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# Operators and strong versions of sentential logics in Abstract Algebraic Logic

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*À Patrícia*

## Abstract

This dissertation presents the results of our research on some recent developments in Abstract Algebraic Logic (AAL), namely on the Suszko operator [**24**], the Leibniz filters [**37**], and truth-equational logics [**55**]. Part I builts and develops an abstract framework which unifies under a common treatment the study of the Leibniz, Suszko, and Frege operators in AAL. Part II generalizes the theory of the strong version of protoalgebraic logics, started in [**37**], to arbitrary sentential logics.

The interplay between several Leibniz- and Suszko-related notions led us to consider a general framework based upon the notion of $\mathcal{S}$-operator (inspired by that of "mapping compatible with $\mathcal{S}$-filters" [**24**, p. 199]), which encompasses the Leibniz, Suszko, and Frege operators. In particular, when applied to the Leibniz and Suszko operators, new notions of Leibniz and Suszko $\mathcal{S}$-filters arise as instances of more general concepts inside the abstract framework built. The former generalizes the existing notion of Leibniz filter for protoalgebraic logics [**37**] to arbitrary logics, while the latter is introduced here for the first time. Several results, both known and new, follow quite naturally inside this framework, again by instantiating it with the Leibniz and Suszko operators. Among the main new results, we prove a General Correspondence Theorem (Theorem 1.38), which generalizes Blok and Pigozzi's well-known Correspondence Theorem for protoalgebraic logics [**10**], as well as Czelakowski's less known Correspondence Theorem for arbitrary logics [**24**]. We characterize protoalgebraic logics in terms of the Suszko operator as those logics in which the Suszko operator commutes with inverse images by surjective homomorphisms (Theorem 3.12). We characterize truth-equational logics in terms of their (Suszko) $\mathcal{S}$-filters (Theorem 2.30), in terms of their full g-models (Corollary 2.31), and in terms of the Suszko operator, a characterization which strengthens that of [**55**], as those logics in which the Suszko operator is a structural representation from the set of $\mathcal{S}$-filters to the set of $\mathsf{Alg}(\mathcal{S})$-relative congruences, on arbitrary algebras (Theorem 4.13). Finally, we prove a new Isomorphism Theorem for protoalgebraic logics (Theorem 3.8), in the same spirit of the famous one for algebraizable logics [**11**] and for weakly algebraizable logics [**25**].

Endowed with a notion of Leibniz filter applicable to any logic, we are able to generalize the theory of the strong version of a protoalgebraic logic developed in [**37**] to arbitrary sentential logics. Given a sentential logic $\mathcal{S}$, its strong version $\mathcal{S}^+$ is the logic induced by the class of matrices whose truth set is Leibniz filter. We study three definability criteria of Leibniz filters: equational, explicit and logical definability. Under (any of) these assumptions, we prove that the $\mathcal{S}^+$-filters coincide with Leibniz $\mathcal{S}$-filters on arbitrary algebras. Finally, we apply the general theory developed to a wealth of non-protoalgebraic logics covered in the literature. Namely, we consider Positive Modal Logic $\mathcal{PML}$ [**28**], Belnap's logic $\mathcal{B}$ [**8**], the subintuitionistic logics $w\mathcal{K}_\sigma$ [**19**] and Visser's logic $\mathcal{VPL}$ [**58**], and Lukasiewicz's infinite-valued logic preserving degrees of truth [**35**]. We also consider the generalization of the last example mentioned to logics preserving degrees of truth from varieties of integral commutative residuated lattices [**17**], and further generalizations to the non-integral case, as well as to the case without multiplicative constant. We classify all the examples investigated inside the Leibniz and Frege hierarchies. While none of the logics studied is protoalgebraic, all the respective strong versions are truth-equational (though this need not hold in general).

# Resum


Aquesta dissertació presenta els resultats de la nostra recerca sobre alguns temes recents en Lògica Algebraica Abstracta (LAA), concretament, l'operador de Suszko [**24**], els filtres de Leibniz [**37**], i les lògiques truth-equacionals [**55**]. La primera part construeix i desenvolupa un marc abstracte que unifica sota un mateix tractament l'estudi dels operadors de Leibniz, Suszko, i Frege en AAL. La segona part generalitza la teoria de la versió forta d'una lògica protoalgebraica, que va començar a [**37**], a lògiques sentencials arbitràries.

La noció abstracta que abasta els operadors de Leibniz, de Suszko, i de Frege, és la de $\mathcal{S}$-*operador* (Definition 1.1). Hem investigat especialment una subclasse de $\mathcal{S}$-operadors, els anomenats $\mathcal{S}$-*operadors de compatibilitat*, que té origen en [**24**, p. 199] sota el nom de "mapping compatible with $\mathcal{S}$-filters". L'operador de Frege no és un $\mathcal{S}$-operador de compatibilitat, mentre que els operadors de Leibniz i de Suszko ho són. De fet, provem que l'operador de Leibniz és l'únic $\mathcal{S}$-operador de compatibilitat que commuta amb imatges inverses d'homomorfismes exhaustius (Theorem 1.24); i que l'operador de Suszko és el més gran $\mathcal{S}$-operador de compatibilitat monòton (Lemma 1.20). D'altra banda, l'operador de Frege és un $\mathcal{S}$-operador de compatibilitat si i només si $\mathcal{S}$ és una lògica plenament Fregeana (Proposition 2.48). Cercant propietats generals comuns als tres $\mathcal{S}$-operadors paradigmàtics en AAL, hem introdüit la nova noció de *coherència* (Definition 1.28), una propietat més feble que la de commutar amb imatges inverses d'homomorfismes exhaustius. Sota la hipòtesi de coherència d'una família de $\mathcal{S}$-operadors, hem establert un Teorema General de la Correspondència (Theorem 1.38), que generalitza altres teoremes de la correspondència coneguts en AAL, concretament el de Blok i Pigozzi per a lògiques protoalgebraiques [**10**, Theorem 2.4], i el de Czelakowski per a lògiques arbitràries [**24**, Proposition 2.3], així com una primera generalització del primer teorema esmentat obtinguda per Font i Jansana [**37**, Corollary 9.1].

Una família de $\mathcal{S}$-operadors $\nabla$ té associades amb ella les nocions de $\nabla$-classe i de $\nabla$-filtre (Definitions 1.12 i 1.15). Quan s'apliquen a les famílies de $\mathcal{S}$-operadors $\boldsymbol{\Omega}$ i $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$, respectivament, el primer concepte origina dues famílies de g-models plens, mentre que el segon origina noves nocions de filtres de Leibniz i de Suszko, com a casos particulars. Proposem com a nova definició de filtre de Leibniz justament la noció de $\boldsymbol{\Omega}$-filtre. Aquesta nova proposta generalitza a lògiques arbitràries la noció ja existent per a lògiques protoalgebraiques [**37**, Definition 1]. A més, el fet que es pugui aplicar a qualsevol lògica obre la possibilitat de generalitzar alguns resultats sobre lògiques protoalgebraiques a lògiques sentencials arbitràries. Per example, donada una lògica sentencial $\mathcal{S}$, els $\mathcal{S}$-filtres de Leibniz són precisament els elements mínims dels g-models plens de $\mathcal{S}$ (Proposition 2.9; compareu amb [**36**, Proposition 3.6]). La noció de filtre de Suszko s'introdueix aquí per primera vegada, i també ha estat investigada en detall. Els $\mathcal{S}$-filtres de Suszko resulten ser els elements mínims dels g-models plens de $\mathcal{S}$ que a més són creixents ("up-sets"). De fet, donada una àlgebra arbitrària $\boldsymbol{A}$ i un $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F$ és un filtre de Suzko de $\boldsymbol{A}$ si i només si $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ és un g-model ple de $\mathcal{S}$ (Theorem 2.29). A més, els filtres de Suszko s'han revelat força conectats ams les lògiques truth-equacionals, introdüides en [**55**]. En efecte, una lògica $\mathcal{S}$ és truth-equacional si i només si tots els $\mathcal{S}$-filtres són de Suszko, per a àlgebres arbitràries (Theorem 2.30).

També s'obtenen de manera natural diferents resultats, tant coneguts com nous, a dins d'aquest marc un cop aplicat als operadors de Leibniz i de Suszko. En particular, hem obtingut noves caracteritzacions d'algunes classes de lògiques de la jerarquía de Leibniz. Per esmentar-ne les principals, caracteritzem les lògiques protoalgebraiques en termes de l'operador de Suszko com aquelles on aquest operador commuta amb imatges inverses d'homomorfismes exhaustius (Theorem 3.12). Caracteritzem les lògiques truth-equacionals en termes dels seus filtres de Suszko com ja hem dit, però també en termes dels seus g-models plens (Corollary 2.31), i en termes de l'operador de Suszko mateix, com les lògiques on aquest operador és una representació estructural del conjunt de $\mathcal{S}$-filtres en el conjunt de congruències $\mathsf{Alg}(\mathcal{S})$-relatives, per a àlgebres arbitràries (Theorem 4.13); aquesta caracterització reforça la de [**55**]. A més, la mateixa condició, imposada només sobre l'àlgebra de les fórmules, caracteritza la veritat equacionalment definible a la classe $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$ (Theorem 4.21), un problema deixat obert a [**55**]. Finalment, provem un nou teorema d'isomorfisme per a lògiques protoalgebraiques (Theorem 3.8), en el mateix espirit que els famosos teoremes d'isomorfisme per a lògiques algebritzables ([**11**, Theorem 3.7]; veure [**48**, Theorem 5.2] per al cas no finitari) i per a lògiques feblement algebritzables ([**25**, Theorem 4.8]).

Un cop dotats d'una noció de filtre de Leibniz aplicable a qualsevol lògica, ens va semblar natural generalitzar la teoria de la versió forta d'una lògica protoalgebraica, desenvolupada en [**37**], a lògiques sentencials arbitràries. Donada una lògica sentencial $\mathcal{S}$, la seva versió forta $\mathcal{S}^+$ és la lògica induïda per la classe de matrius que tenen com a conjunt de veritat un filtre de Leibniz. Ens vam centrar especialment en la interacció entre els $\mathcal{S}$-filtres de Leibniz i els $\mathcal{S}^+$-filtres, mitjançant algunes condicions sota les quals aquestes dues famílies de $\mathcal{S}$-filtres coincideixen. Algunes d'aquestes condicions impliquen a més que les classes de $\mathcal{S}$-àlgebres i de $\mathcal{S}^+$-àlgebres coincideixin, fet que com sabem també passa al cas protoalgebraic, on val $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S}^+) = \mathsf{Alg}(\mathcal{S}^+)$, per a qualsevol lògica protoalgebraica $\mathcal{S}$. Però resulta que això no és un fet general, atés que la classe $\mathsf{Alg}(\mathcal{S}^+)$ pot estar estrictament continguda a la classe $\mathsf{Alg}(\mathcal{S})$, com testimonien alguns dels exemples de lògiques no protoalgebraiques estudiades.

Hem considerat tres criteris de definibilitat dels filtres de Leibniz: definabilitat equacional, definibilitat explícita, i definibilitat lògica. El primer és un nou criteri, mentre que els altres dos són generalitzacions a lògiques arbitràries de les respectives nocions introduïdes per a lògiques protoalgebraiques a [**37**]. Sota qualsevol d'aquestes hipòtesis, els $\mathcal{S}^+$-filtres coincideixen amb els $\mathcal{S}$-filtres de Leibniz en àlgebres arbitràries. Una família gran d'exemples abastada pel primer tipus de definibilitat esmentat és la classe de lògiques basades en semireticles ("semilattice-based") amb teoremes. De fet, aquestes lògiques sempre tenen els seus filtres de Leibniz equacionalment definibles pel conjunt de equacions $\boldsymbol{\tau}(x) = \{x \approx \top\}$, on $\top(x) \in \mathrm{Thm}_{\mathcal{S}}$ (Corollary 6.11). A més, la seva versió forta és la lògica $\boldsymbol{\tau}$-assercional respecte de $\mathsf{Alg}(\mathcal{S})$. Consegüentement, la versió forta de qualsevol lògica basada en semireticles amb teoremes és truth-equacional.

Finalment, hem aplicat la teoria general desenvolupada a un cert nombre de lògiques no protoalgebraiques estudiades a la literatura. Concretament, a la Lògica Positiva Modal $\mathcal{PML}$ [**28**], a la lògica de Belnap $\mathcal{B}$ [**8**], a les lògiques subintuïcionistes $w\mathcal{K}_\sigma$ [**19**] i de Visser $\mathcal{VPL}$ [**58**], i a la lògica infinito-valorada de Łukasiewicz que preserva graus de veritat [**35**]. També hem considerat la generalització del darrer exemple esmentat a lògiques que preserven graus de veritat respecte de varietats de reticles residuats integrals i commutatius [17], així com les generalitzacions als casos no integrals i sense constant multiplicativa. Hem classificat tots els exemples investigats dins de les jerarquies de Leibniz i de Frege. Un resum d'aquests fets es troba a les taules 4 i 5. Cap de les lògiques estudiades és protoalgebraica, mentre que totes les respectives versions fortes són truth-equacionals (fet que no es dóna en general). D'altra banda, les versions fortes obtingudes varien des de no protoalgebraiques a BP-algebritzables. Un cop més, aquesta situació contrasta amb la versió forta d'una lògica protoalgebraica, que sempre és protoalgebraica.

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Nunca tinha provado um teorema. E quando provei, continha um erro, e afinal era só uma proposição. Mas a primeira coisa que fazia depois de provar um lema que fosse, era contar-te. São na verdade todos corolários, porque são só saudades tuas.

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“Que a imaginação te engorde, e a matemática te emagreça.”
Agostinho da Silva

# Contents

# Introduction

Abstract Algebraic Logic (AAL) is a discipline inside Logic which takes a global perspective on the algebraization of different logical systems, mainly propositional, that have been considered in several fields such as Philosophy, Computer Science or the Foundations of Mathematics. Emphasis is put on the general process of associating to a given logic a class of algebras sharing very deep bonds with the logical system itself. The classical example is Classical Propositional Logic ($\mathcal{CPC}$) and its famous algebraic counterpart, the class of Boolean Algebras. Similarly, and to name just a few more well-known examples, Intuitionistic Propositional Logic ($\mathcal{IPC}$) is canonically associated with the class of Heyting Algebras, while the implication fragment of $\mathcal{CPC}$ is canonically associated with Rasiowa's implication algebras [**56**]. These, so-to-speak, "individualized" algebraic studies of particular logics, nowadays are seen as part of the discipline of Algebraic Logic (AL), started still in the XIXth century, and culminated in the so-called *Lindenbaum-Tarski* method, which emerged in the 1920s and was formalized in the 1940s and 1950s. For historical information about AAL, see [**23, 34, 39**].

All the examples of logics mentioned so far are, in AAL terminology, *algebraizable*. The degree to which each particular logic shares strong connections with its algebraic counterpart is one of the core problems addressed in AAL, and it gave rise to the so-called *Leibniz hierarchy*. This hierarchy classifies logics according to the algebraic properties enjoyed by the *Leibniz operator* over the logical filters on arbitrary algebras. The Leibniz operator is one of the cornerstone concepts in AAL, and was first introduced in the seminal work of Blok and Pigozzi [**11**]. It soon acquired the key rôle which still plays today in AAL, and one may safely say that any other operator put forward in AAL will always stand in comparison with the Leibniz operator. In fact, two further operators have been also considered in AAL, though studied to a far less extent. Namely, the *Frege operator* and the *Suszko operator*. Coincidently, the first one also gave rise to a hierarchy bearing its name, this time classifying logics according to some replacement properties they may satisfy. The *Frege hierarchy* was coined in [**32**], but its four classes of logics had already appeared separately in a plethora of different works [**22, 29, 36, 54, 59**]. The Suszko operator, on the other hand, didn't give rise to any (new) hierarchy in AAL, but was soon recognised as a good candidate to extend the Leibniz hierarchy outside the scope of protoalgebraic logics, potentially to *arbitrary* logics. The reason why this was the case is quite simple: protoalgebraic logics are characterized by the monotonicity of the Leibniz operator; the Suszko operator is *always* monotonic; for protoalgebraic logics, both operators coincide. The Suszko operator was formally introduced by Czelakowski in [**22**], though he attributes its invention and first characterization to Suszko, in unpublished lectures.

The original motivation for the work that eventually developed into the present dissertation was to undertake a thorough study of the Suszko operator in AAL. Broadly speaking, the goal was to try to mimic several known properties of protoalgebraic logics in the non-protoalgebraic realm, as well as to generalize some more recent developments in AAL to non-protoalgebraic logics, namely the notion of Leibniz filter [**37**]. The only work in the literature exclusively dedicated to the Suszko operator was — and still is, to the author's knowledge — [**24**]. This seminal paper about the Suszko operator lays the groundwork for the present investigation, and contains already several explicit clues to some of our new notions and results — one can highlight the notion of "mapping compatible with $\mathcal{S}$-filters" [**24**, p. 199] as a predecessor version of an $\mathcal{S}$-compatibility operator (see Definition 1.1), the notion of deductive homomorphism as the particular case of an homomorphism being $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-compatible (see Definition 1.25 and Lemma 1.27), the "Correspondence Property for deductive homomorphisms" [**24**, Proposition 2.3] as an instance of the General Correspondence Theorem 1.38, and its very last result [**24**, Theorem 2.8] which, in the presence of the (unknown at the time) definitions of truth-equational logic and Suszko filter, is remarkably insightful (compare the mentioned result with Theorems 2.30 and 3.11).

Protoalgebraicity is usually thought in AAL as say, the least assumption one can ask for a well-behaved logic. However, we have come to realize that the class of truth-equational logics, introduced by Raftery in [**55**], and independent from that of protoalgebraic logics, still exhibits very well-behaviours, at least with respect to the properties of the Leibniz operator one wished to find parallel in the Suszko operator. Surprisingly enough, or maybe not, the Suszko operator plays a prominent rôle in [**55**]. This paper soon became the main reference in the quest for finding properties of the Suszko operator inside truth-equational logics, which culminated in Section 4.1. Section 4.2 also addresses a problem rose in [**55**]. Furthermore, given Raftery's characterisation of truth-equational logics in terms of the Suszko operator [**55**, Theorem 28], the problem of finding similar characterizations for the remaining classes in the Leibniz hierarchy — answered in Section 3.2 — seemed not only natural, but also an interesting loose end to learn more about the Suszko operator.

The study of the Suszko operator made way to realize that the key points behind the proofs of several results concerning both the Leibniz and the Suszko operator, relied not so much in the definitions of these particular operators, but rather in a few compatibility arguments of congruences in general, and in the behaviour of these operators with respect to inverse images by surjective homomorphisms. In other words, Part I of the present work nourished from a particular instance of $\mathcal{S}$-operator — the Suszko operator.

The second part of this work is easier to track down, as not only it relies heavily on, but actually follows rather closely, the paper [**37**]. The goal of generalizing the protoalgebraic notion of Leibniz filter bifurcated in its generalization to arbitrary logics and the new notion of Suszko filter. The latter is strictly related to truth-equational logics, as shown in Theorem 2.30. The former leads to the definition of strong version of an *arbitrary* sentential logic, which is the core subject of Part II. Indeed, the strong version of a logic, henceforth denoted by $\mathcal{S}^{+}$, is the logic induced by the class of all matrices whose designated set is a Leibniz filter of $\mathcal{S}$. Leibniz filters were originally defined for protoalgebraic logics in [**37**] as the least elements among the class of $\mathcal{S}$-filters which share the same Leibniz congruence.[1] The existence of a minimum element for each such class of $\mathcal{S}$-filters is guaranteed by the protoalgebraic assumption over the underlying logic. However, a new notion of Leibniz filter is proposed in Part I which is applicable to arbitrary logics, and moreover coincides with the known one for protoalgebraic logics. It is only natural then to consider the logic induced by the class of all matrices whose designated set is a Leibniz filter of $\mathcal{S}$, according to our new definition. This is what we propose to do in Part II.

The formalization of the strong version of a logic $\mathcal{S}$ sheds some light on the phænomenon of pairs of logics strongly related found in many areas of non-classical logics. For instance, in [**37**] it is shown that the global modal consequence relation of the class of all Kripke frames is the strong version of the local modal consequence given by that same class, and that the Lukasiewicz's $n$-valued logic is the strong version of the Lukasiewicz's $n$-valued logic preserving degrees of truth. Many interesting non-classical logics are nevertheless not protoalgebraic. For example, Positive Modal Logic [**28**], Belnap's logic [**8**], logics preserving degrees of truth from the varieties of integral commutative residuated lattices [**17**], subintuitionistic logics [**16, 19**], etc. We shall cover these examples, among others, in Chapter 7.

It is worth mentioning that the problem of generalizing "the phænomenon of linked pairs of deductive systems independently of the protoalgebraicity of the weaker member of the pair" had already been tackled in [**38**]. Here, the existence of *enough Leibniz filters* is the assumption upon which the protoalgebraic setting of [**37**] is extended to arbitrary logics. But our new definition of Leibniz filter guarantees the existence of these filters regardless of any further assumption over the underlying logic. Actually, [**38**] exhibits an example of a subintuitionistic logic, concretely $w\mathcal{K}_\sigma$, which although naturally associated to its extension $s\mathcal{K}_\sigma$ under the rule (N), does not form a Leibniz-linked pair with it. As we shall see in Section 7.3, $s\mathcal{K}_\sigma$ is indeed the strong version of $w\mathcal{K}_\sigma$, that is, $(w\mathcal{K}_\sigma)^+ = s\mathcal{K}_\sigma$. In retrospect, and taking Proposition 5.1 into account, one also recognises that in [**31**] another pair of logics whose weak member is non-protoalgebraic was already seen to share the same bonds as those of the strong version for protoalgebraic logics. This pair is composed by the Lukasiewicz infinite logic preserving degrees of truth, $\text{Ł}_\infty^\leq$, and its companion preserving truth, $\text{Ł}_\infty^1$. In Section 7.4, we shall cover this example inside the more general case of integral commutative residuated lattices.

**Summary of contents.** The structure of this thesis is divided in two parts, as we have unfolded already. Both parts aim at extending some traditional, and some more recent, AAL tools, to non-protoalgebraic logics. We now proceed to detail their content.

### Part I

Chapter 1 is devoted to construct a general framework upon which a common study of the Leibniz, Suszko, and Frege operators can be built. The core notion is that of $\mathcal{S}$-*operator* (Definition 1.1), although probably of more relevance is its refinement of $\mathcal{S}$-*compatibility operator* (Definition 1.19). As already mentioned, this latter notion was originally introduced (in rigor, for congruential $\mathcal{S}$-operators)

[1] In fact, these (equivalence) classes of $\mathcal{S}$-filters had already been pointed out in [**36**, p.59] and [**25**, p.650].

under a similar name in [**24**, p. 199]. Another important new concept is that of *coherence*, which may be seen as a weaker property than commutativity with inverse images by surjective homomorphisms. We prove that the three paradigmatic examples of $\mathcal{S}$-operators — Leibniz, Suszko, and Frege — are coherent, while only the Leibniz operator commutes with inverse images by surjective homomorphisms. Furthermore, this notion allows us to prove the main result of Chapter 1 — the General Correspondence Theorem (Theorem 1.38). This result generalizes several known correspondence theorems in AAL, namely Blok and Pigozzi's well-known Correspondence Theorem for protoalgebraic logics [**10**, Theorem 2.4], Czelakowski's less known Correspondence Theorem [**24**, Proposition 2.3] for arbitrary logics, and it also generalizes the first strengthening obtained for protoalgebraic logics by Font and Jansana [**37**, Corollary 9.1].

In Chapter 2, the general framework just built is instantiated with the three main examples of $\mathcal{S}$-operators. Some new concepts arising from these particular instances will turn out to be quite relevant, especially those of *Leibniz filter* and *Suszko filter*. A wealth of both known and new results in AAL emerges rather naturally inside the general framework of $\mathcal{S}$-operators, of which we may point out a characterization of truth-equational logics in terms of Suszko filters (Theorem 2.30), as well as another characterization for this class of logics in terms of their full g-models (Corollary 2.31).

The two main results of Part I, however, appear in Chapter 3. Namely, a new Isomorphism Theorem for protoalgebraic logics (Theorem 3.8) in the same spirit of the famous one for algebraizable logics ([**11**, Theorem 3.7]; see also [**48**, Theorem 5.2] for the non-finitary case) and for weakly algebraizable logics ([**25**, Theorem 4.8]); as a corollary, another isomorphism theorem characterizing equivalential logics is obtained (Corollary 3.9); and finally, following the path set by Raftery's characterization of truth-equational logics in terms of the Suszko operator, we characterize protoalgebraic and equivalential logics in terms of this operator as well. Together with [**55**, Theorem 28], similar characterizations for weakly algebraizable and algebraizable logics follow as corollaries.

Finally, in Chapter 4, we undertake a small detour on truth-equational logics, providing some new contributions to the study of this class of logics. The main result is a new characterisation of the Suszko operator inside this class of logics (Theorem 4.2). We present yet another family of coherent $\mathcal{S}$-compatibility operators for logics having $\mathsf{Alg}(\mathcal{S})$ as an algebraic semantics (or equivalently, $\mathsf{Alg}^*(\mathcal{S})$), of which truth-equational logics are a (proper) subclass.

### Part II

We begin Chapter 5 by introducing the definition of the strong version $\mathcal{S}^+$ of an arbitrary sentential logic $\mathcal{S}$ and proving some rather general properties about $\mathcal{S}^+$. In particular, we shall characterize the $\mathcal{S}^+$-full g-models in terms of those of $\mathcal{S}$, and prove that the strong(er) version of $\mathcal{S}^+$, i.e., $(\mathcal{S}^+)^+$, is still $\mathcal{S}^+$.

In Chapter 6 we investigate some conditions one may impose on $\mathcal{S}$, in order to find general results which encompass several of the forthcoming examples. Do notice that placing $\mathcal{S}$ inside the Leibniz hierarchy, either makes $\mathcal{S}^+$ collapse into $\mathcal{S}$ (assuming $\mathcal{S}$ truth-equational), or makes our study converge with the one in [**37**] (assuming $\mathcal{S}$ protoalgebraic). So, we shall need to impose some condition(s) over

$\mathcal{S}$, but one(s) necessarily weaker than protoalgebraicity, and/or weaker than truth-equationality. We will do this through some definability criteria of the Leibniz filters of $\mathcal{S}$, namely explicit, logical, and equational definability. The two first criteria had already been considered in [**37**], and a generalization with parameters of the first one appears in [**51**], but always within the scope of protoalgebraic logics. The last criterion is new, and derives from the equational definability of $\mathcal{S}$-filters that characterizes truth-equational logics [**55**].

Finally, in Chapter 7, we apply the general results previously established to a plethora of (non-protoalgebraic) examples. For each logic considered, we will find the respective strong version, and characterize its Leibniz and Suszko filters on the $\mathcal{S}$-algebras. Furthermore, we investigate explicit, logical, and equational definability of the Leibniz filters. Surprisingly enough, all examples considered will turn out to have its Leibniz filters equationally and logically definable, but not all of them will have its Leibniz filters explicitly definable.

A final word of notice is in order here. This dissertation is, one may say, an expanded version of two rather long papers yet to appear at the time of writing, with some additional material (mainly, that concerning the Frege operator[2], and all of Chapter 4). The content of Part I is based on [**2**], while that of Part II is based on [**3**]. Their structure differs of course from the present one, but the main notions and results are the same, and arose from joint work with Josep Maria Font and Ramon Jansana.

[2]A distinction between congruential $\mathcal{S}$-operators is made in Part I to encompass the Frege operator, while in [**2**] the $\mathcal{S}$-operators are, by definition, congruential.

CHAPTER 0

# Preliminaries

## 0.1. Foundations

**Set Theory.** We will be working within the standard theory of Zermelo-Fraenkel with the Axiom of Choice (ZFC). We assume that the reader is familiar with the basic notions of set theory (see [46], for instance), and we will focus here on fixing some notation.

Given a map $f\colon A \to B$, we denote its extension to power sets with the same symbol; that is, we consider $f\colon \mathscr{P}(A) \to \mathscr{P}(B)$ defined, for each $X \subseteq A$, by $f(X) \coloneqq \{f(a) : a \in X\} \subseteq B$. The associated "inverse image" map, which is usually denoted as $f^{-1}\colon \mathscr{P}(B) \to \mathscr{P}(A)$, is defined, for each $Y \subseteq B$, by $f^{-1}(Y) = \{a \in A : f(a) \in Y\} \subseteq A$; this map is not the set-theoretic inverse of the extended map $f$, but rather its *residuum*, because it satisfies, for every $X \subseteq A$ and every $Y \subseteq B$, that $X \subseteq f^{-1}(Y)$ if and only if $f(X) \subseteq Y$. The extension construction will be iterated in a natural way, still keeping the same symbol; for instance, for a family $\mathscr{C} \subseteq \mathscr{P}(A)$, we define $f(\mathscr{C}) \coloneqq \big\{f(X) : X \in \mathscr{C}\big\}$, and for $\mathscr{D} \subseteq \mathscr{P}(Y)$, $f^{-1}(\mathscr{D}) \coloneqq \big\{f^{-1}(Y) : Y \in \mathscr{D}\big\}$. Similarly, $f$ is extended to cartesian products component-wise; in particular, $f\colon A \times A \to B \times B$ is defined as $f\big(\langle a, a'\rangle\big) \coloneqq \langle f(a), f(a')\rangle$ for every $a, a' \in A$. This map can itself be extended to power sets as before.

Let $f\colon A \to B$ be a map. The residuum condition between $f$ and $f^{-1}$ implies that for every $X \subseteq A$, $X \subseteq f^{-1}\big(f(X)\big)$, and for every $Y \subseteq B$, $f\big(f^{-1}(Y)\big) \subseteq Y$. As for the converse inclusions, one should keep in mind the following basic facts, which we shall henceforth use without any explicit mention:

1. $f$ is surjective if and only if $f(f^{-1}(Y)) = Y$, for every $Y \subseteq B$;
2. $f$ is injective if and only if $f^{-1}(f(X)) = X$, for every $X \subseteq B$.

A final word on notation: from this point on, we shall refrain from using parenthesis when denoting images and inverse images by maps, whenever its usage results too heavy, and/or the context is clear. For example, we shall write $fX$ instead of $f(X)$, and $f^{-1}Y$ instead of $f^{-1}(Y)$.

**First-order structures.** A *similarity type*, or *logical language*, is a tuple $\mathcal{L} = \langle \mathcal{F}, \mathcal{R}\rangle$, where $\mathcal{F} = \langle f_i\rangle_{i\in I}$ is to be understood as a family of function symbols, with each $f_i$ associated to a finite arity $\geq 0$, and $\mathcal{R} = \langle r_j\rangle_{j\in J}$ is to be understood as a family of relation symbols, with each $r_i$ associated to a finite arity $> 0$. A function symbol of arity 0 is called a *constant symbol*. Both families $\mathcal{F}$ and $\mathcal{R}$ can be empty, and can be finite or infinite. A similarity type is called *algebraic*, if $\mathcal{R} = \varnothing$; and it is called *(purely) relational*, if $\mathcal{F} = \varnothing$.

Let $\mathcal{L} = \langle \mathcal{F}, \mathcal{R}\rangle$ be a similarity type. A *structure of type* $\mathcal{L}$, or simply an $\mathcal{L}$-*structure*, is a tuple $\mathcal{M} = \langle M, \mathcal{F}_{\mathcal{M}}, \mathcal{R}_{\mathcal{M}}\rangle$, where:

- $M$ is a non-empty set, called the *universe*, or *domain*, of $\mathcal{M}$;

- $\mathcal{F}_{\mathcal{M}}$ is a family of functions on $M$ indexed by $\mathcal{F}$ and such that arities are preserved; i.e., each $n$-ary function symbol $f \in \mathcal{L}$ has a corresponding interpretation $f^{\mathcal{M}} : M^n \to M$ in $\mathcal{F}_{\mathcal{M}}$;
- $\mathcal{R}_{\mathcal{M}}$ is a family of relations on $M$ indexed by $\mathcal{R}$ and such that arities are preserved; i.e., each $n$-ary relation symbol $r \in \mathcal{R}$ has a corresponding interpretation $r^{\mathcal{M}} \subseteq M^n$ in $\mathcal{R}_{\mathcal{M}}$.

It is common practice to denote the domain of a given structure with a capital italic letter, namely by that corresponding to the structure's denotational symbol, regardless of this last being calligraphic (e.g., a first-order structure $\mathcal{A}$ and its domain $A$), double-struck (e.g., a lattice $\mathbb{L}$ and its domain $L$), or boldface (e.g., an algebra $\boldsymbol{A}$ and its domain $A$). Given a finite similarity type $\mathcal{L}$, say with $\mathcal{F} = \langle f_1, \ldots, f_n \rangle$ and $\mathcal{R} = \langle r_1, \ldots, r_m \rangle$, we shall denote the $\mathcal{L}$-structure $\mathcal{M}$ simply by

$$\mathcal{M} = \langle M, f_1^{\mathcal{M}}, \ldots, f_n^{\mathcal{M}}, r_1^{\mathcal{M}}, \ldots, r_m^{\mathcal{M}} \rangle.$$

Apart from the cornerstone notion of $\mathcal{L}$-structure, we only need to introduce one further concept concerning first-order logic, namely that of homomorphism between $\mathcal{L}$-structures. We do so, to stress that all the forthcoming notions of homomorphisms (e.g., order homomorphism, lattice homomorphism, algebraic homomorphism, matrix homomorphism) are particular instances of the this more general case, suitably restricted to the underlying language.

Let $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures. A map $h\colon A \to B$ is an *($\mathcal{L}$-)homomorphism*, if:

- $h(c^{\mathcal{A}}) = c^{\mathcal{B}}$, for every constant symbol $c \in \mathcal{L}$;
- $h(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(h(a_1), \ldots, h(a_n))$, for every $n$-ary function symbol $f \in \mathcal{L}$, with $n > 0$, and every $a_1, \ldots, a_n \in A$;
- if $\langle a_1, \ldots, a_n \rangle \in r^{\mathcal{A}}$, then $\langle h(a_1), \ldots, h(a_n) \rangle \in r^{\mathcal{B}}$, for every $n$-ary relation symbol $r \in \mathcal{L}$, with $n > 0$, and every $a_1, \ldots, a_n \in A$.

Finally, we will sporadically make use of two first-order connectives, namely the first-order conjunction and implication, which will be denoted by & and $\rightarrow$, respectively. The familiar symbols $\forall$ and $\exists$ are to be understood here as part of the meta-language, meaning *for all* and *there exists*, respectively. The symbols $\Rightarrow$ and $\Leftrightarrow$ stand for *if . . . then* and *if and only if*, respectively.

## 0.2. Lattice Theory

**Posets.** Let $X$ be a set. A *partial order on* $X$ is a reflexive, anti-symmetric, and transitive, binary relation $\leq$ on $X$. A *partially ordered set* (*poset* for short) is a relational structure $\mathbb{P} = \langle P, \leq \rangle$, where $\leq$ is a partial order on $P$. Given a poset $\langle P, \leq \rangle$, we take for granted that the reader is familiar with the notions of *upper bound* (dually, *lower bound*), *maximal element* (dually, *minimal element*), *maximum* (dually, *minimum*), and *supremum* (dually, *infimum*), of a subset $Y \subseteq X$, all with respect to the order $\leq$. We shall denote the infimum and supremum of a given subset $Y \subseteq X$ by $\bigwedge Y$ and $\bigvee Y$, respectively.

Let $\mathbb{P}_1 = \langle P_1, \leq_1 \rangle$ and $\mathbb{P}_2 = \langle P_2, \leq_2 \rangle$ be two posets. A map $f\colon P_1 \to P_2$ is:

1. *order preserving*, or an *order homomorphism*, if for every $x, y \in P_1$,
$$x \leq_1 y \Rightarrow f(x) \leq_2 f(y).$$
2. *order reversing*, if for every $x, y \in P_1$,
$$x \leq_1 y \Rightarrow f(y) \leq_2 f(x).$$

3. *order reflecting*, if for every $x, y \in P_1$,

$$f(x) \leq_2 f(y) \Rightarrow x \leq_1 y.$$

4. an *order embedding*, if for every $x, y \in P_1$,

$$x \leq_1 y \Leftrightarrow f(x) \leq_2 f(y).$$

5. an *order isomorphism*, if it is a surjective order embedding.

If a map is order reflecting, then it is injective. In particular, every order embedding is injective and every order isomorphism is bijective.

Given a poset $\mathbb{P} = \langle P, \leq \rangle$, a map $f\colon P \to P$ is:

1. *expansive*, if for every $x \in P$, $x \leq f(x)$;
2. *idempotent*, if for every $x \in P$, $f(f(x)) = f(x)$;
3. a *closure on* $\mathbb{P}$ if it is expansive, order preserving and idempotent.

**Lattices.** Lattices can be introduced via relational structures or via algebraic structures. We present here both definitions, not so much for the sake of completeness, but rather because we will use both of them in an exhaustive and indistinguishable manner.

A *lattice* (viewed as poset) is a relational structure $\mathbb{L} = \langle L, \leq \rangle$, where $\leq$ is a partial order on $L$ such that, for every $a, b \in L$, both the infimum and supremum of $a$ and $b$ exist; and, a *lattice* (viewed as an algebra) is an algebraic structure $\mathbb{L} = \langle L, \wedge, \vee \rangle$, where $\wedge$ and $\vee$ are two binary operations on $L$ such that, for every $a, b, c \in L$,

- *Idempotency:* $a \wedge a = a$ and $a \vee a = a$ ;
- *Commutativity:* $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ ;
- *Associativity:* $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$ ;
- *Absorption:* $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$ .

Both definitions are equivalent, in the following sense: given a lattice (as a poset), say $\mathbb{L} = \langle L, \leq \rangle$, then $\mathbb{L}^a = \langle L, \wedge, \vee \rangle$, where

$$a \wedge b = \inf\{a, b\} \quad \text{and} \quad a \vee b = \sup\{a, b\},$$

is a lattice (as an algebra). Conversely, given a lattice (as an algebra), say $\mathbb{L} = \langle L, \wedge, \vee \rangle$, then $\mathbb{L}^p = \langle L, \leq \rangle$, where

$$a \leq b \quad \text{iff} \quad a \wedge b = a,$$

is a lattice (as a poset). Furthermore, it holds $(\mathbb{L}^a)^p = \mathbb{L}$ and $(\mathbb{L}^p)^a = \mathbb{L}$. In light of these facts, we can (and will) speak interchangeably of a lattice $\mathbb{L} = \langle L, \wedge, \vee \rangle$ and its partial order $\leq$, as well of the lattice $\mathbb{L} = \langle L, \leq \rangle$ and its meet and join operations $\wedge$ and $\vee$, respectively.

A poset $\mathbb{L} = \langle L, \leq \rangle$ is a *meet-semilattice*, if for every $a, b \in L$, the infimum of $a$ and $b$ exists; dually, it is a *join-semilattice*, if for every $a, b \in L$, the supremum of $a$ and $b$ exists. Unless explicitly stated (as, for instance, in Lemma 2.27), all semilattices in this thesis will be meet-semilattices.

A lattice $\mathbb{L}_2$ is a *sublattice* of a lattice $\mathbb{L}_1$, if $L_2 \subseteq L_1$ and the meet and join operations of $\mathbb{L}_2$ are the restriction of the meet and join operations of $\mathbb{L}_1$, respectively. The definitions of *meet-sub-semilattice* and *join-sub-sublattice* are the expected ones.

The maximum element in a lattice, if it exists, is called the *top element*; dually, the minimum element of a lattice, if it exists, is called the *bottom element*. A lattice is *bounded*, if it has a top and a bottom element. A lattice $\mathbb{L}$ is *distributive*, if for every $a, b, c \in L$, the distributive laws hold:

- *Distributivity of* $\wedge$ *w.r.t.* $\vee$*:* $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ ;
- *Distributivity of* $\vee$ *w.r.t.* $\wedge$*:* $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ .

Each identity implies the other.

Let $\mathbb{L}_1 = \langle L_1, \wedge_1, \vee_1 \rangle$ and $\mathbb{L}_2 = \langle L_2, \wedge_2, \vee_2 \rangle$ be two lattices. A map $f\colon L_1 \to L_2$ is a *lattice homomorphism*, if for every $a, b \in L_1$,

$$h(a \wedge_1 b) = h(a) \wedge_2 h(b) \quad \text{and} \quad h(a \vee_1 b) = h(a) \vee_2 h(b).$$

A map satisfying only the first condition is called a *meet-homomorphism* and a map satisfying only the second condition is called a *join-homomorphism*. An injective lattice homomorphism is called an *embedding*. A surjective embedding is called a lattice *isomorphism*.

Since lattices can be be seen simultaneously as posets and algebras, it is natural to relate the concepts of order homomorphism and lattice homomorphism. In fact, we shall deal quite often with order isomorphisms, and then speak of the respective algebraic structures as isomorphic. Proposition 0.1.3 justifies this apparent abuse.

**Proposition 0.1.** *Let $\mathbb{L}_1$ and $\mathbb{L}_2$ be two lattices and let $h\colon L_1 \to L_2$ be a map.*

1. *If $h$ is a lattice homomorphism, then it is an order homomorphism.*
2. *If $h$ is a lattice embedding, then it is an order embedding.*
3. *The map $h$ is a lattice isomorphism if and only if it is an order isomorphism.*

So, to retain, although lattices can be seen interchangeably as ordered structures and algebraic structures, the respective notions of structure homomorphisms do not coincide.

**Complete and algebraic lattices.** A lattice $\mathbb{L}$ is *complete*, if for every $X \subseteq L$ there exists its infimum and its supremum, denoted by $\bigwedge X$ and $\bigvee X$, respectively. Comparing with the (algebraic) definition of lattice, we are furthermore imposing the existence of *arbitrary* meets and joins. An easy induction argument establishes the existence of meets and joins of finite non-empty sets in lattices. Hence, every finite lattice is complete. More interesting examples of complete lattices will appear throughout the text.

A sublattice where arbitrary meets and joins exist, and moreover coincide with those taken over the original lattice, is called a *complete sublattice*. A meet sub-semilattice (respectively, join sub-semilattice) where arbitrary meets exist, and moreover coincide with those taken over the original lattice, is called a *meet-complete sub-semilattice* (respectively, *join-complete sub-semilattice*).

Let $\mathbb{L}$ be a lattice. An element $a \in L$ is *compact*, if for every $X \subseteq L$ such that $a \leq \bigvee X$ (in particular, $\bigvee X$ must exist), there exists a finite subset $Y \subseteq X$ such that $a \leq \bigvee Y$. A lattice is *compactly generated*, if every element is a supremum of compact elements. Notice that if we assume $\mathbb{L}$ to be complete, we may forget the proviso that $\bigvee X$ must exist. A lattice is *algebraic*, if it is both complete and compactly generated.

**Lattice filters, prime filters and ultrafilters.** Let $\mathbb{L} = \langle L, \vee, \wedge \rangle$ be a lattice. A non-empty subset $F \subseteq L$ is a (lattice) *filter* of $\mathbb{L}$, if it satisfies the following conditions:

- $F$ is closed under meets, i.e., if $a, b \in F$, then $a \wedge b \in F$;

- $F$ is upwards-closed, i.e., if $a \in F$ and $b \in L$ is such that $a \leq b$, then $b \in F$.

The dual notion of filter is that of *ideal*. That is, a non-empty subset $I \subseteq L$ is an *ideal* of $\mathbb{L}$, if $I$ is closed under joins and downwards-closed.

We shall denote the set of all filters of a lattice $\mathbb{L}$ by $\mathrm{Filt}(\mathbb{L})$. The poset $\langle \mathrm{Filt}(\mathbb{L}) \cup \{\varnothing\}, \subseteq \rangle$ is an algebraic lattice with infima and suprema given by

$$\bigwedge_{i \in I} F_i = \bigcap_{i \in I} F_i \text{ and } \bigvee_{i \in I} F_i = \Big[\bigcup_{i \in I} F_i\Big),$$

where for every $H \subseteq L$,

$$\big[H\big) \coloneqq \bigcap \big\{F \in \mathrm{Filt}(\mathbb{L}) \cup \{\varnothing\} : H \subseteq F\big\}.$$

If $H$ is non-empty, then $\big[H\big)$ is the least filter of $\mathbb{L}$ containing $H$, also called the *filter generated by* $H$. The filter generated by the singleton $\{a\} \subseteq A$ will be simply denoted by $[a)$, and is usually called the *principal filter generated by* $a$; dually, $(a]$ denotes the *principal ideal generated by* $a$.

The following proposition, and subsequent corollary, will be used either explicitly or implicitly in all the examples of Chapter 7.

**Proposition 0.2.** *Let $\mathbb{L}$ be a lattice and $H \subseteq L$ non-empty. The filter generated by $H$ exists and is given by*

$$\big[H\big) = \big\{b \in L : a_1 \wedge \ldots \wedge a_n \leq b, \text{ for some } a_1, \ldots, a_n \in H \text{ and some } n > 0\big\}.$$

The filter generated by $\varnothing$ exists if and only if $\mathbb{L}$ has a maximum element, say $\top \in L$, and in this case $\big[\varnothing\big) = \{\top\}$.

**Corollary 0.3.** *Let $\mathbb{L}$ be a lattice and $H \subseteq L$. If $H$ is a filter and $a \notin H$, then*

$$\big[H, a\big) = \big\{b \in L : a \wedge c \leq b, \text{ for some } c \in H\big\}.$$

Notice that $L$ is always a filter of $\mathbb{L}$. A filter is *proper*, if it is not $L$. A proper filter $F \subseteq L$ is said to be *prime*, if for every $a, b \in L$, it holds

$$a \vee b \in F \Rightarrow a \in F \text{ or } b \in F.$$

We shall denote the set of all prime filters of a lattice $\mathbb{L}$ by $\mathrm{PrFilt}(\mathbb{L})$. Notice that neither $\varnothing$ nor $L$ are prime filters of $\mathbb{L}$.

In Chapter 7, we will sometimes need to extend lattice filters to prime filters. A famous result allow us to do this.

**Theorem 0.4** (Prime Filter Theorem)**.** *Let $\mathbb{L}$ be a distributive lattice. If $I \subseteq L$ is an ideal and $F \subseteq L$ a proper filter such that $I \cap F = \varnothing$, then there exists a prime filter $P$ such that $F \subseteq P$ and $I \cap P = \varnothing$.*

**Galois connections.** Let $\mathbb{P}_1 = \langle P_1, \leq_1 \rangle$ and $\mathbb{P}_2 = \langle P_2, \leq_2 \rangle$ be posets. A pair $\langle f, g \rangle$ of maps $f \colon P_1 \to P_2$ and $g : P_2 \to P_1$ establishes a *Galois connection* between $\mathbb{P}_1$ and $\mathbb{P}_2$, if for every $x \in P_1$ and every $y \in P_2$,

$$x \leq_1 g(y) \quad \Leftrightarrow \quad y \leq_2 f(x).$$

Galois connections entail several consequences, which we next compile in a single result. All the proofs can be found in [**27**, Chapter 7].

**Proposition 0.5.** *Let $\mathbb{P}_1 = \langle P_1, \leq_1 \rangle$ and $\mathbb{P}_2 = \langle P_2, \leq_2 \rangle$ be posets and let $f \colon P_1 \to P_2$ and $g \colon P_2 \to P_1$ establish a Galois connection between $\mathbb{P}_1$ and $\mathbb{P}_2$.*

1. *$f$ and $g$ are both order reversing.*

2. *The composition function* $g \circ f$ *is a closure on* $\mathbb{P}_1$*.*
3. *The composition function* $f \circ g$ *is a closure on* $\mathbb{P}_2$*.*
4. *The set of fixed points of* $g \circ f$ *is* $\mathrm{Ran}(g)$*.*
5. *The set of fixed points of* $f \circ g$ *is* $\mathrm{Ran}(f)$*.*
6. *The maps* $f$ *and* $g$ *restrict to mutually inverse dual order isomorphisms between the set of fixed points of* $g \circ f$ *and the set of fixed points of* $f \circ g$*.*

## 0.3. Closure operators

This is a core topic when working with sentential logics. Though we will not give a detailed treatment of it, understanding the (omitted) proofs should be amenable to the reader.

**Closure relations, closure operators and closure systems.** Let $A$ be a set. A *closure relation on* $A$ is a relation $\vdash \subseteq \mathscr{P}(A) \times A$ such that for every $X \cup Y \subseteq A$ and every $x, y \in A$, the following conditions hold:

- *Extensivity:* if $x \in X$, then $X \vdash x$;
- *Monotonicity:* if $X \subseteq Y$ and $X \vdash y$, then $Y \vdash y$;
- *Cut:* if $Y \vdash x$, for every $x \in X$, and $X \vdash y$, then $Y \vdash y$.

A closure relation $\vdash$ is *finitary*, if moreover it satisfies the additional condition:

- *Finitarity:* if $X \vdash x$, then there exists a finite subset $Y \subseteq X$ such that $Y \vdash x$.

It is not difficult to show that monotonicity holds in the presence of extensivity and cut, but tradition keeps all three conditions together.

A *closure operator over* $A$ is a map $C : \mathscr{P}(A) \to \mathscr{P}(A)$ such that for every $X, Y \subseteq A$,

- *Extensivity:* $X \subseteq C(X)$;
- *Monotonicity:* if $X \subseteq Y$, then $C(X) \subseteq C(Y)$;
- *Idempotency:* $C(C(X)) = C(X)$.

A closure operator $C$ is *finitary*, if it satisfies moreover the additional condition:

- *Finitarity:* $C(X) = \bigcup \{C(Y) : Y \subseteq X, Y \text{ finite}\}$.

A subset $X \subseteq A$ is said to be $C$*-closed*, if $C(X) = X$. Notice that a closure operator *over* $A$ is precisely a closure *on* $A$. Finally, a *closure system on* $A$ is a collection $\mathscr{C} \subseteq \mathscr{P}(A)$ such that

- $A \in \mathscr{C}$;
- $\mathscr{C}$ is closed under arbitrary intersections of non-empty families.

The following notation will be used quite often and in a rather essential way. Given a family $\mathscr{C} \subseteq \mathscr{P}(A)$ and a subset $F \subseteq A$, we define $\mathscr{C}^F := \{G \in \mathscr{C} : F \subseteq G\}$. Note that such a family is always an up-set in the poset $\langle \mathscr{C}, \subseteq \rangle$, and if $\mathscr{C}$ is a closure system, so is $\mathscr{C}^F$.

Every closure operator induces a closure system, and vice-versa. Indeed, given a closure system $\mathscr{C}$ on a set $A$, the map $C : \mathscr{P}(A) \to \mathscr{P}(A)$, defined by

$$C(X) = \bigcap \{Y \in \mathscr{C} : X \subseteq Y\},$$

is a closure operator over $A$. Conversely, given a closure operator $C$ on a set $A$, the collection

$$\mathscr{C} = \{X \subseteq A : C(X) = X\}$$

is a closure system on $A$. Moreover, denoting by $\mathrm{CO}(A)$ and $\mathrm{CS}(A)$ the sets of all closure operators and closure systems on a set $A$, respectively, the posets

$\langle \mathrm{CO}(A), \subseteq \rangle$ and $\langle \mathrm{CS}(A), \subseteq \rangle$ are dually order isomorphic under the mappings $C \mapsto \mathscr{C}$ and $\mathscr{C} \mapsto C$.

Similarly, every closure relation induces a closure operator, and vice-versa. Indeed, given a closure relation $\vdash$ on $A$, the map $C_{\vdash} : \mathscr{P}(A) \to \mathscr{P}(A)$, defined by

$$C_{\vdash}(X) = \{x \in A : X \vdash x\},$$

is a closure operator over $A$. Conversely, given a closure operator $C$ on a set $A$, the relation $\vdash_C \subseteq \mathscr{P}(A) \times A$, defined by

$$X \vdash_C x \quad \text{iff} \quad x \in C(X),$$

is a closure relation on $A$.

It should be clear that finitary closure operators induce finitary closure relations, and vice-versa, through the same maps as above. It only remains to see how does this condition translates to closure systems. This is the content of the famous Schmidt's Theorem.

Let $\lambda$ be any cardinal. A family $\mathcal{D} = \{X_i : i \in I\} \subseteq \mathscr{P}(A)$ is $\lambda$-*directed*, if for every subfamily $\{X_j : j \in J\} \subseteq \mathcal{D}$ of cardinal $< \lambda$ there exists $X_k \in \mathcal{D}$ such that $X_j \subseteq X_k$, for every $j \in J$. In particular, $\omega$-directed families are also called simply *directed*, or *upwards-directed*. A closure system $\mathscr{C}$ on $A$ is *inductive*, if it is closed under unions of non-empty directed families.

**Theorem 0.6** (Schmidt's Theorem)**.** *A closure system $\mathscr{C}$ is inductive if and only if its associated closure operator $C$ if finitary.*

**Closure systems and complete lattices.** There is a close connection between closure systems and complete lattices.

**Theorem 0.7.** *If $\mathscr{C}$ is a closure system, then $\langle \mathscr{C}, \subseteq \rangle$ is a complete lattice, with infima and suprema given by*

$$\bigwedge_{i\in I}^{\mathscr{C}} T_i = \bigcap_{i \in I} T_i \text{ and } \bigvee_{i \in I}^{\mathscr{C}} T_i = C\Big(\bigcup_{i\in I} T_i\Big),$$

*for every $\{T_i : i \in I\} \subseteq \mathscr{C}$.*

Actually, up to isomorphism, the converse is also true. That is, *every complete lattice is order isomorphic to the lattice of closed sets of some closure operator.* Algebraic lattices (which are, by definition, complete lattices) also stand in bijection with a (sub-)family of closure systems.

**Theorem 0.8.** *If $\mathscr{C}$ is an inductive closure system, then $\langle \mathscr{C}, \subseteq \rangle$ is an algebraic lattice.*

Again, up to isomorphism (!), the converse is also true. That is, *every algebraic lattice is order isomorphic to the lattice of closed sets of some inductive closure system.*

We finally state some useful lemmas for dealing with closure operators, which we shall make use of further ahead.

**Lemma 0.9.** *If $C$ is a closure operator over a set $A$, then for every $X \subseteq A$,*

$$C(X) = \bigvee^{\mathscr{C}} \{C(Y) : Y \subseteq X, Y \text{ finite}\} = \bigvee^{\mathscr{C}} \{C(\{x\}) : x \in X\}.$$

Comparing Lemma 0.9 with the defining condition of finitary closure operator, one sees that "finitarity is an essentially set theoretical property rather than a lattice theoretical one" [**34**, p. 36].

### 0.4. Universal Algebra

It is often said that Abstract Algebraic Logic is to Algebraic Logic as Universal Algebra is to Algebra. The whole topic of Universal Algebra is far beyond the scope of the present work, and we shall only cover here the needed material. We will however look with some detail at relative congruences, as they will be transversal to all our work. We refer the reader to the classical reference [**18**] for the most common constructions regarding algebras, such as subalgebras, quotient algebras, homomorphic images, direct, subdirect, and reduced products (in particular, ultraproducts), and free algebras.

**Algebras and the formula algebra.** An *algebra* is an $\mathcal{L}$-structure where $\mathcal{L}$ is an algebraic similarity type. That is, an algebra $\boldsymbol{A}$ is a tuple

$$\boldsymbol{A} = \big\langle A, \langle f^{\boldsymbol{A}} \rangle_{f \in \mathcal{L}} \big\rangle,$$

where $A$ is a non-empty set, and each element $f^{\boldsymbol{A}}$ indexed by the $n$-ary symbol $f \in \mathcal{L}$ is an $n$-ary function in $A$. It is usual to drop the superscript $^{\boldsymbol{A}}$ in the algebra operations. An algebra is *trivial*, if its universe has a single element.

Unless otherwise stated, we henceforth assume fixed an arbitrary algebraic similarity type $\mathcal{L}$.

Let $\boldsymbol{A}, \boldsymbol{B}$ be algebras. A map $h \colon A \to B$ is an (algebraic) *homomorphism*, if

$$h\big(f^{\boldsymbol{A}}(a_1, \dots, a_n)\big) = f^{\boldsymbol{B}}\big(h(a_1), \dots, h(a_n)\big),$$

for every $n$-ary operation symbol $f \in \mathcal{L}$ and every $a_1, \dots, a_n \in A$. We write $h \in \operatorname{Hom}(\boldsymbol{A}, \boldsymbol{B})$, or just $h \colon \boldsymbol{A} \to \boldsymbol{B}$, to indicate that $h$ is a homomorphism from $\boldsymbol{A}$ to $\boldsymbol{B}$. An homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{A}$ is called an *endomorphism*. A bijective homomorphism is called an (algebraic) *isomorphism*.

Let us fix a countably infinite set of variables Var, disjoint from $\mathcal{L}$. The *algebra of terms*, or *formula algebra*[1], $\boldsymbol{Fm}$ is the absolutely free algebra generated by the set Var over the language $\mathcal{L}$. Its universe is denoted by $\mathrm{Fm}_{\mathcal{L}}$, and its members are called ($\mathcal{L}$-)*terms* or ($\mathcal{L}$-)*formulas*. We write $\varphi(x_1, \dots, x_n)$ when we want to stress that every variable occurring in the $\mathcal{L}$-formula $\varphi$ occurs in $\{x_1, \dots, x_n\}$. It follows by the universal mapping property of $\boldsymbol{Fm}$ that every map from Var to $\mathrm{Fm}_{\mathcal{L}}$ can be uniquely extended to an endomorphism of $\boldsymbol{Fm}$; such a map is called a *substitution*.

Given $\varphi(x_1, \dots, x_n) \in \mathrm{Fm}_{\mathcal{L}}$ and $a_1, \dots, a_n \in A$, we denote by $\varphi^{\boldsymbol{A}}(a_1, \dots, a_n)$ the interpretation on $\boldsymbol{A}$ of the formula $\varphi$ under any homomorphism $h \in \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h(x_i) = a_i$, for every $1, \dots, n$. We extend this notation to subsets of formulas $\Gamma(x_1, \dots, x_n) \subseteq \mathrm{Fm}_{\mathcal{L}}$ as expected, i.e., $\Gamma^{\boldsymbol{A}}(a_1, \dots, a_n) = \{\gamma^{\boldsymbol{A}}(a_1, \dots, a_n) : \gamma \in \Gamma\}$; and also to the cartesian product $\mathrm{Fm}_{\mathcal{L}} \times \mathrm{Fm}_{\mathcal{L}}$, so that given $\boldsymbol{\tau}(x_1, \dots, x_n) \subseteq \mathrm{Fm}_{\mathcal{L}} \times \mathrm{Fm}_{\mathcal{L}}$, $\boldsymbol{\tau}^{\boldsymbol{A}}(a_1, \dots, a_n) = \big\{\langle \delta^{\boldsymbol{A}}(a_1, \dots, a_n), \epsilon^{\boldsymbol{A}}(a_1, \dots, a_n)\rangle : \langle \delta, \epsilon \rangle \in \boldsymbol{\tau}\big\}$.

The next proposition tell us that "the interpretation of formulas behave like fundamental operations insofar as homomorphisms (...) are concerned" [**18**, p. 63]; this fact will be repeatedly used in the sequel without any explicit mention.

[1] In the context of Algebraic Logic, the terms of an algebraic similarity type can be considered as the formulas of a propositional logic. In a first-order context however, the $\mathcal{L}$-formulas here defined would be just the $\mathcal{L}$-terms.

**Proposition 0.10.** *Let $\boldsymbol{A}$ be an algebra and $h\colon \boldsymbol{A} \to \boldsymbol{B}$ a homomorphism. For every $\mathcal{L}$-formula $\varphi(x_1,\dots,x_n) \in \mathrm{Fm}_{\mathcal{L}}$ and every $a_1,\dots,a_n \in A$, it holds*

$$h\big(\varphi^{\boldsymbol{A}}(a_1,\dots,a_n)\big) = \varphi^{\boldsymbol{B}}\big(h(a_1),\dots,h(a_n)\big).$$

Given a set $A$, let $A^*$ denote the set of all finite sequences of elements in $A$. It is known that, if $A$ is an infinite set, then $A^*$ has the same cardinality as $A$. Now, on the one hand, every formula in the language $\mathcal{L}$ is a finite sequence of symbols of the set $\mathrm{Var}\cup\mathcal{L}$, so $|\mathrm{Fm}_{\mathcal{L}}| \leq |\mathrm{Var}\cup\mathcal{L}|^* = |\mathrm{Var}\cup\mathcal{L}|$. On the other hand, there exists an injective map from $\mathrm{Var}\cup\mathcal{L}$ to $\mathrm{Fm}_{\mathcal{L}}$ (just assign to each variable itself, and to each $n$-ary function symbol $f \in \mathcal{L}$ the formula $fx_1\dots x_n$, with $x_1,\dots,x_n \in \mathrm{Var}$), so $|\mathrm{Var}\cup\mathcal{L}| \leq |\mathrm{Fm}_{\mathcal{L}}|$. Hence,

$$|\mathrm{Fm}_{\mathcal{L}}| = |\mathrm{Var}\cup\mathcal{L}|.$$

In particular, and since we have fixed an infinite countable set of variables Var, if the language $\mathcal{L}$ is finite (and this will be the case in all the examples covered in Chapter 7), then $|\mathrm{Fm}_{\mathcal{L}}| = \omega$.

**Equational logic.** An $\mathcal{L}$-*equation* is a pair of formulas $\langle\varphi,\psi\rangle \in \mathrm{Fm}_{\mathcal{L}} \times \mathrm{Fm}_{\mathcal{L}}$. We shall denote the set of all $\mathcal{L}$-equations by $\mathrm{Eq}_{\mathcal{L}}$, and we will usually write $\varphi \approx \psi$ instead of $\langle\varphi,\psi\rangle$ in order to stress the equational setting rather than the set theoretical one.

An homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$ *satisfies* an equation $\varphi \approx \psi \in \mathrm{Eq}_{\mathcal{L}}$, if $h(\varphi) = h(\psi)$; this fact is sometimes denoted by $\boldsymbol{A} \vDash \varphi \approx \psi[\![h]\!]$. A particular case often used is the following: an element $a \in A$ *satisfies* an equation $\varphi(x) \approx \psi(x) \in \mathrm{Eq}_{\mathcal{L}}$, if $\varphi^{\boldsymbol{A}}(a) = \psi^{\boldsymbol{A}}(a)$; fact which is denoted by $\boldsymbol{A} \vDash \varphi \approx \psi[\![a]\!]$.

Let $\boldsymbol{A}$ be an algebra and $\varphi(x_1,\dots,x_n), \psi(x_1,\dots,x_n) \in \mathrm{Fm}_{\mathcal{L}}$. The equation $\varphi \approx \psi$ *holds in* $\boldsymbol{A}$, or $\varphi \approx \psi$ *is valid in* $\boldsymbol{A}$, fact which we shall denote by $\boldsymbol{A} \vDash \varphi \approx \psi$, if $\varphi^{\boldsymbol{A}}(a_1,\dots,a_n) = \psi^{\boldsymbol{A}}(a_1,\dots,a_n)$, for every $a_1,\dots,a_n \in A$. Given a class of algebras $\mathsf{K}$, we denote by $\mathsf{K} \vDash \varphi \approx \psi$ the fact that the equation $\varphi \approx \psi$ holds in every algebra of $\mathsf{K}$.

An $\mathcal{L}$-*quasi-equation* is a first-order formula of the form

$$\alpha_1 \approx \beta_1 \;\;\&\;\; \dots \;\;\&\;\; \alpha_n \approx \beta_n \longrightarrow \alpha \approx \beta,$$

where $\alpha,\beta \in \mathrm{Fm}_{\mathcal{L}}$ and $\alpha_i,\beta_i \in \mathrm{Fm}_{\mathcal{L}}$, for every $i = 1,\dots n$. Equations are to be understood as particular cases of quasi-equations with an empty antecedent.

Let $\boldsymbol{A}$ be an algebra. A quasi-equation $\alpha_1 \approx \beta_1 \;\&\; \dots \;\&\; \alpha_n \approx \beta_n \longrightarrow \alpha \approx \beta$ *holds in* $\boldsymbol{A}$, or *is true in* $\boldsymbol{A}$, fact which we shall denote by $\boldsymbol{A} \vDash \alpha_1 \approx \beta_1 \;\&\; \dots \;\&\; \alpha_n \approx \beta_n \longrightarrow \alpha \approx \beta$, if every homomorphism $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$ which satisfies all the equations $\alpha_i \approx \beta_i$, with $i = 1,\dots,n$, also satisfies the equation $\alpha \approx \beta$.

A *generalized* $\mathcal{L}$-*quasi-equation* is a (possibly infinitary) first-order formula of the form

$$\mathop{\&}_{i\in I} \alpha_i \approx \beta_i \longrightarrow \alpha \approx \beta,$$

where $\alpha,\beta \in \mathrm{Fm}_{\mathcal{L}}$ and $\alpha_i,\beta_i \in \mathrm{Fm}_{\mathcal{L}}$, for every $i \in I$, for some (possibly infinite) set $I$. In other words, generalized quasi-equations admit an infinite antecedent; and in doing so, they are indeed more general than quasi-equations. The notion of validity of a generalized quasi-equation is the expected one.

Let $\mathsf{K}$ be a class of $\mathcal{L}$-algebras. The *equational (consequence) relation relative to* $\mathsf{K}$, hereby denoted by $\vDash^{\mathrm{eq}}_{\mathsf{K}} \subseteq \mathscr{P}(\mathrm{Eq}_{\mathcal{L}}) \times \mathrm{Eq}_{\mathcal{L}}$, is defined as follows:

$$\Pi \vDash^{\mathrm{eq}}_{\mathsf{K}} \varphi \approx \psi \quad \text{iff} \quad \forall \boldsymbol{A} \in \mathsf{K}\ \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$$
$$\forall \delta \approx \epsilon \in \Pi\ h(\delta) = h(\epsilon) \ \Rightarrow\ h(\varphi) \approx h(\psi)$$

for every $\Gamma \cup \{\varphi \approx \psi\} \subseteq \mathrm{Eq}_{\mathcal{L}}$. We write simply $\vDash^{\mathrm{eq}}_{\boldsymbol{A}}$ to denote $\vDash^{\mathrm{eq}}_{\{\boldsymbol{A}\}}$. It should be clear that[2]

$$\boldsymbol{A} \vDash \varphi \approx \psi \quad \text{iff} \quad \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\ h(\varphi) = h(\psi)$$
$$\text{iff} \quad \varnothing \vDash^{\mathrm{eq}}_{\boldsymbol{A}} \varphi \approx \psi,$$

and that

$$\boldsymbol{A} \vDash \alpha_1 \approx \beta_1 \ \&\ \dots\ \&\ \alpha_n \approx \beta_n \to \alpha \approx \beta \quad \text{iff} \quad \{\alpha_1 \approx \beta_1, \dots, \alpha_n \approx \beta_n\} \vDash^{\mathrm{eq}}_{\boldsymbol{A}} \alpha \approx \beta.$$

**Varieties, quasi-varieties and generalized quasi-varieties.** A class operator maps classes of algebras to classes of algebras, all of the same similarity type. Let $\mathsf{K}$ be a class of algebras and let $\boldsymbol{A}$ be an algebra. We define the following class operators:

- $\boldsymbol{A} \in \mathbb{I}(\mathsf{K})$ iff $\boldsymbol{A}$ is isomorphic to some member of $\mathsf{K}$;
- $\boldsymbol{A} \in \mathbb{S}(\mathsf{K})$ iff $\boldsymbol{A}$ is a subalgebra of some member of $\mathsf{K}$;
- $\boldsymbol{A} \in \mathbb{H}(\mathsf{K})$ iff $\boldsymbol{A}$ is a homomorphic image of some member of $\mathsf{K}$;
- $\boldsymbol{A} \in \mathbb{P}(\mathsf{K})$ iff $\boldsymbol{A}$ is a direct product of a non-empty family of members of $\mathsf{K}$;
- $\boldsymbol{A} \in \mathbb{P}_{\mathrm{U}}(\mathsf{K})$ iff $\boldsymbol{A}$ is an ultraproduct of a non-empty family of members of $\mathsf{K}$;
- $\boldsymbol{A} \in \mathbb{P}_{\mathrm{R}}(\mathsf{K})$ iff $\boldsymbol{A}$ is a reduced product of a non-empty family of members of $\mathsf{K}$;
- $\boldsymbol{A} \in \mathbb{P}_{\lambda-\mathrm{R}}(\mathsf{K})$ iff $\boldsymbol{A}$ is an $\lambda$-reduced product[3] of a non-empty family of members of $\mathsf{K}$;
- $\boldsymbol{A} \in \mathbb{P}_{\mathrm{S}}(\mathsf{K})$ iff $\boldsymbol{A}$ is a subdirect product of a non-empty family of members of $\mathsf{K}$;
- $\boldsymbol{A} \in \mathbb{U}(\mathsf{K})$ iff every subalgebra of $\boldsymbol{A}$ countably generated is a member of $\mathsf{K}$.

Notice that reduced products are the $\omega$-reduced products. We say that a class of algebras $\mathsf{K}$ is *closed under* a class operator $\mathbb{O}$, if $\mathbb{O}(\mathsf{K}) \subseteq \mathsf{K}$. Given two class operators $\mathbb{O}_1$ and $\mathbb{O}_2$, we write $\mathbb{O}_1 \leq \mathbb{O}_2$ to denote the fact $\mathbb{O}_1(\mathsf{K}) \subseteq \mathbb{O}_2(\mathsf{K})$, for every class of algebras $\mathsf{K}$.

Of course, trivial algebras are all isomorphic. We shall denote by $\boldsymbol{Triv}$ an arbitrary, but fixed, trivial algebra. The constant map from any algebra to any trivial algebra is obviously a homomorphism, therefore any non-empty class closed under $\mathbb{H}$ contains all trivial algebras.

A non-empty class of algebras $\mathsf{K}$ is a *variety*, if it is closed under subalgebras, homomorphic images and direct products. Hence, varieties contain all trivial algebras. Given a class of algebras $\mathsf{K}$, the *variety generated by* $\mathsf{K}$, which we shall denote by $\mathbb{V}(\mathsf{K})$, is the least variety containing the class $\mathsf{K}$. The existence of such variety is justified by the following famous result:

[2] We have chosen to add the supscript $^{\mathrm{eq}}$ to the relative equational consequence relation $\vDash^{\mathrm{eq}}_{\mathsf{K}}$ just to emphasize that it is a closure relation on $\mathrm{Eq}_{\mathcal{L}}$ rather than $\mathrm{Fm}_{\mathcal{L}}$.

[3] A $\lambda$-*reduced product* is a reduced products modulo a $\lambda$-complete filter; a $\lambda$-*complete filter* on a set $I$ is a filter of $\mathscr{P}(I)$ closed under intersections of families of cardinal $< \lambda$.

**Theorem 0.11** (Tarski)**.** *For every class of algebras* $\mathsf{K}$*,*

$$\mathbb{V}(\mathsf{K}) = \mathbb{HSP}(\mathsf{K}).$$

A non-empty class of algebras $\mathsf{K}$ is a *quasivariety*, if it is closed under subalgebras, isomorphisms, reduced products and contains a trivial algebra. Every variety is a quasivariety, since $\mathbb{P}_{\mathrm{R}} \leq \mathbb{P}_{\mathrm{U}} \leq \mathbb{HP}$. Given a class of algebras $\mathsf{K}$, the *quasivariety generated by* $\mathsf{K}$, which we shall denote by $\mathbb{Q}(\mathsf{K})$, is the least quasivariety containing the class $\mathsf{K}$. The existence of such quasivariety is ensured by another famous result:

**Theorem 0.12** (Mal'cev)**.** *For every class of algebras* $\mathsf{K}$*,*

$$\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K} \cup \{\boldsymbol{Triv}\}) = \mathbb{ISP}_{\mathrm{R}}(\mathsf{K} \cup \{\boldsymbol{Triv}\}).$$

If we admit direct products of empty families, defining them as $\prod_{\varnothing} \boldsymbol{A}_i \cong \boldsymbol{Triv}$, then we can state a more elegant version of Mal'cev theorem as $\mathbb{Q}(\mathsf{K}) = \mathbb{ISPP}_{\mathrm{U}}(\mathsf{K}) = \mathbb{ISP}_{\mathrm{R}}(\mathsf{K})$.

A non-empty class of algebras $\mathsf{K}$ is a *generalized quasivariety*, if it is closed under subalgebras, isomorphisms, direct products, closed under the operator $\mathbb{U}$, and contains a trivial algebra. A result from Universal Algebra tells us that every algebra can be embedded into an ultraproduct of its finitely generated subalgebras (see, for instance, [**18**, Theorem 2.14]). So, if a class of algebras is closed under $\mathbb{ISP}_{\mathrm{U}}$, then it is closed under $\mathbb{U}$. As a consequence, every quasivariety is a generalized quasivariety.

Given a class of algebras $\mathsf{K}$, the *generalized quasivariety generated by* $\mathsf{K}$, which we shall denote by $\mathbb{GQ}(\mathsf{K})$, is the least generalized quasivariety containing the class $\mathsf{K}$. The following theorem is not so well known as the previous analogous ones for varieties and quasivarieties; to the author's knowledge, it was formally stated for the first time in [**9**, Corollary 8.2].

**Theorem 0.13.** *For every class of algebras* $\mathsf{K}$*,*

$$\mathbb{GQ}(\mathsf{K}) = \mathbb{UISP}(\mathsf{K} \cup \{\boldsymbol{Triv}\}).$$

Again, if we admit direct products over an empty family of indexes, we can write simply $\mathbb{GQ}(\mathsf{K}) = \mathbb{UISP}(\mathsf{K})$.

**Subdirectly irreducible algebras.** Let $\{\boldsymbol{A}_i : i \in I\}$ be a family of algebras. An embedding $\alpha : \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{A}_i$ is said to be *subdirect*, if the image $\alpha(\boldsymbol{A})$ is a subdirect product of the family $\{\boldsymbol{A}_i\}_{i\in I}$. In this case, $\boldsymbol{A}$ is said to be *subdirectly embeddable into* $\prod_{i\in I} \boldsymbol{A}_i$, or a *subdirect embedding of* $\prod_{i\in I} \boldsymbol{A}_i$. An algebra $\boldsymbol{A}$ is *subdirectly irreducible*, if for every subdirect embedding $\alpha : \boldsymbol{A} \to \prod_{i\in I} \boldsymbol{A}_i$, there exists $i \in I$ such that $\pi_i \circ \alpha : \boldsymbol{A} \to \boldsymbol{A}_i$ is an isomorphism, where $\pi_i : \boldsymbol{A} \to \boldsymbol{A}_i$ is the $i$-th projection of $\boldsymbol{A}$. A classical result from Birkhoff (which we shall need to make use of once — at Proposition 7.34) is the following:

**Theorem 0.14** (Birkhoff)**.** *Every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.*

Notice that, in particular, if $\mathbb{V}$ is a variety, then every algebra $\boldsymbol{A} \in \mathbb{V}$ is isomorphic to a subdirect product of subdirectly irreducible algebras $\{\boldsymbol{A}_i : i \in I\}$ such that $\boldsymbol{A}_i \in \mathbb{V}$, for every $i \in I$. Indeed, by definition of subdirect product, the $i$-th projection of $\boldsymbol{A}$ is surjective, that is, $\pi_i(\boldsymbol{A}) = \boldsymbol{A}_i$, and since $\mathbb{V}$ is closed under homomorphic images, it follows that $\boldsymbol{A}_i \in \mathbb{V}$, for every $i \in I$.

**Congruences.** Let $A$ be a set. A relation $R \subseteq A \times A$ is an *equivalence relation on* $A$, if it is reflexive, symmetric and transitive. That is, if for every $a, b, c \in A$,

- $\langle a, a\rangle \in R$;
- if $\langle a, b\rangle \in R$, then $\langle b, a\rangle \in R$;
- if $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$, then $\langle a, c\rangle \in R$.

We shall denote the set of all equivalence relations on $A$ by Eqr$A$. The poset $\langle \mathrm{Eqr}A, \subseteq\rangle$ is a complete lattice, with infima and suprema given by

$$\bigwedge_{i\in I}\theta_i = \bigcap_{i\in I}\theta_i \quad \text{and} \quad \bigvee_{i\in I}\theta_i = \bigcup\big\{\theta_{i_0}\circ\ldots\circ\theta_{i_k} : i_0,\ldots,i_k \in I, k<\infty\big\},$$

where $\circ$ denotes the relational composition of two relations, which we assume the reader to be familiar with.

Given $\theta \in \mathrm{Eqr}A$ and $a \in A$, the *equivalence class of* $a$ *under* $\theta$ is defined by $a/\theta \coloneqq \{b \in A : \langle a, b\rangle \in \theta\}$. Also, given $F \subseteq A$, we write $F/\theta \coloneqq \{a/\theta : a \in F\}$.

Now, let $\boldsymbol{A}$ be an algebra. A relation $\theta \subseteq A \times A$ is a *congruence relation on* $\boldsymbol{A}$, if it is an equivalence relation on $A$ and moreover it is compatible with language operations on $\boldsymbol{A}$, that is, for every $n$-ary operation symbol $f \in \mathcal{L}$ and every $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

- if $\langle a_i, b_i\rangle \in \theta$, for every $i = 1, \ldots, n$, then $\langle f^{\boldsymbol{A}}(a_1, \ldots, a_n), f^{\boldsymbol{A}}(b_1, \ldots, b_n)\rangle \in \theta$.

We shall denote the set of all congruence relations on $\boldsymbol{A}$ by Con$\boldsymbol{A}$. The poset $\langle \mathrm{Con}\boldsymbol{A}, \subseteq\rangle$ is a complete sublattice of $\langle \mathrm{Eqr}A, \subseteq\rangle$. Its least element is the identity map on $A$, which will be denoted by $id_{\boldsymbol{A}}$, and its largest element is $A \times A$. An algebra $\boldsymbol{A}$ is *simple*, if $\mathrm{Con}\boldsymbol{A} = \{id_{\boldsymbol{A}}, A \times A\}$. The poset $\langle \mathrm{Con}\boldsymbol{A}, \subseteq\rangle$ is an algebraic lattice. Given $X \subseteq A \times A$, we shall denote the least congruence containing $X$ by $\Theta^{\boldsymbol{A}}(X)$, and refer to it as the *congruence generated by* $X$. That is,

$$\Theta^{\boldsymbol{A}}(X) \coloneqq \bigcap\big\{\vartheta \in \mathrm{Con}\boldsymbol{A} : X \subseteq \vartheta\big\}.$$

When $X = \{\langle a, b\rangle\}$, we write simply $\Theta^{\boldsymbol{A}}(a, b)$, and call it the *principal congruence generated by the pair* $\langle a, b\rangle$. Applying Lemma 0.9 to the closure operator $\Theta^{\boldsymbol{A}}$, we get:

**Lemma 0.15.** *For every* $\boldsymbol{A}$ *and every* $X \subseteq A \times A$,

$$\Theta^{\boldsymbol{A}}(X) = \bigvee\big\{\Theta^{\boldsymbol{A}}(a, b) : \langle a, b\rangle \in X\big\}.$$

Let $\boldsymbol{A}$ be an algebra and $F \subseteq A$. A congruence $\theta \in \mathrm{Con}\boldsymbol{A}$ is *compatible with* $F$, if for every $a, b \in A$, if $\langle a, b\rangle \in \theta$ and $a \in F$, then $b \in F$. That is, $\theta$ is compatible with $F$, if it does not identify elements in $F$ with elements outside $F$. The following characterizations of compatibility should be borne in mind, as we will make use of them thoroughly without any explicit mention.

**Lemma 0.16.** *Let* $\boldsymbol{A}$ *be an algebra,* $\theta \in \mathrm{Con}\boldsymbol{A}$ *and* $F \subseteq A$. *The following conditions are equivalent:*

*(i)* $\theta$ *is compatible with* $F$*;*
*(ii)* $a \in F \Leftrightarrow a/\theta \in F/\theta$, *for every* $a \in A$*;*
*(iii)* $F = \pi^{-1}\pi F$, *where* $\pi : \boldsymbol{A} \to \boldsymbol{A}/\theta$ *is the canonical map;*
*(iv)* $F = \bigcup_{a\in F} a/\theta$*; in other words,* $F$ *is a union of equivalence classes of* $\theta$.

A very special congruence associated to any given homomorphism is its *kernel*. The kernel of $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is the congruence defined by

$$\operatorname{Ker} h := \big\{ \langle a, b \rangle \in A \times A : h(a) = h(b) \big\}.$$

It will be useful to record here two of its elementary properties.

**Lemma 0.17.** *Let $h\colon \boldsymbol{A} \to \boldsymbol{B}$.*

1. *For every $F \subseteq A$, $\operatorname{Ker} h$ is compatible with $F$ if and only if $h^{-1}hF = F$.*
2. *For every $\theta \in \operatorname{Eqr} A$, $\operatorname{Ker} h \subseteq \theta$ if and only if $h^{-1}h\theta = \theta$.*

It is easy to check that congruences are preserved by inverse images of arbitrary homomorphisms. Preservation under direct images requires some conditions upon the homomorphisms.

**Proposition 0.18.** *Let $\boldsymbol{A}, \boldsymbol{B}$ be algebras and $h\colon \boldsymbol{A} \to \boldsymbol{B}$.*

1. *If $\theta \in \operatorname{Con} \boldsymbol{B}$, then $h^{-1}\theta \in \operatorname{Con} \boldsymbol{A}$.*
2. *If $\theta \in \operatorname{Con} \boldsymbol{A}$, $h$ is surjective and $\operatorname{Ker} h \subseteq \theta$, then $h\theta \in \operatorname{Con} \boldsymbol{B}$.*

**Relative congruences.** Let $\mathsf{K}$ be a class of algebras and $\boldsymbol{A}$ an algebra (not necessarily in $\mathsf{K}$). A congruence $\theta \in \operatorname{Con} \boldsymbol{A}$ is a $\mathsf{K}$-*congruence*, or a *congruence relative to* $\mathsf{K}$, if $\boldsymbol{A}/\theta \in \mathsf{K}$. We shall denote the set of all congruences of $\boldsymbol{A}$ relative to a class of algebras $\mathsf{K}$ by $\operatorname{Con}_{\mathsf{K}} \boldsymbol{A}$. Notice that if $\mathsf{K}$ is closed under $\mathbb{H}$ (for instance, if $\mathsf{K}$ is a variety) and $\boldsymbol{A} \in \mathsf{K}$, then $\operatorname{Con}_{\mathsf{K}} \boldsymbol{A} = \operatorname{Con} \boldsymbol{A}$.

In general, the poset $\langle \operatorname{Con}_{\mathsf{K}} \boldsymbol{A}, \subseteq \rangle$ need not be a complete lattice. We next state a sufficient condition to be so, as well as a (stronger) sufficient condition to be an algebraic lattice.

**Proposition 0.19.** *Let $\boldsymbol{A}$ be an algebra. If $\mathsf{K}$ is closed under isomorphisms and subdirect products, then $\operatorname{Con}_{\mathsf{K}} \boldsymbol{A}$ is closed under non-empty arbitrary intersections. If moreover $\mathsf{K}$ contains a trivial algebra, then $\operatorname{Con}_{\mathsf{K}} \boldsymbol{A}$ is a closure system; hence, $\langle \operatorname{Con}_{\mathsf{K}} \boldsymbol{A}, \subseteq \rangle$ is a complete lattice.*

Given an algebra $\boldsymbol{A}$ and a class of algebras $\mathsf{K}$ closed under $\mathbb{I}$ and $\mathbb{P}_{\mathrm{S}}$, the lattice $\langle \operatorname{Con}_{\mathsf{K}} \boldsymbol{A}, \subseteq \rangle$ is not necessarily a sublattice of $\langle \operatorname{Con} \boldsymbol{A}, \subseteq \rangle$, because joins might not coincide; but in light of Proposition 0.19 it is always a meet-complete subsemilattice. Under the stated assumption on $\mathsf{K}$, and given $X \subseteq A \times A$, we shall denote by $\Theta^{\boldsymbol{A}}_{\mathsf{K}}(X)$ the $\mathsf{K}$-*congruence generated by* $X$, i.e.,

$$\Theta^{\boldsymbol{A}}_{\mathsf{K}}(X) := \bigcap \big\{ \theta \in \operatorname{Con}_{\mathsf{K}} \boldsymbol{A} : X \subseteq \theta \big\}.$$

**Proposition 0.20.** *Let $\boldsymbol{A}$ be an algebra. If $\mathsf{K}$ is a quasivariety, then $\operatorname{Con}_{\mathsf{K}} \boldsymbol{A}$ is an inductive closure system; hence, $\langle \operatorname{Con}_{\mathsf{K}} \boldsymbol{A}, \subseteq \rangle$ is an algebraic lattice.*

Next, and just like we did in Proposition 0.18 for congruences in general, we state sufficient conditions for relative congruences to be preserved by direct and inverse images of homomorphisms.

**Proposition 0.21.** *Let $\mathsf{K}$ be a class of algebras closed under isomorphisms and subdirect products, $\boldsymbol{A}, \boldsymbol{B}$ algebras and $h\colon \boldsymbol{A} \to \boldsymbol{B}$.*

1. *If $\theta \in \operatorname{Con}_{\mathsf{K}} \boldsymbol{B}$, then $h^{-1}\theta \in \operatorname{Con}_{\mathsf{K}} \boldsymbol{A}$.*
2. *If $\theta \in \operatorname{Con}_{\mathsf{K}} \boldsymbol{A}$, $h$ is surjective and $\operatorname{Ker} h \subseteq \theta$, then $h\theta \in \operatorname{Con}_{\mathsf{K}} \boldsymbol{B}$.*

Another technical lemma which will be useful later on relates images of surjective homomorphisms with generated relative congruences.

**Lemma 0.22.** *Let* $\mathsf{K}$ *be a class of algebras closed under isomorphisms, subdirect products, and containing a trivial algebra, and let* $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$*. If* $h\colon \boldsymbol{A} \to \boldsymbol{B}$ *is a surjective homomorphism, then*

$$\Theta_{\mathsf{K}}^{\boldsymbol{B}}\big(h\Theta_{\mathsf{K}}^{\boldsymbol{A}}(\boldsymbol{\tau}^{\boldsymbol{A}}(X))\big) = \Theta_{\mathsf{K}}^{\boldsymbol{B}}\big(\boldsymbol{\tau}^{\boldsymbol{B}}(hX)\big),$$

*for every* $X \subseteq A$*.*

Finally, and similarly to Lemma 0.15 for congruences in general, by applying Lemma 0.9 to the closure operator $\Theta_{\mathsf{K}}^{\boldsymbol{A}}$, we get:

**Lemma 0.23.** *If* $\mathsf{K}$ *is closed under isomorphisms, subdirect products, and contains a trivial algebra, then*

$$\Theta_{\mathsf{K}}^{\boldsymbol{A}}(X) = \bigvee \big\{\Theta_{\mathsf{K}}^{\boldsymbol{A}}(a,b) : \langle a,b\rangle \in X\big\},$$

*for every* $\boldsymbol{A}$ *and every* $X \subseteq A \times A$*.*

## 0.5. Abstract Algebraic Logic

**Sentential logics.** Let $\mathcal{L}$ be an algebraic similarity type. Its elements will be called *connectives*. A closure relation $\vdash$ on the set of $\mathcal{L}$-formulas is *structural*, or *substitution invariant*, if it satisfies the additional condition:

- **Structurality:** If $\Gamma \vdash \varphi$, then $\sigma(\Gamma) \vdash \sigma(\varphi)$, for every substitution $\sigma$ and every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$.

A *consequence relation on* $\mathrm{Fm}_{\mathcal{L}}$ is a structural closure relation over $\mathrm{Fm}_{\mathcal{L}}$. A *(sentential) logic* is a pair $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}}\rangle$, where $\vdash_{\mathcal{S}}$ is a consequence relation on $\mathrm{Fm}_{\mathcal{L}}$. A logic $\mathcal{S}$ is *finitary*, if the consequence relation $\vdash_{\mathcal{S}}$ is finitary. A *Hilbert-style rule* is a pair $\langle \Gamma, \varphi\rangle$, where $\Gamma$ is a (possibly infinite) set of formulas and $\varphi$ is a formula. The *cardinal of a Hilbert-style rule* $\langle \Gamma, \varphi\rangle$ is given by the cardinal of $\Gamma$. Given an algebra $\boldsymbol{A}$ and a set of Hilbert-style rules $\mathcal{H}$, an $\mathcal{S}$-filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is *closed under* $\mathcal{H}$, if for every $\langle \Gamma, \varphi\rangle \in \mathcal{H}$ and every $h : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h(\Gamma) \subseteq F$, it holds $h(\varphi) \in F$. A logic $\mathcal{S}$ is of course determined by the set of all Hilbert-style rules $\langle \Gamma, \varphi\rangle$ such that $\langle \Gamma, \varphi\rangle \in \vdash_{\mathcal{S}}$.

Given a logic $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}}\rangle$, the $\vdash_{\mathcal{S}}$-closed sets of formulas are called $\mathcal{S}$-*theories*. The set of all $\mathcal{S}$-theories shall be denoted by $\mathcal{T}h\mathcal{S}$. Since $\mathcal{T}h\mathcal{S}$ is a closure system, it has a least element, which shall be denoted by $\mathrm{Thm}_{\mathcal{S}}$, and whose elements are called the *theorems of* $\mathcal{S}$, i.e., formulas $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ such that $\varnothing \vdash_{\mathcal{S}} \varphi$. The set of theorems may be empty. A logic $\mathcal{S}$ is *inconsistent*, if every formula is an $\mathcal{S}$-theorem, i.e., $\mathrm{Thm}_{\mathcal{S}} = \mathrm{Fm}_{\mathcal{L}}$. A logic $\mathcal{S}$ is inconsistent if and only if its only $\mathcal{S}$-theory is the set of all formulas, i.e., $\mathcal{T}h\mathcal{S} = \{\mathrm{Fm}_{\mathcal{L}}\}$. A logic $\mathcal{S}$ is *almost inconsistent*, if it does not have theorems and every formula is a consequence of every formula. A logic $\mathcal{S}$ is almost inconsistent if and only if its only $\mathcal{S}$-theories are the empty set and set of all formulas, i.e., $\mathcal{T}h\mathcal{S} = \{\varnothing, \mathrm{Fm}_{\mathcal{L}}\}$.

Let $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}}\rangle$ be a logic in a language $\mathcal{L}$. A logic $\mathcal{S}' = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}'}\rangle$ in the language $\mathcal{L}$ is an *extension of* $\mathcal{S}$, if $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{S}'}$; and it is an *axiomatic extension of* $\mathcal{S}$, if there exists a set of formulas $\mathrm{Ax} \subseteq \mathrm{Fm}_{\mathcal{L}}$ closed under substitutions such that, for every $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ and every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$, it holds

$$\Gamma \vdash_{\mathcal{S}'} \varphi \quad \text{iff} \quad \Gamma \cup \mathrm{Ax} \vdash_{\mathcal{S}} \varphi.$$

So, an extension $\mathcal{S}'$ of a logic $\mathcal{S}$ has the same underlying language as $\mathcal{S}$ and furthermore, if $\langle \Gamma, \varphi\rangle \in \vdash_{\mathcal{S}}$, then $\langle \Gamma, \varphi\rangle \in \vdash_{\mathcal{S}'}$. Notice that, in particular, for every

$\varphi \in \mathrm{Ax}$, it holds $\mathrm{Ax} \vdash_{\mathcal{S}} \varphi$ by extensivity of $\mathcal{S}$, and therefore $\varnothing \vdash_{\mathcal{S}'} \varphi$. Hence, $\mathrm{Ax} \subseteq \mathrm{Thm}_{\mathcal{S}'}$.

Now, let $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ be a logic in a language $\mathcal{L}$ and let $\mathcal{L}'$ be a language such that $\mathcal{L} \subseteq \mathcal{L}'$. A logic $\mathcal{S}' = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}'} \rangle$ in the language $\mathcal{L}'$ is an *expansion of* $\mathcal{S}$, and $\mathcal{S}$ is a *fragment of* $\mathcal{S}'$, if $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{S}'}$; and it is a *conservative expansion of* $\mathcal{S}$, if for every $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ and every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ in the language $\mathcal{L}$, it holds

$$\Gamma \vdash_{\mathcal{S}'} \varphi \quad \text{iff} \quad \Gamma \vdash_{\mathcal{S}} \varphi.$$

So, an expansion $\mathcal{S}'$ of a logic $\mathcal{S}$ has an underlying language containing that of $\mathcal{S}$ and furthermore every consequence in $\mathcal{S}$ holds in $\mathcal{S}'$ as well.

Let $\mathcal{S}$ be a logic. The *cardinal of* $\mathcal{S}$ is the least infinite cardinal $\kappa \leq |\mathrm{Fm}_{\mathcal{L}}|^+$ such that for every set of formulas $\Gamma$ and every formula $\varphi$, if $\Gamma \vdash_{\mathcal{S}} \varphi$, then there exist a set $\Delta \subseteq \Gamma$ with $|\Delta| < \kappa$ such that $\Delta \vdash_{\mathcal{S}} \varphi$. Hence finitary logics are the logics with cardinal $\omega$ and the non-finitary logics with a countable set of connectives have cardinal $\omega_1$.

**$\mathcal{S}$-filters.** Let $\mathcal{S}$ be a logic and $\boldsymbol{A}$ an algebra. An *$\mathcal{S}$-filter of* $\boldsymbol{A}$ is a subset $F \subseteq A$ such that, for every $h\colon \boldsymbol{Fm} \to \boldsymbol{A}$ and every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, if $\Gamma \vdash_{\mathcal{S}} \varphi$ and $h(\Gamma) \subseteq F$, then $h(\varphi) \in F$. The set of all $\mathcal{S}$-filters of $\boldsymbol{A}$ will be denoted by $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. In general, $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a closure system for every algebra $\boldsymbol{A}$. The closure operator associated with $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ will be denoted by $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}$. The set of all $\mathcal{S}$-filters of $\boldsymbol{A}$ containing a given $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ shall be denoted by $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$. Notice that if $\kappa$ is the cardinal of $\mathcal{S}$, then for every algebra $\boldsymbol{A}$ the union of any $\kappa$-directed family of $\mathcal{S}$-filters is still an $\mathcal{S}$-filter. Hence, given any cardinal $\lambda < \kappa$, $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is closed under unions $\lambda$-directed families, for every $\boldsymbol{A}$. An important fact which should always be borne in mind is the following: *The $\mathcal{S}$-filters of the formula algebra $\boldsymbol{Fm}$ are precisely the $\mathcal{S}$-theories.* In symbols, $\mathcal{F}i_{\mathcal{S}}\boldsymbol{Fm} = \mathcal{T}h\mathcal{S}$. Traditionally, the associated closure operator is denoted by $\mathrm{Cn}_{\mathcal{S}}$ instead of $\mathrm{Fg}^{\boldsymbol{Fm}}_{\mathcal{S}}$.

The interplay between $\mathcal{S}$-filters and homomorphisms will be a cornerstone of our work. The next crucial lemma states sufficient conditions for the property of being an $\mathcal{S}$-filter to be preserved under images and inverse images by (surjective) homomorphisms.

**Lemma 0.24.** *Let $\mathcal{S}$ be a logic, $\boldsymbol{A}, \boldsymbol{B}$ algebras, $h\colon \boldsymbol{A} \to \boldsymbol{B}$, and $G \subseteq B$.*

1. *If $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, then $h^{-1}G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*
2. *If $h$ is surjective and $h^{-1}G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, then $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$.*
3. *If $h$ is surjective and $\mathrm{Ker}\, h$ is compatible with $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, then $hF \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$.*

Another technical lemma which will be useful later on relates images of surjective homomorphisms with generated $\mathcal{S}$-filters. Do compare it with Lemma 0.22.

**Lemma 0.25** ([**10**, Lemma 1.1 (v)])**.** *If $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is a surjective homomorphism, then*

$$\mathrm{Fg}^{\boldsymbol{B}}_{\mathcal{S}}\big(h\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X)\big) = \mathrm{Fg}^{\boldsymbol{B}}_{\mathcal{S}}(hX),$$

*for every $X \subseteq A$.*

**Transformers and structural representations.** This topic is developed in the literature at a much more abstract level, but for our purposes, it is enough to introduce it only for the closure systems $\mathcal{T}h\mathcal{S}$ and $\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{Fm}$. For the general theory, see [**9**, **44**, **45**].

Let us start by fixing some notation. Given an algebra $\boldsymbol{A}$ and a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$, we denote by $\boldsymbol{\tau A}$ the set of all $a \in A$ satisfying every equation in $\boldsymbol{\tau}(x)$ when interpreted in $\boldsymbol{A}$, i.e.,

$$\boldsymbol{\tau A} \coloneqq \{a \in A : \boldsymbol{A} \vDash \boldsymbol{\tau}(x)[\![a]\!]\}.$$

Moreover, given $F \subseteq A$, we write

$$\boldsymbol{\tau}^{\boldsymbol{A}}(F) \coloneqq \{\langle \delta^{\boldsymbol{A}}(a), \epsilon^{\boldsymbol{A}}(a)\rangle : \delta \approx \epsilon \in \boldsymbol{\tau}(x), a \in F\} \; .$$

We write simply $\boldsymbol{\tau}^{\boldsymbol{A}}(a)$ instead of $\boldsymbol{\tau}^{\boldsymbol{A}}(\{a\})$ — which makes the present notation agree with the 1-ary case of that introduced on page 9.

A *transformer from formulas to equations* is a map $\boldsymbol{\tau} : \mathscr{P}(\mathrm{Fm}_{\mathcal{L}}) \to \mathscr{P}(\mathrm{Eq}_{\mathcal{L}})$ such that, for every $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\boldsymbol{\tau}^{\boldsymbol{Fm}}(\Gamma) = \bigcup_{\gamma \in \Gamma} \boldsymbol{\tau}^{\boldsymbol{Fm}}(\gamma).$$

A transformer $\boldsymbol{\tau}$ from formulas to equations is *structural* if it commutes with substitutions, i.e., if for every substitution $\sigma : \boldsymbol{Fm} \to \boldsymbol{Fm}$ and every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$, it holds $\boldsymbol{\tau}^{\boldsymbol{Fm}}\big(\sigma(\varphi)\big) = \sigma\big(\boldsymbol{\tau}^{\boldsymbol{Fm}}(\varphi)\big)$ — notice that the $\sigma$ on the right is the extension to the powerset of cartesian products of the $\sigma$ on the left, following the notation introduced on page 1. Structural transformers from formulas to equations are univocally determined by a set of equations in at most one variable. Indeed, a transformer $\boldsymbol{\tau} : \mathscr{P}(\mathrm{Fm}_{\mathcal{L}}) \to \mathscr{P}(\mathrm{Eq}_{\mathcal{L}})$ is structural if and only if there exists a set of equations $E(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ such that $\boldsymbol{\tau}^{\boldsymbol{Fm}}(\varphi) = E^{\boldsymbol{Fm}}(\varphi)$, for every $\varphi$. Since in particular $\boldsymbol{\tau}(x) = E^{\boldsymbol{Fm}}(x)$, it is safe, and notationally simpler, to identify a transformer $\boldsymbol{\tau}$ with the associated set $\tau(x)$ of equations determining it.

Let now $\mathcal{S}$ be a logic, $\mathsf{K}$ a class of algebras closed under isomorphisms and subdirect products, and $\boldsymbol{A}$ an arbitrary algebra. A map $\Psi : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is a *representation*, if it is injective and preserves arbitrary suprema. A representation $\Psi : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{A}$ is *structural*, if it commutes with endomorphisms, in the sense that $\Psi(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(h(F))) = \Theta^{\boldsymbol{A}}_{\mathsf{K}}(h(\Psi(F)))$, for every endomorphism $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{A})$. These notions, despite of being here introduced under the original names, are particular instances of more general concepts. See, for example, [45, Definition 17, Lemma 18, Definition 24]. For future reference, we record here an important result concerning structural representations, which in our setting follows as a particular case of [44, Theorem 5.1] and [44, Corollary 5.9]. See also the proof of [11, Theorem 3.7 (i)].

**Theorem 0.26.** *Let $\mathcal{S}$ be a logic and $\mathsf{K}$ a class of algebras closed under isomorphisms and subdirect products. If $\Phi : \mathcal{T}h\mathcal{S} \to \mathrm{Con}_{\mathsf{K}}\boldsymbol{Fm}$ is a structural representation, then there exists a structural transformer $\boldsymbol{\tau} : \mathscr{P}(\mathrm{Fm}_{\mathcal{L}}) \to \mathscr{P}(\mathrm{Eq}_{\mathcal{L}})$ such that $\Phi\big(\mathrm{Cn}_{\mathcal{S}}(\Gamma)\big) = \Theta^{\boldsymbol{Fm}}_{\mathsf{K}}\big(\boldsymbol{\tau}^{\boldsymbol{Fm}}(\Gamma)\big)$, for every $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$.*

For the sake of completeness, if $\mathsf{K}$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$ (see (1) below), then $\boldsymbol{\tau}$ induces a structural representation $\Phi : \mathcal{T}h\mathcal{S} \to \mathrm{Con}_{\mathbb{GQ}(\mathsf{K})}\boldsymbol{Fm}$, given by $\Phi(T) \coloneqq \Theta^{\boldsymbol{Fm}}_{\mathsf{K}}\big(\boldsymbol{\tau}^{\boldsymbol{Fm}}(T)\big)$, for every $T \in \mathcal{T}h\mathcal{S}$.

**Algebraic semantics.** Let $\mathcal{S}$ be a logic and $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$. A class of algebras $\mathsf{K}$ is a *$\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$*, if

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \Leftrightarrow \quad \boldsymbol{\tau}(\Gamma) \vDash^{\mathrm{eq}}_{\mathsf{K}} \boldsymbol{\tau}(\varphi), \tag{1}$$

for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. A logic defined by condition (1) is called the *$\boldsymbol{\tau}$-assertional logic of* $\mathsf{K}$, and it is sometimes denoted by $\mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$. So, a class $\mathsf{K}$ is a $\boldsymbol{\tau}$-algebraic semantics for a logic $\mathcal{S}$ if and only if $\mathcal{S} = \mathcal{S}(\mathsf{K}, \boldsymbol{\tau})$.

A $\boldsymbol{\tau}$-algebraic semantics for a logic $\mathcal{S}$ need not to be unique. The definition makes it clear that any other class of algebras $\mathsf{K}'$ such that $\vDash^{\mathrm{eq}}_{\mathsf{K}'} = \vDash^{\mathrm{eq}}_{\mathsf{K}}$ is also a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$. Also, a class of algebras $\mathsf{K}$ can be an algebraic semantics for a logic $\mathcal{S}$ witnessed by two different sets of equations. So, although intuitive, the notion of algebraic semantics leaves much to be desired concerning the "uniqueness" one would reasonably imagine an algebraic counterpart of a logic to enjoy. There is, nevertheless, a distinguished $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$, in case one such $\boldsymbol{\tau}$-algebraic semantics actually exists (there are logics without any algebraic semantics; see [**15**, Theorem 2.19]).

**Definition 0.27** ([**15**, Definition 2.7])**.** Let $\mathcal{S}$ be a logic and $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$. The class of *$\boldsymbol{\tau}$-models of $\mathcal{S}$* is defined by

$$\mathsf{K}(\mathcal{S}, \boldsymbol{\tau}) \coloneqq \{\boldsymbol{A} : \Gamma \vdash_{\mathcal{S}} \varphi \;\Rightarrow\; \boldsymbol{\tau}(\Gamma) \vDash^{\mathrm{eq}}_{\boldsymbol{A}} \boldsymbol{\tau}(\varphi)\}\,.$$

Notice that $\mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$ is non-empty, as all trivial algebras belong to it. Actually, $\mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$ is a generalized quasi-variety, axiomatized by the set of quasi-equations

$$\Big\{ \mathop{\&}_{\gamma\in\Gamma} \boldsymbol{\tau}(\gamma) \to \boldsymbol{\tau}(\varphi) : \Gamma \vdash_{\mathcal{S}} \varphi \;,\; \Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}} \Big\}.$$

Notice also that, if $\mathsf{K}$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$, then $\mathsf{K} \subseteq \mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$. But $\mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$ itself need not be, in general, a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$. However,

**Proposition 0.28** ([**15**, Proposition 2.8])**.** *Let $\mathcal{S}$ be a logic. If there exists a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$, then the class $\mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$ is the largest $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$.*

In Chapter 4 we shall have more to say about the largest $\boldsymbol{\tau}$-algebraic semantics of truth-equational logics (see Definition 0.38).

**The Leibniz, Suszko, Tarski, and Frege operators.** Let $\boldsymbol{A}$ be an algebra. The set of all congruences on $\boldsymbol{A}$ compatible with a given $F \subseteq A$[4] forms a complete sublattice of the lattice $\mathrm{Con}\boldsymbol{A}$. Its least element is, of course, the identity congruence on $\boldsymbol{A}$. Its largest element, known as the *Leibniz congruence of* $F$, plays a prominent rôle in Abstract Algebraic Logic, and is denoted by $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Observe that $\theta \in \mathrm{Con}\boldsymbol{A}$ is compatible with $F \subseteq A$ if and only if $\theta \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Another trivial, but useful observation, is that $\boldsymbol{\Omega}^{\boldsymbol{A}}(\varnothing) = \boldsymbol{\Omega}^{\boldsymbol{A}}(A) = A \times A$.

Two further congruences, both relevant to AAL, also arise from the notion of a congruence being compatible with a set. The first, given $F \subseteq A$, is called the *Suszko congruence of* $F$, and it is defined as the largest congruence of $\boldsymbol{A}$ compatible with every $G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$; it is easy to see that one can equivalently define it by

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \coloneqq \bigcap\big\{\boldsymbol{\Omega}^{\boldsymbol{A}}(G) : G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\,, F \subseteq G\big\}. \tag{2}$$

[4] We depart slightly from the usual practice of introducing the Leibniz and Suszko congruences for $\mathcal{S}$-filters, and consider here arbitrary subsets instead.

The second, given $\mathscr{C} \subseteq \mathscr{P}(A)$, is called the *Tarski congruence of* $\mathscr{C}$, and it is defined as the largest congruence compatible with every $G \in \mathscr{C}$, or, equivalently, by

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \coloneqq \bigcap \{ \boldsymbol{\Omega}^{\boldsymbol{A}}(F) : F \in \mathscr{C} \}. \tag{3}$$

From (2) and (3) it follows that the Suszko congruence can be defined in terms of the Tarski congruence by the identity

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big( (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \big). \tag{4}$$

Observe that the Leibniz and Tarski congruences are independent of any logic. Indeed, they depend only on $\boldsymbol{A}$, and $F \subseteq A$ or $\mathscr{C} \subseteq \mathscr{P}(A)$, respectively. In contrast, the Suszko congruence depends on the underlying logic $\mathcal{S}$, fact which is reflected in the notation.

The main characterization of the Leibniz congruence is the following (the proof can be found in [**21**, Theorem 3.2]):

**Proposition 0.29.** *Let $\boldsymbol{A}$ be an algebra and $F \subseteq A$. For every $a, b \in A$,*

$$\langle a, b \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \textit{iff} \quad \forall \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}} \ \forall \overline{c} \in \boldsymbol{A}$$
$$\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F \Leftrightarrow \varphi^{\boldsymbol{A}}(b, \overline{c}) \in F.$$

In fact, the Leibniz congruence gets its name from this characterization, since it can be viewed as the first-order analogue of Leibniz's second-order definition of identity:

*Two objects in the domain of discourse are equal if they share all the properties that can be expressed in the language of discourse.*

Having (2) in mind, a similar characterization for the Suszko congruence follows immediately as corollary, and we will make use of it in Chapter 7.

**Corollary 0.30.** *Let $\mathcal{S}$ be a logic, $\boldsymbol{A}$ an algebra and $F \subseteq A$. For every $a, b \in A$,*

$$\langle a, b \rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \quad \textit{iff} \quad \forall F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \ \forall \varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}} \ \forall \overline{c} \in \boldsymbol{A}$$
$$\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F' \Leftrightarrow \varphi^{\boldsymbol{A}}(b, \overline{c}) \in F'.$$

We can consider the map assigning to each subset $F \subseteq A$ its Leibniz congruence $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$; when restricting its domain to the set of $\mathcal{S}$-filters of $\boldsymbol{A}$, we refer to the map $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ as the *Leibniz operator on* $\boldsymbol{A}$. Similarly, the *Suszko operator on* $\boldsymbol{A}$ is the map $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}\boldsymbol{A}$ defined by $F \mapsto \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. Observe that, given any $X \subseteq A$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(X) = \bigcap \{ \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : X \subseteq F \} = \{ \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X) \subseteq F \} = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X))$. Finally, the *Tarski operator on* $\boldsymbol{A}$ is the map $\widetilde{\boldsymbol{\Omega}} \colon \mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) \to \mathrm{Con}\boldsymbol{A}$ defined by $\mathscr{C} \mapsto \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})$.

Given that these congruences and operators are defined on every algebra, it is natural to consider the family $\boldsymbol{\Omega} \coloneqq \{ \boldsymbol{\Omega}^{\boldsymbol{A}} : \boldsymbol{A} \text{ an algebra} \}$ and call it *the Leibniz operator*. Similarly, we call the family $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}} \coloneqq \{ \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} : \boldsymbol{A} \text{ an algebra} \}$ *the Suszko operator*. This terminology makes it easy to name properties that necessarily involve the whole family, in particular, those that relate the operators on different algebras, or those concerning a single algebra and holding in all of them (see for instance Definition 0.37).

We have already seen how congruences (in general) and $\mathcal{S}$-filters behave with respect to images and inverse images by (surjective) homomorphisms. Lastly, we consider the behaviour of each one of the three distinguished AAL congruences with respect to inverse images by surjective homomorphisms. We sum it up in next proposition, which will turn out to be a most crucial one to our study.

**Proposition 0.31.** *Let $\mathcal{S}$ be a logic, $\boldsymbol{A}, \boldsymbol{B}$ algebras, and $h\colon \boldsymbol{A} \to \boldsymbol{B}$ surjective. For every $\mathscr{C} \cup \{G\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$,*

1. $h^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(G) = \boldsymbol{\Omega}^{\boldsymbol{A}}(h^{-1}G)$;
2. $h^{-1}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(h^{-1}\mathscr{C})$;
3. $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(G)$.

The Suszko operator does not behave as well as the Leibniz and Tarski operators, at least with respect to inverse images by surjective homomorphisms. Indeed, $h^{-1}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(G)$ need not be equal to $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G)$. To support this statement we must wait until Proposition 3.15, but it can already be foreseen that, when working with the Suszko operator, the usual arguments used with the Leibniz operator will not go as smoothly as one could hope. The quest for a weaker commutativity property shared by both the Leibniz and Suszko operators is one of the main goals addressed in Part I of this work.

Since it will be later needed, we record here a very interesting result relating algebraic semantics with the Suszko operator.

**Proposition 0.32** ([55, Corollary 9])**.** *If a logic $\mathcal{S}$ has a $\boldsymbol{\tau}$-algebraic semantics, then for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

$$\boldsymbol{\tau}^{\boldsymbol{A}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F).$$

The *Frege relation of $F \subseteq A$ on $\boldsymbol{A}$* (again, relative to $\mathcal{S}$) is defined by

$$\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \coloneqq \left\{ \langle a, b\rangle \in A \times A : \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, a) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, b) \right\}.$$

Notice that, unlike the previous operators we have seen so far, the equivalence relation $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is not necessarily a congruence. In fact, an important property to keep in mind is that, for every algebra $\boldsymbol{A}$, the largest congruence below $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is the Suszko congruence $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. We call the map given by $F \mapsto \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, restricted to $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, the *Frege operator* on $\boldsymbol{A}$. Similarly to the Suszko operator, observe that given any $X \subseteq A$, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(X) = \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X)\big)$. Moreover, the Frege operator is always order preserving. Finally, the relation $\boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}(\varnothing)$ is called the *interderivability relation*, and traditionally $\langle \varphi, \psi\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}(\varnothing)$ is abbreviated by $\varphi \dashv\vdash_{\mathcal{S}} \psi$.

**Matrices, generalized matrices, (g-)models, and full g-models of a logic.** A (logical) *matrix* is a pair $\langle \boldsymbol{A}, F\rangle$, where $\boldsymbol{A}$ is an $\mathcal{L}$-algebra and $F \subseteq A$. Every matrix $\mathcal{M} = \langle \boldsymbol{A}, F\rangle$ induces a logic whose consequence relation $\vdash_{\mathcal{M}}$ is defined, for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$, by

$$\Gamma \vdash_{\mathcal{M}} \varphi \quad \Leftrightarrow \quad \text{for all } h\colon \boldsymbol{Fm} \to \boldsymbol{A}, \text{ if } h(\Gamma) \subseteq F, \text{ then } h(\varphi) \in F.$$

Similarly, every class $\mathsf{M}$ of matrices induces a logic whose consequence relation $\vdash_{\mathsf{M}}$ is defined by

$$\vdash_{\mathsf{M}} \coloneqq \bigcap_{\mathcal{M} \in \mathsf{M}} \vdash_{\mathcal{M}} . \tag{5}$$

Let $\mathcal{S}$ be a logic. A matrix $\mathcal{M}$ is a *model of* $\mathcal{S}$ if $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathcal{M}}$. It follows from the definition itself that $\langle \boldsymbol{A}, F\rangle$ is a model of $\mathcal{S}$ if and only if $F$ is an $\mathcal{S}$-filter of $\boldsymbol{A}$. The class of all models of a logic $\mathcal{S}$ is denoted by $\mathsf{Mod}(\mathcal{S})$.

Let $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ and $\mathsf{M}$ a class of matrices. We say that *truth is equationally definable in* $\mathsf{M}$ *by* $\boldsymbol{\tau}$, or that $\boldsymbol{\tau}$ *defines truth in* $\mathsf{M}$, if for every $\langle \boldsymbol{A}, F\rangle \in \mathsf{M}$, $F = \boldsymbol{\tau}\boldsymbol{A}$;

and that *truth is implicitly definable in* $\mathsf{M}$, if for every $\langle \boldsymbol{A}, F\rangle, \langle \boldsymbol{A}, G\rangle \in \mathsf{M}$, $F = G$. Clearly, if truth is equationally definable in $\mathsf{M}$, then it is also implicitly definable in $\mathsf{M}$.

Let $\mathcal{M} = \langle \boldsymbol{A}, F\rangle$ and $\mathcal{N} = \langle \boldsymbol{B}, G\rangle$ be matrices. A *matrix homomorphism* from $\mathcal{M}$ to $\mathcal{N}$ is an algebraic homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ such that $F \subseteq h^{-1}G$. Notice that this notion is still a particular case of the first-order definition of homomorphism between structures, considering the first order language $\mathcal{L}' = \langle \mathcal{F}', \mathcal{R}'\rangle$, where $\mathcal{F}' = \mathcal{F}$ and $\mathcal{R}' = \{r\}$, with $r$ a 1-ary relation symbol. A matrix homomorphism $h\colon \mathcal{M} \to \mathcal{N}$ is *strict*, if $h^{-1}G = F$; and it is *deductive*[5], if $h(a) = h(b)$ implies $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, a) = \mathrm{Fg}^{\boldsymbol{B}}_{\mathcal{S}}(F, b)$. A most crucial fact about strict and surjective matrix homomorphisms is the following:

**Proposition 0.33.** *If there is a strict surjective homomorphism between two matrices, then these matrices define the same logic.*

Just like the operators defined for classes of algebras on page 10, similar operators can be defined for classes of matrices (we skip the details here — see [**60**, Chapter 9]). A famous theorem by Czelakowski characterizes the class $\mathsf{Mod}(\mathcal{S})$, for a logic $\mathcal{S}$ with cardinal $\kappa$, as the least class of matrices closed under images and inverse images by strict surjective homomorphisms, submatrices, $\kappa$-reduced products of matrices, and containing a trivial matrix. We record this result here for future reference.

**Theorem 0.34** (Czelakowski)**.** *Let $\mathcal{S}$ be a logic with cardinal $\kappa$[6] in a countable language, and $\mathsf{M}$ a class of $\mathcal{L}$-matrices. The following conditions are equivalent:*

*(i)* $\mathsf{M} = \mathsf{Mod}(\mathcal{S})$*;*
*(ii)* $\mathsf{M}$ *is closed under the operators* $\mathbb{H}_{\mathrm{S}}^{-1}$, $\mathbb{H}_{\mathrm{S}}$, $\mathbb{S}$, $\mathbb{P}_{\kappa-\mathrm{R}}$ *and contains a trivial matrix;*
*(iii)* $\mathsf{M} = \mathbb{H}_{\mathrm{S}}^{-1}\mathbb{H}_{\mathrm{S}}\mathbb{S}\mathbb{P}_{\kappa-\mathrm{R}}(\mathsf{N})$*, for some class of matrices* $\mathsf{N}$ *containing a trivial matrix.*

The notion of a matrix can be seen as a particular case of a more general notion. A *generalized matrix*, or *g-matrix* for short, is a pair $\mathfrak{M} = \langle \boldsymbol{A}, \mathscr{C}\rangle$, where $\boldsymbol{A}$ is an algebra and $\mathscr{C} \subseteq \mathscr{P}(A)$ is a closure system. Every g-matrix $\mathfrak{M} = \langle \boldsymbol{A}, \mathscr{C}\rangle$ induces a consequence relation $\vdash_{\mathfrak{M}}$ as in (5) by taking the class of matrices $\big\{\langle \boldsymbol{A}, F\rangle : F \in \mathscr{C}\big\}$. A g-matrix $\mathfrak{M}$ is a *generalized model* (*g-model* for short) of a logic $\mathcal{S}$ if $\vdash_{\mathcal{S}} \subseteq \vdash_{\mathfrak{M}}$. One can easily check that $\langle \boldsymbol{A}, \mathscr{C}\rangle$ is a g-model of $\mathcal{S}$ if and only if $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Often, for simplicity, the term "g-model" is applied to $\mathscr{C}$ rather than to the pair $\langle \boldsymbol{A}, \mathscr{C}\rangle$. The class of all g-models of a logic $\mathcal{S}$ is denoted by $\mathsf{GMod}(\mathcal{S})$.

Among the g-models of a logic there are some of special importance to AAL. A family $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is *full* if $\mathscr{C} = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}$. This notion is obviously relative to the logic, but in general there will be no need to specify it. Notice that, given an arbitrary family $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, it always holds that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$ for every $G \in \mathscr{C}$. Thus, $\mathscr{C}$ is full when it is exactly the set of *all* the $\mathcal{S}$-filters on $\boldsymbol{A}$ which $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})$ is compatible with. The family $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is obviously full, for every $\boldsymbol{A}$. It is easy to see that every full family of $\mathcal{S}$-filters is a

[5]The notion of deductive matrix homomorphism was first introduced by Czelakowski in [**24**, p. 200].

[6]Notice that, since the cardinal of a logic is infinite (by definition) and the language of $\mathcal{S}$ is countable (by assumption), either $\kappa = \omega$ or $\kappa = \omega_1$.

closure system, because a congruence compatible with every element of a family of subsets is compatible with its intersection. Full families of $\mathcal{S}$-filters are also called *full g-models*. The class of all full g-models of a logic $\mathcal{S}$ is denoted by $\mathsf{FGMod}(\mathcal{S})$.

We have chosen as definition of full g-model one among its many equivalent formulations, by reasons that will become clear later on. Nevertheless, we shall make use of several characterizations, which we next sum up, and among which is the original definition [**36**, Definition 2.8].

**Proposition 0.35.** *Let* $\boldsymbol{A}$ *be an algebra,* $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$*, and let* $\pi \colon \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})$ *be the canonical projection. The following conditions are equivalent:*

*(i)* $\mathscr{C}$ *is full;*
*(ii)* $\pi\mathscr{C} = \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})\big)$*;*
*(iii)* $\mathscr{C} = \pi^{-1}\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})\big)$*;*
*(iv)* $\mathscr{C} = h^{-1}\mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$*, for some algebra* $\boldsymbol{B}$ *and some surjective* $h \colon \boldsymbol{A} \to \boldsymbol{B}$*.*

A matrix $\langle \boldsymbol{A}, F\rangle$ is *Leibniz-reduced*, or simply *reduced*, if $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}}$; and it is *Suszko-reduced* if $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}}$. A g-matrix $\langle \boldsymbol{A}, \mathscr{C}\rangle$ is *reduced* if $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = id_{\boldsymbol{A}}$. The classes of all reduced (g-)models according to these three criteria are denoted by, respectively,

$$\mathsf{Mod}^*(\mathcal{S}) \coloneqq \big\{\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\mathcal{S}) : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}}\big\},$$

$$\mathsf{Mod}^{\mathrm{Su}}(\mathcal{S}) \coloneqq \big\{\langle \boldsymbol{A}, F\rangle \in \mathsf{Mod}(\mathcal{S}) : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}}\big\},$$

$$\mathsf{GMod}^*(\mathcal{S}) \coloneqq \big\{\langle \boldsymbol{A}, \mathscr{C}\rangle \in \mathsf{GMod}(\mathcal{S}) : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = id_{\boldsymbol{A}}\big\}.$$

Two further classes of matrices will be of interest to us, namely:

$$\mathsf{LMod}^*(\mathcal{S}) \coloneqq \big\{\langle \boldsymbol{Fm}/\boldsymbol{\Omega}^{\boldsymbol{Fm}}(T), T/\boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)\rangle : T \in \mathcal{T}h\mathcal{S}\big\},$$

$$\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S}) \coloneqq \big\{\langle \boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T), T/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T)\rangle : T \in \mathcal{T}h\mathcal{S}\big\}.$$

The classical reference for the theory of matrices is [**60**]. Another major reference is [**23**, Chapter 0]; for the theory of generalized matrices, see [**36**, Chapter 1].

**The classes of algebras $\mathsf{Alg}^*(\mathcal{S})$ and $\mathsf{Alg}(\mathcal{S})$.** Two classes of algebras are considered as naturally, and intrinsically, associated with a logic in AAL. They are obtained by considering the algebraic reducts of the classes of reduced (g-)models seen above.

$$\mathsf{Alg}^*(\mathcal{S}) \coloneqq \big\{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}}\big\}, \tag{6}$$

$$\mathsf{Alg}^{\mathrm{Su}}\mathcal{S} \coloneqq \big\{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}}\big\}, \tag{7}$$

$$\mathsf{Alg}(\mathcal{S}) \coloneqq \big\{\boldsymbol{A} : \text{there is } \mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = id_{\boldsymbol{A}}\big\}. \tag{8}$$

Observe that, since the Tarski operator is order reversing and $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is always full, definition (8) is equivalent to:

$$\mathsf{Alg}(\mathcal{S}) = \big\{\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = id_{\boldsymbol{A}}\big\} \tag{9}$$

$$= \big\{\boldsymbol{A} : \text{there is } \mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ full such that } \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = id_{\boldsymbol{A}}\big\}. \tag{10}$$

The next lemma sums up the standard characterizations of these classes, as well as the known relations between them. An important idea to retain is that, for all three operators, the class of algebraic reducts of the *reduced* (g-)models coincides, up to isomorphism, with the class of algebraic reducts of the respective *reductions* of (g-)models of the logic.

**Lemma 0.36.** *Let $\mathcal{S}$ be a logic.*

1. $\mathsf{Alg}^*(\mathcal{S}) = \mathbb{I}\big\{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F) : \boldsymbol{A}$ *an algebra,* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\}$.
2. $\mathsf{Alg}^{\mathrm{Su}}\mathcal{S} = \mathbb{I}\big\{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) : \boldsymbol{A}$ *an algebra,* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\}$.
3. $\mathsf{Alg}(\mathcal{S}) = \mathbb{I}\big\{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) : \boldsymbol{A}$ *an algebra,* $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\}$
   $= \mathbb{I}\big\{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) : \boldsymbol{A}$ *an algebra,* $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *full*$\big\}$
   $= \mathbb{I}\big\{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) : \boldsymbol{A}$ *an algebra*$\big\}$.
4. $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^{\mathrm{Su}}\mathcal{S}$.
5. $\mathsf{Alg}(\mathcal{S}) = \mathbb{P}_{\mathrm{S}}(\mathsf{Alg}^*(\mathcal{S}))$.

Notice that, as a consequence of 5, it always holds

$$\mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{S}).$$

Lastly, we introduce the famous class of *Lindenbaum-Tarski algebras* and its Suszko analogous, obtained by considering the Leibniz- and Suszko-reductions of models over the formula algebra, respectively. That is,

$$\mathsf{LAlg}^*(\mathcal{S}) \coloneqq \{\boldsymbol{Fm}/\boldsymbol{\Omega}^{\boldsymbol{Fm}}(T) : T \in \mathcal{T}h\mathcal{S}\}, \tag{11}$$

$$\mathsf{LAlg}^{\mathrm{Su}}(\mathcal{S}) \coloneqq \{\boldsymbol{Fm}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T) : T \in \mathcal{T}h\mathcal{S}\}. \tag{12}$$

**The Leibniz hierarchy.** The main classification of sentential logics in AAL is the so called *Leibniz hierarchy*, displayed in Figure 1. It places a given logic $\mathcal{S}$ inside a class of logics, according to the algebraic properties enjoyed by the Leibniz operator over the $\mathcal{S}$-filters on arbitrary algebras. In this section we present those classes of logics within the Leibniz hierarchy which we will take more interest in.

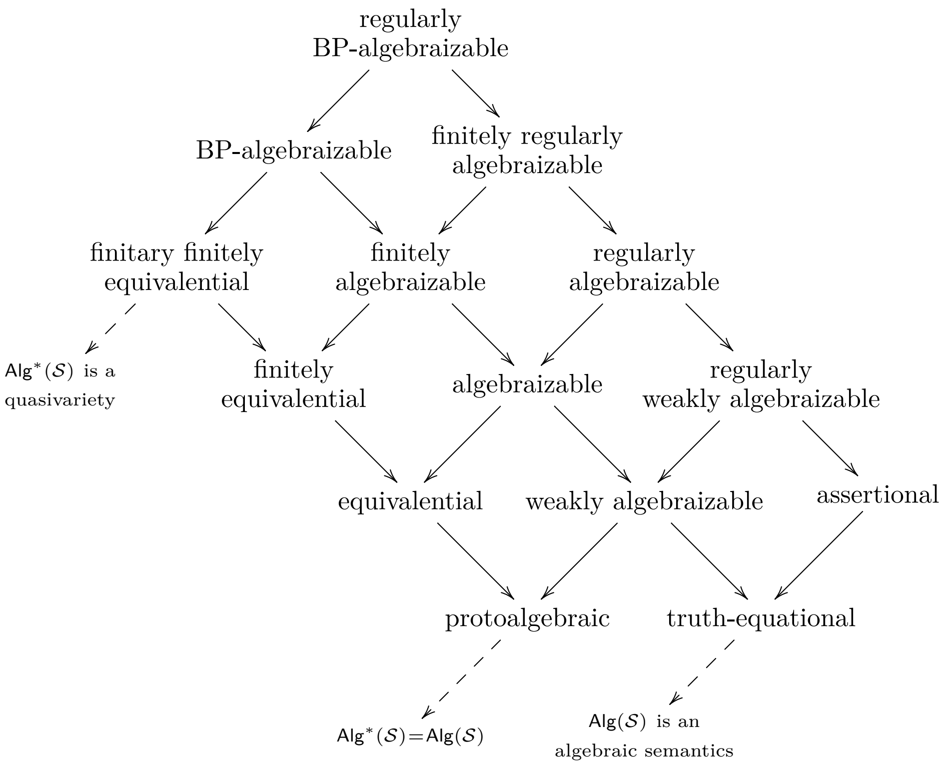



FIGURE 1. The Leibniz hierarchy and some related properties.

Among the many equivalent characterizations of these classes, we have chosen as definitions the ones that fit more naturally within the general framework we intend to settle in Part I of the present work. As we shall make use of several properties concerning the Leibniz operator, we start by introducing them.

**Definition 0.37.** Let $\mathcal{S}$ be a logic. The Leibniz operator:

- is *order preserving*, if for every $\boldsymbol{A}$ and every $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$, it holds $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$;
- is *order reflecting*, if for every $\boldsymbol{A}$ and every $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$, it holds $F \subseteq G$;
- is *completely order reflecting*, if for every $\boldsymbol{A}$ and every $\{F_i : i \in I\} \cup \{G\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\bigcap_{i \in I} \boldsymbol{\Omega}^{\boldsymbol{A}}(F_i) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$, it holds $\bigcap_{i \in I} F_i \subseteq G$.
- *commutes with inverse images by homomorphisms*, if for every $\boldsymbol{A}, \boldsymbol{B}$ and every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, it holds $h^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(G) = \boldsymbol{\Omega}^{\boldsymbol{A}}(h^{-1}G)$;
- is *continuous*, if it commutes with unions of directed families whose union is an $\mathcal{S}$-filter, i.e., if for every $\boldsymbol{A}$ and every directed family $\{F_i : i \in I\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\bigcup_{i \in I} F_i \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, it holds $\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\bigcup_{i \in I} F_i\big) = \bigcup_{i \in I} \boldsymbol{\Omega}^{\boldsymbol{A}}(F_i)$.

**Definition 0.38.** A logic $\mathcal{S}$ is:

- *protoalgebraic*, if the Leibniz operator is order preserving;
- *equivalential*, if it is protoalgebraic and the Leibniz operator commutes with inverse images by homomorphisms;
- *finitely equivalential*, if it is protoalgebraic and the Leibniz operator is continuous;
- *truth-equational*, if the Leibniz operator is completely order reflecting;
- *weakly algebraizable*, if it is protoalgebraic and truth-equational;
- *algebraizable*, if it is equivalential and truth-equational;
- *finitely algebraizable*, if it is finitely equivalential and truth-equational.

We see that some classes of logics were left out of Definition 0.38. We proceed to introduce them. To this end, recall the notion of $\boldsymbol{\tau}$-assertional logic of a class of algebras $\mathsf{K}$, given on page 17. Also, a class of algebras is *pointed* when there is an $\mathcal{L}$-term $c$ that is constant in the class (that is, for each algebra in the class, all the interpretations of the $\mathcal{L}$-term $c$ coincide).

**Definition 0.39.** A logic $\mathcal{S}$ is *assertional*, if it is the $\{x \approx c\}$-assertional logic of some pointed class of algebras $\mathsf{K}$, where $c$ is a constant term in $\mathsf{K}$.

Assertional logics are also called "pointed assertional" (for example, in [**55**, Definition 8]) or "$c$-assertional" (for example, in [**14**, Definition 3.1.1]) in the literature. In [**4**] this class of logics is claimed to legitimately belong to the Leibniz hierarchy. The classes of logics in Figure 1 with the word *regularly* on its name are precisely the intersection of the class of assertional logics with the respective class of logics featuring (the rest of) its name. Actually, following this line of thought, the class of assertional logics could be legitimately called "regularly truth-equational logics" (in fact, the original motivation behind the word "regularly" was to distinguish those algebraizable logics such that $\mathsf{Alg}^*(\mathcal{S})$ is relatively point-regular, and this property also holds for assertional logics, as shown in [**4**, Corollary 9]). Finally, by imposing $\mathcal{S}$ finitary to the (regularly) finitely algebraizable we add the prefix BP, which stands for "Blok and Pigozzi", who first introduced and studied the notion of algebraizability in their famous monograph [11]; the logics they called "algebraizable" correspond to the class of BP-algebraizable logics in the present terminology.

Algebraizable logics are traditionally presented (and apart from terminology and finitariness issues, were also originally defined) via two structural transformers, one from formulas to equations and another from equations to formulas. In fact, we shall make use of this equivalent characterization in several examples of Chapter 7. A logic $\mathcal{S}$ is algebraizable if and only if there exists a class of algebra $\mathsf{K}$, a set of equations in at most one variable $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ and a set of formulas in at most two variables $\boldsymbol{\rho}(x, y) \subseteq \mathrm{Fm}_{\mathcal{L}}$ such that for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ and every $\Pi \cup \{\delta \approx \epsilon\} \subseteq \mathrm{Eq}_{\mathcal{L}}$,

$$\Gamma \vdash_{\mathcal{S}} \varphi \Leftrightarrow \boldsymbol{\tau}^{\boldsymbol{Fm}}(\Gamma) \vDash^{\mathrm{eq}}_{\mathsf{K}} \boldsymbol{\tau}^{\boldsymbol{Fm}}(\varphi), \tag{ALG1}$$

$$\Pi \vDash^{\mathrm{eq}}_{\mathsf{K}} \delta \approx \epsilon \Leftrightarrow \boldsymbol{\rho}^{\boldsymbol{Fm}}(\Pi) \vdash_{\mathcal{S}} \boldsymbol{\rho}^{\boldsymbol{Fm}}(\delta, \epsilon), \tag{ALG2}$$

$$x \approx y \mathrel{=}\!\!\!\mid\!\vDash^{\mathrm{eq}}_{\mathsf{K}} \boldsymbol{\tau}^{\boldsymbol{Fm}}\big(\boldsymbol{\rho}^{\boldsymbol{Fm}}(x, y)\big), \tag{ALG3}$$

$$x \dashv\vdash_{\mathcal{S}} \boldsymbol{\rho}^{\boldsymbol{Fm}}\big(\boldsymbol{\tau}^{\boldsymbol{Fm}}(x)\big). \tag{ALG4}$$

These four conditions hide some redundancy. Indeed, (ALG1) + (ALG3) $\Leftrightarrow$ (ALG2) + (ALG4). The set $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ is called a set of defining equations for $\mathcal{S}$ (which recall, is precisely the terminology introduced for truth-equational logics; this is no coincidence of course, every $\boldsymbol{\tau}$ witnessing the algebraizability of $\mathcal{S}$ witnesses the truth-equationality of $\mathcal{S}$ as well); the set $\boldsymbol{\rho}(x, y) \subseteq \mathrm{Fm}_{\mathcal{L}}$ is called a *set of equivalence formulas for* $\mathcal{S}$; and the class $\mathsf{K}$ is called an *equivalent algebraic semantics for* $\mathcal{S}$. In case $\mathsf{K}$ is a generalized quasi-variety, one speaks of *the* equivalent algebraic semantics for $\mathcal{S}$. Unlike algebraic semantics, equivalent algebraic semantics are unique modulo the respective equational consequence relation; that is, if $\mathsf{K}, \mathsf{K}'$ are two equivalent algebraic semantics for a logic $\mathcal{S}$, then $\vDash^{\mathrm{eq}}_{\mathsf{K}} = \vDash^{\mathrm{eq}}_{\mathsf{K}'}$.

The latest addition to the Leibniz hierarchy is due to Raftery, in the paper [55], where he characterizes the class of truth-equational logics through the completely order reflecting property of the Leibniz operator, which is precisely the algebraic property which we here take as formal definition. For this reason it is, among all classes of logics in Definition 0.38, the least studied in the literature, and the one we will take more interest in. This being said, we record here a few results about truth-equational logics, all established by Raftery in the cited paper. The first one may help to clarify the naming of this class of logics.

**Theorem 0.40.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) $\mathcal{S}$ is truth-equational;*
*(ii) Truth is equationally definable in $\mathsf{LMod}^*(\mathcal{S})$;*
*(iii) Truth is equationally definable in $\mathsf{Mod}^*(\mathcal{S})$;*
*(iv) Truth is equationally definable in $\mathsf{Mod}^{\mathrm{Su}}(\mathcal{S})$.*

A set of equations witnessing the truth-equationality of a logic $\mathcal{S}$ is called a *set of defining equations for* $\mathcal{S}$. The proof of [55, Theorem 27] exhibits a set of defining equations for any given truth-equational logic.

**Proposition 0.41.** *If $\mathcal{S}$ is truth-equational, then $\boldsymbol{\tau}_\infty(x) \coloneqq \sigma_x \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(\{x\})$ is a set of defining equations for $\mathcal{S}$, where $\sigma_x : \boldsymbol{Fm} \to \boldsymbol{Fm}$ is the substitution sending all variables to $x$.*

Two important facts to bear in mind relating assertional logics and truth-equational logics are the following:

**Proposition 0.42.** *Let $\mathcal{S}$ be a logic.*

1. *Every assertional logic $\mathcal{S}\big(\mathsf{K}, \{x \approx \top\}\big)$, where $\top$ is a constant term of* K*, is truth-equational with set of defining equations $\boldsymbol{\tau}(x) = \{x \approx \top\}$.*
2. *Every truth-equational logic with set of defining equations $\boldsymbol{\tau}(x) = \{x \approx \top\}$, where $\top$ is a constant term of $\mathsf{Alg}(\mathcal{S})$, is the assertional logic $\mathcal{S}\big(\mathsf{Alg}(\mathcal{S}), \boldsymbol{\tau}\big)$.*

A key result to *both* parts of our study is the following (we state here a slight enhancement of the cited result, which follows easily from it):

**Proposition 0.43** ([**55**, Proposition 22])**.** *Let $\mathcal{S}$ be a logic and $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$. The following conditions are equivalent:*

*(i) $\mathcal{S}$ is truth-equational with the set of defining equations $\boldsymbol{\tau}(x)$;*

*(ii) For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

$$F = \big\{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big\}. \tag{13}$$

*(iii) For every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

$$F = \big\{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big\}.$$

Finally, we justify the dashed arrows in Figure 1. These conditions are relevant to AAL (and we shall consider them along the exposition), but since they are not characterized in terms of algebraic properties of the Leibniz operator — or at least no such characterization is known — they lie outside the Leibniz hierarchy. Their place in Figure 1 is fairly known: if $\mathcal{S}$ is protoalgebraic, then $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$ [**36**, Proposition 3.2]; and if $\mathcal{S}$ is finitary and finitely equivalential, then $\mathsf{Alg}^*(\mathcal{S})$ is a quasivariety [**47**, p. 426]. Despite not belonging to the Leibniz hierarchy, these conditions are still consistent with the diagram interpretation of seeing converging arrows as the intersection of the involved classes of logics. So, for instance, $\mathcal{S}$ is truth-equational and $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$ if and only if $\mathcal{S}$ is weakly algebraizable (Proposition 3.6); or, $\mathcal{S}$ is assertional and $\mathsf{Alg}^*(\mathcal{S})$ is a quasivariety if and only if $\mathcal{S}$ is regularly BP-algebraizable (Corollary 4.12). Similarly, having an algebraic semantics is not *per se* a condition placing some given logic within the Leibniz hierarchy. However, Raftery proved that:

**Proposition 0.44** ([**55**, Corollary 21])**.** *If $\mathcal{S}$ is truth-equational with defining equations $\boldsymbol{\tau}(x)$, then $\mathsf{Alg}(\mathcal{S})$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$.*

For an exhaustive study of the Leibniz hierarchy and the main results in AAL, see [**13, 23, 34, 36, 39**].

**The Frege hierarchy.** Parallel to the Leibniz hierarchy, there is another important hierarchy in AAL, this time built upon algebraic properties of the Frege operator. Once again, we choose among the known characterizations of the following classes, the one which suits better within the general framework we intend to settle.

**Definition 0.45.** A logic $\mathcal{S}$ is:

- *selfextensional*, if $\boldsymbol{\varLambda}_{\mathcal{S}}^{\boldsymbol{Fm}}(\varnothing) \in \mathrm{Con}\boldsymbol{Fm}$.
- *Fregean*, if for every $T \in \mathcal{T}h\mathcal{S}$, $\boldsymbol{\varLambda}_{\mathcal{S}}^{\boldsymbol{Fm}}(T) \in \mathrm{Con}\boldsymbol{Fm}$.

- *fully selfextensional*, if for every algebra $\boldsymbol{A}$, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\varnothing) \in \mathrm{Con}\boldsymbol{A}$.
- *fully Fregean*, if for every algebra $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathrm{Con}\boldsymbol{A}$.

Notice that, since for arbitrary $\boldsymbol{A}$ and arbitrary $F \subseteq A$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is the largest congruence below $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, one could have equivalently defined:

- $\mathcal{S}$ is *selfextensional*, if $\boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}(\varnothing) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}(\mathcal{T}h\mathcal{S})$.
- $\mathcal{S}$ is *Fregean*, if for every $T \in \mathcal{T}h\mathcal{S}$, $\boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}(T) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T)$.
- $\mathcal{S}$ is *fully selfextensional*, if for every algebra $\boldsymbol{A}$, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\varnothing) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$.
- $\mathcal{S}$ is *fully Fregean*, if for every algebra $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$.

The inclusions between these four classes of logics are straightforward, having in mind that $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\varnothing) = \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(\varnothing)\big) = \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}\big(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big)$. That these inclusions are all strict is far from trivial, but that is indeed the case. Figure 2 displays the so called *Frege hierarchy*. This hierarchy is far less studied than the Leibniz hierarchy.

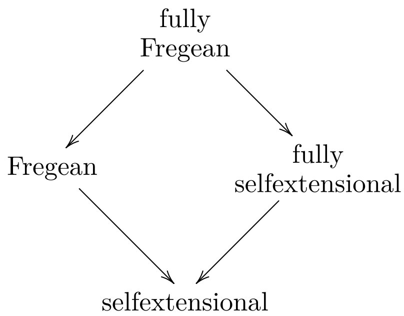



Figure 2. The Frege hierarchy.

As a final remark, and to avoid misunderstandings, the diagram interpretation by which the target of two converging arrows is seen as the intersection of the source classes of logics is still an open problem for the Frege hierarchy, at least in the general case; for logics with theorems however, fully Fregean logics are indeed those logics that are both Fregean and fully selfextensional [4, Theorem 26].

**Semilattice-based logics.** A class of algebras $\mathsf{K}$ (of the same similarity type $\mathcal{L}$) has *semilattice reducts*, if there exists a binary term $\wedge$ (which can be either a primitive connective, i.e., $\wedge \in \mathcal{L}$, or defined by an $\mathcal{L}$-term in two variables) such that for every $\boldsymbol{A} \in \mathsf{K}$, $\langle \boldsymbol{A}, \wedge^{\boldsymbol{A}} \rangle$ is a semilattice. For every $\boldsymbol{A} \in \mathsf{K}$, let $\leq^{\boldsymbol{A}}$ denote the partial order induced by $\wedge^{\boldsymbol{A}}$, that is, $a \leq^{\boldsymbol{A}} b \Leftrightarrow a \wedge^{\boldsymbol{A}} b = a$, for every $a, b \in A$. The *logic preserving degrees of truth* w.r.t. $\mathsf{K}$ is the logic induced by the class of matrices $\big\{\langle \boldsymbol{A}, [a) \rangle : \boldsymbol{A} \in \mathsf{K}, a \in A\big\}$, that is, the pair $\mathcal{S}^{\leq}_{\mathsf{K}} = \langle \mathrm{Fm}_{\mathcal{L}}, \vDash^{\leq}_{\mathsf{K}} \rangle$, where $\vDash^{\leq}_{\mathsf{K}}$ is defined by

$$\begin{aligned} \Gamma \vDash^{\leq}_{\mathsf{K}} \varphi \quad \Leftrightarrow \quad & \forall \boldsymbol{A} \in \mathsf{K}\ \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\ \forall a \in A \\ & \text{if } \forall \gamma \in \Gamma\ a \leq^{\boldsymbol{A}} h(\gamma), \text{ then } a \leq^{\boldsymbol{A}} h(\varphi), \end{aligned}$$

for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. In case $\mathsf{K}$ is a quasivariety, the logic $\mathcal{S}^{\leq}_{\mathsf{K}}$ is finitary because the class of matrices defining it is first-order definable[7] and hence closed under ultraproducts. In the case of a finite set of premisses, the relation $\vDash^{\leq}_{\mathsf{K}}$ can be re-written as follows:

$$\varnothing \vDash^{\leq}_{\mathsf{K}} \varphi \quad \Leftrightarrow \quad \forall \boldsymbol{A} \in \mathsf{K}\ \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\ \ a \leq^{\boldsymbol{A}} h(\varphi),$$

and

$$\{\gamma_1, \dots, \gamma_n\} \vDash^{\leq}_{\mathsf{K}} \varphi \quad \Leftrightarrow \quad \forall \boldsymbol{A} \in \mathsf{K}\ \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\ \ h(\gamma_1) \wedge^{\boldsymbol{A}} \dots \wedge^{\boldsymbol{A}} h(\gamma_n) \leq^{\boldsymbol{A}} h(\varphi).$$

Finitary logics preserving degrees of truth are also called *semilattice-based logics*. Notice that a semilattice-based logic $\mathcal{S}^{\leq}_{\mathsf{K}}$ has theorems if and only if the semilattice reducts in $\mathsf{K}$ have a term-definable maximum. In this case, notice that all theorems can be identified, since they are all interpreted as the maximum element on the algebras in $\mathsf{K}$. Observe also that $\varphi \dashv\vDash^{\leq}_{\mathsf{K}} \psi$ if and only if $\mathsf{K} \vDash \varphi \approx \psi$ if and only if $\mathbb{V}(\mathsf{K}) \vDash \varphi \approx \psi$. As a consequence, the interderivability relation is necessarily a congruence. In other words, every semilattice-based logic is selfextensional. In fact, semilattice-based logics are precisely the selfextensional logics with a conjunction (given a logic $\mathcal{S}$, a binary $\mathcal{L}$-term $\wedge$ is a *conjunction* for $\mathcal{S}$, if $x, y \vdash_{\mathcal{S}} x \wedge y$ and $x \wedge y \vdash_{\mathcal{S}} x, y$):

**Theorem 0.46** ([52, Theorem 3.2]). *A finitary logic $\mathcal{S}$ has a conjunction and is selfextensional if and only if it is a semilattice-based logic.*

Semilattice-based logics with theorems enjoy a very neat characterization of their logical filters on $\mathcal{S}$-algebras. Namely, if $\mathcal{S}^{\leq}_{\mathsf{K}}$ is a semilattice-based logic with theorems, then for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}})$, $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \mathrm{Filt}\boldsymbol{A}$. Since (principal) lattice filters separate points, an important consequence is that:

**Theorem 0.47** ([52, Theorem 3.13]). *Every semilattice-based logic is fully selfextensional.*

Furthermore, for every semilattice-based logic $\mathcal{S}^{\leq}_{\mathsf{K}}$, the class of $\mathcal{S}^{\leq}_{\mathsf{K}}$-algebras is a variety, and is given by $\mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}}) = \mathbb{V}(\mathsf{K})$.

Assume now that each $\boldsymbol{A} \in \mathsf{K}$ is upper-bounded, with its maximum element $1^{\boldsymbol{A}} \in A$ being term-definable by the same $\mathcal{L}$-term. The *logic preserving truth* w.r.t. $\mathsf{K}$ is the $\{x \approx 1\}$-assertional logic of $\mathsf{K}$, that is, the pair $\mathcal{S}^{1}_{\mathsf{K}} = \langle \mathrm{Fm}_{\mathcal{L}}, \vDash^{1}_{\mathsf{K}} \rangle$, where $\vDash^{1}_{\mathsf{K}}$ is defined by

$$\Gamma \vDash^{1}_{\mathsf{K}} \varphi \quad \Leftrightarrow \quad \forall \boldsymbol{A} \in \mathsf{K}\ \forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\ \ \text{if } \forall \gamma \in \Gamma\ h(\gamma) = 1^{\boldsymbol{A}}, \text{ then } h(\varphi) = 1^{\boldsymbol{A}},$$

for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$. It is clear by the definitions involved that $\vDash^{1}_{\mathsf{K}}$ is an extension of $\vDash^{\leq}_{\mathsf{K}}$.

The intuition behind the name "preserving degrees of truth" and "preserving truth" is fairly clear: in $\vDash^{\leq}_{\mathsf{K}}$, each $a \in A$ is to be understood as an attainable degree of truth in $\boldsymbol{A} \in \mathsf{K}$; while in $\vDash^{1}_{\mathsf{K}}$ the truth is to be understood as represented by

[7] A class of $\mathcal{L}$-structures is *elementary*, or *first-order definable*, if it is the class of models of some set of first-order sentences of $\mathcal{L}$. Every elementary class of structures is closed under ultraproducts.

the element $1^{\boldsymbol{A}} \in A$, for each $\boldsymbol{A} \in \mathsf{K}$. In the literature, particular classes of $\mathsf{K}$ are taken and the resulting logic $\vDash_{\mathsf{K}}^{\leq}$ is studied in greater detail. For instance, logics preserving degrees of truth w.r.t. varieties of residuated lattices are covered in [**17**]. The particular case of Lukasiewicz's infinite valued logic preserving degrees of truth, hereby denoted by $Ł_{\infty}^{\leq}$, is treated in [**35**]. For a more philosophical discussion on the whole subject of preserving degrees of truth, see [**33**].

# Part I

# $\mathcal{S}$-operators in Abstract Algebraic Logic

"An algebraic instrument which would make all logics amenable to its methods is available — it is the Suszko operator. For protoalgebraic logics, the Suszko and the Leibniz operator coincide."

[23, p. 9]

CHAPTER 1

# $\mathcal{S}$-operators

## 1.1. $\mathcal{S}$-operators

We wish to settle a general framework upon which a common study of the Leibniz, Suszko and Frege operators can be built. The ground definition is the following:

**Definition 1.1.** An $\mathcal{S}$*-operator on* $\boldsymbol{A}$ is a map $\nabla^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \operatorname{Eqr} A$.

Clearly, the Leibniz, Suszko, and Frege operators, are all $\mathcal{S}$-operators — see page 17. We shall also be interested in the Tarski operator, although it is left out of the scope of Definition 1.1. In order to cope with it, we consider three further maps associated to each $\mathcal{S}$-operator.

**Definition 1.2.** Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$.

(a) The *lifting* of $\nabla^{\boldsymbol{A}}$ to the power set is the map $\widetilde{\nabla}^{\boldsymbol{A}} : \mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) \to \operatorname{Eqr} A$, defined by
$$\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) \coloneqq \bigcap\{\nabla^{\boldsymbol{A}}(F) : F \in \mathscr{C}\},$$
for every $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

(b) The *relativization* of $\nabla^{\boldsymbol{A}}$ (to the logic $\mathcal{S}$) is the map $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \operatorname{Eqr} A$, defined by
$$\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) \coloneqq \bigcap\{\nabla^{\boldsymbol{A}}(F') : F' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\,, F \subseteq F'\} = \widetilde{\nabla}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}\big),$$
for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

(c) The map $\nabla^{\boldsymbol{A}^{-1}} : \operatorname{Eqr} A \to \mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$ is defined by
$$\nabla^{\boldsymbol{A}^{-1}}(\theta) \coloneqq \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \theta \subseteq \nabla^{\boldsymbol{A}}(G)\},$$
for every $\theta \in \operatorname{Eqr} A$.

Notice that the relativization of an $\mathcal{S}$-operator is still an $\mathcal{S}$-operator, since $\langle \operatorname{Eqr} A, \subseteq\rangle$ is a complete lattice. In particular, the relativization of the Leibniz operator is the Suszko operator — see (2) on page 18. Furthermore, the lifting of the Leibniz operator is the Tarski operator. Notice also that $\nabla^{\boldsymbol{A}^{-1}}(\theta) = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \nabla^{\boldsymbol{A}}(G) \in [\theta, A \times A]\big\}$, which somehow justifies the notation chosen, though $\nabla^{\boldsymbol{A}^{-1}}$ is not, of course, the set-theoretical inverse of $\nabla^{\boldsymbol{A}}$. Finally, we may sometimes write $\bigcap_{F' \supseteq F} \nabla^{\boldsymbol{A}}(F)$ instead of $\bigcap\{\nabla^{\boldsymbol{A}}(F') : F' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\,, F \subseteq F'\}$, which is obviously an abuse of notation.

The following elementary relations between an $\mathcal{S}$-operator and its relativization and lifting are immediate consequences of the definitions involved.

**Lemma 1.3.** *Let* $\nabla^{\boldsymbol{A}}$ *be an* $\mathcal{S}$*-operator on* $\boldsymbol{A}$*.*

1. $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \nabla^{\boldsymbol{A}}(F)$ *for every* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$*;*
2. $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \nabla^{\boldsymbol{A}}(F)$ *for every* $F \in \mathscr{C}$*.*

Quite often we shall be interested in the behaviour of the "same" $\mathcal{S}$-operator taken on different algebras. For instance, Proposition 0.31.1 makes use of both $\boldsymbol{\Omega}^{\boldsymbol{A}}$ and $\boldsymbol{\Omega}^{\boldsymbol{B}}$. By *a family of $\mathcal{S}$-operators* we understand a (proper) class $\{\nabla^{\boldsymbol{A}} : \boldsymbol{A}$ an arbitrary algebra$\}$ such that for each $\boldsymbol{A}$, $\nabla^{\boldsymbol{A}}$ is an $\mathcal{S}$-operator on $\boldsymbol{A}$. The whole family will be denoted simply by $\nabla$, following the tradition on the Leibniz and Suszko operators — $\boldsymbol{\Omega}$ and $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$.

We shall also consider several properties that may be enjoyed by the Leibniz, Suszko, and Frege operators. Among these is the monotonicity of each such $\mathcal{S}$-operator. In general, an $\mathcal{S}$-operator $\nabla^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Eqr}A$ is

(i) *order preserving*, if for every $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$, it holds $\nabla^{\boldsymbol{A}}(F) \subseteq \nabla^{\boldsymbol{A}}(G)$;

(ii) *order reflecting*, if for every $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\nabla^{\boldsymbol{A}}(F) \subseteq \nabla^{\boldsymbol{A}}(G)$, it holds $F \subseteq G$;

(iii) *completely order reflecting*, if for every $\{F_i : i \in I\} \cup \{G\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\bigcap_{i\in I} \nabla^{\boldsymbol{A}}(F_i) \subseteq \nabla^{\boldsymbol{A}}(G)$, it holds $\bigcap_{i\in I} F_i \subseteq G$.

It should be clear that the relativization of an $\mathcal{S}$-operator is always order preserving. Interestingly enough, an $\mathcal{S}$-operator is order preserving whenever, and only when, it coincides with its own relativization.

**Lemma 1.4.** *An $\mathcal{S}$-operator $\nabla^{\boldsymbol{A}}$ is order preserving if and only if $\nabla^{\boldsymbol{A}} = \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$.*

Proof. If $\nabla^{\boldsymbol{A}}$ is order preserving, then $\nabla^{\boldsymbol{A}}(F) = \bigcap\{\nabla^{\boldsymbol{A}}(G) : G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}, F \subseteq G\} = \widetilde{\nabla}^{\boldsymbol{A}}(F)$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Conversely, if $\nabla^{\boldsymbol{A}} = \widetilde{\nabla}^{\boldsymbol{A}}$, then for every $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$, it holds $\nabla^{\boldsymbol{A}}(F) = \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\nabla}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big) \subseteq \widetilde{\nabla}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^G\big) = \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(G) = \nabla^{\boldsymbol{A}}(G)$. □

The property of being completely order reflecting can also be characterized using the relativization operator. The next lemma is essentially the generalization of [55, (5) p. 108]) to $\mathcal{S}$-operators.

**Lemma 1.5.** *An $\mathcal{S}$-operator $\nabla^{\boldsymbol{A}}$ is completely order reflecting if and only if, for every $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \nabla^{\boldsymbol{A}}(G)$, then $F \subseteq G$.*

Proof. Suppose $\nabla^{\boldsymbol{A}}$ is a completely order reflecting $\mathcal{S}$-operator on $\boldsymbol{A}$. Let $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \nabla^{\boldsymbol{A}}(G)$. Since $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\nabla}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big)$, it follows by hypothesis that $F = \bigcap(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \subseteq G$. Conversely, let $\{F_i : i \in I\} \cup \{G\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\bigcap_{i\in I} \nabla^{\boldsymbol{A}}(F_i) \subseteq \nabla^{\boldsymbol{A}}(G)$. Then, $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(\bigcap_{i\in I} F_i) \subseteq \bigcap_{i\in I} \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F_i) \subseteq \bigcap_{i\in I} \nabla^{\boldsymbol{A}}(F_i) \subseteq \nabla^{\boldsymbol{A}}(G)$. It follows by hypothesis that $\bigcap_{i\in I} F_i \subseteq F$. □

The following proposition states that the maps $\widetilde{\nabla}^{\boldsymbol{A}}$ and $\nabla^{\boldsymbol{A}^{-1}}$ establish a Galois connection. Given Proposition 0.5, several consequences, and most crucial ones, follow from it.

**Proposition 1.6.** *Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$. The maps $\widetilde{\nabla}^{\boldsymbol{A}}$ and $\nabla^{\boldsymbol{A}^{-1}}$ establish a Galois connection between $\mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$ and $\mathrm{Eq}_{\mathcal{L}}(A)$, both ordered under the subset relation.*

Proof. Let $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\theta \in \mathrm{Eq}_{\mathcal{L}}(A)$. Suppose that $\theta \subseteq \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$. If $F \in \mathscr{C}$, then $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \nabla^{\boldsymbol{A}}(F)$, and hence $\theta \subseteq \nabla^{\boldsymbol{A}}(F)$, that is, $F \in \nabla^{\boldsymbol{A}^{-1}}(\theta)$. Thus, $\mathscr{C} \subseteq \nabla^{\boldsymbol{A}^{-1}}(\theta)$. Conversely, suppose that $\mathscr{C} \subseteq \nabla^{\boldsymbol{A}^{-1}}(\theta)$. Then, $\theta \subseteq \nabla^{\boldsymbol{A}}(G)$, for every $G \in \mathscr{C}$. Thus, $\theta \subseteq \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$. □

**Corollary 1.7.** *Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$.*

1. *The maps $\widetilde{\nabla}^{\boldsymbol{A}}$ and $\nabla^{\boldsymbol{A}^{-1}}$ are order reversing.*
2. *The map $\nabla^{\boldsymbol{A}^{-1}} \circ \widetilde{\nabla}^{\boldsymbol{A}}$ is a closure operator over $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, i.e., is a closure on $\mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$.*
3. *The map $\widetilde{\nabla}^{\boldsymbol{A}} \circ \nabla^{\boldsymbol{A}^{-1}}$ is a closure on* $\mathrm{Eqr}A$.
4. *The set of fixed points of $\nabla^{\boldsymbol{A}^{-1}} \circ \widetilde{\nabla}^{\boldsymbol{A}}$ is* $\mathrm{Ran}(\nabla^{\boldsymbol{A}^{-1}})$.
5. *The set of fixed points of $\widetilde{\nabla}^{\boldsymbol{A}} \circ \nabla^{\boldsymbol{A}^{-1}}$ is* $\mathrm{Ran}(\widetilde{\nabla}^{\boldsymbol{A}})$.
6. *The maps $\widetilde{\nabla}^{\boldsymbol{A}}$ and $\nabla^{\boldsymbol{A}^{-1}}$ restrict to mutually inverse dual order isomorphisms between the set of fixed points of $\nabla^{\boldsymbol{A}^{-1}} \circ \widetilde{\nabla}^{\boldsymbol{A}}$ and the set of fixed points of $\widetilde{\nabla}^{\boldsymbol{A}} \circ \nabla^{\boldsymbol{A}^{-1}}$.*

We shall consider the fixed points of both closures $\nabla^{\boldsymbol{A}^{-1}} \circ \widetilde{\nabla}^{\boldsymbol{A}}$ and $\widetilde{\nabla}^{\boldsymbol{A}} \circ \nabla^{\boldsymbol{A}^{-1}}$ often enough to deserve a proper name.

**Definition 1.8.** Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$. A family $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is $\nabla^{\boldsymbol{A}}$-*full* if $\mathscr{C} = \nabla^{\boldsymbol{A}^{-1}}\big(\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})\big)$, i.e., if $\mathscr{C} \in \mathrm{Ran}(\nabla^{\boldsymbol{A}^{-1}})$. A relation $\theta \in \mathrm{Eqr}A$ is $\nabla^{\boldsymbol{A}}$-*full* if $\theta = \widetilde{\nabla}^{\boldsymbol{A}}\big(\nabla^{\boldsymbol{A}^{-1}}(\theta)\big)$, i.e., if $\theta \in \mathrm{Ran}(\widetilde{\nabla}^{\boldsymbol{A}})$.

Thus, the maps $\widetilde{\nabla}^{\boldsymbol{A}}$ and $\nabla^{\boldsymbol{A}^{-1}}$ restrict to mutually inverse dual order isomorphisms between the sets of all $\nabla^{\boldsymbol{A}}$-full families of $\mathcal{S}$-filters of $\boldsymbol{A}$ and the set of all $\nabla^{\boldsymbol{A}}$-full relations on $\boldsymbol{A}$. The reason behind the terminology "full" will become clear once we arrive at Proposition 2.1. A useful characterization of these $\nabla^{\boldsymbol{A}}$-full objects, which is also a consequence of the Galois connection, is the following.

**Proposition 1.9.** *Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$.*

1. *$\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is $\nabla$-full if and only if it is the largest $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{D}) = \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$.*
2. *$\theta \in \mathrm{Eqr}A$ is $\nabla$-full if and only if it is the largest $\theta' \in A \times A$ such that $\nabla^{\boldsymbol{A}^{-1}}(\theta') = \nabla^{\boldsymbol{A}^{-1}}(\theta)$.*

In particular, both the closure system $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and the congruence $A \times A$ are $\nabla$-full objects, for any $\mathcal{S}$-operator $\nabla^{\boldsymbol{A}}$ and any algebra $\boldsymbol{A}$. Another trivial, yet meaningful, observation is that if $\nabla^{\boldsymbol{A}}$ is order preserving, then every $\nabla$-full family of $\mathcal{S}$-filters of $\boldsymbol{A}$ is an up-set in $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

**Congruential $\mathcal{S}$-operators.** One distinguished feature of the Leibniz and Suszko operators, when seen as $\mathcal{S}$-operators, is that their output is always a congruence on the algebra's domain. This property turns out to be relevant in a great deal of the general results we will establish. The Frege operator, on the other hand, fails to satisfy such property in general.

**Definition 1.10.** An $\mathcal{S}$-operator $\nabla^{\boldsymbol{A}}$ on $\boldsymbol{A}$ is *congruential*, if $\nabla^{\boldsymbol{A}}(F) \in \mathrm{Con}\boldsymbol{A}$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

Although it seems to concern only the Leibniz operator, the following result turns out to be crucial in the study of congruential $\mathcal{S}$-operators, and we shall make use of it innumerous times through the rest of this work.

**Proposition 1.11.** *For every $\theta \in \mathrm{Con}\boldsymbol{A}$,*

$$\boldsymbol{\varOmega}^{\boldsymbol{A}^{-1}}(\theta) = \pi^{-1}\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta) \quad \textit{and} \quad \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta) = \pi\boldsymbol{\varOmega}^{\boldsymbol{A}^{-1}}(\theta),$$

*where* $\pi : \boldsymbol{A} \to \boldsymbol{A}/\theta$ *is the canonical map. Moreover, the extended mappings* $\pi \colon \mathscr{P}(A) \to \mathscr{P}(A/\theta)$ *and* $\pi^{-1} \colon \mathscr{P}(A/\theta) \to \mathscr{P}(A)$ *restrict to order isomorphisms between the sets* $\boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}(\theta)$ *and* $\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$*.*

Proof. Let $F \in \boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}(\theta)$. This means that $\theta$ is compatible with $F$, and hence that $\pi^{-1}\pi F = F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since $\pi$ is surjective, by Lemma 0.24.2, $\pi F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$. So, $F \in \pi^{-1}\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$. Conversely, let $G \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$. It follows by Lemma 0.24.1 that $\pi^{-1}G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Moreover, again by the surjectivity of $\pi$, $\pi\pi^{-1}G = G$. So, $\pi^{-1}\big(\pi\pi^{-1}G\big) = \pi^{-1}G$, which tells us that $\theta$ is compatible with $\pi^{-1}G$. Thus, $\pi^{-1}G \in \boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}(\theta)$. This proves the first equality, and the second follows from it by surjectivity of $\pi$. As to the second part of the statement, observe that we have just seen that both $\pi$ and $\pi^{-1}$ are into (actually, onto) the respective co-domains. Moreover, $\big(\pi \restriction_{\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}}\big) \circ \big(\pi^{-1} \restriction_{\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)}\big) = id_{\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)}$, because $\pi$ is surjective, and $\big(\pi^{-1} \restriction_{\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)}\big) \circ \big(\pi \restriction_{\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}}\big) = id_{\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}}$, by definition of $\boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}(\theta)$. So, they are mutually inverse bijections. Since they are both order preserving, they are in fact order isomorphisms. ☐

### 1.2. $\nabla$-classes and $\nabla$-filters

**Definition 1.12.** Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The $\nabla$*-class of* $F$ is the set

$$[\![F]\!]^{\nabla} \coloneqq \boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}\big(\nabla^{\boldsymbol{A}}(F)\big) = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \nabla^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}.$$

The first basic fact about $\nabla$-classes is that they are closure systems on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

**Proposition 1.13.** *Let* $\nabla^{\boldsymbol{A}}$ *be an* $\mathcal{S}$*-operator on* $\boldsymbol{A}$*. For every* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$*, the* $\nabla$*-class* $[\![F]\!]^{\nabla}$ *is a closure system on* $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$*.*

Proof. Let $\{F_i : i \in I\} \subseteq [\![F]\!]^{\nabla}$. Then,

$$\nabla^{\boldsymbol{A}}(F) \subseteq \bigcap_{i\in I} \boldsymbol{\Omega}^{\boldsymbol{A}}(F_i) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\bigcap_{i\in I} F_i\big).$$

Hence, $\bigcap_{i\in I} F_i \in [\![F]\!]^{\nabla}$. Moreover, since $\boldsymbol{\Omega}^{\boldsymbol{A}}(A) = A \times A$, it trivially holds $A \in [\![F]\!]^{\nabla}$. ☐

Assuming $\nabla$ congruential, Proposition 1.13 can be strengthened as follows:

**Proposition 1.14.** *Let* $\nabla^{\boldsymbol{A}}$ *be a congruential* $\mathcal{S}$*-operator on* $\boldsymbol{A}$*. For every* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$*, the* $\nabla$*-class* $[\![F]\!]^{\nabla}$ *is a full g-model of* $\mathcal{S}$*.*

Proof. By definition, $[\![F]\!]^{\nabla} = \boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}(\nabla^{\boldsymbol{A}}(F))$. Since $\nabla^{\boldsymbol{A}}(F) \in \mathrm{Con}(\boldsymbol{A})$ by hypothesis, it follows by Proposition 1.11 that $[\![F]\!]^{\nabla} = \pi^{-1}\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)\big)$, where $\pi : \boldsymbol{A} \to \boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)$ is the canonical map. It follows by Proposition 0.35 that $[\![F]\!]^{\nabla}$ is a full g-model of $\mathcal{S}$. ☐

Given that every closure system is closed under intersections, it makes sense to consider the smallest element in each $\nabla$-class.

**Definition 1.15.** Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-operator on $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The least element of the $\nabla$-class of $F$ will be denoted by $F^{\nabla}$; i.e., $F^{\nabla} \coloneqq \bigcap [\![F]\!]^{\nabla}$. We say that $F$ is a $\nabla$*-filter* if $F = F^{\nabla}$. The set of all $\nabla$-filters of $\boldsymbol{A}$ will be denoted by $\mathcal{F}i_{\mathcal{S}}^{\nabla}\boldsymbol{A}$.

It is worth noticing that if $\mathcal{S}$ has no theorems, then for any $\boldsymbol{A}$ the only $\nabla$-filter of $\boldsymbol{A}$ is the empty filter, because $\varnothing \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\boldsymbol{\Omega}^{\boldsymbol{A}}(\varnothing) = A \times A$; so for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, it holds $\varnothing \in [\![F]\!]^{\nabla}$ and hence necessarily $\varnothing = \bigcap [\![F]\!]^{\nabla}$. It is therefore clear that the interesting applications of the notions of $\nabla$-class and $\nabla$-filter will concern only logics with theorems; however, technically we need not assume this in any result.

**Proposition 1.16.** *Every $\mathcal{S}$-operator $\nabla^{\boldsymbol{A}}$ on $\boldsymbol{A}$ is order reflecting, and therefore injective, on $\mathcal{F}i^{\nabla}_{\mathcal{S}}\boldsymbol{A}$.*

Proof. Let $F, G \in \mathcal{F}i^{\nabla}_{\mathcal{S}}\boldsymbol{A}$ such that $\nabla^{\boldsymbol{A}}(F) \subseteq \nabla^{\boldsymbol{A}}(G)$. Then $[\![G]\!]^{\nabla} \subseteq [\![F]\!]^{\nabla}$. Thus, $F = \bigcap [\![F]\!]^{\nabla} \subseteq \bigcap [\![G]\!]^{\nabla} = G$. □

In general, an $\mathcal{S}$-operator $\nabla^{\boldsymbol{A}}$ need not be order preserving on $\mathcal{F}i^{\nabla}_{\mathcal{S}}\boldsymbol{A}$. Another useful monotonicity related property, this time concerning the elements of $\nabla$-classes, is the following:

**Lemma 1.17.** *If $\nabla^{\boldsymbol{A}}$ is an order preserving $\mathcal{S}$-operator on $\boldsymbol{A}$, then the map $F \mapsto F^{\nabla}$ is monotonic, i.e., if $F \subseteq G$, then $F^{\nabla} \subseteq G^{\nabla}$.*

Proof. If $F \subseteq G$, then $\nabla^{\boldsymbol{A}}(F) \subseteq \nabla^{\boldsymbol{A}}(G)$ by order preservation, so $[\![G]\!]^{\nabla} \subseteq [\![F]\!]^{\nabla}$, and therefore $G^{\nabla} = \bigcap [\![G]\!]^{\nabla} \subseteq \bigcap [\![F]\!]^{\nabla} = F^{\nabla}$. □

Since $\nabla$-classes are, by definition, sets of $\mathcal{S}$-filters of the form ${\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}(\nabla^{\boldsymbol{A}}(F))$, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, we can apply Proposition 1.11 to $\theta = \nabla^{\boldsymbol{A}}(F)$, provided that $\nabla$ is congruential.

**Proposition 1.18.** *Let $\nabla^{\boldsymbol{A}}$ be a congruential $\mathcal{S}$-operator on $\boldsymbol{A}$. An $\mathcal{S}$-filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a $\nabla$-filter of $\boldsymbol{A}$ if and only if $F/\nabla^{\boldsymbol{A}}(F)$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)$.*

Proof. Let $\pi : \boldsymbol{A} \to \boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)$ be the canonical map. Since ${\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}(\nabla^{\boldsymbol{A}}(F)) = [\![F]\!]^{\nabla}$, it follows by Proposition 1.11 that $\pi$ induces an order isomorphism between $[\![F]\!]^{\nabla}$ and $\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)\big)$. Since order isomorphisms send least elements to least elements, the result should be clear. □

## 1.3. $\mathcal{S}$-compatibility operators

The main notion of this chapter was first introduced in [**24**, p. 199] for the particular case where the $\mathcal{S}$-operators have as outputs congruences on the algebra's domain.

**Definition 1.19.** An *$\mathcal{S}$-compatibility operator on $\boldsymbol{A}$* is an $\mathcal{S}$-operator $\nabla^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Eqr}A$ such that $\nabla^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, for every $F \in \boldsymbol{A}$.

The least $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$ is the map $id^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Eqr}A$ defined by $F \mapsto id_{\boldsymbol{A}}$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The largest $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$ is obviously $\boldsymbol{\Omega}^{\boldsymbol{A}}$. Since for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is also an $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$. Actually, the Suszko operator is the largest order preserving $\mathcal{S}$-compatibility operator:

**Lemma 1.20.** *For every $\boldsymbol{A}$, the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is the largest order preserving $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$.*

Proof. Let $\nabla^{\boldsymbol{A}}$ be an order preserving $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$. For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$\nabla^{\boldsymbol{A}}(F) = \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \bigcap_{F' \supseteq F} \nabla^{\boldsymbol{A}}(F') \subseteq \bigcap_{F' \supseteq F} \boldsymbol{\Omega}^{\boldsymbol{A}}(F') = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F),$$

using Lemma 1.4 and $\mathcal{S}$-compatibility. □

We next state some basic facts concerning $\nabla$-classes and $\nabla$-filters.

**Lemma 1.21.** *Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$. For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

1. *$F \in [\![F]\!]^{\nabla}$;*
2. *$F^{\nabla} \subseteq F$.*

*If moreover $\nabla^{\boldsymbol{A}}$ is order preserving, then*

3. *$(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \subseteq [\![F]\!]^{\nabla}$;*
4. *$[\![F]\!]^{\nabla} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ if and only if $F = F^{\nabla}$, i.e., if and only if $F$ is a $\nabla$-filter.*

Proof. *1.* By $\mathcal{S}$-compatibility, $\nabla^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. *2.* Since $F$ belongs to its own $\nabla$-class by 1, $F^{\nabla} = \bigcap [\![F]\!]^{\nabla} \subseteq F$. *3.* If $F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, then $\nabla^{\boldsymbol{A}}(F) \subseteq \nabla^{\boldsymbol{A}}(F') \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F')$, and therefore $F' \in [\![F]\!]^{\nabla}$. *4.* Suppose that $[\![F]\!]^{\nabla} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$. Then, $F^{\nabla} = \bigcap [\![F]\!]^{\nabla} = \bigcap (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F = F$. Conversely, suppose that $F = F^{\nabla}$. Clearly then, $[\![F]\!]^{\nabla} \subseteq (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$. Moreover, by 3, $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \subseteq [\![F]\!]^{\nabla}$. □

Finally, we state some straightforward consequences of the facts that the Leibniz operator is the largest $\mathcal{S}$-operator, and the Suszko operator is the largest order preserving one (from Chapter 2 on, we shall use a more familiar notation for the $\boldsymbol{\Omega}$- and $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-related notions; for the time being, we use the notation introduced in Definitions 1.12 and 1.15).

**Lemma 1.22.** *Let $\nabla^{\boldsymbol{A}}$ be an $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$. For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

1. *$[\![F]\!]^{\boldsymbol{\Omega}} \subseteq [\![F]\!]^{\nabla}$;*
2. *$F^{\nabla} \subseteq F^{\boldsymbol{\Omega}}$;*
3. *Every $\nabla$-filter is an $\boldsymbol{\Omega}$-filter.*

Proof. *1.* Since $\nabla^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, by $\mathcal{S}$-compatibility. *2.* Just notice that $F^{\nabla} = \bigcap [\![F]\!]^{\nabla} \subseteq \bigcap [\![F]\!]^{\boldsymbol{\Omega}} = F^{\boldsymbol{\Omega}}$, using 1. *3.* Since $F^{\nabla} \subseteq F^{\boldsymbol{\Omega}} \subseteq F$, using 2 and $F \in [\![F]\!]^{\boldsymbol{\Omega}}$, for $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. □

**Lemma 1.23.** *Let $\nabla^{\boldsymbol{A}}$ be an order preserving $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$. For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

1. *$[\![F]\!]^{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}} \subseteq [\![F]\!]^{\nabla}$;*
2. *$F^{\nabla} \subseteq F^{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}}$;*
3. *Every $\nabla$-filter is an $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-filter.*

Proof. *1.* Since $\nabla^{\boldsymbol{A}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, by Lemma 1.20. *2.* Just notice that $F^{\nabla} = \bigcap [\![F]\!]^{\nabla} \subseteq \bigcap [\![F]\!]^{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}} = F^{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}}$, using 1. *3.* Since $F^{\nabla} \subseteq F^{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}} \subseteq F$, using 2 and $F \in [\![F]\!]^{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}}$, for $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. □

## 1.4. Coherent families of $\mathcal{S}$-operators

The main notion of this section — coherence — serves as a generalization of the property of commuting with inverse images by surjective homomorphisms. The reason why we wish to generalize this property is simple. First, as we next show, the only $\mathcal{S}$-compatibility operator commuting with inverse images by surjective homomorphisms is the Leibniz operator, which obviously makes this property of little use for a general treatment. But furthermore, and most importantly, our three paradigmatic examples of $\mathcal{S}$-operators — the Leibniz, Suszko and Frege operators — turn out to be coherent.

Let $\nabla^{\boldsymbol{A}}$ and $\nabla^{\boldsymbol{B}}$ be $\mathcal{S}$-operators on $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. We say that the pair $\langle \nabla^{\boldsymbol{A}}, \nabla^{\boldsymbol{B}} \rangle$ *commutes with inverse images by (surjective) homomorphisms* if for every (surjective) $h \colon \boldsymbol{A} \to \boldsymbol{B}$ and every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$,

$$\nabla^{\boldsymbol{A}}(h^{-1}G) = h^{-1}\nabla^{\boldsymbol{B}}(G).$$

A family $\nabla$ of $\mathcal{S}$-operators *commutes with inverse images by (surjective) homomorphisms* if for all algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ the pair $\langle \nabla^{\boldsymbol{A}}, \nabla^{\boldsymbol{B}} \rangle$ commutes with inverse images by (surjective) homomorphisms in the above sense.

**Theorem 1.24.** *If $\nabla$ is a family of $\mathcal{S}$-compatibility operators that commutes with inverse images by surjective homomorphisms, then $\nabla = \boldsymbol{\Omega}$.*

Proof. Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Let $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ be the canonical map. It is clearly surjective. Moreover, since $\operatorname{Ker}\pi = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is compatible with $F$, we have $F = \pi^{-1}\pi F$ and $\pi F \in \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big)$. Now, it follows by $\mathcal{S}$-compatibility that

$$\nabla^{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)}(F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)}(F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)) = id_{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)}.$$

Hence, $\nabla^{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)}(F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)) = \boldsymbol{\Omega}^{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)}(F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F))$. Applying $\pi^{-1}$ on both sides, using our hypothesis, and the fact that the Leibniz operator commutes with inverse images by surjective homomorphisms,

$$\begin{aligned}\nabla^{\boldsymbol{A}}(F) = \nabla^{\boldsymbol{A}}(\pi^{-1}\pi F) &= \pi^{-1}\nabla^{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)}(F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)) \\ &= \pi^{-1}\boldsymbol{\Omega}^{\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)}(F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)) = \boldsymbol{\Omega}^{\boldsymbol{A}}(\pi^{-1}\pi F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F).\end{aligned}$$

Since we have chosen $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ arbitrary, we conclude that $\nabla = \boldsymbol{\Omega}$. ☐

Do notice that Theorem 1.24 characterizes the Leibniz operator among $\mathcal{S}$-compatibility operators, but not necessarily among the $\mathcal{S}$-operators in general (indeed, it does not, as we shall see in Theorem 2.52 when studying the Frege operator).

An immediate consequence is that the Suszko operator commutes with inverse images by surjective homomorphisms if and only if $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}} = \boldsymbol{\Omega}$, which as we know, is equivalent to protoalgebraicity. This characterization will be the starting point for similiar characterizations of other classes of logics within the Leibniz hierarchy in terms of the Suszko operator (Theorem 3.13). For the time being, it confirms what we had already advanced after Proposition 0.31, namely, that the Suszko operator does not commute, in general, with inverse images by surjective homomorphisms. In order to find a commutativity property suitable for a unified treatment of the remaining two paradigmatic $\mathcal{S}$-operators, we introduce the following technical notion.

**Definition 1.25.** Let $\nabla$ be a family of $\mathcal{S}$-operators. Let $\boldsymbol{A}$ be an algebra, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. A homomorphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$ is $\nabla$*-compatible with* $F$ if $\operatorname{Ker}h \subseteq \nabla^{\boldsymbol{A}}(F)$; and it is $\nabla$*-compatible with* $\mathscr{C}$ if it is $\nabla$-compatible with every member of $\mathscr{C}$, that is, if $\operatorname{Ker}h \subseteq \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$.

Notice that a homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is $\boldsymbol{\Omega}$-compatible with $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ if and only if $\mathrm{Ker}h \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ if and only if $\mathrm{Ker}h$ is compatible (in the usual sense) with $F$. Also, given an $\mathcal{S}$-compatibility operator $\nabla$, observe that if $h$ is $\nabla$-compatible with $F$, then it is also $\boldsymbol{\Omega}$-compatible with $F$. So, by Lemmas 0.17.1 and 0.17.2, $F = h^{-1}hF$ and $\nabla^{\boldsymbol{A}}(F) = h^{-1}h\nabla^{\boldsymbol{A}}(F)$.

Definition 1.25, when instantiated with the Leibniz, Suszko and Frege operators, turns out to be equivalent to two already known notions concerning matrix homomorphisms.

**Lemma 1.26.** *Let $\boldsymbol{A}, \boldsymbol{B}$ algebras, $h : \boldsymbol{A} \to \boldsymbol{B}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The following conditions are equivalent:*

1. *$h$ is $\boldsymbol{\Omega}$-compatible with $F$;*
2. *the matrix homomorphism $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, hF\rangle$ is strict.*

Proof. Having in mind Lemma 0.16, notice that

$$\begin{aligned} h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, hF\rangle \text{ is strict} \quad &\text{iff} \quad h^{-1}hF = F \\ &\text{iff} \quad \mathrm{Ker}h \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F). \end{aligned}$$

□

**Lemma 1.27.** *Let $\boldsymbol{A}, \boldsymbol{B}$ algebras, $h : \boldsymbol{A} \to \boldsymbol{B}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The following conditions are equivalent:*

1. *$h$ is $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-compatible with $F$;*
2. *$h$ is $\boldsymbol{\Lambda}_{\mathcal{S}}$-compatible with $F$;*
3. *the matrix homomorphism $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, hF\rangle$ is deductive.*

Proof. Having in mind that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is the largest congruence on $\boldsymbol{A}$ below $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, notice that

$$\begin{aligned} h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, hF\rangle \text{ is deductive} \quad &\text{iff} \quad \forall a, b \in A \\ &\qquad \text{if } ha = hb, \text{ then } \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, a) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, b) \\ &\text{iff} \quad \mathrm{Ker}h \subseteq \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \\ &\text{iff} \quad \mathrm{Ker}h \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F). \end{aligned}$$

□

Deductive homomorphisms were introduced in [**24**] and used to extend Blok and Pigozzi's Correspondence Theorem to arbitrary logics ([**24**, Proposition 2.3], here stated as Theorem 2.34), which follows in our setting as an instance of the General Correspondence Theorem 1.38.

We are now ready to introduce the main (new) definition of the present section, and probably of the whole Part I.

**Definition 1.28.** A family $\nabla$ of $\mathcal{S}$-operators is *coherent*, if for every surjective homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, if $h$ is $\nabla$-compatible with $h^{-1}G$, then $\nabla^{\boldsymbol{A}}(h^{-1}G) = h^{-1}\nabla^{\boldsymbol{B}}(G)$.

The family $\{id^{\boldsymbol{A}} : \boldsymbol{A} \text{ an algebra}\}$ is trivially a coherent family of $\mathcal{S}$-compatibility operators. For let $\boldsymbol{A}, \boldsymbol{B}$ algebras, $h : \boldsymbol{A} \to \boldsymbol{B}$ surjective, $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and assume $\mathrm{Ker}h \subseteq id^{\boldsymbol{A}}(h^{-1}G) = id_{\boldsymbol{A}}$. Then, $\mathrm{Ker}h = id_{\boldsymbol{A}}$. So, $h$ is injective, and therefore it is an isomorphism. Hence, $h^{-1}id^{\boldsymbol{B}}(G) = h^{-1}id_{\boldsymbol{B}} = id_{\boldsymbol{A}} = id^{\boldsymbol{A}}(h^{-1}G)$.

It should be clear by the definition itself that coherence is a weaker property than commuting with inverse images of surjective homomorphisms. Taking Theorem 1.24 into account, the Leibniz operator is a coherent family of $\mathcal{S}$-operators.

It is also possible, and in fact it will sometimes be rather practical, to use coherence stated in terms of commutativity with images by surjective homomorphisms instead of inverse images. The next lemma holds only for $\mathcal{S}$-compatibility operators.

**Lemma 1.29.** *A family $\nabla$ of $\mathcal{S}$-compatibility operators is coherent if and only if for every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $h$ is $\nabla$-compatible with $F$, then $h\nabla^{\boldsymbol{A}}(F) = \nabla^{\boldsymbol{B}}(hF)$.*

Proof. Suppose $\nabla$ is coherent. Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $h : \boldsymbol{A} \to \boldsymbol{B}$ be surjective and $\nabla$-compatible with $F$. Hence, $F = h^{-1}hF$ and $hF \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. It follows by coherence that $\nabla^{\boldsymbol{A}}(F) = \nabla^{\boldsymbol{A}}\big(h^{-1}hF\big) = h^{-1}\nabla^{\boldsymbol{B}}(hF)$, and hence that $h\nabla^{\boldsymbol{A}}(F) = \nabla^{\boldsymbol{B}}(hF)$ because $h$ is surjective. Conversely, let $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be surjective and $\nabla$-compatible with $h^{-1}G$. Since $h^{-1}G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, it follows by the assumption and the surjectivity of $h$ that

$$h\nabla^{\boldsymbol{A}}(h^{-1}G) = \nabla^{\boldsymbol{B}}(hh^{-1}G) = \nabla^{\boldsymbol{B}}(G).$$

Applying the property in Lemma 0.17.2 to the $\nabla^{\boldsymbol{A}}$-compatibility of $h$ with $h^{-1}G$, we obtain

$$\nabla^{\boldsymbol{A}}(h^{-1}G) = h^{-1}h\nabla^{\boldsymbol{A}}(h^{-1}G) = h^{-1}\nabla^{\boldsymbol{B}}(G),$$

which shows that $\nabla$ is coherent. ☐

Of course, isomorphisms are both surjective and $\nabla$-compatible with any $\mathcal{S}$-filter, as their kernel is the identity.

**Corollary 1.30.** *If $\nabla$ is a coherent family of $\mathcal{S}$-compatibility operators and $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is an isomorphism, then for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, it holds $h\nabla^{\boldsymbol{A}}(F) = \nabla^{\boldsymbol{B}}(hF)$ and $\nabla^{\boldsymbol{A}}(h^{-1}G) = h^{-1}\nabla^{\boldsymbol{B}}(G)$.*

Another interesting characterization of coherence comes in terms of the map ${\nabla^{\boldsymbol{A}}}^{-1}$.

**Proposition 1.31.** *A family $\nabla$ of $\mathcal{S}$-compatibility operators is coherent if and only if, for every surjective homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$,*

$${\nabla^{\boldsymbol{A}}}^{-1}\big(\mathrm{Ker}h\big) = \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : h^{-1}\nabla^{\boldsymbol{B}}(hF) = \nabla^{\boldsymbol{A}}(F)\big\}. \tag{14}$$

Proof. Suppose $\nabla$ is coherent. Let $F \in {\nabla^{\boldsymbol{A}}}^{-1}\big(\mathrm{Ker}h\big)$. Then, $\mathrm{Ker}h \subseteq \nabla^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, using $\mathcal{S}$-compatibility. Therefore, $F = h^{-1}hF$ and hence $hF \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Since $\nabla$ is a coherent by hypothesis, $h^{-1}\nabla^{\boldsymbol{B}}(hF) = \nabla^{\boldsymbol{A}}(F)$. Now, let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $h^{-1}\nabla^{\boldsymbol{B}}(hF) = \nabla^{\boldsymbol{A}}(F)$. Then, since $\mathrm{Ker}h \subseteq h^{-1}\nabla^{\boldsymbol{B}}(hF)$ always holds, it follows that $\mathrm{Ker}h \subseteq \nabla^{\boldsymbol{A}}(F)$ and therefore that $F \in {\nabla^{\boldsymbol{A}}}^{-1}\big(\mathrm{Ker}h\big)$.

Conversely, suppose that surjective homomorphisms satisfy the identity (14). Let $\boldsymbol{A}, \boldsymbol{B}$ any two algebras, let $h \in \boldsymbol{A} \to \boldsymbol{B}$ surjective, let $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and assume $h$ is $\nabla$-compatible with $h^{-1}G$. That is, $\mathrm{Ker}h \subseteq \nabla^{\boldsymbol{A}}(h^{-1}G)$. So, $h^{-1}G \in {\nabla^{\boldsymbol{A}}}^{-1}\big(\mathrm{Ker}h\big)$. It follows by hypothesis that $h^{-1}\nabla^{\boldsymbol{B}}(hh^{-1}G) = \nabla^{\boldsymbol{A}}(h^{-1}G)$. Since $h$ is surjective, $G = hh^{-1}G$, and we therefore obtain $h^{-1}\nabla^{\boldsymbol{B}}(G) = \nabla^{\boldsymbol{A}}(h^{-1}G)$. Thus, $\nabla$ is coherent. ☐

We finish this section by proving that coherence is preserved under relativization.

**Proposition 1.32.** *If $\nabla$ is a coherent family of $\mathcal{S}$-compatibility operators, then the family $\widetilde{\nabla}_{\mathcal{S}} = \{\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}} : \boldsymbol{A}$ an algebra$\}$ is also a coherent family of $\mathcal{S}$-compatibility operators.*

Proof. It is clear by the definition of $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$ that, if $\nabla$ is a family of $\mathcal{S}$-compatibility operators, then so is $\widetilde{\nabla}_{\mathcal{S}}$. Now, let $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be surjective and $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}$-compatible with $h^{-1}G$, i.e., such that $\mathrm{Ker} h \subseteq \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}\big(h^{-1}G\big)$. Let $F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}G}$, i.e., such that $h^{-1}G \subseteq F'$. Then, $\mathrm{Ker} h \subseteq \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G) \subseteq \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F') \subseteq \nabla^{\boldsymbol{A}}(F')$. Hence, $h$ is $\nabla$-compatible with $F'$, and therefore $F' = h^{-1}hF'$ and $hF' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. It follows by hypothesis that

$$\nabla^{\boldsymbol{A}}(F') = \nabla^{\boldsymbol{A}}\big(h^{-1}hF'\big) = h^{-1}\nabla^{\boldsymbol{B}}\big(hF'\big). \tag{15}$$

Next, we claim that:

*Claim.* $h\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}G}\big) = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G}$: Let $F' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ be such that $h^{-1}G \subseteq F'$. We have already seen that under the present assumptions, $hF' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, and obviously $G = hh^{-1}G \subseteq hF'$. Conversely, let $G' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ be such that $G \subseteq G'$. Then we know that $G' = hh^{-1}G'$ and $h^{-1}G' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, and moreover $h^{-1}G \subseteq h^{-1}G'$.

Now, using (15), commutativity of $h^{-1}$ with intersections, and the claim,

$$\begin{aligned}
\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G) &= \bigcap\big\{\nabla^{\boldsymbol{A}}(F') : F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}G}\big\}\\
&= \bigcap\big\{h^{-1}\nabla^{\boldsymbol{B}}\big(hF'\big) : F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}G}\big\}\\
&= h^{-1}\Big(\bigcap\big\{\nabla^{\boldsymbol{B}}\big(hF'\big) : F' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}G}\big\}\Big)\\
&= h^{-1}\Big(\bigcap\big\{\nabla^{\boldsymbol{B}}(G') : G' \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G}\big\}\Big) = h^{-1}\widetilde{\nabla}^{\boldsymbol{B}}_{\mathcal{S}}(G).
\end{aligned}$$

We conclude that the family $\widetilde{\nabla}_{\mathcal{S}}$ is coherent. □

In particular, the Suszko operator, being the relativization of the Leibniz operator, is also a coherent family of $\mathcal{S}$-operators.

**Proposition 1.33.** *Let $\nabla$ be a coherent family of $\mathcal{S}$-compatibility operators, and let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be surjective.*

1. *For any $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, if $h$ is $\nabla$-compatible with $h^{-1}\mathscr{D}$, then $\widetilde{\nabla}^{\boldsymbol{A}}(h^{-1}\mathscr{D}) = h^{-1}\widetilde{\nabla}^{\boldsymbol{B}}(\mathscr{D})$.*
2. *For any $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $h$ is $\nabla$-compatible with $\mathscr{C}$, then $h\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\nabla}^{\boldsymbol{B}}(h\mathscr{C})$.*

Proof. *1.* Assume that $h$ is $\nabla$-compatible with $h^{-1}\mathscr{D}$, i.e., $\mathrm{Ker} h \subseteq \widetilde{\nabla}^{\boldsymbol{A}}(h^{-1}\mathscr{D})$. For each $G \in \mathscr{D}$, $\widetilde{\nabla}^{\boldsymbol{A}}(h^{-1}\mathscr{D}) \subseteq \nabla^{\boldsymbol{A}}(h^{-1}G)$, and hence $h$ is $\nabla$-compatible with $h^{-1}G$. So, by coherence,

$$\widetilde{\nabla}^{\boldsymbol{A}}(h^{-1}\mathscr{D}) = \bigcap_{G\in\mathscr{D}} \nabla^{\boldsymbol{A}}(h^{-1}G) = \bigcap_{G\in\mathscr{D}} h^{-1}\nabla^{\boldsymbol{B}}(G) = h^{-1} \bigcap_{G\in\mathscr{D}} \nabla^{\boldsymbol{B}}(G) = h^{-1}\widetilde{\nabla}^{\boldsymbol{B}}(\mathscr{D}).$$

*2.* Assume now that $h$ is $\nabla$-compatible with $\mathscr{C}$. Thus, if $F \in \mathscr{C}$, then $h$ is $\nabla$-compatible with $F$, which implies that $h^{-1}hF = F$. Therefore, $h^{-1}h\mathscr{C} = \mathscr{C}$, so that we can say that $h$ is $\nabla$-compatible with $h^{-1}h\mathscr{C}$. Moreover, since $\mathrm{Ker} h \subseteq \nabla^{\boldsymbol{A}}(F)$ for each $F \in \mathscr{C}$, we also have that $\mathrm{Ker} h \subseteq \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$, which by Lemma 0.17.2 implies that $h^{-1}h\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$. Then we can apply point 1 to find that

$\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\nabla}^{\boldsymbol{A}}(h^{-1}h\mathscr{C}) = h^{-1}\widetilde{\nabla}^{\boldsymbol{B}}(h\mathscr{C})$ and then by surjectivity of $h$ we conclude that $h\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = hh^{-1}\widetilde{\nabla}^{\boldsymbol{B}}(h\mathscr{C}) = \widetilde{\nabla}^{\boldsymbol{B}}(h\mathscr{C})$. □

**$\nabla$-full objects under coherence.** In this section we show that, for coherent families of congruential $\mathcal{S}$-compatibility operators, the $\nabla$-full objects defined in Section 1.1 can be given finer characterizations. Let us start by pointing out a particular case of Proposition 1.31.

**Proposition 1.34.** *If $\nabla$ is a coherent family of $\mathcal{S}$-compatibility operators, then for every $\theta \in \mathrm{Con}\boldsymbol{A}$,*

$$\begin{aligned}\nabla^{\boldsymbol{A}^{-1}}(\theta) &= \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \pi^{-1}\nabla^{\boldsymbol{A}/\theta}(\pi F) = \nabla^{\boldsymbol{A}}(F)\big\}\\ &= \pi^{-1}\big\{G \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta) : \pi^{-1}\nabla^{\boldsymbol{A}/\theta}(G) = \nabla^{\boldsymbol{A}}(\pi^{-1}G)\big\}.\end{aligned}$$

Proof. For the first equality we apply Proposition 1.31 to the quotient homomorphism $\pi : \boldsymbol{A} \to \boldsymbol{A}/\theta$. To obtain the second note that the inclusion from left to right is clear. The other inclusion follows from the fact that if $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is such that $\pi^{-1}\nabla^{\boldsymbol{A}/\theta}(\pi F) = \nabla^{\boldsymbol{A}}(F)$, then $\mathrm{Ker}\pi$ is compatible with $F$ and therefore $\pi F \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$. □

This allows us to establish the following characterization of $\nabla^{\boldsymbol{A}}$-full families of $\mathcal{S}$-filters:

**Corollary 1.35.** *Let $\nabla$ be a coherent family of congruential $\mathcal{S}$-compatibility operators and $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then $\mathscr{C}$ is a $\nabla^{\boldsymbol{A}}$-full g-model of $\mathcal{S}$ if and only if*

$$\mathscr{C} = \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : h^{-1}\nabla^{\boldsymbol{B}}(hF) = \nabla^{\boldsymbol{A}}(F)\big\},$$

*for some surjective homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$, which can be taken to be the canonical map $\pi : \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$.*

Proof. Suppose $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a $\nabla^{\boldsymbol{A}}$-full g-model of $\mathcal{S}$, i.e., $\mathscr{C} = \nabla^{\boldsymbol{A}^{-1}}\big(\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})\big)$. Let $\boldsymbol{B} := \boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$ and let $\pi : \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$ the quotient homomorphism. Then $\mathscr{C} = \nabla^{\boldsymbol{A}^{-1}}\big(\mathrm{Ker}\pi\big)$. Thus from Proposition 1.31 we obtain $\mathscr{C} = \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \pi^{-1}\nabla^{\boldsymbol{B}}(\pi F) = \nabla^{\boldsymbol{A}}(F)\big\}$. Assume now that $\mathscr{C} = \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : h^{-1}\nabla^{\boldsymbol{B}}(hF) = \nabla^{\boldsymbol{A}}(F)\big\}$ for some surjective homomorphism $h : \boldsymbol{A} \to \boldsymbol{B}$. Then again by Proposition 1.31 we have $\mathscr{C} = \nabla^{\boldsymbol{A}^{-1}}\big(\mathrm{Ker}h\big)$. Thus $\mathscr{C} \in \mathrm{Ran}(\nabla^{\boldsymbol{A}^{-1}})$ and hence it is $\nabla^{\boldsymbol{A}}$-full. □

Considering the proof above and Proposition 1.34, a slightly different corollary can be stated as follows:

**Corollary 1.36.** *Let $\nabla$ be a coherent family of congruential $\mathcal{S}$-compatibility operators, and $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then $\mathscr{C}$ is $\nabla^{\boldsymbol{A}}$-full if and only if*

$$\mathscr{C} = \pi^{-1}\big\{G \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta) : \pi^{-1}\nabla^{\boldsymbol{A}/\theta}(G) = \nabla^{\boldsymbol{A}}(\pi^{-1}G)\big\},$$

*for some $\theta \in \mathrm{Con}\boldsymbol{A}$, which can be taken to be $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$.*

Finally, we address the $\nabla$-full congruences.

**Proposition 1.37.** *Let $\nabla$ be a coherent family of $\mathcal{S}$-compatibility operators and $\theta \in \mathrm{Con}\boldsymbol{A}$. Then $\theta$ is $\nabla^{\boldsymbol{A}}$-full if and only if*

$$\widetilde{\nabla}^{\boldsymbol{A}/\theta}\Big(\big\{G \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta) : \pi^{-1}\nabla^{\boldsymbol{A}/\theta}(G) = \nabla^{\boldsymbol{A}}(\pi^{-1}G)\big\}\Big) = id_{\boldsymbol{A}/\theta}.$$

PROOF. Fix $\mathscr{D} := \{G \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta) : \pi^{-1}\nabla^{\boldsymbol{A}/\theta}(G) = \nabla^{\boldsymbol{A}}(\pi^{-1}G)\}$. Observe that by the comment after Definition 1.28, $\pi$ is $\nabla$-compatible with $\mathscr{D}$, and therefore, using Proposition 1.34 and Lemma 1.33.1, $\theta$ is $\nabla^{\boldsymbol{A}}$-full if and only if $\theta = \widetilde{\nabla}^{\boldsymbol{A}}(\nabla^{\boldsymbol{A}^{-1}}(\theta)) = \widetilde{\nabla}^{\boldsymbol{A}}(\pi^{-1}\mathscr{D}) = \pi^{-1}\widetilde{\nabla}^{\boldsymbol{A}/\theta}(\mathscr{D})$. This implies, by surjectivity of $\pi$, that $\widetilde{\nabla}^{\boldsymbol{A}/\theta}(\mathscr{D}) = \pi\pi^{-1}\widetilde{\nabla}^{\boldsymbol{A}/\theta}(\mathscr{D}) = \pi(\theta) = id_{\boldsymbol{A}/\theta}$. Conversely, if $\widetilde{\nabla}^{\boldsymbol{A}/\theta}(\mathscr{D}) = id_{\boldsymbol{A}/\theta}$, then $\theta = \pi^{-1}id_{\boldsymbol{A}/\theta} = \pi^{-1}\widetilde{\nabla}^{\boldsymbol{A}/\theta}(\mathscr{D})$, which establishes that $\theta$ is $\nabla^{\boldsymbol{A}}$-full by the above consideration. □

Here arrived, observe that by instantiating the above results with the Leibniz operator, which is a coherent family of congruential $\mathcal{S}$-compatibility operators, we find the result proved directly in Proposition 1.11, namely that $\boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}(\theta) = \mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$. Indeed, since the Leibniz operator commutes with inverse images by surjective homomorphisms, the family $\mathscr{D}$ in the above proof is precisely $\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)$.

### 1.5. The General Correspondence Theorem

We are now able to prove the main theorem of the present chapter — the General Correspondence Theorem 1.38. By applying this result to the Leibniz and to the Suszko operators (Theorems 2.12 and 2.34, respectively), we will see that it generalizes and strengthens Blok and Pigozzi's well-known Correspondence Theorem for protoalgebraic logics [**10**, Theorem 2.4], and Czelakowski's less known Correspondence Theorem [**24**, Proposition 2.3] for arbitrary logics, respectively. It also generalizes the strengthening obtained for protoalgebraic logics by Font and Jansana of the first result ([**37**, Corollary 9.1]).

**Theorem 1.38** (General Correspondence Theorem)**.** *Let $\nabla$ be a coherent family of $\mathcal{S}$-compatibility operators. For every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $h$ is $\nabla$-compatible with $F$, then $h$ induces an order isomorphism between $[\![F]\!]^{\nabla^{\boldsymbol{A}}}$ and $[\![hF]\!]^{\nabla^{\boldsymbol{B}}}$, whose inverse is given by $h^{-1}$.*

PROOF. Since $h$ is $\nabla$-compatible with $F$, it is also $\boldsymbol{\Omega}$-compatible with $F$. So, by Lemmas 0.17.1 and 0.24.3, $F = h^{-1}hF$ and $hF \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$.

Take first any $F' \in [\![F]\!]^{\nabla^{\boldsymbol{A}}}$. Then $\operatorname{Ker} h \subseteq \nabla^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F')$ and hence by Lemma 0.24.3, $h^{-1}hF' = F'$ and $hF' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Moreover, since $h$ is both $\boldsymbol{\Omega}$-compatible with $F'$ and $\nabla$-compatible with $F$ and both $\boldsymbol{\Omega}$ and $\nabla$ are coherent, we can apply Lemma 1.29 to both and obtain that $\nabla^{\boldsymbol{B}}(hF) = h\nabla^{\boldsymbol{A}}(F) \subseteq h\boldsymbol{\Omega}^{\boldsymbol{A}}(F') = \boldsymbol{\Omega}^{\boldsymbol{B}}(hF')$. This tells us that $hF' \in [\![hF]\!]^{\nabla^{\boldsymbol{B}}}$.

Now take any $G \in [\![hF]\!]^{\nabla^{\boldsymbol{B}}}$, i.e., such that $\nabla^{\boldsymbol{B}}(hF) \subseteq \boldsymbol{\Omega}^{\boldsymbol{B}}(G)$. We know that $h^{-1}G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and that $hh^{-1}G = G$. Observe that $h$ is $\nabla$-compatible with $h^{-1}hF$, since this is $F$. Then, by coherence, we have

$$\nabla^{\boldsymbol{A}}(F) = \nabla^{\boldsymbol{A}}\big(h^{-1}hF\big) = h^{-1}\nabla^{\boldsymbol{B}}(hF) \subseteq h^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(G) = \boldsymbol{\Omega}^{\boldsymbol{A}}(h^{-1}G).$$

This shows that $h^{-1}G \in [\![F]\!]^{\nabla^{\boldsymbol{A}}}$.

Thus, we have established that $h$ induces a bijection between $[\![F]\!]^{\nabla^{\boldsymbol{A}}}$ and $[\![hF]\!]^{\nabla^{\boldsymbol{B}}}$, whose inverse is given by $h^{-1}$. Since both maps are obviously order preserving, they are in fact order isomorphisms. □

Since order isomorphisms put the least elements of the two ordered sets into correspondence, we obtain:

**Corollary 1.39.** *Under the assumptions of Theorem 1.38, $F$ is a $\nabla$-filter of $\boldsymbol{A}$ if and only if $hF$ is a $\nabla$-filter of $\boldsymbol{B}$.*

Another corollary about $\nabla$-filters, which follows by Theorem 1.38 and will be very useful in Part II, is the following:

**Corollary 1.40.** *Let $\mathcal{S}$ be a logic and $\nabla$ a coherent family of congruential $\mathcal{S}$-compatibility operators. For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

$$F^{\nabla}/\nabla^{\boldsymbol{A}}(F) = \big(F/\nabla^{\boldsymbol{A}}(F)\big)^{\nabla}$$

*and it is the least $\mathcal{S}$-filter of $\boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)$.*

Proof. Let $\pi : \boldsymbol{A} \to \boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)$ be the canonical map. Fix $\boldsymbol{B} := \boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)$. Since $\pi$ is surjective and $\nabla^{\boldsymbol{A}}$-compatible with $F$ (because $\mathrm{Ker} h = \nabla^{\boldsymbol{A}}(F)$), it follows by the General Correspondence Theorem 1.38 that $\pi$ induces an isomorphism between $[\![F]\!]^{\nabla}$ and $[\![\pi F]\!]^{\nabla}$, whose inverse is given by $\pi^{-1}$. As a consequence, since $F^{\nabla}$ is the least element of $[\![F]\!]^{\nabla}$, $\pi(F^{\nabla})$ must be the least element of $[\![\pi F]\!]^{\nabla}$, which is $(\pi F)^{\nabla}$. That is, $\pi(F^{\nabla}) = (\pi F)^{\nabla}$. Finally, notice that $\nabla^{\boldsymbol{B}}(\pi F) = id_{\boldsymbol{B}}$, by Lemma 1.45. So, $[\![\pi F]\!]^{\nabla} = \mathcal{F}i_{\mathcal{S}}(\boldsymbol{B})$. Thus, $(\pi F)^{\nabla}$ is the least $\mathcal{S}$-filter of $\boldsymbol{B}$. □

Given Proposition 1.32, we can apply the General Correspondence Theorem 1.38 to the relativization of an $\mathcal{S}$-operator and obtain:

**Theorem 1.41.** *Let $\nabla$ be a coherent family of $\mathcal{S}$-compatibility operators. For every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $h$ is $\widetilde{\nabla}_{\mathcal{S}}$-compatible with $F$, then $h$ induces an order isomorphism between $[\![F]\!]^{\widetilde{\nabla}_{\mathcal{S}}^{\boldsymbol{A}}}$ and $[\![hF]\!]^{\widetilde{\nabla}_{\mathcal{S}}^{\boldsymbol{B}}}$, whose inverse is given by $h^{-1}$.*

## 1.6. Classes of algebras associated with a family of $\mathcal{S}$-operators

We saw in Lemma 0.36 that the classes of algebras usually associated with a logic through the Leibniz and the Suszko operators can be obtained either by considering reduced models, or by a process of reduction. By analogy, one can apply the first procedure to families of $\mathcal{S}$-operators and the second procedure to families of congruential $\mathcal{S}$-operators.

Throughout this section, we shall assume without any further reference to be dealing with *congruential* $\mathcal{S}$-operators.

**Definition 1.42.** Let $\nabla$ be a family of $\mathcal{S}$-operators. Define

$$\begin{aligned}
\mathsf{Alg}^{\nabla}(\mathcal{S}) &:= \mathbb{I}\{\boldsymbol{A}/\nabla^{\boldsymbol{A}}(F) : \boldsymbol{A} \text{ an algebra}, F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\},\\
\mathsf{Alg}_{\nabla}(\mathcal{S}) &:= \mathbb{I}\{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \nabla^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}}\},\\[1em]
\mathsf{Alg}^{\widetilde{\nabla}_{\mathcal{S}}}(\mathcal{S}) &:= \mathbb{I}\{\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) : \boldsymbol{A} \text{ an algebra}, F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\},\\
\mathsf{Alg}_{\widetilde{\nabla}_{\mathcal{S}}}(\mathcal{S}) &:= \mathbb{I}\{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}}\},\\[1em]
\mathsf{Alg}^{\widetilde{\nabla}}(\mathcal{S}) &:= \mathbb{I}\{\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) : \boldsymbol{A} \text{ an algebra}, \mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\},\\
\mathsf{Alg}_{\widetilde{\nabla}}(\mathcal{S}) &:= \mathbb{I}\{\boldsymbol{A} : \text{there is } \mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = id_{\boldsymbol{A}}\}.
\end{aligned}$$

So, for each family $\nabla$, $\widetilde{\nabla}_{\mathcal{S}}$, and $\widetilde{\nabla}$, we associate two classes of algebras: the class of $\nabla$-*reduced algebras* (respectively, $\widetilde{\nabla}_{\mathcal{S}}$- and $\widetilde{\nabla}$-) and the class of $\nabla$-*reductions* (respectively, $\widetilde{\nabla}_{\mathcal{S}}$- and $\widetilde{\nabla}$-). We have chosen to define all these classes as closed under the operator $\mathbb{I}$, but in fact, having in mind Corollary 1.30, it is easy to see that the reductions' classes could have been defined without it.

**Lemma 1.43.** *The classes*

$$\big\{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \nabla^{\boldsymbol{A}}(F) = id_{\boldsymbol{A}}\big\},$$
$$\big\{\boldsymbol{A} : \text{there is } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}}\big\},$$
$$\big\{\boldsymbol{A} : \text{there is } \mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ such that } \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = id_{\boldsymbol{A}}\big\},$$

*are all closed under isomorphisms.*

Given Lemma 0.36.3, as well as condition (4) on page 136, concerning the Tarski operator (that is, the lifting of the Leibniz operator), let us first observe that the two last classes of algebras in Definition 1.42 can also be given similar characterizations.

**Lemma 1.44.** *Let $\nabla$ be a family of $\mathcal{S}$-operators.*

1. $\mathsf{Alg}^{\widetilde{\nabla}}(\mathcal{S}) = \mathbb{I}\big\{\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) : \boldsymbol{A}$ *an algebra,* $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ $\nabla$*-full*$\big\}$*;*
2. $\mathsf{Alg}_{\widetilde{\nabla}}(\mathcal{S}) = \mathbb{I}\big\{\boldsymbol{A} : \widetilde{\nabla}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = id_{\boldsymbol{A}}\big\}$*;*
3. $\mathsf{Alg}_{\widetilde{\nabla}}(\mathcal{S}) = \mathbb{I}\big\{\boldsymbol{A} :$ *there is a* $\nabla$*-full* $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *such that* $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = id_{\boldsymbol{A}}\big\}$,

PROOF. *1.* The inclusion from right to left is obvious. To prove the converse inclusion, observe that given any $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, by the Galois connection (Proposition 1.6 and related results) the congruence $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$ is a $\nabla$-full congruence and hence there is some $\nabla$-full $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{D}) = \widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})$; therefore, $\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = \boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{D}) \in \mathsf{Alg}^{\widetilde{\nabla}}(\mathcal{S})$.
*2.* The inclusion from right to left is obvious. Conversely, given any $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = id_{\boldsymbol{A}}$, it also holds $\widetilde{\nabla}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = id_{\boldsymbol{A}}$, since $\widetilde{\nabla}$ is order reversing. So, $\mathsf{Alg}_{\widetilde{\nabla}}(\mathcal{S}) \subseteq \mathbb{I}\big\{\boldsymbol{A} : \widetilde{\nabla}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = id_{\boldsymbol{A}}\big\}$.
*3.* The inclusion from right to left is once again obvious. Conversely, just notice that $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is always a $\nabla$-full family of $\mathcal{S}$-filters. So, $\mathsf{Alg}_{\widetilde{\nabla}}(\mathcal{S}) \subseteq \mathbb{I}\big\{\boldsymbol{A}$ : there is a $\nabla$-full $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C}) = id_{\boldsymbol{A}}\big\}$, by 2. □

Our next goal is to see that the two classes of algebras associated with a coherent family of $\mathcal{S}$-compatibility operators coincide; and so do the respective classes associated with its relativization and its lifting. In fact, these last classes all coincide among them. The key point is to see that the "process of $\nabla$-reduction" applied to any model of $\mathcal{S}$ always produces a "$\nabla$-reduced" model.

**Lemma 1.45.** *Let $\nabla$ be a coherent family of $\mathcal{S}$-compatibility operators. For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\theta \in \mathrm{Con}\boldsymbol{A}$, if $\theta \subseteq \nabla^{\boldsymbol{A}}(F)$, then $\nabla^{\boldsymbol{A}/\theta}(F/\theta) = \nabla^{\boldsymbol{A}}(F)/\theta$. In particular,*

$$\nabla^{\boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)}\big(F/\nabla^{\boldsymbol{A}}(F)\big) = id_{\boldsymbol{A}/\nabla^{\boldsymbol{A}}(F)}.$$

PROOF. Consider the canonical projection $\pi : \boldsymbol{A} \to \boldsymbol{A}/\theta$, which is surjective and is $\nabla$-compatible with $F$ by the assumption. Then, by coherence and Lemma 1.29, $\nabla^{\boldsymbol{A}/\theta}\big(F/\theta\big) = \nabla^{\boldsymbol{A}/\theta}\big(\pi F\big) = \pi\nabla^{\boldsymbol{A}}(F) = \nabla^{\boldsymbol{A}}(F)/\theta$. For the last identity, take $\theta = \nabla^{\boldsymbol{A}}(F)$. □

**Proposition 1.46.** *If $\nabla$ is a coherent family of $\mathcal{S}$-compatibility operators, then* $\mathsf{Alg}^{\nabla}\mathcal{S} = \mathsf{Alg}_{\nabla}\mathcal{S}$.

PROOF. The inclusion $\mathsf{Alg}_{\nabla}(\mathcal{S}) \subseteq \mathsf{Alg}^{\nabla}(\mathcal{S})$ holds in general, because $\boldsymbol{A} \cong \boldsymbol{A}/id_{\boldsymbol{A}}$, and the reverse inclusion is a consequence of Lemma 1.45. □

Moreover, we can apply Propositions 1.32 and 1.46 to $\widetilde{\nabla}_{\mathcal{S}}$.

**Corollary 1.47.** *If $\nabla$ is a coherent family of $\mathcal{S}$-compatibility operators, then $\mathsf{Alg}^{\widetilde{\nabla}_{\mathcal{S}}}(\mathcal{S}) = \mathsf{Alg}_{\widetilde{\nabla}_{\mathcal{S}}}(\mathcal{S})$.*

In particular, taking $\nabla = \boldsymbol{\Omega}$ and $\nabla = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$, Proposition 1.46 and Corollary 1.47 yield the equalities $\mathsf{Alg}^{\boldsymbol{\Omega}}(\mathcal{S}) = \mathsf{Alg}_{\boldsymbol{\Omega}}(\mathcal{S}) = \mathsf{Alg}^{*}(\mathcal{S})$, and $\mathsf{Alg}^{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}}(\mathcal{S}) = \mathsf{Alg}_{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}}(\mathcal{S}) = \mathsf{Alg}^{\mathrm{Su}}(\mathcal{S})$, respectively, which are already known (Lemma 0.36).

The proofs of the next two results are completely analogous, modulo Lemma 1.33, to those of Lemma 1.45 and Proposition 1.46, respectively.

**Lemma 1.48.** *Let $\nabla$ be a coherent family of $\mathcal{S}$-compatibility operators. For every $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

$$\widetilde{\nabla}^{\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})}\big(\mathscr{C}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})\big) = id_{\boldsymbol{A}/\widetilde{\nabla}^{\boldsymbol{A}}(\mathscr{C})}.$$

**Proposition 1.49.** *If $\nabla$ is a coherent family of $\mathcal{S}$-compatibility operators, then $\mathsf{Alg}^{\widetilde{\nabla}}(\mathcal{S}) = \mathsf{Alg}_{\widetilde{\nabla}}(\mathcal{S})$.*

Finally, we arrive at:

**Proposition 1.50.** *If $\nabla$ is a coherent family of $\mathcal{S}$-compatibility operators, then $\mathsf{Alg}^{\widetilde{\nabla}}(\mathcal{S}) = \mathsf{Alg}_{\widetilde{\nabla}}(\mathcal{S}) = \mathsf{Alg}^{\widetilde{\nabla}_{\mathcal{S}}}(\mathcal{S}) = \mathsf{Alg}_{\widetilde{\nabla}_{\mathcal{S}}}(\mathcal{S})$.*

PROOF. By definition, for each $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\nabla}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}\big)$. From this it follows that $\mathsf{Alg}_{\widetilde{\nabla}_{\mathcal{S}}}(\mathcal{S}) \subseteq \mathsf{Alg}_{\widetilde{\nabla}}(\mathcal{S})$ and that $\mathsf{Alg}^{\widetilde{\nabla}_{\mathcal{S}}}(\mathcal{S}) \subseteq \mathsf{Alg}^{\widetilde{\nabla}}(\mathcal{S})$. To see the reverse inclusion in the first case, assume that $\boldsymbol{A} \in \mathsf{Alg}_{\widetilde{\nabla}}(\mathcal{S})$. By Lemma 1.44.2, $\widetilde{\nabla}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = id_{\boldsymbol{A}}$. But, fixing $F_0 := \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\widetilde{\nabla}^{\boldsymbol{A}}_{\mathcal{S}}(F_0) = \widetilde{\nabla}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F_0}\big) = \widetilde{\nabla}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = id_{\boldsymbol{A}}$. Therefore, $\boldsymbol{A} \in \mathsf{Alg}_{\widetilde{\nabla}_{\mathcal{S}}}(\mathcal{S})$. Given Corollary 1.47, we are done. □

Since, by Proposition 1.32, coherence is preserved through relativization, it is legitime to apply the results just proved for a coherent $\nabla$ to $\widetilde{\nabla}_{\mathcal{S}}$, and in particular, to consider the classes of algebras associated to the lifting of $\widetilde{\nabla}_{\mathcal{S}}$ and to its relativization to $\mathcal{S}$. But, since $\widetilde{\nabla}_{\mathcal{S}}$ is an order preserving $\mathcal{S}$-operator, in view of Lemma 1.4, the relativization of $\widetilde{\nabla}_{\mathcal{S}}$ is $\widetilde{\nabla}_{\mathcal{S}}$ itself. Therefore, its associated classes of algebras would still be the class $\mathsf{Alg}^{\widetilde{\nabla}}(\mathcal{S})$.

So, given a coherent family of congruential $\mathcal{S}$-compatibility operators, the classes of algebras in Definition 1.42 collapse into just two, namely $\mathsf{Alg}^{\nabla}(\mathcal{S})$ and $\mathsf{Alg}^{\widetilde{\nabla}_{\mathcal{S}}}(\mathcal{S})$. The fact that these classes coincide or not, will be relevant in some results to come — see Proposition 3.4 or Lemma 3.3, for instance.

CHAPTER 2

# The Leibniz, Suszko and Frege operators

## 2.1. The Leibniz operator as an $\mathcal{S}$-compatibility operator

Among all the $\mathcal{S}$-operators we shall consider, the Leibniz operator is by far the most well studied one. As we will soon see, instantiating the results of Chapter 1 with $\nabla = \boldsymbol{\Omega}$ yields both new and familiar notions. But since these later have already well-settled notations and terminology — for instance, $\mathsf{Alg}^*(\mathcal{S})$ — we shall, from now on, write $(\ )^*$ instead of $(\ )^{\boldsymbol{\Omega}}$ in all supscripts concerning the Leibniz operator.

We start our study of this famous operator by viewing it just as an $\mathcal{S}$-operator. From this assumption alone, we will see that some powerful consequences already arise as by-products of the Galois connection established in Proposition 1.6. We then proceed to view the Leibniz operator in its full extension, that is, as a congruential $\mathcal{S}$-compatibility operator.

As already remarked, the lifting of the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is the familiar Tarski operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}$. As to the map ${\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}$, observe that if $\theta \in \operatorname{Con}\boldsymbol{A}$, then

$$\begin{aligned}{\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}(\theta) &= \big\{F \in \mathcal{F}i_{\mathcal{S}}(\mathbf{A}) : \theta \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big\}\\ &= \big\{F \in \mathcal{F}i_{\mathcal{S}}(\mathbf{A}) : \theta \text{ is compatible with } F\big\}.\end{aligned}$$

Let us first characterize the $\boldsymbol{\Omega}$-full objects in terms of some well-known concepts.

**Proposition 2.1.** *A set $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is $\boldsymbol{\Omega}$-full if and only if it is a full g-model of $\mathcal{S}$.*

Proof. It holds, ${\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}\big(\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})\big) = \big\{G \in \mathcal{F}i_{\mathcal{S}}(A) : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}$. Now, by definition 1.8, $\mathscr{C}$ is $\boldsymbol{\Omega}$-full when it equals the left-hand side of the equality; and by definition it is a full g-model of $\mathcal{S}$ when it equals the right-hand side. □

**Proposition 2.2.** *A congruence $\theta \in \operatorname{Con}\boldsymbol{A}$ is $\boldsymbol{\Omega}$-full if and only if $\theta \in \operatorname{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$.*

Proof. By instantiating Proposition 1.37 with the Leibniz operator, and having in mind that this $\mathcal{S}$-operator commutes with inverse images by surjective homomorphisms, we have that $\theta$ is $\boldsymbol{\Omega}$-full if and only if $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}/\theta}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big) = id_{\boldsymbol{A}/\theta}$, which is equivalent to $\boldsymbol{A}/\theta \in \mathsf{Alg}(\mathcal{S})$ by condition (4) on page 136, and equivalent to $\theta \in \operatorname{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$. □

The two preceding results allow us to instantiate Proposition 1.6 and Corollary 1.7.6 in a more familiar form.

**Corollary 2.3.** *The maps $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}$ and ${\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}$ establish a Galois connection between $\mathscr{P}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$ and $\operatorname{Eq}_{\mathcal{L}}(A)$ and restrict to mutually inverse dual order isomorphisms between the poset of all full g-models of $\mathcal{S}$ on $\boldsymbol{A}$ and the poset $\operatorname{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$.*

The second part of this statement is the well-known Isomorphism Theorem [**36**, Theorem 2.30]. We see that it arises here naturally as a by-product of the Galois connection established in Proposition 1.6, taking $\nabla = \boldsymbol{\Omega}$. Finally, from Propositions 1.9 and 2.1, we get the following characterization of the full g-models of $\mathcal{S}$.

**Proposition 2.4.** *A subset $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a full g-model of $\mathcal{S}$ if and only if $\mathscr{C}$ is the largest $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{D})$.*

To finish our study of the Leibniz operator as an $\mathcal{S}$-operator, we state a most crucial fact, originally proved in [**22**, Theorem 1.26], which within our framework follows as an immediate consequence of Lemma 1.4 and Definition 0.38.

**Proposition 2.5.** *A logic $\mathcal{S}$ is protoalgebraic if and only if the Leibniz and the Suszko operators coincide.*

Therefore, when dealing with protoalgebraic logics, all pairs of notions associated with the Leibniz and Suszko operators, such as those of $\boldsymbol{\Omega}$- and $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-classes, those of $\boldsymbol{\Omega}$- and $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-filters, and the respective associated classes of algebras; in particular, Proposition 2.5 directly implies that $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$.

We now introduce the notions of $\boldsymbol{\Omega}$-class and $\boldsymbol{\Omega}$-filter. As we will see, these concepts will play an important rôle in our study. Recall, by Definition 1.12, the $\boldsymbol{\Omega}$-class of $F$, which we shall also call *the Leibniz class of $F$*, is defined by

$$[\![F]\!]^* := {\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}\big(\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big) = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}.$$

By Definition 1.15, $F^*$ denotes the least element of the Leibniz class $[\![F]\!]^*$; we shall call this element *the Leibniz filter of $F$*. We say that $F$ is a *Leibniz filter* if $F = F^*$, and we denote the set of all Leibniz filters of $\boldsymbol{A}$ by $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$. This is the same notation used in [**37**] for protoalgebraic logics.

A clarification is in order here. Leibniz filters were originally introduced in [**37**], within the scope of protoalgebraic logics, as the least elements of the class

$$[F] := \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\} \subseteq [\![F]\!]^*.$$

Indeed, [**37**, Definition 1] is preceded by: "(...) it makes sense to single out a special element of each equivalence class under the kernel of $\boldsymbol{\Omega}^{\boldsymbol{A}}$, namely its least element." In fact, these equivalence classes had already been pointed out in [**36**, p. 59] and explicitly considered in [**25**, p. 650]. Leibniz filters were also studied in [**51**], namely its definability with parameters, but once again within the protoalgebraic setting. Our present definition of Leibniz filter generalizes the former one, as we next prove (Lemma 2.6), in the sense that both definitions coincide for protoalgebraic logics. Furthermore, as we shall see, every Leibniz filter according to our new definition is also a Leibniz filter according to [**37**, Definition 1] (if we apply it to arbitrary logics). But a word of advice must be taken with respect to [**38**, p. 177]. There, a generalization of Leibniz filters for arbitrary logics is also proposed, namely: *an $\mathcal{S}$-filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is Leibniz, if it is the least element of the class $[F]$*. Despite the fact that $[F] \subseteq [\![F]\!]^*$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, this notion does not coincide, in general, with the present one. Indeed, while in general the least element of $[F]$ does not necessarily exist, the least element of $[\![F]\!]^*$ always does.

**Lemma 2.6.** *For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

1. *$F^* \subseteq \bigcap [F] \subseteq F$;*

2. *if $F = F^*$, then $F = \bigcap[F]$;*
3. *if $\mathcal{S}$ is protoalgebraic, then $F = F^*$ (i.e., $F$ is a Leibniz filter) if and only if $F = \bigcap[F]$.*

Proof. Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since $F \in [F]$, it holds $\bigcap[F] \subseteq F$. Moreover, it is clear that $[F] \subseteq [\![F]\!]^*$. Therefore, $F^* = \bigcap[\![F]\!]^* \subseteq \bigcap[F]$. From (i), (ii) follows immediately. Now to prove (iii) assume that $\mathcal{S}$ is protoalgebraic and $F = \bigcap[F]$. Then, since $F^* \subseteq F$ it follows by order preservation of $\boldsymbol{\Omega}^{\boldsymbol{A}}$ that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F^*) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, and since $F^* \in [\![F]\!]^*$, it must also hold $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$. So, $F^* \in [F]$, and hence $F = \bigcap[F] \subseteq F^*$. □

Since the Leibniz operator is a congruential $\mathcal{S}$-operator, we know by Proposition 1.14 that $\boldsymbol{\Omega}$-classes are full g-models of $\mathcal{S}$. But the fact that it is furthermore an $\mathcal{S}$-compatibility operator entails a deeper connection between these classes and Leibniz congruences.

**Proposition 2.7.** *For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $[\![F]\!]^*$ is a full g-model of $\mathcal{S}$. Moreover,*

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big([\![F]\!]^*\big) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F). \tag{16}$$

Proof. By Proposition 1.14, taking $\nabla^{\boldsymbol{A}} = \boldsymbol{\Omega}^{\boldsymbol{A}}$, it follows that $[\![F]\!]^*$ is a full g-model of $\mathcal{S}$. Now, on the one hand, since $F \in [\![F]\!]^*$, it holds $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big([\![F]\!]^*\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. On the other hand, for every $G \in [\![F]\!]^*$, it holds $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Therefore, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \bigcap_{G \in [\![F]\!]^*} \boldsymbol{\Omega}^{\boldsymbol{A}}(G) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big([\![F]\!]^*\big)$. □

Recall that, in general, $F^{\nabla}$ need not be a $\nabla$-filter of $\boldsymbol{A}$. The Leibniz filters of $\boldsymbol{A}$, however, are indeed the $\mathcal{S}$-filters of the form $F^*$, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

**Proposition 2.8.** *For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^*$ is a Leibniz filter of $\boldsymbol{A}$.*

Proof. The inclusion $(F^*)^* \subseteq F^*$ follows by Lemma 1.21.2. As for the converse inclusion, since $F^* \in [\![F]\!]^*$, it follows that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$, and hence $[\![F^*]\!]^* \subseteq [\![F]\!]^*$. Thus, $F^* = \bigcap[\![F]\!]^* \subseteq \bigcap[\![F^*]\!]^* = (F^*)^*$. □

Taking Lemma 2.6.3 into account, [**36**, Proposition 3.6] tells us that: *For every protoalgebraic logic $\mathcal{S}$, an $\mathcal{S}$-filter is a Leibniz filter if and only if it is the least element of some full g-model of $\mathcal{S}$.* We can now see that this remains true for arbitrary logics if we replace the notion of Leibniz filter of [**36, 37**] by the present one.

**Proposition 2.9.** *An $\mathcal{S}$-filter $F$ of $\boldsymbol{A}$ is a Leibniz filter if and only if there exists a full g-model $\langle \boldsymbol{A}, \mathscr{C} \rangle$ of $\mathcal{S}$ such that $F = \bigcap \mathscr{C}$.*

Proof. Suppose $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a Leibniz filter. It is, by definition, the least element of its Leibniz class, which we have seen to be a full g-model of $\mathcal{S}$ in Proposition 2.7. Conversely, suppose $F = \bigcap \mathscr{C}$ and $\langle \boldsymbol{A}, \mathscr{C} \rangle$ is a full g-model of $\mathcal{S}$. Since $\bigcap \mathscr{C} \in \mathscr{C}$, it holds $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Hence, $[\![F]\!]^* = {\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}(\boldsymbol{\Omega}^{\boldsymbol{A}}(F)) \subseteq {\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}(\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})) = \mathscr{C}$, where the last equality follows by Proposition 2.1. Thus, $F = \bigcap \mathscr{C} \subseteq \bigcap[\![F]\!]^* = F^*$. Since the converse inclusion always holds, it follows $F = F^*$, i.e., $F$ is a Leibniz filter of $\boldsymbol{A}$. □

Instantiating Proposition 1.18 for the Leibniz operator we obtain the next proposition; one can see that it generalizes [**37**, Proposition 10] and [**36**, Proposition 3.6 (iii)], again taking Lemma 2.6.3 into account.

**Proposition 2.10.** *A filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a Leibniz filter of $\boldsymbol{A}$ if and only if $F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$.*

In case every $\mathcal{S}$-filter turns out to be a Leibniz filter, we have:

**Proposition 2.11.** *The Leibniz operator is order reflecting if and only if, for every $\boldsymbol{A}$, every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Leibniz filter of $\boldsymbol{A}$.*

Proof. The Leibniz operator is order reflecting iff, for every $\boldsymbol{A}$ and every $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$ it holds $F \subseteq G$, iff for every $\boldsymbol{A}$ and every $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $G \in [\![F]\!]^*$ it holds $F \subseteq G$, iff, for every $\boldsymbol{A}$, every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Leibniz filter of $\boldsymbol{A}$. □

Next, we apply the General Correspondence Theorem 1.38 to the Leibniz operator.

**Theorem 2.12** (Correspondence Theorem for Leibniz classes)**.** *For every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $h$ is $\boldsymbol{\Omega}$-compatible with $F$, then $h$ induces an order isomorphism between $[\![F]\!]^*$ and $[\![hF]\!]^*$, whose inverse is given by $h^{-1}$. Moreover, for each $G \in [\![F]\!]^*$, $h$ induces an order isomorphism between $[G]$ and $[hG]$.*

Proof. Since the Leibniz operator is a coherent family of $\mathcal{S}$-compatibility operators, we can apply Theorem 1.38 to it, and obtain the first part of the statement. For the second part, take any $G, H \in [\![F]\!]^*$; note that from $G \in [\![F]\!]^*$ it follows that $[G] \subseteq [\![F]\!]^*$. By the established isomorphism, $h^{-1}hG = G$ and $h^{-1}hH = H$. Now, using Proposition 0.31.1 and the surjectivity of $h$,

$$\begin{aligned} \boldsymbol{\Omega}^{\boldsymbol{A}}(H) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G) \;\text{ iff }\; & \boldsymbol{\Omega}^{\boldsymbol{A}}(h^{-1}hH) = \boldsymbol{\Omega}^{\boldsymbol{A}}(h^{-1}hG) \\ \text{iff }\; & h^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(hH) = h^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(hG) \\ \text{iff }\; & \boldsymbol{\Omega}^{\boldsymbol{B}}(hH) = \boldsymbol{\Omega}^{\boldsymbol{B}}(hG), \end{aligned}$$

which shows that $H \in [G]$ if and only if $hF \in [hG]$. Thus, the order isomorphism induced by $h$ between $[\![F]\!]^*$ and $[\![hF]\!]^*$ restricts to one between $[G]$ and $[hG]$. □

It is not difficult to see that $[\![F]\!]^* = \bigcup_{G \in [\![F]\!]^*}[G]$, that is, the sets $[G]$ divide the Leibniz class $[\![F]\!]^*$ into disjoint "layers" according to the value of the Leibniz operator. Thus, the second part of Theorem 2.12 is telling us that the isomorphism between the two Leibniz classes is the disjoint union of isomorphisms, one for each corresponding pair of "layers".

**Corollary 2.13.** *Under the assumptions of Theorem 2.12, $F$ is a Leibniz filter of $\boldsymbol{A}$ if and only if $hF$ is a Leibniz filter of $\boldsymbol{B}$.*

Bearing in mind Lemma 1.26, Corollary 2.13 can be re-stated as (this alternative formulation will be useful in Part II):

**Proposition 2.14.** *Let $h : \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ be a strict surjective matrix homomorphism. Then $F \in \mathcal{F}i^*_{\mathcal{S}}(\boldsymbol{A})$ if and only if $G \in \mathcal{F}i^*_{\mathcal{S}}(\boldsymbol{B})$.*

Theorem 2.12 generalizes and strengthens the well-known Correspondence Theorem for protoalgebraic logics, as formulated in [**12**, Corollary 7.7], and its strengthening given in [**37**, Corollary 9]. Indeed,

**Theorem 2.15** (Correspondence Theorem for protoalgebraic logics)**.** *A logic $\mathcal{S}$ is protoalgebraic if and only if every strict surjective matrix homomorphism between $\mathcal{S}$-models $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ induces an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^G$, whose inverse is given by $h^{-1}$.*

Proof. Assume $\mathcal{S}$ is protoalgebraic. If $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ is strict and surjective, then $F = h^{-1}G$ and $G = hh^{-1}G = hF$, so that $F = h^{-1}hF$. This means that, viewed as an algebraic homomorphism, $h$ is $\boldsymbol{\Omega}$-compatible with $F$. Therefore, we can apply Theorem 2.12 to obtain that $h$ induces an order isomorphism between $[\![F]\!]^*$ and $[\![G]\!]^*$, with inverse given by $h^{-1}$. This isomorphism restricts to an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^G$, because by protoalgebraicity, these up-sets are contained in $[\![F]\!]^*$ and $[\![G]\!]^*$, respectively, and $F$ and $G$ correspond to each other under $h$ and $h^{-1}$. The converse implication would be proved as in [23], i.e., by showing that the stated condition easily implies that the Leibniz operator is order preserving. □

Let us now address the question of which full g-models have the form of a Leibniz class.

**Proposition 2.16.** *Let $\langle \boldsymbol{A}, \mathscr{C}\rangle$ be a full g-model of $\mathcal{S}$. The following conditions are equivalent:*

*(i)* $\mathscr{C} = [\![F]\!]^*$*, for some* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$*;*
*(ii)* $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \in \mathsf{Alg}^*(\mathcal{S})$*.*

Proof. Suppose $\mathscr{C} = [\![F]\!]^*$, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}([\![F]\!]^*) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, by Proposition 2.7, and therefore $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \in \mathsf{Alg}^*(\mathcal{S})$. Conversely, suppose $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \in \mathsf{Alg}^*(\mathcal{S})$. Fix $\boldsymbol{B} := \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})$. By the assumption, there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ such that $\boldsymbol{\Omega}^{\boldsymbol{B}}(G) = id_{\boldsymbol{B}}$. This implies that $[\![G]\!]^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Now, let $\pi\colon \boldsymbol{A} \to \boldsymbol{B}$ be the canonical projection. Since $\mathscr{C}$ is full by assumption, $\mathscr{C} = \pi^{-1}\mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Thus, $\mathscr{C} = \pi^{-1}[\![G]\!]^*$. Finally, $\mathrm{Ker}\pi = \pi^{-1}id_{\boldsymbol{B}} = \pi^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(G) = \boldsymbol{\Omega}^{\boldsymbol{A}}(\pi^{-1}G)$, therefore $\pi$ is $\boldsymbol{\Omega}^{\boldsymbol{A}}$-compatible with $\pi^{-1}G$. Now we can apply the Correspondence Theorem 2.12 for Leibniz classes and conclude that $\mathscr{C} = \pi^{-1}[\![G]\!]^* = [\![\pi^{-1}G]\!]^*$. That is, $F := \pi^{-1}G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ witnesses the desired property. □

What happens then if every full g-model of a logic $\mathcal{S}$ is of the form $[\![F]\!]^*$, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$?

**Proposition 2.17.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i)* *For every $\boldsymbol{A}$, the family of full g-models of $\mathcal{S}$ on $\boldsymbol{A}$ is* $\big\{[\![F]\!]^* : F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\}$*;*
*(ii)* $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S})$*.*

Proof. $(i) \Rightarrow (ii)$: The inclusion $\mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{S})$ holds in general. As for the converse inclusion, let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$ and let $F \in \boldsymbol{A}$ witness this fact, i.e., $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}}$. Now, by assumption, every full g-model of $\mathcal{S}$ is of the form of some Leibniz class. In particular, since Suszko classes are full g-models, it follows that for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $[\![F]\!]^{\mathrm{Su}} = [\![G]\!]^*$. Now, on the one hand, since $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \subseteq [\![F]\!]^{\mathrm{Su}}$, it follows that $\boldsymbol{\Omega}^{\boldsymbol{A}}(G) \subseteq \bigcap_{F' \supseteq F} \boldsymbol{\Omega}^{\boldsymbol{A}}(F') = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. On the other hand, since $G \in [\![G]\!]^*$, it follows that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(G) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}}$, and therefore $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$.
$(ii) \Rightarrow (i)$: If $\mathscr{C}$ is full, then $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \in \mathsf{Alg}(\mathcal{S})$. From the assumption it follows

that $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \in \mathsf{Alg}^*(\mathcal{S})$, and from Proposition 2.16 that $\mathscr{C} = [\![F]\!]^*$, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. ☐

Proposition 2.17 allows us to prove a suggestive characterization of the condition $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S})$ in terms of the Leibniz and Suszko operators, which is readily seen as a weaker property than protoalgebraicity (as depicted in Figure 1).

**Proposition 2.18.** *Let $\mathcal{S}$ be a logic. It holds $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S})$ if and only if, for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$.*

Proof. Suppose $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S})$. It follows by Proposition 2.17 that every full g-model of $\mathcal{S}$ is of the form of some Leibniz class. In particular, since Suszko classes are full g-models of $\mathcal{S}$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $[\![F]\!]^{\mathrm{Su}} = [\![G]\!]^*$. But then, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}([\![F]\!]^{\mathrm{Su}}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}([\![G]\!]^*) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$, as desired. As for the converse, let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$. Since $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^{\mathrm{Su}}(\mathcal{S})$, there is $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{A}}$. It follows by hypothesis that there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G) = id_{\boldsymbol{A}}$. Thus, $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$. This establishes that $\mathsf{Alg}(\mathcal{S}) \subseteq \mathsf{Alg}^*(\mathcal{S})$; the converse inclusion always holds. ☐

Compare Proposition 2.18 with the following rephrasing of Proposition 2.5: *$\mathcal{S}$ is protoalgebraic if and only if, for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$.* Given Proposition 2.17, we get as an immediate corollary:

**Corollary 2.19.** *If a logic $\mathcal{S}$ is protoalgebraic, then every full g-model of $\mathcal{S}$ is of the form $[\![F]\!]^*$, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and some algebra $\boldsymbol{A}$.*

In order to get the converse implication, one must impose $[\![F]\!]^*$ to be precisely the up-set on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ generated by $F^*$.

**Theorem 2.20.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) $\mathcal{S}$ is protoalgebraic.*
*(ii) Every full g-model of $\mathcal{S}$ is of the form $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, for some $\mathcal{S}$-filter $F$ of some algebra $\boldsymbol{A}$;*
*(iii) Every full g-model of $\mathcal{S}$ is of the form $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, for some Leibniz $\mathcal{S}$-filter $F$ of some algebra $\boldsymbol{A}$;*
*(iv) $[\![F]\!]^* = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F^*}$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every algebra $\boldsymbol{A}$.*

Proof. $(i) \Rightarrow (ii)$: Let $\langle \boldsymbol{A}, \mathscr{C}\rangle$ be a full g-model of $\mathcal{S}$. So, $\mathscr{C} = \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\}$. Since by the assumption the Leibniz operator is order preserving, it trivially follows that $\mathscr{C}$ is an up-set. Since $\mathscr{C}$ is moreover a closure system, it must be of the form $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, for some $\mathcal{S}$-filter $F$ of $\boldsymbol{A}$, namely $F = \bigcap \mathscr{C}$.

$(ii) \Leftrightarrow (iii)$: This should be clear, given Proposition 2.9.

$(iii) \Rightarrow (iv)$: Clearly, since for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\boldsymbol{A}$, $[\![F]\!]^*$ is a full g-model of $\mathcal{S}$, by Proposition 2.7; $F^*$ is a Leibniz filter of $\boldsymbol{A}$, by Proposition 2.8; and $F^* = \bigcap [\![F]\!]^*$ by definition.

$(iv) \Rightarrow (i)$: Let $\boldsymbol{A}$ be an algebra and let $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$. Then, $F^* \subseteq F \subseteq G$. It follows by hypothesis that $G \in [\![F]\!]^*$. So, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Thus, the Leibniz operator is order preserving on every $\boldsymbol{A}$, and this shows that $\mathcal{S}$ is protoalgebraic. ☐

Notice that we can replace Suszko filter by Leibniz filter in condition (iii). The preceding result extends [**36**, Theorem 3.4], which proves only the equivalence between items (i) and (ii).

It is interesting to compare Proposition 2.20, which characterizes protoalgebraic logics in terms of their $\boldsymbol{\Omega}$-classes, with Corollary 2.31, which characterizes truth-equational logics in terms of their $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-classes.

## 2.2. The Suszko operator as an $\mathcal{S}$-compatibility operator

We now undertake a similar study to the one done in Section 2.1, this time for the Suszko operator. Once again, since the notation $\mathsf{Alg}^{\mathrm{Su}}(\mathcal{S})$ is already well settled in AAL, we shall change all superscripts $(\ )^{\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}}$ to $(\ )^{\mathrm{Su}}$.

Unlike the Leibniz operator, the Suszko operator viewed only as an $\mathcal{S}$-operator yields poor results. For instance, the Galois connection established in Proposition 1.6, in the absence of meaningful characterizations of the $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-full objects, has little consequences.

Let us start by instantiating the notions of $\nabla$-class and $\nabla$-filter for the Suszko operator. Recall that, by Definition 1.12, the $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-class of $F$, which we shall also call *the Suszko class of* $F$, is defined by

$$[\![F]\!]^{\mathrm{Su}} \coloneqq {\boldsymbol{\Omega}^{\boldsymbol{A}}}^{-1}\big(\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big) = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}.$$

By Definition 1.15, $F^{\mathrm{Su}}$ denotes the least element of the Suszko class $[\![F]\!]^{\mathrm{Su}}$. We say that $F$ is a *Suszko filter* if $F = F^{\mathrm{Su}}$, and we denote the set of all Suszko filters of $\boldsymbol{A}$ by $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$.

We next collect some useful basic facts concerning $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-classes and $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-filters, all of them either particular cases, or straightforward consequences, of Lemmas 1.21 and 1.22.

**Lemma 2.21.** *Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then,*

1. *$F^{\mathrm{Su}} \subseteq F^{*} \subseteq F$;*
2. *Every Suszko filter is a Leibniz filter;*
3. *if $F \subseteq G$, then $[\![G]\!]^{\mathrm{Su}} \subseteq [\![F]\!]^{\mathrm{Su}}$ and $F^{\mathrm{Su}} \subseteq G^{\mathrm{Su}}$;*
4. *$(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F} \subseteq [\![F]\!]^{\mathrm{Su}} \subseteq (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F^{\mathrm{Su}}}$;*
5. *$[\![F]\!]^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{F}$ if and only if $F = F^{\mathrm{Su}}$, i.e., if and only if $F$ is a Suszko filter.*

Proof. *1.* The first inclusion holds because $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, so $[\![F]\!]^{*} \subseteq [\![F]\!]^{\mathrm{Su}}$, and therefore $F^{\mathrm{Su}} = \bigcap[\![F]\!]^{\mathrm{Su}} \subseteq \bigcap[\![F]\!]^{*} = F^{*}$; the second inclusion holds because $F \in [\![F]\!]^{*}$. *2.* It follows by Lemma 1.22.3, taking $\nabla = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$. *3.* By monotonicity of the Suszko operator we have $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(G)$, so $[\![G]\!]^{\mathrm{Su}} \subseteq [\![F]\!]^{\mathrm{Su}}$, and therefore $F^{\mathrm{Su}} = \bigcap[\![F]\!]^{\mathrm{Su}} \subseteq \bigcap[\![G]\!]^{\mathrm{Su}} = G^{\mathrm{Su}}$. *4.* The first inclusion follows by Lemma 1.21.3, taking $\nabla = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$; the second inclusion follows by the fact $F^{\mathrm{Su}} = \bigcap[\![F]\!]^{\mathrm{Su}}$. *5.* By Lemma 1.21.4, taking $\nabla = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$. ☐

Suszko classes will be once again full g-models of $\mathcal{S}$, because the Suszko operator is a congruential $\mathcal{S}$-operator. The fact that it is furthermore an $\mathcal{S}$-compatibility operator allows us to establish a nice connection between Suszko classes and Suszko congruences, very much in the same spirit as the one established in (16) for the Leibniz case.

**Proposition 2.22.** *For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $[\![F]\!]^{\mathrm{Su}}$ is a full g-model of $\mathcal{S}$. Moreover,*

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big([\![F]\!]^{\mathrm{Su}}\big) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\,. \qquad (17)$$

Proof. By Proposition 1.14, taking $\nabla^{\boldsymbol{A}} = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$, it follows that $[\![F]\!]^{\mathrm{Su}}$ is a full g-model of $\mathcal{S}$. Now, on the one hand, since $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \subseteq [\![F]\!]^{\mathrm{Su}}$, by Lemma 2.21.4, it holds $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big([\![F]\!]^{\mathrm{Su}}\big) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. On the other hand, for every $G \in [\![F]\!]^{\mathrm{Su}}$, it holds $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Therefore, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \bigcap_{G\in[\![F]\!]^{\mathrm{Su}}} \boldsymbol{\Omega}^{\boldsymbol{A}}(G) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big([\![F]\!]^{\mathrm{Su}}\big)$. □

The similarity of (17) and (16) on page 47, reinforces the parallelism between the Leibniz and the Suszko operators under our general treatment of $\mathcal{S}$-compatibility operators. This parallelism however also has its downfalls. For example, unlike the case of $F^*$, which always is a Leibniz filter, $F^{\mathrm{Su}}$ needs not be a Suszko filter in general. The following example witnessing this fact is due to Tommaso Moraschini.

**Example 2.23.** Consider the language $\mathcal{L} = \langle \Box, \Diamond, c_1, c_2, c_3, \top \rangle$, where $\Box$ and $\Diamond$ are unary function symbols and $c_1, c_2, c_3, \top$ are constant symbols. Consider the set $A = \{a, b, c, d, 1\}$ and the $\mathcal{L}$-algebra $\boldsymbol{A} = \langle A, \Box^{\boldsymbol{A}}, \Diamond^{\boldsymbol{A}}, a, b, d, 1 \rangle$, where the unary operations $\Box^{\boldsymbol{A}}$ and $\Diamond^{\boldsymbol{A}}$ are given by the table below. Consider also the logic $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$ defined by the calculus with axiom and rules displayed below ($x$ is a variable).

| | $\Box^{\boldsymbol{A}}$ | $\Diamond^{\boldsymbol{A}}$ |
|---|---|---|
| $a$ | $a$ | $c$ |
| $b$ | $b$ | $1$ |
| $c$ | $d$ | $d$ |
| $d$ | $d$ | $1$ |
| $1$ | $a$ | $d$ |

Axiom: $\top$

Rule 1: $c_1, c_2 \vdash_{\mathcal{S}} x$

Rule 2: $c_2, c_3 \vdash_{\mathcal{S}} x$

*Fact 1.* Clearly, the proper $\mathcal{S}$-filters of $\boldsymbol{A}$ are the subsets containing 1, not containing $a, b$ simultaneously, and not containing $b, d$ simultaneously. In particular, the set $F := \{1, b, c\}$ is an $\mathcal{S}$-filter of $\boldsymbol{A}$.

*Fact 2.* $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F = \{F, A\}$, because the only proper subsets of $A$ containing $F$ are $\{1, a, b, c\}$ and $\{1, b, c, d\}$, but neither is an $\mathcal{S}$-filter of $\boldsymbol{A}$ by the observation in Fact 1.

*Fact 3.* $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \big\{\{1, c\}, \{a, d\}, \{b\}\big\}$, where for simplicity a congruence is described by its partition. One can check by hand that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \big\{\{1, c\}, \{a, d\}, \{b\}\big\}$. The other equality follows by Fact 2, which implies that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \cap \boldsymbol{\Omega}^{\boldsymbol{A}}(A) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$.

*Fact 4.* $F^{\mathrm{Su}} = \{1, c\}$. To see this, first observe that $[\![F]\!]^{\mathrm{Su}} = [\![F]\!]^* = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}$, which is a direct consequence of Fact 3. But to say that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$ is to say that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is compatible with $G$ or, by Lemma 0.16, that $G$ is a union of blocks of $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Using the description of $\mathcal{S}$-filters in Fact 1 and the description of the blocks of $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ in Fact 3, we conclude that $[\![F]\!]^{\mathrm{Su}} = \big\{\{1, c\}, F, \{1, a, c, d\}, A\big\}$. From this it follows that $F^{\mathrm{Su}} = \{1, c\}$, as claimed.

*Fact 5.* $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F^{\mathrm{Su}}) = id_{\boldsymbol{A}}$. This is because $F$ and $\{1, a, c\}$ are two $\mathcal{S}$-filters of $\boldsymbol{A}$, which contain $F^{\mathrm{Su}} = \{1, c\}$, and it is easy to check that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \cap \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\{1, a, c\}\big) = id_{\boldsymbol{A}}$, using just compatibility arguments and the fact that for any congruence $\equiv$ of this algebra, $1 \equiv c$ if and only if $a \equiv d$.

*Fact 6.* $(F^{\mathrm{Su}})^{\mathrm{Su}} = \{1\}$. It follows by Fact 5 that $[\![F^{\mathrm{Su}}]\!]^{\mathrm{Su}} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Thus, $(F^{\mathrm{Su}})^{\mathrm{Su}} = \min [\![F^{\mathrm{Su}}]\!]^{\mathrm{Su}} = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \{1\}$.

We conclude that $F^{\mathrm{Su}} \neq (F^{\mathrm{Su}})^{\mathrm{Su}}$. That is, $F^{\mathrm{Su}}$ is not a Suszko filter of $\boldsymbol{A}$. ☐

As a consequence, the converse of the implication in Lemma 2.21.2 is false, for $F^{\mathrm{Su}}$ is always a Leibniz filter (by Proposition 2.9 and Proposition 2.22, for it is the least element of the full g-model $[\![F]\!]^{\mathrm{Su}}$), and Example 2.23 exhibits one such $F^{\mathrm{Su}}$ which is not a Suszko filter. However, in case $F^{\mathrm{Su}}$ is indeed a Suszko filter of $\boldsymbol{A}$, then it is the largest one below $F$. In order to see it, we first instantiate Lemma 1.17 with the Suszko operator:

**Lemma 2.24.** *For every $\boldsymbol{A}$, if $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ are such that $F \subseteq G$, then $F^{\mathrm{Su}} \subseteq G^{\mathrm{Su}}$.*

**Lemma 2.25.** *If $F^{\mathrm{Su}}_{\mathcal{S}}$ is a Suszko filter of $\boldsymbol{A}$, then it is the largest Suszko $\mathcal{S}$-filter of $\boldsymbol{A}$ below $F$.*

Proof. Let $G \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}(\boldsymbol{A})$ such that $G \subseteq F$. It follows by Lemma 2.24 that $G = G^{\mathrm{Su}} \subseteq F^{\mathrm{Su}}$. As a consequence, if $F^{\mathrm{Su}}_{\mathcal{S}}$ is a Suszko filter of $\boldsymbol{A}$, then it is necessarily the largest one below $F$. ☐

We next instantiate Proposition 1.18 with the Suszko operator, and state some algebraic properties of the set $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}(\boldsymbol{A})$ which will be later useful.

**Proposition 2.26.** *A filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a Suszko filter of $\boldsymbol{A}$ if and only if $F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$.*

**Lemma 2.27.** *For every $\boldsymbol{A}$, $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}(\boldsymbol{A})$ is a join-complete sub-semilattice of $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*

Proof. Let $\{F_i : i \in I\} \subseteq \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}(\boldsymbol{A})$. Since $F_i \subseteq \bigvee_{i\in I} F_i$, for every $i \in I$, it follows by Lemma 2.24 that $F_i = F_i^{\mathrm{Su}} \subseteq (\bigvee_{i\in I} F_i)^{\mathrm{Su}}$, for every $i \in I$. Thus, $\bigvee_{i\in I} F_i \subseteq (\bigvee_{i\in I} F_i)^{\mathrm{Su}}$, by definition of supremum. The converse inclusion always holds. Therefore, $\bigvee_{i\in I} F_i = (\bigvee_{i\in I} F_i)^{\mathrm{Su}}$, and hence $\bigvee_{i\in I} F_i \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}(\boldsymbol{A})$. ☐

**Lemma 2.28.** *For every $\boldsymbol{A}$, $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}(\boldsymbol{A})$ is closed under unions of $\kappa$-directed families, where $\kappa$ is the cardinal of $\mathcal{S}$.*

Proof. Let $\mathcal{F} \subseteq \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ be a $\kappa$-directed family of Suszko filters of $\boldsymbol{A}$. Recall that $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is always closed under unions of $\kappa$-directed families (see page 15). Hence, $\bigcup \mathcal{F} \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. But then, $\bigcup \mathcal{F}$ must be the supremum of the family $\mathcal{F}$. It follows by Lemma 2.27 that $\bigcup \mathcal{F}$ is a Suszko filter of $\boldsymbol{A}$. ☐

We have seen in Proposition 2.9 that Leibniz filters are precisely the least elements of full g-models. Since we have just seen that every Suszko filter is a Leibniz filter, in particular they are also least elements of full g-models. A natural question is: which full g-models of $\mathcal{S}$ have Suszko filters as least elements?

**Theorem 2.29.** *For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, the following conditions are equivalent:*

*(i)* *$F$ is a Suszko filter of $\boldsymbol{A}$;*
*(ii)* *$\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$ is a full g-model of $\mathcal{S}$;*
*(iii)* *$F = \bigcap \mathscr{C}$, for some full g-model $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\mathscr{C}$ is an up-set.*

Proof. $(i) \Rightarrow (ii)$: This follows from Lemma 2.21.6, which tells us that $[\![F]\!]^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, and Proposition 2.22, which tells us that $[\![F]\!]^{\mathrm{Su}}$ is always full.

$(ii) \Rightarrow (iii)$: This is because $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ is an up-set and $F = \bigcap(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$.

$(iii) \Rightarrow (i)$: On the one hand, observe that $F \in \mathscr{C}$ because $\mathscr{C}$, being full, is a closure system. This implies that $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \subseteq \mathscr{C}$, because $\mathscr{C}$ is an up-set. On the other hand, since $F = \bigcap\mathscr{C}$ by assumption, clearly $\mathscr{C} \subseteq (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$. That is, $\mathscr{C} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, which proves the final assertion. Moreover, since $\mathscr{C}$ is full, $\mathscr{C} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F = \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\}$. But, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. So, $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F = [\![F]\!]^{\mathrm{Su}}$, and therefore $F = \bigcap[\![F]\!]^{\mathrm{Su}}$ is a Suszko filter. □

The next result gives an important answer to the following natural question: what happens if every $\mathcal{S}$-filter is Suszko?

**Theorem 2.30.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) $\mathcal{S}$ is truth-equational;*

*(ii) For every algebra $\boldsymbol{A}$, every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Suszko filter;*

*(iii) For every algebra $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Suszko filter.*

Proof. $(i) \Rightarrow (iii)$: Let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. By hypothesis, taking Definition 0.38 into account, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is completely order reflecting. Now, let $G \in [\![F]\!]^{\mathrm{Su}}$. Then, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. It follows by Lemma 1.5 that $F \subseteq G$. Hence, $F$ is a Suszko filter of $\boldsymbol{A}$.

$(iii) \Rightarrow (ii)$: Let $\boldsymbol{A}$ be an arbitrary algebra and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Fix $\boldsymbol{B} := \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ and let $\pi\colon \boldsymbol{A} \to \boldsymbol{B}$ be the canonical map. Fix $F_0 := \bigcap\mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Notice that $\pi F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, by Lemma 0.24.3. Since $F_0 \subseteq \pi F$, using that the Suszko operator is order preserving and Lemma 1.45 we obtain that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(F_0) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(\pi F) = id_{\boldsymbol{B}}$. Hence, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(F_0) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(\pi F)$. Now, by hypothesis, both $F_0$ and $\pi F$ are Suszko filters of $\boldsymbol{B}$, because $\boldsymbol{B} \in \mathsf{Alg}^{\mathrm{Su}}(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$. Since by Proposition 1.16 the Suszko operator is always injective over Suszko filters, it follows that $F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = F_0 = \bigcap\mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. By Proposition 2.26, this establishes that $F$ is a Suszko filter.

$(ii) \Rightarrow (i)$: Let $\boldsymbol{A}$ and $G, F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Then, $G \in [\![F]\!]^{\mathrm{Su}}$. Since $F = F^{\mathrm{Su}}$ by hypothesis, it follows that $F \subseteq G$. It follows by Lemma 1.5 that the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is completely order reflecting. Again, taking Definition 0.38 into account, $\mathcal{S}$ is truth-equational. □

An analogous condition to (ii) stated with Leibniz filters rather than Suszko filters does not suffice to establish truth-equationality. A counter-example is [**55**, Example 2]. There, a logic $\mathcal{S}$ in the language $\mathcal{L} = \{\top, \Box, \Diamond\}$ is presented such that, for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$a \in F \quad \Leftrightarrow \quad [\langle a, \Box^{\boldsymbol{A}} a\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \text{ or } \langle a, \top^{\boldsymbol{A}}\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)].$$

It is easily seen then that the Leibniz operator is order reflecting. As a consequence, every $\mathcal{S}$-filter is Leibniz, by Proposition 2.11. Nevertheless, Raftery proves that $\mathcal{S}$ is not truth-equational.

As a corollary of Theorem 2.30, we obtain a characterization of truth-equational logics in terms of their full g-models.

**Corollary 2.31.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) $\mathcal{S}$ is truth-equational.*

*(ii) $\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\rangle$ is a full g-model of $\mathcal{S}$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\boldsymbol{A}$.*

*(iii)* $[\![F]\!]^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$*, for every* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *and every* $\boldsymbol{A}$*.*

Proof. It follows by Lemma 2.21.5 and in Theorem 2.29 to Theorem 2.30. ☐

But we had already seen that, for protoalgebraic logics, every full g-model is of the form of a Leibniz class (Corollary 2.19). Moreover, for these logics, the $\boldsymbol{\Omega}$- and $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-classes coincide. Therefore, taking Definition 0.38 into account, we confirm a known characterization of weakly algebraizable logics in terms of their full g-models:

**Corollary 2.32** ([**36**, Theorem 3.8 (iii)])**.** *A logic* $\mathcal{S}$ *is weakly algebraizable if and only if the full g-models of* $\mathcal{S}$ *are exactly all the g-matrices of the form* $\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$ *for any algebra* $\boldsymbol{A}$ *and any* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$*.*

It is also possible to obtain a characterization of weakly algebraizable logics solely in terms of notions related to the Suszko operator.

**Proposition 2.33.** *A logic* $\mathcal{S}$ *is weakly algebraizable if and only if all its full g-models are Suszko classes and all its* $\mathcal{S}$*-filters are Suszko filters.*

Proof. Suppose $\mathcal{S}$ is weakly algebraizable. Since in particular it is protoalgebraic, it follows by Theorem 2.20 that every full g-model of $\mathcal{S}$ is of the form $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, for some algebra $\boldsymbol{A}$ and some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since $\mathcal{S}$ is moreover truth-equational, it follows by Corollary 2.31 that $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F = [\![F]\!]^{\mathrm{Su}}$. Thus, every full g-model of $\mathcal{S}$ is a Suszko class, and by Theorem 2.30 every $\mathcal{S}$-filter is a Suszko filter.

Conversely, suppose the two properties hold. It follows by the second property and Theorem 2.30 that $\mathcal{S}$ is truth-equational. Also, by Corollary 2.31, every Suszko class is of the form $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, for some algebra $\boldsymbol{A}$ and some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since $F = F^* = F^{\mathrm{Su}}$ under truth-equationality, it follows by the first property and Theorem 2.20 that $\mathcal{S}$ is protoalgebraic. Thus, $\mathcal{S}$ is weakly algebraizable. ☐

So far we have explored the notions of $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-class, $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-filter and $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-full g-model. We now proceed to study the notion of coherence for the Suszko operator. Recall that by Proposition 1.32 the Suszko operator is a coherent family of $\mathcal{S}$-compatibility operators. We can, as a consequence, apply the General Correspondence Theorem 1.38 to it, or given the fact that $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$ is the relativization of $\boldsymbol{\Omega}$, apply instead Theorem 1.41.

**Theorem 2.34** (Correspondence Theorem for Suszko classes)**.** *For every surjective* $h\colon \boldsymbol{A} \to \boldsymbol{B}$ *and every* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$*, if* $h$ *is* $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$*-compatible with* $F$*, then* $h$ *induces an order isomorphism between* $[\![F]\!]^{\mathrm{Su}}$ *and* $[\![hF]\!]^{\mathrm{Su}}$*, whose inverse is given by* $h^{-1}$*.*

Let us see that Theorem 2.34 strengthens Czelakowski's Correspondence Theorem for deductive homomorphisms [**24**, Proposition 2.3], which states (in the present terminology) that $h$ is an isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{hF}$ under the assumption that $h$ is a surjective and deductive matrix homomorphism between $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, hF\rangle$. Now, we know by Lemma 1.27 that $h$ is a deductive matrix homomorphism if and only if it is $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-compatible with $F$, viewed as an algebraic homomorphism. Thus, compared with [**24**, Proposition 2.3], Theorem 2.34 extends the isomorphism to the whole Suszko classes $[\![F]\!]^{\mathrm{Su}}$ and $[\![hF]\!]^{\mathrm{Su}}$, which contain $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{hF}$ respectively, by Lemma 2.21.4.

**Corollary 2.35.** *Under the assumptions of Theorem 2.34,* $F$ *is a Suszko filter of* $\boldsymbol{A}$ *if and only if* $hF$ *is a Suszko filter of* $\boldsymbol{B}$*.*

It so happens that, just like Blok and Pigozzi did for protoalgebraic logics, one can state a correspondence theorem characterizing the class of truth-equational logics.

**Theorem 2.36** (Correspondence Theorem for truth-equational logics)**.** *A logic $\mathcal{S}$ is truth-equational if and only if every strict surjective matrix homomorphism between $\mathcal{S}$-models $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ that is $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-compatible with $F$ induces an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{G^{\mathrm{Su}}}$, whose inverse is given by $h^{-1}$.*

Proof. Suppose $\mathcal{S}$ is truth-equational. Let $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ be a strict surjective matrix homomorphism between $\mathcal{S}$-models that is $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-compatible with $F$. It follows by Theorem 2.34 that $h$ induces an order isomorphism between $[\![F]\!]^{\mathrm{Su}}$ and $[\![hF]\!]^{\mathrm{Su}}$, whose inverse is given by $h^{-1}$. But, $[\![F]\!]^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $[\![G]\!]^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^G$, by Corollary 2.31 and $G = G^{\mathrm{Su}}$ by Theorem 2.30.
Conversely, assume the stated property. We shall prove that every $\mathcal{S}$-filter is a Suszko filter, from which the desired conclusion will follow by Theorem 2.30. Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and fix $\boldsymbol{B} := \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. Then, the canonical projection $\pi\colon \boldsymbol{A} \to \boldsymbol{B}$ is a strict and surjective matrix homomorphism between the $\mathcal{S}$-models $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{B}, F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\rangle$, and it is clearly $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-compatible with $F$. Therefore, by the assumption, $\pi$ induces an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{\pi F^{\mathrm{Su}}}$, with inverse given by $\pi^{-1}$. Now, on the one hand, the Suszko operator is a coherent family of $\mathcal{S}$-compatible operators, therefore by Lemma 1.29, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(\pi F) = \pi \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = id_{\boldsymbol{B}}$. This implies that $[\![\pi F]\!]^{\mathrm{Su}} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and hence that $\pi F^{\mathrm{Su}} = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{\pi F^{\mathrm{Su}}} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. On the other hand, we can apply Theorem 2.34 to $\pi$, and we find that it induces an order isomorphism between $[\![F]\!]^{\mathrm{Su}}$ and $[\![\pi F]\!]^{\mathrm{Su}} = \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$, with inverse given by $\pi^{-1}$ as well. Thus, necessarily $[\![F]\!]^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$. It follows by Corollary 2.31 that $F$ is a Suszko filter of $\boldsymbol{A}$. □

Finally, by just extending the scope of the order isomorphism in the last result to all strict and surjective matrix homomorphisms, we reach weakly algebraizable logics.

**Theorem 2.37.** *A logic $\mathcal{S}$ is weakly algebraizable if and only if every strict surjective matrix homomorphism between $\mathcal{S}$-models $h\colon \langle \boldsymbol{A}, F\rangle \to \langle \boldsymbol{B}, G\rangle$ induces an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{G^{\mathrm{Su}}}$, whose inverse is given by $h^{-1}$.*

Proof. If $\mathcal{S}$ is weakly algebraizable, in particular it is protoalgebraic and by Theorem 2.15 we obtain the order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^G$; but since $\mathcal{S}$ is also truth-equational, every $\mathcal{S}$-filter is a Suszko filter and hence $G^{\mathrm{Su}} = G$, which produces the desired result. Conversely, assume the stated property, and observe that in particular it holds for all surjective homomorphisms $h$ that are $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-compatible with $F$. Therefore, by Theorem 2.36, $\mathcal{S}$ is truth-equational. But then all $\mathcal{S}$-filters will be Suszko, so that $G^{\mathrm{Su}} = G$, and the assumed condition establishes, for all the $h$ described, an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^G$. Thus we can apply Theorem 2.15 and conclude that $\mathcal{S}$ is protoalgebraic as well. That is, $\mathcal{S}$ is weakly algebraizable. □

Finally, we investigate the case where all full g-models of a logic are of the form of some Suszko class, just like we did in Proposition 2.17 for Leibniz classes.

**Proposition 2.38.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) Every full g-model of $\mathcal{S}$ is of the form $[\![F]\!]^{\mathrm{Su}}$, for some algebra $\boldsymbol{A}$ and some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$;*

*(ii) The Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{A})$ is surjective, for every $\boldsymbol{A}$; that is, $\mathrm{Ran}(\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}) = \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$.*

Proof. $(i) \Rightarrow (ii)$: Let $\boldsymbol{A}$ arbitrary and $\theta \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{A})$. Then, $\theta = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C})$, for some full g-model $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, by Corollary 2.3. It follows by hypothesis that there exists $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\mathscr{C} = [\![F]\!]^{\mathrm{Su}}$. So,

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big([\![F]\!]^{\mathrm{Su}}\big) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F),$$

using Proposition 2.22. Thus, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{A})$ is surjective.

$(ii) \Rightarrow (i)$: Let $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ be a full g-model of $\mathcal{S}$. Since $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{A})$, it follows by hypothesis that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big([\![F]\!]^{\mathrm{Su}}\big) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ and $[\![F]\!]^{\mathrm{Su}}$ is a full g-model of $\mathcal{S}$, by Proposition 2.22, it follows by the Isomorphism Theorem for full g-models (Corollary 2.3) that $\mathscr{C} = [\![F]\!]^{\mathrm{Su}}$. □

Contrast Proposition 2.38 with the Leibniz operator case, where in general, $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is always onto $\mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$. Moreover, assuming $\mathcal{S}$ to be truth-equational, gives us a more meaningful consequence:

**Proposition 2.39.** *Let $\mathcal{S}$ be a truth-equational logic. The following conditions are equivalent:*

*(i) $\mathcal{S}$ is weakly algebraizable;*

*(ii) Every full g-model of $\mathcal{S}$ is of the form $[\![F]\!]^{\mathrm{Su}}$, for some algebra $\boldsymbol{A}$ and some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*

Proof. $(i) \Rightarrow (ii)$: Since $\mathcal{S}$ is in particular protoalgebraic, all the Suszko and Leibniz related notions coincide, by Proposition 2.5. In particular, $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$. The result now follows by Proposition 2.17.

$(ii) \Rightarrow (i)$: Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Since $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \in \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A} \subseteq \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$, it follows by hypothesis (havind in mind Proposition 2.38) that there exists $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(G)$. As a consequence, $[\![F]\!]^{*} = [\![G]\!]^{\mathrm{Su}}$. Now, since $\mathcal{S}$ is truth-equational by assumption, every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Suszko filter (and hence a Leibniz filter as well), by Theorem 2.30. Therefore, $F = F^{*} = G^{\mathrm{Su}} = G$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. It follows by Proposition 2.5 that $\mathcal{S}$ is protoalgebraic. Since $\mathcal{S}$ is moreover truth-equational by assumption, we conclude that $\mathcal{S}$ is weakly algebraizable. □

**Suszko-full g-models of $\mathcal{S}$.** A thourough study of the closure properties of the class of (Leibniz-)full g-models of a logic can be found in [40]. Although we have not undertaken such an exhaustive study for the class of Suszko-full g-models of a logic, we record here the results obtained for this new class of g-models.

In Corollary 2.31 we have characterized truth-equational logics in terms of their (Leibniz-) full g-models (recall, every Suszko class is a full g-model of $\mathcal{S}$). We now wish to characterize truth-equational logics in terms of their Suszko-full g-models as well. Of course, in Proposition 2.20 we did implicitly characterize protoalgebraic logics in terms of their Suszko-full g-models, as these coincide with the (Leibniz-) full g-models under protoalgebraicity. Here however, such characterization is not immediate, and in fact makes use of several results from [55] (some of them will be discussed in more detail in Chapter 4). While we are at it, we use it to prove that the Suszko operator is order reflecting if and only if $\mathcal{S}$ is truth-equational, and therefore this seemingly stronger property does not take us any further than injectivity of the Suszko operator (bearing in mind Raftery's [55, Theorem 28]).

**Proposition 2.40.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i)* $\mathcal{S}$ *is truth-equational;*

*(ii)* $\langle \boldsymbol{A}, (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \rangle$ *is a Suszko-full g-model of* $\mathcal{S}$, *for every* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *and every* $\boldsymbol{A}$.

*(iii) The Suszko operator* $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ *is order reflecting, for every algebra* $\boldsymbol{A}$.

Proof. Notice that

$$\widetilde{\widetilde{\boldsymbol{\Omega}}}^{\boldsymbol{A}}_{\mathcal{S}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big) = \bigcap_{F \subseteq F'} \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F') = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F),$$

by monotonicity of the Suszko operator.

*(ii)* ⇔ *(iii)*: Suppose $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ is a Suszko-full g-model, for every $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then,

$$\begin{aligned}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F &= \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}\,-1}_{\mathcal{S}}\Big(\widetilde{\widetilde{\boldsymbol{\Omega}}}^{\boldsymbol{A}}_{\mathcal{S}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big)\Big)\\ &= \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}\,-1}_{\mathcal{S}}\big(\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big) = \big\{F' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F')\big\}.\end{aligned}$$

So, given $G, G' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(G) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(G')$, it must hold $G \subseteq G'$.

Conversely, suppose the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is order reflecting, for every algebra $\boldsymbol{A}$. Then,

$$\begin{aligned}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}\,-1}_{\mathcal{S}}\Big(\widetilde{\widetilde{\boldsymbol{\Omega}}}^{\boldsymbol{A}}_{\mathcal{S}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big)\Big) &= \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}\,-1}_{\mathcal{S}}\big(\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)\\ &= \big\{F' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F')\big\} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F.\end{aligned}$$

Hence, $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ is a Suszko-full g-model, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

*(iii)* ⇒ *(i)*: Our hypothesis is stronger than injectivity of the Suszko operator, which in turn is equivalent to truth-equationality, by Theorem 3.11.

*(i)* ⇒ *(iii)*: Suppose $\mathcal{S}$ is truth-equational, say witnessed by $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$. Then, for every algebra $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$F = \big\{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big\},$$

by Proposition 0.43. Now, let $F, F' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F')$. Then,

$$\boldsymbol{\tau}^{\boldsymbol{A}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F') \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F'),$$

where the first inclusion holds by Proposition 0.44 and [55, Corollary 9]. It follows again by Proposition 0.43 that $F \subseteq F'$. □

Moreover, the coincidence of Leibniz-full g-models and Suszko-full g-models characterizes protoalgebraicity:

**Proposition 2.41.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent.*

*(i)* $\mathcal{S}$ *is protoalgebraic.*

*(ii) The full g-models of* $\mathcal{S}$ *coincide with its Suszko-full g-models.*

*(iii)* $[\![F]\!]^* = [\![F]\!]^{\mathrm{Su}}$ *for every* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ *and every* $\boldsymbol{A}$.

Proof. The implications from (i) to (ii) and to (iii) are a direct consequence of Proposition 2.5. Now assume (ii). Since every Suszko-full g-model is always an up-set, the condition implies that the full g-models of $\mathcal{S}$ are all up-sets, and by Theorem 2.20 this implies that $\mathcal{S}$ is protoalgebraic. Finally, assume (iii) and consider any $\boldsymbol{A}$ and any $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$. Then by Lemma 2.21.4, $G \in [\![G]\!]^{\mathrm{Su}} \subseteq [\![F]\!]^{\mathrm{Su}} = [\![F]\!]^{*}$, which implies that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. This shows that the Leibniz operator is order preserving on every $\boldsymbol{A}$, which implies that $\mathcal{S}$ is protoalgebraic. □

By contrast, the coincidence of the following Leibniz- and Suszko-related notions does not characterize protoalgebraicity.

- $F^{*} = F^{\mathrm{Su}}$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\boldsymbol{A}$.
- $F$ is a Suszko filter if and only if $F$ is a Leibniz filter, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\boldsymbol{A}$.

The reason is that these two properties hold (vacuously) in all truth-equational logics, because as we have seen in Theorem 2.30, in them all filters are Suszko filters, and hence also Leibniz filters. In Chapter 7, we will also find non-protoalgebraic and non-truth-equational logics $\mathcal{S}$ such that Leibniz and Suszko filters coincide on the $\mathcal{S}$-algebras (for example, $\mathcal{PML}$ and $w\mathcal{K}_{\sigma}$).

## 2.3. The Frege operator as an $\mathcal{S}$-operator

In this section we undertake the study of the Frege operator as an $\mathcal{S}$-operator. We know that it is not, in general, a congruential $\mathcal{S}$-operator. In fact, it is clear from the definitions involved, that $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}}$ is a congruential $\mathcal{S}$-operator on $\boldsymbol{Fm}$ if and only if $\mathcal{S}$ is Fregean; and that $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}$ is a congruential $\mathcal{S}$-operator on $\boldsymbol{A}$, for every $\boldsymbol{A}$, if and only if $\mathcal{S}$ is fully Fregean. Surprisingly enough, as well shall see further ahead, the same characterizations hold when imposing $\mathcal{S}$-compatibility rather than congruentiality.

Let us start by considering the notion of $\boldsymbol{\Lambda}_{\mathcal{S}}$-class. By Definition 1.12, the $\boldsymbol{\Lambda}_{\mathcal{S}}$-class of $F$, which we shall also call *the Frege class of* $F$, is defined by

$$[\![F]\!]^{\boldsymbol{\Lambda}} := \boldsymbol{\Omega}^{\boldsymbol{A}^{-1}}\big(\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(F)\big) = \big\{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\big\}.$$

This time we cannot guarantee that the $\boldsymbol{\Lambda}_{\mathcal{S}}$-classes are full g-models of $\mathcal{S}$, as we lack $\mathcal{S}$-compatibility. Still, by Proposition 1.13, we do have:

**Proposition 2.42.** *For every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $[\![F]\!]^{\boldsymbol{\Lambda}}$ is a closure system on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*

By Definition 1.15, $F^{\boldsymbol{\Lambda}}$ denotes the least element of the Frege class $[\![F]\!]^{\boldsymbol{\Lambda}}$. We say that $F$ is a *Frege filter* if $F = F^{\boldsymbol{\Lambda}}$, and we denote the set of all Frege filters of $\boldsymbol{A}$ by $\mathcal{F}i_{\mathcal{S}}^{\boldsymbol{\Lambda}}\boldsymbol{A}$. Recall that, in general, whenever working with the notion of $\nabla$-filter we assume that $\mathcal{S}$ has theorems, otherwise the least element of any $\nabla$-class is the empty filter. A rather useful characterization of $\boldsymbol{\Lambda}_{\mathcal{S}}$-filters is the following:

**Lemma 2.43.** *Let $\mathcal{S}$ be a logic and $\boldsymbol{A}$ an algebra. An $\mathcal{S}$-filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a Frege filter of $\boldsymbol{A}$ if and only if $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ if and only if $F \in [\![F]\!]^{\boldsymbol{\Lambda}}$.*

Proof. The last equivalence holds by definition of Frege class. Now, suppose $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a Frege filter of $\boldsymbol{A}$. That is, $F = F^{\boldsymbol{\Lambda}}$. Since $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\boldsymbol{\Lambda}})$, we are done. Conversely, suppose $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Let $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that

$G \in [\![F]\!]^{\boldsymbol{A}}$. That is, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Since $\mathcal{S}$ has theorems, $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \neq \varnothing$. Let $b \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Let $a \in F$. Then, $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, b) = F = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, a)$. So,

$$\langle b, a\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G).$$

Since $b \in G$, it follows by compatibility that $a \in G$. Thus, $F \subseteq G$. Since moreover $F \in [\![F]\!]^{\boldsymbol{A}}$ by assumption, it follows that $F$ is a $\boldsymbol{\Lambda}_{\mathcal{S}}$-filter of $\boldsymbol{A}$. □

Assuming $\mathcal{S}$ protoalgebraic, we immediately get:

**Corollary 2.44.** *Let $\mathcal{S}$ be a protoalgebraic logic and $\boldsymbol{A}$ an algebra. An $\mathcal{S}$-filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a Frege filter of $\boldsymbol{A}$ if and only if $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$.*

An interesting property enjoyed by Frege filters is the following:

**Proposition 2.45.** *If $F$ is a Frege filter of $\boldsymbol{A}$, then $F = a/\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, for any $a \in F$.*

Proof. Let $F \in \mathcal{F}i^{\boldsymbol{A}}_{\mathcal{S}}\boldsymbol{A}$ and $a \in F$. Let $b \in a/\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. Then, $\langle a, b\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, by Lemma 2.43. It follows by compatibility that $b \in F$. Hence, $a/\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq F$. The converse inclusion holds in general. Indeed, given any other element $c \in F$, we have $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, a) = F = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, c)$, and therefore $\langle a, c\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, i.e., $a/\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) = c/\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. Now, since trivially $c \in c/\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) = a/\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, it follows that $F \subseteq a/\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. □

Another interesting fact is that every Frege filter is a Leibniz filter. Recall that given an $\mathcal{S}$-compatibility operator $\nabla$, every $\nabla$-filter is a Leibniz filter (Lemma 1.22.3). But the Frege operator is *not*, in general, an $\mathcal{S}$-compatibility operator. Nevertheless,

**Lemma 2.46.** *Every Frege filter is a Leibniz filter.*

Proof. Let $\boldsymbol{A}$ arbitrary. Let $F \in \mathcal{F}i^{\boldsymbol{A}}_{\mathcal{S}}\boldsymbol{A}$. Then, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$, it follows that $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$, i.e., $F^* \in [\![F]\!]^{\boldsymbol{A}}$, and hence by assumption $F = F^{\boldsymbol{\Lambda}} \subseteq F^*$. □

A final remark about $F^{\boldsymbol{\Lambda}}$, and perhaps an unexpected one, is that it always contains $F$. This contrasts with $F^{\mathrm{Su}}$ and $F^*$, which are always contained in $F$.

**Lemma 2.47.** *For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F \subseteq F^{\boldsymbol{\Lambda}}$.*

Proof. It holds, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\boldsymbol{\Lambda}})$, because $F^{\boldsymbol{\Lambda}} \in [\![F]\!]^{\boldsymbol{A}}$. Since $\mathcal{S}$ has theorems, $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \neq \varnothing$. Let $b \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Let $a \in F$. Then, $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, b) = F = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F, a)$. So, $\langle b, a\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\boldsymbol{\Lambda}})$. Since $b \in F^{\boldsymbol{\Lambda}}$, it follows by compatibility that $a \in F^{\boldsymbol{\Lambda}}$. Thus, $F \subseteq F^{\boldsymbol{\Lambda}}$. □

In general, the Frege operator fails to be a congruential $\mathcal{S}$-operator, as well as an $\mathcal{S}$-compatibility operator. Clearly, given the definitions involved, a logic $\mathcal{S}$ is Fregean if and only if $\boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}$ is a congruential $\mathcal{S}$-operator on $\boldsymbol{Fm}$; and it is fully Fregean if and only if, for every $\boldsymbol{A}$, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}$ is a congruential $\mathcal{S}$-operator on $\boldsymbol{A}$. More interestingly though,

**Proposition 2.48.** *A logic $\mathcal{S}$ is Fregean if and only if $\boldsymbol{\Lambda}^{\boldsymbol{Fm}}_{\mathcal{S}}$ is an $\mathcal{S}$-compatibility operator on $\boldsymbol{Fm}$; and it is fully Fregean if and only if, for every $\boldsymbol{A}$, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}$ is an $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$.*

Proof. Suppose $\mathcal{S}$ is Fregean. Clearly then, $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}}(T) = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}^{\boldsymbol{Fm}}(T) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)$, for every $T \in \mathcal{T}h\mathcal{S}$. Conversely, suppose $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}}(T) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)$, for every $T \in \mathcal{T}h\mathcal{S}$. Since $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}}$ is order preserving, we have

$$\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}}(T) = \bigcap_{T' \supseteq T} \boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}}(T') \subseteq \bigcap_{T' \supseteq T} \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T') = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}^{\boldsymbol{Fm}}(T),$$

for every $T \in \mathcal{T}h\mathcal{S}$. The converse inclusion always holds. We conclude that $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}} = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}^{\boldsymbol{Fm}}$. Thus, $\mathcal{S}$ is Fregean. The second statement is proved similarly, for arbitrary $\boldsymbol{A}$. □

Rephrasing Proposition 2.48: *A logic $\mathcal{S}$ is Fregean if and only if $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}}(T) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)$, for every $T \in \mathcal{T}h\mathcal{S}$.* One cannot avoid to compare this with the original definition of protoalgebraic logics [**10**, Definition 2.1]: *A logic is protoalgebraic if and only if $\boldsymbol{\Omega}^{\boldsymbol{Fm}}(T) \subseteq \boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}}(T)$, for every $T \in \mathcal{T}h\mathcal{S}$.* But unlike the protoalgebraic scenario (recall, a logic is protoalgebraic if and only if $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(F)$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\boldsymbol{A}$), the inclusion $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}}(T) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)$ does not lift from the formula algebra to arbitrary algebras. Indeed, again by Proposition 2.48, a logic is fully Fregean if and only if $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$; and it is well-known that there are Fregean logics which are not fully Fregean (as first shown in [**6**]).

Given Lemma 2.43, we get as immediate corollaries:

**Corollary 2.49.** *A logic $\mathcal{S}$ is Fregean if and only if every $\mathcal{S}$-theory is a Frege theory.*

**Corollary 2.50.** *A logic $\mathcal{S}$ is fully Fregean if and only if, for every $\boldsymbol{A}$, every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Frege filter of $\boldsymbol{A}$.*

We have studied so far the consequences of imposing the Frege operator to be a congruential $\mathcal{S}$-operator, or an $\mathcal{S}$-compatibility operator. We are left to study coherence for this operator. This is what we do next.

**Theorem 2.51.** *The Frege operator $\boldsymbol{\Lambda}_{\mathcal{S}}$ is a coherent family of $\mathcal{S}$-operators.*

Proof. Let $\boldsymbol{A}, \boldsymbol{B}$ two algebras and $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Let $h : \boldsymbol{A} \to \boldsymbol{B}$ surjective and $\boldsymbol{\Lambda}_{\mathcal{S}}$-compatible with $h^{-1}G$, that is, such that $\operatorname{Ker} h \subseteq \boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(h^{-1}G)$. Having in mind Lemma 1.27, it follows by Czelakowski's Correspondence Theorem for deductive homomorphisms [**24**, Proposition 2.3] that $h$ induces an order isomorphism between $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}G}$ and $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G}$, whose inverse is given by $h^{-1}$. As a consequence, for every $F \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}G}$, it holds that $hF \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G}$ and $h^{-1}hF = F$.

First, notice that for every $c \in A$,

$$\begin{aligned} \operatorname{Fg}_{\mathcal{S}}^{\boldsymbol{B}}(G, hc) &= \operatorname{Fg}_{\mathcal{S}}^{\boldsymbol{B}}(hh^{-1}G, hc) \\ &= \operatorname{Fg}_{\mathcal{S}}^{\boldsymbol{B}}\big(h\operatorname{Fg}_{\mathcal{S}}^{\boldsymbol{A}}(h^{-1}G, c)\big) \\ &= h\operatorname{Fg}_{\mathcal{S}}^{\boldsymbol{A}}(h^{-1}G, c), \end{aligned}$$

using the fact that $h$ is surjective, Lemma 0.25 (taking $X = \{h^{-1}G, c\}$), and having in mind that $\operatorname{Fg}_{\mathcal{S}}^{\boldsymbol{A}}(h^{-1}G, c) \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}G}$, so $h\operatorname{Fg}_{\mathcal{S}}^{\boldsymbol{A}}(h^{-1}G, c) \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{B})^{G}$.

Now, for every $a, b \in A$,

$$\begin{aligned}\langle a,b\rangle \in h^{-1}\boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(G) &\Leftrightarrow \langle ha,hb\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(G)\\ &\Leftrightarrow \mathrm{Fg}^{\boldsymbol{B}}_{\mathcal{S}}(G,ha) = \mathrm{Fg}^{\boldsymbol{B}}_{\mathcal{S}}(G,hb)\\ &\Leftrightarrow h\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G,a) = h\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G,b)\\ &\Leftrightarrow \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G,a) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G,b)\\ &\Leftrightarrow \langle a,b\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G),\end{aligned}$$

having in mind that $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G,a) \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{h^{-1}G}$, and hence $h^{-1}h\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G,a) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G,a)$; and similarly for $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G,b)$.

Thus,

$$h^{-1}\boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(G) = \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G),$$

as desired. □

Theorem 2.51 definitely stands in favor of the new notion of coherence. For it remarkably captures the three main operators in AAL. Recall, we knew already that the Leibniz and Suszko operators were coherent families of $\mathcal{S}$-operators. The former, because it commutes with inverse images by surjective homomorphisms, and the later because coherence is preserved under relativization (Proposition 1.32). Actually, given this fact, one could try to find a new coherent $\mathcal{S}$-operator $\widetilde{\boldsymbol{\Lambda}}_{\mathcal{S}}$. But bear in mind that the Frege operator is order preserving, and therefore $\boldsymbol{\Lambda}_{\mathcal{S}} = \widetilde{\boldsymbol{\Lambda}}_{\mathcal{S}}$, by Lemma 1.4.

Since coherence is a weaker property of commutativity with inverse images by surjective homomorphisms, it is natural to ask: when does the Frege operator commute with inverse images by surjective homomorphisms? Surprisingly enough:

**Theorem 2.52.** *A logic $\mathcal{S}$ is protoalgebraic if and only if the Frege operator $\boldsymbol{\Lambda}_{\mathcal{S}}$ commutes with inverse images by surjective homomorphisms.*

Proof. ⇒: Let $\boldsymbol{A}, \boldsymbol{B}$ algebras, $h \in \mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$ surjective, $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and $a, b \in A$. Notice that $h^{-1}G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. It holds,

$$\begin{aligned}\langle a,b\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G) \quad &\text{iff} \quad \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G,a) = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G,b)\\ &\text{iff} \quad h^{-1}\mathrm{Fg}^{\boldsymbol{B}}_{\mathcal{S}}(G,ha) = h^{-1}\mathrm{Fg}^{\boldsymbol{B}}_{\mathcal{S}}(G,hb)\\ &\text{iff} \quad \mathrm{Fg}^{\boldsymbol{B}}_{\mathcal{S}}(G,ha) = \mathrm{Fg}^{\boldsymbol{B}}_{\mathcal{S}}(G,hb)\\ &\text{iff} \quad \langle ha,hb\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(G)\\ &\text{iff} \quad \langle a,b\rangle \in h^{-1}\boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(G),\end{aligned}$$

where we have used [34, Corollary 6.21.2] and the fact that $h$ is surjective.
⇐: Let $\boldsymbol{A}, \boldsymbol{B}$ algebras, $h \in \mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$ surjective and $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Since $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(G) \subseteq \boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(G)$, it follows that

$$h^{-1}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(G) \subseteq h^{-1}\boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(G) = \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G).$$

Now, $h^{-1}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(G) \in \mathrm{Con}(\boldsymbol{A})$ and $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G)$ is the largest congruence on $\boldsymbol{A}$ below $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G)$. Hence,

$$h^{-1}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(G) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}G).$$

The converse inclusion holds in general, by 0.31.3. We conclude that the Suszko operator commutes with inverse images by surjective homomorphisms. It follows by Theorem 1.24 that $\boldsymbol{\Omega} = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$, and hence $\mathcal{S}$ is protoalgebraic. □

Do notice that the same condition imposed upon the Suszko operator also characterizes protoalgebraic logics, and which will be formally stated in Theorem 3.12.

Since there exist protoalgebraic logics which are not Fregean (any protoalgebraic logic which is not selfextensional witnesses this fact; for instance, Lukasiewicz's infinite valued logic), a consequence of Theorem 2.52 is that commuting with inverse images by surjective homomorphisms does not characterize the Leibniz operator among the $\mathcal{S}$-operators. Although it does so among the $\mathcal{S}$-compatibility operators, as we have seen in Theorem 1.24.

Although the Frege operator is not, in general, an $\mathcal{S}$-compatibility operator, the defining condition of coherence for this $\mathcal{S}$-operator also holds with direct images rather than with inverse images by surjective homomorphisms. Recall that this is always the case for an $\mathcal{S}$-compatibility operator $\nabla$, as we saw in Lemma 1.29. But unlike the $\mathcal{S}$-compatibility operators, here it does not characterize coherence of the Frege operator.

**Lemma 2.53.** *For every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $h$ is $\boldsymbol{\Lambda}_{\mathcal{S}}$-compatible with $F$, then $h\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}\big(hF\big)$.*

Proof. Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be surjective and $\boldsymbol{\Lambda}_{\mathcal{S}}$-compatible with $F$. Since $\mathrm{Ker}h \subseteq \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ and $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is the largest congruence below $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, it follows that $\mathrm{Ker}h \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Hence, $F = h^{-1}hF$ and $hF \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. It follows by coherence of the Frege operator that $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}\big(h^{-1}hF\big) = h^{-1}\boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(hF)$, and hence that $h\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(hF)$ because $h$ is surjective. □

We are now able to prove a correspondence theorem for the Frege operator.

**Theorem 2.54** (Correspondence Theorem for the Frege operator)**.** *For every surjective $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, if $h$ is $\boldsymbol{\Lambda}_{\mathcal{S}}$-compatible with $F$, then $h$ induces an order isomorphism between $[\![F]\!]^{\boldsymbol{\Lambda}}$ and $[\![hF]\!]^{\boldsymbol{\Lambda}}$, whose inverse is given by $h^{-1}$.*

Proof. From the assumption that $h$ is $\boldsymbol{\Lambda}_{\mathcal{S}}$-compatible with $F$, that is $\mathrm{Ker}h \subseteq \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, and the fact that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is the largest congruence below $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, it follows that $\mathrm{Ker}h \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Therefore, $h^{-1}hF = F$, and that $hF \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$.

Take first any $F' \in [\![F]\!]^{\boldsymbol{\Lambda}}$. Then $\mathrm{Ker}h \subseteq \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F')$ and hence $h^{-1}hF' = F'$ and $hF' \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Moreover, since $h$ is both $\boldsymbol{\Omega}$-compatible with $F'$ and $\boldsymbol{\Lambda}_{\mathcal{S}}$-compatible with $F$ and both $\boldsymbol{\Omega}$ and $\boldsymbol{\Lambda}_{\mathcal{S}}$ are coherent, we can apply Lemmas 1.29 and 2.53 and obtain that $\boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(hF) = h\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq h\boldsymbol{\Omega}^{\boldsymbol{A}}(F') = \boldsymbol{\Omega}^{\boldsymbol{B}}(hF')$. This tells us that $hF' \in [\![hF]\!]^{\boldsymbol{\Lambda}}$.

Now take any $G \in [\![hF]\!]^{\boldsymbol{\Lambda}}$, i.e., such that $\boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(hF) \subseteq \boldsymbol{\Omega}^{\boldsymbol{B}}(G)$. We know that $h^{-1}G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and that $hh^{-1}G = G$. Observe that $h$ is $\boldsymbol{\Lambda}_{\mathcal{S}}$-compatible with $h^{-1}hF$, since this is $F$. Then, by coherence, we have

$$\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(h^{-1}hF) = h^{-1}\boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(hF) \subseteq h^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(G) = \boldsymbol{\Omega}^{\boldsymbol{A}}(h^{-1}G).$$

This shows that $h^{-1}G \in [\![F]\!]^{\boldsymbol{\Lambda}}$.

Thus, we have established that $h$ induces a bijection between $[\![F]\!]^{\boldsymbol{\Lambda}}$ and $[\![hF]\!]^{\boldsymbol{\Lambda}}$, whose inverse is given by $h^{-1}$. Since both maps are obviously order preserving, they are in fact order isomorphisms. □

Notice that, for the first time so far, we have stated a correspondence theorem which is not an instance of the General Correspondence Theorem 1.38.

Two interesting corollaries of Theorem 2.51 are the following (known) characterizations of full selfextensionality and full Fregeanity.

**Corollary 2.55.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) For every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = id_{\boldsymbol{A}}$;*

*(ii) For every $\boldsymbol{A}$, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$, i.e, $\mathcal{S}$ is fully selfextensional.*

Proof. $(ii) \Rightarrow (i)$: Just notice that, for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = id_{\boldsymbol{A}}$.
$(i) \Rightarrow (ii)$: Let $\boldsymbol{A}$ arbitrary. Fix $\boldsymbol{B} \coloneqq \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$. Consider the canonical map $\pi : \boldsymbol{A} \to \boldsymbol{B}$. Notice that $\mathrm{Ker}\pi$ is compatible with $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Therefore, $\pi \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and $\pi^{-1}\pi \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Also, since $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is a Suszko filter of $\boldsymbol{A}$, it follows by Proposition 2.26 that $\pi \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ is the least $\mathcal{S}$-filter of $\boldsymbol{B}$, that is, $\pi \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. Moreover, $\mathrm{Ker}\pi = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) \subseteq \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$. So, $\pi$ is a surjective homomorpism $\boldsymbol{\Lambda}_{\mathcal{S}}$-compatible with $\pi^{-1}\pi \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Finally, since $\boldsymbol{B} \in \mathsf{Alg}(\mathcal{S})$, it follows by hypothesis $id_{\boldsymbol{B}} = \boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}) = \boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(\pi \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$. Therefore,

$$\begin{aligned}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) &= \mathrm{Ker}\pi = \pi^{-1}id_{\boldsymbol{B}} = \pi^{-1}\boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(\pi \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) \\ &= \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\pi^{-1}\pi \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}),\end{aligned}$$

using coherence of the Frege operator. □

**Corollary 2.56.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) For every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$;*

*(ii) For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, i.e, $\mathcal{S}$ is fully Fregean.*

Proof. $(ii) \Rightarrow (i)$: Trivial.
$(i) \Rightarrow (ii)$: Let $\boldsymbol{A}$ arbitrary. Fix $\boldsymbol{B} \coloneqq \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. Consider the canonical map $\pi : \boldsymbol{A} \to \boldsymbol{B}$. Notice that $\mathrm{Ker}\pi$ is compatible with $F$. Therefore, $\pi F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and $\pi^{-1}\pi F = F$. Moreover, $\mathrm{Ker}\pi = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. So, $\pi$ is a surjective homomorpism $\boldsymbol{\Lambda}_{\mathcal{S}}$-compatible with $\pi^{-1}\pi F$. Finally, since $\boldsymbol{B} \in \mathsf{Alg}(\mathcal{S})$, it follows by hypothesis that $\boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(\pi F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}(\pi F) = id_{\boldsymbol{B}}$. Therefore,

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \mathrm{Ker}\pi = \pi^{-1}id_{\boldsymbol{B}} = \pi^{-1}\boldsymbol{\Lambda}^{\boldsymbol{B}}_{\mathcal{S}}(\pi F) = \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\pi^{-1}\pi F) = \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F),$$

using coherence of the Frege operator. □

We finish our study about the Frege operator by addressing the injectivity of this $\mathcal{S}$-operator. Given the logical relevance the property has for the Suszko operator, it seems quite natural to consider it for the Frege operator as well.

**Proposition 2.57.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) $\mathcal{S}$ has theorems;*

*(ii) There exists $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ such that for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

$$F = \big\{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big\}. \tag{18}$$

*(iii) There exists $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ such that for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

$$F = \big\{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big\}.$$

*If (any) of these conditions hold, then $\boldsymbol{\tau}(x)$ can be taken to be $\boldsymbol{\tau}(x) = \{x \approx \varphi(x)\}$, where $\varphi$ is a theorem of $\mathcal{S}$ with at most the variable $x$.*

Proof. $(i) \Rightarrow (ii)$: Suppose $\mathcal{S}$ has theorems. Then, it has a theorem with at most the variable $x$, say $\varphi(x) \in \mathrm{Thm}_{\mathcal{S}}$. Let $\boldsymbol{A}$ arbitrary, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $a \in A$. Consider $\boldsymbol{\tau}(x) := \{x \approx \varphi(x)\}$. It holds,

$$\begin{aligned}
\boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(F) &\Leftrightarrow \langle a, \varphi^{\boldsymbol{A}}(a)\rangle \in \boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(F)\\
&\Leftrightarrow \mathrm{Fg}_{\mathcal{S}}^{\boldsymbol{A}}(F, a) = \mathrm{Fg}_{\mathcal{S}}^{\boldsymbol{A}}(F, \varphi^{\boldsymbol{A}}(a))\\
&\Leftrightarrow \mathrm{Fg}_{\mathcal{S}}^{\boldsymbol{A}}(F, a) = F\\
&\Leftrightarrow a \in F,
\end{aligned}$$

noticing that $\varphi^{\boldsymbol{A}}(a) \in F$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

$(ii) \Rightarrow (iii)$: Trivial.

$(iii) \Rightarrow (i)$: Let $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ as given by the hypothesis. Suppose, towards an absurd, that $\mathcal{S}$ has no theorems. Then, $\varnothing \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$. Consider a trivial algebra $\boldsymbol{A}$ with universe $A = \{a\}$. Notice then that $\boldsymbol{\tau}^{\boldsymbol{A}}(a) = \{\langle a, a\rangle\}$. Moreover, $\langle a, a\rangle \in \boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(\varnothing) \in \mathrm{Eqr}A$. It follows by hypothesis that $a \in \varnothing$, which is absurd.

The last statement is justified by the proof of $(i) \Rightarrow (ii)$. □

Notice that taking $\boldsymbol{\tau}(x) = \varnothing$ above forces $\mathcal{S}$ to be the inconsistent logic, which trivially has theorems (every formula is a theorem). Proposition 2.57 and Proposition 2.48 allows us to give an easy proof of a very recent result concerning Fregean logics [**4**, Corollary 12]:

**Corollary 2.58.** *If $\mathcal{S}$ is Fregean with theorems, then $\mathcal{S}$ is assertional.*

Proof. Since $\mathcal{S}$ has theorems, it has a theorem with at most the variable $x$, say $\top(x) \in \mathrm{Thm}_{\mathcal{S}}$. Fix $\boldsymbol{\tau}(x) := \{x \approx \top(x)\}$. Let $T \in \mathcal{T}h\mathcal{S}$ arbitrary. It follows by (the last statement of) Proposition 2.57 that $T = \{\varphi \in \mathrm{Fm}_{\mathcal{L}} : \boldsymbol{\tau}^{\boldsymbol{Fm}}(\varphi) \subseteq \boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}}(T)\}$. Since $\boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{Fm}}(T) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)$, by Proposition 2.48, it follows that $T \subseteq \{\varphi \in \mathrm{Fm}_{\mathcal{L}} : \boldsymbol{\tau}^{\boldsymbol{Fm}}(\varphi) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)\}$. Conversely, let $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ such that $\boldsymbol{\tau}^{\boldsymbol{Fm}}(\varphi) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)$. That is, $\langle \varphi, \top(\varphi)\rangle \in \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)$. Since $\top(\varphi) \in T$, it follows by compatibility that $\varphi \in T$. Hence, truth is equationally definable in $\mathsf{LMod}^*(\mathcal{S})$ with a set of defining equations $\boldsymbol{\tau}(x) = \{x \approx \top(x)\}$. Together with Theorem 0.40, it follows by definition that $\mathcal{S}$ is assertional. □

Condition (18) ressembles condition (13) from Proposition 0.43, which is stated with the Leibniz operator instead of the Frege operator. And the analogy does not end here. As we know, the property of the Leibniz operator being completely order reflecting characterizes truth-equationality of the underlying logic. Similarly, the same property imposed on the Frege operator characterizes the logic having theorems. Actually, injectivity of the Frege operator suffices to force the existence of theorems; however, injectivity of the Leibniz operator does not suffice to establish truth-equationality, as Raftery shows in [**55**, Example 2].

**Corollary 2.59.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) $\mathcal{S}$ has theorems;*

*(ii) The Frege operator is injective;*

*(iii) The Frege operator is completely order reflecting.*

Proof. $(i) \Rightarrow (iii)$: Let $\boldsymbol{A}$ arbitrary and $\{F_i : i \in I\} \cup \{G\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\bigcap_{i\in I} \boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(F_i) \subseteq \boldsymbol{\Lambda}_{\mathcal{S}}^{\boldsymbol{A}}(G)$. Notice that $\bigcap_{i\in I} F_i \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Under our hypothesis, let

$\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ be as given by Proposition 2.57. Then,

$$\boldsymbol{\tau}^{\boldsymbol{A}}\Big(\bigcap_{i\in I} F_i\Big) \subseteq \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}\Big(\bigcap_{i\in I} F_i\Big) \subseteq \bigcap_{i\in I} \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F_i) \subseteq \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(G).$$

by Proposition 2.57 (notice that condition (18) implies $\boldsymbol{\tau}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, for any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$), monotonicity of the Frege operator, and the assumption, respectively. So, $\bigcap_{i\in I} F_i \subseteq G$, again by Proposition 2.57.

$(iii) \Rightarrow (ii)$: Trivial.

$(ii) \Rightarrow (i)$: Suppose, towards an absurd, that $\mathcal{S}$ has no theorems. Then, $\varnothing \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$. Consider a trivial algebra $\boldsymbol{A}$ with universe $A = \{a\}$. Notice that $\langle a, a\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\varnothing)$ and $\langle a, a\rangle \in \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(a))$. So necessarily, $\boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\varnothing) = \boldsymbol{\Lambda}^{\boldsymbol{A}}_{\mathcal{S}}(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(a))$. It follows by hypothesis that $\varnothing = \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(a)$, which is absurd, since $a \in \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(a)$. Thus, $\mathcal{S}$ must have theorems. ☐

An interesting corollary is that the injectivity of the three main (families of) $\mathcal{S}$-operators is closely related.

**Corollary 2.60.** *If the family $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$ is injective, then the family $\boldsymbol{\Omega}$ is injective; and, if the family $\boldsymbol{\Omega}$ is injective, then the family $\boldsymbol{\Lambda}_{\mathcal{S}}$ is injective.*

Proof. Injectivity of the Suszko operator on arbitrary algebras is equivalent to truth-equationality, by Theorem 3.11, which under Definition 0.38 clearly implies injectivity of the Leibniz operator on arbitrary algebras. Injectivity of the Leibniz operator on any algebra forces the existence of theorems, since $\boldsymbol{\Omega}^{\boldsymbol{A}}(\varnothing) = \boldsymbol{\Omega}^{\boldsymbol{A}}(A)$. Finally, the existence of theorems is equivalent to injectivity of the Frege operator on arbitrary algebras, by Corollary 2.59. ☐

Consequently, given Raftery's [**55**, Theorem 28], if a logic is truth-equational, then the Suszko, Leibniz and Frege operators are all injective on arbitrary algebras.

CHAPTER 3

# The Leibniz hierarchy revisited

In this chapter we give two (partial) presentations of the Leibniz hierarchy, one in terms of order isomorphisms between the set of $\mathcal{S}$-filters and the set of $\mathsf{Alg}(\mathcal{S})$-congruences on arbitrary algebras (Theorem 3.10), and another in terms of the Suszko operator (Theorems 3.12 and 3.13). In both cases, the new characterizations extend, or complete, the already existing ones to larger classes of logics within the Leibniz hierarchy.

## 3.1. An isomorphism theorem for protoalgebraic logics

In the previous chapter we have looked at the three main $\mathcal{S}$-operators separately. We have considered the notions of $\nabla$-class, $\nabla$-filter and coherence for $\nabla = \boldsymbol{\Omega}, \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}, \boldsymbol{\Lambda}_{\mathcal{S}}$ and obtained a wealth of characterizations, several known and a few new, of the main classes of logics within the Leibniz hierarchy, as well as a plethora of correspondence theorems resulting from the notion of coherence and the General Correspondence Theorem 1.38. All this was done without using the notions involved for the different $\mathcal{S}$-operators simultaneasly. It turns out that the interplay of these notions also gives raise to some new results in AAL. Namely, a new isomorphism theorem for protoalgebraic logics (Theorem 3.8) in the same spirit of the famous one for algebraizable logics ([**11**, Theorem 3.7]; see also [**48**, Theorem 5.2] for the non-finitary case; and [**39**, Corollary 3.14] for a presentation which resembles more ours) and for weakly algebraizable logics ([**25**, Theorem 4.8]). As a corollary, another isomorphism theorem characterizing equivalential logics is obtained (Corollary 3.9).

Our starting point is a (known) result that states an isomorphism theorem for protoalgebraic logics, but unlike the previous mentioned ones, it does not characterize this class of logics.

**Proposition 3.1** ([**37**, Theorem 3])**.** *If $\mathcal{S}$ is protoalgebraic, then for every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^{*}(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism.*

The converse of Proposition 3.1 is false, as we will see in Chapter 7. Under truth-equationality however, it does hold (Proposition 3.5). In order to see it, we first prove some auxiliary results.

**Proposition 3.2.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) For every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^{*}_{\mathcal{S}}(\boldsymbol{A}) \to \mathrm{Con}_{\mathsf{Alg}^{*}(\mathcal{S})}(\boldsymbol{A})$ is an order isomorphism;*

*(ii) For every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^{*}_{\mathcal{S}}(\boldsymbol{A}) \to \mathrm{Con}_{\mathsf{Alg}^{*}(\mathcal{S})}(\boldsymbol{A})$ is an order isomorphism.*

Proof. We prove the non-trivial implication only. Let $\boldsymbol{A}$ be an arbitrary algebra. Let $F, G \in \mathcal{F}i^{*}_{\mathcal{S}}(\boldsymbol{A})$ such that $F \subseteq G$. Consider the canonical map $\pi : \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. Fix $\boldsymbol{B} \coloneqq \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. Since $\mathrm{Ker}\,\pi = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(G) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$ and

$\text{Ker}\pi = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, it follows by Corollary 2.13 that $\pi G \in \mathcal{F}i^*_{\mathcal{S}}(\boldsymbol{B})$ and $\pi F \in \mathcal{F}i^*_{\mathcal{S}}(\boldsymbol{B})$. But $\boldsymbol{B} \in \mathsf{Alg}(\mathcal{S})$. So, it follows by hypothesis that $\boldsymbol{\Omega}^{\boldsymbol{B}}$ is monotone over $\mathcal{F}i^*_{\mathcal{S}}(\boldsymbol{B})$. Therefore, since $\pi F \subseteq \pi G$, it follows

$$\boldsymbol{\Omega}^{\boldsymbol{B}}(\pi F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{B}}(\pi G).$$

Since the Leibniz operator commutes with inverse images by surjective homomorphisms, and $\text{Ker}\pi$ is compatible with both $F$ and $G$, it follows that

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(\pi^{-1}\pi F) = \pi^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(\pi F) \subseteq \pi^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(\pi G) = \boldsymbol{\Omega}^{\boldsymbol{A}}(\pi^{-1}\pi G) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G).$$

Hence, we have established monotonicity. The injectivity is trivial, since the Leibniz operator is always injective on the Leibniz filters, by Proposition 1.16. Finally, we prove surjectivity. Let $\theta \in \text{Con}_{\mathsf{Alg}^*(\mathcal{S})}(\boldsymbol{A})$. So, $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, for some $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Consider this time the canonical map $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Fix $\boldsymbol{B} := \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since $\text{Ker}\pi = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ and $\pi$ is surjective, it holds $\pi F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$. But $\boldsymbol{B} \in \mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{S})$. So, it follows by surjectivity of $\boldsymbol{\Omega}^{\boldsymbol{B}}$ on $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{B}$ that

$$\boldsymbol{\Omega}^{\boldsymbol{B}}(\pi F) = \boldsymbol{\Omega}^{\boldsymbol{B}}(G),$$

for some $G \in \mathcal{F}i^*_{\mathcal{S}}(\boldsymbol{B})$. Since $\pi$ is surjective, we have $G = \pi\pi^{-1}G$. Moreover, $\pi^{-1}G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Now, since the Leibniz operator commutes with inverse images by surjective homomorphisms, it holds

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(\pi^{-1}\pi F) = \pi^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(\pi F) = \pi^{-1}\boldsymbol{\Omega}^{\boldsymbol{B}}(G) = \boldsymbol{\Omega}^{\boldsymbol{A}}(\pi^{-1}G).$$

We are left to see that $\pi^{-1}G$ is a Leibniz filter of $\boldsymbol{A}$. But notice that, as a consequence of the above expression, $\text{Ker}\pi = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is compatible with $\pi^{-1}G$. So, since $\pi\pi^{-1}G = G \in \mathcal{F}i^*_{\mathcal{S}}(\boldsymbol{B})$, it follows by Corollary 2.13 that $\pi^{-1}G \in \mathcal{F}i^*_{\mathcal{S}}(\boldsymbol{A})$. Thus,

$$\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(\pi^{-1}G),$$

with $\pi^{-1}G \in \mathcal{F}i^*_{\mathcal{S}}(\boldsymbol{A})$. □

**Lemma 3.3.** *If for every* $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$*,* $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \to \text{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ *is an order isomorphism, then* $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S})$*.*

Proof. Let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$. Consider the $\mathcal{S}$-filter $F_0 := \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. It is clearly the smallest Leibniz filter. Since we are assuming that $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is order preserving on Leibniz filters, it follows that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F_0) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ for every $F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$. So, $[\![F]\!]^* \subseteq [\![F_0]\!]^*$, for every $F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$. Now, let $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ be arbitrary. Since $\boldsymbol{\Omega}^{\boldsymbol{A}}(G) \in \text{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$, it follows by the assumption (surjectivity) that there exists some $F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(G) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$; so, $G \in [\![G]\!]^* = [\![F]\!]^* \subseteq [\![F_0]\!]^*$. Thus, $[\![F_0]\!]^* = \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. It follows by Proposition 2.16 that $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) \in \mathsf{Alg}^*(\mathcal{S})$. But since $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = id_{\boldsymbol{A}}$ and $\boldsymbol{A} \cong \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$. Therefore, $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$. This shows that $\mathsf{Alg}(\mathcal{S}) \subseteq \mathsf{Alg}^*(\mathcal{S})$. The converse inclusion always holds. □

It is well-known that, if $\mathcal{S}$ is protoalgebraic, then the classes $\mathsf{Alg}^*(\mathcal{S})$ and $\mathsf{Alg}(\mathcal{S})$ coincide (see the remarks after Proposition 2.5). We are now able to see that, under truth-equationality, the converse also holds.

**Proposition 3.4.** *Let* $\mathcal{S}$ *be a truth-equational logic. If* $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$*, then* $\mathcal{S}$ *is protoalgebraic.*

Proof. If $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$, then by Proposition 2.17 every full g-model of $\mathcal{S}$ is of the form $[\![G]\!]^*$, for some $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and some algebra $\boldsymbol{A}$. In particular, so are Suszko classes. Take any $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, for an arbitrary $\boldsymbol{A}$. Then, $[\![F]\!]^{\mathrm{Su}} = [\![G]\!]^*$, for some $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Hence, $F^{\mathrm{Su}} = G^*$. But, since $\mathcal{S}$ is truth-equational by hypothesis, by Theorem 2.30 every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Suszko filter, and in general every Suszko filter is a Leibniz filter, by Lemma 2.21.2. Therefore, $F = F^{\mathrm{Su}} = G^* = G$. Thus, $[\![F]\!]^{\mathrm{Su}} = [\![F]\!]^*$. Since this has been proved for all $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and all $\boldsymbol{A}$, this implies protoalgebraicity by Proposition 2.41. □

Given Lemma 3.3 and Proposition 3.4, it is clear that the converse of Proposition 3.1 does indeed hold under truth-equationality, as we had previously claimed.

**Proposition 3.5.** *Let $\mathcal{S}$ be a truth-equational logic. If for every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism, then $\mathcal{S}$ is protoalgebraic.*

Another consequence of Proposition 3.4 is the following:

**Corollary 3.6.** *A logic $\mathcal{S}$ is weakly algebraizable if and only if it is truth-equational and $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$.*

Going back to Proposition 3.1, as it turns out, it still holds if we replace the set of Leibniz filters by the set of Suszko filters; and this time the converse also holds! Our next goal is to prove this refinement of Proposition 3.1. First, observe that the proof of Lemma 3.3 works, *mutatis mutandis*, for Suszko filters, since $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$. Therefore:

**Lemma 3.7.** *If for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism, then $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S})$.*

However, the proof of Proposition 3.2 does not. For, when establishing surjectiveness, one cannot apply Corollary 2.35 to the canonical map $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, as it is not $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-compatible with $F$. Anyway, with Lemma 3.7 at hand, we are now able to prove the refinement of Proposition 3.1 we are looking for.

**Theorem 3.8.** *A logic $\mathcal{S}$ is protoalgebraic if and only if for every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism.*

Proof. The direct implication is just a rephrasing of Proposition 3.1, because under protoalgebraicity the Leibniz filters and the Suszko filters coincide. Now assume the stated condition. We will prove separately that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}})$ and that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}})$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$; this will imply that the Leibniz and the Suszko operators coincide, which is equivalent to protoalgebraicity by Proposition 2.5. So, let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. To prove the first equality note that since $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$, it follows by Lemma 3.7 and the surjectivity of $\boldsymbol{\Omega}^{\boldsymbol{A}}$ that there exists $G \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. This implies that $[\![F]\!]^{\mathrm{Su}} = [\![G]\!]^*$, and hence that $F^{\mathrm{Su}} = G^* = G$, because every Suszko filter is a Leibniz filter. Thus, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}})$. As to the second equality, since $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \in \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$, it follows from the assumption that there exists $H \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(H)$. Then, $[\![F]\!]^* = [\![H]\!]^*$, and hence $F^* = H^* = H$, again because every Suszko filter is a Leibniz filter. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$. Moreover, $F^* = H$ is a Suszko filter. That is, $(F^*)^{\mathrm{Su}} = F^*$. Now, since $F^* \subseteq F$, it holds $[\![F]\!]^{\mathrm{Su}} \subseteq [\![F^*]\!]^{\mathrm{Su}}$, and therefore $(F^*)^{\mathrm{Su}} \subseteq F^{\mathrm{Su}}$. So, $F^* \subseteq F^{\mathrm{Su}}$. The converse inclusion always holds, by Lemma 2.21.1. Thus, $F^* = F^{\mathrm{Su}}$, which implies that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}})$. □

Here arrived, and taking Definition 0.38 into account, we readily obtain an isomorphism theorem characterizing equivalential logics.

**Corollary 3.9.** *A logic $\mathcal{S}$ is equivalential if and only if the Leibniz operator commutes with inverse images by homomorphisms and for every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\mathcal{S}}^{\mathrm{Su}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism.*

It is not difficult to see that Theorem 3.8 and Corollary 3.8 provide alternative proofs for the known isomorphism theorems for algebraizable and weakly algebraizable logics previously mentioned. Just notice that, bringing truth-equationality into the picture, every $\mathcal{S}$-filter is a Suszko filter, by Theorem 2.30. We state them altogether for the sake of completeness:

**Theorem 3.10.** *Let $\mathcal{S}$ be a logic.*

1. *$\mathcal{S}$ is protoalgebraic if and only if for every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\mathcal{S}}^{\mathrm{Su}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism.*
2. *$\mathcal{S}$ is equivalential if and only if the Leibniz operator commutes with inverse images by homomorphisms and for every $\boldsymbol{A}$, the operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ restricts to an order isomorphism between $\mathcal{F}i_{\mathcal{S}}^{\mathrm{Su}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$.*
3. *$\mathcal{S}$ is weakly algebraizable if and only for every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism.*
4. *$\mathcal{S}$ is algebraizable if and only if the Leibniz operator commutes with inverse images by homomorphisms and for every $\boldsymbol{A}$, the operator $\boldsymbol{\Omega}^{\boldsymbol{A}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism.*

Proof. Theorem 3.8, Corollary 3.9, [**39**, Corollary 3.14] and [**11**, Theorem 3.7] (for $\mathcal{S}$ finitary) or [**48**, Theorem 5.2] (for $\mathcal{S}$ arbitrary), respectively. □

### 3.2. The Leibniz hierarchy via the Suszko operator

This section is devoted to characterize several classes of logics within the Leibniz hierarchy in terms of the Suszko operator. Our starting point is the only known such characterization (to our knowledge), namely Raftery's characterization of truth-equational logics [**55**, Theorem 28] in terms of global injectivity of the Suszko operator. We take the chance to prove it directly within our framework and furthermore to show that one only needs to demand injectivity of the Suszko operator over the class of $\mathcal{S}$-algebras.

**Theorem 3.11** ([**55**, Theorem 28])**.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i)* *$\mathcal{S}$ is truth-equational;*
*(ii)* *The Suszko operator $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}^{\boldsymbol{A}}$ is injective, for every $\boldsymbol{A}$;*
*(iii)* *The Suszko operator $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}^{\boldsymbol{A}}$ is injective, for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$.*

Proof. $(i) \Rightarrow (ii)$: By Proposition 1.16, for every $\boldsymbol{A}$, the Suszko operator $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}^{\boldsymbol{A}}$ is injective on the Suszko filters $\mathcal{F}i_{\mathcal{S}}^{\mathrm{Su}}\boldsymbol{A}$. Now, if $\mathcal{S}$ is truth-equational, then by Theorem 2.30 every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a Suszko filter of $\boldsymbol{A}$. Thus, for every $\boldsymbol{A}$, the Suszko operator $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}^{\boldsymbol{A}}$ is injective.

$(ii) \Rightarrow (iii)$: Trivial.

*(iii)* ⇒ *(i)*: Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Fix $F_0 \coloneqq \bigcap \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)$. Notice that $\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathsf{Alg}^{\mathrm{Su}}(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$. Moreover, by Lemma 0.24.3, $F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)$. Necessarily then, $F_0 \subseteq F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. On the other hand, since the Suszko operator is always order preserving, and using Lemma 1.45, we have $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)}_{\mathcal{S}}(F_0) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)}_{\mathcal{S}}\big(F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big) = id_{\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)}$. It follows by hypothesis that $F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = F_0 = \bigcap \mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)$. By Proposition 2.26, it follows that $F$ is a Suszko filter of $\boldsymbol{A}$. Since both $\boldsymbol{A}$ and $F$ were chosen arbitrarily, every $\mathcal{S}$-filter is a Suszko filter. It follows by Theorem 2.30 that $\mathcal{S}$ is truth-equational. ☐

Next, we know by Theorem 1.24 that the Leibniz operator is the only $\mathcal{S}$-compatibility operator commuting with inverse images by surjective homomorphisms. Therefore:

**Theorem 3.12.** *A logic $\mathcal{S}$ is protoalgebraic if and only if the Suszko operator commutes with inverse images by surjective homomorphisms.*

Proof. It follows immediately by Theorem 1.24, having in mind that the Leibniz operator is order preserving if and only if $\boldsymbol{\Omega} = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$, by Lemma 1.4. ☐

With Theorems 3.11 and 3.12 at hand, we readily get characterizations for other classes of logics within the Leibniz hierarchy.

**Theorem 3.13.** *Let $\mathcal{S}$ be a logic.*

1. *$\mathcal{S}$ is equivalential if and only if the Suszko operator commutes with inverse images by homomorphisms.*
2. *$\mathcal{S}$ is weakly algebraizable if and only if the Suszko operator is injective and commutes with inverse images by surjective homomorphisms.*
3. *$\mathcal{S}$ is algebraizable if and only if the Suszko operator is injective and commutes with inverse images by homomorphisms.*
4. *$\mathcal{S}$ is finitely algebraizable if and only if the Suszko operator is injective, continuous, and commutes with inverse images by homomorphisms.*

Proof. *1.* Suppose $\mathcal{S}$ is equivalential. Then, by Definition 0.38, $\mathcal{S}$ is protoalgebraic and the Leibniz operator commutes with inverse images by homomorphisms. But protoalgebraicity implies the coincidence of the Suszko and Leibniz operators. Thus the Suszko operator commutes with inverse images by homomorphisms. Conversely, suppose that the Suszko operator commutes with inverse images by homomorphisms. In particular, it commutes with inverse images by surjective homomorphisms. So, by Theorem 3.12, $\mathcal{S}$ is protoalgebraic. Consequently, the Leibniz and Suszko operators coincide, and therefore the Leibniz operator commutes with inverse images by homomorphisms. Thus, $\mathcal{S}$ is equivalential.

*2.* It follows by Theorems 3.11 and 3.12.

*3.* It follows by 1 and Theorem 3.11.

*4.* Suppose $\mathcal{S}$ is finitely algebraizable. Then, by Definition 0.38, $\mathcal{S}$ is finitely equivalential and truth-equational. In particular, it is protoalgebraic, and therefore the Leibniz and Suszko operators coincide. Hence, under our hypothesis, the Suszko operator is continuous. Moreover, by 1, the Suszko operator commutes with inverse images by homomorphisms. Finally, it follows by Theorem 3.11 that the Suszko operator is injective. Conversely, suppose the Suszko operator is injective, continuous, and commutes with inverse images by homomorphisms. In particular, it commutes with inverse images by surjective homomorphisms. So, by Theorem 3.12, $\mathcal{S}$ is protoalgebraic. Consequently, the Leibniz and Suszko operators coincide, and therefore the Leibniz operator is continuous and injective. Given Definition 0.38, $\mathcal{S}$ is finitely algebraizable. □

Do notice that the analogous characterizations to those of 1 and 3 stated for the Leibniz operator require moreover monotonicity; see [**39**, Theorems 3.13.2 and 3.13.5].

Next, we show that Theorem 3.8 can also be stated with the Suszko operator. Let us start by applying Proposition 1.16 taking $\nabla = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$:

**Proposition 3.14.** *For every $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\colon \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ is an order embedding.*

Proof. The Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is order preserving on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, hence so is its restriction to $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$. Moreover, by Proposition 1.16, the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is order reflecting on $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$. Finally, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is into $\mathrm{Con}_{\mathsf{Alg}^{\mathrm{Su}}(\mathcal{S})}\boldsymbol{A}$ and $\mathsf{Alg}^{\mathrm{Su}}(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$, by Lemma 0.36.4. □

It is therefore natural to ask under what assumptions does the operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\colon \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ become an order isomorphism. That is, to consider Theorem 3.8 stated with the Suszko operator instead of the Leibniz operator.

**Theorem 3.15.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) $\mathcal{S}$ is protoalgebraic;*

*(ii) For every $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\colon \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism;*

*(iii) For every $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\colon \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ is surjective.*

Proof. *(i)* ⇒ *(ii)*: By Proposition 2.5 and Corollary 2.19, if $\mathcal{S}$ is protoalgebraic, then $\boldsymbol{\Omega}^{\boldsymbol{A}} = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ and $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S})$. Therefore $\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A} = \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$, and then Theorem 3.8 establishes (ii).

*(ii)* ⇒ *(iii)*: Trivial.

*(iii)* ⇒ *(i)*: Let $\boldsymbol{A}$ arbitrary and $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$. Since $\boldsymbol{\Omega}^{\boldsymbol{A}} \in \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}(\boldsymbol{A}) \subseteq \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$, it follows by hypothesis that there exists $H \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(H) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Then,

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\llbracket H \rrbracket^{\mathrm{Su}}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\llbracket F \rrbracket^*),$$

using Propositions 2.7 and 2.22. It follows by the well-known isomorphism theorem for full g-models, here Corollary 2.3 (of course, bearing in mind that $\boldsymbol{\Omega}$- and $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$−classes are full g-models of $\mathcal{S}$), that $\llbracket H \rrbracket^{\mathrm{Su}} = \llbracket F \rrbracket$. Now, since $H$ is a Suszko filter, it follows by Theorem 2.29 that $\llbracket H \rrbracket^{\mathrm{Su}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^H$. Since $F \in \llbracket F \rrbracket$, we have $H \subseteq F \subseteq G$. Therefore, $G \in \llbracket F \rrbracket$, that is, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Since $\boldsymbol{A}$ is arbitrary, we conclude that the Leibniz operator is order preserving. □

Here arrived, we are able to give another (partial) presentation of the Leibniz hierarchy in terms of the Suszko operator, this time highlighting order theoretical properties. To this end, and similarly to the definition of continuity for the Leibniz operator on page 23, the Suszko operator is *continuous*, if it commutes with unions of upwards-directed families of $\mathcal{S}$-filters whose union is an $\mathcal{S}$-filter.

**Theorem 3.16.** *Let $\mathcal{S}$ be a logic.*

1. *$\mathcal{S}$ is protoalgebraic if and only if for every $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} \colon \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism.*
2. *$\mathcal{S}$ is truth-equational if and only if for every $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ is an order embedding.*
3. *$\mathcal{S}$ is weakly algebraizable if and only if for every $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism.*
4. *$\mathcal{S}$ is algebraizable if and only if the Suszko operator commutes with inverse images by homomorphisms and for every $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism.*
5. *$\mathcal{S}$ is finitely algebraizable if and only if the Suszko operator is continuous and for every $\boldsymbol{A}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} \colon \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism.*

Proof. *1.* This is contained in Theorem 3.15.

*2.* The operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is always an order embedding of $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$ into $\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$, by Proposition 3.14. Moreover, truth-equationality implies that $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$, by Theorem 2.30. So, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is actually an order embedding of $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ into $\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$. Conversely, the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is an order isomorphism, and hence in particular injective, for every $\boldsymbol{A}$. It follows by Theorem 3.11 that $\mathcal{S}$ is truth-equational.

*3.* Suppose $\mathcal{S}$ is weakly algebraizable. By Theorems 3.8 and Theorem 2.30, $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is an isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$, for every $\boldsymbol{A}$. Moreover, since $\mathcal{S}$ is protoalgebraic, the Suszko operator and the Leibniz operator coincide, $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$, and we obtain the desired isomorphism. Conversely, suppose that the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is an isomorphism between $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and $\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$, for every $\boldsymbol{A}$. In particular, it is injective on $\mathcal{S}$-filters, and hence by Theorem 3.11 $\mathcal{S}$ is truth-equational. Moreover, every $\mathcal{S}$-filter is a Suszko filter, by Theorem 2.30, so that the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is actually surjective over the Suszko filters of $\boldsymbol{A}$. It follows from Theorem 3.15 that $\mathcal{S}$ is protoalgebraic. Altogether, $\mathcal{S}$ is weakly algebraizable.

*4.* It follows by 3 and Theorem 3.15.2.

*5.* It follows by 5 and [**39**, Theorem 3.13], having in mind that the Suszko and Leibniz operators coincide under both hypothesis. □

CHAPTER 4

# Truth-equational logics revisited

In this chapter, we give some contributions to the study of truth-equational logics, which started in [**55**]. Namely, we prove a new characterization of the Suszko operator in terms of (any) defining set of equations witnessing truth-equationality (Proposition 4.2), which will allow us to arrive at a strenghtning of Raftery's [**55**, Theorem 28] by characterizing truth-equational logics as those logics where the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ is a structural representation, for every algebra $\boldsymbol{A}$ (Theorem 4.13). With this characterization at hand, we prove that definability of truth on the class $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$ is equivalent to the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}} : \mathcal{T}h\mathcal{S} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{Fm}$ being a structural operator on the formula algebra, thus unifying this weaker definability property with the rest of the theory developed for truth-equational logics — a problem left unsolved in [**55**]. We also give a necessary condition for the continuity of the Suszko operator, a property already considered in [**23**, Section 7]; and finally, we present yet another coherent family of $\mathcal{S}$-compatibility operators for truth-equational logics.

## 4.1. The Suszko operator for truth-equational logics

Let us start with a technical lemma, which arises by putting Propositions 0.43 and 1.11 together.

**Lemma 4.1.** *Let $\mathcal{S}$ be a truth-equational logic with a set of defining equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$. For every $\boldsymbol{A}$ and every $X \subseteq A$,*

$$\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\theta\big) = \pi\,\big(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big)^{X} \quad \textit{and} \quad \pi^{-1}\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\theta\big) = \big(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big)^{X},$$

*where $\theta \coloneqq \Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(X))$ and $\pi : \boldsymbol{A} \to \boldsymbol{A}/\theta$ is the canonical map.*

Proof. First, let us fix $\mathscr{C} \coloneqq \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(X)) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big\}$ and $\theta \coloneqq \Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(X))$. It follows by Proposition 1.11 that

$$\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\theta\big) = \pi\mathscr{C} \quad \text{and} \quad \pi^{-1}\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{A}/\theta\big) = \mathscr{C}.$$

Now, since $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$, it holds

$$\mathscr{C} = \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \theta \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big\} = \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\tau}^{\boldsymbol{A}}(X) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big\}.$$

But, by Proposition 0.43,

$$\big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\tau}^{\boldsymbol{A}}(X) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big\} = \big\{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : X \subseteq F\big\} = \big(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big)^{X}.$$

□

**Proposition 4.2.** *Let $\mathcal{S}$ be a truth-equational logic with a set of defining equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$. For every $\boldsymbol{A}$ and every $X \subseteq A$,*

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X)) = \Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(X)).$$

*In particular, if $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, then*

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(F)).$$

Proof. Let $X \subseteq A$. Fix $\theta := \Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(X))$. First of all, since $\theta \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$, it holds $\boldsymbol{A}/\theta \in \mathsf{Alg}(\mathcal{S})$, and therefore $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}/\theta}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big) = id_{\boldsymbol{A}/\theta}$, by (9) on page 21. Now, let $\pi : \boldsymbol{A} \to \boldsymbol{A}/\theta$ be the canonical map. Then,

$$\begin{aligned}
\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X)) &= \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(X) \\
&= \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^{X}\big) \\
&= \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big(\pi^{-1}\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big) \\
&= \pi^{-1}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}/\theta}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{A}/\theta)\big) \\
&= \pi^{-1}(id_{\boldsymbol{A}/\theta}) \\
&= \mathrm{Ker}\pi \\
&= \Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(X)),
\end{aligned}$$

using Lemma 4.1 and Proposition 0.31.2. □

Notice that, given an arbitrary logic $\mathcal{S}$ (not necessarily truth-equational), for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, it always holds that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$. This fact is an easy consequence of Lemma 0.36. Therefore, $\mathrm{Ran}(\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}) \subseteq \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$. So, the meaningful part of Proposition 4.2 is not saying that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ is an $\mathsf{Alg}(\mathcal{S})$-congruence of $\boldsymbol{A}$, but rather determining it as the *least* $\mathsf{Alg}(\mathcal{S})$-congruence of $\boldsymbol{A}$ containing $\boldsymbol{\tau}^{\boldsymbol{A}}(F)$.

Several consequences follow from Proposition 4.2. Some immediate corollaries are:

**Corollary 4.3.** *If $\mathcal{S}$ is truth-equational with a set of defining equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$, then*

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}\big(\mathrm{Cn}_{\mathcal{S}}(\{x\})\big) = \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}(x)).$$

Recall the notation introduced in Proposition 0.41,

$$\boldsymbol{\tau}_{\infty}(x) := \sigma_x \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}\big(\mathrm{Cn}_{\mathcal{S}}(\{x\})\big),$$

where $\sigma_x : \boldsymbol{Fm} \to \boldsymbol{Fm}$ is the substitution sending all variables to $x$. The following fact is contained in [**55**, Proposition 32], and will be useful later on.

**Corollary 4.4.** *If $\mathcal{S}$ is truth-equational with a set of defining equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$, then $\boldsymbol{\tau} \subseteq \boldsymbol{\tau}_{\infty}$.*

Proof. Since $\boldsymbol{\tau}(x) \subseteq \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}(x)) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}\big(\mathrm{Cn}_{\mathcal{S}}(\{x\})\big)$, by Corollary 4.3, it follows that

$$\boldsymbol{\tau}(x) = \sigma_x \boldsymbol{\tau}(x) \subseteq \sigma_x \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}\big(\mathrm{Cn}_{\mathcal{S}}(\{x\})\big) = \boldsymbol{\tau}_{\infty}(x).$$

□

Another interesting consequence is the following:

**Proposition 4.5.** *Let $\mathcal{S}$ be a truth-equational logic. For every $\boldsymbol{A}$ and every $X \subseteq A$,*

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X)) = \bigvee_{a \in X}^{\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}} \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(a)).$$

Proof. Let $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ be a defining set of equations for $\mathcal{S}$, which exists under our hypothesis. Notice that

$$\Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(X)) = \bigvee^{\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}} \{\Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(a)) : a \in X\},$$

by Lemma 0.23, since $\mathsf{Alg}(\mathcal{S})$ is closed under $\mathbb{IP}_{\mathrm{S}}$ by Lemma 0.36.5. But under our hypothesis, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X)) = \Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(X))$ and $\Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}\big(\boldsymbol{\tau}^{\boldsymbol{A}}(a)\big) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(a)\big)$, by Proposition 4.2. □

Notice that the relevant inclusion is $\subseteq$, as the converse one always holds, by monotonicity of the Suszko operator, the fact that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X)) \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ and definition of supremum. In fact, Proposition 4.5 can be generalized as follows:

**Proposition 4.6.** *If $\mathcal{S}$ is truth-equational, then the Suszko operator preserves suprema, i.e.,*

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\Big(\bigvee_{i\in I}^{\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}} F_i\Big) = \bigvee_{i\in I}^{\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}} \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F_i),$$

*for every algebra $\boldsymbol{A}$ and arbitrary families $\{F_i \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : i \in I\}$.*

Proof. Let $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ be a defining set of equations for $\mathcal{S}$, which exists under our hypothesis. It holds,

$$\begin{aligned}
\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\Big(\bigvee_{i\in I}^{\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}} F_i\Big) &= \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\Big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}\big(\bigcup_{i\in I} F_i\big)\Big)\\
&= \Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}\Big(\boldsymbol{\tau}^{\boldsymbol{A}}\big(\bigcup_{i\in I} F_i\big)\Big)\\
&= \bigvee_{i\in I}^{\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}} \Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(F_i))\\
&= \bigvee_{i\in I}^{\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}} \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F_i),
\end{aligned}$$

using Proposition 4.2 (twice) and Lemma 0.23, having in mind that $\mathsf{Alg}(\mathcal{S})$ is closed under $\mathbb{IP}_{\mathrm{S}}$ by Lemma 0.36.5. □

Proposition 4.2 also makes possible to investigate two further algebraic properties enjoyed by the Suszko operator for truth-equational logics, namely its commutativity with direct images of surjective homomorphisms and with unions of upwards-directed families whose union is an $\mathcal{S}$-filter. Recall, commutativity with inverse images by surjective homomorphisms characterizes protoalgebraicity, by Theorem 3.12. The property of commuting with direct images of surjective homomorphisms turns out to be just one aspect of a strenghtening of Raftery's characterization for truth-equational logics in terms of the Suszko operator.

**Proposition 4.7.** *If $\mathcal{S}$ is truth-equational, then the Suszko operator commutes with images by surjective homomorphisms, in the sense that, for every $\boldsymbol{A}, \boldsymbol{B}$, every surjective $h : \boldsymbol{A} \to \boldsymbol{B}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}\big(\mathrm{Fg}^{\boldsymbol{B}}_{\mathcal{S}}(hF)\big) = \Theta^{\boldsymbol{B}}_{\mathsf{Alg}(\mathcal{S})}\big(h\,\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big).$$

Proof. Let $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ be a defining set of equations for $\mathcal{S}$, which exists under our hypothesis. Notice that

$$\begin{aligned}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{B}}_{\mathcal{S}}\big(\mathrm{Fg}^{\boldsymbol{B}}_{\mathcal{S}}(hF)\big) &= \Theta^{\boldsymbol{B}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{B}}(hF)) \\ &= \Theta^{\boldsymbol{B}}_{\mathsf{Alg}(\mathcal{S})}(h\Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{A}}(F))) \\ &= \Theta^{\boldsymbol{B}}_{\mathsf{Alg}(\mathcal{S})}\big(h\,\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\big)\end{aligned}$$

using Proposition 4.2 (twice) and Lemma 0.22. □

**Lemma 4.8.** *If $\mathcal{S}$ is truth-equational and $\mathsf{Alg}(\mathcal{S})$ is a quasivariety, then*

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \bigcup\{\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(X) : X \subseteq F \textit{ finite}\},$$

*for every algebra $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.*

Proof. First of all, notice that $\langle \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}, \subseteq\rangle$ is an algebraic lattice, by Proposition 0.20. So,

$$\begin{array}{rcl}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) & = & \Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}\big(\boldsymbol{\tau}^{\boldsymbol{A}}(F)\big) \\ & = & \bigcup\Big\{\Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}(\vartheta) : \vartheta \subseteq \boldsymbol{\tau}^{\boldsymbol{A}}(F) \text{ finite}\Big\} \\ & \subseteq & \bigcup\Big\{\Theta^{\boldsymbol{A}}_{\mathsf{Alg}(\mathcal{S})}\big(\boldsymbol{\tau}^{\boldsymbol{A}}(X)\big) : X \subseteq F \text{ finite}\Big\} \\ & = & \bigcup\Big\{\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(X)\big) : X \subseteq F \text{ finite}\Big\} \\ & = & \bigcup\Big\{\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(X) : X \subseteq F \text{ finite}\Big\} \\ & \subseteq & \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F),\end{array}$$

using Proposition 4.2 (twice) and monotonicity of the Suszko operator. The middle inclusion is justified as follows: if $\vartheta \subseteq A \times A$ is finite and such that $\vartheta \subseteq \boldsymbol{\tau}^{\boldsymbol{A}}(F)$, then necessarily a finite number of elements of $F$ is involved in those equations, and at most all equations in $\boldsymbol{\tau}(x)$ for each one of these elements are involved. □

**Proposition 4.9.** *If $\mathcal{S}$ is truth-equational and $\mathsf{Alg}(\mathcal{S})$ is a quasivariety, then the Suszko operator is continuous.*

Proof. Let $\boldsymbol{A}$ be an algebra and $\{F_i : i \in I\} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ be an upwards-directed family of $\mathcal{S}$-filters of $\boldsymbol{A}$ such that $\bigcup_{i\in I} F_i \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The inclusion

$$\bigcup_{i\in I}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F_i) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\Big(\bigcup_{i\in I} F_i\Big)$$

always holds, by monotonicity of the Suszko operator. Conversely, it follows by Lemma 4.8 that

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}\Big(\bigcup_{i\in I} F_i\Big) = \bigcup\Big\{\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F') : F' \subseteq \bigcup_{i\in I} F_i \text{ finite}\Big\}.$$

Now, let $F' \subseteq \bigcup_{i\in I} F_i$ finite, say $F' = \{a_{i_1}, \ldots, a_{i_n}\}$. For each $a_{i_j}$ there exists $F_j \in \bigcup_{i\in I} F_i$ such that $a_{i_j} \in F_j$, with $j = 1, \ldots, n$. Since the family is upwards-directed, there exists $k \in I$ such that $F_1 \cup \ldots \cup F_n \subseteq F_k$. So, $F' \subseteq F_k$. It follows by monotonicity of the Suszko operator that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F') \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F_k)$. Thus,

$$\bigcup\Big\{\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F') : F' \subseteq \bigcup_{i\in I} F_i \text{ finite}\Big\} \subseteq \bigcup_{i\in I}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F_i).$$

□

Proposition 4.9 generalizes [**25**, Theorem 5.7], which tells us that for logics such that $\mathsf{Alg}^*(\mathcal{S})$ is a quasivariety,

$$\mathcal{S} \text{ is weakly algebraizable if and only if } \mathcal{S} \text{ is finitely algebraizable.}$$

Indeed, adding protoalgebraicity to the assumptions of Proposition 4.9, implies that the Leibniz and Suszko operators coincide, and that $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$. So, it follows by Proposition 4.9 that the Leibniz operator is continuous, and therefore, according to Definition 0.38, $\mathcal{S}$ is finitely algebraizable. The converse implication holds in general.

We now undertake a short detour: if we replace the assumption $\mathsf{Alg}(\mathcal{S})$ being a quasivariety in Proposition 4.9 by $\mathsf{Alg}^*(\mathcal{S})$ being a quasivariety instead, then necessarily $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$, and much stronger consequences follow. Indeed, if the truth-equationality of $\mathcal{S}$ is witnessed by a finite set of defining equations, then $\mathcal{S}$ is actually BP-algebraizable.

**Lemma 4.10.** *If a logic $\mathcal{S}$ is truth-equational with a finite set of defining equations and $\mathsf{Alg}(\mathcal{S})$ is a quasivariety, then $\mathcal{S}$ is finitary.*

Proof. On the one hand, if $\mathcal{S}$ is truth-equational with a set of defining equations $\boldsymbol{\tau}(x)$ (finite or not), then $\mathsf{Alg}(\mathcal{S})$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$, by Proposition 0.44. On the other hand, since $\mathsf{Alg}(\mathcal{S})$ is a quasivariety, $\vDash^{\mathrm{eq}}_{\mathsf{Alg}(\mathcal{S})}$ is finitary. Finally, since the equational translation witnessing the completeness of $\mathcal{S}$ w.r.t. $\vDash^{\mathrm{eq}}_{\mathsf{Alg}(\mathcal{S})}$ is finitary by hypothesis, it follows that $\mathcal{S}$ is finitary. □

**Proposition 4.11.** *Let $\mathcal{S}$ be truth-equational with a finite set of defining equations. The following conditions are equivalent:*

*(i)* $\mathsf{Alg}^*(\mathcal{S})$ *is a quasivariety;*
*(ii)* $\mathcal{S}$ *is BP-algebraizable.*

Proof. $(i) \Rightarrow (ii)$: Since $\mathsf{Alg}^*(\mathcal{S})$ is a quasivariety by hypothesis, in particular, it is closed under $\mathbb{P}_{\mathrm{S}}$. Hence, $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$. This fact together with truth-equationality implies that $\mathcal{S}$ is weakly algebraizable, by Corollary 3.6. Now, $\mathcal{S}$ is finitary by Lemma 4.10. Finally, for finitary logics, weakly algebraizability together with $\mathsf{Alg}^*(\mathcal{S})$ being a quasivariety implies that $\mathcal{S}$ is BP-algebraizable, by [**25**, Theorem 5.7].

$(ii) \Rightarrow (i)$: Finitary and finitely equivalential logics are such that $\mathsf{Alg}^*(\mathcal{S})$ is a quasivariety [**47**, p. 426]. □

**Corollary 4.12.** *Let $\mathcal{S}$ be an assertional logic. The following conditions are equivalent:*

*(i)* $\mathsf{Alg}^*(\mathcal{S})$ *is a quasivariety;*
*(ii)* $\mathcal{S}$ *is regularly BP-algebraizable.*

This corollary may be rephrased as: *A logic $\mathcal{S}$ is regularly BP-algebraizable if and only if $\mathcal{S}$ is assertional and $\mathsf{Alg}^*(\mathcal{S})$ is a quasivariety.* This tells us that the topmost class of Figure 1 is the intersection of the classes at the left and right ends.

Going back to the main goal of this section, by putting together Raftery's [**55**, Theorem 28] with Proposition 4.6 and Proposition 4.7, we arrive at:

**Theorem 4.13.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i)* *$\mathcal{S}$ is truth-equational;*
*(ii)* *The Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ is a structural representation, for every algebra $\boldsymbol{A}$;*
*(iii)* *The Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$ is a structural representation, for every algebra $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$.*

Proof. $(i) \Rightarrow (ii)$: Suppose $\mathcal{S}$ is truth-equational. Then, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is injective, for every $\boldsymbol{A}$, by Raftery's [**55**, Theorem 28]; $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ preserves suprema, for every $\boldsymbol{A}$, by Proposition 4.6; and $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ commutes with endomorphisms, for every $\boldsymbol{A}$, by Proposition 4.7 (having in mind that endomorphisms are surjective homomorphisms).

$(ii) \Rightarrow (iii)$: Trivial.

$(iii) \Rightarrow (i)$: Since by hypothesis, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is injective for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, the result follows by Theorem 3.11. □

Notice that an analogous condition stated just for the formula algebras would not hold under the same proof. For injectivity of the Suszko operator on the formula algebras does not suffice to establish truth-equationality, as Raftery shows in [**55**, Example 1].

Here arrived, it is natural to consider the same condition imposed upon the Leibniz operator. Appart from the terminology, and for the particular case of finitary logics, such result is contained in [**11**, Theorem 3.7]. We next state it for arbitrary logics.

**Theorem 4.14.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i)* *$\mathcal{S}$ is algebraizable;*
*(ii)* *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ is a structural representation, for every algebra $\boldsymbol{A}$;*
*(iii)* *The Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{Fm}} : \mathcal{T}h\mathcal{S} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{Fm}$ is a structural representation.*

Proof. $(i) \Rightarrow (ii)$: If $\mathcal{S}$ is algebraizable, then the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$ is an order isomorphism which commutes with inverse images of homomorphisms, for arbitrary $\boldsymbol{A}$. So, it preserves arbitrary suprema, it is injective. We are left to see it commutes with endomorphisms. Let $h \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{A})$ be a homomorphism and let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ be an $\mathcal{S}$-filter. On the one hand,

$$\begin{aligned}
F \subseteq h^{-1}\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(hF) \quad &\Rightarrow \quad \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(h^{-1}\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(hF))\\
&\Rightarrow \quad \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq h^{-1}\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(hF)\big)\\
&\Rightarrow \quad h\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq hh^{-1}\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(hF)\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(hF)\big)\\
&\Rightarrow \quad \Theta^{\boldsymbol{A}}_{\mathsf{Alg}^*(\mathcal{S})}\big(h\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(hF)\big)
\end{aligned}$$

using protoalgebracity and the hypothesis. On the other hand, since the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is (always) onto $\mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$, let $G \in \mathcal{F}i_{\mathcal{S}}(\boldsymbol{B})$ be such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(G) =$

$\Theta^{\boldsymbol{A}}_{\mathsf{Alg}^*(\mathcal{S})}\big(h\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big)$. Then,

$$\begin{aligned}\boldsymbol{\Omega}^{\boldsymbol{A}}(G) = \Theta^{\boldsymbol{A}}_{\mathsf{Alg}^*(\mathcal{S})}\big(h\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big) &\Rightarrow h\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G),\\ &\Rightarrow \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq h^{-1}\boldsymbol{\Omega}^{\boldsymbol{A}}(G)\\ &\Rightarrow \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(h^{-1}G)\\ &\Rightarrow F \subseteq h^{-1}G,\\ &\Rightarrow hF \subseteq hh^{-1}G \subseteq G\\ &\Rightarrow \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(hF) \subseteq G,\\ &\Rightarrow \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(hF)\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G) = \Theta^{\boldsymbol{A}}_{\mathsf{Alg}^*(\mathcal{S})}\big(h\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\big)\end{aligned}$$

using the hypothesis, injectivity of $\boldsymbol{\Omega}^{\boldsymbol{A}}$ (which holds by truth-equationality of $\mathcal{S}$), and protoalgebraicity.

$(ii) \Rightarrow (iii)$: Trivial.

$(iii) \Rightarrow (i)$: Let $T, T' \in \mathcal{T}h\mathcal{S}$ such that $T \subseteq T'$. Since $\boldsymbol{\Omega}^{\boldsymbol{Fm}}$ preserves suprema by hypothesis, it follows that

$$\boldsymbol{\Omega}^{\boldsymbol{Fm}}(T) \overset{\mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{Fm}}{\bigvee} \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T') = \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T \overset{\mathcal{T}h\mathcal{S}}{\bigvee} T') = \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T').$$

Hence, $\boldsymbol{\Omega}^{\boldsymbol{Fm}}(T) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T')$. This establishes protoalgebraicity (see for instance, [**23**, Theorem 1.1.3], where it is also proved that monotonicity of $\boldsymbol{\Omega}^{\boldsymbol{Fm}}$ is equivalent to meet-continuity). Together with the fact that $\boldsymbol{\Omega}^{\boldsymbol{Fm}}$ is injective, it follows that $\mathcal{S}$ is weakly algebraizable (see for instance, [**25**]). Finally, let us see that $\boldsymbol{\Omega}^{\boldsymbol{Fm}}$ commutes with *inverse* images by substitutions. Let $\sigma : \boldsymbol{Fm} \to \boldsymbol{Fm}$ be a substitution and let $T \in \mathcal{T}h\mathcal{S}$. We must prove that

$$\boldsymbol{\Omega}^{\boldsymbol{Fm}}(\sigma^{-1}T) = \sigma^{-1}\boldsymbol{\Omega}^{\boldsymbol{Fm}}(T).$$

It is easy to check that $\sigma^{-1}\boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)$ is a congruence on $\boldsymbol{Fm}$, and that furthermore it is compatible with $\sigma^{-1}(T)$. So the inclusion $\sigma^{-1}\boldsymbol{\Omega}^{\boldsymbol{Fm}}(T) \subseteq \boldsymbol{\Omega}^{\boldsymbol{B}}\big(\sigma^{-1}T\big)$ is clear. As for the converse inclusion, we have

$$\begin{aligned}\Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}^*(\mathcal{S})}\big(\sigma\boldsymbol{\Omega}^{\boldsymbol{Fm}}(\sigma^{-1}T)\big) &= \boldsymbol{\Omega}^{\boldsymbol{Fm}}\big(\mathrm{Cn}_{\mathcal{S}}(\sigma\sigma^{-1}T)\big)\\ &\subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T),\end{aligned}$$

using the fact that $\boldsymbol{\Omega}^{\boldsymbol{Fm}} : \mathcal{T}h\mathcal{S} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{Fm}$ commutes with images of substitutions by hypothesis, and that we have just seen it to be moreover order-preserving (bear in mind that $\sigma\sigma^{-1}T \subseteq T$, so $\mathrm{Cn}_{\mathcal{S}}(\sigma\sigma^{-1}T) \subseteq \mathrm{Cn}_{\mathcal{S}}(T) = T$). Then,

$$\begin{aligned}\boldsymbol{\Omega}^{\boldsymbol{Fm}}(\sigma^{-1}T) &\subseteq \sigma^{-1}\sigma\boldsymbol{\Omega}^{\boldsymbol{Fm}}(\sigma^{-1}T)\\ &\subseteq \sigma^{-1}\Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}^*(\mathcal{S})}\big(\sigma\boldsymbol{\Omega}^{\boldsymbol{Fm}}(\sigma^{-1}T)\big) \subseteq \sigma^{-1}\boldsymbol{\Omega}^{\boldsymbol{Fm}}(T).\end{aligned}$$

Altogether, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{Fm}} : \mathcal{T}h\mathcal{S} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{Fm}$ is meet-continuous, injective and commutes with inverse images of substitutions. This establishes that $\mathcal{S}$ is algebraizable, by [**48**, Theorem 5.2]. □

## 4.2. Truth definability in $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$

In this section we adress truth-definability on the class $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$. Recall that by Theorem 0.40 truth definability on any of the classes $\mathsf{LMod}^*(\mathcal{S})$, $\mathsf{Mod}^*(\mathcal{S})$, and $\mathsf{Mod}^{\mathrm{Su}}(\mathcal{S})$ is equivalent to truth-equationality of the underlying logic $\mathcal{S}$. But, *"in* *contrast, the equational definability of truth in* $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$ *does not imply its equational (or even implicit) definability of truth in any of the other matrix semantics mentioned (...).*" [**55**, p.121] So, we pick up this "loose end" of [**55**] and, under the light of Theorem 4.13, unify it with the truth-definability in the remaining matrix semantics.

We start with an auxiliary lemma, whose proof follows that of [**55**, Proposition 22], *mutatis mutandis* for Suszko congruences.

**Lemma 4.15.** *Let* $\mathcal{S}$ *be a logic,* $\boldsymbol{A}$ *an algebra and* $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$*. A set of equations* $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ *defines the* $\mathcal{S}$*-filter of the matrix* $\langle \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F), F/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\rangle$ *if and only if*

$$F = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\}.$$

Proof. Let $\boldsymbol{B} = \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$ and let $\pi : \boldsymbol{A} \to \boldsymbol{B}$ be the canonical projection. Let $a \in A$. Notice that

$$\begin{aligned} \pi(a) \in \boldsymbol{\tau}\boldsymbol{B} \quad &\text{iff} \quad \forall \delta \approx \epsilon \in \boldsymbol{\tau}(x)\ \delta^{\boldsymbol{B}}\big(\pi(a)\big) = \epsilon^{\boldsymbol{B}}\big(\pi(a)\big) \\ &\text{iff} \quad \forall \delta \approx \epsilon \in \boldsymbol{\tau}(x)\ \pi\big(\delta^{\boldsymbol{A}}(a)\big) = \pi\big(\epsilon^{\boldsymbol{A}}(a)\big) \\ &\text{iff} \quad \forall \delta \approx \epsilon \in \boldsymbol{\tau}(x)\ \langle \delta^{\boldsymbol{A}}(a), \epsilon^{\boldsymbol{A}}(a)\rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \\ &\text{iff} \quad \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F). \end{aligned}$$

So, $\boldsymbol{\tau}\boldsymbol{B} = \pi(F)$ if and only if $F = \pi^{-1}\boldsymbol{\tau}\boldsymbol{B} = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\}$. The result should now be clear. □

**Corollary 4.16.** *Truth is equationally definable in* $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$ *if and only if, for every* $T \in \mathcal{T}h\mathcal{S}$*,*

$$T = \{\varphi \in \mathrm{Fm}_{\mathcal{L}} : \boldsymbol{\tau}^{\boldsymbol{Fm}}(\varphi) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T)\}.$$

Next, we prove that the particular case of Lemma 4.1 and Proposition 4.2 for the formula algebras $\boldsymbol{Fm}$ holds under the hypothesis of truth definability in the class $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$. The proofs are entirely analogous to the respective ones for arbitrary algebras.

**Lemma 4.17.** *If truth is equationally definable in* $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$*, then for every* $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$*,*

$$\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{Fm}/\theta\big) = \pi\ \big(\mathcal{T}h\mathcal{S}\big)^{\Gamma} \quad \textit{and} \quad \pi^{-1}\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{Fm}/\theta\big) = \big(\mathcal{T}h\mathcal{S}\big)^{\Gamma}.$$

*where* $\theta := \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{Fm}}(\Gamma))$ *and* $\pi : \boldsymbol{Fm} \to \boldsymbol{Fm}/\theta$ *is the canonical map.*

Proof. First, let us fix $\mathscr{C} := \big\{T \in \mathcal{T}h\mathcal{S} : \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}\big(\boldsymbol{\tau}^{\boldsymbol{Fm}}(\Gamma)\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)\big\}$ and $\theta := \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}\big(\boldsymbol{\tau}^{\boldsymbol{Fm}}(\Gamma)\big)$. It follows by Proposition 1.11 that

$$\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{Fm}/\theta\big) = \pi\mathscr{C} \quad \text{and} \quad \pi^{-1}\mathcal{F}i_{\mathcal{S}}\big(\boldsymbol{Fm}/\theta\big) = \mathscr{C}.$$

Now, since $\boldsymbol{\Omega}^{\boldsymbol{Fm}}(T) \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{Fm}$, it holds

$$\mathscr{C} = \big\{T \in \mathcal{T}h\mathcal{S} : \theta \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)\big\} = \big\{T \in \mathcal{T}h\mathcal{S} : \boldsymbol{\tau}^{\boldsymbol{Fm}}(\Gamma) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)\big\}.$$

But, by Proposition 0.43,

$$\big\{T \in \mathcal{T}h\mathcal{S} : \boldsymbol{\tau}^{\boldsymbol{Fm}}(\Gamma) \subseteq \boldsymbol{\Omega}^{\boldsymbol{Fm}}(T)\big\} = \big\{T \in \mathcal{T}h\mathcal{S} : \Gamma \subseteq T\big\} = \big(\mathcal{T}h\mathcal{S}\big)^{\Gamma}.$$

□

**Proposition 4.18.** *If truth is equationally definable in* $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$*, then for every* $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(\mathrm{Cn}_{\mathcal{S}}(\Gamma)) = \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{Fm}}(\Gamma)).$$

Proof. Let $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$. Fix $\theta \coloneqq \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{Fm}}(\Gamma))$. First, since $\theta \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{Fm}$, it holds $\boldsymbol{Fm}/\theta \in \mathsf{Alg}(\mathcal{S})$, and therefore $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}/\theta}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{Fm}/\theta)\big) = id_{\boldsymbol{Fm}/\theta}$, by (9) on page 21. Now, let $\pi : \boldsymbol{Fm} \to \boldsymbol{Fm}/\theta$ be the canonical map. Then,

$$\begin{aligned}
\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}\big(\mathrm{Cn}_{\mathcal{S}}(\Gamma)\big) &= \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(\Gamma)\\
&= \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}\big((\mathcal{T}h\mathcal{S})^{\Gamma}\big)\\
&= \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}\big(\pi^{-1}\mathcal{F}i_{\mathcal{S}}(\boldsymbol{Fm}/\theta)\big)\\
&= \pi^{-1}\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}/\theta}\big(\mathcal{F}i_{\mathcal{S}}(\boldsymbol{Fm}/\theta)\big)\\
&= \pi^{-1}(id_{\boldsymbol{Fm}/\theta})\\
&= \mathrm{Ker}\,\pi\\
&= \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}\big(\boldsymbol{\tau}^{\boldsymbol{Fm}}(\Gamma)\big),
\end{aligned}$$

using Lemma 4.17 and Proposition 0.31.2. □

Here arrived, we can apply the proofs of Proposition 4.6 and Theorems 4.7, done for an arbitrary algebra $\boldsymbol{A}$, to the formula algebra $\boldsymbol{Fm}$, and obtain:

**Proposition 4.19.** *If truth is equationally definable in* $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$*, then the Suszko operator* $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}$ *preserves suprema, i.e.,*

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}\Big(\bigvee_{i\in I}^{\mathcal{T}h\mathcal{S}} T_i\Big) = \bigvee_{i\in I}^{\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{Fm}} \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(T_i),$$

*for arbitrary families* $\{T_i \in \mathcal{T}h\mathcal{S} : i \in I\}$.

**Proposition 4.20.** *If truth is equationally definable in* $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$*, then the Suszko operator* $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}$ *commutes with substitutions, in the sense that, for every substitution* $\sigma : \boldsymbol{Fm} \to \boldsymbol{Fm}$ *and every* $T \in \mathcal{T}h\mathcal{S}$,

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}\big(\mathrm{Cn}_{\mathcal{S}}(\sigma T)\big) = \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}\big(\sigma\,\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T)\big).$$

Finally, we are able to prove the main result of this section.

**Theorem 4.21.** *Let* $\mathcal{S}$ *be a logic. The following conditions are equivalent:*

*(i) Truth is equationally definable in* $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$*;*

*(ii) The Suszko operator* $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}} : \mathcal{T}h\mathcal{S} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{Fm})$ *is a structural representation.*

Proof. $(i) \Rightarrow (ii)$: Suppose truth is equationally definable in $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$. Then, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}$ is clearly injective. Moreover, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}$ preserves suprema and commutes with substitutions, by Proposition 4.19 and Proposition 4.20.

$(ii) \Rightarrow (i)$: Under the hypothesis, it follows by Theorem 0.26 that there exists $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T) = \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{Fm}}(T))$, for every $T \in \mathcal{T}h\mathcal{S}$. Now, let $T \in \mathcal{T}h\mathcal{S}$ arbitrary and $\varphi \in \mathrm{Fm}_{\mathcal{L}}$. Assume first $\varphi \in T$. Clearly then, $\boldsymbol{\tau}^{\boldsymbol{Fm}}(\varphi) \subseteq \boldsymbol{\tau}^{\boldsymbol{Fm}}(T) \subseteq \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{Fm}}(T)) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T)$. Conversely, assume that $\boldsymbol{\tau}^{\boldsymbol{Fm}}(\varphi) \subseteq$

$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T)$. Then, $\Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{Fm}}(\varphi)) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T) = \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{Fm}}(T))$. Hence,

$$\begin{aligned}
\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}\big(\mathrm{Cn}_{\mathcal{S}}(T,\varphi)\big) &= \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{Fm}}(T,\varphi)) \\
&= \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{Fm}}(T)) \bigvee^{\mathcal{T}h\mathcal{S}} \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{Fm}}(\varphi)) \\
&\subseteq \Theta^{\boldsymbol{Fm}}_{\mathsf{Alg}(\mathcal{S})}(\boldsymbol{\tau}^{\boldsymbol{Fm}}(T)) \\
&= \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T).
\end{aligned}$$

Thus, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}\big(\mathrm{Cn}_{\mathcal{S}}(T,\varphi)\big) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}(T)$. It now follows by injectivity of $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}}$ that $\mathrm{Cn}_{\mathcal{S}}(T,\varphi) = T$, and therefore $\varphi \in T$. Finally, it follows by Corollary 4.16 that truth is equationally definable in $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$. □

As a consequence, since Raftery shows in [**55**, Example 1] that equational definability in the class $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$ does not suffice to ensure that $\mathcal{S}$ is truth-equational, the property of $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{Fm}}_{\mathcal{S}} : \mathcal{T}h\mathcal{S} \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{Fm}$ being a structural representation does *not* lift to arbitrary algebras, bearing in mind Theorem 4.13.

As a final remark, we observe that for a very special set of defining equations, it does hold the equivalence between truth-equationality of a logic $\mathcal{S}$ and the equational definability of truth in the class of matrices $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$.

**Proposition 4.22.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i) $\mathcal{S}$ is truth-equational with a set of defining equations $\{x \approx \top\}$, where $\top$ is a constant term of $\mathsf{LAlg}^{\mathrm{Su}}(\mathcal{S})$;*
*(ii) Truth is equationally definable in $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$ by $\{x \approx \top\}$, where $\top$ is a constant term of $\mathsf{LAlg}^{\mathrm{Su}}(\mathcal{S})$;*

PROOF. $(i) \Rightarrow (ii)$: It follows by [**55**, Proposition 18 (ii)] that truth is equationally definable in $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$ by the equational translation $\boldsymbol{\tau}(x) = \{x \approx \top\}$.
$(ii) \Rightarrow (i)$: It follows by [**55**, Corollary 21] that $\mathsf{LAlg}^{\mathrm{Su}}(\mathcal{S})$ is a $\{x \approx \top\}$-algebraic semantics for $\mathcal{S}$. So, $\mathcal{S}$ is the assertional logic $\mathcal{S}\big(\mathsf{LAlg}^{\mathrm{Su}}(\mathcal{S}), \{x \approx \top\}\big)$. It follows by Proposition 0.42.1 that $\mathcal{S}$ is truth-equational with a set of defining equations $\{x \approx \top\}$, where $\top$ is a constant term of $\mathsf{LAlg}^{\mathrm{Su}}(\mathcal{S})$. □

### 4.3. The largest algebraic semantics for truth-equational logics

Let us recall Definition 0.27. Given a logic $\mathcal{S}$ and $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$, the class of $\boldsymbol{\tau}$-models of $\mathcal{S}$ is defined by

$$\mathsf{K}(\mathcal{S},\boldsymbol{\tau}) \coloneqq \{\boldsymbol{A} : \Gamma \vdash_{\mathcal{S}} \varphi \;\Rightarrow\; \boldsymbol{\tau}(\Gamma) \vDash^{eq}_{\boldsymbol{A}} \boldsymbol{\tau}(\varphi)\}\,.$$

Also, by Proposition 0.28 this class is the largest among all the $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$, if there is one. Notice of course that $\mathsf{K}(\mathcal{S},\boldsymbol{\tau})$ depends explicitly on $\boldsymbol{\tau}$, and that given two sets of equations $\boldsymbol{\tau}, \boldsymbol{\tau}'$, the classes $\mathsf{K}(\mathcal{S},\boldsymbol{\tau})$ and $\mathsf{K}(\mathcal{S},\boldsymbol{\tau}')$ need not coincide.

By simply manipulating the definitions involved, one can re-write the identity above as follows:

**Proposition 4.23.** *Let $\mathcal{S}$ be a logic and $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$. It holds,*

$$\mathsf{K}(\mathcal{S},\boldsymbol{\tau}) = \big\{\boldsymbol{A} : \boldsymbol{\tau}(\boldsymbol{A}) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\}.$$

Another immediate consequence of Proposition 0.28, is that whenever $\mathsf{Alg}(\mathcal{S})$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$, it is necessarily contained in $\mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$. Moreover, Proposition 0.44 tells us that $\mathsf{Alg}(\mathcal{S})$ is always a $\boldsymbol{\tau}$-algebraic semantics for truth-equatinal logics, with $\boldsymbol{\tau}(x)$ a set of defining equations for $\mathcal{S}$. Therefore:

**Corollary 4.24.** *If $\mathcal{S}$ is a truth-equational logic with a set of defining equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$, then*

$$\mathsf{Alg}(\mathcal{S}) \subseteq \mathsf{K}(\mathcal{S}, \boldsymbol{\tau}).$$

It is worth mentioning that these two classes of algebras need not coincide, even at the top most class of logics in the Leibniz hierarchy. In fact, a regularly BP-algebraizable counter-example is $\mathcal{CPC}$, as shown in [**11**, pp. 15-16], having in mind that $\mathsf{Alg}^*(\mathcal{CPC}) = \mathsf{Alg}(\mathcal{CPC}) = \mathsf{BA}$.

We next characterize the least $\mathcal{S}$-filter of an algebra in terms of the defining equations witnessing truth-equationality.

**Lemma 4.25.** *Let $\mathcal{S}$ be a truth-equational logic with a set of defining equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$.*

1. *For every $\boldsymbol{A}$, the least $\mathcal{S}$-filter of $\boldsymbol{A}$ is the $\mathcal{S}$-filter generated by $\boldsymbol{\tau}(\boldsymbol{A})$;*
2. *For $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, the least $\mathcal{S}$-filter of $\boldsymbol{A}$ is exactly $\boldsymbol{\tau}(\boldsymbol{A})$.*

Proof. *1.* Since $\boldsymbol{\tau}^{\boldsymbol{A}}(\boldsymbol{\tau}(\boldsymbol{A})) \subseteq id_{\boldsymbol{A}} \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A})$, it follows by Proposition 0.43 that $\boldsymbol{\tau}(\boldsymbol{A}) \subseteq \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Hence, $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(\boldsymbol{\tau}(\boldsymbol{A})) = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. *2.* Now, let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$. Let $a \in \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then, $a \in F$, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. It follows again by Proposition 0.43 that $\boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \widetilde{\boldsymbol{\Omega}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = id_{\boldsymbol{A}}$. Thus, $a \in \boldsymbol{\tau}(\boldsymbol{A})$. □

Notice that, Proposition 4.23 and Lemma 4.25 provide an alternative proof of Corollary 4.24. Notice also that, for truth-equational logics, the fact that the least $\mathcal{S}$-filter of $\boldsymbol{A}$ is exactly $\boldsymbol{\tau}(\boldsymbol{A})$ does not characterize the algebras in $\mathsf{Alg}(\mathcal{S})$, but rather the algebras in $\mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$. Indeed, in light of Lemma 4.25, Proposition 4.23 can be re-written for truth-equational logics as:

**Proposition 4.26.** *Let $\mathcal{S}$ be a truth-equational logic with a set of defining equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$. It holds,*

$$\mathsf{K}(\mathcal{S}, \boldsymbol{\tau}) = \{\boldsymbol{A} : \boldsymbol{\tau}(\boldsymbol{A}) = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\}.$$

Two interesting corollaries are the following:

**Corollary 4.27.** *If $\mathcal{S}$ is truth-equational witnessed by two sets of defining equations $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$, then*

$$\boldsymbol{\tau}(x) \dashv\vDash^{\mathrm{eq}}_{\mathsf{Alg}(\mathcal{S})} \boldsymbol{\tau}'(x)$$

Proof. Let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$. We know that $\mathsf{Alg}(\mathcal{S}) \subseteq \mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$ and $\mathsf{Alg}(\mathcal{S}) \subseteq \mathsf{K}(\mathcal{S}, \boldsymbol{\tau}')$, by Corollary 4.24. So,

$$\boldsymbol{\tau}(\boldsymbol{A}) = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \boldsymbol{\tau}'(\boldsymbol{A}),$$

by Proposition 4.26. So, given $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $\epsilon^{\boldsymbol{A}}(hx) = \delta^{\boldsymbol{A}}(hx)$, for every $\epsilon \approx \delta \in \boldsymbol{\tau}(x)$, that is, $hx \in \boldsymbol{\tau}(\boldsymbol{A})$, it necessarily holds $hx \in \boldsymbol{\tau}'(\boldsymbol{A})$, that is, $\epsilon'^{\boldsymbol{A}}(hx) = \delta'^{\boldsymbol{A}}(hx)$, for every $\epsilon' \approx \delta' \in \boldsymbol{\tau}(x)$. Hence, $\boldsymbol{\tau}(x) \vDash^{\mathrm{eq}}_{\boldsymbol{A}} \boldsymbol{\tau}'(x)$. Similarly, $\boldsymbol{\tau}'(x) \vDash^{\mathrm{eq}}_{\boldsymbol{A}} \boldsymbol{\tau}(x)$. Thus,

$$\boldsymbol{\tau}(x) \dashv\vDash^{\mathrm{eq}}_{\mathsf{Alg}(\mathcal{S})} \boldsymbol{\tau}'(x).$$

□

So, altough the classes $\mathsf{K}(\mathcal{S},\boldsymbol{\tau})$ and $\mathsf{K}(\mathcal{S},\boldsymbol{\tau}')$ need not coincide, the defining equations involved are interderivable w.r.t. to the equational consequence relation relative to $\mathsf{Alg}(\mathcal{S})$. An example of this situation is deduced from [**34**, Exercise 3.3]. Indeed, since $\mathsf{HA}$ is a $\{\neg\neg x \approx \top\}$-algebraic semantics for $\mathcal{CPC}$, we have $\mathsf{HA} \subseteq \mathsf{K}(\mathcal{CPC},\{\neg\neg x \approx \top\})$, On the other hand, given any $\boldsymbol{A} \in \mathsf{HA}\backslash\mathsf{BA}$, it holds $\boldsymbol{A} \notin \mathsf{K}(\mathcal{CPC},\{x \approx \top\})$, otherwise $\boldsymbol{A} \vDash \neg\neg x \approx \top \rightarrow x \approx \top$. Nevertheless, it does hold $\neg\neg x \approx \top \dashv\vDash^{\mathrm{eq}}_{\mathsf{BA}} x \approx \top$, and of course, $\mathsf{Alg}(\mathcal{CPC}) = \mathsf{BA}$.

**Corollary 4.28.** *If $\mathcal{S}$ is truth-equational with a set of defining equations $\boldsymbol{\tau}(x)$, then*

$$\mathsf{K}(\mathcal{S},\boldsymbol{\tau}_\infty) \subseteq \mathsf{K}(\mathcal{S},\boldsymbol{\tau}).$$

Proof. Let $\boldsymbol{A} \in \mathsf{K}(\mathcal{S},\boldsymbol{\tau}_\infty)$. Since $\boldsymbol{\tau} \subseteq \boldsymbol{\tau}_\infty$, by Corollary 4.4, it follows that

$$\boldsymbol{\tau}_\infty(\boldsymbol{A}) \subseteq \boldsymbol{\tau}(\boldsymbol{A}).$$

But, $\boldsymbol{\tau}_\infty(\boldsymbol{A}) = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, by Proposition 4.26 and assumption. Moreover, $\boldsymbol{\tau}(\boldsymbol{A}) \subseteq \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, by Lemma 4.25.1. So,

$$\boldsymbol{\tau}(\boldsymbol{A}) = \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}.$$

Hence, $\boldsymbol{A} \in \mathsf{K}(\mathcal{S},\boldsymbol{\tau})$, again by Proposition 4.26. □

Thus, $\mathsf{K}(\mathcal{S},\boldsymbol{\tau}_\infty)$ can be seen as a distinguished algebraic semantics, for truth-equational logics. Indeed, it is the least among all the algebraic semantics of the form $\mathsf{K}(\mathcal{S},\boldsymbol{\tau})$, with $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ a defining set of equations for $\mathcal{S}$. Another curiosity concerning the class $\mathsf{K}(\mathcal{S},\boldsymbol{\tau}_\infty)$ is the following:

**Proposition 4.29.** *If $\mathcal{S}$ is truth-equational with a set of defining equations $\boldsymbol{\tau}(x)$, then $\mathsf{K}(\mathcal{S},\boldsymbol{\tau}_\infty)$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$.*

Proof. The result follows from the fact that

$$\mathsf{Alg}(\mathcal{S}) \subseteq \mathsf{K}(\mathcal{S},\boldsymbol{\tau}_\infty) \subseteq \mathsf{K}(\mathcal{S},\boldsymbol{\tau}),$$

and that both classes $\mathsf{Alg}(\mathcal{S})$ and $\mathsf{K}(\mathcal{S},\boldsymbol{\tau})$ are $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$. Indeed, the first inclusion follows by Proposition 0.41 and Corollary 4.24, while the second inclusion follows by Corollary 4.28. That both classes $\mathsf{Alg}(\mathcal{S})$ and $\mathsf{K}(\mathcal{S},\boldsymbol{\tau})$ are $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$ follows by Proposition 0.44 and Proposition 0.28, respectively. □

Compare Proposition 4.29 with Proposition 0.44. Both classes $\mathsf{Alg}(\mathcal{S})$ and $\mathsf{K}(\mathcal{S},\boldsymbol{\tau}_\infty)$ are $\boldsymbol{\tau}$-algebraic semantics, for every set of defining equations $\boldsymbol{\tau}(x)$ witnessing the truth-equationality of $\mathcal{S}$.

## 4.4. Another coherent family of $\mathcal{S}$-compatibility operators

**Definition 4.30.** Let $\mathcal{S}$ be a logic with a $\boldsymbol{\tau}$-algebraic semantics $\mathsf{K}$ such that $\mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{K}$. For every $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, define

$$\Psi^{\boldsymbol{A}}_{\boldsymbol{\tau},\mathsf{K}}(F) := \Theta^{\boldsymbol{A}}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}\big(\boldsymbol{\tau}^{\boldsymbol{A}}(F)\big).$$

Bear in mind that the $\mathbb{IP}_{\mathrm{S}}(\mathsf{K})$-relative congruences of $\boldsymbol{A}$ form a closure system, by Proposition 0.19, and therefore $\Psi^{\boldsymbol{A}}_{\boldsymbol{\tau},\mathsf{K}}$ is well-defined, for every $\boldsymbol{A}$. Also, notice that if $\mathsf{K}$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$, then so is $\mathbb{IP}_{\mathrm{S}}(\mathsf{K})$ (because the equational consequence relations relative to each class coincide). So, another option would be assuming that $\mathsf{K}$ is an algebraic semantics closed under $\mathbb{IP}_{\mathrm{S}}(\mathsf{K})$ — which is *not* a stonger assumption than the plain existence of an algebraic semantics — and work with $\mathsf{K}$-relative congruences instead. We choose a slightly heavier notation in favour of a less lengthy hypothesis.

Let us first see that we are indeed in the presence of a family of $\mathcal{S}$-compatibility operators.

**Proposition 4.31.** *Let $\mathcal{S}$ be a logic with a $\boldsymbol{\tau}$-algebraic semantics $\mathsf{K}$ such that $\mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{K}$. For every $\boldsymbol{A}$, the map $\Psi^{\boldsymbol{A}}_{\boldsymbol{\tau},\mathsf{K}} : \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}(\boldsymbol{A})$ is a congruential order preserving $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$.*

PROOF. Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Notice that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A} \subseteq \mathrm{Con}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}(\boldsymbol{A})$, because $\mathsf{Alg}(\mathcal{S}) = \mathbb{IP}_{\mathrm{S}}(\mathsf{Alg}^*(\mathcal{S})) \subseteq \mathbb{IP}_{\mathrm{S}}(\mathsf{K})$. Moreover, $\boldsymbol{\tau}^{\boldsymbol{A}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, by Proposition 0.32. So,

$$\Theta^{\boldsymbol{A}}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}\big(\boldsymbol{\tau}^{\boldsymbol{A}}(F)\big) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F).$$

Thus, $\Psi^{\boldsymbol{A}}_{\boldsymbol{\tau},\mathsf{K}}$ is an $\mathcal{S}$-compatibility operator on $\boldsymbol{A}$. By definition, it is congruential; and it is clearly order preserving. □

More interestingly, $\Psi_{\boldsymbol{\tau},\mathsf{K}} = \{\Psi^{\boldsymbol{A}}_{\boldsymbol{\tau},\mathsf{K}} : \boldsymbol{A} \text{ an algebra}\}$ is a coherent family of $\mathcal{S}$-operators. We need some auxiliary lemmas to establish it.

**Lemma 4.32.** *Let $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$. For every surjective $h : \boldsymbol{A} \to \boldsymbol{B}$ and every $G \subseteq B$,*

1. $\boldsymbol{\tau}^{\boldsymbol{B}}(G) = h\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G)$;
2. $\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G) \subseteq h^{-1}\boldsymbol{\tau}^{\boldsymbol{B}}(G)$.

PROOF. *1.* For every $X \subseteq A$, $h\boldsymbol{\tau}^{\boldsymbol{A}}(X) = \boldsymbol{\tau}^{\boldsymbol{B}}(hX)$. So, taking $X = h^{-1}G$, we have $h\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G) = \boldsymbol{\tau}^{\boldsymbol{B}}(hh^{-1}G) = \boldsymbol{\tau}^{\boldsymbol{B}}(G)$, using surjectiveness of $h$.
*2.* In general, it holds $\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G) \subseteq h^{-1}h\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G)$. But, $h\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G) = \boldsymbol{\tau}^{\boldsymbol{B}}(G)$, by 1. So, $\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G) \subseteq h^{-1}\boldsymbol{\tau}^{\boldsymbol{B}}(G)$. □

**Lemma 4.33.** *Let $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ and $\mathsf{K}$ be a class of algebras closed under isomorphisms and subdirect products. For every surjective $h : \boldsymbol{A} \to \boldsymbol{B}$ and every $G \subseteq B$,*

$$\big\{\theta \in \mathrm{Con}_{\mathsf{K}}(\boldsymbol{A}) : \mathrm{Ker}h \subseteq \theta \textit{ and } \boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G) \subseteq \theta\big\} = \\ = \big\{h^{-1}\theta' : \theta' \in \mathrm{Con}_{\mathsf{K}}(\boldsymbol{B}) \textit{ and } \boldsymbol{\tau}^{\boldsymbol{B}}(G) \subseteq \theta'\big\}.$$

PROOF. Let $\theta \in \mathrm{Con}_{\mathsf{K}}(\boldsymbol{A})$ such that $\mathrm{Ker}h \subseteq \theta$ and $\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G) \subseteq \theta$. It follows by Lemma 4.32.1 that $\boldsymbol{\tau}^{\boldsymbol{B}}(G) = h\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G) \subseteq h\theta$. Moreover, it follows by Lemma 0.21.2 that $h\theta \in \mathrm{Con}_{\mathsf{K}}(\boldsymbol{B})$. Also, $\theta = h^{-1}h\theta$, by Lemma 0.17.2, since $\mathrm{Ker}h \subseteq \theta$. So, take $\theta' = h\theta$. Conversely, let $\theta' \in \mathrm{Con}_{\mathsf{K}}(\boldsymbol{B})$ such that $\boldsymbol{\tau}^{\boldsymbol{B}}(G) \subseteq \theta'$. It follows by Lemma 4.32.2 that $\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G) \subseteq h^{-1}\boldsymbol{\tau}^{\boldsymbol{B}}(G) \subseteq h^{-1}\theta'$. Also, notice that $\mathrm{Ker}h \subseteq h^{-1}\theta'$, since $id_{\boldsymbol{B}} \subseteq \theta'$. So, take $\theta = h^{-1}\theta'$. □

**Theorem 4.34.** *Let $\mathcal{S}$ be a logic with a $\boldsymbol{\tau}$-algebraic semantics $\mathsf{K}$ such that $\mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{K}$. The family $\Psi_{\boldsymbol{\tau},\mathsf{K}}$ is a coherent family of $\mathcal{S}$-compatibility operators.*

PROOF. Let $\boldsymbol{A}, \boldsymbol{B}$ be any two algebras, $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{B}$ and $h : \boldsymbol{A} \to \boldsymbol{B}$ a surjective homomorphism $\Psi_{\boldsymbol{\tau},\mathsf{K}}$-compatible with $h^{-1}G$. That is, $\mathrm{Ker}h \subseteq \Psi^{\boldsymbol{A}}_{\boldsymbol{\tau},\mathsf{K}}(h^{-1}G) =$

$\Theta^{\boldsymbol{A}}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}\big(\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G)\big)$. Then,

$$\begin{aligned}
\Psi^{\boldsymbol{A}}_{\boldsymbol{\tau},\mathsf{K}}(h^{-1}G) &= \Theta^{\boldsymbol{A}}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}\big(\boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G)\big) \\
&= \bigcap\big\{\theta \in \mathrm{Con}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}(\boldsymbol{A}) : \mathrm{Ker}\,h \subseteq \theta \text{ and } \boldsymbol{\tau}^{\boldsymbol{A}}(h^{-1}G) \subseteq \theta\big\} \\
&= \bigcap\big\{h^{-1}\theta' : \theta' \in \mathrm{Con}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}(\boldsymbol{B}) \text{ and } \boldsymbol{\tau}^{\boldsymbol{B}}(G) \subseteq \theta'\big\} \\
&= h^{-1}\bigcap\big\{\theta' \in \mathrm{Con}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}(\boldsymbol{B}) : \boldsymbol{\tau}^{\boldsymbol{B}}(G) \subseteq \theta'\big\} \\
&= h^{-1}\Theta^{\boldsymbol{B}}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}\big(\boldsymbol{\tau}^{\boldsymbol{B}}(G)\big) \\
&= h^{-1}\Psi^{\boldsymbol{B}}_{\boldsymbol{\tau},\mathsf{K}}(G),
\end{aligned}$$

using Lemma 4.33. □

So, to every logic $\mathcal{S}$ with a $\boldsymbol{\tau}$-algebraic semantics containing $\mathsf{Alg}^*(\mathcal{S})$, one can associate a coherent family of congruential order preserving $\mathcal{S}$-compatibility operators. In particular, for every logic having $\mathsf{Alg}^*(\mathcal{S})$, or equivalently $\mathsf{Alg}(\mathcal{S})$, as a $\boldsymbol{\tau}$-algebraic semantics, there exists a coherent family of congruential order preserving $\mathcal{S}$-compatibility operators. Recall that this is the case for truth-equational logics, by Proposition 0.44.

We finish our study of these families of $\mathcal{S}$-compatibility operators by studying its associated classes of algebras. Let $\mathcal{S}$ be a logic with a $\boldsymbol{\tau}$-algebraic semantics $\mathsf{K}$ such that $\mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{K}$. Recall, by Definition 1.42, that

$$\begin{aligned}
\mathsf{Alg}_{\Psi_{\boldsymbol{\tau},\mathsf{K}}}(\mathcal{S}) &:= \big\{\boldsymbol{A} : \Psi^{\boldsymbol{A}}_{\boldsymbol{\tau},\mathsf{K}}(F) = id_{\boldsymbol{A}}, \text{ for some } F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\} \\
\mathsf{Alg}^{\Psi_{\boldsymbol{\tau},\mathsf{K}}}(\mathcal{S}) &:= \big\{\boldsymbol{A}/\Psi^{\boldsymbol{A}}_{\boldsymbol{\tau},\mathsf{K}}(F) : F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\}
\end{aligned}$$

Since $\Psi_{\boldsymbol{\tau},\mathsf{K}}$ is a coherent family of $\mathcal{S}$-compatibility operators, we know *a priori* by Proposition 1.46 that $\mathsf{Alg}_{\Psi_{\boldsymbol{\tau},\mathsf{K}}}(\mathcal{S}) = \mathsf{Alg}^{\Psi_{\boldsymbol{\tau},\mathsf{K}}}(\mathcal{S})$. But we can in fact give a nicer characterization.

**Proposition 4.35.** *Let $\mathcal{S}$ be a logic with a $\boldsymbol{\tau}$-algebraic semantics $\mathsf{K}$ such that $\mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{K}$. It holds,*

$$\mathsf{Alg}_{\Psi_{\boldsymbol{\tau},\mathsf{K}}}(\mathcal{S}) = \mathsf{Alg}^{\Psi_{\boldsymbol{\tau},\mathsf{K}}}(\mathcal{S}) = \mathbb{IP}_{\mathrm{S}}(\mathsf{K}).$$

Proof. Let $\boldsymbol{A} \in \mathbb{IP}_{\mathrm{S}}(\mathsf{K})$. On the one hand, $\mathsf{K} \subseteq \mathsf{K}(\mathcal{S},\boldsymbol{\tau})$, by Proposition 0.28. On the other hand, $\mathsf{K}(\mathcal{S},\boldsymbol{\tau})$ is a generalized quasivariety, and hence closed under $\mathbb{IP}_{\mathrm{S}}$, by Theorem 0.13. So, $\boldsymbol{A} \in \mathsf{K}(\mathcal{S},\boldsymbol{\tau})$. Therefore, $\boldsymbol{\tau}(\boldsymbol{A}) \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, by Proposition 4.23. Moreover, necessarily $id_{\boldsymbol{A}} \in \mathrm{Con}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}(\boldsymbol{A})$. Hence,

$$\Psi^{\boldsymbol{A}}_{\boldsymbol{\tau},\mathsf{K}}(\boldsymbol{\tau}(\boldsymbol{A})) = \Theta^{\boldsymbol{A}}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}(\boldsymbol{\tau}^{\boldsymbol{A}}(\boldsymbol{\tau}(\boldsymbol{A}))) \subseteq \Theta^{\boldsymbol{A}}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}(id_{\boldsymbol{A}}) = id_{\boldsymbol{A}}.$$

Thus, $\boldsymbol{A} \in \mathsf{Alg}_{\Psi_{\boldsymbol{\tau},\mathsf{K}}}(\mathcal{S})$.
Conversely, let $\boldsymbol{A} \in \mathsf{Alg}_{\Psi_{\boldsymbol{\tau},\mathsf{K}}}(\mathcal{S})$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\Psi^{\boldsymbol{A}}_{\boldsymbol{\tau},\mathsf{K}}(F) = id_{\boldsymbol{A}}$. That is, $\Theta^{\boldsymbol{A}}_{\mathbb{IP}_{\mathrm{S}}(\mathsf{K})}(\boldsymbol{\tau}^{\boldsymbol{A}}(F)) = id_{\boldsymbol{A}}$. Therefore, $id_{\boldsymbol{A}}$ is a $\mathbb{IP}_{\mathrm{S}}(\mathsf{K})$-congruence of $\boldsymbol{A}$. Hence, $\boldsymbol{A} \cong \boldsymbol{A}/id_{\boldsymbol{A}} \in \mathbb{IP}_{\mathrm{S}}(\mathsf{K})$. Since $\mathbb{IP}_{\mathrm{S}}(\mathsf{K})$ is closed under isomorphisms, it follows that $\boldsymbol{A} \in \mathbb{IP}_{\mathrm{S}}(\mathsf{K})$. □

As a final remark, observe that under the assumptions of Definition 4.30, it always holds $\mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{K} \subseteq \mathsf{K}(\mathcal{S},\boldsymbol{\tau})$, bearing in mind Proposition 0.28. The limit cases are therefore $\mathsf{K} = \mathsf{Alg}^*(\mathcal{S})$ and $\mathsf{K} = \mathsf{K}(\mathcal{S},\boldsymbol{\tau})$. In case $\mathcal{S}$ is truth-equational (and consequently has an algebraic semantics containing $\mathsf{Alg}^*(\mathcal{S})$, by Proposition 0.44), taking $\mathsf{K} := \mathsf{Alg}^*(\mathcal{S})$ would lead us to $\Psi = \widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$, in light of Proposition 4.2 and since

$\mathbb{IP}_S(\mathsf{Alg}^*(\mathcal{S})) = \mathsf{Alg}(\mathcal{S})$. Still in case $\mathcal{S}$ is truth-equational, taking $\mathsf{K} \coloneqq \mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$, which complies with the assumption $\mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$ by Corollary 4.24, and with the assumption of being an algebraic semantics by Proposition 0.28, gives us a coherent family of $\mathcal{S}$-compatibility operators $\Psi$ such that $\mathsf{Alg}_\Psi = \mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$, by Proposition 4.35 and the fact that $\mathsf{K}(\mathcal{S}, \boldsymbol{\tau})$ is a generalized quasivariety.

# Part II

# The strong version of a sentential logic

"It has often been observed that some sentential logics come naturally in pairs, one stronger than the other but with the same theorems, and with the peculiarity that the theories of the stronger logic are exactly the theories of the weaker one that are closed under some additional inference rule."

[**37**, p. 2]

CHAPTER 5

# The strong version of a sentential logic

## 5.1. The strong version of a sentential logic

In [**37**] the notion of the strong version of a protoalgebraic logic is introduced, built upon the original definition of Leibniz filter for protoalgebraic logics, also introduced in the cited paper and which we have made reference to on page 48. Now that we have defined a new notion of Leibniz filter for *arbitrary* logics (also on page 48), it is only natural to consider the notion of strong version of an *arbitrary* sentential logic along the same lines of [**37**]. This is what we propose to do in the present section.

Of course, we have introduced not only a new notion of Leibniz filter for arbitrary sentential logics, but also a new, and stronger, notion of Suszko filter. For protoalgebraic logics both concepts happen to coincide with the notion of Leibniz filter given in [**37**]. So, to start with, if we follow the idea developed in [**37**] of considering the strong version of a protoalgebraic logic as the logic induced by its Leibniz filters, we have now two legitimate candidates to consider when defining the strong version of an arbitrary sentential logic. Namely, the logic induced by the class of matrices $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \text{ an algebra, } F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}\}$ of the Leibniz filters, and the logic induced by the class of matrices $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \text{ an algebra, } F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}\}$ of the Suszko filters. But, as we will see, both choices induce the same logic, and moreover the logic so defined has a much simpler definition, independent of the notions of Leibniz and Suszko filters: it is the logic induced by the class of the matrices $\langle \boldsymbol{A}, F\rangle$ where $F$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}$.

**Proposition 5.1.** *Let $\mathcal{S}$ be a logic. The classes of matrices*

$$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \textit{ is an algebra, } F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}\}$$

*and*

$$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \textit{ is an algebra and } F \textit{ is its least } \mathcal{S}\textit{-filter}\}$$

*induce the same logic.*

Proof. First of all recall that on every algebra $\boldsymbol{A}$, the least $\mathcal{S}$-filter is Leibniz. Therefore the second class of matrices is included in the first. This implies that the logic induced by the first class is an extension of the one induced by the second. To prove that the logics are equal, let us see that for every matrix $\langle \boldsymbol{A}, F\rangle$ with $F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ there exists a matrix $\langle \boldsymbol{B}, G\rangle$ where $G$ is the least $\mathcal{S}$-filter of $\boldsymbol{B}$ that induces the same logic as the one induced by $\langle \boldsymbol{A}, F\rangle$. Consider an $\mathcal{L}$-matrix $\langle \boldsymbol{A}, F\rangle$ such that $F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ and let $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ be the canonical quotient homomorphism. By Corollary 1.40, $\pi F^*$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, and since $F$ is Leibniz, $F = F^*$, hence $\pi F = \pi F^*$. Moreover, $\pi$ is a strict surjective homomorphism from $\langle \boldsymbol{A}, F\rangle$ to $\langle \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F), \pi F\rangle$; thus, as it is well known (recall Proposition 0.33), both matrices induce the same logic. $\square$

**Corollary 5.2.** *Let $\mathcal{S}$ be a logic. The classes of matrices*

$$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \text{ is an algebra, } F \in \mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A}\}$$

*and*

$$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \text{ is an algebra, } F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}\}$$

*induce the same logic.*

Proof. Just bear in mind that every Suszko $\mathcal{S}$-filter is a Leibniz $\mathcal{S}$-filter and that the least $\mathcal{S}$-filter of any algebra is a Suszko $\mathcal{S}$-filter. □

Proposition 5.1 and Corollary 5.2 motivate the next definition.

**Definition 5.3.** Let $\mathcal{S}$ be a logic. The *strong version of* $\mathcal{S}$, denoted by $\mathcal{S}^{+}$, is the logic induced by the class of matrices

$$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \text{ an algebra, } F \in \mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A}\};$$

or equivalently, the logic induced by the class of matrices

$$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \text{ an algebra, } F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}\};$$

or equivalently, the logic induced by the class of matrices

$$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \text{ an algebra and } F \text{ is its least } \mathcal{S}\text{-filter}\}.$$

We can restrict the classes of matrices in Definition 5.3 to matrices whose algebras are in $\mathsf{Alg}^{*}(\mathcal{S})$ or $\mathsf{Alg}(\mathcal{S})$.

**Proposition 5.4.** *Let $\mathsf{K}$ be any of the classes of algebras $\mathsf{Alg}^{*}(\mathcal{S})$ or $\mathsf{Alg}(\mathcal{S})$. The logic $\mathcal{S}^{+}$ is induced by any of the classes of matrices $\{\langle \boldsymbol{A}, \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle : \boldsymbol{A} \in \mathsf{K}\}$, $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{K}, F \in \mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A}\}$, and $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{K}, F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}\}$.*

Proof. Let $\vdash$ denote the consequence relation of any of the logics induced by any of the classes of matrices above. In the six cases, it is clear that $\vdash_{\mathcal{S}^{+}} \subseteq \vdash$. Conversely, let $\boldsymbol{A}$ be an arbitrary algebra and $F \in \mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A}$. Then $F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ (which is always a Leibniz $\mathcal{S}$-filter), by Corollary 1.40, and $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \in \mathsf{Alg}^{*}(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{S})$, by Lemma 0.36.1. Moreover the logic induced by $\langle \boldsymbol{A}, F\rangle$ and the logic induced by $\langle \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F), F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\rangle$ are the same, by Proposition 0.33. It follows that $\vdash \subseteq \vdash_{\mathcal{S}^{+}}$. The same reasoning holds with $F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$. □

In the next proposition we collect some obvious consequences of the definition of the strong version for further reference.

**Proposition 5.5.** *Let $\mathcal{S}$ be a logic.*

1. *$\mathcal{S}^{+}$ is an extension of $\mathcal{S}$.*
2. *$\mathcal{F}i_{\mathcal{S}^{+}}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$.*
3. *The Leibniz and Suszko $\mathcal{S}$-filters are $\mathcal{S}^{+}$-filters.*
4. *If the Leibniz operator is order reflecting, then $\mathcal{S}^{+} = \mathcal{S}$. In particular, if $\mathcal{S}$ is truth-equational, then $\mathcal{S}^{+} = \mathcal{S}$.*

Proof. Clearly, $\mathcal{S}^{+}$ is an extension of $\mathcal{S}$. As a consequence, $\mathcal{F}i_{\mathcal{S}^{+}}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$. Also, as $\mathcal{S}^{+}$ is induced by all matrices whose distinguished set is a Leibniz $\mathcal{S}$-filter, as well as by all matrices whose distinguished set is a Suszko $\mathcal{S}$-filter, these special $\mathcal{S}$-filters will always be $\mathcal{S}^{+}$-filters. Thus, $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}^{+}}\boldsymbol{A}$. Finally, assume that the Leibniz operator is order reflecting. It follows by Proposition 2.11 that every $\mathcal{S}$-filter, on an arbitrary algebra, is a Leibniz filter. It should be clear that $\mathcal{S}^{+}$ collapses into $\mathcal{S}$. □

A particular case of the situation considered in the last item of Proposition 5.5 is when $\mathcal{S}$ is Fregean and has theorems. For [4, Corollary 11] shows that in this case $\mathcal{S}$ is truth-equational. Of course, assuming $\mathcal{S}$ protoalgebraic, even if this does not make $\mathcal{S}^+$ collapse into $\mathcal{S}$, it transfers us back to the scope of [37], and so it would be a rather redundant assumption to consider. Thus, when analysing examples, we will concentrate mainly in discussing the strong version of non truth-equational and non protoalgebraic logics. Moreover we will also concentrate on logics with theorems since, as follows from the next proposition, the logics without theorems have, in each type, the same strong version, namely the almost inconsistent logic.

**Proposition 5.6.** *If $\mathcal{S}$ has no theorems, then $\mathcal{S}^+$ is almost inconsistent.*

Proof. If $\mathcal{S}$ has no theorems, then $\varnothing \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$. Necessarily then $\varnothing$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}$, for every $\boldsymbol{A}$. Therefore, $\mathcal{S}^+$ is the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \varnothing\rangle : \boldsymbol{A} \text{ an algebra}\}$. Now, let $\varphi \in \mathrm{Fm}_{\mathcal{L}}$ arbitrary. Notice that, for every $\psi \in \mathrm{Fm}_{\mathcal{L}}$, it vacuously holds $\varphi \vdash_{\mathcal{S}^+} \psi$. Hence, any non-empty $\mathcal{S}^+$-theory is $\mathrm{Fm}_{\mathcal{L}}$. Moreover, $\mathcal{S}^+$ clearly does not have theorems. Thus, $\mathrm{Fm}_{\mathcal{L}}$ and $\varnothing$ are the only $\mathcal{S}^+$-theories. Hence, $\mathcal{S}^+$ is almost inconsistent. □

We state a basic lemma and some of its consequences.

**Lemma 5.7.** *For every $\boldsymbol{A}$, the least $\mathcal{S}$-filter of $\boldsymbol{A}$ and the least $\mathcal{S}^+$-filter of $\boldsymbol{A}$ are the same. In particular, $\mathcal{S}$ and $\mathcal{S}^+$ have the same set of theorems.*

Proof. Let $\boldsymbol{A}$ arbitrary. Notice that the least $\mathcal{S}$-filter of $\boldsymbol{A}$ is always a Leibniz $\mathcal{S}$-filter of $\boldsymbol{A}$. Hence, it is an $\mathcal{S}^+$-filter of $\boldsymbol{A}$, and necessarily the least one, since $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. □

An immediate consequence of this fact is:

**Corollary 5.8.** *For every logic $\mathcal{S}$, $(\mathcal{S}^+)^+ = \mathcal{S}^+$.*

Proof. The logic $(\mathcal{S}^+)^+$ is induced by the class of matrices

$$\big\{\langle \boldsymbol{A}, F\rangle : \ F \text{ is the least } \mathcal{S}^+\text{-filter of } \boldsymbol{A}\big\},$$

which, by Lemma 5.7, is precisely the logic induced by the class of matrices $\{\langle \boldsymbol{A}, F\rangle : F \text{ is the least } \mathcal{S}\text{-filter of } \boldsymbol{A}\}$. □

As a matter of fact, Lemma 5.7 already enables us to establish a criterion which will reveal to be most useful when looking for the strong version of a given logic $\mathcal{S}$.

**Proposition 5.9.** *Let $\mathcal{S}$ be a logic. If $\mathcal{S}'$ is a logic such that*

1. *$\mathcal{S}'$ is truth-equational;*
2. *$\mathsf{Alg}(\mathcal{S}') = \mathsf{Alg}(\mathcal{S})$;*
3. *the least $\mathcal{S}$-filter and the least $\mathcal{S}'$-filter on $\boldsymbol{A}$ coincide, for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}')$;*

*then $\mathcal{S}' = \mathcal{S}^+$.*

Proof. Let $\mathcal{S}'$ be a logic satisfying the conditions 1, 2 and 3. Let $\boldsymbol{\tau}$ be a set of defining equations for $\mathcal{S}'$. On the one hand, it follows by Lemma 4.25 that $\boldsymbol{\tau A}$ is the least $\mathcal{S}'$-filter of $\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}')$. On the other hand, $\mathsf{Alg}(\mathcal{S}')$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}'$, by [55, Corollary 26]. So, $\mathcal{S}'$ is the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \boldsymbol{\tau A}\rangle : \boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}')\}$, by [15, Theorem 2.3]. But, $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}(\mathcal{S}')$ and the least $\mathcal{S}$-filter and the least $\mathcal{S}'$-filter on $\boldsymbol{A}$ coincide, for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}')$, by hypothesis. So, $\mathcal{S}'$ is the logic induced by the class of matrices

$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$ and $F$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}\}$. Now, this logic is precisely $\mathcal{S}^+$, by Proposition 5.4. ☐

The converse of Proposition 5.9 is false, for the strong version of a logic need not be truth-equational. In order to see it, suppose, towards an absurd, that the strong version of every logic is truth-equational. Let $\mathcal{S}$ be any logic such that the Leibniz operator is order reflecting. It follows by Proposition 5.5.4 that $\mathcal{S} = \mathcal{S}^+$. It now follows by hypothesis that $\mathcal{S}$ is truth-equational. We conclude that the order reflecting property of the Leibniz operator suffices to establish truth-equationality. We reach an absurd, for Raftery shows that this is not the case in general. Indeed, [**55**, Example 2] provides a counter-example, as already observed on page 56.

It is also not true, in general, that $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}(\mathcal{S}^+)$. We shall see in Chapter 7 that Positive Modal Logic $\mathcal{PML}$, and the subintuitionistic logics $\mathcal{S}^{\leq}_{\mathsf{WH}}$ and $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{RT})}}$, are logics $\mathcal{S}$ such that $\mathsf{Alg}(\mathcal{S}^+) \subsetneq \mathsf{Alg}(\mathcal{S})$. This contrasts with the protoalgebraic scenario where $\mathsf{Alg}^*(\mathcal{S}^+) = \mathsf{Alg}(\mathcal{S}^+) = \mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S})$ always holds.

**5.1.1. Leibniz and Suszko $\mathcal{S}^+$-filters.** We now briefly study the Leibniz and Suszko $\mathcal{S}^+$-filters. As we shall see, the Leibniz $\mathcal{S}^+$-filters coincide with the Leibniz $\mathcal{S}$-filters. As to Suszko filters, one must pay careful attention and distinguish between Suszko $\mathcal{S}$-filters and Suszko $\mathcal{S}^+$-filters.

First, it is easy to see that Leibniz (respectively, Suszko) $\mathcal{S}$-filters are always Leibniz (respectively, Suszko) $\mathcal{S}^+$-filters:

**Proposition 5.10.** *For every $\boldsymbol{A}$, $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^+}\boldsymbol{A}$ and $\mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i^{*}_{\mathcal{S}^+}\boldsymbol{A}$*

Proof. We had already observed that $[\![F]\!]^*_{\mathcal{S}^+} \subseteq [\![F]\!]^*_{\mathcal{S}}$ and $[\![F]\!]^{\mathrm{Su}}_{\mathcal{S}^+} \subseteq [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}}$. Hence, if $F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$, i.e., $F = \bigcap [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}}$, then necessarily $F = \bigcap [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}^+}$, i.e., $F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^+}\boldsymbol{A}$. The same reasoning holds for Leibniz filters. ☐

But in fact, Leibniz $\mathcal{S}$-filters do coincide with Leibniz $\mathcal{S}^+$-filters. In order to see it, we first prove an auxiliary lemma.

**Lemma 5.11.** *For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, $F^*_{\mathcal{S}} = F^*_{\mathcal{S}^+}$.*

Proof. Since $[\![F]\!]^*_{\mathcal{S}^+} \subseteq [\![F]\!]^*_{\mathcal{S}}$, it is clear that $F^*_{\mathcal{S}} = \bigcap [\![F]\!]^*_{\mathcal{S}} \subseteq \bigcap [\![F]\!]^*_{\mathcal{S}^+} = F^*_{\mathcal{S}^+}$. But also, since $F^*_{\mathcal{S}} \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$ and moreover $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*_{\mathcal{S}})$, because $F^*_{\mathcal{S}} \in [\![F]\!]^*_{\mathcal{S}}$, then $F^*_{\mathcal{S}} \in [\![F]\!]^*_{\mathcal{S}^+}$. Therefore, it must hold $F^*_{\mathcal{S}^+} \subseteq F^*_{\mathcal{S}}$. ☐

Given Lemma 5.11, we shall henceforth denote the least element of any Leibniz class of an $\mathcal{S}^+$-filter, whether considered over $\mathcal{S}$ or $\mathcal{S}^+$, simply by $F^*$. Nonetheless, we shall still have to distinguish between the Suszko filters over these two logics, and we shall do this by explicitly referring to the underlying logic, i.e., using $F^{\mathrm{Su}}_{\mathcal{S}}$ and $F^{\mathrm{Su}}_{\mathcal{S}^+}$. This situation is somehow similar to the one we find in the Leibniz and Suszko operators, where the later is dependent on the logic, which is reflected in the notation $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$, as opposed to $\boldsymbol{\Omega}$.

**Corollary 5.12.** *For every $\boldsymbol{A}$, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}^+}\boldsymbol{A}$.*

Proof. Let $F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$. Then, $F \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$ and moreover $F = F^*_{\mathcal{S}} = F^*_{\mathcal{S}^+}$. Conversely, let $F \in \mathcal{F}i^*_{\mathcal{S}^+}\boldsymbol{A}$. Then, $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and moreover $F = F^*_{\mathcal{S}^+} = F^*_{\mathcal{S}}$. ☐

### 5.2. Leibniz/Suszko $\mathcal{S}$-filters vs. $\mathcal{S}^+$-filters

Following [37], we want to answer the question: When are the $\mathcal{S}^+$-filters on an arbitrary algebra exactly the Leibniz $\mathcal{S}$-filters on it? Actually, in our general setting, it is also reasonable to ask: When are the $\mathcal{S}^+$-filters on an arbitrary algebra exactly the Suszko $\mathcal{S}$-filters on it? The answer for protoalgebraic logics is given in [37, Theorem 19]. Based on this result, two natural conjectures arise. The first is that for any logic $\mathcal{S}$, $\mathcal{S}^+$ is truth-equational if and only if for every algebra $\boldsymbol{A}$ the $\mathcal{S}^+$-filters of $\boldsymbol{A}$ are exactly the Leibniz $\mathcal{S}$-filters of $\boldsymbol{A}$. The second is that $\mathcal{S}^+$ is truth-equational if and only if for every algebra $\boldsymbol{A}$ the $\mathcal{S}^+$-filters of $\boldsymbol{A}$ are exactly the Suszko $\mathcal{S}$-filters of $\boldsymbol{A}$. We record these conjectures here for future reference.

**Conjecture A.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i)* $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$*, for every algebra* $\boldsymbol{A}$*;*
*(ii)* $\mathcal{S}^+$ *is truth-equational.*

**Conjecture B.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i)* $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$*, for every algebra* $\boldsymbol{A}$*;*
*(ii)* $\mathcal{S}^+$ *is truth-equational.*

One of each implications above is easily seen to be true. In the case of Conjecture B, it always holds $(ii) \Rightarrow (i)$.

**Proposition 5.13.** *If, for every algebra $\boldsymbol{A}$, $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$, then $\mathcal{S}^+$ is truth-equational.*

Proof. Let $\boldsymbol{A}$ arbitrary. By Proposition 5.10, $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^+}\boldsymbol{A}$. Hence, under the hypothesis, every $\mathcal{S}^+$-filter of $\boldsymbol{A}$ is an $\mathcal{S}^+$-Suszko filter of $\boldsymbol{A}$. It follows by Theorem 2.30 that $\mathcal{S}^+$ is truth-equational. □

A counter-example to the converse of Proposition 5.13 is the Lukasiewicz's infinite valued logic preserving degrees of truth. We shall see in due time that $Ł^{\leq}_{\infty}$ satisfies that, for every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^*_{Ł^{\leq}_{\infty}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(Ł^{\leq}_{\infty})}\boldsymbol{A}$ is an order-isomorphism (see Proposition 7.60). Now, if the converse implication of Proposition 5.13 were true, then $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^{\mathrm{Su}}_{Ł^{\leq}_{\infty}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(Ł^{\leq}_{\infty})}\boldsymbol{A}$ would be an order-isomorphism, for every $\boldsymbol{A}$. But this condition is equivalent to protoalgebraicity, by Theorem 3.8, and it is known that $Ł^{\leq}_{\infty}$ is not protoalgebraic [35, Theorem 3.11]. We conclude that Conjecture B is false.

As to Conjecture A, it always holds $(i) \Rightarrow (ii)$.

**Proposition 5.14.** *If $\mathcal{S}^+$ is truth-equational, then $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$, for every algebra $\boldsymbol{A}$.*

Proof. Let $\boldsymbol{A}$ arbitrary. Under the hypothesis, it follows by Theorem 2.30 that $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^+}\boldsymbol{A}$. But, $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^+}\boldsymbol{A} \subseteq \mathcal{F}i^*_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$, using Corollary 5.12. Thus, $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} \subseteq \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$. The converse inclusion holds in general. □

We are left with the converse implication of Proposition 5.14. In general, assuming the $\mathcal{S}$-filters of an arbitrary algebra to be exactly the Leibniz $\mathcal{S}$-filters of that same algebra, is equivalent to the order reflecting property of the Leibniz operator over the $\mathcal{S}$-filters (Proposition 5.15). In particular, applied to the logic $\mathcal{S}^+$:

**Proposition 5.15.** *Let $\mathcal{S}$ be a logic. The following conditions are equivalent:*

*(i)* $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$*, for every* $\boldsymbol{A}$*;*
*(ii) The Leibniz operator* $\boldsymbol{\Omega}^{\boldsymbol{A}}$ *is order reflecting on* $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$*, for every* $\boldsymbol{A}$*.*

Proof. Since $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}^+}\boldsymbol{A}$, for every $\boldsymbol{A}$, by Lemma 5.12, the result follows from Proposition 2.11. □

But Raftery has proved that truth-equationality is equivalent to the *completely* order reflecting property of the Leibniz operator. In fact, the logic $\mathcal{S}$ presented in [**55**, Example 2] is a counter-example to the converse of Proposition 5.14. As we have seen already on page 56, the Leibniz operator is order reflecting over the $\mathcal{S}$-filters. As a consequence, $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, for every algebra $\boldsymbol{A}$, using Proposition 2.11 on the first equality. Consequently, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, for every $\boldsymbol{A}$. Moreover, $\mathcal{S} = \mathcal{S}^+$, by Proposition 5.5.4. Nevertheless, Raftery proves that $\mathcal{S}$ is not truth-equational. We conclude that Conjecture A is false as well.

The rest of this section is devoted to investigate two sufficient conditions under which the conditions in Conjecture A are indeed equivalent. The first condition is imposed on the logic $\mathcal{S}^+$, and it is therefore of a more theoretical interest rather than of a practical usage, since we usually do not know *a priori* how does the strong version of a given logic $\mathcal{S}$ behaves. The second condition however is imposed on the logic $\mathcal{S}$, and it will not only be very useful in Chapter 7, but also appears often enough to justify an abstract study of it.

Let us start by proving that requiring $\mathcal{S}^+$ to be protoalgebraic suffices to fill the gap between the property of being order reflecting and that of being *completely* order reflecting of the Leibniz operator on the $\mathcal{S}^+$-filters. Of course, under this assumption, trivially $\mathcal{S}^+$ is truth-equational if and only if it is weakly algebraizable.

**Proposition 5.16.** *Let* $\mathcal{S}$ *be a logic in a countable language. If* $\mathcal{S}^+$ *is protoalgebraic, then the following conditions are equivalent:*

*(i)* $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$*, for every* $\boldsymbol{A}$*;*
*(ii)* $\mathcal{T}h\mathcal{S}^+ = \mathcal{T}h^*\mathcal{S}$*;*
*(iii)* $\mathcal{S}^+$ *is weakly algebraizable;*
*(iv)* $\mathcal{S}^+$ *is truth-equational;*
*(v) For every* $\boldsymbol{A}$*,* $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ *is closed under intersections.*

Proof. *(i)* ⇒ *(ii)*: Trivial.

*(ii)* ⇒ *(iii)*: The Leibniz operator is always injective over Leibniz filters. So, under the hypothesis, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{Fm}} : \mathcal{T}h\mathcal{S}^+ \to \mathrm{Con}_{\mathsf{Alg}(\mathcal{S}^+)}\boldsymbol{Fm}$ is injective. Since it is also surjective (always) and order-preserving (by hypothesis), it follows by [**25**, Theorem 4.8] that $\mathcal{S}^+$ is weakly algebraizable.

*(iii)* ⇒ *(iv)*: This holds by definition.

*(iv)* ⇒ *(v)*: If $\mathcal{S}^+$ is truth-equational, then for every $\boldsymbol{A}$, $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^+}\boldsymbol{A}$, by Theorem 2.30. Moreover, since $\mathcal{S}^+$ is protoalgebraic by hypothesis, for every $\boldsymbol{A}$, $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$, using Corollary 5.12. Thus, for every $\boldsymbol{A}$, $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$, and hence $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ is closed under intersections.

*(v)* ⇒ *(i)*: Since $\mathcal{S}^+$ is protoalgebraic by hypothesis, for every $\boldsymbol{A}$, $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$, using Corollary 5.12. Let $\kappa$ be the cardinal of $\mathcal{S}$. Consider the class of matrices

$$\mathrm{M} = \mathrm{Matr}^*(\mathcal{S}) = \{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \text{ an algebra}, F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}\}$$

as a class of first-order structures. Notice that M is closed under images and inverse images by strict surjective homomorphisms, by Proposition 2.14. Also, by assumption, the family $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ is closed under intersections and, by Lemma 2.28, it is closed under $\kappa$-directed families. It follows by [**26**, Theorem 3] that M is closed under substructures and $\kappa$-reduced products. Finally, M contains all trivial matrices. Since $\vdash_{\mathrm{M}} = \vdash_{\mathcal{S}^+}$, it follows by Czelakowski's Theorem 0.34 that M $=$ Matr($\mathcal{S}^+$). Thus, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, for every $\boldsymbol{A}$. □

Proposition 5.16 generalizes [**37**, Theorem 19], because if $\mathcal{S}$ is protoalgebraic, then so is $\mathcal{S}^+$. An example that is now captured (and that wasn't previously) is Lukasiewicz's infinite valued logic preserving degrees of truth. For $\text{Ł}^{\leq}_{\infty}$ is not protoalgebraic, but its strong version, which is Lukasiewicz's infinite valued logic $\text{Ł}^{1}_{\infty}$, is so.

Next, we move on to the second sufficient condition, this time upon the original logic $\mathcal{S}$, making the conditions in Conjecture A equivalent. We shall henceforth say that a given logic $\mathcal{S}$ enjoys property ($\star$) if and only if

$$\forall \boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}) \quad \boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A} \text{ is an order isomorphism.} \qquad (\star)$$

That ($\star$) holds for any protoalgebraic logic was shown in Proposition 3.1. That ($\star$) is *strictly* weaker than protoalgebraicity is witnessed by Lukasiewicz's infinite valued logic preserving degrees of truth $\text{Ł}^{\leq}_{\infty}$, as we will se in Section 7.4.

We have already seen that ($\star$) can be extended to arbitrary algebras (Proposition 3.2); and that ($\star$) implies $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$ (Lemma 3.3). But in fact, the classes of algebras associated with $\mathcal{S}^+$ also collapse into these two under the property ($\star$).

**Lemma 5.17.** *If a logic $\mathcal{S}$ satisfies ($\star$), then* $\mathsf{Alg}(\mathcal{S}^+) = \mathsf{Alg}^*(\mathcal{S}^+) = \mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$.

Proof. The inclusion $\mathsf{Alg}^*(\mathcal{S}^+) \subseteq \mathsf{Alg}^*(\mathcal{S})$ holds in general, since $\mathcal{S}^+$ is an extension of $\mathcal{S}$. As for the converse inclusion, let $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$. Then, $id_{\boldsymbol{A}} \in \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S})}\boldsymbol{A}$. It follows by hypothesis that $id_{\boldsymbol{A}} = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$, for some $G \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$. Hence, $\mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{Alg}^*(\mathcal{S}^+)$. Next, under our hypothesis, we know by Lemma 3.3 that $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S})$. Then, $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S}^+) \subseteq \mathsf{Alg}(\mathcal{S}^+)$. Finally, the inclusion $\mathsf{Alg}(\mathcal{S}^+) \subseteq \mathsf{Alg}(\mathcal{S})$ holds in general, again because $\mathcal{S}^+$ is an extension of $\mathcal{S}$. □

Notice that Lemma 5.17 might be useful for testing if a given logic satisfies condition ($\star$).

**Proposition 5.18.** *Let $\mathcal{S}$ be a logic satisfying ($\star$). For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*),$$

*and*

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}}_{\mathcal{S}}).$$

Proof. First of all, recall that the property ($\star$) lifts to arbitrary algebras, by Proposition 3.2. So, let $\boldsymbol{A}$ arbitrary. Given $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, there exists $G \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$, by ($\star$). Hence, $[\![F]\!]^* = [\![G]\!]^*$ and therefore $F^* = G^* = G$. Next, we know that $\mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}(\mathcal{S})$, by Lemma 3.3. So, since $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$, it follows by hypothesis that there exists $H \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$

such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(H)$. As a consequence, $[\![F]\!]^{\mathrm{Su}} = [\![H]\!]^{*}$, and therefore $F^{\mathrm{Su}}_{\mathcal{S}} = H^{*} = H$. The result now follows. □

Recall that if $\mathcal{S}$ is protoalgebraic, then it satisfies property $(\star)$. Moreover, the Leibniz and Suszko operators coincide, and $F^{\mathrm{Su}}_{\mathcal{S}} = F^{*}$, for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Hence, the identities in Propositions 5.18 collapse into one. Another immediate consequence of these two propositions is the following:

**Corollary 5.19.** *A logic satisfying $(\star)$ is protoalgebraic if and only if its Leibniz and Suszko filters coincide.*

Proof. Necessity should be clear. As to sufficiency, assume that the Leibniz filters and Suszko filters coincide on arbitrary algebras. As a consequence, for every $\boldsymbol{A}$, $F^{\mathrm{Su}}$ is a Suszko filter of $\boldsymbol{A}$, for it is always a Leibniz one. It follows by Lemma 2.25 and the assumption that, for every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^{\mathrm{Su}}_{\mathcal{S}}$ is the largest Leibniz filter below $F$. But, $F^{\mathrm{Su}}_{\mathcal{S}} \subseteq F^{*} \subseteq F$. Hence, necessarily $F^{\mathrm{Su}}_{\mathcal{S}} = F^{*}$. It now follows by Proposition 5.18 that for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}}_{\mathcal{S}}) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{*}) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F).$$

Thus, $\mathcal{S}$ is protoalgebraic by Proposition 2.5. □

As we have advanced already, property $(\star)$ makes the two conditions in Conjecture A equivalent. We now proceed to prove this fact (Proposition 5.21). To this end, let us first see that the property $(\star)$ is inherited by the strong version $\mathcal{S}^{+}$.

**Lemma 5.20.** *If $\mathcal{S}$ satisfies property $(\star)$, then so does $\mathcal{S}^{+}$.*

Proof. Just notice that $\mathcal{F}i^{*}_{\mathcal{S}^{+}}\boldsymbol{A} = \mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A}$ for every $\boldsymbol{A}$, by Corollary 5.12, and moreover $\mathsf{Alg}^{*}(\mathcal{S}) = \mathsf{Alg}^{*}(\mathcal{S}^{+})$, by Lemma 5.17. □

**Proposition 5.21.** *Let $\mathcal{S}$ be a logic satisfying $(\star)$. The following conditions are equivalent:*

- *(i)* $\mathcal{F}i_{\mathcal{S}^{+}}\boldsymbol{A} = \mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A}$*, for every* $\boldsymbol{A}$*;*
- *(ii)* $\mathcal{T}h\mathcal{S}^{+} = \mathcal{T}h^{*}\mathcal{S}$*;*
- *(iii)* $\mathcal{S}^{+}$ *is weakly algebraizable;*
- *(iv)* $\mathcal{S}^{+}$ *is truth-equational;*
- *(v) Truth is implicitly definable in* $\mathsf{Mod}^{*}(\mathcal{S}^{+})$*.*

Proof. *(i)* ⇒ *(ii)* Trivial.

*(ii)* ⇒ *(iii)*: Our hypothesis, together with property $(\star)$ and Lemma 5.17, gives us that the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{Fm}} : \mathcal{T}h\mathcal{S}^{+} \to \mathrm{Con}_{\mathsf{Alg}^{*}(\mathcal{S}^{+})}\boldsymbol{Fm}$ is an order isomorphism. Thus, $\mathcal{S}^{+}$ is weakly algebraizable, by [**25**, Theorem 4.8].

*(iii)* ⇒ *(iv)* and *(iv)* ⇒ *(v)*: These implications hold in general.

*(v)* ⇒ *(i)*: Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}^{+}}\boldsymbol{A}$. Fix $\boldsymbol{B} \coloneqq \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Let $\pi : \boldsymbol{A} \to \boldsymbol{B}$ be the canonical map. Since $\mathrm{Ker}\pi = \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{*})$, $\mathrm{Ker}\pi$ is compatible with both $F$ and $F^{*}$. Consequently, $F = \pi^{-1}\pi F$ and $F^{*} = \pi^{-1}\pi F^{*}$; moreover, since both $F, F^{*} \in \mathcal{F}i_{\mathcal{S}^{+}}\boldsymbol{A}$, also $\pi F, \pi F^{*} \in \mathcal{F}i_{\mathcal{S}^{+}}\boldsymbol{B}$. Now, it follows by Corollary 1.40 that $\pi F^{*} = (\pi F)^{*}$. Notice that $\mathcal{S}^{+}$ satisfies property $(\star)$, by Lemma 5.20. Therefore, it follows by Proposition 5.18 applied to $\mathcal{S}^{+}$ that $\boldsymbol{\Omega}^{\boldsymbol{B}}(\pi F) = \boldsymbol{\Omega}^{\boldsymbol{B}}\big((\pi F)^{*}\big) = \boldsymbol{\Omega}^{\boldsymbol{B}}(\pi F^{*})$. Moreover, $\boldsymbol{\Omega}^{\boldsymbol{B}}(\pi F) = id_{\boldsymbol{B}}$, by Lemma 1.45. So, both $\langle \boldsymbol{B}, \pi F\rangle, \langle \boldsymbol{B}, \pi F^{*}\rangle \in \mathsf{Mod}^{*}(\mathcal{S})$. It follows by assumption that $\pi F = \pi F^{*}$.

Finally, $F = \pi^{-1}\pi F = \pi^{-1}\pi F^* = F^*$. Therefore, $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} \subseteq \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$. The converse inclusion holds in general. □

Once again, Proposition 5.21 generalizes [**37**, Theorem 19], since every protoalgebraic logic satisfies property $(\star)$.

## 5.3. Full g-models of $\mathcal{S}^+$

We next characterize the full g-models of the strong version $\mathcal{S}^+$ in terms of those of $\mathcal{S}$.

**Proposition 5.22.** *Let $\mathcal{S}$ be a logic. Then $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$ is a full g-model of $\mathcal{S}^+$ if and only if there exists a full model $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ of $\mathcal{S}$ such that $\mathscr{C} = \mathscr{D} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$. In other words,*

$$\mathsf{FGMod}(\mathcal{S}^+) = \{\langle \boldsymbol{A}, \mathscr{C} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}\rangle : \langle \boldsymbol{A}, \mathscr{C}\rangle \in \mathsf{FGMod}(\mathcal{S})\}.$$

Proof. Let $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$ a full g-model of $\mathcal{S}^+$. Take $\mathscr{D} = \{F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$. Clearly, $\mathscr{C} \subseteq \mathscr{D} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, because $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Conversely, let $F \in \mathscr{D} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$. Since $F \in \mathscr{D}$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since $F \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$ and $\mathscr{C} \subseteq \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$ is a full g-model of $\mathcal{S}^+$ by hypothesis, it follows that $F \in \mathscr{C}$. Finally, we show that $\mathscr{D}$ is a full g-model of $\mathcal{S}$. Let $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{D}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Notice that

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \bigcap_{F\in\mathscr{D}} \boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{D}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G).$$

Hence, $G \in \mathscr{D}$. To prove the converse implication, let $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ be a full g-model of $\mathcal{S}$ such that $\mathscr{C} = \mathscr{D} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$. First of all, notice that $\mathscr{D} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} \neq \varnothing$, because $\bigcap \mathscr{D} \in \mathscr{D}$ (since $\mathscr{D}$ is a closure system) and moreover $\bigcap \mathscr{D} \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$ (since the least element of a full g-model of $\mathcal{S}$ is always a Leibniz $\mathcal{S}$-filter). Now, let $G \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$ such that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{D} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Then,

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{D}) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{D} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G).$$

Since $\mathscr{D}$ is a full g-model of $\mathcal{S}$ and $G \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, it follows that $G \in \mathscr{D}$. So, $G \in \mathscr{D} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$. Thus, $\mathscr{D} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$ is a full g-model of $\mathcal{S}^+$. □

If the logic $\mathcal{S}$ and its strong version $\mathcal{S}^+$ share the same algebraic counterpart, that is $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}(\mathcal{S}^+)$, and bearing in mind Corollary 2.3, then for every algebra $\boldsymbol{A}$ the lattice of the full models of $\mathcal{S}$ on $\boldsymbol{A}$ and the lattice of the full models of $\mathcal{S}^+$ on $\boldsymbol{A}$ are isomorphic, because both are isomorphic to the lattice of congruences $\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A}$. Furthermore, the isomorphism has a nice and natural description.

**Proposition 5.23.** *Let $\mathcal{S}$ be a logic such that $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}(\mathcal{S}^+)$. For every $\boldsymbol{A}$ and all full g-models $\mathscr{C}$, $\mathscr{C}'$ of $\mathcal{S}$ on $\boldsymbol{A}$, if $\mathscr{C} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathscr{C}' \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, then $\mathscr{C} = \mathscr{C}'$.*

Proof. Let $\boldsymbol{A}$ arbitrary. Under the assumption, $\mathrm{Con}_{\mathsf{Alg}(\mathcal{S})}\boldsymbol{A} = \mathrm{Con}_{\mathsf{Alg}(\mathcal{S}^+)}\boldsymbol{A}$. Having in mind Corollary 2.3, the poset of full-models of $\mathcal{S}$ on $\boldsymbol{A}$ is order isomorphic to the poset of full-models of $\mathcal{S}^+$ on $\boldsymbol{A}$, under the map $\mathscr{D} \mapsto \overline{\mathscr{D}} := \{F \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{D}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$ — this map is in fact the composition of two isomorphisms $\mathscr{D} \overset{\alpha}{\mapsto} \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{D}) \overset{\beta}{\mapsto} \overline{\mathscr{D}}$, where $\alpha(\mathscr{D}) := \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{D})$ and $\beta(\theta) := \{F \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} : \theta \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$, for every full g-model $\mathscr{D} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $\theta \in \mathrm{Con}_{\mathsf{Alg}(\mathcal{S}^+)}\boldsymbol{A}$. Now, let $\mathscr{C}$, $\mathscr{C}'$ be two full g-models of $\mathcal{S}$ on $\boldsymbol{A}$. Assume $\mathscr{C} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathscr{C}' \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$. We claim that $\overline{\mathscr{C}} = \mathscr{C} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$. Let $F \in \overline{\mathscr{C}}$. Then, $F \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$ and $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since $\mathscr{C}$ is a full g-model of $\mathcal{S}$, it follows that $F \in \mathscr{C}$. Hence, $F \in \mathscr{C} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$. We have established that $\overline{\mathscr{C}} \subseteq \mathscr{C} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$; the converse inclusion should be clear. So, $\overline{\mathscr{C}} = \mathscr{C} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathscr{C}' \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \overline{\mathscr{C}'}$. Next, we claim that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\overline{\mathscr{C}})$. Since $\overline{\mathscr{C}}$ is a full g-model of $\mathcal{S}^+$, we have $\overline{\mathscr{C}} = \{F \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\overline{\mathscr{C}}) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$. So, $\beta\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \beta(\alpha(\mathscr{C})) = \overline{\mathscr{C}} = \beta\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\overline{\mathscr{C}})$. Since $\beta$ is injective, it follows that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\overline{\mathscr{C}})$. Similarly, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}') = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\overline{\mathscr{C}'})$. Hence,

$$\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\overline{\mathscr{C}}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\overline{\mathscr{C}'}) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathscr{C}').$$

It now follows by Proposition 2.4 that $\mathscr{C} = \mathscr{C}'$. ☐

A corollary of the two previous propositions is:

**Corollary 5.24.** *Let $\mathcal{S}$ be a logic such that $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}(\mathcal{S}^+)$. For every algebra $\boldsymbol{A}$, the set of all full g-models of $\mathcal{S}$ on $\boldsymbol{A}$ is order isomorphic to the set of all full g-models of $\mathcal{S}^+$ on $\boldsymbol{A}$, both sets ordered under inclusion. The isomorphism is given by the map $\mathscr{C} \mapsto \mathscr{C} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$.*

It is also natural to consider the special full $\mathcal{S}^+$-models of the form of some Leibniz or Suszko $\mathcal{S}^+$-class. But one must carefully distinguish between Leibniz, or Suszko, classes, when taken over $\mathcal{S}$ and $\mathcal{S}^+$. In general, the later are contained in the former. Indeed,

$$\begin{aligned} [\![F]\!]^*_{\mathcal{S}^+} &= \{G \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\} \\ &\subseteq \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\} \\ &= [\![F]\!]^*_{\mathcal{S}}. \end{aligned}$$

and, since $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}(F) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\big) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}\big((\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A})^F\big) = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}^+}(F)$,

$$\begin{aligned} [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}^+} &= \{G \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}^+}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\} \\ &\subseteq \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\} \\ &= [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}}. \end{aligned}$$

The following two lemmas summarise the situation.

**Lemma 5.25.** *Let $\mathcal{S}$ be a logic and $\mathcal{S}'$ one of its extensions. Then for every algebra $\boldsymbol{A}$ and every $\mathcal{S}'$-filter $F$ of $\boldsymbol{A}$,*

$$[\![F]\!]^*_{\mathcal{S}'} = [\![F]\!]^*_{\mathcal{S}} \cap \mathcal{F}i_{\mathcal{S}'}\boldsymbol{A} \quad \text{and} \quad [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}'} \subseteq [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}} \cap \mathcal{F}i_{\mathcal{S}'}\boldsymbol{A}.$$

*In particular, for every $F \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, $[\![F]\!]^*_{\mathcal{S}^+} = [\![F]\!]^*_{\mathcal{S}} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$ and $[\![F]\!]^{\mathrm{Su}}_{\mathcal{S}^+} \subseteq [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$.*

Proof. The equality is obvious from the definitions. The inclusion follows from the fact that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}'}(F)$. ☐

**Lemma 5.26.** *If $F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$, then $[\![F]\!]^{\mathrm{Su}}_{\mathcal{S}^+} = [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$.*

Proof. If $F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$, then $[\![F]\!]^{\mathrm{Su}}_{\mathcal{S}} = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$, by Lemma 2.21.5. Moreover, since $F = \bigcap [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}}$ and $[\![F]\!]^{\mathrm{Su}}_{\mathcal{S}^+} \subseteq [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}}$, it must be the case $F = \bigcap [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}^+}$ (because $F$ is also an $\mathcal{S}^+$-filter). That is, $F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^+}\boldsymbol{A}$. So, $[\![F]\!]^{\mathrm{Su}}_{\mathcal{S}^+} = (\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A})^F$, again by Lemma 2.21.5. Thus,

$$[\![F]\!]^{\mathrm{Su}}_{\mathcal{S}^+} = (\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A})^F = (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = [\![F]\!]^{\mathrm{Su}}_{\mathcal{S}} \cap \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}.$$

☐

CHAPTER 6

# Definability of Leibniz filters

We have already mentioned that placing $\mathcal{S}$ inside the Leibniz hierarchy, either makes $\mathcal{S}^+$ collapse into $\mathcal{S}$ (assuming $\mathcal{S}$ truth-equational), or makes our study converge with the one in [**37**] (assuming $\mathcal{S}$ protoalgebraic). In order to establish general results which allow us to encompass a wealth of non-protoalgebraic examples, we shall need to impose some condition(s) over $\mathcal{S}$, but one(s) necessarily weaker than protoalgebraicity, and/or weaker than truth-equationality. We will do this through three definability criteria of the Leibniz filters of $\mathcal{S}$ — explicit, logical, and equational definability — , all of which weaker conditions than truth-equationality, as well as through property ($\star$), which we have seen already to be a weaker condition than protoalgebraicity.

## 6.1. Leibniz filters equationally definable

In this section we shall consider a new definability criterion for Leibniz filters. One might say it is an equational analogous to the explicit definability of Leibniz filters considered in [**37**].

Truth-equational logics are characterized by the existence of a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ such that for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F = \{a \in A : \boldsymbol{\tau}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$, that is, by the existence of a set of equations $\boldsymbol{\tau}(x)$ that defines the filters of the logic out of their Leibniz congruence in the way just described (see Proposition 0.43). We will consider this kind of definability of filters enjoyed by truth-equational logics but only for the Leibniz filters and study properties that follow from having the Leibniz filters defined in this way.

**Definition 6.1.** A logic $\mathcal{S}$ *has its Leibniz filters equationally definable*, if there exists a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ such that, for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$F^* = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}.$$

Note that if $\mathcal{S}$ has its Leibniz filters equationally definable, then for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F$ is a Leibniz $\mathcal{S}$-filter if and only if $F = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$. This justifies the name. But our definition provides a definition of the Leibniz filter associated with a filter of the logic out of the Leibniz congruence of the later one.

All truth-equational logics have their Leibniz filters equationally definable because these logics have all the logical filters equationally definable. But the property for a logic of having its Leibniz filters equationally definable is *strictly* weaker than truth-equationality, as it will be witnessed by any of the logics studied in Chapter 7.

**Proposition 6.2.** *Let $\mathcal{S}$ be a logic with its Leibniz filters equationally definable. For every $\boldsymbol{A}$, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ is closed under intersections of arbitrary families. Henceforth, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ is a closure system.*

PROOF. Let $\{F_i : i \in I\} \subseteq \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$. We know that $\bigcap_{i\in I} F_i \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Moreover,

$$\begin{aligned}\textstyle\bigcap_{i\in I} F_i &= \textstyle\bigcap_{i\in I} F_i^* \\ &= \textstyle\bigcap_{i\in I}\{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F_i)\} \\ &\subseteq \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}(\textstyle\bigcap_{i\in I} F_i)\} \\ &= \big(\textstyle\bigcap_{i\in I} F_i\big)^*,\end{aligned}$$

using our hypothesis twice and the fact that $\bigcap_{i\in I} \boldsymbol{\Omega}(F_i) \subseteq \boldsymbol{\Omega}(\bigcap_{i\in I} F_i)$. The converse inclusion holds in general. □

If $\mathcal{S}$ has its Leibniz filters equationally definable, not only for every algebra $\boldsymbol{A}$ the set $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ is a closure system; it is, as we will see in Corollary 6.9, the closure system $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$.

The next characterization of when a logic has its Leibniz filters equationally definable restricts the condition in Definition 6.1 to the algebras in $\mathsf{Alg}^*(\mathcal{S})$.

**Proposition 6.3.** *Let $\mathcal{S}$ be a logic. Then $\mathcal{S}$ has its Leibniz filters equationally definable if and only if there is a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ such that for every $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^* = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$.*

PROOF. The implication from left to right follows from the definition. Assume there is a set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$ such that for every $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^* = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$. Let $\boldsymbol{A}$ be arbitrary and let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Let in addition $\boldsymbol{B} := \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ and $\pi : \boldsymbol{A} \to \boldsymbol{B}$ the canonical map. By Corollary 1.40, $\pi F^* = (\pi F)^*$ and this set is the least $\mathcal{S}$-filter of $\boldsymbol{B}$. Moreover $\boldsymbol{B} \in \mathsf{Alg}^*(\mathcal{S})$. From the assumption it follows that

$$\begin{aligned}(\pi F)^* &= \{\pi(a) : a \in A, \boldsymbol{\tau}^{\boldsymbol{B}}(\pi(a)) \subseteq \boldsymbol{\Omega}^{\boldsymbol{B}}(\pi F)\} \\ &= \{\pi(a) : a \in A, \boldsymbol{\tau}^{\boldsymbol{B}}(\pi(a)) \subseteq id_{\boldsymbol{B}}\}.\end{aligned}$$

Thus we have, $a \in F^*$ if and only if $\pi(a) \in \pi F^* = (\pi F)^*$ if and only if $\boldsymbol{\tau}^{\boldsymbol{B}}(\pi(a)) \subseteq id_{\boldsymbol{B}}$ if and only if $\boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \mathrm{Ker}\pi = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. □

Note that the proof above also works if we take the class of algebras $\mathsf{Alg}(\mathcal{S})$ instead of $\mathsf{Alg}^*(\mathcal{S})$, because the first is include d in the second. A more interesting result is the following.

**Proposition 6.4.** *Let $\mathcal{S}$ be a logic. For any set of equations $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$, the following conditions are equivalent:*

*(i) $\mathcal{S}$ has its Leibniz filters equationally definable by $\boldsymbol{\tau}$.*
*(ii) For every $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$, $\boldsymbol{\tau}\boldsymbol{A}$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}$.*
*(iii) For every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, $\boldsymbol{\tau}\boldsymbol{A}$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}$.*

PROOF. $(i) \Rightarrow (iii)$: Assume that $\mathcal{S}$ has its Leibniz filters equationally definable by $\boldsymbol{\tau}(x)$. Let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$ and let $F$ be its least $\mathcal{S}$-filter. This $\mathcal{S}$-filter is Leibniz. Note now that $\boldsymbol{\tau}^{\boldsymbol{A}}(\boldsymbol{\tau}\boldsymbol{A}) \subseteq id_{\boldsymbol{A}} \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Therefore the assumption implies that $\boldsymbol{\tau}\boldsymbol{A} \subseteq F$. Now let $a \in F$. Then, since $F$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}$, for every $G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $a \in G^*$ and therefore the assumption implies that $\boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)$. Hence, $\boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}) = id_{\boldsymbol{A}}$. Thus, $a \in \boldsymbol{\tau}\boldsymbol{A}$.

$(iii) \Rightarrow (ii)$: This is immediate since $\mathsf{Alg}^*(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{S})$.

$(ii) \Rightarrow (i)$: Assume that for every $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$, $\boldsymbol{\tau}\boldsymbol{A}$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}$. Let

$\boldsymbol{A}$ be arbitrary and let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Consider the algebra $\boldsymbol{B} := \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ and the canonical quotient map $\pi : \boldsymbol{A} \to \boldsymbol{B}$. Then $\boldsymbol{B} \in \mathsf{Alg}^*(\mathcal{S})$. By Corollary 1.40, $\pi F^* = (\pi F)^*$ and it is the least $\mathcal{S}$-filter of $\boldsymbol{B}$. From the assumption we have $\pi F^* = \boldsymbol{\tau}\boldsymbol{B}$. Then, $a \in F^*$ if and only if $\pi(a) \in \pi F^* = \boldsymbol{\tau}\boldsymbol{B}$ if and only if $\boldsymbol{\tau}^{\boldsymbol{B}}(\pi(a)) \subseteq id_{\boldsymbol{B}}$ if and only if $\boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \mathrm{Ker}\pi = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. □

Compare Proposition 6.4 with Lemma 4.25. Assuming $\mathcal{S}^+$ truth-equational with a set of defining equations $\boldsymbol{\tau}$ only gives us that $\boldsymbol{\tau}\boldsymbol{A}$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}^+)$. And recall, $\mathsf{Alg}(\mathcal{S}^+) \subseteq \mathsf{Alg}(\mathcal{S})$. The key point here is that equational definability of Leibniz filters extends this property to the larger class $\mathsf{Alg}(\mathcal{S})$.

Next, we exhibit a large family of logics having its Leibniz filters equationally definable.

**Proposition 6.5.** *If $\mathcal{S}$ is a semilattice-based logic with theorems, then its Leibniz filters are equationally definable by $\boldsymbol{\tau}(x) = \{x \approx \top(x)\}$, with $\top(x) \in \mathrm{Thm}_{\mathcal{S}}$.*

Proof. First, recall from the preliminaries (see page 27) that if $\mathcal{S}$ is a semilattice-based logic with theorems, then every theorem of $\mathcal{S}$ is interpreted as the maximum element for each algebra in $\mathsf{K}$. Let $\top \in \mathrm{Thm}_{\mathcal{S}}$. We can assume, without loss of generality, that $\top$ has at most one variable, say $x \in \mathrm{Var}$. Since $\mathcal{S}$ is a semilattice-based logic with theorems, for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \mathrm{Filt}\boldsymbol{A}$, and hence the least $\mathcal{S}$-filter of $\boldsymbol{A}$ is $\{\top^{\boldsymbol{A}}\}$. Thus, $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \boldsymbol{\tau}\boldsymbol{A}$, with $\boldsymbol{\tau}(x) = \{x \approx \top(x)\}$. The result now follows by Proposition 6.4. □

Since not all logics covered in Chapter 7 will be semilattice-based, we state yet another sufficient condition to cope with the remaining logics.

**Proposition 6.6.** *If $\mathcal{S}^+$ is truth-equational with a set of defining equations $\boldsymbol{\tau}$ and $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}(\mathcal{S}^+)$, then $\mathcal{S}$ has its Leibniz filters equationally definable by $\boldsymbol{\tau}$.*

Proof. Let $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$. We show that $\boldsymbol{\tau}\boldsymbol{A}$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}$. Then Proposition 6.4 implies the result. From the assumption follows that $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}^+)$. Then since $\mathcal{S}^+$ is truth-equational with a set of defining equations $\boldsymbol{\tau}$, we have that $\boldsymbol{\tau}\boldsymbol{A}$ is the least $\mathcal{S}^+$-filter of $\boldsymbol{A}$. But by Lemma 5.7, the least $\mathcal{S}^+$-filter of $\boldsymbol{A}$ equals the least $\mathcal{S}$-filter of $\boldsymbol{A}$. Hence we obtain the desired conclusion. □

We now wish to prove that, under the assumption of equational definability of Leibniz $\mathcal{S}$-filters, the logic $\mathcal{S}^+$ is truth-equational (Corollary 6.8). The next theorem, though it may seem slightly off the topic, turns out to provide the right setting to the establish goal we are after.

**Theorem 6.7.** *If $\mathcal{S}$ is a logic with its Leibniz filters equationally definable by $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$, then all the classes $\mathsf{Alg}^*(\mathcal{S}^+)$, $\mathsf{Alg}(\mathcal{S}^+)$, $\mathsf{Alg}^*(\mathcal{S})$ and $\mathsf{Alg}(\mathcal{S})$ are a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}^+$.*

Proof. Let $\mathsf{K}$ be any of the classes $\mathsf{Alg}^*(\mathcal{S})$ and $\mathsf{Alg}(\mathcal{S})$. We know by Proposition 5.4 that $\mathcal{S}^+$ is the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\rangle : \boldsymbol{A} \in \mathsf{K}\}$. But $\bigcap \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} = \boldsymbol{\tau}\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{K}$, by Proposition 6.4. Hence, $\mathcal{S}^+$ is complete w.r.t. a matrix semantics where truth is equationally definable by $\boldsymbol{\tau}$. It follows by [**15**, Theorem 2.3] that $\mathsf{K}$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}^+$. As for the classes

$\mathsf{Alg}^*(\mathcal{S}^+)$ and $\mathsf{Alg}(\mathcal{S}^+)$, just observe that, under the hypothesis, the Leibniz $\mathcal{S}^+$-filters are also equationally definable by $\boldsymbol{\tau}$, using Lemma 5.11. Since $(\mathcal{S}^+)^+ = \mathcal{S}^+$, by Corollary 5.8, we can apply the proof just done to the logic $\mathcal{S}^+$. □

As a consequence, given a logic $\mathcal{S}$ with its Leibniz filters equationally definable by $\boldsymbol{\tau}$, $\mathsf{Alg}(\mathcal{S})$ is a $\boldsymbol{\tau}$-algebraic semantics for $\mathcal{S}$ if and only if $\mathcal{S} = \mathcal{S}^+$. In particular, since all the examples covered in Chapter 7 will have its Leibniz filters equationally definable by some $\boldsymbol{\tau}$, and none of them coincide with its own strong version, $\mathsf{Alg}(\mathcal{S})$ will not be a $\boldsymbol{\tau}$-algebraic semantics for each such $\mathcal{S}$.

An important consequence of Theorem 6.7 is that the equational definability of the Leibniz filters of $\mathcal{S}$ suffices to ensure the equational definability of (all) $\mathcal{S}^+$-filters, under the same set of equations. That is,

**Corollary 6.8.** *If $\mathcal{S}$ is a logic that has its Leibniz filters equationally definable by $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$, then $\mathcal{S}^+$ is truth-equational with a set of defining equations $\boldsymbol{\tau}$.*

Proof. Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$. Let $a \in F$. Since $\mathcal{S}^+$ has a $\boldsymbol{\tau}$-algebraic semantics by Theorem 6.7, it follows by Proposition 0.32 that $\boldsymbol{\tau}(F) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}^+}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. So, $\boldsymbol{\tau}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Conversely, let $a \in A$ such that $\boldsymbol{\tau}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since $F \in \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, it follows by hypothesis that $a \in F^* \subseteq F$. Thus, $F = \{a \in A : \boldsymbol{\tau}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$. It follows by Proposition 0.43 that $\mathcal{S}^+$ is truth-equational with a set of defining equations $\boldsymbol{\tau}$. □

Notice that Proposition 6.6 establishes a sufficient condition for the converse to hold.

**Corollary 6.9.** *If $\mathcal{S}$ is a logic with its Leibniz filters equationally definable, then $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$.*

Proof. It follows by Corollary 6.8 and Proposition 5.14. □

An immediate consequence of Corollary 6.8 is that, if $\mathcal{S}$ has its Leibniz filters equationally definable by the set of equations $\boldsymbol{\tau}(x) = \{x \approx \top\}$, where $\top$ is a constant term of $\mathsf{Alg}(\mathcal{S})$, then the strong version $\mathcal{S}^+$ is the $\{x \approx \top\}$-assertional logic of $\mathsf{Alg}(\mathcal{S})$. Let us record this fact:

**Corollary 6.10.** *If $\mathcal{S}$ is a logic with its Leibniz filters equationally definable by $\boldsymbol{\tau}(x) = \{x \approx \top\}$, where $\top$ is a constant term of $\mathsf{Alg}(\mathcal{S})$, then $\mathcal{S}^+$ is an assertional logic. Moreover,*

$$\vdash_{\mathcal{S}^+} = \vDash^{\top}_{\mathsf{Alg}^*(\mathcal{S})} = \vDash^{\top}_{\mathsf{Alg}(\mathcal{S})} .$$

Proof. Under the hypothesis, $\mathsf{Alg}(\mathcal{S}^+)$ is a $\{x \approx \top\}$-algebraic semantics for $\mathcal{S}^+$, by Theorem 6.7. The result follows by [**4**, Theorem 7]. The identities follow immediately by Theorem 6.7 and the equational set of equations involved. □

Given Proposition 6.5, we can already summarise the situation for the majority of the examples studied in Chapter 7.

**Corollary 6.11.** *If $\mathcal{S}$ is a semilattice-based logic with theorems, then:*

1. *$\mathcal{S}^+$ is an assertional logic; in particular, $\mathcal{S}^+$ is truth-equational.*
2. $\vdash_{\mathcal{S}^+} = \vDash^{\top}_{\mathsf{Alg}^*(\mathcal{S})} = \vDash^{\top}_{\mathsf{Alg}(\mathcal{S})}$.
3. *For every $\boldsymbol{A}$, $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$.*

Our next goal is to arrive at a general characterization of $F^{\mathrm{Su}}_{\mathcal{S}}$ in terms of the Leibniz $\mathcal{S}$-filters extending $F$, for arbitrary $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, under the assumption of equational definability of Leibniz filters (Corollary 6.13).

Let $\mathcal{S}$ be a logic with Leibniz filters equationally definable by $\boldsymbol{\tau}$. Since for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^{\mathrm{Su}}_{\mathcal{S}}$ is always a Leibniz $\mathcal{S}$-filter (having in mind Proposition 2.9, and the fact $F^{\mathrm{Su}}_{\mathcal{S}}$ is by definition the least element of the $\mathcal{S}$-full model $[\![F]\!]^{\mathrm{Su}}_{\mathcal{S}}$), definability of Leibniz filters immediately implies that $F^{\mathrm{Su}}_{\mathcal{S}} = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}}_{\mathcal{S}})\}$. But as it turns out, for these special Leibniz $\mathcal{S}$-filters, a different equational characterization is also valid, this time using the Suszko operator.

**Proposition 6.12.** *Let $\mathcal{S}$ be a logic with its Leibniz filters equationally definable, say by $\boldsymbol{\tau}(x) \subseteq \mathrm{Eq}_{\mathcal{L}}$. For every $\boldsymbol{A}$, every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ and every $a \in A$,*

$$F^{\mathrm{Su}}_{\mathcal{S}} = \{a \in A : \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)\}.$$

Proof. Consider the canonical map $\pi : \boldsymbol{A} \to \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. Fix $\boldsymbol{B} = \boldsymbol{A}/\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \in \mathsf{Alg}(\mathcal{S})$. By Corollary 1.40, $\pi F^{\mathrm{Su}} = (\pi F)^{\mathrm{Su}}$ and it is the least $\mathcal{S}$-filter of $\boldsymbol{B}$. By Proposition 6.4, $\pi F^{\mathrm{Su}} = \boldsymbol{\tau}\boldsymbol{B}$. Then, $a \in F^{\mathrm{Su}}$ if and only if $\pi(a) \in \pi F^{\mathrm{Su}} = \boldsymbol{\tau}\boldsymbol{B}$ if and only if $\boldsymbol{\tau}^{\boldsymbol{B}}(\pi(a)) \subseteq id_{\boldsymbol{B}}$ if and only if $\boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \mathrm{Ker}\pi = \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F)$. □

**Corollary 6.13.** *Let $\mathcal{S}$ be a logic with its Leibniz filters equationally definable. For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,*

$$F^{\mathrm{Su}}_{\mathcal{S}} = \bigcap_{G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F} G^*.$$

Proof. Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. It holds,

$$\begin{aligned} a \in F^{\mathrm{Su}}_{\mathcal{S}} &\Leftrightarrow \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) \\ &\Leftrightarrow \forall G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F\ \boldsymbol{\tau}^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G) \\ &\Leftrightarrow a \in \textstyle\bigcap_{G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F} G^*, \end{aligned}$$

using Proposition 6.12 and the hypothesis. □

**Corollary 6.14.** *Let $\mathcal{S}$ be a logic with its Leibniz filters equationally definable. For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F$ is a Suszko filter of $\boldsymbol{A}$ if and only if $F \subseteq G^*$, for every $G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$.*

Finally, we consider equational definability of Leibniz filters together with condition $(\star)$.

**Proposition 6.15.** *Let $\mathcal{S}$ be a logic with its Leibniz filters equationally definable. The following conditions are equivalent:*

*(i) $\mathcal{S}^+$ satisfies property $(\star)$;*
*(ii) $\mathcal{S}^+$ is weakly algebraizable.*

Proof. $(i) \Rightarrow (ii)$: By Corollary 5.12 and Corollary 6.9, respectively, for every $\boldsymbol{A}$, $\mathcal{F}i^*_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$. So, by hypothesis, for every $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S}^+)}\boldsymbol{A}$ is an order-isomorphism. This implies that $\mathcal{S}^+$ is weakly algebraizable.

$(ii) \Rightarrow (i)$: This holds in general, since every weakly algebraizable logic is protoalgebraic, and every protoalgebraic logic satisfies property $(\star)$. □

**Corollary 6.16.** *If $\mathcal{S}$ is a logic satisfying* $(\star)$ *and with its Leibniz filters equationally definable, then* $\mathcal{S}^+$ *is weakly algebraizable.*

Proof. Notice that under the assumption, $\mathcal{S}^+$ satisfies property $(\star)$, by Lemma 5.20. The result now follows by Proposition 6.15. □

### 6.2. Leibniz filters explicitly definable

Following [37], we now address explicit definability of Leibniz filters by a set of formulas in at most one variable. We will start by proving that this assumption taken together with condition $(\star)$ actually implies that $\mathcal{S}$ is protoalgebraic. Given that every logic $\mathcal{S}$ studied in the examples is non-protoalgebraic, we know *a priori* that, either $\mathcal{S}$ does not satisfy $(\star)$, or it does not have its Leibniz filters explicitly definable (a non-exclusive disjunction, of course). The main result of the section is Theorem 6.23, where we will see that explicit definability of Leibniz filters implies that the Leibniz $\mathcal{S}$-filters on arbitrary algebras coincide with the $\mathcal{S}^+$-filters.

Let us start by the following definition from [37, Definition 28] but now extended to arbitrary logics and to our notion of Leibniz filter:

**Definition 6.17.** A logic $\mathcal{S}$ *has its Leibniz filters explicitly definable*, if there exists a set of formulas $\Gamma(x) \subseteq \mathrm{Fm}_{\mathcal{L}}$ such that, for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$F^* = \{a \in A : \Gamma^{\boldsymbol{A}}(a) \subseteq F\}.$$

In practice, we might find an explicit characterization of the Leibniz filters of $\mathcal{S}$ only for algebras in $\mathsf{Alg}^*(\mathcal{S})$. But Proposition 6.18 ensures us that it does extend to arbitrary algebras.

**Proposition 6.18.** *Let $\mathcal{S}$ be a logic. Then $\mathcal{S}$ has its Leibniz filters explicitly definable if and only if there is a set of formulas $\Gamma(x) \subseteq \mathrm{Fm}_{\mathcal{L}}$ such that for every $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^* = \{a \in A : \Gamma^{\boldsymbol{A}}(a) \subseteq F\}$.*

Proof. The implication from left to right follows from the definition. Assume there is a a set of formulas $\Gamma(x) \subseteq \mathrm{Fm}_{\mathcal{L}}$ such that, for every $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^* = \{a \in A : \Gamma^{\boldsymbol{A}}(a) \subseteq F\}$. Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Let $\pi : \boldsymbol{A} \to \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ be the canonical map. Fix $\boldsymbol{B} = \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \in \mathsf{Alg}^*(\mathcal{S})$. Notice that $\mathrm{Ker}\pi = \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$. Now, using compatibility arguments, Corollary 1.40, and the hypothesis, $a \in F^*$ if and only if $\pi(a) \in \pi F^*$ if and only if $\pi(a) \in (\pi F)^*$ if and only if $\Gamma^{\boldsymbol{B}}(\pi(a)) \subseteq \pi F$ if and only if $\pi\Gamma^{\boldsymbol{A}}(a) \subseteq \pi F$ if and only if $\Gamma^{\boldsymbol{A}}(a) \in \pi^{-1}(\pi F) = F$. □

A simple observation that will turn out to be quite relevant for proving Theorem 6.23 is that, under the assumption of explicit definability of Leibniz filters, the map $F \mapsto F^*$ from $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ to $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ is monotonic, for every $\boldsymbol{A}$.

**Lemma 6.19.** *Let $\mathcal{S}$ be a logic with Leibniz filters explicitly definable. For every $\boldsymbol{A}$, if $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ are such that $F \subseteq G$, then $F^* \subseteq G^*$.*

Proof. It is quite obvious, because if $F \subseteq G$ and $a \in A$ is such that $\Gamma^{\boldsymbol{A}}(a) \subseteq F$, then $\Gamma^{\boldsymbol{A}}(a) \subseteq G$. □

**Corollary 6.20.** *Let $\mathcal{S}$ be a logic with Leibniz filters explicitly definable. For every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^*$ is the largest Leibniz filter below $F$.*

PROOF. Let $G \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ such that $G \subseteq F$. Then $G = G^* \subseteq F^* \subseteq F$ and we are done. □

A consequence of the monotonicity of the map $F \mapsto F^*$ is the following:

**Proposition 6.21.** *If a logic satisfies condition* $(\star)$ *and has its Leibniz filters explicitly definable, then it is protoalgebraic.*

PROOF. Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Then,

$$\begin{aligned} \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}(F) &= \textstyle\bigcap_{G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F} \boldsymbol{\Omega}^{\boldsymbol{A}}(G) \\ &= \textstyle\bigcap_{G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F} \boldsymbol{\Omega}^{\boldsymbol{A}}(G^*) \\ &= \boldsymbol{\Omega}^{\boldsymbol{A}}(\textstyle\bigcap_{G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F} G^*) \\ &= \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*) \\ &= \boldsymbol{\Omega}^{\boldsymbol{A}}(F), \end{aligned}$$

using Proposition 5.18 (twice), property $(\star)$ and Corollary 6.20. The result now follows by Proposition 2.5. □

By Proposition 6.21, failure of protoalgebraicity must be due to failure of (at least) one of the two conditions, property $(\star)$ or the Leibniz filters being explicitly definable. These two conditions can fail independently of one another. Indeed, as we shall see in Chapter 7, Lukasiewicz's infinite valued logic preserving degrees of truth $\text{Ł}^{\leq}_{\infty}$, satisfies condition $(\star)$ and does not have its Leibniz filters explicitly definable (see Proposition 7.60 and Proposition 7.61, respectively); on the other hand, Positive Modal Logic $\mathcal{PML}$, has its Leibniz filters explicitly definable and does not satisfy property $(\star)$ (see Proposition 7.3.4 and Corollary 7.11, respectively).

Since the map $F \mapsto F^*$ is monotonic under explicit definability of the Leibniz filters (Lemma 6.19), the proofs of [**37**, Proposition 13, Corollary 14] can be replicated under this assumption. We do it in the general setting, without assuming the language to be countable, something assumed in [**37**].

**Proposition 6.22.** *Let $\mathcal{S}$ be a logic with Leibniz filters explicitly definable.*

1. *For every $\boldsymbol{A}$, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ is closed under intersections.*
2. *For every $\boldsymbol{A}$, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ is closed under unions of $\kappa$-directed families, where $\kappa$ is the cardinal of $\mathcal{S}$.*

PROOF. *1.* Let $\Gamma(x) \subseteq \mathrm{Fm}_{\mathcal{L}}$ witness the assumption. Let $\{F_i : i \in I\} \subseteq \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ be a family of Leibniz filters of $\boldsymbol{A}$. We know that $\bigcap_{i\in I} F_i \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Moreover,

$$\begin{aligned} \bigcap_{i\in I} F_i = \bigcap_{i\in I} F_i^* &= \bigcap_{i\in I}\{a \in A : \Gamma^{\boldsymbol{A}}(a) \subseteq F_i\} \\ &= \{a \in A : \Gamma^{\boldsymbol{A}}(a) \subseteq \bigcap_{i\in I} F_i\} = \Big(\bigcap_{i\in I} F_i\Big)^*, \end{aligned}$$

using our hypothesis twice.

*2.* Let $\{F_i : i \in I\} \subseteq \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ be a $\kappa$-directed family. First recall that $\bigcup_{i\in I} F_i$ is an $\mathcal{S}$-filter. Hence, it is necessarily the supremum $\bigvee_{i\in I} F_i$. To prove that it is a Leibniz $\mathcal{S}$-filter, notice that since the map $F \mapsto F^*$ is monotone under the assumption of explicit definability of the Leibniz filters, and $F_i \subseteq \bigcup_{i\in I} F_i$, it holds $F_i = F_i^* \subseteq (\bigcup_{i\in I} F_i)^*$. Hence $(\bigcup_{i\in I} F_i)^* = \bigcup_{i\in I} F_i$. Therefore, $\bigcup_{i\in I} F_i = \bigvee_{i\in I} F_i \subseteq (\bigcup_{i\in I} F_i)^*$. The converse inclusion holds in general. □

**Theorem 6.23.** *Let $\mathcal{S}$ be a logic in a countable language and with its Leibniz filters explicitly definable. For every $\boldsymbol{A}$,*

$$\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}.$$

PROOF. Let $\kappa$ be the cardinal of $\mathcal{S}$. Consider the class of matrices

$$\mathsf{M} = \{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \text{ an algebra}, F \in \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}\}$$

as a class of first-order structures. The class $\mathsf{M}$ is closed under $\mathbb{H}_s$ and $\mathbb{H}_s^{-1}$, by Propositions 2.14. Also, the family $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ is closed under intersections and $\kappa$-directed families, by Propositions 6.22. It follows by [**26**, Theorem 3] that M is closed under substructures and $\kappa$-reduced products. Finally, $\mathsf{M}$ contains all trivial matrices. Since $\vdash_{\mathsf{M}} = \vdash_{\mathcal{S}^+}$, it follows by Czelakowski's Theorem 0.34 that $\mathsf{M} = \mathrm{Matr}(\mathcal{S}^+)$. Thus, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, for every $\boldsymbol{A}$. □

Let $\Gamma(x)$ be a set of formulas in one variable $x$. We use the notation $x \vdash \Gamma(x)$ to refer collectively to the set of rules

$$x \vdash \gamma, \text{ with } \gamma \in \Gamma(x).$$

**Corollary 6.24.** *If $\mathcal{S}$ is a logic in a countable language with its Leibniz filters explicitly definable by $\Gamma(x) \subseteq \mathrm{Fm}_{\mathcal{L}}$, then $\mathcal{S}^+$ is the extension of $\mathcal{S}$ by the additional rules*

$$x \vdash \Gamma(x),$$

PROOF. If $\mathcal{S}$ has its Leibniz filters explicitly definable, then for every $T \in \mathcal{T}h\mathcal{S}$, $T$ is closed under the rules $x \vdash \Gamma(x)$ if and only if $T = T^*$. Since by Theorem 6.23, $\mathcal{T}h\mathcal{S}^+ = \mathcal{T}h^*\mathcal{S}$, the result follows. □

Not only is property $(\star)$ equivalent to $\mathcal{S}$ being protoalgebraic under the assumption of explicit definability of Leibniz filters, by Proposition 6.21, but with Theorem 6.23 at hand, it is also fairly easy to see that it is equivalent to $\mathcal{S}^+$ being protoalgebraic, and even to $\mathcal{S}^+$ weakly algebraizable.

**Corollary 6.25.** *Let $\mathcal{S}$ be a logic in a countable language with Leibniz filters explicitly definable. The following conditions are equivalent:*

*(i)* *$\mathcal{S}^+$ is protoalgebraic;*
*(ii)* *$\mathcal{S}$ is protoalgebraic;*
*(iii)* *$\mathcal{S}$ satisfies property $(\star)$;*
*(iv)* *$\mathcal{S}^+$ satisfies property $(\star)$;*
*(v)* *$\mathcal{S}^+$ is weakly algebraizable.*

PROOF. $(i) \Rightarrow (ii)$: Let $\boldsymbol{A}$ arbitrary and $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$. It follows by Lemma 6.19 that $F^* \subseteq G^*$. Moreover,

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G^*) = \boldsymbol{\Omega}^{\boldsymbol{A}}(G),$$

using Proposition 5.18 (twice) and the hypothesis. Thus, $\mathcal{S}$ is protoalgebraic.

$(ii) \Rightarrow (iii)$: This holds in general, by Proposition 3.1.

$(iii) \Rightarrow (iv)$: This holds in general, by Lemma 5.20.

$(iv) \Rightarrow (v)$: By Corollary 5.12 and Theorem 6.23, respectively, $\mathcal{F}i^*_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, for every $\boldsymbol{A}$. So, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S}^+)}\boldsymbol{A}$ is an order-isomorphism, for every $\boldsymbol{A}$. This implies that $\mathcal{S}^+$ is weakly algebraizable.

$(v) \Rightarrow (i)$: This holds in general. □

We end up this section with a result putting together explicit and equational definability.

**Proposition 6.26.** *If $\mathcal{S}$ has its Leibniz filters both explicit and equationally definable, then for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^* = F^{\mathrm{Su}}_{\mathcal{S}}$.*

Proof. We know by Lemma 6.19 that for every $\boldsymbol{A}$ and every $F, G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ such that $F \subseteq G$, it holds $F^* \subseteq G^*$. So, it follows by Corollary 6.13 that for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^{\mathrm{Su}}_{\mathcal{S}} = \bigcap_{G \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F} G^* = F^*$. □

**Corollary 6.27.** *If $\mathcal{S}$ has its Leibniz filters both explicit and equationally definable, then for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S})$, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$.*

Proof. The inclusion $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ always holds. Let $F \in \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}}\boldsymbol{A}$. Then $F = F^{\mathrm{Su}}_{\mathcal{S}}$. Therefore, by last proposition, $F = F^{\mathrm{Su}}_{\mathcal{S}} = F^*$. Hence $F$ is a Leibniz $\mathcal{S}$-filter. □

Note that the corollary implies that if $\mathcal{S}$ has its Leibniz filters both explicitly and equationally definable, then for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^{\mathrm{Su}}_{\mathcal{S}}$ is a Suszko $\mathcal{S}$-filter. For recall, $F^{\mathrm{Su}}_{\mathcal{S}}$ is always a Leibniz filter, for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$ (although in general, it need not be Suszko filter).

## 6.3. Leibniz filters logically definable

Finally, we consider yet another type of syntactical definability of Leibniz filters, called logical definability, which as we will see is a weaker property than the explicit definability of Leibniz filters. Still, it is enough to guarantee that the Leibniz $\mathcal{S}$-filters, where $\mathcal{S}$ is any logic whose Leibniz filters are logically definable, coincide with the $\mathcal{S}^+$-filters, on arbitrary algebras (Theorem 6.32).

The original motivation behind the following definition is the paragraph after [**37**, Definition 28], where the definability of Leibniz filters closed under a set of logical rules in at most one variable is considered.

**Definition 6.28.** A logic $\mathcal{S}$ *has its Leibniz filters logically definable*, if there exists a set of Hilbert-style rules $\mathcal{H}$, such that, for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F$ is Leibniz if and only if $F$ is closed under the rules in $\mathcal{H}$.

Let us first check that explicit definability is indeed a stronger property than logical definability.

**Lemma 6.29.** *If $\mathcal{S}$ has its Leibniz filters explicitly definable by a set of formulas $\Gamma(x) \subseteq \mathrm{Fm}_{\mathcal{L}}$, then $\mathcal{S}$ has its Leibniz filters logically definable by the set of rules $x \vdash \Gamma(x)$.*

Proof. Assume $\mathcal{S}$ has its Leibniz filters explicitly definable by a set of formulas $\Gamma(x) \subseteq \mathrm{Fm}_{\mathcal{L}}$. Let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. If $F$ is a Leibniz filter of $\boldsymbol{A}$, then $F = F^* = \{a \in F : \Gamma^{\boldsymbol{A}}(a) \subseteq F\}$. Clearly then, $F$ is closed under the set of rules $x \vdash \Gamma(x)$. Conversely, if $F$ is closed under the set of rules $x \vdash_{\mathcal{S}} \Gamma(x)$, then for every $a \in F$, $\Gamma^{\boldsymbol{A}}(a) \subseteq F$. So, under the hypothesis, $F \subseteq F^*$. Therefore, $F$ is a Leibniz filter of $\boldsymbol{A}$. □

The converse is false. That is, logical definability of Leibniz filters does not imply explicit definability of Leibniz filters. A counter-example will appear in Section 7.4.

Similarly to the case of equational and explicit definability of Leibniz filters, logical definability of Leibniz filters on the class $\mathsf{Alg}^*(\mathcal{S})$ suffices to extended the property to arbitrary algebras.

**Proposition 6.30.** *Let $\mathcal{S}$ be a logic. Then $\mathcal{S}$ has its Leibniz filters logically definable if and only if if there exists a set of Hilbert-style rules $\mathcal{H}$, such that, for every $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S})$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F$ is Leibniz if and only if $F$ is closed under the rules in $\mathcal{H}$.*

Proof. Necessity is trivial. As to sufficiency, let $\boldsymbol{A}$ arbitrary and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The matrices $\langle \boldsymbol{A}, F\rangle$ and $\langle \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F), F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)\rangle$ induce the same logic, by Proposition 0.33. Hence, they satisfy the same Hilbert-style rules. In particular, $F$ is closed under the rules in $\mathcal{H}$ if and only $F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is closed under the rules in $\mathcal{H}$. Moreover, $F$ is a Leibniz filter of $\boldsymbol{A}$ if and only if $F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is a Leibniz filter of $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, by Corollary 1.39. Altogether, $F$ is a Leibniz filter of $\boldsymbol{A}$ if and only if $F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is a Leibniz filter of $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ if and only if (using the assumption here) $F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is closed under the rules in $\mathcal{H}$ if and only if $F$ is closed under the rules in $\mathcal{H}$. □

Next, we wish to find analogous results to Theorem 6.23 and Corollary 6.24, this time stated with logical definability of Leibniz filters as hypothesis.

**Proposition 6.31.** *Let $\mathcal{S}$ be a logic with Leibniz filters logically definable by a set of Hilbert-style rules, all of which of cardinality $< \kappa$.*

1. *For every $\boldsymbol{A}$, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ is closed under intersections.*
2. *For every $\boldsymbol{A}$, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ is closed under unions of $\kappa$-directed families.*

Proof. *1.* Let $\{F_i : i \in I\} \subseteq \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ be a family of Leibniz filters of $\boldsymbol{A}$. We know that $\bigcap_{i\in I} F_i \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. Let $\langle \Gamma, \varphi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is a set of Hilbert-style rules witnessing the hypothesis. Let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h(\Gamma) \subseteq \bigcap_{i\in I} F_i$. Since $\bigcap_{i\in I} F_i \subseteq F_i$ and $F_i$ is a Leibniz filter, for every $i \in I$, it follows by hypothesis that $h(\varphi) \in F_i$, for every $i \in I$. Thus, $h(\varphi) \in \bigcap_{i\in I} F_i$. We conclude that $\bigcap_{i\in I} F_i$ is closed under the rules in $\mathcal{H}$. It follows again by hypothesis that $\bigcap_{i\in I} F_i$ is a Leibniz filter of $\boldsymbol{A}$.

*2.* Let $\{F_i : i \in I\} \subseteq \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ be a $\kappa$-directed family of Leibniz filters of $\boldsymbol{A}$. Recall that $\bigcup_{i\in I} F_i$ is an $\mathcal{S}$-filter (see page 15). To prove that it is a Leibniz $\mathcal{S}$-filter, let $\langle \Gamma, \varphi\rangle \in \mathcal{H}$, where $\mathcal{H}$ is a set of Hilbert-style rules witnessing the hypothesis. Let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ such that $h(\Gamma) \subseteq \bigcup_{i\in I} F_i$. Then, for each $\gamma \in \Gamma$, there exists $j_\gamma \in I$ such that $h(\gamma) \subseteq F_{j_\gamma}$. Since $|\Gamma| < \kappa$ and $\{F_i : i \in I\}$ is a $\kappa$-directed family, there exists $j \in I$ such that $F_{j_\gamma} \subseteq F_j$, for every $\gamma \in \Gamma$. Hence, $h(\Gamma) \subseteq F_j$. Since $F_j$ is a Leibniz filter of $\boldsymbol{A}$, it follows by hypothesis that $h(\varphi) \in F_j$. Necessarily then, $h(\varphi) \in \bigcup_{i\in I} F_i$. We conclude that $\bigcup_{i\in I} F_i$ is closed under the rules in $\mathcal{H}$. It follows by hypothesis that $\bigcup_{i\in I} F_i$ is a Leibniz filter of $\boldsymbol{A}$. □

Here arrived, we can mimic the proof of Theorem 6.23, and obtain:

**Theorem 6.32.** *Let $\mathcal{S}$ be a logic in a countable language with Leibniz filters logically definable by a set of Hilbert-style rules. For every $\boldsymbol{A}$,*

$$\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}.$$

We are left to prove an analogous result to Corollary 6.24 for logical definability of Leibniz filters.

**Corollary 6.33.** *If $\mathcal{S}$ is a logic in a countable language with its Leibniz filters logically definable by a set of Hilbert-style rules $\mathcal{H}$, then $\mathcal{S}^+$ is the extension of $\mathcal{S}$ by the additional rules in $\mathcal{H}$.*

Proof. Let $\mathcal{S}'$ denote the extension of $\mathcal{S}$ by the rules in $\mathcal{H}$. On the one hand, $\mathcal{S}^+$ extends $\mathcal{S}$ and moreover every $\mathcal{S}^+$-filter of an arbitrary algebra $\boldsymbol{A}$ is closed under the rules in $\mathcal{H}$, because $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ by Theorem 6.32; hence, $\mathcal{S}' \leq \mathcal{S}^+$. On the other hand, since $\mathcal{S}'$ extends $\mathcal{S}$, it holds $\mathcal{F}i_{\mathcal{S}'}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$; since moreover every $\mathcal{S}'$-filter is closed under the rules in $\mathcal{H}$, it follows by our hypothesis that $\mathcal{F}i_{\mathcal{S}'}\boldsymbol{A} \subseteq \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, for every $\boldsymbol{A}$; therefore, $\mathcal{S}^+ \leq \mathcal{S}'$. □

Notice that Theorem 6.32 generalizes Theorem 6.23 and Corollary 6.33 generalizes Corollary 6.24, in light of Lemma 6.29.

CHAPTER 7

# Examples of non-protoalgebraic logics

In this chapter, we study several examples of non-protoalgebraic and non-truth-equational logics. For each logic considered, we wish to find its strong version, and characterize its Leibniz and Suszko filters. We shall also apply the definability results of Chapter 6 to each example. Sometimes the strong version will turn out to be a well known logic in the literature, while in some cases, at least as far as we know, a logic not previously considered. In the latter situation, we will classify the new logic within the Leibniz hierarchy.

A word on notation. In the following, whenever dealing with Suszko filters, we shall drop the subscript of the underlying logic, as we will always be referring to Suszko filters over the original logic $\mathcal{S}$ and not over its strong version $\mathcal{S}^+$. Actually, since all strong versions covered here will turn out to be truth-equational, there is no risk of misunderstanding, as every $\mathcal{S}^+$-filter is a Suszko $\mathcal{S}^+$-filter, by Theorem 2.30.

## 7.1. Positive Modal Logic

Positive modal Logic, hereby denoted by $\mathcal{PML}$, is the negation-free (or positive) fragment of the local consequence of the least normal modal logic $\mathcal{K}$, which corresponds to the local consequence of the class of all Kripke frames, named $w\mathcal{K}$ in [**37**] (while the global consequence of the class of all Kripke frames is denoted by $s\mathcal{K}$). For information on $\mathcal{PML}$ we address the reader to [**28, 50**].

Consider the modal language $\mathcal{L}' = \{\wedge, \vee, \rightarrow, \neg, \Box, \Diamond, \top, \bot\}$, where we assume the logics $w\mathcal{K}$ and $s\mathcal{K}$ to be formalized. Consider also the positive fragment of $\mathcal{L}'$, given by $\mathcal{L} = \{\wedge, \vee, \Box, \Diamond, \top, \bot\}$. It is well-known that the logic $w\mathcal{K}$ is equivalential, witnessed by the set of congruence formulas $\{\Box^n(p \leftrightarrow q) : n \in \omega\}$, and that $s\mathcal{K}$ is algebraizable, witnessed by the set of congruence formulas $\{p \leftrightarrow q\}$ and the set of defining equations $\{x \approx \top\}$. Its equivalent algebraic semantics is the class of (normal) modal algebras $\mathsf{NMA}$. Since the logic $w\mathcal{K}$ is protoalgebraic, the pair $w\mathcal{K}$ and $s\mathcal{K}$ falls into the scope of [**37**]. It turns out that $(w\mathcal{K})^+ = s\mathcal{K}$ and that $\mathcal{F}i^*_{w\mathcal{K}}\boldsymbol{A} = \mathcal{F}i_{s\mathcal{K}}\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{NMA}$. Furthermore, the $s\mathcal{K}$-filters coincide with the *open* lattice filters on (normal) modal algebras, i.e., the lattice filters closed under the interpretation of $\Box$ (see [**37**, 13ff.], under the notation $\mathsf{K}^w$ and $\mathsf{K}^s$).

Our study of the first non-protoalgebraic example $\mathcal{PML}$ will follow closely the study undertaken in [**37**] of $w\mathcal{K}$ and its strong version $s\mathcal{K}$. The intuitive candidates for both the strong version of $\mathcal{PML}$ and the Leibniz $\mathcal{PML}$-filters will be the ones expected. First, let us introduce the class of algebras which will play the rôle of normal modal algebras when we restrict ourselves to the positive fragment of $w\mathcal{K}$.

**Definition 7.1.** An algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \Box^{\boldsymbol{A}}, \Diamond^{\boldsymbol{A}}, 1, 0\rangle$ is a *positive modal algebra*, if $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, 1, 0\rangle$ is a bounded distributive lattice and $\Box^{\boldsymbol{A}}, \Diamond^{\boldsymbol{A}}$ are two unary modal operations satisfying, for every $a, b \in A$:

1. $\Box^{\boldsymbol{A}}(a \wedge^{\boldsymbol{A}} b) = \Box^{\boldsymbol{A}} a \wedge^{\boldsymbol{A}} \Box^{\boldsymbol{A}} b$;
2. $\Diamond^{\boldsymbol{A}}(a \vee^{\boldsymbol{A}} b) = \Diamond^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} \Diamond^{\boldsymbol{A}} b$;
3. $\Box^{\boldsymbol{A}} a \wedge^{\boldsymbol{A}} \Diamond^{\boldsymbol{A}} b \leq \Diamond^{\boldsymbol{A}}(a \wedge^{\boldsymbol{A}} b)$;
4. $\Box^{\boldsymbol{A}}(a \vee^{\boldsymbol{A}} b) \leq \Box^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} \Diamond^{\boldsymbol{A}} b$;
5. $\Box^{\boldsymbol{A}} 1 = 1$;
6. $\Diamond^{\boldsymbol{A}} 0 = 0$.

The class of all positive modal algebras will be denoted by $\mathsf{PMA}$. The set of lattice filters of a positive modal algebra $\boldsymbol{A}$ will be denoted by $\mathrm{Filt}\boldsymbol{A}$. A lattice filter $F$ of a positive modal algebra $\boldsymbol{A}$ is *open* if it is closed under $\Box^{\boldsymbol{A}}$. The set of open lattice filters of $\boldsymbol{A}$ will be denoted by $\mathrm{Filt}_{\Box}\boldsymbol{A}$.

Notice that distributive lattices, and in particular Boolean algebras, can be expanded as positive modal algebras. Indeed, given a distributive lattice $\boldsymbol{B}$ we can define two unary modal operations $\Box^{\boldsymbol{B}} : B \to B$ and $\Diamond^{\boldsymbol{B}} : B \to B$ both as the identity map on $B$. The algebra we obtain trivially satisfies all the conditions in Definition 7.1.

Let us start by collecting some known facts about the logic $\mathcal{PML}$, which can all be found in [**50**].

**Theorem 7.2.**

1. *$\mathcal{PML}$ is not protoalgebraic.*
2. *$\mathcal{PML}$ is fully selfextensional.*
3. *$\mathcal{PML} = \mathcal{S}^{\leq}_{\mathsf{PMA}}$.*
4. *For every $\boldsymbol{A} \in \mathsf{PMA}$, $\mathrm{Filt}\boldsymbol{A} = \mathcal{F}i_{\mathcal{PML}}\boldsymbol{A}$.*
5. *$\mathsf{Alg}^*(\mathcal{PML}) \subsetneq \mathsf{Alg}(\mathcal{PML}) = \mathsf{PMA}$.*

In particular, given $\boldsymbol{A} \in \mathsf{PMA}$, it follows by 4 above that $\{1\}$ is the least $\mathcal{PML}$-filter of $\boldsymbol{A}$, and hence it is necessarily a Leibniz and Suszko $\mathcal{S}$-filter of $\boldsymbol{A}$.

Since $\mathcal{PML}$ is a semilattice-based logic with theorems, we know in advance several facts about both $\mathcal{PML}$ and $\mathcal{PML}^+$.

**Proposition 7.3.**

1. *$\mathcal{PML}^+$ is assertional, and $\mathcal{PML}^+ = \mathcal{S}^{\top}_{\mathsf{Alg}^*(\mathcal{PML})} = \mathcal{S}^{\top}_{\mathsf{Alg}(\mathcal{PML})} = \mathcal{S}^{\top}_{\mathsf{PMA}}$.*
2. *For every algebra $\boldsymbol{A}$, $\mathcal{F}i_{\mathcal{PML}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{PML}}\boldsymbol{A}$.*
3. *$\mathcal{PML}$ has its Leibniz filters equationally definable by $\boldsymbol{\tau}(x) = \{x \approx \top\}$.*
4. *$\mathcal{PML}$ does not satisfy $(\star)$.*

Proof. Theorem 7.2.3 implies that $\mathcal{PML}$ is semilattice-based, hence 1 and 2 follow by Corollary 6.11. 3 follows by Proposition 6.5. Finally, 4 follows by Lemma 5.17 and Theorem 7.2.5. □

Our next goal is to find an algebraic characterization of the Leibniz $\mathcal{PML}$-filters on positive modal algebras.

**Proposition 7.4.** *Let $\boldsymbol{A} \in \mathsf{PMA}$. Every Leibniz $\mathcal{PML}$-filter of $\boldsymbol{A}$ is an open lattice filter.*

Proof. Let $\boldsymbol{A} \in \mathsf{PMA}$. Since $\mathcal{PML}$ has its Leibniz filters equationally definable by $\boldsymbol{\tau}(x) = \{x \approx \top\}$, if $F \in \mathcal{F}i^*_{\mathcal{PML}}\boldsymbol{A}$, then $F = \{a \in A : \langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$. Let us see that $F$ is closed under $\Box^{\boldsymbol{A}}$. If $a \in F$, then $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, and therefore $\langle \Box^{\boldsymbol{A}} a, \Box^{\boldsymbol{A}} 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Now, since $\Box^{\boldsymbol{A}} 1 = 1 \in F$, it follows that $\Box^{\boldsymbol{A}} a \in F$. □

We could try to show directly that every open lattice filter is a Leibniz filter but we will follow a different, quite informative, path. We will show that every open lattice filter on a positive modal algebra is also a Suszko filter. Therefore it will follow that the Leibniz and Suszko $\mathcal{PML}$-filters coincide on positive modal algebras and that these are precisely the open lattice filters.

**Lemma 7.5.** *Let* $\boldsymbol{A} \in \mathsf{PMA}$ *and* $F \in \mathrm{Filt}_{\Box}\boldsymbol{A}$. *For every* $a, b \in F$,

$$\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{PML}}(F).$$

Proof. Let $F \in \mathrm{Filt}_{\Box}\boldsymbol{A}$. Since, in particular, it is a lattice filter, $F \in \mathcal{F}i_{\mathcal{PML}}\boldsymbol{A}$. Let $a, b \in F$. Let $\varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}$ and $\overline{c} \in A$ arbitrary. We claim that

$$\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F' \Leftrightarrow \varphi^{\boldsymbol{A}}(b, \overline{c}) \in F', \tag{19}$$

for every $F' \in (\mathcal{F}i_{\mathcal{PML}}\boldsymbol{A})^F$. The proof goes by induction on $\varphi \in \mathrm{Fm}_{\mathcal{L}}$.

- $\varphi(x, \overline{z}) = x \in \mathrm{Var}$: Let $F' \in (\mathcal{F}i_{\mathcal{PML}}\boldsymbol{A})^F$. We have $\varphi^{\boldsymbol{A}}(a, \overline{c}) = a$ and $\varphi^{\boldsymbol{A}}(b, \overline{c}) = b$. Since both $a, b \in F \subseteq F'$ by assumption, (19) holds.
- $\varphi(x, \overline{z}) = \top$: Let $F' \in (\mathcal{F}i_{\mathcal{PML}}\boldsymbol{A})^F$. We have $\varphi^{\boldsymbol{A}}(a, \overline{c}) = 1$ and $\varphi^{\boldsymbol{A}}(b, \overline{c}) = 1$. Since $1 \in F'$, (19) holds trivially.
- $\varphi(x, \overline{z}) = \bot$: Let $F' \in (\mathcal{F}i_{\mathcal{PML}}\boldsymbol{A})^F$. We have $\varphi^{\boldsymbol{A}}(a, \overline{c}) = 0$ and $\varphi^{\boldsymbol{A}}(b, \overline{c}) = 0$. Since $0 \notin F'$, (19) holds vacuously.
- $\varphi(x, \overline{z}) = \psi(x, \overline{z}) \wedge \xi(x, \overline{z})$: The inductive hypothesis tells us that (19) holds for $\psi$ and $\xi$. Let $F' \in (\mathcal{F}i_{\mathcal{PML}}\boldsymbol{A})^F$. Assume $\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F'$. Since $\varphi^{\boldsymbol{A}}(a, \overline{c}) = \psi^{\boldsymbol{A}}(a, \overline{c}) \wedge^{\boldsymbol{A}} \xi^{\boldsymbol{A}}(a, \overline{c}) \leq \psi^{\boldsymbol{A}}(a, \overline{c}), \xi(a, \overline{c})$, and $F'$ is upwards-closed, it follows that $\psi^{\boldsymbol{A}}(a, \overline{c}) \in F'$ and $\xi^{\boldsymbol{A}}(a, \overline{c}) \in F'$. It follows by the inductive hypothesis that $\psi^{\boldsymbol{A}}(b, \overline{c}) \in F'$ and $\xi^{\boldsymbol{A}}(b, \overline{c}) \in F'$. Since $F'$ is closed under meets, it follows that $\varphi^{\boldsymbol{A}}(b, \overline{c}) = \psi^{\boldsymbol{A}}(b, \overline{c}) \wedge^{\boldsymbol{A}} \xi^{\boldsymbol{A}}(b, \overline{c}) \in F'$. Similarly, one proves that $\varphi^{\boldsymbol{A}}(b, \overline{c}) \in F'$ implies $\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F'$.
- $\varphi(x, \overline{z}) = \psi(x, \overline{z}) \vee \xi(x, \overline{z})$: The inductive hypothesis tells us that (19) holds for $\psi$ and $\xi$. Let $F' \in (\mathcal{F}i_{\mathcal{PML}}\boldsymbol{A})^F$. Since $\mathsf{PMA}$ is a distributive lattice, it follows as a consequence of the Prime Filter Theorem 0.4, that every lattice filter of $\boldsymbol{A}$ is the intersection of the prime lattice filters containing it. In particular, $F' = \bigcap\{P \in \mathrm{PrFilt}\boldsymbol{A} : F' \subseteq P\}$. Clearly then, $\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F'$ if and only if $\varphi^{\boldsymbol{A}}(a, \overline{c}) \in P$, for every $P \in (\mathrm{PrFilt}\boldsymbol{A})^{F'}$, if and only if $\psi^{\boldsymbol{A}}(a, \overline{c}) \in P$ or $\xi^{\boldsymbol{A}}(a, \overline{c}) \in P$, for every $P \in (\mathrm{PrFilt}\boldsymbol{A})^{F'}$, if and only if $\psi^{\boldsymbol{A}}(b, \overline{c}) \in P$ or $\xi^{\boldsymbol{A}}(b, \overline{c}) \in P$ (using the inductive hypothesis, since $F \subseteq F' \subseteq P$), for every $P \in (\mathrm{PrFilt}\boldsymbol{A})^{F'}$, if and only if $\varphi^{\boldsymbol{A}}(b, \overline{c}) \in P$, for every $P \in (\mathrm{PrFilt}\boldsymbol{A})^{F'}$, if and only if $\varphi^{\boldsymbol{A}}(b, \overline{c}) \in F'$.
- $\varphi(x, \overline{z}) = \Box\psi(x, \overline{z})$: The inductive hypothesis tells us that (19) holds for $\psi$. Let $F' \in (\mathcal{F}i_{\mathcal{PML}}\boldsymbol{A})^F$. Assume $\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F'$. Consider

  $$\Box^{-1}(F') = \{d \in A : \Box^{\boldsymbol{A}} d \in F'\}.$$

  *Claim.* $\Box^{-1}(F')$ *is a lattice filter extending* $F$*:* Since $\Box^{\boldsymbol{A}}1 = 1$, it holds $1 \in \Box^{-1}(F')$. Let $d, e \in \Box^{-1}(F')$. Then, $\Box^{\boldsymbol{A}} d \in F'$ and $\Box^{\boldsymbol{A}} e \in F'$. Since $F'$ is closed under meets, it follows that $\Box^{\boldsymbol{A}} d \wedge^{\boldsymbol{A}} \Box^{\boldsymbol{A}} e \in F'$. But, $\Box^{\boldsymbol{A}} d \wedge^{\boldsymbol{A}} \Box^{\boldsymbol{A}} e = \Box^{\boldsymbol{A}}(d \wedge^{\boldsymbol{A}} e)$, because $\boldsymbol{A} \in \mathsf{PMA}$. So, $d \wedge^{\boldsymbol{A}} e \in \Box^{-1}(F')$. Now, let $d \in \Box^{-1}(F')$ and $d \leq e$. Then $\Box^{\boldsymbol{A}} d \leq \Box^{\boldsymbol{A}} e$, because $\boldsymbol{A} \in \mathsf{PMA}$. Since $\Box^{\boldsymbol{A}} d \in F'$ and $F'$ is upwards-closed, it follows that $\Box^{\boldsymbol{A}} e \in F'$. So, $e \in \Box^{-1}(F')$. Finally, let $d \in F$. Since $F$ is open, it follows that $\Box^{\boldsymbol{A}} d \in F$. Since $F \subseteq F'$, it follows that $\Box^{\boldsymbol{A}} d \in F'$. So, $d \in \Box^{-1}(F')$. Thus, $F \subseteq \Box^{-1}(F')$.

  Now, notice that $\psi^{\boldsymbol{A}}(a, \overline{c}) \in \Box^{-1}(F')$, because $\varphi^{\boldsymbol{A}}(a, \overline{c}) = \Box^{\boldsymbol{A}}\psi^{\boldsymbol{A}}(a, \overline{c}) \in F'$. Since $\Box^{-1}(F') \in (\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ by the Claim, it follows by the inductive hypothesis that $\psi^{\boldsymbol{A}}(b,\overline{c}) \in \Box^{-1}(F')$. That is, $\varphi^{\boldsymbol{A}}(b,\overline{c}) = \Box^{\boldsymbol{A}}\psi^{\boldsymbol{A}}(b,\overline{c}) \in F'$. Similarly one proves that $\varphi^{\boldsymbol{A}}(b,\overline{c}) \in F'$ implies $\varphi^{\boldsymbol{A}}(a,\overline{c}) \in F'$.

- $\varphi(x,\overline{z}) = \Diamond\psi(x,\overline{z})$: The inductive hypothesis tells us that (19) holds for $\psi$. Let $F' \in (\mathcal{F}i_{\mathcal{PML}}\boldsymbol{A})^F$. Assume $\varphi^{\boldsymbol{A}}(a,\overline{c}) \in F'$. Suppose, towards a contradiction, that $\varphi^{\boldsymbol{A}}(b,\overline{c}) \notin F'$. We claim that

$$\psi^{\boldsymbol{A}}(b,\overline{c}) \notin \mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{S}}(F,\psi^{\boldsymbol{A}}(a,\overline{c})).$$

For if not, let $d \in F$ such that $d \wedge^{\boldsymbol{A}} \psi^{\boldsymbol{A}}(a,\overline{c}) \leq \psi^{\boldsymbol{A}}(b,\overline{c})$. Notice that $\Box d \in F \subseteq F'$, because $F$ is open. Also, $\Diamond^{\boldsymbol{A}}(\psi^{\boldsymbol{A}}(a,\overline{c})) = \varphi^{\boldsymbol{A}}(a,\overline{c}) \in F'$, by assumption. So, $\Box^{\boldsymbol{A}} d \wedge^{\boldsymbol{A}} \Diamond^{\boldsymbol{A}}\big(\psi^{\boldsymbol{A}}(a,\overline{c})\big) \in F'$, because $F'$ is closed under meets. Hence,

$$\Box^{\boldsymbol{A}} d \wedge^{\boldsymbol{A}} \Diamond^{\boldsymbol{A}}\big(\psi^{\boldsymbol{A}}(a,\overline{c})\big) \leq \Diamond^{\boldsymbol{A}}\big(d \wedge^{\boldsymbol{A}} \psi^{\boldsymbol{A}}(a,\overline{c})\big) \leq \Diamond^{\boldsymbol{A}}\psi^{\boldsymbol{A}}(b,\overline{c}),$$

using the fact that $\boldsymbol{A} \in \mathsf{PMA}$ and the monotonicity of $\Diamond$. Since $F'$ is upwards-closed, it follows that $\varphi^{\boldsymbol{A}}(b,\overline{c}) = \Diamond^{\boldsymbol{A}}\psi^{\boldsymbol{A}}(b,\overline{c}) \in F'$, which contradicts our assumption. But then, $\mathrm{Fg}^{\boldsymbol{A}}_{\mathcal{PML}}\big(F,\psi^{\boldsymbol{A}}(a,\overline{c})\big)$ is lattice filter extending $F$ which contains $\psi^{\boldsymbol{A}}(a,\overline{c})$ but does not contain $\psi^{\boldsymbol{A}}(b,\overline{c})$. This contradicts our inductive hypothesis.

From (19) and Corollary 0.30 it follows that $\langle a,b\rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{PML}}(F)$. □

**Proposition 7.6.** *Let $\boldsymbol{A} \in \mathsf{PMA}$. Every open lattice filter of $\boldsymbol{A}$ is a Suszko $\mathcal{PML}$-filter.*

Proof. Let $\boldsymbol{A} \in \mathsf{PMA}$ and $F \in \mathrm{Filt}_{\Box}\boldsymbol{A}$. Let $a \in F$. Since also $1 \in F$, it follows by Lemma 7.5 that $\langle 1,a\rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{PML}}(F)$. Now, since $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{PML}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}})$, and moreover $1 \in F^{\mathrm{Su}}$ (bear in mind that $F^{\mathrm{Su}} \in \mathcal{F}i_{\mathcal{PML}}\boldsymbol{A} = \mathrm{Filt}\boldsymbol{A}$), it follows that $a \in F^{\mathrm{Su}}$. So, $F \subseteq F^{\mathrm{Su}}$. Thus, $F$ is a Suszko filter of $\boldsymbol{A}$. □

**Theorem 7.7.** *Let $\boldsymbol{A} \in \mathsf{PMA}$. The Leibniz and Suszko $\mathcal{PML}$-filters of $\boldsymbol{A}$ coincide with the open lattice filters of $\boldsymbol{A}$. That is,*

$$\mathcal{F}i^{*}_{\mathcal{PML}}\boldsymbol{A} = \mathcal{F}i^{\mathrm{Su}}_{\mathcal{PML}}\boldsymbol{A} = \mathrm{Filt}_{\Box}\boldsymbol{A}.$$

Proof. Just notice that $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{PML}}\boldsymbol{A} \subseteq \mathcal{F}i^{*}_{\mathcal{PML}}\boldsymbol{A} \subseteq \mathrm{Filt}_{\Box}\boldsymbol{A} \subseteq \mathcal{F}i^{\mathrm{Su}}_{\mathcal{PML}}\boldsymbol{A}$, using Propositions 7.4 and 7.6. □

We next address the explicit definability of the Leibniz $\mathcal{PML}$-filters. Recall that in general, given an arbitrary logic $\mathcal{S}$, an algebra $\boldsymbol{A}$, and an $\mathcal{S}$-filter $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^{\mathrm{Su}}_{\mathcal{S}}$ is a Leibniz filter of $\boldsymbol{A}$. But in general, $F^{\mathrm{Su}}_{\mathcal{S}}$ needs not be a Suszko filter of $\boldsymbol{A}$, as witnessed by Example 2.23. However, for the logic $\mathcal{PML}$, it follows by Theorem 7.7 that:

**Lemma 7.8.** *Let $\boldsymbol{A} \in \mathsf{PMA}$. For every $F \in \mathcal{F}i_{\mathcal{PML}}\boldsymbol{A}$, $F^{\mathrm{Su}}$ is a Suszko filter of $\boldsymbol{A}$.*

Let us abbreviate $\Box^{\boldsymbol{A}}(\Box^{\boldsymbol{A}}(\ldots\Box^{\boldsymbol{A}}a)\ldots)$, where the operation $\Box^{\boldsymbol{A}}$ appears $n$ times, with $n \in \mathbb{N}$, simply by $\Box^{n}a$. Next, it is easy to check that:

**Lemma 7.9.** *Let $\boldsymbol{A} \in \mathsf{PMA}$. For every $F \in \mathcal{F}i_{\mathcal{PML}}\boldsymbol{A}$, the set*

$$F_{\Box} = \{a \in A : \Box^{n}a \in F, \textit{ for every } n \in \mathbb{N}\}$$

*is the largest open filter included in $F$.*

Proof. Clearly, $1 \in F_\Box$, since $\Box^{\boldsymbol{A}}1 = 1 \in F$. Now, let $a, b \in F_\Box$. Then, $\Box^n a \in F$ and $\Box^n \in F$, for every $n \in \mathbb{N}$. But,

$$\Box^n(a \wedge b) = \Box^n a \wedge \Box^n b,$$

because $\boldsymbol{A} \in \mathsf{PMA}$. Since $F$ is closed under meets, it follows that $\Box^n(a \wedge b) \in F$, for every $n \in \mathbb{N}$. Hence, $a \wedge b \in F_\Box$. Next, let $a \in F_\Box$ and let $b \in A$ such that $a \leq b$. Then, $\Box^n a \in F$, for every $n \in \mathbb{N}$. Since $a \leq b$ and $\boldsymbol{A} \in \mathsf{PMA}$, it follows that $\Box^n a \leq \Box^n b$, for every $n \in \mathbb{N}$. Since $F$ is upwards-closed, it follows that $\Box^n b \in F$, for every $n \in \mathbb{N}$. So, $b \in F_\Box$. So far we have seen that $F_\Box$ is a lattice filter. To see that $F_\Box$ is open, let $a \in F_\Box$. Then, $\Box^n a \in F$, for every $n \in \mathbb{N}$. Clearly then, $\Box^n(\Box a) \in F$, for every $n \in \mathbb{N}$. So, $\Box^{\boldsymbol{A}} a \in F_\Box$. To see that $F_\Box$ extends $F$, let $a \in F$. Taking $n = 0$, it is immediate that $a \in F_\Box$. Finally, to prove the maximality condition, let $F' \subseteq A$ be an open filter below $F$ and let $a \in F'$. Since $F'$ is open, it follows that $\Box^n a \in F' \subseteq F$, for every $n \in \mathbb{N}$. Thus, $a \in F_\Box$. □

**Proposition 7.10.** *Let $\boldsymbol{A} \in \mathsf{PMA}$. For every $F \in \mathcal{F}i_{\mathcal{PML}}\boldsymbol{A}$,*

$$F^* = F^{\mathrm{Su}} = F_\Box.$$

Proof. Let $\boldsymbol{A} \in \mathsf{PMA}$ and $F \in \mathcal{F}i_{\mathcal{PML}}\boldsymbol{A}$. On the one hand, since $F^{\mathrm{Su}}$ is a Suszko $\mathcal{PML}$-filter of $\boldsymbol{A}$, by Lemma 7.8, it follows by Lemma 2.25 that $F^{\mathrm{Su}}$ is the largest Suszko $\mathcal{PML}$-filter below $F$. On the other hand, $F_\Box$ is the largest open lattice filter below $F$, by Lemma 7.9. It follows by Theorem 7.7 that $F^{\mathrm{Su}} = F_\Box$. As to $F^*$, it is also an open filter below $F$ and moreover $F^{\mathrm{Su}} \subseteq F^*$. Therefore, $F^* = F_\Box$. □

**Corollary 7.11.** *The logic $\mathcal{PML}$ has its Leibniz filters explicitly definable by the set of formulas $\Gamma(x) = \{\Box^n x : n \in \mathbb{N}\}$.*

Proof. Since $\mathsf{Alg}^*(\mathcal{PML}) \subseteq \mathsf{PMA}$, by Theorem 7.2.5, the result follows from Proposition 7.10 and Lemma 6.18. □

Hence, $\mathcal{PML}$ has its Leibniz filters both explicitly and equationally definable. This being the case, notice that the first equality in Proposition 7.10 agrees with Proposition 6.26.

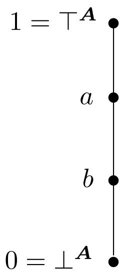
Let us fix the Necessitation rule:

$$\text{(N)}: \quad x \vdash \Box x\ .$$

Since we have seen in Corollary 7.11 that $\mathcal{PML}$ has its Leibniz filters explicitly definable by $\Gamma(x) = \{\Box^n x : n \in \mathbb{N}\}$, it easily follows by Corollary 6.24 that:

**Theorem 7.12.** *The logic $\mathcal{PML}^+$ is the inferential extension of $\mathcal{PML}$ by the rule* (N).

We finish our study of $\mathcal{PML}$ by showing it lies outside the classes of logics in Figure 1 (so far, we only know that $\mathcal{PML}$ is not protoalgebraic, by Theorem 7.2.1), and by completing its classification inside the Frege hierarchy (so far, we know that $\mathcal{PML}$ is fully selfextensional, by Theorem 7.2.2). To this end, we explore in further detail an example of a positive modal algebra taken from [**50**, p. 438].




| $F \in \mathcal{F}i_{\mathcal{PML}}\boldsymbol{A}$ | $\theta = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ | blocks[1] of $\theta$ |
|---|---|---|
| $A = \{0, a, b, 1\}$ | $A \times A$ | $\{0, a, b, 1\}$ |
| $\{1\}$ | $\theta_1$ | $\{1\}$ $\{0, a, b\}$ |
| $\{1, a\}$ | $id_{\boldsymbol{A}}$ | $\{0\}$ $\{a\}$ $\{b\}$ $\{1\}$ |
| $\{1, a, b\}$ | $\theta_2$ | $\{0\}$ $\{1, a, b\}$ |

FIGURE 3 & TABLE 1. The 4-element chain, its $\mathcal{PML}$-filters, and their Leibniz congruences.

**Example 7.13.** Consider the 4-element chain $A = \{0, a, b, 1\}$, ordered by $0 < b < a < 1$, and the algebra $\boldsymbol{A} = \{A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \Box^{\boldsymbol{A}}, \Diamond^{\boldsymbol{A}}, 1, 0\}$, where the meet and join operations are defined as the infimum and supremum of this order, respectively, and the two modal-like operations are defined by:

$$\Box^{\boldsymbol{A}} x = \begin{cases} x, & \text{if } x \in \{0, 1\} \\ b, & \text{if } x \in \{a, b\} \end{cases} \qquad \Diamond^{\boldsymbol{A}} x = \begin{cases} x, & \text{if } x \in \{0, 1\} \\ a, & \text{if } x \in \{a, b\} \end{cases}$$

It is routine to check that $\boldsymbol{A}$ is indeed a positive modal algebra. As a consequence, $\mathcal{F}i_{\mathcal{PML}}\boldsymbol{A} = \mathrm{Filt}\boldsymbol{A}$. The Leibniz operator on these filters is described in Table 1.

**Proposition 7.14.** *$\mathcal{PML}$ is neither truth-equational nor Fregean.*

PROOF. Having in mind Theorem 7.7, $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{PML}}\boldsymbol{A} = \mathrm{Filt}_{\Box}\boldsymbol{A} = \big\{\{1\}, \{1, a, b\}, A\big\} \subsetneq \mathrm{Filt}\boldsymbol{A} = \mathcal{F}i_{\mathcal{PML}}\boldsymbol{A}$. Hence, it follows by Theorem 2.30 that $\mathcal{PML}$ is not truth-equational. Finally, suppose towards an absurd, that $\mathcal{PML}$ is Fregean. Since moreover it has theorems, it follows by [4, Corollary 11] that $\mathcal{PML}$ is truth-equational, which we have just seen to be false. $\Box$

An interesting consequence is that the logic $w\mathcal{K}$ is not Fregean, for Fregeanity is preserved by fragments, and we have just seen that that $\mathcal{PML}$ is not Fregean.

As for the strong version $\mathcal{PML}^+$, we prove that it is neither protoalgebraic nor selfextensional. Moreover, the class of $\mathcal{PML}^+$-algebras is strictly included in the class of $\mathcal{PML}$-algebras.

**Proposition 7.15.** $\mathsf{Alg}(\mathcal{PML}^+) \subsetneq \mathsf{PMA}$.

PROOF. On the one hand, since $\mathcal{PML} \leq \mathcal{PML}^+$, $\mathsf{Alg}(\mathcal{PML}^+) \subseteq \mathsf{Alg}(\mathcal{PML}) = \mathsf{PMA}$. On the other hand, consider $\boldsymbol{A} \in \mathsf{PMA}$ as given in the Example 7.13. Notice that $\mathcal{F}i_{\mathcal{PML}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathrm{Filt}_{\Box}\boldsymbol{A} = \big\{\{1\}, \{1, a, b\}, A\big\}$, by Theorem 7.7. Now, from Table 1 it follows that $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{PML}^+}(A) = A \times A$, $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{PML}^+}\big(\{1, a, b\}\big) = \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\{1, a, b\}\big) \cap \boldsymbol{\Omega}^{\boldsymbol{A}}(A) = \theta_2 \neq id_{\boldsymbol{A}}$, and finally $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{PML}^+}\big(\{1\}\big) = \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\{1\}\big) \cap \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\{1, a, b\}\big) \cap \boldsymbol{\Omega}^{\boldsymbol{A}}(A) = \theta_1 \cap \theta_2 \neq id_{\boldsymbol{A}}$. Indeed, $\langle a, b\rangle \in \theta_1 \cap \theta_2$. Therefore, $\boldsymbol{A} \notin \mathsf{Alg}^{\mathrm{Su}}(\mathcal{PML}^+) = \mathsf{Alg}(\mathcal{PML}^+)$. $\Box$

We are therefore in the presence of a logic $\mathcal{S}$ such that $\mathsf{Alg}(\mathcal{S}^+) \subsetneq \mathsf{Alg}(\mathcal{S})$. Indeed, $\mathsf{Alg}(\mathcal{PML}^+) \subsetneq \mathsf{PMA} = \mathsf{Alg}(\mathcal{PML})$. Furthermore, it must also be the case that $\mathsf{Alg}^*(\mathcal{PML}^+) \subsetneq \mathsf{Alg}^*(\mathcal{PML})$, for otherwise $\mathsf{Alg}(\mathcal{PML}) = \mathbb{IP}_{\mathrm{S}}\mathsf{Alg}^*(\mathcal{PML}) =$

[1] A *block* of $\theta$ is an equivalence class under $\theta$.

$\mathbb{IP}_{\mathrm{S}}\mathsf{Alg}^*(\mathcal{PML}^+) = \mathsf{Alg}(\mathcal{PML}^+)$. This situation contrasts with the protoalgebraic scenario, where in general, for every protoalgebraic logic $\mathcal{S}$, $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S}) = \mathsf{Alg}^*(\mathcal{S}^+) = \mathsf{Alg}(\mathcal{S}^+)$.

**Proposition 7.16.** *$\mathcal{PML}^+$ is not protoalgebraic.*

Proof. This follows from Corollary 6.25, since $\mathcal{PML}$ has its Leibniz filters explicitly definable, by Corollary 7.11, but as we have observed already, $\mathcal{PML}$ does not satisfy property $(\star)$. □

**Example 7.17.** Consider the 3-element chain $B = \{0, a, 1\}$, ordered by $0 < a < 1$, and the algebra $\boldsymbol{B} = \{B, \wedge^{\boldsymbol{B}}, \vee^{\boldsymbol{B}}, \Box^{\boldsymbol{B}}, \Diamond^{\boldsymbol{B}}, 1, 0\}$, where the meet and join operations are defined as the infimum and supremum of this order, respectively, and the two modal-like operations are defined by:

$$\Box^{\boldsymbol{A}} x = \begin{cases} x, & \text{if } x = 1 \\ 0, & \text{if } x \in \{0, a\} \end{cases} \qquad \Diamond^{\boldsymbol{A}} x = \begin{cases} x, & \text{if } x = 0 \\ 1, & \text{if } x \in \{a, 1\} \end{cases}$$

It is routine to check that $\boldsymbol{B}$ is indeed a positive modal algebra.

**Proposition 7.18.** *$\mathcal{PML}^+$ is not selfextensional.*

Proof. Recall that $\mathcal{PML}^+ = \mathcal{S}^{\top}_{\mathsf{PMA}}$. On the one hand, $x \dashv\vdash_{\mathcal{PML}^+} x \wedge \Box x$, because for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{PML}^+) \subseteq \mathsf{PMA}$ and every $h : \boldsymbol{Fm} \to \boldsymbol{A}$, $h(x) = 1$ if and only if $h(x \wedge \Box x) = 1$. On the other hand, we claim that $\Diamond x \nvdash_{\mathcal{PML}^+} \Diamond(x \wedge \Box x)$. Indeed, consider $\boldsymbol{B} \in \mathsf{PMA}$, as given in Example 7.17. Let $h : \boldsymbol{Fm} \to \boldsymbol{B}$ such that $h(x) = a$. Then, $h(\Diamond x) = \Diamond^{\boldsymbol{B}} h(x) = \Diamond^{\boldsymbol{B}} a = 1$, but $h\big(\Diamond(x \wedge \Box x)\big) = \Diamond^{\boldsymbol{B}}(a \wedge^{\boldsymbol{B}} \Box^{\boldsymbol{B}} a) = \Diamond^{\boldsymbol{B}} 0 = 0$. Thus, $\boldsymbol{\Lambda}(\mathcal{PML}^+) \notin \mathrm{Con}\boldsymbol{Fm}$. □

As a final remark on the logic $\mathcal{PML}^+$, we record here that $\mathcal{PML}^+$ is the positive modal fragment of $s\mathcal{K}$, a situation similar to that of $\mathcal{PML}$ and $w\mathcal{K}$. The proof of this fact is outside the scope of the present thesis, and should appear in [3].

## 7.2. Belnap's logic

Our next example is Belnap's four-valued logic, widely known in the literature after the work [8]. For a study of Belnap's logic from an AAL perspective, see [30]. Recall that a logic without theorems has as strong version the almost inconsistent logic. So, in order to use the results of [30] in a meaningful way, we shall add a constant term to the language there considered, thus forcing Belnap's logic to have theorems, and only affecting the results by minor changes (namely, by disregarding the empty set as a $\mathcal{B}$-filter). That is, we shall be working within the language $\mathcal{L} = \langle \wedge, \vee, \neg, \top, \bot \rangle$. We will also use the abbreviation $\varphi \to \psi$ for $\neg\varphi \vee \psi$.

**Definition 7.19.** A *De Morgan algebra* is an algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \neg^{\boldsymbol{A}}, 0, 1 \rangle$ such that:

1. The reduct $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, 1, 0 \rangle$ is a bounded distributive lattice;
2. The De Morgan laws hold, that is, $\neg^{\boldsymbol{A}}(a \vee^{\boldsymbol{A}} b) = \neg^{\boldsymbol{A}} a \wedge^{\boldsymbol{A}} \neg^{\boldsymbol{A}} b$ and $\neg^{\boldsymbol{A}}(a \wedge^{\boldsymbol{A}} b) = \neg^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} \neg^{\boldsymbol{A}} b$, for every $a, b \in A$;
3. The unary operation $\neg^{\boldsymbol{A}}$ is idempotent, that is, $\neg^{\boldsymbol{A}}\neg^{\boldsymbol{A}} a = a$ for every $a \in A$.

The class of all De Morgan algebras will be denoted by $\mathsf{DMA}$.

A reference for De Morgan algebras is [7, Chapter XI]. On every De Morgan algebra $\boldsymbol{A}$, we define the binary operation $\to^{\boldsymbol{A}}$ by setting $a \to^{\boldsymbol{A}} b := \neg^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} b$. Given $\boldsymbol{A} \in \mathsf{DMA}$, we shall denote by $\mathrm{Filt}\boldsymbol{A}$ the set of all lattice filters of $\boldsymbol{A}$, and by $\mathrm{PrFilt}\boldsymbol{A}$ the set of all prime lattice filters of $\boldsymbol{A}$. A lattice filter $F \in \mathrm{Filt}\boldsymbol{A}$ is *implicative*, if for every $a, b \in A$ such that $a, a \to^{\boldsymbol{A}} b \in F$, it holds $b \in F$. The set of all implicative lattice filters of $\boldsymbol{A} \in \mathsf{DMA}$ will be denoted by $\mathrm{Filt}_{\to}\boldsymbol{A}$. It is easily seen that $\{1\}$ is the least (implicative) lattice filter of any $\boldsymbol{A} \in \mathsf{DMA}$.

We next compile some basic properties that hold in all De Morgan algebras, and which we shall make use of, sometimes without any explicit reference.

**Lemma 7.20.** *Let* $\boldsymbol{A} \in \mathsf{DMA}$. *For every* $a, b \in A$,

1. $1 \to^{\boldsymbol{A}} a = a$;
2. $\neg^{\boldsymbol{A}} a = a \to^{\boldsymbol{A}} 0$;
3. *If* $a \leq b$, *then* $\neg^{\boldsymbol{A}} b \leq \neg^{\boldsymbol{A}} a$;
4. $(a \wedge^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} c = a \to^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} c)$.

The class $\mathsf{DMA}$ is a variety. This variety is generated by the four-element De Morgan algebra, which shall be denoted by $\mathfrak{M}_4$. It has universe $M_4 = \{0, a, b, 1\}$, and the lattice operations and the negation operation defined as depicted in Figure 4.

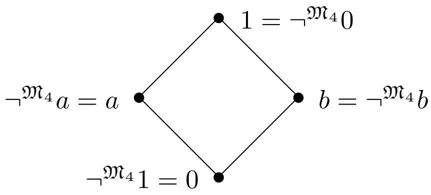
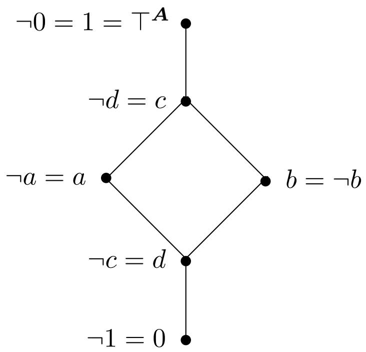



FIGURE 4. The lattice $\mathfrak{M}_4$.

**Definition 7.21.** *Belnap's logic* $\mathcal{B}$ is the semilattice-based logic $\mathcal{S}^{\leq}_{\mathsf{DMA}}$.

Let us first collect some known facts about the logic $\mathcal{B}$. To this end, we exhibit an auxiliary example of a De Morgan algebra taken from [30] (but adding $\top$ to the signature there considered).

**Example 7.22.** Consider the 6-element De Morgan lattice $\mathfrak{M}_6$, with universe $M_6 = \{0, a, b, c, d, 1\}$, sometimes called "the crystal lattice", and whose structure is described in Figure 5. By direct inspection of the table it is clear that the Leibniz $\mathcal{B}$-filters of $\mathfrak{M}_6$ are $\{1\}$ , $\{1, c\}$ and $M_6$. Now, $\boldsymbol{\Omega}^{\boldsymbol{A}}(\{1\}) = \theta_1$ and $\boldsymbol{\Omega}^{\boldsymbol{A}}(\{1, c\}) = \theta_2$, but $\theta_1$ and $\theta_2$ are not comparable. Thus, the Leibniz operator is not order preserving on the Leibniz filters of this algebra. Moreover, it is easy to see that the Suszko $\mathcal{B}$-filters of $\mathfrak{M}_6$ are here $\{1\}$ and $M_6$. Thus, this example also shows that not every Leibniz filter is a Suszko filter; the converse implication does indeed hold, as seen in Lemma 2.21.




| $F \in \mathcal{F}i_{\mathcal{B}}(\mathfrak{M}_6)$ | $\theta = \boldsymbol{\Omega}^{\mathfrak{M}_6}(F)$ | blocks of $\theta$ |
|---|---|---|
| $M_6$ | $M_6 \times M_6$ | $M_6$ |
| $\{1\}$ | $\theta_1$ | $\{1\}\ \{0\}$ |
| $\{1, a, b, c, d\}$ | | $\{a, b, c, d\}$ |
| $\{1, c\}$ | $\theta_2$ | $\{1, c\}$ |
| $\{1, c, a\}$ | | $\{a\}\ \ \{b\}$ |
| $\{1, c, b\}$ | | $\{d, 0\}$ |

FIGURE 5 & TABLE 2. The algebra $\mathfrak{M}_6$, its $\mathcal{B}$-filters, and their Leibniz congruences.

**Theorem 7.23.**

1. *$\mathcal{B}$ is fully selfextensional.*
2. *$\mathcal{B}$ is not protoalgebraic.*
3. *$\mathcal{B}$ is not truth-equational.*
4. *$\mathcal{B}$ is not Fregean.*
5. $\mathcal{S}^{\leq}_{\mathfrak{M}_4} = \mathcal{S}^{\leq}_{\mathsf{DMA}}$.
6. *For every* $\boldsymbol{A} \in \mathsf{DMA}$, $\mathcal{F}i_{\mathcal{B}}\boldsymbol{A} = \mathrm{Filt}\boldsymbol{A}$.
7. $\mathsf{Alg}^*(\mathcal{B}) \subsetneq \mathsf{Alg}(\mathcal{B}) = \mathsf{DMA}$.

PROOF. *1.* Since $\mathcal{B}$ is semilattice-based, it follows by Theorem 0.46. *2.* In Example 7.22, the two comparable $\mathcal{B}$-filters of $\mathfrak{M}_6$, $\{1\}$ and $\{1, c\}$, are such that $\boldsymbol{\Omega}^{\boldsymbol{A}}(\{1\}) \not\subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(\{1, c\})$. *3.* As seen in Example 7.22, not every $\mathcal{B}$-filter of $\mathfrak{M}_6$ is a Suszko filter; therefore, $\mathcal{B}$ is not-truth-equational, by Theorem 2.30. *3.* Suppose, towards an absurd, that $\mathcal{B}$ is Fregean. Since moreover it has theorems, it follows by [4, Corollary 11] that $\mathcal{B}$ is truth-equational, which we have just seen to be false. *5.* Proved in [30, Proposition 2.5]. *6–7.* Since $\mathcal{B}$ is semillatice-based, both 6 and the equality in 7 follow by the general theory seen in the preliminaries (see page 27). The strict inclusion of 7 is proved in [30, p. 16]. ☐

Some consequences of Theorem 7.23 and general facts of the theory developed so far are:

**Proposition 7.24.**

1. *$\mathcal{B}^+$ is assertional, and* $\mathcal{B}^+ = \mathcal{S}^{\top}_{\mathsf{Alg}^*(\mathcal{B})} = \mathcal{S}^{\top}_{\mathsf{Alg}(\mathcal{B})} = \mathcal{S}^{\top}_{\mathsf{DMA}}$.
2. *For every algebra* $\boldsymbol{A}$, $\mathcal{F}i_{\mathcal{B}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{B}}\boldsymbol{A}$.
3. *$\mathcal{B}$ has its Leibniz filters equationally definable by* $\boldsymbol{\tau}(x) = \{x \approx \top\}$.
4. *$\mathcal{B}$ does not satisfy* $(\star)$.

PROOF. By definition $\mathcal{B}$ is semilattice-based, hence by Corollary 6.11 it follows 1 and 2. By Proposition 6.5, it follows 3. Finally, using Lemma 5.17 and Theorem 7.23.7, it follows 4. ☐

We aim at finding an algebraic characterization of the Leibniz $\mathcal{B}$-filters on De Morgan algebras. To this end, let us recall the characterization of the Leibniz operator on De Morgan algebras provided in [30, Proposition 3.13]. For every

$\boldsymbol{A} \in \mathsf{DMA}$, every $F \in \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$, and every $a, b \in A$,

$$\begin{aligned} \langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \text{iff} \quad \forall c \in A, \quad & a \vee^{\boldsymbol{A}} c \in F \Leftrightarrow b \vee^{\boldsymbol{A}} c \in F \\ & \neg^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} c \in F \Leftrightarrow \neg^{\boldsymbol{A}} b \vee^{\boldsymbol{A}} c \in F. \end{aligned} \tag{20}$$

Bear in mind that adding a constant to the underlying language does not affect congruences in general, and therefore (20) is still valid in our setting.

**Theorem 7.25.** *Let $\boldsymbol{A} \in \mathsf{DMA}$. The Leibniz $\mathcal{B}$-filters of $\boldsymbol{A}$ coincide with the implicative lattice filters of $\boldsymbol{A}$. That is,*

$$\mathcal{F}i^{*}_{\mathcal{B}}\boldsymbol{A} = \mathrm{Filt}_{\rightarrow}\boldsymbol{A}.$$

Proof. Let $F \in \mathcal{F}i^{*}_{\mathcal{B}}\boldsymbol{A}$. Then $F = \{a \in A : \langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$ because $\{x \approx \top\}$ defines the Leibniz filters. Assume that $a, a \rightarrow^{\boldsymbol{A}} b \in F$. Then $\langle a, 1\rangle, \langle a \rightarrow^{\boldsymbol{A}} b, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Therefore, $\langle a \rightarrow^{\boldsymbol{A}} b, 1 \rightarrow^{\boldsymbol{A}} b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since $1 \rightarrow^{\boldsymbol{A}} b = b$, $\langle a \rightarrow^{\boldsymbol{A}} b, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since $a \rightarrow^{\boldsymbol{A}} b \in F$, it follows that $b \in F$.

Conversely, let $F \in \mathrm{Filt}_{\rightarrow}\boldsymbol{A}$. Since $\mathcal{B}$ has its Leibniz filters equationally definable by $\{x \approx \top\}$ it will be enough to prove that $F = \{a \in A : \langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$. By compatibility of $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ and the fact that $1 \in F$ we have that $\{a \in A : \langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\} \subseteq F$. Conversely, let $a \in F$. To prove that $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, by (20), we must prove that, for every $c \in A$, it holds

$$a \vee^{\boldsymbol{A}} c \in F \Leftrightarrow 1 \vee^{\boldsymbol{A}} c \in F \tag{a}$$

and

$$\neg^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} c \in F \Leftrightarrow \neg^{\boldsymbol{A}} 1 \vee^{\boldsymbol{A}} c \in F. \tag{b}$$

Now, (a) always holds, since on the one hand $a \leq a \vee^{\boldsymbol{A}} c$ and $F$ is upwards-closed, and on the other hand $1 \vee^{\boldsymbol{A}} c = 1 \in F$. As to (b), by definition of $\rightarrow^{\boldsymbol{A}}$ and since $\neg^{\boldsymbol{A}} 1 = 0$, it amounts to

$$a \rightarrow^{\boldsymbol{A}} c \in F \Leftrightarrow c \in F.$$

Now, if $c \in F$, then $a \rightarrow^{\boldsymbol{A}} c = \neg^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} c \in F$, because $F$ is upwards-closed. If $a \rightarrow^{\boldsymbol{A}} c \in F$, then since $a \in F$ and $F$ is implicative, it follows that $c \in F$. Thus, indeed $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. ☐

As for the $\mathcal{B}$-Suszko filters on De Morgan algebras, given the general theory of Chapter 6, we immediately get:

**Corollary 7.26.** *Let $\boldsymbol{A} \in \mathsf{DMA}$. For every $F \in \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$,*

$$F^{\mathrm{Su}} = \bigcap_{G \in (\mathcal{F}i_{\mathcal{B}}\boldsymbol{A})^{F}} G^{*}.$$

*As a consequence, a $\mathcal{B}$-filter $F$ of $\boldsymbol{A}$ is a Suszko $\mathcal{B}$-filter if and only if $F \subseteq G^{*}$, for every $G \in (\mathcal{F}i_{\mathcal{B}}\boldsymbol{A})^{F}$.*

We now turn our attention to the explicit and logical definability of the Leibniz $\mathcal{B}$-filters. Belnap's logic does not have its Leibniz filters explicitly definable by any set of formulas $\Gamma(x) \subseteq \mathrm{Fm}_{\mathcal{L}}$. In order to see it, we use Proposition 6.27. Since $\mathcal{B}$ has its Leibniz filters equationally definable, it suffices to exhibit $\boldsymbol{A} \in \mathsf{DMA}$ such that $\mathcal{F}i^{*}_{\mathcal{B}}\boldsymbol{A} \neq \mathcal{F}i^{\mathrm{Su}}_{\mathcal{B}}\boldsymbol{A}$. Take the 6-element De Morgan lattice $\mathfrak{M}_6$, as described in Example 7.22. As mentioned there, the Suszko filters of $\mathfrak{M}_6$ do not coincide with the Leibniz ones. Therefore, and in contrast with the case of $\mathcal{PML}$, Belnap's logic does not have its Leibniz filters explicitly definable.

**Proposition 7.27.** *The logic $\mathcal{B}$ does not have its Leibniz filters explicitly definable.*

Consequently, we cannot use Corollary 6.24 in order to find an axiomatization for $\mathcal{B}^+$. Nevertheless, since $\mathcal{B}$ has its Leibniz $\mathcal{B}$-filters logically definable, as we next show, we will still be able to find one. To this end, let us fix the rule *Modus Ponens*:

$$\text{(MP):} \quad x, y \to x \vdash y \ .$$

**Proposition 7.28.** *The logic $\mathcal{B}$ has its Leibniz filters logically definable by the rule Modus Ponens.*

Proof. By Theorems 7.23.6 and 7.25, for every $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{B}) \subseteq \mathsf{DMA}$ and every $F \in \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$, $F$ is a Leibniz $\mathcal{B}$-filter of $\boldsymbol{A}$ if and only if $F$ is an implicative lattice filter of $\boldsymbol{A}$ if and only if $F$ is closed under *Modus Ponens*. Hence, the result follows from Proposition 6.30. □

**Corollary 7.29.** *The logic $\mathcal{B}^+$ is the inferential extension of $\mathcal{B}$ by the rule Modus Ponens.*

Proof. The result follows by Corollary 6.33, since $\mathcal{B}$ has its Leibniz filters logically definable by *Modus Ponens*, by Proposition 7.28. □

We now wish to characterize the map $F \mapsto F^*$, given $\boldsymbol{A} \in \mathsf{DMA}$ and $F \in \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$. To this end, we introduce a generalization of the transformation $\Phi$ considered in [**30**, pp. 16,19], which in turn is a generalization of the so called "Birula-Rasiowa transformation" in [**53**, Definição 7.2, p. 15].

**Definition 7.30.** Let $\boldsymbol{A} \in \mathsf{DMA}$. We define

$$\Phi(F) := \{a \in A : \neg^{\boldsymbol{A}} a \notin F\},$$

for every $F \subseteq A$. We also define

$$\Psi(F) := \{a \in A : \forall b \in A \text{ if } a \to^{\boldsymbol{A}} b \in F, \text{ then } b \in F\},$$

for every $F \subseteq A$.

The transformation $\Psi$ can be seen as a generalization of $\Phi$, because $\neg^{\boldsymbol{A}} a = a \to^{\boldsymbol{A}} 0$ and $0 \notin F$. In fact, for every proper lattice filter $F$ of a $\boldsymbol{A} \in \mathsf{DMA}$, $\Psi(F) \subseteq \Phi(F)$. But the inclusion may be strict. Consider again the De Morgan algebra $\mathfrak{M}_6$ depicted in Example 7.22. Take $F := \{1, c\}$. It is easy to see that $\Phi(F) = \{1, a, b, c\}$, because $\neg^{\boldsymbol{A}} a = a \notin F$, $\neg^{\boldsymbol{A}} b = b \notin F$, $\neg^{\boldsymbol{A}} c = d \notin F$, and $\neg^{\boldsymbol{A}} 1 = 0 \notin F$. On the other hand, $b \notin F$ but $a \to^{\boldsymbol{A}} b = \neg^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} b = c \in F$. So, $a \notin \Psi(F)$. Similarly, $b \notin \Psi(F)$. Indeed, $\Psi(F) = \{1, c\} \subsetneq \{1, a, b, c\} = \Phi(F)$. Interestingly enough, both transformations coincide over prime lattice filters.

**Lemma 7.31.** *Let $\boldsymbol{A} \in \mathsf{DMA}$. For every $P \in \mathrm{PrFilt}\boldsymbol{A}$,*

$$\Phi(P) = \Psi(P).$$

Proof. Let $a \in \Psi(P)$. Since $0 \notin P$ (because $P$ is proper), $\neg^{\boldsymbol{A}} a = a \to^{\boldsymbol{A}} 0 \notin P$. So, $a \in \Phi(P)$. Conversely, let $a \in \Phi(P)$. So, $\neg^{\boldsymbol{A}} a \notin P$. Let $a \to^{\boldsymbol{A}} c \in P$. That is, $\neg^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} c \in P$. Since $P$ is prime and $\neg^{\boldsymbol{A}} a \notin P$, it follows that $c \in P$. Hence, $a \in \Psi(P)$. □

The next result sheds some light on why we are here considering the transformation $\Psi$ rather than the original $\Phi$.

**Proposition 7.32.** *Let* $\boldsymbol{A} \in \mathsf{DMA}$*. If* $F \in \mathrm{Filt}\boldsymbol{A}$*, then* $\Psi(F) \in \mathrm{Filt}\boldsymbol{A}$*.*

PROOF. $1 \in \Psi(F)$, because $1 \to^{\boldsymbol{A}} b = \neg^{\boldsymbol{A}} 1 \vee^{\boldsymbol{A}} b = 0 \vee^{\boldsymbol{A}} b = b$, for every $b \in A$. Let $a \in \Psi(F)$ and $b \in A$ such that $a \leq b$. Then, $\neg^{\boldsymbol{A}} b \leq \neg^{\boldsymbol{A}} a$, by Lemma 7.20.3. Let $c \in A$ such that $b \to^{\boldsymbol{A}} c \in F$. Notice that $b \to^{\boldsymbol{A}} c = \neg^{\boldsymbol{A}} b \vee^{\boldsymbol{B}} c \leq \neg^{\boldsymbol{A}} a \vee^{\boldsymbol{B}} c = a \to^{\boldsymbol{A}} c$. Since $F$ is upwards-closed, it follows that $a \to^{\boldsymbol{A}} c \in F$. Since $a \in \Psi(F)$, it follows that $b \in F$. Next, let $a, b \in \Psi(F)$. Let $c \notin F$. Then, $b \to^{\boldsymbol{A}} c \notin F$, because $b \in \Psi(F)$. Then, $a \to^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} c) \notin F$, because $a \in \Psi(F)$. But, $(a \wedge^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} c = a \to^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} c)$, by Lemma 7.20.4. So, $(a \wedge^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} c \notin F$. Thus, $a \wedge^{\boldsymbol{A}} b \in F$. □

We are now ready to provide a characterization of the operation that turns a lattice filter $F$ of a De Morgan algebra into its associated Leibniz filter $F^*$.

**Proposition 7.33.** *Let* $\boldsymbol{A} \in \mathsf{DMA}$*. For every* $F \in \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$*,*

$$F^* = \Psi(F) \cap F.$$

PROOF. We first check that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is compatible with $\Psi(F)$. Let $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ and let $a \in \Psi(F)$. Let $c \in A$ such that $c \notin F$. We have $\langle a \to^{\boldsymbol{A}} c, b \to^{\boldsymbol{A}} c\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$, because $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \in \mathrm{Con}\boldsymbol{A}$. Since $a \to^{\boldsymbol{A}} c \notin F$, because $a \in \Psi(F)$, it follows by compatibility that $b \to^{\boldsymbol{A}} c \notin F$. That is, $b \in \Psi(F)$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is compatible with $\Psi(F)$. That is, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$. So $\Psi(F) \in \llbracket F\rrbracket^*$, having in mind that $\Psi(F) \in \mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$, by Proposition 7.32 and Theorem 7.23.6. Therefore, $F^* \subseteq \Psi(F)$. Also, in general, $F^* \subseteq F$. Hence, $F^* \subseteq \Psi(F) \cap F$.

Conversely, let $a \in \Psi(F) \cap F$. We claim that $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since $a \in F$ and $F$ is upwards-closed, it trivially holds

$$\forall c \in A \quad a \vee c \in F \Leftrightarrow 1 \vee^{\boldsymbol{A}} c = 1 \in F.$$

Moreover, since $a \in \Psi(F)$ and $c \leq a \to^{\boldsymbol{A}} c$, it holds

$$\forall c \in A \quad a \to^{\boldsymbol{A}} c \in F \Leftrightarrow c \in F.$$

That is,

$$\forall c \in A \quad \neg^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} c \in F \Leftrightarrow \neg^{\boldsymbol{A}} 1 \vee^{\boldsymbol{A}} c \in F.$$

It follows by (20) that $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. But $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$. Since $1 \in F^*$, it follows by compatibility that $a \in F^*$. □

We finish our study of Belnap's logic by proving that $\mathcal{B}^+$ is neither protoalgebraic nor selfextensional. First, we show that $\mathcal{B}^+$ and $\mathcal{B}$ have the same algebraic counterpart.

**Proposition 7.34.** $\mathsf{Alg}(\mathcal{B}^+) = \mathsf{DMA} = \mathsf{Alg}(\mathcal{B})$.

PROOF. Since $\mathcal{B} \leq \mathcal{B}^+$, it is clear that $\mathsf{Alg}(\mathcal{B}^+) \subseteq \mathsf{Alg}(\mathcal{B}) = \mathsf{DMA}$. As for the converse inclusion, we observe that every subdirectly irreducible De Morgan algebra — there are only three, namely, $\mathfrak{M}_4$, $\mathfrak{Z}$ and $\mathfrak{2}$ [7, XI.2, Theorem 6] — belongs to $\mathsf{Alg}^*(\mathcal{B}^+)$. Indeed, since $\{1\}$ is always an implicative lattice filter (see page 124), and the three algebras are simple, $\langle \mathfrak{M}_4, \{1\}\rangle, \langle \mathfrak{Z}, \{1\}\rangle, \langle \mathfrak{2}, \{1\}\rangle \in \mathsf{Mod}^*(\mathcal{B}^+)$. It follows by Birkhoff's subdirect representation theorem (see Theorem 0.14 and the comments after it) that $\mathsf{DMA} = \mathbb{IP}_{\mathrm{S}}\big(\{\mathfrak{M}_4, \mathfrak{Z}, \mathfrak{2}\}\big) \subseteq \mathbb{IP}_{\mathrm{S}}\big(\mathsf{Alg}^*(\mathcal{B}^+)\big) = \mathsf{Alg}(\mathcal{B}^+)$. □

**Corollary 7.35.** *The logic* $\mathcal{B}^+$ *is not protoalgebraic.*

Proof. Suppose, towards an absurd, that $\mathcal{B}^+$ is protoalgebraic. Then, $\mathsf{Alg}^*(\mathcal{B}^+) = \mathsf{Alg}(\mathcal{B}^+) = \mathsf{DMA}$, using Proposition 7.34. But, $\mathsf{Alg}^*(\mathcal{B}^+) \subseteq \mathsf{Alg}^*(\mathcal{B})$, because $\mathcal{B} \leq \mathcal{B}^+$. It follows that $\mathsf{DMA} = \mathsf{Alg}^*(\mathcal{B}^+) \subseteq \mathsf{Alg}^*(\mathcal{B}) \subseteq \mathsf{Alg}(\mathcal{B}) = \mathsf{DMA}$. This contradicts the fact that $\mathsf{Alg}^*(\mathcal{B}) \subsetneq \mathsf{Alg}(\mathcal{B})$ in Theorem 7.23.7. □

**Corollary 7.36.** *The logic $\mathcal{B}^+$ is not selfextensional.*

Proof. Suppose, towards an absurd, that $\mathcal{B}^+$ is selfextensional. Then, since $\mathcal{B}^+$ has a conjunction, it follows by Theorem 0.46 that $\mathcal{B}^+$ is semilattice-based. Then, it is semilattice-based of $\mathsf{Alg}(\mathcal{B}) = \mathsf{DMA}$, using Proposition 7.34. Consequently, $\mathcal{B}^+ = \mathcal{B}^{\leq}_{\mathsf{DMA}} = \mathcal{B}$, and we reach an absurd (for instance, $\mathcal{B}^+$ is truth-equational, while $\mathcal{B}$ is not). □

## 7.3. Subintuitionistic logics

Subintuitionistic logics are logics in the language of intuitionistic logic that have Intuitionistic Propositional Logic as an extension. In this section, we shall be working in the language $\mathcal{L} = \langle \wedge, \vee, \to, \top, \bot \rangle$, and also make use of two non-primitive unary operators defined by $\neg\varphi := \varphi \to \bot$ and $\Box\varphi := \top \to \varphi$, for every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$. Subintuitionistic logics usually enjoy a relational semantics given by classes of Kripke models where the implication $\to$ is interpreted as the strict implication in modal logic. That is,

$$\varphi \to_{\text{subint.}} \psi = \Box(\varphi \to_{\text{modal}} \psi).$$

Such approach to subintuitionistic logics is, for example, the one undertaken in [19] and [16]. Following the previous examples however, we choose to approach subintuitionistic logics by semantically defining them as the semilattice-based logic of some variety having as subvariety the class of Heyting algebras. To this end, we start by introducing the class of *weakly Heyting algebras* [20, Definition 3.1].

**Definition 7.37.** A *weakly Heyting algebra* is an algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, 1, 0 \rangle$ such that $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, 1, 0 \rangle$ is a bounded distributive lattice and $\to^{\boldsymbol{A}}$ is a binary connective satisfying:

1. $(a \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} c) = a \to^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} c)$;
2. $(a \to^{\boldsymbol{A}} c) \wedge^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} c) = (a \vee^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} c$;
3. $(a \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} c) \leq a \to^{\boldsymbol{A}} c$;
4. $a \to^{\boldsymbol{A}} a = 1$.

The class of all weakly Heyting algebras will be denoted by $\mathsf{WH}$. As usual, for any weakly Heyting algebra $\boldsymbol{A}$, we denote by $\mathrm{Filt}\boldsymbol{A}$ the set of lattice filters of $\boldsymbol{A}$ and by $\mathrm{Filt}_{\Box}\boldsymbol{A}$ the set of lattice filters of $\boldsymbol{A}$ closed under the operation $\Box^{\boldsymbol{A}}$, given by $\Box^{\boldsymbol{A}} a := 1 \to^{\boldsymbol{A}} a$; this filters will be called *open*. We shall abbreviate $\Box^{\boldsymbol{A}}(\Box^{\boldsymbol{A}}(\ldots \Box^{\boldsymbol{A}} a)\ldots)$, where the operation $\Box^{\boldsymbol{A}}$ appears $n$ times, with $n \in \mathbb{N}$, simply by $\Box^n a$. We next collect some basic properties valid in any weakly Heyting algebra.

**Lemma 7.38.** *Let $\boldsymbol{A} \in \mathsf{WH}$. For every $a, b, c, \in A$,*

1. *If $a \leq b$, then $a \to^{\boldsymbol{A}} b = 1$;*
2. *If $a \leq b$, then $\Box^{\boldsymbol{A}} a \leq \Box^{\boldsymbol{A}} b$.*

We will consider the following equations:

(eq-N): $x \wedge \Box x \approx x$ $(x \leq \Box x)$;

(eq-MP): $x \wedge (x \to y) \wedge y \approx x \wedge (x \to y)$ $\big(x \wedge (x \to y) \leq y\big)$;

(eq-RT): $(x \to y) \wedge \Box(x \to y) \approx x \to y$ $\big(x \to y \leq \Box(x \to y)\big)$.

We shall denote by $\mathsf{WH}_{(\mathrm{N})}$, $\mathsf{WH}_{(\mathrm{MP})}$, and $\mathsf{WH}_{(\mathrm{RT})}$, the subvarieties of $\mathsf{WH}$ axiomatized by the equations defining $\mathsf{WH}$ plus the equation (eq-N), (eq-MP), and (eq-RT), respectively. For the sake of completeness, we can also consider the subvarieties of $\mathsf{WH}$ axiomatized by the equations defining $\mathsf{WH}$ plus any combination of two of the above equations. However, we will not consider the subvarieties $\mathsf{WH}_{(\mathrm{RT},\mathrm{MP})}$ and $\mathsf{WH}_{(\mathrm{N},\mathrm{MP})}$ because they induce BP-algebraizable logics. A detailed study of all possible combinations can be found in [**16**]. The subvariety axiomatized by the equations defining $\mathsf{WH}$ plus the equations (MP) and (N) is the variety $\mathsf{HA}$ of all Heyting algebras. These varieties are related as follows [**20**, Fig. 1]:

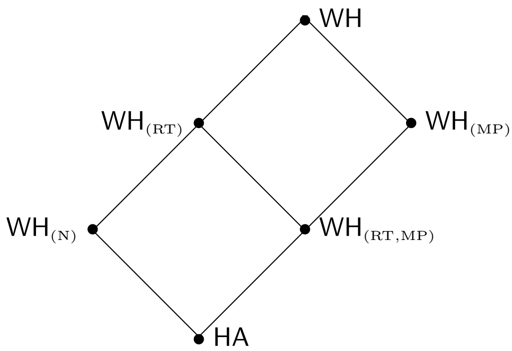



FIGURE 6. Some subvarieties of the variety of Heyting algebras.

Going upwards in the diagram the lines depict strict inclusions of the classes of algebras. The semillatice-based logic of $\mathsf{WH}$, $\mathcal{S}^{\leq}_{\mathsf{WH}}$, is sometimes denoted by $w\mathcal{K}_\sigma$ (for instance, in [**19**]), and is called the strict implicational fragment of $w\mathcal{K}$. The logic $w\mathcal{K}_\sigma$ is a paradigmatic example in our new proposal for a strong version of a sentential logic, since it had been already observed in [**38**, Example 49] that the pair of logics composed of $w\mathcal{K}_\sigma$ and its extension by the Necessitation rule (N), say $s\mathcal{K}_\sigma$, share several properties which resemble the well behaved pair $w\mathcal{K}$ and $s\mathcal{K}$. Nevertheless, $w\mathcal{K}_\sigma$ and $s\mathcal{K}_\sigma$ do not constitute a Leibniz-linked pair, as observed in the cited example. However, as we shall see in Theorem 7.40, $(w\mathcal{K}_\sigma)^+ = s\mathcal{K}_\sigma$. Therefore, our new approach towards a strong version of a (non-protoalgebraic) logic encompasses pairs of logics which were already recognised to be somehow strongly related, but whose relation failed to be formally captured under a general theory in AAL.

There are many subintuitionistic logics studied in the literature. For references, we address the reader to [**16, 19**] and the papers there cited. We shall be interested in studying the semilattice-based logics of $\mathsf{WH}$, $\mathsf{WH}_{(\mathrm{N})}$, $\mathsf{WH}_{(\mathrm{MP})}$, and $\mathsf{WH}_{(\mathrm{RT})}$ (hereby denoted by $\mathcal{S}^{\leq}_{\mathsf{K}}$, for appropriate $\mathsf{K}$), as well as the $\{x \approx \top\}$-assertional logics of the same classes of algebras (hereby denoted by $\mathcal{S}^{\top}_{\mathsf{K}}$, for appropriate $\mathsf{K}$). These logics stand in relation according to the following diagram, which is obtained from [**16**, Theorem 2.55], having in mind [**16**, Definition 4.5].

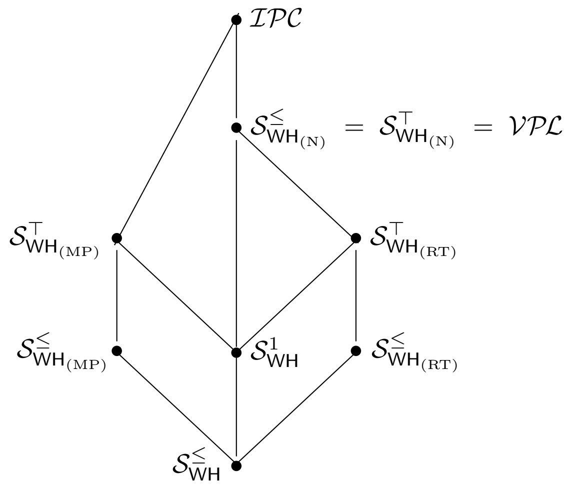



FIGURE 7. Relations between the subintuitionistic logics to be studied.

Going upwards in the diagram the lines depict strict extensions of the logics. The logic $\mathcal{IPC}$ denotes of course Intuitionistic Propositional Logic, while the logic $\mathcal{VPL}$ denotes Visser's Propositional Logic (sometimes also called Basic Propositional Logic, and denoted by $\mathcal{BPL}$) [**58**]. Let us collect some facts about the logics depicted above, all of which proved in [**16, 19**].

**Theorem 7.39.**

1. *None of the logics* $\mathcal{S}^{\leq}_{\mathsf{WH}}$, $\mathcal{S}^{\top}_{\mathsf{WH}}$, $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{RT})}}$, $\mathcal{S}^{\top}_{\mathsf{WH}_{(\mathrm{RT})}}$, *and* $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{N})}} = \mathcal{S}^{\top}_{\mathsf{WH}_{(\mathrm{N})}}$ *is protoalgebraic; the logic* $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{MP})}}$ *is equivalential; the logic* $\mathcal{S}^{\top}_{\mathsf{WH}_{(\mathrm{MP})}}$ *is algebraizable;*
2. *None of the logics* $\mathcal{S}^{\leq}_{\mathsf{WH}}$, $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{RT})}}$, *and* $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{MP})}}$ *is Fregean; the logic* $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{N})}}$ *is Fregean.*
3. $\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A} = \mathrm{Filt}\boldsymbol{A}$, *with* $\mathsf{K} \in \{\mathsf{WH}, \mathsf{WH}_{(\mathrm{N})}, \mathsf{WH}_{(\mathrm{MP})}, \mathsf{WH}_{(\mathrm{RT})}\}$ *and* $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}})$*;*
4. $\mathcal{F}i_{\mathcal{S}^{\top}_{\mathsf{K}}}\boldsymbol{A} = \mathrm{Filt}_{\Box}\boldsymbol{A}$, *with* $\mathsf{K} \in \{\mathsf{WH}, \mathsf{WH}_{(\mathrm{N})}, \mathsf{WH}_{(\mathrm{MP})}, \mathsf{WH}_{(\mathrm{RT})}\}$ *and* $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}})$*;*
5. $\mathsf{Alg}(\mathcal{S}^{\top}_{\mathsf{K}}) = \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}}) = \mathsf{K}$, *with* $\mathsf{K} \in \{\mathsf{WH}_{(\mathrm{N})}, \mathsf{WH}_{(\mathrm{MP})}\}$*;*
6. $\mathsf{Alg}(\mathcal{S}^{\top}_{\mathsf{K}}) \subsetneq \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}}) = \mathsf{K}$, *with* $\mathsf{K} \in \{\mathsf{WH}, \mathsf{WH}_{(\mathrm{RT})}\}$*;*
7. $\mathsf{Alg}^*(\mathcal{S}^{\leq}_{\mathsf{K}}) \subsetneq \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}})$, *with* $\mathsf{K} \in \{\mathsf{WH}, \mathsf{WH}_{(\mathrm{N})}, \mathsf{WH}_{(\mathrm{RT})}\}$.

We also known that all the logics $\mathcal{S}^{\leq}_{\mathsf{WH}}$, $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{RT})}}$, $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{MP})}}$, and $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{N})}}$ are fully selfextensional, by Theorem 0.47. Moreover, $\mathsf{Alg}^*(\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{MP})}}) = \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{MP})}})$, since the logic $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{MP})}}$ is protoalgebraic. As a consequence, the logic $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{MP})}}$ satisfies property ($\star$). On the contrary, a consequence of item 6 above is that none of the logics $\mathcal{S}^{\leq}_{\mathsf{WH}}$, $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{N})}}$, and $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{RT})}}$ satisfy property ($\star$).

With the information of Theorem 7.39 at hand, we can already establish that $\mathcal{S}^{\top}_{\mathsf{K}}$ is the strong version of the logic $\mathcal{S}^{\leq}_{\mathsf{K}}$, for $\mathsf{K} \in \{\mathsf{WH}, \mathsf{WH}_{(\mathrm{N})}, \mathsf{WH}_{(\mathrm{MP})}, \mathsf{WH}_{(\mathrm{RT})}\}$. Indeed, for each such $\mathsf{K}$, $\mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}}) = \mathsf{K}$ and since $\mathcal{S}^{\leq}_{\mathsf{K}}$ is a semilattice-based logic, it follows immediately by Corollary 6.11 that $\mathcal{S}^{\top}_{\mathsf{K}}$ is the strong version of $\mathcal{S}^{\leq}_{\mathsf{K}}$.

**Theorem 7.40.** *The logic* $\mathcal{S}^{\top}_{\mathsf{K}}$ *is the strong version of* $\mathcal{S}^{\leq}_{\mathsf{K}}$.

Once again, we are in the presence of a logic $\mathcal{S}$ such that $\mathsf{Alg}(\mathcal{S}^{+}) \subsetneq \mathsf{Alg}(\mathcal{S})$. Indeed, [**16**, Theorem 4.41.1] tells us that $\mathsf{Alg}\big((\mathcal{S}^{\leq}_{\mathsf{WH}})^{+}\big) = \mathsf{Alg}(\mathcal{S}^{1}_{\mathsf{WH}}) \subsetneq \mathsf{WH} =$

$\mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{WH}})$. The same remarks are true for the logic $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{RT})}}$, also by the cited result.

In the following, let $\mathsf{K} \in \{\mathsf{WH}, \mathsf{WH}_{(\mathrm{N})}, \mathsf{WH}_{(\mathrm{MP})}, \mathsf{WH}_{(\mathrm{RT})}\}$. We now wish to characterize the Leibniz and Suszko $\mathcal{S}^{\leq}_{\mathsf{K}}$-filters on $\mathcal{S}^{\leq}_{\mathsf{K}}$-algebras. To start with, we know that $\mathcal{S}^{\leq}_{\mathsf{K}}$ has its Leibniz filters equationally definable by $\boldsymbol{\tau}(x) = \{x \approx \top\}$, by Proposition 6.5, for it is a semilattice-based logic.

**Proposition 7.41.** *Let $\boldsymbol{A} \in \mathsf{K}$. Every Leibniz $\mathcal{S}^{\leq}_{\mathsf{K}}$-filter $F$ of $\boldsymbol{A}$ is an open lattice filter of $\boldsymbol{A}$.*

Proof. Let $\boldsymbol{A} \in \mathsf{K}$ and $F \in \mathcal{F}i^{*}_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A}$. Since the Leibniz filters of $\mathcal{S}^{\leq}_{\mathsf{K}}$ are equationally definable by $\{x \approx \top\}$, we have $F = \{a \in A : \langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}$. Let $a \in F$. Then $\langle a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Therefore $\langle \Box^{\boldsymbol{A}} a, \Box^{\boldsymbol{A}} 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since $\Box^{\boldsymbol{A}} 1 = 1$, because $\boldsymbol{A} \in \mathsf{K} \subseteq \mathsf{WH}$, we have $\langle \Box^{\boldsymbol{A}} a, 1\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. It follows by compatibility that $\Box^{\boldsymbol{A}} a \in F$. □

**Lemma 7.42.** *Let $\boldsymbol{A} \in \mathsf{K}$ and $F \in \mathrm{Filt}_{\Box}\boldsymbol{A}$. For every $a, b \in F$,*

$$\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}^{\leq}_{\mathsf{K}}}(F).$$

Proof. Let $\boldsymbol{A} \in \mathsf{WH}$ and $F \in \mathrm{Filt}_{\Box}\boldsymbol{A}$. Since, in particular, $F$ is a lattice filter, it holds $F \in \mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A}$. Let $a, b \in F$. Let $\varphi(x, \overline{z}) \in \mathrm{Fm}_{\mathcal{L}}$ and $\overline{c} \in A$ arbitrary. We claim that

$$\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F' \Leftrightarrow \varphi^{\boldsymbol{A}}(b, \overline{c}) \in F', \tag{21}$$

for every $F' \in (\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A})^{F}$. The proof goes by induction on $\varphi \in \mathrm{Fm}_{\mathcal{L}}$.

- $\varphi(x, \overline{z}) = x \in \mathrm{Var}$: Let $F' \in (\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A})^{F}$. We have $\varphi^{\boldsymbol{A}}(a, \overline{c}) = a$ and $\varphi^{\boldsymbol{A}}(b, \overline{c}) = b$. Since both $a, b \in F \subseteq F'$ by assumption, (21) holds.
- $\varphi(x, \overline{z}) = \top$: Let $F' \in (\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A})^{F}$. We have $\varphi^{\boldsymbol{A}}(a, \overline{c}) = 1$ and $\varphi^{\boldsymbol{A}}(b, \overline{c}) = 1$. Since $1 \in F'$, (21) holds trivially.
- $\varphi(x, \overline{z}) = \bot$: Let $F' \in (\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A})^{F}$. We have $\varphi^{\boldsymbol{A}}(a, \overline{c}) = 0$ and $\varphi^{\boldsymbol{A}}(b, \overline{c}) = 0$. Since $0 \notin F'$, (21) holds vacuously.
- $\varphi(x, \overline{z}) = \psi(x, \overline{z}) \wedge \xi(x, \overline{z})$: The inductive hypothesis tell us that (21) holds for $\psi$ and $\xi$. Let $F' \in (\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A})^{F}$. Assume $\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F'$. Since $\varphi^{\boldsymbol{A}}(a, \overline{c}) = \psi^{\boldsymbol{A}}(a, \overline{c}) \wedge^{\boldsymbol{A}} \xi^{\boldsymbol{A}}(a, \overline{c}) \leq \psi^{\boldsymbol{A}}(a, \overline{c}), \xi(a, \overline{c})$, and $F'$ is upwards-closed, it follows that $\psi^{\boldsymbol{A}}(a, \overline{c}) \in F'$ and $\xi^{\boldsymbol{A}}(a, \overline{c}) \in F'$. It follows by the inductive hypothesis that $\psi^{\boldsymbol{A}}(b, \overline{c}) \in F'$ and $\xi^{\boldsymbol{A}}(b, \overline{c}) \in F'$. Since $F'$ is closed under meets, it follows that $\varphi^{\boldsymbol{A}}(b, \overline{c}) = \psi^{\boldsymbol{A}}(b, \overline{c}) \wedge^{\boldsymbol{A}} \xi^{\boldsymbol{A}}(b, \overline{c}) \in F'$. Similarly, one proves that $\varphi^{\boldsymbol{A}}(b, \overline{c}) \in F'$ implies $\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F'$.
- $\varphi(x, \overline{z}) = \psi(x, \overline{z}) \vee \xi(x, \overline{z})$: The inductive hypothesis tell us that (21) holds for $\psi$ and $\xi$. Let $F' \in (\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A})^{F}$. Since $\mathsf{K}$ is a distributive lattice, it follows as a consequence of the Prime Filter Theorem 0.4, that every lattice filter of $\boldsymbol{A}$ is the intersection of the prime lattice filters containing it. In particular, $F' = \bigcap\{P \in \mathrm{PrFilt}\boldsymbol{A} : F' \subseteq P\}$. Clearly then, $\varphi^{\boldsymbol{A}}(a, \overline{c}) \in F'$ if and only if $\varphi^{\boldsymbol{A}}(a, \overline{c}) \in P$, for every $P \in (\mathrm{PrFilt}\boldsymbol{A})^{F'}$, if and only if $\psi^{\boldsymbol{A}}(a, \overline{c}) \in P$ or $\xi^{\boldsymbol{A}}(a, \overline{c}) \in P$, for every $P \in (\mathrm{PrFilt}\boldsymbol{A})^{F'}$, if and only if $\psi^{\boldsymbol{A}}(b, \overline{c}) \in P$ or $\xi^{\boldsymbol{A}}(b, \overline{c}) \in P$ (using the inductive hypothesis, since $F \subseteq F' \subseteq P$), for every $P \in (\mathrm{PrFilt}\boldsymbol{A})^{F'}$, if and only if $\varphi^{\boldsymbol{A}}(b, \overline{c}) \in P$, for every $P \in (\mathrm{PrFilt}\boldsymbol{A})^{F'}$, if and only if $\varphi^{\boldsymbol{A}}(b, \overline{c}) \in F'$.

- $\varphi(x,\overline{z}) = \psi(x,\overline{z}) \to \xi(x,\overline{z})$: The inductive hypothesis tell us that (21) holds for $\psi$ and $\xi$. Let $F' \in (\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A})^F$. Assume $\varphi^{\boldsymbol{A}}(a,\overline{c}) \in F'$. Fix $d \coloneqq \psi^{\boldsymbol{A}}(a,\overline{c})$. Consider the set
$$H = \{e \in A : d \to^{\boldsymbol{A}} a \in F'\}.$$
*Claim. $H$ is a lattice filter extending $F$:* First, $1 \in H$, because $d \to^{\boldsymbol{A}} d = 1 \in F'$, since $\boldsymbol{A} \in \mathsf{K} \subseteq \mathsf{WH}$. Next, let $e, f \in H$. Notice that
$$d \to^{\boldsymbol{A}} (e \wedge^{\boldsymbol{A}} f) = (d \to^{\boldsymbol{A}} e) \wedge^{\boldsymbol{A}} (d \to^{\boldsymbol{A}} f) \in F',$$
since $\boldsymbol{A} \in \mathsf{K} \subseteq \mathsf{WH}$ and $F'$ is closed under meets. Hence, $H$ is closed under meets. Now, let $e \in H$ and $f \in A$ such that $e \leq f$. Then, $d \to^{\boldsymbol{A}} e \leq d \to^{\boldsymbol{A}} f$, because $\boldsymbol{A} \in \mathsf{K} \subseteq \mathsf{WH}$. Since $F'$ is upwards-closed, it follows that $d \to^{\boldsymbol{A}} f \in F'$. Hence, $H$ is upwards-closed. Finally, let $e \in F$. Then, $\Box^{\boldsymbol{A}} e = 1 \to^{\boldsymbol{A}} e \in F \subseteq F'$, using the hypothesis ($F$ open). Moreover, since $d \leq 1$, it holds $1 \to^{\boldsymbol{A}} e \leq d \to^{\boldsymbol{A}} a$, because $\boldsymbol{A} \in \mathsf{K} \subseteq \mathsf{WH}$. Since $F'$ is upwards-closed, it follows that $e \in H$. Thus, $F \subseteq H$.
Now, since $\varphi^{\boldsymbol{A}}(a,\overline{c}) = \psi^{\boldsymbol{A}}(a,\overline{c}) \to^{\boldsymbol{A}} \xi^{\boldsymbol{A}}(a,\overline{c}) \in F'$, we have $\xi^{\boldsymbol{A}}(a,\overline{c}) \in H$. It follows by the inductive hypothesis that $\xi^{\boldsymbol{A}}(b,\overline{c}) \in H$. That is,
$$\psi^{\boldsymbol{A}}(a,\overline{c}) \to^{\boldsymbol{A}} \xi^{\boldsymbol{A}}(b,\overline{c}) \in F'. \tag{i}$$
This time, fix $d \coloneqq \psi^{\boldsymbol{A}}(b,\overline{c})$, and consider the set
$$G = \{e \in A : d \to^{\boldsymbol{A}} e \in F'\}$$
Similarly, one proves that $G$ is a lattice filter extending $F$. Moreover, $d \to d = 1 \in F'$. So, $d = \psi^{\boldsymbol{A}}(b,\overline{c}) \in G$. It follows by the inductive hypothesis that $\psi^{\boldsymbol{A}}(a,\overline{c}) \in G$. That is,
$$\psi^{\boldsymbol{A}}(b,\overline{c}) \to^{\boldsymbol{A}} \psi^{\boldsymbol{A}}(a,\overline{c}) \in F'. \tag{ii}$$
But,
$$(x \to y) \wedge (y \to z) \leq (x \to z)$$
holds in every $\boldsymbol{A} \in \mathsf{K} \subseteq \mathsf{WH}$. Thus, it follows by (i) and (ii), together with $F'$ being closed under meets and upwards-closed, that $\varphi^{\boldsymbol{A}}(b,\overline{c}) = \psi^{\boldsymbol{A}}(b,\overline{c}) \to^{\boldsymbol{A}} \xi^{\boldsymbol{A}}(b,\overline{c}) \in F'$. Similarly one proves that $\varphi^{\boldsymbol{A}}(b,\overline{c}) \in F'$ implies $\varphi^{\boldsymbol{A}}(a,\overline{c}) \in F'$.

From (21) and Corollary 0.30 it follows that $\langle a, b\rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}^{\leq}_{\mathsf{K}}}(F)$. □

**Proposition 7.43.** *Let $\boldsymbol{A} \in \mathsf{K}$. Every open filter of $\boldsymbol{A}$ is a Suszko $\mathcal{S}^{\leq}_{\mathsf{K}}$-filter of $\boldsymbol{A}$.*

Proof. Let $\boldsymbol{A} \in \mathsf{WH}$ and $F \in \mathrm{Filt}_{\Box}\boldsymbol{A}$. Let $a \in F$. Since also $1 \in F$, it follows by Lemma 7.42 that $\langle 1, a\rangle \in \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}^{\leq}_{\mathsf{K}}}(F)$. Since $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}^{\leq}_{\mathsf{K}}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}})$ and moreover $1 \in F^{\mathrm{Su}}$ (bear in mind that $F^{\mathrm{Su}} \in \mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A} = \mathrm{Filt}\boldsymbol{A}$), it follows that $a \in F^{\mathrm{Su}}$. So, $F \subseteq F^{\mathrm{Su}}$. Thus, $F$ is a Suszko filter of $\boldsymbol{A}$. □

**Theorem 7.44.** *Let $\boldsymbol{A} \in \mathsf{K}$. The Leibniz and Suszko $\mathcal{S}^{\leq}_{\mathsf{K}}$-filters of $\boldsymbol{A}$ coincide with the open lattice filters of $\boldsymbol{A}$. That is,*
$$\mathcal{F}i^{*}_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A} = \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A} = \mathrm{Filt}_{\Box}\boldsymbol{A}.$$

Proof. Just notice that $\mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A} \subseteq \mathcal{F}i^{*}_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A} \subseteq \mathrm{Filt}_{\Box}\boldsymbol{A} \subseteq \mathcal{F}i^{\mathrm{Su}}_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A}$. □

We are now able to see that apart from Visser's logic, none of the subintuitionistic logics covered is truth-equational.

**Theorem 7.45.** *None of the logics $\mathcal{S}^{\leq}_{\mathsf{WH}}$, $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{RT})}}$ and $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{MP})}}$ is truth-equational. The logic $\mathcal{VPL}$ is truth-equational.*

Proof. Let $\mathsf{K} \in \{\mathsf{WH}, \mathsf{WH}_{(\mathrm{RT})}, \mathsf{WH}_{(\mathrm{MP})}\}$. Suppose, towards an absurd, that $\mathcal{S}^{\leq}_{\mathsf{K}}$ is truth-equational. Then, $\mathcal{S}^{\leq}_{\mathsf{K}} = (\mathcal{S}^{\leq}_{\mathsf{K}})^{+} = \mathcal{S}^{\top}_{\mathsf{K}}$, by Theorems 5.5.4 and 7.40. Since the inclusions in Figure 7 are strict, we reach an absurd. We are left to prove that $\mathcal{VPL}$ is truth-equational. Since for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{VPL}) \subseteq \mathsf{WH}$, $\mathcal{F}i_{\mathcal{VPL}}\boldsymbol{A} = \mathrm{Filt}_{\Box}\boldsymbol{A} = \mathcal{F}i^{\mathrm{Su}}_{\mathcal{VPL}}\boldsymbol{A}$, by Theorems 7.39.4 and 7.44, the result follows from Theorem 2.30. $\Box$

As a consequence:

**Corollary 7.46.** *None of the logics $\mathcal{S}^{\leq}_{\mathsf{WH}}$, $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{RT})}}$, and $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{MP})}}$ is Fregean. The logic $\mathcal{VPL}$ is fully Fregean.*

Proof. Let $\mathsf{K} \in \{\mathsf{WH}, \mathsf{WH}_{(\mathrm{RT})}, \mathsf{WH}_{(\mathrm{MP})}\}$. Suppose towards an absurd, that $\mathcal{S}^{\leq}_{\mathsf{K}}$ is Fregean. Since moreover it has theorems, it follows by [4, Corollary 11] that $\mathcal{S}^{\leq}_{\mathsf{K}}$ is truth-equational, which we have just seen to be false. As for Visser's logic, we have seen it already to be both fully selfextensional and Fregean. Since moreover it has theorems, it follows by [4, Theorem 24] that it is fully Fregean. $\Box$

In the following, let $\mathsf{K} \in \{\mathsf{WH}, \mathsf{WH}_{(\mathrm{N})}, \mathsf{WH}_{(\mathrm{RT})}, \mathsf{WH}_{(\mathrm{MP})}\}$. Having found the strong version of $\mathcal{S}^{\leq}_{\mathsf{K}}$ and characterized its Leibniz and Suszko $\mathcal{S}^{\leq}_{\mathsf{K}}$-filters, we now turn our attention to the explicit definability of Leibniz $\mathcal{S}^{\leq}_{\mathsf{K}}$-filters. Recall that in general, given an arbitrary logic $\mathcal{S}$, an algebra $\boldsymbol{A}$, and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F^{\mathrm{Su}}$ is always a Leibniz filter of $\boldsymbol{A}$. So, it follows by Theorem 7.44 that:

**Lemma 7.47.** *Let $\boldsymbol{A} \in \mathsf{K}$. For every $F \in \mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A}$, $F^{\mathrm{Su}}$ is a Suszko filter of $\boldsymbol{A}$.*

Moreover,

**Lemma 7.48.** *Let $\boldsymbol{A} \in \mathsf{K}$. For every $F \in \mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}}\boldsymbol{A}$, the set*

$$F_{\Box} = \{a \in A : \Box^{n} a \in F, \textit{ for every } n \in \mathbb{N}\}$$

*is the largest open filter included in $F$.*

Proof. Clearly, $1 \in F_{\Box}$, since $\Box^{\boldsymbol{A}}1 = 1 \to^{\boldsymbol{A}} 1 = 1 \in F$. Now, let $a, b \in F_{\Box}$. Then, $\Box^{n} a \in F$ and $\Box^{n} b \in F$, for every $n \in \mathbb{N}$. Now, by induction on $n \in \mathbb{N}$, one proves that

$$\Box^{n}(a \wedge^{\boldsymbol{A}} b) = \Box^{n} a \wedge^{\boldsymbol{A}} \Box^{n} b\,,$$

using the fact that $a \to^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} c) = (a \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} c)$, since $\boldsymbol{A} \in \mathsf{K} \subseteq \mathsf{WH}$. We show the case $n = 2$ to give an idea of the arguments used in the induction proof.

$$\begin{aligned}
\Box^{2} a \wedge^{\boldsymbol{A}} \Box^{2} b &= \big(1 \to^{\boldsymbol{A}} (1 \to^{\boldsymbol{A}} a)\big) \wedge^{\boldsymbol{A}} \big(1 \to^{\boldsymbol{A}} (1 \to^{\boldsymbol{A}} b)\big) \\
&= 1 \to^{\boldsymbol{A}} \big((1 \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (1 \to^{\boldsymbol{A}} b)\big) \\
&= 1 \to^{\boldsymbol{A}} \big(1 \to^{\boldsymbol{A}} (a \wedge^{\boldsymbol{A}} b)\big) \\
&= \Box^{2}(a \wedge^{\boldsymbol{A}} b)\,.
\end{aligned}$$

Since $F$ is closed under meets, it follows that $\Box^{n}(a \wedge^{\boldsymbol{A}} b) \in F$, for every $n \in \mathbb{N}$. Hence, $a \wedge^{\boldsymbol{A}} b \in F_{\Box}$. Next, let $a \in F_{\Box}$ and let $b \in A$ such that $a \leq b$. Then, $\Box^{n} a \in F$, for every $n \in \mathbb{N}$. Since $a \leq b$ and $\boldsymbol{A} \in \mathsf{K} \subseteq \mathsf{WH}$, it easily follows by Lemma 7.38.4 that $\Box^{n} a \leq \Box^{n} b$, for every $n \in \mathbb{N}$. Since $F$ is upwards-closed, it follows that $\Box^n b \in F$, for every $n \in \mathbb{N}$. So, $b \in F_\Box$. To see that $F_\Box$ is open, let $a \in F_\Box$. Then, $\Box^n a \in F$, for every $n \in \mathbb{N}$. Clearly then, $\Box^n(\Box a) \in F$, for every $n \in \mathbb{N}$. So, $\Box^{\boldsymbol{A}} a \in F_\Box$. To see that $F_\Box$ extends $F$, let $a \in F$. Taking $n = 0$, it is immediate that $a \in F_\Box$. Finally, to prove the maximality condition, let $F' \subseteq A$ be an open filter below $F$ and let $a \in F'$. Since it is open, it follows that $\Box^n a \in F$, for every $n \in \mathbb{N}$. Thus, $a \in F_\Box$. ☐

**Proposition 7.49.** *Let $\boldsymbol{A} \in \mathsf{K}$. For every $F \in \mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}} \boldsymbol{A}$,*

$$F^* = F^{\mathrm{Su}} = \{a \in A : \Box^n a \in F, \textit{ for every } n \in \mathbb{N}\}$$

Proof. We know by Lemma 2.25 that if $F^{\mathrm{Su}}$ is a Suszko filter, then it is the largest one below $F$. And this is indeed the case for $\mathcal{S}^{\leq}_{\mathsf{K}}$ and algebras in $\mathsf{K}$, by Lemma 7.47. Since open filters coincide with Suszko filters on weakly Heyting algebras and $F_\Box$ is the largest open filter below $F$, the result follows. As to $F^*$, it is also an open filter below $F$ and moreover $F^{\mathrm{Su}} \subseteq F^*$. ☐

**Corollary 7.50.** *The logic $\mathcal{S}^{\leq}_{\mathsf{K}}$ has its Leibniz filters explicitly definable by the set of formulas $\Gamma(x) = \{\Box^n x : n \in \mathbb{N}\}$.*

Proof. Since $\mathsf{Alg}^*(\mathcal{S}^{\leq}_{\mathsf{K}}) \subseteq \mathsf{K}$, the result follows from Lemma 6.18. ☐

Let us fix the Necessitation rule:

$$(\mathrm{N}): \quad x \vdash \Box x \ .$$

Another consequence of the general theory of Chapter 6, particularly of Corollary 6.24, is the following:

**Corollary 7.51.** *The logic $\mathcal{S}^{\top}_{\mathsf{K}}$ is the inferential extension of $\mathcal{S}^{\leq}_{\mathsf{K}}$ by the rule* (N)*.*

But this comes with no surprise, as it had already been established in [**16**, Lemma 2.35]. We finish our study of subintuitionistic logics, by proving that none of the strong versions studied, save Visser's logic, is selfextensional.

**Proposition 7.52.** *None of the logics $\mathcal{S}^{\top}_{\mathsf{WH}}$, $\mathcal{S}^{\top}_{\mathsf{WH}_{(\mathrm{RT})}}$, and $\mathcal{S}^{\top}_{\mathsf{WH}_{(\mathrm{MP})}}$ is selfextensional.*

Proof. Let $\mathsf{K} \in \{\mathsf{WH}, \mathsf{WH}_{(\mathrm{RT})}, \mathsf{WH}_{(\mathrm{MP})}\}$. Suppose, towards an absurd, that $\mathcal{S}^{\top}_{\mathsf{K}}$ is selfextensional. Then, since $\mathcal{S}^{\top}_{\mathsf{K}}$ has a conjunction, it follows by Theorem 0.46 that $\mathcal{S}^{\top}_{\mathsf{K}}$ is semilattice-based. Now, we have two cases. In case $\mathsf{Alg}(\mathcal{S}^{\top}_{\mathsf{K}}) = \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}})$ (that is, if $\mathsf{K} = \mathsf{WH}_{(\mathrm{MP})}$), then $\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{K}}} \boldsymbol{A} = \mathrm{Filt}\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}}) = \mathsf{Alg}(\mathcal{S}^{\top}_{\mathsf{K}})$. Consequently, $\mathcal{S}^{\top}_{\mathsf{K}} = \mathcal{S}^{\leq}_{\mathsf{K}}$, and we reach an absurd, as the inclusions in Figure 7 are strict. In case $\mathsf{Alg}\big((\mathcal{S}^{\leq}_{\mathsf{K}})^+\big) \subsetneq \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}})$ (that is, if $\mathsf{K} = \mathsf{WH}$ or $\mathsf{K} = \mathsf{WH}_{(\mathrm{RT})}$), it follows by Theorem 0.47 that $\mathsf{Alg}(\mathcal{S}^{\top}_{\mathsf{K}})$ is a variety; but this contradicts [**16**, Teorema 4.41.2], which tell us that $\mathsf{Alg}(\mathcal{S}^{\top}_{\mathsf{K}})$ is not even a quasivariety. ☐

## 7.4. Semilattice-based logic of CIRL

In this section we study the semilattice-based logic $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ of the variety of commutative integral residuated lattices (the class $\mathsf{CIRL}$ will be formally introduced in Definition 7.53), together with $\mathcal{S}^{1}_{\mathsf{CIRL}}$, the $\{x \approx 1\}$-assertional logic of that same class, under the light of the general results established in Chapters 5 and 6. In particular, we aim at characterizing the Leibniz and Suszko $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$-filters, as well as finding the strong version $(\mathcal{S}^{\leq}_{\mathsf{CIRL}})^+$. For a thorough study of the logic $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$, see

[17]. Moreover, particular examples of multi-valued logics, such as Hájek's Basic Logic $\mathcal{BL}$, Lukasiewicz's infinite valued logic $Ł_\infty$, Product Logic $\Pi$, and Gödel's Logic $\mathcal{GL}$[2], are axiomatic extensions[3] of the logic $\mathcal{S}^1_{\mathsf{CIRL}}$. At the end of the section we explain how to obtain similar results for these particular logics.

The underlying language throughout this section will be $\mathcal{L} = \langle \wedge, \vee, \rightarrow, \odot, 1\rangle$. Our starting point is the definition of commutative residuated lattice.

**Definition 7.53.** An algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \rightarrow^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}, 1^{\boldsymbol{A}}\rangle$ is a *commutative residuated lattice* if:

1. $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}\rangle$ is a lattice;
2. $\langle A, \odot^{\boldsymbol{A}}, 1^{\boldsymbol{A}}\rangle$ is a commutative monoid[4];
3. $\rightarrow^{\boldsymbol{A}}$ is the residuum of $\odot^{\boldsymbol{A}}$, that is, for every $a, b \in A$, $a \odot^{\boldsymbol{A}} c \leq b$ iff $c \leq a \rightarrow^{\boldsymbol{A}} b$, where $\leq^{\boldsymbol{A}}$ is the lattice order.

A commutative residuated lattice is *integral*, if it satisfies additionally:

4. $1^{\boldsymbol{A}}$ is the top element of $\boldsymbol{A}$ of $\leq^{\boldsymbol{A}}$.

The class of all (respectively, integral) commutative residuated lattices will be denoted by $\mathsf{CRL}$ (respectively, $\mathsf{CIRL}$)[5].

The class of (commutative integral) residuated lattices is a variety; an equational axiomatization can be found in [43, Theorem 2.7]. Given $\boldsymbol{A} \in \mathsf{CRL}$, we shall denote by $\mathrm{Filt}\boldsymbol{A}$ the set of the lattice filters of $\boldsymbol{A}$ and by $\mathrm{Filt}_\rightarrow\boldsymbol{A}$ the subset of *implicative lattice filters*, i.e., the set of all lattice filters $F \in \mathrm{Filt}\boldsymbol{A}$ such that whenever $a, a \rightarrow^{\boldsymbol{A}} b \in F$, then $b \in F$.

We next compile some useful properties known to hold on the algebras in $\mathsf{CRL}$ (see for example, [43, Lemma 2.6][6]), all of which we will make use of at some point along the exposition.

**Lemma 7.54.** *Let* $\boldsymbol{A} \in \mathsf{CRL}$. *For every* $a, b, c \in A$,

1. $a \odot^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} b) \leq b$;
2. $a \rightarrow (b \rightarrow c) = (a \odot^{\boldsymbol{A}} b) \rightarrow^{\boldsymbol{A}} c$;
3. $a \leq (a \rightarrow^{\boldsymbol{A}} b) \rightarrow^{\boldsymbol{A}} b$;
4. $a \leq (b \rightarrow^{\boldsymbol{A}} (a \odot^{\boldsymbol{A}} b))$;
5. *If* $a \leq b$, *then* $b \rightarrow^{\boldsymbol{A}} c \leq a \rightarrow^{\boldsymbol{A}} c$;
6. *If* $a \leq b$, *then* $c \rightarrow^{\boldsymbol{A}} a \leq c \rightarrow^{\boldsymbol{A}} a$;
7. *If* $a \leq b$, *then* $a \odot^{\boldsymbol{A}} c \leq b \odot^{\boldsymbol{A}} c$;
8. $(a \rightarrow^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} c) \leq a \rightarrow^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} c)$;

---

[2] All these logics are BP-algebraizable having as equivalent algebraic semantics a subvariety of $\mathsf{CIRL}$.

[3] In rigor, are axiomatic extensions of the expansion of the logic $\mathcal{S}^1_{\mathsf{CIRL}}$ by the constant 0.

[4] A *monoid* is an algebra $\langle A, \circ, e\rangle$, where $\circ$ is a binary operation on $A$ which is associative and with a (left and right) identity $e$.

[5] We follow the notation of [43, p. 96]. The class of algebras having as defining conditions those of $\mathsf{CRL}$, but considered over the language $\mathcal{L} = \langle \wedge, \vee, \rightarrow, \odot, 1, 0\rangle$, is denoted in the literature by $\mathsf{FL_e}$. Similarly, the class of algebras having as defining conditions those of $\mathsf{CRL}$ and satisfying moreover that 0 is the bottom element of $\boldsymbol{A}$, also considered over the language $\mathcal{L} = \langle \wedge, \vee, \rightarrow, \odot, 1, 0\rangle$, is denoted in the literature by $\mathsf{FL_{ew}}$. See [43, Table 3.1, p. 188]. In general, residuated lattices are the 0-free reducts of FL-algebras.

[6] In [43, Lemma 2.6] the residuated lattices are not assumed to be commutative, and therefore the properties of Lemma 7.54 are stated with the left and right division operations, denoted by \ and / respectively, which in our setting both collapse into the operation $\rightarrow$.

9. $(a \to^{\boldsymbol{A}} c) \wedge^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} c) \leq (a \vee^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} c$;
10. $a \to^{\boldsymbol{A}} b \leq (b \to^{\boldsymbol{A}} c) \to^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} c)$;
11. $a \to^{\boldsymbol{A}} b \leq (c \to^{\boldsymbol{A}} a) \to^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} b)$;
12. $(a \to^{\boldsymbol{A}} b) \odot^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} c) \leq (a \to^{\boldsymbol{A}} c)$.

We usually refer to property (5) as *suffixing*; and to property (6) as *prefixing*. Also, since the operation $\odot^{\boldsymbol{A}}$ is commutative, property (7) can be applied on the left as well.

**Lemma 7.55.** *Let $\boldsymbol{A} \in \mathsf{CRL}$. A lattice filter $F \in \mathrm{Filt}\boldsymbol{A}$ is implicative if and only if it is closed under the operation $\odot^{\boldsymbol{A}}$.*

Proof. Assume that $F$ is an implicative lattice filter. Let $a, b \in F$. Notice that $a \leq b \to^{\boldsymbol{A}} (a \odot^{\boldsymbol{A}} b)$, by Lemma 7.54.4. Since $a \in F$ and $F$ is upwards-closed, it follows that $b \to^{\boldsymbol{A}} (a \odot^{\boldsymbol{A}} b) \in F$. Since $b \in F$ and $F$ is implicative, it follows that $a \odot^{\boldsymbol{A}} b \in F$. Conversely, assume that $F$ is closed under the operation $\odot^{\boldsymbol{A}}$. Let $a, a \to^{\boldsymbol{A}} b \in F$. It follows by assumption that $a \odot^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} b) \in F$. Moreover, $a \odot^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} b) \leq b$, by Lemma 7.54.1. Hence, since $F$ is upwards-closed, $b \in F$. ☐

Notice that none of the properties stated in Lemma 7.54 (neither the respective proofs, for what matters) make reference to the constant 1. Notice also that, given $\boldsymbol{A} \in \mathsf{CIRL}$, $1^{\boldsymbol{A}}$ plays two important rôles simultaneously: it is the multiplicative constant of the operation $\odot^{\boldsymbol{A}}$, and it is also the top element w.r.t. the order induced by $\wedge^{\boldsymbol{A}}$. Either considered separately, or taken together, these two conditions allow us to prove some more useful properties.

**Lemma 7.56.** *Let $\boldsymbol{A} \in \mathsf{CRL}$. For every $a, b, c \in A$,*

1. $a \leq b$ *iff* $1^{\boldsymbol{A}} \leq a \to^{\boldsymbol{A}} b$;
2. $1^{\boldsymbol{A}} \to^{\boldsymbol{A}} a = a$;
3. $1^{\boldsymbol{A}} \leq a \to^{\boldsymbol{A}} a$.

*In addition, if $\boldsymbol{A} \in \mathsf{CIRL}$,*

4. $a \odot^{\boldsymbol{A}} b \leq a \wedge^{\boldsymbol{A}} b$.

We are interested in the semilattice-based logic $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ and in the $\{x \approx 1\}$-assertional logic $\mathcal{S}^{1}_{\mathsf{CIRL}}$. In the literature, these logics are known under the terminology of “preserving degrees of truth” and “preserving truth”, respectively. The main reference for logics preserving degrees of truth from varieties of commutative integral residuated lattices is [**17**]; the particular case of Lukasiewicz’s logic preserving degrees of truth can be found in [**35**].

**Theorem 7.57.**

1. $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ *is not protoalgebraic.*
2. $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ *is not truth-equational.*
3. $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ *is not Fregean.*
4. $\mathcal{S}^{1}_{\mathsf{CIRL}}$ *is BP-algebraizable, witnessed by the set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y\}$ and the set of defining equations $\boldsymbol{\tau}(x) = \{x \approx 1\}$; its equivalent algebraic semantics is* $\mathsf{CIRL}$.
5. *For every $\boldsymbol{A} \in \mathsf{CIRL}$, $\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{CIRL}}}\boldsymbol{A} = \mathrm{Filt}\boldsymbol{A}$.*
6. *For every $\boldsymbol{A} \in \mathsf{CIRL}$, $\mathcal{F}i_{\mathcal{S}^{1}_{\mathsf{CIRL}}}\boldsymbol{A} = \mathrm{Filt}_{\to}\boldsymbol{A}$.*
7. $\mathsf{Alg}^*(\mathcal{S}^{\leq}_{\mathsf{CIRL}}) = \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{CIRL}}) = \mathsf{Alg}^*(\mathcal{S}^{1}_{\mathsf{CIRL}}) = \mathsf{Alg}(\mathcal{S}^{1}_{\mathsf{CIRL}}) = \mathsf{CIRL}$.

PROOF. *1.* It is well-known that every extension of a protoalgebraic logic is still protoalgebraic. Moreover, $Ł_\infty^{\leq}$ is not protoalgebraic, and $\mathcal{S}_{\mathsf{CIRL}}^{\leq} \leq Ł_\infty^{\leq}$. *2.* It is not difficult to see that every extension of a truth-equational logic is still truth-equational (if $\mathcal{S} \leq \mathcal{S}'$, then every $\mathcal{S}'$-filter is an $\mathcal{S}$-filter, and these last are equationally definable by assumption). Again, it is known that $Ł_\infty^{\leq}$ is not truth-equational. *3.* Suppose, towards an absurd, that $\mathcal{S}_{\mathsf{CIRL}}^{\leq}$ is Fregean. Since moreover it has theorems, it follows by [**4**, Corollary 11] that $\mathcal{S}_{\mathsf{CIRL}}^{\leq}$ is truth-equational, which we have just seen to be false. *4.–7.* [**17**, p. 1036, p. 1040 and Propositions 2.9,3.1,3.4]. □

For the first time in our examples, the strong version happens to be a fairly well studied logic in the literature, whose properties allow us to spot it right away as the strong version we are after. Indeed, it follows immediately by Corollary 6.11 that:

**Theorem 7.58.** *The logic $\mathcal{S}_{\mathsf{CIRL}}^{1}$ is the strong version of $\mathcal{S}_{\mathsf{CIRL}}^{\leq}$.*

Furthermore, for arbitrary $\boldsymbol{A}$, $\mathcal{F}i_{(\mathcal{S}_{\mathsf{CIRL}}^{\leq})^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}_{\mathsf{CIRL}}^{\leq}}\boldsymbol{A}$, again by Corollary 6.11. Therefore, without any further effort, we get:

**Theorem 7.59.** *Let $\boldsymbol{A} \in \mathsf{CIRL}$. The Leibniz $\mathcal{S}_{\mathsf{CIRL}}^{\leq}$-filters of $\boldsymbol{A}$ coincide with the implicative lattice filters of $\boldsymbol{A}$. That is,*

$$\mathcal{F}i^*_{\mathcal{S}_{\mathsf{CIRL}}^{\leq}}\boldsymbol{A} = \mathrm{Filt}_{\rightarrow}\boldsymbol{A}.$$

Another result which follows almost effortless, given the properties known about $\mathcal{S}_{\mathsf{CIRL}}^{1}$, is the following:

**Proposition 7.60.** *The logic $\mathcal{S}_{\mathsf{CIRL}}^{\leq}$ satisfies ($\star$).*

PROOF. Since $\mathcal{S}_{\mathsf{CIRL}}^{1}$ is algebraizable, the Leibniz operator

$$\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}_{\mathsf{CIRL}}^{1}}\boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S}_{\mathsf{CIRL}}^{1})}\boldsymbol{A}$$

is an order-isomorphism, for every $\boldsymbol{A}$. But, $\mathsf{Alg}^*(\mathcal{S}_{\mathsf{CIRL}}^{1}) = \mathsf{CIRL} = \mathsf{Alg}^*(\mathcal{S}_{\mathsf{CIRL}}^{\leq})$ and $\mathcal{F}i_{\mathcal{S}_{\mathsf{CIRL}}^{1}}\boldsymbol{A} = \mathrm{Filt}_{\rightarrow}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}_{\mathsf{CIRL}}^{\leq}}\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{CIRL}$, by Theorems 7.57 and 7.59. □

An important consequence is:

**Proposition 7.61.** *The logic $\mathcal{S}_{\mathsf{CIRL}}^{\leq}$ does not have its Leibniz filters explicitly definable.*

PROOF. It follows by Propositions 7.60 and 6.21, having in mind that $\mathcal{S}_{\mathsf{CIRL}}^{\leq}$ is not protoalgebraic. □

Although $\mathcal{S}_{\mathsf{CIRL}}^{\leq}$ does not have its Leibniz filters explicitly definable, it does have its Leibniz filters logically definable.

**Proposition 7.62.** *The logic $\mathcal{S}_{\mathsf{CIRL}}^{\leq}$ has its Leibniz filters logically definable by the rule Modus Ponens.*

PROOF. Just notice that, in light of Theorem 7.59, for every $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S}_{\mathsf{CIRL}}^{\leq})$ and every $F \in \mathcal{F}i_{\mathcal{S}_{\mathsf{CIRL}}^{\leq}}\boldsymbol{A}$, $F$ is a Leibniz $\mathcal{S}_{\mathsf{CIRL}}^{\leq}$-filter of $\boldsymbol{A}$ if and only if $F$ is an implicative lattice filter if and only if is closed under *Modus Ponens*. Hence, the result follows from Proposition 6.30. □

Consequently [17, Corollary 2.11],

**Corollary 7.63.** *The logic* $\mathcal{S}^{1}_{\mathsf{CIRL}}$ *is the inferential extension of* $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ *by the rule Modus Ponens.*

Proof. The result follows by Corollary 6.33, since $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ has its Leibniz filters logically definable by *Modus Ponens*, by Proposition 7.62. □

Moreover, since $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ is semilattice-based, it follows by Proposition 6.5 that:

**Proposition 7.64.** *The logic* $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ *has its Leibniz filters equationally definable by* $\boldsymbol{\tau}(x) = \{x \approx \top\}$.

Notice that the same result follows by Proposition 6.6, because $(\mathcal{S}^{\leq}_{\mathsf{CIRL}})^{+} = \mathcal{S}^{1}_{\mathsf{CIRL}}$ is truth-equational and moreover $\mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{CIRL}}) = \mathsf{Alg}(\mathcal{S}^{1}_{\mathsf{CIRL}})$.

Applying one more time the results of Chapter 6, we obtain:

**Corollary 7.65.** *Let* $\boldsymbol{A} \in \mathsf{CIRL}$. *For every* $F \in \mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{CIRL}}}\boldsymbol{A}$,

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{*}) \quad \textit{and} \quad \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}^{\leq}_{\mathsf{CIRL}}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}}).$$

*Moreover,*

$$F^{\mathrm{Su}} = \bigcap_{G \in (\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{CIRL}}}\boldsymbol{A})^{F}} G^{*}.$$

*As a consequence,* $F$ *is a Suszko* $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$*-filter of* $\boldsymbol{A}$ *if and only if* $F \subseteq G^{*}$, *for every* $G \in (\mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{CIRL}}}\boldsymbol{A})^{F}$.

Finally, and similarly to Belnap's logic, we will give a characterization of the operation that turns a $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$-filter $F$ into its associated Leibniz $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$-filter $F^{*}$, for commutative integral residuated lattices, inspired once again by the Birula-Rasiowa transformation.

**Definition 7.66.** Let $\boldsymbol{A} \in \mathsf{CIRL}$. For every $F \in \mathrm{Filt}\boldsymbol{A}$, define

$$\Psi(F) := \{a \in A : \forall b \in A \text{ if } a \to^{\boldsymbol{A}} b \in F, \text{ then } b \in F\}.$$

**Proposition 7.67.** *Let* $\boldsymbol{A} \in \mathsf{CIRL}$. *If* $F \in \mathrm{Filt}\boldsymbol{A}$, *then* $\Psi(F) \in \mathrm{Filt}_{\to}\boldsymbol{A}$.

Proof. First note that $1 \in \Psi(F)$, because $1 \to^{\boldsymbol{A}} b = b$, by Lemma 7.56.2, for every $b \in A$. Let $a \in \Psi(F)$ and $b \in A$ such that $a \leq b$. Then, $a \to^{\boldsymbol{A}} b = 1 \in F$, by Lemma 7.56.1. Since $a \in \Psi(F)$, it follows that $b \in F$. Next, let $a, b \in \Psi(F)$. We claim that $a \odot^{\boldsymbol{A}} b \in \Psi(F)$. Let $c \notin F$. Then, $b \to^{\boldsymbol{A}} c \notin F$, because $b \in \Psi(F)$. Then, $a \to^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} c) \notin F$, because $a \in \Psi(F)$. But,

$$(a \odot^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} c = a \to^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} c),$$

by Lemma 7.54.2. So, $(a \odot^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} c \notin F$. Thus, $a \odot^{\boldsymbol{A}} b \in \Psi(F)$. Since $a \odot^{\boldsymbol{A}} b \leq a \wedge^{\boldsymbol{A}} b$, by Lemma 7.56.4, and we have seen already $\Psi(F)$ to be upwards-closed, it follows that $a \wedge^{\boldsymbol{A}} b \in F$. Finally, let $a, a \to^{\boldsymbol{A}} b \in \Psi(F)$. Since $\Psi(F) \subseteq F$, $a \to^{\boldsymbol{A}} b \in F$. Since $a \in \Psi(F)$, $b \in F$. □

**Proposition 7.68.** *Let* $\boldsymbol{A} \in \mathsf{CIRL}$. *For every* $F \in \mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{CIRL}}}\boldsymbol{A}$,

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big).$$

Proof. We claim that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is compatible with $\Psi(F)$. Let $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ and $a \in \Psi(F)$. Let $c \in A$ such that $b \to^{\boldsymbol{A}} c \in F$. Then, $\langle a \to^{\boldsymbol{A}} c, b \to^{\boldsymbol{A}} c\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since $b \to^{\boldsymbol{A}} c \in F$, it follows that $a \to^{\boldsymbol{A}} c \in F$. Since $a \in \Psi(F)$, it follows that $c \in F$. So, $b \in \Psi(F)$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$.

Conversely, we claim that $\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$ is compatible with $F$. Let $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$ and let $a \in F$. Then, $\langle a \to^{\boldsymbol{A}} b, b \to^{\boldsymbol{A}} b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$. Since $b \to^{\boldsymbol{A}} b = 1 \in \Psi(F)$, it follows that $a \to^{\boldsymbol{A}} b \in \Psi(F)$. Also, $a \leq (a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} b$, by Lemma 7.54.3. So, $(a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} b \in F$, because $F$ is upwards-closed. Since $a \to^{\boldsymbol{A}} b \in \Psi(F)$, it follows that $b \in F$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. □

**Corollary 7.69.** *Let $\boldsymbol{A} \in \mathsf{CIRL}$. For every $F \in \mathcal{F}i_{\mathcal{S}^{\leq}_{\mathsf{CIRL}}}\boldsymbol{A}$,*

$$F^* = \Psi(F).$$

Proof. On the one hand, since $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$, we have $\Psi(F) \in [\![F]\!]^*$, and hence $F^* \subseteq \Psi(F)$. On the other hand, since $\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$, we have $F^* \in [\![\Psi(F)]\!]^*$, and hence $\Psi(F)^* \subseteq F^*$. But $\Psi(F) = \Psi(F)^*$, because $\Psi(F)$ is an implicative lattice filter of $\boldsymbol{A}$, by Proposition 7.67, and the Leibniz $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$-filters of $\boldsymbol{A}$ coincide with the implicative lattice filters of $\boldsymbol{A}$, by Theorem 7.59. □

We finish our study of the logic $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ by proving that its strong version is not selfextensional (this result is not new however, it follows from [**17**, Theorem 4.12]).

**Proposition 7.70.** *The logic $\mathcal{S}^{1}_{\mathsf{CIRL}}$ is not selfextensional.*

Proof. Suppose, towards an absurd, that $\mathcal{S}^{1}_{\mathsf{CIRL}}$ is selfextensional. Then, since $\mathcal{S}^{1}_{\mathsf{CIRL}}$ has a conjunction, it follows by Theorem 0.46 that $\mathcal{S}^{1}_{\mathsf{CIRL}}$ is semilattice-based. Then, it is semilattice-based of $\mathsf{Alg}(\mathcal{S}^{1}_{\mathsf{CIRL}}) = \mathsf{CIRL}$, using Theorem 7.57.7. Consequently, $\mathcal{S}^{1}_{\mathsf{CIRL}} = \mathcal{S}^{\leq}_{\mathsf{CIRL}}$, and we reach an absurd (for instance, $\mathcal{S}^{1}_{\mathsf{CIRL}}$ is truth-equational, while $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ is not). □

As final remarks, we explain how the results of the present section apply to some particular semilattice-based logics of subvarieties of commutative integral residuated lattices, whose strong versions turn out be well-known multi-valued logics. In the following, let $\mathcal{L}'$ be the expansion of $\mathcal{L}$ by the constant 0, i.e., $\mathcal{L}' = \langle \wedge, \vee, \to, \odot, 1, 0\rangle$. Let us also define the unary operation $\neg^{\boldsymbol{A}} a := a \to^{\boldsymbol{A}} 0$, for every $\mathcal{L}'$-algebra $\boldsymbol{A}$ and every $a \in A$.

### Hájek's basic logic $\mathcal{BL}$

A *BL-algebra* is an $\mathcal{L}'$-algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}, 1, 0\rangle$, where

1. The reduct $\langle \wedge, \vee, \to, \odot, 1\rangle$ belongs to $\mathsf{CIRL}$;
2. $a \wedge^{\boldsymbol{A}} b = a \odot^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} b)$, for every $a, b \in A$;
3. $(a \to^{\boldsymbol{A}} b) \vee^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} a) = 1$, for every $a, b \in A$.

Let us denote the semilattice-based logic of the class of all BL-algebras by $\mathcal{BL}^{\leq}$. The $\{x \approx 1\}$-assertional logic of $\mathcal{BL}$ is usually known as Hájek's basic logic $\mathcal{BL}$. It is known that $\mathcal{BL}$ is BP-algebraizable witnessed by the set of equivalence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y\}$ and the set of defining equations $\boldsymbol{\tau}(x) = \{x \approx \top\}$, and that $Ł^{\leq}_{\infty}$ is an extension of $\mathcal{BL}^{\leq}$. From this latter fact follows 1 and 2 below, reasoning similarly as in Theorem 7.57.1 and 2. Also, since $\mathcal{BL}^{\leq}$ is a semilattice-based logic with theorems, items 3 and 4 below follow by Corollary 6.11.

1. $\mathcal{BL}^{\leq}$ is not protoalgebraic.

2. $\mathcal{BL}^{\leq}$ is not truth-equational.
3. $(\mathcal{BL}^{\leq})^{+} = \mathcal{BL}$;
4. $\mathcal{F}i_{\mathcal{BL}}\boldsymbol{A} = \mathcal{F}i^{*}_{\mathcal{BL}^{\leq}}\boldsymbol{A}$, for every BL-algebra $\boldsymbol{A}$.

**Lukasiewicz infinite valued logic $\mathbf{L}_{\infty}$**

An *MV-algebra* is an $\mathcal{L}'$-algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \rightarrow^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}, 1, 0\rangle$, where

1. $\boldsymbol{A}$ is a BL-algebra;
2. $\neg^{\boldsymbol{A}}\neg^{\boldsymbol{A}}a = a$, for every $a \in A$.

Let us denote the class of all MV-algebras algebras by MV. The semilattice-based logic of MV is usually known as the Lukasiewicz's infinite valued logic preserving degrees of truth $Ł_{\infty}^{\leq}$ [35], while the $\{x \approx 1\}$-assertional logic of MV is the famous Lukasiewicz's infinite valued logic $Ł_{\infty}$. It is known that $Ł_{\infty}^{\leq}$ is not protoalgebraic [35, Theorem 3.11], and that $Ł_{\infty}$ is BP-algebraizable witnessed by the set of equivalence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y\}$ and the set of defining equations $\boldsymbol{\tau}(x) = \{x \approx \top\}$ [35, Theorem 2.1]. Again, since $L_{\infty}^{\leq}$ is a semilattice-based logic with theorems, items 2 and 3 below follow by Corollary 6.11.

1. $Ł_{\infty}^{\leq}$ is not truth-equational.
2. $(Ł_{\infty}^{\leq})^{+} = Ł_{\infty}$;
3. $\mathcal{F}i_{Ł_{\infty}}\boldsymbol{A} = \mathcal{F}i^{*}_{Ł_{\infty}^{\leq}}\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{MV}$.

That $L_{\infty}^{\leq}$ is not truth-equational follows from the proof of [35, Theorem 3.10], where an MV-algebra is exhibited such that the Leibniz operator is not injective over its $L_{\infty}^{\leq}$-filters. So, in fact, truth is not even implicitly definable in $\mathsf{Mod}^{*}(L_{\infty}^{\leq})$.

**Product Logic $\Pi$**

A *product algebra* is an $\mathcal{L}'$-algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \rightarrow^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}, 1, 0\rangle$, where

1. $\boldsymbol{A}$ is a BL-algebra;
2. $\neg^{\boldsymbol{A}}\neg^{\boldsymbol{A}}c \leq \big((a \odot^{\boldsymbol{A}} c) \rightarrow^{\boldsymbol{A}} (b \odot^{\boldsymbol{A}} c)\big) \rightarrow^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} b)$, for every $a, b, c \in A$;
3. $a \wedge^{\boldsymbol{A}} \neg^{\boldsymbol{A}}a = 0$, for every $a \in A$.

Let us denote the semilattice-based logic of the class of all product algebras by $\Pi^{\leq}$. The $\{x \approx 1\}$-assertional logic of the class of all product algebras is usually known as the Product Logic $\Pi$. It is known that $\Pi^{\leq}$ is not protoalgebraic [17, Example B.3], while $\Pi$ is BP-algebraizable witnessed by the set of equivalence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y\}$ and the set of defining equations $\boldsymbol{\tau}(x) = \{x \approx \top\}$. Again, since $\Pi^{\leq}$ is a semilattice-based logic with theorems, items 2 and 3 below follow by Corollary 6.11.

1. $\Pi^{\leq}$ is not truth-equational.
2. $(\Pi^{\leq})^{+} = \Pi$;
3. $\mathcal{F}i_{\Pi}\boldsymbol{A} = \mathcal{F}i^{*}_{\Pi^{\leq}}\boldsymbol{A}$, for every product algebra $\boldsymbol{A}$.

To see 1, we reason as follows: suppose, towards an absurd, that $\Pi^{\leq}$ is truth-equational; then, it coincides with its own strong version, by Proposition 5.5.4; hence, by 2 above, $\Pi^{\leq} = \Pi$; as a consequence $\Pi^{\leq}$ is protoalgebraic; we reach an absurd.

**Gödel's Logic $\mathcal{GL}$**

A *Gödel algebra* is an $\mathcal{L}'$-algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \rightarrow^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}, 1, 0\rangle$, where

1. $\boldsymbol{A}$ is a BL-algebra;
2. $a \odot^{\boldsymbol{A}} a = a$, for every $a \in A$.

Let us denote the semilattice-based logic of the class of all product algebras by $\mathcal{GL}^{\leq}$. The $\{x \approx 1\}$-assertional logic of the class of all Gödel algebras is usually known as the Gödel Logic's $\mathcal{GL}$. It is known that $\mathcal{GL}$ is BP-algebraizable witnessed by the set of equivalence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y\}$ and the set of defining equations $\boldsymbol{\tau}(x) = \{x \approx \top\}$. But in this case, the operations $\wedge^{\boldsymbol{A}}$ and $\odot^{\boldsymbol{A}}$ coincide, and therefore Gödel algebras form a subvariety of generalized Heyting algebras [**17**, p. 1046]. It follows by [**17**, Theorem 4.12] that $\mathcal{GL}^{\leq} = \mathcal{GL}$. So, having in mind Corollary 6.11, $\mathcal{GL}^{\leq}$ coincides with its own strong version.

## 7.5. An intermediate logic between the semilattice-based logic of $\mathsf{CRL}$ and the $\{x \wedge 1 \approx 1\}$-assertional logic of $\mathsf{CRL}$

We now wish to generalize the results of the previous section by considering commutative residuated lattices not necessarily integral. The motivation is to capture some more examples of substructural logics covered in the literature — most notably, the Classical and Intuitionistic Linear Logics without exponentials — as the strong versions of two new (at least to our knowledge) non-protoalgebraic logics.

The semilattice-based logic of $\mathsf{CRL}$, $\mathcal{S}^{\leq}_{\mathsf{CRL}}$, does not have theorems, because there are commutative residuated lattices which are not integral (recall that a semilattice-based logic $\mathcal{S}^{\leq}_{\mathsf{K}}$ has theorems if and only if the semilattice reducts in $\mathsf{K}$ have a term-definable maximum element). Consequently, the strong version $(\mathcal{S}^{\leq}_{\mathsf{CRL}})^{+}$ is the almost inconsistent logic. This being the case, and for the first time so far, we shall work with an extension of the semilattice-based logic under consideration, one which has as theorems a distinguished set of formulas whose interpretation in the associated semilattice reducts is algebraically meaningful. The underlying language in the present section will (still) be $\mathcal{L} = \langle \wedge, \vee, \to, \odot, 1\rangle$.

Recall that the logic $\mathcal{S}^{\leq}_{\mathsf{CRL}}$ is induced by the class of matrices $\{\langle \boldsymbol{A}, [a) \rangle : \boldsymbol{A} \in \mathsf{CRL}, a \in A\}$ and therefore also by the class of matrices $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{CRL}, F \in \mathrm{Filt}\boldsymbol{A}\}$. The substructural logic usually associated with $\mathsf{CRL}$ is the logic induced by the class of matrices $\{\langle \boldsymbol{A}, [1^{\boldsymbol{A}})\rangle : \boldsymbol{A} \in \mathsf{CRL}\}$, where $[1^{\boldsymbol{A}})$ is the up-set of $1^{\boldsymbol{A}}$; in other words, it is the $\boldsymbol{\tau}$-assertional logic of the class of algebras $\mathsf{CRL}$, where $\boldsymbol{\tau}(x) \coloneqq \{x \wedge 1 \approx 1\}$. According to the notation on page 17, such logic is denoted by $\mathcal{S}\big(\mathsf{CRL}, \{x \wedge 1 \approx 1\}\big)$; for ease of notation however, we shall denote it by $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}$. So, the consequence relation $\vdash_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}}$ is defined by

$$\Gamma \vdash_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}} \varphi \quad \text{iff} \quad \forall \boldsymbol{A} \in \mathsf{CRL}\ \forall h \in \mathrm{Hom}(\mathrm{Fm}_{\mathcal{L}}, \boldsymbol{A}) \\ \text{if } \forall \gamma \in \Gamma\ 1^{\boldsymbol{A}} \leq h(\gamma), \text{ then } 1^{\boldsymbol{A}} \leq h(\varphi),$$

for every $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ and every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$. We next collect some (known) facts about the logic $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}$. These results can be found in [**49**, Chapter 6] (stated explicitly for the $\{x \wedge 1 \approx 1\}$-assertional logic of $\mathsf{CRL}$), but also follow from [**43**, Section 2.6] (stated more generally for the $\{x \wedge 1 \approx 1\}$-assertional logic of the variety $\mathsf{FL}$ of FL-algebras).

**Theorem 7.71.**

1. *$\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}$ is BP-algebraizable, witnessed by the set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y\}$ and the set of defining equations $\boldsymbol{\tau}(x) = \{x \wedge 1 \approx 1\}$; its equivalent algebraic semantics is $\mathsf{CRL}$.*
2. *For every $\boldsymbol{A} \in \mathsf{CRL}$, $\mathcal{F}i_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}}\boldsymbol{A} = \{F \in \mathrm{Filt}_{\to}\boldsymbol{A} : 1^{\boldsymbol{A}} \in F\}$.*
3. *$\mathsf{Alg}^{*}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}) = \mathsf{Alg}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}) = \mathsf{CRL}$.*

For the sake of simplicity, from this point on we shall refrain from distinguishing notationally the $\mathcal{L}$-term 1 and its interpretation $1^{\boldsymbol{A}}$, whenever the context is clear.

We are interested in finding a logic as close as possible to the semilattice-based logic of $\mathsf{CRL}$ and whose strong version happens to be the logic $\mathcal{S}^{\tau}_{\mathsf{CRL}}$. Since it must have the same theorems as $\mathcal{S}^{\tau}_{\mathsf{CRL}}$, the natural candidate is the least logic in between $\mathcal{S}^{\leq}_{\mathsf{CRL}}$ and $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ with the same theorems as the latter. The logic with these properties is defined by the class of matrices

$$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{CRL}, F \in \mathrm{Filt}\boldsymbol{A}, 1 \in F\},$$

as we will show in Proposition 7.73. We denote the consequence relation induced by the class of matrices above by $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$. Before going any further note that this consequence relation is finitary because the class of matrices above is first-order definable and hence closed under ultraproducts. Moreover, since for every $\boldsymbol{A} \in \mathsf{CRL}$, $\{a \in A : 1^{\boldsymbol{A}} \leq a\}$ is a lattice filter that contains $\{1^{\boldsymbol{A}}\}$, it follows by the definitions involved that $\mathcal{S}^{\leq}_{\mathsf{CRL}} \leq \mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}} \leq \mathcal{S}^{\tau}_{\mathsf{CRL}}$.

Let us first determine the class of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$-algebras.

**Proposition 7.72.** $\mathsf{CRL} = \mathsf{Alg}^*(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}) = \mathsf{Alg}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}})$.

Proof. Just notice that, since $\mathcal{S}^{\leq}_{\mathsf{CRL}} \leq \mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}} \leq \mathcal{S}^{\tau}_{\mathsf{CRL}}$,

$$\mathsf{CRL} = \mathsf{Alg}^*(\mathcal{S}^{\tau}_{\mathsf{CRL}}) \subseteq \mathsf{Alg}^*(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}) \subseteq \mathsf{Alg}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}) \subseteq \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{CRL}}) = \mathsf{CRL},$$

using Theorem 7.71.3, and the fact that $\mathcal{S}^{\leq}_{\mathsf{CRL}}$ is, by definition, the semilattice-based logic of $\mathsf{CRL}$. □

**Proposition 7.73.** *The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ is the least logic in between $\mathcal{S}^{\leq}_{\mathsf{CRL}}$ and $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ with the same theorems that $\mathcal{S}^{\tau}_{\mathsf{CRL}}$.*

Proof. Let us first show that the theorems of $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ and $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ are the same. Since $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}} \leq \mathcal{S}^{\tau}_{\mathsf{CRL}}$, it is clear that if $\varnothing \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \varphi$, then $\varnothing \vdash_{\mathcal{S}^{\tau}_{\mathsf{CRL}}} \varphi$. Conversely, assume $\varnothing \vdash_{\mathcal{S}^{\tau}_{\mathsf{CRL}}} \varphi$. Let $\boldsymbol{A} \in \mathsf{CRL}$ and $F \in \mathrm{Filt}\boldsymbol{A}$ such that $1 \subseteq F$. It follows by assumption that $1 \leq h(\varphi)$. Since $1 \in F$ and $F$ is upwards-closed, it follows that $h(\varphi) \in F$. Thus, $\varnothing \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \varphi$.

Assume now that $\mathcal{S}$ is a logic such that $\mathcal{S}^{\leq}_{\mathsf{CRL}} \leq \mathcal{S} \leq \mathcal{S}^{\tau}_{\mathsf{CRL}}$ and with the same theorems that $\mathcal{S}^{\tau}_{\mathsf{CRL}}$. Then, $\mathsf{CRL} = \mathsf{Alg}(\mathcal{S}^{\tau}_{\mathsf{CRL}}) \subseteq \mathsf{Alg}(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{CRL}}) = \mathsf{CRL}$, using Theorem 7.71.3 and the fact that $\mathcal{S}^{\leq}_{\mathsf{CRL}}$ is semilattice-based of $\mathsf{CRL}$. Thus, for every $\boldsymbol{A} \in \mathsf{CRL}$, every $\mathcal{S}$-filter of $\boldsymbol{A}$ is an $\mathcal{S}^{\leq}_{\mathsf{CRL}}$-filter of $\boldsymbol{A}$, and therefore it is a lattice filter of $\boldsymbol{A}$. Moreover, since 1 is a theorem of $\mathcal{S}^{\tau}_{\mathsf{CRL}}$, $1^{\boldsymbol{A}}$ belongs to every $\mathcal{S}$-filter of $\boldsymbol{A}$. Therefore $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S}), F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\} \subseteq \{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{CRL}, F \in \mathrm{Filt}\boldsymbol{A}, 1^{\boldsymbol{A}} \in F\}$. This implies that $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}} \leq \mathcal{S}$. □

The consequence relation $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ has a useful and enlightening characterization.

**Proposition 7.74.** *For every $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ and every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$*

$$\Gamma \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \varphi \quad \textit{iff} \quad \forall \boldsymbol{A} \in \mathsf{CRL}\ \forall a \in A\ \forall h \in \mathrm{Hom}(\mathrm{Fm}_{\mathcal{L}}, \boldsymbol{A})$$
$$\textit{if } \forall \gamma \in \Gamma\ 1 \wedge^{\boldsymbol{A}} a \leq h(\gamma), \textit{ then } 1 \wedge^{\boldsymbol{A}} a \leq h(\varphi).$$

Proof. Let us temporarily denote by $\vDash$ the consequence relation defined by the condition on the right hand side of the 'iff'. Note that $\vDash$ is the consequence relation induced by the class of matrices $\{\langle \boldsymbol{A}, [1 \wedge^{\boldsymbol{A}} a)\rangle : \boldsymbol{A} \in \mathsf{CRL}, a \in A\}$. This class is easily seen to be first-order definable. Therefore it is closed under ultraproducts, and this implies that the induced consequence relation is finitary. Moreover, since this class is included in $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{CRL}, F \in \mathrm{Filt}\boldsymbol{A}, 1 \in F\}$, then $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}} \leq \vDash$.

Conversely, suppose that $\Gamma \vDash \varphi$. Let $\boldsymbol{A} \in \mathsf{CRL}$ and $F$ a lattice filter of $\boldsymbol{A}$ such that $1 \in F$. Let $h \in \mathrm{Hom}(\mathrm{Fm}_{\mathcal{L}}, \boldsymbol{A})$ be such that $h(\Gamma) \subseteq F$. Now since $\vDash$ is finitary, let $\Gamma' \subseteq \Gamma$ be a finite set such that $\Gamma' \vDash \varphi$. Then $h(\Gamma) \subseteq F$. Since $F$ is a lattice filter there is $a \in F$ such that $a \leq h(\gamma)$ for every $\gamma \in \Gamma'$. Therefore, $1 \wedge^{\boldsymbol{A}} a \leq h(\gamma)$ for every $\gamma \in \Gamma'$. Thus, $1 \wedge^{\boldsymbol{A}} a \leq h(\varphi)$. Since $1 \in F$, we have $h(\varphi) \in F$. It follows that $\Gamma' \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \varphi$. Therefore $\Gamma \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \varphi$. $\square$

Let us now see that the logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ falls outside the classes of logics in Figure 1.

**Theorem 7.75.**

1. *$\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ is not protoalgebraic.*
2. *$\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ is not truth-equational.*

Proof. Notice that $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}} \leq \mathcal{S}^{\leq}_{\mathsf{CIRL}}$, because $\mathsf{CIRL} \subseteq \mathsf{CRL}$, and every lattice filter $F \in \mathrm{Filt}\boldsymbol{A}$, with $\boldsymbol{A} \in \mathsf{CIRL}$, is necessarily such that $1 \in F$. Now, we know that $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ is neither protoalgebraic nor truth-equational, by Theorem 7.57.1 and 2, respectively. $\square$

Unlike in the integral case, we have now left the semilattice-based setting, and we can no longer apply the general results of Chapter 5 concerning this family of logics. So, our strategy to find the strong version of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ will be different. For the logics $\mathcal{PML}$ and $\mathcal{B}$, we first characterized the Leibniz filters in order to find its respective strong version. For the subintuitionistic logics and the logic $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ it followed straightforwardly by Corollary 6.11. This time we will use Proposition 5.9, using as candidate for the strong version the logic $\mathcal{S}^{\tau}_{\mathsf{CRL}}$. We know already that $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ is truth-equational, for it is algebraizable. We also know that $\mathsf{Alg}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}) = \mathsf{Alg}(\mathcal{S}^{\tau}_{\mathsf{CRL}})$. It remains to be checked that the least $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ and the least $\mathcal{S}^{\tau}_{\mathsf{CRL}}$-filter coincide for every $\boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}^{\tau}_{\mathsf{CRL}}) = \mathsf{CRL}$. This is what we do next, by first characterizing the $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$-filters of the algebras in $\mathsf{CRL}$.

**Proposition 7.76.** *For every $\boldsymbol{A} \in \mathsf{CRL}$, the $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$-filters coincide with the lattice filters of $\boldsymbol{A}$ containing 1. That is,*

$$\mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \boldsymbol{A} = \{F \in \mathrm{Filt}\boldsymbol{A} : 1 \in F\}.$$

Proof. Let $\boldsymbol{A} \in \mathsf{CRL}$ and $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \boldsymbol{A}$. Notice that using the definition of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ it easily follows that

$$\varnothing \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} 1, \qquad x, y \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} x \wedge y, \qquad x \wedge y \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} x, y.$$

This implies that $F \in \mathrm{Filt}\boldsymbol{A}$ and $1 \in F$. From the definition of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ it follows immediately that if $F \in \mathrm{Filt}\boldsymbol{A}$ is such that $1 \in F$, then $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \boldsymbol{A}$. $\square$

Hence, for every $\boldsymbol{A} \in \mathsf{CRL}$, the least $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$-filter on $\boldsymbol{A}$ is $[1)$, which is exactly the least $\mathcal{S}^{\tau}_{\mathsf{CRL}}$-filter on $\boldsymbol{A}$. We are now able to apply Proposition 5.9:

**Theorem 7.77.** *The logic $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ is the strong version of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$.*

Having in mind that $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ is algebraizable and thus truth-equational, it readily follows from Proposition 5.14 that:

**Theorem 7.78.** *Let $\boldsymbol{A} \in \mathsf{CRL}$. The Leibniz $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$-filters of $\boldsymbol{A}$ coincide with the implicative lattice filters of $\boldsymbol{A}$ containing* 1*. That is,*

$$\mathcal{F}i^{*}_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \boldsymbol{A} = \{F \in \mathrm{Filt}_{\rightarrow} \boldsymbol{A} : 1 \in F\}.$$

Although not semilattice-based, $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ still has its Leibniz filters equationally definable. For we know that $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}$ is truth-equational witnessed by the set of defining equations $\boldsymbol{\tau}(x) = \{x \wedge 1 \approx 1\}$ and moreover $\mathsf{Alg}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}) = \mathsf{CRL} = \mathsf{Alg}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}})$. Hence, it follows by Proposition 6.6 that:

**Proposition 7.79.** *The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ has its Leibniz filters equationally definable by $\boldsymbol{\tau}(x) = \{x \wedge 1 \approx 1\}$.*

Furthermore, the fact that $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}$ is algebraizable also allows us to prove:

**Proposition 7.80.** *The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ satisfies* ($\star$).

Proof. Since $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}$ is algebraizable, $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}} \boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^{*}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}})} \boldsymbol{A}$ is an order-isomorphism, for every $\boldsymbol{A}$. But, $\mathsf{Alg}^{*}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}) = \mathsf{CRL} = \mathsf{Alg}^{*}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}})$ and $\mathcal{F}i_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}} \boldsymbol{A} = \{F \in \mathrm{Filt}_{\rightarrow} \boldsymbol{A} : 1 \in F\} = \mathcal{F}i^{*}_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \boldsymbol{A}$, for $\boldsymbol{A} \in \mathsf{CRL}$, by Theorems 7.75 and 7.78, respectively. □

We can therefore apply the general theory of Chapter 6 and obtain:

**Corollary 7.81.** *Let $\boldsymbol{A} \in \mathsf{CRL}$. For every $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \boldsymbol{A}$,*

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{*}) \quad \text{and} \quad \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}}).$$

*Moreover,*

$$F^{\mathrm{Su}} = \bigcap_{G \in (\mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \boldsymbol{A})^{F}} G^{*}.$$

*As a consequence, $F$ is a Suszko $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$-filter of $\boldsymbol{A}$ if and only if $F \subseteq G^{*}$, for every $G \in (\mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \boldsymbol{A})^{F}$.*

Another interesting consequence is the following:

**Proposition 7.82.** *The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ does not have its Leibniz filters explicitly definable.*

Proof. It follows by Propositions 7.80 and 6.21, having in mind that $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ is not protoalgebraic. □

Just like the integral case, although $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ does not have its Leibniz filters explicitly definable, it does have its Leibniz filters logically definable. Indeed, given Theorem 7.78, it easily follows that $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ has its Leibniz filters logically definable by the rule *Modus Ponens*.

**Proposition 7.83.** *The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ has its Leibniz filters logically definable by the rule Modus Ponens.*

Proof. Just notice that, in light of Theorem 7.78, for every $\boldsymbol{A} \in \mathsf{Alg}^{*}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}) = \mathsf{CRL}$ and every $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \boldsymbol{A}$, $F$ is a Leibniz $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$-filter of $\boldsymbol{A}$ if and only if $F$ is an implicative lattice filter if and only if is closed under *Modus Ponens*. Hence, the result follows from Proposition 6.30. □

Consequently,

**Corollary 7.84.** *The logic $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ is the inferential extension of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ by the rule Modus Ponens.*

PROOF. The result follows by Corollary 6.33, since $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ has its Leibniz filters logically definable by the rule *Modus Ponens*, by Proposition 7.83. □

Our final goal is to find a characterization of $F^*$ using again some Birula-Rasiowa style transformation, as we did for the integral case. The natural candidate for the transformation $\Psi$ is now:

**Definition 7.85.** Let $\boldsymbol{A} \in \mathsf{CRL}$. For every $F \in \mathrm{Filt}\boldsymbol{A}$, define

$$\Psi(F) := \{a \in A : \forall b \in A \text{ if } (a \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} b \in F, \text{ then } b \in F\}.$$

**Proposition 7.86.** *Let $\boldsymbol{A} \in \mathsf{CRL}$. For every $F \in \mathrm{Filt}\boldsymbol{A}$, $\Psi(F) \in \mathrm{Filt}_{\to}\boldsymbol{A}$ and $1 \in \Psi(F)$.*

PROOF. First of all, $1 \in \Psi(F)$, because $(1 \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} b = 1 \to^{\boldsymbol{A}} b = b$, using Lemma 7.56.2. Next, let $a \in \Psi(F)$ and $b \in A$ such that $a \leq b$. Let $c \in A$ be such that $(b \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} c \in F$. Since $a \wedge^{\boldsymbol{A}} 1 \leq b \wedge^{\boldsymbol{A}} 1$, we have $(b \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} c \leq (a \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} c$, by suffixing. Since $F$ is upwards-closed, it follows that $(a \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} c \in F$. Since $a \in \Psi(F)$, it follows that $c \in F$. Hence, $b \in \Psi(F)$. This shows that $\Psi(F)$ is an up-set. To prove that it is closed under meets, let $a, b \in \Psi(F)$. Suppose that $c \notin F$. Then, $(b \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} c \notin F$, because $b \in \Psi(F)$, and therefore $(a \wedge^{\boldsymbol{A}} 1) \to \big((b \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} c\big) \notin F$, because $a \in \Psi(F)$. Thus, $\big((a \wedge^{\boldsymbol{A}} 1) \odot^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} 1)\big) \to^{\boldsymbol{A}} c \notin F$, by Lemma 7.54.2. Now, notice that $(a \wedge^{\boldsymbol{A}} 1) \odot^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} 1) \leq (a \wedge^{\boldsymbol{A}} 1) \odot^{\boldsymbol{A}} 1 = a \wedge^{\boldsymbol{A}} 1$ and similarly that $(a \wedge^{\boldsymbol{A}} 1) \odot^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} 1) \leq b \wedge^{\boldsymbol{A}} 1$, by Lemma 7.54.7. Therefore, $(a \wedge^{\boldsymbol{A}} 1) \odot^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} 1) \leq (a \wedge^{\boldsymbol{A}} 1) \wedge^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} 1) = (a \wedge^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} 1$. So, $\big((a \wedge^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} 1\big) \to^{\boldsymbol{A}} c \leq \big((a \wedge^{\boldsymbol{A}} 1) \odot^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} 1)\big) \to^{\boldsymbol{A}} c$, by suffixing. Since $F$ is upwards-closed, it follows that $\big((a \wedge^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} 1\big) \to^{\boldsymbol{A}} c \notin F$. Thus, $a \wedge^{\boldsymbol{A}} b \in \Psi(F)$. Finally we prove that $\Psi(F)$ is implicative. To this end, and given Lemma 7.55, it is enough to prove that $\Psi(F)$ is closed under $\odot^{\boldsymbol{A}}$. Let $a, b \in \Psi(F)$. Let $c \notin F$. Then, $(b \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} c \notin F$, because $b \in \Psi(F)$. Then, $(a \wedge^{\boldsymbol{A}} 1) \to \big((b \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} c\big) \notin F$, because $a \in \Psi(F)$. Hence, $\big((a \wedge^{\boldsymbol{A}} 1) \odot^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} 1)\big) \to^{\boldsymbol{A}} c \notin F$, by Lemma 7.54.2. Now, notice that $(a \wedge^{\boldsymbol{A}} 1) \odot^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} 1) \leq a \odot^{\boldsymbol{A}} b$ and $(a \wedge^{\boldsymbol{A}} 1) \odot^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} 1) \leq 1 \odot^{\boldsymbol{A}} 1 = 1$, by Lemma 7.54.7. So, $(a \wedge^{\boldsymbol{A}} 1) \odot^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} 1) \leq 1 \wedge^{\boldsymbol{A}} (a \odot^{\boldsymbol{A}} b)$. Therefore, $\big(1 \wedge^{\boldsymbol{A}} (a \odot^{\boldsymbol{A}} b)\big) \to^{\boldsymbol{A}} c \leq \big((a \wedge^{\boldsymbol{A}} 1) \odot^{\boldsymbol{A}} (b \wedge^{\boldsymbol{A}} 1)\big) \to^{\boldsymbol{A}} c$, by suffixing. Since $F$ is an up-set and $\big((a \wedge^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} 1\big) \to^{\boldsymbol{A}} c \notin F$, we have $\big(1 \wedge^{\boldsymbol{A}} (a \odot^{\boldsymbol{A}} b)\big) \to^{\boldsymbol{A}} c \notin F$. Thus, $a \odot^{\boldsymbol{A}} b \in \Psi(F)$. □

**Proposition 7.87.** *Let $\boldsymbol{A} \in \mathsf{CRL}$. For every $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}}\boldsymbol{A}$,*

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big).$$

PROOF. We claim that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is compatible with $\Psi(F)$. Let $\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ and $a \in \Psi(F)$. Then, $\langle (a \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} b, (b \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since

$$1 \leq b \to^{\boldsymbol{A}} b \leq (b \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} b,$$

by Lemma 7.56.3 and suffixing, respectively, and $F$ is upwards-closed, it follows that $(b \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} b \in F$. By compatibility, we have $(a \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} b \in F$. Since $a \in \Psi(F)$, it follows that $b \in F$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$.

Conversely, we claim that $\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$ is compatible with $F$. Let $\langle a,b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$ and let $a \in F$. Then, $\langle a \to^{\boldsymbol{A}} b, b \to^{\boldsymbol{A}} b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$. Since $1 \leq b \to^{\boldsymbol{A}} b$, and $1 \in \Psi(F)$, and $\Psi(F)$ is upwards-closed, it follows that $b \to^{\boldsymbol{A}} b \in \Psi(F)$. Then, by compatibility, $a \to^{\boldsymbol{A}} b \in \Psi(F)$. Now, $a \leq (a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} b$, by Lemma 7.54.3. Since $a \in F$ and $F$ is upwards-closed, it follows that $(a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} b \in F$. Also, since $(a \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} 1 \leq a \to^{\boldsymbol{A}} b$, it follows by suffixing that

$$(a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} b \leq ((a \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} b.$$

Therefore, $((a \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} 1) \to^{\boldsymbol{A}} b \in F$. Since, as we have seen, $a \to^{\boldsymbol{A}} b \in \Psi(F)$, it follows that $b \in F$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. ☐

**Corollary 7.88.** *Let $\boldsymbol{A} \in \mathsf{CRL}$. For every $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}} \boldsymbol{A}$,*

$$F^* = \Psi(F).$$

Proof. On the one hand, since $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$, we have $\Psi(F) \in [\![F]\!]^*$, and hence $F^* \subseteq \Psi(F)$. On the other hand, since $\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$, we have $F^* \in [\![\Psi(F)]\!]^*$, and hence $\Psi(F)^* \subseteq F^*$. But $\Psi(F) = \Psi(F)^*$, because we have seen that $\Psi(F)$ is an implicative lattice filter of $\boldsymbol{A}$ containing $\{1\}$, by Proposition 7.86, and the Leibniz $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$-filters of $\boldsymbol{A}$ are precisely these filters, by Theorem 7.78. ☐

We finish our study by showing that neither $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ nor $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ belong to any of the classes of the Frege hierarchy. This contrasts with the previous example, where the semilattice-based logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CIRL}}$ is of course fully selfextensional. The two proofs are very similar, but both are necessary, as selfextensionality is *not* preserved by extensions, and therefore we cannot use a contra-positive argument here.

**Proposition 7.89.** *The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ is not selfextensional.*

Proof. Suppose, towards an absurd, that $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ is selfextensional. Then, since $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ has a conjunction, it follows by Theorem 0.46 that $\mathcal{S}^{\leq}_{\mathsf{CRL}}$ is semilattice-based. Then, it is semilattice-based of $\mathsf{Alg}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}) = \mathsf{CRL}$, using Proposition 7.72. Consequently, $\mathcal{S}^{\tau}_{\mathsf{CRL}} = \mathcal{S}^{\leq}_{\mathsf{CRL}}$, and we reach an absurd (for instance, $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ has theorems, while $\mathcal{S}^{\leq}_{\mathsf{CRL}}$ has not). ☐

**Proposition 7.90.** *The logic $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ is not selfextensional.*

Proof. Suppose, towards an absurd, that $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ is selfextensional. Then, since $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ has a conjunction, it follows by Theorem 0.46 that $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ is semilattice-based. Then, it is semilattice-based of $\mathsf{Alg}(\mathcal{S}^{\tau}_{\mathsf{CRL}}) = \mathsf{CRL}$, using Theorem 7.71.3. Consequently, $\mathcal{S}^{\tau}_{\mathsf{CRL}} = \mathcal{S}^{\leq}_{\mathsf{CRL}}$, and we reach an absurd (for instance, $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ has theorems, while $\mathcal{S}^{\leq}_{\mathsf{CRL}}$ has not). ☐

As a final remark, we explain how to apply the results of the present section to Classical and Intuitionistic Linear Logic without exponentials, hereby denoted by $\mathcal{CLL}$ and $\mathcal{ILL}$, respectively. The underlying language for linear logic without exponentials is $\mathcal{L}' = \langle \wedge, \vee, \to, \odot, 1, 0, \top, \bot \rangle$, that is, the expansion of $\mathcal{L}$ by the constants $0, \top, \bot$. An *IL-algebra*[7] is an algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}, 1^{\boldsymbol{A}}, 0^{\boldsymbol{A}}, \top^{\boldsymbol{A}}, \bot^{\boldsymbol{A}} \rangle$, where

1. The reduct $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}, 1^{\boldsymbol{A}} \rangle$ belongs to $\mathsf{CRL}$;

[7] Following the terminology of [57, Definition 8.2, p. 71].

2. $\bot^{\boldsymbol{A}}$ is the bottom element of $A$, that is, for every $a \in A$, $\bot^{\boldsymbol{A}} \leq a$;
3. $\top^{\boldsymbol{A}}$ is the top element of $A$, that is, for every $a \in A$, $a \leq \top^{\boldsymbol{A}}$.

Let us denote by $\mathsf{IL}$ the class of all IL-algebras. Consider once again the equational transformer $\boldsymbol{\tau}(x) = \{x \wedge 1 \approx 1\}$. Define $\mathcal{S}_{\mathsf{IL}}^{\boldsymbol{\tau}}$ as the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \boldsymbol{\tau}\boldsymbol{A}\rangle : \boldsymbol{A} \in \mathsf{IL}\}$. Then, $\mathcal{ILL} = \mathcal{S}_{\mathsf{IL}}^{\boldsymbol{\tau}}$ and moreover $\mathcal{ILL}$ is BP-algebraizable witnessed by the set of congruence formulas $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y\}$ and the set of defining equations $\boldsymbol{\tau}(x)$ [**49**, Section 6.3]. Now, define the logic $\mathcal{S}_{\mathsf{IL}}^{\preccurlyeq}$ as the logic induced by the class of matrices $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{IL}, F \in \mathrm{Filt}\boldsymbol{A}, \boldsymbol{\tau}\boldsymbol{A} \subseteq F\}$. Under similar proofs to those undertaken in the present section, one shows that:

1. $\mathcal{S}_{\mathsf{IL}}^{\preccurlyeq}$ is not protoalgebraic, nor truth-equational, nor selfextensional.
2. $(\mathcal{S}_{\mathsf{IL}}^{\preccurlyeq})^{+} = \mathcal{ILL}$;
3. $\mathcal{ILL}$ is the inferential extension of $\mathcal{S}_{\mathsf{IL}}^{\preccurlyeq}$ by the rule *Modus Ponens*;
4. $\mathcal{F}i_{\mathcal{ILL}}\boldsymbol{A} = \mathcal{F}i^{*}_{\mathcal{S}_{\mathsf{IL}}^{\preccurlyeq}}\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{IL}$.

The classical case is carried out similarly, with the help of a non-primitive binary connective $\neg$, defined by $\neg\varphi := \varphi \to 0$. A *CL-algebra* is an algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}, 1^{\boldsymbol{A}}, 0^{\boldsymbol{A}}, \top^{\boldsymbol{A}}, \bot^{\boldsymbol{A}}\rangle$, where

1. $\boldsymbol{A} \in \mathsf{IL}$;
2. $\neg^{\boldsymbol{A}}\neg^{\boldsymbol{A}}a = a$, for every $a \in A$.

Let us denote by $\mathsf{CL}$ the class of all CL-algebras. Define $\mathcal{S}_{\mathsf{IL}}^{\boldsymbol{\tau}}$ as the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \boldsymbol{\tau}\boldsymbol{A}\rangle : \boldsymbol{A} \in \mathsf{CL}\}$. Then, $\mathcal{CLL} = \mathcal{S}_{\mathsf{CL}}^{\boldsymbol{\tau}}$ and moreover $\mathcal{CLL}$ is BP-algebraizable witnessed by the set of congruence formulas $\boldsymbol{\rho}(x, y)$ and the set of defining equations $\boldsymbol{\tau}(x)$, both given as above [**49**, Section 6.4]. Now, define the logic $\mathcal{S}_{\mathsf{CL}}^{\preccurlyeq}$ as the logic induced by the class of matrices $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{CL}, F \in \mathrm{Filt}\boldsymbol{A}, \boldsymbol{\tau}\boldsymbol{A} \subseteq F\}$. Under similar proofs to those undertaken in the present section, one shows that:

1. $\mathcal{S}_{\mathsf{CL}}^{\preccurlyeq}$ is not protoalgebraic, nor truth-equational, nor selfextensional.
2. $(\mathcal{S}_{\mathsf{CL}}^{\preccurlyeq})^{+} = \mathcal{CLL}$;
3. $\mathcal{CLL}$ is the inferential extension of $\mathcal{S}_{\mathsf{CL}}^{\preccurlyeq}$ by the rule *Modus Ponens*;
4. $\mathcal{F}i_{\mathcal{CLL}}\boldsymbol{A} = \mathcal{F}i^{*}_{\mathcal{S}_{\mathsf{CL}}^{\preccurlyeq}}\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{CL}$.

## 7.6. An intermediate logic between the semilattice-based logic of $\mathsf{CRLr}$ and the $\{x \wedge (x \to x) \approx x \to x\}$-assertional logic of $\mathsf{CRLr}$

We finish our examples of non-protoalgebraic logics with yet another generalization of the previous section, namely that obtained by dropping the existence of the multiplicative constant on Definition 7.53.

Throughout the present section, we shall be working within the language $\mathcal{L} = \langle \wedge, \vee, \to, \odot\rangle$. Let us start by introducing the class of residuated lattices without multiplicative constant which we shall be interested in.

**Definition 7.91.** An algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}\rangle$ is a *commutative residuated lattice without multiplicative constant*, if:

1. $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}\rangle$ is a lattice;
2. $\langle A, \odot^{\boldsymbol{A}}\rangle$ is a commutative semigroup[8];
3. $\to^{\boldsymbol{A}}$ is the residuum of $\odot^{\boldsymbol{A}}$, that is, for every $a, b \in A$, $a \odot^{\boldsymbol{A}} c \leq b$ iff $c \leq a \to^{\boldsymbol{A}} b$.

[8]A *semigroup* is an algebra $\langle A, \circ\rangle$, where $\circ$ is a binary operation on $A$ which is associative.

A commutative residuated lattice without multiplicative constant is *relevant*, if:

4. for every $a, b, c \in A$, $\big((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} b)\big) \to^{\boldsymbol{A}} c \leq c$[9].

The class of all relevant commutative residuated lattices without multiplicative constant will be denoted by $\mathsf{CRLr}$.

So, when compared with Definition 7.53, we are here relaxing condition 2, demanding only the presence of a commutative semigroup, rather than a commutative monoid. In other words, we drop the existence of the multiplicative constant 1, not only from the language but by allowing the semigroup not to have a unit.

Notice that the reducts of the algebras in $\mathsf{CRL}$ to the language $\mathcal{L}$ are algebras in $\mathsf{CRLr}$. Indeed, let $\boldsymbol{A} \in \mathsf{CRL}$ and consider its $\mathcal{L}$-reduct, say $\boldsymbol{A}{\restriction_{\mathcal{L}}}$. It is clear that $\boldsymbol{A}{\restriction_{\mathcal{L}}}$ satisfies conditions 1–3 of Definition 7.91. As to condition 4, since by Lemma 7.56.3, $1 \leq d \to^{\boldsymbol{A}} d$, for every $d \in A$, it follows by that

$$1 = 1 \wedge^{\boldsymbol{A}} 1 \leq (a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} b),$$

for every $a, b \in A$, and therefore by suffixing

$$\big((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (b \to^{\boldsymbol{A}} b)\big) \to^{\boldsymbol{A}} c \leq 1 \to^{\boldsymbol{A}} c = c,$$

for every $a, b, c \in A$. So, $\boldsymbol{A}{\restriction_{\mathcal{L}}}$ is indeed a relevant commutative residuated lattice without multiplicative constant. Thus, $\mathsf{CRL}{\restriction_{\mathcal{L}}} = \{\boldsymbol{A}{\restriction_{\mathcal{L}}} \colon \boldsymbol{A} \in \mathsf{CRL}\} \subseteq \mathsf{CRLr}$. This fact motivates the notation chosen for $\mathsf{CRLr}$, despite the fact that, in rigor, these algebras are *not* commutative residuated lattices according to Definition 7.53.

It is important to realize that all the conditions stated in Lemma 7.54 still hold in $\mathsf{CRLr}$, as none of them relies on the multiplicative constant of the underlying residuated lattice. We shall make use of these properties throughout the present section, although they are formally stated for algebras in $\mathsf{CRL}$. We next state some more useful inequalities which hold in all relevant commutative residuated lattices without multiplicative constant.

**Lemma 7.92.** *Let $\boldsymbol{A} \in \mathsf{CRLr}$. For every $a, b, c \in A$,*

1. $(a \to^{\boldsymbol{A}} a) \to^{\boldsymbol{A}} b \leq b$;
2. $(a \to^{\boldsymbol{A}} a) \to^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} a) \leq (a \to^{\boldsymbol{A}} a)$;
3. $(a \to^{\boldsymbol{A}} a) \odot^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} a) \leq (a \to^{\boldsymbol{A}} a)$;
4. *if for every $i = 1, \dots, n$, with $n \in \mathbb{N}$, $a_i \to^{\boldsymbol{A}} a_i \leq a_i$, then* $\bigwedge^{\boldsymbol{A}}_{i=1,\dots,n} a_i \to^{\boldsymbol{A}} \bigwedge^{\boldsymbol{A}}_{i=1,\dots,n} a_i \leq \bigwedge^{\boldsymbol{A}}_{i=1,\dots,n} a_i$.

Proof. *1.* Take $b = a$ in the relevance condition of Definition 7.91. *2.* Take $b = a \to^{\boldsymbol{A}} a$ in 1. *3.* It holds $(a \to^{\boldsymbol{A}} a) \leq (a \to^{\boldsymbol{A}} a) \to^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} a)$, by Lemma 7.54.10, taking $a = b = c$. Hence, it follows by residuation that $(a \to^{\boldsymbol{A}} a) \odot^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} a) \leq (a \to^{\boldsymbol{A}} a)$. *4.* The proof goes by induction on $n \in \mathbb{N}$. The basis case follows immediately from the assumption $a_1 \to^{\boldsymbol{A}} a_1 \leq a_1$. Now, assume that the stated property holds for $n > 1$. Let $a_1, \dots, a_{n+1} \in A$ such that $a_i \to^{\boldsymbol{A}} a_i \leq a_i$, for every $i = 1, \dots, n+1$. It follows by the inductive hypothesis that

$$\bigwedge_{i=1,\dots,n}^{\boldsymbol{A}} a_i \to^{\boldsymbol{A}} \bigwedge_{i=1,\dots,n}^{\boldsymbol{A}} a_i \leq \bigwedge_{i=1,\dots,n}^{\boldsymbol{A}} a_i.$$

[9]This condition appears in [5, pp. 275,321], but it is its algebraic treatment in [41] for the Relevance Logic $\mathcal{R}$ that lays the groundwork for the results of this section. It also appears in [42, p. 373], where in fact a first study of $\mathcal{S}^{\tau}_{\mathsf{CRL}}$ seen as a strong version is carried out, and the semilattice-based logic $\mathcal{S}^{\leq}_{\mathsf{CRL}}$ is proposed as "weak version" of $\mathcal{S}^{\tau}_{\mathsf{CRL}}$. For the sake of completeness, the condition also appears explicitly in [1, p. 22].

Hence,

$$\Big(\bigwedge_{i=1,\dots,n}^{\boldsymbol{A}} a_i \to^{\boldsymbol{A}} \bigwedge_{i=1,\dots,n}^{\boldsymbol{A}} a_i\Big) \wedge^{\boldsymbol{A}} a_{n+1} \leq \bigwedge_{i=1,\dots,n}^{\boldsymbol{A}} a_i \wedge^{\boldsymbol{A}} a_{n+1} = \bigwedge_{i=1,\dots,n+1}^{\boldsymbol{A}} a_i.$$

Now, since $a_{n+1} \to^{\boldsymbol{A}} a_{n+1} \leq a_{n+1}$, it follows that

$$\Big(\bigwedge_{i=1,\dots,n}^{\boldsymbol{A}} a_i \to^{\boldsymbol{A}} \bigwedge_{i=1,\dots,n}^{\boldsymbol{A}} a_i\Big) \wedge^{\boldsymbol{A}} (a_{n+1} \to^{\boldsymbol{A}} a_{n+1}) \leq \bigwedge_{i=1,\dots,n+1}^{\boldsymbol{A}} a_i.$$

Finally, by suffixing and using the relevance condition, we have

$$\begin{aligned} &\bigwedge_{i=1,\dots,n+1}^{\boldsymbol{A}} a_i \to^{\boldsymbol{A}} \bigwedge_{i=1,\dots,n+1}^{\boldsymbol{A}} a_i \\ &\leq \Big[\Big(\bigwedge_{i=1,\dots,n}^{\boldsymbol{A}} a_i \to^{\boldsymbol{A}} \bigwedge_{i=1,\dots,n}^{\boldsymbol{A}} a_i\Big) \wedge^{\boldsymbol{A}} (a_{n+1} \to^{\boldsymbol{A}} a_{n+1})\Big] \to^{\boldsymbol{A}} \bigwedge_{i=1,\dots,n+1}^{\boldsymbol{A}} a_i \\ &\leq \bigwedge_{i=1,\dots,n+1}^{\boldsymbol{A}} a_i. \end{aligned}$$

□

Once again, the semilattice-based logic of $\mathsf{CRLr}$, $\mathcal{S}^{\leq}_{\mathsf{CRLr}}$, does not have theorems. Consequently, its strong version is the almost inconsistent logic. Recall that $\mathcal{S}^{\leq}_{\mathsf{CRLr}}$ is induced by the class of matrices $\{\langle \boldsymbol{A}, [a)\rangle : \boldsymbol{A} \in \mathsf{CRLr}, a \in A\}$ and therefore also by the class of matrices $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{CRLr}, F \in \mathrm{Filt}\boldsymbol{A}\}$.

The fragment of intuitionistic linear logic associated with the class $\mathsf{CRLr}$ is the $\boldsymbol{\tau}$-assertional logic of $\mathsf{CRLr}$, with $\boldsymbol{\tau}(x) \coloneqq \big\{x \wedge (x \to x) \approx x \to x\big\}$. Following the notation introduced on page 17, such logic is denoted by $\mathcal{S}\big(\mathsf{CRLr}, \big\{x \wedge (x \to x) \approx x \to x\big\}\big)$; once again for ease of notation, we shall denote it by $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$. By definition, $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ is the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \boldsymbol{\tau}\boldsymbol{A}\rangle : \boldsymbol{A} \in \mathsf{CRLr}\}$, where (recall the notation introduced on page 16)

$$\boldsymbol{\tau}\boldsymbol{A} \coloneqq \{a \in A : \boldsymbol{A} \vDash \boldsymbol{\tau}(x)[\![a]\!]\} = \{a \in A : a \to^{\boldsymbol{A}} a \leq a\}.$$

In other words, its consequence relation $\vdash_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}}$ is defined by

$$\begin{aligned} \Gamma \vdash_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} \varphi \quad \text{iff} \quad & \forall \boldsymbol{A} \in \mathsf{CRLr}\ \forall h \in \mathrm{Hom}(\mathrm{Fm}_{\mathcal{L}}, \boldsymbol{A}) \\ & \text{if } \forall \gamma \in \Gamma\ h(\gamma) \in \boldsymbol{\tau}\boldsymbol{A}, \text{ then } h(\varphi) \in \boldsymbol{\tau}\boldsymbol{A}, \end{aligned}$$

or equivalently,

$$\begin{aligned} \Gamma \vdash_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} \varphi \quad \text{iff} \quad & \forall \boldsymbol{A} \in \mathsf{CRLr}\ \forall h \in \mathrm{Hom}(\mathrm{Fm}_{\mathcal{L}}, \boldsymbol{A}) \\ & \text{if } \forall \gamma \in \Gamma\ h(\gamma) \to^{\boldsymbol{A}} h(\gamma) \leq h(\gamma), \text{ then } h(\varphi) \to^{\boldsymbol{A}} h(\varphi) \leq h(\varphi), \end{aligned}$$

for every $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ and every $\varphi \in \mathrm{Fm}_{\mathcal{L}}$. It is clear that $\boldsymbol{\tau}\boldsymbol{A}$ is now playing the rôle of the up-set $[1)$ in the non-integral case, or if one prefers, the rôle of the singleton $\{1\}$ in the integral case. We next collect some (not-so-known) properties about the logic $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ in Theorem 7.98. These results can be found in [**49**, Section 6.5], but given the fact that this is an unpublished reference, and therefore may be of difficult access to the reader, we exhibit the proofs. To this end, we first prove some auxiliary lemmas regarding the distinguished set $\boldsymbol{\tau}\boldsymbol{A}$, with $\boldsymbol{A} \in \mathsf{CRLr}$, which in any case are very insightful.

**Lemma 7.93.** *For every $\boldsymbol{A} \in \mathsf{CRLr}$, $\boldsymbol{\tau}\boldsymbol{A}$ is an implicative lattice filter of $\boldsymbol{A}$.*

Proof. First of all, $\boldsymbol{\tau}\boldsymbol{A}$ is non-empty, because for every $c \in A$, $c \to^{\boldsymbol{A}} c \in \boldsymbol{\tau}\boldsymbol{A}$ by Lemma 7.92.2. To see that $\boldsymbol{\tau}\boldsymbol{A}$ is an up-set, let $a \in \boldsymbol{\tau}\boldsymbol{A}$ and $b \in A$ such that $a \leq b$. So, $a \to^{\boldsymbol{A}} a \leq a \leq b$. It follows by suffixing and Lemma 7.92.1 that $b \to^{\boldsymbol{A}} b \leq (a \to^{\boldsymbol{A}} a) \to^{\boldsymbol{A}} b \leq b$. Thus, $b \in \boldsymbol{\tau}\boldsymbol{A}$. Next, we prove that $\boldsymbol{\tau}\boldsymbol{A}$ is closed under meets. Let $a, b \in \boldsymbol{\tau}\boldsymbol{A}$. So, $a \to^{\boldsymbol{A}} a \leq a$ and $b \to^{\boldsymbol{A}} b \leq b$. It follows by Proposition 7.92.4 that $(a \wedge^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} (a \wedge^{\boldsymbol{A}} b) \leq a \wedge^{\boldsymbol{A}} b$. So, $a \wedge^{\boldsymbol{A}} b \in \boldsymbol{\tau}\boldsymbol{A}$. We are left to see that $\boldsymbol{\tau}\boldsymbol{A}$ is implicative. Let $a, b$ such that $a \to^{\boldsymbol{A}} a \leq a$ and $(a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} b) \leq a \to^{\boldsymbol{A}} b$. It holds,

$$(a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} b) = \Big(\big(a \to^{\boldsymbol{A}} b\big) \odot^{\boldsymbol{A}} a\Big) \to^{\boldsymbol{A}} b$$

by Lemma 7.54.2. But,

$$\big(a \to^{\boldsymbol{A}} b\big) \odot^{\boldsymbol{A}} a = a \odot^{\boldsymbol{A}} \big(a \to^{\boldsymbol{A}} b\big) \leq b,$$

by Lemma 7.54.1. So,

$$b \to^{\boldsymbol{A}} b \leq \Big(\big(a \to^{\boldsymbol{A}} b\big) \odot^{\boldsymbol{A}} a\Big) \to^{\boldsymbol{A}} b,$$

by suffixing. So, using the second assumption,

$$b \to^{\boldsymbol{A}} b \leq (a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} b) \leq a \to^{\boldsymbol{A}} b.$$

But, since $a \to^{\boldsymbol{A}} a \leq a$ by the first assumption,

$$a \to^{\boldsymbol{A}} b \leq \big(a \to^{\boldsymbol{A}} a\big) \to^{\boldsymbol{A}} b \leq b,$$

by suffixing and Lemma 7.92.1, respectively. Altogether,

$$b \to^{\boldsymbol{A}} b \leq a \to^{\boldsymbol{A}} b \leq b.$$

Hence, $\boldsymbol{\tau}\boldsymbol{A}$ is implicative. □

**Lemma 7.94.** *For every $\boldsymbol{A} \in \mathsf{CRLr}$,*

$$\boldsymbol{\tau}\boldsymbol{A} = \mathrm{Filt}^{\boldsymbol{A}}\big(\{a \to^{\boldsymbol{A}} a : a \in A\}\big).$$

Proof. On the one hand, for every $a \in A$, $a \to^{\boldsymbol{A}} a \in \boldsymbol{\tau}\boldsymbol{A}$ by Lemma 7.92.2. Since moreover $\boldsymbol{\tau}\boldsymbol{A}$ is a lattice filter of $\boldsymbol{A}$ by Lemma 7.93, the inclusion $\mathrm{Filt}^{\boldsymbol{A}}\big(\{a \to^{\boldsymbol{A}} a : a \in A\}\big) \subseteq \boldsymbol{\tau}\boldsymbol{A}$ follows. On the other hand, let $b \in \mathrm{Filt}^{\boldsymbol{A}}\big(\{a \to^{\boldsymbol{A}} a : a \in A\}\big)$. So, there exist $a_1, \ldots, a_n \in A$ such that $(a_1 \to^{\boldsymbol{A}} a_1) \wedge^{\boldsymbol{A}} \ldots \wedge^{\boldsymbol{A}} (a_n \to^{\boldsymbol{A}} a_n) \leq b$. Now, since $a_i \to^{\boldsymbol{A}} a_i \in \boldsymbol{\tau}\boldsymbol{A}$, for every $i = 1, \ldots, n$, and moreover $\boldsymbol{\tau}\boldsymbol{A}$ is both closed under meets and an up-set by Lemma 7.93, it follows that $b \in \boldsymbol{\tau}\boldsymbol{A}$. □

**Lemma 7.95.** *For every $\boldsymbol{A} \in \mathsf{CRLr}$ and every $a, b \in A$,*

$$a \leq b \quad \Leftrightarrow \quad a \to^{\boldsymbol{A}} b \in \boldsymbol{\tau}\boldsymbol{A}.$$

Proof. Suppose $a \leq b$. Then, $a \to^{\boldsymbol{A}} a \leq a \to^{\boldsymbol{A}} b$ by suffixing. Since $a \to^{\boldsymbol{A}} a \in \boldsymbol{\tau}\boldsymbol{A}$ and $\boldsymbol{\tau}\boldsymbol{A}$ is upwards-closed, by Lemma 7.94, it follows that $a \to^{\boldsymbol{A}} b \in \boldsymbol{\tau}\boldsymbol{A}$. Conversely, suppose $a \to^{\boldsymbol{A}} b \in \boldsymbol{\tau}\boldsymbol{A}$. Having Lemma 7.94 in mind, there exist $a_1, \ldots, a_n \in A$ such that $\bigwedge^{\boldsymbol{A}}_{i=1,\ldots,n}(a_i \to^{\boldsymbol{A}} a_i) \leq a \to^{\boldsymbol{A}} b$. Since for every $i = 1, \ldots, n$, it holds $(a_i \to^{\boldsymbol{A}} a_i) \to^{\boldsymbol{A}} (a_i \to^{\boldsymbol{A}} a_i) \leq a_i \to^{\boldsymbol{A}} a_i$, by Lemma 7.92.2, it follows by Lemma 7.92.4 that $\bigwedge^{\boldsymbol{A}}_{i=1,\ldots,n}(a_i \to^{\boldsymbol{A}} a_i) \to^{\boldsymbol{A}} \bigwedge^{\boldsymbol{A}}_{i=1,\ldots,n}(a_i \to^{\boldsymbol{A}} a_i) \leq \bigwedge^{\boldsymbol{A}}_{i=1,\ldots,n}(a_i \to^{\boldsymbol{A}} a_i)$. So, $\bigwedge^{\boldsymbol{A}}_{i=1,\ldots,n}(a_i \to^{\boldsymbol{A}} a_i) \to^{\boldsymbol{A}} \bigwedge^{\boldsymbol{A}}_{i=1,\ldots,n}(a_i \to^{\boldsymbol{A}} a_i) \leq a \to^{\boldsymbol{A}} b$. It follows by suffixing, Lemma 7.54.3 and Lemma 7.92.1 that

$$a \leq (a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} b \leq \Big(\bigwedge_{i=1,\ldots,n}^{\boldsymbol{A}}(a_i \to^{\boldsymbol{A}} a_i) \to^{\boldsymbol{A}} \bigwedge_{i=1,\ldots,n}^{\boldsymbol{A}}(a_i \to^{\boldsymbol{A}} a_i)\Big) \to^{\boldsymbol{A}} b \leq b.$$

□

Also, notice that the proof of Lemma 7.55 is done exclusively using properties of Lemma 7.54. Hence, it still holds for algebras in CRLr.

**Lemma 7.96.** *Let $\boldsymbol{A} \in \mathsf{CRLr}$. A lattice filter $F \in \mathrm{Filt}\boldsymbol{A}$ is implicative if and only if it is closed under the operation $\odot^{\boldsymbol{A}}$.*

**Lemma 7.97.** *Let $\boldsymbol{A} \in \mathsf{CRLr}$. For every $F \in \mathrm{Filt}_{\rightarrow}\boldsymbol{A}$ such that $\boldsymbol{\tau A} \subseteq F$,*

$$\forall a, b \in A \quad \langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \quad \textit{iff} \quad a \leftrightarrow^{\boldsymbol{A}} b \in F.$$

Proof. Define $R \subseteq A \times A$ by

$$\forall a, b \in A \quad \langle a, b\rangle \in R \quad \text{iff} \quad a \leftrightarrow^{\boldsymbol{A}} b \in F.$$

We prove that $R$ is the largest congruence on $\boldsymbol{A}$ compatible with $F$.

- *Reflexive:* Since $a \leftrightarrow^{\boldsymbol{A}} a = (a \rightarrow^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} a) = a \rightarrow^{\boldsymbol{A}} a \in \boldsymbol{\tau A} \subseteq F$.
- *Symmetric:* This should be clear, given the definition of $\leftrightarrow$.
- *Transitive:* Let $a, b, c \in A$ such that $a \leftrightarrow^{\boldsymbol{A}} b \in F$ and $b \leftrightarrow^{\boldsymbol{A}} c \in F$. Since $F$ is upwards-closed, it follows that $a \rightarrow^{\boldsymbol{A}} b \in F$ and $b \rightarrow^{\boldsymbol{A}} c \in F$. Since

$$a \rightarrow^{\boldsymbol{A}} b \leq (b \rightarrow^{\boldsymbol{A}} c) \rightarrow^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} c),$$

by Lemma 7.54.10, and moreover $F$ is upwards-closed and implicative, it follows that $a \rightarrow^{\boldsymbol{A}} c \in F$. Similarly, $c \rightarrow^{\boldsymbol{A}} a \in F$. So, $a \leftrightarrow^{\boldsymbol{A}} c \in F$.

- *Compatible with $\wedge$:* Let $a, b \in A$ such that $a_1 \leftrightarrow^{\boldsymbol{A}} b_1 \in F$ and $a_2 \leftrightarrow^{\boldsymbol{A}} b_2 \in F$. Since $F$ is upwards-closed, it follows that $a_1 \rightarrow^{\boldsymbol{A}} b_1 \in F$ and $a_2 \rightarrow^{\boldsymbol{A}} b_2 \in F$. Now, by suffixing,

$$a_1 \rightarrow^{\boldsymbol{A}} b_1 \leq (a_1 \wedge^{\boldsymbol{A}} a_2) \rightarrow^{\boldsymbol{A}} b_1,$$

and

$$a_2 \rightarrow^{\boldsymbol{A}} b_2 \leq (a_1 \wedge^{\boldsymbol{A}} a_2) \rightarrow^{\boldsymbol{A}} b_2.$$

Moreover,

$$\big((a_1 \wedge^{\boldsymbol{A}} a_2) \rightarrow^{\boldsymbol{A}} b_1\big) \wedge^{\boldsymbol{A}} \big((a_1 \wedge^{\boldsymbol{A}} a_2) \rightarrow^{\boldsymbol{A}} b_2\big) \leq (a_1 \wedge^{\boldsymbol{A}} a_2) \rightarrow^{\boldsymbol{A}} (b_1 \wedge^{\boldsymbol{A}} b_2),$$

by Lemma 7.54.8. Since $F$ is closed under meets and upwards-closed, it follows that $(a_1 \wedge^{\boldsymbol{A}} a_2) \rightarrow^{\boldsymbol{A}} (b_1 \wedge^{\boldsymbol{A}} b_2) \in F$. Similarly, $(b_1 \wedge^{\boldsymbol{A}} b_2) \rightarrow^{\boldsymbol{A}} (a_1 \wedge^{\boldsymbol{A}} a_2) \in F$. So, $(a_1 \wedge^{\boldsymbol{A}} a_2) \leftrightarrow^{\boldsymbol{A}} (b_1 \wedge^{\boldsymbol{A}} b_2) \in F$.

- *Compatible with $\vee$:* Let $a, b \in A$ such that $a_1 \leftrightarrow^{\boldsymbol{A}} b_1 \in F$ and $a_2 \leftrightarrow^{\boldsymbol{A}} b_2 \in F$. Since $F$ is upwards-closed, it follows that $a_1 \rightarrow^{\boldsymbol{A}} b_1 \in F$ and $a_2 \rightarrow^{\boldsymbol{A}} b_2 \in F$. Now,

$$a_1 \rightarrow^{\boldsymbol{A}} b_1 \leq a_1 \rightarrow^{\boldsymbol{A}} (b_1 \vee^{\boldsymbol{A}} b_2),$$

and

$$a_2 \rightarrow^{\boldsymbol{A}} b_2 \leq a_2 \rightarrow^{\boldsymbol{A}} (b_1 \vee^{\boldsymbol{A}} b_2),$$

by Lemma 7.54.6. Moreover,

$$\big(a_1 \rightarrow^{\boldsymbol{A}} (b_1 \vee^{\boldsymbol{A}} b_2)\big) \wedge^{\boldsymbol{A}} \big(a_2 \rightarrow^{\boldsymbol{A}} (b_1 \vee^{\boldsymbol{A}} b_2)\big) \leq (a_1 \vee^{\boldsymbol{A}} a_2) \rightarrow^{\boldsymbol{A}} (b_1 \vee^{\boldsymbol{A}} b_2),$$

by Lemma 7.54.9. Since $F$ is closed under meets and upwards-closed, it follows that $(a_1 \vee^{\boldsymbol{A}} a_2) \rightarrow^{\boldsymbol{A}} (b_1 \vee^{\boldsymbol{A}} b_2) \in F$. Similarly, $(b_1 \vee^{\boldsymbol{A}} b_2) \rightarrow^{\boldsymbol{A}} (a_1 \vee^{\boldsymbol{A}} a_2) \in F$. So, $(a_1 \wedge^{\boldsymbol{A}} a_2) \leftrightarrow^{\boldsymbol{A}} (b_1 \wedge^{\boldsymbol{A}} b_2) \in F$.

- *Compatible with $\rightarrow$:* Let $a, b \in A$ such that $a_1 \leftrightarrow^{\boldsymbol{A}} b_1 \in F$ and $a_2 \leftrightarrow^{\boldsymbol{A}} b_2 \in F$.

Since $F$ is upwards-closed, it follows that $a_1 \to^{\boldsymbol{A}} b_1 \in F$ and $a_2 \to^{\boldsymbol{A}} b_2 \in F$. On the one hand,

$$b_1 \to^{\boldsymbol{A}} a_1 \leq (a_1 \to^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \to^{\boldsymbol{A}} a_2),$$

by Lemma 7.54.10. On the other hand,

$$a_2 \to^{\boldsymbol{A}} b_2 \leq (b_1 \to^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \to^{\boldsymbol{A}} b_2),$$

this time by Lemma 7.54.11. Since $F$ is upwards-closed, both $(a_1 \to^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \to^{\boldsymbol{A}} a_2) \in F$ and $(b_1 \to^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \to^{\boldsymbol{A}} b_2) \in F$. Moreover,

$$\begin{aligned} &\big[(a_1 \to^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \to^{\boldsymbol{A}} a_2)\big] \odot^{\boldsymbol{A}} \big[(b_1 \to^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \to^{\boldsymbol{A}} b_2)\big] \\ \leq\ & (a_1 \to^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \to^{\boldsymbol{A}} b_2), \end{aligned} \tag{22}$$

by Lemma 7.54.12. Now, since $F$ is closed under $\odot^{\boldsymbol{A}}$ (by Lemma 7.96, because $F$ is implicative), it follows that

$$\big[(a_1 \to^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \to^{\boldsymbol{A}} a_2)\big] \odot^{\boldsymbol{A}} \big[(b_1 \to^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \to^{\boldsymbol{A}} b_2)\big] \in F.$$

Since $F$ is upwards-closed, it follows by (22) that $(a_1 \to^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \to^{\boldsymbol{A}} b_2) \in F$. Similarly, $(b_1 \to^{\boldsymbol{A}} b_2) \to^{\boldsymbol{A}} (a_1 \to^{\boldsymbol{A}} a_2) \in F$. So, $(a_1 \to^{\boldsymbol{A}} a_2) \leftrightarrow^{\boldsymbol{A}} (b_1 \to^{\boldsymbol{A}} b_2) \in F$.

*compatible with* $\odot$*:* Let $a, b \in A$ such that $a_1 \leftrightarrow^{\boldsymbol{A}} b_1 \in F$ and $a_2 \leftrightarrow^{\boldsymbol{A}} b_2 \in F$. Since $F$ is upwards-closed, it follows that $a_1 \to^{\boldsymbol{A}} b_1 \in F$ and $a_2 \to^{\boldsymbol{A}} b_2 \in F$. We claim that

$$(a_1 \to^{\boldsymbol{A}} b_1) \odot^{\boldsymbol{A}} (a_2 \to^{\boldsymbol{A}} b_2) \leq (a_1 \odot^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \odot^{\boldsymbol{A}} b_2). \tag{23}$$

By residuation (right-to-left), this amounts to

$$a_1 \odot^{\boldsymbol{A}} a_2 \odot^{\boldsymbol{A}} \big((a_1 \to^{\boldsymbol{A}} b_1) \odot^{\boldsymbol{A}} (a_2 \to^{\boldsymbol{A}} b_2)\big) \leq b_1 \odot^{\boldsymbol{A}} b_2.$$

But indeed,

$$\begin{aligned} & a_1 \odot^{\boldsymbol{A}} a_2 \odot^{\boldsymbol{A}} \big((a_1 \to^{\boldsymbol{A}} b_1) \odot^{\boldsymbol{A}} (a_2 \to^{\boldsymbol{A}} b_2)\big) \\ = \ & \big(a_1 \odot^{\boldsymbol{A}} (a_1 \to^{\boldsymbol{A}} b_1)\big) \odot^{\boldsymbol{A}} \big(a_2 \odot^{\boldsymbol{A}} (a_2 \to^{\boldsymbol{A}} b_2)\big) \\ \leq\ & b_1 \odot^{\boldsymbol{A}} b_2. \end{aligned}$$

Since $F$ is closed under $\odot^{\boldsymbol{A}}$ (by Lemma 7.96, because $F$ is implicative), it follows that $(a_1 \to^{\boldsymbol{A}} b_1) \odot^{\boldsymbol{A}} (a_2 \to^{\boldsymbol{A}} b_2) \in F$. Since $F$ is upwards-closed, it follows by (23) that $(a_1 \odot^{\boldsymbol{A}} a_2) \to^{\boldsymbol{A}} (b_1 \odot^{\boldsymbol{A}} b_2) \in F$. Similarly, $(b_1 \odot^{\boldsymbol{A}} b_2) \to^{\boldsymbol{A}} (a_1 \odot^{\boldsymbol{A}} a_2) \in F$. So, $(a_1 \odot^{\boldsymbol{A}} a_2) \leftrightarrow^{\boldsymbol{A}} (b_1 \odot^{\boldsymbol{A}} b_2) \in F$.

- *Compatible with* $F$*:* Let $a, b \in A$ such that $a \leftrightarrow^{\boldsymbol{A}} b \in F$ and $a \in F$. Since $F$ is upwards-closed, it follows that $a \to^{\boldsymbol{A}} b \in F$. Since $F$ is implicative, it follows that $b \in F$.
- *Largest compatible with* $F$*:* Let $\theta \in \operatorname{Con}\boldsymbol{A}$ compatible with $F$. Let $\langle a, b\rangle \in \theta$. Since $\theta \in \operatorname{Con}\boldsymbol{A}$, it follows that $\langle a \leftrightarrow^{\boldsymbol{A}} b, b \leftrightarrow^{\boldsymbol{A}} b\rangle \in \theta$. But, $b \leftrightarrow^{\boldsymbol{A}} b = b \to^{\boldsymbol{A}} b \in \boldsymbol{\tau}\boldsymbol{A} \subseteq F$. Since $\theta$ is compatible with $F$, it follows that $a \leftrightarrow^{\boldsymbol{A}} b \in F$. □

**Theorem 7.98.**

1. $\mathsf{Alg}^*(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}) = \mathsf{Alg}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}) = \mathsf{CRLr}$.
2. *For every* $\boldsymbol{A} \in \mathsf{CRLr}$*,* $\mathcal{F}i_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}}\boldsymbol{A} = \{F \in \operatorname{Filt}_{\to}\boldsymbol{A} : \boldsymbol{\tau}\boldsymbol{A} \subseteq F\}$.
3. $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ *is BP-algebraizable, witnessed by the set of congruence formulas* $\boldsymbol{\rho}(x, y) = \{x \leftrightarrow y\}$ *and the set of defining equations* $\boldsymbol{\tau}(x) = \big\{x \wedge (x \to x) \approx x \to x\big\}$*; its equivalent algebraic semantics is* $\mathsf{CRLr}$.

PROOF. *1.* Since for every $\boldsymbol{A} \in \mathsf{CRLr}$, $\boldsymbol{\tau A} \in \mathrm{Filt}\boldsymbol{A}$, by Lemma 7.93, and the logic $\mathcal{S}^{\leq}_{\mathsf{CRLr}}$ is induced by the class of matrices $\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{CRLr}, F \in \mathrm{Filt}\boldsymbol{A}\}$, it is clear by definition of $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ that $\mathcal{S}^{\leq}_{\mathsf{CRLr}} \leq \mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$, and therefore $\mathsf{Alg}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}) \subseteq \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{CRLr}}) = \mathsf{CRLr}$ (recall that, given a semilattice logic $\mathcal{S}^{\leq}_{\mathsf{K}}$, it holds $\mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}}) = \mathbb{V}(\mathsf{K})$). Conversely, let $\boldsymbol{A} \in \mathsf{CRLr}$. Notice that the matrix $\langle \boldsymbol{A}, \boldsymbol{\tau A}\rangle$ is reduced. Indeed, given Lemmas 7.93, 7.95, and 7.97, we have

$$\langle a, b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(\boldsymbol{\tau A}) \quad \text{iff} \quad a \leftrightarrow^{\boldsymbol{A}} b \in \boldsymbol{\tau A} \quad \text{iff} \quad a = b,$$

for every $a, b \in A$. Now, clearly by definition of $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$, $\boldsymbol{\tau A} \in \mathcal{F}i_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}}\boldsymbol{A}$. So, $\mathsf{CRLr} \subseteq \mathsf{Alg}^*(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}) \subseteq \mathsf{Alg}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}})$.

*2.* Since for every $\boldsymbol{A} \in \mathsf{CRLr}$, $\boldsymbol{\tau A} \in \mathrm{Filt}_{\rightarrow}\boldsymbol{A}$, by Lemma 7.93, and moreover $\{a \rightarrow^{\boldsymbol{A}} a : a \in A\} \subseteq \boldsymbol{\tau A}$, by Lemma 7.94, it is clear by definition of $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ that

$$\varnothing \vdash_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} x \rightarrow x, \quad x, y \vdash_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} x \wedge y, \quad x \wedge y \vdash_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} x, y, \quad x, x \rightarrow y \vdash_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} y.$$

This implies that every $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$-filter of $\boldsymbol{A} \in \mathsf{CRLr}$ is non-empty, closed under meets, upwards-closed and implicative. Hence, $\mathcal{F}i_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}}\boldsymbol{A} \subseteq \{F \in \mathrm{Filt}_{\rightarrow}\boldsymbol{A} : \boldsymbol{\tau A} \subseteq F\}$. Conversely, let $\boldsymbol{A} \in \mathsf{CRLr}$ and $F \in \mathrm{Filt}_{\rightarrow}\boldsymbol{A}$ such that $\boldsymbol{\tau A} \subseteq F$. Consider the algebra $\boldsymbol{B} = \boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Let $\pi : \boldsymbol{A} \rightarrow \boldsymbol{B}$ be the canonical map.

*Claim.* $\boldsymbol{\tau B} = \pi F$*:* Let $b \in \pi F$. So, there exists $a \in F$ such that $b = \pi(a)$. Let us see that

$$\big(a \wedge^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} a)\big) \leftrightarrow^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} a) \in F.$$

On the one hand, since $(a \rightarrow^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} a \leq a \rightarrow^{\boldsymbol{A}} a$, it follows by Lemma 7.95 that $\big((a \rightarrow^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} a\big) \rightarrow^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} a) \in \boldsymbol{\tau A} \subseteq F$. On the other hand, since $a \odot^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} a) \leq a$, by Lemma 7.54.1, it follows by residuation that $a \leq (a \rightarrow^{\boldsymbol{A}} a) \rightarrow^{\boldsymbol{A}} a$. Since $a \in F$ and $F$ is upwards-closed, it follows that $(a \rightarrow^{\boldsymbol{A}} a) \rightarrow^{\boldsymbol{A}} a \in F$. Now,

$$\begin{aligned} &\big((a \rightarrow^{\boldsymbol{A}} a) \rightarrow^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} a)\big) \wedge^{\boldsymbol{A}} \big((a \rightarrow^{\boldsymbol{A}} a) \rightarrow^{\boldsymbol{A}} a\big) \\ &\leq (a \rightarrow^{\boldsymbol{A}} a) \rightarrow^{\boldsymbol{A}} \big((a \rightarrow^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} a\big), \end{aligned}$$

by Lemma 7.54.8. Since both $(a \rightarrow^{\boldsymbol{A}} a) \rightarrow^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} a) \in \boldsymbol{\tau A} \subseteq F$ and $(a \rightarrow^{\boldsymbol{A}} a) \rightarrow^{\boldsymbol{A}} a \in F$ and $F$ is closed under meets and upwards-closed, it follows that $(a \rightarrow^{\boldsymbol{A}} a) \rightarrow^{\boldsymbol{A}} \big((a \rightarrow^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} a\big) \in F$. Hence indeed,

$$\big(a \wedge^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} a)\big) \leftrightarrow^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} a) \in F.$$

It follows by Lemma 7.97 that

$$\langle a \wedge^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} a), a \rightarrow^{\boldsymbol{A}} a\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F).$$

So, $\pi\big(a \wedge^{\boldsymbol{A}} (a \rightarrow^{\boldsymbol{A}} a)\big) = \pi(a \rightarrow^{\boldsymbol{A}} a)$. That is, $\pi(a \rightarrow^{\boldsymbol{A}} a) \leq \pi(a)$. So,

$$b \rightarrow^{\boldsymbol{B}} b = \pi(a) \rightarrow^{\boldsymbol{B}} \pi(a) = \pi(a \rightarrow^{\boldsymbol{A}} a) \leq \pi(a) = b.$$

Thus, $b \in \boldsymbol{\tau B}$.

Conversely, let $b \in \boldsymbol{\tau B}$. Since $\pi$ is surjective, there exists $a \in A$ such that $b = \pi(a)$. Since, $b \rightarrow^{\boldsymbol{B}} b \leq b$, it holds

$$\begin{aligned} \pi\big((a \rightarrow^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} a\big) &= \big(\pi(a) \rightarrow^{\boldsymbol{B}} \pi(a)\big) \wedge^{\boldsymbol{B}} \pi(a) \\ &= (b \rightarrow^{\boldsymbol{B}} b) \wedge^{\boldsymbol{A}} b = b \rightarrow^{\boldsymbol{B}} b = \pi(a \rightarrow^{\boldsymbol{A}} a). \end{aligned}$$

So, $\langle (a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} a, a \to^{\boldsymbol{A}} a \rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Since $a \to^{\boldsymbol{A}} a \in \boldsymbol{\tau A} \subseteq F$, it follows by compatibility that $(a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} a \in F$. Since $F$ is upwards-closed, it follows that $a \in F$. Hence, $b = \pi(a) \in \pi F$.

Finally, since $\mathrm{Ker}\pi = \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is compatible with $F$, it follows that

$$F = \pi^{-1}\pi F = \pi^{-1}(\boldsymbol{\tau B}).$$

But, $\boldsymbol{\tau B} \in \mathcal{F}i_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} \boldsymbol{B}$. Thus, $F \in \mathcal{F}i_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} \boldsymbol{A}$, by Lemma 0.24.1.

*3.* Fix $\boldsymbol{\rho}(x,y) := \{x \leftrightarrow y\}$ and $\boldsymbol{\tau}(x) := \{x \wedge (x \to x) \approx x \to x\}$. By definition, $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ is the $\boldsymbol{\tau}$-assertional logic of CRLr. So, condition (ALG1) on page 24 holds taking $\mathsf{K} = \mathsf{CRLr}$. We claim that condition (ALG3) holds as well for CRLr, that is, $x \approx y \dashv\vDash^{\mathrm{eq}}_{\mathsf{CRLr}} \boldsymbol{\tau}(\boldsymbol{\rho}(x,y))$. Assume $\boldsymbol{A} \in \mathsf{CRLr}$ and let $h : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h(x) = h(y) = a$. On the one hand, $h\big[(x \leftrightarrow y) \wedge \big((x \leftrightarrow y) \to (x \leftrightarrow y)\big)\big] = (a \leftrightarrow^{\boldsymbol{A}} a) \wedge \big((a \leftrightarrow^{\boldsymbol{A}} a) \to^{\boldsymbol{A}} (a \leftrightarrow^{\boldsymbol{A}} a)\big) = (a \leftrightarrow^{\boldsymbol{A}} a) \to^{\boldsymbol{A}} (a \leftrightarrow^{\boldsymbol{A}} a)$, using Lemma 7.92.2. On the other hand, $h\big((x \leftrightarrow y) \to (x \leftrightarrow y)\big) = (a \leftrightarrow^{\boldsymbol{A}} a) \to^{\boldsymbol{A}} (a \leftrightarrow^{\boldsymbol{A}} a)$. Hence, $\mathsf{CRLr} \vDash \boldsymbol{\tau}(\boldsymbol{\rho}(x,y))$. Conversely, assume $\boldsymbol{A} \in \mathsf{CRLr}$ and let $h : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h\big[(x \leftrightarrow y) \wedge \big((x \leftrightarrow y) \to (x \leftrightarrow y)\big)\big] = h\big((x \leftrightarrow y) \to (x \leftrightarrow y)\big)$. Since $h\big((x \leftrightarrow y) \to (x \leftrightarrow y)\big) = h(x \leftrightarrow y) \to^{\boldsymbol{A}} h(x \leftrightarrow y) \in \boldsymbol{\tau A}$ and $\boldsymbol{\tau A}$ is upwards-closed, it follows that $h(x \leftrightarrow y) \in \boldsymbol{\tau A}$. So, both $h(x) \to^{\boldsymbol{A}} h(y) \in \boldsymbol{\tau A}$ and $h(y) \to^{\boldsymbol{A}} h(x) \in \boldsymbol{\tau A}$. It now follows by Lemma 7.95 that $h(x) \leq h(y)$ and $h(y) \leq h(x)$. So, $h(x) = h(y)$. Thus, $\mathsf{CRLr} \vDash x \approx y$. We conclude that (ALG3) holds for CRLr, as claimed. Therefore, $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ is finitely algebraizable witnessed by $\boldsymbol{\rho}$ and $\boldsymbol{\tau}$. Finally, as we had seen already that $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ is finitary, it is in fact BP-algebraizable. The last statement follows by 1. □

We now turn our attention to a logic not previously considered in the literature, at least to our knowledge. Following the previous example, we consider the least logic in between $\mathcal{S}^{\leq}_{\mathsf{CRLr}}$ and $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ with the same theorems as $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$. The logic with these properties is defined by the class of matrices

$$\big\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{CRLr}, F \in \mathrm{Filt}\boldsymbol{A}, \boldsymbol{\tau A} \subseteq F\big\},$$

as we will show in Proposition 7.100. We denote the consequence relation induced by the class of matrices above by $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$. Once again, this consequence relation is finitary because the class of matrices above is first-order definable and hence closed under ultraproducts. Moreover, since for every $\boldsymbol{A} \in \mathsf{CRLr}$, $\boldsymbol{\tau A}$ is a lattice filter of $\boldsymbol{A}$, by Lemma 7.93, it follows from the definitions involved that $\mathcal{S}^{\leq}_{\mathsf{CRLr}} \leq \mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}} \leq \mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$.

Having defined both $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRL}}$ and $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$, it is worth mentioning that the these logics are expansions of the logics $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ and $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ (by a constant 1), respectively. Indeed, recall that $\mathsf{CRL} \restriction_{\mathcal{L}} \subseteq \mathsf{CRLr}$, and notice that furthermore, given a lattice filter $F \in \mathrm{Filt}\boldsymbol{A}$ such that $1 \in F$, with $\boldsymbol{A} \in \mathsf{CRL}$, it is necessarily the case that $\boldsymbol{\tau}(\boldsymbol{A}) \subseteq F$, because $1 \leq a \to^{\boldsymbol{A}} a$, for every $a \in A$, by Lemma 7.56.3.

Let us now determine the class of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$-algebras.

**Proposition 7.99.** $\mathsf{CRLr} = \mathsf{Alg}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}) = \mathsf{Alg}^*(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}})$.

Proof. Just notice that, since $\mathcal{S}^{\leq}_{\mathsf{CRLr}} \leq \mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}} \leq \mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$,

$$\mathsf{CRLr} = \mathsf{Alg}^*(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}) \subseteq \mathsf{Alg}^*(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}) \subseteq \mathsf{Alg}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}) \subseteq \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{CRLr}}) = \mathsf{CRLr},$$

using Theorem 7.98.1, and the fact that $\mathcal{S}^{\leq}_{\mathsf{CRLr}}$ is, by definition, the semilattice-based logic of CRLr. □

**Proposition 7.100.** *The logic* $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ *is the least logic in between* $\mathcal{S}^{\leq}_{\mathsf{CRLr}}$ *and* $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ *with the same theorems that* $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$.

Proof. Let us first show that the theorems of $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ and $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ are the same. Since $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}} \leq \mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$, it is clear that if $\varnothing \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} \varphi$, then $\varnothing \vdash_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} \varphi$. Conversely, assume $\varnothing \vdash_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} \varphi$. Let $\boldsymbol{A} \in \mathsf{CRLr}$, $F \in \mathrm{Filt}\boldsymbol{A}$ such that $\boldsymbol{\tau}\boldsymbol{A} \subseteq F$ and $h : \boldsymbol{Fm} \to \boldsymbol{A}$. It follows by assumption that $h(\varphi) \in \boldsymbol{\tau}\boldsymbol{A}$. Clearly then, $h(\varphi) \in F$. Thus, $\varnothing \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} \varphi$.

Assume now that $\mathcal{S}$ is a logic such that $\mathcal{S}^{\leq}_{\mathsf{CRLr}} \leq \mathcal{S} \leq \mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ and with the same theorems that $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$. Then, $\mathsf{CRLr} = \mathsf{Alg}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}) \subseteq \mathsf{Alg}(\mathcal{S}) \subseteq \mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{CRLr}}) = \mathsf{CRLr}$. Thus for every $\boldsymbol{A} \in \mathsf{CRLr}$, every $\mathcal{S}$-filter of $\boldsymbol{A}$ is a $\mathcal{S}^{\leq}_{\mathsf{CRLr}}$-filter and therefore, since it is non-empty, it is a lattice filter of $\boldsymbol{A}$. Moreover, since all formulas of the form $x \to x$ are theorems of $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$, and having Lemma 7.94 in mind, $\boldsymbol{\tau}\boldsymbol{A}$ is included in every $\mathcal{S}$-filter of $\boldsymbol{A}$. Therefore, $\big\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{Alg}(\mathcal{S}), F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}\big\} \subseteq \big\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{CRLr}, F \in \mathrm{Filt}\boldsymbol{A}, \boldsymbol{\tau}\boldsymbol{A} \subseteq F\big\}$. This implies that $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}} \leq \mathcal{S}$. □

Once again, let us see that the logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ falls outside the classes of logics in Figure 1.

**Theorem 7.101.**

1. $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ *is not protoalgebraic.*
2. $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ *is not truth-equational.*

Proof. The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ is an expansion of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$. Now, both protoalgebraicity and truth-equationality are preserved through expansions. And we know that $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ is neither protoalgebraic nor truth-equational, by Theorem 7.75.1 and 2, respectively. □

In order to find the strong version of the logic $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$, we follow the same strategy as we did for the non-integral case. That is, we first prove that $\mathsf{Alg}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}) = \mathsf{Alg}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}})$, and that moreover for any $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$-algebra, the least $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$-filter and the least $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ coincide. To this end, we next characterize the $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ filters of algebras in $\mathsf{CRLr}$.

**Proposition 7.102.** *Let* $\boldsymbol{A} \in \mathsf{CRLr}$. *The* $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$*-filters of* $\boldsymbol{A}$ *coincide with the lattice filters of* $\boldsymbol{A}$ *containing* $\boldsymbol{\tau}\boldsymbol{A}$. *That is,*

$$\mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}}\boldsymbol{A} = \{F \in \mathrm{Filt}\boldsymbol{A} : \boldsymbol{\tau}\boldsymbol{A} \subseteq F\}.$$

Proof. On the one hand, using the definition of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ and Lemma 7.94, it easily follows that

$$\varnothing \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} x \to x, \qquad x, y \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} x \wedge y, \qquad x \wedge y \vdash_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} x, y.$$

So, given $\boldsymbol{A} \in \mathsf{CRLr}$ and $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}}\boldsymbol{A}$, it must hold $F \in \mathrm{Filt}\boldsymbol{A}$ and $\boldsymbol{\tau}\boldsymbol{A} \subseteq F$. On the other hand, from the definition of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ it follows immediately that if $F \in \mathrm{Filt}\boldsymbol{A}$ is such that $\boldsymbol{\tau}\boldsymbol{A} \subseteq F$, then $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}}\boldsymbol{A}$. □

Here arrived, and unlike the logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$, it is not immediate to see that the least $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$-filter of an algebra in $\mathsf{CRLr}$ coincides with its least $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$-filter. Of course, the natural candidate is $\boldsymbol{\tau}\boldsymbol{A}$. To see that it fulfils our requirements, notice that it is an implicative lattice filter of $\boldsymbol{A} \in \mathsf{CRLr}$, by Lemma 7.93, and that moreover, the

$\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$-filters of any $\boldsymbol{A} \in \mathsf{CRLr}$ are precisely those lattice filters which contain $\boldsymbol{\tau A}$, by Proposition 7.102. Therefore, for every $\boldsymbol{A} \in \mathsf{CRLr}$,

$$\bigcap \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} \boldsymbol{A} = \bigcap \mathcal{F}i_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} \boldsymbol{A} = \boldsymbol{\tau A}.$$

We are now able to apply Proposition 5.9:

**Theorem 7.103.** *The logic $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ is the strong version of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$.*

It readily follows by Proposition 5.14 that:

**Theorem 7.104.** *Let $\boldsymbol{A} \in \mathsf{CRLr}$. The Leibniz $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$-filters of $\boldsymbol{A}$ coincide with the implicative lattice filters of $\boldsymbol{A}$ which contain $\boldsymbol{\tau A}$. That is,*

$$\mathcal{F}i^*_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} \boldsymbol{A} = \{F \in \mathrm{Filt}_{\rightarrow} \boldsymbol{A} : \boldsymbol{\tau A} \subseteq F\}.$$

Once again, although not semilattice-based, $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ still has its Leibniz filters equationally definable. For we know that $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ is truth-equational, and moreover $\mathsf{Alg}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}) = \mathsf{CRLr} = \mathsf{Alg}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}})$. Hence, it follows by Proposition 6.6 that:

**Proposition 7.105.** *The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ has its Leibniz filters equationally definable by $\boldsymbol{\tau}(x) = \big\{x \wedge (x \to x) \approx x \to x\big\}$.*

Furthermore, the fact that $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ is algebraizable gives us:

**Proposition 7.106.** *The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ satisfies ($\star$).*

Proof. Since $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ is algebraizable, $\boldsymbol{\Omega}^{\boldsymbol{A}} : \mathcal{F}i_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} \boldsymbol{A} \to \mathrm{Con}_{\mathsf{Alg}^*(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}})} \boldsymbol{A}$ is an order-isomorphism, for every $\boldsymbol{A}$. But, $\mathsf{Alg}^*(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}) = \mathsf{CRLr} = \mathsf{Alg}^*(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}})$ and $\mathcal{F}i_{\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}} \boldsymbol{A} = \{F \in \mathrm{Filt}_{\rightarrow} \boldsymbol{A} : \boldsymbol{\tau A} \subseteq F\} = \mathcal{F}i^*_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} \boldsymbol{A}$, for every $\boldsymbol{A} \in \mathsf{CRLr}$, by Theorem 7.98, Proposition 7.99 and Theorem 7.104. □

We can therefore apply the results of Chapter 6 and get:

**Corollary 7.107.** *Let $\boldsymbol{A} \in \mathsf{CRLr}$. For every $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} \boldsymbol{A}$,*

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*) \quad \textit{and} \quad \widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}(F^{\mathrm{Su}}).$$

*Moreover,*

$$F^{\mathrm{Su}} = \bigcap_{G \in (\mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} \boldsymbol{A})^F} G^*.$$

*As a consequence, $F$ is a Suszko $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$-filter of $\boldsymbol{A}$ if and only if $F \subseteq G^*$, for every $(\mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} \boldsymbol{A})^F$.*

Just like the non-integral case, the logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ fails to satisfy the explicit definability of its Leibniz filters.

**Proposition 7.108.** *The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ does not have its Leibniz filters explicitly definable.*

Proof. It follows by Propositions 7.106 and 6.21, having in mind that $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ is not protoalgebraic. □

Nevertheless, $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ does admit the logical definability of its Leibniz filters.

**Proposition 7.109.** *The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ has its Leibniz filters logically definable by the rule Modus Ponens.*

Proof. Just notice that, in light of Theorem 7.104, for every $\boldsymbol{A} \in \mathsf{Alg}^*(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}})$ and every $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} \boldsymbol{A}$, $F$ is a Leibniz $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$-filter of $\boldsymbol{A}$ if and only if $F$ is an implicative lattice filter if and only if is closed under *Modus Ponens*. Hence, the result follows from Proposition 6.30. □

Consequently,

**Corollary 7.110.** *The logic $\mathcal{S}^{\tau}_{\mathsf{CRLr}}$ is the inferential extension of $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ by the rule Modus Ponens.*

Proof. The result follows by Corollary 6.33, since $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ has its Leibniz filters logically definable by the rule *Modus Ponens*, by Proposition 7.109. □

We now wish to find a Birula-Rasiowa style characterization of $F^*$. Not surprisingly, the set $\boldsymbol{\tau A}$ is also reflected in the definition of the transformation $\Psi$.

**Definition 7.111.** Let $\boldsymbol{A} \in \mathsf{CRLr}$. For every $F \in \mathrm{Filt}\boldsymbol{A}$ such that $\boldsymbol{\tau A} \subseteq F$, define

$$\Psi(F) := \big\{a \in A : \forall b \in A\ \forall c \in \boldsymbol{\tau A} \text{ if } (a \wedge^{\boldsymbol{A}} c) \to^{\boldsymbol{A}} b \in F, \text{ then } b \in F\big\}.$$

**Proposition 7.112.** *Let $\boldsymbol{A} \in \mathsf{CRLr}$. For every $F \in \mathrm{Filt}\boldsymbol{A}$ such that $\boldsymbol{\tau A} \subseteq F$, $\Psi(F)$ is an implicative lattice filter of $\boldsymbol{A}$ and such that $\boldsymbol{\tau A} \subseteq \Psi(F)$.*

Proof. First, let $a \in \Psi(F)$ and $b \in A$ such that $a \leq b$. Let $c \in \boldsymbol{\tau A}$ and $d \in A$ such that $(b \wedge^{\boldsymbol{A}} c) \to^{\boldsymbol{A}} d \in F$. Notice that $a \wedge^{\boldsymbol{A}} c \leq b \wedge^{\boldsymbol{A}} c$. Hence, by suffixing, it follows that

$$(b \wedge^{\boldsymbol{A}} c) \to^{\boldsymbol{A}} d \leq (a \wedge^{\boldsymbol{A}} c) \to^{\boldsymbol{A}} d.$$

Since $F$ is upwards-closed, it follows that $(a \wedge^{\boldsymbol{A}} c) \to^{\boldsymbol{A}} d \in F$. Since $a \in \Psi(F)$, it follows that $d \in F$. Hence, $b \in \Psi(F)$.

Now, let $a, b \in \Psi(F)$. Let $d \in A$ such that $d \notin F$. Let $c \in \boldsymbol{\tau A}$. Then, $\big[b \wedge^{\boldsymbol{A}} \big((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big)\big] \to^{\boldsymbol{A}} d \notin F$, because $b \in \Psi(F)$ and $(a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c) \in \boldsymbol{\tau A}$. Then, $\big[a \wedge^{\boldsymbol{A}} \big((b \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big)\big] \to^{\boldsymbol{A}} \big[\big(b \wedge^{\boldsymbol{A}} \big((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big)\big) \to^{\boldsymbol{A}} d\big] \notin F$, because $a \in \Psi(F)$ and $(b \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c) \in \boldsymbol{\tau A}$. That is,

$$\Big[\big[a \wedge^{\boldsymbol{A}} \big((b \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big)\big] \odot^{\boldsymbol{A}} \big[b \wedge^{\boldsymbol{A}} \big((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big)\big]\Big] \to^{\boldsymbol{A}} d \notin F,$$

by Lemma 7.54.2. Now,

$$\begin{aligned}
&\big[a \wedge^{\boldsymbol{A}} \big((b \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big)\big] \odot^{\boldsymbol{A}} \big[b \wedge^{\boldsymbol{A}} \big((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big)\big]\\
&\leq a \odot^{\boldsymbol{A}} (a \to^{\boldsymbol{A}} a)\\
&\leq a.
\end{aligned}$$

using Lemma 7.54.7 twice and 1, respectively. Similarly,

$$\begin{aligned}
&\big[a \wedge^{\boldsymbol{A}} \big((b \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big)\big] \odot^{\boldsymbol{A}} \big[b \wedge^{\boldsymbol{A}} \big((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big)\big]\\
&\leq (b \to^{\boldsymbol{A}} b) \odot^{\boldsymbol{A}} b\\
&\leq b.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\left[a \wedge^{\boldsymbol{A}} \left((b \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right)\right] \odot^{\boldsymbol{A}} \left[b \wedge^{\boldsymbol{A}} \left((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right)\right] \\
&\leq (c \to^{\boldsymbol{A}} c) \odot^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c) \\
&\leq (c \to^{\boldsymbol{A}} c) \\
&\leq c,
\end{aligned}$$

using Lemma 7.54.7 twice, Lemma 7.92.3, and the fact that $c \in \boldsymbol{\tau A}$, respectively. So, by definition of infimum,

$$\left[a \wedge^{\boldsymbol{A}} \left((b \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right)\right] \odot^{\boldsymbol{A}} \left[b \wedge^{\boldsymbol{A}} \left((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right)\right] \leq (a \wedge^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} c.$$

So, by suffixing,

$$\begin{aligned}
&\left((a \wedge^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} c\right) \to^{\boldsymbol{A}} d \\
&\leq \Big[\left[a \wedge^{\boldsymbol{A}} \left((b \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right)\right] \odot^{\boldsymbol{A}} \left[b \wedge^{\boldsymbol{A}} \left((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right)\right]\Big] \to^{\boldsymbol{A}} d.
\end{aligned}$$

Since $F$ is upwards-closed, it follows that $\left((a \wedge^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} c\right) \to^{\boldsymbol{A}} d \notin F$, for every $c \in \boldsymbol{\tau A}$. Thus, $a \wedge^{\boldsymbol{A}} b \in \Psi(F)$.

Next, let $a, b \in \Psi(F)$. Let $d \in A$ such that $d \notin F$. Let $c \in \boldsymbol{\tau A}$. Then, $\left(b \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right) \to^{\boldsymbol{A}} d \notin F$, because $b \in \Psi(F)$. Then, $\left(a \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right) \to^{\boldsymbol{A}} \left[\left(b \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right) \to^{\boldsymbol{A}} d\right] \notin F$, because $a \in \Psi(F)$. That is,

$$\left[\left(a \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right) \odot^{\boldsymbol{A}} \left(b \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right)\right] \to^{\boldsymbol{A}} d \notin F,$$

by Lemma 7.54.2. Now,

$$\left(a \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right) \odot^{\boldsymbol{A}} \left(b \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right) \leq a \odot^{\boldsymbol{A}} b,$$

using Lemma 7.54.7 twice. Moreover,

$$\left(a \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right) \odot^{\boldsymbol{A}} \left(b \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right) \leq (c \to^{\boldsymbol{A}} c) \odot^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c) \leq c \to^{\boldsymbol{A}} c \leq c,$$

using Lemma 7.54.7 twice again, Lemma 7.92.3, and the fact that $c \in \boldsymbol{\tau A}$, respectively. So, by definition of infimum,

$$\left(a \wedge^{\boldsymbol{A}} c\right) \odot^{\boldsymbol{A}} \left(b \wedge^{\boldsymbol{A}} c\right) \leq (a \odot^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} c.$$

So, by suffixing,

$$\left((a \odot^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} c\right) \to^{\boldsymbol{A}} d \leq \left[\left(a \wedge^{\boldsymbol{A}} c\right) \odot^{\boldsymbol{A}} \left(b \wedge^{\boldsymbol{A}} c\right)\right] \to^{\boldsymbol{A}} d \notin F.$$

Since $F$ is upwards-closed, it follows that $\left((a \odot^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} c\right) \to^{\boldsymbol{A}} d \notin F$, for every $c \in A$. Thus, $a \odot^{\boldsymbol{A}} b \in \Psi(F)$.

So far, we have proved that $\Psi(F)$ is an implicative lattice filter of $\boldsymbol{A}$. We are left to prove that it contains $\boldsymbol{\tau A}$. Let $a \in \boldsymbol{\tau A}$. Let $b \in A$ and $c \in \boldsymbol{\tau A}$ such that $\left(a \wedge^{\boldsymbol{A}} c\right) \to^{\boldsymbol{A}} b \in F$. Since $a \to^{\boldsymbol{A}} a \leq a$ and $c \to^{\boldsymbol{A}} c \leq c$, it holds $(a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c) \leq a \wedge^{\boldsymbol{A}} c$. It follows by suffixing that $(a \wedge^{\boldsymbol{A}} c) \to^{\boldsymbol{A}} b \leq \left((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right) \to^{\boldsymbol{A}} b$. Since $F$ is upwards-closed, it follows that $\left((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right) \to^{\boldsymbol{A}} b \in F$. But $\left((a \to^{\boldsymbol{A}} a) \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\right) \to^{\boldsymbol{A}} b \leq b$, by the relevance condition. Again, since $F$ is upwards-closed, it follows that $b \in F$. Hence, $a \in \Psi(F)$. Thus, $\boldsymbol{\tau A} \subseteq \Psi(F)$. □

**Proposition 7.113.** *Let $\boldsymbol{A} \in \mathsf{CRLr}$. For every $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}} \boldsymbol{A}$,*

$$\boldsymbol{\Omega}^{\boldsymbol{A}}(F) = \boldsymbol{\Omega}^{\boldsymbol{A}}\left(\Psi(F)\right).$$

Proof. We claim that $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is compatible with $\Psi(F)$. Let $\langle a,b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ and $a \in \Psi(F)$. Let $c \in \boldsymbol{\tau A}$. It holds, $\big\langle \big(a \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big) \to^{\boldsymbol{A}} b, \big(b \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big) \to^{\boldsymbol{A}} b\big\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. Moreover, by suffixing,

$$b \to^{\boldsymbol{A}} b \leq \big(b \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big) \to^{\boldsymbol{A}} b.$$

Since $b \to^{\boldsymbol{A}} b \in \boldsymbol{\tau A} \subseteq F$ and $F$ is upwards-closed, it follows that $\big(b \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big) \to^{\boldsymbol{A}} b \in F$. It follows by compatibility that $\big(a \wedge^{\boldsymbol{A}} (c \to^{\boldsymbol{A}} c)\big) \to^{\boldsymbol{A}} b \in F$. Since $a \in \Psi(F)$, it follows that $b \in F$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$.

Conversely, we claim that $\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$ is compatible with $F$. Let $\langle a,b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$ and let $a \in F$. Then, $\langle a \to^{\boldsymbol{A}} b, b \to^{\boldsymbol{A}} b\rangle \in \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$. Since $b \to^{\boldsymbol{A}} b \in \boldsymbol{\tau A} \subseteq \Psi(F)$, it follows by compatibility that $a \to^{\boldsymbol{A}} b \in \Psi(F)$. Now, $a \leq (a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} b$, by Lemma 7.54.3. Since $a \in F$ and $F$ is upwards-closed, it follows that $(a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} b \in F$. Let $c \in \boldsymbol{\tau A}$. Since $(a \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} c \leq a \to^{\boldsymbol{A}} b$, it follows by suffixing that

$$(a \to^{\boldsymbol{A}} b) \to^{\boldsymbol{A}} b \leq \big((a \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} c\big) \to^{\boldsymbol{A}} b.$$

So, $\big((a \to^{\boldsymbol{A}} b) \wedge^{\boldsymbol{A}} c\big) \to^{\boldsymbol{A}} b \in F$. Since $a \to^{\boldsymbol{A}} b \in \Psi(F)$, it follows that $b \in F$. Thus, $\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)$. □

**Corollary 7.114.** *Let $\boldsymbol{A} \in \mathsf{CRLr}$. For every $F \in \mathcal{F}i_{\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}}\boldsymbol{A}$,*

$$F^* = \Psi(F).$$

Proof. On the one hand, since $\boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big)$, we have $\Psi(F) \in [\![F]\!]^*$, and hence $F^* \subseteq \Psi(F)$. On the other hand, since $\boldsymbol{\Omega}^{\boldsymbol{A}}\big(\Psi(F)\big) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F^*)$, we have $F^* \in [\![\Psi(F)]\!]^*$, and hence $\Psi(F)^* \subseteq F^*$. But $\Psi(F) = \Psi(F)^*$, because we have seen that $\Psi(F)$ is an implicative lattice filter of $\boldsymbol{A}$ containing $\boldsymbol{\tau A}$, by Proposition 7.112, and the Leibniz $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$-filters of $\boldsymbol{A}$ are precisely these filters, by Theorem 7.104. □

We finish our study by showing that neither $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ nor $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ belong to any of the classes of the Frege hierarchy.

**Proposition 7.115.** *The logic $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ is not selfextensional.*

Proof. Suppose, towards an absurd, that $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ is selfextensional. Then, since $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ has a conjunction, it follows by Theorem 0.46 that $\mathcal{S}^{\leq}_{\mathsf{CRLr}}$ is semilattice-based. Then, it is semilattice-based of $\mathsf{Alg}(\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}) = \mathsf{CRLr}$, using Proposition 7.99. Consequently, $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}} = \mathcal{S}^{\leq}_{\mathsf{CRLr}}$, and we reach an absurd (for instance, $\mathcal{S}^{\preccurlyeq}_{\mathsf{CIRL}}$ has theorems, while $\mathcal{S}^{\leq}_{\mathsf{CRLr}}$ has not). □

**Proposition 7.116.** *The logic $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ is not selfextensional.*

Proof. Suppose, towards an absurd, that $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ is selfextensional. Then, since $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ has a conjunction, it follows by Theorem 0.46 that $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ is semilattice-based. Then, it is semilattice-based of $\mathsf{Alg}(\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}) = \mathsf{CRLr}$, using Theorem 7.98.1. Consequently, $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}} = \mathcal{S}^{\leq}_{\mathsf{CRLr}}$, and we reach an absurd (for instance, $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}}$ has theorems, while $\mathcal{S}^{\leq}_{\mathsf{CRLr}}$ has not). □

As final remarks, we make a few comments on how the results of the present section relate to the system $\mathcal{R}$ of Relevance Logic. Consider the class of $\mathcal{L}-$algebras in $\mathsf{CRLr}$ which satisfy moreover the additional condition:

1. $\odot^{\boldsymbol{A}}$ is square increasing, that is, for every $a \in A$, $a \leq a \odot^{\boldsymbol{A}} a$.

Let us denote such class by $\mathsf{CRLr}_{\mathrm{sq}}$. All the auxiliary results we have proved concerning the set $\boldsymbol{\tau}\boldsymbol{A}$, with $\boldsymbol{A} \in \mathsf{CRLr}_{\mathrm{sq}}$, are still valid. In particular, the logic $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}_{\mathrm{sq}}}$ induced by the class of matrices $\{\langle \boldsymbol{A}, \boldsymbol{\tau}\boldsymbol{A}\rangle : \boldsymbol{A} \in \mathsf{CRLr}_{\mathrm{sq}}\}$ is BP-algebraizable witnessed by the set of equivalence formulas $\boldsymbol{\rho}(x,y) = \{x \leftrightarrow y\}$ and the set of defining equations $\boldsymbol{\tau}(x) = \big\{x \wedge (x \to x) \approx x \to x\big\}$. Given $\boldsymbol{A} \in \mathsf{CRLr}_{\mathrm{sq}}$, the $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}_{\mathrm{sq}}}$-filters of $\boldsymbol{A}$ are the implicative lattice filters containing $\boldsymbol{\tau}\boldsymbol{A}$. Also, let $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}_{\mathrm{sq}}}$ be the logic induced by the class of matrices $\big\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{CRLr}_{\mathrm{sq}}, F \in \mathrm{Filt}\boldsymbol{A}, \boldsymbol{\tau}\boldsymbol{A} \subseteq F\big\}$. Given $\boldsymbol{A} \in \mathsf{CRLr}_{\mathrm{sq}}$, the $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}_{\mathrm{sq}}}$-filters of $\boldsymbol{A}$ are the lattice filters containing $\boldsymbol{\tau}\boldsymbol{A}$. Now, it can be proved that, for every $\boldsymbol{A} \in \mathsf{CRLr}_{\mathrm{sq}}$ and every $a, b \in A$, $a \wedge^{\boldsymbol{A}} b \leq a \odot^{\boldsymbol{A}} b$. As a consequence, and having in mind Lemma 7.96, every lattice filter of $\boldsymbol{A} \in \mathsf{CRLr}_{\mathrm{sq}}$ is implicative. Consequently, $\mathcal{S}^{\boldsymbol{\tau}}_{\mathsf{CRLr}_{\mathrm{sq}}} = \mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}_{\mathrm{sq}}}$.

Consider the language $\mathcal{L}' = \langle \wedge, \vee, \to, \odot, 0\rangle$, that is, the expansion of $\mathcal{L}$ by the constant 0. Consider moreover the unary operation $\neg^{\boldsymbol{A}} a \coloneqq a \to^{\boldsymbol{A}} 0$, for every $\mathcal{L}'$-algebra $\boldsymbol{A}$ and $a \in A$. An $\mathcal{R}$-*algebra* is an $\mathcal{L}'$-algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}, 0\rangle$, where

1. The reduct $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \to^{\boldsymbol{A}}, \odot^{\boldsymbol{A}}\rangle$ belongs to $\mathsf{CRLr}_{\mathrm{sq}}$;
2. $\neg^{\boldsymbol{A}}\neg^{\boldsymbol{A}} a \leq a$, for every $a \in A$.

Let us denote the class of all $\mathcal{R}$-algebras by $\mathsf{R}$. All the auxiliary results we have proved concerning the set $\boldsymbol{\tau}\boldsymbol{A}$, restricted to $\boldsymbol{A} \in \mathsf{R}$, can be found in [42]. In particular, $\mathcal{R}$ is the logic induced by the class of matrices $\{\langle \boldsymbol{A}, \boldsymbol{\tau}\boldsymbol{A}\rangle : \boldsymbol{A} \in \mathsf{R}\}$. It is well-known that $\mathcal{R}$ is BP-algebraizable witnessed by the set of equivalence formulas $\boldsymbol{\rho}(x,y) = \{x \leftrightarrow y\}$ and the set of defining equations $\boldsymbol{\tau}(x) = \big\{x \wedge (x \to x) \approx x \to x\big\}$ [11, Theorem 5.8]. Now, define the logic $\mathcal{R}^{\preccurlyeq}$ as the logic induced by the class of matrices $\big\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \in \mathsf{R}, F \in \mathrm{Filt}\boldsymbol{A}, \boldsymbol{\tau}\boldsymbol{A} \subseteq F\big\}$. By the same considerations as above, we have $\mathcal{R}^{\preccurlyeq} = \mathcal{R}$.

# Conclusions

In this dissertation we have aimed at extending the traditional AAL tools to non-protoalgebraic logics. The two main concepts investigated to this end were the Suszko operator [**24**] and the Leibniz filters [**37**]. Each of these concepts motivated a broader and independent study that eventually culminated in the two parts of the present dissertation. Part I builts and develops an abstract framework which intends to unify under a common treatment the study of the Leibniz, Suszko, and Frege operators in AAL. Part II generalizes the theory of the strong version of protoalgebraic logics, started in [**37**], to arbitrary sentential logics.

The abstract notion which ecompasses the Leibniz, Suszko, and Frege, operators is that of $\mathcal{S}$-*operator* (Definition 1.1). Its origin roots back to [**24**, p. 199], under the name of "mapping compatible with $\mathcal{S}$-filters". In the quest of finding general properties common to the three paradigmatic AAL $\mathcal{S}$-operators, we have introduced the new notion of *coherence* (Definition 1.28), a weaker property than commuting with inverse images by surjective homomorphisms. Under the assumption of coherence of a family of $\mathcal{S}$-operators, we established a General Correspondence Theorem (Theorem 1.38), which generalizes several known correspondence theorems in AAL, namely Blok and Pigozzi's well-known Correspondence Theorem for protoalgebraic logics [**10**, Theorem 2.4], Czelakowski's less known Correspondence Theorem [**24**, Proposition 2.3] for arbitrary logics, and also the first strengthening obtained for protoalgebraic logics by Font and Jansana [**37**, Corollary 9.1].

A family of $\mathcal{S}$-operators $\nabla$ has associated to it the notions of $\nabla$-class and $\nabla$-filter. We propose as new notion of Leibniz filter precisely that of $\boldsymbol{\Omega}$-filter (see page 48). Our new definition of Leibniz filter coincides with the previous one for protoalgebraic logics ([**37**, Definition 1]), and furthermore it is applicable to arbitrary sentential logics. This fact pathed the way to generalize several known results for protoalgebraic logics, to arbitrary sentential logics. For instance, given any sentential logic $\mathcal{S}$, the Leibniz $\mathcal{S}$-filters are precisely the least elements of the full g-models of $\mathcal{S}$ (Proposition 2.9; compare with [**36**, Proposition 3.6]). The notion of $\widetilde{\boldsymbol{\Omega}}_{\mathcal{S}}$-filter was also thoroughly investigated. The Suszko $\mathcal{S}$-filters turn out to be the least elements of the full g-models of $\mathcal{S}$ which are moreover up-sets. In fact, given an arbitrary algebra $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, $F$ is a Suszko filter of $\boldsymbol{A}$ if and only if $(\mathcal{F}i_{\mathcal{S}}\boldsymbol{A})^F$ is a full g-model of $\mathcal{S}$ (Theorem 2.29). Suszko filters revealed to be deeply connected with the class of truth-equational logics, introduced in [**55**]. Indeed, a logic $\mathcal{S}$ is truth-equational if and only if every $\mathcal{S}$-filter is a Suszko filter (Theorem 2.30). Furthermore, with the notion of Suszko filter at hand, a new Isomorphism Theorem for protoalgebraic logics was proved (Theorem 3.8), very much in the same spirit of the famous one for algebraizable logics ([**11**, Theorem 3.7]; see [**48**, Theorem 5.2] for the non-finitary case) and for weakly algebraizable logics ([**25**, Theorem 4.8]).

Following the characterization of truth-equational logics in terms of the Suszko operator given in [**55**], new characterizations in terms of the Suszko operator for other classes of logics belonging to the Leibniz hierarchy were proved (Theorems 3.13 and 3.16). To mention just one, a logic is protoalgebraic if and only if the Suszko operator commutes with inverse images by surjective homomorphisms (Theorem 3.12).

Some new contributions to the study of truth-equational logics were also put forward, specially concerning the behaviour of Suszko operator inside this class of logics. These results are collected in Chapter 4. In particular, we have established that a logic is truth-equational if and ony if the Suszko operator $\widetilde{\boldsymbol{\Omega}}^{\boldsymbol{A}}_{\mathcal{S}}$ is a structural representation, for every $\boldsymbol{A}$ (Theorem 4.13). The same condition imposed only over the formulas algebra $\boldsymbol{Fm}$ turns out to characterize truth definability in the class $\mathsf{LMod}^{\mathrm{Su}}(\mathcal{S})$ (Theorem 4.21), a problem left open in [**55**].

Chapter 5 was devoted to developing a general theory of the strong version $\mathcal{S}^{+}$ of a sentential logic $\mathcal{S}$. We payed special attention to the interplay between the Leibniz $\mathcal{S}$-filters and the $\mathcal{S}^{+}$-filters, namely by investigating several conditions upon which these two families of $\mathcal{S}$-filters coincide. Some of these conditions force moreover the classes of $\mathcal{S}$-algebras and $\mathcal{S}^{+}$-algebras to coincide. For as it turns out, the class $\mathsf{Alg}(\mathcal{S}^{+})$ may be strictly contained in the class $\mathsf{Alg}(\mathcal{S})$. This situation contrasts with the protoalgebraic scenario, where in general, for every protoalgebraic logic $\mathcal{S}$, $\mathsf{Alg}(\mathcal{S}) = \mathsf{Alg}^{*}(\mathcal{S}) = \mathsf{Alg}^{*}(\mathcal{S}^{+}) = \mathsf{Alg}(\mathcal{S}^{+})$.

In Chapter 6 we considered three definability criteria for the Leibniz $\mathcal{S}$-filters — equational, explicit, and logical. The first one is a new criterion, while the latter two are generalizations to arbitrary sentential logics of the respective notions introduced for protoalgebraic logics in [**37**]. Table 3 summarizes the situation for the examples covered in Chapter 7.

| | **Definability of Leibniz filters** | | |
|---|---|---|---|
| | **Equational** | **Explicit** | **Logical** |
| $\mathcal{PML}$ | Yes | Yes | Yes |
| $\mathcal{B}$ | Yes | No | Yes |
| $\mathcal{S}^{\leq}_{\mathsf{WH}} = w\mathcal{K}_{\sigma}$ | Yes | Yes | Yes |
| $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{RT})}}$ | Yes | Yes | Yes |
| $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{N})}} = \mathcal{VPL}$ | Yes | Yes | Yes |
| $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{MP})}}$ | Yes | Yes | Yes |
| $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ | Yes | No | Yes |
| $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ | Yes | No | Yes |
| $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ | Yes | No | Yes |

TABLE 3. Leibniz filters' definability criteria of the logics covered in Chapter 7

In Chapter 7 we applied the general results of Chapters 5 and 6 to a wealth of non-protoalgebraic logics covered in the literature. Namely, Positive Modal Logic

[28], Belnap's logic [8], some subintuitionistic logics studied in [16], logics preserving degrees of truth w.r.t. varieties of integral commutative residuated lattices [17], as well as two logics not previously considered (at least to our knowledge), which are intermediate logics between the semilattice-based logics and the algebraizable logics usually associated with the class of commutative residuated lattices and commutative residuated lattices without multiplicative constant, respectively. For each particular logic $\mathcal{S}$, we have characterized its Leibniz and Suszko $\mathcal{S}$-filters, as well as determined its strong version $\mathcal{S}^+$. Both $\mathcal{S}$ and $\mathcal{S}^+$ were classified inside the Leibniz and Frege hierarchies. We summarize the situation in Tables 4 and 5.

| | **Leibniz hierarchy** | | **Frege hierarchy** | | | |
|---|---|---|---|---|---|---|
| | **Proto.** | **Truth-eq.** | **Fregean** | **Self.** | **Fully Self.** | **Fully Freg.** |
| $\mathcal{PML}$ | No | No | No | Yes | Yes | No |
| $\mathcal{B}$ | No | No | No | Yes | Yes | No |
| $w\mathcal{K}_\sigma$ | No | No | No | Yes | Yes | No |
| $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{RT})}}$ | No | No | No | Yes | Yes | No |
| $\mathcal{VPL}$ | No | Yes | Yes | Yes | Yes | Yes |
| $\mathcal{S}^{\leq}_{\mathsf{WH}_{(\mathrm{MP})}}$ | Yes | No | No | Yes | Yes | No |
| $\mathcal{S}^{\leq}_{\mathsf{CIRL}}$ | No | No | No | Yes | Yes | No |
| $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ | No | No | No | No | No | No |
| $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ | No | No | No | No | No | No |

TABLE 4. Classification of the logics covered in Chapter 7 inside the Leibniz and Frege hierarchies.

| | **Leibniz hierarchy** | | | **Frege hierarchy** |
|---|---|---|---|---|
| | **Proto.** | **Truth-eq.** | **BP-algebraizable** | **Selfextensional** |
| $\mathcal{PML}^+$ | No | Yes | No | No |
| $\mathcal{B}^+$ | No | Yes | No | No |
| $w\mathcal{K}_\sigma{}^+$ | No | Yes | No | No |
| $\mathcal{S}^{\top}_{\mathsf{WH}_{(\mathrm{RT})}}$ | No | Yes | No | No |
| $\mathcal{VPL}$ | No | Yes | Yes | Yes |
| $\mathcal{S}^{\top}_{\mathsf{WH}_{(\mathrm{MP})}}$ | Yes | Yes | Yes | No |
| $\mathcal{S}^{1}_{\mathsf{CIRL}}$ | Yes | Yes | Yes | No |
| $\mathcal{S}^{\mathcal{T}}_{\mathsf{CRL}}$ | Yes | Yes | Yes | No |
| $\mathcal{S}^{\mathcal{T}}_{\mathsf{CRLr}}$ | Yes | Yes | Yes | No |

TABLE 5. Classification of the strong versions of the logics covered in Chapter 7 inside the Leibniz and Frege hierarchies.

It should come with no surprise that the vast majority of the logics studied is neither protoalgebraic nor truth-equational. In fact, these two conditions were *a priori* requesites for their study in the first place. Also, observe that the logics $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRL}}$ and $\mathcal{S}^{\preccurlyeq}_{\mathsf{CRLr}}$ fall outside both hierarchies. Interestingly enough, all the strong versions studied turned out to be truth-equational (altough this is not a general fact, as observed on page 122). It is worth adding that the strong versions which are not protoalgebraic (namely, $\mathcal{PML}^{+}$, $\mathcal{B}^{+}$, $w\mathcal{K}_{\sigma}{}^{+}$, and $\mathcal{S}^{1}_{\mathsf{WH}_{(\mathrm{RT})}}$) constitute new examples of “strictly” truth-equational logics. As for the Frege hierarchy, appart from Visser’s logic, all the strong versions studied fall outside the Frege hierarchy.

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