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Departament de Matemàtica Aplicada III

Doctoral Thesis

**On the consistency of hysteresis
models**

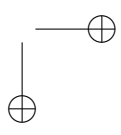
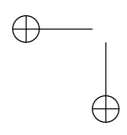
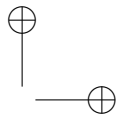
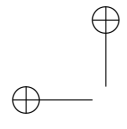
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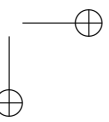
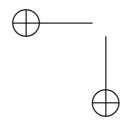
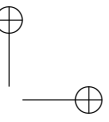
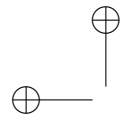


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Abstract

Hysteresis is a nonlinear behavior encountered in a wide variety of processes including biology, optics, electronics, ferroelectricity, magnetism, mechanics, structures, among other areas. The presence of hysteresis can be detected experimentally in physical systems by doing the following: Consider as input of the hysteresis system the signal $u_\gamma = U_0 \sin(t/\gamma)$ with amplitude U_0 and frequency $1/\gamma$. Let y_γ be the corresponding system output. When y_γ is plotted against u_γ , we get a curve G_γ parametrized with time. When this experiment is repeated with different frequencies, and when $\gamma \rightarrow \infty$, it is observed that the sequence of graphs G_γ converges to some curve G^* [15]. It is also observed that G^* converges asymptotically to a periodic orbit G° which is commonly called hysteresis loop. If the curve G_γ is independent of γ , then the hysteretic system is rate-independent like the ones described in [66]. Otherwise, we get G_γ depends on γ for the rate-dependent hysteresis systems like the ones described in [42, 37, 32]. For linear systems G° is a line segment, and for hysteresis systems G° is a non-trivial curve [15].

The detailed modeling of hysteresis systems using the laws of Physics is an arduous task, and the obtained models are often too complex to be used in applications. For this reason, alternative models of these complex systems have been proposed [98, 23, 62, 53, 69]. These models do not come, in general, from the detailed analysis of the physical behavior of the systems with hysteresis.

Instead, they combine some physical understanding of the system along with some kind of black-box modeling.

In the current literature, these models are mostly used within the following black-box approach: given a set of experimental input-output data, how to adjust the hysteresis model so that the output of the model matches the experimental data? The use of system identification techniques is one practical way to perform this task. Once an identification method has been applied to tune the hysteresis model, the resulting model is considered as a “good” approximation of the true hysteresis when the error between the experimental data and the output of the model is small enough. Then this model is used to study the behavior of the true hysteresis under different excitations. By doing this, it is important to consider the following remark. It may happen that a hysteresis model presents a good match with the experimental real data for a specific input, but does not necessarily keep significant physical properties which are inherent to the real data, independently of the exciting input. In the current literature, this issue has been considered in [58, 76] regarding the passivity/dissipativity of Duhem model and in [53] regarding the stability of the Bouc-Wen model.

In this thesis, we investigate the conditions under which the Duhem model and the LuGre model are consistent with the hysteresis behavior. The concept of consistency is formalized in [52] where a general class of hysteresis operators is considered. The class of operators that are considered in [52] are the causal ones, with the additional condition that a constant input leads to a constant output. For these classes of systems, consistency has been defined formally. Consider for example the Duhem model described by $\dot{x} = f(x, u)g(\dot{u})$, where u is the input and x is the state or the output [78] along with the sequence of inputs $u_\gamma(t) = u(t/\gamma), t \geq 0, \gamma > 0$, and the corresponding sequence of outputs x_γ with $\dot{x}_\gamma = f(x_\gamma, u_\gamma)g(\dot{u}_\gamma)$. Consistency means that the sequence of functions $t \rightarrow x_\gamma(\gamma t)$ converges uniformly when $\gamma \rightarrow \infty$. In this thesis, we seek necessary conditions and sufficient ones for this uniform convergence to hold for both Duhem model and LuGre friction model.

Thesis layout

This thesis focuses on the consistency of the LuGre friction model and the generalized Duhem model with hysteresis and is organized in four chapters. In each chapter, all obtained results are illustrated by numerical simulations.

Chapter 1 presents background results that are needed throughout the thesis and is divided into two sections. In **Section 1.1**, some mathematical results are given. **Section 1.2** summarizes the findings obtained by [52], where a new understanding of hysteresis is introduced and the notions of consistency and strong consistency are formulated and explained.

Chapter 2 investigates the strong consistency of the LuGre friction model and consists of two sections. **Section 2.1** presents a literature review of the friction and the LuGre model. The consistency and strong consistency are studied in **Section 2.2**.

Chapter 3 focuses on the consistency of the generalized Duhem model and is divided into four sections. **Section 3.1** gives an introduction for the model. **Section 3.2** presents the problem statement. A classification of this model is introduced in **Section 3.3**. Depending on that classification, necessary conditions and sufficient ones are derived for the consistency to hold. This is done

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in **Sections 3.4** and **3.5**.

Chapter 4 presents the main conclusions.

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Background results

The present chapter gives some mathematical results that are needed in this work.

1.1 Mathematical preliminaries

A real number x is said positive when $x > 0$, negative when $x < 0$, non-positive when $x \leq 0$, and non-negative when $x \geq 0$. A function $h : \mathbb{R} \rightarrow \mathbb{R}$ is said increasing when $t_1 < t_2 \Rightarrow h(t_1) < h(t_2)$, decreasing when $t_1 < t_2 \Rightarrow h(t_1) > h(t_2)$, non-increasing when $t_1 < t_2 \Rightarrow h(t_1) \geq h(t_2)$, and non-decreasing when $t_1 < t_2 \Rightarrow h(t_1) \leq h(t_2)$.

The Lebesgue measure on \mathbb{R} is denoted μ . A subset of \mathbb{R} is said measurable when it is Lebesgue measurable. Consider a function $p : I \subset \mathbb{R}_+ = [0, \infty) \rightarrow \mathbb{R}^m$ where I is some interval and m a positive integer; the function p is said measurable when p is (M, B) -measurable where B is the class of Borel sets of \mathbb{R}^m and M is the class of measurable sets of \mathbb{R}_+ . For a measurable function $p : I \subset \mathbb{R}_+ \rightarrow \mathbb{R}^m$, $\|p\|_{\infty, I}$ denotes the essential supremum of the function $|p|$ on I where $|\cdot|$ is the Euclidean norm on \mathbb{R}^m . When $I = \mathbb{R}_+$, it is denoted simply $\|p\|_{\infty}$.

Definition 1.1.1. [88, p. 24] For any measurable sets $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, we define $L^1(X, Y)$ to be the collection of measurable functions $f : X \rightarrow Y$ for

which

$$\int_X |f| d\mu < \infty.$$

The members of $L^1(X, Y)$ are called Lebesgue integrable or summable functions from X to Y . Now, we present the Dominated Lebesgue Theorem:

Theorem 1.1.1. [88, p. 26] *Let X be a measurable subset of \mathbb{R}_+ . Assume that we have a sequence of measurable functions $f_n : X \rightarrow \mathbb{R}^m$; for any positive integer n , such that there exists a function $f : X \rightarrow \mathbb{R}^m$ with*

$$\lim_{n \rightarrow \infty} f_n(\vartheta) = f(\vartheta), \forall \vartheta \in X.$$

If there exists a function $h \in L^1(X, \mathbb{R}_+)$ such that

$$|f_n(\vartheta)| \leq h(\vartheta), \forall \vartheta \in X, \forall \text{ positive integer } n,$$

then $f \in L^1(X, \mathbb{R}^m)$ and

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu.$$

Definition 1.1.2. [87, p. 104] *A function $h : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^m$ (I is an interval) is said to be absolutely continuous on I if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\sum_{i=1}^k |h(\vartheta'_i) - h(\vartheta_i)| < \varepsilon,$$

for every k non-overlapping finite collection of $\{(\vartheta_i, \vartheta'_i)\}_{i=1}^k$ of intervals which are subsets of I with

$$\sum_{i=1}^k |\vartheta'_i - \vartheta_i| < \delta.$$

Each absolutely continuous function is continuous and differentiable almost everywhere on its domain [87].

1.1 Mathematical preliminaries

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Consider the system

$$\dot{x}(t) = h(t, x(t)), \quad (1.1)$$

$$x(0) = x_0, \quad (1.2)$$

where $x(t) \in \mathbb{R}^m$ is the state vector at instant t , m is some positive integer, and $h : \mathbb{R}_+ \times \mathbb{R}^m \rightarrow \mathbb{R}^m$.

Definition 1.1.3. [39, p. 4] *Let $I \subseteq \mathbb{R}_+$ be any interval. A function $x : I \rightarrow \mathbb{R}^m$ is said to be Carathéodory solution for (1.1) if*

- *The function x is absolutely continuous on each closed interval that is subset of I ,*
- *The function x satisfies (1.1) almost everywhere on I .*

The following theorems state the existence and uniqueness of Carathéodory solutions for (1.1)-(1.2).

Theorem 1.1.2. [39, p. 4] *Assume that for each positive numbers a and b , we have*

- *For almost all $t \in [0, a]$, the function $h(t, \cdot)$ is continuous on the set $\{\alpha \in \mathbb{R}^m / |\alpha - x_0| < b\}$.*
- *For each $\alpha \in \{\alpha \in \mathbb{R}^m / |\alpha - x_0| < b\}$, the function $h(\cdot, \alpha)$ is measurable on $[0, a]$.*
- *There exists a function $m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that is summable on $[0, a]$ such that $|h(t, \alpha)| \leq m(t)$ for all $t \in [0, a]$ and for all $\alpha \in \mathbb{R}^m$ that satisfy $|\alpha - x_0| < b$.*

Then there exists some $d > 0$ such that the system (1.1)-(1.2) admits a Carathéodory solution that is defined on $[0, d]$.

Definition 1.1.4. [39, p. 5] A function $h : D \subseteq \mathbb{R}_+ \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is Lipschitz if there exists a summable function $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$|h(t, \alpha_1) - h(t, \alpha_2)| \leq l(t) |\alpha_1 - \alpha_2|,$$

for almost all $t \geq 0$ and for all $\alpha_1, \alpha_2 \in \mathbb{R}^m$ with $(t, \alpha_1), (t, \alpha_2) \in D$.

Theorem 1.1.3. [39, p. 5] Assume that the system (1.1)-(1.2) has a solution. Assume that the function $h : \mathbb{R}_+ \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is Lipschitz as in Definition 1.1.4, then the solution of system (1.1)-(1.2) is unique.

Note that if the solution of (1.1)-(1.2) is bounded, its solution is global, i.e. is defined on \mathbb{R}_+ (see [90, p. 71]).

Theorem 1.1.4. [39, p. 5] Let \mathcal{M} be the set of $m \times m$ matrices. Assume that all components of the matrices

$$A : \mathbb{R}_+ \rightarrow \mathcal{M},$$

$$b : \mathbb{R}_+ \rightarrow \mathbb{R}^m,$$

are summable on each subset of \mathbb{R}_+ . Then the linear time-varying system $\dot{x}(t) = A(t)x + b(t)$, $t \geq 0$ has a unique solution that is defined on \mathbb{R}_+ , for each initial condition $x(0) = x_0 \in \mathbb{R}$.

Definition 1.1.5. [100, p. 235] The function $\Gamma : \mathbb{R} \setminus \{0, -1, -2, \dots\} \rightarrow \mathbb{R}$ is defined as

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt, \forall \alpha \in \mathbb{R} \setminus \{0, -1, -2, \dots\}. \quad (1.3)$$

Observe that the gamma function is defined on the whole \mathbb{R} except at the non-positive integers (see Figure 1.1) because the improper integral in (1.3) diverges at that points. Furthermore, it can be shown that $\Gamma(1/2) = \sqrt{\pi}$ and that the gamma function satisfies $\Gamma(k+1) = k\Gamma(k)$ for each positive integer k .

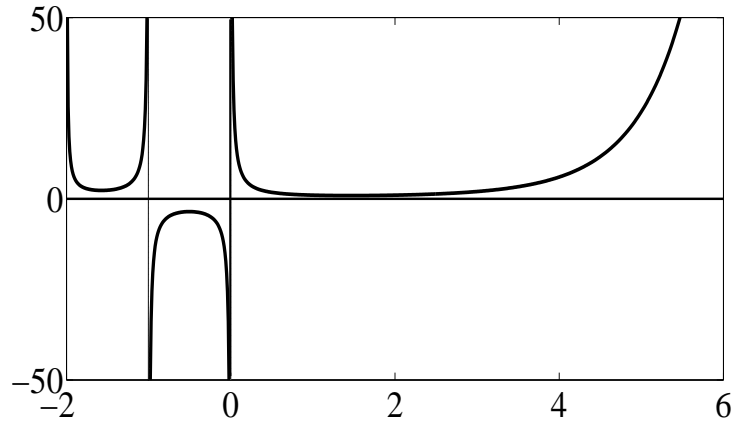


Figure 1.1: The gamma function Γ

1.2 Characterization of hysteresis

This section summarizes the results obtained in [52].

1.2.1 Class of inputs

Consider the Sobolev space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ of absolutely continuous functions $u : \mathbb{R}_+ \rightarrow \mathbb{R}^n$, where n is a positive integer. For this class of functions, the derivative \dot{u} is defined a.e, and we have $\|u\|_\infty < \infty$, $\|\dot{u}\|_\infty < \infty$. Endowed with the norm $\|u\|_{1,\infty} = \max(\|u\|_\infty, \|\dot{u}\|_\infty)$, $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is a Banach space [1]. For $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$, let $\rho_u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the total variation of u on $[0, t]$, that is $\rho_u(t) = \int_0^t |\dot{u}(\tau)| d\tau \in \mathbb{R}_+$. The function ρ_u is well defined as $\dot{u} \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ ¹. It is non-decreasing and absolutely continuous. Denote $\rho_{u,\max} = \lim_{t \rightarrow \infty} \rho_u(t)$ and let

- $I_u = [0, \rho_{u,\max}]$ if $\rho_{u,\max} = \rho_u(t)$ for some $t \in \mathbb{R}_+$ (in this case, $\rho_{u,\max}$ is necessarily finite).
- $I_u = [0, \rho_{u,\max})$ if $\rho_{u,\max} > \rho_u(t)$ for all $t \in \mathbb{R}_+$ (in this case, $\rho_{u,\max}$ may be finite or infinite).

¹ $L^1_{loc}(\mathbb{R}_+, \mathbb{R}^n)$ is the space of locally integrable functions $\mathbb{R}_+ \rightarrow \mathbb{R}^n$.

Lemma 1.2.1. *Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be non-constant so that the interval I_u is not reduced to a single point. Then there exists a unique function $\psi_u \in W^{1,\infty}(I_u, \mathbb{R}^n)$ that satisfies $\psi_u \circ \rho_u = u$.*

Consider the linear time scale change $s_\gamma(t) = t/\gamma$, for any $\gamma > 0$ and $t \geq 0$.

Lemma 1.2.2. *For all $\gamma > 0$, we have $I_{u \circ s_\gamma} = I_u$ and $\psi_{u \circ s_\gamma} = \psi_u$.*

1.2.2 Class of operators

Let Ξ be a set of initial conditions. Let \mathcal{H} be an operator that maps the input function $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ and initial condition $\xi^0 \in \Xi$ to an output in $L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. That is $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \rightarrow L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. The operator \mathcal{H} is said to be causal if the following holds [98, p.60]: $\forall (u_1, \xi^0), (u_2, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$, if $u_1 = u_2$ in $[0, \tau]$, then $\mathcal{H}(u_1, \xi^0) = \mathcal{H}(u_2, \xi^0)$ in $[0, \tau]$.

Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$ and $y = \mathcal{H}(u, \xi^0) \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$. In the rest of this work, only causal operators are considered. Additionally, we consider that the operator \mathcal{H} satisfies the following.

Assumption 1.2.1. *Let $(u, \xi^0) \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$; if there exists a time instant $\theta \in \mathbb{R}_+$ such that u is constant in $[\theta, \infty)$, then the corresponding output $\mathcal{H}(u, \xi^0)$ is constant in $[\theta, \infty)$.*

Assumption 1.2.1 is verified by all causal and rate-independent hysteresis operators (see for example [65, Proposition 2.1] for a proof). This includes relay hysteresis, Ishlinskii model, Preisach model, Krasnosel’skii and Pokrovskii hysteron and generalized play [66]. Assumption 1.2.1 is also verified by some causal and rate-dependent hysteresis models like the generalized Duhem model [72].

Lemma 1.2.3. *There exists a unique function $\varphi_u \in L^\infty(I_u, \mathbb{R}^m)$ that satisfies $\varphi_u \circ \rho_u = y$. Moreover, we have $\|\varphi_u\|_{\infty, I_u} \leq \|y\|_\infty$. If y is continuous on \mathbb{R}_+ , then φ_u is continuous on I_u and we have $\|\varphi_u\|_{\infty, I_u} = \|y\|_\infty$.*

1.2 Characterization of hysteresis

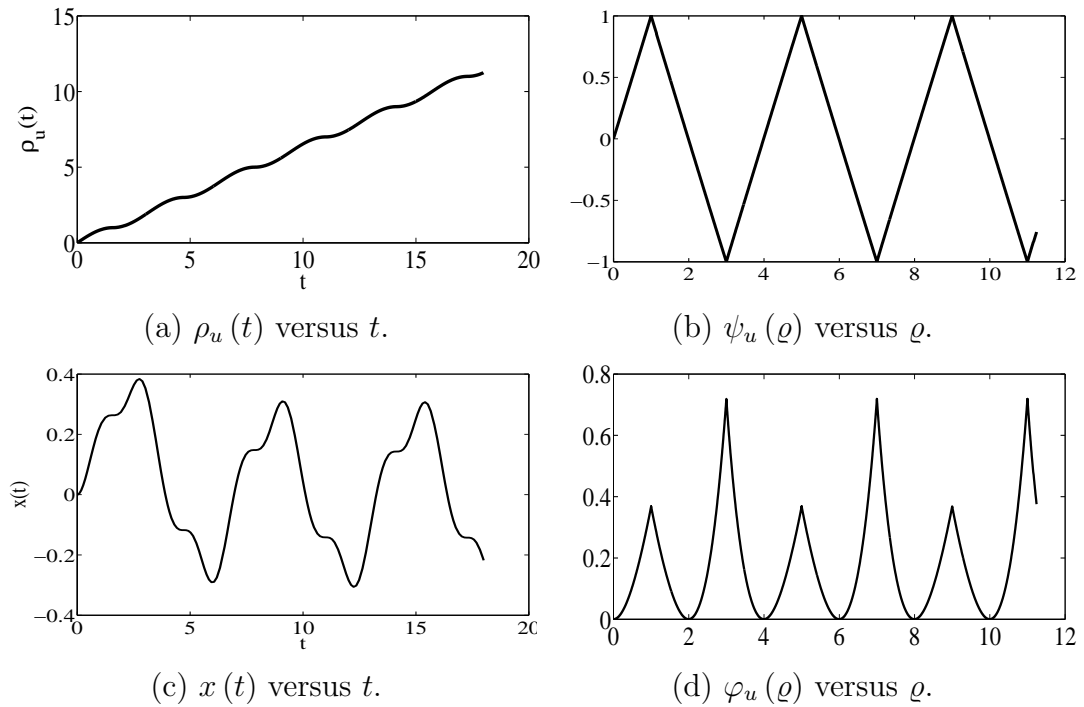


Figure 1.2: Simulations of Example 1.2.1

Example 1.2.1. Let the input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ be defined for all $t \geq 0$ as $u(t) = \sin(t)$. Figure 1.2a shows the total variation ρ_u of the input u . Lemma 1.2.1 states that there exists a unique function $\psi_u \in W^{1,\infty}(I_u, \mathbb{R}^n)$ that satisfies $\psi_u \circ \rho_u = u$. The graph of ψ_u is presented in Figure 1.2b. Now, consider the system

$$\dot{x}(t) = (-x(t) + u(t)) (\dot{u}(t))^2, \quad t \geq 0, \quad (1.4)$$

$$x(0) = 0. \quad (1.5)$$

where $x \in \mathbb{R}$ is the output and $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ is the input of the system. Equation (1.4) can be written as the linear time-varying system of Theorem 1.1.4. It is easy to show that all assumptions of Theorem 1.1.4 are satisfied. Thus, system (1.4)-(1.5) admits a unique solution that is defined on \mathbb{R}_+ . This solution is given in Figure 1.2c.

The operator \mathcal{H} which maps the input function $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ and initial

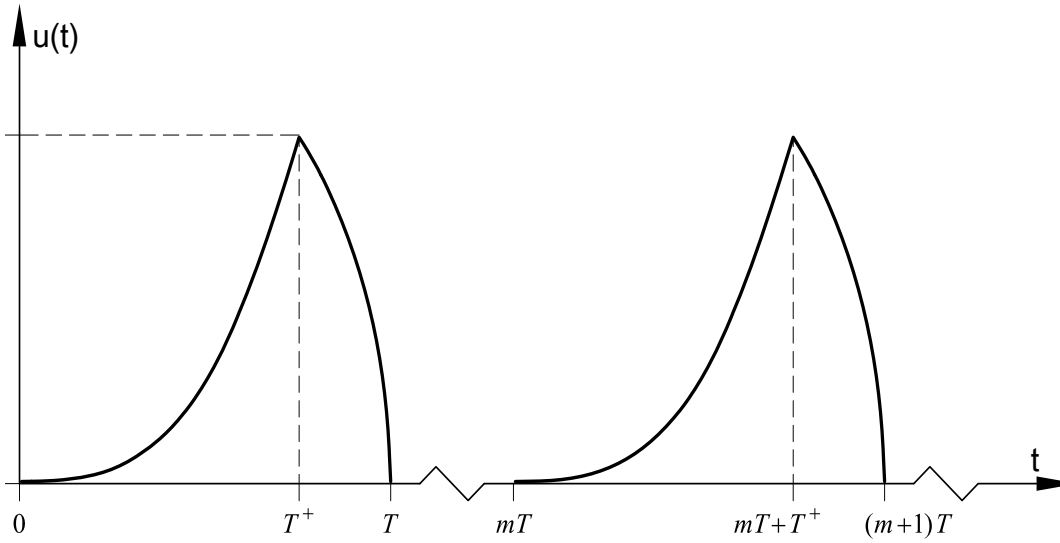


Figure 1.3: A wave periodic signal

condition $\xi^0 \in \mathbb{R}$ to output y in $L^\infty(\mathbb{R}_+, \mathbb{R})$ is well defined. Moreover, the operator \mathcal{H} is causal because if we pick two input signals which are equal on some interval $[0, \tau]$, $\tau > 0$, their outputs are also equal on the same interval since the initial condition is the same for both inputs. On the other hand, Assumption 1.2.1 is satisfied because of (1.4).

All the assumptions in this section are satisfied and thus we conclude from Lemma 1.2.1 that there exists a unique function $\varphi_u \in L^\infty(I_u, \mathbb{R})$ that satisfies $\varphi_u \circ \rho_u = y$ (see the graph of φ_u in Figure 1.2d).

1.2.3 Definition of consistency and strong consistency

Definition 1.2.1. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ and initial condition $\xi^0 \in \Xi$ be given. Consider an operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \Xi \rightarrow L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that is causal and that satisfies Assumption 1.2.1. The operator \mathcal{H} is said to be consistent with respect to input u and initial condition ξ^0 if and only if the sequence of functions $\{\varphi_{u \circ s_\gamma}\}_{\gamma > 0}$ converges in $L^\infty(I_u, \mathbb{R}^m)$ as $\gamma \rightarrow \infty$.

For any causal model of hysteresis that satisfies Assumption 1.2.1, consistency should hold.

Let $T > 0$. In what follows we consider that the input u is T -periodic.

1.2 Characterization of hysteresis

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Definition 1.2.2. A T -periodic function $w : \mathbb{R}_+ \rightarrow \mathbb{R}$ is said to be wave periodic if there exists some $T^+ \in (0, T)$ such that

- The function w is continuous on \mathbb{R}_+
- The function w is continuously differentiable on $(0, T^+)$ and on (T^+, T)
- The function w is increasing on $(0, T^+)$ and is decreasing on (T^+, T)

An example of a wave periodic function is given in Figure 1.3.

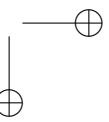
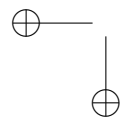
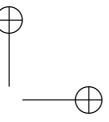
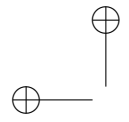
Lemma 1.2.4. If the input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is non-constant and T -periodic, then $I_u = \mathbb{R}_+$ and $\psi_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ is $\rho_u(T)$ -periodic. Furthermore, if $n = 1$ and u is wave periodic, then ψ_u is also wave periodic and $\dot{\psi}_u(\varrho) = 1$ for almost all $\varrho \in (0, \rho_u(T^+))$ and $\dot{\psi}_u(\varrho) = -1$ for almost all $\varrho \in (\rho_u(T^+), \rho_u(T))$.

For any positive integer k , define $\varphi_{u,k}^* \in L^\infty([0, \rho_u(T)], \mathbb{R}^m)$ as

$$\varphi_{u,k}^*(\varrho) = \varphi_u^*(\rho_u(T)k + \varrho), \forall \varrho \in [0, \rho_u(T)].$$

Definition 1.2.3. The operator \mathcal{H} is said to be strongly consistent with respect to input u and initial condition ξ^0 if and only if it is consistent with respect to u and ξ^0 , and the sequence of functions $\varphi_{u,k}^*$ converges in $L^\infty([0, \rho_u(T)], \mathbb{R}^m)$ as $k \rightarrow \infty$.

If the operator \mathcal{H} is strongly consistent with respect to input u and initial condition ξ^0 , then $\{(\varphi_u^\circ(\varrho), \psi_u(\varrho)), \varrho \in [0, \rho_u(T)]\}$ represents the so-called hysteresis loop, where $\varphi_u^\circ = \lim_{k \rightarrow \infty} \varphi_{u,k}^*$.



The LuGre friction model

2.1 Literature review on friction models

Friction is a force of resistance encountered when one object is moved while it is in contact with another [8, 94]. It is derived from the Latin word *fricare*, “to rub”. Although the objects might look smooth, they are very rough and jagged at the molecular level. Therefore, no wonder that friction exists in almost all mechanical systems [17].

Leonardo da Vinci (1452-1519) is considered the first who studied friction systematically [8, 103, 9, 33]. He realized how important friction is for the working of machines. Da Vinci stated the two basic laws of friction 200 years before Newton even defined what force is. The first law is: the areas in contact have no effect on friction. The second law is: friction produces double the amount of effort if the weight be doubled. He did not publish his theories. The only evidence of their existence is in his vast collection of journals. Two centuries after Da Vinci’s discoveries, the French physicist Guillaume Amontons (1663-1705) rediscovered the two basic laws of friction first put forward by Da Vinci. He believed that friction was a result of the work done to lift one surface over the roughness of the other or from the deforming or the wearing of the other surface [33]. These results were then verified mathematically by Charles-Augustin de Coulomb (1736-1806) [22]. Coulomb published his work referring to Amontons and deduced the following law: “the friction between



Figure 2.1: Leonardo da Vinci (1452-1519)

two surfaces is proportional to the force pressing one to the other”. The previous law of friction is known as the “Amontons-Coulomb Law” referring to work done by the two scientists.

Since the discoveries of the centuries 13-16, researchers have been studying friction in many contexts, including identification [19, 10, 60] and compensation [41, 67, 24, 60]. Friction compensated system would be of interest to many companies, especially to those whose primary industry deals with systems that require precise motion control. Detailed studies about friction and its modeling techniques can be found in [8, 94].

There are many types of friction, depending upon the nature of materials, that it is created between. Fluid friction, lubricated friction and dry friction are the most popular types of friction [14]. Fluid friction occurs in fluids that are moving relative to each other [14]. In the case of fluid friction, if a fluid separates two solid surfaces, then the friction is said to be lubricated [49]. Finally, dry friction is the friction force created between solid surfaces in contact with each other [11].

Dry friction is divided into static and dynamic friction [68]. Static friction is friction between two or more solid objects that are not moving relative to each other. For example between a cement block and a wooden floor. The static

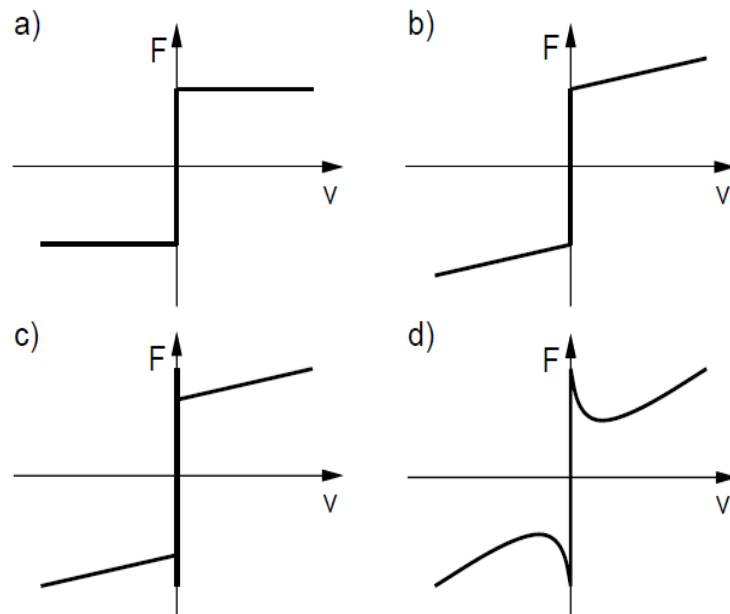


Figure 2.2: Static friction models: a) Coulomb friction; b) Coulomb + viscous; c) Coulomb + viscous + stiction; d) Coulomb + viscous + Stribeck

friction force must be overcome by an applied force before an object can move. It is measured as the maximum force the bodies will sustain before motion occurs. In the static models, friction depends only on the relative velocity between two bodies in contact [68, 97, 31]. Dynamic friction occurs between objects that are moving relative to each other and rub together, as for example the force that works against sliding a cement block along a wooden floor.

Both dynamic friction and static friction, depend on the nature of the surfaces in contact and the magnitude of the force that is acted upon the body in motion. However, static friction is usually greater than dynamic friction for the same surfaces in contact [70]. In some models, static friction can be less than dynamic friction [81].

Static friction models

The first mathematical friction model was proposed by Coulomb in 1773 AD [48]. This model, despite its simplicity, is able to capture the basic physical

behavior of frictionally induced vibrations. The main idea behind this model is that the friction force F , opposes motion and that it is independent of the magnitude of the velocity v and contact area [28, 2]. It can therefore be described as

$$F = F_C \cdot \text{sgn}(v), \quad (2.1)$$

where the friction force $F_C > 0$ is proportional to the normal load $F_N \in \mathbb{R}$, i.e. $F_C = \mu |F_N|$, $\mu > 0$ being the friction coefficient of the normal load F_N . The signum function $\text{sgn} : \mathbb{R} \rightarrow \mathbb{R}$ is such that

$$\text{sgn}(\vartheta) = \begin{cases} 1 & \vartheta > 0 \\ -1 & \vartheta < 0 \end{cases}$$

The description of friction given in (2.1) is termed Coulomb friction (see Figure 2.2a). In this model, friction force is not specified for zero velocity. It may take any value in the interval $[-F_C, F_C]$, depending on how the sgn function is defined. In fact, all the classical static models of friction have the same problem. The Coulomb friction model, because of its simplicity, is used for friction compensation [41, 13].

In the 19th century, the theory of hydrodynamics was developed leading to expressions for the friction force caused by the viscosity of lubricants [82, 80]. The viscous friction can be described by

$$F = \sigma \cdot v,$$

where $\sigma > 0$ in this equation represents a proportionality constant and v is the relative velocity between the moving bodies. Unlike Coulomb friction model, the viscous friction force depends on the magnitude of the velocity (see Figure 2.2b).

Stiction is the short term that describes the static friction force at rest. In 1833, Mortin observed in [71] that stiction is higher than Coulomb friction level [18]. Friction at rest (stiction) can be described as a function of an external

2.1 Literature review on friction models

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force F_e in addition to velocity as follows [83]:

$$F = \begin{cases} F_e & \text{if } v = 0 \text{ and } |F_e| < F_S \\ F_S \operatorname{sgn}(F_e) & \text{if } v = 0 \text{ and } |F_e| \geq F_S \end{cases} \quad (2.2)$$

where F_S is the static (breakaway) force. We observe from (2.2) that stiction at zero velocity can take any value in the interval $[-F_S, F_S]$ as illustrated in Figure 2.2c.

At the beginning of the 20th century, Stribeck described the friction coefficient in lubricated bearings, and observed that the friction force does not decrease discontinuously as in Figure 2.2c, but that the velocity dependence is continuous as shown in Figure 2.2d [93, 8, 25, 94, 36]. In his honor, the deep drop of friction while increasing the relative velocity is known as Stribeck effect. A general description of friction is

$$F = \begin{cases} F_*(v) & \text{if } v \neq 0 \\ F_e & \text{if } v = 0 \text{ and } |F_e| < F_S \\ F_S \operatorname{sgn}(F_e) & \text{if } v = 0 \text{ and } |F_e| \geq F_S \end{cases}$$

where $F_*(v)$ is an arbitrary function that looks like the plot in Figure 2.2d.

Dynamic friction models

Dynamic friction occurs when two objects are moving relative to each other, with their surfaces in contact. It is the frictional force that slows down moving object until they stop [91, 70].

Many properties that cannot be captured by typical static friction models, may be described by dynamic friction models [31]. Presliding displacement, frictional lag, varying break-away force, and stick-slip motion are examples of such properties. More details the previous properties come next.

Presliding displacement refers to displacement that occurs just before a complete slip between two contacting surfaces takes place [93]. It is due to limited

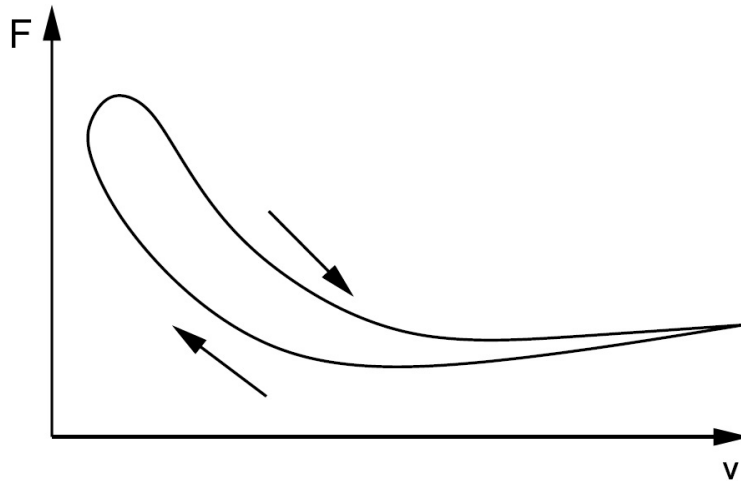


Figure 2.3: Frictional lag

stiffness of the contact asperities (roughness features) on the surfaces [45]. In presliding regimes, the friction force appears to be a function of displacement rather than velocity [9]. Studying presliding friction is useful in control applications because a hysteresis behavior may exist between the presliding friction force input and the displacement output [85, 94, 4].

Experimental results show that relative displacement between two bodies in contact may occur when the applied tangential force is less than the static friction (stiction) [27, 56]. As the displacement increases the tangential force increases more and more slowly, approaching asymptotically to the stiction force [27]. On the other hand, as the displacement increases more and more asperity junctions ¹ will deform elastically and then plastically [27, 7]. When the applied tangential force reaches the stiction level, the asperity junctions break and the sliding begins [27, 56]. This elastic-plastic process cause increases and decreases in the applied tangential force which causes hysteresis.

Frictional lag; also called hysteresis in the velocity, refers to the delay between a change in velocity and the corresponding change in friction [47]. The origin of friction lag in lubricated friction relates to the time required before the friction force changes with changing sliding velocity. Experiments on fric-

¹asperity junctions are the load bearing interfaces between rubbing surfaces

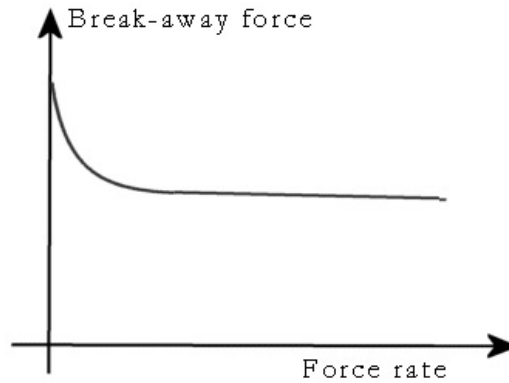


Figure 2.4: Varying break-away force

tion lag ensure that hysteresis may occur between friction force and velocity, where the friction force for increasing velocities is larger than the friction force for decreasing velocities [25]. Frictional lag is presented in Figure 2.3.

The break-away force is the force required to overcome the stiction and initiate motion [8]. Varying break-away force is the dependence of the break-away force on the rate of increase of the applied force [27, 79] (see Figure 2.4).

Stick-slip concept has been introduced for the first time in [16]. It is a repeated sequences of sticking spontaneous jerking motion that can occur when two objects are sliding over each other [7], where in a first phase of relatively small displacement (stick) strain energy is accumulated which in a second phase (slip) transforms into kinetic energy [22, 35]. In other words, stick-slip motion occurs when friction at rest is larger than during motion [25].

LuGre friction model

Each surface has a number of asperities at the microscopic level. This is pictured by Haessig et al. [43] as a contact through elastic bristles. Haessig et al. [43] introduced a system that describes the friction between two surfaces

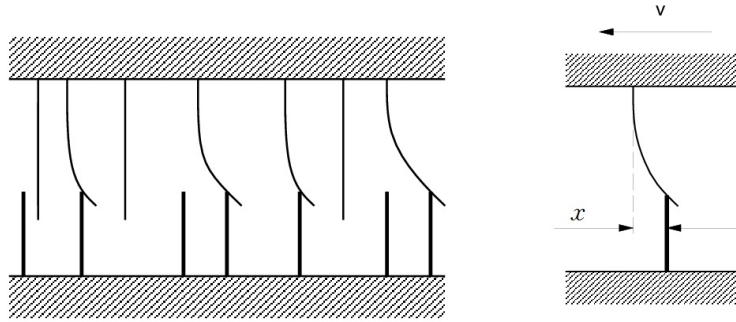


Figure 2.5: LuGre model

in contact depending upon the idea that friction is caused by a large number of interacting bristles. This idea was then employed by Canudas de Wit et al. [25] to derive a new friction model which treats the low velocity friction using a model of the deflection of elastic bristles as in Figure 2.5 [59, 31, 25, 75]. This model is called LuGre because it is resulted from a collaboration between control groups in Lund and Grenoble [25]. The LuGre model has the form [59]:

$$\dot{x}(t) = -\sigma_0 \frac{|\dot{u}(t)|}{\mu(\dot{u}(t))} x(t) + \dot{u}(t), \forall t \geq 0, \quad (2.3)$$

$$x(0) = x_0, \quad (2.4)$$

$$F(t) = \sigma_0 x(t) + \sigma_1 \dot{x}(t) + \nu(\dot{u}(t)). \quad (2.5)$$

where

- $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ is the relative displacement and is the input of the system;
- $x(t) \in \mathbb{R}$ is the average deflection of the bristles and is the inner state of the model;
- $x_0 \in \mathbb{R}$ is the initial state;
- $F(t) \in \mathbb{R}$ is the friction force and is the output of the system;
- $\sigma_0 > 0$ is stiffness of the bristles;

2.2 Consistency of the LuGre model

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- $\sigma_1 > 0$ is the viscous friction coefficient;
- $\mu \in C^0(\mathbb{R}, \mathbb{R})$ ² represents the macro-damping friction with $\mu(\vartheta) > 0, \forall \vartheta \in \mathbb{R}$. This function may model the Stribeck effect;
- $\nu \in C^0(\mathbb{R}, \mathbb{R})$ is a memoryless velocity-dependent function that represents the viscous friction.

Typically, the function $\nu(\vartheta)$ takes the form $\nu(\vartheta) = \sigma_2 \vartheta, \forall \vartheta \in \mathbb{R}$, where the parameter $\sigma_2 > 0$ is the viscous friction coefficient. A reasonable choice of $\mu(\vartheta)$ giving a good approximation of the Stribeck effect is (see [25]):

$$\mu(\vartheta) = F_C + (F_S - F_C) e^{-|\vartheta/v_s|^\alpha}, \quad (2.6)$$

for all $\vartheta \in \mathbb{R}$, where $F_C > 0$ is the Coulomb friction force, $F_S > 0$ is the stiction force, $v_s \in \mathbb{R}$ is the Stribeck velocity, and α is a positive constant.

The LuGre model is able to capture hysteresis, Stribeck effect, stick-slip motion, presliding displacement, varying break-away force and frictional lag effects [59, 85, 46]. Thus, it is widely used to describe the friction phenomenon [78, 59]. The LuGre model behaves like a linear spring/damper pair when it is linearized at rest [25]. Algebraic necessary and sufficient conditions for the dissipativity of the LuGre model are obtained in [12]. The model is used for friction compensation [40, 57, 105, 102, 89, 64, 95], and it is studied in terms of parameter identification [101, 84, 92].

2.2 Consistency of the LuGre model

In this section, we study the consistency and the strong consistency of the LuGre model (2.3)-(2.5). To this end, we need to prove first the existence of a solution for the system. This comes next.

In Equation (2.3), the function $g(\dot{u})$ is measurable [88, Theorem 1.12(d)]. Thus, the differential equation (2.3) can be seen as a linear time-varying system

² $C^0(\mathbb{R}, \mathbb{R})$ is the space of continuous functions $\mu : \mathbb{R} \rightarrow \mathbb{R}$, with the norm $\|\cdot\|_\infty$.

that satisfies all assumptions of [39, Theorem 3]. This implies that a unique absolutely continuous solution of (2.3) exists on \mathbb{R}_+ (see Theorem 1.1.4).

Now we present some mathematical preliminaries that will be used in Lemma 2.2.2 which includes the main results of this Chapter.

We define some operators for the equations (2.3)-(2.5) as follows:

- The operator $\mathcal{H}_s : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \rightarrow L^\infty(\mathbb{R}_+, \mathbb{R})$ such that $\mathcal{H}_s(u, x_0) = x$
- The operator $\mathcal{H}_o : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \rightarrow L^\infty(\mathbb{R}_+, \mathbb{R})$ such that $\mathcal{H}_o(u, x_0) = F$

Now consider the following system.

$$\dot{x}(t) = -\sigma_0 \frac{|v(t)|}{\mu(v(t))} x(t) + v(t), \quad (2.7)$$

$$x(0) = x_0, \quad (2.8)$$

$$F(t) = \sigma_0 x(t) + \sigma_1 \dot{x}(t) + \nu(v(t)). \quad (2.9)$$

in which $v \in L^\infty(\mathbb{R}_+, \mathbb{R})$. In equations (2.7)-(2.9), consider the following operators:

- The operator $\mathcal{H}'_s : L^\infty(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \rightarrow L^\infty(\mathbb{R}_+, \mathbb{R})$ such that $\mathcal{H}'_s(v, x_0) = x$
- The operator $\mathcal{H}'_o : L^\infty(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \rightarrow L^\infty(\mathbb{R}_+, \mathbb{R})$ such that $\mathcal{H}'_o(v, x_0) = F$

Observe that the operators \mathcal{H}'_s and \mathcal{H}'_o are causal due to the uniqueness of the solutions of Equation (2.3).

Consider the left-derivative operator Δ_- defined on $W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ by

$$[\Delta_-(u)](t) = \lim_{\tau \uparrow t} \frac{u(\tau) - u(t)}{\tau - t}$$

The operator Δ_- is causal as $[\Delta_-(u)](t)$ depends only on values of $u(\tau)$ for $\tau \leq t$, and we have $\Delta_-(u) = \dot{u}$ a.e. as $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ so that $\Delta_-(u) \in$

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$L^\infty(\mathbb{R}_+, \mathbb{R})$.

Note that $\mathcal{H}_s = \mathcal{H}'_s \circ (\Delta_-, x_0)$ and $\mathcal{H}_o = \mathcal{H}'_o \circ (\Delta_-, x_0)$ so that the operators \mathcal{H}_s and \mathcal{H}_o are causal. Observe also that \mathcal{H}_s and \mathcal{H}_o satisfy Assumption 1.2.1.

Proposition 2.2.1. *Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$. There exists a unique function $v_u \in L^\infty(I_u, \mathbb{R})$ that is defined by $v_u \circ \rho_u = \dot{u}$. Moreover, $\|v_u\|_{\infty, I_u} \leq \|\dot{u}\|_\infty$. Assume that \dot{u} is nonzero on a set $A \subseteq \mathbb{R}$ that satisfies $\mu(\rho_u(\mathbb{R} \setminus A)) = 0$. Then, v_u is nonzero almost everywhere.*

Proof. The operator $\Delta_- : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \rightarrow L^\infty(\mathbb{R}_+, \mathbb{R})$ is causal and satisfies Assumption 1.2.1. The first part of Proposition 2.2.1 follows immediately from Lemma 1.2.3. Now, let $B = \{\varrho \in I_u / v_u(\varrho) = 0\}$, then $B \subseteq \rho_u(\mathbb{R} \setminus A)$ which implies that $\mu(B) = 0$. \square

In general, \dot{u} does not need to be nonzero almost everywhere to make v_u non-zero. This fact is illustrated in Example 2.2.1.

Example 2.2.1. *Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ be a periodic function of period 3 such that*

$$u(t) = \begin{cases} -t & 0 \leq t < 1 \\ -1 & 1 \leq t < 2 \\ t - 3 & 2 \leq t < 3 \end{cases}$$

Functions u , \dot{u} , and ρ_u are shown respectively in Figures 2.6a, 2.6b, and 2.6c. Let us focus on the interval $[0, 3]$. The function u is zero on $[1, 2]$, and the total variation ρ_u maps the interval $[1, 2]$ into the singleton $\{1\}$. Therefore, v_u is non-zero almost everywhere as we observe in Figure 2.6d.

In the rest of the section, we consider that the input u satisfies the conditions of Proposition 2.2.1.

Consider the time scale change $s_\gamma(t) = t/\gamma, \gamma > 0, t \geq 0$. When the input

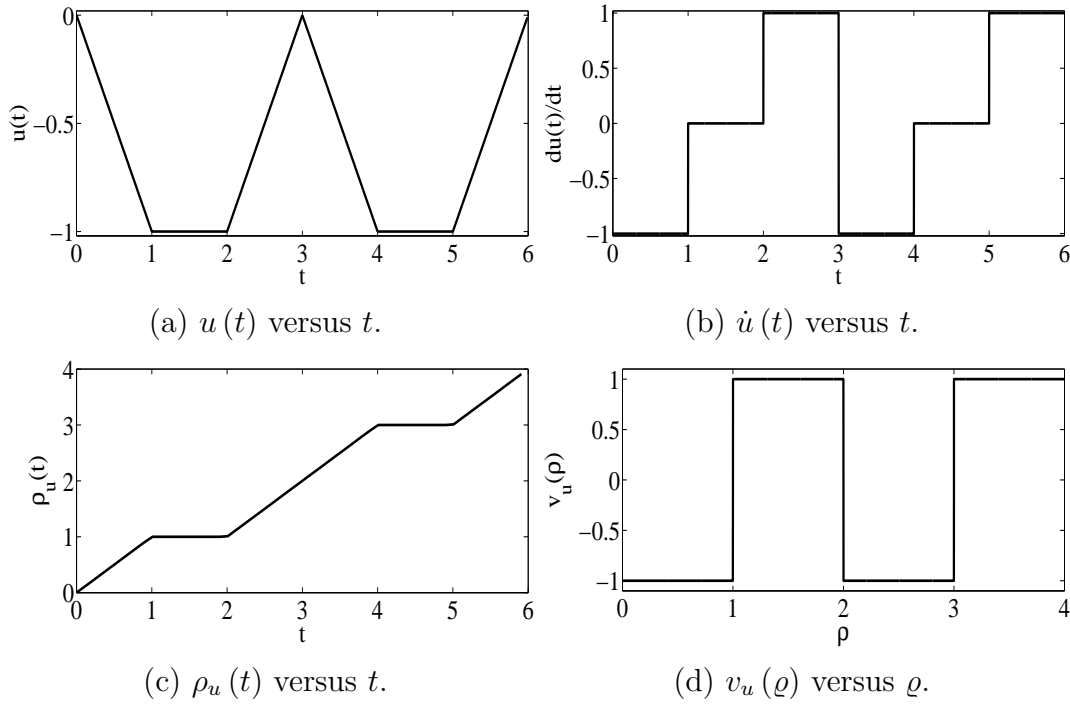


Figure 2.6: Simulations of Example 2.2.1

$u \circ s_\gamma$ is used instead of u , system (2.3)-(2.5) becomes

$$\dot{x}_\gamma(t) = -\sigma_0 \frac{\left| \frac{\dot{u} \circ s_\gamma(t)}{\gamma} \right|}{\mu \left(\frac{\dot{u} \circ s_\gamma(t)}{\gamma} \right)} x_\gamma(t) + \frac{\dot{u} \circ s_\gamma(t)}{\gamma}, \quad (2.10)$$

$$x_\gamma(0) = x_0, \quad (2.11)$$

$$F_\gamma(t) = \sigma_0 x_\gamma(t) + \sigma_1 \dot{x}_\gamma(t) + \nu \left(\frac{\dot{u} \circ s_\gamma(t)}{\gamma} \right). \quad (2.12)$$

When $\gamma = 1$, system (2.10)-(2.12) reduces to (2.3)-(2.5).

Lemma 1.2.3 shows that for any $\gamma > 0$, there exists a unique function $x_{u \circ s_\gamma} \in L^\infty(I_u, \mathbb{R})$ such that $x_{u \circ s_\gamma} \circ \rho_{u \circ s_\gamma} = x_\gamma$, and a unique function $\varphi_{u \circ s_\gamma} \in L^\infty(I_u, \mathbb{R})$ such that $\varphi_{u \circ s_\gamma} \circ \rho_{u \circ s_\gamma} = F_\gamma$. Using the change of variables $\varrho = \rho_{u \circ s_\gamma}(t)$, it follows from Equations (2.10)-(2.12), Lemma 1.2.2 and

2.2 Consistency of the LuGre model

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Proposition 2.2.1 that

$$\dot{x}_{u \circ s_\gamma}(\varrho) = -\frac{\sigma_0}{\mu\left(\frac{v_u(\varrho)}{\gamma}\right)} x_{u \circ s_\gamma}(\varrho) + \dot{\psi}_u(\varrho), \quad (2.13)$$

$$x_{u \circ s_\gamma}(0) = x_0, \quad (2.14)$$

$$\varphi_{u \circ s_\gamma}(\varrho) = \sigma_0 x_{u \circ s_\gamma}(\varrho) + \frac{\sigma_1}{\gamma} |v_u(\varrho)| \dot{x}_{u \circ s_\gamma}(\varrho) + \nu \left(\frac{v_u(\varrho)}{\gamma} \right), \quad (2.15)$$

for all $\gamma > 0$ and for almost all $\varrho \in I_u$.

Problem statement: The aim of this section is to analyze the convergence properties of the sequence of functions $\varphi_{u \circ s_\gamma}$ in order to study the consistency and strong consistency of the operator \mathcal{H}_o .

The following lemma generalizes Theorem 4.18 in [61, p.172]. Indeed, in [61], continuous differentiability is needed, while in Lemma 2.2.1, we only need absolute continuity. Also, in [61], the inequality on the derivative of the Lyapunov function is needed everywhere, while in Lemma 2.2.1 it is needed only almost everywhere.

Lemma 2.2.1. *Consider a function $z : [0, \omega) \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+$, where $\omega > 0$ is finite or infinite. Assume the following*

1. *The function z is absolutely continuous on each compact interval of $[0, \omega)$.*
2. *There exist $z_1 \geq 0$ and $z_2 > 0$ such that $z_1 < z_2$, $z(0) < z_2$ and*

$$\dot{z}(t) \leq 0 \text{ for almost all } t \in [0, \omega) \text{ that satisfy } z_1 < z(t) < z_2. \quad (2.16)$$

Then, $z(t) \leq \max(z(0), z_1)$, $\forall t \in [0, \omega)$.

Proof. We discuss two cases, CASE I: $z(0) \leq z_1$ and CASE II: $z_1 < z(0) < z_2$.

CASE I: $z(0) \leq z_1$.

The objective of what follows is to prove that $\forall t \in [0, \omega)$ we have $z(t) \leq z_1$. To this end, assume that $\exists t_1 \in (0, \omega)$ such that $z(t_1) > z_1$. Put $C = \{\tau \in [0, \omega) / z(\tau) \leq z_1, \forall \tau \in [0, \tau]\}$. The set C is nonempty because $0 \in C$. Define $t_2 := \text{Sup } C$, then there exists a real sequence $\{\tau_n \in C\}_{n=1}^{\infty}$ such that $\lim_{n \rightarrow \infty} \tau_n = t_2$. By the continuity of z , we have $z(t_2) = \lim_{n \rightarrow \infty} z(\tau_n) \leq z_1$. This fact implies that $t_2 \in C$ leading to $t_2 < t_1$. Also, there exists a real sequence $\{\tau'_n > t_2\}_{n=1}^{\infty}$ such that $z(\tau'_n) > z_1, \forall n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \tau'_n = t_2$. Since z is continuous we get $z(t_2) \geq z_1$ which leads to $z(t_2) = z_1$. Let $D = \{t \in [t_2, t_1] / z(t) = z_1\}$. The set D is nonempty since $t_2 \in D$. Define $t_3 := \text{Sup } D$, then using a similar argument as above we get $t_3 \in D$ which implies that $z(t_3) = z_1$ and $t_3 < t_1$.

Claim 1. $z(t) > z_1, \forall t_3 < t \leq t_1$.

Proof. Assume that $\exists t_4 \in (t_3, t_1]$ such that $z(t_4) \leq z_1$. By definition of t_3 , we get $z(t_4) < z_1$. Since z is continuous and $z(t_4) < z_1 < z(t_1)$, we can use the Intermediate Value Theorem [6] to find $t_5 \in (t_4, t_1)$ such that $z(t_5) = z_1$ which implies that $t_5 \in D$ and $t_5 > t_3$ which is a contradiction. \square

Let $E = \{t \in [t_3, t_1] / z(t) = z_2\}$. We consider the following sub-cases $E = \emptyset$ and $E \neq \emptyset$. If $E = \emptyset$, then $z(t_1) < z_2$ and hence Claim 1 and (2.16) imply that

$$\dot{z}(t) \leq 0, \text{ for almost all } t \in (t_3, t_1].$$

Thus, the absolute continuity of the function z implies

$$\int_{t_3}^{t_1} \dot{z}(t) dt = z(t_1) - z(t_3) \leq 0,$$

which contradicts the fact that $z(t_3) = z_1 < z(t_1)$. Now, if $E \neq \emptyset$, let $t_6 = \text{Inf } E$. It can be shown that $t_6 \in E$ and $t_3 < t_6$. Thus, Claim 1 and (2.16)

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give

$$\dot{z}(t) \leq 0, \quad \text{for almost all } t \in (t_3, t_6),$$

which also contradicts the facts that z is absolutely continuous on each compact interval and $z(t_3) = z_1 < z_2 = z(t_6)$.

We have thus proved that, in CASE I,

$$z(t) \leq z_1, \quad \forall t \in [0, \omega) \text{ whenever } z(0) \leq z_1. \quad (2.17)$$

CASE II: $z_1 < z(0) < z_2$.

Assume that $\exists t_1 \in (0, \omega)$ such that $z(t_1) = z_1$. Let $t_2 > 0$ be the smallest real number such that $z(t_2) = z_1$ (it exists due to the continuity of z). Then, seeing t_2 as an initial time, and z_1 as an initial condition, it follows from CASE I that $\forall t \geq t_2$ we have $z(t) \leq z_1$. So we have just to analyze what happens in the interval $[0, t_2]$ and discuss the case $\forall t \in [0, \omega), z(t) > z_1$. The analysis of both situations is the same so that we focus on the case $\forall t \in [0, \omega), z(t) > z_1$.

Assume that $\exists t_3 \in (0, \omega)$ such that $z(t_3) = z_2$. Let $t_4 > 0$ be the smallest real number such that $z(t_4) = z_2$ (it exists due to the continuity of z). Then, $\forall t \in [0, t_4)$ we have $z_1 < z(t) < z_2$ which implies that for almost all $t \in [0, t_4)$ we have $\dot{z}(t) \leq 0$. Since z is absolutely continuous, it follows that $\int_0^{t_4} \dot{z}(t) dt = z(t_4) - z(0) = z_2 - z(0) \leq 0$. This contradicts the fact that $z_2 > z(0)$ which means that $\forall t \geq 0, z(t) < z_2$. Since $\forall t \in [0, \omega), z(t) > z_1$, it follows that $\forall t \in [0, \omega), \dot{z}(t) \leq 0$ so that $\forall t \in [0, \omega), \int_0^t \dot{z}(\tau) d\tau = z(t) - z(0) \leq 0$.

As a conclusion, we have proved that in CASE II that $\forall t \in [0, \omega), z(t) \leq z(0)$. \square

Example 2.2.2. *We want to study the stability of the following system*

$$\dot{x}(t) = -x^3(t) + u(t), \quad (2.18)$$

$$x(0) = x_0, \quad (2.19)$$

where x_0 and state x take values in \mathbb{R} , and input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$. System (2.18)-(2.19) has an absolutely continuous solution that is defined on an interval of the form $[0, \omega)$ [39, p.4].

Let $z : [0, \omega) \rightarrow \mathbb{R}_+$ be such that $z(t) = x^2(t), \forall t \in [0, \omega)$. The function z is absolutely continuous on each compact subset of $[0, \omega)$ because x is absolutely continuous. Thus, Condition 1 in Lemma 2.2.1 is satisfied.

We have for almost all $t \in [0, \omega)$ that

$$\begin{aligned} \dot{z}(t) &= 2x(t) \cdot \dot{x}(t) \\ &= 2x(t) (-x^3(t) + u(t)) \\ &\leq -2z^2(t) + 2\|u\|_\infty \sqrt{z(t)}. \end{aligned}$$

Thus,

$$\dot{z}(t) \leq 0 \text{ for almost all } t \in [0, \omega) \text{ that satisfy } \|u\|_\infty^{2/3} < z(t).$$

Therefore, Condition 2 in Lemma 2.2.1 is satisfied with $z_1 = \|u\|_\infty^{2/3}$ and z_2 can be any positive real number such that

$$z_2 > \max(z(0), z_1) = \max(x_0^2, \|u\|_\infty^{2/3}).$$

Thus, we deduce from Lemma 2.2.1 that

$$z(t) \leq \max(z(0), \|u\|_\infty^{2/3}) = \max(x_0^2, \|u\|_\infty^{2/3}), \forall t \in [0, \omega),$$

so that

$$|x(t)| \leq \max(|x_0|, \sqrt[3]{\|u\|_\infty}), \forall t \in [0, \omega).$$

Corollary 2.2.1. Consider a function $z : [0, \omega) \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}_+$, where $\omega > 0$ may be infinite. Assume the following

1. The function z is absolutely continuous on each compact subset of $[0, \omega)$.

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2. There exist a class \mathcal{K}_∞ function $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $z_1, z_2, z_3 \geq 0$ such that

$$\max(\beta^{-1}(z_3), z_1, z(0)) < z_2,$$

and

$$\begin{cases} \dot{z}(t) \leq -\beta(z(t)) + z_3 \text{ for almost all} \\ t \in [0, \omega) \text{ that satisfy } z_1 < z(t) < z_2. \end{cases} \quad (2.20)$$

Then, $z(t) \leq \max(z(0), z_1, \beta^{-1}(z_3))$, $\forall t \in [0, \omega)$.

Proof. We have from (2.20) that

$$\begin{cases} \dot{z}(t) \leq 0 \text{ for almost all } t \in [0, \omega) \text{ that} \\ \text{satisfy } \max(\beta^{-1}(z_3), z_1) < z(t) < z_2, \end{cases}$$

and hence the result follows directly from Lemma 2.2.1. \square

The following Lemma is the main result of this chapter, where it proves the consistency and the strong consistency of the LuGre model (2.3)-(2.5).

Lemma 2.2.2. *Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ be such that u is non-zero on a set $A \subseteq \mathbb{R}$ that satisfies $\mu(\rho_u(\mathbb{R} \setminus A)) = 0$. Then the following holds:*

- There exist $E, \gamma_1 > 0$ such that $\|F_\gamma\|_\infty \leq E$, $\forall \gamma > \gamma_1$.³
- The operator \mathcal{H}_o is consistent with respect to input u and initial condition x_0 , that is there exists a unique function $\varphi_u^* \in W^{1,\infty}(I_u, \mathbb{R})$ such that

$$\lim_{\gamma \rightarrow \infty} \|\varphi_{u \circ s_\gamma} - \varphi_u^*\|_{\infty, I_u} = 0,$$

where

$$\varphi_u^*(\varrho) = \sigma_0 e^{\frac{-\sigma_0 \varrho}{\mu(0)}} \left[x_0 + \int_0^{\varrho} e^{\frac{\sigma_0 \tau}{\mu(0)}} \dot{\psi}_u(\tau) d\tau \right] + \nu(0), \forall \varrho \in I_u. \quad (2.21)$$

³ F_γ is given in (2.12).

Moreover, if u is T -periodic, then the operator \mathcal{H}_o is strongly consistent with respect to input u and initial condition x_0 . That is, there exists a unique function $\varphi_u^\circ \in W^{1,\infty}([0, \rho_u(T)], \mathbb{R})$ such that

$$\lim_{k \rightarrow \infty} \|\varphi_{u,k}^* - \varphi_u^\circ\|_{\infty, [0, \rho_u(T)]} = 0,$$

where

$$\varphi_u^\circ(\varrho) = \sigma_0 h_\infty(\varrho) + \nu(0), \forall \varrho \in [0, \rho_u(T)],$$

$$\dot{h}_\infty(\varrho) = -\frac{\sigma_0}{\mu(0)} h_\infty(\varrho) + \dot{\psi}_u(\varrho), \text{ for almost all } \varrho \in [0, \rho_u(T)].$$

In this case, $h_\infty(0)$ may be different than x_0 . Additionally, if the input u is wave periodic (see Definition 1.2.2), then we have

$$\varphi_u^\circ(0) = \mu(0) \left(2e^{\frac{\sigma_0 \rho_u(T^+)}{\mu(0)}} - 1 - e^{\frac{\sigma_0 \rho_u(T)}{\mu(0)}} \right) \frac{1}{e^{\frac{\sigma_0 \rho_u(T)}{\mu(0)}} - 1} + \nu(0), \quad (2.22)$$

and

$$\varphi_u^\circ(\varrho) = \begin{cases} Q_1 e^{\frac{-\sigma_0 \varrho}{\mu(0)}} + \mu(0) + Q_3(\varrho) & \forall \varrho \in [0, \rho_u(T^+)] \\ Q_2 e^{\frac{-\sigma_0 \varrho}{\mu(0)}} - \mu(0) + Q_3(\varrho) & \forall \varrho \in [\rho_u(T^+), \rho_u(T)] \end{cases}$$

where

$$\begin{aligned} Q_1 &= \varphi_u^\circ(0) - \mu(0) \\ Q_2 &= \varphi_u^\circ(0) + \mu(0) \left(2e^{\frac{\sigma_0 \rho_u(T^+)}{\mu(0)}} - 1 \right) \\ Q_3(\varrho) &= \nu(0) \left(1 - e^{\frac{-\sigma_0 \varrho}{\mu(0)}} \right), \forall \varrho \in [0, \rho_u(T)] \end{aligned}$$

Proof. Consider the linear system

$$\dot{h}(\varrho) = -\frac{\sigma_0}{\mu(0)} h(\varrho) + \dot{\psi}_u(\varrho), \forall \varrho \in I_u \quad (2.23)$$

$$h(0) = x_0, \quad (2.24)$$

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where state $h \in \mathbb{R}$. The solution of (2.23)-(2.24) is given by

$$h(\varrho) = \frac{\sigma_0}{\mu(0)} e^{-\sigma_0 \varrho / \mu(0)} \left[x_0 + \int_0^{\varrho} e^{\sigma_0 \tau / \mu(0)} \dot{\psi}_u(\varrho) d\tau \right], \forall \varrho \in I_u. \quad (2.25)$$

Consider the Lyapunov function $W : I_u \rightarrow \mathbb{R}_+$ with $W = h^2$. From (2.23)-(2.24) it follows that $W(0) = x_0^2$ and

$$\dot{W} = -\frac{2\sigma_0}{\mu(0)} W + 2\dot{\psi}_u \sqrt{W}$$

By Lemma 1.2.1 we have $\|\dot{\psi}_u\|_{\infty, I_u} = 1$ so that $\dot{W} \leq -\frac{2\sigma_0}{\mu(0)} W + 2\sqrt{W}$. This leads to $\dot{W} \leq 0$, whenever $W > \left(\frac{\mu(0)}{\sigma_0}\right)^2$. Using Lemma 2.2.1 it comes that $W \leq \max\left(x_0^2, \left(\frac{\mu(0)}{\sigma_0}\right)^2\right)$ so that

$$|h(\varrho)| \leq \max\left(|x_0|, \frac{\mu(0)}{\sigma_0}\right), \forall \varrho \in I_u. \quad (2.26)$$

Claim 2. $\lim_{\gamma \rightarrow \infty} \|\chi_\gamma\|_{\infty, I_u} = 0$, where $\chi_\gamma : I_u \rightarrow \mathbb{R}$ is defined a.e. as

$$\chi_\gamma(\varrho) = \frac{1}{\mu\left(\frac{v_u(\varrho)}{\gamma}\right)} - \frac{1}{\mu(0)}, \text{ for almost all } \varrho \in I_u.$$

Proof. Let $\varepsilon > 0$. Since g is continuous and non-zero, we have $\lim_{\vartheta \rightarrow 0} \frac{1}{g(\vartheta)} = \frac{1}{g(0)}$. Hence there exists some $\delta_\varepsilon > 0$ that depend solely on ε , such that

$$\left| \frac{1}{\mu(\vartheta)} - \frac{1}{\mu(0)} \right| < \varepsilon, \text{ whenever } \vartheta \in (-\delta_\varepsilon, \delta_\varepsilon). \quad (2.27)$$

By Proposition 2.2.1 we have $\|v_u\|_{\infty, I_u} \leq \|\dot{u}\|_{\infty}$. Thus there exists $\gamma_* > 0$ such that $\|v_u\|_{\infty, I_u} / \gamma < \delta_\varepsilon, \forall \gamma > \gamma_*$. Thus, we conclude from (2.27) that

$$\left| \frac{1}{\mu(v_u(\varrho)/\gamma)} - \frac{1}{\mu(0)} \right| < \varepsilon, \text{ for almost all } \varrho \in I_u, \forall \gamma > \gamma_*$$

so that

$$|\chi_\gamma(\varrho)| < \varepsilon, \text{ for almost all } \varrho \in I_u, \forall \gamma > \gamma_*,$$

which completes the proof. \square

For any $\gamma > 0$, let $y_\gamma : I_u \rightarrow \mathbb{R}$ be defined as $y_\gamma = x_{u \circ s_\gamma} - h$, where $x_{u \circ s_\gamma}$ is given in (2.13)-(2.14) and h is given in (2.23)-(2.24). Since $x_{u \circ s_\gamma}(0) = h(0) = x_0$, we have for all $\gamma > 0$ and for almost all $\varrho \in I_u$ that

$$\dot{y}_\gamma(\varrho) = \frac{-\sigma_0}{\mu\left(\frac{v_u(\varrho)}{\gamma}\right)} y_\gamma(\varrho) - \sigma_0 \chi_\gamma(\varrho) h(\varrho), \quad (2.28)$$

$$y_\gamma(0) = 0. \quad (2.29)$$

For any $\gamma > 0$, consider the Lyapunov function $V_\gamma : I_u \rightarrow \mathbb{R}_+$ with $V_\gamma(\varrho) = y_\gamma^2(\varrho), \forall \varrho \in I_u$. By (2.28), the boundedness of v_u and (2.26), we have

$$\begin{cases} \dot{V}_\gamma(\varrho) \leq -2 \frac{\sigma_0}{\mu\left(\frac{v_u(\varrho)}{\gamma}\right)} V_\gamma(\varrho) + D_1 \|\chi_\gamma\|_{\infty, I_u} \sqrt{V_\gamma(\varrho)} \\ \forall \gamma > 0, \text{ for almost all } \varrho \in I_u. \end{cases} \quad (2.30)$$

where $D_1 = \sigma_0 \|h\|_{\infty, I_u} > 0$. On the other hand, since v_u is bounded and the function g is continuous and positive, there exists $M > 0$ such that

$$\mu(v_u(\varrho)/\gamma) > M, \text{ for almost all } \varrho \in I_u, \forall \gamma > 1. \quad (2.31)$$

Thus, we obtain from (2.30) that

$$\begin{cases} \dot{V}_\gamma(\varrho) \leq -\frac{2\sigma_0}{M} V_\gamma(\varrho) + D_1 \|\chi_\gamma\|_{\infty, I_u} \sqrt{V_\gamma(\varrho)}, \\ \forall \gamma > 1, \text{ for almost all } \varrho \in I_u. \end{cases}$$

so that

$$\begin{cases} \dot{V}_\gamma(\varrho) \leq 0, \text{ for almost all } \varrho \in I_u, \forall \gamma > 1 \\ \text{that satisfy } V_\gamma(\varrho) > \left(\frac{D_1 M \|\chi_\gamma\|_{\infty, I_u}}{2\sigma_0}\right)^2. \end{cases} \quad (2.32)$$

Therefore, the fact that $V_\gamma(0) = 0$ along with Lemma 2.2.1 imply that $V_\gamma(\varrho) \leq$

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$\left(\frac{D_1 M \|\chi_\gamma\|_{\infty, I_u}}{2\sigma_0}\right)^2$ for all $\gamma > 1$ and almost all $\varrho \in I_u$, and hence we obtain

$$\begin{cases} |y_\gamma(\varrho)| = |x_{u \circ s_\gamma}(\varrho) - h(\varrho)| \leq \frac{M D_1}{2\sigma_0} \|\chi_\gamma\|_{\infty, I_u}, \\ \forall \gamma > 1, \text{ for almost all } \varrho \in I_u. \end{cases}$$

Thus we conclude from Claim 2 that

$$\lim_{\gamma \rightarrow \infty} \|y_\gamma\|_{\infty, I_u} = \lim_{\gamma \rightarrow \infty} \|x_{u \circ s_\gamma} - h\|_{\infty, I_u} = 0. \quad (2.33)$$

On the other hand, we deduce from the continuity of x_γ that $\|x_{u \circ s_\gamma}\|_{\infty, I_u} = \|x_\gamma\|_{\infty}, \forall \gamma > 0$ (see Lemma 1.2.3). Thus (2.33) implies that there exists $\gamma_1 > 0$ with

$$\|x_\gamma\|_{\infty} = \|x_{u \circ s_\gamma}\|_{\infty, I_u} < 1 + \|h\|_{\infty, I_u} = E_1, \forall \gamma > \gamma_1. \quad (2.34)$$

Thus, we get from equation (2.10), inequality (2.31) and the boundedness of \dot{u} , that \dot{x}_γ is bounded by a number that does not depend on γ . Therefore, by (2.12) and the continuity of ν , there exists some $E > 0$ such that

$$\|F_\gamma\|_{\infty} \leq E, \forall \gamma > \gamma_1. \quad (2.35)$$

On the other hand, by (2.28), inequality (2.31) and the boundedness of h , there exist $D_2 > 0$ and $D_3 > 0$ such that for all $\gamma > \gamma_1$ we have

$$\|\dot{y}_\gamma\|_{\infty, I_u} \leq D_2 \|y_\gamma\|_{\infty, I_u} + D_3 \|\chi_\gamma\|_{\infty, I_u}.$$

Thus, Claim 2 and (2.33) implies that

$$\lim_{\gamma \rightarrow \infty} \|\dot{y}_\gamma\|_{\infty, I_u} = \lim_{\gamma \rightarrow \infty} \|\dot{x}_{u \circ s_\gamma} - \dot{h}\|_{\infty, I_u} = 0.$$

This means that $x_{u \circ s_\gamma}$ converges to h in $W^{1, \infty}(I_u, \mathbb{R})$ as $\gamma \rightarrow \infty$ because of (2.33). Hence, (2.15), the boundedness of v_u and the continuity of ν imply that $\lim_{\gamma \rightarrow \infty} \|\varphi_{u \circ s_\gamma} - \varphi_u^*\|_{\infty, I_u} = 0$ (convergence in $L^\infty(I_u, \mathbb{R})$), where $\varphi_u^* = \sigma_0 h + \nu(0) \in W^{1, \infty}(I_u, \mathbb{R})$. This fact proves the first part of Lemma 2.2.2.

Now, assume that the input u is T -periodic, then ψ_u and $\dot{\psi}_u$ are $\rho_u(T)$ -periodic (see Lemma 1.2.4). For any positive integer k , let

$$h_k(\varrho) = h(\rho_u(T)k + \varrho), \forall \varrho \in [0, \rho_u(T)].$$

The periodicity of $\dot{\psi}_u$ implies that the function h_k satisfies (2.23) for each k , with initial condition $h_k(0) = h(\rho_u(T)k)$. Let k_1, k_2 be positive integers. Let $V_{k_1, k_2} = (h_{k_1} - h_{k_2})^2$, then we have $\dot{V}_{k_1, k_2} = \frac{-2\sigma_0}{\mu(0)} V_{k_1, k_2}$, so that

$$V_{k_1, k_2}(\varrho) = V_{k_1, k_2}(0) e^{\frac{-2\sigma_0}{\mu(0)} \varrho}, \forall \varrho \in [0, \rho_u(T)]. \quad (2.36)$$

Therefore, we get

$$V_{k_1, k_2}(\rho_u(T)) = V_{k_1, k_2}(0) \beta,$$

where $\beta = e^{-2\rho_u(T)\sigma_0/\mu(0)} \in (0, 1)$. Observe that $V_{k_1, k_2}(\rho_u(T)) = V_{k_1+1, k_2+1}(0)$ so that $V_{k_1+1, k_2+1}(0) = \beta V_{k_1, k_2}(0)$. Therefore, it can be verified by induction that

$$V_{k_1, k_2}(0) \leq \beta^{\min\{k_1, k_2\}} D_4,$$

where D_4 is a positive constant that depends on $\|h\|_\infty$. This means that $V_{k_1, k_2}(0)$ converges to 0 as k_1 and k_2 go to ∞ . Thus, we obtain from (2.36) that

$$\|V_{k_1, k_2}\|_{\infty, [0, \rho_u(T)]} \rightarrow 0, \text{ as } k_1, k_2 \rightarrow \infty,$$

which means that the sequence of functions h_k is a Cauchy sequence in the Banach space $C^0([0, \rho_u(T)], \mathbb{R})$. This implies the sequence h_k converges with respect to the norm $\|\cdot\|_{\infty, [0, \rho_u(T)]}$ to a continuous function h_∞ . By applying the Dominated Lebesgue Convergence Theorem (see Theorem 1.1.1) in equation (2.23) written in the integral form, we deduce that h_∞ satisfies the same equation (2.23). Hence $\varphi_u^\circ \in W^{1, \infty}([0, \rho_u(T)], \mathbb{R})$. Observe that $h_\infty(0)$ may be different than x_0 . As a conclusion we obtain

$$\lim_{k \rightarrow \infty} \|\varphi_{u, k}^* - \varphi_u^\circ\|_{\infty, [0, \rho_u(T)]} = 0,$$

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where $\varphi_u^\circ(\varrho) = \sigma_0 h_\infty(\varrho) + \nu(0)$, $\forall \varrho \in [0, \rho_u(T)]$ which ends the proof of the second part of Lemma 2.2.2. On the other hand, if the input u is wave periodic, Lemma 1.2.4 states that ψ_u is also wave periodic and $\dot{\psi}_u(\varrho) = 1$ for almost all $\varrho \in (0, \rho_u(T^+))$ and $\dot{\psi}_u(\varrho) = -1$ for almost all $\varrho \in (\rho_u(T^+), \rho_u(T))$. Thus, we obtain from (2.25) for any positive integer k that

$$h(\rho_u(T)k) = e^{-\sigma_0 \rho_u(T)k/\mu(0)} [x_0 + R_k], \quad (2.37)$$

where

$$\begin{aligned} R_k &= \sum_{j=1}^k \left[\int_{\rho_u(T)(j-1)}^{\rho_u(T)(j-1)+\rho_u(T^+)} e^{\sigma_0 \tau/\mu(0)} d\tau - \int_{\rho_u(T)(j-1)+\rho_u(T^+)}^{\rho_u(T)j} e^{\sigma_0 \tau/\mu(0)} d\tau \right] \\ &= \frac{\mu(0)}{\sigma_0} \sum_{j=1}^k \left[e^{c(j-1)} \left(2e^{\frac{\sigma_0+\rho_u(T^+)}{\mu(0)}} - 1 \right) - e^{cj} \right]. \end{aligned}$$

with $c = \sigma_0 \rho_u(T) / \mu(0) > 0$. Hence we deduce from (2.37) that

$$\begin{aligned} h(\rho_u(T)k) &= x_0 e^{-ck} + \frac{\mu(0)}{\sigma_0} \left[\left(2e^{\frac{\sigma_0+\rho_u(T^+)}{\mu(0)}} - 1 \right) e^{-ck} \sum_{j=1}^k e^{c(j-1)} \right. \\ &\quad \left. - e^{-ck} \sum_{j=1}^k e^{cj} \right]. \end{aligned} \quad (2.38)$$

It can be checked that

$$\lim_{k \rightarrow \infty} e^{-ck} \sum_{j=1}^k e^{c(j-1)} = \frac{1}{e^c - 1}.$$

Therefore, we obtain from (2.38) that

$$h_\infty(0) = \lim_{k \rightarrow \infty} h(\rho_u(T)k) = \frac{\mu(0)}{\sigma_0(e^c - 1)} \left(2e^{\frac{\sigma_0 \rho_u(T^+)}{\mu(0)}} - 1 - e^c \right)$$

From the relation $\varphi^\circ(\varrho) = \sigma_0 h_\infty(\varrho) + \nu(0)$, $\forall \varrho \in I_u$ it follows that

$$\varphi^\circ(0) = \sigma_0 h_\infty(0) + \nu(0) = \frac{\mu(0)}{e^c - 1} \left(2e^{\frac{\sigma_0 \rho_u(T^+)}{\mu(0)}} - 1 - e^c \right) + \nu(0),$$

where $c = \sigma_0 \rho_u(T) / \mu(0)$.

Observe that the function h_k satisfies (2.23) for each k . Thus, using the Dominated Lebesgue Theorem in equation (2.23) written in the integral form, we deduce that h_∞ satisfies the same equation (2.23). Thus, we have

$$h_\infty(\varrho) = e^{\frac{-\sigma_0 \varrho}{\mu(0)}} \left[h_\infty(0) + \int_0^\varrho e^{\frac{\sigma_0 \tau}{\mu(0)}} d\tau \right], \forall \varrho \in [0, \rho_u(T^+)],$$

$$\begin{cases} \forall \varrho \in [\rho_u(T^+), \rho_u(T)] \text{ we have} \\ h_\infty(\varrho) = e^{\frac{-\sigma_0 \varrho}{\mu(0)}} \left[h_\infty(0) + \int_0^{\rho_u(T^+)} e^{\frac{\sigma_0 \tau}{\mu(0)}} d\tau - \int_{\rho_u(T^+)}^\varrho e^{\frac{\sigma_0 \tau}{\mu(0)}} d\tau \right] \end{cases}$$

so that

$$h_\infty(\varrho) = e^{\frac{-\sigma_0 \varrho}{\mu(0)}} \left[h_\infty(0) + \frac{\mu(0)}{\sigma_0} \left(e^{\frac{\sigma_0 \varrho}{\mu(0)}} - 1 \right) \right], \forall \varrho \in [0, \rho_u(T^+)],$$

$$\begin{cases} \forall \varrho \in [\rho_u(T^+), \rho_u(T)] \text{ we have} \\ h_\infty(\varrho) = e^{\frac{-\sigma_0 \varrho}{\mu(0)}} \left[h_\infty(0) + \frac{\mu(0)}{\sigma_0} \left(2e^{\frac{\sigma_0 \rho_u(T^+)}{\mu(0)}} - e^{\frac{\sigma_0 \varrho}{\mu(0)}} - 1 \right) \right] \end{cases}$$

The last part of Lemma 2.2.2 follows from the relation $\varphi^\circ(\varrho) = \sigma_0 h_\infty(\varrho) + \nu(0)$, $\forall \varrho \in I_u$. \square

Example 2.2.3. Consider the LuGre model (2.3)-(2.5) with $\nu(\vartheta)$ taking the form $\nu(\vartheta) = \sigma_2 \vartheta$, $\forall \vartheta \in \mathbb{R}$, where the parameter σ_2 is the viscous friction coefficient. A possible choice for $\mu(\vartheta)$ that leads to a reasonable approximation of the Stribeck effect is [25]:

$$\mu(\vartheta) = F_C + (F_S - F_C) e^{-|\vartheta/v_s|^\alpha}, \forall \vartheta \in \mathbb{R}, \quad (2.39)$$

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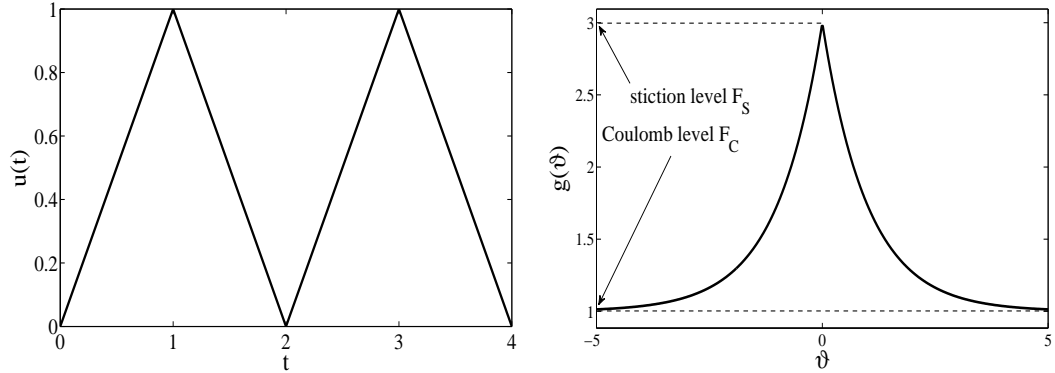


Figure 2.7: *Left* input $u(t)$ versus t . *Right* $g(v)$ versus v in Example 2.

where $F_C > 0$ is the Coulomb friction force, $F_S > 0$ is the stiction force, $v_s > 0$ is the Stribeck velocity, and α is a positive constant.

Take $\sigma_0 = 4$ N/m, $v_s = 0.001$ m/s, $F_S = 3$ N, $F_C = 1$ N, $\sigma_1 = 1$ N.s/m, $\sigma_2 = 1$ N.s/m, and $x(0) = 0$ m. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ be the wave periodic function of period $T = 2$ s and with $T^+ = 1$ s, such that $u(t) = t$ (in meters), $\forall t \in [0, 1]$ s, and $u(t) = 2 - t$, $\forall t \in [1, 2]$ s. Then ρ_u is the identity mapping and hence $I_u = \mathbb{R}_+$, $\psi_u = u$ and $v_u = \dot{u}$ a.e. Note that $T = \rho_u(T) = 2$ and $T^+ = \rho_u(T^+) = 1$. The input u is presented in Figure 2.7 left, and the graph of the function g is given in Figure 2.7 right.

Lemma 2.2.2 implies that the operator \mathcal{H}_o is consistent with respect to input u and initial condition x_0 ; that is

$$\lim_{\gamma \rightarrow \infty} \|\varphi_{u \circ s_\gamma} - \varphi_u^*\|_{\infty, I_u} = 0,$$

where the function $\varphi_u^* \in W^{1,\infty}(I_u, \mathbb{R})$ is defined as

$$\varphi_u^*(\varrho) = 4e^{-\frac{4}{3}\varrho} \int_0^{\varrho} e^{4\tau/3} \dot{\psi}_u(\tau) d\tau, \forall \varrho \in I_u = \mathbb{R}_+.$$

Moreover, the operator \mathcal{H}_o is strongly consistent with respect to input u and

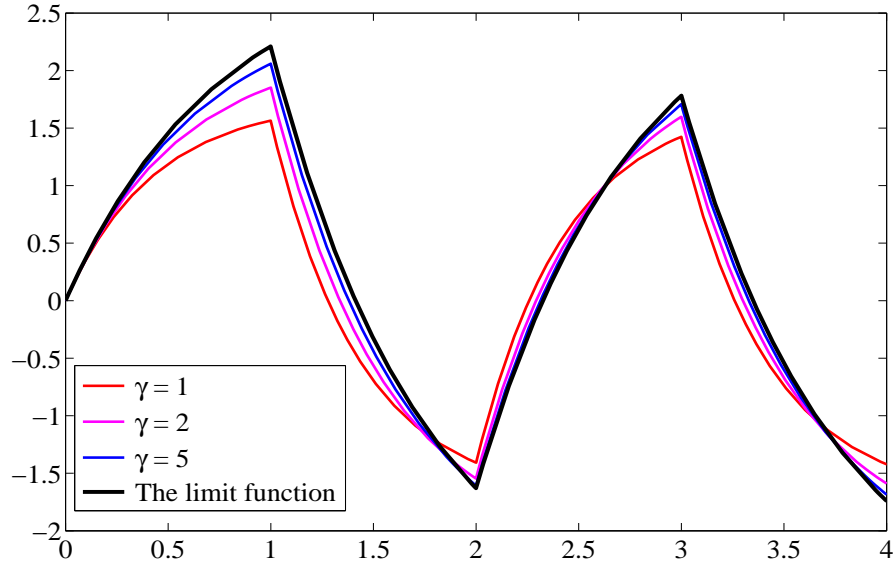


Figure 2.8: $\varphi_{uos_\gamma}(\varrho)$ versus ϱ for different values of γ .

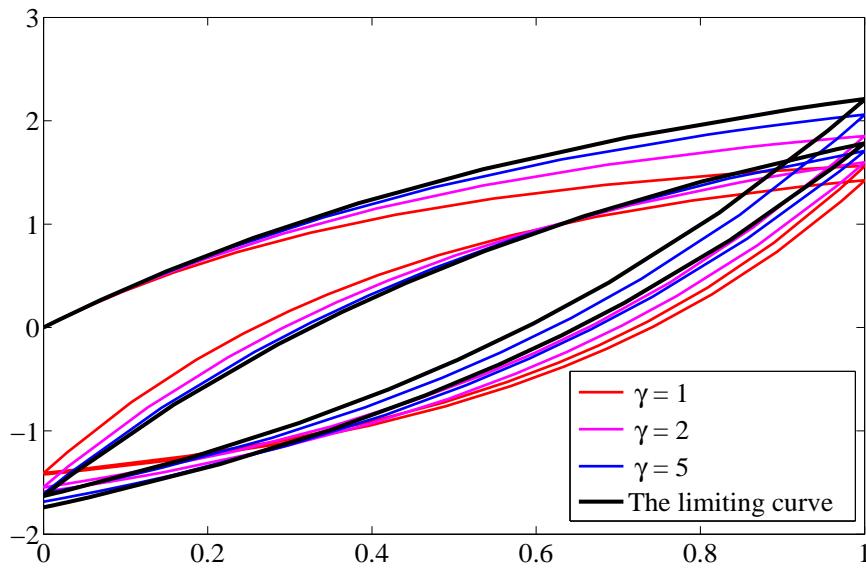
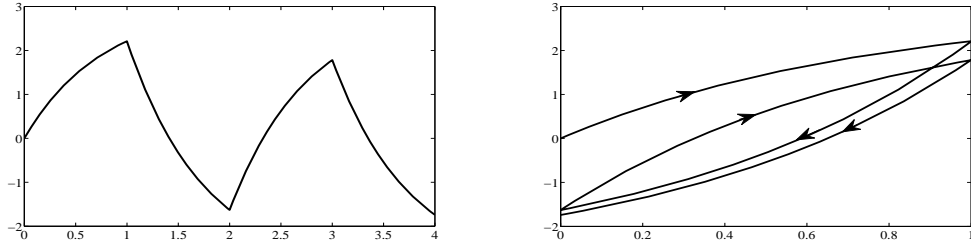


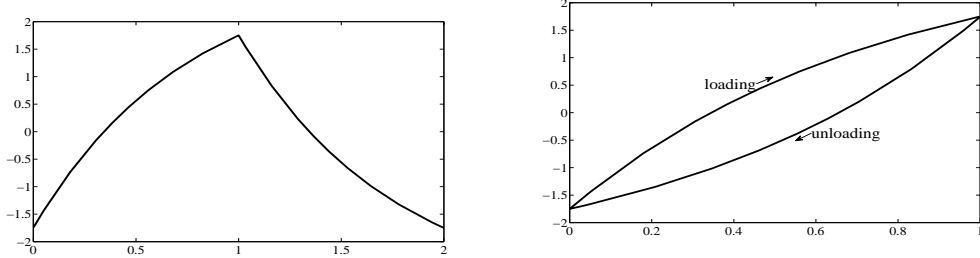
Figure 2.9: $\varphi_{uos_\gamma}(\varrho)$ versus $\psi_u(\varrho)$ for different values of γ .

2.2 Consistency of the LuGre model

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(a) $\varphi_u^*(\varrho)$ versus ϱ (the limit function in Figure 2.8). (b) $\varphi_u^*(\varrho)$ versus $\psi_u(\varrho)$ (the limiting curve in Figure 2.9).



(c) $\varphi_u^o(\varrho)$ versus ϱ .

(d) $\varphi_u^o(\varrho)$ versus $\psi_u(\varrho)$ (the hysteresis loop).

Figure 2.10: LuGre friction model in Example 2.2.3

initial condition x_0 ; that is $\lim_{k \rightarrow \infty} \|\varphi_{u,k}^* - \varphi_u^o\|_{\infty, [0,2]} = 0$, where

$$\varphi_u^o(0) = \frac{3}{e^{\frac{8}{3}} - 1} \left(2e^{\frac{4}{3}} - 1 - e^{\frac{8}{3}} \right) \approx -1.7483488,$$

and

$$\varphi_u^o(\varrho) = \begin{cases} e^{-\frac{4\varrho}{3}} [\varphi_u^o(0) - 3] + 3 & \varrho \in [0, 1] \\ e^{-\frac{4\varrho}{3}} [\varphi_u^o(0) + 6e^{\frac{4}{3}} - 3] - 3 & \varrho \in [1, 2] \end{cases}$$

Figure 2.8 shows the uniform convergence of $\varphi_{u \circ s_\gamma}$ to φ_u^* as $\gamma \rightarrow \infty$. The limit function φ_u^* is plotted alone in Figure 2.10a. Figure 2.9 shows that the graphs $\{(\varphi_{u \circ s_\gamma}(\varrho), \psi_u(\varrho)), \varrho \in I_u = \mathbb{R}_+\}$ converge as $\gamma \rightarrow \infty$ to the set $\{(\varphi_u^*(\varrho), \psi_u(\varrho)), \varrho \in I_u = \mathbb{R}_+\}$. Figure 2.10b presents the limiting curve $\{(\varphi_u^*(\varrho), \psi_u(\varrho)), \varrho \in I_u = \mathbb{R}_+\}$. Figure 2.10d shows the plot of hysteresis loop $\{(\varphi_u^o(\varrho), \psi_u(\varrho)), \varrho \in [0, \rho_u(T)] = [0, 2]\}$. Figure 2.10c presents the function $\varphi_u^o(\varrho)$ versus ϱ . Observe that $\varphi_u^o(0) \approx -1.7483488$ is different than $\varphi_u^*(0) = 0$.

2.3 Conclusions

In this chapter, LuGre model is seen as an operator \mathcal{H} that associates to an input u and initial condition x_0 an output $\mathcal{H}(u, x_0)$, all belonging to some appropriate spaces. Following the research carried out in [52], the consistency and strong consistency of the operator are analyzed. The main results of the paper are given in Lemma 3.5.3. To illustrate these results, numerical simulations are carried out in Example 2.2.3.

The generalized Duhem model

3.1 Introduction

In 1897, Duhem [34] proposed an ordinary differential equation-based model that exhibits hysteresis. This model, widely used in structural, electrical and mechanical engineering, gives an analytical description of a smooth hysteretic behavior. It has been used to represent friction [77], hysteresis in magnetorheological dampers [86], and to represent a jump-resonance hysteresis¹ in Duffing oscillator [77].

In its most general form, Duhem model is given by [78]:

$$\dot{x}(t) = f(x(t), u(t)) g(\dot{u}(t)), \quad (3.1)$$

$$x(0) = x_0, \quad (3.2)$$

for almost all $t \geq 0$, x_0 and state $x(t)$ take values in \mathbb{R}^m for some positive integer m , input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$, function $f : \mathbb{R}^m \times \mathbb{R} \rightarrow \mathbb{R}^{m \times r}$ is continuous, and function $g : \mathbb{R} \rightarrow \mathbb{R}^r$ is continuous with $g(0) = \mathbf{0}$, where r and m are positive integers.

Specializations of this general form are the semilinear Duhem model $\dot{x} = (Ax + Bu) g(\dot{u})$ [72], the model $\dot{x} = f_1(x, u) \max(\dot{u}, 0) + f_2(x, u) \min(\dot{u}, 0)$ [69, 66, 50, 51], and the Bouc-Wen model for hysteresis [99, 21, 20, 55, 53].

¹Jump-resonance is a phenomenon observed in nonlinear systems where the output exhibits abrupt jumps when the frequency of the input signal varies

Other specialization of Duhem model are Dahl model [29, 30], Maxwell-slip model [78, 5, 84], and LuGre model [25, 78].

This chapter gives a characterization of different classes of Duhem models in terms of their consistency with the hysteresis behavior. It is organized as follows. The problem statement is formalized in Section 3.2. A classification of functions g that is used throughout the paper, is introduced in Section 3.3. Sections 3.4 and 3.5 present necessary conditions and sufficient ones for the Duhem model to be consistent with the hysteresis behavior. Conclusions are given in Section 3.6.

3.2 Problem statement

First of all, we need to prove the existence of solution for the system (3.1)-(3.2) which is done in the coming passage.

Since g is continuous and $\dot{u} \in L^\infty(\mathbb{R}_+, \mathbb{R})$, we have $g \circ \dot{u} \in L^\infty(\mathbb{R}, \mathbb{R}^r)$. The differential equation (3.1) satisfies Carathéodory conditions, thus, for each initial state $x_0 \in \mathbb{R}^m$, (3.1) has an absolutely continuous solution that is defined on an interval of the form $[0, T)$, $T > 0$ [39, p.4] (see Theorems 1.1.2 and 1.1.3).

Consider the time scale change $s_\gamma(t) = t/\gamma, \gamma > 0, t \geq 0$. When the input $u \circ s_\gamma$ is used instead of u , the system (3.1)-(3.2) becomes

$$\dot{x}_\gamma(t) = f(x_\gamma(t), u \circ s_\gamma(t)) g\left(\frac{1}{\gamma} \dot{u} \circ s_\gamma(t)\right), \quad (3.3)$$

$$x_\gamma(0) = x_0. \quad (3.4)$$

When $\gamma = 1$, system (3.3)-(3.4) reduces to (3.1)-(3.2). For any $\gamma > 0$, define $\sigma_\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ as $\sigma_\gamma = x_\gamma \circ s_{1/\gamma}$. System (3.3)-(3.4) can be re-written as:

$$\sigma_\gamma(t) = x_0 + \gamma \int_0^t f(\sigma_\gamma(\tau), u(\tau)) g\left(\frac{1}{\gamma} \dot{u}(\tau)\right) d\tau, \quad (3.5)$$

3.2 Problem statement

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for all $\gamma > 0$ and for almost all $t \in [0, \omega_\gamma)$, where $[0, \omega_\gamma)$ is the maximal interval of existence of the solution σ_γ .

Observe that Lemma 1.2.3 implies that for any $\gamma > 0$ there exists a unique function $x_{u \circ s_\gamma} \in L^\infty(I_u, \mathbb{R}^m)$ such that $x_{u \circ s_\gamma} \circ \rho_{u \circ s_\gamma} = x_\gamma$ (when $\gamma = 1$, we get $x_u \circ \rho_u = x$). The latter equality is equivalent to $x_{u \circ s_\gamma} \circ \rho_u = \sigma_\gamma$. According to Definition 1.2.1, the system (3.1)-(3.2) is consistent with respect to (u, x_0) if and only if the sequence of functions $x_{u \circ s_\gamma}$ converges in $L^\infty(I_u, \mathbb{R}^m)$.

Proposition 3.2.1. *The system (3.1)-(3.2) is consistent with respect to (u, x_0) in the sense of Definition (1.2.1) if and only if the sequence of function σ_γ converges in $L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ as $\gamma \rightarrow \infty$.*

Proof. To prove the if part, define the causal operator $\mathcal{H} : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n) \times \mathbb{R}^m \rightarrow L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ that maps (u, x_0) to x , where x is given in (3.1)-(3.2). Assume that there exists $\sigma^* \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ such that $\lim_{\gamma \rightarrow \infty} \|\sigma_\gamma - \sigma^*\|_\infty = 0$. We know from equation (3.5), that σ_γ is a sequence of continuous functions. Thus, the function σ^* is continuous as a uniform limit of continuous function. Lemma 1.2.3 implies that there exists a unique continuous function $x_u^* \in L^\infty(I_u, \mathbb{R}^m)$ such that $x_u^* \circ \rho_u = \sigma^*$. Let $\varrho \in I_u$. Since ρ_u is continuous, there exists some $t \geq 0$ such that $\varrho = \rho_u(t)$. We get from the relation $\sigma_\gamma = x_{u \circ s_\gamma} \circ \rho_u$ that for all $\gamma > 0$: $|x_{u \circ s_\gamma}(\varrho) - x_u^*(\varrho)| = |x_{u \circ s_\gamma} \circ \rho_u(t) - x_u^* \circ \rho_u(t)| \leq \|\sigma_\gamma - \sigma^*\|_\infty$. This implies that

$$\|x_{u \circ s_\gamma} - x_u^*\|_{\infty, I_u} \leq \|\sigma_\gamma - \sigma^*\|_\infty,$$

so that $\lim_{\gamma \rightarrow \infty} \|x_{u \circ s_\gamma} - x_u^*\|_{\infty, I_u} = 0$, which means that the system (3.1)-(3.2) is consistent with respect to (u, x_0) .

To prove the only if part, assume that $\lim_{\gamma \rightarrow \infty} \|x_{u \circ s_\gamma} - x_u^*\|_{\infty, I_u} = 0$, then the relation $x_{u \circ s_\gamma} \circ \rho_u = \sigma_\gamma$ implies that for almost all $t \geq 0$:

$$\begin{aligned} |\sigma_\gamma(t) - x_u^* \circ \rho_u(t)| &= |x_{u \circ s_\gamma} \circ \rho_u(t) - x_u^* \circ \rho_u(t)| \\ &\leq \|x_{u \circ s_\gamma} - x_u^*\|_{\infty, I_u}. \end{aligned}$$

Thus, we have $\|\sigma_\gamma - x_u^* \circ \rho_u\|_\infty \leq \|x_{u \circ s_\gamma} - x_u^*\|_{\infty, I_u}$ so that

$$\lim_{\gamma \rightarrow \infty} \|\sigma_\gamma - x_u^* \circ \rho_u\|_\infty = 0.$$

□

Proposition 3.2.1 implies that the consistency of the system (3.1)-(3.2) can be investigated by studying the uniform convergence of the sequence of functions σ_γ instead of $x_{u \circ s_\gamma}$. Thus we know from Definition (1.2.1) that the system (3.1)-(3.2) is a hysteresis only if σ_γ converges uniformly as $\gamma \rightarrow \infty$.

Problem statement: In this chapter, our objective is to derive necessary conditions and sufficient ones for the uniform convergence of the sequence of functions σ_γ as $\gamma \rightarrow \infty$.

The generalized Duhem model represents a wide class of systems and thus we introduced a classification for this system to study the consistency in all possible cases. This classification is presented in the next section (Section 3.3).

3.3 Classification of the function g

This section introduces a classification for the function g that is used throughout this Chapter.

Definition 3.3.1. Let $G \in C^0([t_1, t_2], \mathbb{R})^2$. The right and left local fractional derivative of G at $t_3 \in (t_1, t_2)$ with respect to order $\lambda > 0$ are defined respectively as: [3]

$$d_+^\lambda G(t_3) = \Gamma(1 + \lambda) \lim_{\kappa \rightarrow t_3^+} \frac{G(\kappa) - G(t_3)}{(\kappa - t_3)^\lambda} \in \mathbb{R},$$

$$d_-^\lambda G(t_3) = \Gamma(1 + \lambda) \lim_{\kappa \rightarrow t_3^-} \frac{G(t_3) - G(\kappa)}{(t_3 - \kappa)^\lambda} \in \mathbb{R},$$

² $C^0([t_1, t_2], \mathbb{R}) = \{p : [t_1, t_2] \rightarrow \mathbb{R} \text{ such that } p \text{ is continuous on } [t_1, t_2]\}$

3.3 Classification of the function g

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where Γ is the gamma function (see Definition 1.1.5).

The local fractional derivative of a vector-valued function is the vector of local fractional derivatives of its components.

Definition 3.3.2. The function $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ is said to be of class $\lambda > 0$ if $g(0) = \mathbf{0}$ and the quantities $d_+^\lambda g(0), d_-^\lambda g(0)$ exist, are finite, and at least one of them is nonzero.

Example 3.3.1. Let k be a positive integer. Define $g \in C^0(\mathbb{R}_+, \mathbb{R})$ as $g(\vartheta) = \vartheta^k, \forall \vartheta \in \mathbb{R}$. The function g is of class $\lambda = k$ because the limit $\lim_{\kappa \rightarrow 0} g(\kappa) / \kappa^k$ exists and equals 1 so that both $d_+^\lambda g(0), d_-^\lambda g(0)$ exist, are finite, and are nonzero.

Example 3.3.2. The zero function $g \in C^0(\mathbb{R}_+, \mathbb{R})$ is not of any class because

$$\lim_{\kappa \rightarrow 0^+} g(\kappa) / \kappa^\lambda = \lim_{\kappa \rightarrow 0^-} g(\kappa) / \kappa^\lambda = 0, \forall \lambda > 0.$$

3.3.1 Determination the class of the function g

In this subsection, we provide a procedure for determining the class of a function g . To this end, let $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ satisfies $g(0) = \mathbf{0}$. Define

$$S = \left\{ j_1 \in \{0, 1, 2, \dots\} / \forall i \in \{0, 1, \dots, j_1\} \right. \\ \left. \text{we have } g^{(i)} \text{ exist and } g^{(i)}(0) = \mathbf{0} \right\}. \quad (3.6)$$

The set S is nonempty because $0 \in S$. Note that when g is the zero function, $S = \mathbb{N}$, where \mathbb{N} is the set of positive integers. We assume that S is finite and hence $j = \text{Sup } S \in S$.

Case I: If $j = 0$.

For this case, we present the following propositions.

Proposition 3.3.1. *The function g is of class $\lambda \in (0, 1)$ iff there exist constants $\nu > 0$, $b_1, b_2 \in \mathbb{R}^r$; at least one of the vectors b_1, b_2 is non-zero, and functions $Q_1, Q_2 : \mathbb{R} \rightarrow \mathbb{R}^r$ such that*

$$g(\vartheta) = \begin{cases} b_1 \vartheta^\lambda + Q_1(\vartheta) & \vartheta \in (0, \nu) \\ b_2 (-\vartheta)^\lambda + Q_2(\vartheta) & \vartheta \in (-\nu, 0) \end{cases}$$

where $\lim_{\kappa \rightarrow 0^+} Q_1(\kappa) / \kappa^\lambda = \lim_{\kappa \rightarrow 0^-} Q_2(\kappa) / (-\kappa)^\lambda = \mathbf{0}$.

Proof. The result follows from the fractional Taylor’s expansion in [3]. \square

Proposition 3.3.2. *For a function $g \in C^0(\mathbb{R}, \mathbb{R}^r)$, assume that $g'_+(0)$ and $g'_-(0)$ exist, are finite, and at least one of them is nonzero. Then the function g is of class $\lambda = 1$, and*

$$\lim_{\gamma \rightarrow \infty} \gamma g\left(\frac{\vartheta}{\gamma}\right) = \begin{cases} g'_+(0) \vartheta & \vartheta \geq 0 \\ g'_-(0) \vartheta & \vartheta < 0 \end{cases}$$

which implies that: if $g'(0)$ exists, then

$$\forall \vartheta \in \mathbb{R}, \text{ we have } \lim_{\gamma \rightarrow \infty} \gamma g\left(\frac{\vartheta}{\gamma}\right) = g'(0) \vartheta.$$

Proof. The result comes directly from Proposition 3.3.7. \square

Case II: If $j > 0$ and $g^{(j+1)}(0)$ exists and finite.

For this case, we introduce the following lemma.

Proposition 3.3.3. *Assume that $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ has finite $n \in \mathbb{N}$ derivative $g^{(n)}$ everywhere in an interval $(-\nu, \nu)$, $\nu > 0$ so that $j + 1 < n$. Then the function g is of class $\lambda = j + 1$, and for all $\vartheta \in \mathbb{R}$, we have*

$$\lim_{\gamma \rightarrow \infty} \gamma^{j+1} g(\vartheta/\gamma) = \frac{g^{(j+1)}(0)}{(j+1)!} \vartheta^{j+1}.$$

3.3 Classification of the function g

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Proof. We get from Taylor expansion with remainder [6] that there exist $\vartheta_1 \in (-\nu, \nu)$ such that $\forall \vartheta \in (-\nu, \nu)$:

$$g(\vartheta) = \frac{g^{(j+1)}(0)}{(j+1)!} \vartheta^{j+1} + \frac{g^{(j+2)}(0)}{(j+2)!} \vartheta^{j+2} \dots + \frac{g^{(n)}(\vartheta_1)}{n!} \vartheta^n,$$

where $g^{(n)}(\vartheta_1)$ is bounded. Proposition 3.3.7 and the fact that $g^{(j+1)}(0) \neq \mathbf{0}$ complete the proof. \square

Case III: If $j > 0$ and $g^{(j+1)}(0)$ does not exist or diverges.

Proposition 3.3.4. *Assume that the function $g^{(j)}$ is of class $\mu \in (0, 1)$ ³; that is there exist constants $\nu > 0$, $b_1, b_2 \in \mathbb{R}^r$; at least one of the vectors b_1, b_2 is non-zero, and functions $Q_1, Q_2 : \mathbb{R} \rightarrow \mathbb{R}^r$ such that*

$$g^{(j)}(\vartheta) = \begin{cases} b_1 \vartheta^\lambda + Q_1(\vartheta) & \vartheta \in (0, \nu) \\ b_2 (-\vartheta)^\lambda + Q_2(\vartheta) & \vartheta \in (-\nu, 0) \end{cases}$$

where $\lim_{\kappa \rightarrow 0^+} Q_1(\kappa) / \kappa^\lambda = \lim_{\kappa \rightarrow 0^-} Q_2(\kappa) / (-\kappa)^\lambda = \mathbf{0}$. Suppose that the quantities $d_+^\mu g^{(j)}(\vartheta)$ and $d_-^\mu g^{(j)}(\vartheta)$ exist on an interval $(-\nu, \nu)$, $\nu > 0$. Then, the function g is of class $\lambda = j + \mu$ and for all $\vartheta \in \mathbb{R}$: $\lim_{\gamma \rightarrow \infty} \gamma^{j+\mu} g(\vartheta/\gamma) = g^*(\vartheta)$, where $g^* \in C^0(\mathbb{R}, \mathbb{R}^r)$ is defined as

$$g^*(\vartheta) = \begin{cases} \frac{d_+^\mu g^{(j)}(0)}{\Gamma(j+1+\mu)} \vartheta^{j+\mu} & \vartheta \geq 0 \\ -\frac{d_-^\mu g^{(j)}(0)}{\Gamma(j+1+\mu)} (-\vartheta)^{j+\mu} & \vartheta < 0 \end{cases}$$

Proof. We obtain from the fractional Taylor's theorem [3] that

$$g(\vartheta) = \begin{cases} \frac{d_+^\mu g^{(j)}(0)}{\Gamma(j+1+\mu)} \vartheta^{j+\mu} + Q_1(\vartheta) & \vartheta \in (0, \nu) \\ -\frac{d_-^\mu g^{(j)}(0)}{\Gamma(j+1+\mu)} (-\vartheta)^{j+\mu} + Q_2(\vartheta) & \vartheta \in (-\nu, 0) \end{cases}$$

³look at Proposition 3.3.1.

for some $\nu > 0$, where

$$\frac{Q_1(\vartheta)}{\vartheta^{j+\mu}} \rightarrow 0 \text{ as } \vartheta \rightarrow 0^+ \text{ and } \frac{Q_2(\vartheta)}{(-\vartheta)^{j+\mu}} \rightarrow 0 \text{ as } \vartheta \rightarrow 0^-.$$

Thus, the function g is of class $j + \mu$, and we have

$$\lim_{\gamma \rightarrow \infty} \gamma^{j+\mu} g\left(\frac{\vartheta}{\gamma}\right) = \begin{cases} \frac{\Gamma(1+\mu)b_1}{\Gamma(j+1+\mu)} \vartheta^{j+\mu} & \vartheta \geq 0 \\ -\frac{\Gamma(1+\mu)b_2}{\Gamma(j+1+\mu)} (-\vartheta)^{j+\mu} & \vartheta < 0 \end{cases}$$

□

Example 3.3.3. Define $g \in C^0(\mathbb{R}, \mathbb{R})$ as $g(\vartheta) = \vartheta^{2/3} + 2\vartheta \cos(\vartheta^2), \forall \vartheta \in \mathbb{R}$. Since $g'(0)$ does not exist, we have $S = \{0\}$ and hence $j = \text{Sup} S = 0$, so we are case I. It can be easily shown that $\lim_{\vartheta \rightarrow 0} 2\kappa \cos(\kappa^2) / |\kappa|^{2/3} = 0$, all conditions of Proposition 3.3.1 are satisfied with $\lambda = 2/3, r = 1, b_1 = b_2 = 1$, and $Q_1(\vartheta) = Q_2(\vartheta) = 2\vartheta \cos(\vartheta^2), \forall \vartheta \in \mathbb{R}$. Thus, the function g is of class $\lambda = 2/3$.

Example 3.3.4. Let $g \in C^0(\mathbb{R}, \mathbb{R})$ be defined as $g(\vartheta) = \sin(\vartheta) + \vartheta, \forall \vartheta \in \mathbb{R}$. We have $g(0) = 0$ and $g'(0) = \cos(0) + 1 = 2$. Thus, we get from (3.6) that $S = \{0\}$ and hence $j = \text{Sup} S = 0$. This means that we are in case I. Since $g'(0)$ exists and is equal to 2, we deduce from Proposition 3.3.2 that g is of class $\lambda = 1$, and

$$\forall \vartheta \in \mathbb{R}, \text{ we have } \lim_{\gamma \rightarrow \infty} \gamma g\left(\frac{\vartheta}{\gamma}\right) = 2\vartheta.$$

Example 3.3.5. Define $g \in C^0(\mathbb{R}, \mathbb{R})$ as $g(\vartheta) = \cos(\vartheta) + \vartheta^2/2 - 1, \forall \vartheta \in \mathbb{R}$. It can be easily verified that $g(0) = g'(0) = g''(0) = g'''(0) = 0$ and $g^{(4)}(0) = 1 \neq 0$. Thus, we get $S = \{0, 1, 2, 3\}$ and hence $j = 3$. Since $g^{(4)}(0)$ exists and finite, we are case II. All conditions of Proposition 3.3.3 are satisfied because the function g is infinitely many differentiable on its domain. Thus, g is of

3.3 Classification of the function g

class $\lambda = j + 1 = 4$, and for all $\vartheta \in \mathbb{R}$, we have

$$\lim_{\gamma \rightarrow \infty} \gamma^4 g(\vartheta/\gamma) = \frac{g^{(4)}(0)}{24} \vartheta^4.$$

Example 3.3.6. Let $g \in C^0(\mathbb{R}, \mathbb{R})$ such that $g(\vartheta) = \frac{3}{5}\vartheta^{5/3} + \sin(\vartheta^2)$, $\forall \vartheta \in \mathbb{R}$. We have $g'(0)$ exists and equals 0 but $g''(0)$ does not exist. Thus, we obtain $S = \{0, 1\}$ so that $j = \text{Sup } S = 1$. This means that we are case III. The derivative function of the g is function studied in Example 3.3.3. Thus, using the result of Example 3.3.3, we get that the function g' is of class $\mu = 2/3$ with $b_1 = b_2 = 1$. It can be easily seen that all conditions of Proposition 3.3.1 are satisfied. Thus, the function g is of class $\lambda = j + \mu = 5/3$ with

$$\lim_{\gamma \rightarrow \infty} \gamma^{5/3} g\left(\frac{\vartheta}{\gamma}\right) = \begin{cases} \frac{\Gamma(\frac{5}{3})}{\Gamma(\frac{8}{3})} \vartheta^{5/3} & \vartheta \geq 0 \\ \frac{-\Gamma(\frac{5}{3})}{\Gamma(\frac{8}{3})} (-\vartheta)^{5/3} & \vartheta < 0 \end{cases} = \begin{cases} \frac{3}{5} \vartheta^{5/3} & \vartheta \geq 0 \\ \frac{-3}{5} (-\vartheta)^{5/3} & \vartheta < 0 \end{cases}$$

Proposition 3.3.5. Let $g, g_1, g_2 \in C^0(\mathbb{R}, \mathbb{R}^r)$ be such that

$$g(\vartheta) = \begin{cases} g_1(\vartheta) & \vartheta \geq 0 \\ g_2(\vartheta) & \vartheta < 0 \end{cases}$$

Define $G_1, G_2 \in C^0(\mathbb{R}, \mathbb{R}^r)$ for any $\vartheta \in \mathbb{R}$ as

$$G_1(\vartheta) = g_1(|\vartheta|) \text{ and } G_2(\vartheta) = g_2(-|\vartheta|).$$

Assume that the functions G_1 and G_2 are respectively of class λ_1 and λ_2 . Then the function g is of class $\min(\lambda_1, \lambda_2)$.

Proof. We have

$$\begin{aligned} \lim_{\kappa \rightarrow 0^-} \frac{G_1(\kappa)}{(-\kappa)^{\lambda_1}} &= \lim_{\kappa \rightarrow 0^-} \frac{g_1(-\kappa)}{(-\kappa)^{\lambda_1}} = \lim_{\kappa \rightarrow 0^+} \frac{g_1(\kappa)}{\kappa^{\lambda_1}} = \lim_{\kappa \rightarrow 0^+} \frac{G_1(\kappa)}{\kappa^{\lambda_1}}, \\ \lim_{\kappa \rightarrow 0^+} \frac{G_2(\kappa)}{\kappa^{\lambda_2}} &= \lim_{\kappa \rightarrow 0^+} \frac{g_2(-\kappa)}{\kappa^{\lambda_2}} = \lim_{\kappa \rightarrow 0^-} \frac{g_2(\kappa)}{(-\kappa)^{\lambda_2}} = \lim_{\kappa \rightarrow 0^-} \frac{G_2(\kappa)}{(-\kappa)^{\lambda_2}}. \end{aligned}$$

Thus, we get

$$\lim_{\kappa \rightarrow 0^+} \frac{g_1(\kappa)}{\kappa^{\lambda_1}} = \lim_{\kappa \rightarrow 0} \frac{G_1(\kappa)}{|\kappa|^{\lambda_1}}, \quad (3.7)$$

$$\lim_{\kappa \rightarrow 0^-} \frac{g_2(\kappa)}{(-\kappa)^{\lambda_2}} = \lim_{\kappa \rightarrow 0} \frac{G_2(\kappa)}{|\kappa|^{\lambda_2}}. \quad (3.8)$$

Without loss of generality, assume that $\lambda_1 < \lambda_2$. Then, we deduce from (3.7)-(3.8) that ⁴

$$\begin{aligned} d_+^{\lambda_1} g(0) &= d_+^{\lambda_1} G_1(0) = d_-^{\lambda_1} G_1(0) \neq \mathbf{0}, \\ d_-^{\lambda_1} g(0) &= \lim_{\kappa \rightarrow 0^-} \frac{g_2(\kappa)}{(-\kappa)^{\lambda_1}} = \lim_{\kappa \rightarrow 0} |\kappa|^{\lambda_2 - \lambda_1} \frac{g_2(\kappa)}{|\kappa|^{\lambda_2}} = 0 \times \lim_{\kappa \rightarrow 0} \frac{g_2(\kappa)}{|\kappa|^{\lambda_2}} = \mathbf{0}. \end{aligned}$$

which completes the proof. □

Example 3.3.7. Define $g \in C^0(\mathbb{R}, \mathbb{R})$ as

$$g(\vartheta) = \begin{cases} \sin(\vartheta) + \vartheta & \vartheta \geq 0 \\ \cos(\vartheta) + \vartheta^2/2 - 1 & \vartheta < 0 \end{cases}$$

then, the functions $g_1, g_2 \in C^0(\mathbb{R}, \mathbb{R})$ are given for any $\vartheta \in \mathbb{R}$ by $g_1(\vartheta) = \sin(\vartheta) + \vartheta$ and $g_2(\vartheta) = \cos(\vartheta) + \vartheta^2/2 - 1$. Thus, the functions $G_1, G_2 \in C^0(\mathbb{R}, \mathbb{R})$ are given for any $\vartheta \in \mathbb{R}$ by

$$G_1(\vartheta) = g_1(|\vartheta|) = \sin(\vartheta) + \vartheta,$$

$$G_2(\vartheta) = g_2(-|\vartheta|) = \cos(\vartheta) + \vartheta^2/2 - 1.$$

Examples 3.3.4 and 3.3.5 proved that the functions G_1 and G_2 are respectively of class $\lambda_1 = 1$ and $\lambda_2 = 4$. Thus, we deduce from Proposition 3.3.5 that the function g is of class $\lambda = \min(\lambda_1, \lambda_2) = 1$.

To determine the class of more complicated functions, we introduce the following lemma:

⁴ $d_+^{\lambda_1} G_1(0) = d_-^{\lambda_1} G_1(0) \neq 0$ because the function G_1 is of class λ_1 and because of the symmetry property the function has.

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Lemma 3.3.1. *Assume that the functions $g_1, g_2 \in C^0(\mathbb{R}, \mathbb{R}^r)$ are of class λ_1, λ_2 respectively. Then, we have*

1. *The functions $|g_1|$ and $b g_1$ are of class λ_1 for any nonzero constant $b \in \mathbb{R}$.*
2. *The function $g_1 + g_2$ is of class $\min(\lambda_1, \lambda_2)$ if $\lambda_1 \neq \lambda_2$.*
3. *The function $g_1 g_2$ is of class $\lambda_1 + \lambda_2$ if at most one of the vectors $d_+^{\lambda_1} g_1(0)$, $d_-^{\lambda_1} g_1(0)$, $d_+^{\lambda_2} g_2(0)$, and $d_-^{\lambda_2} g_2(0)$ is zero.*
4. *The function $g_1 \circ g_2$ is of class $\lambda_1 \lambda_2$ if at most one of the vectors $d_+^{\lambda_1} g_1(0)$, $d_-^{\lambda_1} g_1(0)$, $d_+^{\lambda_2} g_2(0)$, and $d_-^{\lambda_2} g_2(0)$ is zero and if each component of g_2 doesn't change its sign on some right (resp. left) neighborhood about the origin.*

Proof. Items (1)-(3) are similar to the classical rules of local fractional differentiation (see [104, 3]) and can be easily verified from Definition 3.3.2. To prove the fourth item: Assume; without loss of generality, that $r = 1$ and that there exists some $\kappa_0 > 0$ such that $g_2(\kappa) < 0, \forall \kappa \in (0, \kappa_0)$ and $g_2(\kappa) > 0, \forall \kappa \in (-\kappa_0, 0)$. Thus, $d_+^{\lambda_2} g_2(0) \leq 0$ and $d_-^{\lambda_2} g_2(0) \geq 0$. Look at

$$\begin{aligned} \lim_{\kappa \rightarrow 0^+} \frac{g_1 \circ g_2(\kappa)}{\kappa^{\lambda_1 \lambda_2}} &= \lim_{l \rightarrow 0^-} \frac{g_1(l)}{(-l)^{\lambda_1}} \lim_{\kappa \rightarrow 0^+} \left(\frac{-g_2(\kappa)}{\kappa^{\lambda_2}} \right)^{\lambda_1} \\ &= \frac{d_-^{\lambda_1} g_1(0)}{\Gamma(1 + \lambda_1)} \left(-\frac{d_+^{\lambda_2} g_2(0)}{\Gamma(1 + \lambda_2)} \right)^{\lambda_1}, \end{aligned}$$

$$\begin{aligned} \lim_{\kappa \rightarrow 0^-} \frac{g_1 \circ g_2(\kappa)}{(-\kappa)^{\lambda_1 \lambda_2}} &= \lim_{l \rightarrow 0^+} \frac{g_1(l)}{l^{\lambda_1}} \lim_{\kappa \rightarrow 0^-} \left(\frac{g_2(\kappa)}{(-\kappa)^{\lambda_2}} \right)^{\lambda_1} \\ &= \frac{d_+^{\lambda_1} g_1(0)}{\Gamma(1 + \lambda_1)} \left(\frac{d_-^{\lambda_2} g_2(0)}{\Gamma(1 + \lambda_2)} \right)^{\lambda_1}, \end{aligned}$$

which complete the proof. □

Example 3.3.8. Let $g \in C^0(\mathbb{R}, \mathbb{R})$ be defined as

$$g(\vartheta) = (\sin(\vartheta) + \vartheta) (\cos(\vartheta) + \vartheta^2/2 - 1), \forall \vartheta \in \mathbb{R}.$$

Observe that $g = g_1 g_2$, where $g_1, g_2 \in C^0(\mathbb{R}, \mathbb{R})$ are defined as

$$g_1(\vartheta) = (\sin(\vartheta) + \vartheta), \forall \vartheta \in \mathbb{R},$$

$$g_2(\vartheta) = (\cos(\vartheta) + \vartheta^2/2 - 1), \forall \vartheta \in \mathbb{R}.$$

We deduced in Examples 3.3.4 and 3.3.5 that the functions g_1 and g_2 are respectively of class $\lambda_1 = 1$ and $\lambda_2 = 4$. Furthermore, we have

$$\lim_{\vartheta \rightarrow 0^+} \frac{g_1(\vartheta)}{\vartheta} = \lim_{\vartheta \rightarrow 0^+} \frac{\sin(\vartheta) + \vartheta}{\vartheta} = 2,$$

$$\lim_{\vartheta \rightarrow 0^-} \frac{g_1(\vartheta)}{-\vartheta} = \lim_{\vartheta \rightarrow 0^-} \frac{\sin(\vartheta) + \vartheta}{-\vartheta} = -2,$$

so that both quantities $d_+^1 g_1(0)$ and $d_-^1 g_1(0)$ are non-zero. Moreover, at least one of $d_+^4 g_2(0)$ and $d_-^4 g_2(0)$ is non-zero because g_2 is of class $\lambda_2 = 4$. Therefore, we conclude from part 3 of Lemma 3.3.1, that the function g is of class $\lambda_1 + \lambda_2 = 1 + 4 = 5$.

Proposition 3.3.6. Let $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ be such that

$$g = \begin{bmatrix} g_1 & g_2 & \cdots & g_r \end{bmatrix}^T,$$

where $g_1, g_2, \dots, g_r \in C^0(\mathbb{R}, \mathbb{R})$. Then, if for $i \in \{1, 2, \dots, r\}$, the function g_i is of class λ_i , then the function g is of class $\lambda = \min(\lambda_1, \lambda_2, \dots, \lambda_r)$ ⁵.

Proof. Let $i \in \{0, 1, \dots, r\}$. Functions g_i and $\begin{bmatrix} 0 & \cdots & 0 & g_i & 0 & \cdots & 0 \end{bmatrix}^T$ are of the same class because of Definition 3.3.2. Thus, the result follows from part 2 of Lemma 3.3.1 because $g = \sum_{i=1}^r \begin{bmatrix} 0 & \cdots & 0 & g_i & 0 & \cdots & 0 \end{bmatrix}^T$. \square

⁵The converse need not be true, for instance, let $r = 2$, $g_1 \equiv 0$, and g_2 be the identity mapping

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3.3.2 Mathematical preliminaries

Proposition 3.3.7. *The function $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ is of class λ if and only if*

$$\forall \vartheta \in \mathbb{R}, \text{ we have } \lim_{\gamma \rightarrow \infty} \gamma^\lambda g\left(\frac{\vartheta}{\gamma}\right) = g^*(\vartheta),$$

where $g^* \in C^0(\mathbb{R}, \mathbb{R}^r)$ is defined as

$$g^*(\vartheta) = \begin{cases} \vartheta^\lambda \lim_{\kappa \rightarrow 0^+} \frac{g(\kappa)}{\kappa^\lambda} & \vartheta \geq 0 \\ (-\vartheta)^\lambda \lim_{\kappa \rightarrow 0^-} \frac{g(\kappa)}{(-\kappa)^\lambda} & \vartheta < 0 \end{cases} \quad (3.9)$$

Proof. Immediate using of the change of variables $\kappa = \vartheta/\gamma$. □

Proposition 3.3.8. *If the function $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ is of class λ , then*

$$\lim_{\gamma \rightarrow \infty} \left\| \gamma^\lambda g(\dot{u}/\gamma) - g^*(\dot{u}) \right\|_\infty = 0,$$

where $g^* \in C^0(\mathbb{R}, \mathbb{R}^r)$ is defined in (3.9).

Proof. The result is trivial when u is constant. Assume that u is non-constant. Given $\varepsilon > 0$. Since $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ is of class λ , there exists some c_ε ; that depend solely on ε , such that

$$\left| \frac{g(\vartheta)}{\vartheta^\lambda} - \frac{d_+^\lambda g(0)}{\Gamma(1+\lambda)} \right| < \frac{\varepsilon}{\|\dot{u}\|_\infty^\lambda}, \text{ whenever } 0 < \vartheta < c_\varepsilon,$$

$$\left| \frac{g(\vartheta)}{(-\vartheta)^\lambda} - \frac{d_-^\lambda g(0)}{\Gamma(1+\lambda)} \right| < \frac{\varepsilon}{\|\dot{u}\|_\infty^\lambda}, \text{ whenever } -c_\varepsilon < \vartheta < 0.$$

The boundedness of \dot{u} implies that there exists a positive constant γ^* such that $|\dot{u}(t)|/\gamma < c_\varepsilon, \forall \gamma > \gamma^*$. Thus, we have for all $\gamma > \gamma^*$ that

$$\left| \gamma^\lambda \frac{g(\dot{u}/\gamma)}{(\dot{u})^\lambda} - \frac{d_+^\lambda g(0)}{\Gamma(1+\lambda)} \right| < \frac{\varepsilon}{\|\dot{u}\|_\infty^\lambda}, \text{ whenever } \dot{u} > 0,$$

$$\left| \gamma^\lambda \frac{g(\dot{u}/\gamma)}{(-\dot{u})^\lambda} - \frac{d_-^\lambda g(0)}{\Gamma(1+\lambda)} \right| < \frac{\varepsilon}{\|\dot{u}\|_\infty^\lambda}, \text{ whenever } \dot{u} < 0.$$

Thus, we obtain from (3.9) that

$$|\dot{u}|^\lambda \max \left(\left| \gamma^\lambda \frac{g(\dot{u})}{|\dot{u}|^\lambda} - \frac{d_+^\lambda g(0)}{\Gamma(1+\lambda)} \right|, \left| \gamma^\lambda \frac{g(\dot{u})}{|\dot{u}|^\lambda} - \frac{d_-^\lambda g(0)}{\Gamma(1+\lambda)} \right| \right) \leq \left. \vphantom{\max} \right\} < \varepsilon, \forall \gamma > \gamma^*,$$

which completes the proof. \square

Proposition 3.3.9. *If the function $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ is of class λ for some $\lambda > 0$, then it cannot be of any class $\mu > 0$ different than λ .*

Proof. Assume that the function $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ is of class λ and μ with $\lambda < \mu$. Then,

$$\begin{aligned} \frac{d_+^\lambda g(0)}{\Gamma(1+\lambda)} &= \lim_{\kappa \rightarrow 0^+} \frac{g(\kappa)}{\kappa^\lambda} = \lim_{\kappa \rightarrow 0^+} \kappa^{\mu-\lambda} \frac{g(\kappa)}{\kappa^\mu} = 0 \times \frac{d_+^\mu g(0)}{\Gamma(1+\mu)} = \mathbf{0}, \\ \frac{d_-^\lambda g(0)}{\Gamma(1+\lambda)} &= \lim_{\kappa \rightarrow 0^-} \frac{g(\kappa)}{(-\kappa)^\lambda} = \lim_{\kappa \rightarrow 0^-} |\kappa|^{\mu-\lambda} \frac{g(\kappa)}{|\kappa|^\mu} = 0 \times \frac{d_-^\mu g(0)}{\Gamma(1+\mu)} = \mathbf{0}, \end{aligned}$$

which contradicts the fact that g is of class λ . \square

Proposition 3.3.9 states the uniqueness of the class of functions.

Proposition 3.3.10. *If the function $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ is of class $\lambda > 0$, then there exists $g_0 \in C^0(\mathbb{R}, \mathbb{R}_+)$, such that*

$$\left| \gamma^\lambda g\left(\frac{\vartheta}{\gamma}\right) \right| \leq g_0(\vartheta), \forall \gamma > 1, \forall \vartheta \in \mathbb{R}. \quad (3.10)$$

Proof. Without loss of generality, assume that $r = 1$. Definition 3.3.2 implies that there exists some $c_0 > 0$ such that

$$\left| \frac{g(\kappa)}{\kappa^\lambda} - \frac{d_+^\lambda g(0)}{\Gamma(1+\lambda)} \right| < 1, \forall \kappa \in (0, c_0),$$

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$$\left| \frac{g(\kappa)}{(-\kappa)^\lambda} - \frac{d^\lambda g(0)}{\Gamma(1+\lambda)} \right| < 1, \forall \kappa \in (-c_0, 0).$$

Therefore, there exists some $d > 0$ with

$$|g(\kappa)| \leq d |\kappa|^\lambda, \forall |\kappa| < c_0. \quad (3.11)$$

Thus, the substitution $\kappa = \vartheta/\gamma$ implies that

$$\left| \gamma^\lambda g\left(\frac{\vartheta}{\gamma}\right) \right| \leq d |\vartheta|^\lambda, \forall \vartheta \in \mathbb{R}, \forall \gamma > 0 \text{ that satisfy } \frac{|\vartheta|}{\gamma} < c_0. \quad (3.12)$$

Define $g_1 : \mathbb{R} \rightarrow \mathbb{R}_+$, as

$$g_1(\vartheta) = \begin{cases} \sup_{0 \leq s \leq \vartheta} |g(s)| & \vartheta > 0 \\ 0 & \vartheta = 0 \\ \sup_{\vartheta \leq s \leq 0} |g(s)| & \vartheta < 0 \end{cases}$$

The function g_1 is continuous. Moreover, we have $|g(\vartheta)| \leq g_1(\vartheta)$, $\forall \vartheta \in \mathbb{R}$. The function g_1 is non-decreasing on $[0, \infty)$ and non-increasing on $(-\infty, 0]$. This implies that

$$\left| \gamma^\lambda g\left(\frac{\vartheta}{\gamma}\right) \right| \leq \frac{|\vartheta|^\lambda}{c_0^\lambda} g_1(\vartheta), \forall \vartheta \in \mathbb{R}, \forall \gamma > 1 \text{ that satisfy } \frac{|\vartheta|}{\gamma} \geq c_0.$$

Thus, we get from (3.12) for all $\vartheta \in \mathbb{R}$ and all $\gamma > 1$ that

$$\left| \gamma^\lambda g\left(\frac{\vartheta}{\gamma}\right) \right| \leq \max\left(d |\vartheta|^\lambda, \frac{|\vartheta|^\lambda}{c_0^\lambda} g_1(\vartheta)\right) = g_0(\vartheta),$$

which completes the proof. □

3.4 Necessary conditions for consistency

The objective of this section is to derive necessary conditions for the uniform convergence of the sequence of functions σ_γ as $\gamma \rightarrow \infty$.

Lemma 3.4.1. *Assume that the system (3.1)-(3.2) has a unique global solution⁶ for each input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ and initial condition $x_0 \in \mathbb{R}^m$. Assume that the function g is of class $\lambda > 0$. Suppose that there exists a continuous function $Q : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$|x(t)| \leq Q(|x_0|, \|u\|_\infty, \|\dot{u}\|_\infty), \quad \forall t \geq 0, \quad (3.13)$$

for each initial state $x_0 \in \mathbb{R}^m$ and each input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$. Assume that the system (3.1)-(3.2) is consistent with respect to (u, x_0) ; that is there exists $q_u \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ such that (see Proposition 3.2.1)

$$\lim_{\gamma \rightarrow \infty} \|\sigma_\gamma - q_u\|_\infty = 0.$$

Then

if $\lambda = 1$, we have

- $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$.
- We have for all $t \geq 0$ that

$$q_u(t) = x_0 + \int_0^t f(q_u(\tau), u(\tau)) g^*(\dot{u}(\tau)) d\tau, \quad (3.14)$$

where g^* is given in equation (3.9).

if $\lambda \in (0, 1)$, we have

- $q_u \in C^0(\mathbb{R}_+, \mathbb{R}^m) \cap L^\infty(\mathbb{R}_+, \mathbb{R}^m)$.
- $q_u(0) = x_0$.

⁶The standard way to ensure that the system (3.1)-(3.2) admits a unique solution is to prove that the right-hand side of (3.1)-(3.2) is Lipschitz with respect to x . A function $\nu : D \subseteq \mathbb{R}^m \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ is Lipschitz with respect to x if there exists a summable function $l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $|\nu(\alpha_1, t) - \nu(\alpha_2, t)| \leq l(t) |\alpha_1 - \alpha_2|$, for almost all $t \geq 0$ and for all $\alpha_1, \alpha_2 \in \mathbb{R}^m$ that satisfy $(t, \alpha_1), (t, \alpha_2) \in D$ [39]

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- We have for almost all $t \geq 0$ that

$$f(q_u(t), u(t))g^*(\dot{u}(t)) = \mathbf{0}, \quad (3.15)$$

where g^* is defined in equation (3.9).

if $\lambda > 1$, we have $q_u(t) = x_0, \forall t \geq 0$.

Proof. By (3.13), the fact $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$, the continuity of the function Q , and the relation $\|u\|_\infty = \|u \circ s_\gamma\|_\infty, \forall \gamma > 0$, there exists some $a > 0$ independent of γ , and $\gamma_0 > 0$ such that

$$|x_\gamma(t)| \leq Q\left(|x_0|, \|u\|_\infty, \frac{\|\dot{u}\|_\infty}{\gamma}\right) \leq a, \forall t \geq 0, \forall \gamma > \gamma_0,$$

where x_γ is given in (3.3)-(3.4). Thus

$$\|x_\gamma\|_\infty \leq a, \forall \gamma > \gamma_0. \quad (3.16)$$

On the other hand, we conclude from Lemma 1.2.3 that $x_{u \circ s_\gamma} \in C^0(I_u, \mathbb{R}^m) \cap L^\infty(I_u, \mathbb{R}^m)$, and $\|x_{u \circ s_\gamma}\|_{\infty, I_u} = \|x_\gamma\|_\infty$, for all $\gamma > 0$. Hence, the continuity of $x_{u \circ s_\gamma}$ and (3.16) imply that

$$\begin{aligned} |\sigma_\gamma(t)| &= |x_{u \circ s_\gamma} \circ \rho_u(t)| \leq \|x_{u \circ s_\gamma}\|_{\infty, I_u} = \|x_\gamma\|_\infty \\ &\leq a, \forall t \geq 0, \forall \gamma > \gamma_0. \end{aligned} \quad (3.17)$$

Thus, the continuity of f and g , the boundedness of \dot{u} , and Proposition (3.3.10), imply that there exists a constant $b > 0$ independent of γ such that

$$\gamma^\lambda |f(\sigma_\gamma(\tau), u(\tau))g(\dot{u}(\tau)/\gamma)| \leq b, \text{ for almost all } \tau \geq 0, \forall \gamma > 1.$$

Thus, we can apply the Dominated Lebesgue Theorem [88] to get

$$\lim_{\gamma \rightarrow \infty} \gamma^\lambda \int_0^t f(\sigma_\gamma(\tau), u(\tau))g\left(\frac{\dot{u}(\tau)}{\gamma}\right) d\tau$$

$$= \int_0^t f(x_u^* \circ \rho_u(\tau), u(\tau)) g^*(\dot{u}(\tau)) d\tau, \forall t \geq 0. \quad (3.18)$$

On the other hand, since q_u is continuous as a uniform limit of continuous sequence of functions, we have $q_u \in C^0(\mathbb{R}_+, \mathbb{R}^m) \cap L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ and $q_u(0) = x_0$ (note that $\sigma_\gamma(0) = x_0, \forall \gamma > 0$).

When $\lambda = 1$, we obtain from (3.18) and (3.5) that

$$q_u(t) = x_0 + \int_0^t f(q_u(\tau), u(\tau)) g^*(\dot{u}(\tau)) d\tau, \forall t \geq 0.$$

Thus, the continuity of the functions f and g^* along with the boundedness of the functions q_u, u and \dot{u} imply that the function \dot{q}_u is bounded. Therefore, $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ and (3.14) is satisfied.

When $\lambda \in (0, 1)$, we conclude from inequality (3.17) that

$$\lim_{\gamma \rightarrow \infty} \frac{1}{\gamma^{1-\lambda}} \|\sigma_\gamma - x_0\|_\infty = 0.$$

Moreover, equation (3.5) can be written for all $t \geq 0$ as

$$\frac{\sigma_\gamma(t) - x_0}{\gamma^{1-\lambda}} = \gamma^\lambda \int_0^t f(\sigma_\gamma(\tau), u(\tau)) g\left(\frac{\dot{u}(\tau)}{\gamma}\right) d\tau, \quad (3.19)$$

The fact that $\lim_{\gamma \rightarrow 0} \|\sigma_\gamma - x_0\|_\infty / \gamma^{1-\lambda} = 0$, along with (3.18) and (3.19) imply that

$$\int_0^t f(x_u^* \circ \rho_u(\tau), u(\tau)) g^*(\dot{u}(\tau)) d\tau = \mathbf{0}, \forall t \geq 0,$$

which proves (3.15).

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Finally, when $\lambda > 1$, (3.5) implies for all $t \geq 0$ that

$$\sigma_\gamma(t) - x_0 = \gamma^{1-\lambda} \gamma^\lambda \int_0^t f(\sigma_\gamma(\tau), u(\tau)) g\left(\frac{\dot{u}(\tau)}{\gamma}\right) d\tau,$$

and thus, we get from (3.18) that $\lim_{\gamma \rightarrow \infty} \sigma_\gamma(t) = x_0, \forall t \geq 0$. Therefore, the uniqueness of limits and the continuity of q_u imply that $q_u(t) = x_0, \forall t \geq 0$. \square

Remark 3.4.1. Observe that for $\lambda > 1$, the fact that $q_u(t) = x_0, \forall t \geq 0$, means that system (3.1)-(3.2) does not represent a hysteresis behavior [15].

Remark 3.4.2. For the case $\lambda \in (0, 1)$, equation (3.15) and the fact that $q_u(0) = x_0$ imply $f(x_0, u(0)) g(\dot{u}(0)) = \mathbf{0}$, whenever $\dot{u}(0)$ exists.

Now, we provide some examples to illustrate Lemma 3.4.1.

Example 3.4.1. Consider the following LuGre model ⁷

$$\dot{x} = \dot{u} - 10^5 \frac{|\dot{u}|}{\mu(\dot{u})} x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} \dot{u} \\ -10^5 \frac{|\dot{u}|}{\mu(\dot{u})} \end{bmatrix} \quad (3.20)$$

$$x(0) = x_0, \quad (3.21)$$

where $x \in \mathbb{R}$ is the output and the function $\mu : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$\mu(\alpha) = F_C + (F_S - F_C) e^{-|\alpha/v_s|}, \forall \alpha \in \mathbb{R},$$

where $F_C > 0$ is the Coulomb friction force, $F_S > 0$ is the stiction force, and $v_s \in \mathbb{R}$ is the Stribeck velocity.

The sequence of function σ_γ is given by (see equation (3.5))

$$\dot{\sigma}_\gamma(t) = \dot{u}(t) - 10^5 \frac{|\dot{u}(t)|}{\mu\left(\frac{\dot{u}(t)}{\gamma}\right)} \sigma_\gamma(t), \text{ for almost all } t \geq 0.$$

⁷ see system (2.3)-(2.4) in Chapter 2

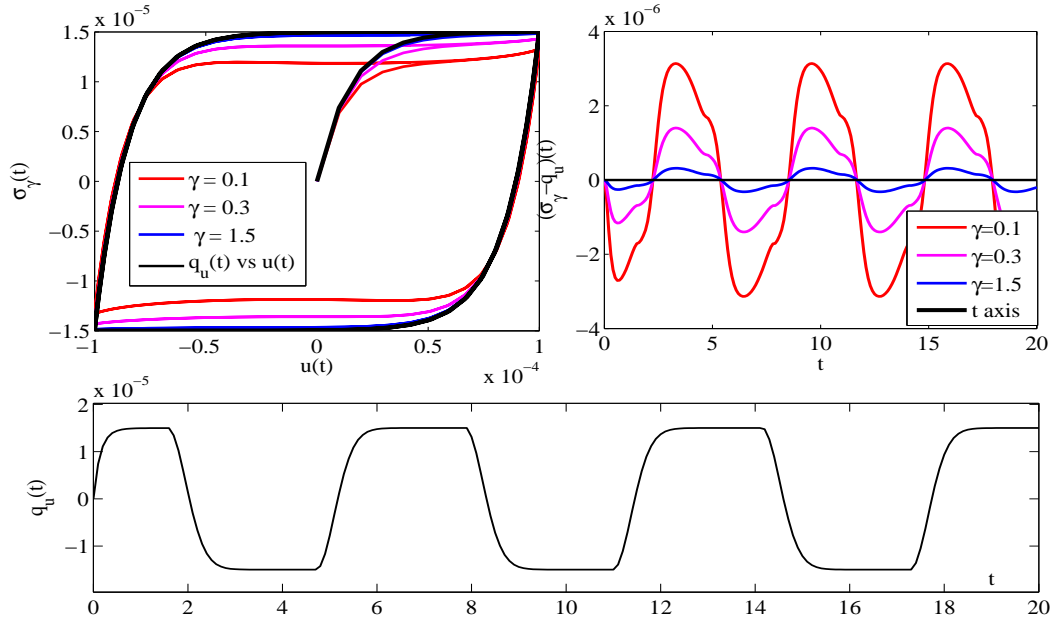


Figure 3.1: Upper left $\sigma_\gamma(t)$ versus $u(t)$ for different values of γ . Upper right $(\sigma_\gamma - q_u)(t)$ versus t for different values of γ . Lower $q_u(t)$ versus t . The plots are for Example 3.4.1, where the solid lines are for q_u .

The following facts follows by Lemma 2.2.2:

- There exist $\gamma_1, E > 0$ such that $\|x_\gamma\|_\infty = \|F_\gamma\|_\infty \leq E, \forall \gamma > \gamma_1$, where $x_\gamma = F_\gamma$ is the output when we use input $u \circ s_\gamma$ instead of u (see system (3.3)-(3.4)).
- $\|\varphi_{u \circ s_\gamma} - \varphi_u^*\|_{\infty, I_u} \rightarrow 0$ as $\gamma \rightarrow \infty$, where the function $\varphi_u^* \in W^{1,\infty}(I_u, \mathbb{R}^m)$ is defined as

$$\varphi_u^*(\varrho) = e^{-\frac{10^5 \varrho}{FS}} \left[x_0 + \int_0^{\varrho} e^{\frac{10^5 \tau}{FS}} \dot{\psi}_u(\tau) d\tau \right], \forall \varrho \in I_u. \quad (3.22)$$

Thus, we conclude from the relation $\sigma_\gamma = \varphi_{u \circ s_\gamma} \circ \rho_u$ that

$$\lim_{\gamma \rightarrow \infty} \frac{1}{\gamma^{1-\lambda}} \|\sigma_\gamma - q_u\|_\infty = 0,$$

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where the function $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ is such that $q_u = \varphi_u^* \circ \rho_u$ (see proposition 3.2.1). Thus, we get from the change of variable $\varrho = \rho_u(t)$ that

$$q_u(t) = e^{\frac{-10^5 \rho_u(t)}{F_S}} \left[x_0 + \int_0^{\rho_u(t)} e^{\frac{10^5 \tau}{F_S}} \dot{\psi}_u(\tau) d\tau \right], \forall t \geq 0,$$

so that

$$q_u(t) = e^{\frac{-10^5 \rho_u(t)}{F_S}} \left[x_0 + \int_0^t e^{\frac{10^5 \rho_u(\tau)}{F_S}} \dot{u}(\tau) d\tau \right], \quad (3.23)$$

where the relation $\psi_u \circ \rho_u = u$ has been used. Thus, all conditions of Lemma 3.4.1 are satisfied.

Now, we have to find the value of λ and the function g^* . We have

$$\lim_{\kappa \rightarrow 0^+} g(\kappa) / \kappa = \begin{bmatrix} 1 \\ \frac{-\theta}{F_S} \end{bmatrix}$$

$$\lim_{\kappa \rightarrow 0^-} g(\kappa) / (-\kappa) = \begin{bmatrix} -1 \\ \frac{-\theta}{F_S} \end{bmatrix}.$$

Thus, the function $g \in C^0(\mathbb{R}_+, \mathbb{R}^r)$ in (3.20) is of class $\lambda = 1$ (see Definition 3.3.2) and the function $g^* \in C^0(\mathbb{R}_+, \mathbb{R}^r)$ in (3.9) is defined as

$$g^*(\vartheta) = \begin{bmatrix} \vartheta \\ \frac{-\theta}{F_S} |\vartheta| \end{bmatrix}, \forall \vartheta \in \mathbb{R}.$$

Therefore, by applying Lemma 3.4.1, it follows that the system (3.23) satisfies equation (3.14).

Simulations: Take $v_S = 0.001$ m/s, $F_S = 1.5$ N, $F_C = 1.0$ N, $x(0) = 0$ N, and $u(t) = 10^{-4} \sin(\vartheta)$, m , $\forall t \geq 0$ (values taken from [78]). Figure 3.1 upper left shows that the graphs $\{(\sigma_\gamma(t), u(t)), t \geq 0\}$ converge to the set

$\{(q_u(t), u(t)), t \geq 0\}$ as $\gamma \rightarrow \infty$. This is the main characteristic of a hysteresis system. Also observe that $\{(\sigma_\gamma(t), u(t)), t \geq 0\}$ are different for different values of γ . This is what is called “rate-dependent” property of the model (3.20)-(3.21). Figure 3.1 upper right presents the graph of $(\sigma_\gamma - q_u)(t)$ versus t ; we observe that $\sigma_\gamma - q_u$ converges uniformly to the zero function as $\gamma \rightarrow \infty$ which means that σ_γ converges uniformly to q_u when $\gamma \rightarrow \infty$. The graph of $q_u(t)$ versus t is presented in Figure 3.1 lower.

For the case $\lambda \in (0, 1)$, when we equal the right-hand side of equation (3.1) to $\mathbf{0}$ and solve it in x , the function q_u in Lemma 3.4.1 should be one of the solutions. In the following example, we present a system in which there exists a function $h \in L^\infty(\mathbb{R}_+, \mathbb{R}^m) \cap C^0(\mathbb{R}_+, \mathbb{R}^m)$; different than q_u , such that $f(h(t), u(t))g(\dot{u}) = \mathbf{0}$, for almost all $t \geq 0$. The function h is not necessarily absolutely continuous.

Example 3.4.2. The Cantor function $C : [0, 1] \rightarrow [0, 1]$ is continuous everywhere, not absolutely continuous, and satisfies $C(0) = 0$ [87]. Let $K : \mathbb{R} \rightarrow \mathbb{R}$ be a function of period 2 such that

$$K(\beta) = \begin{cases} C(-\beta) & : \text{if } -1 \leq \beta \leq 0 \\ C(\beta) & : \text{if } 0 \leq \beta \leq 1 \end{cases}$$

Consider the system

$$\dot{x} = f(x, u)g(\dot{u}) = -|x|(x + K(u))\sqrt[3]{|\dot{u}|}, \quad (3.24)$$

$$x(0) = 0. \quad (3.25)$$

where state $x \in \mathbb{R}$, and input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$. The functions f and g are continuous.

When $x > 0$, we get $\dot{x} = -|x|(x + K(u))\sqrt[3]{|\dot{u}|} \leq 0$. Moreover, when $x \leq -1$, we get $\dot{x} = -|x|(x + K(u))\sqrt[3]{|\dot{u}|} \geq 0$. Thus, we have $|x| \leq \max(|x_0|, 1)$, for any initial state $x_0 \in \mathbb{R}$ and any output $x \in \mathbb{R}$. Therefore, each solution

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of the system (3.24)-(3.25) is global [90].

Since $\lim_{\vartheta \rightarrow 0} g(\vartheta) / \sqrt[3]{|\vartheta|} = 1$, the function g is of class $\lambda = 1/3 \in (0, 1)$.

The right-hand side of (3.24) is Lipschitz with respect to x . Therefore, the system (3.24)-(3.25) has a unique solution (see [39]). In fact, this solution is the zero function, so we obtain $\sigma_\gamma = x_\gamma \circ s_{1/\gamma} \equiv 0, \forall \gamma > 0$, which means that $\lim_{\gamma \rightarrow \infty} \|\sigma_\gamma\| = 0$. Hence the function $q_u \in L^\infty(\mathbb{R}_+, \mathbb{R}) \cap C^0(\mathbb{R}_+, \mathbb{R})$ in Lemma 3.4.1 is the zero function which satisfies (3.15).

The conditions of Lemma 3.4.1 are all satisfied. However, the function $h \in C^0(\mathbb{R}_+, \mathbb{R}) \cap L^\infty(\mathbb{R}_+, \mathbb{R})$ which is defined as $h = -K(u)$ satisfies the conditions $f(h(t), u(t)) = 0, \forall t \geq 0$ and $h(0) = 0$. Note that the function h may be not absolutely continuous. For instance, if there exists $0 < \vartheta < 1$ such that $u(t) = t, \forall t \in (0, \vartheta)$, then $h(t) = -K(u(t)) = -K(t) = -C(t), \forall t \in (0, \vartheta)$, which means that h is not absolutely continuous.

3.5 Sufficient conditions for consistency

This section presents sufficient conditions for the uniform convergence of the sequence of functions σ_γ as $\gamma \rightarrow \infty$ (and hence for consistency of the system (3.1)-(3.2) with respect to (u, x_0)). The main results of this section are given in Lemmas 3.5.1, 3.5.2 and 3.5.3.

3.5.1 Class $\lambda \in (0, 1)$ functions

In this subsection, sufficient conditions for the uniform convergence of σ_γ as $\gamma \rightarrow \infty$, are derived when the function g is of class $\lambda \in (0, 1)$.

Definition 3.5.1. [61] A continuous function $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to belong to class \mathcal{K}_∞ if it is increasing, satisfies $\beta(0) = 0$, and $\lim_{t \rightarrow \infty} \beta(t) = \infty$.

Lemma 3.5.1. *Suppose that the system (3.1)-(3.2) has a unique solution and that the function g is of class $\lambda \in (0, 1)$. Assume that there exists $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ such that for almost all $t \geq 0$*

$$\begin{cases} q_u(0) = x_0, \\ f(q_u(t), u(t)) g^*(\dot{u}(t)) = \mathbf{0}. \end{cases} \quad (3.26)$$

For all $\gamma > 0$, define $y_\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ as

$$y_\gamma(t) = \sigma_\gamma(t) - q_u(t) = x_\gamma(\gamma t) - q_u(t), \quad (3.27)$$

for all $t \in [0, \omega_\gamma)$, where $[0, \omega_\gamma)$ is the maximal interval of existence of solution σ_γ in (3.5). Suppose that we can find a continuously differentiable function $V : \mathbb{R}^m \rightarrow \mathbb{R}_+$ such that

1. There exists a function $\delta_1 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that satisfies

$$\delta_1(\gamma) \rightarrow 0 \text{ as } \gamma \rightarrow \infty, \quad (3.28)$$

2. There exist constants $\gamma_*, \delta_2 > 0$, continuous functions $R_1, R_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and \mathcal{K}_∞ class functions $\beta_1, \beta_2, \beta_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying:

$$\beta_1(|\alpha|) \leq V(\alpha) \leq \beta_2(|\alpha|), \quad \forall \alpha \in \mathbb{R}^m, \quad (3.29)$$

$$\begin{cases} \left. \begin{aligned} & \left. \frac{dV(\alpha)}{d\alpha} \right|_{\alpha=y_\gamma(t)} \cdot f(y_\gamma(t) + q_u(t), u(t)) g\left(\frac{\dot{u}(t)}{\gamma}\right) \\ & \leq -\frac{1}{\gamma^\lambda} \beta_3(|y_\gamma(t)|) + \frac{1}{\gamma} R_1(|y_\gamma(t)|) \end{aligned} \right\} \quad (3.30) \\ & \text{for almost all } t \in [0, \omega_\gamma) \text{ and } \forall \gamma > \gamma_* \\ & \text{that satisfy } \delta_1(\gamma) < |y_\gamma(t)| < \delta_2, \end{cases}$$

$$\left| \frac{dV(\alpha)}{d\alpha} \right| \leq R_2(|\alpha|), \quad \forall \alpha \in \mathbb{R}^m, \quad (3.31)$$

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Then,

- There exist $E, \gamma^* > 0$ such that $\forall \gamma > \gamma^* : \omega_\gamma = +\infty$, and $\|x_\gamma\|_\infty \leq E$, where x_γ is given in (3.3)-(3.4).
- $\lim_{\gamma \rightarrow \infty} \|\sigma_\gamma - q_u\|_\infty = 0$.

Proof. From (3.5) and (3.27), we get for all $\forall \gamma > 0$ and almost all $t \in [0, \omega_\gamma)$ that $y_\gamma(0) = \mathbf{0}$ and

$$\dot{y}_\gamma(t) = \gamma f(y_\gamma(t) + q_u(t), u(t)) g\left(\frac{\dot{u}(t)}{\gamma}\right) - \dot{q}_u(t). \quad (3.32)$$

For any $\gamma > 0$, define $V_\gamma : (0, \omega_\gamma] \rightarrow \mathbb{R}_+$ as $V_\gamma(t) = V(y_\gamma(t)), \forall t \in (0, \omega_\gamma]$. The derivative of V along trajectories (3.32) is given for almost all $t \in [0, \omega_\gamma)$ and all $\gamma > 0$ by

$$\begin{aligned} \dot{V}_\gamma(t) &= \left. \frac{dV(\alpha)}{d\alpha} \right|_{\alpha=y_\gamma(t)} \cdot \dot{y}_\gamma(t) \\ &= \left. \frac{dV(\alpha)}{d\alpha} \right|_{\alpha=y_\gamma(t)} \cdot \left[\gamma f(y_\gamma(t) + q_u(t), u(t)) - \dot{q}_u(t) \right]. \end{aligned} \quad (3.33)$$

By (3.28), there exists some $\gamma_1 > \gamma_*$ such that $\beta_2 \circ \delta_1(\gamma) < \beta_1(\delta_2), \forall \gamma > \gamma_1$. Let $\Omega_\gamma = (\beta_2 \circ \delta_1(\gamma), \beta_1(\delta_2))$. By (3.29), we have for any $\gamma > \gamma_1$, for almost all $t \in [0, \omega_\gamma)$ that

$$V_\gamma(t) \in \Omega_\gamma \Rightarrow \delta_1(\gamma) < |y_\gamma(t)| < \delta_2. \quad (3.34)$$

Thus, we deduce from (3.30), (3.31), (3.33), and (3.34) that

$$\begin{cases} \dot{V}_\gamma(t) \leq -\gamma^{1-\lambda} \beta_3(|y_\gamma(t)|) + R_1(|y_\gamma(t)|) + \|\dot{q}_u\|_\infty R_2(|y_\gamma(t)|) \\ \text{for almost all } t \in [0, \omega_\gamma), \forall \gamma > \gamma_1 \text{ that satisfy } V_\gamma(t) \in \Omega_\gamma. \end{cases}$$

Therefore, (3.34) and the continuity of the functions R_1, R_2 imply that there

exists a constant $a > 0$ that does not depend on γ , such that

$$\begin{cases} \dot{V}_\gamma(t) \leq -\gamma^{1-\lambda} \beta_3(|y_\gamma(t)|) + a, \\ \text{for almost all } t \in [0, \omega_\gamma), \forall \gamma > \gamma_1 \text{ that satisfy } V_\gamma(t) \in \Omega_\gamma. \end{cases} \quad (3.35)$$

Thus, we deduce from (3.29) that

$$\begin{cases} \dot{V}_\gamma(t) \leq -\beta_\gamma(V_\gamma(t)) + a, \\ \text{for almost all } t \in [0, \omega_\gamma), \forall \gamma > \gamma_1 \text{ that satisfy } V_\gamma(t) \in \Omega_\gamma. \end{cases} \quad (3.36)$$

where $\beta_\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is defined as $\beta_\gamma = \gamma^{1-\lambda} \beta_3 \circ \beta_2^{-1}$.

On the other hand, since $\lambda \in (0, 1)$, there exists $\gamma_2 > \gamma_1$ such that $\beta_\gamma^{-1}(a) = \beta_2 \circ \beta_3^{-1}(a\gamma^{\lambda-1}) < \beta_1(\delta_2)$, $\forall \gamma > \gamma_2$. Hence, Corollary 2.2.1 and the fact that $V_\gamma(0) = 0, \forall \gamma > 0$ ⁸, imply that

$$\begin{cases} V_\gamma(t) \leq \max(\beta_\gamma^{-1}(a), \beta_2 \circ \delta_1(\gamma)) \\ \forall t \in [0, \omega_\gamma), \forall \gamma > \gamma_2. \end{cases}$$

so that

$$\begin{cases} V_\gamma(t) \leq \max(\beta_2 \circ \beta_3^{-1}(a\gamma^{\lambda-1}), \beta_2 \circ \delta_1(\gamma)) \\ \forall t \in [0, \omega_\gamma), \forall \gamma > \gamma_2. \end{cases}$$

Therefore, (3.29) implies $\forall \gamma > \gamma_2$ and $\forall t \in [0, \omega_\gamma)$ that

$$|y_\gamma(t)| \leq \max(\beta_1^{-1} \circ \beta_2 \circ \beta_3^{-1}(a\gamma^{\lambda-1}), \beta_1^{-1} \circ \beta_2 \circ \delta_3(\gamma)) \quad (3.37)$$

Thus, $\omega_\gamma = +\infty, \forall \gamma > \gamma_2$. Furthermore, (3.28), (3.37), and the fact that $\lambda \in (0, 1)$ imply that $\|y_\gamma\|_\infty = \|\sigma_\gamma - q_u\|_\infty \rightarrow 0$ as $\gamma \rightarrow \infty$. This proves the consistency with respect to (u, x_0) because of Proposition 3.2.1.

⁸Note that function V_γ is absolutely continuous on each compact subset of $[0, \omega_\gamma)$ as a composition of a continuously differentiable function V and an absolutely continuous function y_γ

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Moreover, by (3.37), there exist some $E > 0, \gamma^* > \gamma_2$ such that

$$\|\sigma_\gamma\|_\infty \leq E, \forall \gamma > \gamma^*. \quad (3.38)$$

On the other hand, let $\gamma > \gamma^*$. Since x_γ is continuous, Lemma 1.2.3 ensures that $\|x_{u \circ s_\gamma}\|_{\infty, I_u} = \|x_\gamma\|_\infty$. Let $\varrho \in I_u$. Due to the continuity of ρ_u , there exists some $t \geq 0$ such that $\varrho = \rho_u(t)$ and thus (3.38) and the continuity of σ_γ lead to $|x_{u \circ s_\gamma}(\varrho)| = |x_{u \circ s_\gamma} \circ \rho_u(t)| = |\sigma_\gamma(t)| \leq \|\sigma_\gamma\|_\infty \leq E, \forall \gamma > \gamma^*$. Therefore, $\|x_{u \circ s_\gamma}\|_\infty = \|x_\gamma\|_\infty \leq E, \forall \gamma > \gamma^*$, which completes the proof. \square

Remark 3.5.1. For $\lambda \in (0, 1)$, if the function q_u in Lemma 3.5.1 is such that $q_u = R(u)$ for some $R: \mathbb{R} \rightarrow \mathbb{R}^m$, then the graphs $\{(\sigma_\gamma(t), u(t)), t \geq 0\}$ converge to the curve R as $\gamma \rightarrow \infty$. Hence (3.1)-(3.2) is not a hysteresis because the hysteresis loop cannot be a function [15]. This fact is illustrated in Example 3.5.1.

Example 3.5.1. Consider the semilinear Duhem model:

$$\begin{aligned} \dot{x} &= (Ax + Bu)g(\dot{u}), \\ x(0) &= x_0, \end{aligned} \quad (3.39)$$

where A is a Hurwitz $m \times m$ matrix (that is, every eigenvalue of A has negative real part), vector B and state x taking values in \mathbb{R}^m . The right-hand side of (3.39) is Lipschitz and thus the system has a unique solution [39]. Take an input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ such that $A^{-1}Bu(0) = -x_0$, and that $|\dot{u}(t)| \geq M$ for almost all $t \in \mathbb{R}$ and for some $M > 0$. Assume that the function $g: \mathbb{R} \rightarrow \mathbb{R}_+$ is of class $\lambda \in (0, 1)$ and that $d_+^\lambda g(0), d_-^\lambda g(0) > 0$. Thus, there exists $L > 0$ such that $g^*(\vartheta) \geq L|\vartheta|^\lambda, \forall \vartheta \in \mathbb{R}$, where the function g^* is defined in (3.9). On the other hand, Proposition 3.3.8 states that $\lim_{\gamma \rightarrow \infty} \|\gamma^\lambda g(\dot{u}/\gamma) - g^*(\dot{u})\|_\infty = 0$. This means that there exists $\gamma_1 > 0$ such that we get for almost all $t \geq 0$, and

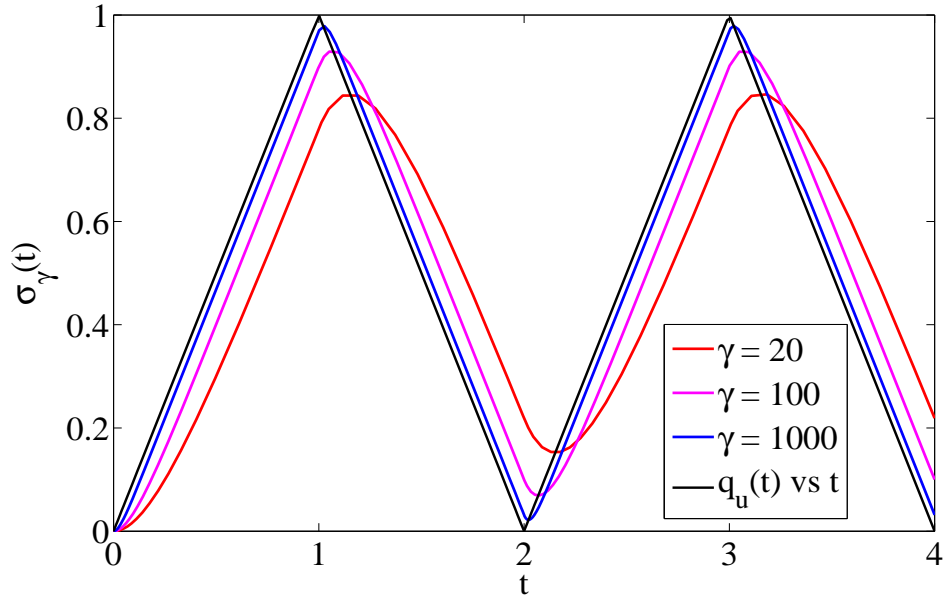


Figure 3.2: $\sigma_\gamma(t)$ versus t for the semilinear Duhem model (3.39).

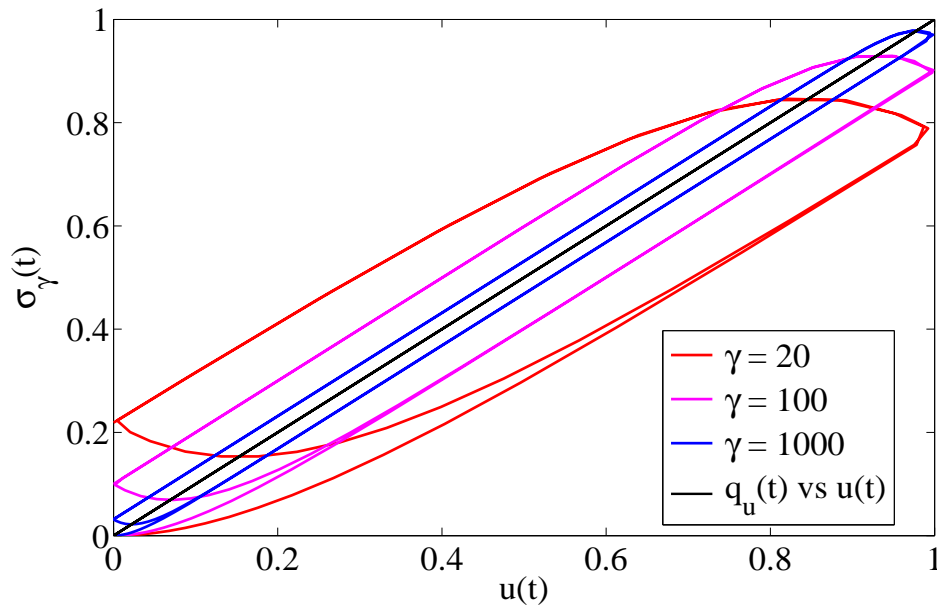


Figure 3.3: $\sigma_\gamma(t)$ versus $u(t)$ for the semilinear Duhem model (3.39).

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all $\gamma > \gamma_1$ that

$$\gamma^\lambda g\left(\frac{\dot{u}(t)}{\gamma}\right) > g^*(\dot{u}(t)) - \frac{LM^\lambda}{2}.$$

Thus, the facts $|\dot{u}| \geq M$ and $g^*(\cdot) \geq L|\cdot|^\lambda$ imply that

$$g\left(\frac{\dot{u}(t)}{\gamma}\right) > \frac{LM^\lambda}{2\gamma^\lambda}, \text{ for almost all } t \geq 0, \forall \gamma > \gamma_1. \quad (3.40)$$

The function $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ which is defined as $q_u = -A^{-1}Bu$ satisfies (3.26) because $A^{-1}Bu(0) = -x_0$.

Since A is Hurwitz, there exists a $m \times m$ positive-definite matrix P such that [61, p.136]

$$PA + A^T P = -I, \quad (3.41)$$

where I is the identity matrix. Consider the continuously differentiable quadratic Lyapunov function candidate $V : \mathbb{R}^m \rightarrow \mathbb{R}$ such that $V(\alpha) = \alpha^T P \alpha$, $\forall \alpha \in \mathbb{R}^m$. Since P is symmetric, we have $\forall \alpha \in \mathbb{R}^m$ that

$$\lambda_{\min}(P) |\alpha|^2 \leq V(\alpha) = \alpha^T P \alpha \leq \lambda_{\max}(P) |\alpha|^2,$$

where $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are, respectively, the maximum and minimum eigenvalues of the matrix P . Hence, (3.29) is satisfied with $\beta_1(\eta) = \lambda_{\min}(P) \eta^2$ and $\beta_2(\eta) = \lambda_{\max}(P) \eta^2$ for all $\eta > 0$. Since P is symmetric, we have the following matrices derivation:

$$\frac{dV(\alpha)}{d\alpha} = (P + P^T) \alpha = 2P\alpha, \forall \alpha \in \mathbb{R}^m. \quad (3.42)$$

Thus, we get

$$\left| \frac{dV(\alpha)}{d\alpha} \right| = 2|P\alpha| \leq 2|P||\alpha|, \forall \alpha \in \mathbb{R}^m,$$

where $|P|$ is the induced 2-norm for the matrix P and hence (3.31) is satisfied

with $R_2(\eta) = 2|P|\eta$, $\forall \eta \geq 0$. From (3.42), we have $\forall \alpha \in \mathbb{R}^m$ that

$$\frac{dV(\alpha)}{d\alpha} \cdot A\alpha = 2P\alpha \cdot A\alpha = \alpha^T (PA + A^T P) \alpha = -|\alpha|^2. \quad (3.43)$$

Therefore (3.40) implies that for almost all $t \in \mathbb{R}_+$ and $\gamma > 0$ that

$$\begin{aligned} \frac{dV(\alpha)}{d\alpha} \Big|_{\alpha=y_\gamma(t)} \cdot [A(y_\gamma(t) + q_u(t)) + B u(t)] g\left(\frac{\dot{u}(t)}{\gamma}\right) \\ = -g\left(\frac{\dot{u}(t)}{\gamma}\right) |y_\gamma(t)|^2 \leq -\frac{LM^\lambda}{2\gamma^\lambda} |y_\gamma(t)|^2, \end{aligned}$$

where y_γ is defined in (3.27). Thus, (3.30) is satisfied with $R_1(\eta) = 0$, $\forall \eta \geq 0$ and $\beta_3(\eta) = \frac{LM^\lambda}{2}\eta^2$, $\forall \eta \geq 0$.

Let δ_1 be the zero function. Then (3.28) is verified. Take δ_2, γ_* arbitrary in \mathbb{R}_+ (say $\delta_2 = 1$, $\gamma_* = 1$). Hence, all conditions of Lemma 3.5.1 are satisfied. Thus, it follows from Lemma 3.5.1 that there exist some $E, \gamma^* > 0$ such that $\forall \gamma > \gamma^*$, the solution of (3.39) is global with $|x_\gamma(t)| \leq E$, $\forall t \geq 0$. Moreover, the operator which maps (u, x_0) to x is consistent. In particular, we have $\|\sigma_\gamma - q_u\|_\infty = \|\sigma_\gamma + A^{-1}B u\|_\infty \rightarrow 0$ as $\gamma \rightarrow \infty$.

As a conclusion, the graphs $\{(\sigma_\gamma(t), u(t)), t \geq 0\}$ converge to the graph of the linear function $R: \mathbb{R} \rightarrow \mathbb{R}^m$, which is defined as $R(\alpha) = -A^{-1}B\alpha$, $\forall \alpha \in \mathbb{R}$. This means that for $\lambda \in (0, 1)$, the model (3.39) does not represent a hysteresis (see Remark 3.5.1).

Simulations: Take $m = 1$, $B = 1.0$, $A = -1.0$, and $x_0 = 0$. Let $g(\vartheta) = \sqrt{|\vartheta|}$, $\forall \vartheta \in \mathbb{R}$, then $d_+^\lambda g(0) = d_-^\lambda g(0) = \Gamma(3/2) = \Gamma(1/2)/2 = \sqrt{\pi}/2 > 0$. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ be the function of period 2 such that $u(t) = t$, $\forall t \in [0, 1]$, and $u(t) = 2 - t$, $\forall t \in [1, 2]$. Then, we have $|\dot{u}(t)| = 1$, for almost all $t \geq 0$. We also have $q_u = -A^{-1}B u = u$. Figure 3.3 shows that the graph $\{(\sigma_\gamma(t), u(t)) / t \geq 0\}$ collapses into the identity function when $\gamma \rightarrow \infty$. This happens because of the fact that $q_u = u$ and Remark 3.5.1. Figure 3.2 shows

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that the sequence of functions σ_γ converges uniformly to $q_u = u$ as $\gamma \rightarrow \infty$.

3.5.2 Class $\lambda = 1$ functions

In this subsection, we consider class $\lambda = 1$ functions⁹. The main results of this subsection are given in Lemmas 3.5.2 and 3.5.3.

Lemma 3.5.2. *Assume the following*

1. *The system (3.1)-(3.2) has a unique global solution.*
2. *For the function g in system (3.1)-(3.2), there exist $a_1, a_2 \in \mathbb{R}^r$ such that*
¹⁰

$$g(\vartheta) = \begin{cases} a_1 \vartheta & \vartheta \geq 0 \\ a_2 \vartheta & \vartheta < 0 \end{cases}$$

Then the sequence of functions σ_γ of (3.5) is independent of γ and the operator which maps (u, x_0) to x is consistent.

Proof. By condition 2, the right-hand side of (3.5) is independent of γ . Thus, the solution σ_γ of (3.5) is independent of γ . Since $x_{u \circ \sigma_\gamma} \circ \rho_u = \sigma_\gamma$, the function $x_{u \circ \sigma_\gamma}$ is also independent of γ (this is the so-called “rate-independent hysteresis”) and hence consistency holds. \square

Example 3.5.2. *Consider Bouc’s hysteresis model [21]:*

$$\dot{x} = -c |\dot{u}| x + \Phi'(u) \dot{u}, \tag{3.44}$$

where $c > 0$, $\Phi \in C^1(\mathbb{R}, \mathbb{R})$, input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$, and $\Phi'(u) = \frac{d\Phi(\alpha)}{d\alpha}|_{\alpha=u}$.

⁹A function $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ is of class $\lambda = 1$ if $g(0) = \mathbf{0}$ and the limits $\lim_{\kappa \rightarrow 0^+} g(\kappa) / |\kappa|$ and $\lim_{\kappa \rightarrow 0^-} g(\kappa) / |\kappa|$ exist, are finite, and at least one of them is nonzero. (see Definition 3.3.2)

¹⁰In this case, the function g is of class 1 (see Definition 3.3.2)

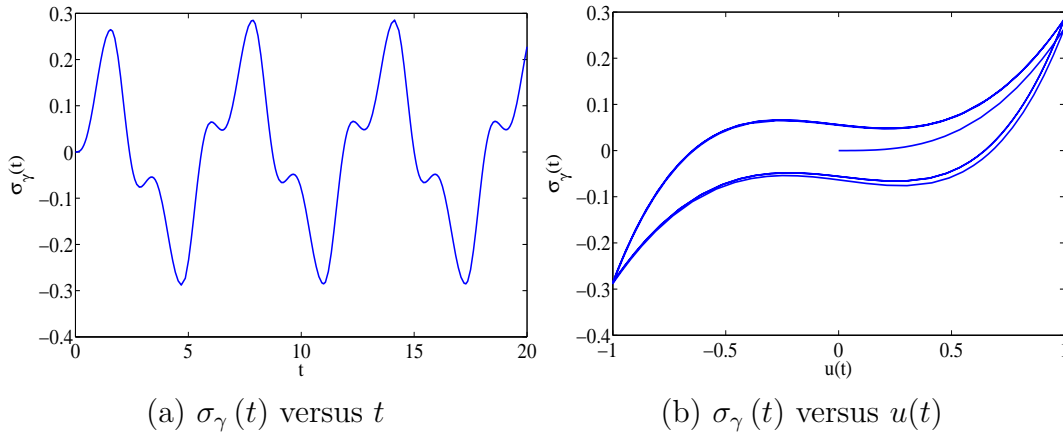


Figure 3.4: Bouc's hysteresis model (3.44).

The right-hand side of (3.44) is Lipschitz with respect to x . Thus, the system has a unique solution. Furthermore, we have

$$\dot{x} \leq -\left| \Phi'(u) \dot{u} \right| + \Phi'(u) \dot{u} \leq 0, \text{ when } x \geq \frac{1}{c} \left| \Phi'(u) \right|,$$

$$\dot{x} \geq \left| \Phi'(u) \dot{u} \right| + \Phi'(u) \dot{u} \geq 0, \text{ when } x \leq -\frac{1}{c} \left| \Phi'(u) \right|.$$

Thus, $|x| \leq \max\left(\left| \Phi'(u) \right| / c, |x(0)|\right)$, for each initial state $x(0) \in \mathbb{R}$ and each input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$. Since u is bounded and $\Phi'(u)$ is continuous, the solution x of (3.44) is bounded and hence global. Hence Condition 1 in Lemma 3.5.2 is satisfied. Equation (3.44) can be written as

$$\begin{aligned} \dot{x} &= f(x, u) g(\dot{u}) \\ &= \begin{bmatrix} -cx + \Phi'(u) & cx + \Phi'(u) \end{bmatrix} \begin{bmatrix} \max(0, \dot{u}) \\ \min(0, \dot{u}) \end{bmatrix}. \end{aligned}$$

Clearly, the function g is of class $\lambda = 1$ and satisfies Condition 2 in Lemma 3.5.2. This fact implies that the operator which maps $(u, x(0))$ to x is consistent and σ_γ is independent of γ .

Simulations: Let $c = 1$, $\Phi(\alpha) = \alpha^3/3, \forall \alpha \in \mathbb{R}$, $x(0) = 0$, and input $u(t) =$

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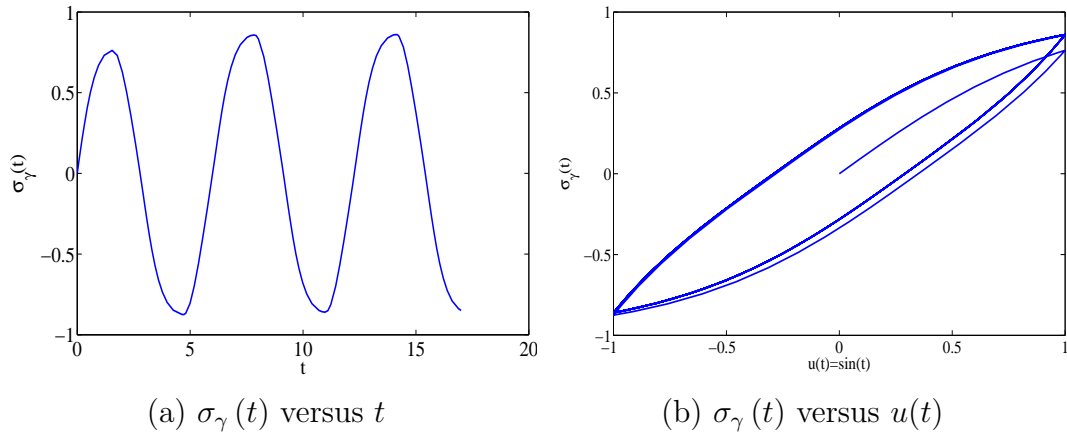


Figure 3.5: Bouc-Wen model (3.45)

$\sin(t), \forall t \geq 0$. The function σ_γ is independent of γ and is plotted in Figure 3.4a. Furthermore, Figure 3.4b shows a rate-independent hysteresis behavior; that is graphs $\{(\sigma_\gamma(t), u(t)), t \geq 0\}$ are the same for different values of γ .

Example 3.5.3. Consider the Bouc-Wen model [53]:

$$\dot{x} = d_1 \dot{u} - d_2 |\dot{u}| |x|^{d_4-1} x - d_3 |x|^{d_4} \dot{u}, \quad (3.45)$$

where input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$, state $x \in \mathbb{R}$, and parameters $d_4 > 2, d_1, d_2, d_3 \in \mathbb{R}$ satisfy $d_1 \neq 0$ and

$$d_2 + d_3 > 0 \text{ and } d_2 - d_3 \geq 0 \text{ whenever } d_1 > 0,$$

$$d_2 + d_3 \geq 0 \text{ and } d_2 - d_3 > 0 \text{ whenever } d_1 < 0.$$

The system (3.45) has a unique global solution for each initial state $x(0) \in \mathbb{R}$ [54]. System (3.45) can be rewritten into the form

$$\dot{x} = f(x) g(\dot{u}),$$

where

$$f(x) = \begin{bmatrix} d_1 - d_2 |x|^{d_4-1} x - d_3 |x|^{d_4} & d_1 + d_2 |x|^{d_4-1} x - d_3 |x|^{d_4} \end{bmatrix},$$

$$g(\dot{u}) = \begin{bmatrix} \max(0, \dot{u}) \\ \min(0, \dot{u}) \end{bmatrix}.$$

Thus, the function g is of class $\lambda = 1$ and satisfies condition 2 in Proposition 3.5.2 and thus the operator which maps $(u, x(0))$ to x is consistent.

Simulations: Let $d_1 = d_2 = 1$, $d_3 = 0$, $d_4 = 2$, initial state $x(0) = 0$, and input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ be defined as $u(t) = \sin(t)$, $\forall t \geq 0$. The function σ_γ ; which is independent of γ and equal to the output x of the system (3.45), is plotted in Figure 3.5.

Example 3.5.4. Consider the Dahl friction model [78]:

$$\dot{x} = \theta \left| 1 - \frac{x}{F_C} \operatorname{sgn}(\dot{u}) \right|^p \operatorname{sgn} \left(1 - \frac{x}{F_C} \operatorname{sgn}(\dot{u}) \right) \dot{u}, \quad (3.46)$$

where the output x is the friction force, $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ is the relative displacement between the two surfaces in contact (which is the input of the system), $F_C > 0$ is the Coulomb friction force, $p \geq 1$ is a parameter that determines the shape of the force-deflection curve, and $\theta > 0$ is the rest stiffness; that is, the slope of the force-deflection curve when $x = 0$. The right-hand side of (3.46) is continuous and Lipschitz [78] and hence (3.46) admits a unique solution for each initial state $x(0) \in \mathbb{R}$. We have

$$\begin{aligned} \dot{x} &< 0, \text{ whenever } x > \operatorname{sgn}(\dot{u}) F_C, \\ \dot{x} &> 0, \text{ whenever } x < \operatorname{sgn}(\dot{u}) F_C. \end{aligned}$$

Thus,

$$|x| \leq \max(F_C, |x(0)|),$$

3.5 Sufficient conditions for consistency

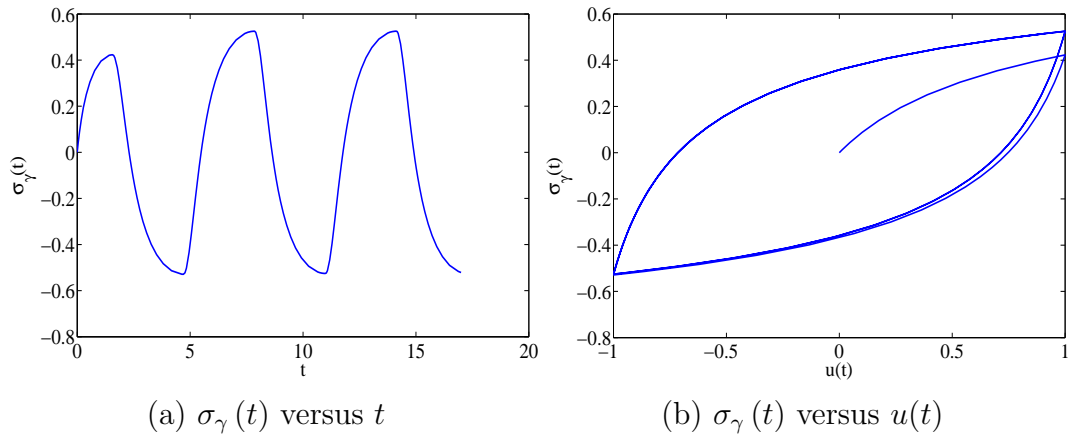


Figure 3.6: Dahl friction model (3.46)

for each initial state $x(0) \in \mathbb{R}$. Therefore, the solution of (3.46) is global. System (3.46) can be rewritten into the form

$$\dot{x} = f(x)g(\dot{u}) = \theta \begin{bmatrix} X_+ & X_- \end{bmatrix} \begin{bmatrix} \max(0, \dot{u}) \\ \min(0, \dot{u}) \end{bmatrix},$$

where [78]

$$X_+ = \left| 1 - \frac{x}{F_C} \right|^p \operatorname{sgn} \left(1 - \frac{x}{F_C} \right),$$

$$X_- = \left| 1 + \frac{x}{F_C} \right|^p \operatorname{sgn} \left(1 + \frac{x}{F_C} \right).$$

Thus, the function g is of class $\lambda = 1$ and satisfies condition 2 in Proposition 3.5.2 which implies that the operator which maps $(u, x(0))$ to x is consistent.

Simulations: Let $\theta = 1$, $p = 3$, $F_C = 1$, and $x(0) = 0$. Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ be such that $u(t) = \sin(t)$, $\forall t \geq 0$. The function σ_γ ; which is independent of γ and equal to the output x of the system (3.46), is plotted in Figure 3.6a. Moreover, the Figure 3.6b shows the rate-independent hysteresis behavior between output σ_γ and input u .

Proposition 3.5.1. *Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be a non-constant function. There exists a unique function $v_u \in L^\infty(I_u, \mathbb{R}^n)$ that is defined by $v_u \circ \rho_u = \dot{u}$. Moreover, $\|v_u\|_{\infty, I_u} \leq \|\dot{u}\|_{\infty}$. Assume that \dot{u} is nonzero on a set $A \subseteq \mathbb{R}$ that satisfies $\mu(\rho_u(\mathbb{R} \setminus A)) = 0$. Then, v_u is nonzero almost everywhere.*

Proof. See the proof of Proposition 2.2.1. □

Lemma 3.5.3. *Let $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ be such that \dot{u} is nonzero on a set $A \subseteq \mathbb{R}$ that satisfies $\mu(\rho_u(\mathbb{R} \setminus A)) = 0$. Consider the semilinear Duhem model with $m = 1$ and $\lambda = 1$*

$$\dot{x} = (Ax + Bu + C)g(\dot{u}), \quad (3.47)$$

$$x(0) = x_0. \quad (3.48)$$

where $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_r \end{bmatrix} \neq \mathbf{0}$, B , and C are $1 \times r$ row vectors, state $x \in \mathbb{R}$, function $g \in C^0(\mathbb{R}, \mathbb{R}^r)$ is of class $\lambda = 1$, and non-constant input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$.

Denote

$$\lim_{\kappa \rightarrow 0^+} \frac{g(\kappa)}{|\kappa|} = G_+^* = \begin{bmatrix} G_{+,1}^* \\ G_{+,2}^* \\ \vdots \\ G_{+,r}^* \end{bmatrix}, \quad (3.49)$$

$$\lim_{\kappa \rightarrow 0^-} \frac{g(\kappa)}{|\kappa|} = G_-^* = \begin{bmatrix} G_{-,1}^* \\ G_{-,2}^* \\ \vdots \\ G_{-,r}^* \end{bmatrix}. \quad (3.50)$$

For any $i \in \{1, 2, \dots, r\}$, assume that

$$G_{+,i}^*, G_{-,i}^* \geq 0 \text{ whenever } a_i < 0, \quad (3.51)$$

$$G_{+,i}^*, G_{-,i}^* \leq 0 \text{ whenever } a_i > 0. \quad (3.52)$$

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Suppose that there exists some $i_0 \in \{1, 2, \dots, r\}$ such that

$$a_{i_0} \neq 0 \text{ and } |G_{+,i_0}^*| + |G_{-,i_0}^*| > 0. \quad (3.53)$$

Then

- There exist $E, \gamma_1 > 0$ such that $\|x_\gamma\|_\infty \leq E, \forall \gamma > \gamma_1$.¹¹
- There exists a function $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ such that

$$\lim_{\gamma \rightarrow \infty} \|\sigma_\gamma - q_u\|_{1,\infty} = 0,$$

where $\|\cdot\|_{1,\infty}$ is the norm of the Banach space $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ (and hence the consistency of the system (3.47)-(3.48) is guaranteed with respect to input u and initial condition x_0).

- $q_u(0) = x_0$. Furthermore, for almost all $t \geq 0$, we have

$$\dot{q}_u(t) = (Aq_u(t) + Bu(t) + C)g^*(\dot{u}(t)),$$

where the function $g^* \in C^0(\mathbb{R}, \mathbb{R})$ is defined as in equation (3.9), i.e.

$$g^*(\vartheta) = \begin{cases} \vartheta G_+^* & \vartheta \geq 0 \\ -\vartheta G_-^* & \vartheta < 0 \end{cases}$$

Proof. The semilinear Duhem model has a unique global solution due to the Lipschitz property of the right-hand-side [39, p.5].

For any $i \in \{1, 2, \dots, r\}$ and any $\kappa \in \mathbb{R} \setminus \{0\}$, let $G_i(\kappa)$ to be the i -th component of the function $g(\kappa)/|\kappa|$. From (3.49)-(3.50), there exists some constant $\zeta > 0$ such that $\forall i \in \{1, 2, \dots, r\}$:

$$G_{+,i}^* - |G_{+,i}^*|/2 < G_i(\kappa) < G_{+,i}^* + |G_{+,i}^*|/2, \text{ whenever } \kappa \in (0, \zeta), \quad (3.54)$$

¹¹ x_γ is the output of the system (3.47)-(3.48) when we use the input $u \circ s_\gamma$ instead of the input u (see system (3.3)-(3.4))

$$G_{-,i}^* - |G_{-,i}^*|/2 < G_i(\kappa) < G_{-,i}^* + |G_{-,i}^*|/2, \text{ whenever } \kappa \in (-\zeta, 0). \quad (3.55)$$

Let $i \in \{1, 2, \dots, r\}$. We get from (3.51)-(3.52) and (3.54)-(3.55) that

$$\begin{cases} a_i G_i(\kappa) \leq -\max\left(\frac{|a_i| |G_{+,i}^*|}{2}, \frac{|a_i| |G_{-,i}^*|}{2}\right) = -M_i \\ \text{whenever } \kappa \in (-\zeta, \zeta) \setminus \{0\} \text{ and } a_i \neq 0. \end{cases} \quad (3.56)$$

For the case $a_i = 0$, take $M_i = 0$. Let $M = \sum_{i=1}^r M_i$. From (3.51)-(3.53), we get $M > 0$. Thus, we obtain from (3.56) that

$$A \frac{g(\kappa)}{|\kappa|} = \sum_{i=1}^r a_i G_i(\kappa) \leq -M < 0, \quad \forall \kappa \in (-\zeta, \zeta) \setminus \{0\}. \quad (3.57)$$

By Proposition 3.5.1, a function $v_u \in L^\infty(I_u, \mathbb{R})$ can be defined a.e. as $v_u \circ \rho_u = \dot{u}$ with $v_u \neq \mathbf{0}$ a.e. The boundedness of v_u implies that there exists $\gamma_0 > 0$ such that $v_u(\varrho)/\gamma \in (-\zeta, \zeta)$ for almost all $\varrho \in I_u$ and all $\gamma > \gamma_0$. Thus we deduce from (3.57) that for all $\gamma > \gamma_0$:

$$\gamma A \frac{g(v_u(\varrho)/\gamma)}{|v_u(\varrho)|} \leq -M, \text{ for almost all } \varrho \in I_u. \quad (3.58)$$

Let $H : \mathbb{R} \rightarrow \mathbb{R}^r$ be a function such that

$$H(\kappa) = \begin{cases} G_+^* & \kappa > 0 \\ G_-^* & \kappa < 0 \end{cases}$$

For any $\gamma > 0$, define $\chi_\gamma : I_u \rightarrow \mathbb{R}^r$ as

$$\chi_\gamma(\varrho) = \begin{cases} \gamma \frac{g(v_u(\varrho)/\gamma)}{|v_u(\varrho)|} - H(v_u(\varrho)) & v_u(\varrho) \neq 0 \\ \mathbf{0} & \text{Otherwise} \end{cases}$$

Since $v_u \in L^\infty(I_u, \mathbb{R})$, relations (3.49)-(3.50) imply that ¹²

¹²This result can be easily checked using the same techniques used in the proof of Proposition 3.3.8

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$$\lim_{\gamma \rightarrow \infty} \|\chi_\gamma\|_\infty = 0. \quad (3.59)$$

Now, consider the system

$$\dot{h} = (Ah + B\psi_u + C)(H \circ v_u), \quad (3.60)$$

$$h(0) = x_0. \quad (3.61)$$

where state $h \in \mathbb{R}$. The differential equation (3.60) verifies Carathéodory conditions with Lipschitz property with respect to h . Thus, system (3.60)-(3.61) has a unique absolutely continuous local solution [39, p.4] (see Theorems 1.1.2 and 1.1.3). Consider the Lyapunov function $W = h^2$. We deduce from (3.60)-(3.61) that $W(0) = x_0^2$ and

$$\dot{W} = 2A(H \circ v_u)W + 2(B\psi_u + C)(H \circ v_u)\sqrt{W}. \quad (3.62)$$

We get from (3.51)-(3.53) that

$$AH(v_u(\varrho)) \leq -\sum_{i=1}^r \max(|a_i| |G_{+,i}^*|, |a_i| |G_{-,i}^*|) = -2M,$$

for almost all $\varrho \in I_u$. Thus, the boundedness of the function ψ_u (see Lemma 1.2.1) along with the equation (3.62) imply that $\dot{W} \leq -4MW + D_1\sqrt{W}$, for some $D_1 > 0$. This leads to $\dot{W} \leq 0$, whenever $W \geq (D_1/4M)^2$. Therefore, Lemma 2.2.1 and the fact $W(0) = x_0^2$ imply that $W \leq \max(x_0^2, (D_1/4M)^2)$ which means that the solution of the system (3.60)-(3.61) is bounded and hence is global (i.e. is defined on I_u). On the other hand, The relation $\sigma_\gamma = x_{u \circ s_\gamma} \circ \rho_u$ implies that $\dot{\sigma}_\gamma = |\dot{u}| \dot{x}_{u \circ s_\gamma} \circ \rho_u$. Thus, we obtain from systems (3.47)-(3.48) and (3.5), and the relations $v_u \circ \rho_u = \dot{u}$, and $\psi_u \circ \rho_u = u$, that $x_{u \circ s_\gamma}(0) = x_0$, and

$$\dot{x}_{u \circ s_\gamma}(\varrho) = \gamma \left(Ax_{u \circ s_\gamma}(\varrho) + B\psi_u(\varrho) + C \right) \frac{g\left(\frac{v_u(\varrho)}{\gamma}\right)}{|v_u(\varrho)|} \quad (3.63)$$

for all $\gamma > 0$, for almost all $\varrho \in [0, \tau_\gamma) \subseteq I_u$, where $[0, \tau_\gamma)$ is the maximal

interval of existence [39, p.4]. For any $\gamma > 0$, let $y_\gamma : I_u \rightarrow \mathbb{R}$ be defined as $y_\gamma = x_{uos_\gamma} - h$. Since $x_{uos_\gamma}(0) = h(0) = x_0$, the system (3.63) can be written for all $\gamma > 0$, for almost all $\varrho \in [0, \tau_\gamma)$ as

$$\begin{aligned} \dot{y}_\gamma(\varrho) &= \gamma A \frac{g\left(\frac{v_u(\varrho)}{\gamma}\right)}{|v_u(\varrho)|} y_\gamma(\varrho) \\ &\quad + (A h(\varrho) + B \psi_u(\varrho) + C) \chi_\gamma(\varrho), \end{aligned} \quad (3.64)$$

$$y_\gamma(0) = 0. \quad (3.65)$$

For any $\gamma > 0$, consider the Lyapunov function $V_\gamma : [0, \tau_\gamma) \rightarrow \mathbb{R}_+$ with $V_\gamma(\varrho) = y_\gamma^2(\varrho)$, $\forall \varrho \in [0, \tau_\gamma)$. By (3.64), and the boundedness of both ψ_u and the solution of (3.60)-(3.61), there exists some $D_2 > 0$ such that for almost all $\varrho \in [0, \tau_\gamma)$ and all $\gamma > \gamma_0$:

$$\dot{V}_\gamma(\varrho) \leq 2\gamma A \frac{g\left(\frac{v_u(\varrho)}{\gamma}\right)}{|v_u(\varrho)|} V_\gamma(\varrho) + D_2 \|\chi_\gamma\|_\infty \sqrt{V_\gamma(\varrho)}. \quad (3.66)$$

Thus, we obtain from (3.58) that

$$\begin{cases} \dot{V}_\gamma(\varrho) \leq 0, \text{ for almost all } \varrho \in [0, \tau_\gamma), \forall \gamma > \gamma_0, \\ \text{that satisfy } V_\gamma(\varrho) > \left(\frac{D_2 \|\chi_\gamma\|_\infty}{2M}\right)^2. \end{cases} \quad (3.67)$$

Therefore, we deduce from Lemma 2.2.1 and the fact $V_\gamma(0) = 0$ that

$$V_\gamma(\varrho) \leq (D_2 \|\chi_\gamma\|_\infty / 2M)^2,$$

for all $\gamma > \gamma_0$ and almost all $\varrho \in [0, \tau_\gamma)$, and hence we obtain, for almost all $\varrho \in [0, \tau_\gamma)$ that

$$|y_\gamma(\varrho)| = |x_{uos_\gamma}(\varrho) - h(\varrho)| \leq \frac{D_2}{2M} \|\chi_\gamma\|_\infty, \forall \gamma > \gamma_0,$$

which implies that $[0, \tau_\gamma) = I_u, \forall \gamma > \gamma_0$ and (3.59) implies that

3.5 Sufficient conditions for consistency

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$$\lim_{\gamma \rightarrow \infty} \|x_{u \circ s_\gamma} - h\|_{\infty, I_u} = 0. \quad (3.68)$$

On the other hand, the continuity of x_γ implies that $\|x_{u \circ s_\gamma}\|_{\infty, I_u} = \|x_\gamma\|_\infty$, for any $\gamma > \gamma_0$ (see Lemma 1.2.3). Thus there exists some $E > 0$ and $\gamma_1 > \gamma_0$ with

$$\|x_{u \circ s_\gamma}\|_\infty = \|x_\gamma\|_\infty \leq E, \quad \forall \gamma > \gamma_1. \quad (3.69)$$

Moreover, we get from (3.60) and (3.63) for all $\gamma > \gamma_1$ that

$$\begin{aligned} \dot{x}_{u \circ s_\gamma} - \dot{h} &= (A x_{u \circ s_\gamma} + B \psi_u + C) \chi_\gamma \\ &\quad + A (x_{u \circ s_\gamma} - h) (H \circ v_u). \end{aligned}$$

Thus, by the boundedness of functions \dot{u} and ψ_u , and the relation (3.69), there exist positive constants D_3 and D_4 independent of γ such that

$$|\dot{x}_{u \circ s_\gamma} - \dot{h}| \leq D_3 \|\chi_\gamma\|_\infty + D_4 \|x_{u \circ s_\gamma} - h\|_\infty, \quad \forall \gamma > \gamma_1, \quad (3.70)$$

which means that $x_{u \circ s_\gamma}$ converges to h in $W^{1,\infty}(I_u, \mathbb{R})$ as $\gamma \rightarrow \infty$ because of (3.59) and (3.68). Define $q_u \in C^0(\mathbb{R}_+, \mathbb{R})$ as $q_u = h \circ \rho_u$. Since for all $\dot{q}_u = |\dot{u}| \dot{h} \circ \rho_u$, relations (3.60)-(3.61) imply for all $t \geq 0$ that

$$q_u(t) = x_0 + \int_0^t (A q_u(\tau) + B u(\tau) + C) g^*(\dot{u}(\tau)) d\tau.$$

Moreover, using the relation $\sigma_\gamma = x_{u \circ s_\gamma} \circ \rho_u$, it can be easily verified that

$$\lim_{\gamma \rightarrow \infty} \|\sigma_\gamma - q_u\|_{1,\infty} = 0.$$

□

Remark 3.5.2. *In Lemma 3.5.3, the sequence of functions σ_γ converges in $W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ as $\gamma \rightarrow \infty$. This result is stronger than the one obtained in Lemma 3.5.1, where the convergence is only in $L^\infty(\mathbb{R}_+, \mathbb{R})$.*

Example 3.5.5. Consider the LuGre model [25]:

$$\dot{x} = \dot{u} - \theta \frac{|\dot{u}|}{\mu(\dot{u})} x = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} \dot{u} \\ -\theta \frac{|\dot{u}|}{\mu(\dot{u})} \end{bmatrix} = f(x) g(\dot{u}) \quad (3.71)$$

$$x(0) = x_0, \quad (3.72)$$

where parameters $\theta, c_1, c_2 > 0$, $x \in \mathbb{R}$ is the output, $x_0 \in \mathbb{R}$ is the initial state, and input $u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$. The function $\mu : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$\mu(\alpha) = F_C + (F_S - F_C) e^{-|\alpha/v_s|}, \forall \alpha \in \mathbb{R},$$

where F_C, F_S, v_s are positive parameters.

The LuGre model can be written in the form of the system (3.47)-(3.48) with $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$. We have

$$\lim_{\kappa \rightarrow 0^+} g(\kappa) / |\kappa| = G_+^* = \begin{bmatrix} G_{+,1}^* \\ G_{+,2}^* \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-\theta}{F_S} \end{bmatrix},$$

$$\lim_{\kappa \rightarrow 0^-} g(\kappa) / |\kappa| = G_-^* = \begin{bmatrix} G_{-,1}^* \\ G_{-,2}^* \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{-\theta}{F_S} \end{bmatrix}.$$

Clearly, Conditions (3.51)-(3.53) are satisfied. Thus, Lemma 3.5.3 implies that $\|\sigma_\gamma - q_u\|_{1,\infty} \rightarrow 0$, as $\gamma \rightarrow \infty$, where the functions $\sigma_\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $q_u \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ are defined for all $t \geq 0$ as

$$\sigma_\gamma(t) = x_0 + \int_0^t \left(\dot{u}(\tau) - \frac{\theta |\dot{u}(\tau)|}{\mu(\dot{u}(\tau)/\gamma)} q_u(\tau) \right) d\tau,$$

and

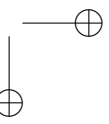
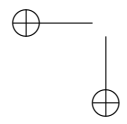
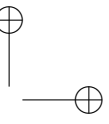
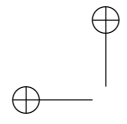
$$q_u(t) = x_0 + \int_0^t \left(\dot{u}(\tau) - \frac{\theta |\dot{u}(\tau)|}{F_S} q_u(\tau) \right) d\tau.$$

Also, there exist some $E, \gamma_1 > 0$ such that for all $\gamma > \gamma_1$, the solution of

(3.71)-(3.72) is global with $\|x_\gamma\|_\infty \leq E$.

3.6 Conclusions

This chapter has investigated the consistency of the generalized Duhem model with hysteresis. To this end, a classification has been introduced for the system in relation with a parameter λ related to the description of the field. Three classes of models have been considered: $\lambda > 1$, $0 < \lambda < 1$, and $\lambda = 1$. For $\lambda > 1$, it has been shown that the generalized Duhem model does not exhibit hysteresis. For $0 < \lambda < 1$, it has been shown that the semilinear Duhem model is not compatible with a hysteresis behavior. For the case $\lambda = 1$, necessary conditions and sufficient ones for the consistency of the Duhem model have been derived.



Conclusions and future work

Conclusions

This thesis has focused on the study of consistency of some hysteresis models. More precisely, Chapter 2 has considered LuGre friction model under minimal conditions. It has been shown that the model is both consistent and strongly consistent. Explicit expressions have been derived for the hysteresis loop as a result of the study.

Chapter 3 has investigated the consistency of the generalized Duhem model with hysteresis. To this end, a classification has been introduced for the system in relation with a parameter λ related to the description of the field. Three classes of models have been considered: $\lambda > 1$, $0 < \lambda < 1$, and $\lambda = 1$. For $\lambda > 1$, it has been shown that the generalized Duhem model does not exhibit hysteresis. For $0 < \lambda < 1$, it has been shown that the semilinear Duhem model is not compatible with a hysteresis behavior. For the case $\lambda = 1$, necessary conditions and sufficient ones for the consistency of the Duhem model have been derived.

Working on this thesis gave rise to the following publications:

- M.F.M. Naser, F. Ikhouane, “Consistency of the Duhem Model with Hysteresis”, *Mathematical Problems in Engineering*, vol. 2013, ID 586130,

2013, doi: 10.1155/2013/586130. (impact factor of the journal=1.383).

- M.F.M. Naser, F. Ikhouane, “On LuGre Friction Model”, in review process.
- M.F.M. Naser, F. Ikhouane, “Characterization of the Hysteresis Duhem Model”, *5th IFAC International Workshop on Periodic Control Systems (PSYCO 2013)*, Caen, France, pp. 29-34, 2013.
- M.F.M. Naser, F. Ikhouane, “Advances on LuGre Friction Model”, *International Conference on Applied Mathematics (ICAM 2013)*, vol. 7, no. 10, pp. 97-103, Paris, France, 2013.

Future work

One of the possible future research line is to design an identification algorithm for the LuGre model using the analytical description of its hysteresis loop. Another possible research line is to extend the results obtained in Lemma 2.2.1 to study the stability of slowly varying systems.

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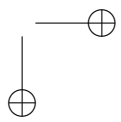
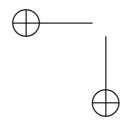
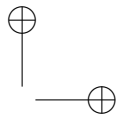
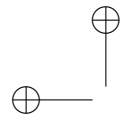
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