorem reads as follows. For every $(q,p) \in M$ there is a chart (U,ψ), with $\psi(m) = (x,y) \in \mathbb{R}^{2n}$, $m \in U$, such that $\psi(q,p) = 0$ and

$$\omega|_U = dx \wedge dy.$$

The above chart (U,ψ) is called canonical coordinates. By using these coordinates, the integral curves associated to the Hamiltonian vector field $X_{\mathcal{H}}$ given in definition 1.12, are determined by the classic Hamilton equations, see definition 1.1.

1.1.2 Poisson manifolds. Hamilton equations

A Poisson manifold is a smooth manifold with a Poisson bracket defined on its function space, which appears as a natural generalization of symplectic manifolds. They are endowed with the minimum features inherited from the symplectic manifold to define a Hamiltonian system. In this section we gather some basic facts and concepts taken from [Abraham and Marsden, 1985, Marsden and Ratiu, 1999, Meyer et al., 2009].

Definition 1.14. *A **Poisson bracket** (or **Poisson structure**) on a manifold M is a bilinear operation $\{\ ,\ \}$ on $\mathcal{F}(M) = C^\infty(M)$ such that:*

(i) *It is skew-symmetric, $\{F, G\} = -\{G, F\}$*

(ii) *The Jacobi identity holds $\{F,\{G, H\}\} + \{G, \{F, H\}\} + \{H, \{F, G\}\} = 0$*

(iii) *$\{\ ,\ \}$ is a derivation in each factor, that is, Leibniz rule is satisfied in each factor*

$$\{FG, H\} = \{F, H\}G + F\{G, H\}, \qquad \forall\, F, G, H \in \mathcal{F}(M).$$

A manifold M endowed with a Poisson bracket is called a Poisson manifold.

It can be shown that a symplectic manifold (M, ω) is a Poisson manifold. The Poisson bracket in M is defined by the symplectic form in the following way

$$\{F,\, G\} = \omega(X_F,\, X_G) = X_G\,[F] \qquad \forall\; F,\, G \in \mathcal{F}(M).$$

Note that in canonical coordinates, the Poisson bracket obtaines the traditional form

$$\{F, G\} = \sum_{i=1}^{n} \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p^i} - \frac{\partial G}{\partial q^i}\frac{\partial F}{\partial p^i}.$$

Definition 1.15. *If $(M, \{\ ,\ \})$ is a Poisson manifold, due to the derivation property of the Poisson bracket, the value of the bracket $\{F, G\}$ at $z \in M$ depends on F only through $dF(z)$, idem for G. Thus, there exists a contravariant antisymmetric 2-tensor*

$$W : T^*M \times T^*M \longrightarrow \mathbb{R} : (dF, dG) \longrightarrow \{F, G\}.$$

A symplectic form is a covariant antisymmetric two tensor which is nondegenerate. The tensor W is called a ***cosymplectic*** *or* ***Poisson structure****. In local coordinates $(x_1, \cdots, x_n)$, W is determined by its matrix elements $W_{ij} = \{x_i, x_j\}$, and the bracket becomes*

$$\{F, G\} = \sum (\frac{\partial F}{\partial x_i})(\frac{\partial G}{\partial x_j})W_{ij}$$

Definition 1.16 (Poisson morphisms)**.** *Let $(M, \{\cdot,\, \cdot\}_M)$ and $(N, \{\cdot,\, \cdot\})_N$ be two Poisson manifolds. A map $f : M \to N$ is called a Poisson morphism or Poisson map if it preserves the Poisson bracket,*

$$\{f^*g,\, f^*h\}_M = f^*\{g,\, h\}_N, \quad \forall g, h \in \mathcal{C}^\infty(N).$$

A **Lie algebra** is a vector space V with a bilinear form satisfying properties **(i)** and **(ii)** from definition 1.14. Thus $\mathcal{F}(M)$ endowed with a Poisson bracket is a Lie algebra, which is called a **Poisson algebra**.

Next we extend the notion of a Hamiltonian vector field from the symplectic to Poisson context and we establish the equations of motion. By fixing one of the factors in the Poisson bracket, we obtain the following linear application

$$ad_F : \mathcal{F}(M) \longrightarrow \mathcal{F}(M) \;:\; G \longrightarrow \{F,\, G\},$$

because of the **(ii)** property of definition 1.14. This application is a derivation called the adjoint representation of $\mathcal{F}(M)$. By means of the identification between vector fields on M and derivations there is a unique element $X_F \in TM$ such that

$$X_F = ad_F,$$

this vector field is called the **Hamiltonian vector field of** H. Let φ_t be the flow of X_F, then the variation of any function $F \in \mathcal{F}(M)$ along the flow φ_t is given by

$$\frac{d}{dt}(F \circ \varphi_t) = \{F, H\} \circ \varphi_t = \{F \circ \varphi_t, H\},$$

which are the equations in Poisson bracket form often written as $\dot{F} = \{F, H\}$. Therefore, $H \circ \varphi_t = H$ and $F \in \mathcal{F}(M)$ is constant along the integral curves of X_H if and only if $\{F, H\} = 0$. The triplet $(M, \{\ ,\ \}, H)$ is called a Poisson dynamical system.

Definition 1.17. *A function $C \in \mathcal{F}(M)$ such that $\{C, F\} = 0$ for all $F \in \mathcal{F}(M)$ is said to be a* ***Casimir function*** *of the Poisson structure. The space of Casimir functions is going to be denoted by $\mathcal{C}(\mathcal{M})$ and they form the center of the Poisson algebra.*

Remark 1.2. *The unit element of the algebra $(\mathcal{F}(M), \{\ ,\ \})$ is always a Casimir. If all the constant functions in the algebra $(\mathcal{F}(M), \{\ ,\ \})$ are Casimirs, we say that such an algebra is a non degenerate algebra.*

Remark 1.3. *Every Casimir C is constant along the flow of all Hamiltonian vector fields, that is, $X_C = 0$.*

It happens that a symplectic manifold induces a Poisson structure, through the symplectic form, thus any Hamiltonian system is provided with a natural Poisson structure. The following proposition analyses the process in the inverse direction.

Proposition 1.2. *A Poisson manifold M is symplectic if and only if the Poisson structure matrix is invertible.*

Symplectic stratification theorem. The following statement shows that every Poisson manifolds is a union of symplectic manifolds, each of which is a Poisson submanifold called symplectic leaf.

Definition 1.18. *Let M be a Poisson manifold. Two points $p_1, p_2 \in M$ are equivalent if they can be connected by a trajectory of a locally Hamiltonian vector field. The corresponding equivalence class is called a symplectic leaf.*

Thus a finite dimensional Poisson manifold M can be obtained as a disjoint union of its symplectic leaves. Each leaf is a symplectically immersed Poisson submanifold, and the induced Poisson structure on the leaf is symplectic. The dimension of the leaf through $m \in M$ equals the rank of the Poisson structure at m.

Casimir functions are constant on every symplectic leaf of the Poisson manifold M, which may be obtained as the level set of C. Notice that, even if all the Casimir functions are constant, the Poisson structure can still be degenerate.

1.2 On Lie Groups, Lie Algebras and Actions

Lie group theory is a wide field in the contemporary mathematics. Here only a few facts and concepts are presented. Further details are found in several references. A detailed treatment of this subject is given in [Arnold, 1989, Marsden and Ratiu, 1999, Weyl, 1939]. For a very practical approach see [Shapukow, 1989], where a complete collection of exercises with solutions can be found.

1.2.1 Lie groups and Lie algebras

Definition 1.19. *A Lie group is a group that is also differentiable manifold G, and for which the following operations, as a group, are smooth*

$$G \times G \longrightarrow G,\ (g_1, g_2) \mapsto g_1 g_2,$$

$$G \longrightarrow G,\ g \mapsto g^{-1}.$$

In what follows we assume the same notation than in [Marsden and Ratiu, 1999]. Let L_g and R_h be denote *left and right translation maps*, which are given by

$$L_g : G \longrightarrow G, h \mapsto gh, \text{ and } R_h : G \longrightarrow G, g \mapsto gh.$$

Definition 1.20. *A Lie algebra is a vector space V together with a bilinear skew-symmetric operation $[\ ,\ ] : V \times V \longrightarrow V$, which satisfies the Jacobi identity*

$$[[u,v]\,,w] + [[v,w]\,,u] + [[w,u]\,,v] = 0.$$

The operation $[\ ,\ ]$ is often called the Lie or Poisson bracket.

Every Lie group has associated a Lie algebra. It consists of a vectorial space $V = T_eG$ together with a bracket operation. In order to obtain this bracket on T_eG we first introduce the *left invariant vectors on G*. A vector field X is said to be left invariant when, for every $g \in G$ we have that $L_g^* X = X$ or equivalently when the following equality holds for every $h \in G$

$$T_h L_g X(h) \;=\; X(gh).$$

The commutative diagram holds

$$\begin{array}{ccc} G & \xrightarrow{\;L_g\;} & G \\ {\scriptstyle X}\big\downarrow & & \big\downarrow{\scriptstyle X} \\ TG & \xrightarrow[\;TL_g\;]{} & TG \end{array}$$

then, we denote the set of left invariant vector fields on G by $\mathfrak{X}_L(G)$. Note that if $g \in G$ and $X,\ Y \in \mathfrak{X}_L(G)$, then

$$L_g^*\,[X,Y] = \left[L_g^*X, L_g^*Y\right] = [X,Y]\,,$$

so $[X,Y] \in \mathfrak{X}_L(G)$ and $\mathfrak{X}_L(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$. This is the key to induce a structure of Lie algebra on T_eG. By mean of the isomorphism between the vectorial spaces $\mathfrak{X}_L(G)$ and T_eG through the aplication

$$\begin{array}{lll} T_eG: & \longrightarrow & \mathfrak{X}_L(G) \\ \xi & \longrightarrow & X_\xi(g) = T_eL_g(\xi) \end{array}$$

we may define the brackets in T_eG as $[\xi,\eta] = [X_\xi, Y_\eta]\,(e)$. Note that by construction we have that

$$[X_\xi, Y_\eta] = X_{[\xi,\eta]}.$$

Definition 1.21. *The above construction provides a structure of Lie algebra on T_eG, then $(T_eG, [\ ,\ ])$ is called the Lie algebra of G and is denoted by $\mathfrak{g}$.*

Summarizing it up, we have for each $\xi \in \mathfrak{g}$ an invariant vector field associated X_ξ, and by mean of the integral curves of X_ξ we obtain a one parameter subgroup in G, it is

$$\xi \to X_\xi \to [\gamma_\xi : \mathbb{R} \to G],$$

where γ_ξ is the unique integral curve through $e \in G$, then $\{\gamma_\xi(t)\}_{t\in\mathbb{R}}$ is a one parameter subgroup.

Definition 1.22. *The exponential map $exp : \mathfrak{g} \longrightarrow G$ is defined by*

$$exp(\xi) = \gamma_\xi(1).$$

Note that $exp(s\xi) = \gamma_\xi(s)$.

Definition 1.23. *Let M and N be manifolds and let G be a Lie group that acts on M by Φ_g and on N by Ψ_g. A smooth map $f : M \longrightarrow N$ is called equivariant with respect to these actions if for all $g \in G$,*

$$f \circ \Phi_g = \Psi_g \circ f,$$

that is, the diagram commutes.

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ \Big\downarrow{\scriptstyle \Phi_g} & & \Big\downarrow{\scriptstyle \Psi_g} \\ M & \xrightarrow{\ f\ } & N \end{array}$$

Among the groups that most commonly appear acting on manifolds in classical mechanics, we found the matrix groups, also called the classical groups. Here we introduce these groups and calculate their Lie algebras associated.

$GL(n, \mathbb{R})$ The general linear group is the group of all invertible, $n \times n$ real matrices. The Lie algebra associated is $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$, the space of all $n \times n$ real matrices.

$GL^+(n, \mathbb{R})$ The $n \times n$ real matrices with positive determinant, isomorphic to $GL^-(n, \mathbb{R})$ and double covered by $GL(n, \mathbb{R})$. The Lie algebra associated is $\mathfrak{g} = \mathfrak{gl}^+(n, \mathbb{R})$, the space of al square, $n \times n$ and real matrices.

$SL(n, \mathbb{R})$ The special linear group, real group of matrices with determinant one. Let us study the Lie algebra associated. Let us consider $X \in SL(n, \mathbb{R})$, then $X = \exp(x)$ for some $x \in \mathfrak{gl}(n, \mathbb{R})$. The condition that $X \in SL(n, \mathbb{R})$ is equivalent to $detX = det \exp(x) = 1$. By finding a basis in which x is upper-triangular, it can be proved that $det \exp(x) = \exp(tr(x))$, therefore $\exp(x) \in SL(n, \mathbb{R})$ if and only if $tr(x) = 0$. Thus, we have that $\mathfrak{sl}(n, \mathbb{R})$ is given by

$$\mathfrak{sl} = \{x \in \mathfrak{gl}(n, K) | tr(x) = 0\}.$$

$O(n, \mathbb{R})$ The orthogonal group, is the group of real orthogonal matrices, *i.e.* $X\,X^t = Id_n$. This is the symmetry group of the sphere ($n = 3$) or hypersphere. For the Lie algebra we consider the defining property, then $X,\ X^t$ commute. Writing $X = \exp(x),\ X^t = \exp(x^t)$, we see that $x,\ x^t$ also commute, and thus $\exp(x) \in O(n)$ implies $\exp(x)\exp(x^t) = \exp(x + x^t) = 1$, so $x + x^t = 0$; conversely, if $x + x^t = 0$, then $x,\ x^t$ commute, so we can reverse the argument to get $\exp(x) \in O(n, \mathbb{R})$. Thus, in this case the theorem also holds, with

$$\mathfrak{o} = \{x | x + x^t = 0\},$$

the space of skew-symmetric matrices.

$SO(n, \mathbb{R})$ The special orthogonal group, is the group or real orthogonal matrices with determinant one. $SO(2)$ is isomorphic to the circle group, $SO(3)$ is the rotation group of the sphere. For the Lie algebra In this case, we should add to the condition $X\,X^t = 1$ (which gives $x + x^t = 0$) also the condition $\det(X) = 1$, which gives $tr(x) = 0$. However, this last condition is unnecessary, because $x + x^t = 0$ implies that all diagonal entries of x are zero. So both $O(n)$ and $SO(n)$ correspond to the same space of matrices

$$\mathfrak{o} = \mathfrak{so} = \{x | x + x^t = 0\}.$$

This might seem confusing until one realizes that $SO(n)$ is exactly the connected component of identity in $O(n)$. Thus, the neighborhood of 1 in $O(n)$ coincides with the neighborhood of 1 in $SO(n)$.

$U(n, \mathbb{R})$, $SU(n, \mathbb{R})$ Group of unitary matrices and special unitary group, unitary matrices with determinant 1. A similar argument shows that $\exp x \in U(n) \Leftrightarrow x + x^* = 0$, (where $x^* = \bar{x}^t$) and $\exp x \in SU(n) \Leftrightarrow x + x^* = 0,\ tr(x) = 0$. Note that in this case, $x + x^*$ does not imply that x has zeros on the diagonal: it only implies that the diagonal entries are purely imaginary. Thus, $tr(x) = 0$ does not follow automatically from $x + x^* = 0$, so in this case the tangent spaces for $U(n)$, $SU(n)$ are different.

$Sp(2n, \mathbb{R})$ Symplectic group; real symplectic matrices. Similar argument shows that $\exp(x) \in Sp(2n, K) \Leftrightarrow x + JxtJ^{-1} = 0$; thus, in this case the theorem also holds.

1.2.2 Actions of Lie groups

Definition 1.24. *Let M be a manifold and let G be a Lie group. A left action of a Lie group G on M is a smooth mapping $\Phi : G \times M \longrightarrow M$ such that:*

(i) $\Phi(e, x) = x$ *for all* $x \in M$

(ii) $\Phi(g, \Phi(h, x)) = \Phi(gh, x)$ *for all* $g,\ h \in G$ *and* $x \in M$.

in a analogous way can be defined a right action.

Important examples of group action are the following actions of G on itself:

- $L_g\ :\ G \longrightarrow G\ ,\ h \mapsto gh$, the left action map.
- $R_{g^{-1}}\ :\ G \longrightarrow G\ ,\ g \mapsto hg^{-1}$, the right action map.
- $Ad_g\ :\ G \longrightarrow G\ ,\ h \mapsto ghg^{-1}$, the adjoint action.

Remark 1.4. *Left and right actions commute and* $Ad_g = L_g R_{g^{-1}}$.

Since the adjoint action preserves the identity element $e \in G$, it also defines an action of G on the space T_eG, which is also named the adjoint action. Abusing the notation, this action is denoted by

$$Ad_g\ :\ T_eG \longrightarrow T_eG,\ \xi \mapsto \frac{\partial Ad_{g|h=e}(\xi)}{\partial g}.$$

Then, Ad_g denotes the adjoint action of G, but the context is what determines where G is acting. Considering the dual aplication $Ad_g^*\ :\ T_e^*G \longrightarrow T_e^*G$ we obtain an action over the cotangent space at $e \in G$, it is called the co-adjoint action.

For each element $m \in M$ the *orbit of m under the action of Φ* is defined as the set

$$\mathcal{O}_m = \{\Phi_g(m) |\, g \in G\},$$

the tangent space of $\mathcal{O}_m$ is given by $T_e\phi_m(\mathfrak{g})$. There are also two important concepts to define, the *isotropy group* and the *invariant group of an action*. Given $m \in M$

$$G_m = \{g \in G | \Phi_g(m) = m\},$$

is the isotropy group relative to m. The invariant group of an action is the kernel of the action, that is, the intersection of all the isotropy groups

$$G' = \bigcap_{m \in M} G_m = \{g \in G \,|\, \Phi_g(m) = m,\ \forall m \in M\}.$$

The geometric reduction is reviewed in the following section. It is performed by moving to the quotient space of orbits through the momentum map, which in order to provide a reduction has to be equivariant. In general M/G is just a topologic space. To assure that the quotient space is a manifold we have to impose some conditions to the action.

Definition 1.25. *With the above notation, an action is said to be:*

1. *Proper if the mapping $\tilde{\Phi} : G \times M \longrightarrow M \times M$ defined by $\tilde{\Phi}(g,x) = (x, \Phi(g,x))$, is proper,* i. e.*, $\tilde{\Phi}$ is continuous and the anti-image of a compact set is compact.*
2. *Transitive if there is only one orbit or, equivalently, if for every $x,\ y \in M$ there is a $g \in G$ such that $g \cdot x = y$. One easily sees that left and right actions are transitive.*
3. *Effective or faithful, if $\Phi_g = id_M$ implies $g = e$, that is, $g \mapsto \Phi_g$ is one to one.*
4. *Free if it has no fixed points, that is, $\Phi_g(x) = x$ implies $g = e$. Note that an action is free iff $G_x = \{e\}$, for all $x \in M$ and every free action is faithful.*

Theorem 1.3. *If Φ is free and proper, then the orbital space M/G is a manifold and $\pi : M \longrightarrow M/G$ is a smooth submersion.*

Definition 1.26. *Let $\Phi : G \times M \longrightarrow M$ be an action. For $\xi \in \mathfrak{g}$, the map $\Phi^\xi : \mathbb{R}^+ \times M \longrightarrow M$ defined by*

$$\Phi^\xi(t,x) = \Phi(\exp\, t\xi,\ x)$$

is an $\mathbb{R}$-action on M. In other words, $\Phi_{\exp\, t\xi} : M \longrightarrow M$ is a flow on M. The corresponding vector field on M given by

$$\xi_M(x) := \frac{d}{dt}|_{t=0}\Phi_{\exp\, t\xi}(x)$$

is called the infinitesimal generator of the action corresponding to ξ.

Remark 1.5. *The infinitesimal generator of the action corresponding to ξ corresponds with the differentiate of the action Φ with respect to g at the identity in the direction ξ.*

1.3 Reduction Theory

The word reduction is used in a wide sense. However, it is always related with a simplification of the system at hand in terms of a decrease in the number of variables needed. Next, we explore two different approaches. There are many books where this material can be found, some basic references are the following [Abraham and Marsden, 1985, Arnold, 1989, Marsden and Ratiu, 1999].

1.3.1 Conserved quantities and Momentum maps. Noether theorem

Symmetries associated to Hamiltonian system are exhibited by means of Lie groups actions and they lead to the construction of momentum mapping, or equivalently, conserved quantities. They are the ingredients to perform reductions on the original system.

Definition 1.27. *Let M be a Poisson manifold, G a Lie group and $\Phi : G \times M \longrightarrow M$ a smooth left action of G on M. We say the action is **canonical** or **Poisson** if*

$$\Phi_g^*\{F, H\} = \{\Phi_g^* F, \Phi_g^* H\}.$$

If M is symplectic with symplectic form ω, then the action is canonical iff $\Phi_g^*\omega = \omega$.

The elements $\phi \in \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ associated to the Lie group G determine flows $\Phi_{\exp t\xi}$ and vector fields ξ_M on the manifold M. Then one may wonder if the vector fields ξ_M are globally Hamiltonian. From now on we assume that there is a global Hamiltonian $J(\xi) \in \mathcal{F}(M)$ for ξ_M, that is

$$X_{J(\xi)} = \xi_M. \tag{1.6}$$

Unicity of $J(\xi)$ is often a problem for M a Poisson manifold. In other words, if both $J_1(\xi)$ and $J_2(\xi)$ satisfy (1.6), then

$$X_{J_1(\xi) - J_2(\xi)} = 0,$$

therefore, $J_1(\xi) - J_2(\xi)$ is a Casimir on M. For M symplectic and connected, $J(\xi)$, when exists, is determined up to a constant.

Definition 1.28. *An action Φ is a Hamiltonian G-action if for all $\xi \in \mathfrak{g}$, the infinitesimal generator associated X_ξ is a Hamiltonian vector field. That is, for all $\xi \in \mathfrak{g}$ there exists a differentiable function $J(\xi) : M \to \mathbb{R}$ such that*

$$X_{J(\xi)} = \xi_M.$$

Therefore, any Hamiltonian action is associated with several functions (the Hamiltonians), which generate the same vector fields as the action does from the elements of the Lie algebra.

Definition 1.29. *Let Φ be a Hamiltonian G-action on M, then the map $\boldsymbol{J} : M \longrightarrow \mathfrak{g}^*$ defined by*

$$\boldsymbol{J}_\xi(z) = J(\xi)(z),$$

*for all $\xi \in \mathfrak{g}$ and $z \in M$ is called a **momentum mapping** of the action. The momentm mapping is Ad^* equivariant if $J(\Phi_g(m)) = Ad^*_{g^{-1}} J(m)$.*

The following result shows why momentum mappings are so relevant. It happens that they are conserved quantities along the flow of the system. Since the above development can be generalized to Poisson manifolds, the Noether's theorem is presented in the most general possible context, that is to say, in the context of a Poisson manifold.

Theorem 1.4 (Noether's Theorem. Poisson version)**.** *If the Lie Group G with Lie algebra $\mathfrak{g}$ acts on the Poisson manifold M and admits a momentum mapping $\boldsymbol{J}$, and if $H \in \mathcal{F}(M)$ is $\mathfrak{g}$-invariant, that is, $\xi_M[H] = 0$ for all $\xi \in \mathfrak{g}$, then $\boldsymbol{J}$ is a constant of the motion for H.*

Remark 1.6. *If the Lie algebra action comes from a canonical left group action Φ, then the invariance hypothesis on H is implied.*

Next, we are concerned with the reduction theorems on symplectic and Poisson manifolds. They allow us to make use of symmetries in the form of group actions and momentum maps, in order to change our former system into a simpler one, the reduced system.

1.3.2 Geometric reduction

Theorem 1.5 (Classical regular reduction theorem)**.** *Let (M, ω, G) be a Hamiltonian G-space with Ad^*-equivariant momentum mapping $\boldsymbol{J} \; :\longrightarrow \mathfrak{g}^*$. Let $\mu \in \mathfrak{g}^*$ be a regular value of $\boldsymbol{J}$ and let the isotropy subgroup G_μ for the coadjoint action on $\mathfrak{g}^*$ act freely and properly on $\boldsymbol{J}^{-1}(\mu)$. Then $M_\mu = \boldsymbol{J}^{-1}(\mu)/G_\mu$ has a unique symplectic form ω_μ making (M_μ, ω_μ) into a symplectic manifold.*

This theorem also known as the Marsden-Weinstein-Meyer theorem, establishes when an action leads to a reduced space, which is also a symplectic manifold. The following one relates the Hamiltonian of the reduced and the original spaces, that is to say, the reduced Hamiltonian flow is induced by the full Hamiltonian flow in M.

Theorem 1.6. *Under the hypotheses of Theorem 1.5 and assuming that $\mathbb{H} : M \longrightarrow \mathbb{R}$ is preserved under the action of the group G, there is a function $\mathbb{H}_\mu : M_\mu \longrightarrow \mathbb{R}$ such that $\mathbb{H}_\mu \circ \pi = \mathbb{H}$, where π denotes the quotient map from $\Phi^{-1}(\mu)$ to M_μ. Furthermore, if $f(t, m)$ denotes the flow of $\mathbb{H}$ in (M, ω) and $f_\mu(t, [m])$ the flow of $\mathbb{H}_\mu$ in (M_μ, ω_μ) we have $\pi \circ f(t, m) = f_\mu(t, [m])$ for every $m \in \Phi^{-1}(\mu)$.*

Finally the reduction theorem for the case of Poisson manifolds reads as follows

Theorem 1.7. *Consider a Lie group G acting canonically on a Poisson manifold M by Φ. Suppose the action is free and proper. Then M/G is a manifold and $\pi : M \longrightarrow M/G$ is a submersion. Moreover there exists a unique Poisson structure on M/G such that π is a Poisson map. If H is a G-invariant Hamiltonian on M, there is a function H_G on M/G such that $H = H_G \circ \pi$. Moreover π transforms X_H on M to X_{H_G} on M/G.*

1.3.3 Constructive geometric reduction

An equivalent construction of the reduced space can be achieved through invariants. This approach is called the constructive geometric reduction.

Theorem 1.8 (Hilbert)**.** *The algebra of polynomials over $\mathbb{C}$ of degree d in n variables which are invariant under $GL(n, \mathbb{C})$, acting by substitution of variables, is finitely generated.*

Theorem 1.9 (Weyl)**.** *The algebra of invariants is finitely generated for any representation of a compact Lie group or a complex semi-simple Lie group.*

Corollary 1.10. *Consider a compact Lie group G acting linearly on $\mathbb{R}^n$. Then there exist finitely many polynomials $\rho_1, \rho_2, \ldots, \rho_k \in \mathbb{R}\left[x\right]_G$, where $x \in \mathbb{R}^n$ and $\mathbb{R}\left[x\right]_G$ denote the space of G-invariant real polynomials in n variables, which generate $\mathbb{R}\left[x\right]_G$ as an $\mathbb{R}$ algebra. These generators can be chosen to be homogeneous of degree greater than zero. We call $\rho_1, \rho_2, \ldots, \rho_k$ a Hilbert basis for $\mathbb{R}\left[x\right]_G$.*

Theorem 1.11 (Schwarz, 1975)**.** *Consider a compact Lie group G acting linearly on $\mathbb{R}^n$. Let $\rho_1, \rho_2, \ldots, \rho_k \in \mathbb{R}\left[x\right]_G$ be a Hilbert basis for $\mathbb{R}\left[x\right]_G$, and let $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}^k;\ x \to (\rho_1, \rho_2, \ldots, \rho_k)$. Then $\rho^* : C^\infty(\mathbb{R}^k, \mathbb{R}) \to C^\infty(\mathbb{R}^n, \mathbb{R})_G$ is surjective, with ρ^* the pull-back of ρ.*

Theorem 1.12 (Poenaru, 1976)**.** *The map ρ is proper, that is, inverse images of compact subsets are compact, and it separates the orbits of G. Moreover the following diagram commutes, with $\tilde{\rho}$ a homomorphism*

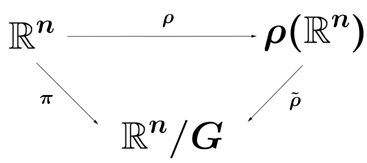


Figure 1.1: Constructive geometric reduction diagram.

This theorem states that we can take $\rho(\mathbb{R}^n)$ as a model for the orbit space and provides a method to reduce Poisson manifolds. Consider a Poisson bracket on $\mathbb{R}^n$ and a Lie group G acting canonically on it. If we consider on $\mathbb{R}^k$ the Poisson structure induced by ρ by taking as structure matrix $W_{ij} = \{\rho_i, \rho_j\}$, then $(\mathbb{R}^k, \{\ ,\ \}_W)$ is also a Poisson manifold and ρ is a Poisson map. We have a Poisson reduction if we restrict the bracket on $\mathbb{R}^k$ to $\rho(\mathbb{R}^n)$.

1.3.4 Symplectic reduction

In sections (1.3.2) and (1.3.3) it is shown how to take advantage of the conserved quantities associated with symmetries. Now symmetries are used in a different way. The associated integrals are used to design a new set of symplectic variables in which the system is expressed in a simplified way. This technique is the oldest one, in fact it is as old as the *Principia* of Newton, where the symplectic change of variables moving the origin to the center of mass performs a reduction in the system.

This technique is profusely used in the field of astronomy. For example, in the study of the free rigid body, astronomers pick a set of variables that relies on the invariance of the perpendicular plane to the angular momentum. Those variables were named the Serret-Andoyer variables by Deprit. Therefore, the system in these new variables incorporates integrals that lead to cyclic variables.

The applicability of symplectic reduction hinges in the existence of any physical knowledge of the system that relates partially the old and new variables, usually it gives rise to a configuration space transformation that should be extended to a transformation in the phase space. Later on, this approach will be fully exploited to get a reduction of the degrees of freedom. The change of variables is given by means of a symplectomorphism, thus we study the generating functions (not only for configuration transformations), which are the key to obtain a full transformation of the phase space.

Let $Q = Q(q, p)$, $P = P(q, p)$ be a change of variables defined in a ball in $\mathbb{R}^{2n}$. It is a symplectic transformation if and only if

$$dq \wedge dp = dQ \wedge dP.$$

This condition may be rewritten as $d(q\,dp - Q\,dP) = 0$, that is to say $\sigma_1 = q\,dp - Q\,dP$ is exact. Proceeding in analogous way, the symplectic character is equivalent to σ_2, σ_3 or σ_4 to be exact, where

$$\begin{aligned} \sigma_1 &= p\,dq - Q\,dP, \quad & \sigma_2 &= q\,dp + P\,dQ, \\ \sigma_3 &= q\,dp - P\,dQ, \quad & \sigma_4 &= p\,dq + Q\,dP. \end{aligned} \tag{1.7}$$

Thus, assuming that there exist four functions S_1, S_2, S_3 and S_4 satisfying that their differentials are equal to σ_1, σ_2, σ_3 and σ_4 respectively, and identifying terms, one can write

$$q = \frac{\partial S_1}{\partial p}(p, P), \quad Q = -\frac{\partial S_1}{\partial P}(p, P).$$

If the Hessian of S_1 is nonsingular, then the above equation is solvable for P as $P(q, p)$ and for p as $p(Q, P)$. Thus, proceeding with all the functions in a similar manner we have the following result.

Theorem 1.13. *The following equalities define a local symplectic change of vari-*

ables:

$$\begin{aligned} q &= \frac{\partial S_1}{\partial p}(p,P), & Q &= -\frac{\partial S_1}{\partial P}(p,P) & &\textit{when } \frac{\partial^2 S_1}{\partial p \partial P} \textit{ is no singular,} \\ q &= \frac{\partial S_2}{\partial p}(p,Q), & P &= \frac{\partial S_2}{\partial Q}(p,Q) & &\textit{when } \frac{\partial^2 S_2}{\partial p \partial Q} \textit{ is no singular,} \\ p &= \frac{\partial S_3}{\partial q}(q,Q), & P &= -\frac{\partial S_3}{\partial Q}(q,Q) & &\textit{when } \frac{\partial^2 S_3}{\partial q \partial Q} \textit{ is no singular,} \\ p &= \frac{\partial S_4}{\partial q}(p,Q), & Q &= -\frac{\partial S_4}{\partial P}(q,P) & &\textit{when } \frac{\partial^2 S_4}{\partial q \partial P} \textit{ is no singular.} \end{aligned} \tag{1.8}$$

The functions S_i $i = 1, 2, 3, 4$ are called generating functions. We will make special use of the **Mathieu Transformations** in the case that we are given a point transformation $Q = f(q)$, with $\partial f / \partial q$ invertible, then the transformation can be extended to a symplectic transformation by defining $S_4(q, P) = f(q)^T P$ and

$$p = \frac{\partial f}{\partial q}(q)^T P, \quad Q = f(q).$$

The extension of a point transformation leads us to the concept of cotangent lift, extracted from [Marsden and Ratiu, 1999].

Definition 1.30. *Given two manifolds M and N with coordinates (q,p) and (Q,P) respectively and a diffeomorphism $f : M \to N$, the* cotangent lift *of f is a symplectomorphism $T^*f : T^*N \to T^*M$, which is defined by*

$$\langle T^* f(\alpha_Q), v\rangle = \langle \alpha_Q, (Tf \cdot v)\rangle,$$

where $\alpha_Q \in T^*_Q N$, $v \in TM$ and $Q = f(q)$.

1.4 Stability in Hamiltonian Dynamical Systems

Some basics on theory of the stability related to equilibrium points in Hamiltonian systems are presented. These notes are organized following several references [Marsden, 1992, Ortega, 1998, Ortega, 2014]. We start by recalling some basic terminology about stability to fix notation, next we specialize it in the case of Hamiltonian systems.

1.4.1 Basics on stability

Definition 1.31. *Let $X \in \mathfrak{X}(M)$, let $m \in M$ be an equilibrium of (M, X) and let γ be a periodic orbit such that $m \in \gamma$. Then*

(i) m is stable or nonlinearly stable or Lyapunov stable, if for any open neighborhood U of m in M, there is an open neighborhood V of m such that if F_t is the flow associated to X, then $F_t(z) \in U$, for any $z \in V$ and for all $t > 0$. This definition of stability is also known as positively stable or negatively stable if the final condition turns to $t < 0$ and the word stable is applied when a point is both positively and negatively stable.

(ii) m is unstable if it is not stable. (The adjectives "positively", "negatively" can be also used with "unstable".)

(iii) m is asymptotically stable if there is a neighborhood V of m such that $F_t(V) \subset F_s(V)$ when $t > s$ and $\lim_{t\to+\infty} F_t(V) = \{m\}$.

(iv) γ is orbitally stable, or m is a stable periodic point, if for any open neighborhood U of γ in M, there is an open neighborhood V of m such that $F_t(z) \in U$, for any $z \in V$ and for all $t > 0$.

It is worth remarking that, since the flow given by $F_t : M \to M$ defines a parametric family of symplectic maps in M, Hamiltonian systems can not present asymptotically stable points. Thus, the Liouville's theorem establishes that F_t preserves volume, but flows having asymptotically stable points "loose" volume near that point. This can also be shown by the Hamiltonian eigenvalues theorem.

Although there are many definitions for the stability of an equilibrium, one of the most popular is stability in the Lyapunov sense. Intuitively this kind of stability means that solutions starting close enough to the equilibrium point do not go away from it. Next we give a classic sufficient, but not necessary, condition which guarantees stability.

Theorem 1.14 (Lagrange)**.** *Let $(M, \omega, \mathcal{H})$ a Hamiltonian dynamical system, and $m \in M$ an equilibrium of $X_{\mathcal{H}}$. Suppose that $d^2\mathcal{H}(m)$ is definite; that is, for all $v \in T_mM$, $v \neq 0$*

$$d^2\mathcal{H}(m)(v, v) \gtrless 0.$$

Then, m is stable.

Some of the tools to establish stability of the above types hinge on the eigenvalues of the linearized system. Hence, in our application the study of stability points starts with an analysis of the linearized system. In this regard, we define the *linearization* at the equilibrium m of the general system (1.3) given by $X \in \mathfrak{X}(M)$ as

$$X'(m) : T_m M \to T_m M, \tag{1.9}$$

where

$$X'(m) \cdot v := \frac{d}{dt}\Big|_{t=0} (T_m F_t(m) \cdot v), \; v \in T_m M.$$

We say that the equilibrium m is *linearly stable* if $X'(m)$ is stable and we say that it is *spectrally stable* if all the eigenvalues of $X'(m)$, called characteristic exponents, have non-positive real parts. Linear and spectral stability coincide iff the Jordan blocks are all one dimensional.

$$\begin{array}{ccccccc} Asymptotic & \not\Leftarrow & Nonlinear & \not\Leftarrow & Linear & \not\Leftarrow & Spectral \\ stability & \Rightarrow & stability & \Rightarrow & stability & \Rightarrow & stability \end{array} \tag{1.10}$$

Also a classic theorem from Lyapunov ensures that spectral stability with strictly negative real parts in the eigenvalues implies asymptotic stability. In (1.10) we collect the stability relations types.

1.4.2 Hamiltonian linear systems and linearizations

In the particular case of a Hamiltonian dynamical system (M, ω, H) the linearization is done as follows. Let X_H be a Hamiltonian vector field on (M, ω, H) which exhibits an equilibrium at m, that is, $d\,H(m) = 0$.

- The linearization $X'(m) : T_m M \to T_m M$ of X_H at m is a linear Hamiltonian vector field on $(T_m M, \omega_m)$ with quadratic Hamiltonian function

$$Q(v) := \frac{1}{2} d^2 H(m)(v, v), \quad v \in T_m. \tag{1.11}$$

- Let $dim(T_m M) = 2n$. If we use Darboux coordinates in $(T_m M, \omega_m)$, the linearization $X'_H(m)$ is an element of the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ of the symplectic group $Sp(2n, \mathbb{R})$, that is, it satisfies

$$X'_H(m)^T J + J X'_H(m) = 0,$$

where

$$J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}.$$

For the Hamiltonian case there are some restriction in the spectra of the linearized system. It is characterized in the Hamiltonian eigenvalues theorem, that we give after the following necessary definition.

Definition 1.32. *A polynomial $p(y) = a_n y^n + a_{n-1}y^{n-1} + \cdots + a_0$ is even if $p(-y) = p(y)$, which implies that $a_k = 0$ for every odd k. If y_0 is a root of an even polynomial, then so is $-y_0$. That is, the zeros of a real, even polynomial are symmetric about the real and imaginary axes.*

Definition 1.33. *A matrix $A \in gl(2n)$ is called Hamiltonian or infinitesimally symplectic if*

$$A^T J + JA = 0.$$

Theorem 1.15 (Hamiltonian Eigenvalues Theorem)**.** *The characteristic polynomial of a real Hamiltonian matrix is an even polynomial. Thus, if λ is an eigenvalue of a Hamiltonian matrix, then so are $-\lambda$, $\bar{\lambda}$ and $-\bar{\lambda}$.*

Corollary 1.16. *If $\lambda \in \mathbb{C}$ is an eigenvalue of the linearization $X'_H(m) \in \mathfrak{sp}(2n, \mathbb{R})$ of multiplicity k, then so are $-\lambda$, $\bar{\lambda}$ and $-\bar{\lambda}$. If 0 is an eigenvalue it has necessarily even multiplicity.*

Note that asymptotic instability can be concluded out of the spectral instability and that Lyapunov stability is only possible in the purely elliptic case, that is, when all the eigenvalues are in the imaginary axis.

Energy-Casimir method

The Energy-Casimir method is a generalization of the classical Lagrange-Dirichlet method. Let $(M, \{.,.\}, H)$ be a Poisson system, given $m \in M$ an equilibrium of the Hamiltonian vector field X_H, it proceeds in the following tree steps.

(i) Find a set of conserved quantities $C_1, \ldots, C_n \in C^\infty(M)$ (C^i are typically a Casimir function plus other conserved quantities) such that the first variation vanishes

$$d(H + C_1 + \cdots + C_n)(m) = 0.$$

(ii) Calculate the second variation

$$d^2(H + C_1 + \cdots + C_n)(m)|_{W \times W}.$$

(iii) If the second variation is definite for W defined by

$$W = ker(dC_1(m) \cap \ldots \cap dC_n(m)),$$

then, m is stable. If $W = \{0\}$, m is always stable. W is called the stability space.

The definiteness character of a matrix and the properties of those kind of matrices is studied in [Johnson, 1970]

1.5 On the Constrained Flows

In this section we review some results concerning to constrained Hamiltonian systems. The proofs of the results presented here are in the paper of van der Meer and Cushman [van der Meer and Cushman, 1986], where the ideas about constrained systems were first introduced.

Suppose we are given a Hamiltonian system $(\mathcal{H}, M, \omega)$ of dimension $2m$ and we want to study the flow when it is restricted to a certain $2n$-submanifold N, where $n < m$. It is also supposed that we have C_1, ..., $C_{2k} \in C^\infty(M)$ smooth functions, where $m - n = k$, such that $N = \{m \in M | C_1(m) = \cdots = C_{2k}(m) = 0\}$ and the differentials DC_1, ..., DC_{2k} are independent on N, that is, N is a *smoothly embedded submanifold* of M. In [Cushman and Bates, 1997], N is called the *constraint manifold* with *constraint functions* C_1, ..., C_{2k}. Furthermore suppose that the matrix $C = (c_{ij}) = (\{C_i, C_j\})$ is non singular at every point of N, then N is called a *cosymplectic submanifold* of M.

Theorem 1.17. *If N is a cosymplectic submanifold of a symplectic manifold (M, ω), then $\omega_{|N}$ is a symplectic form on N.*

Proof. See details in [Cushman and Bates, 1997]. □

Let $X_\mathcal{H}$ be the Hamiltonian field on M generated by the Hamiltonian $\mathcal{H}$, thus the restricted field $X_\mathcal{H}|N$ does not belong in general to the tangent bundle of N. It is a desirable feature of the original Hamiltonian. Next we give a characterization of when it does happen.

Lemma 1.1. *The following statements are equivalent:*

1. *$X_\mathcal{H}$ is tangent to N at each point of N.*
2. *N is an invariant manifold of $X_\mathcal{H}$.*

3. $\exp L_{\mathcal{H}}(\mathcal{I}) \subseteq \mathcal{I}$, *where* $\mathcal{I}$ *is the ideal of* $C^\infty(M)$ *generated by* C_1, ..., C_{2k}

4. $\{\mathcal{H}, C_i\}_{|N} = 0$ *for* $i = 1, \cdots, 2k$.

5. $X_{\mathcal{H}|N} = X_{\mathcal{H}}$ *on* N.

In practice, when we have a cosymplectic submanifold N of M and $X_{\mathcal{H}}$ is tangent to N, we are lucky of work out any computation of the constrained system in the ambient space, because $X_{\mathcal{H}}|N$ is a Hamiltonian field on N with associated Hamiltonian $\mathcal{H}|N$ and $\omega_N = \omega_{|N}$.

For the case in which $X_{\mathcal{H}}$ is no longer tangent to N the above setting is complete spoiled. The strategy is to construct a function $\mathcal{H}^*$ featuring that $\mathcal{H}^*|N = \mathcal{H}|N$, $X_{\mathcal{H}^*}$ is tangent to N and $X_{\mathcal{H}^*}|N = X_{\mathcal{H}|N}$. By this procedure we are constructing a vector field in M, $X_{\mathcal{H}^*}$, whose restriction to N is the projection of $X_{\mathcal{H}}$ on the tangent space of N, by mean of this setting, we will be able to compute in the ambient space as before.

Lemma 1.2. *Let* $C^{-1} = (c^{ij})$ *be the inverse of the matrix* C *and let* $\mathcal{H}^* : M \longrightarrow \mathbb{R}$ *defined as follows*

$$\mathcal{H}^* = \mathcal{H} + \sum_{i,j}(\{\mathcal{H}, C_i\} + F_i)c^{ij}C_j, \tag{1.12}$$

where F_i *belong to the ideal* $\mathcal{I}$ *generated by* C_1, ..., C_{2k}. *Thus* $\mathcal{H}^*|N = \mathcal{H}|N$, $X_{\mathcal{H}^*}$ *is tangent to* N *and* $X_{\mathcal{H}^*}|N = X_{\mathcal{H}|N}$.

Note that in concrete applications it is often necessary to use the freedom in the choice of F_i to simplify $\mathcal{H}^*$.

For any two functions $F, G \in C^\infty(M)$ we can compute the Poisson bracket $\{F|N, G|N\}^N$ on N by mean of the “ * ” procedure in the following way

$$\{F|N, G|N\}^N = \{F^*, G^*\}|N,$$

so that, the Poisson bracket on N can be computed in terms of the Poisson bracket on M.

Chapter 2

The Extended Euler System

In this chapter we put forward the study of the family of differential equations, which will be of interest in the study of the quartic parametric polynomial model presented in Chapter 4

$$\frac{\mathrm{d}\Pi_1}{\mathrm{d}s} = a_1\,\Pi_2\Pi_3, \quad \frac{\mathrm{d}\Pi_2}{\mathrm{d}s} = a_2\,\Pi_1\Pi_3, \quad \frac{\mathrm{d}\Pi_3}{\mathrm{d}s} = a_3\,\Pi_1\Pi_2, \tag{2.1}$$

The results obtained here has been sent for publication, see [Crespo and Ferrer, 2014b]. Let the initial values of the above equations be Π_1^0, Π_2^0 and Π_3^0, where a_i are arbitrary real parameters. A simple computation shows that this system is endowed with the following integrals

$$h_1 = a_2\Pi_3^2 - a_3\Pi_2^2, \quad h_2 = a_3\Pi_1^2 - a_1\Pi_3^2, \quad h_3 = a_1\Pi_2^2 - a_2\Pi_1^2, \tag{2.2}$$

those integrals are three hyperbolic cylinders (HC) when the parameters a_1, a_2 and a_3 has the same sign; otherwise, we have two elliptic cylinders (EC) and a hyperbolic one.

Note that in those systems there is always the possibility of taking one of the a_i equal to one by rescaling the independent variable, but thinking on the applications we will maintain all the parameters.

It is well known that, Euler equations are a dynamical system fitting (2.1), *i.e.*, the reduced system associated to the motion of a rigid body with a fixed point in the absence of external forces. Then, the variables Π_i are the three components of the angular momentum. The parameters a_i $(1 \leq i \leq 3)$ depend on the *principal moments of inertia* of the body and they satisfy that the sum $a_1 + a_2 + a_3 = 0$. The approach of this Chapter is to release the system (2.1) from any physical background and consider it as an initial value problem.

Moreover, the powerful presence of the rigid body problem has constrained the study of the system (2.1) to bounded solutions, while the dynamic corresponding with system (2.1) also provides unbounded ones. In the work of [Holm and Marsden, 1991] and [Iwai and Tarama, 2010] the authors gave the exhaustive list of Lie-Poisson structures associated to the family (2.1). Here, we consider a six-dimensional Poisson structure that fits all of them together. That Poisson structure is not of Lie-Poisson type.

In the literature there are several studies of this generalization. For instance, in [Iwai and Tarama, 2010], the extended free rigid body is analyzed in classical and quantum mechanics, while in [Holm and Marsden, 1991], [Puta, 1993], [Puta, 1997] or [Puta and Casu, 1999], the authors are concerned with the inclusion of one, two or even three rotors that turn up as quadratic controls. Also, we could mention a proposal for a regularization of the flow in [Molero, 2013].

We follow and extend the approach of Meyer in [Meyer, 2001], where he introduces the basic Jacobi elliptic functions as the solutions of a particular differential system of the family (2.1). As such, the twelve Jacobi elliptic functions are shown in a unified way as the family of solutions for the (EES). In addition, some important properties of the Jacobi elliptic functions, see [Lawden, 1989, Whittaker and Watson, 1940], are obtained easily in Section 2.5

2.1 Deformation of the Extended Euler System Poisson Structure.

The family of differential equations given in (2.1) may be rendered as a Hamiltonian family when a suitable Poisson structure is defined in $\mathcal{C}^\infty(\mathbb{R}^3)$. For this purpose, we recall the Nambu [Nambu, 1973, Holm, 2008b] type bracket on $\mathcal{C}^\infty(\mathbb{R}^3)$

$$\begin{aligned} \{\,,\}_{\mathbf{c}} : \mathcal{C}^\infty(\mathbb{R}^3) \times \mathcal{C}^\infty(\mathbb{R}^3) &\longrightarrow \mathcal{C}^\infty(\mathbb{R}^3) \\ (F, G) \qquad\qquad &\longrightarrow \{F, G\}_{\mathbf{c}} = -\nabla \mathbf{c} \cdot (\nabla F \times \nabla G), \end{aligned} \tag{2.3}$$

where $\mathbf{c} \in \mathcal{C}^2(\mathbb{R}^3)$. Thus, $(\mathcal{C}^\infty(\mathbb{R}^3), \{\,,\}_{\mathbf{c}})$ is a Lie-Poisson manifold and $\mathbf{c}(\Pi)$ is a Casimir. We will restrict the Casimir function to be in the following fashion,

$$\mathbf{c}(\Pi) = \frac{1}{2}(c_1\Pi_1^2 + c_2\Pi_2^2 + c_3\Pi_3^2), \tag{2.4}$$

under this assumption, the Nambu bracket defined above provides a linear Poisson structure given by means of the following relations

$$\{\Pi_1, \Pi_2\}_{\mathbf{c}} = c_3\,\Pi_3, \quad \{\Pi_3, \Pi_1\}_{\mathbf{c}} = c_2\,\Pi_2, \quad \{\Pi_2, \Pi_3\}_{\mathbf{c}} = c_1\,\Pi_1. \tag{2.5}$$

where Π_i, $i = 1, 2, 3$ are the coordinate functions. Therefore, according to the signs of (c_1, c_2, c_3), the Poisson structures defined by the above bracket corresponds, up to a canonical map, to any of the ones given next

$$\begin{array}{lll} \mathfrak{so}^*(3) & (+,+,+), & (-,-,-) \\ \mathfrak{so}^*(2,1) & (+,+,-), & (+,-,-) \\ \mathfrak{so}^*(2)\times\mathbb{R}^2 & (+,+,0), & (-,-,0) \\ \mathfrak{so}^*(1,1)\times\mathbb{R}^2 & (+,-,0) & \\ \mathfrak{h}_1^* & (+,0,0), & (-,0,0) \\ \mathbb{R}^{3*} & (0,0,0) & \end{array} \tag{2.6}$$

where $\mathfrak{h}_1$ is the Heisenberg group.

Following [Weinstein, 1983], it is shown that the Poisson manifolds $(\mathcal{C}^\infty(\mathbb{R}^3), \{\,,\}_{\mathbf{c}})$ fit together in a larger Poisson manifold, which is not of a Lie-Poisson type. In other words, the inclusion of $(\mathcal{C}^\infty(\mathbb{R}^3), \{\,,\}_{\mathbf{c}})$ in $\mathbb{R}^6 = \{(\Pi_1, \Pi_2, \Pi_3, c_1, c_2, c_3)/\Pi_i, c_i \in \mathbb{R}\}$ with the following Poisson structure is a canonical map

$$\{\Pi_i, \Pi_j\} = \epsilon_{ijk} c_k \Pi_k, \quad \{\Pi_i, c_j\} = 0, \quad \{c_i, c_j\} = 0, \qquad i = 1, 2, 3. \tag{2.7}$$

The leaves of the symplectic stratification are given by the Casimir functions $\frac{1}{2}(c_1 x^2 + c_2 y^2 + c_3 z^2)$, c_1, c_2 and c_3. Therefore, by means of the identification $\mathbb{R}^6 \equiv \mathbb{R}^3 \times \mathbb{R}^3$, the symplectic leaves may be rendered as the product of the two-dimensional quadric $\mathbf{c}(\Pi) \subset \mathbb{R}^3$ times a single point $(c_1, c_2, c_3) \subset \mathbb{R}^3$.

Proposition 2.1. *Let us consider the functions given by*

$$\begin{aligned} \mathbf{c}(\mathbf{\Pi}) &= \frac{h_1 - h_3}{2} = \frac{1}{2}\left(a_2\Pi_1^2 - (a_1 + a_3)\Pi_2^2 + a_2\Pi_3^2\right), \\ \mathbf{h}(\mathbf{\Pi}) &= \frac{-h_2}{2\,(a_1 + a_3)} = \frac{1}{2}\left(\frac{a_1}{a_1 + a_3}\Pi_3^2 - \frac{a_3}{a_1 + a_3}\Pi_1^2\right). \end{aligned} \tag{2.8}$$

Thus system (2.1) may be expressed in Hamiltonian form as

$$\dot{\Pi}_i(\mathbf{\Pi}) = \{\Pi_i, \mathbf{h}\}_{\mathbf{c}}(\mathbf{\Pi}), \tag{2.9}$$

together with the initial conditions $\mathbf{\Pi}(0) = (\Pi_1^0, \Pi_2^0, \Pi_3^0)$. This system will be named as the Extended Euler system *(EES).*

Proof. The direct computation of the Poisson bracket $\{\Pi_i, \mathbf{h}\}_{\mathbf{c}}(\mathbf{\Pi})$ leads us to the differential equations defining system (2.1). □

Notice that Holm and Marsden shown in [Holm and Marsden, 1991] that any linear combination

$$K = a\,\mathbf{c} + b\,\mathbf{h}, \quad N = c\,\mathbf{c} + d\,\mathbf{h}, \qquad ad - cb = 1, \tag{2.10}$$

may replace $\mathbf{c}$ and $\mathbf{h}$ in Proposition 2.1. That is to say

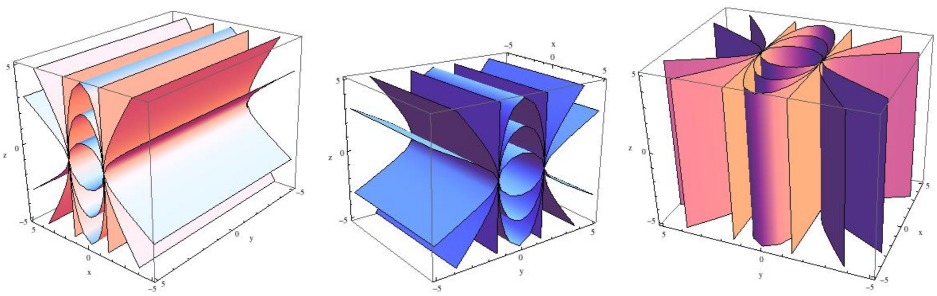
$$\dot{\Pi}_i(\mathbf{\Pi}) = \{\Pi_i, \mathbf{h}\}_{\mathbf{c}}(\mathbf{\Pi}) = \{\Pi_i, N\}_K(\mathbf{\Pi}). \tag{2.11}$$

Moreover, by a rescaling of the independent variable in system (2.1) if it is needed, we can replace the condition $ad - cb = 1$ in (2.10) by $ad - cb \neq 0$. Therefore, we are allowed to carry out our study using the original integrals given in (2.2), which provides a simply presentation of the general analytic solutions given in Section 2.4. Moreover, although two of them are enough since the following relation holds

$$a_1 h_1 + a_2 h_2 + a_3 h_3 = 0, \tag{2.12}$$

we will show that the third one also play a key role in the analytic solutions formulas, detecting heteroclinic orbits and equilibria.

Under the assumption $K = h_i$, $i = 1, 2, 3$ in (2.11), the symplectic stratification is not given by an arbitrary quadric in $\mathbb{R}^3$, but for elliptic (EC) and hyperbolic (HC) cylinders, see Figure 2.1.



(a) Evolution of h_1. (b) Evolution of h_2. (c) Evolution of h_3.

Figure 2.1: **Symplectic stratification:** Symplectic leaves associated to $K = h_1$ 2.1(a), $K = h_2$ 2.1(b) and $K = h_3$ 2.1(c), where $a_2 \in \{1, 0.2, 0.08, 0, -0.5, -1\}$ for h_1 and h_3 and $a_3 \in \{1, 0.2, 0.08, 0, -0.5, -1\}$ for h_2.

Observe that the Poisson structure has an influence in geometry of the solutions, since the choice of the Casimir corresponding with $\mathfrak{so}^*(3)$ or $\mathfrak{so}^*(2) \times \mathbb{R}^2$ leads to bounded trajectories.

2.2 Basic Features of the Extended System.

Equilibria. For the general case when a_1, a_2, $a_3 \neq 0$, the three coordinate axes correspond to equilibria points. Those points are arranged in pairs, each one corresponds with the zero values of each of the integrals. In other words those equilibria (Π_1, Π_2, Π_3) are given by

$$h_1 = 0: \ (\pm\sqrt{|h_2/a_3|}, 0, 0); \ h_2 = 0: \ (0, \pm\sqrt{|h_3/a_1|}, 0); \ h_3 = 0: \ (0, 0, \pm\sqrt{|h_1/a_2|}) \tag{2.13}$$

For the case in which one of the parameter a_i is equal to zero, the equilibria are the following

$$a_1 = 0: \ (\pm\sqrt{|h_2/a_3|}, 0, 0); \ a_2 = 0: \ (0, \pm\sqrt{|h_3/a_1|}, 0); \ a_3 = 0: \ (0, 0, \pm\sqrt{|h_1/a_2|}) \tag{2.14}$$

together with

$$a_1 = 0: \ (0, x_2^0, x_3^0); \qquad a_2 = 0: \ (x_1^0, 0, x_3^0); \qquad a_3 = 0: \ (x_1^0, x_2^0, 0). \tag{2.15}$$

Our family depends on two sets of values, the initial conditions and the parameters, as we have seen in Section 2.1; the integrals depend on both. In this Section we will present the geometry of the solutions. They are defined by the intersection of the integral surfaces. The germinal paper of this approach comes from Holm and Marsden [Holm and Marsden, 1991]. Also Meyer in [Meyer, 2001] expresses that the solutions of a differential system of the Jacobi type are given as the intersection of the cylinders defined by the integrals (2.2). We take benefit of both papers.

Following Meyer we give a simple characterization of the bounded and unbounded solutions of system (2.1). Firstly we define the concept of *standard system* in order to avoid repetition.

Definition 2.1 (Standard form of the extended Euler system)**.** *By interchanging variables and or equations and by choosing a suitable orientation of the independent variable, if it is needed, system (2.1) may be arranged in such a way that the parameter a_1 is strictly positive, a_3 is positive or zero, and $a_2 \in \mathbb{R}$. Then we say that it is in* standard form.

Proposition 2.2. *Given initial conditions such that $h_1 \neq 0$, $h_2 \neq 0$ and $h_3 \neq 0$, the solutions of the standard system are equal to each of the connected component of the intersection of the integrals. For the case $a_2 < 0$ we obtain periodic and bounded* *solutions, and* $a_2 > 0$ *leads to the non-bounded ones. If any of the parameters vanishes, we have the following cases;* $a_j = 0$ *and* $a_i a_k > 0$ *leads to unbounded solutions, while* $a_j = 0$, $a_i a_k < 0$ *to the bounded ones. The case in which two parameters vanish corresponds to straight trajectories.*

Proof. We study the case $j = 2$ and $i = 1$, $k = 3$, different combinations are analogous. Along this proof we denote the intersection $h_1 \cap h_3$ by $\mathcal{C}$. For $a_2 < 0$, the integral surfaces give place to the intersection of a pair of EC, therefore $\mathcal{C}$ is made up of two closed symmetric curves and the corresponding solutions are bounded. When $a_2 > 0$, the intersection $\mathcal{C}$ is given by a pair of HC, so that $\mathcal{C}$ is built up of four symmetric curves. All those curves are free of equilibria points, see (2.13). It is also satisfied that any solution starting at a connected component of $\mathcal{C}$ remains in that component for all s in the domain of the solution. Thus, by mean of the continuation theorem for differential equations [Hale, 1969], pp. 16-17, we obtain that solutions starting in a connected component of $\mathcal{C}$ must traverse all of it. Therefore, solutions corresponding to $a_2 > 0$ are unbounded and for $a_2 < 0$, keeping in mind that system (2.1) does not dependent on the independent variable, we have that solutions are periodic.

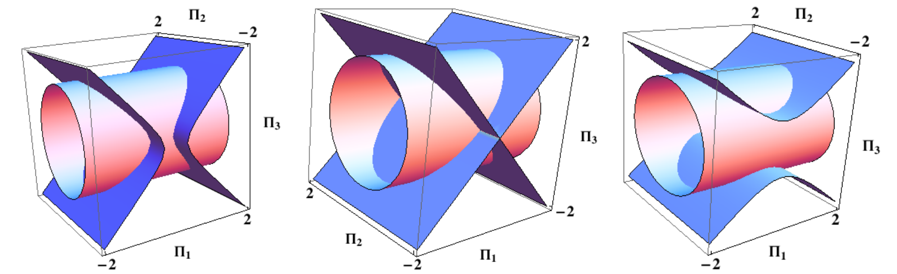
For $a_2 = 0$ we have that h_1 and h_3 are two planes, and solutions are given by their intersection with h_2. Therefore we obtain an hyperbola for $a_3 \neq 0$. Analogously, $a_2 = 0$ and $a_3 \neq 0$ lead to elliptic solutions. □

2.3 Geometric Description of the Solutions

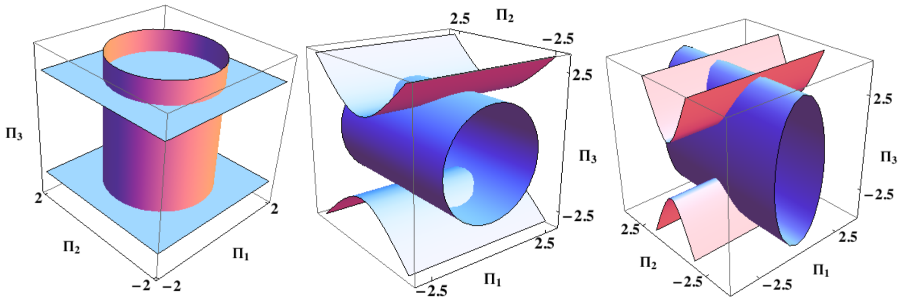
Along this section we distinguish the effects that changes in the parameters of system (2.1) has on the solutions, as well as the influence of the initial conditions.

2.3.1 Parameters dependency

In [Holm and Marsden, 1991], the authors show how to control the stability properties of an equilibria, by using a_3 as a control parameter. We extend the analysis of the parameters evolution given in [Holm and Marsden, 1991] by also studying the variation of a_2. Thus, we show how the change in the sign of those parameters has to do with the stability, and also how it determines whether the solutions orbits are bounded or not.



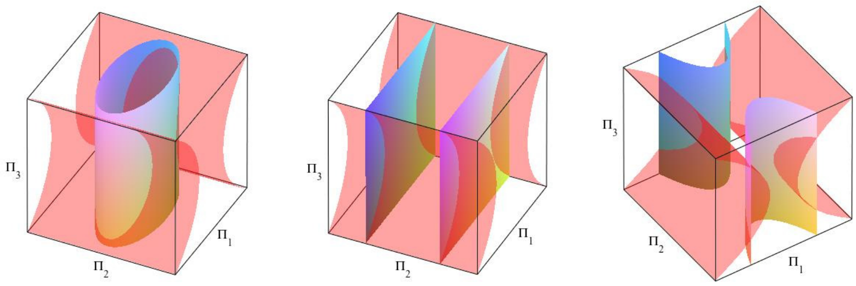
(a) $a_3 = 1.05$. (b) $a_3 = 1$. (c) $a_3 = 0.75$.



(d) $a_3 = 0$. (e) $a_3 = -1$. (f) $a_3 = -6$.

Figure 2.2: **The Holm and Marsden stability evolution:** For $a_3 > 0$ the integral surfaces h_1 and h_2 correspond to an EC and HC, see figures 2.2(a) and 2.2(c). In 2.2(b) h_2 become a pair of planes that intersects with h_1 along heteroclinic orbits and a pair of unstable equilibria in the Π_2 axis. In 2.2(d) $a_3 = 0$, thus h_1 and h_2 are completely flat. Therefore, we incorporate h_3 to show the trajectory solution as $h_1 \cap h_2 \cap h_3$. For $a_3 < 0$ the stability of the equilibria in the axis Π_1 and Π_2 interchange their roles.

Secondly, two of the parameters are fixed and have the same sign, in this case the sign of the third one characterizes whether the solutions are bounded or not see Figure 2.3.

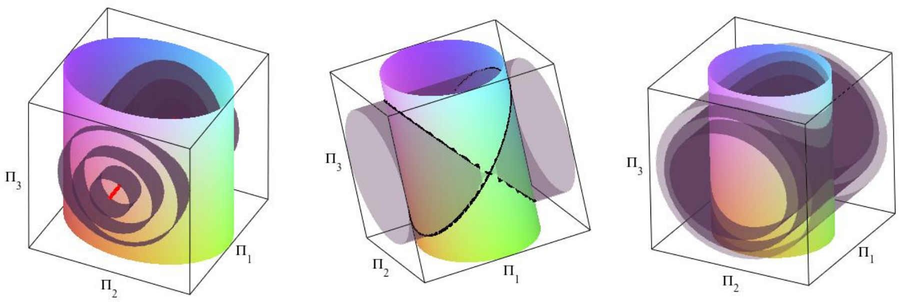


(a) $a_2 < 0$. (b) $a_2 = 0$. (c) $a_2 > 0$.

Figure 2.3: **Global bifurcation from bounded to unbounded trajectories:** Intersection of h_2 and h_3 as $a_2 \in \mathbb{R}$ varies. For $a_2 < 0$, in 2.3(a), we obtain bounded solutions. As the value of a_2 tends to zero the EC associated to h_3 becomes more and more eccentric, till it breaks for $a_2 = 0$ into two planes perpendicular to Π_2 axis, see 2.3(b). For $a_2 > 0$ we obtain a pair of HC that intersect along unbounded orbits.

2.3.2 Initial conditions dependency

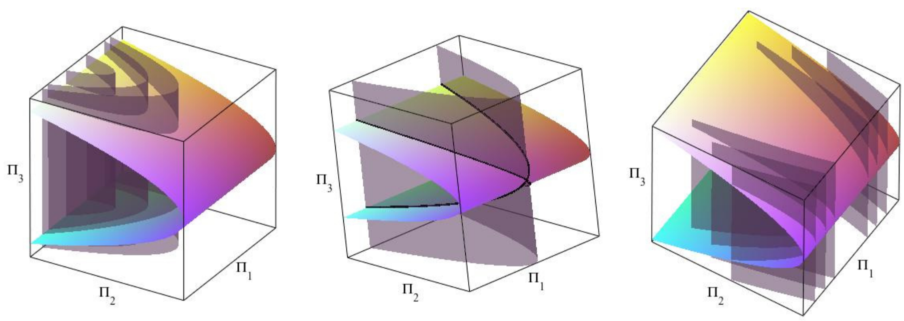
When we fix the parameters, initial conditions play a role in the final solutions. The trajectories are organized in connected components which borders are given by the heteroclinic orbits. Those orbits are characterized by $h_i = 0$, while the positive and negative values leads to different connected components. Those situations are illustrated in Fig. 2.4 and Fig. 2.5.



(a) $h_2 < 0$. (b) $h_2 = 0$. (c) $h_2 > 0$.

Figure 2.4: **Bounded case:** The negative and positive values for h_2 correspond to surfaces plotted in figures 2.4(a) and 2.4(c) respectively, $h_2 = 0$ is represented by 2.4(b), this value lead us to a pair of unstable equilibria points and four heteroclinic orbits.

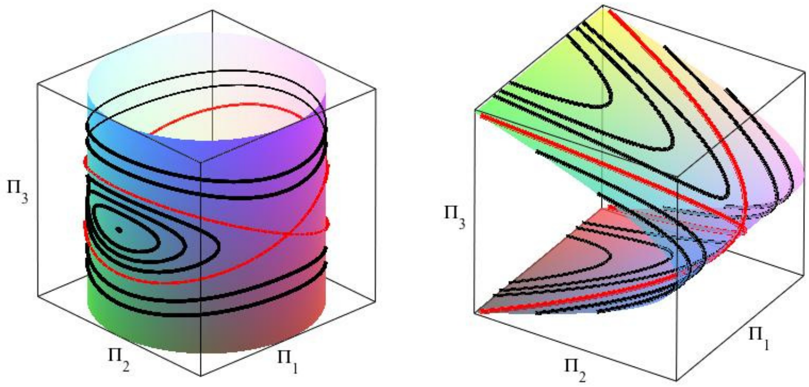
Notice that in Fig. 2.5, in order to get a clearer figure, we only plot one of the two sheets of the (HC) corresponding to the integrals. Therefore, the heteroclinic orbits and the equilibria shown there have twins that are not plotted.



(a) $h_2 < 0$. (b) $h_2 = 0$. (c) $h_2 > 0$.

Figure 2.5: **Unbounded case:** Figures 2.5(a) and 2.5(c) correspond to the positive and negative values of h_2 respectively. In 2.5(b), it is we obtain four heteroclinic orbits and an equilibria.

In Fig. 2.6 we fix the value of one of the integrals and plot the intersection for the variation of the remaining one. It is shown how the orbits are organized in connected components determined by the heteroclinic orbits plotted in red color.



(a) Bounded resume. (b) Unbounded resume

Figure 2.6: **Resume of bounded and unbounded orbits.**. For the sake of clarity, in Fig 2.6(b), only one of the sheets of the hyperboloids is plotted.

2.4 Qualitative Characterization of the Solutions. The Canonical Form.

Along this section we focus on the qualitative behavior of system (2.1). Symmetries relative to the family of differential systems are investigated and the period of the solutions is calculated.

In what follows, in order to avoid repetition and without loss of generality, we assume that system (2.1) is always presented in *standard form*.

2.4.1 Intrinsic symmetries to the system

The set of initial conditions may be reduced to the first octant by considering the three axial symmetries that the system is endowed with. This property is given in Proposition 2.3 following an analogous result of Meyer in [Meyer, 2001], see Proposition 2.3.

Proposition 2.3. *If* $\big(\Pi_1(s), \Pi_2(s), \Pi_3(s)\big)$ *is a solution of the initial value problem given by (2.1), then so are the following ones*

$$\begin{aligned} &\big(\Pi_1(-s), -\Pi_2(-s), \Pi_3(-s)\big), && \big(-\Pi_1(s), -\Pi_2(s), \Pi_3(s)\big),\\ &\big(-\Pi_1(-s), \Pi_2(-s), \Pi_3(-s)\big), && \big(\Pi_1(s), -\Pi_2(s), -\Pi_3(s)\big),\\ &\big(\Pi_1(-s), \Pi_2(-s), -\Pi_3(-s)\big), && \big(-\Pi_1(s), \Pi_2(s), -\Pi_3(s)\big). \end{aligned}$$

Proof. It is a straightforward computation to check that those triads also satisfy the original system. □

From now on we assume the initial conditions to be in the first octant, that is, Π_1^0, Π_2^0 and Π_3^0 always be positive numbers or one of them zero. Any other arrangement could be obtained by applying the above proposition.

Corollary 2.4. *Let* $\Pi(s) = \big(\Pi_1(s), \Pi_2(s), \Pi_3(s)\big)$ *be a solution of system (2.1) with initial condition* $\Pi(s_0) = \big(\Pi_1^0, \Pi_2^0, \Pi_3^0\big)$. *Then,* $\Pi_i^0 = 0$ *and* $\Pi_j^0\Pi_k^0 \neq 0$ *implies* $\Pi_i(s)$ *is an odd function and* $\Pi_j(s)$ *and* $\Pi_k(s)$ *are even functions for* $j, k \neq i$.

Proof. Let $\big(\Pi_1(s), \Pi_2(s), \Pi_3(s)\big)$ be the solution of system (2.1) satisfying $\Pi_i(0) = 0$ and let us assume $i = 1$, for $i = 2, 3$ the proof is analogus. By Proposition 2.3 we have that $\big(-\Pi_1(-s), \Pi_2(-s), \Pi_3(-s)\big)$ is also a solution for system (2.1) and it also satifies the same initial condition, therefore by the uniqueness theorem for ODE we obtain that

$$\Pi_1(s) = -\Pi_1(-s), \quad \Pi_2(s) = \Pi_2(-s), \quad \Pi_3(s) = \Pi_3(-s),$$

that is, $\Pi_1(s)$ is an odd function and $\Pi_2(s)$ and $\Pi_3(s)$ are even functions. □

Note that if we change the initial condition from $s_0 = 0$ to any arbitrary s_0, the odd and even symmetry on zero are translated to a symmetry about s_0.

The following proposition shows a remarkable feature of the solutions. That is, the ratios of the solutions are also the solutions of a *extended Euler system*. The following theorem reflects the definition of the twelve elliptic functions as a natural feature of the family.

Theorem 2.5. *Let* $\Pi_i(s)$, $(1 \leq i \leq 3)$ *be a solution of the system (2.1). Then, the following ratios*

$$z_k^0(s) = \frac{1}{\Pi_k(s)}, \quad z_k^i(s) = \frac{\Pi_i(s)}{\Pi_k(s)}, \quad z_k^j(s) = \frac{\Pi_j(s)}{\Pi_k(s)}, \qquad i, j, k \in \{1, 2, 3\}. \tag{2.16}$$

in their domain of existence, are the solution of the system given by

$$\frac{\mathrm{d}z_k^0}{\mathrm{d}s} = \bar{a}_{kk}\, z_k^j\, z_k^i, \quad \frac{\mathrm{d}z_k^i}{\mathrm{d}s} = \bar{a}_{jk}\, z_k^j\, z_k^0, \quad \frac{\mathrm{d}z_k^j}{\mathrm{d}s} = \bar{a}_{ik}\, z_k^i\, z_k^0, \tag{2.17}$$

with associated coefficients and integrals

$$\begin{aligned} \bar{a}_{ik} &= \sigma_{ijk} h_i, & \bar{a}_{jk} &= \sigma_{jik} h_j, & \bar{a}_{kk} &= -a_k, \\ H_{ik} &= \sigma_{ijk} a_i, & H_{jk} &= \sigma_{jik} a_j, & H_{kk} &= -h_k, \end{aligned} \tag{2.18}$$

where σ_{ijk} correspond to the signature of the permutation given by $(1,2,3) \to (i,j,k)$, and the values of h_1, h_2 and h_3 are determined by mean of the initial conditions of (2.1).

Proof. Using system (2.1) by a straight computation we check that (2.16) satisfy equations (2.17). □

Proposition 2.5 shows that every system (2.1) is associated to the following three systems (given in rows), which solutions are the ratios given in (2.16):

$$\dot{z}_1^0 = -a_1\, z_1^3\, z_1^2, \quad \dot{z}_1^2 = -h_3\, z_1^3\, z_1^0, \quad \dot{z}_1^3 = h_2\, z_1^2\, z_1^0. \tag{2.19}$$

$$\dot{z}_2^0 = -a_2\, z_2^3\, z_2^1, \quad \dot{z}_2^1 = h_3\, z_2^3\, z_2^0, \quad \dot{z}_2^3 = -h_1\, z_2^1\, z_2^0, \tag{2.20}$$

and

$$\dot{z}_3^0 = -a_3\, z_3^1\, z_3^2, \quad \dot{z}_3^1 = -h_2\, z_3^2\, z_3^0, \quad \dot{z}_3^2 = h_1\, z_3^1\, z_3^0, \tag{2.21}$$

and the integrals associated to systems (2.31), (2.32) and (2.33) are given respectively by

$$\begin{aligned} &H_1 = a_1,\ H_2 = -a_2,\ H_3 = -h_3, \\ &H_1 = -a_1,\ H_2 = -h_2,\ H_3 = a_3, \\ &H_1 = -h_1,\ H_2 = a_2,\ H_3 = -a_3. \end{aligned} \tag{2.22}$$

Therefore, Proposition 2.5 relates every unbounded system with a bounded one in the following sense.

Corollary 2.6. *Let us consider a extended Euler system in the standard form, with* a_3, $a_2 \neq 0$, $h_i \neq 0$ $i = 1, 2, 3$. *Thus only one of the associated systems given in (2.19), (2.20) and (2.21) give rise to bounded solutions . More precisely, if* $h_i h_j > 0$ *the bounded system associated is given by the ratios*

$$z_k^0(s) = \frac{1}{\Pi_k(s)}, \quad z_k^i(s) = \frac{\Pi_i(s)}{\Pi_k(s)}, \quad z_k^j(s) = \frac{\Pi_j(s)}{\Pi_k(s)}. \tag{2.23}$$

Proof. Condition $h_i h_j > 0$ together with $a_2 > 0$ and the expression of the integrals (2.2) ensure that $\Pi_k(s) \neq 0$ in its domain of existence.

Notice also that if h_i and h_j has the same sign, then $\sigma_{ijk} h_i$ and $\sigma_{jik} h_j$ has not. Therefore, the associated system to $z_k^0(s)$, $z_k^i(s)$ and $z_k^j(s)$, see (2.17), has its three parameters with different signs, such a system corresponds to the bounded type. □

2.4.2 Periodic solutions. Computing the period

Corollary 2.6 allows us to study the unbounded system through the bounded ones, in other words, we can restrict ourselves to the study of bounded system and relate any unbounded with its correspondent bounded one by means of the new functions given in (2.23). Following Meyer [Meyer, 2001], we study the periodicity for the bounded case. Therefore, periodicity of the bounded solutions will imply the same for the unbounded systems.

Lemma 2.1. *Let us consider an extended Euler system in standard form, such that* $a_2 < 0$ *and let*

$$\begin{aligned} \mathrm{A} &= \left(\sqrt{\left|\frac{h_3}{a_2}\right|}, 0, \sqrt{\left|\frac{h_1}{a_2}\right|}\right), \quad \mathrm{B} = \left(\sqrt{\left|\frac{h_2}{a_3}\right|}, -\sqrt{\left|\frac{h_1}{a_3}\right|}, 0\right), \\ \mathrm{C} &= \left(\sqrt{\left|\frac{h_3}{a_2}\right|}, 0, -\sqrt{\left|\frac{h_1}{a_2}\right|}\right), \quad \mathrm{D} = \left(\sqrt{\left|\frac{h_2}{a_3}\right|}, \sqrt{\left|\frac{h_1}{a_3}\right|}, 0\right), \end{aligned} \tag{2.24}$$

if $h_2 > 0$ *and*

$$\begin{aligned} \mathrm{A} &= \left(\sqrt{\left|\frac{h_3}{a_2}\right|}, 0, \sqrt{\left|\frac{h_1}{a_2}\right|}\right), \quad \mathrm{B} = \left(0, -\sqrt{\left|\frac{h_3}{a_1}\right|}, \sqrt{\left|\frac{h_2}{a_1}\right|}\right), \\ \mathrm{C} &= \left(\sqrt{\left|\frac{h_3}{a_2}\right|}, 0, -\sqrt{\left|\frac{h_1}{a_2}\right|}\right), \quad \mathrm{D} = \left(0, \sqrt{\left|\frac{h_3}{a_1}\right|}, \sqrt{\left|\frac{h_2}{a_1}\right|}\right), \end{aligned} \tag{2.25}$$

for $h_2 < 0$. *Thus we claim that* $\Pi(s_0) = (\Pi_1^0, \Pi_2^0, \Pi_3^0) = A$, $\Pi(s_1) = B$, $\Pi(s_2) = C$ *and* $\Pi(s_3) = D$, *for certain* $s_0, s_1, s_2, s_3 \in \mathbb{R}$.

Proof. A straightforward computation shows that A, B, C and D belong to the integral surfaces given in (2.2). □

The following proposition shows that $s_1 = s_0 + P$, $s_2 = s_0 + 2P$ and $s_3 = s_0 + 3P$, for certain $P \in \mathbb{R}$.

Proposition 2.7. *Let us consider the extended Euler system expressed in standard form with $a_2 < 0$ and $h_1h_2h_3 \neq 0$. Thus, its solutions are periodic functions and satisfy*

$$\begin{array}{ll} \text{if } h_2 < 0: & \text{if } h_2 > 0: \\ \Pi_1(P+s) = \Pi_1(P-s), & \Pi_1(P+s) = -\Pi_1(P-s), \\ \Pi_2(P+s) = \Pi_2(P-s), & \Pi_2(P+s) = \Pi_2(P-s), \\ \Pi_3(P+s) = -\Pi_3(P-s), & \Pi_3(P+s) = \Pi_3(P-s). \end{array} \tag{2.26}$$

Furthermore, $\Pi_3(s)$ and $\Pi_2(s)$ are $4P$ periodic and $\Pi_1(s)$ is $2P$ periodic if $h_2 < 0$. The case $h_2 > 0$ implies $\Pi_1(s)$ and $\Pi_2(s)$ are $4P$ periodic and $\Pi_3(s)$ is $2P$ periodic. Finally P may be calculated as the integral given by

$$\int_0^1 \frac{\sqrt{|a_1/h_1|}\,dz}{\sqrt{(1-z^2)(1-\Omega_{3,1}^2 z^2)}} \quad or \quad \int_0^1 \frac{\sqrt{|a_3/h_3|}\,dz}{\sqrt{(1-z^2)(1-\Omega_{1,3}^2 z^2)}} \tag{2.27}$$

for $h_2 < 0$ and $h_2 > 0$ respectively. Where

$$\Omega_{i,j}^2 = \left|\frac{a_i h_i}{a_j h_j}\right| \in (0,1). \tag{2.28}$$

Proof. We assume $h_2 > 0$, the remaining case, $h_2 < 0$, is completely analogous. By Proposition 2.2, we can assure that the solution is built up of three periodic functions with rational ratios between their periods.

Let P be the increment of the independent variable needed to go from A to B, thus since the system is autonomous and proposition (2.3), we have that $\big(\Pi_1(P+s), \Pi_2(P+s), -\Pi_3(P+s)\big)$ and $\big(\Pi_1(P-s), \Pi_2(P-s), \Pi_3(P-s)\big)$ are both solutions of system (2.1) with initial conditions B, therefore (2.26) follows.

The period of each function is obtained taking into account that $\Pi(s_0) = A$. Thus, by assuming without lost of generality that $s_0 = 0$ and keeping in mind Corollary 2.4, we have that Π_2 is an odd function and Π_i, Π_k are even, therefore

$$\begin{aligned} \Pi_1(s+P) &= \Pi_1(-s+P) = \Pi_1(s-P), \\ \Pi_2(s+P) &= \Pi_2(-s+P) = -\Pi_2(s-P), \\ \Pi_3(s+P) &= \Pi_3(-s+P) = -\Pi_3(s-P), \end{aligned} \tag{2.29}$$

thus, $\Pi_1(s+2P) = \Pi_1(s)$, $\Pi_2(s+4P) = \Pi_2(s)$ and $\Pi_3(s+4P) = \Pi_3(s)$.

The value of P is finally obtained taking into account the property that characterizes it, $\Pi_i(s_0) = A$ and $\Pi_i(P) = B$, thus

$$P = \Pi_2^{-1}\left(\sqrt{\left|\frac{h_1}{a_3}\right|}\right) - \Pi_2^{-1}(0),$$

thus, taking into account that the integrals verify $h_1 < 0$ and $h_3 > 0$ when $a_2 < 0$

$$\dot{\Pi}_2 = a_2\,\Pi_1\Pi_3 = -\sqrt{|h_1h_3|}\,\sqrt{\left(1-\left(\sqrt{|a_3/h_1|}\Pi_2\right)^2\right)\left(1-\left(\sqrt{|a_1/h_3|}\Pi_2\right)^2\right)}$$

and the change of variable $z = \sqrt{|a_3/h_1|}\Pi_2$, we obtain

$$P = \int_0^1 \frac{\sqrt{|a_3/h_3|}}{\sqrt{(1-z^2)(1-\Omega_{1,3}^2 z^2)}}\,dz,$$

where

$$\Omega_{13}^2 = \left|\frac{a_1h_1}{a_3h_3}\right|,$$

finally, taking into account relation (2.12) and that $a_2 < 0$ is easy to see that $|a_1h_1| < |a_3h_3|$, therefore $\Omega_{13} \in (0,1)$ □

2.5 Connecting with the Jacobi Elliptic Functions

For historical reasons, the motion of the simple pendulum in particular, there is a system which has received special attention. More precisely the system where the parameters are given by $a_1 = 1$, $a_2 = -1$ and $a_3 = -k^2$, where $k \in (0,1)$, it corresponds with the Jacobi elliptic functions when suited initial conditions are taken into account, that is

$$\begin{aligned}
\frac{d}{ds}\mathrm{sn}(s,k) &= \mathrm{cn}(s,k)\,\mathrm{dn}(s,k),\\
\frac{d}{ds}\mathrm{cn}(s,k) &= -\mathrm{sn}(s,k)\,\mathrm{dn}(s,k),\\
\frac{d}{ds}\mathrm{dn}(s,k) &= -k^2\,\mathrm{sn}(s,k)\,\mathrm{cn}(s,k),
\end{aligned} \tag{2.30}$$

where $\mathrm{sn}(0,k) = 0$, $\mathrm{cn}(0,k) = 1$ and $\mathrm{dn}(0,k) = 1$. By identifying $\Pi_1(s) = \mathrm{sn}(s,k)$, $\Pi_2(s) = \mathrm{cn}(s,k)$, $\Pi_3(s) = \mathrm{dn}(s,k)$ and $\Omega_{3,1} = k$ and applying systematically Proposition 2.5 we obtain the following associated system to (2.30)

$$\dot{z}_1^0 = -z_1^3\,z_1^2, \qquad \dot{z}_1^2 = -z_1^3\,z_1^0, \qquad \dot{z}_1^3 = -z_1^2\,z_1^0. \tag{2.31}$$

$$\dot{z}_2^0 = z_2^3\, z_2^1, \qquad \dot{z}_2^1 = z_2^3\, z_2^0, \qquad \dot{z}_2^3 = k'^2\, z_2^1\, z_2^0, \tag{2.32}$$

and

$$\dot{z}_3^0 = k^2\, z_3^1\, z_3^2, \qquad \dot{z}_3^1 = z_3^2\, z_3^0, \qquad \dot{z}_3^2 = -k'^2\, z_3^1\, z_3^0, \tag{2.33}$$

which solutions are given by the twelve Jacobi elliptic functions defined as usual in the Glaisher notation

$$\begin{aligned}
z_1^0 &= \mathrm{ns}(s,k) = \frac{1}{\mathrm{sn}(s,k)} & z_1^2 &= \mathrm{cs}(s,k) = \frac{\mathrm{cn}(s,k)}{\mathrm{sn}(s,k)} & z_1^3 &= \mathrm{ds}(s,k) = \frac{\mathrm{dn}(s,k)}{\mathrm{sn}(s,k)} \\
z_2^0 &= \mathrm{nc}(s,k) = \frac{1}{\mathrm{cn}(s,k)} & z_2^1 &= \mathrm{sc}(s,k) = \frac{\mathrm{sn}(s,k)}{\mathrm{cn}(s,k)} & z_2^3 &= \mathrm{dc}(s,k) = \frac{\mathrm{dn}(s,k)}{\mathrm{cn}(s,k)} \\
z_3^0 &= \mathrm{nd}(s,k) = \frac{1}{\mathrm{dn}(s,k)} & z_3^1 &= \mathrm{sd}(s,k) = \frac{\mathrm{sn}(s,k)}{\mathrm{dn}(s,k)} & z_3^2 &= \mathrm{cd}(s,k) = \frac{\mathrm{cn}(s,k)}{\mathrm{dn}(s,k)}
\end{aligned} \tag{2.34}$$

As a final remark note that the integrals h_1, h_2 and h_3 associated to the system (2.30) with initial conditions $\mathrm{sn}(0,k) = 0$, $\mathrm{cn}(0,k) = 1$ and $\mathrm{dn}(0,k) = 1$ correspond to the *fundamental elliptic relations*, see Byrd and Friedman [Byrd and Friedman, 1971], page 20. Namely

$$-h_1 = k'^2 = \mathrm{dn}^2 - k^2\mathrm{cn}^2, \quad h_2 = 1 = \mathrm{dn}^2 + k^2\mathrm{sn}^2, \quad h_3 = 1 = \mathrm{sn}^2 + \mathrm{cn}^2, \tag{2.35}$$

where $k'^2 = 1 - k^2$.

2.5.1 Analytic solutions of the extended Euler system

We concern about the explicit formulas giving the solution of system (2.1). They are presented distinguishing between bounded and unbounded solutions. The reader should keep in mind that the system is expressed in *standard form* and taking into account Proposition 2.3, we restrict, without loss of generality, to initial conditions in the first octant.

Theorem 2.8 (Bounded analytic solutions)**.** *The solutions of the bounded extended Euler system expressed in* standard form*, with initial conditions*

$$\Pi_1(s_0) = \Pi_1^0 = \sqrt{\frac{h_3}{a_2}}, \quad \Pi_2(s_0) = \Pi_2^0 = 0, \quad \Pi_3(s_0) = \Pi_3^0 = \sqrt{\frac{h_1}{a_2}},$$

may be expressed equivalently in the following two forms

$$\Pi_1(s+s_0) = \sqrt{\left|\frac{h_3}{a_2}\right|} cn(\mu_1\, s, k_{31}), \quad \Pi_1(s+s_0) = \sqrt{\left|\frac{h_3}{a_2}\right|} dn(\mu_3\, s, k_{13}), \tag{2.36}$$

$$\Pi_2(s+s_0) = -\sqrt{\left|\frac{h_3}{a_1}\right|} sn(\mu_1\, s, k_{31}), \quad \Pi_2(s+s_0) = -\sqrt{\left|\frac{h_1}{a_3}\right|} sn(\mu_3\, s, k_{13}),$$

$$\Pi_3(s+s_0) = \sqrt{\left|\frac{h_1}{a_2}\right|} dn(\mu_1\, s, k_{31}), \quad \Pi_3(s+s_0) = \sqrt{\left|\frac{h_1}{a_2}\right|} cn(\mu_3\, s, k_{13}).$$

where the elliptic modulus is given by

$$k_{ij} = \sqrt{\left|\frac{a_i h_i}{a_j h_j}\right|}, \tag{2.37}$$

and $k_{31} \in (0,1)$ *iff* $a_2 h_2 > 0$*, for the case* $a_2 h_2 < 0$ *we have that* $k_{13} \in (0,1)$.

Proof. By simply substitution of the explicit formulas (2.36) in (2.1), it is shown that they satisfy the equations of the extended Euler system.

Taking into account that $a_1 h_1 < 0$, $a_3 h_3 > 0$ and relation (2.12), we have that $k_{31} \in (0,1)$ iff $a_2 h_2 > 0$ and $k_{13} \in (0,1)$ iff $a_2 h_2 < 0$. □

Corollary 2.9 (Jacobi's Real Transformation)**.** *Let* $k \in \mathbb{R} - \{0\}$ *and* $\kappa = k^{-1}$*. The following relations hold*

$$\mathrm{sn}(s,k) = \sqrt{\kappa}\,\mathrm{sn}(s,\kappa), \quad \mathrm{cn}(s,k) = \mathrm{dn}(s,\kappa), \quad \mathrm{dn}(s,k) = \mathrm{cn}(s,\kappa).$$

Proof. Taking into account that the expressions given in (2.36) are both solutions for system (2.1), the corollary is a direct consequence of Theorem 2.8 and the uniqueness theorem of differential equations. □

Corollary 2.10. *Let* $k > 0$*, the following relations hold*

$$\mathrm{sn}(s-P,-k) = \mathrm{cn}(k\,s,\mu), \quad \mathrm{cn}(s-P,-k) = -\mathrm{sn}(k\,s,\mu), \quad \mathrm{dn}(s-P,-k) = \mu'\,\mathrm{dn}(k\,s,\mu), \tag{2.38}$$

$$\mathrm{sn}(s+P,k) = -\mathrm{cd}(s,k), \quad \mathrm{cn}(s+P,k) = k'\mathrm{sd}(s,k), \quad \mathrm{dn}(s+P,-k) = k'\mathrm{nd}(s,k), \tag{2.39}$$

and

$$\mathrm{sn}(s,-k) = \mu'\mathrm{sd}(k\,s,\mu), \quad \mathrm{cn}(s,-k) = \mathrm{cd}(k\,s,\mu), \quad \mathrm{dn}(s,-k) = \mathrm{nd}(k\,s,\mu), \tag{2.40}$$

where

$$\mu = \sqrt{\frac{k^2}{1+k^2}}, \quad \mu' = \sqrt{1+k^2}. \tag{2.41}$$

Proof. $\mathrm{sn}(s,-k)$, $\mathrm{cn}(s,-k)$ and $\mathrm{dn}(s,-k)$ are defined as the solutions of the extended Euler system

$$\begin{aligned}
\dot{\mathrm{sn}}(s,k) &= \mathrm{cn}(s,k)\,\mathrm{dn}(s,k),\\
\dot{\mathrm{cn}}(s,k) &= -\mathrm{sn}(s,k)\,\mathrm{dn}(s,k),\\
\dot{\mathrm{dn}}(s,k) &= k^2\,\mathrm{sn}(s,k)\,\mathrm{cn}(s,k),
\end{aligned}$$

with initial conditions $\mathrm{sn}(0,-k)=0$, $\mathrm{cn}(0,-k)=1$ and $\mathrm{dn}(0,-k)=1$. Thus, using Lemma 2.1 we have that $\mathrm{sn}(-P,-k)=1$, $\mathrm{cn}(-P,-k)=0$ and $\mathrm{dn}(-P,-k)=1$. Therefore, direct application of the analytic solution given in Theorem 2.8 leads to (2.38). The same argument applied to system

$$\begin{aligned}
\dot{\mathrm{nd}}(s,k) &= k^2\mathrm{cd}(s,k)\,\mathrm{sd}(s,k),\\
\dot{\mathrm{cd}}(s,k) &= -k'^2\mathrm{sd}(s,k)\,\mathrm{nd}(s,k),\\
\dot{\mathrm{sd}}(s,k) &= \mathrm{cd}(s,k)\,\mathrm{nd}(s,k),
\end{aligned}$$

gives (2.38). Finally (2.40) follows from the combination of (2.38) and (2.39). □

Lemma 2.2. *Let us consider a extended Euler system in* standard form*, such that* $a_2 > 0$ *and* $h_i \neq 0$ *for* $i = 1,2,3$. *The initial conditions in the first octant are characterized as follows*

$$h_2 < 0, h_3 > 0 \Leftrightarrow (\Pi_1(s_0), \Pi_2(s_0), \Pi_3(s_0)) = (0, \sqrt{\frac{h_3}{a_1}}, \sqrt{\frac{h_2}{a_1}}) = A_1 \tag{2.42}$$

$$h_3 < 0, h_1 > 0 \Leftrightarrow (\Pi_1(s_0), \Pi_2(s_0), \Pi_3(s_0)) = (\sqrt{\frac{h_3}{a_2}}, 0, \sqrt{\frac{h_1}{a_2}}) = A_2 \tag{2.43}$$

$$h_1 < 0, h_2 > 0 \Leftrightarrow (\Pi_1(s_0), \Pi_2(s_0), \Pi_3(s_0)) = (\sqrt{\frac{h_2}{a_3}}, \sqrt{\frac{h_1}{a_3}}, 0) = A_3 \tag{2.44}$$

on account of relation (2.12), the above cases cover all the possible combinations of integrals' signs.

Proof. For the unbounded case, according to the definition of the integrals (2.2), there is one and only one point in the intersection $h_1 \cap h_2 \cap h_3$ which one of the coordinates vanishes at any $s_0 \in \mathbb{R}$. The above cases are deduced by simply substitution of $(\Pi_1(s_0), \Pi_2(s_0), \Pi_3(s_0))$ in the integrals. □

Theorem 2.11 (Unbounded analytic solutions)**.** *The solutions of the unbounded extended Euler system expressed in* standard form*, with initial conditions* $\Pi(s_0) = A_i$*,* $i \in \{1, 2, 3\}$*, see Lemma 2.2, are given by the following expression*

$$\begin{aligned} \Pi_i(s + s_0) &= \sqrt{\left|\frac{h_k}{a_j}\right|} sc(\mu_j\, s, k_{ij}), \\ \Pi_j(s + s_0) &= \sqrt{\left|\frac{h_k}{a_i}\right|} nc(\mu_j\, s, k_{ij}), \\ \Pi_k(s + s_0) &= \sqrt{\left|\frac{h_j}{a_i}\right|} dc(\mu_j\, s, k_{ij}), \end{aligned} \tag{2.45}$$

where the subindex $i, j, k \in \{1, 2, 3\}$ *are chosen according to the following criteria*

$$\begin{aligned} \Pi(s_0) = A_1, h_1 > 0 &\Rightarrow i = 1,\ j = 2,\ k = 3, \\ \Pi(s_0) = A_1, h_1 < 0 &\Rightarrow i = 1,\ j = 3,\ k = 2 \\ \Pi(s_0) = A_2, h_2 > 0 &\Rightarrow i = 2,\ j = 3,\ k = 1, \\ \Pi(s_0) = A_2, h_2 < 0 &\Rightarrow i = 2,\ j = 1,\ k = 3 \\ \Pi(s_0) = A_3, h_3 > 0 &\Rightarrow i = 3,\ j = 1,\ k = 2, \\ \Pi(s_0) = A_3, h_3 < 0 &\Rightarrow i = 3,\ j = 2,\ k = 1. \end{aligned} \tag{2.46}$$

Proof. Taking into account relation (2.2) and the choice of i, j and k given by the formulas (2.46) we have that the elliptic modulus $k_{ij} \in (0, 1)$. Thus, a straightforward substitution of (2.45) in (2.1) shows that the system is verified. □

Part II

Hopf Reduction on a Quartic Polynomial Model

Chapter 3

The Quaternionic Hopf Fibration

Along this chapter we intend to present the needed elements for the definition of the parametric polynomial model that we present in Chapter 4. First, in order to fix notation, a detailed tour on the field of quaternions is given. Then, a key object in this work is introduced, the quaternionic Hopf fibration. It is a generalization of the classic Hopf fibration, that supplies the ingredients to define the named model, induces its Poisson reduction and gives the geometry of the reduced space in which the reduced model lives.

3.1 Quaternions and Rotations

Quaternion are the generalization of the complex number to a hyper-complex kind. In fact we can consider the field of the real numbers as the hyper-complex of rank 1 and the complex field the hyper-complex of rank 2. However, it turns out that any set of hyper-complex numbers having rank greater than 2 fails to be a field.

As a group, the unit quaternions have the same algebra as the three dimensional rotations so it is reasonable to assume that they can somehow be used to rotate vectors. In fact the quaternions primary application is the quaternion rotation operator, which plays an important role in classical mechanics.

3.1.1 The field of quaternions

In 1843 William Rowan Hamilton [Hamilton, 1844] invented the hyper-complex numbers of rank 4, which he gave the name *quaternion*, it is a division ring $\mathbb{H}$ (or, by abuse of language, a non-commutative field), which elements are denoted by

$\mathbf{q} = (q_1, q) \in \mathbb{H}$, and may be regarded as a real part q_1 plus the imaginary vector part $q = (q_2, q_3, q_4) \in \mathbb{R}^3$. Quaternions having zero scalar part are called *pure quaternions* and in the following will be regarded both as a vector in $\mathbb{R}^3$ and as a quaternion. Therefore there is a bijective identification between $\mathbb{R}^3$ and the pure quaternions.

The quaternions, together with the operations of addition $(+)$ by components and scalar multiplication $(\cdot)$, may be identified with $\mathbb{R}^4$ as a vector space. Quaternion multiplication $(\circ)$ will provide $\mathbb{H}$ with a division ring structure, due to the non-commutative feature of this operation.

It is customary to use the notation $\{1, i, j, k\}$ for a basis in $(\mathbb{H}, +, \cdot)$, that is, $1 = (1, 0, 0, 0)$, $i = (0, 1, 0, 0)$, $j = (0, 0, 1, 0)$ and $k = (0, 0, 0, 1)$, thus elements in $\mathbb{H}$ may be expressed as follows

$$\mathbb{H} := \{\mathbf{q} = q_1 \cdot 1 + q_2 \cdot i + q_3 \cdot j + q_4 \cdot k,\ q_1,\ q_2,\ q_3,\ q_4 \in \mathbb{R}\},$$

quaternionic multiplication is performed in the usual manner, like polynomial multiplication, taking the following relations into account

$\circ$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Note that the relations given in the table for the multiplication may be deduced directly from the ones that Hamilton gave initially, namely $i^2 = j^2 = k^2 = ijk = -1$.

Also we can think of $\{1, i, j, k\}$ as the four roots of unity. An alternative way of defining the quaternionic product making use of the "dot" and "cross" product in $\mathbb{R}^3$ is given by

$$\mathbf{q} \circ \mathbf{Q} = (q_1 Q_1 - q \cdot Q\,,\, q_1 Q\,+\,Q_1 q\,+\,q \times Q). \tag{3.1}$$

For the sake of a cleaner notation, we will drop the symbols $(\cdot)$ and $(\circ)$, they will be used just in case of possible confusion. Note also that "dot" is used to denote two different products, we distinguish them by the context.

Definition (3.1) is, of course, directly deduced from the relations given above and may be written explicitly in term of coordinates as follows

$$\begin{aligned}\mathbf{qQ} &= q_1Q_1 - (q_2Q_2 + q_3Q_3 + q_4Q_4) \\ &\quad +(q_1Q_2 + q_2Q_1 + q_3Q_4 - q_4Q_3)\, i \\ &\quad +(q_1Q_3 - q_2Q_4 + q_3Q_1 + q_4Q_2)\, j \\ &\quad +(q_1Q_4 + q_2Q_3 - q_3Q_2 + q_4Q_1)\, k,\end{aligned} \tag{3.2}$$

or written in matrix notation as

$$\mathbf{qQ} = M_L(\mathbf{q})\mathbf{Q} = \mathbf{q}^t M_R(\mathbf{Q}), \tag{3.3}$$

where $\mathbf{q}$ and $\mathbf{Q}$ are regarded as column vectors and the matrices $M_L(\mathbf{q})$ and $M_R(\mathbf{Q})$ are built in the following manner,

$$M_L(\mathbf{z}) = \begin{pmatrix} z_1 & -z_2 & -z_3 & -z_4 \\ z_2 & z_1 & -z_4 & z_3 \\ z_3 & z_4 & z_1 & -z_2 \\ z_4 & -z_3 & z_2 & z_1 \end{pmatrix} \qquad M_R(\mathbf{z}) = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 \\ -z_2 & z_1 & -z_4 & z_3 \\ -z_3 & z_4 & z_1 & -z_2 \\ -z_4 & -z_3 & z_2 & z_1 \end{pmatrix}, \tag{3.4}$$

note that the subindex L and R in the matrices above stand for left and right respectively, in reference to the fact that $M_L(\mathbf{z})$ is the matrix form for the quaternion located on the left in (3.3) and $M_R(\mathbf{z})$ refers to the one located on the right.

In addition, every quaternion $\mathbf{q} = (q_1, q)$ has a *conjugate* $\bar{\mathbf{q}} = (q_1, -q)$, that is, the real numbers are fixed by the conjugation and $\bar{i} = -i$, $\bar{j} = -j$, $\bar{k} = -k$. Note that $\overline{\mathbf{qQ}} = \bar{\mathbf{Q}}\bar{\mathbf{q}}$.

The usual *hermitian inner product* is defined in $\mathbb{H}$ as

$$\langle \mathbf{q}, \mathbf{Q} \rangle = \mathbf{q}\bar{\mathbf{Q}}, \tag{3.5}$$

such a inner product is extended in a natural way to $\mathbb{H}^2$, the vectorial space made of column vectors $(\mathbf{q},\mathbf{Q})^T$, where $\mathbf{q},\mathbf{Q} \in \mathbb{H}$

$$\langle (\mathbf{q}, \mathbf{Q}), (\mathbf{p}, \mathbf{P}) \rangle = \mathbf{q}\bar{\mathbf{Q}} + \mathbf{p}\bar{\mathbf{P}}. \tag{3.6}$$

Note that the hermitian inner product may also be defined by

$$\langle \mathbf{q}, \mathbf{Q} \rangle = \bar{\mathbf{q}}\mathbf{Q}. \tag{3.7}$$

Another important concept is the *norm*, it is denoted by $\|\mathbf{q}\|$, sometimes called the length of $\mathbf{q}$, is the scalar defined by

$$\|\mathbf{q}\| = \sqrt{\mathbf{q}\bar{\mathbf{q}}} = \sqrt{\langle \mathbf{q}, \mathbf{q} \rangle},$$

notice that this definition is the same as that for the length of a vector in $\mathbb{R}^4$, or equivalently the Euclidean norm. In the following section we will deal with quaternions satisfying $\|\mathbf{q}\| = 1$, such a quaternion is named the *unit or normalized quaternion*. The set of unit quaternions will be noted by $\mathbb{S}^3$.

Finally we give the expression for the *inverse*, for every $\mathbf{q} \neq 0$ there exists another quaternion $\mathbf{q}^{-1}$, which will be noted as the inverse and is given by

$$\mathbf{q}^{-1} = \frac{\bar{\mathbf{q}}}{\|\mathbf{q}\|^2}.$$

3.1.2 Quaternions as rotations

Our target here is to show the well known relation between the unit quaternions and rotations in the three dimensional space. On the contrary to what one may think, it is not the usual multiplication which gives the named relation, but the triple multiplication operator that will be defined next.

Let us introduce some notation for the group of rotations. Every rotation maps an orthonormal basis of $\mathbb{R}3$ to another orthonormal basis. Like any linear transformation of finite-dimensional vector spaces, a rotation can always be represented by a matrix. Let R be a given rotation. With respect to the standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}3$ the columns of R are given by $(R\,e_1, R\,e_2, R\,e_3)$. Since the standard basis is orthonormal, the columns of R form another orthonormal basis, that is to say $R^T R = Id_3$. Matrices for which this property holds are called orthogonal matrices. The group of all 3×3 orthogonal matrices is denoted $O(3)$, and consists of all proper and improper rotations.

In addition to preserving length, proper rotations must also preserve orientation. A matrix will preserve or reverse orientation according to whether the determinant of the matrix is positive or negative. For an orthogonal matrix R, note that $det(R^T) = det(R^{-1})$ implies $\big(det(R)\big)^2 = 1$ so that $det(R) = \pm 1$. The subgroup of orthogonal matrices with determinant $+1$ is called the special orthogonal group, denoted $SO(3)$.

Thus every rotation can be represented uniquely by an orthogonal matrix with unit determinant. Moreover, since composition of rotations corresponds to matrix multiplication, the rotation group is isomorphic to the special orthogonal group $SO(3)$. From now on, we denote by $SO(3)$ both the rotation group and the special orthogonal group.

Every rotation in the space is determined by a unitary vector $\vec{u}$ and an angle α. Therefore, it is natural to expect that we should codify that vector and the angle with a quaternion, then, by means of some kind of operator acting on the pure imaginary quaternions, to obtain the original rotation.

Consider the rotation $\mathbf{R}(\alpha, \vec{u})$ of angle α around the axis determined by the unitary vector $\vec{u}$. Thus, the following theorem states the correspondence between $SO(3)$ and the unit quaternions.

Theorem 3.1. *Given a rotation $\mathbf{R}(\alpha, \vec{u})$ in $SO(3)$ determined by a unitary vector $\vec{u}$ and an angle α, let $\mathbf{q}$ be the following unitary quaternion*

$$\mathbf{q} = (\cos\frac{\alpha}{2}, \vec{u}\sin\frac{\alpha}{2}) \in \mathbb{S}^3, \tag{3.8}$$

and the operator $L_{\mathbf{q}}$ defined as

$$L_{\mathbf{q}} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3,\ \vec{x} \longrightarrow \mathbf{q}\,\vec{x}\,\bar{\mathbf{q}}, \tag{3.9}$$

thus the following properties are satisfied:

1. *For any $\vec{x} \in \mathbb{R}^3$, $L_{\mathbf{q}}(\vec{x}) \in \mathbb{R}^3$.*
2. *$L_{\mathbf{q}}$ is a linear application.*
3. *$L_{\mathbf{q}}$ is a rotation of angle α around the axis determined by the unitary vector $\vec{u}$*
4. *The associated matrix to $L_{\mathbf{q}}$ is given by*

$$M_{L_{\mathbf{q}}} = \begin{pmatrix} q_1^2+q_2^2-q_3^2-q_4^2 & 2(q_2q_3-q_1q_4) & 2(q_2q_4+q_1q_3) \\ 2(q_2q_3+q_1q_4) & q_1^2-q_2^2+q_3^2-q_4^2 & 2(q_3q_4-q_2q_1) \\ 2(q_2q_4-q_1q_3) & 2(q_3q_4+q_2q_1) & q_1^2-q_2^2-q_3^2+q_4^2 \end{pmatrix}. \tag{3.10}$$

Proof. See Kuipers [Kuipers, 1999], pages 124 to 134. □

Note that in order to provide a coherent definition in (3.9), it is crucial to use the equivalence between $\mathbb{R}^3$ and pure imaginary quaternions.

The operator $L_{\mathbf{q}}$ defined in Theorem 3.1 induces a mapping between the unit quaternions denoted by $\mathbb{S}^3$ and the group of rotations $SO(3)$, which is given by

$$\mathbf{R} : \mathbb{S}^3 \longrightarrow SO(3),\ \mathbf{q} \longrightarrow \mathbf{R}(\mathbf{q}) = M_{L_{\mathbf{q}}} \tag{3.11}$$

Corollary 3.2. *The map* $\boldsymbol{R}$ *defined above is a surjective two to one map. Equivalently,* $SO(3)$ *is doubly covered by* $\mathbb{S}^3$.

Proof. This result is obtained taking into account that $L_{\mathbf{q}}$ and $L_{\text{-}\mathbf{q}}$ are equal, that is, $\mathbf{q}$ and $-\mathbf{q}$ determine the same rotation. □

Corollary 3.3. $\mathbb{S}^3$ *and* $SO(3)$ *are groups with the quaternion and matrix multiplication respectively and* $\boldsymbol{R}$ *satisfies that*

$$\boldsymbol{R}(\boldsymbol{qr}) = \boldsymbol{R}(\boldsymbol{q})\boldsymbol{R}(\boldsymbol{r}). \tag{3.12}$$

In other words, the map $\boldsymbol{R}$ *is a group homomorphism.*

Proof. After some computations and taking into account that $\mathbf{R}(q) = M_{L\mathbf{q}}$, the equality (3.12) is obtained. □

As the groups $\mathbb{S}^3$ and $SO(3)$ are also Lie groups, it is possible to establish a relation between their corresponding Lie algebras. All those relations are summarized up in the figure 3.1.

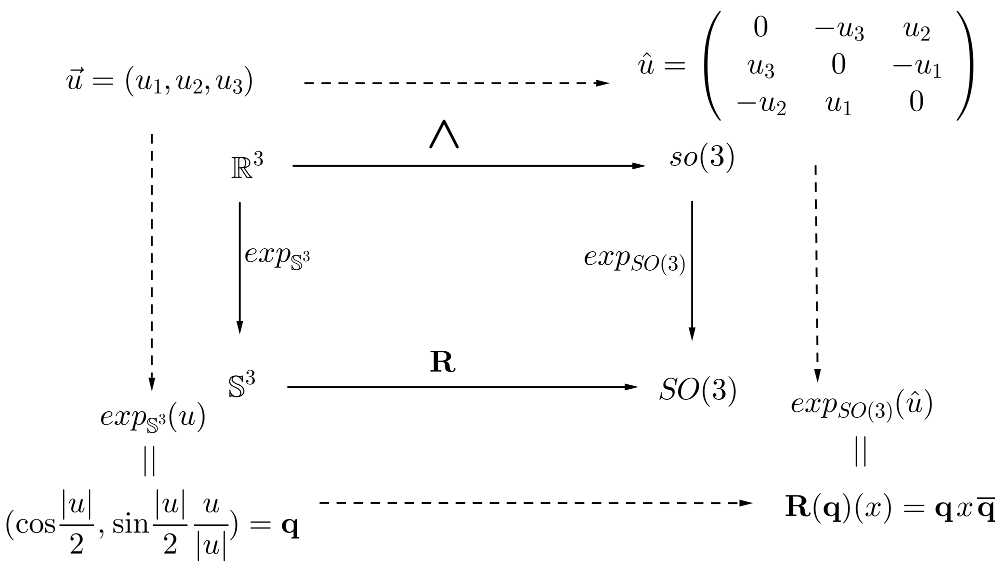


Figure 3.1: Double covering of $SO(3)$ by S^3 and the Lie algebras associated.

Note that the wedge map $\wedge$ is a isomorphism between the Lie-algebras associated.

3.2 Hopf Fibrations

In this section we start recalling the classical Hopf fibration and then we focus in a quaternionic generalization, which plays a key role in our study. Furthermore, it is well known that the Hopf fibration and its generalizations have many implications, some purely attractive, others deeper, see [Urbantke, 2003]. There are many ways in which the Hopf fibration may be generalizated, see for example [Gluck et al., 1986]. The unit sphere in complex coordinate space $\mathbb{C}^{n+1}$ fibers naturally over the complex projective space $\mathbb{CP}^n$ with circles as fibers, and there are also real, quaternionic, and octonionic versions of these fibrations. In particular, the Hopf fibration belongs to a family of fiber bundles in which the total space, base space, and fiber space are all spheres.

3.2.1 The classical Hopf fibration

The Hopf fibration, named after Heinz Hopf who studied it in 1931 [Hopf, 1931], it is an influential early example of a fiber bundle, the Hopf application maps each $\mathbb{S}^1$ circle in the hypersphere $\mathbb{S}^3$ to a point in $\mathbb{S}^2$. Thus the hypersphere $\mathbb{S}^3$ is made up of $\mathbb{S}^1$ fibers, one for each point on $\mathbb{S}^2$. The Hopf fibration, like any fiber bundle, has the important property that it is locally a product space. However it is not a trivial fiber bundle, i.e., $\mathbb{S}^3$ is not globally a product of $\mathbb{S}^2$ and $\mathbb{S}^1$.

In the classical Hopf fibration the ambient space is the real vector space $\mathbb{R}^4$. The identification between $\mathbb{R}^4$ and $\mathbb{C}^2$ allows to consider the complex vectorial space $\mathbb{C}^2$ as the ambient space, together with the usual hermitian inner product. Let $z \in \mathbb{C}^2$ and $w \in \mathbb{C}^2$, where $z = (z_1, z_2)$, $w = (w_1, w_2)$ and $z_1, z_2, w_1, w_2 \in \mathbb{C}$, then

$$\langle z, w\rangle = \bar{z}w = \bar{z}_1 w_1 + \bar{z}_2 w_2,$$

thus $\mathbb{S}^3$ can be shown as a subset in $\mathbb{C}^2$ given by $\mathbb{S}^3 = \{z \in \mathbb{C}^2 \mid \rangle z, z\langle = 1\}$, and defining a equivalence relation on $\mathbb{S}^3$ by $z \sim \omega$ iff $\omega = \lambda z$, $\lambda \in \mathbb{S}^1$ we get that

$$\mathbb{S}^3/\sim \; \cong P(\mathbb{C}^2) = \mathbb{CP}^1.$$

Therefore, by means of the stereographic projection $\mathbb{C}P^1 \to \mathbb{S}^2$, the classic Hopf fibration $\Pi : \mathbb{S}^3 \to \mathbb{S}^2$ is built in a constructive process that we describe next by mean of the commutative diagram 3.2.1. where Π is the stereographic projection, P_r is the application from $\mathbb{C}^2 - \{0\}$ to $\mathbb{S}^3(r)$ that match each semi-ray through the origin with the corresponding element of module equal to r, $P_\sim$ from $\mathbb{S}^3(r)$ into $\mathbb{C}P^1$ identify each point in the sphere to its corresponding class of equivalence, $P^{\rho}_{1,2,3}$ is

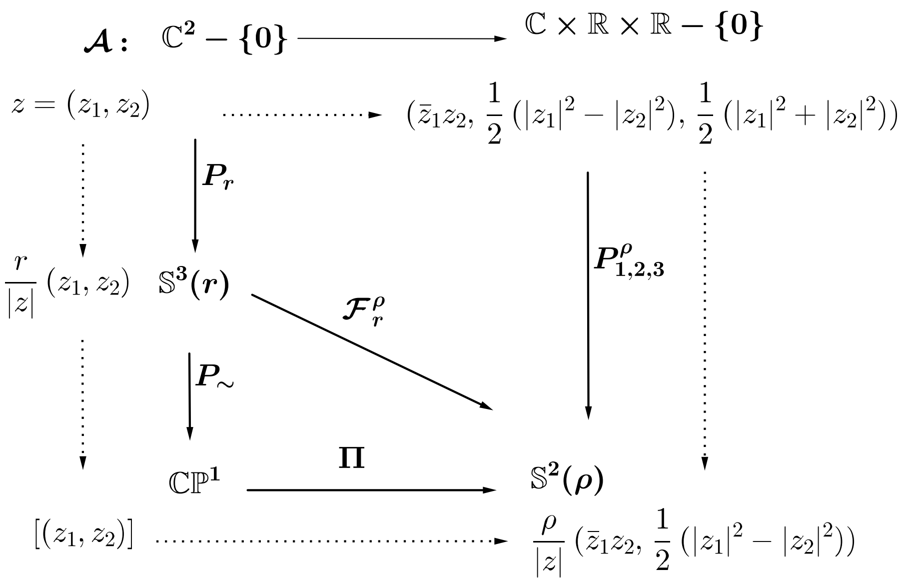


Figure 3.2: Hopf application and Hopf classic fibration diagram.

the elimination of the fourth component factor ρ and $\mathcal{A}$ is the Hopf application. The Hopf fibration $\mathcal{F}^{\rho}_r$ from $\mathbb{S}^3(r)$ to $\mathbb{S}^2(\rho)$ is given by composing Π and $P_\sim$, also it can be obtained by means of the restriction of $\mathcal{A}$, the Hopf application, to $\mathbb{S}^3(r)$ and composing it with $P^{\rho}_{1,2,3}$.

Notice that $\Pi : \mathbb{C}P^1 \longrightarrow \mathbb{S}^2(\rho)$ is based on the classic stereographic projection from $\mathbb{S}^2(\rho) - N$ onto the real plane, but in this case the north pole is covered by the infinite point $[(1, 0)]$.

Moving to cartesian coordinates, and taking into account the natural relation between $\mathbb{C} = \{x - yi/\, x, y \in \mathbb{R}\}$ and $\mathbb{R}^2 = \{(x, y)/\, x, y \in \mathbb{R}\}$ given by $x + yi \to (x, y)$, we lead to:

$$\mathcal{A}(q_1, q_2, Q_1, Q_2) = 2(\omega_1(q, Q), \omega_2(q, Q), \omega_3(q, Q), \omega_4(q, Q)).$$

where

$$\omega_1 = q_1 Q_2 - q_2 Q_1, \qquad \omega_3 = \frac{1}{2}(q_1^2 + q_2^2 - Q_1^2 - Q_2^2),$$
$$\omega_2 = q_1 Q_1 + q_2 Q_2, \qquad \omega_4 = \frac{1}{2}(q_1^2 + q_2^2 + Q_1^2 + Q_2^2).$$

Proposition 3.4. *The following relations between the ω_i's hold*

$$\mathcal{C}(q_1, q_2, Q_1, Q_2) = \omega_1^2 + \omega_2^2 + \omega_3^2 - \omega_4^2 = 0 \tag{3.13}$$

Notice that the image of $\mathbb{C}^2 - \{0\}$ by the Hopf application is given by the algebraic manifold defined in 3.13 by $\mathcal{C}$, it is a 3-dimensional cone with vertex in the coordinate origin.

Theorem 3.5. *The Hopf fibration $\mathcal{F}_r^\rho$ satisfy the following properties*

1. *$\mathcal{F}_r^\rho$ is a proper submersion.*
2. *For each $\omega \in \mathbb{S}^2(\rho)$, the fiber ${\mathcal{F}_r^\rho}^{-1}(\omega)$ is a circumference $\mathbb{S}^1(r)$.*
3. *$\mathbb{S}^3(r)$ is a principal bundle over the base space $\mathbb{S}^2(\rho)$ together with the structural group $\mathbb{S}^1$.*

Proof. The first claim is derived from the fact that $Dim(\mathbb{S}^3(r)) > Dim(\mathbb{S}^2(\rho))$ and the differential $D\mathcal{F}_r^\rho : T_p\mathbb{S}^3(r) \longrightarrow \mathbb{S}^2(\rho)$ is a surjective map between tangent spaces, i. e., $\mathcal{F}_r^\rho$ is a proper submersion.

In the second claim, we know from the definition of the Hopf fibration that $\mathcal{F}_r^\rho = P_\sim \circ \Pi$, see figure 3.2.1. As the stereographic projection Π is a diffeomorphism, we have that the pre-image on any point in $\mathbb{S}^1(r)$ is a single class of equivalence in $\mathbb{C}P^1$ $\Pi^{-1}(\omega) = [(z_1, z_2)]$, and the pre-image by $P_\sim$ is given by

$$P_\sim([(z_1, z_2)]) = \{(a, b)/\lambda(a, b) = (z_1, z_2),\ \lambda \in \mathbb{S}^1,\ |a| + |b| = r\} \equiv \mathbb{S}^1(r),$$

thus ${\mathcal{F}_r^\rho}^{-1}(\omega) = \mathbb{S}^1(r)$ for each $\omega \in \mathbb{S}^2(\rho)$.

Finally the third claim is derived from 1 and 2. □

3.2.2 The quaternionic Hopf fibration

Our aim in this section is to introduce a 4-D generalization of the classical Hopf fibration by mean of a increasing on the manifold's dimension. Next we are going to follow the same methodology than in the classic Hopf fibration $\mathcal{F}_r^\rho : \mathbb{S}^3(r) \to \mathbb{S}^2(\rho)$, but instead of the complex numbers use quaternions $\mathbb{H}$. From now on, we will use the subindex $_\mathbb{H}$ to distinguish between the classic and generalized Hopf fibration, application, etc, but, every time it does not mean any confusion, we will drop the mentioned subindex for the shake of a cleaner notation.

The classic Hopf application $\mathcal{A}$, could be extended from the complex plane $\mathbb{C}^2$ to the quaternionic plane $\mathbb{H}^2$ together with the inner product

$$\langle(\mathbf{q}, \mathbf{Q}), (\mathbf{p}, \mathbf{P})\rangle = \mathbf{q}\bar{\mathbf{p}} + \mathbf{Q}\bar{\mathbf{P}},$$

where $(\mathbf{q},\mathbf{Q})$, $(\mathbf{p},\mathbf{P}) \in \mathbb{H}^2$. Here we define the extension of the classical Hopf application to $\mathbb{H}^2$, given by $\mathcal{A}_{\mathbb{H}} : \mathbb{H}^2 \equiv \mathbb{R}^8 \longrightarrow \mathbb{R}^6 \equiv \mathbb{H} \times \mathbb{R} \times \mathbb{R} - \{0\}$, see the following scheme.

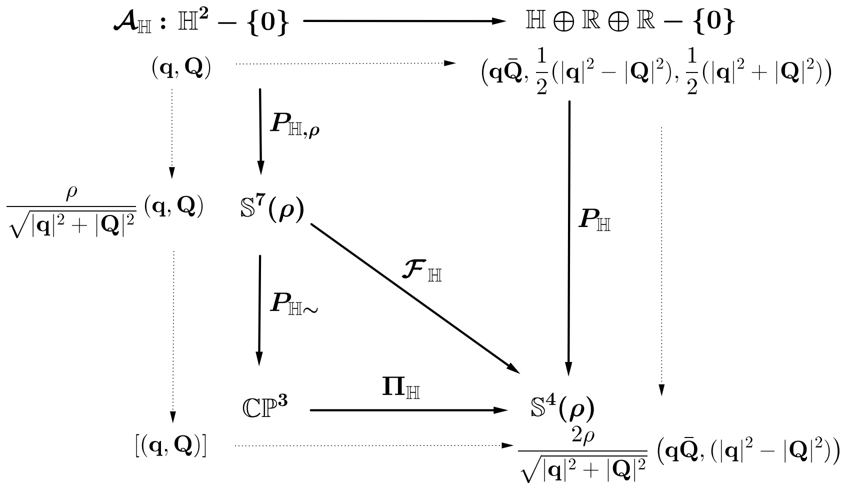


Figure 3.3: Generalized Hopf application and fibration.

Moving to Euclidean coordinates $\mathbf{q} = (q_1,\, q_2,\, q_3,\, q_4)$, after the natural equivalence $\mathbb{H} \equiv \mathbb{R}^4$, the Hopf application is given by

$$\begin{aligned} \mathcal{A}_{\mathbb{H}}(\mathbf{q},\mathbf{Q}) &= \left(\langle \mathbf{q}\mathbf{Q}\rangle, \frac{1}{2}(\mathbf{q}\bar{\mathbf{q}} - \mathbf{Q}\bar{\mathbf{Q}}), \frac{1}{2}(\mathbf{q}\bar{\mathbf{q}} + \mathbf{Q}\bar{\mathbf{Q}})\right) \qquad (3.14) \\ &= (\omega_1(\mathbf{q},\mathbf{Q}), \omega_2(\mathbf{q},\mathbf{Q}), \omega_3(\mathbf{q},\mathbf{Q}), \omega_4(\mathbf{q},\mathbf{Q}), \omega_5(\mathbf{q},\mathbf{Q}), \omega_6(\mathbf{q},\mathbf{Q})), \end{aligned}$$

where

$$\begin{aligned} \omega_1(\mathbf{q},\mathbf{Q}) &= q_1Q_1 + q_2Q_2 + q_3Q_3 + q_4Q_4, \\ \omega_2(\mathbf{q},\mathbf{Q}) &= q_1Q_2 - q_2Q_1 - q_3Q_4 + q_4Q_3, \\ \omega_3(\mathbf{q},\mathbf{Q}) &= q_1Q_3 - q_3Q_1 + q_2Q_4 - q_4Q_2, \\ \omega_4(\mathbf{q},\mathbf{Q}) &= q_1Q_4 - q_4Q_1 - q_2Q_3 + q_3Q_2, \\ \omega_5(\mathbf{q},\mathbf{Q}) &= \frac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2 - Q_1^2 - Q_2^2 - Q_3^2 - Q_4^2) = \frac{1}{2}(\|\mathbf{q}\|^2 - \|\mathbf{Q}\|^2), \\ \omega_6(\mathbf{q},\mathbf{Q}) &= \frac{1}{2}(q_1^2 + q_2^2 + q_3^2 + q_4^2 + Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2) = \frac{1}{2}(\|\mathbf{q}\|^2 + \|\mathbf{Q}\|^2), \end{aligned} \qquad (3.15)$$

and $\langle \mathbf{qQ} \rangle$ is the Hermitic product defined in (3.5). A different choice is possible, that is, the Hermitic product may also be defined by $\langle \mathbf{q}, \mathbf{Q} \rangle = \bar{\mathbf{q}}\mathbf{Q}$. This alternative commutes ω_2, ω_3 and ω_4 into ω_7, ω_8 and ω_9 respectively

$$\begin{aligned} \omega_7(\mathbf{q}, \mathbf{Q}) &= q_2 Q_1 - q_1 Q_2 + q_4 Q_3 - q_3 Q_4, \\ \omega_8(\mathbf{q}, \mathbf{Q}) &= q_3 Q_1 - q_1 Q_3 - q_4 Q_2 + q_2 Q_4, \\ \omega_9(\mathbf{q}, \mathbf{Q}) &= q_3 Q_2 - q_2 Q_3 + q_4 Q_1 - q_1 Q_4. \end{aligned} \tag{3.16}$$

The following identities relating the functions defined above holds

$$4M^2 = \omega_7^2 + \omega_8^2 + \omega_9^2 = \omega_6^2 - \omega_5^2 - \omega_1^2 = \omega_2^2 + \omega_3^2 + \omega_4^2, \tag{3.17}$$

where is defined by

$$M = \frac{1}{2}\sqrt{\|\mathbf{q}\|\, \|\mathbf{Q}\| - \langle \mathbf{q}, \mathbf{Q} \rangle}. \tag{3.18}$$

It is easy to check that assuming $q_3 = q_4 = Q_3 = Q_4 = 0$, we get the classic Hopf application. Note also that the components of the Hopf map, ω_i, $i = 1 \ldots, 6$, satisfy the following relation

$$\mathcal{C}(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6) = \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 - \omega_6^2 = 0, \;\; \omega_6 \geq 0, \tag{3.19}$$

that is, they are restricted to a cone in $\mathbb{R}^6$ given by $\mathcal{C}$. Note that $\mathcal{C}$ is also used in Proposition 3.4, we distinguish both of them by the context in which they are used. By eliminating the vertex $\mathcal{C}^* = \mathcal{C} - \{0\}$, it can be proven that $\mathcal{A}_{\mathbb{H}}(\mathbb{H}^2) = \mathcal{C}^*$. Now if we restrict $\mathcal{A}_{\mathbb{H}}$ to the 7-dimensional sphere of radius ρ

$$\mathbb{S}^7(\rho) = \{(\mathbf{q}, \mathbf{Q}) \in \mathbb{R}^8 \mid q_1^2 + q_2^2 + q_3^2 + q_4^2 + Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 = \rho^2 = 2\omega_6\}$$

we obtain the generalized Hopf fibration

$$\begin{array}{rccc} \mathcal{F}_{\mathbb{H}}(\mathbf{q}, \mathbf{Q}): & \mathbb{S}^7(\rho) & \longrightarrow & \mathbb{S}^4(\rho) \\ & (\mathbf{q}, \mathbf{Q}) & \rightsquigarrow & \dfrac{\rho}{\sqrt{\omega_6}}(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) \end{array} \tag{3.20}$$

where $\mathbb{S}^4(\rho)$ is the 4-dimensional sphere of radius ρ.

Theorem 3.6. *The Hopf application $\mathcal{F}$ satisfies the following properties*

1. *$\mathcal{F}$ is a proper submersion.*
2. *For each $\omega \in \mathbb{S}^4(\rho)$, the fiber $\mathcal{F}^{-1}(\omega)$ is a 3-sphere $\mathbb{S}^3(\rho)$.*
3. *$\mathbb{S}^7$ is a principal bundle over the base space $\mathbb{S}^4$ together with the structural group $\mathbb{S}^3$.*

Proof. Is the same reasoning followed in the proof of the analogous theorem given for the classical Hopf fibration. □

Chapter 4

The Quartic Polynomial Model

A parametric family of Hamiltonian functions in $\mathbb{T}^*\mathbb{R}^4$ is proposed and examined, related to the components of the quaternionic Hopf mapping. This family is a homogeneous quartic polynomial with six parameters, defining an integrable family of non-natural Hamiltonian systems. Additionally, a detailed study will reveal that it is an integrable system in the Liouville sense. The key feature of the model is its nested Hamiltonian-Poisson structure, which appears as two extended Euler systems in the reduced equations. This is fully exploited in the process of integration, where we find two 1-DOF subsystems and a quadrature involving both of them. The solution is quasi-periodic, expressed by means of Jacobi elliptic functions and integrals, based on two periods. Some remarkable classical models such as the Kepler, geodesic flow, isotropic oscillator and free rigid body systems are obtained as constrained flows for particular choices of the parameters and using a suitable set of variables. In this regard, we set a framework to study, in a unified way, these classical integrable models in mechanics. This idea goes back to the work of [Ferrer, 2010] and the references therein.

Let us refer now to the motivation of this model. Along the last century the connection between different classical models has been studied in detail; the reader will find in Cushman and Bates [Cushman and Bates, 1997] a panorama about this, although its echo even reaches until today [Saha, 2009, Waldvogel, 2006, Waldvogel, 2008]. More precisely we refer to the geodesic, Kepler and isotropic oscillator systems. Either by the KS [Kustaanheimo and Stiefel, 1965] or the stereographic transformation [Moser, 1970] the 3 dimensional Kepler system is brought into systems on submanifolds of $\mathbb{R}^4$. Moreover the rigid body dynamics admits two representations, either by the $\mathbb{SO}(3)$ group materialized by Euler angles or by means of unit quaternions based on Euler parameters, which may be considered also a submanifold in $\mathbb{R}^4$. Thus, one may expect that a system including all those classical ones should be defined as a 4-DOF system. Our main goal is to develop a generic scheme in order to deal with perturbation theories based on it.

For an earlier version of this study we refer to [Ferrer and Crespo, 2015]. Later on, we address two applications focused in the roto-translatory problem. Let us define this family as follows

Definition 4.1 (The quartic model)**.** *We consider the parametric family $\mathcal{F}_a : \mathbb{T}^*\mathbb{R}^4 \to \mathbb{R}$ of quartic Hamiltonians systems defined by*

$$\begin{aligned} \mathcal{F}_a(\boldsymbol{q}, \boldsymbol{Q}) \;=\; & a_1 \left(q_1 Q_1 + q_2 Q_2 + q_3 Q_3 + q_4 Q_4\right)^2 \\ & + a_2 \left(q_1 Q_2 - q_2 Q_1 - q_3 Q_4 + q_4 Q_3\right)^2 \\ & + a_3 \left(q_1 Q_3 - q_3 Q_1 + q_2 Q_4 - q_4 Q_2\right)^2 \\ & + a_4 \left(q_1 Q_4 - q_4 Q_1 - q_2 Q_3 + q_3 Q_2\right)^2 \\ & + a_5 \frac{1}{2} \left(\|\boldsymbol{q}\|^2 - \|\boldsymbol{Q}\|^2\right)^2 \\ & + a_6 \frac{1}{2} \left(\|\boldsymbol{q}\|^2 + \|\boldsymbol{Q}\|^2\right)^2, \end{aligned} \tag{4.1}$$

where the vector of parameters $a = (a_1, \ldots, a_6) \in \mathbb{R}^6$.

This family of Hamiltonians (4.1), together with the standard symplectic form $\omega = d\mathbf{q} \wedge d\mathbf{Q}$, determines a symplectic flow on $\mathbb{T}^*\mathbb{R}^4 \equiv \mathbb{R}^8$

$$\dot{\mathbf{q}} = \frac{\partial \mathcal{F}_a}{\partial \mathbf{Q}}, \qquad \dot{\mathbf{Q}} = -\frac{\partial \mathcal{F}_a}{\partial \mathbf{q}}. \tag{4.2}$$

Due to the symmetries we will show that the Hamiltonian system (4.2) is integrable. Extending the work done by van der Meer *et al.* [van der Meer et al., 2014], we make use of the $\mathrm{SO}(3)$ reduction as an essential part of this study. In fact, the system is separable and so, we take advantage of this feature by means of a generalization of the symplectic Andoyer variables.

4.1 Hopf-Poisson Reduction of $\mathbb{R}^8$. The Regular and Singular Cases

In this part we are concerned with the components of the quaternionic Hopf mapping (see 3.15). In this light, it is shown that this components are considered as functions on $\mathcal{C}^\infty(\mathbb{R}^8)$, so as they span several Lie algebras, which commute between them. That is to say, the following proposition holds

Proposition 4.1. *In $T^*\mathbb{R}^4 \equiv \mathbb{R}^8$ with the standard Poisson bracket $\{,\}$, the following sets of functions*[1] $\omega_{789} = \{\omega_7, \omega_8, \omega_9\}$, $\omega_{156} = \{\omega_1, \omega_5, \omega_6\}$ *and* $\omega_{234} = \{\omega_2, \omega_3, \omega_4\}$ *commute between each other and span a Lie algebras in $C^\infty(\mathbb{R}^8)$ isomorphic to $\mathfrak{so}(3)$, $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$ respectively. Therefore, $\Omega_{789} \equiv \Omega_{234} \equiv SO(3)$ and $\Omega_{156} \equiv SL(2,\mathbb{R})$, where Ω_{789}, Ω_{234} and Ω_{156} denote the groups generated by the flows of the corresponding functions. Moreover, the function M given in (3.18) is the centralizer of Ω_{789}, Ω_{234} and Ω_{156}.*

Proof. Straightforward computations show that, $\{\omega_i, \omega_j\} = \{\omega_i, \omega_k\} = \{\omega_j, \omega_k\} = 0$ for any $i = 2, 3, 4$, $j = 1, 5, 6$ and $k = 2, 3, 4$. Furthermore $\{\omega_i, \omega_j\} = \epsilon_{ijk} 2\omega_k$ for $i, j, k \in \{2, 3, 4\}$ and for $i, j, k \in \{7, 8, 9\}$, thus, $\Omega_{789} \equiv \Omega_{234} \equiv SO(3)$.

The computation of the Poisson bracket for ω_1, ω_5, ω_6 yields $\{\omega_1, \omega_5\} = -2\omega_6$, $\{\omega_1, \omega_6\} = -2\omega_5$, $\{\omega_5, \omega_6\} = 2\omega_1$, that is, those functions span a Lie algebra isomorphic to $\mathfrak{su}(1,1) \cong \mathfrak{sl}(2,\mathbb{R})$ and $\Omega_{156} \equiv SL(2,\mathbb{R})$.

Finally, a direct computation is just needed to show that $\{M, F\} = 0$ for any F in Ω_{789}, Ω_{234} or Ω_{156} and consequently M is the centralizer. □

Coming back to the family defined in (4.1), we have that it is made up of the components of the quaternionic Hopf fibration given in (3.15). That is to say, it may be expressed in the more compact way

$$\mathcal{F}_a := a_1\omega_1^2 + a_2\omega_2^2 + a_3\omega_3^2 + a_4\omega_4^2 + a_5\omega_5^2 + a_6\omega_6^2. \tag{4.3}$$

It suggests that those Hamiltonian systems could be reduced to the six dimensional space given by the omegas. Furthermore, all those Hamiltonian systems are endowed with the following integrals and symmetry.

Corollary 4.2. *The functions ω_7, ω_8, ω_9, M given in (3.16) and (3.18) respectively and $\mathcal{F}_{156}$, $\mathcal{F}_{234}$ defined as*

$$\begin{aligned} \mathcal{F}_{156} &= a_1\omega_1^2 + a_5\omega_5^2 + a_6\omega_6^2 \\ \mathcal{F}_{234} &= a_2\omega_2^2 + a_3\omega_3^2 + a_4\omega_4^2, \end{aligned} \tag{4.4}$$

are integrals of the system defined by $\mathcal{F}_a$. Moreover, ω_7, ω_8 and ω_9 give rise to a $SO(3)$ symmetry.

[1]The reader should note that in [Ferrer and Crespo, 2015] ω_{789} are referred as $\{G_1, G_2, G_3\}$. Here for a more compact notation we have found convenient to denote them by $\{\omega_7, \omega_8, \omega_9\}$ respectively.

Proof. It is a direct consequence of Proposition 4.1. □

Proposition 4.1 implies that ω_1, ω_2, ω_3, ω_4, ω_5 and ω_6 are invariants for the Ω_{789}-action and are generators for the space of Ω_{789}-invariant polynomials. Consequently the orbit map for the Ω_{789}-action is given by W, which is equivalent to the real representation of the Hopf map given in (3.14)

$$W : T^*\mathbb{R}^4 \to \mathcal{M} \subset \mathbb{R}^6 : (\mathbf{q}, \mathbf{Q}) \to (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6).$$

Relation (3.19) implies that the orbit space $W(T^*\mathbb{R}^4)$ is five dimensional, or equivalently we have that $\mathcal{M} \equiv T^*\mathbb{R}^4/\Omega_{789}$.

On the other hand, Ω_{789} induces a free and proper Hamiltonian action with Ad-equivariant moment-map J

$$J : T^*\mathbb{R}^4 \to \mathfrak{so}(3)^* : (\mathbf{q}, \mathbf{Q}) \to (\omega_7, \omega_8, \omega_9).$$

Then the moment-map J and the orbit map W form a dual pair.

Following [van der Meer et al., 2014], we claim that the reduced phase spaces for the Ω_{789}-action are given as the co-adjoint orbits for the action of $SO(3)$ on $\mathfrak{so}(3) \times \mathfrak{sl}(2, R)$, but also they are determined by the $\mathfrak{so}(3)$ Casimir $4M^2(q, Q) = \Omega_{789}(q, Q) = \omega_7^2 + \omega_8^2 + \omega_9^2$. Therefore the reduced phase spaces are $W(\Omega_{789}^{-1}(\omega))$, i.e. the subset of $\mathbb{R}^6$ determined by

$$\begin{aligned} \omega_2^2 + \omega_3^2 + \omega_4^2 = \omega = 4M^2\,, \\ \omega_1^2 + \omega_5^2 - \omega_6^2 = \omega = 4M^2\,. \end{aligned}$$

Thus, if $\omega \neq 0$, the reduced phase space $\mathcal{M}$ is the four dimensional product of a two-sphere and a two-sheeted hyperboloid. If $\omega = 0$ we have a critical two dimensional reduced phase space which is a cone times a point. It is, $\omega = 0$ induces a singular reduction in the sense of [Ortega and Ratiu, 2004].

The expression of the reduced system is given by studying the variation of s, the independent variable, in the components ω_i, $i = 1, \ldots, 6$. The new Hamiltonian in the reduced variables is given by (6.5) and system (4.2) becomes the reduced system given by mean of the following computations

$$\dot{\mathbf{w}}^T = \{\mathbf{w}, \mathcal{F}_a\} = (\{\omega_i, \omega_j\})_{6\times 6} \nabla \mathcal{F}_a,$$

computing the brackets, the Poisson structure associated to the system is obtained, it is given by Table 4.1

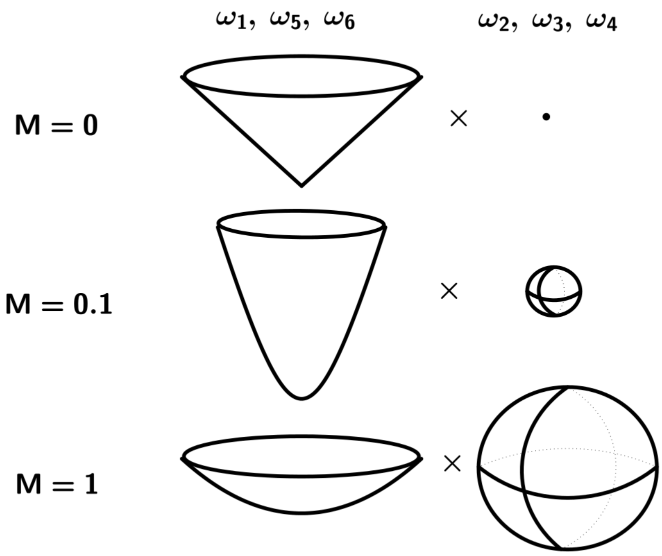


Figure 4.1: **The reduced space.** The value of $M \in \mathbb{R}^+ \cup \{0\}$ determines the geometry of the components. For $M = 0$ we obtain a cone times one single point. The positive values of M give a two sheeted hyperboloid and a sphere which radius is M. The variables ω_1, ω_5, ω_6 are restricted to the upper sheet of the hyperboloid since $\omega_6 \geq 0$.

The Poisson structure matrix (4.1) shows that the system is separable in the sets of variables $\omega_1, \omega_5, \omega_6$ and $\omega_2, \omega_3, \omega_4$. The equations of motion for the separate systems are given then in a very familiar way, that is

$$\begin{array}{lllll} \dot{\omega}_1 & = & \{\omega_1, \mathcal{F}_a\} = \bar{a}_1\omega_5\,\omega_6, & \dot{\omega}_2 & = & \{\omega_2, \mathcal{F}_a\} = \bar{a}_2\omega_3\,\omega_4, \\ \dot{\omega}_5 & = & \{\omega_5, \mathcal{F}_a\} = \bar{a}_5\,\omega_1\,\omega_6, & \dot{\omega}_3 & = & \{\omega_3, \mathcal{F}_a\} = \bar{a}_3\omega_2\,\omega_4, \\ \dot{\omega}_6 & = & \{\omega_6, \mathcal{F}_a\} = \bar{a}_6\omega_1\,\omega_5, & \dot{\omega}_4 & = & \{\omega_4, \mathcal{F}_a\} = \bar{a}_4\omega_2\,\omega_3, \end{array} \tag{4.5}$$

those systems are studied in Chapter 2, where the authors investigate the classical Euler equations with arbitrary coefficients $\bar{a}_i$, $\bar{a}_j$ and $\bar{a}_k$, it is the so called extended

$\{,\}$	ω_2	ω_3	ω_4	ω_1	ω_5	ω_6
ω_2	0	$-2\omega_4$	$2\omega_3$	0	0	0
ω_3	$2\omega_4$	0	$-2\omega_2$	0	0	0
ω_4	$-2\omega_3$	$2\omega_2$	0	0	0	0
ω_1	0	0	0	0	$-2\omega_6$	$-2\omega_5$
ω_5	0	0	0	$2\omega_6$	0	$2\omega_1$
ω_6	0	0	0	$2\omega_5$	$-2\omega_1$	0

.

Table 4.1: Poisson structure (ω_i).

Euler system (EES). The Jacobi elliptic functions are characterized as the general solution of the (EES) and explicit formulas of the analytic solutions are also provided. Later on, we make use of those formulas in the integration of the above systems. The coefficients $\bar{a}_i$ are obtained from the original a_i, $i = 1, \ldots, 6$ by the relations

$$\begin{array}{lll} \bar{a}_1 = -4(a_5 + a_6), & \bar{a}_5 = 4(a_1 + a_6), & \bar{a}_6 = 4(a_1 - a_5), \\ \bar{a}_2 = -4(a_3 - a_4), & \bar{a}_3 = 4(a_2 - a_4), & \bar{a}_4 = -4(a_2 - a_3), \end{array} \tag{4.6}$$

each subsystem is endowed with the following set of quadratic integrals

$$\begin{array}{lllllll} h_1 & = & \bar{a}_5\omega_6^2 - \bar{a}_6\omega_5^2, & \quad & h_2 & = & \bar{a}_3\omega_4^2 - \bar{a}_4\omega_3^2, \\ h_5 & = & \bar{a}_6\omega_1^2 - \bar{a}_1\omega_6^2, & & h_3 & = & \bar{a}_4\omega_2^2 - \bar{a}_2\omega_4^2, \\ h_6 & = & \bar{a}_1\omega_5^2 - \bar{a}_5\omega_1^2, & & h_4 & = & \bar{a}_2\omega_3^2 - \bar{a}_3\omega_2^2, \end{array} \tag{4.7}$$

they correspond to elliptic cylinders (EC) or hyperbolic cylinders (HC) depending on the signs of the coefficients. In Chapter 2, the authors found more convenient to consider those integrals. Notice that the classical one given by the angular momentum is not an integral when the sum of the parameters $\bar{a}_i$ does not vanish.

From the definition of the new coefficients, see (4.6), it is clear that the following relations hold

$$\bar{a}_2 + \bar{a}_3 + \bar{a}_4 = 0, \quad \bar{a}_1 + \bar{a}_5 - \bar{a}_6 = 0, \tag{4.8}$$

thus system $\omega_2\omega_3\omega_4$ can be regarded as a classical Euler system of the free rigid body, but system $\omega_1\omega_5\omega_6$ does not satisfies the condition on the parameters, that is, for this system $\bar{a}_1 + \bar{a}_5 + \bar{a}_6 \neq 0$ in general.

This new context for the Euler equations, where the condition on the parameters $\bar{a}_i + \bar{a}_j + \bar{a}_k \neq 0$ is eliminated, makes that the classical integrals for this system are not valid any more.

4.1.1 Geometric description of the solutions in the reduced space

The geometry of the solutions is studied, they are obtained as the intersection between the reduced space manifold $\mathcal{M}$ and the Hamiltonian given by $\mathcal{F}_a$.

Although the reduced space is a four dimensional manifold, we will be able to visualize the intersections because of the separation of variables ω_1, ω_5, ω_6 and ω_2, ω_3, ω_4.

As it is shown above, each triad of variables corresponds with an extended generalized Euler system, thus, solution trajectories are given as the intersection of two pairs of quadrics. In addition, it is shown in Chapter 2 that solutions of the subsystem ω_1, ω_5, ω_6 traverse all the intersection of the surfaces given by

$$4M^2 = -\omega_1^2 - \omega_5^2 + \omega_6^2, \qquad \mathcal{F}_{156} = a_6\omega_6^2 + a_5\omega_5^2 + a_1\omega_1^2, \tag{4.9}$$

the same happens to solutions associated to subsystem ω_2, ω_3, ω_4, they are given by the intersection of

$$4M^2 = \omega_2^2 + \omega_3^2 + \omega_4^2, \qquad \mathcal{F}_{234} = a_2\omega_2^2 + a_3\omega_3^2 + a_4\omega_4^2. \tag{4.10}$$

Following the same approach than in Chapter 2 in the geometric study of the trajectories, we replace (4.9) and (4.10) by the (EC) and (HC) given in (4.7). Those integrals are related by the following linear combinations

$$h_1 = 4(\mathcal{F}_{156} + a_1\, 4M^2), \quad h_5 = 4(\mathcal{F}_{156} + a_5\, 4M^2),$$

for subsystem 156 and

$$h_2 = -4(\mathcal{F}_{234} - a_2\, 4M^2), \quad h_4 = -4(\mathcal{F}_{234} - a_4\, 4M^2),$$

for 234. The coefficients $\bar{a}_i$ defined in (4.6), together with the value of $\mathcal{F}_{156}$ and $\mathcal{F}_{234}$, determines the trajectory of the solutions. A summary of the possible scenarios is given next

Table 4.2: Subsystem 156 summary.

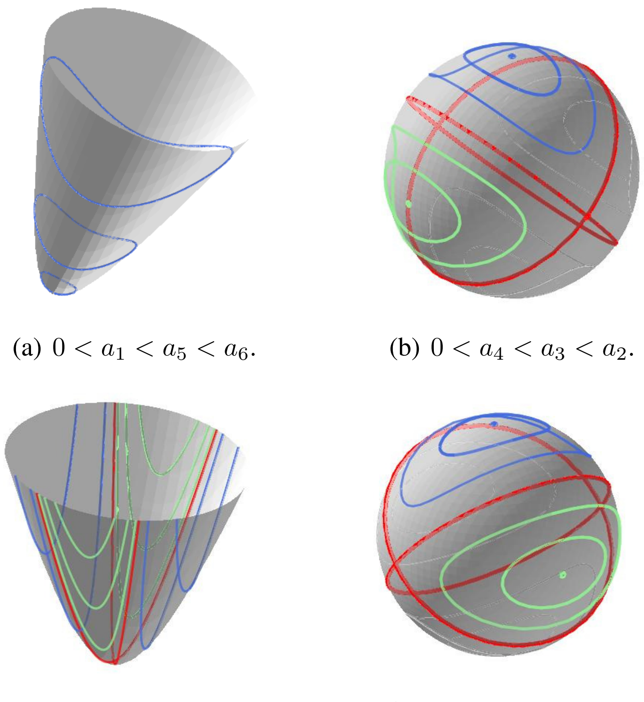
$\bar{a}_1$	$\bar{a}_5$	$\bar{a}_6$	$\mathcal{F}_{156}$	Trajectory summary, $h_1 \cap h_5$
+	+	+	$\mathcal{F}_{156} \in \mathbb{R}$	Unbounded curve resulting of the intersection (HC) -(HC).
+	+	-		Impossible combination according to relation (4.8).
+	-	+	$\mathcal{F}_{156} \geq -a_1\, 4M^2$	Bounded curve resulting of the intersection (EC)-(HC).
+	-	-	$\mathcal{F}_{156} \leq -a_5\, 4M^2$	Bounded curve resulting of the intersection (HC)-(EC).

Note that some cases still remain uncovered, that is, we have included here half of the possible cases. Those cases with the opposite combination of signs are completely analogous. They can be derived from the above just by interchanging the independent variable s by $-s$ in the corresponding subsystem.

Finally, we include a view of the bounded and unbounded trajectories. In Fig 4.2 it is shown the intersections of the reduced space with several level surfaces given by the Hamiltonians $\mathcal{F}_{156}$ and $\mathcal{F}_{234}$.

Table 4.3: Subsystem 234 summary.

$\bar{a}_2$	$\bar{a}_3$	$\bar{a}_4$	$\mathcal{F}_{234}$	Trajectory summary, $h_2 \cap h_4$
+	+	+	$\mathcal{F}_{234} \in \mathbb{R}$	Impossible combination according to relation (4.8).
+	+	-	$\mathcal{F}_{234} \leq a_2\,4M^2$	Bounded curve resulting of the intersection (EC)-(HC).
+	-	+	$a_2\,4M^2 \leq \mathcal{F}_{234} \leq a_4\,4M^2$	Bounded curve resulting of the intersection (EC)-(EC).
+	-	-	$\mathcal{F}_{234} \leq a_2\,4M^2$	Bounded curve resulting of the intersection (EC)-(HC).



(a) $0 < a_1 < a_5 < a_6$. (b) $0 < a_4 < a_3 < a_2$.

(c) $a_1 < 0 < a_5 < a_6$. (d) $a_4 < 0 < a_3 < a_2$.

Figure 4.2: **Trajectories in the reduced space.** Red color is used for homoclinic orbits and equilibria in the intersection. Green and blue correspond correspond to trajectories that belong to different connected components of the reduced space. Those components are determined by the homoclinic orbits.

4.1.2 On the integration of the reduced subsystems

Integration of the reduced systems (4.5) is readily obtained from Chapter 2, where the case of bounded and unbounded solutions are considered separately. Thinking in physical application we only consider bounded solutions. Therefore we obtain the following formulas for the solutions

$$\begin{aligned} \omega_1 &= \sqrt{\left|\frac{h_6}{\bar a_5}\right|}\mathrm{cn}(\mu_1\, s, k_{61}), & \omega_2 &= \sqrt{\left|\frac{h_4}{\bar a_3}\right|}\mathrm{cn}(\mu_2\, s, k_{42}),\\ \omega_5 &= \sqrt{\left|\frac{h_6}{\bar a_1}\right|}\mathrm{sn}(\mu_1\, s, k_{61}), & \omega_3 &= \sqrt{\left|\frac{h_4}{\bar a_2}\right|}\mathrm{sn}(\mu_2\, s, k_{42}),\\ \omega_6 &= \sqrt{\left|\frac{h_1}{\bar a_5}\right|}\mathrm{dn}(\mu_1\, s, k_{61}), & \omega_4 &= \sqrt{\left|\frac{h_2}{\bar a_3}\right|}\mathrm{dn}(\mu_2\, s, k_{42}), \end{aligned} \tag{4.11}$$

where

$$k_{ij} = \sqrt{\left|\frac{\bar a_i h_i}{\bar a_j h_j}\right|}, \qquad \mu_i = \sqrt{|\bar a_i h_i|}.$$

For the case in which the elliptic modulus k_{ij} is not in the interval $(0, 1)$ we have also the alternative formulas

$$\begin{aligned} \omega_1 &= \sqrt{\left|\frac{h_6}{\bar a_5}\right|}\mathrm{dn}(\mu_6\, s, k_{16}), & \omega_2 &= \sqrt{\left|\frac{h_4}{\bar a_3}\right|}\mathrm{dn}(\mu_4\, s, k_{24}),\\ \omega_5 &= \sqrt{\left|\frac{h_1}{\bar a_6}\right|}\mathrm{sn}(\mu_6\, s, k_{16}), & \omega_3 &= \sqrt{\left|\frac{h_2}{\bar a_4}\right|}\mathrm{sn}(\mu_4\, s, k_{24}),\\ \omega_6 &= \sqrt{\left|\frac{h_1}{\bar a_5}\right|}\mathrm{cn}(\mu_6\, s, k_{16}), & \omega_4 &= \sqrt{\left|\frac{h_2}{\bar a_3}\right|}\mathrm{cn}(\mu_4\, s, k_{24}). \end{aligned} \tag{4.12}$$

To ensure that $k_{ij} \in (0, 1)$, we have to use formulas (6.37) if $\bar a_5 h_5 > 0$ and formulas (4.12) if $\bar a_5 h_5 < 0$. The case $\bar a_5 h_5 = 0$ leads to unbounded solutions.

4.1.3 Poisson reconstruction

Complementing the previous approach, making use of the quadratic functions ω_i (3.15) defined above, the canonical equations (4.2) may also be regarded in the following matrix form

$$\begin{pmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \\ \dot q_4 \\ \dot Q_1 \\ \dot Q_2 \\ \dot Q_3 \\ \dot Q_4 \end{pmatrix} = 2 \begin{pmatrix} a_1\omega_1 & a_2\omega_2 & a_3\omega_3 & a_4\omega_4 & \Delta & 0 & 0 & 0 \\ -a_2\omega_2 & a_1\omega_1 & a_4\omega_4 & -a_3\omega_3 & 0 & \Delta & 0 & 0 \\ -a_3\omega_3 & -a_4\omega_4 & a_1\omega_1 & a_2\omega_2 & 0 & 0 & \Delta & 0 \\ -a_4\omega_4 & a_3\omega_3 & -a_2\omega_2 & a_1\omega_1 & 0 & 0 & 0 & \Delta \\ -\Delta^* & 0 & 0 & 0 & -a_1\omega_1 & a_2\omega_2 & a_3\omega_3 & a_4\omega_4 \\ 0 & -\Delta^* & 0 & 0 & -a_2\omega_2 & -a_1\omega_1 & a_4\omega_4 & -a_3\omega_3 \\ 0 & 0 & -\Delta^* & 0 & -a_3\omega_3 & -a_4\omega_4 & -a_1\omega_1 & a_2\omega_2 \\ 0 & 0 & 0 & -\Delta^* & -a_4\omega_4 & a_3\omega_3 & -a_2\omega_2 & -a_1\omega_1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} \tag{4.13}$$

where

$$\Delta = -a_5\omega_5 + a_6\omega_6, \qquad \Delta^* = a_5\omega_5 + a_6\omega_6$$

therefore, $(\mathbb{T}^*\mathbb{R}^4, \mathcal{F}_a, \omega)$ is a homogeneous cubic polynomial system. As we show below, associated with this system we have the reduced system defined on $SO(3) \times SL(2,\mathbb{R})$

$$\dot\omega_i = \alpha_{jk}\,\omega_j\,\omega_k, \qquad i,j,k \in \text{Per}\{1,5,6\} \quad \text{and} \quad i,j,k \in \text{Per}\{2,3,4\}, \quad \alpha_{jk} \in \mathbb{R}. \tag{4.14}$$

Then, we may approach our problem (4.2) considering an alternative system of differential equations defined by (4.13) and (4.14) in the space of the variables

$$q_1, q_2, q_3, q_4, \omega_1, \omega_5, \omega_6, \omega_2, \omega_3, \omega_4.$$

Indeed, note that from the relations defining ω_i, $(i = 1, \ldots, 4)$ we may write

$$\begin{aligned} Q_1 &= \frac{1}{|q|^2}(\omega_1 q_1 + \omega_2 q_2 + \omega_3 q_3 + \omega_4 q_4), \\ Q_2 &= \frac{1}{|q|^2}(-\omega_2 q_1 + \omega_1 q_2 + \omega_4 q_3 - \omega_3 q_4), \\ Q_3 &= \frac{1}{|q|^2}(-\omega_3 q_1 - \omega_4 q_2 + \omega_1 q_3 + \omega_2 q_4), \\ Q_4 &= \frac{1}{|q|^2}(-\omega_4 q_1 + \omega_3 q_2 - \omega_2 q_3 + \omega_1 q_4), \end{aligned} \tag{4.15}$$

where $|q|$ is the modulus of the configuration variables, that is, $|q|^2 = \sum_{i=1}^4 q_i^2$. Note also that (4.15) may be expressed in a more compact way by considering the quaternionic notation, thus we obtain

$$Q = \frac{1}{|q|^2}\, q * \bar\omega,$$

where $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$.

Then, replacing in the system of equations (4.13) we may give the system the following form

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{pmatrix} = 2 \begin{pmatrix} a_1^*\,\omega_1 & a_2^*\,\omega_2 & a_3^*\,\omega_3 & a_4^*\,\omega_4 \\ -a_2^*\,\omega_2 & a_1^*\,\omega_1 & a_4^*\,\omega_4 & -a_3^*\,\omega_3 \\ -a_3^*\,\omega_3 & -a_4^*\,\omega_4 & a_1^*\,\omega_1 & a_2^*\,\omega_2 \\ -a_4^*\,\omega_4 & a_3^*\,\omega_3 & -a_2^*\,\omega_2 & a_1^*\,\omega_1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \tag{4.16}$$

with

$$a_i^* = a_i + \frac{\Delta}{|q|^2} = a_i + \frac{-a_5\omega_5 + a_6\omega_6}{|q|^2}, \qquad i = 1, \ldots, 4 \tag{4.17}$$

In short, the alternative system is given by (4.16) and (4.14). Several studies suggest that, in contrast to the first impression due to the large dimension of the system, this formulation could be more efficient from the numerical point of view. In this sense note that apart from the Hamiltonian function, as we will see later, the system has the constraints defined by the integrals

$$4M^2 = \omega_6^2 - \omega_5^2 - \omega_1^2 = \omega_2^2 + \omega_3^2 + \omega_4^2, \tag{4.18}$$

and

$$\Lambda = \frac{1}{2}(q_1Q_4 - q_4Q_1 + q_2Q_3 - q_3Q_2), \tag{4.19}$$

which will be relevant when implementing and controlling the numerical integration method used. Moreover, for some specific models, like the rigid body, we have the constraint of the configuration manifold $\mathbb{S}^3$ (see for instance Fukushima [Fukushima, 2008]). In this chapter we focus on the analytic and qualitative aspects of the model.

4.2 Variables that Perform Reductions

In this section two different sets of symplectic variables are considered. Our target is to incorporate integrals of system (4.1) among the new variables in order to perform the reduction on the system. It is well known in astrodynamic that the Euler and Andoyer variables carry out this task for classical problem as Kepler and the rigid body. Thus, we extend those variables to the four dimensional case by means of the Projective Euler and Projective Andoyer variables, which were first defined in

[Ferrer, 2010], see also the references therein to track earlier versions of these variables.

Although the Projective Andoyer variables perform a 2-DOF reduction in the system, we first consider the Projective Euler variables. From this effort we obtain some gain, since in Projective Euler variables, it is shown that the free rigid body Hamiltonian belongs to the family $\mathcal{F}_a$.

Before we start to develop our plan, we have to make a stop and reconsider our methodology in this section. Although, it has been said that we are going to define new variables to reduce our system and we have related it to the Kepler and free rigid body problems, it is necessary to consider the following transformations just by what the are, that is, symplectic maps from $T^*\mathbb{R}^4$, that do not depend of the problem in which they will be applied later on.

4.2.1 Projective Euler variables. 1-DOF reduction

To start defining the Projective Euler variables, we first consider a map on the configuration space $\mathbb{R}^4$. After that, this map is extended canonically to the phase space $T^*\mathbb{R}^4$ giving a symplectic transformation. Let us fix some notation and conventions.

The Euler angles are one way of endowing $SO(3)$ with coordinates. It hinges in the fact that any rotation in $\mathbb{R}^3$ is the result of the composition of three basic rotations. Now we recall the expression of the basic rotation matrices $R_3(\alpha)$ and $R_1(\beta)$, that are used in the Euler chart of $SO(3)$. Note that, regarding to the Euler angles, there is not an agreement in the literature. In what follows we follow [Arnold, 1989, Goldstein et al., 2002], which are different from [Whittaker, 1937].

$$R_3(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad R_1(\beta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix}. \tag{4.20}$$

Then we consider two reference frames in $\mathbb{R}^3$ given by two orthonormal basis $\mathcal{B}^s = \{X_1^s, X_2^s, X_3^s\}$ and $\mathcal{B}^E = \{X_1^E, X_2^E, X_3^E\}$, which relative position in space is given by Fig. 4.3. Let (x, y, z) and (x''', y''', z''') the coordinates with respect to the basis $\mathcal{B}^s$ and $\mathcal{B}^E$ respectively. Thinking in the applications, we express the "new" coordinates in terms of the "old" ones, that is to say

$$\begin{bmatrix} x''' \\ y''' \\ z''' \end{bmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4.21}$$

where the rotation matrix $R = (r_{ij})_{ij\in\{1,2,3\}} = R_3(-\psi)R_1(-\theta)R_3(-\phi)$, that sends $\mathcal{B}^s$ to $\mathcal{B}^E$, is given by

$$R = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\psi\sin\phi & \cos\theta\sin\psi\cos\phi + \cos\psi\sin\phi & \sin\theta\sin\psi \\ -\cos\theta\cos\psi\sin\phi - \sin\psi\cos\phi & \cos\theta\cos\psi\cos\phi - \sin\psi\sin\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

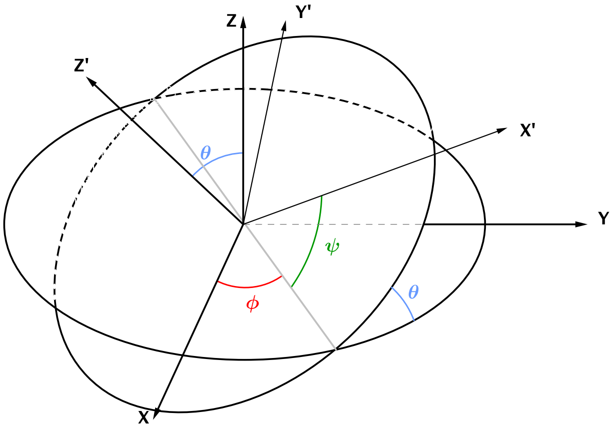


Figure 4.3: Euler angles. The rotation transforming the reference system $\{X, Y, Z\}$ into $\{X', Y', Z'\}$ is decomposed in three basic rotations, which angles are ϕ, θ and ψ.

A minimal atlas for $\mathbb{R}^+ \times SO(3)$. Euler charts

Along this section we follow [Grafarend and Kuhnel, 2011], where they give a complete minimal atlas for $SO(3)$. This work refines the claim of [Cushman and Bates, 1997], page 402, where it is said that at least three chart are needed to cover $SO(3)$ and they prove that the minimum number of charts needed is four. We are also investigating is there is a minimal atlas made of four charts for the case of the Andoyer angles. In this light, we give minimal atlas in terms of the Euler angles for $\mathbb{R}^+ \times SO(3)$.

Proposition 4.3. *The following set of maps constitute a minimal atlas for* $\mathbb{R}^+ \times SO(3)$

denoted by $\mathcal{A}_{\mathbb{R}\times SO(3)}$.

$$\begin{aligned}
\mathcal{P}e_1 &: U_1 \subset \mathbb{R}^4 \longrightarrow \mathbb{R}^+ \times SO(3),\ \ (\rho,\phi,\theta,\psi) \to \Big(F(\rho), \big(\ R_3(-\psi)R_1(-\theta)R_3(-\phi)\big)\Big),\\
\mathcal{P}e_2 &: U_2 \subset \mathbb{R}^4 \longrightarrow \mathbb{R}^+ \times SO(3),\ \ (\rho,\phi,\theta,\psi) \to \Big(F(\rho), \big(\ R_3(-\psi)R_1(-\theta)R_3(-\phi)\big)\Big),\\
\mathcal{P}e_3 &: U_3 \subset \mathbb{R}^4 \longrightarrow \mathbb{R}^+ \times SO(3),\ \ (\rho,\phi,\theta,\psi) \to \Big(F(\rho), \big(\Omega\ R_3(-\psi)R_1(-\theta)R_3(-\phi)\big)\Big),\\
\mathcal{P}e_4 &: U_4 \subset \mathbb{R}^4 \longrightarrow \mathbb{R}^+ \times SO(3),\ \ (\rho,\phi,\theta,\psi) \to \Big(F(\rho), \big(\Omega\ R_3(-\psi)R_1(-\theta)R_3(-\phi)\big)\Big),
\end{aligned} \tag{4.22}$$

with their domains being the following open sets in $\mathbb{R}^4$

$$\begin{aligned}
U_1 = U_3 &= \mathbb{R}^+ \times (0,2\pi) \times (0,\pi) \times (0,2\pi),\\
U_2 = U_4 &= \mathbb{R}^+ \times (-\pi,\pi) \times (0,\pi) \times (-\pi,\pi),
\end{aligned} \tag{4.23}$$

and

$$\Omega = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \tag{4.24}$$

Proof. Analogous to the proof given by [Grafarend and Kuhnel, 2011] for the case of $SO(3)$. □

Every basic rotation matrix may be represented by mean of a unitary pure quaternion. Following Kuipers' [Kuipers, 1999] notation we have

$$\begin{aligned}
R_3(\phi) \sim q_{k,\phi} &= (\cos(\phi) + k\sin(\phi)) = (\cos(\phi),0,0,\sin(\phi))\\
R_1(\theta) \sim q_{i,\theta} &= (\cos(\theta) + i\sin(\theta)) = (\cos(\theta),\sin(\theta),0,0)\\
R_3(\psi) \sim q_{k,\psi} &= (\cos(\psi) + k\sin(\psi)) = (\cos(\psi),0,0,\sin(\psi)),\\
\Omega \sim \omega &= (0,\sqrt{1/3},\sqrt{1/3},\sqrt{1/3}),
\end{aligned} \tag{4.25}$$

therefore, the unit quaternion $q_{\psi\theta\phi} = q_{k,\psi}\, q_{i,\theta}\, q_{k,\phi} = (q^1_{\psi\theta\phi}, q^2_{\psi\theta\phi}, q^3_{\psi\theta\phi}, q^4_{\psi\theta\phi})$ is the quaternion associated to the whole rotation and scaling by $F(\rho)$ we obtain an atlas in terms of the Euler angles for $\mathbb{H} - \{0\}$ (or $\mathbb{S}^3$ if $F(\rho) = \pm 1$)

$$\begin{aligned}
\mathcal{PE}_1 &: V_1 \subset \mathbb{R}^4 \longrightarrow \mathbb{H} - \{0\},\ \ (\rho,\phi,\theta,\psi) \to \mathbf{q},\\
\mathcal{PE}_2 &: V_2 \subset \mathbb{R}^4 \longrightarrow \mathbb{H} - \{0\},\ \ (\rho,\phi,\theta,\psi) \to \mathbf{q},\\
\mathcal{PE}_3 &: V_3 \subset \mathbb{R}^4 \longrightarrow \mathbb{H} - \{0\},\ \ (\rho,\phi,\theta,\psi) \to \omega\cdot\mathbf{q},\\
\mathcal{PE}_4 &: V_4 \subset \mathbb{R}^4 \longrightarrow \mathbb{H} - \{0\},\ \ (\rho,\phi,\theta,\psi) \to \omega\cdot\mathbf{q},
\end{aligned} \tag{4.26}$$

where their domains are given by the following open sets in $\mathbb{R}^4$

$$\begin{aligned} V_1 = V_3 &= \mathbb{R} - \{0\} \times (0, 2\pi) \times (0, \pi) \times (0, 2\pi), \\ V_2 = V_4 &= \mathbb{R} - \{0\} \times (-\pi, \pi) \times (0, \pi) \times (-\pi, \pi), \end{aligned} \tag{4.27}$$

and **q** $= (q_1, q_2, q_3, q_4)$ may be considered as a quaternion as well as a vector in $\mathbb{R}^4$ with coordinates

$$\begin{aligned} q_1 &= F(\rho)\, \cos(\frac{\theta}{2}) \cos(\frac{\phi+\psi}{2}), \quad q_3 = F(\rho)\, \sin(\frac{\theta}{2}) \sin(\frac{\phi-\psi}{2}), \\ q_2 &= F(\rho)\, \sin(\frac{\theta}{2}) \cos(\frac{\phi-\psi}{2}), \quad q_4 = F(\rho)\, \cos(\frac{\theta}{2}) \sin(\frac{\phi+\psi}{2}), \end{aligned} \tag{4.28}$$

This change of variables in $\mathbb{R}^4$ is dubbed as Projective Euler variables, since ϕ, θ and ψ are the well known Euler angles representing a triad of basic rotations 3-1-3. In the following theorem we state the relation between the atlas given above.

Definition 4.2. *We define the map $\boldsymbol{L_q}$ as the extension of the corresponding rotation operator $L_q : \mathbb{R}^3 \to \mathbb{R}^3$, defined in Theorem 3.1, that is*

$$\begin{aligned} \boldsymbol{L_q} : \mathbb{H} - \{0\} &\longrightarrow \mathbb{R}^+ \times SO(3), \\ \boldsymbol{q} \qquad &\longrightarrow \quad (\|\boldsymbol{q}\|, L_{\frac{\boldsymbol{q}}{\|\boldsymbol{q}\|}}). \end{aligned} \tag{4.29}$$

From now on we use the convention $\mathbb{H} - \{0\} = \mathbb{H}^*$.

Theorem 4.4. *We have that the following statements hold*

(i) The following diagram is commutative for $i = 1, 2, 3, 4$

$$\begin{array}{ccc} \mathbb{R}^4 - \{0\} \cong \mathbb{H}^* & \xrightarrow{\boldsymbol{L_q}} & \mathbb{R}^+ \times SO(3) \\ & {}_{\mathcal{PE}_i}\nwarrow \quad \nearrow_{\mathcal{P}e_i} & \\ & U_i & \end{array}$$

(ii) By choosing $F(\rho) = 1$ and $F(\rho) = -1$ in $\mathcal{A}_{\mathbb{H}}$, we obtain that

$$\begin{aligned} \mathcal{A}_{\mathbb{S}^3} \;=\; &\{(\mathcal{PE}_1^+, W_1), (\mathcal{PE}_2^+, W_2), (\mathcal{PE}_3^+, W_3), (\mathcal{PE}_4^+, W_4), \\ &(\mathcal{PE}_1^-, W_1), (\mathcal{PE}_2^-, W_2), (\mathcal{PE}_3^-, W_3), (\mathcal{PE}_4^-, W_4)\}, \end{aligned} \tag{4.30}$$

is an atlas for $\mathbb{S}^3$, where $\mathcal{PE}_i^+ = \mathcal{PE}_i$, $\mathcal{PE}_i^- = -\mathcal{PE}_i$ and $W_i = U_i \cap (\{1\} \times \mathbb{R}^3)$.

(iii) By fixing $F(\rho) = 1$ *in* $\mathcal{A}_{\mathbb{R}\times SO(3)}$*, we obtain that*

$$\mathcal{A}_{SO(3)} = \{(\mathcal{P}e_1^+, U_1), (\mathcal{P}e_2^+, U_2), (\mathcal{P}e_3^+, U_1), (\mathcal{P}e_4^+, U_2)\} \tag{4.31}$$

is a minimal atlas for $SO(3)$*, where* $\mathcal{P}e_i^+ = \mathcal{P}e_i$*.*

Proof. It is a consequence of (4.25,4.26) and Definition 4.2. □

Extended atlas in the cotangent bundle

The following result says that the diagram given in the above theorem may be lifted to the cotangent bundles of the manifolds involved. Moreover, the lifted maps give rise to symplectic maps.

Theorem 4.5. *Consider the cotangent lift to the respective cotangent bundles of the applications given in Theorem 4.4. Thus the corresponding diagram is commutative and the maps involved are symplectic.*

$$\begin{array}{ccc} T^*\mathbb{R}^4 \cong \mathbb{H}\times\mathbb{H} & \xrightarrow{\quad T^*\boldsymbol{L_q}\quad} & T^*(\mathbb{R}^+ \times SO(3)) \\ & {\scriptstyle T^*\mathcal{P}\mathcal{E}_i}\nwarrow \qquad \nearrow{\scriptstyle T^*\mathcal{P}e_i} & \\ & T^*U_i & \end{array}$$

Proof. It follows from the fact that a diffeomorphism between manifolds may be lifted to their corresponding cotangent bundles, see Definition 1.30. □

In what follows we denote the lifted maps $T^*\mathcal{P}\mathcal{E}_i$, $T^*\mathcal{P}e_i$ and $T^*\mathcal{P}e_i$ by $\mathcal{P}\mathcal{E}_i$, $\mathcal{P}e_i$ and $\mathcal{P}e_i$, since by the context one can easily distinguish the applications between the configuration and the phase spaces.

We plan to use the Projective Euler variables to express the model given in Definition 4.1. Thus, we provide to the reader with plenty of details related to this transformation. To be more precise, we focus on the canonical character of $\mathcal{P}\mathcal{E}_i$. Coming back to (4.28), the expression of the momenta is readily obtained as a Mathieu transformation, which satisfies $\sum Q_i\,dq_i = P\,d\rho + \Theta\,d\theta + \Psi\,d\psi + \Phi\,d\phi$, thus

$$
\begin{aligned}
P &= \frac{F'(\rho)}{F(\rho)}(q_1Q_1 + q_2Q_2 + q_3Q_3 + q_4Q_4),\\
\Theta &= \frac{(q_2Q_2 + q_3Q_3)(q_1^2 + q_4^2) - (q_1Q_1 + q_4Q_4)(q_2^2 + q_3^2)}{2\sqrt{(q_1^2 + q_4^2)(q_2^2 + q_3^2)}},\\
\Phi &= \frac{1}{2}(q_1Q_4 - q_4Q_1 + q_2Q_3 - q_3Q_2),\\
\Psi &= \frac{1}{2}(q_1Q_4 - q_4Q_1 - q_2Q_3 + q_3Q_2),
\end{aligned}
\tag{4.32}
$$

The whole transformation takes place in $\mathbb{T}^*\mathbb{R}^4$, hence our transformation $\mathcal{PE}_F$ should be extended also to the momenta. Taking into account that the new momenta P, Θ, Φ and Ψ are linear in the old ones, we have that the complete transformation given by

$$\mathcal{PE}_F : (\rho, \phi, \theta, \psi, P, \Psi, \Theta, \Phi) \to (q_1, q_2, q_3, q_4, Q_1, Q_2, Q_3, Q_4),$$

is obtained just by expressing (4.32) in matrix form and solving for Q_1, Q_2, Q_3 and Q_4. Namely

$$
\begin{aligned}
Q_1 &= \frac{1}{F'(\rho)}\cos\frac{\theta}{2}\cos\frac{\phi+\psi}{2}\,P - \frac{\sin\frac{\phi+\psi}{2}}{F(\rho)\cos\frac{\theta}{2}}(\Phi+\Psi) + \frac{\cos\frac{\phi+\psi}{2}\sin\theta}{F(\rho)\cos\frac{\theta}{2}}\,\Theta,\\
Q_2 &= \frac{1}{F'(\rho)}\sin\frac{\theta}{2}\cos\frac{\phi-\psi}{2}\,P - \frac{\sin\frac{\phi-\psi}{2}}{F(\rho)\cos\frac{\theta}{2}}(\Phi-\Psi) + \frac{\cos\frac{\phi-\psi}{2}\sin\theta}{F(\rho)\cos\frac{\theta}{2}}\,\Theta,\\
Q_3 &= \frac{1}{F'(\rho)}\sin\frac{\theta}{2}\sin\frac{\phi-\psi}{2}\,P + \frac{\sin\frac{\phi-\psi}{2}}{F(\rho)\sin\frac{\theta}{2}}(\Phi-\Psi) + \frac{\sin\frac{\phi-\psi}{2}\sin\theta}{F(\rho)\cos\frac{\theta}{2}}\,\Theta,\\
Q_4 &= \frac{1}{F'(\rho)}\cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}\,P + \frac{\cos\frac{\phi+\psi}{2}}{F(\rho)\sin\frac{\theta}{2}}(\Phi+\Psi) - \frac{\sin\frac{\phi+\psi}{2}\sin\theta}{F(\rho)\cos\frac{\theta}{2}}\,\Theta,
\end{aligned}
\tag{4.33}
$$

Also we give here the inverse transformation $\mathcal{PE}_F^{-1} : (q_1, q_2, q_3, q_4) \to (\rho, \phi, \theta, \psi)$, by using (4.32) and the following relations

$$
\begin{aligned}
&F(\rho) = \sqrt{q_1^2+q_2^2+q_3^2+q_4^2},\\
&\sin\phi = \frac{q_2q_4+q_1q_3}{\sqrt{(q_1^2+q_3^2)(q_2^2+q_4^2)}}, \qquad \cos\phi = \frac{q_1q_2-q_3q_4}{\sqrt{(q_1^2+q_3^2)(q_2^2+q_4^2)}},\\
&\sin\theta = \frac{2\sqrt{(q_1^2+q_4^2)(q_2^2+q_3^2)}}{q_1^2+q_2^2+q_3^2+q_4^2}, \qquad \cos\theta = \frac{q_1^2+q_4^2-q_2^2-q_3^2}{q_1^2+q_2^2+q_3^2+q_4^2},\\
&\sin\psi = \frac{q_2q_4-q_1q_3}{\sqrt{(q_1^2+q_3^2)(q_2^2+q_4^2)}}, \qquad \cos\psi = \frac{q_1q_2+q_3q_4}{\sqrt{(q_1^2+q_3^2)(q_2^2+q_4^2)}}.
\end{aligned}
\tag{4.34}
$$

Remark 4.1. *Note that $\omega_9 = 2\Phi$ and $\omega_4 = -2\Psi$. In other words, the system defined by (4.1), when expressed in Projective Euler variables, reveal one of the conjugate momenta as an integral.*

Proposition 4.6. *For $F(\rho) = \rho$, the family of Hamiltonians $\mathcal{F}_a$ expressed in Projective Euler variables takes the form*

$$
\begin{aligned}
\mathcal{F}(\rho,\theta,\phi,\_,P,\Phi,\Theta,\Psi) = {} & a_1\,P^2\rho^2\\
&+ a_5\left(\frac{\rho^2-P^2}{2} - \frac{2}{\rho^2}\left(\Theta^2 + \frac{\Phi^2 - 2\,\Psi\,\Phi\,\cos\theta + \Psi^2}{\sin^2\theta}\right)\right)^2,\\
&+ a_6\left(\frac{\rho^2+P^2}{2} + \frac{2}{\rho^2}\left(\Theta^2 + \frac{\Phi^2 - 2\,\Psi\,\Phi\,\cos\theta + \Psi^2}{\sin^2\theta}\right)\right)^2,\\
&+ 4a_2\left(\frac{\Phi-\Psi\cos\theta}{\sin\theta}\sin\psi + \Theta\cos\psi\right)^2,\\
&+ 4a_3\left(\frac{\Phi-\Psi\cos\theta}{\sin\theta}\cos\psi - \Theta\sin\psi\right)^2,\\
&+ 4a_4\;\Psi^2.
\end{aligned}
\tag{4.35}
$$

and for the case of ω_7, ω_8 and ω_9 we have

$$
\begin{aligned}
\omega_7 &= 2\left(\frac{\Phi\,\cos(\theta)\sin(\phi)}{\sin(\theta)} - \frac{\Psi\,\sin(\phi)}{\sin(\theta)} - \Theta\,\cos(\phi)\right),\\
\omega_8 &= -2\left(\frac{\Psi\,\cos(\theta)\cos(\phi)}{\sin(\theta)} - \frac{\Psi\,\cos(\phi)}{\sin(\theta)} + \Theta\,\sin(\phi)\right),\\
\omega_9 &= -2\Phi,
\end{aligned}
\tag{4.36}
$$

Proof. By using the above transformations formulae and after some algebraic and trigonometric manipulations, we obtain the following expression for the components of the family $\mathcal{F}_a$ that reads as follows

$$\begin{aligned}
\omega_1 &= \frac{PF(\rho)}{F'(\rho)},\\
\omega_5 &= \frac{F'^2(\rho)F^2(\rho)-P^2}{F'^2(\rho)} - \frac{2}{F^2(\rho)}\left(\Theta^2+\frac{\Phi^2-2\,\Psi\,\Phi\,\cos\theta+\Psi^2}{\sin^2\theta}\right),\\
\omega_6 &= \frac{F'^2(\rho)F^2(\rho)-P^2}{F'^2(\rho)} + \frac{2}{F^2(\rho)}\left(\Theta^2+\frac{\Phi^2-2\,\Psi\,\Phi\,\cos\theta+\Psi^2}{\sin^2\theta}\right),\\
\omega_2 &= 2\left(\frac{\Phi-\Psi\cos\theta}{\sin\theta}\sin\psi+\Theta\cos\psi\right),\\
\omega_3 &= -2\left(\frac{\Phi-\Psi\cos\theta}{\sin\theta}\cos\psi-\Theta\sin\psi\right),\\
\omega_4 &= 2\,\Psi,
\end{aligned} \tag{4.37}$$

thus, by simply substitution we obtain (4.35) and (4.36). □

Although Projective Euler variables do perform a 1-DOF reduction, they are not our choice. In order to integrate the system, we can simplify the problem furthermore because we know it is endowed with more integrals, see Proposition 4.1.

4.2.2 Projective Andoyer variables. 2-DOF reduction

In this section we go further in the reduction process changing to the Projective Andoyer variables $(\rho, \lambda, \mu, \nu)$, which are a generalization to four dimensions of the Andoyer variables. Precise details can be found in [Andoyer, 1923] and [Deprit, 1967]. As in the case of the Euler angles λ, μ and ν represent angles and ρ is a scaling. The geometric relations of those three angles with the Euler ones (ϕ, θ, ψ) is illustrated in Fig. 4.4 and Fig. 4.5 and hinges on the existence of the *Andoyer vector* $\vec{a}$ that we define below. After that, the symplectic character of the new variables is obtained in a similar way as in [Heard, 2006]. Those variables provide us with the 2-DOF reduction by incorporating two of the integrals among the variables, which renders two cyclic variables.

On the geometry of the Projective Andoyer variables

Usually, the construction of the Andoyer and Euler variables is related to the study of the free rigid body. Maybe for that reason, it is customary in the literature [Deprit, 1967, Heard, 2006, Ferrer and Molero, 2014a], to define the Andoyer variables using the angular momentum vector. Here we modify slightly this approach and construct the Andoyer variables hinging on the *Andoyer vector* $\vec{a}$, this is the mathematical abstraction of the physic magnitude angular momentum vector, which emphasises that there is no need to deal with a rigid body to construct the Andoyer variables.

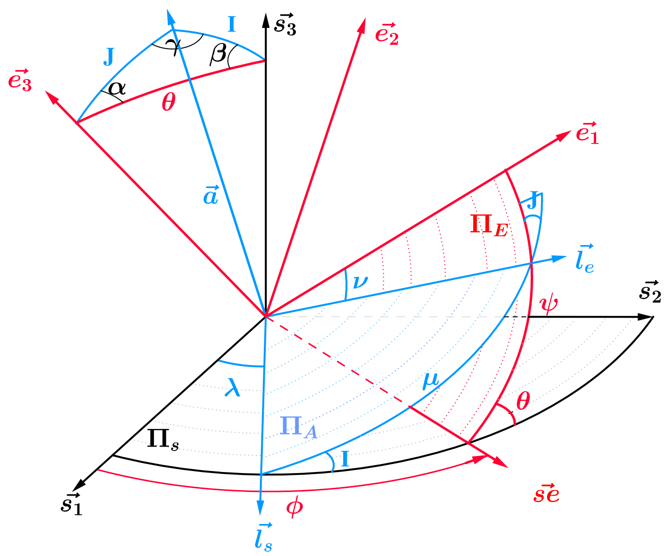


Figure 4.4: Geometric relations between the two triads of angles (ϕ, θ, ψ) and (λ, μ, ν). Andoyer angles rely on the intermediate plane determined by the *Andoyer vector* represented by $\vec{a}$. The labels Π_s Π_E and Π_A correspond to the xy-planes given by the spatial, Euler and Andoyer reference systems respectively. The angles α, β and γ are given by $\psi - \nu$, $\phi - \lambda$ and μ respectively.

Fig. 4.4 shows three reference systems; the spatial $\mathcal{B}^s = \{\vec{s_1}, \vec{s_2}, \vec{s_3}\}$, the Euler $\mathcal{B}^E = \{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$ and the Andoyer one $\mathcal{B}^A = \{\vec{l_s}, \vec{l_e}, \vec{a}\}$. Nothing left to be said for $\mathcal{B}^s$ and $\mathcal{B}^E$, just recall that both are orthonormal basis of $\mathbb{R}^3$. Then we define the vectors belonging to $\mathcal{B}^A$ starting with the key object of the Andoyer angles, the *Andoyer vector* $\vec{a}$. Going back to the Euler angles we consider the basis of $\mathbb{R}^3$ given by the vectors $\mathcal{B} = \{\vec{s_3}, \vec{se}, \vec{e_3}\}$, where

$$\vec{se} = \frac{\vec{s_3} \times \vec{e_3}}{\|\vec{s_3} \times \vec{e_3}\|}. \tag{4.38}$$

Then, the *Andoyer vector* is given as follows

$$\vec{a} = \Phi\vec{s_3} + \Theta\vec{se} + \Phi\vec{e_3}. \tag{4.39}$$

In other words, vector $\vec{a}$ coordinates are the old momenta associated to the Euler angles in the basis $\mathcal{B}$, see Fig. 4.5. This vector is precisely the angular momentum of the free rigid body, see [Gurfil et al., 2007, Heard, 2006], but it is defined with no mention to it.

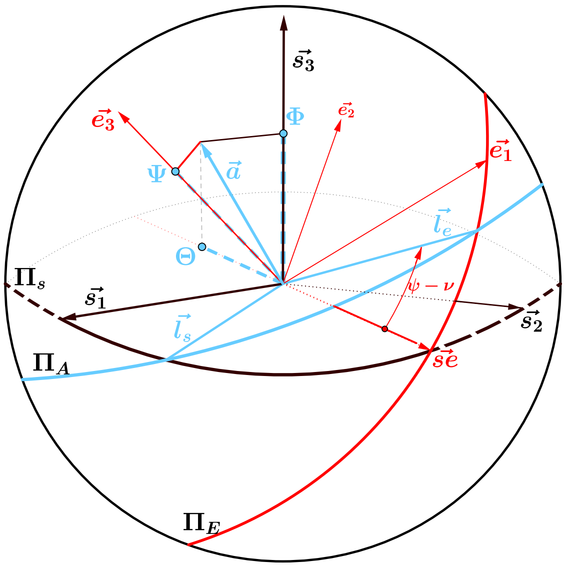


Figure 4.5: Andoyer angles rely on the intermediate plane determined by the vector G. For the family $\mathcal{F}_a$, this vector is given by $(\omega_7, \omega_8, \omega_9)$, which provides an orthogonal plane through the origin that relates the Euler angles to the new Andoyer angles.

The expressions for the nodes, as they are used to be referred in astronomy and astrodynamics, $\vec{l_s}$ and $\vec{l_e}$ are given below. Note that the modulus of those vectors are also given, they are obtained taking into account that $|\vec{s_3}| = |\vec{e_3}| = 1$ and the angles between $\widehat{(\vec{s_3}, \vec{a})} = I$ and $\widehat{(\vec{e_3}, \vec{a})} = J$, then

$$\begin{aligned} \vec{l_s} \sin I &= \vec{s_3} \times \frac{\vec{a}}{\|\vec{a}\|} \\ \vec{l_e} \sin J &= \frac{\vec{a}}{\|\vec{a}\|} \times \vec{e_3} \end{aligned} \tag{4.40}$$

where we denote by $I,\ J \in (0, \pi)$ the angles between the planes Π_S-Π_A and Π_A-Π_E respectively, see Fig. 4.5.

Note also that other definitions for the nodes $\vec{l_s}$ and $\vec{l_e}$ are possible. Alternative choices lead to sign differences in the objects involved, see for instance [Ferrer and Molero, 2014a].

At this point the new Andoyer angles are determined by specifying their sines and cosines.

$$\begin{aligned} \sin\lambda &= \|\vec{s_1} \times \vec{l_s}\|, \quad \cos\lambda = \|\vec{s_1} \cdot \vec{l_s}\|, \\ \sin\mu &= \|\vec{l_e} \times \vec{e_1}\|, \quad \cos\mu = \|\vec{e_1} \cdot \vec{l_s}\|, \\ \sin\nu &= \|\vec{l_e} \times \vec{l_s}\|, \quad \cos\nu = \|\vec{l_e} \cdot \vec{l_s}\|. \end{aligned} \tag{4.41}$$

That is, λ is the angle measured from $\vec{s_1}$ to $\vec{l_s}$, μ the angle from $\vec{l_s}$ to $\vec{l_e}$ and ν from $\vec{l_e}$ to $\vec{e_1}$. We follow obtaining more information relating the old momenta with the new variables, which is derived just from basic vectorial calculus,

$$\begin{aligned} \Phi &= \vec{a} \cdot \vec{s_3} = \|\vec{a}\|\,\|\vec{s_3}\| \cos(a,\hat{} s_3) \Rightarrow \cos I = \frac{\Phi}{\|\vec{a}\|} \\ \Psi &= \vec{a} \cdot \vec{e_3} = \|\vec{a}\|\,\|\vec{e_3}\| \cos(a,\hat{} e_3) \Rightarrow \cos J = \frac{\Psi}{\|\vec{a}\|}. \end{aligned} \tag{4.42}$$

Finally we reach an important relation between Θ and $\|\vec{a}\|$. Let us consider the plane Π_E in Fig. 4.5 and the reference system given by

$$\mathcal{B}_1 = \{\vec{se}, \vec{v} = \frac{\vec{e_3} \times \vec{se}}{\|\vec{e_3} \times \vec{se}\|}, \vec{w} = \vec{e_3}\},$$

then, since $\vec{se}, \vec{e_3} \times \vec{se} \in \Pi_E$, the pair of vectors $\mathcal{B}_2 = \{\vec{se}, \vec{v}\}$ give also a reference

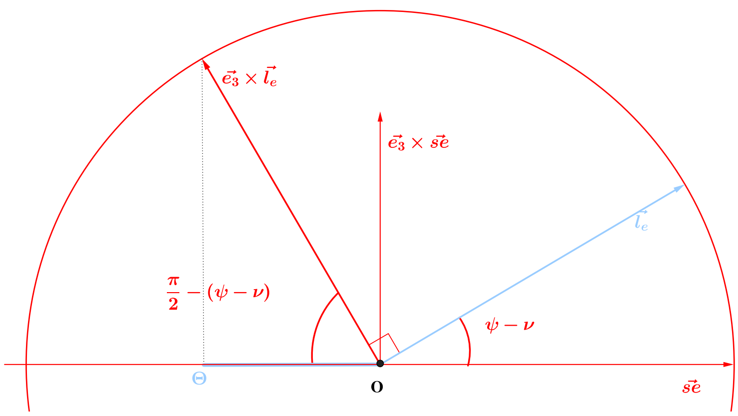


Figure 4.6: Intersection of plane Π_E with the imaginary sphere of Fig. 4.5. The point of view of the observer is from the vector $\vec{e_3}$, which projection is the point **O**.

system in Π_E, see Fig. 4.6. Note that the orthogonal projection of $\vec{a}$ in Π_E is $\|\vec{a}\|(\vec{e_3} \times \vec{l_e})$ and therefore the following relation hold

$$\Theta = \vec{a} \cdot \vec{se} = \|\vec{a}\| \sin J(\vec{e_3} \times \vec{l_e} \cdot \vec{se}),$$

after some computation the last factor becomes

$$\sin J(\vec{e_3} \times \vec{l_e} \cdot \vec{se}) = -\sin J \cos(\frac{\pi}{2} - (\psi - \nu)) = \sin J \sin(\nu - \psi)$$

thus we have that

$$\Theta = \|\vec{a}\| \sin J \sin(\nu - \psi). \tag{4.43}$$

4.2.3 Symplectic Character of the Projective Andoyer Transformation

In Section 4.2.2 we have derived the relations between the Euler and Andoyer angles, that is we focused in the configuration space although the moments were also involved in the formulas. Nevertheless, the transformation from Projective Euler variables to Projective Andoyer takes place in $\mathbb{T}^*\mathbb{R}^4$, that is

$$\mathcal{PA}_E : (\rho, \phi, \theta, \psi, P, \Phi, \Theta, \Psi) \to (\rho, \lambda, \mu, \nu, P, \Lambda, M, N),$$

in the case of the Andoyer map it is no longer obtained as a canonical extension of the configuration space. Rather than this, we follow [Heard, 2006] and the Projective Andoyer varibles are derived by specifying the momenta and evaluating a generating function which produces a symplectic transformation. Namely, the momenta are given by

$$P = P, \quad \Lambda = \Phi, \quad N = \Psi, \quad M = \|\vec{a}\|. \tag{4.44}$$

Taking into account the definition of the Andoyer vector (4.39), we have that the new momenta M is related with the Euler variables as follows

$$M = \sqrt{\Theta^2 + \Psi^2 + \Big(\frac{\Phi - \Psi\cos\theta}{\sin\theta}\Big)^2}, \tag{4.45}$$

and using (4.43) we have the inverse relations

$$
\begin{aligned}
P &= P, \\
\Phi &= \Lambda, \\
\Psi &= N, \\
\Theta &= M \sin J \sin(\nu - \psi) \\
&= \pm\sqrt{M^2 - N^2 - \big(\frac{\Lambda - N\cos\theta}{\sin\theta}\big)^2} \\
&= \pm\sqrt{[M^2\cos^2\theta - 2N\Lambda\cos\theta + N^2 + \Lambda^2 - M^2]\,/\,\sin^2\theta}.
\end{aligned}
\tag{4.46}
$$

Let us abbreviate the Euler angles by $\mathbf{e} = (\rho, \phi, \theta, \psi)$, the momenta conjugate to the Euler angles by $\mathbf{E} = (P, \Phi, \Theta, \Psi)$, the Andoyer momenta by $\mathbf{a} = (\rho, \lambda, \mu, \nu)$, and the conjugate coordinates by $\mathbf{A} = (P, \Lambda, M, N)$. Since the transformation from $(\mathbf{e}, \mathbf{E})$ to $(\mathbf{a}, \mathbf{A})$ is to be symplectic the differential forms $\mathbf{e}\,d\mathbf{E}$ and $\mathbf{a}\,d\mathbf{A}$ can differ only by a closed form. According to Theorem 1.13, this introduces the generating function $S(\mathbf{e}, \mathbf{A})$, that satisfies

$$
\mathbf{E}\,d\mathbf{e} + \mathbf{a}\,d\mathbf{A} = dS(\mathbf{e}, \mathbf{A}),
\tag{4.47}
$$

and provides the generating equations

$$
\mathbf{E} = \frac{\partial S}{\partial \mathbf{e}}(\mathbf{e}, \mathbf{A}) \qquad \mathbf{a} = \frac{\partial S}{\partial \mathbf{A}}(\mathbf{e}, \mathbf{A}).
\tag{4.48}
$$

By using the first equation of (4.48) we obtain the generating function itself

$$
\begin{aligned}
S(\mathbf{e}, \mathbf{A}) &= \int \mathbf{E}\,d\mathbf{e} \\
&= \int P(\mathbf{a}, \mathbf{A})d\rho + \Phi(\mathbf{a}, \mathbf{A})d\phi + \Theta(\mathbf{a}, \mathbf{A})d\theta + \Psi(\mathbf{a}, \mathbf{A})d\psi \\
&= P\rho + \Lambda\phi + N\psi + \int_{\theta_0}^{\theta} \sqrt{[M^2\cos^2\theta - 2N\Lambda\cos\theta + N^2 + \Lambda^2 - M^2]\,/\,\sin^2\theta}\,d\theta
\end{aligned}
\tag{4.49}
$$

where θ_0 is the larger root of the polynomial $M^2\cos^2\theta - 2N\Lambda\cos\theta + N^2 + \Lambda^2 - M^2$, then, keeping in mind that $\|\vec{a}\| = M$ and the expressions for the cosines of I and J given in (4.42) we obtain that the above polynomial can be rewritten as follows

$$
\cos^2\theta - 2\cos I\cos J\cos\theta + \cos^2 I + \cos^2 J - 1 = 0
$$

which larger root is $\theta_0 = I + J$. Once we have obtain S we can use the second differential equation in (4.48) to obtain $\mathbf{a} = (\rho, \lambda, \mu, \nu)$:

$$
\begin{aligned}
\rho &= \frac{\partial S}{\partial P}(\mathbf{e}, \mathbf{A}) = \rho, \\
\lambda &= \frac{\partial S}{\partial \Lambda}(\mathbf{e}, \mathbf{A}) \\
&= \phi + \int_{\theta_0}^{\theta} \frac{\Lambda - N\cos\theta \, d\theta}{\sin^2\theta\sqrt{[M^2\cos^2\theta - 2N\Lambda\cos\theta + N^2 + \Lambda^2 - M^2]\,/\sin^2\theta}} \\
&= \phi + \int_{\theta_0}^{\theta} \frac{\cos I - \cos J\cos\theta \, d\theta}{\sin\theta\sqrt{\sin^2\theta + 2\cos I\cos J\cos\theta - \cos^2 I - \cos^2 J}} \\
&= \phi + \arcsin\left(\frac{\cos J - \cos I\cos\theta}{\sin I\sin\theta}\right) - \frac{\pi}{2}, \\
\mu &= \frac{\partial S}{\partial M}(\mathbf{e}, \mathbf{A}) \\
&= \int_{\theta_0}^{\theta} \frac{\sin\theta \, d\theta}{\sqrt{\sin^2\theta + 2\cos I\cos J\cos\theta - \cos^2 I - \cos^2 J}} \\
&= \arcsin\left(\frac{\cos J\cos I - \cos\theta}{\sin I\sin J}\right) - \frac{\pi}{2}, \\
\nu &= \frac{\partial S}{\partial N}(\mathbf{e}, \mathbf{A}) \\
&= \psi + \int_{\theta_0}^{\theta} \frac{\cos I\cos\theta - \cos J \, d\theta}{\sin\theta\sqrt{\sin^2\theta + 2\cos I\cos J\cos\theta - \cos^2 I - \cos^2 J}} \\
&= \psi + \frac{\pi}{2} - \arcsin\left(\frac{\cos I - \cos J\cos\theta}{\sin I\sin\theta}\right).
\end{aligned}
\tag{4.50}
$$

The whole change of variables may be summarized in a more compact way by using the following notation

$$
c_1 = \cos\frac{I}{2}, \quad c_2 = \cos\frac{J}{2}, \quad s_1 = \sin\frac{I}{2}, \quad s_2 = \sin\frac{J}{2}. \tag{4.51}
$$

Thus, in the domain $(\rho, \lambda, \mu, \nu) \in \mathbb{R}^+ \times (0, 2\pi) \times (0, 2\pi) \times (0, 2\pi)$, those new variables are related to the Euler angles by mean of the following trigonometric relations

$$
\begin{aligned}
&\rho = \rho, && P = P, \\
&\cos\theta = c_1 c_2 + s_1\, s_2\, \cos 2\mu, && \Psi = N, \\
&\sin(\psi - \nu) = \frac{\sin\mu}{\sin\theta} s_2, && \Phi = \Lambda, \\
&\sin(\phi - \lambda) = \frac{\sin\mu}{\sin\theta}\, s_1, && \Theta = M\sqrt{1 - \frac{c_1^2 + c_2^2 - 2\, c_1\, c_2\, \cos\theta}{\sin^2\theta}},
\end{aligned}
\tag{4.52}
$$

The final expressions for the integrals and the family components in the Projective Andoyer variables are given as follows

$$
\begin{aligned}
&\omega_2 = 2\sqrt{M^2 - N^2}\sin\nu, && \omega_7 = 2\sqrt{M^2 - \Lambda^2}\sin\lambda,\\
&\omega_3 = -2\sqrt{M^2 - N^2}\cos\nu, && \omega_8 = 2\sqrt{M^2 - \Lambda^2}\cos\lambda,\\
&\omega_4 = 2N, && \omega_9 = -2\lambda.\\
&\omega_1 = \frac{PF(\rho)}{F'(\rho)}, \\
&\omega_5 = \frac{1}{2}\big(F^2(\rho) - \frac{P^2}{F'^2(\rho)} - \frac{4M^2}{F^2(\rho)}\big),\\
&\omega_6 = \frac{1}{2}\big(F^2(\rho) + \frac{P^2}{F'^2(\rho)} + \frac{4M^2}{F^2(\rho)}\big),
\end{aligned}
\tag{4.53}
$$

Therefore, the family of Hamiltonians yields

$$
\mathcal{F}_a(\rho, -, -, \nu; P, -, M, N) = \mathcal{F}_{156} \ + \ \mathcal{F}_{234} \tag{4.54}
$$

with

$$
\begin{aligned}
\mathcal{F}_{156}(\rho, -, -, -; P, -, M, -) = {} & a_1\Big(\frac{PF(\rho)}{F'(\rho)}\Big)^2\\
& + \frac{a_5}{4}\,\big(F^2(\rho) - \frac{P^2}{F'^2(\rho)} - \frac{4M^2}{F^2(\rho)}\big)^2\\
& + \frac{a_6}{4}\,\big(F^2(\rho) + \frac{P^2}{F'^2(\rho)} + \frac{4M^2}{F^2(\rho)}\big)^2,\\
\mathcal{F}_{234}(-, -, -, \nu; -, -, M, N) = {} & \ 4((a_2\,\sin^2\nu + a_3\,\cos^2\nu)(M^2 - N^2) + a_4\,N^2).
\end{aligned}
\tag{4.55}
$$

From Cartesian to Andoyer variables

The relation of the Projective Andoyer variables with the Cartesian variables is also provided. In this case we proceed in a similar way as in the preceding section, but now for the Andoyer variables we have the composition of five rotations instead of the three corresponding to the Euler angles. That is to say, the new variables are given by the components of the corresponding associated quaternion $q_{\nu J \mu I \lambda}$ =

$q_{k,\nu}\, q_{i,J}\, q_{k,\mu}\, q_{i,I}\, q_{k,\lambda}$, multiplied times ρ. The explicit change is given by the canonical mapping

$$\mathcal{PA}_C : (q_1, q_2, q_3, q_4, Q_1, Q_2, Q_3, Q_4) \to (\rho, \lambda, \mu, \nu, P, \Lambda, M, N),$$

the configuration space is transformed as follows

$$\begin{aligned} q_1 &= F(\rho)\, c_1 c_2 \cos\frac{\lambda+\mu+\nu}{2} - F(\rho)\, s_1 s_2 \cos\frac{\lambda-\mu+\nu}{2}, \\ q_2 &= F(\rho)\, c_2 s_1 \cos\frac{\lambda-\mu-\nu}{2} + F(\rho)\, c_1 s_2 \cos\frac{\lambda+\mu-\nu}{2}, \\ q_3 &= F(\rho)\, c_2 s_1 \sin\frac{\lambda-\mu-\nu}{2} + F(\rho)\, c_1 s_2 \sin\frac{\lambda+\mu-\nu}{2}, \\ q_4 &= F(\rho)\, c_1 c_2 \sin\frac{\lambda+\mu+\nu}{2} - F(\rho)\, s_1 s_2 \sin\frac{\lambda-\mu+\nu}{2}, \end{aligned} \tag{4.56}$$

The momenta are found taking into account relations (4.52) and imposing the condition $\sum Q_i\, dq_i = P\, d\rho + \Lambda\, d\lambda + M\, d\mu + N\, d\nu$

$$\begin{aligned} P &= \frac{F'(\rho)}{F(\rho)}(q_1Q_1 + q_2Q_2 + q_3Q_3 + q_4Q_4), \\ \Lambda &= \frac{1}{2}(-q_1Q_4 + q_4Q_1 - q_2Q_3 + q_3Q_2), \\ M &= \frac{1}{2}\sqrt{|q|^2|Q|^2 - \langle q, Q\rangle^2}, \\ N &= \frac{1}{2}(q_1Q_4 - q_4Q_1 - q_2Q_3 + q_3Q_2). \end{aligned} \tag{4.57}$$

In other words, Projective Andoyer variables incorporate two integrals among the new momenta, M and $\omega_9 = 2\Phi = 2\Lambda$, thus the 2-DOF symplectic reduction is carried out.

The whole transformation (4.56), in terms of the Projective Andoyer variables, is not completed until the momenta Q_i for $i = 1, 2, 3, 4$ are found. They are readily obtained by taking into account relations (4.15) and (4.53). Note also that (4.15) may be expressed in a more compact way by considering the quaternionic notation, thus we obtain

$$Q = \frac{F'^2(\rho)}{F^2(\rho)}\, q * \bar{\omega}, \tag{4.58}$$

where $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)$.

Finally we also give the complete inverse transformation

$$\mathcal{PA}_F^{-1} : (q_1, q_2, q_3, q_4, Q_1, Q_2, Q_3, Q_4) \to (\rho, \lambda, \mu, \nu, P, \Lambda, M, N)$$

by the combination of (4.57) and the following expressions for the Andoyer angles and ρ

$$\begin{aligned} &\rho = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}, \\ &\cos\mu = \frac{q_2^2 + q_3^2 - q_1^2 - q_4^2}{2\rho^2\, c_1 c_2 s_1 s_2}, \quad \sin\mu = \frac{q_3 q_4(\omega_2 + \omega_3) - q_1 q_2(\omega_2 - \omega_3)}{\rho^2 M\, c_2 s_2^2} \\ &\cos\lambda = \frac{-\omega_8}{\sqrt{\omega_8^2 + \omega_7^2}}, \qquad \sin\lambda = \frac{\omega_7}{\sqrt{\omega_8^2 + \omega_7^2}}, \\ &\cos\nu = \frac{-\omega_3}{\sqrt{\omega_2^2 + \omega_3^2}}, \qquad \sin\nu = \frac{\omega_2}{\sqrt{\omega_2^2 + \omega_3^2}}, \end{aligned} \tag{4.59}$$

As a final remark, we refer to the singularities of the Andoyer angles and therefore also for the Projective Andoyer angles. More precisely, for the particular case in which the invariant angular momentum is parallel to the s_3-axis (the third axis of the spatial frame) or parallel to b_3-axis (the third axis of the body frame), those variables are not defined. Equivalently, singularities arise when $I = 0$, $I = \pi$, $J = 0$ and $J = \pi$ because the node lines vanish. One alternative set of variables, can be found in Sidorenko [Sidorenko, 2014]

4.3 Projective Andoyer Variables and the M-Orbit Map

The quadratic functions ω_i, $i = 1, \dots, 9$ describe the reduced space for the M-action. More precisely, Proposition 4.1 implies that ω_1, ω_2, ω_3, ω_4, ω_5, ω_6, ω_7, ω_8 and ω_9 are invariants for the M-action and are generators for the space of M-invariant polynomials. Consequently the orbit map for the M-action, which is an $\mathbb{S}^1$-action, is given by

$$W : T^*\mathbb{R}^4 \to \mathcal{M} \subset \mathbb{R}^9 : \, (\mathbf{q}, \mathbf{Q}) \to (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9).$$

Relation (3.19) implies that the orbit space $W(T^*\mathbb{R}^4) = \mathcal{M}$ is six dimensional, or equivalently we have that $\mathcal{M} \equiv T^*\mathbb{R}^4/M$.

On the other hand, M induces a free and proper Hamiltonian action with Ad-equivariant moment-map J

$$J : T^*\mathbb{R}^4 \to \mathbb{R} : \, (\mathbf{q}, \mathbf{Q}) \to M.$$

Then the moment-map J and the orbit map W form a dual pair.

The Projective Andoyer variables provide the M-reduced space $\mathcal{M}$ with coordinates. Note that the M-reduced space is described with nine "coordinates" and three restrictions given by the relation (3.17). Therefore, the reduced space is a Poisson manifold $(\mathcal{M}, \{.,.\})$ foliated by six dimensional symplectic manifolds. Where the Poisson structure matrix is given by

$$P = \begin{pmatrix} 0 & 2\omega_9 & -2\omega_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2\omega_9 & 0 & 2\omega_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\omega_8 & -2\omega_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\omega_4 & -2\omega_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\omega_4 & 0 & 2\omega_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\omega_3 & -2\omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2\omega_6 & -2\omega_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\omega_6 & 0 & 2\omega_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2\omega_5 & -2\omega_1 & 0 \end{pmatrix}. \tag{4.60}$$

and Casimirs

$$C_1 = \omega_7^2 + \omega_8^2 + \omega_9^2, \quad C_2 = \omega_2^2 + \omega_3^2 + \omega_4^2, \quad C_3 = \omega_6^2 - \omega_5^2 - \omega_1^2. \tag{4.61}$$

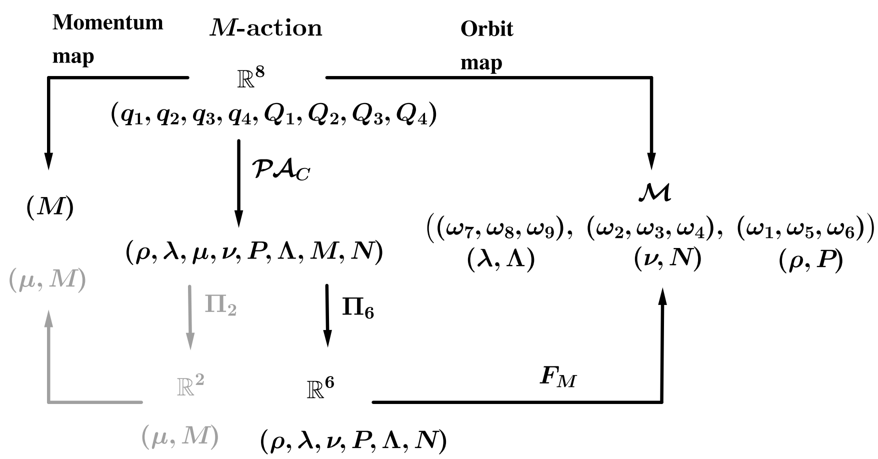


Figure 4.7: Andoyer coordinates and the M-reduced space.

This reduced space is closely related to the Andoyer chart coordinates. Let us consider the diagram given in Fig. 4.7. Π_6 and Π_2 are the usual projections from $\mathbb{R}^8$ to $\mathbb{R}^6$ and $\mathbb{R}^2$ respectively, the symplectic map $\mathcal{PA}_C$, and also its inverse, are given in

Section 4.2.3. Thus, we focus on F_M, this map is defined, depending in the parameter M as follows

$$F_M : D \subset (\mathbb{R}^6, \omega) \to \mathbb{S}^{2*}_M \times \mathbb{S}^{2*}_M \times \mathcal{C}^{2*} \subset (\mathbb{R}^9, \{\, ,\, \}), \tag{4.62}$$

the domain is given by $D = \{(\rho, \lambda, \nu, P, \Lambda, N) \in \mathbb{R}^6 / \ \rho > 0,\ \lambda, \nu \in (0, 2\pi),\ |N| < M,\ |\Lambda| < M\}$ and ω is the standard symplectic form. On the other hand $\mathbb{S}^{2*}_M = \mathbb{S}^2_M - \{(0,0,\pm M)\}$ (denoting $\mathbb{S}^2_M$ the sphere of radius M), $\mathcal{C}^{2*}$ is the upper sheet of the hyperboloid defined by $C_3 = \omega_6^2 - \omega_5^2 - \omega_1^2 = M^2$ minus the point $(0,0,M)$ and the map is given by

$$F_M(\rho, \lambda, \nu, P, \Lambda, N) = (F^1_M, F^2_M, F^3_M, F^4_M, F^5_M, F^6_M, F^7_M, F^8_M, F^9_M), \tag{4.63}$$

where

$$\begin{aligned}
F^1_M &= 2\sqrt{M^2 - \Lambda^2}\sin\lambda, & F^7_M &= \frac{PF(\rho)}{F'(\rho)},\\
F^2_M &= -2\sqrt{M^2 - \Lambda^2}\cos\lambda, & F^8_M &= \frac{1}{2}\big(F^2(\rho) - \frac{P^2}{F'^2(\rho)} - \frac{4M^2}{F^2(\rho)}\big),\\
F^3_M &= -2\Lambda, & F^9_M &= \frac{1}{2}\big(F^2(\rho) + \frac{P^2}{F'^2(\rho)} + \frac{4M^2}{F^2(\rho)}\big).\\
F^4_M &= 2\sqrt{M^2 - N^2}\sin\nu, & &\\
F^5_M &= -2\sqrt{M^2 - N^2}\cos\nu, & &\\
F^6_M &= 2N, & &
\end{aligned} \tag{4.64}$$

The following proposition allows us to claim that Projective Andoyer variables give coordinates to the M-reduced space.

Proposition 4.7. *The map F_M is endowed with the following features:*

(i) F_M is a diffeomorphism from D to its image in $\mathbb{S}^{2*}_M \times \mathbb{S}^{2*}_M \times \mathcal{C}^{2*}$,

ii) $\mathbb{S}^2_M \times \mathbb{S}^2_M \times \mathcal{C}^2$ corresponds to the level manifold of the Casimirs $C_1 = C_2 = C_3 = M^2$, in other words $\mathcal{M}$ is foliated by the symplectic leaves

$$\mathbb{S}^2_M \times \mathbb{S}^2_M \times \mathcal{C}^2 = C_1^{-1}(M) \cap C_2^{-1}(M) \cap C_3^{-1}(M),$$

iii) F_M satisfies that

$$\{F^*_M f, F^*_M g\}_{\omega|D} = F^*_M\{f, g\}, \qquad \forall f, g \in C^\infty(\mathcal{M})$$

where $\{.,.\}_{\omega|D}$ is the Poisson structure induced by the symplectic form ω_D on the open manifold D. Thus, is F_M a Poisson map.

Proof. (i) After some computations it is shown that the rank of the Jacobian matrix of $F : M$ is equal to six.

(ii) Is a direct consequence of the fact that C_1, C_2 and C_3 are Casimirs of the Poisson structure.

(iii) Again after some rather long computations with the Poisson brackets, one checks that

$$\begin{aligned} \{F_M^* G_i, F_M^* G_j\}_{\omega|D} &= F_M^*\{G_i, G_j\}, \\ \{F_M^* \omega_i, F_M^* \omega_j\}_{\omega|D} &= F_M^*\{\omega_i, \omega_j\}, \\ \{F_M^* \omega_i, F_M^* G_j\}_{\omega|D} &= 0 = F_M^*\{\omega_i, G_j\}, \end{aligned} \tag{4.65}$$

thus, it follows for all f, g in $C^\infty(\mathcal{M})$.

□

4.4 Integration in Projective Andoyer Variables

In this section we prove the integrability of the family of Hamiltonians systems given by (4.1), and also provide the analytic solution formulas.

4.4.1 Integrability of the system

Up to now we did not say anything about a remarkable feature of the system, that is, it is an integrable system in the Liouville sense. The reason why we wait till this moment is that Projective Andoyer variables are the appropriate frame to perform this task.

Definition 4.3. *Let (M, ω) be a 2n-dimensional symplectic manifold. The Hamiltonian $\mathcal{H}$ is said to be completely integrable if there exists $n-1$ independent functions $f_1, \ldots, f_{n-1}$ (independent in the sense that the differentials $d_m f_1, \ldots, d_m f_{n-1}$ are linearly independent at almost every point $m \in M$) that are first integrals of $\mathcal{H}$ and that pairwise Poisson commute, i.e. $\{H, f_i\} = 0$ and $\{f_i, f_j\} = 0$.*

Theorem 4.8. *The family of Hamiltonian systems given by (4.1) is endowed with the following first integrals in involution*

$$\mathcal{F}_{234},\ \mathcal{F}_{156},\ M^2,\ \omega_9. \tag{4.66}$$

Therefore, this family of Hamiltonians defines a set of completely integrable systems in the sense of Liouville.

Proof. The involution condition of the integrals is obtained from the computation of the brackets, for this task we are going to use the Projective Andoyer expression of the functions involved. Thus we have that $\{\omega_9, M^2\} = \{\omega_9, \mathcal{F}_{156}\} = \{\omega_9, \mathcal{F}_{234}\} = \{M^2, \mathcal{F}_{156}\} = \{M^2, \mathcal{F}_{234}\} = \{\mathcal{F}_{156}, \mathcal{F}_{234}\}$, trivially vanish since those functions does not depend on whole set of new variables, in others words, we have that

$$\mathcal{F}_{234}(\nu, N, M), \quad \mathcal{F}_{156}(\rho, P, M), \quad M^2, \quad \omega_9 = -2\Lambda$$

The independence condition of $d\mathcal{F}_{234}$, $d\mathcal{F}_{156}$, dM^2 and $d\omega_9$ is checked out now.

$$\begin{pmatrix} D\mathcal{F}_{156} \\ D\mathcal{F}_{234} \\ DM^2 \\ D\omega_9 \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{F}_{156}}{\partial \rho} & 0 & 0 & 0 & \frac{\partial \mathcal{F}_{156}}{\partial P} & 0 & \frac{\partial \mathcal{F}_{156}}{\partial M} & 0 \\ 0 & 0 & 0 & \frac{\partial \mathcal{F}_{234}}{\partial \nu} & 0 & 1 & \frac{\partial \mathcal{F}_{234}}{\partial M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2M & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial \mathcal{F}_{234}}{\partial N} \end{pmatrix} \begin{pmatrix} dq_1 \\ dq_2 \\ dq_3 \\ dq_4 \\ dQ_1 \\ dQ_2 \\ dQ_3 \\ dQ_4 \end{pmatrix}, \tag{4.67}$$

we focus on the last four columns, then the rank of A is four, except for the values that satisfy

$$\frac{\partial \mathcal{F}_{156}}{\partial P} = 0, \quad \frac{\partial \mathcal{F}_{234}}{\partial M} = 0, \quad 2M = 0, \quad \frac{\partial \mathcal{F}_{234}}{\partial \nu},$$

the above conditions defines four 7-dimensional manifolds on the phase space $\mathbb{T}^*\mathbb{R}^4$. So the rank of A is four in a set of full Lebesgue measure. □

In another work, currently in progress (see [Crespo and Ferrer, 2014a]), we study the energy-momentum mapping

$$EM : \mathbb{T}^*\mathbb{R}^4 \to \Delta \subset \mathbb{R}^4;\ EM(q, Q) = (\mathcal{F}_{156},\ \mathcal{F}_{156},\ M^2,\ \omega_9)$$

4.4.2 Analytic integration. Case $F(\rho) = \rho$

In what follows we assume $F(\rho) = \rho$ unless the contrary is said. Although Theorem 4.8 ensures that the systems given by $\mathcal{F}_a$ are integrable, Cartesian variables lead to a partial system of differential equations, see (4.2), which is not easy to handle at all. The symplectic change of variables performed above turned the Hamiltonian $\mathcal{F}_a$ into a 2-DOF system and the expression of the differential equations are given by

$$\begin{aligned}
\dot{\rho} &= \frac{1}{4}\big((\bar{a}_6 + \bar{a}_5)\rho^2 - \bar{a}_1(P^2 + \frac{4M^2}{\rho^2})\big)P, \qquad (4.68)\\
\dot{P} &= \frac{1}{4}\big(-\bar{a}_1(\frac{16M^4}{\rho^6} + \frac{P^2\,4M^2}{\rho^4} - \rho^2) - (\bar{a}_6 + \bar{a}_5)P^2\big)\rho,\\
\dot{\nu} &= 2N\big(\bar{a}_2 + \bar{a}_4\sin^2\nu\big),\\
\dot{N} &= \bar{a}_4(M^2 - N^2)\sin 2\nu,\\
\dot{\mu} &= M\big(4(a_6 - a_5 + 2a_2) - \frac{\bar{a}_1}{\rho^2}(\frac{4M^2}{\rho^2} + P^2) + 2\bar{a}_4\cos^2\nu\big),\\
\dot{M} &= 0,\\
\dot{\lambda} &= 0,\\
\dot{\Lambda} &= 0,
\end{aligned}$$

i. e., we have two different 1-DOF system in the variables (ρ, P) and (ν, N), and a quadrature that gives μ. Even now when the system has reduced its complexity, it is not painless to tackle it. Therefore, in order to solve the system, we use the fact that the system is explicitly solved for the ω_1,..., ω_6 in Section4.1, thus, by finding the inverse relations of those giving in (4.53), system (6.4) is solved in terms of the omegas

$$\begin{aligned}
&\rho = \sqrt{\omega_5 + \omega_6}, \quad P = \frac{\omega_1}{\sqrt{\omega_5 + \omega_6}}, \qquad (4.69)\\
&\cos\nu = \frac{-\omega_3}{\sqrt{\omega_2^2 + \omega_3^2}}, \qquad \sin\nu = \frac{\omega_2}{\sqrt{\omega_2^2 + \omega_3^2}}, \quad N = \frac{\omega_4}{2},\\
&\mu = (a_6 - a_5 + 2a_4)Ms\, - \bar{a}_1 M I_1\, - 2\bar{a}_2 M I_2,\\
&2M = \sqrt{\omega_6^2 - \omega_5^2 - \omega_1^2} = \sqrt{\omega_2^2 + \omega_3^2 + \omega_4^2} = \sqrt{\omega_7^2 + \omega_8^2 + \omega_9^2},\\
&\cos\lambda = \frac{-\omega_8}{\sqrt{\omega_8^2 + \omega_9^2}}, \qquad \sin\lambda = \frac{\omega_9}{\sqrt{\omega_8^2 + \omega_9^2}}, \quad \Lambda = \frac{-\omega_7}{2},
\end{aligned}$$

where

$$I_1(s) = \int_0^s \frac{1}{\rho^2}(\frac{4M^2}{\rho^2} + P^2)\,dr = \int_0^s \frac{\omega_6 - \omega_5}{\omega_6 + \omega_5}\,dr, \qquad (4.70)$$

and

$$I_2(s) = \int_0^s \cos^2 \nu \, dr = \int_0^s \frac{\omega_3^2}{\omega_2^2 + \omega_3^2} \, dr. \tag{4.71}$$

Computing I_1 and I_2

Note that the integrand of I_1 does not present singularities, since the denominator $\omega_6 + \omega_5 = \|\mathbf{q}\|^2$. The expressions giving I_1 and I_2 depend on the final expression of the ω_i $i = 2, 3, 5, 6$, which are obtained from the integration formulas given in Chapter 2. Here we only consider the bounded case. Furthermore, we assume initial conditions and parameters such that the solutions for the reduced system are given by

$$\omega_1 = \sqrt{\left|\frac{h_6}{\bar{a}_5}\right|} \mathrm{cn}(\mu_1 \, s, k_{61}), \;\; \omega_5 = \sigma_{156} \sqrt{\left|\frac{h_6}{\bar{a}_1}\right|} \mathrm{sn}(\mu_1 \, s, k_{61}), \;\; \omega_6 = \sqrt{\left|\frac{h_1}{\bar{a}_5}\right|} \mathrm{dn}(\mu_1 \, s, k_{61}), \tag{4.72}$$

$$\omega_2 = \sqrt{\left|\frac{h_4}{\bar{a}_3}\right|} \mathrm{cn}(\mu_2 \, s, k_{42}), \;\; \omega_3 = \sigma_{234} \sqrt{\left|\frac{h_4}{\bar{a}_2}\right|} \mathrm{sn}(\mu_2 \, s, k_{42}), \;\; \omega_4 = \sqrt{\left|\frac{h_2}{\bar{a}_3}\right|} \mathrm{dn}(\mu_2 \, s, k_{42}), \tag{4.73}$$

with $\sigma_{156} = \sigma_{234} = 1$.

Next we give the quadratures I_1 and I_2 under the above assumptions, any other arrangement could be carried out analogously. Thus, assuming that all the coefficients and parameters do not vanish, and by substitution of (4.72) and (4.73) in (4.70) and (4.71) we get

$$I_1(s) = \int_0^s \frac{\sqrt{\left|\frac{h_1}{\bar{a}_5}\right|} \mathrm{dn}(\mu_1 \, r, k_{61}) - \sqrt{\left|\frac{h_6}{\bar{a}_1}\right|} \mathrm{sn}(\mu_1 \, r, k_{61})}{\sqrt{\left|\frac{h_1}{\bar{a}_5}\right|} \mathrm{dn}(\mu_1 \, r, k_{61}) + \sqrt{\left|\frac{h_6}{\bar{a}_1}\right|} \mathrm{sn}(\mu_1 \, r, k_{61})} \, dr, \tag{4.74}$$

and

$$I_2(s) = \int_0^s \frac{\sqrt{\left|\frac{h_4}{\bar{a}_2}\right|} \mathrm{sn}^2(\mu_2 \, r, k_{42})}{\sqrt{\left|\frac{h_4}{\bar{a}_2}\right|} \mathrm{sn}^2(\mu_2 \, r, k_{42}) + \sqrt{\left|\frac{h_2}{\bar{a}_3}\right|} \mathrm{dn}^2(\mu_2 \, r, k_{42})} \, dr, \tag{4.75}$$

and after some algebraic computations and the rescaling $\mu_i r = z$ in the independent variable, the above integrals may be expressed in standard form as follows

$$\begin{aligned} I_1(s) &= \frac{1}{\mu_1}\int_0^{\mu_1 s}\frac{1}{1-\alpha_1^2\mathrm{sn}^2(r,k_{61})}dr \quad \text{B\&F, 431.01 to 436.01} \\ &+ \frac{\beta_1}{\mu_1}\int_0^{\mu_1 s}\frac{\mathrm{sn}^2(r,k_{61})}{1-\alpha_1^2\mathrm{sn}^2(r,k_{61})}dr \quad \text{B\&F, 431.02 to 436.02} \\ &- \frac{\gamma_1}{\mu_1}\int_0^{\mu_1 s}\frac{\mathrm{dn}(r,k_{61})\,\mathrm{sn}(r,k_{61})}{1-\alpha_1^2\mathrm{sn}^2(r,k_{61})}dr, \end{aligned} \tag{4.76}$$

where

$$\alpha_1^2 = \frac{(\bar{a}_5+\bar{a}_6)h_6}{\bar{a}_1 h_1}, \quad \beta_1^2 = \frac{(\bar{a}_5-\bar{a}_6)h_6}{\bar{a}_1 h_1}, \quad \gamma_1^2 = \frac{\bar{a}_5 h_6}{\bar{a}_1 h_1}. \tag{4.77}$$

With respect to the third integral of the expression above, the change $\mathrm{cn}(r,k_{61}) = z$ reduces it to a quadrature which can be expressed by elementary functions. It has to be distinguished three cases depending on the value of the parameter α_1. The second quadrature I_2 is given by

$$I_2(s) = \frac{1}{\beta_2\mu_2}\int_0^{\mu_2 s}\frac{\mathrm{sn}^2(r,k_{42})}{1-\alpha_2^2\mathrm{sn}^2(r,k_{42})}dr, \quad \text{B\&F, 431.02 to 436.02} \tag{4.78}$$

where

$$\beta_2 = \sqrt{\frac{\bar{a}_2 h_2}{\bar{a}_3 h_4}} \quad \alpha_2^2 = \frac{\beta_2 k_{42}-1}{\beta_2}, \tag{4.79}$$

4.5 Invariant Manifolds and Constrained Flows

Given a Hamiltonian system on a manifold M and imposing the restriction to N, a submanifold of M obtained through several constraints, does not lead to a Hamiltonian system on N in general. In this section, we concern about what happens if we restrict system (4.1) to the sphere the manifolds $\mathbb{S}^3$ and $M = 0$. That is, we wonder whether the Hamiltonian formulation of the problem still maintains after assuming some algebraic restrictions. The ideas about constrained Hamiltonian systems were first introduced by J.C. van der Meer and R. Cushman in [van der Meer and Cushman, 1986], from which we extract a resume in Section 1.5.

4.5.1 Spherical solutions

In this section we will prove that the constrained flow of system (4.1) to the tangent bundle of the sphere of radius ρ, $T\mathbb{S}^3_\rho = \{(q,p) \in \mathbb{R}^8/\langle q,q\rangle = \rho,\ \langle q,p\rangle = 0\}$ defines a Hamiltonian flow, which Hamiltonian function is given by the restriction of (4.1) to $T\mathbb{S}^3_\rho$. It will be of crucial importance in our searching of the free rigid body system as a subsystem of the family defined by (4.1).

Theorem 4.9. *$T\mathbb{S}^3_\rho$ is an invariant manifold of the system defined by the family $\mathcal{F}_a$.*

Proof. For the proof of this statement we use the Lemma 1.1. Consider the smooth functions $c_1(q,p) = \langle q, q\rangle - \rho$ and $c_2(q,p) = \langle q, p\rangle$, they are the constrains that define $T\mathbb{S}^3_\rho$. It is a straightforward computation to show that $\{\mathcal{F}_a, c_i\} = 0$ for $i = 1, 2$, thus $T\mathbb{S}^3_\rho$ is an invariant manifold of $X_{\mathcal{F}_a}$.

□

Theorem 4.10. *The Hamiltonian system $(T\mathbb{S}^3, \omega_{|T\mathbb{S}^3}, \mathcal{F}_{a|T\mathbb{S}^3})$ lives in the ambient Hamiltonian system $(T^*\mathbb{R}^4, \omega, \mathcal{F}_a)$, with ω the standard symplectic 2-form, and $\omega_{|T\mathbb{S}^3}$ the symplectic form on $T\mathbb{S}^3$ given by the restriction of ω.*

Proof. By theorem 4.9 we know that $T\mathbb{S}^3$ is an invariant manifold under the flow of the system defined by $\mathcal{F}_a$. Thus we consider the ambient space $(T^*\mathbb{R}^4, \omega, \mathcal{F}_a)$, with ω the standard 2-form, together with the smooth functions $c_1(q,p) = \langle q, q\rangle - 1$ and $c_2(q,p) = \langle q, p\rangle$, for the function given by

$$\mathcal{C} : T^*\mathbb{R}^4 \longrightarrow \mathbb{R}^2 : \ (q,p) \to (c_1, c_2),$$

the value (0,0) is a regular one, then $T\mathbb{S}^3 = c^{-1}(0,0)$ is a smooth submanifold of $T^*\mathbb{R}^4$, called the constrained manifold defined by the constrained functions c_1, c_2. Moreover, $T\mathbb{S}^3$ is a cosymplectic submanifold, that is, the matrix $(\{c_i, c_j\}(q,p))$ of Poisson brackets is invertible for every $(q,p) \in T\mathbb{S}^3$. Hence $\omega_{|T\mathbb{S}^3}$ is a symplectic form on $T\mathbb{S}^3$. □

Next we give a brief discussion of the solutions living in the manifold $T\mathbb{S}^3_\rho$. Imposing the condition ρ constant on system (6.4), we obtain a nonlinear system of two equations corresponding to $\dot\rho$ and $\dot P$. Thus, excluding $\rho = 0$, that leads to the equilibria at the origin, we obtain the following expressions for ρ and P.

1. *Case $P = 0$:* The equation corresponding to $\dot\rho$ vanishes and $\dot P$ provides the value of ρ, that is

$$\rho = |\mathbf{q}(s)| = \sqrt[4]{\frac{\bar a_1}{\bar a_5 + \bar a_6} 4M^2} = \sqrt[4]{\frac{a_6 + a_5}{a_5 - 2a_1 - a_6} 4M^2}, \tag{4.80}$$

for those values of the parameters that implies $\frac{\bar{a}_1}{\bar{a}_5+\bar{a}_6} \geq 0$. When $\bar{a}_5 + \bar{a}_6 = 0$, we are led to

$$\rho = \sqrt{2M}. \tag{4.81}$$

Under the assumption that $P = 0$, $|\mathbf{Q}(s)|$ may also be calculated. Namely, $P = 0$ implies that $\omega_1 = \rho P = \langle \mathbf{q}(s), \mathbf{Q}(s) \rangle = 0$, thus $\mathbf{q}(s)$ and $\mathbf{Q}(s)$ are always perpendicular, that is to say $(\mathbf{q}(s),\ \mathbf{Q}(s)) \in T^*\mathbb{S}^3$. Furthermore, $|\mathbf{Q}(s)|$ is obtained by using M

$$2M = \sqrt{|\mathbf{q}(s)|^2|\mathbf{Q}(s)|^2 - \langle \mathbf{q}(s), \mathbf{Q}(s) \rangle^2} = |\mathbf{q}(s)||\mathbf{Q}(s)|,$$

thus

$$|\mathbf{Q}(s)| = \sqrt{\frac{\bar{a}_5 + \bar{a}_6}{-\bar{a}_1}} = \sqrt{\frac{a_5 - 2a_1 - a_6}{a_5 + a_6}}. \tag{4.82}$$

2. *Case $P \neq 0$:* In this case $\rho \in \mathbb{R}^+$ and the value of P is given by

$$P = \sqrt{\frac{-\bar{a}_1 M^2 + (\bar{a}_5 + \bar{a}_6)\rho^4}{\bar{a}_1 \rho^2}}, \tag{4.83}$$

but we only obtain solutions if $\bar{a}_6 = 0$ or $\bar{a}_5 = 0$. Therefore, keeping in mind relation (4.8), we obtain that

$$P = \sqrt{\rho^4 - \frac{4M^2}{\rho^2}}, \tag{4.84}$$

4.5.2 Orbits on the surface $M = 0$

M is defined by means of (3.17), solutions having $M = 0$ are special, they are usually excluded in the study of several systems included in the family. For example, when we deal with the rigid body system, $M = 0$ means that the total angular momentum vanishes. Thus we study separately the manifold $M = 0$ following the preceding methodology. In what follows, we denote $M = 0$ by M_0

Theorem 4.11. *M_0 is an invariant manifold of the system defined by the family $\mathcal{F}_a$.*

Proof. Following the proof given for Theorem 4.9 and taking into account Proposition 4.1 we readily obtain that $\{\mathcal{F}_a, M\} = 0$ □

Theorem 4.12. *The Hamiltonian system* $(M_0, \omega_{|M_0}, \mathcal{F}_{a|M_0})$ *lives in the ambient Hamiltonian system* $(T^*\mathbb{R}^4, \omega, \mathcal{F}_a)$*, with* ω *the standard symplectic 2-form, and* $\omega_{|M_0}$ *the symplectic form on* M_0 *given by the restriction of* ω*.*

Proof. Analogous to proof of Theorem 4.10 □

Let us now study the constrained flow on M_0. The solution vector $(\mathbf{q}(s), \mathbf{Q}(s)) \in \mathbb{R}^8$ may be split into the position and momenta vectors, it gives rise to a pair of vectors $\mathbf{q}(s) \in \mathbb{R}^4$, $\mathbf{Q}(s) \in \mathbb{R}^4$, which, under the assumption $M = 0$, are collinear considered as vectors in $\mathbb{R}^4$. Therefore, $M = 0$ implies that we are looking for solutions in the following fashion

$$\mathbf{q}(s) = \alpha(s)\mathbf{u}_0, \qquad \mathbf{Q}(s) = \beta(s)\mathbf{u}_0, \tag{4.85}$$

where $\mathbf{u}_0 \in \mathbb{R}^4$ is a unitary vector. Note that the reciprocal is also true, it is, any solution expressed in the above form satisfies $M = 0$.

Functions $\alpha(s)$ and $\beta(s)$ have already been calculated, since they are related to the solutions of the reduced system (4.5). To be precise, the following relations hold

$$\alpha(s) = \sqrt{\omega_6(s) + \omega_5(s)} = \rho(s), \quad \beta(s) = \sqrt{\omega_6(s) - \omega_5(s)} = P(s). \tag{4.86}$$

According to the expressions obtained in (4.86) for α and β, we claim that solutions in the manifold $M = 0$ are bounded if and only if the coefficients $\bar{a}_1$, $\bar{a}_5$ and $\bar{a}_6$ are not all with the same sign. This statement follows from the formulas given in Chapter 2 applied to the reduced subsystem 156, which also give the explicit expression of the functions $\alpha(s)$ and $\beta(s)$.

On top of this and recalling that ω_5 and ω_6 depend exclusively on the parameters $\bar{a}_1$, $\bar{a}_5$, $\bar{a}_6$ and the modulus of the initial conditions $\mathbf{q}_0 = \alpha(0)$ and $\mathbf{Q}_0 = \beta(0)$, see Chapter 2. We have that solutions in the manifold $M = 0$ are independent of the chosen direction $\mathbf{u}_0$.

4.6 Distinguished Systems

Systems defined by $\mathcal{F}_a$ correspond to an integrable family of systems. At first sight, it may not generate a great expectancy, but the real value of this family is that it encapsulates several classical models, namely the free rigid body, the harmonic oscillator, the Kepler system and the geodesic flow are the most remarkable ones.

4.6.1 The free rigid body

The usual formulation of the free rigid body (FRB), takes place with phase space $T^*SO(3)$, together with an atlas expressed in Euler or Andoyer angles. Nevertheless, this is not the only possibility, the quaternionic formulation in $T\mathbb{S}^3$ has gained popularity for the case of numeric integrations, because this formulation has not singularities, that is, there is no "gimbal lock" for the quaternionic formulation. We show that both formulations are included in our model.

Quaternionic formulation of the free rigid body

Here a formulation of (4.2) in terms of quaternions is carried out. The motivation for this is well known, in attitude dynamics it has some advantages versus the Euler angles, (see [Altmann, 1986] Ch. 12., or [Kuipers, 1999] Ch. 8). Quaternions are easier and more efficient to compute and also they are free of the ambiguity associated to the Euler angles.

Recall that under the natural identification of the quaternions with $\mathbb{R}^4$ and regarding $\mathbf{q}$ and $\mathbf{Q}$ as elements in $\mathbb{H}$, we have that $\mathbf{q}\bar{\mathbf{Q}} = (\omega_1, \omega_2, \omega_3, \omega_4)$. Taking into account the matrix form of the quaternionic product through the matrices given in (3.4), it will not be very difficult to express (4.1) in terms of quaternions. First note that the system may be split up as follows

$$\mathcal{F}_a(\mathbf{q}, \mathbf{Q}) = T_a^{1234}(\mathbf{q}, \mathbf{Q}) + V_a^{56}(\mathbf{q}, \mathbf{Q}), \tag{4.87}$$

where $T_a^{1234}(\mathbf{q}, \mathbf{Q})$ and $V_a^{56}(\mathbf{q}, \mathbf{Q})$ are given by

$$\begin{aligned} T_a^{1234}(\mathbf{q}, \mathbf{Q}) = {} & a_1 \left(q_1Q_1 + q_2Q_2 + q_3Q_3 + q_4Q_4\right)^2 \\ & + a_2 \left(q_1Q_2 - q_2Q_1 - q_3Q_4 + q_4Q_3\right)^2 \\ & + a_3 \left(q_1Q_3 + q_2Q_4 - q_3Q_1 - q_4Q_2\right)^2 \\ & + a_4 \left(q_1Q_4 - q_2Q_3 + q_3Q_2 - q_4Q_1\right)^2, \end{aligned} \tag{4.88}$$

and

$$V_a^{56}(\mathbf{q}, \mathbf{Q}) := a_5(\frac{1}{2}\,\|\mathbf{q}\|^2 - \frac{1}{2}\|\mathbf{Q}\|^2)^2 \;+\; a_6(\frac{1}{2}\,\|\mathbf{q}\|^2 + \frac{1}{2}\|\mathbf{Q}\|^2)^2, \tag{4.89}$$

then we obtain a "kinetic" and "potential" part. From the fact that

$$T_a^{1234}(\mathbf{q}, \mathbf{Q}) = a_1\omega_1^2 + a_2\omega_2^2 + a_3\omega_3^2 + a_4\omega_4^2 \;\text{ and }\; (\omega_1, \omega_2, \omega_3, \omega_4) = \mathbf{q}\bar{\mathbf{Q}}$$

and taking into account the matrix expression of the quaternionic product, we can reformulate the $\mathcal{F}_a$ family into the quaternion fashion as follows

$$T_a^{1234}(\mathbf{q},\mathbf{Q}) := \mathbf{Q}^T\, M^*(\mathbf{q})\, \mathbf{Q} \;=\; \mathbf{q}^T\, M^*(\mathbf{Q})\, \mathbf{q} \tag{4.90}$$

$$V_a^{56}(\mathbf{q},\mathbf{Q}) := a_5(\frac{1}{2}\,\|\mathbf{q}\|^2 - \frac{1}{2}\|\mathbf{Q}\|^2)^2 \;+\; a_6(\frac{1}{2}\,\|\mathbf{q}\|^2 + \frac{1}{2}\|\mathbf{Q}\|^2)^2, \tag{4.91}$$

in (4.90) $\mathbf{q}$ and $\mathbf{Q}$ are regarded as four dimensional vectors and matrices $M^*(\mathbf{q})$ and $M^*(\mathbf{Q})$ are defined through $M_L(\mathbf{z})$ given in (3.4)

$$M^*(\mathbf{q}) = M_L(\mathbf{q})\, A\, M_L(\mathbf{q})^T, \quad M^*(\mathbf{Q}) = M_L(\mathbf{Q})\, A\, M_L(\mathbf{Q})^T,$$

where

$$A = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{pmatrix}$$

But this is not enough to define a Hamiltonian system over $\mathbb{H}^2$, we need a symplectic form $\omega_{\mathbb{H}^2}$. Then, taking into account the identification $\mathbb{H}^2 \simeq \mathbb{R}^8 \simeq T^*\mathbb{R}^4$, any given two elements $\mathbb{Q}_1, \mathbb{Q}_2 \in \mathbb{H}^2$ may be considered both as elements in $\mathbb{H}^2$ and $T^*\mathbb{R}^4$. Thus, we can define $\omega_{\mathbb{H}^2}$ as follows

$$\omega_{\mathbb{H}^2}(\mathbb{Q}_1, \mathbb{Q}_2) = \omega(\mathbb{Q}_1, \mathbb{Q}_2) \tag{4.92}$$

where ω is the standard symplcetic form in $T^*\mathbb{R}^4$. Therefore, the canonical equations for the system defined by $(\mathbb{H}^2, \omega_{\mathbb{H}^2}, \mathcal{F}_a(\mathbf{q},\mathbf{Q}))$ are given by

$$\begin{aligned} \dot{\mathbf{q}} &= \;\;\nabla_{\mathbf{Q}} T_a^{1234} \;+\; \nabla_{\mathbf{Q}} V_a^{56}, \\ \dot{\mathbf{Q}} &= -\nabla_{\mathbf{q}} T_a^{1234} \;-\; \nabla_{\mathbf{q}} V_a^{56}, \end{aligned}$$

which in matrix notation reads as follows

$$\begin{aligned} \dot{\mathbf{q}} &= \;\;2\, M_L(\mathbf{q})\, A\, M_L(\mathbf{q})^T\mathbf{Q} \;+\; \nabla_{\mathbf{Q}} V_a^{56}, \\ \dot{\mathbf{Q}} &= -2\, M_L(\mathbf{Q})\, A\, M_L(\mathbf{Q})^T\mathbf{q} \;-\; \nabla_{\mathbf{q}} V_a^{56}. \end{aligned} \tag{4.93}$$

The equation (4.93) represent a more compact expression that the ones given in (4.13). Matrix expression may also be written in the following way

$$\begin{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{Q}} \end{pmatrix} = \begin{pmatrix} M_L(\bar{\omega}_a) & \Delta\, Id_4 \\ -\Delta^*\, Id_4 & M_L(-\omega_a) \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \mathbf{Q} \end{pmatrix},$$

where $\omega_a = (a_1\omega_1, a_2\omega_2, a_3\omega_3, a_4\omega_4)$, Id_4 is the identity matrix of dimension four and Δ, Δ^* are defined as in (4.13).

Note that restricting to the case $a_5 = a_6 = 0$, terms of the "potential" part vanish, thus we obtain that our canonical equations are expressed as the quaternionic equations for the free rigid body given in [Betsch and Siebert, 2009]

$$\begin{aligned} \dot{\mathbf{q}} &= 2\, M_L(\mathbf{q})\, A\, M_L(\mathbf{q})^T \mathbf{Q}, \\ \dot{\mathbf{Q}} &= -2\, M_L(\mathbf{Q})\, A\, M_L(\mathbf{Q})^T \mathbf{q}. \end{aligned} \tag{4.94}$$

Then, restriction of system $(\mathbb{H}^2, \omega_{\mathbb{H}^2}, \mathcal{F}_a(\mathbf{q}, \mathbf{Q}))$ to $T\mathbb{S}^3_\rho$, which is given by

$$T\mathbb{S}^3_\rho = \{(\mathbf{q}, \mathbf{Q}) \in \mathbb{H}^2 / \|\mathbf{q}\| = \rho,\ \sum_{i,j=1}^{4} q_i Q_i = 0\}, \tag{4.95}$$

leads us to the desired system. That is, $(T\mathbb{S}^3_\rho, \omega_{\mathbb{H}^2|T\mathbb{S}^3_\rho}, \mathcal{F}_a(\mathbf{q}, \mathbf{Q}))$ together with $a_5 = a_6 = 0$ is the free rigid body system.

The rigid body in $T^*SO(3)$

First we concentrate in searching the phase space. In this second formulation of the rigid body in $T^*SO(3)$ and by means of the identification between $\mathbb{H}$ and $\mathbb{R}^4$, we will consider the set $\mathbb{S}^3$ both as a set in $\mathbb{R}^4$, the unit sphere, and as a set in $\mathbb{H}$, the unit quaternions, therefore we may also consider $T\mathbb{S}^3$ as a set in $\mathbb{H}^2$ and $\mathbb{R}^8 \simeq T^*\mathbb{R}^4$. Moreover, by means of the equivalence between ω and $\omega_{\mathbb{H}^2}$, given in (4.92), we can identify the symplectic spaces $(T\mathbb{S}^3, \omega_{|T\mathbb{S}^3})$ and $(T\mathbb{S}^3, \omega_{\mathbb{H}^2|T\mathbb{S}^3})$.

Lemma 4.1. *$SO(3)$ is diffeomorphic to $\mathbb{RP}^3$.*

Proof. We claim that every transformation from $SO(3))$ is a rotation about some axis through some angle. Indeed, let $A \in SO(3)$. It suffices to show that A has a unit eigenvalue, then A is a rotation about the respective eigenspace. Consider the eigenproblem: $det(A - \lambda I_3) = 0$. This cubic equation has a real root that must be ± 1. If it is 1 we are done. If it is -1 then A preserves the plane, perpendicular to the eigendirection, and is a reflection in this plane, then the third eigenvalue of A equals to 1. We may consider only rotations through the angles from 0 to π, the direction of the rotation gives the axis an orientation (by the left-hand rule). Encode such a rotation by the vector whose direction is that of the axis and whose magnitude is the angle of rotation. We obtain a ball of radius π, but the points on its boundary should be identified with the antipodal point, the rotations through π and $-\pi$ coincide. The result is the real projective space. $\square$

The rotation group is double covered by the unit quaternions, see Section 3.1.2, where the explicit two to one correspondence between quaternions and the group of rotations $SO(3)$ is shown. Thus, using the fact that $SO(3)$ is diffeomorphic to the real projective three space $\mathbb{RP}^3$, or equivalently, to the space formed by identifying antipodal points on $\mathbb{S}^3$, we move from $T\mathbb{S}^3$ to $T^*SO(3)$ by a $\mathbb{Z}_2$ action. That is to say, the following proposition holds

Proposition 4.13. *Let us consider the $\mathbb{Z}_2$-action ψ given by*

$$\psi : \mathbb{Z}_2 \times \mathbb{S}^3 \longrightarrow \mathbb{S}^3 : (i, \boldsymbol{q}) \longrightarrow \Psi(i, \boldsymbol{q}) = (-1)^i \boldsymbol{q}, \quad i \in \mathbb{Z}_2. \tag{4.96}$$

Then the manifold $T^*SO(3)$ is obtained as the orbit space of the cotangent lifted action of Ψ on $T\mathbb{S}^3 \sim T^*\mathbb{S}^3$.

Proof. ψ is a free proper action, that sends the point **q** into **-q** and the reduce space is given by $\mathbb{S}^3/\psi \cong \mathbb{RP}^3$. Therefore, the reduced space obtained from $T\mathbb{S}^3$ (keeping in mind that $T\mathbb{S}^3 \sim T^*\mathbb{S}^3$), by the corresponding cotangent lifted action Φ is $T^*SO(3)$. $\square$

Proposition 4.14. *A particular choice of the parameters of the family $\mathcal{F}_a$ given by $a_1 = a_5 = a_6 = 0$ and a_2, a_3, $a_4 > 0$ leads us to the Hamiltonian of the free rigid body.*

Proof. Let us set the parameters of the family as

$$a_2 \geq a_3 \geq a_4, \qquad a_3 + a_2 > \frac{a_2 a_3}{a_4}, \qquad a_1 = a_5 = a_6 = 0, \tag{4.97}$$

leads to

$$\begin{aligned} \mathcal{F}(\_, \theta, \phi, \_, \_, \Theta, \Phi, \Psi) &= 4a_2 \left(\frac{\Phi - \Psi \cos\theta}{\sin\theta} \sin\psi + \Theta \cos\psi \right)^2, \\ &+ 4a_3 \left(\frac{\Phi - \Psi \cos\theta}{\sin\theta} \cos\psi - \Theta \sin\psi \right)^2, \\ &+ 4a_4\, \Psi^2. \end{aligned} \tag{4.98}$$

which is the Hamiltonian of the free rigid body, see [Marsden and Ratiu, 1999] page 496, where the principal moments of inertia of the rigid body are given by I_1, I_2 and

I_3 are identified in the above formula with the inverse of the parameters $8a_2$, $8a_3$ and $8a_4$, namely

$$I_1 = \frac{1}{8a_2}, \ I_2 = \frac{1}{8a_3}, \ I_3 = \frac{1}{8a_4}. \tag{4.99}$$

Note that (4.97) ensures the needed relation for the principal moments of inertia given by

$$I_1 < I_2 < I_3, \quad I_1 + I_2 > I_3.$$

□

Corollary 4.15. *The free rigid body system with phase space $T^*SO(3)$ is obtained as a constrained and reduced flow for a suitable choice of the parameters in the family of Hamiltonian systems defined by (4.1).*

Proof. By imposing the quadratic condition $\langle \mathbf{q}, \mathbf{Q} \rangle = 0$ we obtain the constrained subsystem $(T\mathbb{S}^3_\rho, \omega_{|T\mathbb{S}^3_\rho}, \mathcal{F}_a(\mathbf{q}, \mathbf{Q}))$, see Theorem 4.10. Therefore, this result follows from Proposition 4.13, Proposition 4.14 and the fact that the Hamiltonians $\mathcal{F}_a$ are invariant under the action Ψ. □

A treatment of this model in the context of a hyperkahler manifold deserve to be explored in future works, see [Gaeta and Rodriguez, 2014], where quaternionic integrable systems are investigated.

4.6.2 Recovering the harmonic oscillator

As has been announced some classical models are included among the flow of the family $\mathcal{F}_a$. The harmonic oscillator is one of them and this model is easily identified by choosing the parameters

$$a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_{\mathcal{F}}, \tag{4.100}$$

under this assumptions, and taking into account relation (3.19), the Hamiltonian written in Projective Andoyer variables becomes into

$$\mathcal{F}_a = a_{\mathcal{F}} \, (\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 + \omega_5^2 + \omega_6^2) = 2a_{\mathcal{F}} \, \omega_6^2 = \frac{a}{2} \left(\rho^2 + P^2 + \frac{4M^2}{\rho^2} \right)^2 \tag{4.101}$$

thus, after the change of the independent variable given by $r = 2a_{\mathcal{F}} \, (\rho^2 + P^2 + \frac{4M^2}{\rho^2})s$ and the symplectic transformation $(\rho, \mu, P, M) \longrightarrow (\rho, \sigma, P, \Sigma)$, where $\mu = 2\sigma$ and

$2M = \Sigma$, the equations of the system associated are

$$\begin{aligned} \dot{\rho} &= P \qquad \dot{\sigma} = \frac{\sigma}{\rho^2} \\ \dot{P} &= (-\frac{\Sigma^2}{\rho^3} + \rho) \qquad \dot{\Sigma} = 0 \end{aligned} \tag{4.102}$$

with associated Hamiltonian $\frac{1}{2}\Big(\rho^2 + P^2 + \frac{\Sigma^2}{\rho^2}\Big)$.

This system was tackled by Deprit in [Deprit, 1991], it corresponds to the harmonic oscillator. Next we are going to check out our integration with the Hamiltonian provided in [Deprit, 1991].

The integration of system (4.102) is already done, since the variables are expressed in terms of the omegas. Therefore, solving system 156 given in (4.5) under the assumption (4.100), we can construct the solutions for (4.102) using the formulas (4.69). Namely, the omegas are given by

$$\omega_1 = A\cos(2r), \quad \omega_5 = A\sin(2r), \quad \omega_6 = \text{constant}, \tag{4.103}$$

where $A = \sqrt{\omega_5^2 + \omega_1^2} = \omega_6^2 - 4M^2$. Thus, the solution of system (4.102) is given by

$$\begin{aligned} &\rho(r) = A\sin(2r) + \omega_6, \quad P(r) = \frac{A\cos(2r)}{\rho(r)}, \quad M(r) = \frac{\Sigma}{2} \\ &\mu(r) = 2\sigma(r) = \frac{2\omega_6}{M}\arctan\Big(\frac{\omega_6}{2M}(\tan(r) + A)\Big) - 2r, \end{aligned} \tag{4.104}$$

after some algebraic and trigonometric manipulations, this solution may be expressed in the more familiar way

$$\begin{aligned} &\rho^2(\varepsilon) = a^2\sin^2(\varepsilon) + b^2\cos^2(\varepsilon), \quad P(\varepsilon) = \frac{(a^2 - b^2)\sin(2\varepsilon)}{2\rho(\varepsilon)}, \\ &\mu(r) = \frac{2\omega_6}{M}\arctan\Big(\frac{\omega_6}{2M}(\tan(r) + A)\Big) - 2r, \quad M(r) = \frac{\Sigma}{2} \end{aligned} \tag{4.105}$$

where $a^2 = (A + \omega_6)$, $b^2 = (A - \omega_6)$ and $2\varepsilon = \frac{\pi}{2} + 2r$.

That is to say, the solution describes an ellipse of semimajor axis a and semiminor axis b, see Deprit [Deprit, 1991], where the same expressions for ρ and P are obtained. However, note that we provide explicit formulas for $\mu(r)$.

4.6.3 Geodesic flow on the sphere

There are several Hamiltonians that give the geodesic flow when restricted to the cotangent bundle of the sphere. For example, one of them is obtained from the family $\mathcal{F}_a$ by mean of the values $a_2 = a_3 = a_4 = 0$ and $-a_1 = -a_5 = a_6 = \frac{1}{2}$. Namely, it is obtained

$$\mathcal{F}_a(q, Q) = \frac{1}{2}\,\|q\|^2\,\|Q\|^2 - \langle q, Q\rangle. \tag{4.106}$$

Hamiltonian (4.106) was used by Moser in [Moser, 1970, Moser and Zehnder, 2005] to show how the Kepler system is included in the geodesic flow on the sphere up to a regularization of the independent variable. Next we compare the solution obtained from our system with the one given by Moser, which is given by

$$\phi(s) = \begin{pmatrix} q_0 \cos s + Q_0 \sin s \\ -q_0 \sin s + Q_0 \cos s \end{pmatrix},$$

for initial conditions $(q(0), Q(0)) = (q_0, Q_0)$. On the other hand, using Projective Andoyer variables, the expression of the Hamiltonian becomes very simple

$$\mathcal{H}_2(\rho, \lambda, \mu, \nu; P, \Lambda, M, N) = 2M^2,$$

from the expression of the Hamiltonian above, we can see that ρ, λ, ν, P, Λ, M, N are all integrals of the system and μ is a linear function of the time

$$\mu(r) = \mu_0 + 2M^2 r,$$

the geometric interpretation of the Andoyer variables gives a precise description of the situation, ρ is the module of the particle and ν, μ, λ are angles, two of them give the position of the fixed plane of motion and the other one describes how the particle rotates around the origin of the named plane.

Therefore, the system in Andoyer variables is trivially integrated as kind of "polar coordinates" with constant radius ρ and argument μ. It is, the particle moves on circles with constant angular velocity and frequency equal to $\frac{4\pi}{M^2}$.

The Andoyer expression of the solutions is readily related to the corresponding Cartesian expression by means of (4.56) and (4.57). Namely,

$$\begin{aligned} s &= \mu(\tau) = \mu_0 + 2M^2\tau, \\ q_0 &= \rho\,(a\,\cos\alpha, b\,\cos\beta, b\,\sin\beta, a\,\sin\alpha)\,, \\ Q_0 &= \rho\,(A\,\cos\alpha, B\,\cos\beta, B\,\sin\beta, A\,\sin\alpha)\,, \end{aligned}$$

where

$$a = c_1c_2 - s_1s_2, \quad b = c_1s_2 + c_2s_1,$$
$$A = c_1c_2 + s_1s_2, \quad B = c_1s_2 - c_2s_1,$$
$$\alpha = \frac{\lambda + \nu}{2}, \quad \beta = \frac{\lambda - \nu}{2}.$$

4.6.4 The Kepler system

The Kepler system is not obtained directly from our model. It is connected with two of the systems included in the quartic family, that is, the geodesic flow and the harmonic oscillator. The links arises from the regularization of the collision orbits. This issue goes back to [Levi-Civita, 1920], [Moser, 1970] and [Ligon and Schaaf, 1976] in one hand and the KS transformation [Kustaanheimo and Stiefel, 1965] on the other hand. The regularization of the collision orbits through the method of Moser has one disadvantage, the negative energy level must be fixed before the Kepler flow is transformed into the geodesic flow by the stereographic projection. An alternative giving a fully symplectic map between the negative energy part of the phase space of the Kepler Hamiltonian into the punctured cotangent bundle of $\mathbb{S}^n$ was proposed by Ligon and Schaaf in [Ligon and Schaaf, 1976]. A simplified treatment was given by Cushman and Duistermaat in [Cushman and Duistermaat, 1997] and even more by Heckman and de Laat in [Heckman and Laat, 2012]. Unfortunately, we did not success in the application of the Ligon-Schaaf map to the treatment of the perturbed case and we have used the Kustaanheimo-Stiefel transformation. Then, the usage of the Ligon-Schaaf transformation under perturbations is left for future works.

In our model, initial conditions leading to collision orbits are restricted to be in the manifold $M = 0$, see Section(4.5.2). The KS-transformation has the advantage that there is no need to a special treatment for the collision orbits. Nevertheless, in the approach given here, using the Projective Andoyer variables, the constraint of the ”bilinear relation” is no longer needed. Next, we give the explicit formulas defining the Kustaanheimo-Stiefel transformation, a geometric interpretation of this map in terms of quaternions given by Saha [Saha, 2009] is also provided.

The KS-connection between Kepler and the oscillator. A geometric insight

The Kustaanheimo-Stiefel transform goes from $\mathbb{R}^3$ to $\mathbb{R}^4$, but we reinterpret real vectors in terms of quaternions. More precisely, a point in $\mathbb{R}^3$ is identified as pure imaginary quaternion

$$\mathbf{q} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

The KS-transform of $\mathbf{q}$ is the quaternion $\mathbf{Q} = Q_1 + Q_2\mathbf{i} + Q_3\mathbf{j} + Q_3\mathbf{k}$ satisfying

$$\mathbf{q} = \bar{\mathbf{Q}}\mathbf{k}\mathbf{Q}. \tag{4.107}$$

By quaternionic multiplication rules it is easy to verify that this equation has infinitely many solutions given by

$$\mathbf{Q} = (\cos\psi, -\sin\psi\,\mathbf{k})\mathbf{Q}^I \tag{4.108}$$

where

$$\mathbf{Q}^I = \frac{x\mathbf{i} + y\mathbf{j} + Z\mathbf{k}}{\sqrt{2Z}}, \quad Z = z + \sqrt{x^2 + y^2 + z^2} \tag{4.109}$$

Note that the expression (4.107) amounts to a rotation of angle ω about a unit vector $\mathbf{n}$ combined with scalar multiplication, that is

$$\mathbf{Q} = \|\mathbf{Q}\|(\cos\frac{1}{2}\omega + \cos\frac{1}{2}\omega\,\mathbf{n}), \qquad \|\mathbf{n}\| = 1 \tag{4.110}$$

Everything so far is already in the classic literature related to the KS-transform. The new result in the paper of Saha [Saha, 2009] is that he provides a visualization of Q, including its non-uniqueness.

Comparing (4.107) and Theorem 3.1, it is evident that $\mathbf{Q}$ is a rotator that takes the z axis to $\mathbf{q}$, see Fig 4.6.4. To visualize $\mathbf{Q}$, let us rewrite $\mathbf{q}$ as

$$\mathbf{q} = r(\sin\theta\cos\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\theta\mathbf{k}) \tag{4.111}$$

where r, θ, ϕ are the usual polar coordinates. Rewriting $\mathbf{Q}^I$ in the solution (4.108) and simplifying, we have

$$\mathbf{Q}^I = \sqrt{r}\,(\sin\frac{1}{2}\theta\cos\phi\,\mathbf{i} + \sin\frac{1}{2}\theta\sin\phi\,\mathbf{j} + \cos\frac{1}{2}\theta\,\mathbf{k}) \tag{4.112}$$

In other words, the zenith distance of $\mathbf{Q}^I$ is halfway along the great circle from $\mathbf{k}$ to $\mathbf{q}$. From (4.110) we see the rotation angle $\omega = \pi$. Now let us apply the transformation (4.108) with $\psi = \dfrac{\pi}{2}$ to $\mathbf{Q}^I$. This gives

$$\mathbf{Q}^{II} = \sqrt{r}\,(\cos\frac{1}{2}\theta\,\mathbf{i} + \sin\frac{1}{2}\theta\sin\phi\,\mathbf{j} + \sin\frac{1}{2}\theta\cos\phi\,\mathbf{k}), \tag{4.113}$$

in this case, rendering $\mathbf{Q}^{II}$ as the rotator given by (4.110), the implied rotation is by θ, about an axis perpendicular to both $\mathbf{k}$ and $\mathbf{q}$. In general, we can write

$$\mathbf{Q} = \cos\phi\mathbf{Q}^I - \sin\phi\mathbf{Q}^{II}, \tag{4.114}$$

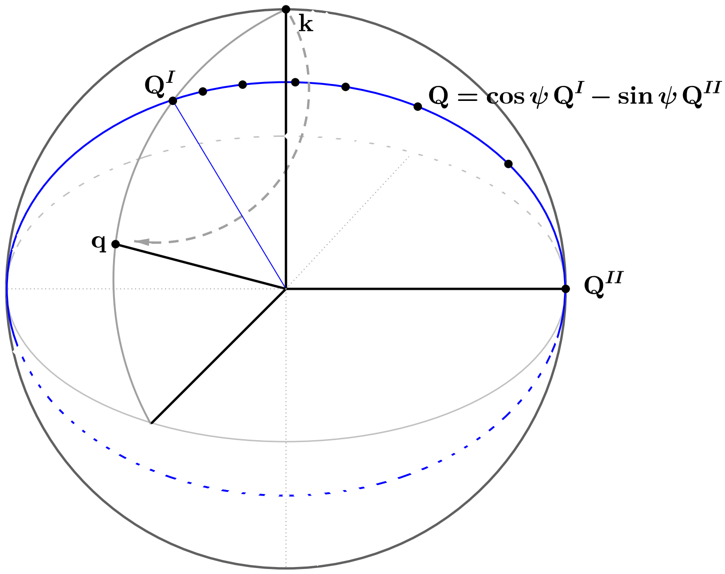


Figure 4.8: $\mathbf{Q}$ is a rotator that takes the z axis to $\mathbf{q}$. The non unicity character of this rotator is showed in the figure. Namely, $\mathbf{Q}$ is any point located in the blue maximal circunference.

which is to say, $\mathbf{Q}$ could be anywhere on the great circle joining $\mathbf{Q}^I$ and $\mathbf{Q}^{II}$.

Let us extend the above configuration space transformation to a canonical one in the phase space. For this purpose we give the relation between the respective conjugate momenta

$$\mathbf{p} = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k} \longrightarrow \mathbf{P} = P_o + P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k},$$

where the old and new conjugate momenta are related as follows

$$\mathbf{P} = -2\mathbf{kQp}, \qquad \mathbf{p} = \frac{\bar{\mathbf{Q}}\mathbf{kP}}{2Q^2}. \tag{4.115}$$

This transformation leads to the following well known result:

Theorem 4.16. *The system defined by (4.1), provided a suitable set of parameters a_i and properly regularized, is related to the 3-D Kepler system.*

Proof. Let the 3-D Kepler system given in Cartesian variables

$$\mathcal{H}_K = \frac{1}{2m}\|\mathbf{p}\|^2 - \frac{\mathcal{G}M_\odot}{\|\mathbf{q}\|}. \tag{4.116}$$

Then, applying the KS-transformation we obtain a 4-D harmonic oscillator. The correspondence $(\|\mathbf{q}\|, \|\mathbf{p}\|) \to (\|\mathbf{Q}\|, \|\mathbf{P}\|)$ maps $T^*\mathbb{R}^3$ into the manifold in $T^*\mathbb{R}^4$ given by the "bilinear relation" $q_1Q_4 - q_4Q_1 + q_2Q_3 - q_3Q_2 = 0$.

$$\mathcal{H} = \frac{\|\mathbf{P}\|^2}{8m\|\mathbf{Q}\|^2} - \frac{\mathcal{G}M_\odot}{\|\mathbf{Q}\|^2} = -h, \tag{4.117}$$

where $h > 0$, $\mathcal{G}$ is the gravitational constant and $M_\odot$, m are the masses of the primary and secondary bodies respectively. After a change of independent variable according to Poincar technique, $\mathcal{H}_o = g(\|\mathbf{Q}\|)(\mathcal{H} + h)$ and $g(\|\mathbf{Q}\|) = \|\mathbf{Q}\|^2$ we obtain

$$\mathcal{H} = \frac{1}{8m}\|\mathbf{P}\|^2 + h\|\mathbf{Q}\|^2 - \mathcal{G}M_\odot, \tag{4.118}$$

by means of the rescaling $\mathbf{Q}_i = 1/\sqrt{8mh}\,\mathbf{Q}'_i$, $\mathbf{P}_i = \sqrt{8mh}\,\mathbf{P}'_i$ we finally obtain

$$\mathcal{H} = \omega(\|\mathbf{P}'\|^2 + \|\mathbf{Q}'\|^2) - \mathcal{G}M_\odot, \tag{4.119}$$

where $\omega = \sqrt{(}h/8m)$. This Hamiltonian is related to the one given in (4.1) $\mathcal{F}_a$ with the parameters $a_1 = \cdots = a_6 = \omega$. That is, the following equality holds

$$\mathcal{F}_a = \omega\mathcal{H}^2 + \mathcal{G}M_\odot.$$

Therefore, both Hamiltonians define the same flow up to the following independent change of variable applied to the system associated to $\mathcal{F}_a$

$$t = \frac{1}{\omega}(\|\mathbf{P}\|^2 + \|\mathbf{Q}\|^2)s.$$

□

Kepler system, Geodesic flow and the Harmonic Oscillator connection through the Projective Andoyer variables

In the preceding section we recalled the classical approach that shows how the 3-D Kepler system is introduced in the 4-D oscillator together with the "bilinear relation". Our approach is the opposite, we will show that the 3-D Kepler system is found by constraining a possible realization of the quartic family, the 4-D harmonic oscillator, to each lever of energy. A similar approach is presented in [Cushman and Bates, 1997], the geodesic flow on $\mathbb{R}^4$ is constrained to the 3-sphere, then after a regularization it is connected to the Delaunay flow, which leads to the Kepler system through the Ligon-Schaaf map, we also set a parallel construction to that one hinging in an alternative proposal for the family (4.1).

Theorem 4.17. *In Projective Andoyer variables, the system defined by (4.1) and $a_i = a \neq 0$ for $i = 1, \ldots 6$, properly regularized, includes the Keplerian system for any value of the integral $N = q_1Q_4 - q_4Q_1 + q_2Q_3 - q_3Q_2$.*

Proof. The harmonic oscillator is obtained from (4.1) after an independent variable rescaling and by setting $a_i = a \neq 0$ for $i = 1, \ldots 6$. That is

$$\mathcal{F}_a(\mathbf{q}, \mathbf{Q}) = a(\|\mathbf{q}\|^2 + \|\mathbf{Q}\|^2)^2. \tag{4.120}$$

Then, let us consider the function $g(\mathbf{q}, \mathbf{Q}) = 1/(8a(\|\mathbf{q}\|^2 + \|\mathbf{Q}\|^2)\|\mathbf{q}\|^2)$ and the change of independent variable given by $t = g(\mathbf{q}, \mathbf{Q})\, s$. As a result, the above Hamiltonian takes the form $\mathcal{K}_a(\mathbf{q}, \mathbf{Q}) = g(\mathbf{q}, \mathbf{Q})(\mathcal{F}_a - h)$ in the manifold $\mathcal{K}_a = 0$, *i.e.*

$$\mathcal{K}_a(\mathbf{q}, \mathbf{Q}) = \frac{\|\mathbf{Q}\|^2 - \sqrt{h}/a}{8\|\mathbf{q}\|^2} = -\frac{1}{8}. \tag{4.121}$$

At first saw, this Hamiltonian does not seem to be very familiar, but when expressed in Projective Andoyer variables with $F(\rho) = \sqrt{\rho}$, it becomes into the form

$$\mathcal{K}_a(\rho, \lambda, \mu, \nu, P, \Lambda, M, N) = \frac{1}{2}\big(P^2 + \frac{M^2}{\rho^2} - \frac{\tilde{h}}{\rho}\big), \tag{4.122}$$

in the manifold $\mathcal{K}_a = -1/8$, where $\tilde{h} = \sqrt{h}/(4a)$. This Hamiltonian corresponds to the Kepler system in 3-D given in polar-nodal canonical variables extended to four dimensions. Indeed, let us consider the following canonical transformation of Mathieu type: $X dx + Y dy + Z dz + N d\nu = P d\rho + \Lambda d\lambda + M d\mu + N d\nu$, which is not a canonical extension.

$$(\rho, \lambda, \mu, \nu, P, \Lambda, M, N) \to (x, y, z, \nu, X, Y, Z, N),$$

given by

$$\begin{aligned}
x &= \rho(\cos\mu\cos\lambda - \sin\mu\sin\lambda\cos I),\\
y &= \rho(\cos\mu\sin\lambda + \sin\mu\cos\lambda\cos I),\\
z &= \rho\sin\mu\sin I,\\
X &= P(\cos\mu\cos\lambda - \sin\mu\sin\lambda\cos I) + \frac{M}{\rho}(\cos\mu\sin\lambda + \sin\mu\cos\lambda\cos I),\\
Y &= P(\cos\mu\sin\lambda + \sin\mu\cos\lambda\cos I) + \frac{M}{\rho}(\sin\mu\sin\lambda + \cos\mu\cos\lambda\cos I),\\
Z &= (P - \frac{M}{\rho})\sin\mu\sin I.
\end{aligned} \tag{4.123}$$

where $\cos I = \Lambda/M$ Then, the Hamiltonian (4.122) in the variables $(x, y, z, \nu, X, Y, Z, N)$ takes the form of the Cartesian 3-D Keplerian system

$$\mathcal{K}_a(x, y, z, \nu, X, Y, Z, N) = \frac{1}{2}\|X\|^2 - \frac{\gamma}{\|x\|}, \tag{4.124}$$

where $\|X\| = (X, Y, Z)$, $\|x\| = (x, y, z)$. Moreover, (ν, N) are integrals that take any value. □

Part III

Application To Roto-Orbital Dynamics

Chapter 5

On The Roto-Orbital Problem. Intermediaries

The application we are interested in is the roto-orbital dynamics of the satellite and asteroid within the full two-body problem (F2BP) [Vereshchagin et al., 2010], which continues to be one of the most challenging problems of Astrodynamics, see for instance [Scheeres, 2012]. In those kind of problems two types of dynamic are identified, *i.e.*, the orbital and the rotational motions. Usually, both dynamics are analyzed separately by identifying them with perturbed Kepler and free rigid body (FRB) systems respectively.

In our study we consider a particular case of the F2BP, which is the result of several simplification due to physical assumptions. Nevertheless, these restrictions are not sufficient enough and we have to deal with non-integrable systems. Therefore, the analysis of such systems is performed with a perturbation theory scheme in mind, where the Hamiltonian $\mathcal{H}$, which defines the system, is split in two parts; the zero order $\mathcal{H}_0$ (integrable) plus a perturbation $\mathcal{H}_1$

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1.$$

These systems may be tackled in several different ways. On one hand we have the numerical methods approach, on which we do not deepen in this work. On the other hand, reduction by means of normalization may be implemented in Poisson or symplectic formalisms. This method relies on two pillars: the existence of invariants and the use of Lie-Deprit transformation [Deprit, 1969, Meyer et al., 2009], but also the possible combination of both. The first technique is based on the existence of invariants associated to the symmetries of the system, and is carried out using Gröbner bases, see [Egea, 2007]. The Lie-Deprit transformation needs the homological equation to be solved in the context of constrained and non-constrained systems. In [Crespo et al., 2009], we initiate the comparison between the reduction of a system having exact symmetries by means of Gröbner bases and the parallel procedure through Lie-Deprit transformations. In the perturbation theory scheme we focus on the choice of the zero order $\mathcal{H}_0$. Nevertheless, the quartic family set a framework in which the classification of the possible normal forms should be investigated, this is in fact one of our current research lines. We point out that several works has been made in this sense and we consider a valuable input to extend to the quartic four dimensional case the construction and classification of normal forms of quadratic polynomial, [Palacián and Yanguas, 2000a, Palacián and Yanguas, 2000b].

Traditionally, the role of the zero order, in which the perturbation theory hinges, is embodied by the free rigid body and the Kepler system. Within the context of Poincaré [Poincaré, 1899] and the later refinement given by Arnold in KAM theory, see [Arnold et al., 1993] page 185, a perturbation theory should be developed upon an *integrable and non-degenerate* zero order. In our study, instead of the classical zero orders, which are degenerated (superintegrable [Fassò, 2005]), we also study another candidates, the *intermediaries* (see [Deprit, 1981] and the references therein). The idea of the intermediary is to define a simplified integrable system of the problem at stake, by adding some terms coming from the perturbation and use it as the new zero order.

Although we do not have any specific problem in mind, we will study two restricted models in the subsequent applications, Chapter 6 and Chapter 7. More specifically, we suggest that these intermediaries should be tested as a zero order in the cases when we are modeling a satellite in a low or a medium orbit around the Earth (or any other principal body).

5.1 Roto-Orbital Problem as a Double Quartic Realization

In this Part we study the dynamics associated to a system made of two rigid bodies $\mathcal{B}_1$ and $\mathcal{B}_2$ with masses m_1 and m_2, which are under mutual gravitational attraction and the effects of their corresponding rotational motions. Moreover, this problem leads to a high dimensional and non-integrable dynamical system, which energy may be expressed in the following fashion

$$\mathcal{H} = T_O + T_R + \mathcal{P}, \tag{5.1}$$

where we denote by T_O, T_R the orbital and rotational kinetic energies and $\mathcal{P}$ is the perturbing potential containing the coupling between the orbital and rotational motions. In our study we always assume that one of the two bodies, named secondary, has a negligible mass when compared with the primary. In this case, the expression of the full perturbing potential $\mathcal{P}$ as well as the way in which it is split lies in the particular problem to be studied. In the applications, we focus in systems with the following physical assumptions:

1. We assume that the dimensions of the rigid body are small when compared with the distance between the two bodies and that the primary object generating the gravitational field is considered to be a sphere, which allow us to truncate the perturbing potential to the MacCullagh's term [MacCullagh, 1840].
2. The non-sphericity of the secondary rigid body may or not affect its orbital motion about the distant body. Therefore, it is assumed to be Keplerian for the satellite case and rosette-like orbits for the case of the asteroid.
3. The orbital plane is chosen as the inertial reference frame.

Therefore, we are left with a system in which the main body is assumed to be at rest and generates a central force field. That is to say, by means of the physical assumptions we obtain a new system consisting in a rotating rigid body, which orbits in a central force field. As a result, the perturbing potential $\mathcal{P}$, may be rewritten as follows

$$\mathcal{P} = -\frac{\mathcal{G}M_\odot}{r} + V, \tag{5.2}$$

where $\mathcal{G} = 6.67384 \times 10^{-11}\, m^3/s^2\, kg$ is the universal gravitational constant, r denotes the distant between the center of masss of the bodies and $M_\odot = m_1 + m_2$ and V is given by

$$V = -\frac{\mathcal{G}M_\odot}{2r^3}\,(A + B + C - 3\,\mathcal{D})\,, \tag{5.3}$$

where $\{A, B, C\}$ are the three principal moments of inertia with $A < B < C$, and

$$\mathcal{D} = A\,\gamma_1^2 + B\,\gamma_2^2 + C\,\gamma_3^2 \tag{5.4}$$

is the moment of inertia of the rigid body with respect to an axis in the direction of the line joining its center of mass with the perturber, of direction cosines γ_1, γ_2, and γ_3. Therefore, (5.1) is given by

$$\begin{aligned} \mathcal{H} &= T_O + T_R + \mathcal{P} \\ &= T_O + T_R - \frac{\mathcal{G}M_\odot}{r} + V \\ &= \mathcal{H}_K + \mathcal{H}_R + V. \end{aligned} \tag{5.5}$$

Where we have that $\mathcal{H}_K = T_O - \mathcal{G}M_\odot/r$ is the Hamiltonian of Kepler system, $\mathcal{H}_R = T_R$ is the FRB system and V is given in (5.3).

5.1.1 Phase space and Hamiltonian formulation

A rigid body is a system of point masses, constrained by holonomic relations expressed by the fact that the distance between points is constant. The most general movement of a rigid body can be described as the composition of the translation of the center of mass of the body and the rotation about an axis through the named center of mass. This motion decomposition is described by mean of three reference systems. Namely, the material, spatial and body coordinates, which correspond with three possible observers. When we make measurements from the origin of an inertial reference system and we see the body translating and rotating, we obtain material coordinates. If we translate with the center of mass of the body we have spatial coordinates and finally, if we set our reference system in such a way that the orthonormal basis vectors point in the same direction than the principal directions of inertia, we are led to the body or convected coordinates.

The evolution with time t of any point in the rigid body may be the result of a translation of vector $v(t)$ and a rotation $R(t)$. That is, if $x_m = (x_1, x_2, x_3)$ represents the material coordinates and $X_b = (X_1, X_2, X_3)$ the body or convected coordinates, we have that

$$x_m(t) = M(t) \begin{pmatrix} X_b^T \\ 1 \end{pmatrix} = \begin{pmatrix} R_t & v_t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_b^T \\ 1 \end{pmatrix} = \begin{pmatrix} R_t X_b^T + v_t \\ 1 \end{pmatrix}, \tag{5.6}$$

where $M(t) \in SE(3)$. As a result, the study of the motion of a rigid body is equivalent to the study of $M(t)$ and consequently the configuration space and the phase space are given by $SE(3)$ and $T^*SE(3)$ respectively. The special Euclidean group $SE(3)$, as a set, is the Cartesian product of $\mathbb{R}^3 \times SO(3)$, which acts by rotations and translations on $\mathbb{R}^3$, $x \to Rx + v$, where $R \in SO(3)$ and $v \in \mathbb{R}^3$. This action may be represented by the multiplication from the left of 4×4 block matrices given in (5.6).

Now we give coordinates to the phase space $T^*SE(3)$, we use polar-nodal variables referred to the material frame for the orbital part, see Appendix A, and Andoyer variables referred to the spatial frame for the rotational part. In Fig 5.1.1 we also provide a geometric view of the relation between both sets of variables. Thus, generically we obtain the expression of the Hamiltonian $\mathcal{H} = \mathcal{H}_K + \mathcal{H}_R + V$ as follows

$$\begin{aligned} \mathcal{H}_K &= \frac{1}{2m}\left(R^2 + \frac{\Xi^2}{r^2}\right) - \frac{\mathcal{G}M_\odot}{r}, \\ \mathcal{H}_R &= \frac{1}{2}\left(\frac{\sin^2\nu}{A} + \frac{\cos^2\nu}{B}\right)(M^2 - N^2) + \frac{1}{2C}N^2, \\ V &= V. \end{aligned} \tag{5.7}$$

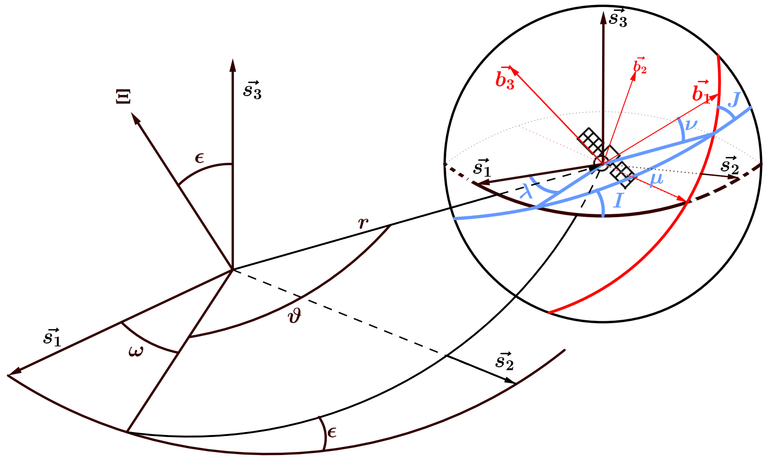


Figure 5.1: In this figure we collect the geometrical interpretation of the angles involved in the rotational and orbital frames, when polar-nodal and Andoyer variables are used for the orbital and rotational motions respectively.

We recall that the Kepler system is regularized, using the KS-transformation (KS) or the Ligon-Schaaf map (LS), by an increase in the involved dimension. As such, the usual itinerary start with the Kepler system with coordinates in $T^*\mathbb{R}^3$. On the one hand and by means of the KS, this system is related to the 4-D isotropic oscillator, together with the restriction given by the bilinear relation. On the other hand, using the LS map, the Kepler system is related to the geodesic flow in the sphere $\mathbb{S}^3$. Observe that, in both cases, we obtain the regularization as the restriction of a flow in $T^*\mathbb{R}^4$ to a 3-D submanifold, for that reason our approach in this Chapter is to consider the Kepler system and the free rigid body as subsystems of a flow in $T^*\mathbb{R}^4$. Moreover, we study the roto-orbital problem as a perturbed double realization of the polynomial family given in (4.1).

5.1.2 Double Projective Andoyer chart

The variables that we have choose in the previous section has some advantages in our system representation, since they express the Hamiltonian system associated as a one degree of freedom systems. Observe that the variables used in both systems has a different geometric meaning. Nevertheless, in the previous Chapter we generalized the Andoyer angles to the Projective Andoyer variables by adding a radius, as well as the polar-nodal variables by adding an angle. Let us also recall that we provided a polynomial family of integrable systems and we identified the Kepler and the FRB systems as constrained flows for particular choices among the polynomial family. These facts set a common framework to study the rotational and orbital dynamics. Therefore, we propose to study the roto-orbital problem in $T^*\mathbb{R}^8$, together with suitable constrains, instead of $T^*SE(3)$, using the following double Projective Andoyer chart

$$(a, A) = (\rho_K, \lambda_K, \mu_K, \nu_K, \rho_R, \lambda_R, \mu_R, \nu_R, P_K, \Lambda_K, M_K, N_K, P_R, \Lambda_R, M_R, N_R),$$

where $(a, A) = (a_K, a_R, A_K, A_R)$. The explicit relation between the polar-nodal-Andoyer chart in $T^*SE(3)$ and the double Projective Andoyer chart in $T^*\mathbb{R}^8$ read as follows

$$\begin{array}{cccccccc} \rho_K = r, & \lambda_K = \omega, & \mu_K = \vartheta, & \nu_K & P_K = R, & \Lambda_K = \Omega, & M_K = \Xi, & N_K \\ \rho_R, & \lambda_R = \lambda, & \mu_R = \mu, & \nu_R = \nu, & P_R, & \Lambda_R = \Lambda, & M_R = M, & N_R = N. \end{array} \tag{5.8}$$

Finally, in the double Projective Andoyer chart we have that the Hamiltonian (5.5) becomes

$$\begin{aligned} \mathcal{H}_K(a_K, \_, A_K, \_) &= \frac{1}{2}\left(P_K^2 + \frac{M_K^2}{\rho_K^2}\right) - \frac{\mathrm{G}}{\rho_K}, \\ \mathcal{H}_R(\_, a_R, \_, A_R) &= \frac{1}{2}\left(\frac{\sin^2\nu_R}{A} + \frac{\cos^2\nu_R}{B}\right)(M_R^2 - N_R^2) + \frac{1}{2C}N_R^2, \\ V &= V(a_K, a_R, A_K, A_R). \end{aligned} \tag{5.9}$$

The canonical differential system associated to the Hamiltonian (5.9) in the double Projective Andoyer chart is determined, up to the potential V, by the following differential equations in $T^*\mathbb{R}^8$

$$\begin{aligned}
\dot\rho_K &= P_K + \frac{\partial V}{\partial P_K}, & \dot\rho_R &= \frac{\partial V}{\partial P_R},\\
\dot P_K &= \frac{M_K^2}{\rho_K^3} - \frac{\partial V}{\partial \rho_K}, & \dot P_R &= -\frac{\partial V}{\partial \rho_R},\\
\dot\lambda_K &= \frac{\partial V}{\partial \Lambda_K}, & \dot\lambda_R &= \frac{\partial V}{\partial \Lambda_R},\\
\dot\Lambda_K &= -\frac{\partial V}{\partial \lambda_K}, & \dot\Lambda_R &= -\frac{\partial V}{\partial \lambda_R},\\
\dot\mu_K &= \frac{M_K}{\rho_K^2} + \frac{\partial V}{\partial M_K}, & \dot\mu_R &= M\left(\frac{\sin^2\nu_R}{A} + \frac{\cos^2\nu_R}{B}\right) + \frac{\partial V}{\partial M_R},\\
\dot M_K &= -\frac{\partial V}{\partial \mu_K}, & \dot M_R &= -\frac{\partial V}{\partial \mu_R},\\
\dot\nu_K &= \frac{\partial V}{\partial N_K}, & \dot\nu_R &= N_R\left(\frac{1}{C} - \frac{\sin^2\nu_R}{A} - \frac{\cos^2\nu_R}{B}\right) + \frac{\partial V}{\partial N_R},\\
\dot N_K &= -\frac{\partial V}{\partial \nu_K}, & \dot N_R &= \frac{(A-B)}{2AB}(M_R^2 - N_R^2)\sin 2\nu_R - \frac{\partial V}{\partial \nu_R}.
\end{aligned} \tag{5.10}$$

Remark 5.1. *Observe that $\mathcal{H}_K$ and $\mathcal{H}_R$ are both considered as a realization of the polynomial family given in (4.1). Namely, let the Hamiltonians $\mathcal{F}_R(\boldsymbol{q}_R, \boldsymbol{Q}_R)$ and $\mathcal{F}_K(\boldsymbol{q}_K, \boldsymbol{Q}_K)$ be the polynomial family realizations giving the FRB, see Corollary 4.15, and the 4-D harmonic oscillator, which is connected to the Kepler system through Theorem 4.17. Therefore, the system given by $\mathcal{H}_K + \mathcal{H}_R$ lives in a submanifold in $T^*\mathbb{R}^8$ determined by the constraints*

$$N_K = 0, \quad \rho_R = \rho, \quad P_R = 0. \tag{5.11}$$

Duplication of the Projective Andoyer chart allows us to take advance of the M-reduction performed in $T^*\mathbb{R}^4$ by means of the change of variables generated by the Poisson map F_M, see 4.62. We duplicate here this map, that is, we consider the map F_M^{RK} given by

$$\begin{aligned}
F_M^{RK} : T^*\mathbb{R}^8 &\longrightarrow (\mathbb{S}_M^2 \times \mathbb{S}_M^2 \times \mathcal{C}^2)^K \times (\mathbb{S}_M^2 \times \mathbb{S}_M^2 \times \mathcal{C}^2)^R\\
(a_K, a_R, A_K, A_R) &\quad\rightsquigarrow\quad (G_{123}^K, \omega_{234}^K, \omega_{156}^K, G_{123}^R, \omega_{234}^R, \omega_{156}^R)
\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}
\omega^K_{789} &= (F^1_M(a_K, A_K), F^2_M(a_K, A_K), F^3_M(a_K, A_K)),\\
\omega^R_{789} &= (F^1_M(a_R, A_R,), F^2_M(a_R, A_R,), F^3_M(a_R, A_R)),\\
\omega^K_{234} &= (F^2_M(a_K, A_K), F^3_M(a_K, A_K), F^4_M(a_K, A_K)),\\
\omega^R_{234} &= (F^2_M(a_R, A_R), F^3_M(a_R, A_R), F^4_M(a_R, A_R)),\\
\omega^K_{156} &= (F^1_M(a_K, A_K), F^5_M(a_K, A_K), F^6_M(a_K, A_K)),\\
\omega^R_{156} &= (F^1_M(a_R, A_R), F^5_M(a_R, A_R), F^6_M(a_R, A_R)),
\end{aligned} \tag{5.13}$$

The double map F^{RK}_M is also Poisson and performs, through the change of variables it defines, a $M_K M_R$-reduction in $T^*\mathbb{R}^8$. The reduced phase space is the product of the rotational and orbital M-reduced phase spaces.

The MacCullagh Term in Projective Andoyer coordinates

At this point, we complete this section by looking for the expression of the potential V in (5.5), which is given by Mac Cullagh term. It will allows us to give the explicit expressions of the differential system of equations (5.10), but also is required to generate the different ways in which V is split. Details are not needed for our purposes, the reader interested may found further explanation in [Cid and Ferrer, 1997], Chapter 7.

By replacing Eq. (5.4) in Eq. (5.3) and taking into account that the direction cosines satisfies the quadratic relation $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$, we get

$$V = -\frac{\mathcal{G}M_\odot}{2\rho_K^3}\left[(C-B)(1-3\gamma_3^2) - (B-A)(1-3\gamma_1^2)\right]. \tag{5.14}$$

The expression of the direction cosines is obtained from the relation matching the spatial frame and the body frame. Following [Vallejo, 1995, Kinoshita, 1972] we have that

$$\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix} = R_R \circ R_K \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}, \tag{5.15}$$

where

$$R_R = R_3(\nu_R)\, R_1(J)\, R_3(\mu_R)\, R_1(I)\, R_3(\lambda_R), \quad R_K = R_3(\lambda_K)\, R_1(\epsilon)\, R_3(\mu_K) \tag{5.16}$$

and

$$\cos I = \frac{\Lambda_R}{M_R}, \quad \cos J = \frac{N_R}{M_R}, \quad \cos\epsilon = \frac{\Lambda_K}{M_K}. \tag{5.17}$$

If the orbital plane is chosen as the inertial reference frame, then the orbital reference frame is related to the body frame by the following composition of rotations:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix} = R_3(\nu_R)\, R_1(J)\, R_3(\mu_R)\, R_1(I)\, R_3(\phi) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{5.18}$$

where the new variable introduced ϕ is given by

$$\phi = \lambda_R - \mu_K. \tag{5.19}$$

Later on we will come back to ϕ, this variable plays an important role when we deal with the circular intermediaries, since the assumption of a circular orbit introduces the time in the Hamiltonian through ϕ.

Then, by replacing γ_1 and γ_3 as given by Eq. (5.18) in the disturbing potential (5.14), after some calculations we get that,

$$V = -\frac{\mathcal{G}M_\odot}{32r^3}\left[(2C - B - A)V_1 + \frac{3}{2}(B - A)V_2\right]. \tag{5.20}$$

Next, using Andoyer variables, we explicitly give the expression of V for the case in which the orbital plane is chosen as the inertial reference frame. This potential is made of V_1, the "axisymmetric part", which is independent of ν, is given by

$$\begin{aligned} V_1 \;=\; & (4 - 6s_J^2)\,(2 - 3s_I^2 + 3s_I^2\,C_{2,0,0}) \\ & -12 s_J c_J s_I\,[(1 - c_I)\,C_{-2,1,0} + 2c_I\,C_{0,1,0} - (1 + c_I)\,C_{2,1,0}] \\ & +3s_J^2\,[(1 - c_I)^2\,C_{-2,2,0} + 2s_I^2\,C_{0,2,0} + (1 + c_I)^2\,C_{2,2,0}]\,, \end{aligned} \tag{5.21}$$

and the "triaxiality part" V_2, which carries the ν_R contribution to the perturbation, reads as follows

$$\begin{aligned} V_2 \;=\; & 6s_I^2 s_J^2\,(C_{2,0,-2} + C_{2,0,2}) - 4(1 - 3c_I^2)s_J^2\,C_{0,0,2} \\ & +(1 + c_J)^2\,[(1 - c_I)^2\,C_{-2,2,2} + 2s_I^2\,C_{0,2,2} + (1 + c_I)^2\,C_{2,2,2}] \\ & +(1 - c_J)^2\,[(1 - c_I)^2\,C_{-2,2,-2} + 2s_I^2\,C_{0,2,-2} + (1 + c_I)^2\,C_{2,2,-2}] \\ & +4s_I s_J\,(1 + c_J)\,[(1 - c_I)\,C_{-2,1,2} + 2c_I\,C_{0,1,2} - (1 + c_I)\,C_{2,1,2}] \\ & -4s_I s_J\,(1 - c_J)\,[(1 - c_I)\,C_{-2,1,-2} + 2c_I\,C_{0,1,-2} - (1 + c_I)\,C_{2,1,-2}]\,. \end{aligned} \tag{5.22}$$

Note that $C_{i,j,k} \equiv \cos(i\phi + j\mu_R + k\nu_R)$ and the notation has been abbreviated by writing $c_I \equiv \cos I$, $s_I \equiv \sin I$, $c_J \equiv \cos J$, and $s_J \equiv \sin J$.

5.2 Intermediaries Approach. Gravity-Gradient Type

Yet, the difficulties associated with the equations (5.10) lead to perturbation theories, since it defines a non integrable system. Therefore, the Hamiltonian $\mathcal{H}$ is split in two part as follows

$$\mathcal{H} = H_0 + H_1, \tag{5.23}$$

where H_0 denotes the zero order and H_1 is considered as the perturbation. In this light, the classical way to proceed is by choosing the zero order being the sum of the Kepler and FRB systems

$$H_0 = \mathcal{H}_K + \mathcal{H}_R.$$

Nevertheless, following Poincaré [Poincaré, 1899] other choices for the zero order are possible. In other words, the Hamiltonian may be split to develop a perturbation theory in such a way that the zero order H_0 is integrable

$$\mathcal{H} = H_0 + H_1, \tag{5.24}$$

Arnold, in the context of KAM theory [Arnold et al., 1993], added the condition of non-degenerancy to the zero order. Here we change the zero order by making $V = V_0 + V_1$ and adding V_0 to the classic zero order

$$\mathcal{H}_0 = \mathcal{H}_K + \mathcal{H}_R + V_0. \tag{5.25}$$

In our applications we check alternative choices for H_0. The suitability of them depends on the scope of the study, ranging from autonomous navigation algorithms and control strategies to long-term dynamics surveys. The requirements for those perturbative schemes may be quite different as well, being critical the choice made for the zero order model on which the rest of the perturbation process hinges. However, the searching for a better zero order has given place to the concept of *intermediary*, that we define later on, it is a classic in both astronomy and astrodynamics (see [Deprit, 1981]).

The basic idea of the intermediaries consists in defining a simplified integrable and non degenerate system of the problem at stake, where the work of Hill on the Moon motion [Wilson, 2010] is, perhaps, the best known example. In our applications we focus on the roto-orbital dynamics of a triaxial rigid body under gravity-gradient torque. The simplified system includes part of the full potential where the roto-orbital coupling is present and the new zero order is given by (5.25) satisfying that the system defined by $\mathcal{H}_0$ is integrable. Thus, we obtain some advantages versus the use of the Kepler and free rigid body models. On the one hand, it allows to identify special solutions that could become nominal trajectories in designed missions whereas it alleviates usual heavy computations when the perturbation approach is built. On the other hand, it can be used to build a perturbation theory, in the sense that the new unperturbed part avoids the degenerate character inherent to the classical superintegrable models (Kepler or free rigid body systems in astrodynamics). In other words, a first order perturbed solution based on the intermediaries might be accurate enough for tracking purposes. In astrodynamics, when dealing with orbital dynamics applied to artificial satellites, some lines of research on intermediaries arose during the seventies by Garfinkel, Aksnes, Cid, Sterne, etc. (see Deprit's review [Deprit, 1981] for further details), whose benefits are now seen in areas such as the relative motion in formation flights [Lara and Gurfil, 2012]. Nevertheless, less work has been done when dealing with attitude dynamics, where the proposal of intermediaries is more recent (see Arribas [Arribas, 1989] and Ferrándiz and Sansaturio[Ferrándiz and Sansaturio, 1989]) and, to our knowledge, no systematic study has been done on them.

5.2.1 A New Set of Intermediaries

By observation of the MacCullagh potential Ferrer and Molero identify, when Andoyer variables are used, five new intermediaries in [Ferrer and Molero, 2014b]. For the benefit of the reader, we have listed all of them here keeping the original notation, and we have classified them in three categories. Indeed, while the Hamiltonian $\mathcal{H}_\nu$ and $\mathcal{H}_{\nu,\phi}$ relate to the generic triaxial case, the other pair $\mathcal{H}_\phi$ and $\mathcal{H}_\mu$ fit the almost symmetric bodies, and for both pairs we assume the satellite to be in a circular orbit. Finally, with respect to the first intermediary listed, $\mathcal{H}_r$, it is the only integrable case giving orbits different from the circular one.

• **Elliptic type intermediary:** $\mathcal{H}_r$**.** This intermediary seems to be the only integrable case giving a good approximation when the satellite moves in an elliptic orbit, it is defined by the following Hamiltonian function

$$\mathcal{H}_r = \mathcal{H}_K + \mathcal{H}_R + V_0$$

where the perturbing potential V_0 is a function of the radial distance and two of the rotational momenta. More precisely we have

$$V_0 \equiv V_r = \frac{\mathcal{G}M_\odot}{4\rho_K^3}(B + A - 2C)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right)\left(1 - 3\frac{\Lambda_K^2}{M_K^2}\right). \tag{5.26}$$

For the particular case in which the orbital plane is chosen as the inertial reference frame we have the following simplified version

$$V_0 \equiv V_r = \frac{\mathcal{G}M_\odot}{4\rho_K^3}(B + A - 2C)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right). \tag{5.27}$$

For details on the dynamics of this system see Chapter 7.

In many applications we may assume the orbit to be circular, hence the radius is constant $r = a$. As a consequence we simplify the previous expressions introducing n, the mean orbital motion, and we write

$$\mathcal{G}M_{\odot} = n^2 a^3. \tag{5.28}$$

Then we will drop from the Hamiltonian the Keplerian term and we will not consider the equations of the system related to the orbital part.

• **Circular generic triaxial intermediaries: $\mathcal{H}_\nu$ and $\mathcal{H}_{\nu,\phi}$.** Assuming a circular orbit for the satellite and the orbital plane chosen as the inertial reference frame, we consider the second and third intermediaries defined by the following Hamiltonian functions

$$\mathcal{H}_\nu = \mathcal{H}_R + V_\nu(-, -, \nu_R, \Phi, M_R, N_R), \tag{5.29}$$

where the potential V_ν is a function of the variable ν_R and three rotational momenta. From now on, and also in the following intermediaries defined, we drop the subindex $_R$ since there is no possibility of confusion and for the sake of a clearer notation.

$$V_\nu = -\frac{n^2}{16}\left[(2C - B - A)(2 - 3s_J^2)(2 - 3s_I^2) - 3(B - A)(1 - 3c_I^2)s_J^2 \cos 2\nu\right] \tag{5.30}$$

and

$$\mathcal{H}_{\nu,\phi} = \mathcal{H}_R + V_{\nu,\phi}(\phi, -, \nu, \Phi, M, N) \tag{5.31}$$

where the potential $V_{\nu,\phi}$ is now a function of the variables ν and ϕ together with the three rotational momenta. We will see below in Section 5.3.1 that $\Phi = \Lambda$. More precisely

$$V_{\nu,\phi} = -\frac{n^2}{8}\left\{(2C - B - A)[(2 - 3s_I^2 + 3s_I^2 \cos 2\phi) - 3s_J^2] - \frac{3}{2}(B - A)s_J^2 \cos 2\nu\right\} \tag{5.32}$$

Note that, with respect to V_ν, the perturbation $V_{\nu,\phi}$ does not include the secular term $18(2C - B - A)s_I^2 s_J^2$ coming from the axisymmetric part of the potential, which is one of the most important differences between these two intermediaries.

In [Ferrer et al., 2014], some preliminary numerical comparisons with the full problem shown that $V_{\nu,\phi}$ behaves worse than V_ν. Then, we devote Chapter 6 to study V_ν.

On the other hand, considering quasi-symmetric bodies, as an alternative to the classical expression of the rotational kinetic energy, since the time of Andoyer the function

$\mathcal{H}_R$ used to be rearranged as follows

$$\mathcal{H}_R = \frac{1}{4}\left(\frac{1}{A}+\frac{1}{B}\right)(M^2-N^2)+\frac{1}{2C}N^2-\frac{1}{4}\left(\frac{1}{A}-\frac{1}{B}\right)(M^2-N^2)\cos 2\nu, \quad (5.33)$$

including the last term as part of the perturbation.

With this in mind, we may consider again a reordering of the gravity-gradient perturbation (5.20) leading to two intermediaries for quasi-symmetric bodies:

• **Quasi-symmetric intermediaries: $\mathcal{H}_\phi$ and $\mathcal{H}_\mu$.** In this case we also assume a circular orbit for the satellite and the orbital plane chosen as the inertial reference frame, then the two last intermediaries are the systems defined by the following Hamiltonian functions

$$\begin{aligned}\mathcal{H}_\phi &= \frac{1}{4}\left(\frac{1}{A}+\frac{1}{B}\right)(M^2-N^2)+\frac{1}{2C}N^2 \\ &\quad -\frac{n^2}{16}(2C-B-A)[(4-6s_J^2)\left(2-3s_I^2+3s_I^2\,\cos 2\phi\right)].\end{aligned} \quad (5.34)$$

$$\begin{aligned}\mathcal{H}_\mu &= \frac{1}{4}\left(\frac{1}{A}+\frac{1}{B}\right)(M^2-N^2)+\frac{1}{2C}N^2 \\ &\quad -\frac{n^2}{16}(2C-B-A)[(2-3s_J^2)(2-3s_I^2)-12s_Jc_Js_Ic_I\,\cos\mu+3s_J^2s_I^2\,\cos 2\mu].\end{aligned} \quad (5.35)$$

From the numerical and analytical study worked out in [Ferrer et al., 2014], that compares the above Hamiltonians with the full gravity-gradient torque, we conclude that they enjoy some benefits: (i) the dynamics of the new models is undoubtedly closer to that given by the full problem. (ii) All models allow to study the coupling between the orbital mean motion and the rotational variables. (iii) All the proposed triaxial models are built without involving μ-terms, hence the modulus of the angular momentum vector is still an integral, which leads to a $\mathbb{S}^2_M \times \mathbb{S}^2_M$ reduced space. (iv) Despite that the triaxial models maintain the magnitude of the angular momentum, its characteristic vector and plane are generically not fixed in the space rotating frame, hence we are breaking the degeneracy of the torque free motion and therefore taking a more realistic model to address perturbation strategies.

Note that in [Ferrer et al., 2014] those intermediaries are labeled as Intermediary 1, Intermediary 2, Intermediary 3, Intermediary 4 and Intermediary 5 corresponding with the same order given here.

5.3 Methodology

In this section we gather, from a theoretical point of view, some of the common techniques applied to the study of the intermediaries in the following chapters.

5.3.1 Transformations in the extended phase space

The extended phase space is commonly used when we have a non autonomous Hamiltonian system. The new phase space increases dimension by two and lead to an autonomous Hamiltonian, see [Lanczos, 1970]. In this section we develop a modified construction to handle the non autonomous issue, but also the Poincaré time regularizations.

Consider a Hamiltonian system $(M, \omega, \mathcal{H}(q,p))$, where M is a $2n$-dimensional manifold and $q = (q_1, \ldots, q_n)$. Usually, the extended phase space is associated to the case in which $\mathcal{H}$ is a non autonomous Hamiltonian, but we do not impose any assumption on the system at stake. Following [Lanczos, 1970], the extended phase space is obtained by letting the time t become one of the mechanical variables. Instead of considering the position q_i coordinates, we consider the position coordinates and the time t as variables given as functions of some unspecified parameter τ. Hence, we obtain a new Hamiltonian system $(M', \omega', \mathcal{K}(Q,P))$, where $M' = M \times \mathbb{R}^2$,

$$Q = (q, q_{n+1}) = (q, t), \qquad P = (p, p_{n+1}) = (p, -\mathcal{H}),$$

and $\omega' = \omega + (dQ_{n+1} \wedge dP_{n+1})$. For the special choice of the new Hamiltonian

$$\mathcal{K} = f(q_1, \ldots, q_n, p_1, \ldots, p_n)(P_{n+1} + \mathcal{H}), \tag{5.36}$$

where $f(q_1, \ldots, q_n, p_1, \ldots, p_n) = f(Q_1, \ldots, Q_n, P_1, \ldots, P_n) = f(q,p)$ is a strictly positive and continuously differentiable function in the domain of $\mathcal{H}$. If the original system was autonomous, we have not obtain any gain. But if it was not, the new system is now autonomous. Next we show the relation between the old system and the extended one.

The canonical equations associated to the new autonomous system are given by

$$\left.\begin{aligned} \dot{Q}_k &= \frac{\partial \mathcal{H}}{\partial P_k} f(q,p) + (P_{n+1} + \mathcal{H}) \frac{\partial f(q,p)}{\partial P_k}, \\ \dot{P}_k &= -\frac{\partial \mathcal{H}}{\partial Q_k} f(q,p) - (P_{n+1} + \mathcal{H}) \frac{\partial f(q,p)}{\partial Q_k}, \end{aligned}\right\} \tag{5.37}$$

for $k = 1, \ldots, n$, and

$$\left.\begin{array}{rcl} \dot{Q}_{n+1} & = & f(q,p), \\ \dot{P}_{n+1} & = & -\dfrac{\partial \mathcal{H}}{\partial Q_{n+1}} f(q,p), \end{array}\right\} \tag{5.38}$$

the next to last equation implies that $dt = f(q,p)\,d\tau$, for the particular case in which $f(q,p) = 1$ $\tau = t - t_0$ and the last one gives the law according to which the negative of the total energy, that is P_{n+1}, changes with the time. This equation is now included in the parametric set of equations as an independent equation, since P_{n+1} is one of the independent variables.

Poincaré time transformation

A remarkable result is obtained when we restrict to the manifold $\mathcal{K} = 0$ and the Hamiltonian $\mathcal{H}$ is autonomous. In this case system (5.43) becomes

$$\left.\begin{array}{rcl} \dot{Q}_k & = & \dfrac{\partial \mathcal{K}}{\partial P_k} = f(q,p)\dfrac{\partial \mathcal{H}}{\partial P_k}, \\ \dot{P}_k & = & -\dfrac{\partial \mathcal{K}}{\partial Q_k} = -f(q,p)\dfrac{\partial \mathcal{H}}{\partial Q_k}, \end{array}\right\} \tag{5.39}$$

for $k = 1, \ldots, n$, and

$$\left.\begin{array}{rcl} dt & = & f(q,p)\,d\tau, \\ \dot{P}_{n+1} & = & -\dfrac{\partial \mathcal{H}}{\partial Q_{n+1}} f(q,p), \end{array}\right\}, \tag{5.40}$$

which corresponds to the well known Poincaré time transformation of a Hamiltonian system. That is, a regularization in the independent variable t given by $dt = f(q,p)\,d\tau$, that transform the Hamiltonian defining the system $\mathcal{H} \to f(q,p)\mathcal{H}$.

Avoiding the time in the circular intermediaries

Due to the consideration of a circular orbit in the triaxial and quasi-symmetric intermediaries given in Section 5.2.1 and taking into account the system of canonical differential equations (5.10), the polar coordinate of the orbital motion is given by $\mu_K = \mu_{K0} + n\,t$, where $n = M_K/\rho_K^2$. Thus, μ_K introduces the time in the Hamiltonian function through ϕ, see (5.19), and the system is not autonomous any more, since the Hamiltonian is expressed in the variables $\mathcal{H}(\rho_R, \lambda_R, \mu_R, \nu_R, P_R, \Lambda_R, M_R, N_R; t)$. Let us consider the following identification in the above generalized construction of the extended phase space

$$\begin{array}{lllll} Q_1 = \rho, & Q_2 = \lambda, & Q_3 = \mu, & Q_4 = \nu, & Q_5 = t, \\ P_1 = P, & P_2 = \Lambda, & P_3 = M, & P_4 = N, & P_5 = T = -\mathcal{H}, \end{array} \tag{5.41}$$

again we have drop the subindex $_R$ and let the function given in (5.36) constant $f(q,p) = 1$. Thus, the Hamiltonian given in (5.36) becomes

$$\mathcal{K}(Q_1, Q_2, Q_3, Q_4, Q_5, P_1, P_2, P_3, P_4 P_5) = \mathcal{K}(\rho, \lambda, \mu, \nu, t, P, \Lambda, M, N, T) = T + \mathcal{H},$$

then, $\mathcal{K}$ is autonomous and the associated system (5.42) expressed in the variables (5.41) is given by

$$\left. \begin{array}{lllll} \dot{\rho} = \dfrac{\partial \mathcal{H}}{\partial P}, & \dot{\lambda} = \dfrac{\partial \mathcal{H}}{\partial \Lambda}, & \dot{\mu} = \dfrac{\partial \mathcal{H}}{\partial M}, & \dot{\nu} = \dfrac{\partial \mathcal{H}}{\partial N}, & \dot{t} = 1, \\ \dot{P} = -\dfrac{\partial \mathcal{H}}{\partial \rho}, & \dot{\Lambda} = -\dfrac{\partial \mathcal{H}}{\partial \lambda}, & \dot{M} = -\dfrac{\partial \mathcal{H}}{\partial \mu}, & \dot{N} = -\dfrac{\partial \mathcal{H}}{\partial \nu}, & \dot{T} = -\dfrac{\partial \mathcal{H}}{\partial t}, \end{array} \right\} \tag{5.42}$$

Let us consider the following linear symplectic change of variables in the extended phase space

$$\begin{aligned} \phi &= \lambda - nt \\ \psi &= t \\ \Phi &= \Lambda \\ \Psi &= n\Lambda + T, \end{aligned} \tag{5.43}$$

then,

$$\tilde{\mathcal{K}}(\rho, \phi, \psi, \mu, \nu, P, \Phi, \Psi, M, N) = \Psi - n\Phi + \mathcal{H}(\rho, \phi, \_, \mu, \nu, P, \Phi, \_, M, N),$$

which gives place to a separable system

$$\psi = t, \quad \Psi = \Psi_0, \tag{5.44}$$

and the remaining variables are given by the new zero order Hamiltonian

$$\tilde{\mathcal{K}}(\rho, \phi, \mu, \nu, P, \Phi, M, N) = -n\Phi + \mathcal{H}. \tag{5.45}$$

Some authors avoid the time by moving to a rotating frame at the same rotation rate as the orbital motion see [San-Juan J.F., 2012, Ferrer et al., 2014], the new term $-n\Phi$ is named as the Coriolis term.

5.3.2 Poisson Reduction of the Intermediaries

All the proposed triaxial models are built without involving μ-terms, hence the modulus of the angular momentum vector M is still an integral, which leads to a $\mathbb{S}^2_M \times \mathbb{S}^2_M \times \mathcal{C}^2$ reduced space, where $\mathbb{S}^2_M$ is the sphere with radius M and $\mathcal{C}^2$ is the upper sheet of the hyperboloid defined by the Casimir $C_3 = \omega_6^2 - \omega_5^2 - \omega_1^2 = M^2$, see Section 4.3. Further on, we focus in the intermediaries of the elliptic and triaxial types.

In this section we suggest a generic treatment for Hamiltonian systems having the M-symmetry. Let $(T^*\mathbb{R}^4, \omega, \mathcal{H})$ be a Hamiltonian dynamical system, where ω is the standard symplectic form and the Hamiltonian function will usually be expressed in Projective Andoyer variables $(\rho, \lambda, \mu, \nu, P, \Lambda, M, N)$. In addition, let us assume that $\mathcal{H}$ is a M-invariant Hamiltonian. Thus, the variable μ is not in the Projective Andoyer expression of the Hamiltonian, that is, $\mathcal{H} = \mathcal{H}(\rho, \lambda, \_, \nu, P, \Lambda, M, N)$, and the system may be reduced using the M-invariance of the Hamiltonian to the six dimensional space given by $\mathbb{S}^2_M \times \mathbb{S}^2_M \times \mathcal{C}^{2*}$.

The canonical equations associated with $\mathcal{H}(\rho, \lambda, \_, \nu, P, \Lambda, M, N)$ are given by

$$\begin{aligned} \dot{\rho} &= \frac{\partial \mathcal{H}}{\partial P}, & \dot{\lambda} &= \frac{\partial \mathcal{H}}{\partial \Lambda}, & \dot{\mu} &= \frac{\partial \mathcal{H}}{\partial M}, & \dot{\nu} &= \frac{\partial \mathcal{H}}{\partial N}, \\ \dot{P} &= -\frac{\partial \mathcal{H}}{\partial \rho}, & \dot{\Lambda} &= -\frac{\partial \mathcal{H}}{\partial \lambda}, & \dot{M} &= -\frac{\partial \mathcal{H}}{\partial \mu} = 0, & \dot{N} &= -\frac{\partial \mathcal{H}}{\partial \nu}. \end{aligned} \tag{5.46}$$

This system can be reduced by fixing M. The reconstruction depends on the following quadrature

$$\mu(t) = \int_{t_0}^{t} \frac{\partial \mathcal{H}}{\partial M} ds,$$

to the 3-DOF Hamiltonian dynamical system $(T^*\mathbb{R}^3, \omega, \mathcal{H}_M)$, where we also use ω to denote the standard symplectic form and the Hamiltonian depends on the six variables $(\rho, \lambda, \nu, P, \Lambda, N)$ and a parameter M, that is $\mathcal{H}_M = \mathcal{H}(\rho, \lambda, \_, \nu, P, \Lambda, N; M)$.

Under certain circumstances, one may need to avoid Andoyer variables. For instance, a high complexity of the canonical system expressed in these variables or the well known fact that Andoyer variables are not defined for the particular case in which the invariant angular momentum is parallel to the s_3-axis (the third axis of the spatial frame) or parallel to b_3-axis (the third axis of the body frame). In other words, those variables are not defined for $I = 0$, $I = \pi$, $J = 0$ and $J = \pi$ because the node lines vanish. Therefore, we propose now a new set of Poisson variables rather than symplectic

$$(\rho, \lambda, \nu, P, \Lambda, N) \rightarrow (G_1, G_2, G_3, M_1, M_2, M_3, W_1, W_2, W_3),$$

we explore this possibility in the following chapters. Henceforth, we have found convenient for the sake of more compact formulas, the following rescaling in the Poisson variables of the reduced space

$$\begin{aligned} M_1 &= \frac{1}{2}\,\omega_2, & M_2 &= -\frac{1}{2}\,\omega_3, & M_3 &= \frac{1}{2}\,\omega_4, \\ G_1 &= \frac{1}{2}\,\omega_7, & G_2 &= -\frac{1}{2}\,\omega_8, & G_3 &= \frac{1}{2}\,\omega_9 \\ W_1 &= \frac{1}{2}\,\omega_1, & W_2 &= \frac{1}{2}\,\omega_5, & W_3 &= \frac{1}{2}\,\omega_6. \end{aligned} \tag{5.47}$$

This change is also made in order to be in consonance with the related literature, where it is common to use the notation $\mathbf{M} = (M_1, M_2, M_3)$ and $\mathbf{G} = (G_1, G_2, G_3)$ to designate the angular momentum vector referred to the body and the spatial frames respectively, see for instance [Gurfil et al., 2007]. The new expression of the Poisson matrix in these variables is given by

$$P = \begin{pmatrix} 0 & G_3 & -G_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -G_3 & 0 & G_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ G_2 & -G_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_3 & -M_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -M_3 & 0 & M_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2 & -M_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -W_3 & -W_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_3 & 0 & W_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_2 & -W_1 & 0 \end{pmatrix} \tag{5.48}$$

and Casimirs

$$C_1 = G_1^2 + G_2^2 + G_3^2, \quad C_2 = M_1^2 + M_2^2 + M_3^2, \quad C_3 = W_3^2 - W_2^2 - W_1^2. \tag{5.49}$$

The Poisson map F_M given in (4.63) connects the symplectic $(\rho, \lambda, \nu, P, \Lambda, N)$ and Poisson variables $(G_1, G_2, G_3, M_1, M_2, M_3, W_1, W_2, W_3)$. It is expressed in the rescaled variables as follows

$$
\begin{aligned}
F_M^1 &= \sqrt{M^2-\Lambda^2}\sin\lambda, & F_M^6 &= N,\\
F_M^2 &= \sqrt{M^2-\Lambda^2}\cos\lambda, & F_M^7 &= \frac{PF(\rho)}{F'(\rho)},\\
F_M^3 &= \Lambda, & F_M^8 &= F^2(\rho)-\frac{P^2}{F'^2(\rho)}-\frac{4M^2}{F^2(\rho)},\\
F_M^4 &= \sqrt{M^2-N^2}\sin\nu, & F_M^9 &= F^2(\rho)+\frac{P^2}{F'^2(\rho)}+\frac{4M^2}{F^2(\rho)}.\\
F_M^5 &= \sqrt{M^2-N^2}\cos\nu,
\end{aligned}
\tag{5.50}
$$

Let us consider the change of variables given by F_M. The system of canonical differential equations generated by $\mathcal{H}$ is expressed in the new variables by taking into account the Poisson structure matrix given in (5.48), that is

$$
\left(\dot G_1,\dot G_2,\dot G_3,\dot M_1,\dot M_2,\dot M_3,\dot W_1,\dot W_2,\dot W_3\right)^T = P\,\nabla_{(G,M,W)}\mathcal{H},
$$

where $(G,M,W)=(G_1,G_2,G_3,M_1,M_2,M_3,W_1,W_2,W_3)$. This system of differential equations is endowed with the integrals C_1, C_2 and C_3 defined in (5.49).

Remark 5.2. *Note that for the circular intermediaries the Hamiltonian time dependency is avoided by the canonical transformation in the extended phase space given in Section 5.3.1. Thus, the variables (λ,Λ) must be replaced by the pair (ϕ,Φ) in all the formulas concerning these circular intermediaries.*

Remark 5.3. *Particular cases of interest, are when $F(\rho)=\rho$ and $F(\rho)=\sqrt{\rho}$. More specifically we obtain*

$$
\begin{aligned}
F_M^1 &= \sqrt{M^2-\Lambda^2}\sin\lambda, & F_M^4 &= \sqrt{M^2-N^2}\sin\nu, & F_M^7 &= \rho P,\\
F_M^2 &= \sqrt{M^2-\Lambda^2}\cos\lambda, & F_M^5 &= \sqrt{M^2-N^2}\cos\nu, & F_M^8 &= \rho^2-P^2-\frac{4M^2}{\rho^2},\\
F_M^3 &= \Lambda, & F_M^6 &= N, & F_M^9 &= \rho^2+P^2+\frac{4M^2}{\rho^2},
\end{aligned}
\tag{5.51}
$$

for $F(\rho) = \rho$ *and*

$$F_M^1 = \sqrt{M^2 - \Lambda^2}\sin\lambda, \quad F_M^4 = \sqrt{M^2 - N^2}\sin\nu, \quad F_M^7 = \frac{\rho P}{2},$$
$$F_M^2 = \sqrt{M^2 - \Lambda^2}\cos\lambda, \quad F_M^5 = \sqrt{M^2 - N^2}\cos\nu, \quad F_M^8 = \rho - 4P^2\rho - \frac{4M^2}{\rho},$$
$$F_M^3 = \Lambda, \qquad F_M^6 = N, \qquad F_M^9 = \rho + 4P^2\rho + \frac{4M^2}{\rho}, \tag{5.52}$$

for $F(\rho) = \sqrt{\rho}$.

Chapter 6

A Circular Intermediary

In this application model we assume the secondary body moving in a circular orbit. Although it is a very strong assumption, we found that many artificial satellites have quasi-circular orbits. That is, we can always assume that the orbit is circular when the satellite is in a low orbit. In that case, real examples have shown that the gravitational effect is the most important one to take into account. Precisely, in Fig. 6 we can see the Envisat, an artificial satellite located in a quasi-circular orbit at a height of 640 km above the earth. In this example, given the ordered list of perturbation effects, the gravity gradient perturbation is in the first position, with a magnitude around 10^{-4} vs 10^{-6} of the second perturbation.

Figure 6.1: Envisat, a satellite at a low orbit out of control. Further details in [Virgili et al., 2014].

In practical terms, for the situation described above, the very low eccentricity of the orbit allows to use models with the assumption of a circular orbit. This is the very same context in which the circular intermediaries should be investigated as an alternative zero order for the perturbation theories. However, prior to the study of perturbation, the candidate to be the new zero order has to be analyzed in detail. As such, this chapter is devoted to that purpose.

In particular, we further explain the previous terminology: Low Earth Orbit (LEO. $180-2000$ km): Most scientific satellites and many weather satellites are in a low Earth orbit. Satellites in this category have near-circular orbits with eccentricity values $e \approx 0.1$. Medium Earth Orbit (MEO. $2000-35.780$ km): There are two notable types of MEO: the semi-synchronous orbit is a near-circular orbit 26.560 kilometers from the center of the Earth, and the Molniya orbit, which is highly eccentric, $e \in (0.5, 0.7)$ and it was invented by the Russians, works well for observing high latitudes. Since a MEO is higher than a LEO, the satellite has a larger communication foot print on the Earth's surface so fewer satellites are needed to cover the whole Earth. GPS satellites use a MEO semi-synchronous orbit. High Earth Orbit (HEO. ≥ 35.780 km): When a satellite reaches exactly 42.164 kilometers from the center of the Earth (about 36.000 kilometers from Earths surface), its orbit matches Earths rotation. This special high Earth orbit is called geosynchronous.

6.1 The ν-Gravity Gradient Intermediary and the Reduced Space

In this chapter we study one of the two circular intermediaries defined in Section 5.2.1. Namely, we focus on the intermediary defined by means of the following Hamiltonian

$$\mathcal{H}_\nu = \mathcal{H}_K + V_\nu(\_, \_, \nu, \Phi, M, N)$$

where V_ν is given in (5.30), and after some calculations the expression for the potential is given by

$$V_\nu = -\frac{n^2}{8}(2-3\sin^2 I)\{(2C-B-A)-3\sin^2 J[(C-A)-(B-A)\cos^2\nu]\}. \quad (6.1)$$

This intermediary is analyzed in [Molero et al., 2014], where numerical comparisons show that this one is the closer choice to the full model. Furthermore, a partial communication focusing on the geometric aspects was given in [Crespo et al., 2014]. In what follows, we use the term ν-gravity gradient intermediary to refer to the Hamiltonian $\mathcal{H}_\nu$ as well as to the Hamiltonian dynamical system that it gives place. The expression for $\mathcal{H}_\nu$ in Andoyer variables is given by

$$\begin{aligned}\mathcal{H}_\nu &= \frac{1}{2}\left(\frac{\sin^2\nu}{A}+\frac{\cos^2\nu}{B}\right)(M^2-N^2)+\frac{N^2}{2C}-n\Phi \quad (6.2)\\ &+n^2\Delta\left[\left(\frac{2}{3}-\sin^2 I\right)\left(\frac{2}{3}-\sin^2 J\right)+f_3\left(\frac{2}{3}-\sin^2 I\right)\sin^2 J\cos 2\nu\right]\end{aligned}$$

where we have used the notations introduced by Kinoshita and Andoyer

$$f_3 = \frac{B-A}{2C-B-A} > 0, \qquad \Delta = -\frac{9}{16}(2C-B-A) < 0, \tag{6.3}$$

and the system of differential equations that it gives rise is

$$\begin{aligned}
&\dot\rho = 0, \quad \dot P = 0,\\
&\dot\phi = -n + \frac{3\Phi n^2}{8M^2}\left(3(A-B)\cos 2\nu\, s_J^2 + (2C-A-B)\left(1-3c_J^2\right)\right),\\
&\dot\Phi = 0,\\
&\dot\mu = M\left(\frac{\sin^2(\nu)}{A} + \frac{\cos^2(\nu)}{B}\right) + \frac{3n^2}{8M}\left[\left(-6c_I^2c_J^2 + c_I^2 + c_J^2\right)(A+B-2C)\right.\\
&\qquad\left. +(B-A)\left(-6c_I^2c_J^2 + 3c_I^2 + c_J^2\right)\cos(2\nu)\right],\\
&\dot M = 0,\\
&\dot\nu = N\left(\frac{1}{C} - \frac{\sin^2(\nu)}{A} - \frac{\cos^2(\nu)}{B}\right) + \frac{3n^2}{8M}\left[c_J\left(1-3c_I^2\right)((2C-A-B)\right.\\
&\qquad\left. +(A-B)\cos(2\nu))\right],\\
&\dot N = \frac{A-B}{\mathrm{AB}}\left(M^2-N^2\right)\sin(\nu)\cos(\nu) + \frac{3n^2(B-A)}{8}\left(1-3c_I^2\right)\left(1-c_J^2\right)\sin(2\nu).
\end{aligned} \tag{6.4}$$

The reader should appreciate how the Andoyer variables incorporate two integrals among the conjugate momenta leading us to a 1-DOF separable system. Thus, it is easy to see that it is still integrable. Although the system enjoys of the named features, one may find rather complicated the integration of the subsystem (ν, N). Another drawback is given by the singularities of the variables. For that reason we change to the Poisson variables suggested in Section 5.3.2.

6.1.1 $\mathcal{H}_\nu$ as a perturbed quartic realization

By means of the rescaling given in (5.47), the family of Hamiltonians defined in (4.1) is expressed in the following fashion

$$\mathcal{F}_{\alpha\beta} := \alpha_1 M_1^2 + \alpha_2 M_2^2 + \alpha_3 M_3^2 + \beta_1 W_1^2 + \beta_2 W_2^2 + \beta_3 W_3^2. \tag{6.5}$$

where $\alpha_i = 2a_i$ and $\beta_i = 2a_{i+3}$ for $i = 1,2,3$. We recall that for the particular value of the parameters $\beta_1 = \beta_2 = \beta_3 = 0$ and $\alpha_1 = 1/(2A)$, $\alpha_2 = 1/(2B)$ and

$\alpha_3 = 1/(2C)$, we obtain the Hamiltonian of the free rigid body. Then, let us add a perturbation $\mathcal{P}_0$, such that the Hamiltonian is given as follows:

$$\mathcal{F} = \mathcal{F}_\alpha(M_1, M_2, M_3) + \mathcal{P}_0(M_1, M_2, M_3; G_3), \tag{6.6}$$

where

$$\mathcal{F}_\alpha(M_1, M_2, M_3) = \frac{1}{2}\left(\frac{M_1^2}{A} + \frac{M_2^2}{B} + \frac{M_3^2}{C}\right). \tag{6.7}$$

We choose the perturbation added to the quartic realization above $\mathcal{P}_0(M_1, M_2, M_3; G_3)$ as given by

$$\begin{aligned}\mathcal{P}_0 = &- nG_3\\ &\frac{n^2}{8}\left(1 - 3\frac{G_3^2}{M^2}\right)\left[\frac{3}{M^2}(AM_1^2 + BM_2^2 + CM_3^2) - (A+B+C)\right].\end{aligned} \tag{6.8}$$

Note that this Hamiltonian is (W_1, W_2, W_3)-invariant. Therefore, the flow of the Hamiltonian $\mathcal{F}$ is restricted to be in the reduced space given by $\mathcal{M}_\nu = \mathbb{S}^2_M \times \mathbb{S}^2_M$. Next, we connect the Hamiltonian $\mathcal{H}_\nu$ with $\mathcal{F}$. This connection is obtained through the following modification of the map F_M given in (5.52)

$$F_M : D \subset (\mathbb{R}^4, \omega) \to S^{2*}_M \times S^{2*}_M \subset (\mathbb{R}^6, \{\,,\,\}), \tag{6.9}$$

where $S^{2*}_M = S^2_M - \{(0, 0, \pm M)\}$ and the map is given by

$$\begin{aligned} F_M^1 &= \sqrt{M^2 - N^2}\sin\nu, & F_M^4 &= \sqrt{M^2 - \Phi^2}\sin\phi,\\ F_M^2 &= \sqrt{M^2 - N^2}\cos\nu, & F_M^5 &= \sqrt{M^2 - \Phi^2}\cos\phi,\\ F_M^3 &= N, & F_M^6 &= \Phi, \end{aligned} \tag{6.10}$$

Another remark is that we have abuse of notation by using the same symbol F_M for the adapted Poisson map. The domain now is given by $D = \{(\nu, \phi, N, \Phi) \in \mathbb{R}^4 / \ |N| < M, |\Phi| < M\}$ and ω is the standard symplectic form. In other sense, $\{\,,\,\}$ is the Poisson bracket on $\mathbb{R}^6$ with structure matrix

$$P = \begin{pmatrix} 0 & M_3 & -M_2 & 0 & 0 & 0\\ -M_3 & 0 & M_1 & 0 & 0 & 0\\ M_2 & -M_1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & G_3 & -G_2\\ 0 & 0 & 0 & -G_3 & 0 & G_1\\ 0 & 0 & 0 & G_2 & -G_1 & 0 \end{pmatrix}. \tag{6.11}$$

and Casimirs

$$C_1 = M_1^2 + M_2^2 + M_3^2 = M^2, \quad C_2 = G_1^2 + G_2^2 + G_3^2 = M^2. \tag{6.12}$$

Proposition 6.1. *The map F_M is endowed with the following features:*

i) F_M *is a diffeomorphism from* D *to* $S^{2*}_M \times S^{2*}_M$*,*

ii) $S^2_M \times S^2_M$ *corresponds to the level manifold of the Casimirs* $C_1 = C_2 = M^2$*, in other words*

$$S^2_M \times S^2_M = C_1^{-1}(M) \cap C_2^{-1}(M),$$

iii) F_M *satisfies that*

$$\{F^*_M f, F^*_M g\}_{\omega|D} = \{f, g\}$$

where $\{.,.\}_{\omega|D}$ *is the Poisson structure induced by the symplectic form* ω_D *on the open manifold* D*. Thus, is* F_M *a Poisson map.*

Proof. Analogous to the proof of Proposition 4.7. □

After we carry out this change of variables, the Hamiltonian $\mathcal{H}_\nu : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ given in (6.2) takes the form

$$\begin{aligned}\mathcal{F} = \mathcal{H}_\nu &= \mathcal{H}_\nu(M_1, M_2, M_3, G_1, G_2, G_3;\ A, B, C, n)\\ &= \frac{1}{2}\left(\frac{M_1^2}{A^*} + \frac{M_2^2}{B^*} + \frac{M_3^2}{C^*}\right) - nG_3 - \frac{n^2}{8}(A+B+C)\left(1 - 3\frac{G_3^2}{M^2}\right),\end{aligned} \tag{6.13}$$

where

$$\frac{1}{A^*} = \frac{1}{A} + n^2\frac{3A}{4M^2}\left(1 - \frac{3G_3^2}{M^2}\right),$$

$$\frac{1}{B^*} = \frac{1}{B} + n^2\frac{3B}{4M^2}\left(1 - \frac{3G_3^2}{M^2}\right),$$

$$\frac{1}{C^*} = \frac{1}{C} + n^2\frac{3C}{4M^2}\left(1 - \frac{3G_3^2}{M^2}\right).$$

After substitution of A^*, B^* and C^* in (6.13), and some easy computations we obtain the equality $\mathcal{F} = \mathcal{H}_\nu$ in the Poisson variables.

6.2 General Flow of the Reduced System

Henceforth, we focus in the study of the principal features of this intermediary. We compute the equilibria of the system and give and analysis of the linear stability and bifurcations. Several scenarios arises for different kinds of triaxiality, we focus on one of the possibilities, precisely T_2. For this case, explicit integration formulas are provided in terms of Jacobi elliptic functions, the remaining cases can be studied analogously.

Taking into account the Poisson structure of the reduced space, the canonical equations are readily obtained by means of the following computations

$$\dot{x}_i = \sum_{j=1}^{6} \{x_i, x_j\} \frac{\partial \mathcal{H}_\nu}{\partial x_j} \tag{6.14}$$

where $x = (x_1, x_2, \ldots, x_6) = (M_1, M_2, M_3, G_1, G_2, G_3)$ and the Poisson's bracket are the elements of the Poisson structure matrix P given in (6.11) $\{x_i, x_j\} = P_{ij}$. Therefore, the system reads as follows

$$\dot{M}_1 = a_1 M_2 M_3, \tag{6.15}$$

$$\dot{M}_2 = a_2 M_1 M_3, \tag{6.16}$$

$$\dot{M}_3 = a_3 M_1 M_2, \tag{6.17}$$

$$\dot{G}_1 = -\Delta_G G_2, \tag{6.18}$$

$$\dot{G}_2 = \quad \Delta_G G_1, \tag{6.19}$$

$$\dot{G}_3 = 0, \tag{6.20}$$

where $a_1(M, G_3)$, $a_2(M, G_3)$, $a_3(M, G_3)$ and $\Delta_G(\mathbf{M}, G_3)$ are given by the following expressions

$$a_1 = \frac{B-C}{BC}[1 - n^2 \frac{3BC}{4M^2}(1 - 3\frac{G_3^2}{M^2})], \tag{6.21}$$

$$a_2 = \frac{C-A}{AC}[1 - n^2 \frac{3AC}{4M^2}(1 - 3\frac{G_3^2}{M^2})], \tag{6.22}$$

$$a_3 = \frac{A-B}{AB}[1 - n^2 \frac{3AB}{4M^2}(1 - 3\frac{G_3^2}{M^2})], \tag{6.23}$$

and

$$\Delta_G(\mathbf{M}) = \frac{n}{4M^4}\{4M^4 + 3G_3 n[3(AM_1^2(t) + BM_2^2(t) + CM_3^2(t)) - M^2(A+B+C)]\}. \tag{6.24}$$

Let us analyze the structure of this system. At first sight of the equations (6.15)-(6.20), it seems that the system is separable in two subsystems. On one side, equations for M_1, M_2 and M_3 constitute an Euler-type system. On the other side, for G_1,

G_2 and G_3 we have a non-autonomous Hamiltonian system, which Hamiltonian is given by

$$\mathcal{H}(G_1, G_2, G_3) = \frac{1}{2}\Delta_G(\mathbf{M})(G_1^2 + G_2^2). \tag{6.25}$$

Nevertheless there is a coupling between both subsystems, since G_3 appear in the coefficients a_1, a_2 and a_3 and $\Delta_G(\mathbf{M})$ depends on the M_i for $i = 1, 2, 3$. Therefore, in order to solve it, we propose the following strategy; once the subsystem in the M_i is solved in terms of elliptic functions for each fixed value of G_3, the system in G_i may also be solved, after the substitution of the M_i in $\Delta_G(\mathbf{M})$. Explicit solutions following this plan are given in Theorem 6.2 and Corollary 6.3. In this chapter we are

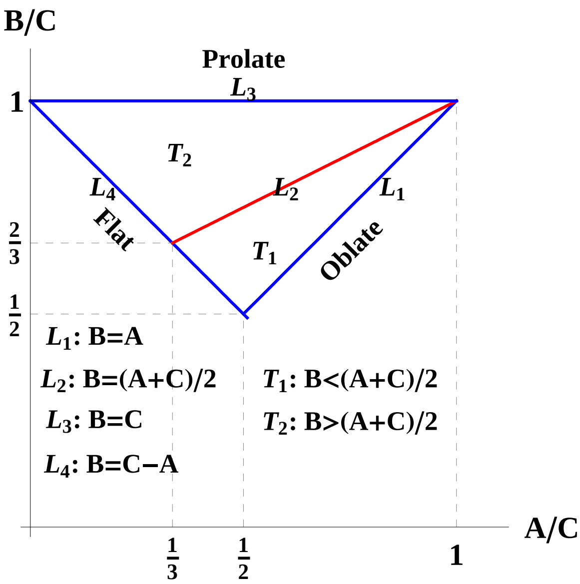


Figure 6.2: **Triaxiality regions:** The regions and lines on this figure are derived from the study of the relative equilibria in section 6.3. That is to say, the relative equilibria changes as we move from one triaxiality region to another.

going to identify the relative equilibria of the system. In order to fulfil this objective, we collect the way in which we have organized our study in Fig 6.2 and Fig 6.3 . Let us explain how to interpret them.

It is well known that the principal moments of inertia are not arbitrary. Thus, we denote the domain for real solids by D, which is defined as follows

$$D = \{A, B, C \in \mathbb{R}^+; A \leq B \leq C,\ A + B > C\}, \tag{6.26}$$

among the valid terns in the domain D we distinguish the oblate symmetric bodies given by $L_1 = \{(A, B, C) \in D : A = B\}$, the prolate symmetric bodies $L_3 =$

$\{(A, B, C) \in D : B = C\}$, and the flat bodies L_4. Finally we consider the generic triaxial cases $T_1 = \{(A, B, C) \in D : 2B < A + C\}$, $T_2 = \{(A, B, C) \in D : 2B > A + C\}$ and the special one $L_2 = \{(A, B, C) \in D : 2B = A + C\}$, see Fig 6.2. The

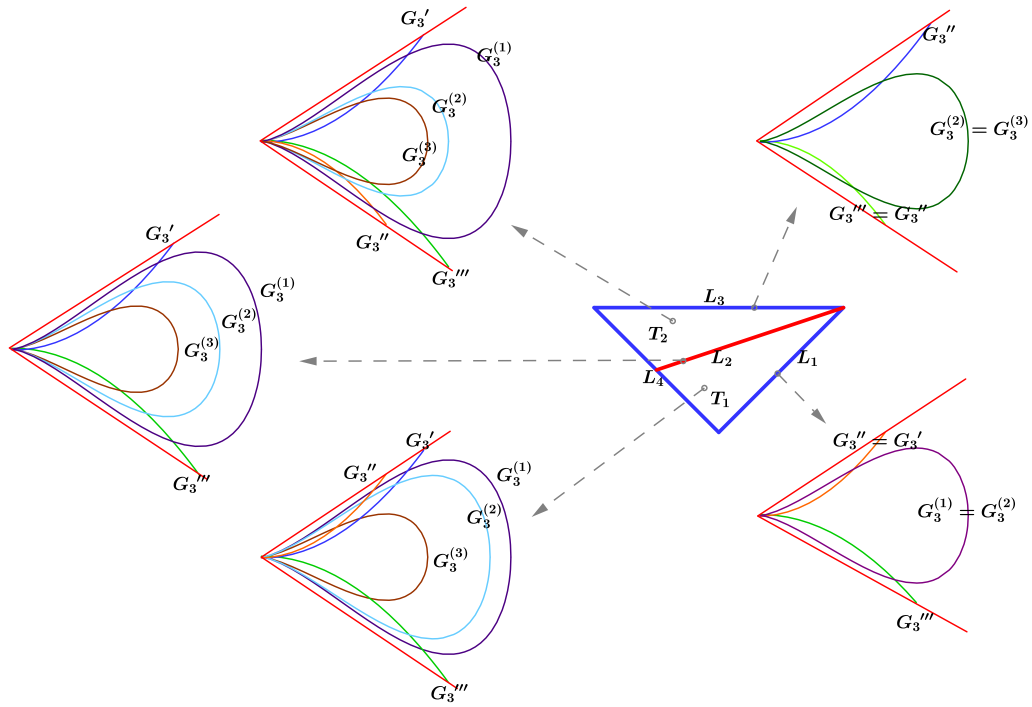


Figure 6.3: **Momentum mapping related to the physical parameters:** This figure presents the relation among the two fundamental planes of physical parameters (moments of inertia) and the dynamical parameters (integrals) of the problem.

reason why we distinguish several "types of triaxiality" is because of the key role that the parameters A, B and C play in the study of the relative equilibria. Section 6.3 shows that there are several curves in the integral plane M-G_3 leading to parametric families of relative equilibria. Those curves depend precisely upon the triaxiality types given above giving rise to different scenarios, see Fig 6.3. In other words, we have already said that relative equilibria occurs along certain curves in the M-G_3 plane, then Fig 6.3 match each type of triaxiality with a cualitative representation of the named curves. In what follows we focus on region T_2.

In Fig. 6.4 we portrait the general flow for the region T_2 of the system given by (6.15)–(6.20). Next, Theorem 6.2 and Corollary 6.3 give the analytic formulas for the general case. In order to alleviate the expressions involved, we have introduced

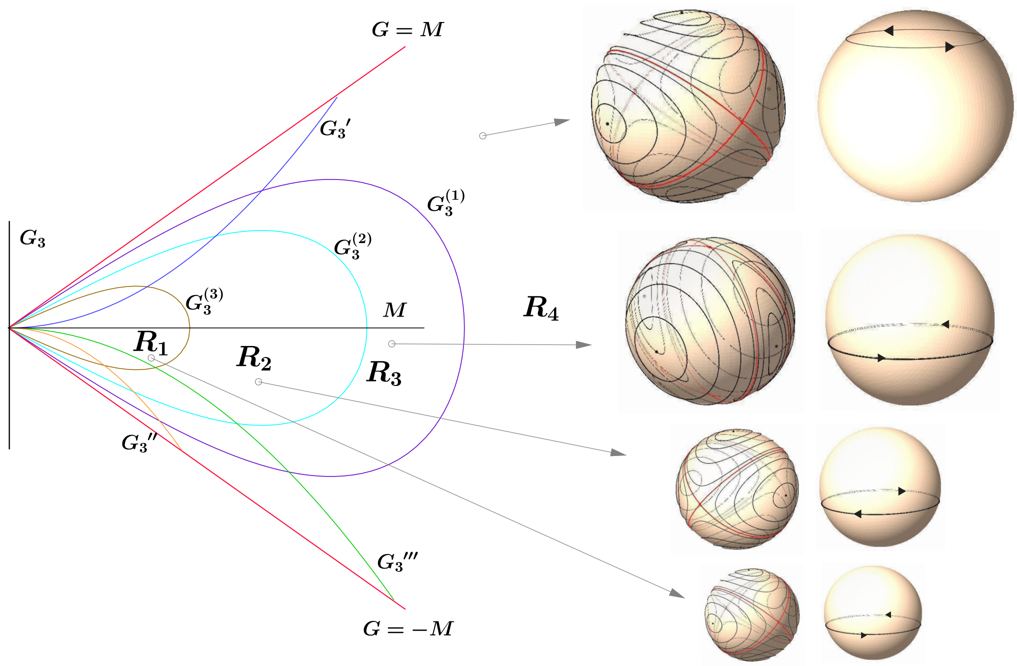


Figure 6.4: **Generic flows on the reduced space $\mathbb{S}^2 \times \mathbb{S}^2$, over the plane of integrals in region T_2:** Note that the pattern is always the same. Nevertheless, the projection of the flow in the M-sphere (left) changes the stability as (M, G_3) passes through the regions R_1, R_2, R_3 and R_4. In the G-sphere (right), the flow take place along a parallel at G_3 constant, the specific character of this motion depends on $\Delta_G(\mathbf{M})$. The parabolas G_3', G_3'' and G_3''' correspond to families of relative equilibria and as (M, G_3) passes through them, the flow in the G-sphere moves counterclockwise or clockwise.

the following notation; h_2, h_2 and h_3 are integrals of the subsystem (6.15)–(6.17) given by

$$
\begin{aligned}
&h_1 = a_2 M_3^2 - a_3 M_2^2, \quad h_2 = a_3 M_1^2 - a_1 M_3^2, \quad h_3 = a_1 M_2^2 - a_2 M_1^2, \\
&k_{ij} = \sqrt{\left|\frac{a_i h_i}{a_j h_j}\right|}, \quad \mu_i = \sqrt{|a_i h_i|}.
\end{aligned}
\tag{6.27}
$$

and

$$
\begin{aligned}
\delta_{ij}(t) &= -\frac{a}{\mu_i}\left[E(\mathrm{am}(\mu_j(t-t_0)|k_{ij})|k_{ij}) - (\mu_j - b\mu_i)t + t_0\right],\\
\delta_1(t) &= \left(M_1^2 + \frac{a_3-a_2}{2a_2a_3}h_1\right)s + \frac{a_3+a_2}{4\,a_2a_3\,\mu_3}\sin(2\mu_3\,s),\\
\delta_2(t) &= \left(M_2^2 + \frac{a_3-a_1}{2a_1a_3}h_1\right)s + \frac{a_3+a_1}{4\,a_3a_1\,\mu_3}\sin(2\mu_3\,s),\\
\delta_3(t) &= \left(M_3^2 + \frac{a_1-a_2}{2a_2a_1}h_3\right)s + \frac{a_1+a_2}{4\,a_2a_1\,\mu_1}\sin(2\mu_1\,s),
\end{aligned}
\tag{6.28}
$$

where

$$
\begin{aligned}
a &= \frac{9G_3n^2}{4M^2}\Xi, \qquad b = \frac{3G_3n^2}{4M^2}\big(3\Gamma - M^2(A+B+C)\big) + n,\\
\Xi &= \frac{(A-C)(C-B)(B-A)}{ABC}, \qquad \Gamma = A\,M_1^2(t_0) + C\,M_3^2(t_0),
\end{aligned}
\tag{6.29}
$$

Besides of the above notation introduced, for the case T_2, we also consider four regions R_1, R_2, R_3 and R_4 on the plane MG_3-plane. These regions already appear in Fig. 6.4 and now we define them more precisely. For that purpose, let us consider the coefficients in the equations (6.15)–(6.17)

$$
a_1(M, G_3; A, B, C, n),\;\; a_2(M, G_3; A, B, C, n),\;\; a_3(M, G_3; A, B, C, n)
$$

considered as real parametric functions defined in the MG_3-plane. Then we define the curves $G_3^{(i)}$ for $i = 1, 2, 3$ as the geometric places in the MG_3-plane where $a_i(M, G_3; A, B, C, n) = 0$ respectively,. The explicit formulas for each $G_3^{(i)}$ are given later on in (6.41), (6.41) and (6.41), also the curves themselves appear in Fig. 6.4, by shape they have we usually refer to them as the *petals*. Now we resume our first task and define R_i as follows

(i) R_1 is the region in the MG_3-plane inside of the closed curve $G_3^{(3)}$.

(ii) R_2 is the region in the MG_3-plane between $G_3^{(3)}$ and $G_3^{(2)}$.

(iii) R_3 is the region in the MG_3-plane between $G_3^{(2)}$ and $G_3^{(1)}$.

(iv) R_4 is the region in the MG_3-plane outside of $G_3^{(1)}$.

Theorem 6.2 (Global flow)**.** *Let $s = (t - t_0)$. Thus, analytic solutions to the system (6.15)–(6.20) depend on the regions R_i and read as follows*

•Analytic solutions in R_1

$$\begin{array}{lll}
\textit{Case: } a_2h_2 > 0 & \textit{Case: } a_2h_2 < 0 & \\
M_1 = \sqrt{\left|\frac{h_3}{a_2}\right|} cn(\mu_1\, s, k_{31}), & M_1 = \sqrt{\left|\frac{h_3}{a_2}\right|} dn(\mu_3\, s, k_{13}), & G_1 = \sin(\delta_{ij}(t)), \\
M_2 = \sqrt{\left|\frac{h_3}{a_1}\right|} sn(-\mu_1\, s, k_{31}), & M_2 = \sqrt{\left|\frac{h_1}{a_3}\right|} sn(-\mu_3\, s, k_{13}), & G_2 = \cos(\delta_{ij}(t)), \\
M_3 = \sqrt{\left|\frac{h_1}{a_2}\right|} dn(\mu_1\, s, k_{31}), & M_3 = \sqrt{\left|\frac{h_1}{a_2}\right|} cn(\mu_3\, s, k_{13}), & G_3 = G_3,
\end{array} \tag{6.30}$$

where $a_2h_2 > 0 \Rightarrow [i = 3, j = 1]$ *and* $a_2h_2 < 0 \Rightarrow [i = 1, j = 3]$.

•Analytic solutions in R_2

$$\begin{array}{lll}
\textit{Case: } a_1h_1 < 0 & \textit{Case: } a_1h_1 > 0 & \\
M_1 = \sqrt{\left|\frac{h_3}{a_2}\right|} sn(-\mu_2\, s, k_{32}), & M_1 = \sqrt{\left|\frac{h_2}{a_3}\right|} sn(-\mu_3\, s, k_{23}), & G_1 = \cos(\delta_{ij}(t)), \\
M_2 = \sqrt{\left|\frac{h_3}{a_1}\right|} cn(\mu_2\, s, k_{32}), & M_2 = \sqrt{\left|\frac{h_3}{a_1}\right|} dn(\mu_3\, s, k_{23}), & G_2 = \sin(\delta_{ij}(t)), \\
M_3 = \sqrt{\left|\frac{h_1}{a_1}\right|} dn(\mu_2\, s, k_{32}), & M_3 = \sqrt{\left|\frac{h_2}{a_2}\right|} cn(\mu_3\, s, k_{23}), & G_3 = G_3,
\end{array} \tag{6.31}$$

where $a_1h_1 < 0 \Rightarrow [i = 3, j = 2]$, *and* $a_1h_1 > 0 \Rightarrow [i = 2, j = 3]$.

•Analytic solutions in R_3

$$\begin{array}{lll}
\textit{Case: } a_3h_3 < 0 & \textit{Case: } a_3h_3 > 0 & \\
M_1 = \sqrt{\left|\frac{h_2}{a_3}\right|} cn(\mu_1\, s, k_{21}), & M_1 = \sqrt{\left|\frac{h_2}{a_3}\right|} dn(\mu_2\, s, k_{12}), & G_1 = \sin(\delta_{ij}(t)), \\
M_2 = \sqrt{\left|\frac{h_1}{a_3}\right|} dn(\mu_1\, s, k_{21}), & M_2 = \sqrt{\left|\frac{h_1}{a_3}\right|} cn(\mu_3\, s, k_{12}), & G_2 = G_2, \\
M_3 = \sqrt{\left|\frac{h_2}{a_1}\right|} sn(\mu_1\, s, k_{21}), & M_3 = \sqrt{\left|\frac{h_1}{a_2}\right|} sn(\mu_2\, s, k_{12}), & G_3 = \cos(\delta_{ij}(t)),
\end{array} \tag{6.32}$$

where $a_3h_3 < 0 \Rightarrow [i = 2, j = 1]$ *and* $a_3h_3 > 0 \Rightarrow [i = 1, j = 2]$.

•Analytic solutions in R_4

$$\begin{array}{lll} \textit{Case: } a_2h_2>0 & \textit{Case: } a_2h_2<0 & \\ M_1=\sqrt{\left|\frac{h_3}{a_2}\right|}cn(\mu_1\, s,k_{31}), & M_1=\sqrt{\left|\frac{h_3}{a_2}\right|}dn(\mu_3\, s,k_{13}), & G_1=\sin(\delta_{ij}(t)), \\ M_2=\sqrt{\left|\frac{h_3}{a_1}\right|}sn(-\mu_1\, s,k_{31}), & M_2=\sqrt{\left|\frac{h_1}{a_3}\right|}sn(-\mu_3\, s,k_{13}), & G_2=\cos(\delta_{ij}(t)), \\ M_3=\sqrt{\left|\frac{h_1}{a_2}\right|}dn(\mu_1\, s,k_{31}), & M_3=\sqrt{\left|\frac{h_1}{a_2}\right|}cn(\mu_3\, s,k_{13}), & G_3=G_3, \end{array} \tag{6.33}$$

where $a_2h_2>0 \Rightarrow [i=3,j=1]$ *and* $a_2h_2<0 \Rightarrow [i=1,j=3]$

Proof. From Theorem 2.8 we readily obtain the expressions for M_i $i=1,2,3$. Then, we look for a compact expression of $AM_1^2(t)+BM_2^2(t)+CM_3^2(t)$, which appear in $\Delta_G(\mathbf{M})$, in order to integrate the G_1, G_2, G_3 subsystem. Since the flow of the subsystem M_i is Eulerian, multiplication in succession (6.15), (6.16) and (6.17) by AM_1, BM_2 and CM_3 and addition, after some calculations leads us to

$$AM_1^2(t)+BM_2^2(t)+CM_3^2(t)=\Xi\,\mathrm{sn}^2(\mu_3\, s,k_{13})+\Gamma. \tag{6.34}$$

Thus, replacing (6.34) in (6.24), a simple quadrature leads to the formulas given for G_1 and G_2.

□

Corollary 6.3 (Flow along the petals). *Let* $s=(t-t_0)$. *Thus, analytic solutions to the system (6.15)–(6.20) depend on the regions* R_i *and read as follows*

•Analytic solutions in $G_3^{(1)}$

$$\begin{array}{lll} M_1=M_1, & M_2=\sqrt{\left|\frac{h_1}{a_3}\right|}\sin(-\mu_3\, s), & M_3=\sqrt{\left|\frac{h_1}{a_2}\right|}\cos(\mu_3\, s), \\ G_1=\sin(\delta_1(t)), & G_2=\cos(\delta_1(t)), & G_3=G_3, \end{array} \tag{6.35}$$

•Analytic solutions in $G_3^{(2)}$

$$\begin{array}{lll} M_1=\sqrt{\left|\frac{h_2}{a_3}\right|}\sin(-\mu_3\, s), & M_2=M_2, & M_3=\sqrt{\left|\frac{h_2}{a_2}\right|}\cos cn(\mu_3\, s), \\ G_1=\cos(\delta_2(t)), & G_2=\sin(\delta_2(t)), & G_3=G_3, \end{array} \tag{6.36}$$

•Analytic solutions in $G_3^{(3)}$

$$\begin{aligned} &M_1=\sqrt{\left|\frac{h_3}{a_2}\right|}\cos(\mu_1\, s) \quad M_2=\sqrt{\left|\frac{h_3}{a_1}\right|}\sin(-\mu_1\, s), \quad M_3=M_3, \\ &G_1=\sin(\delta_3(t)), \qquad G_2=\cos(\delta_3(t)), \qquad G_3=G_3, \end{aligned} \tag{6.37}$$

Proof. Analogous to Theorem 6.2. □

6.3 Searching for $\mathbb{S}^1$-Relative Equilibria

The search or relative equilibria corresponding with the first reduced space is arranged in two categories, isolated and parametric families of relative equilibria. The isolated equilibria are sextuples that, when projected on the G-sphere gives the north and south pole and when they are projected in the M-sphere lead to the classical equilibria of the free rigid body. The case of the parametric families of equilibria presents a more complex panorama, we deal with two families *principal direction equilibria* and *quasi-Euler equilibria*. The first one does not introduce changes in the M-sphere from the isolated case, but in the G-sphere a circumference of equilibria appears given rise the product of both projection to a family of sextuples. For the *quasi-Euler equilibria* the G-sphere remain the same that in the *principal direction equilibria*, but equilibria different from the classical picture of the rigid body system come into sight.

6.3.1 Isolated relative equilibria

The relative equilibria described in this section do not depend on the values of the physical parameters A, B and C, neither in the value of n, but occur for $|G_3| = M$. Those equilibria are dubbed as permanent, since they exist for every value of M. There are twelve of them and they correspond with the principal directions in the M-space and the north and south poles in the G-space. That is to say, each relative equilibria is a sextuple, which is made up of three coordinates coming from the M-sphere and three coming from the G-sphere, see Fig. 6.5.

In order to identify along the text the permanent relative equilibria and their respective energy values, the following notation has been introduced in Table 6.3.1 where the value of the energies is given by

$$h_{E1}=\frac{M^2}{2A}+nM-\frac{n^2}{4}(2A-B-C), \quad h_{E4}=\frac{M^2}{2A}-nM-\frac{n^2}{4}(2A-B-C),$$

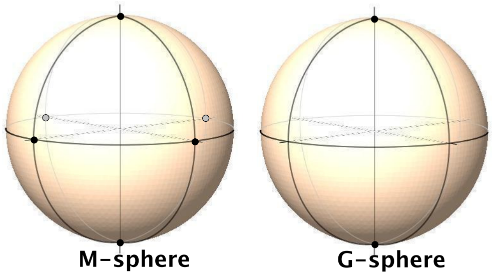


Figure 6.5: **Permanent equilibria:** The sextuples of those equilibria are obtained as a combination of a pair of tridimensional points in the M-sphere times the G-sphere

Permanent equilibria notation	
Equilibria	Energy
$E_1 = (\pm M, 0, 0, 0, 0, M)$	h_{E1}
$E_2 = (0, \pm M, 0, 0, 0, M)$	h_{E2}
$E_3 = (0, 0, \pm M, 0, 0, M)$	h_{E3}
$E_4 = (\pm M, 0, 0, 0, 0, -M)$	h_{E4}
$E_5 = (0, \pm M, 0, 0, 0, -M)$	h_{E5}
$E_6 = (0, 0, \pm M, 0, 0, -M)$	h_{E6}

Table 6.1: Note that each E_i corresponds with two relative equilibria with the aim of alleviate the notation, since each of those pairs E_i are always manipulated together and have the same energy level.

$$h_{E2} = \frac{M^2}{2B} + nM - \frac{n^2}{4}(2B - A - C), \quad h_{E5} = \frac{M^2}{2B} - nM - \frac{n^2}{4}(2B - A - C),$$
$$h_{E3} = \frac{M^2}{2C} + nM - \frac{n^2}{4}(2C - A - B), \quad h_{E6} = \frac{M^2}{2C} - nM - \frac{n^2}{4}(2C - A - B).$$

Theorem 6.4 (Permanent equilibria.)**.** *The set of sextuples given in Table 6.3.1 correspond to equilibria of the system defined by the equations (6.15)–(6.20). The value of the energy associated to each equilibria is given also in Table 6.3.1.*

Proof. By simple substitution on the system made of (6.15)–(6.20) of the sextuples given in Table 6.3.1, it follows that they are equilibria. The values h_0^1, h_0^2 and h_0^3 are obtained by the evaluation of the equilibria in the Hamiltonian. □

To summarize, we gather together all the permanent equilibria of a generic body in the following Table 6.2.

Permanent equilibria			
Region	M	Equilibria	Energy
D	$(0, +\infty)$	E_1	h_{E1}
D	$(0, +\infty)$	E_2	h_{E2}
D	$(0, +\infty)$	E_3	h_{E3}
D	$(0, +\infty)$	E_4	h_{E4}
D	$(0, +\infty)$	E_5	h_{E5}
D	$(0, +\infty)$	E_6	h_{E6}

Table 6.2: Permanent equilibria for a general body.

6.3.2 Two parametric families of relative equilibria

The relative equilibria that we study in this section do depend on the value of n, M and G_3, as well as on the value of the physical parameters A, B and C. The dependency on the principal moments of inertia determines the regions that we give in Fig. 6.3, see Remark 6.1 for a complete explanation. There is also one remarkable difference with the permanent equilibria studied before, now the bifurcation phenomena takes place and the relative equilibria are no longer a finite set of sextuples, but a non discrete parametric family of points in the MG-space.

Our strategy to obtain those equilibria is given next. Once the triaxiality type is fixed, we look for special values of M and G_3 that provide zeros in the coefficients of the system of differential equations (6.15)–(6.20). In this regard, we distinguish between zeros in the subsystem (6.18,6.19) giving place to the *principal direction equilibria* and zeros in the subsystem (6.15,6.16,6.17), that lead to the equilibria named as *quasi-Euler equilibria*.

Principal direction equilibria

Let us start with the *principal direction equilibria*. Proceeding in the same way than before we introduce some notation in Table 6.3.2

In Fig 6.6 we illustrate the equilibria E_7, E_8 and E_9. The coordinates from the above Table 6.3.2 corresponding with the G-space (G_1, G_2, G_3'), (G_1, G_2, G_3'') and

Principal direction equilibria notation	
Equilibria	Energy
$E_7 = (\pm M, 0, 0, G_1, G_2, G_3')$	h_{E7}
$E_8 = (0, \pm M, 0, G_1, G_2, G_3'')$	h_{E8}
$E_9 = (0, 0, \pm M, G_1, G_2, G_3''')$	h_{E9}

Table 6.3: Note that each E_i corresponds with two families of relative equilibria with the aim of alleviating the notation.

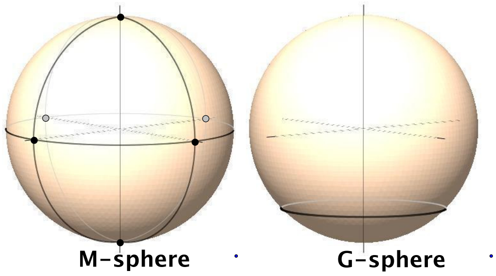


Figure 6.6: **Principal direction equilibria:** The sextuples of those equilibria are obtained by the product of a point in any of the principal directions in the M-sphere times a point in the parallel located in the G-sphere

(G_1, G_2, G_3''') are given as follows

$$\begin{aligned} G_1^2 + G_2^2 &= M^2 - G_3'^2, \quad M \in (0, \mathcal{M}'), \quad \mathcal{M}' = \frac{3n}{4}|2A - B - C|, \\ G_1^2 + G_2^2 &= M^2 - G_3''^2, \quad M \in (0, \mathcal{M}''), \quad \mathcal{M}'' = \frac{3n}{4}|2B - A - C|, \\ G_1^2 + G_2^2 &= M^2 - G_3'''^2, \quad M \in (0, \mathcal{M}'''), \quad \mathcal{M}''' = \frac{3n}{4}(2C - A - B), \end{aligned} \tag{6.38}$$

and

$$G_3' = \frac{4M^2}{3n(B + C - 2A)}, \quad G_3'' = \frac{4M^2}{3n(A + C - 2B)}, \quad G_3''' = \frac{4M^2}{3n(A + B - 2C)}, \tag{6.39}$$

where the last coordinates are parabolas along the MG_3-plane satisfy

$$G_3' > 0, \qquad G_3'' \lessgtr 0, \qquad G_3''' < 0.$$

The coefficient $\Delta_G(\mathbf{M})$ given in (6.24) vanishes along them. Therefore, those curves gives place to relative equilibria. The values of the energy at these points are given by

$$
\begin{aligned}
h_{E7} &= \frac{M^2(B+C+2A)}{2A\,(2\,A-B-C)} + \frac{n^2}{8}\,(2\,A-B-C)\,, \\
h_{E8} &= \frac{M^2(A+C+2B)}{2B\,(2\,B-A-C)} + \frac{n^2}{8}\,(2\,B-A-C)\,, \\
h_{E9} &= \frac{M^2(A+B+2C)}{2C\,(2\,C-B-A)} + \frac{n^2}{8}\,(2\,C-B-A)\,.
\end{aligned}
\tag{6.40}
$$

Remark 6.1. *Note that along the line L_2, see Fig 6.2, the denominators of G_3'' and h_{E8} in (6.39) and (6.40) are not defined. Thus the signs of G_3'' and h_{E8} depend on the region T_1 and T_2. Those features may also be observed in Fig 6.3, where we gave in advance a qualitative scheme of the curves in which principal direction equilibria and quasi-Euler equilibria take place. Note also that formulas (6.38) show how principal direction equilibria are only valid for a finite domain of M.*

Theorem 6.5 (Principal direction equilibria)**.** *The set of sextuples given in Table 6.3.2 correspond to equilibria of the system defined by the equations (6.15)–(6.20). The value of the energy associated to each equilibria is given also in Table 6.3.1.*

Proof. We consider the case of $E_7 = (\pm M, 0, 0, G_1, G_2, G_3')$, the other ones are analogous. Taking into account that the particular value of $G_3 = G_3'$ together with $M_1 = \pm M$, $M_2 = M_3 = 0$ imply $a_1 = 0$ and $\Delta_G(\mathbf{M}) = 0$, we have that this set of equilibria follows by simple substitution on the system made of (6.15)–(6.20). The values h_{E7}, h_{E8} and h_{E9} are obtained by the evaluation of the equilibria in the Hamiltonian (6.13).

□

Principal direction equilibria			
Region	M	Equilibria	Energy
D	$(0, \mathcal{M}')$	E_7	h_{E7}
D	$(0, \mathcal{M}'')$	E_8	h_{E8}
D	$(0, \mathcal{M}''')$	E_9	h_{E9}

Table 6.4: Principal direction equilibria for T_2.

Quasi-Euler equilibria

Up to now, all the equilibria may be identified with the classical ones coming from the free rigid body, since they are organized in pairs along the principal directions of the M-sphere, see Table 6.2 and Table 6.4. The next family of relative equilibria are located along circumferences in the G-space, but we detect a different behavior in the M-sphere, since four equilibria emerge from the principal directions

This new set of relative equilibria is obtained by looking for particular values of G_3, namely $G_3 = G_3^{(1)}$, $G_3 = G_3^{(2)}$ and $G_3 = G_3^{(3)}$ that make $a_1 = 0$, $a_2 = 0$ and $a_3 = 0$ respectively in equations (6.15), (6.16) or (6.17), together with the especial values in the M-space, $M_i = 0$, $M_j = M_j^{(l)}$ and $M_k = M_k^{(l)}$ that imply $\Delta_G(\mathbf{M}) = 0$. All those values for the coordinates of the sextuples are given in (6.41), (6.43) and (6.45).

Those relative equilibria have been named as *quasi-Euler equilibria* and they are structured in three groups, each one is located in the intersections between the coordinate planes M_i-M_j and the M-sphere together with the circle in the G-space given by G_3 constant. Relative equilibria in the M_1-M_2 plane are illustrated in Fig. 6.7, the remaining groups of equilibria in M_1-M_3 plane and M_3-M_2 plane are analogous.

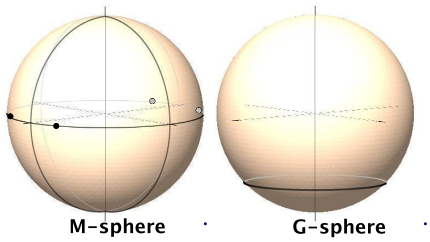


Figure 6.7: **Equilibria in the M_1-M_2 plane:** The sextuples of those equilibria are obtained by the product of a point in the parallel located in the G-sphere times any of the four points in the equator of the M-sphere.

At this point we introduce all the notation needed for this section. First we give all the relative equilibria families dubbed as quasi-Euler equilibria in Table 6.6 and then the detail of the components of the sextuples in (6.41), (6.43) and (6.45).

Notation for the expressions related with the equilibria in the M_2-M_3 coordinate

Quasi-Euler equilibria	
Equilibria	Energy
$E_{10} = (0, \pm M_2^{(1)}, \pm M_3^{(1)}, G_1, G_2, -G_3^{(1)})$	h_{E10}
$E_{11} = (0, \pm M_2^{(1)}, \pm M_3^{(1)}, G_1, G_2, G_3^{(1)})$	h_{E11}
$E_{12} = (\pm M_1^{(2)}, 0, \pm M_3^{(2)}, G_1, G_2, -G_3^{(2)})$	h_{E12}
$E_{13} = (\pm M_1^{(2)}, 0, \pm M_3^{(2)}, G_1, G_2, -G_3^{(2)})$	h_{E13}
$E_{14} = (\pm M_1^{(3)}, \pm M_2^{(3)}, 0, G_1, G_2, G_3^{(3)})$	h_{E14}
$E_{15} = (\pm M_1^{(3)}, \pm M_2^{(3)}, 0, G_1, G_2, -G_3^{(3)})$	h_{E15}

Table 6.5: Quasi-Euler equilibria.

plane

$$
\begin{aligned}
&G_3^{(1)} = M\sqrt{\frac{3BCn^2 - 4M^2}{9n^2BC}}, \quad M_{31} = M\sqrt{\frac{3nG_3^{(1)}\,(-2B + A + C) - 4M^2}{9nG_3^{(1)}(C - B)}},\\
&M_{21}^2 + M_{31}^2 = M^2,\\
&\mathcal{M}_1^{(1)} = \frac{1}{2}\sqrt{\frac{3BCn^2(2B - A - C)^2}{(2B - A - C)^2 + 4BC}}, \qquad \mathcal{M}_1^{(2)} = \frac{1}{2}\sqrt{\frac{3BCn^2(2C - B - A)^2}{(2C - B - A)^2 + 4BC}},
\end{aligned}
\tag{6.41}
$$

and energies

$$
\begin{aligned}
h_{E10} &= \frac{(-A + 2B + 2C)}{6\,CB}M^2 + \frac{n}{2}G_3^{(1)} + \frac{4M^4 - 3M^2nBC}{18G_3nBC},\\
h_{E11} &= \frac{(-A + 2B + 2C)}{6\,CB}M^2 - \frac{n}{2}G_3^{(1)} + \frac{4M^4 - 3M^2nBC}{18G_3nBC}.
\end{aligned}
\tag{6.42}
$$

Notation for the expressions related with the equilibria in the M_1-M_3 coordinate plane

$$G_3^{(2)} = M\sqrt{\frac{3ACn^2 - 4M^2}{9n^2AC}}, \quad M_{32} = M\sqrt{\frac{3nG_3^{(2)}(-2A+B+C) - 4M^2}{9nG_3^{(2)}(C-A)}},$$
$$M_{12}^2 + M_{32}^2 = M^2,$$
$$\mathcal{M}_2^{(1)} = \frac{1}{2}\sqrt{\frac{3ACn^2(2A-B-C)^2}{(2A-B-C)^2+4AC}}, \quad \mathcal{M}_2^{(2)} = \frac{1}{2}\sqrt{\frac{3ACn^2(2C-B-A)^2}{(2C-B-A)^2+4AC}}, \tag{6.43}$$

and energies

$$\begin{aligned} h_{E12} &= \frac{(-B+2A+2C)}{6\,AC}M^2 + \frac{n}{2}G_3^{(3)} + \frac{4M^4 - 3M^2nAC}{18G_3nAC}, \\ h_{E13} &= \frac{(-B+2A+2C)}{6\,AC}M^2 - \frac{n}{2}G_3^{(3)} + \frac{4M^4 - 3M^2nAC}{18G_3nAC}. \end{aligned} \tag{6.44}$$

Notation for the expressions related with the equilibria in the M_1-M_2 coordinate plane

$$G_3^{(3)} = M\sqrt{\frac{3ABn^2 - 4M^2}{9n^2AB}}, \quad M_{23} = M\sqrt{\frac{3nG_3^{(3)}(-2A+B+C) - 4M^2}{9nG_3^{(3)}(B-A)}},$$
$$M_{13}^2 + M_{23}^2 = M^2,$$
$$\mathcal{M}_3^{(1)} = \frac{1}{2}\sqrt{\frac{3ABn^2(2B-A-C)^2}{(2B-C-A)^2+4AB}}, \quad \mathcal{M}_3^{(2)} = \frac{1}{2}\sqrt{\frac{3ABn^2(2A-B-C)^2}{(2A-B-C)^2+4AB}}, \tag{6.45}$$

and energies

$$\begin{aligned} h_{E14} &= \frac{(-C+2B+2A)}{6\,AB}M^2 + \frac{n}{2}G_3^{(3)} + \frac{4M^4 - 3M^2nAB}{18G_3nAB}, \\ h_{E15} &= \frac{(-C+2B+2A)}{6\,AB}M^2 - \frac{n}{2}G_3^{(3)} + \frac{4M^4 - 3M^2nAB}{18G_3nAB}. \end{aligned} \tag{6.46}$$

Results of Theorems 6.6, 6.7 and 6.8 are summarized in Table 6.6

Quasi-Euler equilibria in the M_i-M_j planes			
Region	M	Equilibria	Energy
T_1	$(0, \mathcal{M}_1^{(2)})$	$E_{10} = (0, \pm M_2^{(1)}, \pm M_3^{(1)}, G_1, G_2, -G_3^{(1)})$	h_{E10}
T_2	$(\mathcal{M}_1^{(1)}, \mathcal{M}_1^{(2)})$	$E_{10} = (0, \pm M_2^{(1)}, \pm M_3^{(1)}, G_1, G_2, -G_3^{(1)})$	h_{E10}
L_2	$(0, \mathcal{M}_1^{(2)})$	$E_{10} = (0, \pm M_2^{(1)}, \pm M_3^{(1)}, G_1, G_2, -G_3^{(1)})$	h_{E10}
T_1	$(0, \mathcal{M}_1^{(1)})$	$E_{11} = (0, \pm M_2^{(1)}, \pm M_3^{(1)}, G_1, G_2, G_3^{(1)})$	h_{E11}
I_D	$(0, \mathcal{M}_2^{(2)})$	$E_{12} = (\pm M_1^{(2)}, 0, \pm M_3^{(2)}, G_1, G_2, -G_3^{(2)})$	h_{E12}
I_D	$(0, \mathcal{M}_2^{(1)})$	$E_{13} = (\pm M_1^{(2)}, 0, \pm M_3^{(2)}, G_1, G_2, G_3^{(2)})$	h_{E13}
L_2	$(0, \mathcal{M}_3^{(2)})$	$E_{14} = (\pm M_1^{(3)}, \pm M_2^{(3)}, 0, G_1, G_2, G_3^{(3)})$	h_{E14}
T_1	$(\mathcal{M}_3^{(1)}, \mathcal{M}_3^{(2)})$	$E_{14} = (\pm M_1^{(3)}, \pm M_2^{(3)}, 0, G_1, G_2, G_3^{(3)})$	h_{E14}
T_2	$(0, \mathcal{M}_3^{(2)})$	$E_{14} = (\pm M_1^{(3)}, \pm M_2^{(3)}, 0, G_1, G_2, G_3^{(3)})$	h_{E14}
T_2	$(0, \mathcal{M}_3^{(1)})$	$E_{15} = (\pm M_1^{(3)}, \pm M_2^{(3)}, 0, G_1, G_2, -G_3^{(3)})$	h_{E15}

Table 6.6: **Quasi-Euler equilibria:** Equilibria along the coordinate planes in the M-sphere and a circumference in the G-sphere.

Theorem 6.6 (Quasi-Euler equilibria in the M_2-M_3 plane)**.** *The sextuples given by E_{10} and E_{11} with associated energies h_{E10} and h_{E11} are relative equilibria if any of the following cases occur*

$$\begin{array}{llll} T_1, & G_3 = -G_3^{(1)}, & M \in (0, \mathcal{M}_1^{(1)}), \\ T_1, & G_3 = G_3^{(1)}, & M \in (0, \mathcal{M}_1^{(2)}), \\ T_2, & G_3 = G_3^{(1)}, & M \in (\mathcal{M}_1^{(1)}, \mathcal{M}_1^{(2)}), \\ L_2, & G_3 = G_3^{(1)}, & M \in (0, \mathcal{M}_1^{(2)}), \end{array} \tag{6.47}$$

For the extremums, $M = \mathcal{M}_1^{(1)}$ and $M = \mathcal{M}_1^{(2)}$, we have

$$M_2^{(1)} = M,\ M_3^{(1)} = 0 \quad \textit{and} \quad M_2^{(1)} = 0,\ M_3^{(1)} = M. \tag{6.48}$$

Proof. After some algebraic manipulations one can verify that $G_3 = G_3^{(1)}$ and $G_3 = -G_3^{(1)}$ imply $a_1 = 0$ in the equation (6.15) and that $M_1 = 0$, $M_2 = \pm M_{21}$ and $M_3 = \pm M_{31}$ imply $\Delta_G(\mathbf{M}) = 0$ in equations (6.18) and (6.19). The conditions (6.47) for the existence of equilibria comes from the definition of $G_3^{(1)}$, which involves a square root with positive radicand when conditions (6.47) are verified. Finally, (6.52) is obtained by simple substitution of $M = \mathcal{M}_1^{(1)}$ and $M = \mathcal{M}_1^{(2)}$.

□

Theorem 6.7 (Quasi-Euler equilibria in the M_1-M_3 plane)**.** *The sextuples given by E_{12} and E_{13} with associated energies h_{E12} and h_{E13} are relative equilibria if any of the following cases occur*

$$\begin{aligned} G_3 &= -G_3^{(2)}, \quad M \in (0, \mathcal{M}_2^{(1)}), \\ G_3 &= G_3^{(2)}, \quad M \in (0, \mathcal{M}_2^{(2)}), \end{aligned} \tag{6.49}$$

For the extremums, $M = \mathcal{M}_2^{(1)}$ and $M = \mathcal{M}_2^{(2)}$, we have

$$M_1^{(2)} = M,\ M_3^{(2)} = 0 \quad \textit{and} \quad M_1^{(2)} = 0,\ M_3^{(2)} = M. \tag{6.50}$$

Proof. Analogous to proof of Theorem 6.6

□

Theorem 6.8 (Quasi-Euler equilibria in the M_1-M_2 plane)**.** *The sextuples given by E_{10} and E_{11} with associated energies h_{E14} and h_{E15} are relative equilibria if any of the following cases occur*

$$\begin{aligned} T_1, &\quad G_3 = -G_3^{(3)}, \quad M \in (\mathcal{M}_3^{(1)}, \mathcal{M}_3^{(2)}), \\ T_2, &\quad G_3 = G_3^{(3)}, \quad M \in (0, \mathcal{M}_3^{(1)}), \\ T_2, &\quad G_3 = -G_3^{(3)}, \quad M \in (0, \mathcal{M}_3^{(2)}), \\ L_2, &\quad G_3 = G_3^{(3)}, \quad M \in (0, \mathcal{M}_3^{(2)}), \end{aligned} \tag{6.51}$$

For the extremums, $M = \mathcal{M}_3^{(1)}$ and $M = \mathcal{M}_3^{(2)}$, we have

$$M_1^{(3)} = M,\ M_2^{(3)} = 0 \quad \textit{and} \quad M_1^{(3)} = 0,\ M_2^{(3)} = M. \tag{6.52}$$

Proof. Analogous to the proof of Theorem 6.6.

□

6.4 Linear Stability and Bifurcation of the $\mathbb{S}^1$-Relative Equilibria

Among the local bifurcations from an equilibrium in Hamiltonian systems with parameters, there are two well known scenarios:

(i.1) steady-state (splitting case) bifurcation when at the equilibrium the linearized vector field has a zero eigenvalue of multiplicity two. After the multiplicity the eigenvalues move into the reals.

(i.2) steady-state (passing case) bifurcation when at the equilibrium the linearized vector field has a zero eigenvalue of multiplicity two. After the multiplicity the eigenvalues continue moving in the imaginary axis.

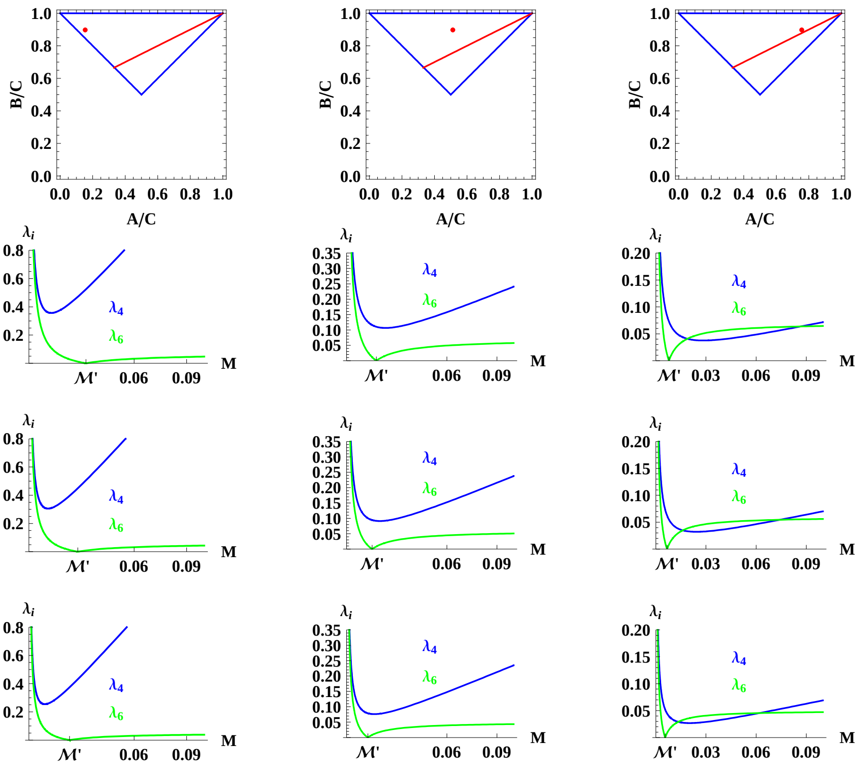
(ii.1) 1-1 resonance (splitting case) when the linearization has a pair of purely imaginary eigenvalues of multiplicity two. After the multiplicity they move into the complex domain.

(ii.2) 1-1 resonance (passing case) when the linearization has a pair of purely imaginary eigenvalues of multiplicity two. After the multiplicity all remain pure imaginary.

These bifurcations have received different names in the literature. Details about the features of each of them may be found in several publications (see for instance van der Meer [van der Meer, 1985], Dellnitz *et al.* [Dellnitz et al., 1992] and Bouno *et al.* [Buono et al., 2005] and references therein). Note that although the previous list suggests that both scenarios are separate, we show below that it is not the case, see Fig. 6.8.

In our model, within the T_2 triaxiality region, we identify several of them. In that respect it is worth to point out that, in agreement with the generic result of Galin [Galin, 1982], our system satisfies the required condition in order to be at the passing scenario: to have at least three parameters.

In this section we obtain the linear stability of a representative set of each $\mathbb{S}^1$-relative equilibria given in the previous section. The calculations for the rest of sextuples are left to the reader. We have structured this section in three cases according with the zero eigenvalue multiplicity. Permanent equilibria with multiplicity two, principal direction equilibria with multiplicity four and the degenerate last case, the quasi-Euler equilibria where the eigenvalues are all zero.



(a) $B >> (A+C)/2$ (b) $B > (A+C)/2$ (c) $B \approx (A+C)/2$

Figure 6.8: Evolution with M of the positive imaginary part of the eigenvalues λ_4 and λ_6 for three different mean motions $n = \{0.07, 0.06, 0.05\}$ $(\mathrm{rad/min})$ corresponding respectively with each one of the rows. Note that the first row only shows the location of the selected moments of inertia over the parameter plane. (a) Evolution for values $\{A, B, C\} = \{0.06, 0.35, 0.39\}$ $(\mathrm{kg} \cdot \mathrm{km}^2)$. (b) Evolution for values $\{A, B, C\} = \{0.2, 0.35, 0.39\}$ $(\mathrm{kg} \cdot \mathrm{km}^2)$. (c) Evolution for values $\{A, B, C\} = \{0.295, 0.35, 0.39\}$ $(\mathrm{kg} \cdot \mathrm{km}^2)$. A deeper insight of one of these graphs is given in Fig. 6.9.

6.4.1 Permanent equilibria

In order to show the linear stability of the permanent equilibria, we start with the study in detail of the stability of the sextuple: $(M, 0, 0, 0, 0, M)$ given in Tab. 6.2. To do this, we first obtain the Jacobian of the differential equations (6.15)-(6.20) for the corresponding sextuple and then compute their eigenvalues. Note that the two integrals of this system imply the existence of two eigenvalues $\lambda_{1,2} = 0$; the other four of them are given by

$$\lambda_{3,4} = \pm \frac{i}{2AM}\sqrt{\frac{(B-A)(C-A)(2M^2+3ABn^2)(2M^2+3ACn^2)}{BC}} \tag{6.53}$$

$$\lambda_{5,6} = \pm i\frac{n}{4M}[4M - 3n(C+B-2A)] \tag{6.54}$$

whose evolution with M is given in Fig. 6.8. In this figure we show how different scenarios arise as we vary the principal moments of inertia along the region T_2. More precisely, we can see that columns 6.8(a) and 6.8(b) give place to steady-state bifurcations, while in column 6.8(c) the 1-1 resonance (passing case) also occurs. In this regard, we have illustrated in Fig 6.9 the behavior of the eigenvalues in the case 6.8(c), it shows that the apparently separated scenarios of the steady-state and 1-1 resonance bifurcations that we have listed at the beginning of this section may also appear together.

6.4.2 Principal direction equilibria

We study now the linear stability of the sextuple: $(0,0,M,G_1,G_2,G_3''')$ given in Tab. 6.4. Following the same procedure we have again $\lambda_{1,2}=0$ together with

$$\lambda_{3,4} = 0 \tag{6.55}$$

$$\begin{aligned}\lambda_{5,6} = &\pm \frac{1}{4C(2C-B-A)\sqrt{ABM}} \\ &\times\sqrt{(C-A)[4M^2(A^2+2AB+(B-2C)^2)-3ACn^2(2C-B-A)^2]} \\ &\times\sqrt{(C-B)[3BCn^2(2C-B-A)^2-4M^2((A+B)^2-4AC+4C^2)]}\end{aligned} \tag{6.56}$$

From these expressions it is easy to see that $\{\lambda_{3,4}\}\in\mathbb{C}$, but $\{\lambda_{5,6}\}$ can be real or complex depending on the parameters and can also be zero if

$$M^2 = \frac{3ACn^2(2C-B-A)^2}{4[A^2+2AB+(B-2C)^2]} \quad \text{or} \quad M^2 = \frac{3BCn^2(2C-B-A)^2}{4[(A+B)^2-4AC+4C^2]} \tag{6.57}$$

The evolution with M of these eigenvalues is shown in Fig. 6.10, where we have a zero eigenvalue of multiplicity four and two more eigenvalues, $\lambda_{5,6}$, that give place to a steady-state (splitting case) bifurcation.

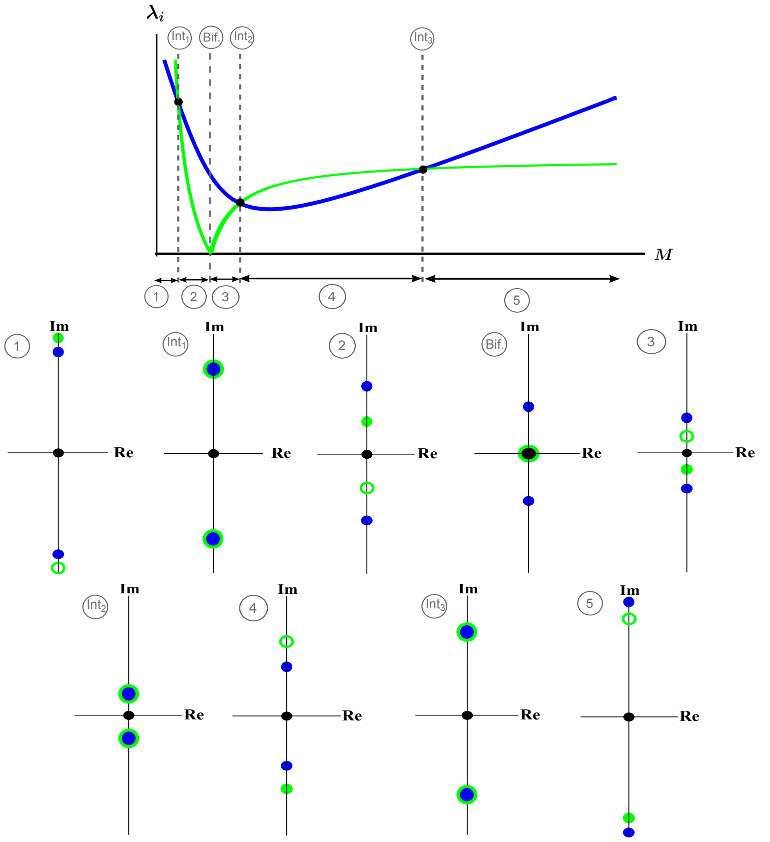


Figure 6.9: Example of the evolution with M of the eigenvalues corresponding to Fig. 6.8(c) shown by means of their variations over the complex plane. The different intervals of M are labeled with a number. The labels "$\mathrm{Int_i}$" indicate values of M where the eigenvalues come together. The label "Bif." indicates the value $M = \mathcal{M}'$ where the north pole of the $G-$sphere bifurcates in a circumference of equilibria.

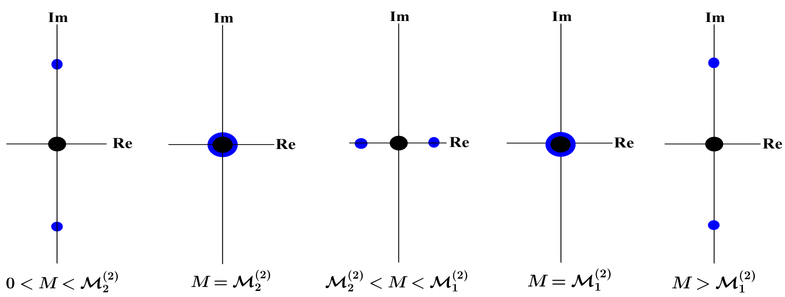


Figure 6.10: Evolution with M of the eigenvalues λ_3 and λ_4. It corresponds to a double steady-state splitting bifurcation.

6.4.3 Bifurcations: From the border to the crossing of the parabolas with the petals

This section is about the explanation of how equilibria and bifurcations are distributed along the parabolas and petals given in (6.39), (6.41), (6.43) and (6.45). For this purpose Fig 6.11 is crucial and encapsulates all the information related to equilibria and bifurcation for the region T_2. In the elaboration of this figure we have change our methodology in the following sense. Up to now sextuples of equilibria are projected in the M-sphere and the G-sphere, thus they are represented in a product of two spheres. Nevertheless, in this figure we have omitted most of the G-spheres just to alleviate the amount of information contained. Note also that there is one more abuse in the construction of Fig 6.11, with the aim of provide a clearer presentation, the radius of the spheres are scaled to be the same, i. e., those radius do not change in the figure with M.

All the points collected in Fig 6.11(a) correspond with bifurcations. For A_0', A_0'' and A_0''' we observe how the isolated equilibria E_1, E_5 and E_6 bifurcate into the families of equilibria E_7, E_8 and E_9. In particular we have studied in Section 6.4.1 the eigenvalues of E_9 obtaining from expressions (6.53), that $\{\lambda_{3,4}; \lambda_{5,6}\} \in \mathbb{C}$, except for the value in which $\lambda_{5,6}$ cancels out

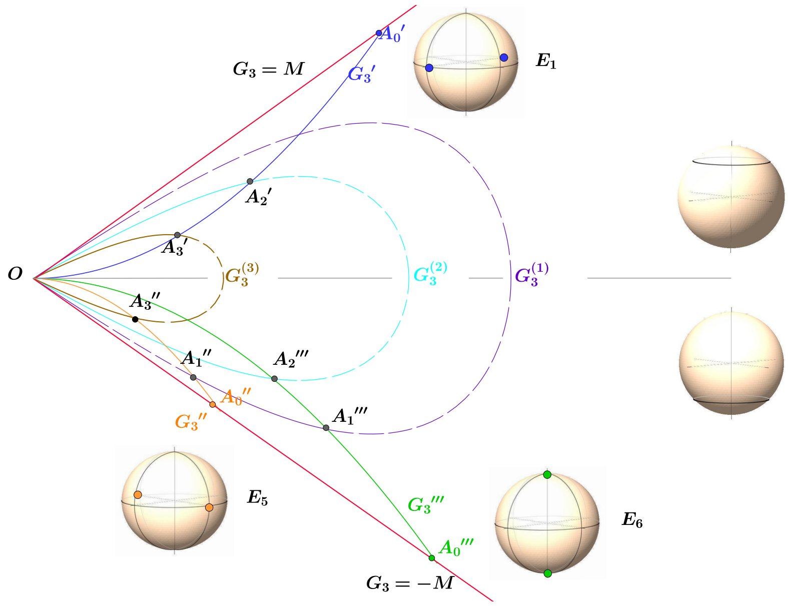
$$M = \mathcal{M}' = \frac{3n}{4}(C + B - 2A)$$

which correspond exactly with the bifurcation from E_6 to E_9 at A_0''' in Fig 6.11(a) .

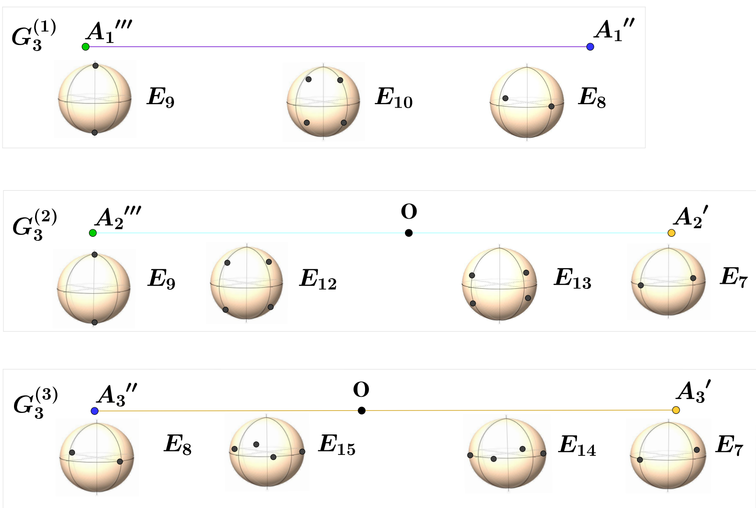
Analogously, we have obtained in Section 6.4.2 that $\lambda_{5,6}$ vanish for certain values of M given in (6.57). They correspond to the bifurcation points identified in Fig 6.11(a) by A_1'' and A_1''', where E_9 and E_8 bifurcate in E_{10}. This phenomena and the evolutions from E_9 to E_{10} and finally to E_8 along an arc in the petal $G_3^{(1)}$ is illustrated in Fig 6.11(b), where we also give a detailed picture of what happen in $G_3^{(2)}$ and $G_3^{(3)}$.

6.5 Second Reduction: $\mathbb{T}^2$-Relative Equilibria

A second reduction can be performed by using the integral G_3. The relative equilibria of the second reduced space correspond with orbits contained in a 2-dimensional torus.



(a) MG_3-Bifurcation plane.



(b) Itineraries detail.

Figure 6.11: Bifurcation lines and detailed tours along them showing the relation between the families of relative equilibria. In Fig 6.11(a) we have used continuous lines in $G_3^{(1)}$, $G_3^{(2)}$ and $G_3^{(3)}$ to distinguish the zones where quasi-Euler equilibria occur. Note that the spheres have been scaled to the same radius.

Recalling the Poisson structure of the first reduced space given in (4.60), it is clear that the second reduce space $\mathbb{S}^2(M) = \{(M_1, M_2, M_3)/\ M_1^2 + M_2^2 + M_3^2 = M^2\}$ is endowed with the following Poisson structure

$$\{M_1, M_2\} = -M_3, \quad \{M_1, M_3\} = M_2, \quad \{M_3, M_2\} = M_1, \tag{6.58}$$

which provides the associated Poisson algebra $(C^\infty(\mathbb{S}^2(M)), \{,\})$ with the structure of $\mathfrak{so}(3)$. Thus, the equations of the second reduced system are obtained by mean of the following computations

$$(\dot{\mathbf{M}}_1, \dot{\mathbf{M}}_2, \dot{\mathbf{M}}_3) = (\{M_i, M_j\})_{3\times 3}\nabla\mathcal{H}_0^*,$$

which lead to the system

$$\dot{M}_1 = a_1 M_2 M_3, \tag{6.59}$$

$$\dot{M}_2 = a_2 M_1 M_3, \tag{6.60}$$

$$\dot{M}_3 = a_3 M_1 M_2, , \tag{6.61}$$

where a_1, a_2 and a_3 are given by (6.21), (6.22) and (6.23). Next we study the relative equilibria of the above system of differential equations, we follow a similar methodology to the one applied in the first reduced space, distinguishing between permanent equilibria and quasi-Euler equilibria.

Theorem 6.9 (Permanent equilibria)**.** *The following triads are relative equilibria of the second reduced system*

$$\{E_{17} = (\pm M, 0, 0), \quad E_{18} = (0, \pm M, 0), \quad E_{19} = (0, 0, \pm M)\}, \tag{6.62}$$

Proof. By simple substitution on the system made of (6.59), (6.60) and (6.61) of the sextuples given in (6.62), it follows that they are equilibria.

□

The energy associated to each equilibria is readily obtained using the Hamiltonian $\mathcal{H}_0^*$, they are given by

$$\begin{aligned} h_{E17} =& \frac{(-A+2B+2C)}{6\,CB}M^2 + \frac{n}{2}G_3 + \frac{4M^4 - 3M^2 nBC}{18G_3 nBC}, \\ h_{E18} =& \frac{(-B+2A+2C)}{6\,CA}M^2 + \frac{n}{2}G_3 + \frac{4M^4 - 3M^2 nAC}{18G_3 nAC}, \\ h_{E19} =& \frac{(-C+2B+2A)}{6\,AB}M^2 + \frac{n}{2}G_3 + \frac{4M^4 - 3M^2 nBA}{18G_3 nBA}, \end{aligned} \tag{6.63}$$

Theorem 6.10 (Quasi-Euler equilibria)**.** *Consider the following set of triads*

$$\begin{aligned} E_{20} &= \{(0, \pm M_2, \pm M_3), \quad M_2^2 + M_3^2 = m^2\}, \\ E_{21} &= \{(\pm M_1, 0, \pm M_3), \quad M_1^2 + M_3^2 = m^2\}, \\ E_{22} &= \{(\pm M_1, \pm M_2, 0), \quad M_1^2 + M_2^2 = m^2\}, \end{aligned} \tag{6.64}$$

Then, if $G_3 = \pm G_3^{(i)}$, we have that the sextuples E_{16+i} and E_{19+i} are made of relative equilibria for $i = 1, 2, 3$.

Proof. After some algebraic manipulations one can verify that $G_3 = \pm G_3^{(1)}$ imply $a_1 = 0$ in the equation (6.59), thus the set E_{20} is a circumference full of equilibria and by simple substitution we obtain that E_{17} is also a pair of equilibria. Analogously for $G_3 = G_3^{(2)}$ or $G_3 = -G_3^{(2)}$ and E_{21} or E_{22} respectively. □

Analogously we have for the energies

$$\begin{aligned} h_{20} =& \frac{(-A + 2B + 2C)}{6\,CB} M^2 + \frac{n}{2} G_3 + \frac{4M^4 - 3M^2 nBC}{18 G_3 nBC}, \\ h_{21} =& \frac{(-B + 2A + 2C)}{6\,CA} M^2 + \frac{n}{2} G_3 + \frac{4M^4 - 3M^2 nAC}{18 G_3 nAC}, \\ h_{22} =& \frac{(-C + 2B + 2A)}{6\,AB} M^2 + \frac{n}{2} G_3 + \frac{4M^4 - 3M^2 nBA}{18 G_3 nBA}. \end{aligned} \tag{6.65}$$

In Fig.6.12 we describe the flow and the equilibria on the second reduced space. On the right, the spheres show how the unstable point switches its position as we move in the MG_3-plane. On the left, we can see how the equilibria becomes degenerate, along the curve $G_3^{(1)}$ we get a maximal circumference on the M_2M_3 coordinate plane of equilibria. The same occurs for $G_3^{(2)}$ in the M_1M_3 coordinate plane and for $G_3^{(3)}$ in the M_1M_2 coordinate plane.

6.6 Stability of the $\mathbb{T}^2$-Relative Equilibria

In this section we study the stability of the $\mathbb{T}^2$-relative equilibria. We start by checking the stability of the so called permanent equilibria following the 3-step energy-Casimir method (see [Marsden and Ratiu, 1999]).

We recall that the differential system given by (6.15)-(6.20) is Hamiltonian in the Poisson structure of $\mathbb{S}^2$ defined in (6.58), the Hamiltonian function is $\mathcal{F}_a^*$. In addition,

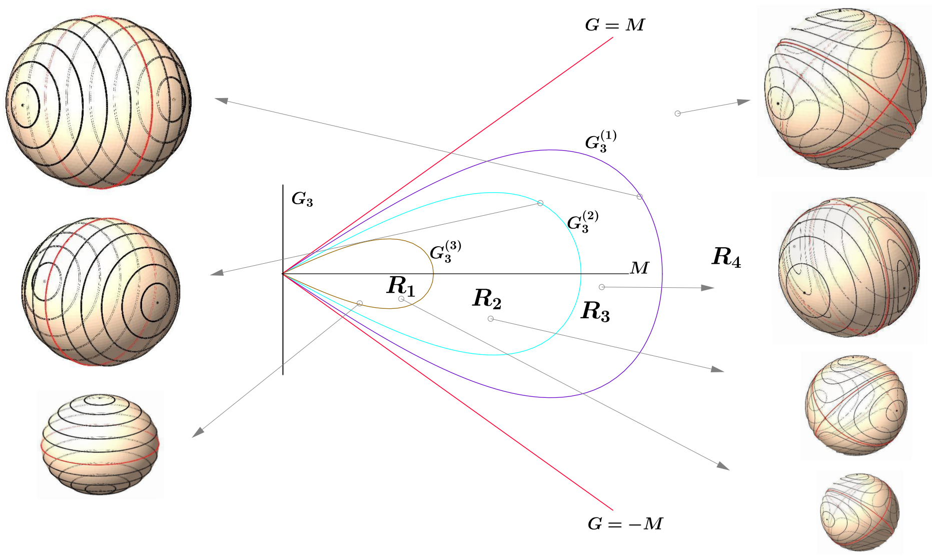


Figure 6.12: MG_3**-momentum-momentum plane:** Equilibria and general flow on the second reduced space.

we have that $f(M_M)$ is a Casimir, where $M_M = M_1^2 + M_2^2 + M_3^2$ and f is any smooth function $f : \mathbb{R} \to \mathbb{R}$. More precisely, let us consider

$$f_A(x) = -\left(\frac{1}{2A} + \frac{3An^2\left(M^2 - 3\mathrm{G3}^2\right)}{2M^4}\right)x + (x-1)^2,$$

$$f_B(x) = -\left(\frac{1}{2B} + \frac{3Bn^2\left(M^2 - 3\mathrm{G3}^2\right)}{2M^4}\right)x + (x-1)^2,$$

$$f_C(x) = -\left(\frac{1}{2C} + \frac{3Cn^2\left(M^2 - 3\mathrm{G3}^2\right)}{2M^4}\right)x + (x-1)^2,$$

then, we define the energy-Casimir functions

$$\begin{aligned} C_{fA}(M_1, M_2, M_3) &= \mathcal{F}_a(M_1, M_2, M_3, G_1, G_2, G_3) + f_A(M_M), \\ C_{fB}(M_1, M_2, M_3) &= \mathcal{F}_a(M_1, M_2, M_3, G_1, G_2, G_3) + f_B(M_M), \\ C_{fC}(M_1, M_2, M_3) &= \mathcal{F}_a(M_1, M_2, M_3, G_1, G_2, G_3) + f_C(M_M), \end{aligned} \tag{6.66}$$

a straightforward computation shows that the derivative of C_{fA}, C_{fB} and C_{fC} evaluated at $(\pm M, 0, 0)$, $(0, \pm M, 0)$ and $(0, 0, \pm M)$ respectively equal to zero. Let us now investigate the definiteness of the second derivatives at the equilibria

$$D^2 C_{fA} = \begin{pmatrix} -8M^2 & 0 & 0 \\ 0 & \Delta_{AB} & 0 \\ 0 & 0 & \Delta_{AC} \end{pmatrix} \tag{6.67}$$

where

$$\Delta_{AB} = \frac{1}{B} - \frac{1}{A} + \frac{3Bn^2\left(M^2 - 3\text{G3}^2\right)}{4M^4} - \frac{3An^2\left(M^2 - 3\text{G3}^2\right)}{4M^4}$$

and

$$\Delta_{AC} = \frac{1}{C} - \frac{1}{A} + \frac{3Cn^2\left(M^2 - 3\text{G3}^2\right)}{4M^4} - \frac{3An^2\left(M^2 - 3\text{G3}^2\right)}{4M^4},$$

$$D^2C_{fB} = \begin{pmatrix} \Delta_{BA} & 0 & 0 \\ 0 & -8M^2 & 0 \\ 0 & 0 & \Delta_{BC} \end{pmatrix} \tag{6.68}$$

where

$$\Delta_{BA} = \frac{1}{A} - \frac{1}{B} + \frac{3An^2\left(M^2 - 3\text{G3}^2\right)}{4M^4} - \frac{3Bn^2\left(M^2 - 3\text{G3}^2\right)}{4M^4}$$

and

$$\Delta_{BC} = \frac{1}{C} - \frac{1}{B} + \frac{3Cn^2\left(M^2 - 3\text{G3}^2\right)}{4M^4} - \frac{3Bn^2\left(M^2 - 3\text{G3}^2\right)}{4M^4},$$

$$D^2C_{fC} = \begin{pmatrix} \Delta_{CA} & 0 & 0 \\ 0 & \Delta_{CB} & 0 \\ 0 & 0 & -8M^2 \end{pmatrix} \tag{6.69}$$

where

$$\Delta_{CA} = \frac{1}{A} - \frac{1}{C} + \frac{3An^2\left(M^2 - 3\text{G3}^2\right)}{4M^4} - \frac{3Cn^2\left(M^2 - 3\text{G3}^2\right)}{4M^4}$$

and

$$\Delta_{CB} = \frac{1}{B} - \frac{1}{C} + \frac{3Bn^2\left(M^2 - 3\text{G3}^2\right)}{4M^4} - \frac{3Cn^2\left(M^2 - 3\text{G3}^2\right)}{4M^4},$$

Thus, taking into account that

$$\begin{aligned}
&\Delta_{AB} < 0,\ \Delta_{AC} < 0 \ \Leftrightarrow\ (M, G_3) \notin R2 \Rightarrow (\pm M, 0, 0) \text{ is Liapunov stable,} \\
&\Delta_{AB} < 0,\ \Delta_{AC} < 0 \ \Leftrightarrow\ (M, G_3) \notin R1 \cup R4 \Rightarrow (0, \pm M, 0) \text{ is Liapunov stable,} \\
&\Delta_{AB} < 0,\ \Delta_{AC} < 0 \ \Leftrightarrow\ (M, G_3) \notin R3 \Rightarrow (0, 0, \pm M) \text{ is Liapunov stable,}
\end{aligned} \tag{6.70}$$

where the regions R_i in the M-G_3 plane, see Fig. 6.12, are limited by the curves $G_3^{(1)}$, $G_3^{(2)}$ and $G_3^{(3)}$, see (6.41), (6.43) and (6.45). Combining this study with the computation of the eigenvalues of the linearized system we characterize the stability of the system in Tab. .6.7

Equilibria	M	Stability
$(\pm M, 0, 0)$	R_1, R_3, R_4	stable
	R_2	unstable
$(0, \pm M, 0)$	R_3, R_2	stable
	R_1, R_4	unstable
$(0, 0, \pm M)$	R_1, R_2, R_4	stable
	R_3	unstable

Table 6.7: Stability in the second reduced space.

Chapter 7

On Elliptic Type Intermediaries

In the model that we consider in this chapter, we examine a body moving in a rosette-like orbit. More precisely we are thinking in two type of application, namely to artificial satellites or asteroids around a planet. In other words, we consider perturbed elliptic orbits in general; we also investigate conditions for which this model admits the circular ones. This scenario leads to medium orbits rather than to the low type of orbits studied in the preceding chapter. A very well-known example of these are the Molniya orbits, see Fig 7.1, which are candidates to use the elliptic gravity-gradient as a zero order in the context of perturbation theory.

Figure 7.1: **NEO Asteroid:** An illustration of a hypothetical near earth orbit asteroid.

7.1 The r-Gravity Gradient Intermediary

Let us consider the elliptic intermediary defined in Section 5.2.1. This is the only intermediary with non-constant radius we have found and it is given by the following Hamiltonian function

$$\mathcal{H}_r = \mathcal{H}_K + \mathcal{H}_R + V_r(\rho_K, M_K, \Lambda_K, M_R, \Lambda_R)$$

where the perturbing potential V_r is a function of the radial distance and two of the rotational and orbital momenta. Along this chapter, we will use the double Projective Andoyer chart. The Hamiltonian function is given by

$$\begin{aligned} \mathcal{H}_K =& \frac{1}{2m}\left(P_K^2 + \frac{M_K^2}{\rho_K 2}\right) - \frac{\mathcal{G}M_\odot}{\rho_K}, \\ \mathcal{H}_R =& \frac{1}{2}\left(\frac{\sin^2\nu}{A} + \frac{\cos^2\nu}{B}\right)(M_R^2 - N_R^2) + \frac{1}{2C}N_R^2, \\ V_r(\rho_K, M_K, \Lambda_K, M_R, \Lambda_R) =& \frac{\mathcal{G}M_\odot}{8\rho_K^3}(2C - B - A)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right)\left(1 - 3\frac{\Lambda_K^2}{M_K^2}\right), \end{aligned} \tag{7.1}$$

The tern $(T^*\mathbb{R}^6, \omega, \mathcal{H}_r)$ is a Hamiltonian dynamical system. The following propo characterizes two of the main features of the system. Moreover, the V_r encapsulates the coupling that the system contains.

Proposition 7.1. *The functions $\mathcal{H}_R$ and $\mathcal{H}_O = \mathcal{H}_K + V_r$ are conserved quantities along the flow of $\mathcal{H}_r$.*

Proof. By computing $\{\mathcal{H}_r, \mathcal{H}_R\} = \{\mathcal{H}_r, \mathcal{H}_K^V\} = \{\mathcal{H}_R, \mathcal{H}_K^V\} = 0$ the claim is obtained. □

Proposition 7.2. *The Hamiltonians $\mathcal{H}_R$ and $\mathcal{H}_O$ define two integrable 1-DOF systems.*

Proof. The claim is readily obtained by expressing these Hamiltonians in the double Projective Andoyer chart. □

This model has been already considered in the literature [Ferrer and Lara, 2012a]. They simplify the model carrying out the elimination of the parallax as a first step and after that they build up the associated action-angle variables. The content of this Chapter may be seen as a complement of this work. We proceed studying the system without the elimination of the parallax presenting the analytical solutions of it including the circular case. Moreover, that is the main aspect, we recover the model in four dimension together with a constraint. Note that our model include what in literature has been named dubbed as critical inclination. In order to simplify the formulas we decided to consider only the equatorial case. The full analysis of the Hamiltonian system defined by (7.1) is in progress and will be published elsewhere [Crespo et al., 2015].

7.1.1 On the existence of constant radius. Equatorial orbits case

In this section we analyze if this elliptic intermediary also admits initial conditions for which the eccentricity of the orbit becomes equal to zero. Henceforth, we assume initial conditions such that $(1 - 3\Lambda_K^2/M_K^2) = 1$. Thus, the Hamiltonian (7.2) becomes

$$\begin{aligned} \mathcal{H}_K &= \frac{1}{2m}\left(P_K^2 + \frac{M_K^2}{\rho_K 2}\right) - \frac{\mathcal{G}M_\odot}{\rho_K}, \\ \mathcal{H}_R &= \frac{1}{2}\left(\frac{\sin^2\nu}{A} + \frac{\cos^2\nu}{B}\right)(M_R^2 - N_R^2) + \frac{1}{2C}N_R^2, \\ V_r(\rho_K, M_R, \Lambda_R) &= \frac{\mathcal{G}M_\odot}{8\rho_K^3}(2C - B - A)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right). \end{aligned} \tag{7.2}$$

Then, the differential equations for the variables (ρ_K, P_K) read as follows

$$\begin{aligned} \dot{\rho}_K &= P_K/m, \\ \dot{P}_K &= -\frac{\mathcal{G}M_\odot}{\rho_K^2} + \frac{M_K^2}{m\,\rho_K^3}, + \frac{3\mathcal{G}M_\odot}{8\rho_K^4}(2C - A - B)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right), \end{aligned} \tag{7.3}$$

looking for circular orbits is equivalent to impose $\dot{\rho}_K = 0$ and $\dot{P}_K = 0$. Thus, we get that the following relations hold

$$P_K = 0, \qquad -\mathcal{G}M_\odot\rho_K^2 + \frac{M_K^2}{m}\rho_K + \frac{3\mathcal{G}M_\odot}{8}(2C - A - B)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right) = 0, \tag{7.4}$$

the radius ρ_K is readily obtained by solving the above quadratic equation

$$\rho_{K\pm} = \frac{1}{2}\left[\frac{M_K^2}{\mathcal{G}M_\odot m} \pm \sqrt{\frac{M_K^4}{\mathcal{G}M_\odot^2 m^2} + \frac{3}{2}(2C - A - B)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right)}\,\right]. \tag{7.5}$$

Note that not all the solutions given above are valid values of the radius. Namely, only positive real roots bigger than the Earth radius are allowed and also it must be satisfied that $-1 < \cos I = \Lambda_R/M_R < 1$. That is to say

$$|\Lambda_R| < M_R, \qquad \rho_{K\pm} > \rho_{Kt} \tag{7.6}$$

where $\rho_{Kt} = 6.371$ Km is the Earth radius. Furthermore, we are going to show that only one of the roots in (7.5) is valid. That is, the "positive" root given by

$$\rho_{K+} = \frac{1}{2}\left[\frac{M_K^2}{\mathcal{G}M_\odot m} + \sqrt{\frac{M_K^4}{\mathcal{G}M_\odot^2 m^2} + \frac{3}{2}(2C - A - B)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right)}\right]. \tag{7.7}$$

provides the radius of a circular orbit just by imposing $M_K^2 \geq \mathcal{G}M_\odot m\, \rho_{Kt}$, since the conditions (7.6) are satisfied. The remaining root in (7.5) is

$$\rho_{K-} = \frac{1}{2}\left[\frac{M_K^2}{\mathcal{G}M_\odot m} - \sqrt{\frac{M_K^4}{\mathcal{G}M_\odot^2 m^2} + \frac{3}{2}(2C - A - B)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right)}\right],$$

which is not a possible value for a radius since it does not satisfies $\rho_{K-} > \rho_{Kt}$. Let us check this claim in two steps. For the case $\alpha \geq 0$, where

$$\alpha = \frac{3}{2}(2C - A - B)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right)$$

we have that $\rho_{K-} < 0$ and the second condition in (7.6) is not satisfied. On the other hand, when $\alpha < 0$ we proceed assuming that $\rho_K > \rho_{Kt}$. Thus, it is clear that

$$\frac{M_K^2}{\mathcal{G}M_\odot m} > \rho_{Kt},$$

and considering the Taylor expansion of first order of the function

$$f(x) = \frac{M_K^2}{\mathcal{G}M_\odot m} - \sqrt{\frac{M_K^4}{\mathcal{G}M_\odot^2 m^2} + x},$$

together with the Lagrange remainder we obtain the following limit to the value of ρ_{K-}

$$0 < \frac{M_K^2}{\mathcal{G}M_\odot m} - \sqrt{\frac{M_K^4}{\mathcal{G}M_\odot^2 m^2} + \alpha} < \frac{-\alpha}{2a} + \frac{1}{8(a^2 + \alpha)^{3/2}} \ll 1,$$

where $a = M_K^2/\mathcal{G}M_\odot m$.

7.2 Analytic Integration

The system of canonical differential equations associated with the Hamiltonian defined by (7.2) reads as follows

$$
\begin{aligned}
\dot\rho_K &= P_K/m, & \dot\rho_R &= 0,\\
\dot P_K &= -\frac{\mathcal{G}M_\odot}{\rho_K^2} + \frac{M_K^2}{\rho_K^3 m}, & \dot P_R &= 0,\\
&\quad + \frac{3\mathcal{G}M_\odot}{8\rho_K^4}(2C-A-B)\left(1-3\frac{\Lambda_R^2}{M_R^2}\right), & &\\
\dot\lambda_K &= 0, & \dot\lambda_R &= -\frac{3\mathcal{G}M_\odot}{4M_R^2\rho_K^3}\Lambda_R(2C-A-B),\\
\dot\Lambda_K &= 0, & \dot\Lambda_R &= 0,\\
\dot\mu_K &= \frac{M_K}{\rho_K^2}, & \dot\mu_R &= M_R\left(\frac{\sin^2\nu_R}{A}+\frac{\cos^2\nu_R}{B}\right)\\
& & &\quad + \frac{3\mathcal{G}M_\odot}{4M_R^3\rho_K^3}\Lambda_R^2(2C-A-B),\\
\dot M_K &= 0, & \dot M_R &= 0,\\
\dot\nu_K &= 0, & \dot\nu_R &= N_R\left(\frac{1}{C}-\frac{\sin^2\nu_R}{A}-\frac{\cos^2\nu_R}{B}\right),\\
\dot N_K &= 0, & \dot N_R &= \frac{(A-B)}{2AB}(M_R^2-N_R^2)\sin 2\nu_R.
\end{aligned}
\tag{7.8}
$$

Remark 7.1 (On special types of trajectories. Critical Inclination)**.** *Without entering in this Chapter in what might be the concept of critical inclination in roto-orbital dynamics, we would like to point out that there are particular values of the integrals of our model, that we identify as what other authors have named by critical inclination, see for example [Chernousko, 1963, Lara et al., 2010]. By simply observation one can realize that there are two critical inclinations of the plane defined by the rotational angular momentum. More precisely, we recall that the angle between the rotational angular momentum vector and $\vec{s}_3$ the third vector in the spatial frame was named by I, see Fig. 4.4, we also have that $\cos I = \Lambda_R/M_R$, see (4.42). Thus, for the particular values of I making $\cos I = \pm\sqrt{1/3}$, we have that $V_r(\rho_K, M_R, \Lambda_R) = 0$, which implies that the orbital variables behave as the pure Kepler system, while the perturbation persists in the rotational part, see equations (7.8).*

This set of differential equations determine a strategy in order to solve them. There are three trivial subsystems (λ_K, Λ_K), (ν_K, N_K) and (ρ_R, P_R). Besides that, the system is endowed with three more integrals M_K, Λ_R and M_R and it is almost separable in the following way. By simply observation one can identify five ”almost” separate subsystems, namely (ρ_K, P_K),(μ_K, M_K), (λ_R, Λ_R), (μ_R, M_R) and (ν_R, N_R). Therefore, if we solve first systems (ν_R, N_R) and (ρ_K, P_K), we are just three cuadratures far from the solution by injecting $\rho_K(t)$ in $\dot{\lambda}_R$, $\dot{\mu}_K$ and $\dot{\mu}_R$.

Others authors follow an alternative treatment for intermediaries having a cubic perturbing term, which is based on the elimination of the parallax, rather than the linearizing change of the independent variable. This technique goes back to Deprit [Deprit, 1981], and recently has been successfully used to simplify the Hamiltonian associated to roto-translatory problems, see for example [Ferrer and Lara, 2012b, Gurfil and Lara, 2014].

• *Subsystem* (ρ_K, P_K): These integration is performed by following the approach given in [Ferrándiz, 1986], where a linearizing transformation procedure of the independent variable is found to solve perturbed Keplerian motions. Without lost of generality, in order to study the subsystem (ρ_K, P_K), we consider the partial Hamiltonian $\mathcal{H} = \mathcal{H}_K + V_r(\rho_K, \Lambda_R, M_R)$, rather than the Hamiltonian given in (7.2). It may be split as $\mathcal{H} = \frac{1}{2}P_K^2 + V(r)$, where $V(r)$ is the effective potential given by

$$V(\rho_K) = \frac{1}{2}\frac{M_K^2}{\rho_K^2} - \frac{\mathcal{G}M_\odot}{\rho_K} + \frac{\mathcal{G}M_\odot}{8\rho_K^3}(2C - B - A)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right). \tag{7.9}$$

Then, we fix the total energy value h and a new independent variable s is introduced by means of the differential relation

$$dt = g(\rho_K)ds, \tag{7.10}$$

$g(\rho_K)$ being strictly positive and continuously differentiable along trajectories. The equation for the radial variable, ρ_K, in the new independent variable is replaced by

$$\frac{d^2\rho_K}{ds^2} = \frac{\partial}{\partial r}\left[g^2(\rho_K)\big(h - V(\rho_K)\big)\right]. \tag{7.11}$$

Therefore, the new time s leads to a linear equation for r iff there exists constants c_1, c_2 and c_3 such that

$$g^2(\rho_K)\big(h - V(\rho_K)\big) = \frac{1}{2}c_1\rho_K^2 + c_2\rho_K + c_3. \tag{7.12}$$

Since we have that $h - V(\rho_K) = 1/2P_K^2$, all the orbits are confined to the region defined by $V(\rho_K) \leq h$. The equality holds for the values of r satisfying the cubic equation

$$h\,\rho_K^3 + \mu\,\rho_K^2 - \frac{M_K^2}{2}\rho_K + d = 0, \tag{7.13}$$

where $d = \mathcal{G}M_\odot/8(2C - B - A)\left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right)$. Thus, we are interested in the positive domain of the polynomial defining the above equation. We start studying the case of the critical inclination $\cos^2 I = \Lambda_R^2/M_R^2 = 1/3$, which eliminates the perturbation and lead to the unperturbed Keplerian motion. Under this assumption, condition (7.12) holds for $g(\rho_K) = \rho_K$ (Sundman's transformation), and the roots of the equation (7.13) become

$$\rho_K^0 = 0, \qquad \rho_K^1 = -\frac{\mu}{2h} + \sqrt{\frac{\mu^2}{4h^2} + \frac{\Theta^2}{2h}}, \qquad \rho_K^2 = -\frac{\mu}{2h} - \sqrt{\frac{\mu^2}{4h^2} + \frac{\Theta^2}{2h}},$$

Then, ρ_K^1 and ρ_K^2 are positive real numbers if $\mu^2 > 2|h|\Theta^2$ and they have astrodynamics sense for the case in which $\rho_K^1, \rho_K^2 > r_t$. Then, for the particular values

$$c_1 = 2h, \qquad c_2 = \mu, \qquad c_3 = \frac{-\Theta^2}{2}, \tag{7.14}$$

and assuming elliptic motion, $h < 0$, we obtain the well known solution

$$\begin{aligned} \rho_K &= a(1 - e\cos E),\\ \frac{d\theta}{ds} &= \frac{\theta}{a(1 - e\cos E)},\\ \frac{d\nu}{ds} &= 0, \end{aligned} \tag{7.15}$$

where $a = -c_2/c_1$, $1 - e^2 = 2c_1c_3/c_2^2$ and $E = \sqrt{-c_1}$ is the eccentric anomaly. Note that the differential equation corresponding to $d\theta/ds$ is equivalent to the law of areas when expressed in terms of the old independent variable t.

Finally we focus in the elliptic-type motion, that is the case in which $\rho_K \in (\rho_K^1, \rho_K^2)$, where ρ_K^1 and ρ_K^2 are two positive real roots of (7.13) bigger than r_t and satisfying that $V(\rho_K^1) = V(\rho_K^2) = h$ and $\partial V/\partial\rho_K \neq 0$ at ρ_K^1 and ρ_K^2. Then, the linearizing function is given by

$$g(\rho_K) = \rho_K^{3/2}(\rho_K + \alpha)^{-1/2}, \tag{7.16}$$

where

$$\begin{aligned} h &= \frac{c_1}{2}, & \frac{\Theta^2}{2} &= -c_3 - \alpha c_2,\\ \mu &= c_2 + \alpha, \frac{c_1}{2} & d &= -\alpha c_3. \end{aligned} \tag{7.17}$$

Thus, the solution for the radial distance in the new time variable s, has the form

$$\rho_K = a(1 - e\cos E), \tag{7.18}$$

where $E = s\sqrt{-2h}$, $2a = r_1 + r_2$ and $a^2(1-e^2) = r_1 r_2$.

• *Subsystem* (μ_K, M_K): The differential equation corresponding to μ_K reads as follows

$$\frac{d\mu_K}{ds} = g(\rho_K)\frac{\partial \mathcal{H}}{\partial M_K} = \frac{-\rho_K^{3/2}}{\sqrt{\rho_K + \alpha}}\frac{M_K}{\rho_K^2} = \frac{-M_K}{\sqrt{\rho_K(\rho_K+\alpha)}} \tag{7.19}$$

By substitution of (7.18) in (7.19) and fixing the time origin such as, for $s = 0$, $\rho_K = \rho_K^0 = a(1-e)$ and $\mu_K = 0$, we have

$$\mu_K = \int_0^s \frac{-M_K\, ds}{\sqrt{\rho_K(\rho_K+\alpha)}} = \frac{-M_K}{a^2}\int_0^s \frac{ds}{\sqrt{(1-e\cos E)(1+\alpha/a - e\cos E)}} \tag{7.20}$$

by means of the change o variable $x = \cos E$ we obtain

$$\mu_K = \frac{-M_K}{a^2}\int_{\cos E}^1 \frac{dx}{\sqrt{(1+\alpha/a - ex)(1-ex)(1-x)(x+1)}}. \tag{7.21}$$

The roots of the polynomial under the square root are $x_1 = (a+\alpha)/(ea)$, $x_2 = 1/e$, $y = 1$, $z = -1$. By using formula (253.00) from [Byrd and Friedman, 1971], and taking into account that $x_i > y > z$ and assuming $\alpha/a < 0$. The case $\alpha/a > 0$ is similar. We obtain the following expression for the quadrature (7.29)

$$\mu_K(s) = \frac{-M_K}{a^2}\sqrt{\frac{4e^2 a}{(a+\alpha-ea)(1+e)}}F(\varphi, k), \tag{7.22}$$

where

$$k^2 = \frac{2\alpha e}{(a+\alpha-ea)(1+e)}, \quad \sin\varphi = \mathrm{sn}(u,k) = \sqrt{\frac{(1+e)(1-\cos E)}{2(1-e\cos E)}}. \tag{7.23}$$

• *Subsystem* (ν_R, N_R): These equations does not contain any perturbed term. Thus the integration formulas for the pure free rigid body in Andoyer variables given by Molero [Molero, 2013] are valid

$$N_R(t) = R\,\mathrm{dn}(s\,t, k),$$

$$\sin\nu_R(t) = \frac{\mathrm{cn}(s\,t,k)}{\sqrt{1-n\,\mathrm{sn}^2(s\,t,k)}}, \quad \cos\nu_R(t) = \sqrt{\frac{B(C-A)}{A(C-B)}}\frac{\mathrm{sn}(s\,t,k)}{\sqrt{1-n\,\mathrm{sn}^2(s\,t,k)}}, \tag{7.24}$$

where,

$$
\begin{aligned}
k &= \frac{(B-A)(2hC-M^2)}{(C-B)(M^2-2hA)}, & s^2 &= \frac{(C-B)(M^2-2hA)}{ABC}, \\
R^2 &= \frac{C(M^2-2hA)}{C-A}, & n &= \frac{C(B-A)}{A(C-B)}.
\end{aligned}
\tag{7.25}
$$

• *Subsystem* (λ_R, Λ_R): Once we have the expression for the radius, the quadrature giving λ_R may be calculated. For this task we use the alternative independent variable s. Thus the differential equation corresponding to λ_R reads as follows

$$
\frac{d\lambda_R}{ds} = g(\rho_K)\frac{\partial \mathcal{H}}{\partial \Lambda_R} = \frac{-\rho_K^{3/2}}{\sqrt{\rho_K+\alpha}}\frac{\Delta_\lambda}{\rho_K^3} = \frac{-\Delta_\lambda}{\rho_K\sqrt{\rho_K(\rho_K+\alpha)}} \tag{7.26}
$$

where

$$
\Delta_\lambda = -\frac{3\mathcal{G}M_\odot}{4M_R^2}\Lambda_R(2C-A-B). \tag{7.27}
$$

By substitution of (7.18) in (7.26) and fixing the time origin such as, for $s = 0$, $\rho_K = \rho_K^0 = a(1-e)$ and $\lambda_R = 0$, we have

$$
\begin{aligned}
\lambda_R &= \int_0^s \frac{-\Delta_\lambda\, ds}{\rho_K\sqrt{\rho_K(\rho_K+\alpha)}} \\
&= \frac{-\Delta_\lambda}{a^2}\int_0^s \frac{ds}{(1-e\cos E)\sqrt{(1-e\cos E)(1+\alpha/a-e\cos E)}}
\end{aligned}
\tag{7.28}
$$

by means of the change o variable $x = \cos E$ we obtain

$$
\lambda_R = \frac{\Delta_\lambda}{a^2}\int_{\cos E}^1 \frac{dx}{(1-ex)\sqrt{(1+\alpha/a-ex)(1-ex)(1-x)(x+1)}}. \tag{7.29}
$$

The roots of the polynomial under the square root are $x_1 = (a+\alpha)/(ea)$, $x_2 = 1/e$, $y = 1$, $z = -1$. By using formula (253.39) from [Byrd and Friedman, 1971], and taking into account that $x_i > y > z$ and assuming $\alpha/a < 0$. The case $\alpha/a > 0$ is similar. We obtain the following expression for the quadrature (7.29)

$$
\lambda_R = \frac{\Delta_\lambda}{a^2}\frac{1-e}{\sqrt{1-e^2+\alpha(1+e)/a}}\int_0^u (1-\alpha_2^2\mathrm{sn}^2 u)du \tag{7.30}
$$

where,

$$
k^2 = \frac{2\alpha e}{(a+\alpha-ea)(1+e)}, \quad \alpha_2^2 = \frac{2e}{1-e}, \quad \sin\varphi = \mathrm{sn}(u,k) = \sqrt{\frac{(1+e)(1-\cos E)}{2(1-e\cos E)}}. \tag{7.31}
$$

Finally, from [Byrd and Friedman, 1971] formula 331.0, we get the expression for λ_R

$$\lambda_R(s) = \frac{\Delta_\lambda(1-e)}{a^2\sqrt{1-e^2+\alpha(1+e)/a}} \left[F(\varphi,k) + \frac{(a+\alpha-ea)(1+e)}{\alpha(1-e)}\big(E(\varphi,k)-F(\varphi,k)\big)\right], \tag{7.32}$$

where the functions $F(\varphi, k)$ and $E(\varphi, k)$ are the normal elliptic integrals of the first and second kind respectively.

• *Subsystem* (μ_R, M_R): Considering the differential equation corresponding to μ_R in the system (7.8) we are lead to the following quadratures

$$\mu_R(t) = I_1(t) + I_2(t), \tag{7.33}$$

where

$$I_1(t) = \int_0^t \frac{3\mathcal{G}M_\odot}{4M_R^3\rho_K^3(\mathrm{t})}\Lambda_R^2(2C-A-B)dt, \tag{7.34}$$

and

$$I_2(t) = \int_0^t M_R\left(\frac{\sin^2\nu_R}{A} + \frac{\cos^2\nu_R}{B}\right)\mathrm{d}t, \tag{7.35}$$

Proceeding with $I_1(t)$ in the same way than in the subsystem (λ_R, Λ_R), we change the independent variable t to s and we obtain the following quadrature where

$$I_1(s) = \int_0^s \frac{-\Delta_\mu\, ds}{\rho_K\sqrt{\rho_K(\rho_K+\alpha)}}, \tag{7.36}$$

with

$$\Delta_\mu = \frac{3\mathcal{G}M_\odot}{4M_R^3}\Lambda_R^2(2C-A-B). \tag{7.37}$$

Therefore, taking into account the expression for λ_R given in (7.28), $I_1(s)$ is obtained replacing Δ_λ by Δ_μ in (7.32). Then, we focus in the second quadrature $I_2(s)$. After some easy computations we have

$$I_2(t) = M_R\left[\frac{t}{A} + \frac{A-B}{AB}\int_0^t \cos^2\nu_R(t)\,\mathrm{d}t\right], \tag{7.38}$$

and recalling the formula (7.24) I_2 turns to

$$I_2(t) = M_R\left[\frac{t}{A} + \frac{(A-B)(C-A)}{A^2(C-B)}\int_0^t \frac{\mathrm{sn}^2(s\,t,k)}{1-n\,\mathrm{sn}^2(s\,t,k)}\,\mathrm{d}t\right]. \tag{7.39}$$

Finally, using formula (337.01) from [Byrd and Friedman, 1971], we get that

$$I_2(t) = M_R\left[\frac{t}{A} + \frac{(A-B)(C-A}{n^2A^2(C-B)}\big(\Pi(\varphi,n^2,k) - F(\varphi,k)\big)\right], \tag{7.40}$$

where $\sin\varphi = \mathrm{sn}(u,k)$ and $\Pi(\varphi,n^2,k)$, $F(\varphi,k)$ are the normal elliptic integrals of the third and second kind respectively.

7.3 Connecting with the Quartic Model

From now on, we make use of the "almost separability" feature of the system (7.1) and we study the orbital and rotational subsystems separately. In this Section, we connect the intermediary with the family given in (4.1) in two stages. In oder words, the particular "separable coupled" structure of system (7.8) allows us to split the Hamiltonian (7.1) in two parts

$$\begin{aligned}\mathcal{H}_O &= \mathcal{H}_K + V_r = \frac{1}{2m}\left(P_K^2 + \frac{M_K^2}{\rho_K^2}\right) - \frac{\mathcal{G}M_\odot}{\rho_K} + \frac{\beta\left(1-3\frac{\Lambda_R^2}{M_R^2}\right)}{\rho_K^3},\\ \mathcal{H}_R &= \frac{1}{2}\left(\frac{\sin^2\nu_R}{A} + \frac{\cos^2\nu_R}{B}\right)(M_R^2 - N_R^2) + \frac{1}{2C}N_R^2,\end{aligned} \tag{7.41}$$

where

$$\beta = \frac{\mathcal{G}M_\odot}{8}(2C - B - A). \tag{7.42}$$

The orbital part $\mathcal{H}_O = \mathcal{H}_K + V_r$ contains the perturbing potential and the pure rotational part $\mathcal{H}_R$ is the Hamiltonian of the free rigid body in Andoyer angles, see [Molero, 2013]. Nevertheless, for the case of the orbital part, the quartic Hamiltonian will be replaced by its square root following the approach of Moser in [Moser, 1970].

Proposition 7.3. *The subsystems defined by $\mathcal{H}_O$ and $\mathcal{H}_R$, provided a suitable set of parameters a_i, properly regularized, are related to a constrained flow of a perturbed double copy of the family of Hamiltonians (4.1).*

Proof. Let us consider the square root of the following quartic realization with parameters $a_1 = \cdots = a_5 = 0$ and $a_6 = 1/4m^2$. Thus, $H_\omega = \sqrt{\mathcal{F}_a} = \omega_6$ and let us perturb this Hamiltonian by means of

$$\mathcal{H}_\omega = \frac{1}{2m}\big(\omega\|\mathbf{q}\|^2 + \|\mathbf{Q}\|^2 + \frac{4\alpha m}{\|\mathbf{q}\|^4}\big) = h_\omega, \tag{7.43}$$

where α is a small parameter depending on the physical parameters and the rotational integrals

$$\alpha = \beta \left(1 - 3\frac{\Lambda_R^2}{M_R^2}\right). \tag{7.44}$$

The connection between the harmonic oscillator and the Kepler system is made through a regularization of the independent variable. The same applies for this perturbed case, namely we have the following change of independent variable

$$dt = \frac{1}{4m\|\mathbf{q}\|^2} ds, \tag{7.45}$$

it leads us to the following Hamiltonian $\mathcal{K} = 1/(4m\|\mathbf{q}\|^2)(\mathcal{H}_\omega - h_\omega)$ for each fixed energy level $\mathcal{K} = 0$. Next, we rewrite this Hamiltonian in Projective Andoyer variables with $F(\rho) = \sqrt{\rho}$

$$\mathcal{K} = \frac{1}{2m}\left(P^2 + \frac{M^2}{\rho^2}\right) - \frac{h_\omega/4m}{\rho} + \frac{\alpha}{\rho^3} + \frac{\omega}{8m}. \tag{7.46}$$

Therefore, by choosing the energy level of the oscillator as

$$h_\omega = 4m\mathcal{G}M_\odot \tag{7.47}$$

and redefining the Hamiltonian $\mathcal{K}$ we obtain that the Hamiltonian of the 4-D perturbed isotropic oscillator is given by

$$\tilde{\mathcal{K}} = \frac{1}{2m}\left(P^2 + \frac{M^2}{\rho^2}\right) - \frac{h_\omega/4m}{\rho} + \frac{\alpha}{\rho^3} = \mathcal{H}_O, \tag{7.48}$$

in the manifold $\tilde{\mathcal{K}} = -\dfrac{\omega}{8m}$. This Hamiltonian, when constrained to $N = 0$, leads to the Kepler system given by the orbital part in (7.41).

Therefore, for $m\sqrt{f_a}/4 = \mathcal{G}M_\odot$ and $\delta/(8m) = \beta\,(1 - 3\Lambda_R^2/M_R^2)$ we obtain that $\mathcal{K} = \mathcal{H}_O$ defined in (7.41). In addition, taking into account that $\tilde{\mathcal{K}} = \mathcal{H}_O = h_O$, we obtain that

$$\omega = -8mh_O > 0 \tag{7.49}$$

where the energy level h_O is assumed to be negative. In other words, we only consider closed Keplerian orbital.

For the rotational part we have a free rigid body (FRB). Thus, by using the Projective Andoyer variables and choosing the parameters

$$a_1 = a_5 = a_6 = 0, \quad a_2 = \frac{1}{8A}, \; a_3 = \frac{1}{8B}, \; a_4 = \frac{1}{8C}. \tag{7.50}$$

we readily obtain the Hamiltonian $\mathcal{H}_R$. This given quartic family realization, together with the restrictions $\|\mathbf{q}\| = c$ and $\langle \|\mathbf{q}\|, \|\mathbf{Q}\| \rangle = 0$, leads to the (FRB) system, see Section 4.6.1. □

7.4 Analysis of the $\mathbb{T}^2$-Relative Equilibria

Rather than in the circular intermediary case, the triaxiality does not play a role in the discussion of this model. As a consequence, the analysis of the relative equilibria becomes simpler. From the expression of the system of differential equations (7.8) it is clear that there is not absolute equilibria of the system. More precisely, the equations corresponding with $\dot{\mu}_K$ and $\dot{\mu}_R$ do not vanish. Then, we focus in the $M_K M_R$-reduced system, which removes the angles μ_K and μ_M and we look for $\mathbb{T}^2$-relative equilibria. For this purpose, we proceed to express the Hamiltonian $\mathcal{H}_r$ in the $M_K M_R$-reduced space given by $(\mathbb{S}^2_M \times \mathbb{S}^2_M \times \mathcal{C}^2)^K \times (\mathbb{S}^2_M \times \mathbb{S}^2_M \times \mathcal{C}^2)^R$, where a generic point will be represented in the following fashion

$$p = (G^R, M^R, W^R, G^K, M^K, W^K), \tag{7.51}$$

and

$$\begin{aligned} G^K &= (G^K_1, G^K_2, G^K_3), \quad M^K = (M^K_1, M^K_2, M^K_3), \quad W^K = (W^K_1, W^K_2, W^K_3),\\ G^R &= (G^R_1, G^R_2, G^R_3), \quad M^R = (M^R_1, M^R_2, M^R_3), \quad W^R = (W^R_1, W^R_2, W^R_3). \end{aligned} \tag{7.52}$$

are the rescaling of the ω_i for $i = 1, \ldots, 9$ given in (5.47). These variables satisfies the following relations that define the reduced space

$$\begin{aligned} {G^K_1}^2 + {G^K_2}^2 + {G^K_3}^2 = {M^K_1}^2 + {M^K_2}^2 + {M^K_3}^2 = {W^K_3}^2 - {W^K_2}^2 - {W^K_1}^2 = M^2_K\\ {G^R_1}^2 + {G^R_2}^2 + {G^R_3}^2 = {M^R_1}^2 + {M^R_2}^2 + {M^R_3}^2 = {W^R_3}^2 - {W^R_2}^2 - {W^R_1}^2 = M^2_R. \end{aligned} \tag{7.53}$$

Finally, we obtain the following expression for the Hamiltonian $\mathcal{H}_r$

$$\begin{aligned} \mathcal{H}_r &= \mathcal{H}_O(G^R, W^K; s) + \mathcal{H}_R(M^R; t)\\ &= (\omega+1)W^K_3 + (\omega-1)W^K_2 + \frac{m\beta\left(1 - 3\frac{{G^R_1}^2}{M^2_R}\right)}{(W^K_2 + W^K_3)^2} + \frac{1}{2}\Big(\frac{{M^R_1}^2}{A} + \frac{{M^R_2}^2}{B} + \frac{{M^R_3}^2}{C}\Big). \end{aligned} \tag{7.54}$$

Observe that in the above reduced Hamiltonian we have carried out a partial regularization, *i. e.*, the orbital part and the rotational part are not expressed in the same independent variable. In this light, the system of differential equations in the reduced space is obtained after some computations and taking into account that the derivative corresponding with d/dt is denoted by a dot and the derivative corresponding with

d/ds is denoted by a tilde. Thus, the differential equations read as follows

$$
\begin{aligned}
\dot{G}_1^R &= \frac{-12\,\beta}{{M_R}^2\left({W_3}^K+{W_2}^K\right)^{3/2}}G_2^R G_3^R, & \dot{M}_1^R &= \frac{B-C}{BC}M_2^R M_3^R, & \dot{W}_1^R &= 0,\\
\dot{G}_2^R &= \frac{12\,\beta}{{M_R}^2\left({W_3}^K+{W_2}^K\right)^{3/2}}G_1^R G_3^R, & \dot{M}_2^R &= \frac{C-A}{AC}M_1^R M_3^R, & \dot{W}_2^R &= 0,\\
\dot{G}_3^R &= 0, & \dot{M}_3^R &= \frac{A-B}{BA}M_1^R M_2^R, & \dot{W}_3^R &= 0,
\end{aligned}
$$

$$
\begin{aligned}
G_1^{K\prime} &= M_1^{K\prime} = 0, & W_1^{K\prime} &= (1-\omega)W_3^K-(1+\omega)W_2^K+\frac{2\alpha\,m}{(W_3^K+W_2^K)^2},\\
G_2^{K\prime} &= M_2^{K\prime} = 0, & W_2^{K\prime} &= W_1^K\left(1+\omega-\frac{2\alpha\,m}{(W_3^K+W_2^K)^3}\right),\\
G_3^{K\prime} &= M_3^{K\prime} = 0, & W_3^{K\prime} &= W_1^K\left(1-\omega+\frac{2\alpha\,m}{(W_3^K+W_2^K)^3}\right),
\end{aligned}
\tag{7.55}
$$

where we recall that $\alpha=\beta\left(1-3G_1^{R2}/M_R^2\right)$. This system is made up of six ”almost separable” subsystems. By simply observation of the equations we see that there are three trivial subsystems and the remaining three ones are separated except for the fact that they share some integrals. The strategy to solve this system is to integrate first the equations corresponding to the subsystems (M_1^R, M_2^R, M_3^R) and (W_1^K, W_2^K, W_3^K). Then, we obtain M_i^R and W_i^K as functions valuated in t and s respectively. Finally, taking into account the relation $s = s(t)$, the subsystem is integrated by injecting $W_2^K(s(t))$ and $W_3^K(s(t))$ in the equations $G_1^{K\prime}$ and $G_2^{K\prime}$.

The particular way in which this system is expressed shows how the perturbation acts. On one hand, the rotational part (G_1^R, G_2^R, G_3^R) represents the body coordinates of the total angular momentum, which are no longer fixed. Instead, they move in circumferences with center in the G_3^R-axis. On the other hand, in the orbital part the angular momentum remain unchanging over time. However, the perturbation affects now to the shape of the orbit, which is related to the variables W_1^K, W_5^K and W_6^K.

Next we study the equilibria for the three non-trivial subsystems given above in (7.55), which correspond to (G_1^R, G_2^R, G_3^R), (M_1^R, M_2^R, M_3^R) and (W_1^K, W_2^K, W_3^K). These partial analysis are summarized up in Theorem 7.4, where we distinguish between isolated equilibria and a one-parameter family of equilibria. The proof is a consequence of the subsequent partial studies of the preceding subsystems.

Theorem 7.4 (Relative equilibria.)**.** *The set of 18-tuples given in Table 7.4 corresponds to equilibria of the system defined by the equations (7.55).*

Isolated Equilibria
$E_1 = (0, 0, \pm M_R,\ \pm M_R, 0, 0,\ W^R, G^K, M^K\, 0, (W_2^K)', (W_3^K)')$
$E_2 = (0, 0, \pm M_R,\ 0, \pm M_R, 0,\ W^R, G^K, M^K\, 0, (W_2^K)', (W_3^K)')$
$E_3 = (0, 0, \pm M_R,\ 0, 0, \pm M_R,\ W^R, G^K, M^K\, 0, (W_2^K)', (W_3^K)')$
One-Parametric Families of Equilibria
$E_4 = (G_1^R, G_2^R, 0,\ \pm M_R, 0, 0,\ W^R, G^K, M^K\, 0, (W_2^K)', (W_3^K)')$
$E_5 = (G_1^R, G_2^R, 0,\ 0, \pm M_R, 0,\ W^R, G^K, M^K\, 0, (W_2^K)', (W_3^K)')$
$E_6 = (G_1^R, G_2^R, 0,\ 0, 0, \pm M_R,\ W^R, G^K, M^K\, 0, (W_2^K)', (W_3^K)')$

Table 7.1: We have used the notation given in (7.51) and (7.57) for W^R, G^K, M^K and $((W_2^K)', (W_3^K)')$ respectively. Note that E_i, for $i = 1, 2, 3$ denote sets of four isolated equilibria and E_i, for $i = 4, 5, 6$ denote one-parametric families of equilibria.

• **Subsystem** (W_1^K, W_2^K, W_3^K)**:** The evolution of ρ^K and P^K in Projective Andoyer variables is related to this subsystem. In this sense, since (G_1^K, G_2^K, G_3^K) and (M_1^K, M_2^K, M_3^K) are constant, the orbital motion is encapsulated in the reduced system given by the reduced variables (W_1^K, W_2^K, W_3^K). The unperturbed case $\alpha = 0$, leads to the pure Kepler system, to which we devote special attention. After that, we explore the way in which this remarkable case is perturbed.

The existence of equilibria in this subsystem is related to the circular orbits and is characterized by the conditions $W_1^K = 0$ and $W_1^{K\prime} = 0$, see (7.55). Taking into account these conditions and the defining relation among the variables of the reduced space, we lead to the following cubic equation characterizing the equilibria

$$4\alpha\, W_3{}^3 + \frac{M_K^4}{m}\, W_3^2 - \frac{M_K{}^2\,(3\,\omega + 1)\,\alpha}{\omega}\, W_3 - \frac{\alpha^2\, m}{\omega} - \frac{M_K^6(\omega+1)^2\alpha}{4\,\omega\, m} = 0. \quad (7.56)$$

Generally, the roots of the cubic equation will be denoted by $(W_3^K)'$ and the equilibria of the subsystem are given by

$$(0, (W_2^K)', (W_3^K)') \quad (7.57)$$

where $(W_2^K)'^2 + (W_3^K)'^2 = M_K^2$.

When $\alpha > 0$ the rule of Descartes, see [Mignotte, 1992] page 197, shows that the cubic equation has only one positive root. For $\alpha < 0$, the same argument shows that there is one negative root and two complex or two positive real roots. However, further analysis should be done to clarify the negative case. Furthermore, the existence of bifurcations depends on the multiplicity of the roots of the cubic equation, it leads to the study of the zeros of the discriminant given by

$$4096\,\omega^2\alpha\,m\left(-27\,\omega\,\alpha^2m^2+M_K^6\right)\left(-4\,\alpha^2m^2+\omega\,M_K^6+\omega^2M_K^6\right)^2. \tag{7.58}$$

This task, together with the study of the characterization of real roots in the case $\alpha < 0$, will be tackled in [Crespo et al., 2015], which is already in progress. Therefore, in what follows, we focus in the cases $\alpha = 0$ and $\alpha > 0$.

$*$ *Case* $\alpha = 0$. *The Kepler like system:* Given a triaxial body, the critical inclination $\cos I = \Lambda_R/M_R = \pm\sqrt{1/3}$ makes $\alpha = 0$ and the cubic perturbation vanishes. Thus, the orbital part of the system becomes Keplerian with Hamiltonian

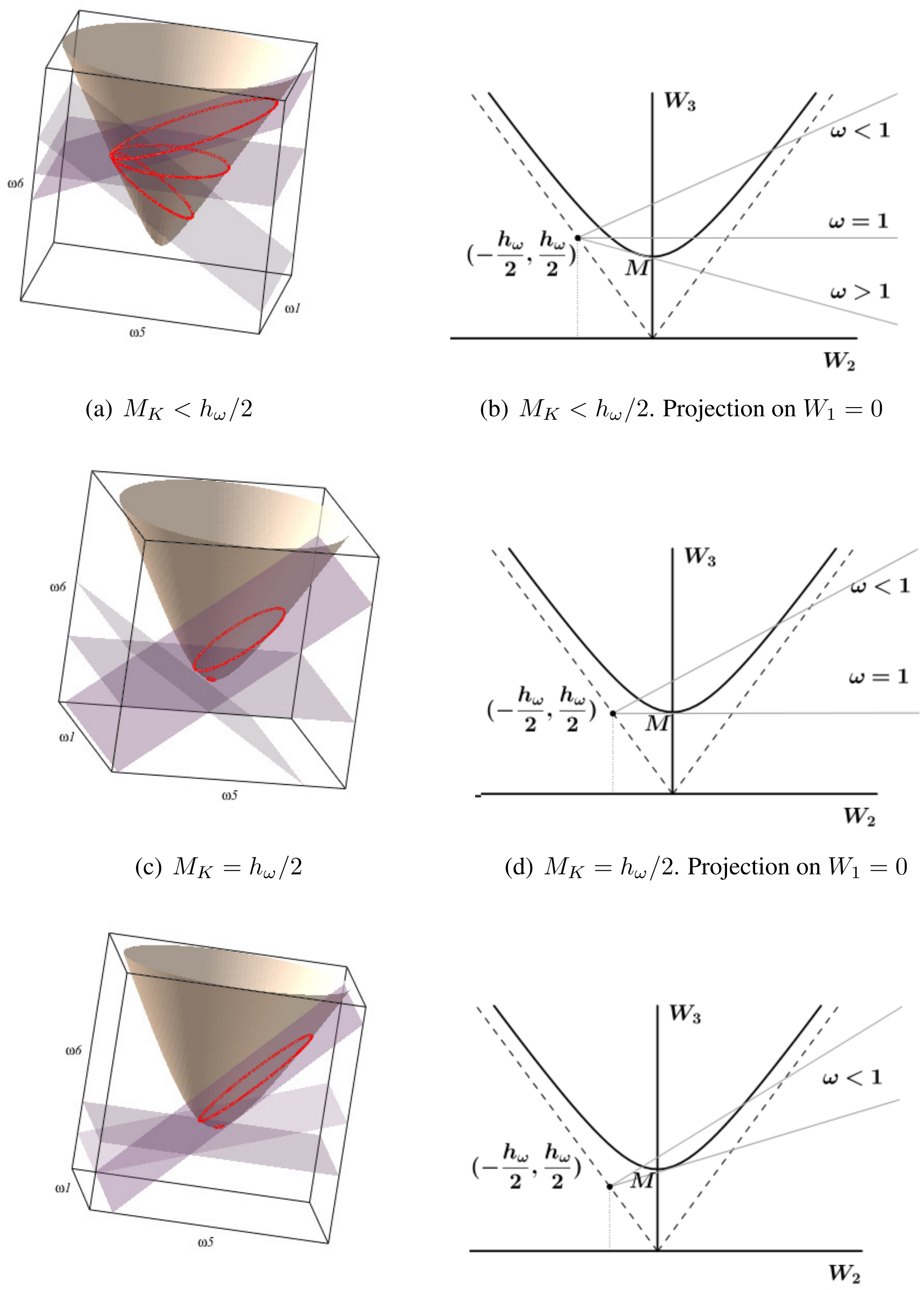
$$\mathcal{H}_O = \frac{1}{2m}\left(P_K^2+\frac{M_K^2}{\rho_K 2}\right)-\frac{\mathcal{G}M_\odot}{\rho_K} = h_O. \tag{7.59}$$

Observe that the physical parameters $\mathcal{G}$, $M_\odot$ and m are fixed all along and recalling the connection of the Kepler and the oscillator systems given in Proposition 7.3, we also have that the energy h_O should be fixed in order to proceed with the regularization that leads us to the reduced Hamiltonian of the oscillator

$$\mathcal{H}_\omega = (\omega+1)W_3^K + (\omega-1)W_2^K = h_\omega. \tag{7.60}$$

Therefore, the energy of the oscillator is fixed by the gravitational constant and the masses of the bodies involved $h_\omega = 4\mathcal{G}M_\odot m$ and the frequency is related to the Keplerian energy $\omega = -8mh_O$. As a consequence, h_ω is fixed for each problem and it does not make sense to study the intersection of the reduced space with the level sets of the oscillator. Instead, variation on the Keplerian energy are related to changes in the frequency ω. In Fig. 7.2 we show the corresponding reduced orbits for several Kepler energy levels, *i.e.* the influence of the parameter ω. In this figure it is shown that there is always a tangent contact between the upper sheet of the hyperboloid and the Hamiltonian surface. Moreover, these contacts presents three possibilities; the case $h_\omega/2 = M_K^2$ implies that the tangent contact occurs at $W_2^K = 0$, for $h_\omega/2 > M_K^2$ and $h_\omega/2 < M_K^2$ we have the contact at $W_2^K < 0$ and $W_2^K > 0$ respectively. Those tangencies are related to equilibria of the system (W_1^K, W_2^K, W_3^K) and their localization are obtained by means of the cubic (7.56). For the case $\alpha = 0$ it becomes the quadratic equation with its corresponding roots

$$M_K^4\,W_3^2 - \frac{M_K^6(\omega+1)^2}{4\,\omega} = 0, \quad \Rightarrow \quad W_3 = \pm\frac{M_K(\omega+1)}{2\sqrt{\omega}}. \tag{7.61}$$



(a) $M_K < h_\omega/2$

(b) $M_K < h_\omega/2$. Projection on $W_1 = 0$

(c) $M_K = h_\omega/2$

(d) $M_K = h_\omega/2$. Projection on $W_1 = 0$

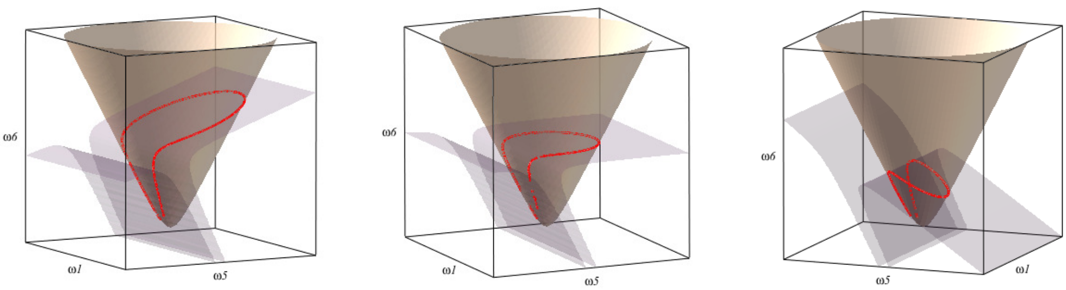
(e) $M_K > h_\omega/2$

(f) $M_K > h_\omega/2$. Projection on $W_1 = 0$

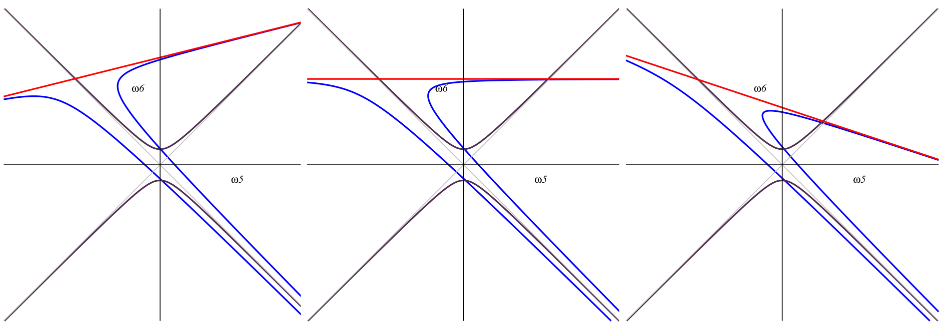
Figure 7.2: Evolution of the solution curves with the value of the parameter ω. Details for the cases $M_K < h_\omega/2$, $M_K = h_\omega/2$ and $M_K > h_\omega/2$. Observe that all the Hamiltonians are given by a bundle of planes through the line $(W_1, -h_\omega/2, h_\omega/2)$.

Therefore we have two equilibria in this case, they correspond to the tangential intersection of the Hamiltonian plane with the sheets of the hyperboloid defined by the relation among the reduced variables. Nevertheless, the negative root is excluded since our problem has the restriction $W_3 \geq M_K$. Observe that the positive root is guaranteed to satisfy this condition since $(\omega+1)/(2\sqrt{\omega}) \geq 1$ for all $\omega > 0$.

$*$ *Case* $\alpha > 0$. *Perturbed Kepler motion:* For this case we have only one positive root of the equation (7.56) and then only one equilibria, which is characterized by the tangential intersection of the reduced orbital Hamiltonian given below and the hyperbolic reduced space



(a) $\omega < 1$ (b) $\omega = 1$ (c) $\omega > 1$



(d) $\omega < 1, W_1 = 0$ (e) $\omega = 1, W_1 = 0$ (f) $\omega > 1, W_1 = 0$

Figure 7.3: The role of the orbital energy in the shape of the reduced orbits. The red straight line corresponds to the level set of $\mathcal{H}_\omega$ for $\alpha = 0$. For $\alpha > 0$ this straight line breaks into the curve plotted in blue color, which approximates asymptotically to the line $\omega_6 = -\omega_5$.

$$\mathcal{H}_O = (\omega+1)W_3^K + (\omega-1)W_2^K + \frac{m\beta\left(1 - 3\frac{G_1^{R\,2}}{M_R^2}\right)}{(W_2^K + W_3^K)^2}. \tag{7.62}$$

We illustrate the way in which the Hamiltonian intersects with the hyperboloid in Fig. 7.4

• **Subsystem** (G_1^R, G_2^R, G_3^R)**:** From the way in which equations are given, equilibria are obtained by simply observation. Thus, we have that the equator $G_3^R = 0$ is a circumference of equilibria and also the poles $(0, 0, \pm M)$, see Fig. 7.4

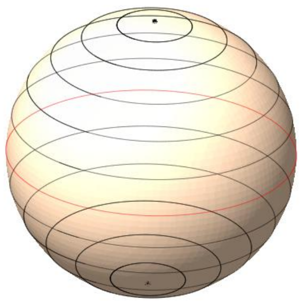


Figure 7.4: G_R**-equilibria:** One-parametric family of equilibria in the intersection with $G_3^R = 0$ and two isolated equilibria located in the poles. Red color is used to denote the set of equilibria.

• **Subsystem** (M_1^R, M_2^R, M_3^R)**:** The coordinates M_i^R for $i = 1, 2, 3$ correspond with the rotational angular momentum in the spatial frame. This subsystem corresponds to the reduced unperturbed free rigid body. Therefore, the equilibria are given by the terns $(\pm M, 0, 0)$, $(0, \pm M, 0)$ and $(0, 0, \pm M)$, see Fig. 7.5.

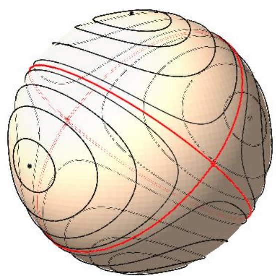


Figure 7.5: M_R**-equilibria:** The equilibria in the M_R subsystem correspond to the classical picture of the rigid body. Therefore, in the triaxial case, we only have isolated equilibria in the intersections of the coordinate axes and the sphere.

Appendices

Appendix A

The Two Body Problem

In this Appendix we present the unperturbed two body system and its reduction by the polar coordinates to the Kepler system. Our aim is just to provide some notation and refresh basic facts. Further details may be found in any book of astrodynamics or celestial mechanics, see for example [Abad, 2012].

A.1 Formulation of the 2-Body Problem

Applying Newton's second law and law of gravity in an inertial system of reference $R_0 = \{O; \vec{i}, \vec{j}, \vec{k}\}$ we obtain

$$\begin{aligned} F_1 =& m_1 \ddot{q}_1 = \dot{p}_1 = \frac{\mathcal{G} m_1 m_2 (q_2 - q_1)}{\|q_1 - q_2\|^3} \\ F_2 =& m_2 \ddot{q}_2 = \dot{p}_2 = \frac{\mathcal{G} m_1 m_2 (q_1 - q_2)}{\|q_1 - q_2\|^3}, \end{aligned} \tag{A.1}$$

where $q_i \in \mathbb{R}^3$ for $i = 0, 1$ are the position vectors of the punctual masses m_i referred to R_0, $p_i \in \mathbb{R}^3$ are the associated momenta and $\mathcal{G} = 6.67384 \times 10^{-11}\ m^3/s^2\,kg$ is the universal gravitational constant. On the other hand, taking into account that by simply definition of the momenta $p_i = m_i \dot{q}_i$, we obtain a system of twelve ordinary first-order differential equations given by

$$\begin{aligned}
\dot{q}_1 &= \frac{p_1}{m_1}\\
\dot{q}_2 &= \frac{p_2}{m_2}\\
\dot{p}_1 &= -\frac{\mathcal{G}m_1m_2(q_1-q_2)}{\|q_1-q_2\|^3}\\
\dot{p}_2 &= -\frac{\mathcal{G}m_1m_2(q_2-q_1)}{\|q_1-q_2\|^3}
\end{aligned} \tag{A.2}$$

Those equations can be rendered as a Hamiltonian system $(T^*\mathbb{R}^6 - \{0\}, \omega, \mathcal{H}_O)$ by considering $(q_1, q_2, p_1, p_2) \in T^*\mathbb{R}^6 - \{0\}$, with the standard symplectic form $\omega = dq \wedge dp$ and the Hamiltonian function

$$\mathcal{H}_O = \frac{\|p_1\|^2}{2m_1} + \frac{\|p_2\|^2}{2m_2} - \frac{\mathcal{G}m_1m_2}{\|q_1-q_2\|} \tag{A.3}$$

this system is coupled and nonlinear, luckily it is provided with several integrals.

Proposition A.1 (The Classical Integrals)**.** *The total linear momentum, the total angular momentum and the energy $\mathcal{H}_O$ are integrals of the system $(T^*\mathbb{R}^6 - \{0\}, \omega, \mathcal{H}_O)$. Moreover, the center of mass moves with uniform rectilinear motion.*

Proof. Let $L = p_1 + p_2$ the total linear momentum, the fact that it is an integral can easily be derived from (A.1), which shows that $\dot{L} = 0$. Then, the total linear momentum is constant and the center of mass of the system defined by $C = m_1q_1 + m_2p_2$ moves with uniform rectilinear motion, that is $C(t) = Lt + C_0$, where $L = (l_1, l_2, l_3)$ and $C_0 = (c_1, c_2, c_3)$ are constant depending on the initial conditions of the system. On the other hand, let the total angular momentum $\mathbf{A} = q_1 \times p_1 + q_2 \times p_2$. Then

$$\begin{aligned}
\frac{d\mathbf{A}}{dt} &= \dot{q}_1 \times p_1 + q_1 \times \dot{p}_1 + \dot{q}_2 \times p_2 + q_2 \times \dot{p}_2\\
&= \dot{q}_1 \times m_1\dot{q}_1 + \dot{q}_2 \times m_2\dot{q}_2 + \frac{\mathcal{G}m_1m_2\, q_1 \times (q_2-q_1)}{\|q_1-q_2\|^3} + \frac{\mathcal{G}m_1m_2\, q_2 \times (q_1-q_2)}{\|q_2-q_1\|^3}\\
&= \vec{0} + \vec{0} + \frac{\mathcal{G}m_1m_2\, q_1 \times q_2}{\|q_1-q_2\|^3} + \frac{\mathcal{G}m_1m_2\, q_2 \times q_1}{\|q_2-q_1\|^3}\\
&= \vec{0}.
\end{aligned} \tag{A.4}$$

□

As a result, we have that the three components of L, C_0 and A plus the energy $\mathcal{H}_O$ are the classical 10 integrals of the 2-body problem. Next we use them to reduce the system.

A.1.1 First reduction. Jacobi coordinates

Modern techniques of reduction take advance of the invariance of the system under the group of translations. The action of this group has an associated momentum map. Thus a geometric reduction in the sense of Marsden-Weinstein-Meyer may be performed, see for example [Abraham and Marsden, 1985, Holm, 2008a, Singer, 2001]. An alternative approach to carry out the reduction is by means of a change of variables, which incorporates the conserved quantities among the variables. As a first step in the reduction process, the Jacobi coordinates incorporates the integrals corresponding with L and C_0 through the following transformation

$$T : (T^*\mathbb{R}^6, \omega) \longrightarrow (T^*\mathbb{R}^6, \omega), \; (q, p) \to (Q, P),$$

where $(q, p) = (q_1, q_2, p_1, p_2)$ and $(Q, P) = (Q_1, Q_2, P_1, P_2)$ are given by

$$\begin{aligned} Q_1 &= q_1 - q_2, \qquad & Q_2 &= \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2}, \\ P_1 &= \frac{m_2 p_1 - m_1 p_2}{m_1 + m_2}, \qquad & P_2 &= p_1 + p_2 \end{aligned} \tag{A.5}$$

Proposition A.2. *The above linear map is a symplectic transformation.*

Proof. A straightforward computation shows that

$$p_1 dq_1 + p_2 dq_2 = P_1 dQ_1 + P_2 dQ_2.$$

□

The expression of the Hamiltonian in the Jacobi coordinates is

$$\mathcal{H}_O = \frac{1}{2}\left[(\frac{1}{m_1} + \frac{1}{m_2})P_1^2 + P_2^2\right] - \frac{\mathcal{G} m_1 m_2}{\|Q_1\|}. \tag{A.6}$$

and the new system of differential equations becomes into the form

$$\begin{aligned} \dot{Q}_1 &= \frac{m_1 m_2}{m_1 + m_2} P_1, & \dot{Q}_2 &= P_2, \\ \dot{P}_1 &= -\frac{2\mathcal{G} m_1 m_2 Q_1}{\|Q_1\|^3}, & \dot{P}_2 &= \vec{0}, \end{aligned} \tag{A.7}$$

that is, a 3 DOF system for Q_1 and P_1 given by $(T^*\mathbb{R}^3 - \{0\}, \omega, \mathcal{H}_O)$. Since the system is separated and trivially integrated for the variables Q_2 and P_2 In what follows we focus on the variables Q_1 and P_1 and drop the subindices. Then, we express $\mathcal{H}_O$ in the more compact form

$$\mathcal{H}_O = \frac{\|P_1\|^2}{2m} - \frac{\mathcal{G} M_\odot}{\|Q_1\|}, \tag{A.8}$$

where $m = \dfrac{m_1 m_2}{m_1 + m_2}$ and $\mathcal{G} M_\odot = \mathcal{G} m_1 m_2$. Then the differential system is finally written as

$$\begin{aligned} \dot{Q} &= mP, \\ \dot{P} &= -\frac{2\mathcal{G} M_\odot}{\|Q\|^3} Q, \end{aligned} \tag{A.9}$$

This system is named as the Kepler system, it may be interpreted as an approximation of the 2-body problem when one body is assumed to be much more massive than the other, therefore fixed in the origin. Note that at this point we have not already take advance of all the integrals provided in Proposition A.1, that this system may also be reduced by means of the axial symmetry of the total angular momentum A. That is to say, as before, $\mathbf{A} = Q \times P$ angular momentum is constant.

Proposition A.3 (Classical integral)**.** *The angular momentum* $\mathbf{A} = Q \times P$ *is a conserved quantity of the system* $(T^*\mathbb{R}^3 - \{0\}, \omega, \mathcal{H}_O)$.

Proof. Analogous to the proof of Proposition A.1.

□

A.1.2 Second reduction. Polar-Nodal variables

Again a reduction process can be carried out by taking into account that the rotation action of the group $SO(3)$ of rotations of three-space acts on $T^*\mathbb{R}^3 - \{0\}$. This is a

Hamiltonian action and has a momentum map associated. Nevertheless, following the same approach than in the first reduction, further simplification of the system is done by choosing suitable coordinates. The second reduction is performed by using polar-nodal variables [Deprit, 1981]. Then we recall to the reader the geometry associated to those variables, see Fig. A.1.

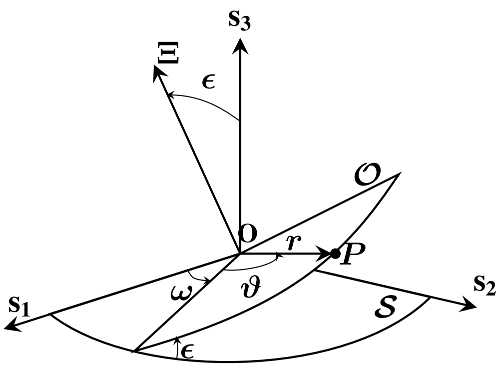


Figure A.1: Space $\mathcal{S}$ and nodal $\mathcal{N}$ frames and the angles relating them.

Polar-nodal coordinates are very well known in celestial mechanics, they have been used in the first of the twenty century by Hill and Whittaker, and also they have been exploited in the artificial satellite theory along the sixties. Since they are based in the existence of the node, that is, the intersection of the invariant plane of movement with one of the coordinate planes, three charts are needed. Each chart corresponds to the case in which the node is obtained as the intersection with the plane s_1s_2, s_1s_3 or s_2s_3. As it is shown in Fig. A.1, here we give the detailed derivation of the polar-nodal variables referred to the plane $s_1 - s_2$.

The configuration space in the first reduced space is $\mathbb{R}^3 - \{0\}$, where we consider a spatial reference system given by $\{\vec{0}; s_1, s_2, s_3\}$. The invariance of the total angular momentum $\mathbf{A} = Q \times P$ define an invariable plane called the orbital plane. We exclude here the case of rectilinear orbits, that is, we restric to the region of the phase space where $\mathbf{A} \neq \vec{0}$, then we can define $\mathbf{n}$ as follows

$$\mathbf{A} = \Xi\mathbf{n}, \quad A > 0, \quad \|n\| = 1,$$

the new vector $\mathbf{n}$ is normal and also perpendicular to the invariant plane of motion, which inclination ϵ respect to the s_1s_2-plane (the equatorial plane) is given by

$$s_3 \cdot \mathbf{n} = \cos\epsilon, \quad 0 < \epsilon < \pi.$$

As the line of the ascending node is defined as the interception of the orbital plane and the horizontal plane s_1s_2 and therefore it has to be perpendicular to both the angular momentum and the s_3 axis, we have that ,calling l the vector identifying the position of the ascending node

$$s_3 \times \mathbf{n} = l \sin \epsilon,$$

where

$$s_3 \times \mathbf{A} = l = (Q_1P_3 - Q_3P_1, Q_2P_3 - Q_3P_2, 0),$$

the longitude of the ascending node is the angle defined by

$$l = \cos\omega s_1 + \sin\omega s_2,$$

$\omega \in (0, 2\pi)$ is therefore found by:

$$\cos\omega = s_1 \cdot \frac{l}{\|l\|} = \frac{Q_1P_3 - Q_3P_1}{\sqrt{(Q_1P_3 - Q_3P_1)^2 + (Q_2P_3 - Q_3P_2)^2}},$$

$$\sin\omega = \|s_2 \times \frac{l}{\|l\|}\| = \frac{Q_2P_3 - Q_3P_2}{\sqrt{(Q_1P_3 - Q_3P_1)^2 + (Q_2P_3 - Q_3P_2)^2}}.$$

We define also the latitude argument, $\vartheta \in (0, 2\pi)$, by means of the vectors **u**, which is a unitary vector pointing in the direction of P and l and the relation between them

$$\mathbf{u} = l\cos\vartheta - \mathbf{n} \times l \sin\vartheta.$$

Finally the relation between the Cartesian and polar-nodal variables reads as follows

$$\begin{aligned} Q &= r\mathbf{u}, \\ P &= R\mathbf{u} + \frac{\Xi}{r}\mathbf{n} \times \mathbf{u}. \end{aligned} \tag{A.10}$$

Bibliography

[Abad, 2012] Abad, A. (2012). *Astrodinámica*. Buboc publishing sl, 1st edition.

[Abraham and Marsden, 1985] Abraham, R. and Marsden, J. (1985). *Foundations of Mechanics*. American Mathematical Society, 2nd edition.

[Abraham et al., 1988] Abraham, R., Marsden, J., and Ratiu, T. (1988). *Manifolds, Tensor Analysis, and Applications*. Applied Mathematical Sciences, Springer-Verlag, 2nd edition.

[Altmann, 1986] Altmann, S. (1986). *Rotations, Quaternions, and Double Groups*. Clarendon Press: Oxford.

[Andoyer, 1923] Andoyer, M. H. (1923). *Cours de Mécanique Céleste*. Gauthier-Villars, Paris.

[Arnold, 1989] Arnold, V. (1989). *Mathematical Methods of Classical Mechanics*. Springer-Verlag, New York.

[Arnold and Givental, 1973] Arnold, V. and Givental, A. (1973). *Dynamical Systems IV, Symplectic Geometry and its Applications*. Encyclopaedia of Mathematical Sciences, Springer.

[Arnold et al., 1993] Arnold, V., Kozlov, V., and Neishtadt, A. (1993). *Dynamical Systems III, Mathematical Aspects of Classical and Celestial Mechanics*. Encyclopaedia of Mathematical Sciences, Springer, Vol. 3, 2nd edition.

[Arribas, 1989] Arribas, M. (1989). *Sobre la dinámica de actitud de satélites artificiales*. PhD, Universidad de Zaragoza.

[Berndt, 2001] Berndt, R. (2001). *An Introduction to Symplectic Geometry*. AMS Providence, Rhode Island.

[Betsch and Siebert, 2009] Betsch, P. and Siebert, R. (2009). Rigid body dinamics in terms of quaternions: Hamiltonian formulation and conserving numerical integration. *Int. J. Numer. Meth. Engng*, 79:444–473.

[Buono et al., 2005] Buono, L., Laurent-Polz, F., and Montaldi, J. (2005). *Symmetric Hamiltonian Bifurcations, in Geometric Mechanics and Symmetry: The Peyresq Lectures*. J. Montaldi and T. S. Ratiu, eds, Cambridge University Press, Berlin.

[Byrd and Friedman, 1971] Byrd, P. F. and Friedman, M. D. (1971). *Handbook of Elliptic Integrals for Engineers and Scientists*. Springer Verlag, Berlin.

[Chernousko, 1963] Chernousko, F. L. (1963). On the motion of a satellite about its center of mass under the action of gravitational moments. *Journal of Applied Mathematics and Mechanics*, 27(3):708–722.

[Cid and Ferrer, 1997] Cid, R. and Ferrer, S. (1997). *Geodesia Geométrica, Física y por Satélites*. Instituto Geográfico Nacional, Ministerio de Fomento, Madrid.

[Crespo et al., 2015] Crespo, F., Díaz, G., and Ferrer, S. (2015). An elliptic type gravity-gradient intermediary in attitude dynamics. A case of double critical inclination. *In progress*.

[Crespo et al., 2009] Crespo, F., Díaz, G., Ferrer, S., and Lara, M. (2009). Poisson and symplectic reductions of 4-dof isotropic oscillators. The van der Waals system as benchmark. *Proceedings of the Workshop on Computer Algebra and Differential Equations CADE 2009. Accepted for publication*.

[Crespo and Ferrer, 2014a] Crespo, F. and Ferrer, S. (2014a). Energy-momentum mapping of a quartic polynomial model. *In preparation*.

[Crespo and Ferrer, 2014b] Crespo, F. and Ferrer, S. (2014b). On the extended Euler system and the jacobi elliptic functions. *Submitted to Journal of Geometric Mechanics*.

[Crespo et al., 2014] Crespo, F., Ferrer, S., Molero, F. J., and van der Meer, J. C. (2014). A gravity-gradient intermediary in attitude dynamics. Relative equilibria and bifurcations. *Proceedings of 10th AIMS Conference on Dynamical Systems, Differential Equations and Applications. 7-12 Julio 2014. Madrid, Spain*.

[Cushman and Bates, 1997] Cushman, R. and Bates, L. (1997). *Global Aspects of Classical Integrable Systems*. Birkhäuser Verlag, Basel.

[Cushman and Duistermaat, 1997] Cushman, R. and Duistermaat, J. (1997). A characterization of the Ligon-Schaaf regularization map. *Comm. Pure and Appl. Math.*, 50(8):773–787.

[Dellnitz et al., 1992] Dellnitz, M., Melbourne, I., and Marsden, J. (1992). Generic bifurcation of Hamiltonian vector fields with symmetry. *Nonlinearity*, 5:979–996.

[Deprit, 1967] Deprit, A. (1967). Free rotation of a rigid body studied in the phase plane. *American Journal of Physics*, 35:424–428.

[Deprit, 1969] Deprit, A. (1969). Canonical transformations depending on a small parameter. *Celest. Mech.*, 1:12–29.

[Deprit, 1981] Deprit, A. (1981). The elimination of the parallax in satellite theory. *Celestial Mechanics*, 24(2):111–153.

[Deprit, 1991] Deprit, A. (1991). The Lissajous transformation I. Basics. *Celest. Mech.*, 51:203–226.

[Egea, 2007] Egea, J. (2007). *Sistemas Hamiltonianos en resonancia 1:1:1:1. Reducciones toroidales y Bifurcaciones de Hopf.* Tesis Doctoral, Universidad de Murcia, 180 pp.

[Fassò, 2005] Fassò, F. (2005). Superintegrable Hamiltonian systems: geometry and applications. *Acta Appl. Math.*, 87:93–121.

[Ferrándiz, 1986] Ferrándiz, J. M. (1986). Linearization in special cases of perturbed Keplerian motions. *Celestial Mechanics*, 39:23–31.

[Ferrándiz and Sansaturio, 1989] Ferrándiz, J. M. and Sansaturio, M. E. (1989). Elimination of the nodes when the satellite is a non spherical rigid body. *Celestial Mechanics*, 46:307–320.

[Ferrer, 2010] Ferrer, S. (2010). The projective Andoyer transformation and the connection between the 4-d isotropic oscillator and Kepler systems. *arXiv:1011.3000v1 [nlin.SI]*.

[Ferrer and Crespo, 2015] Ferrer, S. and Crespo, F. (2015). Parametric quartic Hamiltonian model. A unified treatment of classic integrable systems. *Journal of Geometric Mechanics*, 7(1).

[Ferrer and Lara, 2012a] Ferrer, S. and Lara, M. (2012a). On roto-translatory motion: Reductions and radial intermediaries. *The Journal of the Astronautical Sciences*, 59(1-2):22–40.

[Ferrer and Lara, 2012b] Ferrer, S. and Lara, M. (2012b). On roto-translatory motion: Reductions and radial intermediaries. *The Journal of the Astronautical Sciences,*, 59:21–39.

[Ferrer and Molero, 2014a] Ferrer, S. and Molero, F. J. (2014a). Andoyer variables and phases in the free rigid body. *Journal of Geometric Mechanics*, 6(1):25–37.

[Ferrer and Molero, 2014b] Ferrer, S. and Molero, F. J. (2014b). Intermediaries for gravity-gradient attitude dynamics I. Action-angle variables. IAA-AAS-DyCoSS2-14-04-05, Roma, 23-26 March 2014. *To appear in Advances in the Astronautical Sciences*.

[Ferrer et al., 2014] Ferrer, S., Molero, F. J., and Crespo, F. (2014). Intermediaries for gravity-gradient attitude dynamics. The role of the angular momentum. *Submitted to Celestial Mechanics and Dynamical Astronomy*.

[Fukushima, 2008] Fukushima, T. (2008). Simple, regular, and efficient numerical integration of rotational motion. *The Astronomical Journal*, 135:2298–2322.

[Gaeta and Rodriguez, 2014] Gaeta, G. and Rodriguez, M. (2014). Symmetry and quaternionic integrable systems. *Journal of Geometry and Physics. http://dx.doi.org/10.1016/j.geomphys.2014.05.019*.

[Galin, 1982] Galin, D. (1982). Versal deformations of linear Hamiltonian systems. *AMS Transl.*, 2(118):1–12.

[Gluck et al., 1986] Gluck, H., Warner, F., and Wolfgang, Z. (1986). The geometry of the Hopf fibrations. *L'Enseignement Mathématique*, 32:173–198.

[Goldstein et al., 2002] Goldstein, H., Poole, C., and Safko, J. (2002). *Classical Mechanics*. Addison Wesley, New York, Third edition.

[Grafarend and Kuhnel, 2011] Grafarend, E. and Kuhnel, W. (2011). A minimal atlas for the rotation group $so(3)$. *International Journal of Geomathematics*, 2(1):113–122.

[Gurfil et al., 2007] Gurfil, P., Elipe, A., Tangren, W., and Efroimsky, M. (2007). The Serret-Andoyer formalism in rigid-body dynamics: I. Symmetries and perturbations. *Regular and Chaotic Dynamics*, 12(4):389–425.

[Gurfil and Lara, 2014] Gurfil, P. and Lara, M. (2014). Satellite onboard orbit propagation using Deprit's radial intermediary. *Celestial Mechanics and Dynamical Astronomy*, 120:217–232.

[Hale, 1969] Hale, J. (1969). *Ordinary Differential Equations*. Dover ed.,Wiley-Interscience, New York.

[Hamilton, 1844] Hamilton, W. (1844). On quaternions, or on a new system of imaginaries in algebra. *Philos. Mag.*, 25:489–495.

[Heard, 2006] Heard, W. (2006). *Rigid Body Mechanics*. WILEY-VCH Verlag GmbH & Co. KGaA, Mathematics, Physics and Applications.

[Heckman and Laat, 2012] Heckman, G. and Laat, T. (2012). On the regularization of the Kepler problem. *J. of Symplectic Geometry*, 01:1463–473.

[Hirsch, 1994] Hirsch, M. (1994). *Differential Topology*. Springer-Verlag, New York.

[Holm, 2008a] Holm, D. (2008a). *Geometric Mechanics, Part I: Dynamics and Symmetry*. World Scientific Publishing Co. Pte. Ltd., Singapore.

[Holm, 2008b] Holm, D. (2008b). *Geometric Mechanics, Part II: Rotating, Translating and Rolling*. World Scientific Publishing Co. Pte. Ltd., Singapore.

[Holm and Marsden, 1991] Holm, D. and Marsden, J. (1991). The rotor and the pendulum. *In Symplectic Geometry and Mathematical Physics. Actes du colloque en l'honneur de Jean-Marie Souriau, Ed by P. Donato et al.. Prog. in Math. Birkhäuser*, 99:189–203.

[Hopf, 1931] Hopf, H. (1931). Ber die abbildungen der dreidimensinalen sphere auf die kugelfiche. *Math. Ann.*, 104:637–665.

[Iwai and Tarama, 2010] Iwai, T. and Tarama, D. (2010). Classical and quantum dynamics for an extended free rigid body. *Differential Geometry and its Applications*, 28:501–517.

[Johnson, 1970] Johnson, C. (1970). Positive definite matrices. *The American Mathematical Monthly*, 77(3).

[Kinoshita, 1972] Kinoshita, H. (1972). Stationary motions of a triaxial body and their stabilities. *Publications of the Astronomical Society of Japan*, 24:409–417.

[Kuipers, 1999] Kuipers, J. (1999). *Quaternions and rotation sequences*. Princeton University text, Princeton, New Jersey.

[Kustaanheimo and Stiefel, 1965] Kustaanheimo, P. and Stiefel, E. (1965). Perturbation theory of Kepler motion based on spinor regularization. *J. Reine Angew. Math.*, 218:204–219.

[Lanczos, 1970] Lanczos, C. (1970). *The Variational Principles of Mechanics*. Dover.

[Lara et al., 2010] Lara, M., Fukushima, T., and Ferrer, S. (2010). First-order rotation solution of an oblate rigid body under torque of a perturber in circular orbit. *Astronomy and Astrophysics*, 519.

[Lara and Gurfil, 2012] Lara, M. and Gurfil, P. (2012). Integrable approximation of j_2-perturbed relative orbits. *Celestial Mechanics and Dynamical Astronomy*, 114:229–254.

[Lawden, 1989] Lawden, D. F. (1989). *Elliptic Functions and Applications*. Springer-Verlag, New York.

[Levi-Civita, 1920] Levi-Civita, T. (1920). Sur la régularisation du problème des trois corps. *Acta Mathematica*, 42:99–144.

[Ligon and Schaaf, 1976] Ligon, T. and Schaaf, M. (1976). On the global symmetry of the classical Kepler problem. *Rep. Math. Phys.*, 9:281–300.

[MacCullagh, 1840] MacCullagh, J. (1840). On the rotation of a solid body. *Proceedings of the Royal Irish Academy*, 2:520–545.

[Marsden, 1992] Marsden, J. E. (1992). *Lectures on Mechanics*, volume 174. London Mathematical Society, Lecture Note Series, Cambridge University Press.

[Marsden and Ratiu, 1999] Marsden, J. E. and Ratiu, T. S. (1999). *Introduction to* M*echanics and* S*ymmetry*. Springer-Verlag New York, Inc., 2nd edition.

[Meyer, 2001] Meyer, K. (2001). Jacobi elliptic functions from a dynamical system point of view. *The American Mathematical Monthly*, 108(8):729–737.

[Meyer et al., 2009] Meyer, K., Hall, G. R., and Offin, D. (2009). *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem, 2nd Ed*, volume 90. Applied Mathematical Sciences, Springer, New York.

[Mignotte, 1992] Mignotte, M. (1992). *Mathematics for Computer Algebra*. Springer, New York.

[Molero, 2013] Molero, F. J. (2013). *On the perturbed fast rotating satellite. Closed form vs. series expansion approach*. PhD Thesis, Universidad de Murcia.

[Molero et al., 2014] Molero, F. J., Ferrer, S., Crespo, F., and van der Meer, J. C. (2014). A gravity-gradient intermediary in attitude dynamics. Relative equilibria and complete reduction. *Submitted to Celestial Mechanics and Dynamical Astronomy*.

[Moser, 1970] Moser, J. (1970). Regularization of Kepler's problem and the averaging method on a manifold. *Communication on pure and applied mathematics*, XXIII:609–636.

[Moser and Zehnder, 2005] Moser, J. and Zehnder, E. J. (2005). *Notes on Dynamical Systems*. AMS.

[Nambu, 1973] Nambu, Y. (1973). Generalized Hamiltonian mechanics. *Phys. Rev.*, 7:2405–2412.

[Ortega, 1998] Ortega, J. P. (1998). *Symmetry, reduction and stability in Hamiltonian systems*. PhD, University of California, Santa Cruz.

[Ortega, 2014] Ortega, J. P. (2014). Symmetric Hamiltonian systems. Stability methods and applications. *Summer school ICMAT. La Cristalera*.

[Ortega and Ratiu, 2004] Ortega, J. P. and Ratiu, T. S. (2004). *Momentum maps and Hamiltonian reduction*. Birkhuser.

[Palacián and Yanguas, 2000a] Palacián, J. and Yanguas, P. (2000a). Reduction of polynomial Hamiltonians by the construction of formal integrals. *Nonlinearity*, 13:1021–1054.

[Palacián and Yanguas, 2000b] Palacián, J. and Yanguas, P. (2000b). Reduction of polynomial planar Hamiltonians with quadratic unperturbed part. *SIAM Review*, 42(4):671–691.

[Poincaré, 1899] Poincaré, H. (1892-1899). *Les Méthodes nouvelles de la Mécanique céleste*. American Institute of Physics.

[Puta, 1993] Puta, M. (1993). On the dynamics of the rigid body with two torques. *C. R. Acad.Sci. Paris*, 317:377–380.

[Puta, 1997] Puta, M. (1997). Stability and control in spacecraft dynamics. *Journal of Lie Theory*, 7:269–278.

[Puta and Casu, 1999] Puta, M. and Casu, I. (1999). Geometrical aspects in the rigid body dynamics with three quadratic controls. *Geometry, Integrability and Quantization*, pages 1–10.

[Saha, 2009] Saha, P. (2009). Interpreting the Kustaanheimo-Stiefel transform in gravitational dynamics. *Mon. Not. R. Astron. Soc.*, 400:228–231.

[San-Juan J.F., 2012] San-Juan J.F., López L.M., R. L. (2012). Higher-order analytical attitude propagation of an oblate rigid body under gravity-gradient torque. http://dx.doi.org/10.1155/2012/123138. *Mathematical Problems in Engineering*.

[Scheeres, 2012] Scheeres, D. (2012). *Orbital Motion in Strongly Perturbed Environments: Application to Asteroid, Comet and Planetary Satellite Orbiters*. Jointly published with Praxis Publishing, UK.

[Shapukow, 1989] Shapukow, B. N. (1989). *Grupos y algebras de Lie, en ejercicios y problemas*. URSS, Moscu.

[Sidorenko, 2014] Sidorenko, V. (2014). Rotational dynamics, in "Celestial Mechanics", edited by A. Celletti. *Encyclopedia of Life Support Systems (EOLSS). To appear*.

[Singer, 2001] Singer, S. G. (2001). *Symmetry in mechanics, a gentle, modern introduction*. Birkhäuser.

[Urbantke, 2003] Urbantke, H. (2003). The Hopf fibration seven times in physics. *Journal of geometry and physics*, 46:125–150.

[Vallejo, 1995] Vallejo, M. (1995). *Series de Fourier de funciones elípticas. Aplicación a la precesión terrestre*. PhD, Real Instituto y Observatorio de la Armada y Universidad de Zaragoza, Cádiz, España.

[van der Meer, 1985] van der Meer, J. C. (1985). *The Hamiltonian Hopf Bifurcation*. Springer, LNM 1160.

[van der Meer et al., 2014] van der Meer, J. C., Crespo, F., and Ferrer, S. (2014). Generalized Hopf fibration and geometric SO(3) reduction, the 4-DOF harmonic oscillator. *CASA-Report 14-13*.

[van der Meer and Cushman, 1986] van der Meer, J. C. and Cushman, R. (1986). Constrained normalization of Hamiltonian systems and perturbed Keplerian motion. *Journal of Applied Mathematics and Physics*, 37:402–424.

[Vereshchagin et al., 2010] Vereshchagin, M., Maciejewski, A. J., and Goździewski, K. (2010). Relative equilibria in the unrestricted problem of a sphere and symmetric rigid body. *Monthly Notices of the Royal Astronomical Society*, 403:848–858.

[Virgili et al., 2014] Virgili, B. B., Lemmens, S., and Krag, H. (2014). Investigation on Envisat attitude motion. *e.Deorbit Workshop*.

[Waldvogel, 2006] Waldvogel, J. (2006). Quaternions and the perturbed Kepler problem. *Celest. Mech. Dynamical Astron.*, 95:201–212.

[Waldvogel, 2008] Waldvogel, J. (2008). Quaternions for regularizing celestial mechanics: the right way. *Celest. Mech. Dynamical Astron.*, 102:149–162.

[Weinstein, 1983] Weinstein, A. (1983). The local structure of Poisson manifolds. *J. Differential Geom.*, 18:523–557.

[Weyl, 1939] Weyl, H. (1939). *The classical groups. Their invariants and representations*. Princeton university press.

[Whittaker, 1937] Whittaker, E. T. (1937). *A treatise on the analytical dynamics of particles&rigid bodies*. Cambridge Mathematical Library.

[Whittaker and Watson, 1940] Whittaker, E. T. and Watson, G. N. (1940). *A course of modern analysis*. Cambridge Mathematical Library.

[Wilson, 2010] Wilson, C. (2010). *The Hill-Brown Theory of the Moon's Motion*. Springer.

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Y así, del mucho leer y del poco dormir,
se le secó el celebro de manera que vino
a perder el juicio.

MIGUEL DE CERVANTES SAAVEDRA.