UNIVERSITAT DE BARCELONA

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Certifico que la següent tesi ha estat realitzada per en Rubén Berenguel Montoro sota la meva codirecció
Barcelona, 30 de setembre de 2015

Ernest Fontich Julià

Certifico que la següent tesi ha estat realitzada per en Rubén Berenguel Montoro sota la meva codirecció
Barcelona, 30 de setembre de 2015

Núria Fagella Rabionet

‘Beauty is truth, truth beauty,—that is all
Ye know on earth, and all ye need to know.’

John Keats (1795-1821) *Ode on a Grecian urn*

Acknowledgements

My interest in dynamical systems can be traced back to my undergraduate studies, when I quit from the course *Group Theory* and instead enrolled in *Dynamical Systems*. This is where I started to enjoy the fun (and despair) tied to its study. The course was taught by whom would be my advisor in a few years, Ernest Fontich, handling the theoretical aspects and Immaculada Baldomà, in charge of more applied classes. I still remember clearly how Imma tried to convince me to *not* enroll in any PhD program, just as the course was finishing (even though I was still one year away from graduating.) Ignoring her advice, not only did I start a PhD, but I chose the same advisor she had.

My first contact with holomorphic dynamics happened a year before, and wasn't as brilliant. The last few classes of *Complex Analysis I* had a brief introduction to holomorphic dynamics, and of course I found them interesting. But I didn't prepare enough and got a failing grade. The next semester I attended a few classes to prepare for the second try, and this is where I met my other advisor, Núria Fagella. She was teaching the class, and I managed to get a barely passing grade. Since I'm stubborn, as soon as I could I enrolled in the optional *Complex Analysis II*, where I did way better. And not happy with that, holomorphic dynamics is now part of my PhD dissertation.

Like the song said, the road here was long and winding. After two years, it was clear that my original research project lead nowhere, and I changed my thesis subject from a technique to study separatrix splitting to dynamics on lattices with decay. Those years were not wasted though, because (aside from everything I learnt about dynamics) during that time I did most of the research that forms the holomorphic dynamics part of this thesis.

Life takes many turns, and so far looks like mathematics will not be my main job, but one never knows and I still have a few papers to publish, and a few things I want to prove before I call it quits. And I could always parametrise shoe uppers for The Fancy Puffin.

First and foremost, thanks to my advisors, Núria Fagella and Ernest Fontich for their continued support, help and time invested in this work. Without you this would not have been possible, and without your attention to detail no-one would be able to decipher anything from these pages. Everything that makes sense is thanks to you, everything that seems weird is my only fault.

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Thanks to all the friends that shared in one way or another this road with me.

Thanks to my parents, who always pushed me to reach as high as I could, and helped me as much as they could.

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Ruben Berenguel
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Contents

Preface

In this thesis we consider two different problems in the theory of dynamical systems. Dynamical systems cover a wide array of subjects, from finite dimensional to infinite dimensional, from analytic to statistical viewpoints and through all gradations in-between. No matter the aspect or tool considered, the study of any dynamical system is concerned in some way or another with the evolution of points through the action of a map

$$x \mapsto F(x) \mapsto F(F(x)) \mapsto F(F(F(x))) \mapsto \cdots .$$

The simplest question to ask of a dynamical system is then *which points are invariant*? Once we have an answer to this question we can proceed to study the dynamics in a neighborhood of them. In general we find invariant subsets containing the fixed point which provide very relevant information. Formally a fixed point satisfies

$$x_0 \mapsto F(x_0) = x_0.$$

Under some sufficient conditions on the derivative of F, the invariant object has associated *invariant manifolds*. The fundamental example is here a hyperbolic fixed point, x_0, which has stable (we can denote it by W^s) and unstable (W^u) invariant manifolds associated to it. In this case, the iterates of the points on W^s are always on W^s, and get closer to x_0 as the map is iterated. The unstable manifold W^u works in reverse, getting further away from x_0 as iterates increase. A fundamental result in the study of dynamical systems is that under some suitable non-degeneracy conditions, slight perturbations of F preserve these dynamical properties (the fixed object, hyperbolicity, associated invariant manifolds.)

Real-life dynamical systems can be modeled in a multitude of ways. Some of them can be thought of as being an ensemble of really simple systems, together with a mechanism to propagate changes from one to another. For instance, the human brain can be considered as a dynamical system formed by several hundred billion variables (one for each neuron,) with complex interactions and dynamics among them. Or instead as the ensemble of several hundred billion simple dynamical systems (one for each neuron, or in a general setting, one for each *node*) and a mechanism of interaction between them. The mathematical generalisation is the study of systems in infinite dimensional lattices.

In Part I of this thesis we will study dynamical systems which are skew-products of a lattice and a torus and are perturbations of systems which are uncoupled, with a hyperbolic fixed point in the projection on the lattice. In other words, we will study systems of the form

$$(x, \theta) \mapsto (F(x, \theta), \theta + \omega), \quad x \in \ell^\infty(\mathbb{R}^n),\ \theta \in \mathbb{T}^d,$$

which are perturbations of systems having $\{0\} \times \mathbb{T}^d$ as an invariant hyperbolic torus. The interactions between nodes are governed through a *decay function*, which ensures that further away nodes have increasingly smaller influence.

We will study the persistence of the invariant torus under perturbation and the persistence of its invariant manifolds. The approach we will follow is the *parametrisation method*, which allows us to determine invariant manifolds associated to the torus, as well as to study its decay properties and obtain sharp results on the regularity of the manifolds.

We will also study normal forms of maps with decay in lattices around fixed points, and as an application we will prove Sternberg theorems for attractors in lattices.

In Part II we will turn to a special case of 1-dimensional dynamics, namely holomorphic dynamics. This field is concerned with the study of the iteration of holomorphic maps in $\mathbb{C}$, and its origin can be traced to another kind of fundamental question: *when does the Newton method converge for complex polynomials*? Observe that the question can be thought as *for which points* or *for which polynomials*. This dichotomy appears when considering dynamical systems depending on one parameter: we can talk about the dynamics *for a specific parameter* in the dynamical plane or we can talk about the *general properties of a set of parameters* in the parameter plane.

The study of holomorphic dynamics is also grounded in the study of invariant objects, the foremost being the Fatou and Julia sets. They are complementary sets in the dynamical plane, the Fatou set being an open set formed of open components where orbits behave similarly and the Julia set being a closed set, which defines where orbits stop behaving similarly. A similar dichotomy occurs in the parameter plane, where different behaviors of parameters are separated by a bifurcation set.

In the dynamical plane the Fatou set can in turn be decomposed into several kinds of open sets, depending on the behavior of orbits. For instance, an attracting fixed point would induce an open Fatou component, containing its whole attraction basin. There is a fundamental difference between real (or just higher-dimensional) dynamics and holomorphic dynamics: singularities of the map play a very important role in splitting the Fatou set into smaller invariant sets. Namely, each invariant open set of the Fatou set has "associated" a singularity of the map. Thus, if the map has 1 singularity, the Fatou set will have one basic component. Hence, as the number of singularities increases, the number of dynamically different components increases. And when studying families of functions, the interplay between singularities and Fatou components gets more complicated as their numbers rise. Recall that holomorphic functions have two kind of singularities: critical points, where the derivative vanishes, and asymptotic values, where the inverse is undefined because of the presence of an essential singularity.

In the parameter plane different components appear for similarly behaving parameters, usually meaning parameters whose Fatou set has a specific property. For instance, having an attracting orbit of a specific period could form a component in parameter plane.

Thus, the most basic functions to study are polynomials (because the are the simplest holomorphic functions in many ways) with one critical value, in other words, studying the family $P_c(z) = z^2 + c$. The parameter plane of such a polynomial exhibits a widely known bifurcation set: the *Mandelbrot set* (the bifurcation set is the boundary of the Mandelbrot set, though.) The next step in complexity is studying an entire transcendental function with one asymptotic value. Or in other words, studying the family $E_\lambda(z) = \lambda e^z$. These two families exhibit incredibly intricate parameter spaces and bifurcation sets, and play a fundamental role in the study of more complicated holomorphic dynamical systems.

A natural next step would then be the study of families having two singularities, but there is an intermediate step. Since singular values are always related to specific Fatou components we can study families whose Fatou sets have a persistent component. This means that one of the singular values is always tied to this component. For instance, studying polynomials of degree 3 such that there is always a Siegel disc around 0. This fixes one singular value to "handle" the Siegel disc and leaves one free to generate another Fatou component. A natural extension is then studying a family of entire transcendental functions with a persistent Siegel disc. One such a family is

$$f_a(z) = \lambda a(e^{z/a}(z+1-a) - 1 + a), \quad a \in \mathbb{C} \setminus \{0\}, \quad \lambda = e^{2\pi i\theta}.$$

This family is a good model, in the sense that any other function exhibiting these properties (a persistent Siegel disc, one critical value, one asymptotic value, finite degree) is conjugate to a map of this form, and thus models a wide array of holomorphic families in the simplest possible way. It also includes as extreme cases the exponential family $u \mapsto \lambda(e^u - 1)$ when $a \to 0$, a polynomial-like function of degree 2 when $a \to \infty$ and the family $u \mapsto \lambda u e^u$ when $a = 1$.

In this work we will study the dynamics of f_a and its parameter plane, where we will describe its main stable components as well as parametrise some of them through the techniques of quasiconformal surgery. We will also show several topological properties about the open components in the parameter plane and about the boundaries of the Siegel discs appearing in this family. We will also prove a general result, useful for generating images of bifurcation sets that can be applied to families with one parameter, and thus can be applied to all the families introduced above.

Resum en Català

En aquest treball considerem dos problemes en la teoria dels sistemes dinàmics. El camp dels sistemes dinàmics abarca un ampli espectre de temes, des de sistemes finit dimensionals a infinit dimensionals, des de punts de vista analítics a estadístics, amb totes les possibles gradacions intermitges. Obviant l'aspecte o eina considerats, l'estudi de qualsevol sistema dinàmic es centra, d'una manera o una altra, en l'estudi de l'evolució de punts sota l'acció d'una aplicació

$$x \mapsto F(x) \mapsto F(F(x)) \mapsto F(F(F(x))) \mapsto \cdots .$$

La pregunta més simple que podem fer-li a un sistema dinàmic és llavors *quins punts són invariants*? Un cop en tenim una resposta podem passar a estudiar la dinàmica en un entorn d'ells. En general, hi trobem conjunts invariants que contenen els punts fixos, i que ens proveeixen d'informació molt rellevant. Formalment, un punt fix satisfà

$$x_0 \mapsto F(x_0) = x_0.$$

Requerint certes condicions suficients a la derivada de F en $\mathcal{M}$, el conjunt invariant té associades unes *varietats invariants*. L'exemple fonamental seria un punt fix hiperbòlic, x_0, que té associades una varietat estable (que podem anomenar W^s) i una varietat inestable (W^u). Els iterats dels punts de W^s sempre estan a W^s, i s'apropen a x_0 a mesura que iterem l'aplicació. La varietat inestable W^u es comporta a l'inrevés, allunyant-se de x_0 en iterar. Un resultat fonamental en l'estudi dels sistemes dinàmics és el fet que, sota certes condicions de no-degeneració petites pertorbacions de F conserven aquestes propietats dinàmiques (l'objecte invariant, l'hiperbolicitat, les varietats associades.)

Els sistemes dinàmics al món real es poden modelar de moltes maneres. Alguns d'ells es poden considerar com un gran conjunt de sistemes senzills, juntament amb un mètode per propagar canvis d'un sistema a un altre. Per exemple, el cervell humà es pot considerar un sistema dinàmic de bilions de variables (una per cada neurona,) amb complexos lligams entre les variables. O bé el podem considerar com un conjunt de bilions de sistemes dinàmics simples (un per cada neurona, o bé en general, un per cada *node*) juntament amb un mecanisme per propagar les interaccions. La generalització matemàtica és l'estudi de sistemes dinàmics en reticles infinit dimensionals.

A la primera part d'aquesta tesi estudiarem sistemes dinàmics formats per un skew-product d'un reticle infinit i un tor, i que són pertorbacions de sistemes desacoblats en els que la projecció sobre el reticle té punts fixos hiperbòlics. En altres paraules, estudiarem sistemes del tipus

$$(x, \theta) \mapsto (F(x, \theta), \theta + \omega), \quad x \in \ell^\infty(\mathbb{R}^n),\ \theta \in \mathbb{T}^d,$$

que són pertorbacions de sistemes amb $\{0\} \times \mathbb{T}^d$ com a tor hiperbòlic invariant. La interacció entre els nodes està controlada a través d'una *funció de decaïment*, que assegura que com més llunyà és un node, menor és la seva influència.

Estudiarem la persistència del tor invariant sota pertorbació i la persistència de les seves varietats invariants. El mètode que emprarem és el *mètode de la parametrització*, que ens permetrà determinar les varietats així com estudiar-ne les propietats de decaïment i obtenir resultats fins de regularitat. Més concretament, començarem establint les definicions de funció de decaïment i el sistema a estudiar al Capítol 2. Tot seguit establirem les propietats dels espais de funcions en què treballem, primer definint aplicacions lineals amb decaïment al Capítol 3, després espais de funcions diferenciables amb decaïment al Capítol 4 per acabar amb funcions amb diferenciabilitat anisotròpica al Capítol 5. Tot seguit introduirem el concepte de Γ-espectre i conceptes bàsics de teoria espectral en espais de Banach. Un cop establerts aquests preliminars, determinarem la regularitat i decaïment del tor en el sistema pertorbat al Capítol 7. Continuarem determinant la part lineal de la dinàmica local en les varietats invariants al Capítol 8, per a tot seguit estudiar termes d'ordre superior en varietats fortament estables al Capítol 9, on trobarem resultats de diferenciabilitat fins. Finalment, als Capítols 10 i 11 acabarem la parametrització de les varietats no ressonants, determinant-ne la seva regularitat i decaïment. Per acabar, al Capítol 12 donarem resultats sobre formes normals i probarem els teoremes d'Sternberg en reticles amb decaïment.

També estudiarem formes normals d'aplicacions amb decaïment en reticles al voltant de punts fixos, i com a resultat provarem teoremes d'Sternberg per atractors en reticles.

A la segona part d'aquesta tesi estudiarem dinàmica 1-dimensional, en concret dinàmica holomorfa. Aquest és un camp focalitzat en l'estudi de la dinàmica de funcions holomorfes en $\mathbb{C}$, i el seu origen es pot traçar a un altre tipus de pregunta fonamental: *quan convergeix el mètode de Newton per a polinomis complexos*? Fixem-nos que aquesta pregunta té dos vessants, *per a quins punts* o bé *per a quins polinomis*. Aquesta dicotomia és freqüent quan considerem sistemes dinàmics depenents de paràmetres: podem parlar sobre la dinàmica *per a un paràmetre específic* en el pla dinàmic o bé podem parlar sobre *propietats generals d'un conjunt de paràmetres* en el pla de paràmetres.

L'estudi de la dinàmica holomorfa també es fonamenta en l'estudi d'objectes invariants, els més destacats serien els conjunts de Fatou i Julia. Són conjunts complementaris en el pla dinàmic, el conjunt de Fatou és un obert format per components on les òrbites es comporten de manera similar, en canvi el conjunt de Julia és un conjunt tancat, i defineix on les òrbites deixen de comportar-se de manera similar. Una dicotomia similar apareix en el pla de paràmetres, on comportaments diferents estan separats per un conjunt de bifurcació.

En el pla dinàmic, el conjunt de Fatou es pot descomposar en diferents tipus de conjunts oberts, en funció del comportament de les òrbites. Per exemple, un punt fix atractor formaria un obert en el conjunt de Fatou format per la conca d'atracció. Hi ha una diferència fonamental entre dinàmica real (o simplement, dinàmica en dimensions superiors) i la dinàmica holomorfa: les singularitats de l'aplicació tenen un paper destacat en separar el conjunt de Fatou en conjunts invariants més senzills. Concretament, cada obert invariant del conjunt de Fatou té associat una singularitat de l'aplicació. Així, si l'aplicació té 1 singularitat, el conjunt de Fatou tindrà una component. Per tant, quan augmenta el nombre de singularitats, el nombre de components amb comportaments dinàmics diferents creix. D'aquesta manera quan estudiem famílies de funcions depenents d'un paràmetre les interaccions entre singularitats serà més complicat com major sigui el nombre de singularitats lliures. Recordem que les funcions holomorfes tenen dos tipus de singularitats: punts crítics, on la derivada és zero i valors asimptòtics, on la inversa no és ben definida per la presència d'una singularitat essencial.

En el pla de paràmetres apareixeràn diferents components per paràmetres amb comportaments similars, normalment això implica que són paràmetres on el conjunt de Fatou té una certa propietat. Per exemple, tenir una òrbita atractora d'un cert període podria formar una component en el pla de paràmetres.

Així, les funcions més simples a estudiar serien polinomis (les funcions holomorfes més simples) amb un valor crític, en altres paraules, estudiar la família $P_c(z) = z^2 + c$. El pla de paràmetres d'aquesta família mostra un conjunt de bifurcació àmpliament conegut: *el conjunt de Mandelbrot* (de fet el conjunt de bifurcació és la frontera del conjunt de Mandelbrot.) El següent nivell en complexitat seria l'estudi de funcions enteres transcendents amb un valor asimptòtic. En altres paraules, l'estudi de la família $E_\lambda(z) = \lambda e^z$. Aquestes dos famílies mostren un intrincat pla de paràmetres i complexos conjunts de bifurcacions, i tenen un paper fonamental en l'estudi de sistemes dinàmics holomorfs més complexos.

Un pas següent natural seria doncs estudiar famílies amb dos singularitats, però hi ha un pas intermig. Com els valors singulars sempre estàn lligats a una component de Fatou, podem estudiar famílies de funcions amb una component persistent. Això voldrà dir que un dels valors singulars sempre estarà lligat a aquesta component. Per exemple, podriem estudiar polinomis de grau 3 tals que sempre hi ha un disc de Siegel al voltant del 0. Així fixem un valor crític, que s'haurà d'encarregar del disc i deixarà l'altre lliure per crear una altra component de Fatou. Una complicació natural serà doncs l'estudi d'una família entera transcendent amb un disc de Siegel persistent. Una família així té la forma

$$f_a(z) = \lambda a(e^{z/a}(z + 1 - a) - 1 + a), \quad a \in \mathbb{C} \setminus \{0\} \quad \lambda = e^{2\pi i\theta}.$$

Aquesta família és única (fet que provarem al Capítol 15), en el sentit que qualsevol altra família de funcions amb les mateixes propietats (disc de Siegel persistent, un valor crític, un valor asimptòtic, grau finit) tindrà aquesta mateixa forma i per tant aquesta família modela de la manera més simple una gran quantitat de funcions. També inclou com casos extrems la família exponencial $u \mapsto \lambda(e^u - 1)$ quan $a \to 0$, una funció polynomial-like de grau 2 quan $a \to \infty$ i la família $u \mapsto \lambda u e^u$ quan $a = 1$.

En aquesta part estudiarem la dinàmica d'aquesta família al Capítol 15, així com el seu pla de paràmetres, on descriurem les seves components obertes i parametritzarem algunes d'elles amb la tècnica de la cirurgia quasiconforme, als Capítols 16 i 17. També donarem diversos resultats topològics sobre aquestes components en el pla de paràmetres en aquests dos capítols, i sobre les fronteres dels discs de Siegel que apareixen en aquesta família en el Capítol 18. També provarem un resultat general, útil per generar imatges de conjunts de bifurcació, que es pot aplicar a famílies depenents d'un paràmetre i per tant es pot aplicar a totes les famílies esmentades anteriorment. Aquest resultat es presenta al Capítol 19.

Part I

The Parametrisation Method for Invariant Manifolds of Tori in Skew-Product Lattices

Introduction

In this part we deal with the study of some (invariant) objects and the associated tools and methods related to the dynamics of lattice systems. The origin of the study of lattice systems is found in the first models of the dynamics of chains of particles under the action of some potential, with an interaction to nearest neighbours, models which were first considered by Prandtl (see [Pra28]) and Dehlinger (see [Deh29]). Later these models were also considered by Frenkel and Kontorova for specific cases (see [FK38a], [FK38b], [FK38c] and [FK39]). For one dimensional lattices, chains of particles can be described by a (formal) Hamiltonian:

$$H(p,q) = \sum_{i\in\mathbb{Z}} \left(\frac{1}{2}\|p_i\|^2 + V(q_i) \right) + \sum_{i\in\mathbb{Z}} W(q_{i+1} - q_i), \tag{1.1}$$

where $(q_i, p_i) \in \mathbb{R}^{2n}$ are the position-momentum variables of the i-th particle, $i \in \mathbb{Z}$, V is the potential acting over the i-th particle (which is the same for all particles) and W is the interaction potential with the nearest particle. Although the sums are infinite, the equations of motion are well defined, with the form:

$$\begin{aligned} \dot{q}_i &= p_i \\ \dot{p}_i &= -\nabla V(q_i) + \nabla W(q_{i+1} - q_i) - \nabla W(q_i - q_{i-1}), \quad i \in \mathbb{Z}^d, \end{aligned}$$

or equivalently

$$\ddot{q}_i + \nabla V(q_i) = \nabla W(q_{i+1} - q_i) - \nabla W(q_i - q_{i-1}), \quad i \in \mathbb{Z}.$$

More generally one can consider higher dimensional lattices, that is, consider $i \in \mathbb{Z}^m$ with $m > 1$ and also interactions of every particle with all the others. In this case we should write the second term in (1.1) in the form

$$\sum_{i\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} W_k(q_{i+k} - q_i),$$

where W_k is the potential of the interaction between particles separated k positions.

In this case we have to ask for some decay in the strength of the interaction because, as it is physically natural, the larger the separation between particles is, the smaller the force of interaction should be. Observe that we could also consider $i, k \in \mathbb{Z}^m$.

It is worth mentioning that there is a methodology to convert these systems to their continuous limit, yielding a partial differential equation. In particular, when $V(q) = 1-\cos q$

and $W(q_{i+1}-q_i) = C(q_{i+1}-q_i)^2$ one obtains (after normalisation and ignoring discretisation effects) the Sine-Gordon PDE:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin u = 0.$$

Several problems can be modeled under the Frenkel-Kontorova model or some generalisation, ranging from chains of coupled pendula, dislocation dynamics and surface physics to DNA dynamics. See [BK04] for the description of many applications of the model.

In this work we are concerned with the study of invariant tori and their invariant manifolds, taking into account their decay properties. We consider not only the stable manifold but also the strong stable and more generally non-resonant manifolds (see [CFdlL03a], [CFdlL03b], [CFdlL05], [HdlL06a] and [HdlL06b] for the description of these manifolds in different settings.)

The first versions of invariant manifold theory are found in the works of Poincaré and Lyapunov. In [Poi90] we find convergent series for the manifold of a fixed point (say 0) of a map F associated with a simple eigenvalue λ, with $|\lambda| < 1$, such that there are no eigenvalues of $DF(0)$ which are powers of λ. In [Poi99] we find asymptotic expansions of the solutions around periodic solutions of periodic vector fields. In [Lya92] we find similar results to those of Poincaré (with great differences of style and methods of proof.) In [Lef77] we can find a modern presentation of the work of Lyapunov.

There is a huge amount of literature dealing with invariant manifolds. Among the most classical works we can mention [HP70], [HPS77], [Fen72], [Irw70].

In this work we use the *parametrisation method*. Roughly, this method, when applied to finding invariant manifolds of a fixed point p of a map $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$, consists of looking for immersions $K : U \subset \mathbb{R}^m \to \mathbb{R}^n, 1 \le m \le n$, such that $K(0) = p$ and

$$F \circ K = K \circ R, \tag{1.2}$$

where $R : V \subset \mathbb{R}^m \to \mathbb{R}^m$ is the restriction of F to the manifold we are looking for, expressed in the new variables (the variables of K) which is also an unknown.

Assuming differentiability of the map F, condition (1.2) implies

$$DF(p)DK(0) = DK(0)DR(0).$$

We can see that the image of $DK(0)$ has to be invariant under $DF(p)$, hence the image of the linear part of the manifolds we are looking for has to be a collection of invariant subspaces under the action of $DF(p)$. Thus we can try to look for a manifold associated to each invariant subspace of $DF(p)$, but not all of them have associated (nonlinear) invariant manifolds.

To find invariant manifolds associated to fixed points of vector fields $X : \mathbb{R}^n \to \mathbb{R}^n$ we look instead for immersions $K : U \subset \mathbb{R}^m \to \mathbb{R}^n$ such that $X \circ K = DK \cdot Y$ where Y is a vector field in $\mathbb{R}^m$ which is also unknown.

The papers [FR81] and [FG92] use this method for one-dimensional manifolds, specially in conjunction with numerical calculations. Simó in [Sim90] describes the method to compute stable and unstable invariant manifolds. In [BK98] the Taylor expansion of invariant manifolds for n-dimensional maps is computed, to approximate the manifolds numerically. The papers [CFdlL03a] and [CFdlL03b] systematically study non-resonant manifolds of fixed points of maps and vector fields in general Banach spaces obtaining sharp regularity regularity results for the parametrisation and deals with the dependence of parameters of the manifolds obtained in [CFdlL03a], respectively. Some papers deal with the existence of tori using the parametrisation method. In this case, given a map F on a manifold $\mathcal{M}$, one looks for

$$K : \mathbb{T}^d \to \mathcal{M},$$

a parametrisation of a d-dimensional torus carrying a quasi-periodic motion of rotation vector $\omega \in \mathbb{R}^d$ and satisfying the equation

$$F \circ K = K \circ T_\omega,$$

where $T_\omega(\theta) = \theta + \omega$.

Similarly given a vector field X on a manifold $\mathcal{M}$ we can determine a parametrisation $K : \mathbb{T}^d \to \mathcal{M}$ satisfying

$$X \circ K = DK \cdot \omega.$$

The papers [HdlL06a] and [HdlL06b] use the method to obtain invariant tori and their manifolds in skew-product systems. This method is able to deal with the small divisor problems that appear when determining tori in conservative systems (see [dlLGEJV05], [GEdlL08], [HdLL07] and [GEHdlL14]). In [FdlLS09] the method is used to find invariant tori and its invariant manifolds in finite-dimensional exact symplectic maps and flows and [FdlLS15] determine invariant tori and their invariant manifolds in infinite-dimensional coupled map lattices. In the forthcoming book [H$^+$] the method is applied to finding invariant manifolds of fixed points, invariant tori and their manifolds and normally hyperbolic manifolds. The results provide the theoretical estimates to be readily applicable in a concrete system and obtain validated numerical computations. In this book the power of the parametrisation method for effective computations becomes evident.

Dynamics on lattices appear in physics, biology and mathematics. They model infinite arrays of subystems (called *nodes* or *sites*) which interact among them. Hence, the evolution of each node depends on its state and also the states of all the others. Several surveys are devoted to the study of such systems which are referred to as *coupled map lattices*, *coupled oscillators*, *extended systems*, etc. These surveys study several mathematical aspects of their dynamics, like travelling waves, spatio-temporal chaos, fronts and invariant measures among others. See [Gal07], [BCC03], [MP03], [CF05], [Kan85], [FdlLS15], [Pey04], [BK97], [FP99], [FP04], [JdlL00] for different approaches to this kind of problems. Mathematical models of arrays of cells are also considered in neuroscience ([Hop86], [HI97], [BEFT05], [Izh07]).

Several papers are devoted to the study of hyperbolicity properties in lattice maps, [FdlLM11b] among others. In general they assume finite range interaction or an exponential decay of the interactions among nodes. In [JdlL00] the authors introduce a class of decay functions such that the linear maps which satisfy decay properties according to decay functions of this class have algebra properties which simplify the functional analysis neded in the proofs of the results.

The papers [FdlLM11a], [FdlLM11b] contain a functional analysis background to study certain objects as invariant manifolds modelling the phase space as an ℓ^∞ space. In particular hyperbolic sets and their invariant manifolds for maps with decay properties are considered and their properties are studied. In [FdlLS15] finite and infinite dimensional tori are obtained in lattices where each node is governed by a Hamiltonian system and each node is coupled to all others with some interactions satisfying decay properties. In these papers the spaces of functions considered are such that their derivatives satisfy a decay controlled by decay functions in the class introduced in [JdlL00].

In this thesis we consider lattices indexed by $\mathbb{Z}^m$ over $\ell^\infty(\mathbb{R}^n)$. Concretely, we consider maps $F : \ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d$ having a skew-product structure, that is,

$$(x, \theta) \mapsto (F(x, \theta), \theta)$$

where $\omega \in \mathbb{R}^d$ is a fixed frequency vector. We assume that F is a perturbation of an uncoupled map F_0, and that the perturbation F_1 at each node depends on the other nodes with an interaction whose strength is dominated by a decay function which satisfies some properties.

We study the linear and multilinear maps between ℓ^∞ spaces having decay properties according to some precise definition. From them we introduce C^r decay maps, which roughly means that their derivatives are multilinear maps with decay. These classes of maps were already introduced in [JdlL00], where some properties were already established. A deeper study is found in [FdlLM11a]. Here we present a more complete set of properties and provide more details.

Further we consider spaces of functions with different degrees of regularity with respect to spatial and angular variables. We call them functions with anisotropic differentiability. We also establish a set of properties of this class of functions to be used later within this work.

Writing $F = F_0 + F_1$ with $F_0(0) = 0$ and F_0, F_1 decomposed as the sum of their linear and nonlinear parts, namely

$$F_0(x) = M_0 x + N_0(x), \quad \text{and} \quad F_1(x, \theta) = M_1(\theta)x + N_1(x, \theta),$$

we have results which give, under some suitable hypotheses, the existence of invariant tori close to $\{0\} \times \mathbb{T}^d$, using the parametrisation method. This result gives different decay and regularity of the tori, assuming different decays and regularities for the map. See Chapters 3, 4 and 5 for the definitions of the corresponding function spaces.

Theorem 7.1. *Using the notation introduced above, consider the dynamical system determined by F and assume that $DF_0(0)$ is hyperbolic. Consider the functional equation*

$$F(W_0(\theta), \theta) = W_0(\theta + \omega). \tag{1.3}$$

(i) Assume $M_1 \in C^0\big(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))\big)$ and $N(x, \theta) \in C^{0,0}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with N Lipschitz with respect to x for all $\theta \in \mathbb{T}^d$ and assume $\|F_1\|_{C^0}$ and

$$\mathrm{Lip}_x(N) := \sup_{\theta \in \mathbb{T}^d} \mathrm{Lip}(N(\cdot, \theta))$$

are small enough. Then the functional equation (1.3) has a unique solution $W_0(\theta) \in C^0(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ close to 0.

(ii) Assume $F_0 \in C^t(U, \ell^\infty(\mathbb{R}^n))$, $F_1 \in C^{t,r}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $t \geq r+1$, $r \geq 0$ and $\|F_1\|_{C^{t,r}}$ small enough. Then the functional equation (1.3) has a solution $W_0 \in C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$. Since $C^{t,r}_\Gamma \subset C^{t,r}$, for $F \in C^{t,r}_\Gamma$ we also obtain a solution $W_0 \in C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ close to 0.

(iii) Assume $F_0 \in C^t(U, \ell^\infty(\mathbb{R}^n))$, $F_1 \in C^{t,r}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $t \geq r+2$, $r \geq 0$ and $\|F_1\|_{C^{t,r}_{j,\Gamma}}$ small enough. Then the functional equation (1.3) has a solution $W_0 \in S^r_{j,\Gamma}(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ close to 0.

To obtain the strong stable invariant manifolds and more generally, non-resonant manifolds associated to this invariant torus, first we have to obtain a vector bundle tangent to them, to have a good local coordinate system to work in. Before doing this though, we translate the torus to the origin through a θ-dependent translation.

The invariant bundle associated to a strong stable invariant manifold is determined in Proposition 8.1 and the bundle associated to a non-resonant manifold is obtained in Proposition 8.4.

Concerning the existence and regularity of a strong stable manifold of the invariant torus we have a first result for low degree of regularity.

Theorem 9.1. *Given a dynamical system $F(x, \theta)$ as defined in Section 2.1 and a splitting of $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$ as in Chapter 9 we can determine the unique local strong stable manifold of W_0 tangent to $\mathcal{E}^1$ at 0 as the graph of $\varphi : \mathcal{E}^1 \times \mathbb{T}^d \to \mathcal{E}^2$ in the following regularity cases:*

(i) If $F(x, \theta) \in C^{1,0}\big(\ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)\big)$ and $\|F_1\|_{C^{1,0}}$ is small enough then $\varphi \in C^{1,0}(B(0,1) \times \mathbb{T}^d, \mathcal{E}^2)$.

(ii) If $F(x, \theta) \in C^{1,0}_\Gamma\big(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)\big)$ and $\|F_1\|_{C^{1,0}_\Gamma}$ is small enough then $\varphi \in C^{1,0}_\Gamma(B(0,1) \times \mathbb{T}^d, \mathcal{E}^2)$.

(iii) If $F(x, \theta) \in C^{1,0}_{j,\Gamma}\big(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)\big)$ and $\|F_1\|_{C^{1,0}_{j,\Gamma}}$ is small enough then $\varphi \in C^{1,0}_{j,\Gamma}(B(0,1) \times \mathbb{T}^d, \mathcal{E}^2)$.

This theorem considers a more general setting than Theorems 1.1 and 1.2 in [CFdlL03a] and [CFdlL03b], where the invariant manifolds of a fixed point are considered. Even though, if in Theorem 9.1 we assume that F does not depend on θ we have a comparable situation. In this case Theorem 9.1 covers the case of C^1 maps, which was not covered in [CFdlL03a] because the standing assumptions therein have a C^r map with $r \geq L+1$ and $L \geq 1$. To cover this case we have to deal with some nonlinear operators in function spaces whereas in [CFdlL03a] the analogous operators involved are linear, a technique which does not work directly for the C^1 case.

We have a result which deals with higher orders of regularity.

Theorem 9.14. *Given a dynamical system $F(x, \theta)$ as defined in Section 2.1, we can determine the unique local strong stable manifold of W_0 tangent to $\mathcal{E}^1$ at 0 as a graph $\varphi : \mathcal{E}^1 \times \mathbb{T}^d \to \mathcal{E}^2$ under the following regularity and decay assumptions:*

(i) If $F(x, \theta) \in C^{\Sigma_{s,r}}\big(\ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)\big)$ and $\|F_1\|_{C^{\Sigma_{s,r}}}$ is small enough then $\varphi \in C^{\Sigma_{s,r}}(B(0,1) \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, \mathcal{E}^2)$.

(ii) *If* $F(x,\theta) \in C_\Gamma^{\Sigma_{s,r}}\big(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)\big)$ *and* $\|F_1\|_{C_\Gamma^{\Sigma_{s,r}}}$ *is small enough then* $\varphi \in C_\Gamma^{\Sigma_{s,r}}(B(0,1)\times \mathbb{T}^d, \mathcal{E}^2)$.

(iii) *If* $F(x,\theta) \in C_{j,\Gamma}^{\Sigma_{s,r}}\big(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)\big)$ *and* $\|F_1\|_{C_{j,\Gamma}^{\Sigma_{s,r}}}$ *is small enough then* $\varphi \in C_{j,\Gamma}^{\Sigma_{s,r}}(B(0,1)\times \mathbb{T}^d, \mathcal{E}^2)$.

The standard local stable manifold of the torus is obtained as a particular case of Theorem 9.1. Also strong unstable manifolds as well as unstable manifolds are obtained applying Theorem 9.1 to the map F^{-1}.

As for non-resonant manifolds, the results are more involved since we have to first obtain sufficiently good polynomial approximations of the parametrisation of the manifolds. The main result is the following.

Theorem 10.1. *Let* U *be an open set of* $\ell^\infty(\mathbb{R}^n)$ *such that* $0 \in U$ *and consider a dynamical system* $F : U \times \mathbb{T}^d \subseteq \ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n)$, $F(x,\theta) = M(\theta)x + N_1(x,\theta)$ *with* $M(\theta) = M_0 + \widetilde{M}(\theta)$ *and*

$$M_0 = \begin{pmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{pmatrix}, \ \widetilde{M}(\theta) = \begin{pmatrix} B_{1,1}(\theta) & B_{1,2}(\theta) \\ 0 & B_{2,2}(\theta) \end{pmatrix}, \ M(\theta) = \begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ 0 & M_{2,2}(\theta) \end{pmatrix}.$$

Assume the following hypotheses,

(H1) $F \in C_\Gamma^{t,r}\big(\ell^\infty(\mathbb{R}^n)\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n)\big)$, M_0, $\widetilde{M}(\theta) \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $\sup_{\theta\in\mathbb{T}^d} \|\widetilde{M}(\theta)\|_\Gamma$ *is sufficiently small*

(H2) $\mathcal{A}\,\mathrm{Spec}_\Gamma(A_{1,1}) \subset \mathbb{D}\backslash\{0\}$,

(H3) $0 \notin \mathrm{Spec}_\Gamma(A_{2,2})$,

(H4) $\mathcal{A}\,\mathrm{Spec}_\Gamma(A_{1,1})^{L+1} \cdot \mathcal{A}\,\mathrm{Spec}_\Gamma(M_0^{-1}) \subset \mathbb{D}$,

(H5) $\mathcal{A}\,\mathrm{Spec}_\Gamma(A_{1,1})^i \cap \mathcal{A}\,\mathrm{Spec}_\Gamma(A_{2,2}) = \emptyset$ *for* $2 \le i \le L$,

(H6) $L + 1 \le t$.

Then

(a) *We can determine a polynomial bundle map* $R(s,\theta) : \mathcal{E}^1 \times \mathbb{T}^d \to \mathcal{E}^1$ *of degree not larger than* L *in* $C_\Gamma^{\infty,r}(\mathcal{E}_1 \times \mathbb{T}^d, \mathcal{E}_1)$ *such that* $R(0,\theta) = 0$, $D_sR(0,\theta) = M_{1,1}(\theta)$ *and a bundle map* $W : \mathcal{E}^1 \times \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n)$ *in* $C_\Gamma^{t,r}(\mathcal{E}_1 \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ *such that*

$$F(W(s,\theta),\theta) = W(R(s,\theta), \theta + \omega),$$

where $W(0,\theta) = 0$, $\Pi_{\mathcal{E}^1} D_s W(0,\theta) = \mathrm{Id}_{\mathcal{E}^1}$ *and* $\Pi_{\mathcal{E}^2} D_s W(0,\theta) = 0$.

(b) *Furthermore, if there is* $l \ge 2$ *such that*

$$\mathcal{A}\,\mathrm{Spec}(A_{1,1})^i \cap \mathcal{A}\,\mathrm{Spec}(A_{1,1}) = \emptyset, \quad l \le i \le L,$$

then we can choose R *to be a polynomial bundle map of degree not larger than* $l - 1$.

To obtain the mentioned polynomial approximations of the parametrisation of the invariant manifolds we have to solve "normal form" type of equations, which involve the use of the so-called Sylvester operators. These operators have to be inverted, hence we need to study spectral properties of them. These operators were studied in detail in the setting of Banach spaces in [CFdlL03a] but here, since we are interested in decay properties, we consider them in sections of spaces of k-linear maps in $\ell^\infty(\mathbb{R}^n)$ with decay.

We introduce the notion of Γ-spectrum for linear maps with decay and establish some spectral results for Sylvester operators in terms of it. The theory for the Γ-spectrum is developed in Chapter 6 and appears to be quite similar to the theory for the standard spectrum of linear operators, since we have been able to reproduce the majority of basic facts. To the best of our knowledge this is a new notion which appears to be very well suited for the kind of problems we need to solve in this work.

Finally, concerning the dynamics of lattices, we obtain Sternberg theorems for the conjugation of a map to its linear part or to its normal form, in the case that the linear part is a contraction (Poincaré domain.) Assuming decay properties for the map we obtain decay properties for the conjugating map. For the results where we allow the existence of resonances, we use a normal form theory with decay which we develop here (analogous to the standard normal form theory around a fixed point) and is based on the use of Sylvester operators in spaces of k-linear maps in $\ell^\infty(\mathbb{R}^n)$, introduced in Chapter 6, Section 6.7

We present two cases, the first one concerning maps that are small perturbations of an uncoupled map with equal dynamics in each node. For this class of maps we put conditions on the eigenvalues of the unperturbed map restricted to each node (all are the same). The main results for this case are the two following theorems. The first theorem determines a conjugation to the linear part of the map in the absence of resonances among eigenvalues.

Theorem 12.3. *Let U be an open set of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$. Let $F : U \to \ell^\infty(\mathbb{R}^n)$ be a C^r_Γ map of the form $F = F_0 + F_1$ where F_0 is an uncoupled map and $F_0(0) = F_1(0) = 0$. Let $A = DF_0(0)$, $B = DF_1(0)$ and $M = A+B$. Assume that $A_{ij} = \mathfrak{a}\delta_{ij}$ with $\mathfrak{a} \in L(\mathbb{R}^n, \mathbb{R}^n)$.*

Let $\operatorname{Spec}(\mathfrak{a}) = \{\lambda_1, \dots, \lambda_n\}$. Assume furthermore

(H1) $0 < |\lambda_i| < 1, 1 \le i \le n$,

(H2) $\lambda_i \neq \lambda^k$, $k \in \mathbb{Z}^m$, $|k| \ge 2$, $1 \le i \le n$.

Let $\alpha = \min_i |\lambda_i|$, $\beta = \max_i |\lambda_i|$, $\nu = \frac{\log \alpha}{\log \beta}$ and $r_0 = [\nu] + 1$. Then if $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ with $r \ge r_0$ and $\|B\|_\Gamma$ is small enough then there exists $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that $R(0) = 0$, $DR(0) = \mathrm{Id}$ and

$$R \circ F = MR$$

in some neighborhood $U_1 \subseteq U$ of 0 in $\ell^\infty(\mathbb{R}^n)$.

The second theorem is more general, and conjugates with a normal form in the presence of resonances (i.e. omitting Hypothesis (H2)).

Theorem 12.9. *Under the conditions and notation of Theorem 12.3 except hypothesis (H2), if $F \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ with $r \ge r_0$ and $\|B\|_\Gamma$ is small enough there exists a polynomial $C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ of degree not larger than r_0 and $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that*

$$R(0) = 0, \quad DR(0) = \mathrm{Id}$$

and

$$R \circ F = H \circ R$$

in some neighborhood $U_1 \subset U$*.*

Similar results (see Theorems 12.10 and 12.12) can be proved for maps which are not perturbations of uncoupled maps, requiring conditions over the Γ-spectrum of their linear part.

Observe that we could prove analogous results for flows in lattices using a time-one map argument.

Chapter 2

Lattices, decay functions and dynamical systems

In the first part of this work we will consider dynamical systems in the space of bounded sequences of points of $\mathbb{R}^n$ with indices in $\mathbb{Z}^m$. That is, we will work in the infinite product space $(\mathbb{R}^n)^{\mathbb{Z}^m}$. As usual we will call *node* to each individual $\mathbb{R}^n$. Associated with this space we will consider a decay function, which will control the strength of the interactions between different nodes.

The space of bounded sequences over the infinite product space $(\mathbb{R}^n)^{\mathbb{Z}^m}$ is denoted by $\ell^\infty(\mathbb{R}^n)$, formally defined as

$$\ell^\infty(\mathbb{R}^n) = \left\{ (x_i)_{i\in\mathbb{Z}^m} \,|\, x_i \in \mathbb{R}^n,\ \sup_{i\in\mathbb{Z}^m} \|x_i\| < \infty \right\},$$

where $\|\cdot\|$ is a given norm in $\mathbb{R}^n$. We endow $\ell^\infty(\mathbb{R}^n)$ with the norm $\|x\|_\infty = \sup_{i\in\mathbb{Z}^m} \|x_i\|$ as usual. Note that if we change the norm in $\mathbb{R}^n$ we end up with an equivalent norm in $\ell^\infty(\mathbb{R}^n)$. We denote by $\mathrm{proj}_i : \ell^\infty(\mathbb{R}^n) \to \mathbb{R}^n$ the projection onto the i-th component. We denote by $\mathrm{emb}_i : \mathbb{R}^n \to \ell^\infty(\mathbb{R}^n)$ the i-th embedding such that for every $u \in \mathbb{R}^n$, $\mathrm{proj}_j(\mathrm{emb}_i(u)) = 0$, $i \neq j$, and $\mathrm{proj}_i(\mathrm{emb}_i(u)) = u$. This embedding is an isometry if the norm in $\ell^\infty(\mathbb{R}^n)$ is induced by the norm considered in $\mathbb{R}^n$.

2.1 The dynamical system

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 mapping with a hyperbolic fixed point at 0. We can also consider maps defined in a neighbourhood A of 0 in $\mathbb{R}^n$. Observe that in the differentiable case we can always reduce the latter case to the former one. Indeed, by using cut-off functions of suitable differentiability we can extend f from a neighbourhood $\widetilde{A}$ of 0 such that $\overline{\widetilde{A}} \subsetneq A$ (i.e. a neighbourhood "smaller" than A) to $\mathbb{R}^n$ with the extension having the same regularity as f. For simplicity of notation in this introductory chapter we consider $f : \mathbb{R}^n \to \mathbb{R}^n$. However, later we will write $f : A \to \mathbb{R}^n$.

We will consider dynamical systems in $\mathbb{R}^n$ which have the form

$$F(x, \theta) = F_0(x) + F_1(x, \theta), \qquad x \in \ell^\infty(\mathbb{R}^n),\ \theta \in \mathbb{T}^d.$$

From now on we will consider F_0 as the unperturbed system and we will assume that F_1 is small in a suitable sense.

The unperturbed system F_0 will be an uncoupled system in $\ell^\infty(\mathbb{R}^n)$ such that each node $\mathbb{R}^n$ in the lattice has the same dynamics, given by $f : \mathbb{R}^n \to \mathbb{R}^n$, that is $F_0 : \ell^\infty(\mathbb{R}^n) \to \ell^\infty(\mathbb{R}^n)$ is such that

$$(F_0(x))_i = f(x_i).$$

We will write $F_0(x) = \big(f(x_i)\big)_{i\in\mathbb{Z}^m}$. We will assume quasiperiodic dependence of F with respect to θ. Therefore we can write the system as a skew product from $\ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d$ to itself defined by

$$(x, \theta) \mapsto (F(x, \theta), \theta + \omega).$$

The norm we will use in $\ell^\infty(\mathbb{R}^n)$ is the supremum norm, $\|\cdot\|_\infty$ i.e. $\|x\|_\infty = \sup_{i\in\mathbb{Z}^m} |x_i|$ where $|x_i|$ is a norm in $\mathbb{R}^n$. Later on we will specify concrete norms depending on the problem we deal with. In general the norms will be adapted norms to the decompositions of the spectrum of $D_x f(0)$ in $\mathbb{R}^n$.

For instance, the spectrum of $Df(0)$ can be decomposed into two sets of eigenvalues, namely contracting and expanding eigenvalues. According to this decomposition we can write $\mathbb{R}^n = E^s \oplus E^u$, where E^s is the subspace formed by the eigenspaces corresponding to the eigenvalues of modulus less than 1, and E^u the subspace formed by eigenspaces corresponding to eigenvalues of modulus larger than 1. We denote by π_s, $\pi_u : \mathbb{R}^n \to E^{s,u}$ the linear projections on these eigenspaces. According to this decomposition of $\mathbb{R}^n$ we can write:

$$Df(0) = \begin{pmatrix} \mathfrak{a}_{1,1} & 0 \\ 0 & \mathfrak{a}_{2,2} \end{pmatrix}, \tag{2.1}$$

where $\mathfrak{a}_{1,1}$ is the contracting part and $\mathfrak{a}_{2,2}$ the expanding one. In such case we would consider a norm adapted to both $\mathfrak{a}_{1,1}$ and $\mathfrak{a}_{2,2}$. Also, the splitting $E^s \oplus E^u$ induces a splitting in $\ell^\infty(\mathbb{R}^n)$. To study this we put ourselves in a more general setting where the splitting may be different at each node.

Given a family $(E_i)_{i\in\mathbb{Z}^m}$ of subspaces of $\mathbb{R}^n$ we define $\mathcal{E} \subset \ell^\infty(\mathbb{R}^n)$ by

$$v = (v_i)_{i\in\mathbb{Z}^m} \in \mathcal{E} \quad \Leftrightarrow \quad v_i \in E_i$$

and

$$\|v\|_\infty < \infty.$$

Given two families $(E_i)_{i\in\mathbb{Z}^m}$, $(G_i)_{i\in\mathbb{Z}^m}$ of subspaces of $\mathbb{R}^n$ such that $E_i \oplus G_i = \mathbb{R}^n$ we define $\mathcal{E}$ and $\mathcal{G}$ as above. Let $\Pi_i : \mathbb{R}^n \to E_i$ be projections such that

$$\operatorname{Im}\Pi_i = E_i$$
$$\operatorname{Ker}\Pi_i = G_i.$$

We have that $\widetilde{\Pi} = \operatorname{Id} - \Pi_i$ is a projection onto G_i.

Lemma 2.1. *We have that* $\sup_{i\in\mathbb{Z}^m} \|\Pi_i\| < \infty$ *if and only if* $\ell^\infty(\mathbb{R}^n) = \mathcal{E} \oplus \mathcal{G}$. *In such case, since* $\mathcal{E}$, $\mathcal{G}$ *are Banach spaces,* $\mathcal{E} \oplus \mathcal{G} \simeq \mathcal{E} \times \mathcal{G}$ *and the projections* $\Pi_{\mathcal{E}} : \ell^\infty(\mathbb{R}^n) \to \mathcal{E}$ *and* $\Pi_{\mathcal{G}} : \ell^\infty(\mathbb{R}^n) \to \mathcal{G}$ *defined by* $\Pi_{\mathcal{E}}(u) = (\Pi_i u_i)_{i\in\mathbb{Z}}$ *and* $\Pi_{\mathcal{G}}(u) = ((\operatorname{Id} - \Pi_i)u_i)_{i\in\mathbb{Z}}$ *are continuous. Furthermore,* $\|\Pi_{\mathcal{E}}\| = \sup_{i\in\mathbb{Z}^m} \|\Pi_i\|$.

Proof. Let $u = (u_i)_{i\in\mathbb{Z}^m} \in \ell^\infty(\mathbb{R}^n)$, then we can write $u_i = v_i + w_i$ with $v_i \in E_i$, $w_i \in G_i$. Let $v = (v_i)_{i\in\mathbb{Z}^m}$ and $w = (w_i)_{i\in\mathbb{Z}^m}$. As $v_i = \Pi_i u_i$,

$$\|v_i\| \le \|\Pi_i\|\|u_i\| \le C\|u\|_\infty$$

which implies that $v \in \mathcal{E}$. Since

$$\|\widetilde{\Pi_i}\| \le \|\operatorname{Id}\| + \|\Pi_i\| < 1 + \sup_{i\in\mathbb{Z}^m}\|\Pi_i\| < \infty$$

we get that $w \in \mathcal{G}$. Clearly $u = v + w$ and $\mathcal{E}\cap\mathcal{G} = \{0\}$.

The following easy computation

$$\|\Pi_{\mathcal{E}} u\| = \sup_{i\in\mathbb{Z}^m}\|\Pi_i u_i\| \le \sup_{i\in\mathbb{Z}^m}\|\Pi_i\|\|u_i\| \le \sup_{i\in\mathbb{Z}^m}\|\Pi_i\|\|u\|_\infty$$

proves that $\|\Pi_{\mathcal{E}}\| \le \sup_{i\in\mathbb{Z}^m}\|\Pi_i\|$. Now let $\varepsilon > 0$ and j_0 such that $\|\Pi_{j_0}\| > \sup_{i\in\mathbb{Z}^m}\|\Pi_i\| - \varepsilon/2$. Let $x_0 \in \mathbb{R}^n$ such that $\|x_0\| = 1$ and $\|\Pi_{j_0}x_0\| \ge \|\Pi_{j_0}\| - \varepsilon/2$. By the definition of norm,

$$\|\Pi_{\mathcal{E}}\| = \sup_{\|u\|=1}\|\Pi_{\mathcal{E}}u\| \ge \|\Pi_{\mathcal{E}}u_0\|,$$

where $\operatorname{proj}_{j_0}(u_0) = x_0$ and $\operatorname{proj}_i(u_0) = 0$ for $i \ne j_0$. Then

$$\|\Pi_{\mathcal{E}}\| \ge \|\Pi_{j_0}(x_0)\| \ge \|\Pi_{j_0}\| - \varepsilon/2 \ge \sup_{i\in\mathbb{Z}^m}\|\Pi_i\| - \varepsilon$$

which proves the other inequality.

For the reciprocal let $\Pi_{\mathcal{E}}u = v$. The continuity of $\Pi_{\mathcal{E}}$ implies that

$$\|v\| \le \|\Pi_{\mathcal{E}}\|\|u\|$$

and hence

$$\|v_i\| \le \|\Pi_{\mathcal{E}}\|\|u\|, \qquad i \in \mathbb{Z}^m.$$

Now

$$\|\Pi_i\| = \sup_{\|u_i\|\le 1}\|\Pi_i(u_i)\| = \sup_{\|u_i\|\le 1}\|v_i\| \le \|\Pi_{\mathcal{E}}\|$$

and the result is proved.

□

As an application of Lemma 2.1 we consider the following important example.

Let $\mathfrak{a} \in L(\mathbb{R}^n, \mathbb{R}^n)$ and assume that $\mathbb{R}^n$ has a splitting $\mathbb{R}^n = E \times G$ where $\mathfrak{a}$ is written as

$$\mathfrak{a} = \begin{pmatrix} \mathfrak{a}_{1,1} & 0 \\ 0 & \mathfrak{a}_{2,2} \end{pmatrix}.$$

This splitting may be the one associated to the stable and unstable parts of the spectrum of $\mathfrak{a}$ or any other splitting.

Let $M \in L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ be such that

$$M_{i,i} = \mathfrak{a}, \qquad M_{i,j} = 0, \quad j \ne i.$$

The sequence of subspaces $(E_i)_{i\in\mathbb{Z}^m}$, $(G_i)_{i\in\mathbb{Z}^m}$ with $E_i = E$ and $G_i = G$ define two subspaces $\mathcal{E}$ and $\mathcal{G}$ of $\ell^\infty(\mathbb{R}^n)$ and

$$\ell^\infty(\mathbb{R}^n) = \mathcal{E} \oplus \mathcal{G}.$$

This induces a representation of M of the form

$$M = \begin{pmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{pmatrix}$$

because if $x \in \ell^\infty(\mathbb{R}^n)$, $x = x^1 + x^2 \in \mathcal{E} \oplus \mathcal{G}$ and $x^1 = (x_i^1)_{i\in\mathbb{Z}^m}$, $x^2 = (x_i^2)_{i\in\mathbb{Z}^m}$, thus

$$Mx = Mx^1 + Mx^2 = (\mathfrak{a}x_i^1)_{i\in\mathbb{Z}^m} + (\mathfrak{a}x_i^2)_{i\in\mathbb{Z}^m} = (\mathfrak{a}_{1,1}x_1^1)_{i\in\mathbb{Z}^m} + (\mathfrak{a}_{2,2}x_1^2)_{i\in\mathbb{Z}^m}$$

and hence

$$A_{1,1}x^1 = (\mathfrak{a}_{1,1}x_i^1)_{i\in\mathbb{Z}^m}, \qquad A_{2,2}x^2 = (\mathfrak{a}_{2,2}x_i^2)_{i\in\mathbb{Z}^m}.$$

Assume for instance that $\operatorname{Spec}\mathfrak{a}_{1,1} \subset \{\lambda \mid |\lambda| < 1\}$ and $\operatorname{Spec}\mathfrak{a}_{2,2} \subset \{\lambda \mid |\lambda| > 1\}$. Then we have that M is a hyperbolic linear map. Thus we can consider an adapted norm to $\mathfrak{a}_{1,1}$, $\|\cdot\|_E$ in E, i.e. a norm such that $\|\mathfrak{a}_{1,1}\| < \alpha < 1$ and an adapted norm to $\mathfrak{a}_{2,2}^{-1}$, $\|\cdot\|_G$ in G, i.e., a norm such that $\|\mathfrak{a}_{2,2}^{-1}\| < \beta < 1$. In $\mathbb{R}^n$ we take $\|x\| = \max(\|x_1\|_E, \|x_2\|_G)$ if $x = x_1 + x_2 \in E \oplus G$. In $\ell^\infty(\mathbb{R}^n)$ we take the supremum norm with the norms introduced above in $\mathbb{R}^n$. Then

$$\begin{aligned} \|A_{1,1}\| = \|\, M|_{\mathcal{E}}\,\| &= \sup_{\substack{x\in\mathcal{E}\\ \|x\|\le 1}} \|Mx\| = \sup_{\substack{x\in\mathcal{E}\\ \|x\|\le 1}} \|(\mathfrak{a}_{1,1}x_i)_{i\in\mathbb{Z}^m}\| \\ &\le \sup_{\substack{x\in\mathcal{E}\\ \|x\|\le 1}} \sup_{i\in\mathbb{Z}^m} \|\mathfrak{a}_{1,1}\|\|x_i\| \le \|\mathfrak{a}_{1,1}\| < \alpha. \end{aligned}$$

Clearly, $M|_{\mathcal{G}}$ is invertible, a similar argument follows for $\|A_{2,2}^{-1}\|$ and we get $\|A_{2,2}^{-1}\| < \beta$.

If we have a gap in the splitting of the spectrum of $\mathfrak{a}$, we can write

$$\mathfrak{a} = \begin{pmatrix} \widetilde{\mathfrak{a}}_{1,1} & 0 \\ 0 & \widetilde{\mathfrak{a}}_{2,2} \end{pmatrix}$$

according to some decomposition $\mathbb{R}^n = \widetilde{E} \oplus \widetilde{G}$ with $\|\widetilde{\mathfrak{a}}_{1,1}\| < a$, $\|\widetilde{\mathfrak{a}}_{2,2}\| < b$ and $a < b^{-1}$, a completely analogous construction for the splitting in $\ell^\infty(\mathbb{R}^n)$ can be carried out.

Finally, in a decomposition $\ell^\infty(\mathbb{R}^n) = \mathcal{E} \oplus \mathcal{G}$ as above we will write

$$D_xF_1(0,\theta) = \begin{pmatrix} B_{1,1}(\theta) & B_{1,2}(\theta) \\ B_{2,1}(\theta) & B_{2,2}(\theta) \end{pmatrix}$$

and

$$D_xF(0,\theta) = M(\theta) = \begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ M_{2,1}(\theta) & M_{2,2}(\theta) \end{pmatrix}.$$

2.2 Decay functions in lattices

To be able to define meaningful localised perturbations in this product space, we consider an appropriate set of weighted Banach spaces. The main idea is that the coupling term in the perturbed system belongs to a weighted space, which controls the strength of the interaction between nodes. We should note that nearest-neighbour (or any other finite rank coupling, with similarly appropriate weights) coupling will satisfy these hypotheses.

We will make use of the following decay functions, originally introduced in [JdlL00].

Definition 2.2. *We say that a function* $\Gamma : \mathbb{Z}^m \to \mathbb{R}^+$ *is a decay function when it satisfies:*

1. $\sum_{k\in\mathbb{Z}^m} \Gamma(k) \leq 1$,
2. $\sum_{k\in\mathbb{Z}^m} \Gamma(i-k)\Gamma(k-j) \leq \Gamma(i-j), \quad i, j \in \mathbb{Z}^m$.

The first property ensures energy propagation related to such a decay function is finite, while the second property is akin to a triangular inequality in a discrete lattice. As pointed out by Prof. L. Sadun the second property says that the sum of the interactions between two nodes through the interactions involving third nodes is dominated by the direct interaction between them.

The following proposition can be found in [JdlL00] and provides a family of examples of decay functions satisfying Definition 2.2.

Proposition 2.3. *Given* $\alpha > m, \theta \geq 0$*, there exists* $a > 0$*, depending on* α, θ, m *such that the function defined by*

$$\Gamma(j) = \begin{cases} a, & j = 0, \\ a|j|^{-\alpha} e^{-\theta|j|}, & j \neq 0 \end{cases}$$

is a decay function on $\mathbb{Z}^m$*.*

Note that the standard exponential function $\Gamma(j) = Ce^{-\theta|j|}$ is not a decay function for any $C, \theta > 0$, as is proved in [JdlL00].

Chapter 3

Linear and multilinear maps with decay

To define C^r maps with decay properties between lattices we need to introduce spaces of linear and multilinear mappings with suitable decay properties. Then we can use these definitions to introduce spaces of C^r maps with these predefined decay properties for its derivatives. In this section we will define linear maps with decay and its related norm $\|\cdot\|_\Gamma$. From now on we will use $\|\cdot\|$ to denote the norm induced in the space of linear or multilinear maps by the same norm in $\ell^\infty(\mathbb{R}^n)$. We reproduce some results from [FdlLM11a] and provide more details and some additional results, mainly the proofs that certain spaces are Banach spaces.

3.1 The space of linear maps with decay

One of the simpler ways to define linear maps with decay is to ask for the components of "infinite matrices" to have decay properties with respect to their indices. Given $\|\cdot\|$ a norm in $\mathbb{R}^n$ we define the space of linear mappings with decay as a subset of the space of continuous linear mappings $L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ with a weighted norm depending on a decay function as follows:

$$L_\Gamma = L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) = \big\{A \in L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) \mid \|A\|_\Gamma < \infty\big\},$$

where

$$\|A\|_\Gamma = \max\{\|A\|, \gamma(A)\},$$

with $\|A\|$ the operator norm of A and

$$\gamma(A) = \sup_{i,k\in\mathbb{Z}^m} \sup_{\substack{\|u\|\leq 1,\\ \mathrm{proj}_j u=0, j\neq k}} \|(Au)_i\|\Gamma(i-k)^{-1}.$$

Remark 3.1. *We will use L_Γ for $L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. Although the definition has been written for $\ell^\infty(\mathbb{R}^n)$, we can define linear maps with decay among arbitrary vector subspaces of $\ell^\infty(\mathbb{R}^n)$, $\mathcal{E}, \mathcal{F} \subset \ell^\infty(\mathbb{R}^n)$ as*

$$L_\Gamma = L_\Gamma(\mathcal{E}, \mathcal{F}) = \big\{A \in L(\mathcal{E}, \mathcal{F}) \mid \|A\|_\Gamma < \infty\big\}.$$

Many results stated for $L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *extend straightforwardly to* $L_\Gamma(\mathcal{E}, \mathcal{F})$.

The following lemma formalises the introductory text of this section relating decay properties and "components" of infinite dimensional matrices. We define $A_{ij} = \mathrm{proj}_i A\, \mathrm{emb}_j$. Using this notation then

$$\gamma(A) = \sup_{i,k \in \mathbb{Z}^m} \|A_{ik}\| \Gamma(i-k)^{-1}.$$

Lemma 3.2. *Let* $A \in L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$*, and* $v = (v_i)_{i \in \mathbb{Z}^m} \in \ell^\infty(\mathbb{R}^n)$ *such that*

$$\lim_{|j| \to \infty} \|v_j\| = 0.$$

Then

$$(Av)_i = \sum_{k \in \mathbb{Z}^m} A_{ik} v_k.$$

Proof. Given $v \in \ell^\infty(\mathbb{R}^n)$, let $v^m \in \ell^\infty(\mathbb{R}^n)$, $m \geq 0$, be the vector defined by $v^m_k = v_k$ if $|k| \leq m$, and $v^m_k = 0$ otherwise. Then v^m tends to v as $m \to \infty$ in the ℓ^∞ topology. Indeed, since $\lim_{|k| \to \infty} \|v_k\| = 0$,

$$\|v - v^m\|_\infty = \sup_{|k| > m} \|v_k\|$$

tends to 0 when $m \to \infty$. Moreover, since v^m has only a finite number of components different from 0, we have that $(Av^m)_i = \sum_{|k| \leq m} A_{ik} v_k$. Finally,

$$\|(Av)_i - \sum_{|k| \leq m} A_{ik} v_k\| \leq \|A(v - v^m)\|_\infty \leq \|A\| \|v - v^m\|_\infty$$

tends to 0 when $m \to \infty$.

□

Remark 3.3. *In general, a linear map* $A \in L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *is* not *determined by its matrix components* A_{ij}*. As an example, consider the following space:*

$$E_0 = \{v \in \ell^\infty(\mathbb{R}) \mid \lim_{|j| \to \infty} v_j \textit{ exists}\}$$

and the linear map lim, $\lim E_0 \to \mathbb{R}$ *defined as* $\lim(v) = \lim_{|j| \to \infty} v_j$*. Clearly, the operator norm of* lim *is bounded on* E_0 *by 1, thus from the Hahn-Banach theorem it follows that* lim *admits an extension to* $\ell^\infty(\mathbb{R})$, $\mathcal{L}$ *with the same norm. The matrix elements of* $\mathcal{L}$, $\mathcal{L}_{ij}$ *are given by* $\mathcal{L}_j : \mathbb{R} \to \mathbb{R}$*, defined by* $\mathcal{L}_j(\alpha) = \mathcal{L}(\beta)$*, where* $\beta \in \ell^\infty(\mathbb{R})$ *is such that* $\beta_k = 0$ *if* $k \neq j$ *and* $\beta_j = \alpha$*. Clearly,* $\lim_{|k| \to \infty} \beta_k = 0$*, thus* $\mathcal{L}_j = 0$ *for all* j*. However,* $\mathcal{L}$ *is not the 0 mapping.*

This remark tells us that in general $A \in L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ is not completely determined by its matrix entries.

Remark 3.4. *Observe that given an uncoupled linear map* $A = \mathfrak{a}\delta_{ij}$*, with* $\mathfrak{a} \in L(\mathbb{R}^n, \mathbb{R}^n)$ *we have* $A \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $\gamma(A) = \Gamma(0)^{-1}\|\mathfrak{a}\|$ *and* $\|A\|_\Gamma = \Gamma(0)^{-1}\|\mathfrak{a}\|$.

We begin our study of L_Γ spaces with its completeness properties.

Proposition 3.5. *The space* $(L_\Gamma, \|\cdot\|_\Gamma)$ *is a Banach space.*

Proof. First we have to show that $\|\cdot\|_\Gamma$ is a norm. As $\|\cdot\|_\Gamma$ is defined as the maximum between $\|\cdot\|$ and $\gamma(\cdot)$, we just need to check the norm properties for $\gamma(\cdot)$.

Observe that given $A \in L_\Gamma$, Lemma 3.2 and the definition of $\gamma(\cdot)$ imply $\|A_{ij}\| \leq \|A\|_\Gamma \Gamma(i-j)$.

First we check that $\gamma(\lambda A) = |\lambda|\gamma(A)$.

$$\begin{aligned}\gamma(\lambda A) &= \sup_{i,k\in\mathbb{Z}^m} \sup_{\substack{\|u\|\leq 1,\\ \mathrm{proj}_j u=0,\, j\neq k}} \|(\lambda A u)_i\|\Gamma(i-k)^{-1} \\ &= \sup_{i,k\in\mathbb{Z}^m} \sup_{\substack{\|u\|\leq 1,\\ \mathrm{proj}_j u=0,\, j\neq k}} |\lambda|\|(Au)_i\|\Gamma(i-k)^{-1} = |\lambda|\gamma(A).\end{aligned}$$

Now check that $\gamma(A+B) \leq \gamma(A) + \gamma(B)$.

$$\begin{aligned}\gamma(A+B) &= \sup_{i,k\in\mathbb{Z}^m} \sup_{\substack{\|u\|\leq 1,\\ \mathrm{proj}_j u=0,\, j\neq k}} \|((A+B)u)_i\|\Gamma(i-k)^{-1} \\ &\leq \sup_{i,k\in\mathbb{Z}^m} \sup_{\substack{\|u\|\leq 1,\\ \mathrm{proj}_j u=0,\, j\neq k}} \left(\|(Au)_i\| + \|(Bu)_i\|\right)\Gamma(i-k)^{-1} \\ &\leq \gamma(A) + \gamma(B).\end{aligned}$$

Obviously if $\|A\| = 0$, then $A = 0$, thus if $\|A\|_\Gamma = 0$, $A = 0$. It is however not true that $\gamma(A) = 0$ implies that $A = 0$, as the previous remark shows.

The only thing left is to prove that $(L_\Gamma, \|\cdot\|_\Gamma)$ is a complete space. Let $\{A^n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of L_Γ.

As the $\|\cdot\|_\Gamma$-norm is stronger than the $\|\cdot\|$-norm and the space of linear applications is complete, $\{A^n\}_{n\in\mathbb{N}}$ converges to some linear map $A^* \in L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ in the $\|\cdot\|$-norm.

Moreover, as it is a Cauchy sequence, given $\varepsilon > 0$ there exists n_0 such that for $n > m > n_0$ and for every u such that $\|u\| \leq 1$ and $\mathrm{proj}_k u = 0$, $k \neq i$,

$$\|((A^n - A^m)u)_i\|\Gamma(i-k)^{-1} \leq \gamma(A^n - A^m) < \varepsilon/2.$$

Taking limit with respect to m in this expression,

$$\|((A^n - A^*)u)_i\|\Gamma(i-k)^{-1} \leq \varepsilon/2.$$

Observe that this bound holds for any i, k and u such that $\|u\| = 1$ and $\mathrm{proj}_k u = 0$ if $k \neq i$. Now taking supremum with respect to u and i, k as in the definition of $\gamma(\cdot)$, we get

$$\gamma(A^n - A^*) \leq \varepsilon/2 < \varepsilon.$$

This implies that $A^n - A^* \in L_\Gamma$, thus $A^* \in L_\Gamma$ and that $\{A^n\}_{n\in\mathbb{N}}$ converges to A^* in the $\|\cdot\|_\Gamma$-norm.

□

Now we introduce, the following vector space which plays a significant role in the study of linear maps with decay. Consider the vector subspace of $\ell^\infty(\mathbb{R}^n)$ of *vectors centred around the j-th component*

$$S_{j,\Gamma} = \{v \in \ell^\infty(\mathbb{R}^n) \mid \|v\|_{S_{j,\Gamma}} < \infty\},$$

where

$$\|v\|_{S_{j,\Gamma}} = \sup_{k\in\mathbb{Z}^m} \|v_k\|\Gamma(k-j)^{-1}.$$

Although this space looks dependant on the node j, all these spaces are equivalent (as sets) as the following proposition shows. The intuitive idea is that all nodes look the same from "far enough".

Lemma 3.6. *For all i, $j \in \mathbb{Z}^m$, $S_{i,\Gamma} = S_{j,\Gamma}$ as sets.*

Proof. Let $v \in S_{i,\Gamma}$, that is, $\sup_{k\in\mathbb{Z}^m} \|v_k\|\Gamma(k-i)^{-1} < \infty$. Now, as $\sum_{l\in\mathbb{Z}^m} \Gamma(k-l)\Gamma(l-j) \leq \Gamma(k-j) \leq 1$ and all terms in the sum are positive,

$$\frac{\Gamma(k-i)}{\Gamma(k-j)} \leq \frac{\Gamma(k-i)}{\sum_{l\in\mathbb{Z}^m} \Gamma(k-l)\Gamma(l-j)} \leq \frac{\Gamma(k-i)}{\Gamma(k-i)\Gamma(i-j)} = \frac{1}{\Gamma(i-j)}.$$

Therefore

$$\begin{aligned}\sup_{k\in\mathbb{Z}^m} \|v_k\|\Gamma(k-j)^{-1} &= \sup_{k\in\mathbb{Z}^m} \|v_k\|\Gamma(k-i)^{-1}\frac{\Gamma(k-i)}{\Gamma(k-j)} \\ &\leq \frac{1}{\Gamma(i-j)} \sup_{k\in\mathbb{Z}^m} \|v_k\|\Gamma(k-i)^{-1} < \infty\end{aligned}$$

and this implies $v \in S_{j,\Gamma}$.

□

Note that if $v \in S_{j,\Gamma}$ then

$$\|v\|_\infty \leq \sup_{i\in\mathbb{Z}^m} \|v_i\| \leq \sup_{i\in\mathbb{Z}^m} \|v_i\|\Gamma(i-j)^{-1} = \|v\|_{S_{j,\Gamma}}.$$

And now we prove completeness of this space.

Proposition 3.7. *The space $S_{j,\Gamma}$ is a Banach space.*

Proof. We check the completeness. Consider a Cauchy sequence $\{v^n\}_{n\in\mathbb{N}}$ of elements of $S_{j,\Gamma}$. As $\|\cdot\|_{S_{j,\Gamma}}$ is a stronger norm than $\|\cdot\|_\infty$ the sequence $\{v^n\}_{n\in\mathbb{N}}$ has a limit v^* in $\ell^\infty(\mathbb{R}^n)$.

As the sequence is a Cauchy sequence, for every $\varepsilon > 0$ there exists an n_0 such that $\|v^m - v^n\|_{S_{j,\Gamma}} < \varepsilon/2$ if m, $n \geq n_0$. By definition of the $S_{j,\Gamma}$-norm, this means that

$$\sup_{k\in\mathbb{Z}^m} \|v_k^m - v_k^n\|\Gamma(k-j)^{-1} < \varepsilon/2.$$

We can take limit with respect to m in this last expression. The convergence of $\{v^m\}$ to v^* in $\|\cdot\|_\infty$-norm implies that

$$\sup_{k\in\mathbb{Z}^m} \|v_k^* - v_k^n\|\Gamma(k-j)^{-1} \leq \varepsilon/2 < \varepsilon.$$

Therefore $\|v^* - v^n\|_{S_{j,\Gamma}} < \varepsilon$ implying $(v^* - v^n) \in S_{j,\Gamma}$ and hence $v^* \in S_{j,\Gamma}$. Now for every ε there exists an n_0 such that if $n > n_0$, $\|v^* - v^n\|_{S_{j,\Gamma}} < \varepsilon$, thus the sequence $\{v^n\}_{n\in\mathbb{N}}$ converges to v^* in the $S_{j,\Gamma}$-norm.

□

After this short digression about the properties of the spaces $S_{j,\Gamma}$ we come back to linear mappings with decay proving several properties that lead to a fundamental fact: L_Γ is a Banach algebra. Observe that the identity map Id is a unit of this algebra but $\|\operatorname{Id}\|_\Gamma = \Gamma(0)^{-1} \neq 1$.

The following proposition from [FdlLM11a] characterises linear maps with decay Γ as linear mappings preserving $S_{j,\Gamma}$. This is a key result that will be used several times.

Proposition 3.8. *Let $A \in L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. Then*

1. *If $A \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, then for any $j \in \mathbb{Z}^m$ and for any $v \in S_{j,\Gamma}$, $Av \in S_{j,\Gamma}$ and $\|Av\|_{S_{j,\Gamma}} \leq \gamma(A)\|v\|_{S_{j,\Gamma}}$.*
2. *If there exists $0 < C < \infty$ such that for any $j \in \mathbb{Z}^m$ and for any $v \in S_{j,\Gamma}$, $\|Av\|_{S_{j,\Gamma}} \leq C\|v\|_{S_{j,\Gamma}}$ then $A \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and $\gamma(A) \leq C\Gamma(0)^{-1}$.*

Proof. The proof of Part (1) is standard, and the arguments used in it will appear several times. It follows from

$$\begin{aligned}\|(Av)_i\| &\leq \sum_{k\in\mathbb{Z}^m} \|A_{ik}\|\|v_k\| \leq \gamma(A)\|v\|_{S_{j,\Gamma}} \sum_{k\in\mathbb{Z}^m} \Gamma(i-k)\Gamma(k-j) \\ &\leq \gamma(A)\|v\|_{S_{j,\Gamma}}\Gamma(i-j),\end{aligned}$$

where we have used Lemma 3.2. Therefore

$$\sup_{i\in\mathbb{Z}^m} \|(Av)_i\|\Gamma(i-j)^{-1} \leq \gamma(A)\|v\|_{S_{j,\Gamma}}.$$

To prove Part (2), we only need to use the definition of $\gamma(A)$ as follows

$$\begin{aligned}\gamma(A) &= \sup_{i,k\in\mathbb{Z}^m} \sup_{\substack{\|u\|\leq 1,\\ \operatorname{proj}_j u=0, j\neq k}} \|(Au)_i\|\Gamma(i-k)^{-1} \\ &= \sup_{k\in\mathbb{Z}^m} \sup_{\substack{\|u\|\leq 1,\\ \operatorname{proj}_j u=0, j\neq k}} \sup_{i\in\mathbb{Z}^m} \|(Au)_i\|\Gamma(i-k)^{-1} = \sup_{k\in\mathbb{Z}^m} \sup_{\substack{\|u\|\leq 1,\\ \operatorname{proj}_j u=0, j\neq k}} \|Au\|_{S_{k,\Gamma}} \\ &\leq \sup_{k\in\mathbb{Z}^m} \sup_{\substack{\|u\|\leq 1,\\ \operatorname{proj}_j u=0, j\neq k}} C\,\|u\|_{S_{k,\Gamma}} \leq C\,\Gamma(0)^{-1}.\end{aligned}$$

Observe that a vector $u \in \ell^\infty(\mathbb{R}^n)$ such that $\|u\| \leq 1$, $\operatorname{proj}_j u = 0, j \neq k$ satisfies $u \in S_{k,\Gamma}$.

□

Proposition 3.9 (Algebra properties). *Let A, $B \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. Then $AB \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and*

- $\gamma(AB) \leq \gamma(A)\gamma(B)$,

- $\|AB\|_\Gamma \le \|A\|_\Gamma \|B\|_\Gamma$.

Proof. We just need to check the bounds on the norm, using the properties of $\gamma(\cdot)$ and Proposition 3.8. Since

$$\begin{aligned}
\gamma(AB) &= \sup_{i,\,k\in\mathbb{Z}^m} \sup_{\substack{\mathrm{proj_j} u=0,\\ j\neq k\\ \|u\|\le 1}} \|(ABu)_i\|\Gamma(i-k)^{-1} \\
&= \sup_{i,\,k\in\mathbb{Z}^m} \sup_{\substack{\mathrm{proj_j} u=0,\\ j\neq k\\ \|u\|\le 1}} \|\sum_{p\in\mathbb{Z}^m} A_{ip}(Bu)_p\|\Gamma(i-k)^{-1} \\
&\le \sup_{i,\,k\in\mathbb{Z}^m} \sup_{\substack{\mathrm{proj_j} u=0,\\ j\neq k\\ \|u\|\le 1}} \sum_{p\in\mathbb{Z}^m} \|A_{ip}\|\|(Bu)_p\|\Gamma(i-k)^{-1} \\
&\le \sup_{i,\,k\in\mathbb{Z}^m} \gamma(A) \sum_{p\in\mathbb{Z}^m} \Gamma(i-p)\gamma(B)\Gamma(p-k)\Gamma(i-k)^{-1} \\
&\le \sup_{i,\,k\in\mathbb{Z}^m} \gamma(A)\gamma(B) \le \gamma(A)\gamma(B)
\end{aligned}$$

and

$$\|AB\|_\Gamma = \max(\gamma(AB), \|AB\|) \le \max(\gamma(A)\gamma(B), \|A\|\|B\|),$$

the statement follows.

□

Remark 3.10. *Proposition 3.5 and Proposition 3.9 imply $L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ is a Banach algebra. It has a unit element* Id *but* $\|\mathrm{Id}\|_\Gamma \neq 1$.

Proposition 3.11. *Let $M_0 \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ invertible such that*

$$M_0^{-1} \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$$

(or equivalently $0 \notin \mathrm{Spec}_\Gamma(M_0)$) and $M_1 \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that $\|M_0^{-1}\|_\Gamma \|M_1\|_\Gamma < 1$. Then $M = M_0 + M_1$ is invertible, $M^{-1} \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and

$$|\|M^{-1}\|_\Gamma - \|M_0^{-1}\|_\Gamma| \le \|M^{-1} - M_0^{-1}\|_\Gamma = \mathcal{O}(\|M_1\|_\Gamma).$$

The proof of this proposition is completely analogous to the proof of Lemma 4.18 where a slightly more involved result is considered.

3.2 The space of k-linear maps with decay

With the definition of linear maps with decay we can only characterise first derivatives with decay. But we can inductively define multilinear maps with decay which in turn we can use to characterise higher order derivatives with decay.

Recall that we can define the space of k-linear maps $L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ via the identification

$$L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) = L(\ell^\infty(\mathbb{R}^n), L^{k-1}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))).$$

There are k possible identifications defined by ι_j as follows: let

$$\iota_j : L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) \to L(\ell^\infty(\mathbb{R}^n), L^{k-1}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))), \quad 1 \le j \le k,$$

and given $A \in L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$

$$\iota_j(A)(w)(v_1, \ldots, v_{k-1}) = A(v_1, \ldots, \overbrace{w}^{j}, \ldots, v_{k-1}). \tag{3.1}$$

All these mappings are isometries in the corresponding operator norms. We define

$$\begin{aligned} L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) = \big\{ & A \in L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) \mid \\ & \iota_p(A) \in L_\Gamma(\ell^\infty(\mathbb{R}^n), L^{k-1}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))),\ 1 \le p \le k \big\}, \end{aligned}$$

with the norm

$$\|A\|_\Gamma = \max\{\|A\|, \gamma(A)\},$$

where

$$\gamma(A) = \max_{1 \le p \le k} \{\gamma(\iota_p(A))\}.$$

Note that this definition makes sense because we can identify $L^{k-1}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ with the ℓ^∞ space $\ell^\infty(L^{k-1}(\ell^\infty(\mathbb{R}^n), \mathbb{R}^n))$.

With this norm we can prove that $L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ is a Banach space as in the proof of Proposition 3.5.

The following proposition gives bounds to the norm of multilinear contractions. These bounds are fundamental later on, since multilinear contractions appear naturally when differentiating repeatedly invariance equations.

Proposition 3.12 (Γ norms of contractions)**.** *Let $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $k \ge 2$, and $u \in \ell^\infty(\mathbb{R}^n)$. Then, for any permutation of k elements $\tau \in S_k$ the map $B_{\tau,u} : \ell^\infty(\mathbb{R}^n) \times \overset{(k-1)}{\cdots} \times \ell^\infty(\mathbb{R}^n) \to \ell^\infty(\mathbb{R}^n)$ defined by*

$$B_{\tau,u}(v_1, \ldots, v_{k-1}) = A(\tau(v_1, \ldots, v_{k-1}, u))$$

belongs to $L^{k-1}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. Moreover

$$\gamma(B_{\tau,u}) \le \gamma(A)\|u\|. \tag{3.2}$$

As a consequence

$$\|B_{\tau,u}\|_\Gamma \le \|A\|_\Gamma \|u\|. \tag{3.3}$$

If $\tau = \mathrm{Id}$ we will simply write $B_u = B_{\mathrm{Id},u}$.

Proof. For simplicity, we will only check the case $\tau = \mathrm{Id}$.

Inequality (3.3) is trivial if $u = 0$. If $u \neq 0$ we need some auxiliary vectors to prepare the bound as in the definition of $\gamma(A)$. Let $i, j \in \mathbb{Z}^m$, $1 \leq p \leq k$, $v_2, \ldots v_{k-1} \in \ell^\infty(\mathbb{R}^n)$ such that $\|v_q\| \leq 1$, $2 \leq q \leq k-1$, and $w = (w^l)_{l \in \mathbb{Z}^m}$ such that $w^l = 0$, if $l \neq j$. Then

$$\begin{aligned}
&\|\iota_p(B_u)_i(w)(v_2, \ldots, v_{k-1})\|\Gamma(i-j)^{-1} \\
&\qquad = \|(B_u)_i(v_2, \ldots, \overbrace{w}^{p}, \ldots, v_{k-1})\|\Gamma(i-j)^{-1} \\
&\qquad = \|A_i(v_2, \ldots, w, \ldots, v_{k-1}, u/\|u\|)\| \, \|u\|\Gamma(i-j)^{-1} \\
&\qquad = \|\iota_p(A)_i(w)(v_2, \ldots, v_{k-1}, u/\|u\|)\| \, \|u\|\Gamma(i-j)^{-1} \\
&\qquad \leq \gamma(A)\|u\|.
\end{aligned}$$

Inequality (3.2) follows from taking supremum above. Moreover $\|B_u\| \leq \|A\| \, \|u\|$ and using these bounds

$$\|B_u\|_\Gamma = \max(\gamma(B_u), \|B_u\|) \leq \max(\gamma(A) \, \|u\|, \|A\| \, \|u\|) = \|A\|_\Gamma \|u\|$$

we get that (3.3) holds true.

□

Remark 3.13. *Observe that in the previous proposition we required $u \in \ell^\infty(\mathbb{R}^n)$, not $u \in S_{j,\Gamma}$.*

In the same way we have

Proposition 3.14. *Let $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $k \geq 2$ and $u_1, \ldots, u_p \in \ell^\infty(\mathbb{R}^n)$, $1 \leq p \leq k-1$. Then for any permutation of k elements $\tau \in S_k$ the map*

$$B_{\tau, u_1, \ldots, u_p} : \ell^\infty(\mathbb{R}^n) \times \overset{(k-p)}{\cdots} \times \to \ell^\infty(\mathbb{R}^n)$$

defined by

$$B_{\tau, u_1, \ldots, u_p}(v_1, \ldots, v_{k-p}) = A(\tau(v_1, \ldots, v_{k-p}, u_1, \ldots, u_p))$$

belongs to $L^{k-p}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. Moreover

$$\gamma(B_{\tau, u_1, \ldots, u_p}) \leq \gamma(A)\|u_1\| \cdots \|u_p\|$$

and

$$\|B_{\tau, u_1, \ldots, u_p}\| \leq \|A\|_\Gamma \|u_1\| \cdots \|u_p\|.$$

Proposition 3.12 can be used to bound Γ-norms of contractions in such a way that decay properties can be ignored except for the bound of just one component, as the next proposition shows. Furthermore, such a bound is also a characterisation of multilinear mappings with decay.

Proposition 3.15. *Let $A \in L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. Then*

(i) If $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, then, for any $v_2, \ldots, v_k \in \ell^\infty(\mathbb{R}^n)$, $j \in \mathbb{Z}^m$, $v \in S_{j,\Gamma}$ and $\tau \in S_k$, we have $A(v, v_2, \ldots, v_k) \in S_{j,\Gamma}$ and

$$\|A(\tau(v, v_2, \ldots, v_k))\|_{S_{j,\Gamma}} \leq \gamma(A)\|v\|_{S_{j,\Gamma}} \|v_2\| \ldots \|v_k\|.$$

(ii) If there exists $C > 0$ such that for any $v_2, \ldots, v_k \in \ell^\infty(\mathbb{R}^n)$, $j \in \mathbb{Z}^m$, $v \in S_{j,\Gamma}$, $\tau \in S_k$, we have $A(\tau(v, v_2, \ldots, v_k)) \in S_{j,\Gamma}$ and

$$\|A(\tau(v, v_2, \ldots, v_k))\|_{S_{j,\Gamma}} \le C\|v\|_{S_{j,\Gamma}}\|v_2\| \ldots \|v_k\|,$$

then $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and $\gamma(A) \le C\Gamma(0)^{-1}$.

Proof of Part (i)*:* Proposition 3.8 proves case $k = 1$. By induction assume that the result is true for $k-1 \ge 1$ and let B_{τ,v_k} be defined by $B_{\tau,v_k}(v, v_2, \ldots, v_{k-1}) = A(\tau(v, v_2, \ldots, v_{k-1}, v_k))$. By Proposition 3.12, $B_{\tau,v_k} \in L^{k-1}_\Gamma$ and $\gamma(B_{\tau,v_k}) \le \gamma(A)\|v_k\|$. Now by the induction hypothesis we have

$$\begin{aligned}\|A(\tau(v, v_2, \ldots, v_k))\|_{S_{j,\Gamma}} = \|B_{\tau,v_k}(v, v_2, \ldots, v_{k-1})\|_{S_{j,\Gamma}} &\le \gamma(B_{\tau,v_k})\|v\|_{S_{j,\Gamma}}\|v_2\| \ldots \|v_{k-1}\| \\ &\le \gamma(A)\|v\|_{S_{j,\Gamma}}\|v_2\| \ldots \|v_k\|.\end{aligned}$$

Proof of Part (ii)*:* Given $1 \le p \le k$ we compute

$$\begin{aligned}\gamma(\iota_p(A)) &= \sup_{i,j\in\mathbb{Z}^m} \sup_{\substack{\|v\|\le 1 \\ \mathrm{proj}_l\, v=0,\, l\neq j}} \|[\iota_p(A)(v)]_i\|\Gamma(i-j)^{-1} \\ &= \sup_{i,j\in\mathbb{Z}^m} \sup_{\|v\|\le 1} \sup_{\|v_q\|\le 1} \|[\iota_p(A)(v)(v_2, \ldots, v_k)]_i\|\Gamma(i-j)^{-1} \\ &= \sup_{i,j\in\mathbb{Z}^m} \sup_{\|v\|\le 1} \sup_{\|v_q\|\le 1} \|[A(v_2, \ldots, v_p, v, \ldots, v_k)]_i\|\Gamma(i-j)^{-1} \\ &\le \sup_{i,j\in\mathbb{Z}^m} \sup_{\|v\|\le 1} \sup_{\|v_q\|\le 1} \|[A(\tau(v, v_2, \ldots, v_k))]_i\|\Gamma(i-j)^{-1} \\ &\le \sup_{j\in\mathbb{Z}^m} \sup_{\|v\|\le 1} \sup_{\|v_q\|\le 1} \|A(\tau(v, v_2, \ldots, v_k))\|_{S_{j,\Gamma}} \\ &\le \sup_{j\in\mathbb{Z}^m} \sup_{\|v\|\le 1} \sup_{\|v_q\|\le 1} C\|v\|_{S_{j,\Gamma}}\|v_2\| \cdots \|v_k\| \\ &\le C\Gamma(0)^{-1}\end{aligned}$$

for some permutation $\tau \in S_k$. Recall that since $\mathrm{proj}_l\, v = 0$, $l \neq j$, $\|v\|_{S_{j,\Gamma}} = \|v_j\|\Gamma(0)^{-1}$. This implies $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and $\gamma(A) \le C\Gamma(0)^{-1}$.

□

From Proposition 3.12 and Proposition 3.9 we also obtain the following composition property, which will prove crucial for later developments.

Given $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $B_j \in L^{l_j}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ for $j = 1, \ldots, k$ and $w_{l_j} \in \ell^\infty(\mathbb{R}^n)^{l_j}$, we define the composition $AB_1 \cdots B_k$ by

$$AB_1 \cdots B_k(w_{l_1}, \ldots, w_{l_k}) = A(B_1 w_{l_1}, \ldots, B_k w_{l_k}).$$

Proposition 3.16. *If $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and $B_j \in L^{l_j}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, for $j = 1, \ldots, k$, then the composition $AB_1 \cdots B_k \in L^{l_1+\cdots+l_k}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and*

$$\gamma(AB_1 \cdots B_k) \le \gamma(A)\|B_1\|_\Gamma \cdots \|B_k\|_\Gamma, \tag{3.4}$$

$$\|AB_1 \cdots B_k\|_\Gamma \le \|A\|_\Gamma\|B_1\|_\Gamma \cdots \|B_k\|_\Gamma. \tag{3.5}$$

A consequence of the proof is that if $B_p = B_q$ for all $q \neq p$ the bounds can be written instead as

$$\begin{aligned}\gamma(AB_1 \cdots B_k) \leq& \gamma(A)\|B\|_\Gamma \cdots \|B\|\|B\|,\\ \|AB \cdots B\|_\Gamma \leq& \|A\|_\Gamma \|B\|_\Gamma \|B\| \cdots \|B\|.\end{aligned}$$

Proof. Since $\gamma(A) \leq \|A\|_\Gamma$ and the fact that we have the bound

$$\|AB \cdots B\| \leq \|A\|\|B\| \cdots \|B\|$$

in the operator norms, inequality (3.4) implies (3.5).

For convenience of notation we let $l_0 = 0$. Then, for any $1 \leq s \leq k$ and for any $u_{l_{s-1}+1}, \ldots, u_{l_s} \in \ell^\infty(\mathbb{R}^n)$, we have that $\|B_s u_{l_{s-1}+1} \ldots u_{l_s}\| \leq \|B_s\|_\Gamma \|u_{l_{s-1}+1}\| \cdots \|u_{l_s}\|$. Also, by an iterative application of Proposition 3.12, for any $u_{l_{s-1}+2}, \ldots, u_{l_s} \in \ell^\infty(\mathbb{R}^n)$ and $\tau \in S_{l_s}$, the map $B_{\tau,s} : u \mapsto B_s(\tau(u, u_{l_{s-1}+2}, \ldots, u_{l_s}))$ belongs to L_Γ and $\|B_{\tau,s}\|_\Gamma \leq \|B_s\|_\Gamma \|u_2\| \cdots \|u_{l_s}\|$.

Hence, by Proposition 3.15 for any $u_2, \ldots, u_{l_1+\cdots+l_k} \in \ell^\infty(\mathbb{R}^n)$, $\|u_2\|, \ldots, \|u_{l_1+\cdots+l_k}\| \leq 1$ and $l_{s-1} < p \leq l_s$, the map $\widetilde{A_p} : \ell^\infty(\mathbb{R}^n) \to \ell^\infty(\mathbb{R}^n)$ defined by

$$\widetilde{A_p} : u \mapsto \iota_p(AB_1 \ldots B_k)(u)(u_2, \ldots, u_{l_1+\cdots+l_k}),$$

where $\iota_p(AB_1 \ldots B_k)$ was introduced in (3.1), belongs to L_Γ and

$$\gamma(\widetilde{A_p}) \leq \gamma(A)\|B_1\| \ldots \gamma(B_p) \ldots \|B_k\|.$$

Finally, since $\|B_j\| \leq \|B_j\|_\Gamma$, for all $1 \leq j \leq k$,

$$\gamma(AB_1 \ldots B_k) = \max_{1 \leq p \leq k} \gamma(\widetilde{A_p}) \leq \gamma(A)\|B_1\|_\Gamma \ldots \|B_k\|_\Gamma.$$

□

An important special case of the previous proposition is the following result.

Corollary 3.17. *If $A \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and $B \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ then $A \cdot B \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and*

$$\begin{aligned}\gamma(A\,B) \leq \gamma(A)\gamma(B),\\ \|A\,B\|_\Gamma \leq \|A\|_\Gamma \|B\|_\Gamma.\end{aligned}$$

Chapter 4

Spaces of differentiable functions with decay

With the definition of k-linear mappings with decay, we can now define spaces of differentiable functions with decay between open sets in $\ell^\infty(\mathbb{R}^n)$. We can also define spaces of differentiable maps from the torus to $\ell^\infty(\mathbb{R}^n)$ and spaces of differentiable maps from the torus to $L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$.

Definition 4.1. *Let U be an open set of $\ell^\infty(\mathbb{R}^n)$. We define*

$$\begin{aligned} C^1_\Gamma(U, \ell^\infty(\mathbb{R}^n)) = \{F \in C^1(U, \ell^\infty(\mathbb{R}^n)) \mid \sup_{x\in U} \|F(x)\|_\infty < \infty, \\ DF(x) \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)), \forall x \in U, \\ \sup_{x\in U} \|DF(x)\|_\Gamma < \infty\} \end{aligned}$$

with norm

$$\|F\|_{C^1_\Gamma} = \max\left(\|F\|_{C^0}, \sup_{x\in U} \|DF(x)\|_\Gamma\right),$$

where $\|F\|_{C^0} = \sup_{x\in U} \|F(x)\|_\infty$ as usual. We can thus define

$$\begin{aligned} C^1_\Gamma(U, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))) = \big\{F \in C^1(U, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))) \mid \\ F(x) \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)),\ \forall x \in U, \\ \sup_{x\in U} \|F(x)\|_\Gamma < \infty\big\}. \end{aligned}$$

Based on this we can define spaces of C^r_Γ functions:

$$\begin{aligned} C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n)) = \big\{F \in C^r(U, \ell^\infty(\mathbb{R}^n)) \mid D^k F \in C^1_\Gamma(U, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))), \\ 0 \le k \le r-1\big\} \end{aligned}$$

with norm

$$\|F\|_{C^r_\Gamma} = \max\left(\|F\|_{C^0}, \max_{0\le k\le r-1} \sup_{x\in U} \|DD^k F(x)\|_\Gamma\right).$$

Remark 4.2. *The inclusions* $C^i_\Gamma \subset C^{i-1}_\Gamma$, $1 \le i \le r$, *are satisfied.*

It is easy to check that $C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ is a Banach space. We have the following result concerning the composition of maps which appears in [FdlLM11a].

Proposition 4.3. *Let* U, V *be open sets of* $\ell^\infty(\mathbb{R}^n)$, $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ *and* $G \in C^r_\Gamma(V, \ell^\infty(\mathbb{R}^n))$ *such that* $F(U) \subseteq V$. *Then*

- $G \circ F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$,
- $\|G \circ F\|_{C^r_\Gamma} \le K(1 + \|F\|^r_{C^r_\Gamma})\|G\|_{C^r_\Gamma}$.

Remark 4.4. *An important particular case appears when* G *is a linear map in* $L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. *In this case the estimates in the proof are much easier and the bound is*

$$\|A \circ F\|_{C^r_\Gamma} \le \|A\|_\Gamma \|F\|_{C^r_\Gamma}. \tag{4.1}$$

Theorem 4.5 (Inverse Function Theorem)**.** *Let* U *be an open set of* $\ell^\infty(\mathbb{R}^n)$ *and* $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$, $r \ge 1$. *Let* $p \in U$ *and* $q = F(p)$. *Assume that* $DF(p)$ *is invertible and* $DF(p)^{-1} \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. *Then* F *is locally invertible around* p *and* $F^{-1} \in C^r_\Gamma(V, \ell^\infty(\mathbb{R}^n))$ *where* V *is a neighbourhood of* q.

Proof. From the standard inverse function theorem in Banach spaces, F is locally invertible and F^{-1} is defined in a neighbourhood V of q. Moreover,$DF^{-1}(q) = DF(p)^{-1}$ and by the continuity of DF and Proposition 6.15, $DF^{-1}(x) \in L_\Gamma$ for $x \in V$, provided V is small. Since

$$DF^{-1}(x) = \left(DF(F^{-1}(x))\right)^{-1} \tag{4.2}$$

we can obtain the higher order derivatives of F^{-1} taking derivatives in the right hand side of (4.2). For instance,

$$\begin{aligned} D^2F^{-1}(x) &= -\left(DF(F^{-1}(x))\right)^{-1} D^2F(F^{-1}(x))(DF(F^{-1}(x)))^{-1} \\ &= -DF^{-1}(x)D^2F(F^{-1}(x))DF^{-1}(x). \end{aligned} \tag{4.3}$$

Then, by Proposition 3.16, $D^2F^{-1}(x) \in L^2_\Gamma$. Proceeding in the same way for the other derivatives we get that $F^{-1} \in C^r_\Gamma(V, \ell^\infty(\mathbb{R}^n))$. Alternatively, we can use (4.3) to prove inductively that $F^{-1} \in C^i_\Gamma$ assuming $F^{-1} \in C^{i-1}_\Gamma$, for $i \le r$.

□

Now we define mappings from the torus to an open subset $U \subseteq \ell^\infty(\mathbb{R}^n)$ with centred decay. This definition is motivated by the definition of $S_{j,\Gamma}$ given in Chapter 3.

Definition 4.6. *Given* $j \in \mathbb{Z}^m$, *we define*

$$S^0_{j,\Gamma} = S^0_{j,\Gamma}(\mathbb{T}^d) = \left\{\sigma \in C^0(\mathbb{T}^d, U) \mid \sup_{i \in \mathbb{Z}^m} \sup_{\theta \in \mathbb{T}^d} \|\sigma(\theta)\|\Gamma(i-j)^{-1} < \infty\right\},$$

with norm

$$\|\sigma\|_{S^0_{j,\Gamma}} = \sup_{i \in \mathbb{Z}^m} \|\sigma_i\|_{C^0}\Gamma(i-j)^{-1} = \sup_{\theta \in \mathbb{T}^d} \|\sigma(\theta)\|_{S_{j,\Gamma}} = \sup_{i \in \mathbb{Z}^m} \sup_{\theta \in \mathbb{T}^d} \|\sigma_i(\theta)\|\Gamma(i-j)^{-1}.$$

We also define mappings of higher regularity with centred decay. Let

$$S^r_{j,\Gamma} = S^r_{j,\Gamma}(\mathbb{T}^d) = \big\{\sigma \in C^r(\mathbb{T}^d, U) \mid \frac{\partial^k}{\partial\theta_1^{\ell_1}\cdots\partial\theta_d^{\ell_d}}\sigma \in S^0_{j,\Gamma}(\mathbb{T}^d),\\ \ell_1 + \ldots + \ell_d = k,\, 0 \le k \le r\big\}$$

with norm

$$\|\sigma\|_{S^r_{j,\Gamma}} = \max_{0\le k\le r} \max_{\substack{l_1,\ldots,l_d\ge 0\\ l_1+\ldots+l_d=k}} \left\| \frac{\partial^k}{\partial\theta_1^{\ell_1}\cdots\partial\theta_d^{\ell_d}}\sigma \right\|_{S^0_{j,\Gamma}}.$$

Proposition 4.7. *The space $S^r_{j,\Gamma}$ is a Banach space.*

Proof. The proof for $r = 0$ follows similar arguments to the ones used in the proof of Proposition 3.7. The proof for higher order derivatives is standard, but we will give it in the case $r = 1$ for completeness. A standard induction argument from this case gives the desired result.

Let $\{f^n\}_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of $S^1_{j,\Gamma}$. Since $\|\cdot\|_{S^1_{j,\Gamma}}$ is a stronger norm than the C^1 norm (this is clear from the definition), $\{f^n\}_{n\in\mathbb{N}}$ converges to some function $f^* \in C^1(\mathbb{T}^d, U)$ in C^1-norm. Similar arguments to those used in Proposition 3.7 show that $f^* \in S^0_{j,\Gamma}$. All we need to prove now is that the partial derivatives of the functions converge to the corresponding partial derivatives of the limit function, in the $S^0_{j,\Gamma}$-norm. Given $1 \le p \le d$ let g_p be the limit of the sequence $\{\frac{\partial}{\partial\theta_p} f^n\}_{n\in\mathbb{N}}$ in the C^0-norm.

Define the following auxiliary sequence

$$h^n(\delta,\theta) = \begin{cases} \frac{f^n(\theta+\delta\cdot e_p)-f^n(\theta)}{\delta} & \delta \neq 0,\\ \frac{\partial}{\partial\theta_p} f^n(\theta) & \delta = 0.\end{cases}$$

The mapping h^n is continuous with respect to δ and θ and $h^n(\delta,\cdot)$ belongs to $S^0_{j,\Gamma}$. By an application of the mean value theorem, we can bound $\|h^n - h^m\|_{S^0_{j,\Gamma}}$ as

$$\begin{aligned}\|h^n - h^m\|_{S^0_{j,\Gamma}} &= \Big\| \frac{\int_0^1 D(f^m - f^n)(\theta + \lambda\,\delta\cdot e_p)\cdot\delta e_p}{\delta} d\lambda \Big\|_{S^0_{j,\Gamma}} \le \Big\| \frac{\partial}{\partial\theta_p}(f^n - f^m)\Big\|_{S^0_{j,\Gamma}}\\ &\le \|f^n - f^m\|_{S^1_{j,\Gamma}} < \varepsilon/2,\end{aligned}$$

given $\varepsilon > 0$ and $n > m > n_0$ large enough. Taking limit with respect to n in the previous expression, we get

$$\|h^* - h^m\|_{S^0_{j,\Gamma}} \le \varepsilon/2,$$

where

$$h^*(\delta,\theta) = \begin{cases} \frac{f^*(\theta+\delta\cdot e_p)-f^*(\theta)}{\delta} & \delta \neq 0,\\ \frac{\partial}{\partial\theta_p} f^*(\theta) & \delta = 0.\end{cases}$$

As the convergence is uniform and $\{h^n\}_{n\in\mathbb{N}}$ are continuous, the limit function h^* is also continuous and $\frac{\partial}{\partial\theta_p} f^*(\theta) = g_p(\theta)$ for all $\theta \in \mathbb{T}^d$.

Therefore $\{f^n\}_{n\in\mathbb{N}}$ converges to f^* in $S^1_{j,\Gamma}$-norm and the derivatives of the sequence converge to the derivative of the limit with respect to $S^0_{j,\Gamma}$-norm.

□

The space $S^0_{j,\Gamma}$ has several properties that we can extend to the differentiable case, the most obvious one being the decay at infinity.

Lemma 4.8. *Let $U \subset \ell^\infty(\mathbb{R}^n)$ be an open set and let $\sigma \in S^r_{j,\Gamma}(U)$. Then for all $1 \leq k \leq r$*

$$\lim_{|i|\to\infty} \left\| \frac{\partial^k}{\partial\theta_1^{\ell_1}\cdots\partial\theta_d^{\ell_d}} \sigma_i \right\|_{C^0} \leq \lim_{|i|\to\infty} \left\| \frac{\partial^k}{\partial\theta_1^{\ell_1}\cdots\partial\theta_d^{\ell_d}} \sigma \right\|_{S^0_{j,\Gamma}} \Gamma(i-j) = 0.$$

Proof. The proof is an easy consequence of the definitions of $S^r_{j,\Gamma}$. Observe that the decay speed is controlled by the decay function Γ.

□

Now we define differentiable mappings from the torus $\mathbb{T}^d$ to the space of k-linear maps. These mappings will appear in the study and construction of invariant manifolds found in this work.

Definition 4.9. *We define*

$$\begin{aligned} C^0_{L^k_\Gamma} &= C^0\big(\mathbb{T}^d, L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))\big) \\ &= \{F \in C^0(\mathbb{T}^d, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))) \mid \sup_{\theta\in\mathbb{T}^d} \|F(\theta)\|_\Gamma < \infty\}. \end{aligned}$$

with norm

$$\|F\|_{C^0_{L_\Gamma}} = \sup_{\theta\in\mathbb{T}^d} \|F(\theta)\|_\Gamma.$$

From this we define

$$C^r_{L^k_\Gamma} = \{F \in C^r(\mathbb{T}^d, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))) \mid \|F\|_{C^r_{L^k_\Gamma}} < \infty\}$$

with norm

$$\|F\|_{C^r_{L^k_\Gamma}} = \max_{\substack{0\leq j\leq r \\ l_1+\ldots+l_d=j}} \left\| \frac{\partial^j F(\theta)}{\partial\theta_1^{\ell_1}\cdots\partial\theta_d^{\ell_d}} \right\|_{C^0_{L_\Gamma}}.$$

When there is no possible confusion we will use the notation C^r_Γ instead of $C^r_{L^k_\Gamma}$.

Remark 4.10. *As in the definition of linear and multilinear mappings, the definition can be made general for linear subspaces $\mathcal{E}_i$, $\mathcal{F} \subset \ell^\infty(\mathbb{R}^n), 1 \leq i \leq k$,*

$$C^r_{L^k_\Gamma} = C^r_{L^k_\Gamma}(\mathcal{E}^1 \times \cdots \times \mathcal{E}_k, \mathcal{F}) = \{F \in C^r(\mathbb{T}^d, L^k(\mathcal{E}^1 \times \cdots \times \mathcal{E}_k, \mathcal{F})) \mid \|F\|_{C^r_{L^k_\Gamma}} < \infty\}.$$

Remark 4.11. *Given a linear mapping $A(\theta) : \ell^\infty(\mathbb{R}^n) \to \ell^\infty(\mathbb{R}^n)$ for all $\theta \in \mathbb{T}^d$ depending C^j with respect to θ, $\frac{\partial^j}{\partial\theta_1^{\ell_1}\cdots\partial\theta_d^{\ell_d}} A(\theta)$ is again a mapping of the same type, as we are considering only partial derivatives and we can compute its $C^0_{L^k_\Gamma}$-norm in a straightforward way.*

As usual, after defining each space we check its completeness.

Proposition 4.12. *The space* $(C^r_{L^k_\Gamma}, \|\cdot\|_{C^r_{L^k_\Gamma}})$ *is a Banach space.*

Remark 4.13. *If* $A, B \in C^r_{L^k_\Gamma}$ *we define* AB *by* $(AB)(\theta) = A(\theta)B(\theta)$. *We have that* $AB \in C^r_{L^k_\Gamma}$ *but in general we do not have*

$$\|AB\|_{C^r_{L^k_\Gamma}} \leq \|A\|_{C^r_{L^k_\Gamma}} \|B\|_{C^r_{L^k_\Gamma}}.$$

Hence $C^r_{L^k_\Gamma}$ *is not a Banach algebra. However, see Proposition 4.15 below for a bound of this norm.*

Proof. The proof that this space is a Banach space is very similar to the proof of Propositions 3.5 and 4.7 (L_Γ is a Banach space and $S^r_{j,\Gamma}$ is a Banach space respectively). To prove it is an algebra, we can follow the same ideas as in Propositions 3.9 and 4.14 (Algebra properties for L_Γ and norm of the product of an element of $C^r_{L_\Gamma}$ and one of $S^s_{j,\Gamma}$, respectively).

□

Now that we have defined vector maps from the torus to $\ell^\infty(\mathbb{R}^n)$ and (multi)linear maps over the torus, we can study its interplay.

Proposition 4.14. *Given* $\sigma \in S^r_{j,\Gamma}(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ *and* $A \in C^t_{L_\Gamma}(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$, *then* $A\sigma$ *defined as* $(A\sigma)(\theta) = A(\theta)\sigma(\theta)$ *belongs to* $\in S^m_{j,\Gamma}(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ *with* $m = \min(t, r)$. *Moreover*

$$\|A\sigma\|_{S^m_{j,\Gamma}} \leq 2^m \|A\|_{C^m_{L_\Gamma}} \|\sigma\|_{S^m_{j,\Gamma}}.$$

In the particular case that A *does not depend on* θ *we have*

$$\|A\sigma\|_{S^m_{j,\Gamma}} \leq \|A\|_\Gamma \|\sigma\|_{S^m_{j,\Gamma}}.$$

Proof. It is clear that $A\sigma$ has regularity at most m by an application of Leibniz's rule. Thus we can assume without loss of generality that $t = r = m$ for the rest of the proof. We need to check the statement on the bounds. If $m = 0$, by definition

$$\|A\sigma\|_{S^0_{j,\Gamma}} = \sup_{i\in\mathbb{Z}^m} \| (A\sigma)_i \|_{C^0} \Gamma(i-j)^{-1}.$$

As $A(\theta) \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ for all $\theta \in \mathbb{T}^d$, by Lemma 3.2 we can write

$$(A(\theta)\sigma(\theta))_i = \sum_{k\in\mathbb{Z}^m} A_{ik}(\theta)\sigma_k(\theta)$$

and

$$\begin{aligned}\sup_{\theta\in\mathbb{T}} \|A_{ik}(\theta)\sigma_k(\theta)\| &\leq \sup_{\theta\in\mathbb{T}} \gamma(A(\theta))\Gamma(i-k) \sup_{\theta\in\mathbb{T}} \|\sigma_k(\theta)\| \\ &\leq \|A\|_{C^0_{L_\Gamma}} \|\sigma\|_{S^0_{j,\Gamma}} \Gamma(i-k)\Gamma(k-j).\end{aligned}$$

Therefore

$$\|A\sigma\|_{S^0_{j,\Gamma}} \leq \|A\|_{C^0_{L_\Gamma}} \|\sigma\|_{S^0_{j,\Gamma}}.$$

Assume now that $m > 0$. In this case we can use Leibniz's rule to calculate $\frac{\partial^m}{\partial\theta_1^{\ell_1}\cdots\partial\theta_d^{\ell_d}}(A_{ik} \cdot \sigma_k)(\theta)$ for $\ell_1 + \cdots + \ell_d = m$, and we get 2^m terms, each of these terms can be bounded as in the previous case yielding

$$\|A\sigma\|_{S^m_{j,\Gamma}} \leq 2^m \|A\|_{C^m_{L_\Gamma}} \|\sigma\|_{S^m_{j,\Gamma}}.$$

□

In the following result we bound compositions of linear maps, with and without dependence on θ.

Proposition 4.15. *Given A_i, $i = 1, \dots, j$, with $A_i \in C^{t_i}_{L_\Gamma}(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$, then $A_1 \cdots A_j \in C^m_{L_\Gamma}(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$ with $m = \min_{i=1,\dots,j}\{t_i\}$. Moreover*

$$\|A_1 \cdots A_j\|_{C^m_\Gamma} \leq j^m \|A_1\|_{C^m_\Gamma} \cdots \|A_j\|_{C^m_\Gamma}.$$

If A_{i_l}, $l = 1, \dots, k$, $k \leq j$, do not depend on θ, then

$$\|A_1 \cdots A_j\|_{C^m_\Gamma} \leq (j-k)^t \|A_1\|_{C^m_\Gamma} \cdots \|A_j\|_{C^m_\Gamma}.$$

Proof. The proof is analogous to the proof of Proposition 4.14 above. When we differentiate m times a product of j matrices, we get j^m summands, and each summand can be bounded by $\|A_1\|_{C^m_\Gamma} \cdots \|A_j\|_{C^m_\Gamma}$. If some elements do not depend on θ, they do not add summands and the second statement follows.

□

The next proposition is a version of Proposition 3.12 for maps depending on $\theta \in \mathbb{T}^d$.

Proposition 4.16. *Let $A \in C^t_{L^k_\Gamma}(\mathbb{T}^d, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$ and $u \in C^t(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$. Then for any permutation of k elements $\tau \in S_k$ the map $B_{\tau,u}$ defined by*

$$B_{\tau,u}(v_1, \dots, v_{k-1}) = A(\theta)(\tau(v_1, \dots, v_{k-1}, u(\theta)))$$

satisfies $B_{\tau,u} \in C^t_{L^{k-1}_\Gamma}(\mathbb{T}^d, L^{k-1}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$, and

$$\|B_{\tau,u}\|_{C^t_{L^{k-1}_\Gamma}} \leq 2^t \|A\|_{C^t_{L^k_\Gamma}} \|u\|_{C^t}.$$

Proof. The proof follows an inductive procedure with respect to t. For the case $t = 0$ we write

$$B_{\tau,u}(\theta)(v_1, \dots, v_{k-1}) = A(\theta)(\tau(v_1, \dots, v_{k-1}, u(\theta))).$$

We have

$$\|B_{\tau,u}\|_{C^0_{L^{k-1}_\Gamma}} = \sup_{\theta \in \mathbb{T}^d} \|B_{\tau,u}(\theta)\|_\Gamma.$$

Given $1 \le p \le k-1$ we compute, for θ such that $u(\theta) \neq 0$,

$$\begin{aligned}
\gamma(\iota_p B_{\tau,u}(\theta)) &= \sup_{i,k\in\mathbb{Z}^m} \sup_{\substack{\|w\|\le 1\\ \mathrm{proj}_j\, w=0,\, j\neq k}} \|(\iota_p B_{\tau,u}(\theta))_i(w)\|\Gamma(i-k)^{-1} \\
&= \sup_{i,k\in\mathbb{Z}^m} \sup_{\substack{\|w\|\le 1\\ \mathrm{proj}_j\, w=0,\, j\neq k}} \sup_{\|v_l\|\le 1} \|[B_{\tau,u}(\theta)(v_2,\dots,v_p,w,\dots,v_{k-1})]_i\|\Gamma(i-k)^{-1} \\
&= \sup_{i,k\in\mathbb{Z}^m} \sup_{\substack{\|w\|\le 1\\ \mathrm{proj}_j\, w=0,\, j\neq k}} \sup_{\substack{\|v_l\|\le 1\\ u(\theta)\neq 0}} \|[A(\theta)(\tau(v_2,\dots,v_p,w,\dots,v_{k-1},\frac{u(\theta)}{\|u(\theta)\|}))]_i\|\|u(\theta)\|\Gamma(i-k) \\
&= \sup_{i,k\in\mathbb{Z}^m} \sup_{\substack{\|w\|\le 1\\ \mathrm{proj}_j\, w=0,\, j\neq k}} \sup_{\substack{\|v_l\|\le 1\\ u(\theta)\neq 0}} \|[\iota_p(A(\theta))(w)(v_2,\dots,v_{k-1},\frac{u(\theta)}{\|u(\theta)\|})]_i\|\Gamma(i-k)^{-1}\|u(\theta)\| \\
&\le \gamma(\iota_p(A(\theta)))\|u(\theta)\| \le \|A\|_{C^0_{L^k_\Gamma}}\|u\|_{C^0}.
\end{aligned}$$

If $u(\theta)=0$ we easily see that $\gamma(\iota_p B_{\tau,u}(\theta)) = 0$.

Let $t=1$, now $B_{\tau,u}$ is clearly a $(k-1)$-linear mapping of class C^1, i.e. $B_{\tau,u} \in C^1_{L^{k-1}_\Gamma}(\mathbb{T}^d, L^{k-1}(\ell^\infty(\mathbb{R}^n),\ell^\infty(\mathbb{R}^n)))$. We still have to check the bounds in Γ-norms. By definition,

$$\|B_{\tau,u}\|_{C^1_{L^{k-1}_\Gamma}} = \max_{0\le i\le 1}\left\|\frac{\partial^i B_{\tau,u}}{\partial\theta_i}\right\|_{C^0_{L_\Gamma}},$$

and

$$\left\|\frac{\partial B_{\tau,u}}{\partial\theta_i}\right\|_{C^0_{L_\Gamma}} \le \left\|\left(\frac{\partial B}{\partial\theta_i}\right)_{\tau,u}\right\| + \left\|B_{\tau,\frac{\partial u}{\partial\theta_i}}\right\|_{C^0_{L_\Gamma}}.$$

Since both $u(\theta)$ and $\frac{\partial u}{\partial\theta_i}(\theta)$ are in $S_{j,\Gamma}$ for all $\theta\in\mathbb{T}^d$, we can use the case $t=0$ to get the bound

$$\|B_{\tau,u}\|_{C^1_{L^k_\Gamma}} \le 2\|B\|_{C^1_{L_\Gamma}}\|u\|_{S^1_{j,\Gamma}}.$$

When $t>1$ we can use an induction argument based on the cases $t=0$, $t=1$ and the application of Leibniz's rule.

□

Remark 4.17. *Observe that each additional contraction adds a summand, thus applying p contractions to a multilinear map results in a factor p^t, i.e. given $v_1,\dots,v_p \in C^t(\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))$ and $A\in C^t_{L^k_\Gamma}$, $p\le k-1$, then*

$$\|A(v_1,\dots,v_p)\|_{C^t_{L^{k-p}_\Gamma}} \le p^t\|A\|_{C^t_{L^k_\Gamma}}\|v_1\|_{S^t_{j,\Gamma}}\|v_2\|_{C^t}\cdots\|v_p\|_{C^t}.$$

Lemma 4.18. *Let $M\in C^r_{L_\Gamma}(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n),\ell^\infty(\mathbb{R}^n)))$ of the form $M(\theta)=M_0+M_1(\theta)$, satisfying*

- *M_0 is invertible and $M_0^{-1}\in L_\Gamma(\ell^\infty(\mathbb{R}^n),\ell^\infty(\mathbb{R}^n))$, (or equivalently, $0\notin \mathrm{Spec}_\Gamma(M_0)$)*
- *$\|M_0^{-1}\|_\Gamma\|M_1\|_{C^r_{L_\Gamma}} < 1$.*

Then M is invertible, its inverse M^{-1} belongs to $C^r_{L_\Gamma}(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$ and

$$\|M^{-1} - M_0^{-1}\|_{C^r_{L_\Gamma}} = \mathcal{O}(\|M_1\|_{C^r_{L_\Gamma}}).$$

Proof. Since M_0 is invertible, we can write

$$M(\theta) = M_0\big(\operatorname{Id} + M_0^{-1} M_1(\theta)\big).$$

If $\|M_0^{-1}\|_\Gamma \|M_1\|_{C^r_{L_\Gamma}} < 1$ we can use Neumann's series to invert $M(\theta)$ as

$$M(\theta)^{-1} = \sum_{j=0}^{\infty} (-M_0^{-1} M_1(\theta))^j M_0^{-1} = M_0^{-1} + \sum_{j=1}^{\infty} (-M_0^{-1} M_1(\theta))^j M_0^{-1}$$

which is convergent in $\|\cdot\|_{C^r_{L_\Gamma}}$ if $\|M_0^{-1}\|_\Gamma \|M_1\|_{C^r_{L_\Gamma}} < 1$, since M_0 does not depend on θ (see Proposition 4.15.)

□

Lemma 4.19. *Given $M \in C^r_{L_\Gamma}(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$ such that we can write $M(\theta) = M_0 + \widetilde{M}(\theta)$ where M_0 is a constant uncoupled linear map we have the following bounds for all $m \le r$, $n \ge 2$:*

$$\|M(\theta) \cdots M(\theta + (n-1)\omega)\|_{C^0_{L_\Gamma}} \le \Gamma(0)^{-1} \left(\|M_0\| + \Gamma(0)^{-1} \|\widetilde{M}\|_{C^0_{L_\Gamma}}\right)^n,$$

$$\begin{aligned} &\|D^m_\theta \left(M(\theta) \cdots M(\theta + (n-1)\omega)\right)\|_{C^0_{L_\Gamma}} \le \\ &\qquad \Gamma(0)^{-2} n^m \|\widetilde{M}\|_{C^0_{L_\Gamma}} \left(\|M_0\| + \Gamma(0)^{-1} \|\widetilde{M}\|_{C^0_{L_\Gamma}}\right)^{n-1}, \ 1 \le n \le r. \end{aligned}$$

Remark 4.20. *Remember that from Proposition 4.15*

$$\|A(\theta)B(\theta)\|_{L_\Gamma} \le 2^r \|A(\theta)\|_{C^r_{L_\Gamma}} \|B(\theta)\|_{C^r_{L_\Gamma}},$$

by the combinatorics coming from Leibniz's rule.

Remark 4.21. *Remember also that from the properties of the decay functions,*

$$\begin{aligned} \|AB(\theta)A\|_{C^0_{L_\Gamma}} &\le \Gamma(0)^{-3} \|A\|^2 \|B(\theta)\|_{C^0}, \\ \|AAB(\theta)\|_{C^0_{L_\Gamma}} &\le \Gamma(0)^{-2} \|A\|^2 \|B(\theta)\|_{C^0}. \end{aligned}$$

Proof of Lemma 4.19. We will give details for $m \in \{0, 1\}$ and sketch the general case from it.

For the case $m = 0$ we want a bound for the $C^0_{L_\Gamma}$ norm of $\left(M_0 + \widetilde{M}(\theta)\right)^{[n]}$, where we denote $(A(\theta))^{[n]} = A(\theta)A(\theta+\omega)\cdots A(\theta + (n-1)\omega)$. As the norms we are considering involve taking suprema with respect to θ, we can ignore the angular shifts and just study the combinatorics of the norms of n products of linear maps. We can write $M(\theta)^{[n]} = M_0^n + B(\theta)$, where B is a term with all possible combinations of n products of M_0 and

$\widetilde{M}(\theta)$, except n times M_0. From the previous remarks, we need to collect terms wisely. To do so, consider the groupings of the factor M_0 in the products of n factors chosen from M_0 and $\widetilde{M}$. We can have all these factors M_0 together, completely spread out or arranged in several combinations.

Each term in $B(\theta)$ having $j \leq [n/2]$ factors M_0 (there are $\binom{n}{j}$ such terms) will have a bound having at most j factors $\Gamma(0)^{-1}$. When $j > [n/2]$ (again, there are $\binom{n}{j}$ such terms), there will be at most $n - j + 1$ factors $\Gamma(0)^{-1}$, since there are only $n - j$ factors $\widetilde{M}$ and thus there can only be $n - j + 1$ groupings of factors having only M_0 factors.

After taking norms, we can bound $\|M^{[n]}\|_\Gamma$ by the following expression

$$\sum_{j=0}^{[n/2]} \binom{n}{j} \Gamma(0)^{-j} \|M_0\|^j \|\widetilde{M}\|^{n-j}_{C^0_{L_\Gamma}} \tag{4.4}$$

$$+ \sum_{j=[n/2]+1}^{n-1} \binom{n}{j} \Gamma(0)^{-n+j-1} \|M_0\|^j \|\widetilde{M}\|^{n-j}_{C^0_{L_\Gamma}}. \tag{4.5}$$

Since in (4.4)+(4.5) the index j satisfies $j \leq [n/2]$ we have $j \leq n - j + 1$ and we can bound $\Gamma(0)^{-j}$ by $\Gamma(0)^{-n+j-1}$, hence by the binomial formula (4.4)+(4.5) is bounded by

$$\Gamma(0)^{-1} \left(\|M_0\| + \Gamma(0)^{-1} \|\widetilde{M}\|_{C^0_{L_\Gamma}} \right)^n.$$

Hence

$$\left\| \left(M_0 + \widetilde{M} \right)^{[n]} \right\|_{C^0_{L_\Gamma}} \leq \Gamma(0)^{-1} \left(\|M_0\| + \Gamma(0)^{-1} \|\widetilde{M}\|_{C^0_{L_\Gamma}} \right)^n. \tag{4.6}$$

For the case $m = 1$ notice that differentiating each term in the binomial decomposition we obtain as many new terms as the number of factors $\widetilde{M}$. Using the same trick to bound powers of $\Gamma(0)^{-1}$ as before we get

$$\begin{aligned}
\|D_\theta (M_0 + \widetilde{M})^{[n]}\|_{C^0_{L_\Gamma}} &\leq \sum_{j=0}^{n-1} \Gamma(0)^{-1} \binom{n}{j} (n-j) \|M_0\|^j \left(\Gamma(0)^{-1} \|\widetilde{M}\|_{C^0_{L_\Gamma}} \right)^{n-j} \\
&= \Gamma(0)^{-2} n \|\widetilde{M}\|_{C^0_{L_\Gamma}} \sum_{j=0}^{n-1} \binom{n-1}{j} \|M_0\|^j \left(\Gamma(0)^{-1} \|\widetilde{M}\|_{C^0_{L_\Gamma}} \right)^{n-1-j} \\
&= \Gamma(0)^{-2} n \|\widetilde{M}\|_{C^0_{L_\Gamma}} \left(\|M_0\| + \Gamma(0)^{-1} \|\widetilde{M}\|_{C^0_{L_\Gamma}} \right)^{n-1}.
\end{aligned}$$

For $1 \leq m \leq r$ we bound the number of new terms arising at every step of differentiation from each term by n. The same of type of arguments give the final bound in the statement.
□

Chapter 5

Spaces of differentiable functions with anisotropic differentiability

In this section we will introduce two notions of differentiable functions with anisotropic differentiability, i.e. functions with different regularity in the x and θ directions. The first will be the standard, independent notion of spaces $C^{t,r}$ (and related spaces with decay). To define the other concept we will follow the construction found in [CFdlL03b]. Once we have defined these spaces using the spaces defined in Chapter 4, thus introducing the spaces of anisotropic functions with decay and the spaces of anisotropic functions with j-localised decay.

5.1 The spaces $C^{t,r}_\Gamma$, $C^{t,r}_\Gamma$ and $C^{t,r,L}_\Gamma$

Definition 5.1. *Let U be an open set of $\ell^\infty(\mathbb{R}^n)$. We define*

$$\begin{aligned} C^{t,r}_\Gamma(U\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n)) = \{F\in C^{t,r}(U\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))\mid \\ D^i_x D^j_\theta F\in C^0(U\times\mathbb{T}^d, L^i_\Gamma(\ell^\infty(\mathbb{R}^n),\ell^\infty(\mathbb{R}^n))), \\ 1\le i\le t,\, 0\le j\le r\}, \end{aligned}$$

with norm

$$\|F\|_{C^{t,r}_\Gamma} = \max\big(\|F\|_{C^0}, \max_{\substack{1\le i\le t\\ 0\le j\le r}} \sup_{\substack{x\in U\\ \theta\in\mathbb{T}^d}} \|D^i_x D^j_\theta F(x,\theta)\|_\Gamma\big),$$

where $\|F\|_{C^0} = \sup_{\substack{x\in U\\ \theta\in\mathbb{T}^d}} \|F(x,\theta)\|$ as usual.

Definition 5.2. *Let U be an open set of $\ell^\infty(\mathbb{R}^n)$. We define*

$$C^{t,r}_{j,\Gamma}(U\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n)) = \{F\in C^{t,r}_\Gamma(U\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))\mid D^k_\theta F(\cdot,\theta)\in S^0_{j,\Gamma},\, 0\le k\le r\},$$

with norm

$$\|F\|_{C^{t,r}_{j,\Gamma}} = \max\big(\|F\|_{C^{t,r}_\Gamma}, \max_{0\le k\le r} \sup_{\substack{x\in U\\ \theta\in\mathbb{T}^d}} \|D^k_\theta F(x,\theta)\|_{S^0_{j,\Gamma}}\big).$$

The following results have technically simpler proofs than the equivalent results we will prove in the next section for the spaces $C^{\Sigma_{t,r}}$ (to be defined later), and thus we omit the proofs here.

Proposition 5.3. *Let $F \in C^{t,r}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$, $t \geq r+1$ and $g \in C^r(\mathbb{T}^d, U)$. Then $F \circ (g, \mathrm{Id}) \in C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ and*

$$\partial_\theta^q F(g(\theta), \theta) = \sum_{(b,a) \in \widetilde{\Sigma}_{0,q}} \sum_{\|I\|_1 = q-a} C \partial_\theta^a \partial_x^b F(g(\theta), \theta) \partial_\theta^{i_1} g(\theta) \cdots \partial_\theta^{i_b} g(\theta),$$

with $q \leq r$, where

$$\widetilde{\Sigma}_{0,i} = \{(k,i) \in \mathbb{N}^2 \mid k+i \leq i, k \geq 1\} \cup \{(0,i)\}$$

and $I = (i_1, \ldots, i_b)$ is a multi-index. We have used the notation ∂_θ^j with $j \in \mathbb{N}$ to denote $\frac{\partial^j}{\partial \theta_1^{\ell_1} \ldots \partial_{\theta_d}^{\ell_d}}$ with $\ell_1 + \ldots + \ell_d = j$.

Proposition 5.4. *Let $0 \in U \subseteq \ell^\infty(\mathbb{R}^n)$ be an open set, $F \in C^{t,r}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ and $g(\theta) \in S^r_{j,\Gamma}(\mathbb{T}^d, U)$. If $t \geq r+1$ then $F \circ (g, \mathrm{Id}) \in S^r_{j,\Gamma}$.*

5.2 The spaces $C^{\Sigma_{t,r}}$, $C^{\Sigma_{t,r}}_\Gamma$, $C^{\Sigma_{t,r}}_{j,\Gamma}$ and $C^{\Sigma_{t,r},L}_{j,\Gamma}$

Consider the following subsets of indices

$$\begin{aligned}
\Sigma_{t,r} &= \{(k,i) \mid i \leq r,\, i+k \leq r+t\}, \\
\widetilde{\Sigma}_{t,r} &= \{(k,i) \mid i \leq r,\, i+k \leq r+t,\, k \geq 1\}, \qquad t \geq 1 \\
\widetilde{\Sigma}_{0,r} &= \{(k,i) \in \mathbb{N}^2 \mid k+i \leq r, k \geq 1\} \cup \{(0,r)\}
\end{aligned}$$

and let $U \subset \ell^\infty(\mathbb{R}^n)$ be an open set in the $\|\cdot\|_\infty$ norm. We define the space of anisotropic differentiable functions F with derivatives $D_\theta^i D_x^k F$ with indices (k,i) in the set $\Sigma_{t,r}$. These spaces are well suited to deal with derivatives of compositions of the form $F(W(\theta), \theta)$. The first space is the following.

Definition 5.5. *Given U an open set of $\ell^\infty(\mathbb{R}^n)$ we define*

$$\begin{aligned}
C^{\Sigma_{t,r}}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) = \big\{ & F : U \times \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n) \mid \\
& D_\theta^i D_x^k F \in C^0(U \times \mathbb{T}^d, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))),\, (k,i) \in \Sigma_{t,r}, \\
& \|F\|_{C^{\Sigma_{t,r}}} < \infty \big\},
\end{aligned} \tag{5.1}$$

where

$$\|F\|_{C^{\Sigma_{t,r}}} = \max\left(\|F\|_{C^0}, \max_{(k,i) \in \Sigma_{t,r}} \|D_\theta^i D_x^k F\|_{C^0} \right), \tag{5.2}$$

and

$$\|D_x^k D_\theta^i F\|_i = \max_{i_1 + \ldots + i_j = i} \sup_{(x,\theta) \in U \times \mathbb{T}^d} \left\| D_x^k \frac{\partial^i}{\partial \theta_1^{i_1} \ldots \partial \theta_d^{i_d}} F(x, \theta \right\|_{L^k}.$$

We can define a version of this space with decay as the following

Definition 5.6. *Given U an open set of $\ell^\infty(\mathbb{R}^n)$ we define*

$$\begin{aligned} C_\Gamma^{\Sigma_{t,r}}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) = \big\{ & F : U \times \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n) \,|\, F \in C^{\Sigma_{t,r}}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)), \\ & D_\theta^i D_x^k F \in C_\Gamma^0(U \times \mathbb{T}^d, L_\Gamma^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))),\ (k,i) \in \widetilde{\Sigma}_{t,r}, \\ & \|F\|_{C_\Gamma^{\Sigma_{t,r}}} < \infty \big\}, \end{aligned} \tag{5.3}$$

where

$$\|F\|_{C_\Gamma^{\Sigma_{t,r}}} = \max\left(\|F\|_{C^{\Sigma_{t,r}}},\ \max_{(k,i)\in\widetilde{\Sigma}_{t,r}} \|D_\theta^i D_x^k F\|_{C_\Gamma^0} \right), \tag{5.4}$$

and

$$\|D_x^k D_\theta^i F\|_{C_\Gamma^0} = \max_{i_1+\ldots+i_d=1} \sup_{(x,\theta)\in U\times\mathbb{T}^d} \|D_x^k \frac{\partial^i}{\partial\theta_1^{i_1}\cdots\partial\theta_d^{i_d}} F(x,\theta)\|_{L_\Gamma^k}.$$

We can also define a j-localised version of this space.

Definition 5.7. *Given U an open set of $\ell^\infty(\mathbb{R}^n)$ we define*

$$\begin{aligned} C_{j,\Gamma}^{\Sigma_{t,r}}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) = \big\{ & F \in C_\Gamma^{\Sigma_{t,r}}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)), \\ & |\, D_\theta^i F(\cdot,\theta) \in S_{j,\Gamma}^0,\ 0 \le i \le r,\ \|F\|_{C_{j,\Gamma}^{\Sigma_{t,r}}} < \infty \big\} \end{aligned} \tag{5.5}$$

with the norm

$$\|F\|_{C_{j,\Gamma}^{\Sigma_{t,r}}} = \max\left(\|F\|_{C_\Gamma^{\Sigma_{t,r}}},\ \max_{1\le i\le r} \sup_{(x,\theta)\in U\times\mathbb{T}^d} \|D_\theta^i F\|_{S_{j,\Gamma}} \right) \tag{5.6}$$

Finally, we define spaces of l-flat functions with j-localised decay.

Definition 5.8. *Given U an open set of $\ell^\infty(\mathbb{R}^n)$ we define for $l \le t$*

$$\begin{aligned} C_{j,\Gamma}^{\Sigma t,r,l}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) = \{ & F \in C_{j,\Gamma}^{\Sigma_{t,r}}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) \,| \\ & D_x^j F(0,\theta) = 0,\ \forall\theta \in \mathbb{T}^d,\ 0 \le j \le l, \\ & \|F\|_{C_{j,\Gamma}^{\Sigma_{t,r,l}}} < \infty \}, \end{aligned} \tag{5.7}$$

where

$$\|F\|_{C_{j,\Gamma}^{\Sigma_{t,r,l}}} = \max\left(\|F\|_{C_{j,\Gamma}^{\Sigma_{t,r}}},\ \max_{i\le r} \sup_{x\in U\setminus\{0\}} \frac{\|D_\theta^i D_x^l F(x,\cdot)\|_{C^0_{L_\Gamma^l}}}{\|x\|} \right). \tag{5.8}$$

Remark 5.9. *The definition of $C_{j,\Gamma}^{\Sigma_{t,r}}$ is needed to study j-localised properties of invariant tori, forced by the chain rule. As an example, consider $W(\theta)$ an invariant object under the action of F, r-times differentiable w.r.t. θ with j-localised decay, i.e. $W \in S_{j,\Gamma}^r$. We need*

$F(W(\theta),\theta)$ to be as differentiable as $W(\theta)$ with respect to θ, to study the object as a fixed point. For the specific case of $r=2$,

$$\begin{aligned}D_\theta^2 F(W(\theta),\theta) &= D_x^2 F(W(\theta),\theta)(D_\theta W(\theta))^2 + 2D_\theta D_x F(W(\theta),\theta)D_\theta W(\theta)\\ &\quad + D_x F(W(\theta),\theta)D_\theta^2 W(\theta)\\ &\quad + D_\theta^2 F(W(\theta),\theta).\end{aligned}$$

Since we need $F(W(\theta),\theta)$ to be in $S^2_{j,\Gamma}$, a necessary condition comes now from Propositions 4.14 and 4.16, when $D_x^2 F$ and $D_x F$ are in $C^0_{L^2_\Gamma}$ and $C^0_{L_\Gamma}$ respectively and $W \in S^1_{j,\Gamma}$, then $D_x^2 F(D_\theta W, D_\theta W)$ and $D_\theta D_x F D_\theta W$ are in $S^0_{j,\Gamma}$ thus this is a sufficient condition is that derivatives with respect to x are in $C^0_{L_\Gamma}$ and derivatives with respect to (only) θ (resulting in vectors) are in $S^0_{j,\Gamma}$.

Proposition 5.10. *The space of functions $C^{\Sigma_{t,r}}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ is a Banach space.*

Proof. Since $C^{\Sigma_{t,r}}_{j,\Gamma}$ is a subset of $C^{\Sigma_{t,r}}$ and the latter is a Banach space, we can use Cauchy convergence in this space. Let $F_n(x,\theta)$ be a Cauchy sequence of elements of $C^{\Sigma_{t,r}}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$, thus $F_n(x,\theta)$ converges to $F_*(x,\theta)$ in $C^{\Sigma_{t,r}}(U\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ (as this is a Banach space), moreover $D_\theta^i D_x^k F_n(x,\theta)$ converges to $G^{k,i}(s,\theta)$ in $C^k_{L_\Gamma}$-norm, as $C^k_{L_\Gamma}$ is a Banach space. Now, as we have both uniform convergence for the function and its derivatives, $G^{k,i}(x,\theta) = D_\theta^i D_x^k F_*(x,\theta)$.

This is a well-known result, with a proof similar to the one given for the Banach space properties of $S^r_{j,\Gamma}$ in Proposition 4.7, but we prove it here for the first derivative with respect to x nevertheless.

Let $x_1 \in U$ and

$$G_n(x,\theta) = \begin{cases} \frac{F_n(x,\theta)-F_n(x_1,\theta)}{x-x_1} & x \neq x_1,\\ D_x F_n(x_1,\theta) & x = x_1,\end{cases}$$

for $x \in B(0,r) \subset U$. The function $G_n(x,\theta)$ is well-defined and is continuous, as $F_n \in C^{1,0}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$. Moreover

$$\lim_{n\to\infty} G_n(x,\theta) = \begin{cases} \frac{F_*(x,\theta)-F_*(x_1,\theta)}{x-x_1} & x \neq x_1,\\ G^{1,0}(x_1,\theta) & x = x_1,\end{cases}$$

in L_Γ-norm, taking supreme with respect to θ. Finally we check that the convergence is uniform:

$$(G_n - G_m)(x,\theta) = \begin{cases} \frac{F_n(x,\theta)-F_n(x_1,\theta)+F_m(x_1,\theta)-F_m(x,\theta)}{x-x_1} & x \neq x_1,\\ (D_x F_n - D_x F_m)(x_1,\theta) & x = x_1.\end{cases}$$

We can bound the function $(G_n - G_m)$ for $x \neq x_1$ by using the mean value theorem,

$$\begin{aligned}F_n(x,\theta) - F_n(x_1,\theta) + F_m(x_1,\theta) - F_m(x,\theta) = \int_0^1 \big(&D_x F_n(x+\lambda(x-x_1),\theta)(x-x_1)\\ &- D_x F_m(x+\lambda(x-x_1),\theta)(x-x_1)\big)d\lambda,\end{aligned}$$

which can be bounded by $\varepsilon\|x-x_1\|$, as $\{D_xF_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. When $x=x_1$ the function (G_n-G_m) can be bounded by ε, thus the convergence is uniform.

The norm of the limit function is finite, by using the same argument as in the proof of Proposition 3.7.

The proof can be readily adapted to higher order derivatives, at the expense of longer expressions.

□

We will now prove composition properties of functions of $C^{\Sigma_{t,r}}_{j,\Gamma}$ and $S^r_{j,\Gamma}$. These are crucial to study regularity properties of solutions of the invariance equations.

Proposition 5.11. *Let $0\in U\subseteq \ell^\infty(\mathbb{R}^n)$ be an open set, $F\in C^{\Sigma_{t,r}}_{j,\Gamma}(U\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))$ and $g\in S^r_{j,\Gamma}(\mathbb{T}^d,U)$. If $r\geq 1$ then $F\circ(g,\mathrm{Id})\in S^r_{j,\Gamma}$.*

Before proving this proposition, we need a technical lemma.

Lemma 5.12. *Let $F\in C^{\Sigma_{t,r}}(U\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))$ and $g\in C^r(\mathbb{T}^d,U)$. Then $F\circ(g,\mathrm{Id})\in C^r(\mathbb{T}^d,U)$ and*

$$\partial^q_\theta F(g(\theta),\theta)=\sum_{(b,a)\in\widetilde{\Sigma}_{0,q}}\ \sum_{\|I\|_1=q-a} C\partial^a_\theta\partial^b_x F(g(\theta),\theta)\partial^{i_1}_\theta g(\theta)\cdots\partial^{i_b}_\theta g(\theta),$$

with $q\leq r$ where $I=(i_1,\ldots,i_b)$ is a multi-index. We have used the notation ∂^j_θ with $j\in\mathbb{N}$ to denote $\frac{\partial^j}{\partial\theta_1^{\ell_1}\cdots\partial_{\theta_d}^{\ell_d}}$ with $\ell_1+\ldots+\ell_d=j$.

Proof. We prove this by induction over q. Let $q=1$ and apply the chain rule to get

$$\begin{aligned}\partial^1_\theta(F(g(\theta),\theta))&=D_xF(g(\theta),\theta)\partial^1_\theta g(\theta)+\partial^1_\theta F(g(\theta),\theta),\\ \widetilde{\Sigma}_{0,1}&=\{(0,1),\,(1,0)\}\,,\end{aligned}$$

which coincides with the stated formula. Assume the formula for the derivatives holds for all $k\leq q-1$, thus

$$\partial^{q-1}_\theta F(g(\theta),\theta)=\sum_{(b,a)\in\widetilde{\Sigma}_{0,q-1}}\ \sum_{\|I\|_1=q-1-a} C\partial^a_\theta D^b_x F(g(\theta),\theta)\partial^{i_1}_\theta g(\theta)\cdots\partial^{i_b}_\theta g(\theta). \tag{5.9}$$

Apply again the chain rule to differentiate an arbitrary summand of the sum in 5.9 with respect to θ_p, with $p\in\{1,\ldots,m\}$, yielding

$$\begin{aligned}&\partial^1_\theta\big(C\partial^a_\theta D^b_xF(g(\theta),\theta)\partial^{i_1}_\theta g(\theta)\cdots\partial^{i_b}_\theta g(\theta)\big)\\ &\quad=C\partial^{a+1}_\theta D^b_xF(g(\theta),\theta)\partial^{i_1}_\theta g(\theta)\cdots\partial^{i_b}_\theta g(\theta)\\ &\qquad+C\partial^a_\theta D^{b+1}_xF(g(\theta),\theta)\partial^1_\theta g(\theta)\partial^{i_1}_\theta g(\theta)\cdots\partial^{i_b}_\theta g(\theta)\\ &\qquad+C\partial^a_\theta D^b_xF(g(\theta),\theta)\partial^{i_1+1}_\theta g(\theta)\cdots\partial^{i_b}_\theta g(\theta)\\ &\qquad+\ldots+C\partial^a_\theta D^b_xF(g(\theta),\theta)\partial^{i_1}_\theta g(\theta)\cdots\partial^{i_b+1}_\theta g(\theta).\end{aligned}$$

In the previous expression we have exactly all the terms needed to get from $\widetilde{\Sigma}_{0,q-1}$ to $\widetilde{\Sigma}_{0,q}$, since

$$\widetilde{\Sigma}_{0,q}=\widetilde{\Sigma}_{0,q-1}\cup\{(b+1,a),\,(b,a+1)\,|\,a+b=q-1\}\cup\{(0,q)\},$$

therefore the formula holds for the q-th derivative and the result is proved.

□

Proof of Proposition 5.11. We will prove this result by a repeated application of Lemma 5.12, first by setting $r = 1$. We need to check that $\frac{\partial}{\partial\theta_p} F(g(\theta),\theta) \in S^0_{j,\Gamma}(\ell^\infty(\mathbb{R}^n))$ for any $p \in \{1,\dots,d\}$. We apply the properties stated in Proposition 4.14 to get

$$\left\|\frac{\partial}{\partial\theta_p} F(g(\theta),\theta)_i\right\|_{S^0_{j,\Gamma}} = \left\|D_x F(g(\theta),\theta)\frac{\partial}{\partial\theta_p} g(\theta) + \frac{\partial}{\partial\theta_p} F(g(\theta),\theta)\right\|_{S^0_{j,\Gamma}} \leq$$
$$\|F\|_{C^{\Sigma_{0,1}}_{j,\Gamma}} \|g\|_{S^1_{j,\Gamma}} + \|F\|_{C^{\Sigma_{0,1}}_{j,\Gamma}}$$

as we wanted.

Now differentiate again with respect to θ, with $r \geq 1$, getting the stated formula. To finish the proof we have to bound $\|\partial^m F(g(\theta),\theta)\|_{S^0_{j,\Gamma}}$. To do so we use the formula given in Lemma 5.12:

$$\partial^m_\theta(F(g(\theta),\theta)) = \sum_{(b,a)\in\widetilde{\Sigma_{0,m}}} \sum_{i_1+\ldots+i_b=m-a} C\partial^a_\theta D^b_x F(g(\theta),\theta)\partial^{i_1}_\theta g(\theta)\cdots\partial^{i_b}_\theta g(\theta).$$

Now the b derivatives of F w.r.t. x are in L^b_Γ and all derivatives of g are in $S_{j,\Gamma}$. Hence we can use Proposition 4.16 to get to get

$$\|\partial^m_\theta F(g(\theta),\theta)\|_{S^0_{j,\Gamma}} \leq \sum_{(b,a)\in\widetilde{\Sigma}_{0,m}} \sum_{i_1+\ldots+i_b=m-a} C\|F\|_{C^{\Sigma_{t,r}}_{j,\Gamma}} \|g\|_{S^0_{j,\Gamma}} \cdots \|g\|_{S^0_{j,\Gamma}} < \infty$$

and then the statement follows.

□

Lemma 5.13. *Let $F \in C^{\Sigma_{t,r},l}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ and $G \in C^{\Sigma_{t,r}}_{j,\Gamma}(V \times \mathbb{T}^d, U)$ such that $G(0,\theta) = 0$, $\forall\theta \in \mathbb{T}^d$ and U, V are open subsets of $\ell^\infty(\mathbb{R}^n)$. Then $F(G(x,\theta),\theta) \in C^{\Sigma_{t,r},l}_{j,\Gamma}(V \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ and*

$$\|F\circ G\|_{C^{\Sigma_{t,r},l}_{j,\Gamma}} \leq K\|F\|_{C^{\Sigma_{t,r},l}_{j,\Gamma}} \cdot \sigma^*\big(\|G\|_{C^{\Sigma_{t,r}}_{j,\Gamma}}, r+t\big),$$

where $\sigma^*(t,m) = \sum_{j=1}^m t^j$.

Proof. Since we can write $G(x,\theta) = G_1(\theta)x + \cdots$, it is clear that $D^i_x(F\circ G)(0,\theta) = 0$ by the chain rule, for $i \leq l$. Remember that to determine the $C^{\Sigma_{t,r},l}_{j,\Gamma}$-norm of a mapping H, we need to bound $D^i_\theta D^k_x H$ for $(k,i) \in \widetilde{\Sigma_{t,r}}$ in C^0 and $C^0_{L_\Gamma}$ norms (to find bounds for the expressions in the definition of the $C^{\Sigma_{t,r}}$ norm), bound the $S^0_{j,\Gamma}$-norm of $D^i_\theta H$ for $i \leq r$ (to have bounds in the $C^{\Sigma_{t,r}}_{j,\Gamma}$-norm) and finally bound the expression

$$\max_{i\leq r} \sup_{x\in U\setminus\{0\}} \frac{\|D^i_\theta D^l_x H(x,\theta)\|_{C^0_{L_\Gamma}}}{\|x\|}.$$

We can use an induction procedure similar to the one used in the proof of Lemma 5.12, and obtain the following formula for the derivative of this composition,

$$D^i_\theta D^k_x(F\circ G) = \sum_{(b,a)\in\widetilde{\Sigma}_{k,i}} \sum_{I,J} CD^a_\theta D^b_x F\circ G D^{i_1}_\theta D^{j_1}_x G\cdots D^{i_b}_\theta D^{j_b}_x G,$$

where C is a combinatorial coefficient, $I = (i_1, \dots, i_b)$, $J = (j_1, \dots, j_b)$ are multi-indices with $\|I\|_1 = i - a$ and $\|J\|_1 = k$. This formula also appears in Lemma 2.3 of [CFdlL03b] in an equivalent setting. We can bound this expression in C^0-norm by using the fact that $\|D_\theta^{i_p} D_x^{j_p} G(x,\theta)\|_{C^0} \le \|G\|_{C^{\Sigma_{t,r}}} \le \|G\|_{C^{\Sigma_{t,r}}_{j,\Gamma}}$ and $\|D_\theta^a D_x^b F \circ G\|_{C^0} \le \|F\|_{C^{\Sigma_{t,r}}} \le \|F\|_{C^{\Sigma_{t,r},l}_{j,\Gamma}}$, therefore

$$\begin{aligned}\|D_\theta^i D_x^k (F \circ G)\|_{C^0} &\le \sum_{(b,a)\in\widetilde{\Sigma}_{k,i}} \sum_{I,J} C\|F\|_{C^{\Sigma_{t,r}}_{j,\Gamma}} \|G\|^b_{C^{\Sigma_{t,r}}_{j,\Gamma}} \\ &\le \widetilde{C}\|F\|_{C^{\Sigma_{t,r}}_{j,\Gamma}} \cdot \sigma^*\big(\|G\|_{C^{\Sigma_{t,r}}_{j,\Gamma}}, r+t\big)\end{aligned}$$

as we wanted.

For the derivative $D_\theta^i(F \circ G)$ we can use the formula in Lemma 5.12 to get

$$D_\theta^i(F \circ G) = \sum_{(b,a)\in\widetilde{\Sigma}_{0,i}} \sum_{\|I\|_1 = i-a} C D_\theta^a D_x^b F \circ G D_\theta^{i_1} G \cdots D_\theta^{i_b} G,$$

and use the properties of the product of $C^r_{L^k_\Gamma}$ functions to bound in $S^0_{j,\Gamma}$ norm getting

$$\|D_\theta^i F \circ G\|_{S^0_{j,\Gamma}} \le \sum_{(b,a)\in\widetilde{\Sigma}_{0,i}} \sum_{\|I\|_1 = i-a} C\|F\|_{C^r_{L^b_\Gamma}} \|G\|^b_{S^r_{j,\Gamma}},$$

where C is a combinatorial coefficient depending on r, and we can bound as in the previous case.

Finally, we need to bound the last term in the definition of the $C^{\Sigma_{t,r},l}_{j,\Gamma}$-norm. We can use the previous arguments to likewise bound $\|D_\theta^i D_x^l (F \circ G)(x,\theta)\|_{C^0_{L_\Gamma}} / \|x\|$ by

$$C\|F\|_{C^{\Sigma_{t,r},l}_{j,\Gamma}} \sigma^*(\|G\|_{C^{\Sigma_{t,r}}_{j,\Gamma}}, i + l).$$

□

The following Corollary is straightforward from the previous lemmas.

Corollary 5.14. *Consider $P \in C^r_{L_\Gamma}(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$, $F \in C^{\Sigma_{t,r},l}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ and $G \in C^{\Sigma_{t,r}}_{j,\Gamma}(V \times \mathbb{T}^d, U)$ such that $G(0,\theta) = 0$, $\forall \theta \in \mathbb{T}^d$. Then $P(\theta) \cdot F(G(x,\theta),\theta) \in C^{\Sigma_{t,r},l}_{j,\Gamma}$ and*

$$\|P \cdot F \circ G\|_{C^{\Sigma_{t,r},l}_{j,\Gamma}} \le K\|P\|_{C^r_{L_\Gamma}} \|F\|_{C^{\Sigma_{t,r},l}_{j,\Gamma}} \cdot \sigma^*(\|G\|_{C^{\Sigma_{t,r}}_{j,\Gamma}}, r + t).$$

The following proposition can be proved used the same techniques as in Propositions 5.11 and 5.13.

Proposition 5.15. *(i) Let $F \in C^{\Sigma_{t,r}}(V, \ell^\infty(\mathbb{R}^n))$ and $G \in C^{\Sigma_{t,r}}(U, V)$ where U, V are open sets in $\ell^\infty(\mathbb{R}^n)$. Then $F \circ (G, \mathrm{Id})$ is a function in $C^{\Sigma_{t,r}}(U, \ell^\infty(\mathbb{R}^n))$.*

(ii) Let $F \in C^{\Sigma_{t,r}}_\Gamma(V, \ell^\infty(\mathbb{R}^n))$ and $G \in C^{\Sigma_{t,r}}_{j,\Gamma}(U, V)$ where U, V are open sets in $\ell^\infty(\mathbb{R}^n)$. Then $F \circ (G, \mathrm{Id})$ is a function in $C^{\Sigma_{t,r}}_{j,\Gamma}(U, \ell^\infty(\mathbb{R}^n))$.

Chapter 6

Spectral theory for Γ-coupled maps

In this Chapter we recall some results in spectral theory of linear operators and also introduce a new notion, the Γ-spectrum of a linear operator in a lattice, associated to a decay function Γ satisfying the definition introduced in Chapter 2. It turns out that this definition is very convenient when we deal with linear maps and operators with decay. We develop the basic theory for the Γ-spectrum which will be used to deal with the invariant manifolds we consider in this thesis and a few Sternberg conjugation theorems.

To that end we need to study the so-called Sylvester operators in spaces with decay. We will deal with two slightly different versions of these operators in two ways, depending on the spaces where they are defined. The first type are defined in the spaces $L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and the second one in $C^r_\Gamma(\mathbb{T}^d, L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$. In the latter case the actual way that $\theta \in \mathbb{T}^d$ is involved in the definition of the Sylvester operator leads to dealing with annular extensions of the spectrum of the associated linear operators.

Definition 6.1. *A complex algebra is a $\mathbb{C}$-vector space E with a product $E \times E \to E$ denoted by $(x, y) \mapsto xy$ satisfying*

- $x(yz) = (xy)z,$
- $(x + y)z = xz + yz,\ x(y + z) = xy + xz,$
- $\alpha(xy) = (\alpha x)y = x(\alpha y)$

for all $x, y, z \in E$, $\alpha \in \mathbb{C}$. We call an algebra unitary if it has a unit element $e \in E$ such that

- $ex = xe = x, \quad \forall x \in E,.$

A (complex) Banach algebra is a (complex) algebra such that E is a Banach space with norm $\|\cdot\|$ such that

- $\|xy\| \leq \|x\|\|y\|, \quad \forall x, y \in E,$
- *if the algebra is unitary,* $\|e\| = 1.$

Throughout this appendix E and F will stand for arbitrary complex Banach spaces. If we have to deal with a real Banach space V we can apply the results of this appendix to the complexified space $V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$.

Given a Banach space E the space of continuous linear maps $L(E, E)$ is also a Banach space with the standard operator norm. Moreover it is a Banach algebra with the product given by the composition of maps.

Let $E = \ell^\infty(\mathbb{R}^n)$. Given a decay function Γ as in Definition 2.2 the space $L_\Gamma(E, E)$ introduced in Section 3.1 is a Banach algebra (see Proposition 3.9.)

We have that $L_\Gamma(E, E) \subset L(E, E)$ as sets, but $L_\Gamma(E, E)$ is not a closed subalgebra of $L(E, E)$, so that it is not a Banach subalgebra of $L(E, E)$. Indeed, consider a concrete decay function:

$$\Gamma(j) = a|j|^{-\alpha} e^{-\theta|j|}, \quad j \in \mathbb{Z}^m$$

with $\alpha > m$, $\theta > 0$ and $a > 0$ small enough.

Consider the sequence of linear maps $\{A^k\}_{k\in\mathbb{N}}$ defined by

$$A^k = \begin{cases} A^k_{i,j} = |i-j|\Gamma(i-j), & |i-j| \le k \\ A^k_{i,j} = 0, & \text{otherwise.} \end{cases}$$

Clearly $A^k \in L(E, E) \cap L_\Gamma(E, E)$. Next we check that $\{A^k\}_{k\in\mathbb{N}}$ converges to A^∞ in $L(E, E)$, where $A^\infty_{i,j} = |i-j|\Gamma(i-j)$, $\forall i, j \in \mathbb{Z}^m$. Indeed

$$\begin{aligned} \|A^\infty - A^k\| &= \sup_{\substack{u\in E \\ \|u\|\le 1}} \|(A^\infty - A^k)u\| = \sup_{\|u\|\le 1} \sup_{i\in\mathbb{Z}^m} \| \sum_{|i-j|>k} |i-j|\Gamma(i-j)u_j \| \\ &\le \sum_{|l|>k} |l|\Gamma(l) \end{aligned}$$

which goes to zero as $k \to \infty$ because $\sum_{l\in\mathbb{Z}^m} |l|\Gamma(l)$ is convergent provided either $\theta > 0$ or $\theta = 0$ and $\alpha > m + 1$. However $A^\infty \notin L_\Gamma(E, E)$ because

$$\gamma(A^\infty) = \sup_{i,j\in\mathbb{Z}^m} |A_{i,j}|\Gamma(i-j)^{-1} = \sup_{i,j\in\mathbb{Z}^m} |i-j| = \infty.$$

$L_\Gamma(E, E)$ is a Banach algebra with the identity as unit, but $\|\operatorname{Id}\|_\Gamma = \Gamma(0)^{-1} \neq 1$. To be able to apply the general results of Banach algebras with unit we can introduce an equivalent norm in $L_\Gamma(E, E)$, $\|\cdot\|'$ such that $\|\operatorname{Id}\|' = 1$. The procedure is standard (see [Joh72]). We define

$$\|A\|' = \sup\left\{\|AC\|,\, C \in L_\Gamma(E, E),\, \|C\| \le 1\right\}.$$

The properties of norm are easily checked from the definition. We check the equivalence. On the one hand,

$$\|A\|' = \sup_{\|C\|\le 1} \|AC\| \le \sup_{\|C\|\le 1} \|A\|\|C\| = \|A\|$$

and on the other hand

$$\|A\|' \ge \|A\frac{\operatorname{Id}}{\|\operatorname{Id}\|}\| = \frac{1}{\|\operatorname{Id}\|}\|A\|.$$

Finally,

$$\|\operatorname{Id}\|' = \sup_{\|C\|\le 1} \|\operatorname{Id}\cdot C\| = \sup_{\|C\|\le 1} \|C\| = 1.$$

6.1 Two examples

Let $E = \ell^\infty(\mathbb{C}^n)$ and $A \in L(E, E)$ be such that $A_{ij} = C\delta_{ij}$ with $C \in L(\mathbb{C}^n, \mathbb{C}^n)$, i.e. an uncoupled linear map such that all the dynamics on the nodes are identical. Concretely, A is defined by $(Ax)_i = Cx_i$, $\forall x \in \ell^\infty(\mathbb{C}^n)$. Assume that $\operatorname{Spec} C = \{\lambda_1, \dots, \lambda_n\}$. We claim that

$$\operatorname{Spec} A = \{\lambda_1, \dots, \lambda_n\}.$$

Indeed, first we compute the resolvent of A. Let $\mu \in \mathbb{C}$, $\mu \neq \lambda_j$, $j = 1, \dots, n$. Given $y = (y_i)_{i \in \mathbb{Z}^m} \in \ell^\infty(\mathbb{C}^n)$ we consider the equation

$$(A - \mu \operatorname{Id})x = y$$

for $x = (x_i)_{i \in \mathbb{Z}^m} \in \ell^\infty(\mathbb{C}^n)$. The solution of the equation is straightforward due to the uncoupledness of the system, hence for each node j we have

$$(C - \mu \operatorname{Id})x_j = y_j$$

which has the unique solution $x_i = (C - \mu \operatorname{Id})^{-1} y_i$. Clearly

$$|x_i| \leq \|(C - \mu \operatorname{Id})^{-1}\| |y_i| \leq \|(C - \mu \operatorname{Id})^{-1}\| \|y\|$$

and therefore $x = (x_i)_{i \in \mathbb{Z}^m} \in \ell^\infty(\mathbb{C}^n)$. This proves that the resolvent of C is contained in the resolvent of A. Then we deduce that $\operatorname{Spec} A = \operatorname{Spec} C$, since the relation $\operatorname{Spec} C \subset \operatorname{Spec} A$ is obvious.

Now we provide examples of arbitrary (compact) spectrum. Actually we will show that given a compact set $K \subset \mathbb{C}$ there exists a linear map $A \in L(E, E)$, with $E = \ell^\infty(\mathbb{C}^2)$ over a one dimensional lattice (i.e. the lattice is indexed by $j \in \mathbb{Z}$) such that $\operatorname{Spec} A = K$. For the sake of concreteness, we assume that we are working with the Euclidean norm in $\mathbb{C}^2$, although the spectrum is an intrinsic object that does not depend on the norm.

Indeed, since K is compact there exists a countable dense subset $S \subset K$. We write

$$S = \{\gamma_j \mid j \in \mathbb{Z}\}.$$

Let $C_j \in L(\mathbb{C}^2, \mathbb{C}^2)$ such that γ_j is an eigenvalue of C_j and $\|C_j\| = |\gamma_j|$. In fact, we can explicitly set

$$C_j = \begin{pmatrix} \gamma_j & 0 \\ 0 & \gamma_j \end{pmatrix} \quad \text{if } \gamma_j \in \mathbb{R}$$

and

$$C_j = \begin{pmatrix} \alpha_j & -\beta_j \\ \beta_j & \alpha_j \end{pmatrix} \quad \text{if } \gamma_j = \alpha_j + i\beta_j \in \mathbb{C}$$

otherwise.

Now let $A \in L(E, E)$ be determined by $A_{ij} = C_j \delta_{ij}$, i.e. $(Ax)_i = C_i x_i$ for all $x \in \mathbb{C}^n$. Clearly it is well defined because

$$\|Ax\| = \sup_{j \in \mathbb{Z}} \|C_j x_j\| \leq \sup_{j \in \mathbb{Z}} \|C_j\| |x_j| \leq (\sup_{j \in \mathbb{Z}} \|C_j\|) \|x\|$$

and $\sup_{j \in \mathbb{Z}} \|C_j\| = \sup_{\gamma \in K} |\gamma| < \infty$.

If $\mu \notin K$, $\operatorname{dist}(\mu, K) > 0$. Let $y = (y_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{C}^2)$ and consider again the equation

$$(A - \mu \operatorname{Id})x = y$$

for $x = (x_j)_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{C}^2)$. It is equivalent to

$$(C_j - \mu \operatorname{Id})x_j = y_j, \quad j \in \mathbb{Z}$$

and hence we have that

$$x_j = (C_j - \mu \operatorname{Id})^{-1} y_j$$

and

$$|x_j| \le \|(C_j - \mu \operatorname{Id})^{-1}\| |y_j| \le \frac{1}{\operatorname{dist}(\mu, K)} \|y\|.$$

The last inequality is obvious if C_j is $(\gamma_j \operatorname{Id})$. For the other case we use the identity

$$\|(C_j + \mu \operatorname{Id})^{-1}\| = \left\| \begin{pmatrix} \alpha_j - \mu & -\beta_j \\ \beta_j & \alpha_j - \mu \end{pmatrix}^{-1} \right\| = \frac{1}{|\alpha_j + i\beta_j - \mu|}.$$

However, if $\mu \in S$,

$$(A - \mu \operatorname{Id})x = y \tag{6.1}$$

has no solution and if $\mu \in K \backslash S$ we can solve Equation (6.1) but for $y \neq 0$ the obtained solution x does not belong to $\ell^\infty(\mathbb{C}^2)$ since $\|(C_j - \mu \operatorname{Id})^{-1}\|$ is unbounded with respect to j.

6.2 Γ-spectrum of linear maps on lattices

Consider the lattice $\ell^\infty(\mathbb{R}^n)$ and a decay function Γ. Also consider the complexified space

$$\ell^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C} \sim \ell^\infty(\mathbb{R}^n) \oplus i\ell^\infty(\mathbb{R}^n) \sim \ell^\infty(\mathbb{C}^n).$$

Let $\mathcal{E}$ be a linear subspace of $\ell^\infty(\mathbb{C}^n)$. Given $A \in L_\Gamma(\mathcal{E}, \mathcal{E})$ we define:

- Γ-resolvent of A as

$$\rho_\Gamma(A) = \{\lambda \in \mathbb{C} \mid A - \lambda \operatorname{Id} \text{ is invertible and } (A - \lambda \operatorname{Id})^{-1} \in L_\Gamma(\mathcal{E}, \mathcal{E})\},$$

- Γ-spectrum of A as

$$\operatorname{Spec}_\Gamma A = \mathbb{C} \backslash \rho_\Gamma(A),$$

- Γ-spectral radius of A as

$$r_\Gamma(A) = \sup\{|\lambda| \mid \lambda \in \operatorname{Spec}_\Gamma(A)\}.$$

From the definition it is immediate that

$$\rho_\Gamma(A) \subset \rho(A)$$

and therefore

$$\operatorname{Spec}(A) \subset \operatorname{Spec}_\Gamma(A), \quad r(A) \le r_\Gamma(A).$$

The fact that $L_\Gamma(\mathcal{E}, \mathcal{E})$ is a Banach algebra implies that $\operatorname{Spec}_\Gamma(A)$ is a compact subset of $\mathbb{C}$.

Remark 6.2. *The theory of* Γ*-spectrum is similar to the theory of the spectrum of linear maps between Banach spaces but the proofs have to be adapted to this setting, because the algebra* $L(\mathcal{E},\mathcal{E})$ *is different from* $L_\Gamma(\mathcal{E},\mathcal{E})$*. In the next sections we develop this theory to be used later for Sylvester operators.*

An easy consequence of the definitions is the following lemma.

Lemma 6.3. *Let* $B \in L_\Gamma(\mathcal{E},\mathcal{E})$*. If* $0 \notin \operatorname{Spec}_\Gamma(B)$ *then*

$$\lambda \in \operatorname{Spec}_\Gamma(B) \Leftrightarrow \lambda^{-1} \in \operatorname{Spec}_\Gamma(B^{-1}).$$

Proof. It follows from the identities

$$\begin{aligned} B^{-1} - \lambda^{-1} &= -\lambda^{-1}B^{-1}(B-\lambda), \\ B - \lambda &= -\lambda B(B^{-1} - \lambda^{-1}), \end{aligned}$$

and the algebra properties of $L_\Gamma(\mathcal{E},\mathcal{E})$.

□

To illustrate some features of $L_\Gamma(\mathcal{E},\mathcal{E})$ we present an example of an invertible linear map in $L_\Gamma(\ell^\infty(\mathbb{C}^n),\ell^\infty(\mathbb{C}^n))$ such that its inverse may not be in $L_\Gamma(\ell^\infty(\mathbb{C}^n),\ell^\infty(\mathbb{C}^n))$ depending on the decay function Γ considered.

Let $\ell^\infty(\mathbb{C})$ be a one dimensional lattice ($m = 1$,) $r \in \mathbb{N}$, and $A \in L(\ell^\infty(\mathbb{C}^n),\ell^\infty(\mathbb{C}^n))$ determined by

$$\begin{aligned} A_{ij} &= 0, \quad \text{if either } j < i \text{ or } j > i + r, \\ A_{ii} &= a_0, \\ A_{i,i+1} &= a_1, \\ &\vdots \\ A_{i,i+r} &= a_r, \end{aligned}$$

with $a_k \in \mathbb{C}$.

Clearly $A \in L_\Gamma(\ell^\infty(\mathbb{C}),\ell^\infty(\mathbb{C}))$ for any decay function Γ, since

$$\gamma(A) = \sup_{i,j} |A_{ij}|\Gamma(i-j)^{-1} = \max\{|a_0|\Gamma(0)^{-1}, |a_1|\Gamma(1)^{-1}, \ldots, |a_r|\Gamma(r)^{-1}\} < \infty$$

and

$$\|A\| = \sup_{\|x\|\le 1} \|Ax\| = \sup_{\|x\|\le 1} \|(\ldots, 0, a_0x_0, a_1x_1, \ldots, a_rx_r, 0, \ldots)\| \le \max(|a_0|, \ldots, |a_r|).$$

We look for the inverse B of A assuming "a priori" that it is upper triangular and that it is a band matrix, i.e. $B_{ij} = b_{j-i}$ for some $b_k \in \mathbb{C}$, with $b_k = 0$ if $k < 0$.

Imposing the condition $AB = \mathrm{Id}$, or equivalently

$$\sum_{k\in\mathbb{Z}} A_{ik}B_{kj} = \delta_{ij}$$

we get

$$a_0 b_{j-i} + a_1 b_{j-i-1} + \ldots + a_r b_{j-i-r} = \delta_{ij}.$$

When $i = j$ we have $a_0 b_0 = 1$. This condition implies $a_0 \neq 0$. We assume it from now on. Then we proceed by induction and recursively obtain b_j for $j > 0$. Actually b_j satisfies the r-th order linear difference equation

$$b_j = -\frac{a_1}{a_0} b_{j-1} - \frac{a_2}{a_0} b_{j-2} - \ldots - \frac{a_r}{a_0} b_{j-r}, \quad j \geq 1,$$

with initial conditions $b_0 = 1/a_0$, $b_{-1} = 0, \ldots, b_{-r+1} = 0$.

Using the theory of linear difference equations we can compute b_j in terms of the zeros of the characteristic polynomial of this equation,

$$a_0 x^r + a_1 x^{r-1} + \ldots + a_r = 0.$$

Once we have determined b_j we can check that formally

$$AB = BA = \mathrm{Id}\,.$$

For this to hold it is important that $A_{ik} \in L(\mathbb{R}, \mathbb{R}) \sim \mathbb{R}$. It remains to check that B sends $\ell^\infty(\mathbb{R})$ to itself. This will depend on the choice of the values of a_i.

To work with a concrete example, assume $r = 2$ and $a_0 = 1$. Hence we can determine the zeros of the characteristic polynomial and determine the general solution of the difference equation as

$$b_j = \beta_1 \left(\frac{-a_1 + \sqrt{a_1^2 - 4a_2}}{2} \right)^j + \beta_2 \left(\frac{-a_1 - \sqrt{a_1^2 - 4a_2}}{2} \right)^j, \quad j \geq 1,$$

for suitable values of β_1, β_2.

Now we can choose a_1, a_2 to adjust the growth of the coefficients b_j. For instance, taking $a_1 = -\frac{3}{4}$, $a_2 = \frac{1}{8}$, then $b_j = 2\left(\frac{1}{2}\right)^j - \left(\frac{1}{4}\right)^j$. In this case $B \in L(\ell^\infty(\mathbb{R}), \ell^\infty(\mathbb{R}))$. With the choice of decay function $\Gamma(j) = a|j|^{-\alpha} e^{-\theta|j|}$ we have that

$$\begin{aligned} \gamma(B) &= \sup_{i,j} |B_{ij}| \Gamma(i-j)^{-1} \\ &= \max \left(\frac{1}{a}, \sup_{j-i \geq 1} \left[2\left(\frac{1}{2}\right)^{j-i} - \left(\frac{1}{4}\right)^{j-i} \right] a^{-1} |j-i|^{\alpha} e^{\theta|j-i|} \right) \end{aligned}$$

which is finite provided that $\theta < \log 2$. Hence $B \in L_\Gamma(\ell^\infty(\mathbb{R}), \ell^\infty(\mathbb{R}))$ (with this particular choice of decay function Γ) if and only if $\theta < \log 2$.

6.3 Integration of Banach space valued functions

We introduce the so-called Cauchy-Bochner integral (see [AMR88]) and adapt it to deal with complex line integrals. First we recall the Linear Extension Theorem

Theorem 6.4 (Linear Extension Theorem)**.** *Let E, F be normed vector spaces with $F \subset E$, G a Banach space and $A \in L(F, G)$. Then the closure $\overline{F}$ of F is a normed subspace of E and the map A can be extended in a unique way to a map $\overline{A} \in L(\overline{F}, G)$. Moreover $\|\overline{A}\| = \|A\|$.*

Let $[a, b]$ be an interval of $\mathbb{R}$ and E a Banach space.

Definition 6.5. *A map $f : [a, b] \to E$ is a step function if there exists a partition $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ such that $f|_{[t_{i-1}, t_i)}$ is constant (the values of f at t_i do not matter.) We denote by $S([a, b], E)$ the set of step functions from $[a, b]$ to E.*

Remark 6.6. *It is easy to check that $S([a, b], E)$ is a vector subspace of the Banach space $B([a, b], E)$ of bounded functions from $[a, b]$ to E.*

We define the integral of a step function by

$$\int_a^b f = \int_a^b f(t)\, dt = \sum_{j=0}^{n-1} (t_{j+1} - t_j) f(t_j).$$

It is easy to verify that this definition of integral is independent of the partition. Also note that

$$\left\| \int_a^b f \right\| \leq \int_a^b \|f\| \leq (b - a)\|f\|_\infty,$$

where $\|f\|_\infty = \sup_{a \leq t \leq b} |f(t)|$, that is, the operator

$$\int_a^b : S([a, b], E) \to E$$

is continuous and linear. By the Linear Extension Theorem (Theorem 6.4), it extends to a continuous linear map

$$\int_a^b \in L(\overline{S([a, b], E)}, E).$$

Definition 6.7. *The extended linear map $\int_a^b$ is called the Cauchy-Bochner integral.*

Note that the Cauchy-Bochner integral satisfies

$$\left\| \int_a^b f \right\| \leq \int_a^b \|f\| \leq (b - a)\|f\|_\infty$$

and the usual properties of the integral such as

$$\int_a^b \lambda f = \lambda \int_a^b f$$

and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

as can be readily verified. For $b < a$ we define

$$\int_a^b f = -\int_b^a f.$$

Observe that

$$C^0([a, b], E) \subset \overline{S([a, b], E)} \subset B([a, b], E).$$

We can prove the first inclusion using uniform continuity of continuous functions in the compact set $[a,b]$ and the second inclusion is trivial. Observe also that the Cauchy-Bochner integral defined in this section is less general than the Riemann integral if $E=\mathbb{R}$.

Note that if E and F are Banach spaces, $A \in L(E,F)$ and $f \in \overline{S([a,b],E)}$, we have $A \circ f \in \overline{S([a,b],F)}$ since

$$\|A \circ f_n - A \circ f\|_\infty \leq \|A\| \|f_n - f\|_\infty,$$

where $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of step functions in E converging to f. Moreover,

$$\int_a^b A \circ f = A\left(\int_a^b f\right),$$

since the same relation holds for step functions.

Let $A \in \overline{S([a,b],L(E,F))}$ and $u \in E$. We have

$$\int_a^b A(s)u\, ds = \left(\int_a^b A(s)\, ds\right) u. \tag{6.2}$$

Note that the previous formula involves two different integrals (in E and $L(E,F)$, respectively.) Indeed, it is easily seen that (6.2) is true if $A \in S([a,b],L(E,F))$. Now let $A_k \in S([a,b],L(E,F))$ such that $\{A_k\}_{k\in\mathbb{N}}$ is a sequence converging to A. From the inequality

$$\sup_{s\in[a,b]} \|A_k(s)u - A(s)u\| \leq \sup_{s\in[a,b]} \|A_k(s) - A(s)\| \|u\|$$

we conclude that

$$\lim_{k\to\infty} A_k(s)u = A(s)u.$$

Since $s \mapsto A_k(s)u$ belongs to $S([a,b],F)$ we have that

$$\begin{aligned}\int_a^b A(s)u\, ds &= \int_a^b \left(\lim_k A_k(s)\right) u\, ds = \int_a^b \lim_k A_k(s)u\, ds = \lim_k \int_a^b A_k(s)u\, ds \\ &= \lim_k \left(\int_a^b A_k(s)\, ds\right) u = \left(\int_a^b \lim_k A_k(s)\right) u = \left(\int_a^b A(s)\, ds\right) u.\end{aligned}$$

In the particular case that $E = \ell^\infty(\mathbb{R}^n)$, if $f \in \overline{S([a,b],E)}$, since proj_j is a linear and continuous operator

$$\left(\int_a^b f(t)\, dt\right)_i = \int_a^b f_i(t)\, dt.$$

Let Γ be a decay function. We will consider integrals of functions from $[a,b]$ to $L_\Gamma(\mathcal{E},\mathcal{F})$. Since $L_\Gamma(\mathcal{E},\mathcal{F})$ is a Banach space the general definition of integral applies.

Proposition 6.8. *If* $A \in \overline{S([a,b],L_\Gamma(\mathcal{E},\mathcal{F}))}$ *then*

(i) $\int_a^b A(t)\, dt \in L_\Gamma(\mathcal{E},\mathcal{F})$,

(ii) $\left\|\int_a^b A(t)\, dt\right\|_\Gamma \leq \int_a^b \|A(t)\|_\Gamma\, dt$.

Proof. It is immediate from the definitions that if $A \in S([a,b], L_\Gamma(\mathcal{E},\mathcal{F}))$ then $\int_a^b A(t)\,dt \in L_\Gamma(\mathcal{E},\mathcal{F})$. Now let $A \in \overline{S([a,b], L_\Gamma(\mathcal{E},\mathcal{F}))}$ and $\{A_k\}_{k\in\mathbb{N}}$ be a sequence in $S([a,b], L_\Gamma(\mathcal{E},\mathcal{F}))$ such that $\lim_k A_k = A$. This condition implies

$$\sup_{a\le t\le b} \gamma(A_k(t) - A(t)) = \sup_{a\le t\le b} \sup_{i,j} \sup_{\substack{\|u\|\le 1\\ u_l=0, l\neq j}} \|[A_k(t) - A(t)]_i u\| \Gamma(i-j)^{-1} < \varepsilon$$

and $\sup_t \|A_k(t) - A(t)\| < \varepsilon$ provided $k > k_0$ for some k_0 depending on ε. Then

$$\begin{aligned} &\left\| \left[\int_a^b (A_k(t) - A(t))\ dt\right]_i u\right\| \Gamma(i-j)^{-1} \\ &= \left\|\int_a^b [A_k(t) - A(t)]_i\, u\, dt\right\| \Gamma(i-j)^{-1} \\ \le \int_a^b &\|\,[A_k(t) - A(t)]_i\, u\|\, dt\, \Gamma(i-j)^{-1} \le \varepsilon(b-a) \end{aligned}$$

and

$$\left\|\int_a^b [A_k(t) - A(t)]\ dt\right\| \le \int_a^b \|A_k(t) - A(t)\|\, dt \le \varepsilon(b-a).$$

Hence $\int_a^b A_k(t)\,dt - \int_a^b A(t)\,dt \in L_\Gamma(\mathcal{E},\mathcal{F})$ and therefore $\int_a^b A(t)\,dt \in L_\Gamma(\mathcal{E},\mathcal{F})$.

Moreover, by $\left\|\int_a^b A(t)\,dt\right\| \le \int_a^b \|A(t)\|\,dt$ and

$$\begin{aligned} \gamma\left(\int_a^b A(t)\,dt\right) &= \sup_{i,j} \sup_{\substack{\|u\|\le 1\\ u_l=0, l\neq j}} \left\|\left(\int_a^b A(t)\,dt\right)_i u\right\| \Gamma(i-j)^{-1} \\ &\le \sup_{i,j} \sup_{\substack{\|u\|\le 1\\ u_l=0, l\neq j}} \left\|\int_a^b [A(t)]_i\, u\, dt\right\| \Gamma(i-j)^{-1} \\ &\le \int_a^b \sup_{i,j} \sup_{\substack{\|u\|\le 1\\ u_l=0, l\neq j}} \|[A(t)]_i u\| \Gamma(i-j)\, dt \\ &\le \int_a^b \gamma\left(A(t)\right)\ dt \end{aligned} \tag{6.3}$$

we obtain (ii).

□

Analogously we can prove the following result.

Proposition 6.9. *If* $f \in \overline{S([a,b], S_{j,\Gamma})}$ *then*

(i) $\int_a^b f(t)\,dt \in S_{j,\Gamma}$,

(ii) $\left\|\int_a^b f(t)\,dt\right\|_{S_{j,\Gamma}} \le \int_a^b \|f(t)\|_{S_{j,\Gamma}}\,dt$.

If furthermore $A \in L_\Gamma$ *then*

$\left\| \int_a^b Af(t)\,dt \right\|_{S_{j,\Gamma}} \leq \|A\|_\Gamma \int_a^b \|f(t)\|_{S_{j,\Gamma}}\,dt.$

We also have the following properties.

- if $A \in L_\Gamma(\mathcal{E},\mathcal{F})$ and $g : [a,b] \to \mathcal{E}$ then

$$\int_a^b Ag(t)\,dt = A\int_a^b g(t)\,dt$$

  and

$$\left\| \int_a^b Ag(t)\,dt \right\| \leq \|A\|_\Gamma \left\| \int_a^b g(t)\,dt \right\|,$$

- if $A : [a,b] \to L_\Gamma(\mathcal{E},\mathcal{F})$ belongs to $\overline{S([a,b], L_\Gamma(\mathcal{E},\mathcal{F}))}$ and $u \in \mathcal{E}$

$$\int_a^b A(s)u\,ds = \left(\int_a^b A(s)\,ds\right)u$$

  and

$$\left\| \int_a^b A(s)u\,ds \right\| \leq \left\| \int_a^b A(s)\,ds \right\|_\Gamma \|u\|.$$

6.4 Complex line integrals

If Ω is an open set of $\mathbb{C}$, E is a complex Banach space, $\gamma : [a,b] \to \Omega$ is a curve in Ω and $f : \Omega \to E$ is analytic then we define

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_a^b \mathrm{Re}(f(\gamma)\gamma'(t))\,dt + i\int_a^b \mathrm{Im}(f(\gamma)\gamma'(t))\,dt.$$

From this definition the usual properties of complex integrals follow. In particular

- Linearity of the integral with respect to f.
- $\| \int_\gamma f(z)\,dz \| \leq \sup_{z\in\gamma^*} \|f(z)\| \mathrm{length}(\gamma)$, where $\gamma^* = \gamma([a,b])$.
- If $A \in L(E,F)$, then $\int_\gamma Af(z)\,dz = A\int_\gamma f(z)\,dz$.
- If $A \in C^0(\Omega, L(E,F))$, $u \in E$,

$$\int_\gamma A(z)u\,dz = \left(\int_\gamma A(z)\,dz\right)u$$

and the analogous properties for functions over $L_\Gamma(\mathcal{E},\mathcal{F})$.

Moreover, if $f(z)$ is analytic on a simply connected domain Ω and γ is a closed curve in Ω then

$$\int_\gamma f(z)\,dz = 0.$$

Lemma 6.10. *Let $A \in L_\Gamma(\mathcal{E}, \mathcal{E})$. If $z, w \in \rho_\Gamma(A)$ then*

$$(z - A)^{-1} - (w - A)^{-1} = (w - z)(z - A)^{-1}(w - A)^{-1}. \tag{6.4}$$

Proof. Let $u \in \mathcal{E}$ and $v = (z - A)^{-1}u$. Then

$$(z - A)v = u$$

and

$$(w - A)v = (w - z)v + u.$$

Hence

$$\begin{aligned}(z - A)^{-1}u - (w - A)^{-1}u &= v - (v - (w - z)(w - A)^{-1}v) \\ &= (w - z)(w - A)^{-1}(z - A)^{-1}u\end{aligned}$$

which proves the statement.

□

Lemma 6.11. *Let $A \in L_\Gamma(\mathcal{E}, \mathcal{E})$ and $w \in \rho_\Gamma(A)$. If $|z-w|\|(w-A)^{-1}\|_\Gamma < 1$ then $z \in \rho_\Gamma(A)$ and*

$$(z - A)^{-1} = \sum_{k=1}^{\infty}(w - z)^{-k-1}(w - A)^{-k} \tag{6.5}$$

and the series converges in $L_\Gamma(\mathcal{E}, \mathcal{E})$.

Proof. We write (6.4) in the form

$$(z - A)^{-1} = (w - A)^{-1} + (w - z)(z - A)^{-1}(w - A)^{-1}.$$

Substituting $(z - A)^{-1}$ into the right hand side and iterating the process we obtain

$$(z - A)^{-1} = \sum_{k=1}^{m}(w - z)^{k-1}(w - A)^{-k} + (w - z)^m(z - A)^{-1}(w - A)^{-m}.$$

Expression (6.5) follows from the fact that

$$|w - z|^m\|(z - A)^{-1}\|_\Gamma\|(w - A)^{-1}\|_\Gamma^m \underset{m\to\infty}{\longrightarrow} 0.$$

□

Lemmas 6.10 and 6.11 prove the continuity and analiticity of $z \mapsto (z-A)^{-1}$ respectively.

6.5 Operational calculus

Given $A \in L_\Gamma(\mathcal{E},\mathcal{F})$, let Ω be an open set such that $\mathrm{Spec}_\Gamma(A) \subset \Omega$ and let ω be an open set such that

$$\mathrm{Spec}_\Gamma(A) \subset \omega \subset \overline{\omega} \subset \Omega$$

and $\partial\omega$ is a finite union of closed curves.

Given $f : \Omega \to \mathbb{C}$ analytic we define

$$f(A) = \frac{1}{2\pi i}\int_{\partial\omega} f(z)(z-A)^{-1}\,dz.$$

This definition is independent of the choice of ω provided it satisfies the previous conditions.

Lemma 6.12. *We have that*

$$f(A) \in L_\Gamma(\mathcal{E},\mathcal{F}).$$

Proof. Let $\omega : [a,b] \to \partial\omega$ be a parametrisation of $\partial\omega$ and v an arbitrary element of $S_{j,\Gamma}$. We have

$$\begin{aligned}
\|f(A)v\|_{S_{j,\Gamma}} &= \frac{1}{2\pi}\left\|\int_{\partial\omega} f(z)(z-A)^{-1}v\,dz\right\|_{S_{j,\Gamma}}\\
&\le \frac{1}{2\pi}\int_a^b |f(\gamma(t))|\|(\gamma(t)-A)^{-1}v\|_{S_{j,\Gamma}}|\gamma'(t)|\,dt\\
&\le \frac{1}{2\pi}\sup_{z\in\partial\omega}|f(z)|\sup_{z\in\partial\omega}\|(z-A)^{-1}v\|_{S_{j,\Gamma}}\mathrm{length}(\partial\omega)\\
&\le \frac{\mathrm{length}(\partial\omega)}{2\pi}\sup_{z\in\partial\omega}|f(z)|\sup_{z\in\partial\omega}\|(z-A)^{-1}\|_\Gamma\|v\|_{S_{j,\Gamma}} \approx C\|v\|_{S_{j,\Gamma}},
\end{aligned}$$

where C depends on $\partial\omega$, f and A but not on v. By statement 2 of Proposition 3.8 we have that $f(A) \in L_\Gamma(\mathcal{E},\mathcal{F})$.

□

In the case that f is a polynomial, $f(z) = \sum_{k=0}^m a_k z^k$, the previous definition gives $f(A) = \sum_{k=0}^m a_k A^k$. Indeed, let B_η be a ball of radius $\eta > 0$ around 0, big enough such that $\mathrm{Spec}_\Gamma(A) \subset B_\eta$ and $\eta \ge \|A\|_\Gamma$. If $z \in \partial B_\eta$ we have $(z-A)^{-1} = \sum_{j=0}^\infty z^{-j-1}A^j$ convergent in L_Γ norm. Then

$$\begin{aligned}
f(A) &= \frac{1}{2\pi i}\int_{\partial\omega}\left(\sum_{k=0}^m a_k z^k\right)(z-A)^{-1}\,dz = \frac{1}{2\pi i}\int_{\partial B_\eta}\left(\sum_{k=0}^m a_k z^k\right)(z-A)^{-1}\,dz\\
&= \frac{1}{2\pi i}\int_{\partial B_\eta}\left(\sum_{k=0}^m a_k z^k\right)\left(\sum_{j=0}^\infty z^{-j-1}A^j\right)\,dz = \sum_{k,j} a_k A^j \int_{\partial B_\eta} z^{k-j-1}\,dz\\
&= \sum_{k,j} a_k A^j \delta_{k-j-1,-1} = \sum_{k=1}^m a_k A^k.
\end{aligned}$$

At this point we can prove the following result.

Proposition 6.13. *We have*

$$r_\Gamma(A) = \lim_{n\to\infty} (\|A^n\|_\Gamma)^{1/n} = \inf_{n\geq 1} (\|A^n\|_\Gamma)^{\frac{1}{n}} . \tag{6.6}$$

We need the following auxiliary lemma.

Lemma 6.14. *Given $A \in L_\Gamma(\mathcal{E},\mathcal{E})$,*

(i) If $|\lambda| > \|A\|_\Gamma$ then $\lambda \in \rho_\Gamma(A)$.

(ii) If $\lambda \in \mathrm{Spec}_\Gamma(A)$ then $\lambda^n \in \mathrm{Spec}_\Gamma(A^n)$, for all n.

Proof:

Part (i)*:* Consider the formal identity

$$A - \lambda = -\lambda(\mathrm{Id} - \lambda^{-1}A).$$

If $|\lambda| \geq \|A\|_\Gamma$ then $\|\lambda^{-1}A\|_\Gamma < 1$ and by a particular case of Lemma 4.18 we have that $\mathrm{Id} - \lambda^{-1}A$ is invertible and $(\mathrm{Id} - \lambda^{-1}A)^{-1} \in L_\Gamma(\mathcal{E},\mathcal{E})$. As a consequence, $A - \lambda$ is invertible and $(A-\lambda)^{-1} \in L_\Gamma(\mathcal{E},\mathcal{E})$.

Part (ii)*:* Assume the opposite, i.e. $\lambda^n \in \rho_\Gamma(A^n)$ for some n. From the identity

$$A^n - \lambda^n = (A-\lambda)C = C(A-\lambda),$$

with

$$C = A^{n-1} + \lambda A^{n-2} + \lambda^2 A^{n-3} + \ldots + \lambda^{n-2}A + \lambda^{n-1}$$

we have that if $A^n - \lambda^n$ is invertible then $(A-\lambda)$ is exhaustive and $(A-\lambda)$ is injective, which implies $(A-\lambda)$ is invertible.

Moreover

$$\mathrm{Id} = (A-\lambda)C(A^n - \lambda^n)^{-1}$$

and hence

$$(A-\lambda)^{-1} = C(A^n-\lambda^n)^{-1} \in L_\Gamma(\mathcal{E},\mathcal{E})$$

because $C \in L_\Gamma(\mathcal{E},\mathcal{E})$. This is a contradiction with $\lambda \in \mathrm{Spec}_\Gamma(A)$.

□

Proof of Proposition 6.13. On the one hand from Lemma 6.14, if $\lambda \in \mathrm{Spec}_\Gamma(A)$ we have

$$|\lambda^n| \leq \|A^n\|_\Gamma, \quad \forall n \in \mathbb{N}$$

and

$$|\lambda| \leq (\|A^n\|_\Gamma)^{1/n}, \quad \forall n \in \mathbb{N}$$

which implies

$$r_\Gamma(A) \leq \liminf_{n\to\infty} (\|A^n\|_\Gamma)^{1/n}$$

and

$$r_\Gamma(A) \le \inf_{n\ge 1} (\|A^n\|_\Gamma)\,\frac{1}{n}.$$

On the other hand, from the consequence of Lemma 6.10 we have

$$A^n = \frac{1}{2\pi i}\int_{\partial\omega} z^n (z-A)^{-1}\,dz,$$

where ω is an open set such that $\mathrm{Spec}_\Gamma(A) \subset \omega$. We take ω to be the disc centered at the origin with radius $\alpha+\varepsilon$, where $\alpha = r_\Gamma(A)$ and $\varepsilon > 0$ arbitrary. Then

$$\|A^n\|_\Gamma \le \frac{1}{2\pi}\sup_{z\in\partial\omega}|z^n| \sup_{z\in\partial\omega}\|(z-A)^{-1}\|_\Gamma \,\mathrm{length}(\partial\omega) \le \frac{1}{2\pi}(\alpha+\varepsilon)^n M 2\pi(\alpha+\varepsilon)$$

where $M = \sup_{z\in\partial\omega}\|(z-A)^{-1}\|_\Gamma < \infty$ by compactness.

Then

$$\limsup_{n\to\infty} (\|A^n\|_\Gamma)^{1/n} \le r_\Gamma(A)+\varepsilon \le \liminf_{n\to\infty} (\|A^n\|_\Gamma)^{1/n} + \varepsilon.$$

Since ε is arbitrary $\lim_{n\to\infty}(\|A^n\|_\Gamma)^{1/n}$ exists and we get 6.6.

□

The following proposition is a kind of continuity of $\mathrm{Spec}_\Gamma A$ with respect to A.

Proposition 6.15. *Let $A \in L_\Gamma(\mathcal{E},\mathcal{E})$ and $\mu \in \rho_\Gamma(A)$. Then if $B \in L_\Gamma(\mathcal{E},\mathcal{E})$ and $\|B\|_\Gamma$ is small enough then $\mu \in \rho_\Gamma(A+B)$.*

Proof. From the hypothesis, $(A-\mu\,\mathrm{Id})^{-1} \in L_\Gamma$. By Proposition 3.11, if $\|(A-\mu\,\mathrm{Id})^{-1}\|_\Gamma\|B\|_\Gamma < 1$, $A+B-\mu\,\mathrm{Id}$ is invertible and $(A+B-\mu\,\mathrm{Id})^{-1} \in L_\Gamma$.

□

Lemma 6.16. *Let $A \in L_\Gamma(\mathcal{E},\mathcal{E})$ and Ω be an open set such that $\Omega \supset \mathrm{Spec}_\Gamma(A)$. Let $f,g : \Omega \to \mathbb{C}$ be analytic functions and $h(z) = f(z)g(z)$. Then*

$$h(A) = f(A)g(A).$$

Proof. Let ω_1, ω_2 be two open sets such that

$$\mathrm{Spec}_\Gamma(A) \subset \omega_1 \subset \overline{\omega_1} \subset \omega_2 \subset \overline{\omega_2} \subset \Omega$$

and $\partial\omega_1, \partial\omega_2$ are the finite union of closed curves. Next we are going to use the identity (6.4), $(z-A)^{-1}(w-A)^{-1} = \frac{(z-A)^{-1}-(w-A)^{-1}}{w-z}$ and the fact that if $z \in \partial\omega_1$, $\int_{\partial\omega_2}\frac{g(w)}{w-z}\,dw = 2\pi i g(z)$

and if $w \in \partial\omega_2$, $\int_{\partial\omega_1} \frac{f(z)}{z-w}\, dz = 0$. Thus

$$\begin{aligned}
f(A)g(A) &= \frac{1}{2\pi i}\int_{\partial\omega_1} f(z)(z-A)^{-1}g(A)\, dz \\
&= \frac{-1}{4\pi^2}\int_{\partial\omega_1} f(z)(z-A)^{-1}\int_{\partial\omega_2} g(w)(w-A)^{-1}\, dw\, dz \\
&= \frac{-1}{4\pi^2}\int_{\partial\omega_1}\int_{\partial\omega_2} f(z)g(w)\frac{(z-A)^{-1}-(w-A)^{-1}}{w-z}\, dw\, dz \\
&= \frac{1}{2\pi i}\int_{\partial\omega_1} f(z)(z-A)^{-1}\frac{1}{2\pi i}\int_{\partial\omega_2}\frac{g(w)}{w-z}\, dw\, dz \\
&\quad - \frac{1}{2\pi i}\int_{\partial\omega_2} g(w)(w-A)^{-1}\frac{1}{2\pi i}\int_{\partial\omega_1}\frac{f(z)}{w-z}\, dz\, dw \\
&= \frac{1}{2\pi i}\int_{\partial\omega_1} f(z)g(z)(z-A)^{-1}\, dz = h(A).
\end{aligned}$$

□

Corollary 6.17. *In the conditions of the previous lemma, assume $f : \Omega \to \mathbb{C}$ is not zero. Then $f(A)$ is invertible, $f(A)^{-1} \in L_\Gamma(\mathcal{E}, \mathcal{E})$ and*

$$[f(A)]^{-1} = \frac{1}{2\pi i}\int_{\partial\omega_1}\frac{1}{f(z)}(z-A)^{-1}\, dz.$$

Proof. Let $g(z) = \frac{1}{f(z)}$ in Lemma 6.16. Then $f(z)g(z) = 1$ which means $f(A)g(A) = \mathrm{Id}$, and hence we have the stated formula for $[f(A)]^{-1}$.

□

6.6 Spectral projections associated to a gap in the Γ-spectrum

We can adapt the spectral projection theorem to the setting of Γ-spectrum.

Assume that

$$\mathrm{Spec}_\Gamma(A) = \sigma_1 \cup \sigma_2$$

with

$$\sigma_i \subset \omega_i \subset \overline{\omega_i} \subset \Omega_i, \quad i = 1, 2,$$

where Ω_i are disjoint open sets and ω_i are open sets such that $\partial\omega_i$ are finite union of simple closed curves.

We define

$$P = \frac{1}{2\pi i}\int_{\partial\omega_1}(z-A)^{-1}\, dz.$$

Lemma 6.18. *We have*

(i) $P \in L_\Gamma(\mathcal{E}, \mathcal{E})$,

(ii) $P^2 = P$,

(iii) $P(\mathcal{E})$ *and* $\mathrm{Ker}(P)$ *are closed and invariant.*

Proof. Part (i) follows from the properties of integrals of functions over L_Γ. Part (ii) follows from the fact that P can be written as

$$\frac{1}{2\pi i}\int_{\partial\omega} f(z)(z-A)^{-1}\,dz,$$

with $f:\Omega_1\to\mathbb{C}$, defined by $f(z)=1$. Since $f(z)=f(z)f(z)$, by Lemma 6.16

$$PP=f(A)f(A)=f(A)=P,$$

proving P is a projection.

For Part (iii), $P(\mathcal{E})$ and $\mathrm{Ker}(P)$ are invariant when P is a projection, and $\mathcal{E}=P(\mathcal{E})\oplus\mathrm{Ker}(P)$. Moreover since $P(\mathcal{E})=\mathrm{Ker}(\mathrm{Id}-P)$ and P is continuous, both $P(\mathcal{E})$ and $\mathrm{Ker}(\mathrm{Id}-P)$ are closed. □

We denote $\mathcal{E}^1=P(\mathcal{E})$ and $\mathcal{E}^2=(\mathrm{Id}-P)(\mathcal{E})=\mathrm{Ker}(P)$ and $A_i=A|_{\mathcal{E}_i}$.

Theorem 6.19. *We have that*

$$\mathrm{Spec}_\Gamma(A_i)=\sigma_i,\quad i=1,2.$$

Proof. Let

$$f(z)=\begin{cases}1, & \text{if } z\in\Omega_1\\ 0, & \text{if } z\in\Omega_2\end{cases}.$$

Moreover let $\lambda\notin\sigma_1$ and $g_1(z)=\frac{f(z)}{\lambda-z}$. The function g_1 is analytic in a neighbourhood U of $\mathrm{Spec}_\Gamma A$ and satisfies

$$\begin{aligned} f(z)&=(\lambda-z)g_1(z), && z\in U,\\ f(z)g_1(z)&=g_1(z), && z\in U.\end{aligned}$$

By Lemma 6.16,

$$\begin{aligned} f(A)&=(\lambda-A)g_1(A),\\ f(A)g_1(A)&=g_1(A),\end{aligned}\tag{6.7}$$

and hence

$$P=(\lambda-A)g_1(A)=g_1(A)(\lambda-A),\quad Pg_1(A)=g_1(A)P=g_1(A).$$

If $x\in\mathcal{E}^1$,

$$g_1(A)x=Pg_1(A)x=P(g_1(A)x)\in\mathcal{E}^1.$$

Moreover, from (6.7)

$$g_1(A)=(\lambda-A)^{-1}\big|_{\mathcal{E}^1}=(\lambda-A_1)^{-1}$$

which implies that $\lambda\in\rho_\Gamma(A_1)$. Therefore $\mathrm{Spec}_\Gamma(A_1)\subset\sigma_1$. Since $\mathrm{Id}-P$ is a projection onto $\mathcal{E}^2$ a completely analogous argument shows that $\mathrm{Spec}_\Gamma(A_2)\subset\sigma_2$. Indeed, given $\lambda\notin\sigma_2$, let $g_2(z)=\frac{1-f(z)}{\lambda-z}$. Then

$$(\mathrm{Id}-P)=(\lambda-A)g_2(A)=g_2(A)(\lambda-A),\quad (\mathrm{Id}-P)g_2(A)=g_2(A)(\mathrm{Id}-P)=g_2(A)$$

and

$$g_2(A) = (\lambda - A)^{-1}\big|_{\mathcal{E}^2} = (\lambda - A_2)^{-1}.$$

Now suppose that $\lambda \in \rho_\Gamma(A_1) \cap \rho_\Gamma(A_2)$. Since $g_1(z) + g_2(z) = \frac{1}{\lambda - z}$, Corollary 6.17 shows

$$\begin{aligned}(\lambda - A)^{-1} &= g_1(A) + g_2(A) = g_1(A)P + g_2(A)(\operatorname{Id} - P)\\ &= (\lambda - A_1)^{-1}P + (\lambda - A_2)^{-1}(\operatorname{Id} - P).\end{aligned}$$

Then $\lambda \in \rho_\Gamma(A)$ which implies that

$$\operatorname{Spec}_\Gamma(A) \subset \operatorname{Spec}_\Gamma(A_1) \cup \operatorname{Spec}_\Gamma(A_2),$$

and therefore

$$\sigma_i \subset \operatorname{Spec}_\Gamma(A_i), \quad i = 1, 2.$$

□

6.7 Sylvester operators in L^k_Γ

In this section we will prove some auxiliary results on Sylvester operators in spaces with decay. In the next section we will give results related to Sylvester operators defined on bundles.

Definition 6.20. *Let $E = \ell^\infty(\mathbb{R}^n)$. Given $A, B \in L_\Gamma(E, E)$ we define the operators*

$$\mathcal{R}_{j,A} : L^k_\Gamma(E, E) \to L^k_\Gamma(E, E), \quad 1 \le j \le k,$$

by

$$\mathcal{R}_{j,A}(W)(u_1, \dots, u_k) = W(u_1, \dots, Au_j, \dots, u_k),$$

and $\mathcal{L}_B, \mathcal{S}_{B,A} : L^k_\Gamma(E, E) \to L^k_\Gamma(E, E)$ by

$$\begin{aligned}\mathcal{L}_B(W)(u_1, \dots, u_k) &= BW(u_1, \dots, u_k),\\ \mathcal{S}_{B,A}(W)(u_1, \dots, u_k) &= BW(Au_1, \dots, Au_k),\end{aligned}$$

respectively.

Note that by Proposition 3.16, if $W \in L^k_\Gamma(E, E)$ then $\mathcal{R}_{j,A}(W), \mathcal{L}_B(W)$ and $\mathcal{S}_{B,A}(W)$ are in $L^k_\Gamma(E, E)$ so that the operators are well defined.

Given two subsets X, Y of $\mathbb{C}$ we denote by $X \cdot Y$ the set

$$X \cdot Y = \{x \cdot y \,|\, x \in X,\, y \in Y\}.$$

Analogously we define

$$X^k = X \cdot X \overset{k}{\cdots} X.$$

Proposition 6.21. *We have*

$$\operatorname{Spec}(\mathcal{S}_{B,A}, L^k_\Gamma(E, E)) \subset \operatorname{Spec}_\Gamma(B) \cdot (\operatorname{Spec}_\Gamma(A))^k, \quad k \in \mathbb{N}.$$

The proof of this proposition is a consequence of the following theorem and the next lemma.

Theorem 6.22. *[Theorem 11.23, [Rud91]] Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two commuting elements in a unitary Banach algebra. Then*

$$\mathrm{Spec}(\mathfrak{a}\mathfrak{b}) \subseteq \mathrm{Spec}(\mathfrak{a}) \cdot \mathrm{Spec}(\mathfrak{b}).$$

Lemma 6.23. *Given* $A, B \in L_\Gamma(E,E), k \in \mathbb{N}, 1 \le j \le k$*, then*

$$\mathrm{Spec}(\mathcal{R}_{j,A}, L^k_\Gamma(E,E)) \subset \mathrm{Spec}_\Gamma(A),$$
$$\mathrm{Spec}(\mathcal{L}_B, L^k_\Gamma(E,E)) \subset \mathrm{Spec}_\Gamma(B).$$

Proof. Let $\lambda \in \rho_\Gamma(A)$. This means that $(A - \lambda\,\mathrm{Id})^{-1} \in L_\Gamma(E,E)$. To study the invertibility of $\mathcal{R}_{j,A} - \lambda\,\mathrm{Id}$ we consider the equation

$$W(u_1,\dots,Au_j,\dots,u_k) - \lambda W(u_1,\dots,u_j,\dots,u_k) = H(u_1,\dots,u_j,\dots,u_k),$$

which is equivalent to

$$W(u_1,\dots,(A-\lambda\,\mathrm{Id})u_j,\dots,u_k) = H(u_1,\dots,u_j,\dots,u_k).$$

Then, formally,

$$H = \mathcal{R}_{j,(A-\lambda\,\mathrm{Id})^{-1}} W$$

and hence $H \in L^k_\Gamma(E,E)$ and $\lambda \in \rho(\mathcal{R}_{j,A})$.

The proof of the result for $\mathcal{L}_B$ is completely analogous.

□

Proof of Proposition 6.21. It follows directly from the fact that

$$\mathcal{S}_{B,A} = \mathcal{L}_B \circ \mathcal{R}_{1,A} \circ \ldots \circ \mathcal{R}_{k,A}$$

and the fact that the operators on the r.h.s. commute. Then Theorem 6.22 gives the result.

□

6.8 Sylvester operators on $C^r_\Gamma(\mathbb{T}^d, L^k(\mathcal{E},\mathcal{F}))$

In this section we will prove some auxiliary results on Sylvester operators on bundles used in this thesis. Let $\mathcal{E}$, $\mathcal{F}$ be linear subspaces of $\ell^\infty(\mathbb{R}^n)$. To simplify the notation in this section we will write $C^r_{L^k_\Gamma}$ instead of $C^r_\Gamma(\mathbb{T}^d, L^k(\mathcal{E},\mathcal{F}))$.

Definition 6.24. *Given a linear map* $A \in L_\Gamma(\mathcal{E},\mathcal{E})$*, we define the operator* $\mathcal{R}_{j,A} : C^r_{L^k_\Gamma} \to C^r_{L^k_\Gamma}$*,* $1 \le j \le k$*, by*

$$\mathcal{R}_{j,A}(W)(\theta)(z_1,\dots,z_k) = W(\theta)(z_1,\dots,Az_j,\dots,z_k).$$

Definition 6.25. *Given a linear map $B \in L_\Gamma(\mathcal{F},\mathcal{F})$, we define the operator $\mathcal{L}_B : C^r_{L^k_\Gamma} \to C^r_{L^k_\Gamma}$ by*

$$\mathcal{L}_B(W)(\theta)(z_1,\dots,z_k) = BW(\theta-\omega)(z_1,\dots,z_k).$$

Remark 6.26. *Observe that these two operators trivially commute. This is a key point in the main proof of this section.*

Definition 6.27. *Given $A \in L_\Gamma(\mathcal{E},\mathcal{E})$, $B \in L_\Gamma(\mathcal{F},\mathcal{F})$ we define the Sylvester operator $\mathcal{S}_{B,A} : C^r_{L^k_\Gamma} \to C^r_{L^k_\Gamma}$ by*

$$\mathcal{S}_{B,A}(W)(\theta)(z_1,\dots,z_k) = BW(\theta-\omega)(Az_1,\dots,Az_k).$$

Remark 6.28. *Actually $\mathcal{L}_B$ and $\mathcal{S}_{B,A}$ depend on ω, but since ω will be kept fixed we do not express this dependence explicitly in the notation.*

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. Given a set $S \subset \mathbb{C}$, we define the annulus generated by S as

$$\mathcal{A}S = \{e^{i\theta}s \,|\, \theta \in [0,2\pi),\, s \in S\}.$$

Proposition 6.29. *We have the following inclusions of spectra*

$$\begin{aligned}\operatorname{Spec}\left(\mathcal{S}_{B,A}, C^r_{L^k_\Gamma}(\mathcal{E},\mathcal{F})\right) &\subseteq \operatorname{Spec}\left(\mathcal{L}_B, C^r_{L^k_\Gamma}\right)\cdot\operatorname{Spec}\left(\mathcal{R}_{1,A}, C^r_{L^k_\Gamma}\right)\cdots\operatorname{Spec}\left(\mathcal{R}_{k,A}, C^r_{L^k_\Gamma}\right)\\ &\subseteq \mathcal{A}\operatorname{Spec}_\Gamma(B)\mathcal{A}\left(\operatorname{Spec}_\Gamma(A)\right)^k.\end{aligned}$$

The proof of this proposition follows from the next results and Theorem 6.22.

Remark 6.30. *We will apply Theorem 6.22 when the Banach algebra is $L_\Gamma(\mathcal{E},\mathcal{E})$. As we indicated before $\|\operatorname{Id}\|_\Gamma \neq 1$, but there is an equivalent norm $\|\cdot\|'$ in $L_\Gamma(\mathcal{E},\mathcal{E})$ such that $\|\operatorname{Id}\|' = 1$.*

Using this norm we are able to apply the theorem and obtain the inclusions of spectra. Since the spectra are intrinsic we therefore obtain that if $A, B \in L_\Gamma(\mathcal{E},\mathcal{E})$ commute then

$$\operatorname{Spec}_\Gamma(AB) \subseteq \operatorname{Spec}_\Gamma(A)\cdot\operatorname{Spec}_\Gamma(B).$$

Lemma 6.31. *Given $B \in L_\Gamma(\mathcal{F},\mathcal{F})$ the spectrum of $\mathcal{L}_B$ over $C^r_\Gamma(\mathbb{T}^d, L^k(\mathcal{E},\mathcal{F}))$ is related to the Γ-spectrum of B through*

$$\operatorname{Spec}(\mathcal{L}_B, C^r_{L^k_\Gamma}) \subseteq \mathcal{A}\operatorname{Spec}_\Gamma(B,\mathcal{F}).$$

Proof. We will prove that if

$$\lambda e^{i\varphi} \notin \operatorname{Spec}_\Gamma(B), \quad \forall \varphi \in [0,2\pi) \tag{6.8}$$

then $\lambda \notin \operatorname{Spec}\mathcal{L}_B$. Since $\operatorname{Spec}_\Gamma(B)$ is compact the condition (6.8) implies that there is a gap in the spectrum, i.e. there exists $\eta > 0$ such that

$$\{\mu \in \mathbb{C} \,|\, |\lambda| - \eta \leq |\mu| \leq |\lambda| + \eta\} \cap \operatorname{Spec}_\Gamma(B) \neq \emptyset.$$

By Lemma 6.18 and Theorem 6.19 there exist $\mathcal{F}_1, \mathcal{F}_2$ linear subspaces of $\mathcal{F}$ such that $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ and

$$\operatorname{Spec}_\Gamma(B|_{\mathcal{F}_1}) \subset \{\mu \in \mathbb{C} \,|\, |\mu| > |\lambda| + \eta\},$$
$$\operatorname{Spec}_\Gamma(B|_{\mathcal{F}_2}) \subset \{\mu \in \mathbb{C} \,|\, |\mu| < |\lambda| - \eta\}.$$

Let $P_i : \mathcal{F} \to \mathcal{F}_i$ be the spectral projections. We have that $P_i \in L_\Gamma(\mathcal{F}, \mathcal{F}_i)$. We write $B_i = P_i B$, $i = 1, 2$. By Proposition 6.13 there exist $C_1, C_2 > 0$ such that

$$\|B_1^{-n}\|_\Gamma \le C_1(|\lambda| + \eta)^{-n}, \quad n \ge 0,$$
$$\|B_2^{n}\|_\Gamma \le C_2(|\lambda| - \eta)^{n}, \quad n \ge 0.$$

We want to determine whether $\mathcal{L}_B - \lambda \operatorname{Id}$ is invertible. Given $W \in C^r_{L^k_\Gamma}$ we write

$$W_i = P_i \circ W.$$

Now let $H \in C^r_{L^k_\Gamma}$, $H_i = P_i \circ H$ and consider the equations

$$B_1 W_1(\theta - \omega) - \lambda W_1(\theta) = H_1(\theta), \tag{6.9}$$
$$B_2 W_2(\theta - \omega) - \lambda W_2(\theta) = H_2(\theta) \tag{6.10}$$

which we can solve recursively as

$$W_1(\theta) = \sum_{n=1}^{\infty} B_1^{-n} \lambda^{n-1} H_1(\theta + n\omega), \tag{6.11}$$
$$W_2(\theta) = -\sum_{n=0}^{\infty} B_2^{n} \lambda^{-(n+1)} H_2(\theta - n\omega). \tag{6.12}$$

These two series are absolutely convergent in $\|\cdot\|_{C^r_{L^k_\Gamma}}$. Indeed, we can bound the general term of the series in (6.11) as follows

$$\begin{aligned}\|B_1^{-n}\lambda^{n-1}H_1(\theta + n\omega)\|_{C^r_{L^k_\Gamma}} &\le \sup_{\theta\in\mathbb{T}^d} \max_{\substack{0\le j\le r\\ j_1+\ldots+j_d=j}} \left\| \frac{\partial^j}{\partial\theta_1^{j_1}\cdots\partial\theta_d^{j_d}} B_1^{-n}\lambda^{n-1}H_1(\theta+n\omega)\right\|_{L^k_\Gamma} \\ &\le C_1 \frac{|\lambda|^{n-1}}{(|\lambda|+\eta)^n}\|H\|_{C^r_{L^k_\Gamma}},\end{aligned}$$

and hence

$$\|W_1\|_{C^r_{L^k_\Gamma}} \le C_1 \sum_{n=1}^{\infty} \frac{|\lambda|^{n-1}}{(|\lambda|+\eta)^n}\|H\|_{C^r_{L^k_\Gamma}} = \frac{C_1}{\eta}\|H\|_{C^r_{L^k_\Gamma}}.$$

We can proceed similarly for the series in (6.10). Therefore these series are absolutely convergent in $\|\cdot\|_{C^r_{L^k_\Gamma}}$, and this implies that $\mathcal{L}_B - \lambda \operatorname{Id}$ is invertible in $C^r_{L^k_\Gamma}$ whenever $\lambda \notin \mathcal{A}\operatorname{Spec}_\Gamma(B)$.

The formulas (6.11) and (6.12) determine uniquely the solution $W = (W_1, W_2)^\top$ of (6.9), (6.10), thus $\mathcal{L}_B - \lambda \operatorname{Id}$ is bijective whenever $\lambda \notin \mathcal{A}\operatorname{Spec}_\Gamma(B)$.

□

Remark 6.32. *It is worth to remember the relationship between spectra. In particular,* $\lambda \notin \mathcal{A}\,\mathrm{Spec}_\Gamma B$ *implies* $\lambda \notin \mathrm{Spec}\,\mathcal{L}_B$*, thus* $\mathbb{C}\backslash\mathcal{A}\,\mathrm{Spec}_\Gamma B \subseteq \mathbb{C}\backslash\,\mathrm{Spec}\,\mathcal{L}_B$*, and* $\mathcal{A}\,\mathrm{Spec}_\Gamma B \supset \mathrm{Spec}\,\mathcal{L}_B$*.*

Lemma 6.33. *The spectrum of* $\mathcal{R}_{j,A}$*,* $1 \leq j \leq k$*, over* $C^r_{L^k_\Gamma}$ *is related to the* Γ*-spectrum of* A *through*

$$\mathrm{Spec}(\mathcal{R}_{j,A}, C^r_{L^k_\Gamma}) \subseteq \mathcal{A}\,\mathrm{Spec}_\Gamma(A, \mathcal{E}).$$

Proof. Observe that there are k natural identifications between $L^k(\mathcal{E},\mathcal{F})$ and $L(\mathcal{E}, L^{k-1}(\mathcal{E},\mathcal{F}))$. We denote them as ι_j, then

$$W(z_1,\ldots,z_k) = \iota_j(W)(z_j)(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_k).$$

We can use this notation with $\mathcal{R}_{j,A}$, and write

$$\mathcal{R}_{j,A}(W)(z_1,\ldots,z_k) = \iota_j(W)(Az_j)(z_1,\ldots,z_{j-1},z_{j+1},\ldots,z_k).$$

From now on we will write W instead of $\iota_j(W)$ to simplify the notation.

Let $\lambda \in \mathbb{C}$ such that $\lambda \notin \mathcal{A}\,\mathrm{Spec}_\Gamma(A)$. Then there is a gap in the spectrum of A in the sense that there exists $\eta > 0$ such that

$$\{\mu \in \mathbb{C} \,|\, |\lambda| - \eta \leq |\mu| \leq |\lambda| + \eta\} \cap \mathrm{Spec}_\Gamma A \neq \emptyset.$$

As in the previous lemma, Lemma 6.18 and Theorem 6.19 prove that there exist $\mathcal{E}^1, \mathcal{E}^2$ linear subspaces of $\mathcal{E}$ such that $\mathcal{E} = \mathcal{E}^1 \oplus \mathcal{E}^2$ and

$$\mathrm{Spec}_\Gamma(A|_{\mathcal{E}^1}) \subset \{\mu \in \mathbb{C} \,|\, |\mu| > |\lambda| + \eta\},$$
$$\mathrm{Spec}_\Gamma(A|_{\mathcal{E}^2}) \subset \{\mu \in \mathbb{C} \,|\, |\mu| < |\lambda| - \eta\}.$$

Let $P_i : \mathcal{E} \to \mathcal{E}_i$ be the corresponding spectral projections. We have that $P_i \in L_\Gamma(\mathcal{E}, \mathcal{E}_i)$. We write $A_i = P_i A$, $i = 1, 2$. As in the proof of the previous lemma, there exist $C_1, C_2 > 0$ such that

$$\|A_1^{-n}\|_\Gamma \leq C_1(|\lambda| + \eta)^{-n}, \quad n \geq 0,$$
$$\|A_2^{n}\|_\Gamma \leq C_2(|\lambda| - \eta)^{n}, \quad n \geq 0.$$

Now to study whether $(\mathcal{R}_{j,A} - \lambda\mathrm{Id})$ is invertible, let $H_{1,2} : \mathbb{T}^d \to L_\Gamma(\mathcal{E}_{1,2}, L^{k-1}_\Gamma(\mathcal{E},\mathcal{F}))$ be the induced embeddings of an arbitrary element $H \in C^r_{L^k_\Gamma}$ and consider the following equations

$$W_1(\theta)(A_1) - \lambda W_1(\theta)(\mathrm{Id}_1) = H_1(\theta),$$
$$W_2(\theta)(A_2) - \lambda W_2(\theta)(\mathrm{Id}_2) = H_2(\theta).$$

Observe that these equations can be rewritten as $W_{1,2}(\theta)(A_{1,2} - \lambda\,\mathrm{Id}) = H_{1,2}(\theta)$, and we can solve them just by writing

$$W_i(\theta) = H_i(\theta)(A_i - \lambda\,\mathrm{Id})^{-1}, \quad i = 1,\, 2.$$

Observe that $\|\iota_j(W_p)(A_p)\|_{C^r_{L^{k-1}_\Gamma}} \leq \|W_p\|_{C^r_{L^k_\Gamma}} \cdot \|A_i\|_\Gamma$, $p = 1,\, 2$, by Proposition 3.12. Then this condition and the spectral properties of A ensure that W obtained in this way belongs to $C^r_{L^k_\Gamma}$.

□

Proof of Proposition 6.29. Observe that we can write the Sylvester operator $\mathcal{S}_{B,A}$ as

$$\mathcal{S}_{B,A}(W)(\theta)(z_1, \dots, z_k) = \mathcal{L}_B \circ \mathcal{R}_{1,A} \circ \cdots \circ \mathcal{R}_{k,A}(W)(\theta)(z_1, \dots, z_k).$$

Thus we can apply Theorem 6.22 to get

$$\mathrm{Spec}(\mathcal{S}_{B,A}, C^r_{L^k_\Gamma}) \subseteq \mathrm{Spec}(\mathcal{L}_B, C^r_{L^k_\Gamma}) \cdot \mathrm{Spec}(\mathcal{R}_{1,A}, C^r_{L^k_\Gamma}) \cdots \mathrm{Spec}(\mathcal{R}_{k,A}, C^r_{L^k_\Gamma})$$

and now apply Lemmas 6.31 and 6.33 to get

$$\mathrm{Spec}(\mathcal{S}_{B,A}, C^r_{L^k_\Gamma} \subseteq \mathcal{A}\,\mathrm{Spec}_\Gamma\, B \cdot (\mathcal{A}\,\mathrm{Spec}_\Gamma\, A)^k$$

as desired.

□

Chapter 7

The invariant torus

We want to determine an invariant torus close to $\{0\} \times \mathbb{T}^d$ in $\ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d$ of the perturbed dynamical system $F = F_0 + F_1$, where F_0 is defined in Section 2.1 and F_1 is a small perturbation of it. Once we have obtained this invariant object, we will find a parametrisation of some of its invariant manifolds, namely strong stable manifolds and non-resonant ones. The first step is then to determine the invariant torus, a task that will be done in this section. In the forthcoming sections we will find the parametrisation of the invariant manifolds.

We will denote the invariant torus by its parametrisation, $W_0 : \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n)$. Finding this object is the first step in the parametrisation method as detailed in [CFdlL03a], [HdlL06b] or [FdlLS09]. We will do this (under suitable smallness assumptions on the perturbing function F_1) by solving the functional equation given by the invariance of the torus with respect to the dynamical system

$$F(W_0(\theta), \theta) = W_0(\theta + \omega). \tag{7.1}$$

The solution will be found under different regularity and decay assumptions for F, which will result in different properties for the torus determined by W_0.

We recall that the dynamical system we deal with, as defined in Section 2.1 has the form

$$(x, \theta) \mapsto \Big(F(x, \theta), \theta + \omega\Big), \qquad x \in \ell^\infty(\mathbb{R}^n),\ \theta \in \mathbb{T}^d, \tag{7.2}$$

where $F(x, \theta) = F_0(x, \theta) + F_1(x, \theta)$ and $F_0(x, \theta) = (f(x_i))_{i \in \mathbb{Z}^m}$. The function $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^t in a neighbourhood A of 0. Let $U \subset \ell^\infty(\mathbb{R}^n)$ be a neighbourhood of 0 such that $U \subseteq \prod_{i \in \mathbb{Z}^m} A$. The function $W_0 : \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n)$ parametrising the invariant torus will be a solution of Equation (7.1) and note that due to the particular form of the dynamical system (7.2), $(W_0(\theta), \theta)$ will be an invariant graph of (7.2):

$$\Big(F(W_0(\theta), \theta), \theta + \omega\Big) = \Big(W_0(\theta + \omega), \theta + \omega\Big).$$

We write

$$F(x, \theta) = F_0(x) + F_1(x, \theta),$$

and

$$M(\theta) = M_0 + M_1(\theta),$$
$$N(x,\theta) = N_0(x) + N_1(x,\theta),$$

with $DN_0(0) = D_x N_1(0,\theta) = 0$. We also write

$$F_0(x) = M_0 x + N_0(x),$$
$$F_1(x,\theta) = M_1(\theta)x + N_1(x,\theta).$$

The main result of this section is the following theorem, which determines, under hyperbolicity and several regularity assumptions, the invariant torus as a graph.

Theorem 7.1. *Consider the dynamical system defined in* (7.2) *together with the notation introduced above. Assume that $Df(0)$ is hyperbolic and consider the functional equation*

$$F(W_0(\theta),\theta) = W_0(\theta+\omega). \tag{7.3}$$

(i) Assume $M_1 \in C^0\big(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))\big)$ and $N(x,\theta) \in C^{0,0}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with N Lipschitz with respect to x for all $\theta \in \mathbb{T}^d$ and assume $\|F_1\|_{C^0}$ and

$$\mathrm{Lip}_x(N) := \sup_{\theta\in\mathbb{T}^d} \mathrm{Lip}(N(\cdot,\theta))$$

are small enough. Then the functional equation (7.3) *has a unique solution $W_0(\theta) \in C^0(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ close to 0.*

(ii) Assume $F_0 \in C^t(U, \ell^\infty(\mathbb{R}^n))$, $F_1 \in C^{t,r}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $t \geq r+1$, $r \geq 0$ and $\|F_1\|_{C^{t,r}}$ small enough. Then the functional equation (7.3) *has a solution $W_0 \in C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$. Since $C^{t,r}_\Gamma \subset C^{t,r}$, for $F \in C^{t,r}_\Gamma$ we also obtain a solution $W_0 \in C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ close to 0.*

(iii) Assume $F_0 \in C^t(U, \ell^\infty(\mathbb{R}^n))$, $F_1 \in C^{t,r}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $t \geq r+2$, $r \geq 0$ and $\|F_1\|_{C^{t,r}_{j,\Gamma}}$ small enough. Then the functional equation (7.3) *has a solution $W_0 \in S^r_{j,\Gamma}(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ close to 0.*

As was justified in Section 2.1, $M_0 = DF_0(0)$ is a hyperbolic linear map, and M_0 is uncoupled.

We will start by proving Part (i). The proof of Parts (ii) and (iii) follow a different scheme of proof. To prove this "Lipschitz" case we will use a fixed point theorem, since we cannot use the differentiability of N to apply an implicit function theorem in Banach spaces.

Proof of Part (i) *in Theorem 7.1.* A solution of (7.3) can be obtained as a fixed point of the operator

$$\widetilde{\mathcal{F}} : C^0(\mathbb{T}^d, U) \to C^0(\mathbb{T}^d, U)$$

defined by

$$\widetilde{\mathcal{F}}(W)(\theta) = F(W(\theta-\omega), \theta-\omega).$$

We can write $F(x,\theta) = M(\theta)x + N(x,\theta)$ where $M(\theta) = M_0 + M_1(\theta)$ and use the decomposition

$$\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$$

associated to M_0 introduced in Section 2.1, where the super-indices 1 and 2 stand for stable and unstable respectively, to get the following decompositions

$$M_0 = \begin{pmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{pmatrix}, \quad M_1(\theta) = \begin{pmatrix} B_{1,1}(\theta) & B_{1,2}(\theta) \\ B_{2,1}(\theta) & B_{2,2}(\theta) \end{pmatrix},$$
$$M(\theta) = \begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ M_{2,1}(\theta) & M_{2,2}(\theta) \end{pmatrix}.$$

Remember that the hyperbolicity of M_0 implies $\|A_{1,1}\| < 1$ and $\|A_{2,2}^{-1}\| < 1$ in the norm we are using, based on a suitable adapted norm in $\mathbb{R}^n$, and the norm of M_1 is bounded by the norm of F_1. We also write

$$N(x,\theta) = (N^1(x,\theta), N^2(x,\theta)),$$
$$W(\theta) = (W^1(\theta), W^2(\theta))$$

according to this decomposition of $\ell^\infty(\mathbb{R}^n)$. We can express the operator $\widetilde{\mathcal{F}}$ component-wise as

$$\widetilde{\mathcal{F}}(W)(\theta) = \begin{pmatrix} M_{1,1}(\theta-\omega)W^1(\theta-\omega) + M_{1,2}(\theta-\omega)W^2(\theta-\omega) + N^2(W(\theta-\omega), \theta-\omega) \\ M_{2,1}(\theta-\omega)W^1(\theta-\omega) + M_{2,2}(\theta-\omega)W^2(\theta-\omega) + N^2(W(\theta-\omega), \theta-\omega) \end{pmatrix}^\top.$$

The operator $\widetilde{\mathcal{F}}$ is clearly not a contraction since $\|A_{2,2}\| > 1$, which means the second component is expanding. As $A_{2,2}^{-1}$ exists and $\|A_{2,2}^{-1}\| < 1$, since $\|B_{2,2}\|_{C^0}$ is small enough we can use Neumann's formula to invert $M_{2,2}(\theta) = A_{2,2} + B_{2,2}(\theta) = A_{2,2}(\mathrm{Id} + A_{2,2}^{-1}B_{2,2})$ as

$$M_{2,2}^{-1}(\theta) = \sum_{k=0}^{\infty} (-A_{2,2}B_{2,2}(\theta))^k A_{2,2}^{-1}$$

and hence $\|M_{2,2}^{-1}\| \leq \|A_{2,2}^{-1}\| \, (1 + C\|M_1\|_{C^0})$ (see also Lemma 4.18) and we can rewrite the fixed point equation projected to the unstable directions:

$$M_{2,1}(\theta-\omega)W^1(\theta-\omega) + M_{2,2}(\theta-\omega)W^2(\theta-\omega) + N^2(W(\theta-\omega), \theta-\omega) = W^2(\theta)$$

as

$$M_{2,2}(\theta)^{-1}W^2(\theta+\omega) - M_{2,2}(\theta)^{-1}M_{2,1}(\theta)W^1(\theta) - M_{2,2}(\theta)^{-1}N^2(W(\theta),\theta) = W^2(\theta).$$

Observe that for a fixed W^1, the operator $\mathcal{G}$ defined as

$$\mathcal{G}(W) = M_{2,2}(\theta)^{-1}\left(W(\theta+\omega) - M_{2,1}(\theta)W^1(\theta) - N^2\left((W^1 + W)(\theta), \theta\right)\right)$$

is a contraction by the hyperbolicity of M_0 and the smallness of $M_{2,1}$ and N. Now we can find the invariant torus as a fixed point of $\mathcal{F} : C^0(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) \to C^0(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ defined by

$$\mathcal{F}(W)(\theta)$$
$$= \begin{pmatrix} M_{1,1}(\theta-\omega)W^1(\theta-\omega) + M_{1,2}(\theta-\omega)W^2(\theta-\omega) + N^1(W(\theta-\omega), \theta-\omega) \\ M_{2,2}(\theta)^{-1}W^2(\theta+\omega) - M_{2,2}(\theta)^{-1}M_{2,1}(\theta)W^1(\theta) - M_{2,2}(\theta)^{-1}N^2(W(\theta),\theta) \end{pmatrix}^\top.$$

We need to check this operator is a contraction if $\|M_1\|_{C^0}$ and $\mathrm{Lip}_x(N)$ are small enough. Indeed, given W, V in $C^0(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$, we can bound $\|\mathcal{F}(W) - \mathcal{F}(V)\|_{C^0}$ component-wise:

$$\begin{aligned}\left\|\Big[\mathcal{F}(W) - \mathcal{F}(V)\Big]_1\right\|_{C^0} &= \sup_{\theta\in\mathbb{T}^d} \big\|M_{1,1}(\theta-\omega)\big(W^1(\theta-\omega) - V^1(\theta-\omega)\big) \\ &\qquad + M_{1,2}(\theta-\omega)\big(W^2(\theta-\omega) - V^2(\theta-\omega)\big) \\ &\qquad + N^1(W(\theta-\omega),\theta-\omega) - N^1(V(\theta-\omega),\theta-\omega)\big\| \\ &\le \big(\|A_{1,1}\| + 2\|M_1\|_{C^0}\big)\|W-V\|_{C^0} + \mathrm{Lip}_x(N)\|W-V\|_{C^0} \\ &\le \rho_1\|W-V\|_{C^0},\end{aligned}$$

with $\rho_1 = \|A_{1,1}\| + 2\|M_1\|_{C^0} + \mathrm{Lip}_x(N) < 1$ if $\|M_1\|_{C^0}$ and $\mathrm{Lip}_x(N)$ are small enough. The bound of $\|\big[\mathcal{F}(W) - \mathcal{F}(V)\big]_2\|_{C^0}$ is accordingly

$$\big(\|A_{2,2}^{-1}\| + \mathcal{O}(\|M_1\|_{C^0}) + \mathrm{Lip}_x(N)\big)\|W-V\|_{C^0} \le \rho_2\|W-V\|_{C^0}$$

with $\rho_2 = \|A_{2,2}^{-1}\| + \mathcal{O}(\|M_1\|_{C^0}) + \mathrm{Lip}_x(N) < 1$ if $\|M_1\|_{C^0}$ is small enough.

In this bound we have used the fact that, for $* \in \{1,2\}$,

$$\begin{aligned}\|N^*(W,\cdot) - N^*(V,\cdot)\|_{C^0} &= \sup_{\theta\in\mathbb{T}^d}\|N^*(W(\theta),\theta) - N^*(V(\theta),\theta)\| \\ &\le \sup_{\theta\in\mathbb{T}^d}\mathrm{Lip}(N(\cdot,\theta))\|W(\theta)-V(\theta)\| \\ &\le \mathrm{Lip}_x(N)\|W-V\|_{C^0}.\end{aligned}$$

By the Banach fixed point theorem we get a unique fixed point W_0 which gives the desired invariant torus.

□

The proofs of Parts (ii) and (iii) in Theorem 7.1 are a consequence of the next two Lemmas, which deal with the regularity of an operator related to the invariance equation.

Lemma 7.2. *Let*

$$\begin{aligned}\mathcal{V}_r &= \{W \in C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) \mid W(\theta) \in U\,, \forall\theta\in\mathbb{T}^d\}, \\ \mathcal{V} &= C^{t,r}(U\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) \times \mathcal{V}_r, \quad t \ge r+1.\end{aligned}$$

We introduce the operator $\mathcal{F} : \mathcal{V} \to C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ *by*

$$\mathcal{F}(F,W)(\theta) = F(W(\theta-\omega),\theta-\omega) - W(\theta).$$

The operator $\mathcal{F}$ *is well-defined, jointly* C^1 *in* $\mathcal{V}$ *and*

$$[D_W\mathcal{F}(F,W)]\,\Delta W(\theta) = D_xF(W(\theta-\omega),\theta-\omega)\Delta W(\theta-\omega) - \Delta W(\theta).$$

We will need the following auxiliary result. The proof can be found in [FdlLM11a] (page 2875).

Theorem 7.3. *Let* E, F, G *be Banach spaces and* $U \subset E$, $V \subset F$ *open sets such that* $0 \in V$. *Let* $f : U \times V \to G$ *under the following assumptions*

1. *for all* $y \in V$*,* $f(\cdot, y)$ *is linear continuous,*
2. *for all* $x \in U$*,* $f(x, \cdot)$ *is* $C^r(V, G)$ *and* $\|f(x, \cdot)\|_{C^r(V,G)} \le C$ *for* $x \in B(0, \delta)$ *and* $C, \delta > 0$*.*

Then $f \in C^r(U \times V, G)$*.*

Remark 7.4. *In fact, by the linearity of* f *with respect to* $x \in E$ *we can extend* f *to be* C^r *in* $E \times V$*.*

Proof. First we need to check that the operator is well defined between these spaces. Lemma 5.3 proves that $C^{t,r}$ functions composed with C^r functions are again C^r functions, thus the operator is well-defined.

To study the regularity of $\mathcal{F}$, we only need to study the differentiability of the composition operator:

$$\widetilde{\mathcal{F}}(F, W)(\theta) = F(W(\theta - \omega), \theta - \omega),$$

observe that $\widetilde{\mathcal{F}} = \mathcal{F} + \Pi_2$, where $\Pi_2(F, W) = W$, and $\widetilde{\mathcal{F}}$ is linear with respect to F. Theorem 7.3 proves that an operator in two variables between Banach spaces which is linear with respect to one variable and C^r with respect to the other is jointly C^r. If we can show $\widetilde{\mathcal{F}}$ is C^1 with respect to the first variable (which is W) then the operator $\widetilde{\mathcal{F}}$ is jointly C^1.

Define

$$\mathcal{A}(W)(\Delta W)(\theta) = D_x F(W(\theta - \omega), \theta - \omega)\Delta W(\theta - \omega)$$

with $\Delta W \in C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $\|\Delta W\|$ small enough so that $W(\theta) + \lambda \Delta W(\theta) \in U$ for all $\theta \in \mathbb{T}^d$ and $\lambda \in [0, 1]$. By applying the mean value theorem we can write

$$\begin{aligned} &\widetilde{\mathcal{F}}(F, W + \Delta W)(\theta) - \widetilde{\mathcal{F}}(F, W)(\theta) - \mathcal{A}(W)(\Delta W)(\theta) \\ &\quad = \int_0^1 \left[D_x F((W + \lambda \Delta W)(\theta - \omega), \theta - \omega) - D_x F(W(\theta - \omega), \theta - \omega)\right] \Delta W(\theta - \omega)\, d\lambda. \end{aligned} \tag{7.4}$$

Observe that the graph of W for a fixed W,

$$S_W := \{W(\theta) \,|\, \theta \in \mathbb{T}^d\},$$

is a compact set: a torus. Since $D_x F$ is continuous, it is uniformly continuous in this compact set and since $\lambda \Delta W$ is arbitrarily small, $W + \lambda \Delta W$ is arbitrarily close to the compact set S_W. We are thus working with two sets of points, $x = \big(W + \lambda \Delta W\big)(\theta)$ and $y = W(\theta)$, and with an uniformly continuous function, $D_x F$. The difference $D_x F(W + \lambda \Delta W) - D_x F(W)$ is then a difference of images of points such that one of them is in the set where $D_x F$ is uniformly continuous and thus is bounded by ε if $\|x - y\| < \delta$ for x and y defined as above and $\theta \in \mathbb{T}^d$, $\lambda \in [0, 1]$, in particular when $\|\Delta W\|_{C^0} < \delta$, where ε and δ are the constants used in the definition of uniform continuity of $D_x F$.

Thus given $\varepsilon > 0$ there is $\delta > 0$ such that if $\|\Delta W\|_{C^0} < \delta$, we can bound the integral (7.4) in C^0-norm as

$$\begin{aligned} &\sup_{\theta \in \mathbb{T}^d} \left\| \int_0^1 \left[D_x F((W + \lambda \Delta W)(\theta - \omega), \theta - \omega) - D_x F(W(\theta - \omega), \theta - \omega)\right] \Delta W(\theta - \omega) d\lambda \right\| \\ &\le \int_0^1 \sup_{\theta \in \mathbb{T}^d} \left\| \left[D_x F((W + \lambda \Delta W)(\theta - \omega), \theta - \omega) - D_x F(W(\theta - \omega), \theta - \omega)\right] \Delta W(\theta - \omega) \right\| d\lambda \\ &\le \varepsilon \|\Delta W\|_{C^0}. \end{aligned}$$

Actually we have to bound the C^r norm of (7.4). To this end, consider the integral of the derivatives, which we can write as follows (cf. 5.3) for $1 \le k \le r$:

$$\int_0^1 D_\theta^k \left[D_x F((W+\lambda\Delta W)(u),u) - D_x F(W(u),u)\right]\Delta W(u) d\lambda$$
$$= C\int_0^1 \sum_{\substack{(b,a)\in\Sigma^*_{0,k}\\ \|I\|_1=k-a}} \Big[\partial_\theta^a\partial_x^b D_x F((W+\lambda\Delta W)(u),u)\partial_\theta^{i_1}(W+\lambda\Delta W)(u)\cdots\partial_\theta^{i_b}(W+\lambda\Delta W)(u)$$
$$-\partial_\theta^a\partial_x^b D_x F(W(u),u)\partial_\theta^{i_1}W(u)\cdots\partial_\theta^{i_b}W(u)\Big]\Delta W(u)d\lambda,$$

where $u=\theta-\omega$,

$$\Sigma^*_{0,i} = \{(b,a)\in\mathbb{N}^2 \,|\, a+b\le i, b\ge 1\}\cup\{(0,i)\}$$

and $I=(i_1,\ldots,i_b)$ is a multi-index. We can bound the argument of this integral in C_0-norm grouping terms and using the same technique of uniform continuity of $\partial_x^{b+1}\partial_\theta^a F$ around the graph of W we have used above. This bound for the derivatives implies we have a bound in C^r-norm of the form $C\varepsilon\|\Delta W\|_{C^r}$, where the constant C only depends on t and r. This means we can exchange derivation with respect to θ and integration, thus the C^r-norm of

$$\widetilde{\mathcal{F}}(F,W+\Delta W)(\theta) - \widetilde{\mathcal{F}}(F,W)(\theta) - \mathcal{A}(W)(\Delta W)(\theta)$$

has a bound of the form $C\varepsilon\|\Delta W\|_{C^r}$.

To prove continuity of $\mathcal{A}(W)$ with respect to W we can use a similar argument with the uniform continuity of $D_x^j F$, $1\le j\le r+1$, in a compact set as follows. Given $\varepsilon>0$ and a fixed W_1, there is a $\delta>0$ such that if $\|W_1-W_2\|_{C^r}\le\delta$, the expression

$$\big(\mathcal{A}(W_1)-\mathcal{A}(W_2)\big)\Delta W(\theta)$$
$$= \left(D_x F(W_1(\theta-\omega),\theta-\omega) - D_x F(W_2(\theta-\omega),\theta-\omega)\right)\Delta W(\theta)$$

can be bounded in C^r-norm by $C\varepsilon\|\Delta W\|_{C^r}$ by the uniform continuity of $D_x^j F$, $0\le j\le r+1$ in $W_1(\theta)$, $\theta\in\mathbb{T}^d$ (or $W_2(\theta)$, $\theta\in\mathbb{T}^d$), therefore $\mathcal{A}(W)$ is continuous.

□

The following lemma is analogous to Lemma 7.2 above, but using decay norms.

Lemma 7.5. *Let*

$$\mathcal{V}_{r,\Gamma} = \{W\in S^r_{j,\Gamma}(\mathbb{T}^d,\ell^\infty(\mathbb{R}^n)) \,|\, W(\theta)\in U, \forall\theta\in\mathbb{T}^d\},$$
$$\mathcal{V}_\Gamma = C^{t,r}_\Gamma(U\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))\times\mathcal{V}_{r,\Gamma}, \quad t\ge r+2.$$

We introduce the operator $\mathcal{F}:\mathcal{V}_\Gamma\to S^r_{j,\Gamma}(\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))$ *by*

$$\mathcal{F}(F,W)(\theta) = F(W(\theta-\omega),\theta-\omega) - W(\theta).$$

The operator $\mathcal{F}$ *is well-defined, jointly* C^1 *in* $\mathcal{V}_\Gamma$ *and*

$$[D_W\mathcal{F}(F,W)]\,\Delta W(\theta) = D_x F(W(\theta-\omega),\theta-\omega)\Delta W(\theta-\omega) - \Delta W(\theta).$$

Proof. Proposition 5.4 proves that the composition of a function $F \in C^{t,r}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $W(\theta) \in S^r_{j,\Gamma}(\mathbb{T}^d, U)$ is in $S^r_{j,\Gamma}$. Therefore this operator sends $\mathcal{V}_\Gamma$ to $S^r_{j,\Gamma}(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$.

To study the regularity of $\mathcal{F}$, we only need to study the differentiability of the composition operator:

$$\widetilde{\mathcal{F}}(F, W)(\theta) = F(W(\theta - \omega), \theta - \omega)$$

as we did in the previous case.

Observe that as before $\widetilde{\mathcal{F}} = \mathcal{F} + \Pi_2$, where $\Pi_2(F, W) = W$, and $\widetilde{\mathcal{F}}$ is linear with respect to F, thus Theorem 7.3 proves that if we can show $\widetilde{\mathcal{F}}$ is C^1 with respect to the first variable by the continuity with respect to the other variable the operator $\widetilde{\mathcal{F}}$ will be jointly C^1.

Let $\mathcal{A} : \mathcal{V}_{r,\Gamma} \to L(S^r_{j,\Gamma}, S^r_{j,\Gamma})$ be defined by

$$\mathcal{A}(W)(\Delta W)(\theta) = D_x F(W(\theta - \omega), \theta - \omega)\Delta W(\theta - \omega)$$

with $\Delta W(\theta) \in S^r_{j,\Gamma}(\mathbb{T}^d, U)$ small enough. By means of an application of the mean value theorem

$$\begin{aligned}
&\widetilde{\mathcal{F}}(F, W + \Delta W)(\theta) - \widetilde{\mathcal{F}}(F, W)(\theta) - \mathcal{A}(W)(\Delta W)(\theta) \qquad (7.5)\\
&\quad = \int_0^1 \left[D_x F((W + \lambda\Delta W)(\theta - \omega), \theta - \omega) - D_x F(W(\theta - \omega), \theta - \omega)\right] \Delta W(\theta - \omega) d\lambda.
\end{aligned}$$

As before, we use the fact that the graph of W is a compact set:

$$S_W := \left\{ W(\theta) \,|\, \theta \in \mathbb{T}^d \right\},$$

hence $D_x F$ is uniformly continuous in S_W and the difference $D_x F(W + \lambda\Delta W) - D_x F(W)$ is bounded by ε if $\|x - y\| < \delta$ for all $x = \big(W + \lambda\Delta W\big)(\theta)$, $y = W(\theta)$, $\theta \in \mathbb{T}^d$, $\lambda \in [0, 1]$, which holds if $\|\Delta W\|_{C^0} < \delta$, where ε and δ are the constants given by the definition of uniform continuity of $D_x F$.

Given $\varepsilon > 0$ there is $\delta > 0$ such that if $\|\Delta W\|_{C^0} < \delta$, we can bound the integral in $S^0_{j,\Gamma}$-norm by

$$\begin{aligned}
&\left\| \int_0^1 \left[D_x F((W + \lambda\Delta W)(\theta - \omega), \theta - \omega) - D_x F(W(\theta - \omega), \theta - \omega)\right] \Delta W(\theta - \omega) d\lambda \right\|_{S^0_{j,\Gamma}}\\
&\le \int_0^1 \left\| \left[D_x F((W + \lambda\Delta W)(\theta - \omega), \theta - \omega) - D_x F(W(\theta - \omega), \theta - \omega)\right] \Delta W(\theta - \omega) \right\|_{S^0_{j,\Gamma}} d\lambda\\
&\le \varepsilon \|\Delta W\|_{S^0_{j,\Gamma}}
\end{aligned}$$

by applying Proposition 4.14, which proves that for $\sigma \in S^r_{j,\Gamma}(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ and $A \in C^t_{L_\Gamma}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $A\sigma \in S^m_{j,\Gamma}(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $m = \min(t, r)$ and $\|A(\theta)\sigma(\theta)\|_{S^m_{j,\Gamma}} \le 2^m \|A(\theta)\|_{C^m_{L_\Gamma}} \|\sigma(\theta)\|_{S^m_{j,\Gamma}}$. Hence the result follows with $A(\theta) = D_x F(W(\theta), \theta)$ and $\sigma(\theta) = \Delta W(\theta)$.

We can now write the integral of the derivatives as

$$\int_0^1 D_\theta^r \left[D_x F((W+\lambda\Delta W)(u),u) - D_x F(W(u),u)\right]\Delta W(u) d\lambda$$
$$= C\int_0^1 \sum_{\substack{(b,a)\in\Sigma^*_{0,r}\\ \|I\|_1=r-a}} \Big[\partial_\theta^a\partial_x^b D_x F((W+\lambda\Delta W)(u),u)\partial_\theta^{i_1}(W+\lambda\Delta W)(u)\cdots\partial_\theta^{i_b}(W+\lambda\Delta W)(u)$$
$$- \partial_\theta^a\partial_x^b D_x F(W(u),u)\partial_\theta^{i_1}W(u)\cdots\partial_\theta^{i_b}W(u)\Big]\Delta W(u)d\lambda,$$

where $u=\theta-\omega$,

$$\Sigma^*_{0,i} = \{(b,a)\in\mathbb{N}^2 \,|\, a+b\le i, b\ge 1\}\cup\{(i,0)\}$$

and $I=(i_1,\dots,i_b)$ is a multi-index. We can bound this expression in $S^0_{j,\Gamma}$ norm with the same uniform continuity argument as before and using Proposition 4.16, which proves the contraction properties for multilinear maps with regularity $S^r_{j,\Gamma}$. This bound for the derivatives implies we have a bound in $S^r_{j,\Gamma}$-norm of the type $C\varepsilon\|\Delta W\|_{S^r_{j,\Gamma}}$ with the constant C depending only on and r. This shows that $\mathcal{A}$ is the derivative with respect to W of $\mathcal{F}$ in the spaces we are working with.

To prove the continuity of $\mathcal{A}(W)$ with respect to W we use the fact that F is at least C^2 with respect to x. Given $\varepsilon>0$ and $W_1\in\mathcal{V}_{r,\Gamma}$ there is $\delta>0$ such that given $W_2\in\mathcal{V}_{r,\Gamma}$ with $\|W_1-W_2\|_{S^r_{j,\Gamma}}\le\delta$,

$$\begin{aligned}\|(\mathcal{A}(W_1)-\mathcal{A}(W_2))\Delta W(\theta)\|_{S^r_{j,\Gamma}} &\\ &= \|\left(D_xF(W_1(\theta-\omega),\theta-\omega)-D_xF(W_2(\theta-\omega),\theta-\omega)\right)\Delta W(\theta-\omega)\|_{S^r_{j,\Gamma}}\\ &= \Big\|\int_0^1 D_x^2F(W_2(\theta-\omega)+t(W_1-W_2)(\theta-\omega),\theta-\omega)\\ &\qquad\times((W_1-W_2)(\theta-\omega),\Delta W(\theta-\omega)\,dt\Big\|_{S^r_{j,\Gamma}},\end{aligned}$$

which can be bounded by $C\|W_1-W_2\|_{S^r_{j,\Gamma}}\|\Delta W\|_{S^r_{j,\Gamma}}$ since $D_x^jF(x)\in L_\Gamma$, $0\le j\le r+2$ thus $\mathcal{A}(W)$ is continuous in $\mathcal{V}_{r,\Gamma}$. Observe that the condition $t\ge r+2$ is needed in this step which proves continuity of the derivative.

□

Proof of Parts (ii) *and* (iii) *of Theorem 7.1.* We will prove these two parts using the Implicit Function Theorem in Banach spaces.

Part (ii) is standard, we need to check that $D_W\mathcal{F}(F_0,0)$ is invertible, i.e. we have to uniquely solve

$$D_W\mathcal{F}(F_0,0)\Delta W = g \tag{7.6}$$

given an arbitrary function $g\in C^r(\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))$, determining ΔW.

To do so, consider the decomposition $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$, for which $D_x F_0(0)$ splits as a block-diagonal linear application

$$D_x F_0(0) = \begin{pmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{pmatrix},$$

as detailed in Section 2.1. We can project ΔW according to this decomposition as ΔW^1 and ΔW^2.

In coordinates, Equation (7.6) reads

$$\begin{aligned} A_{1,1}\Delta W^1(\theta-\omega) - \Delta W^1(\theta) &= g^1(\theta), \\ A_{2,2}\Delta W^2(\theta-\omega) - \Delta W^2(\theta) &= g^2(\theta), \end{aligned}$$

for an arbitrary $g = (g^1, g^2) \in C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ also projected as above. This can be solved iteratively, by setting

$$\begin{aligned} \Delta W^1(\theta) &= -\sum_{p=0}^{\infty} (A_{1,1})^p g^1(\theta - p\omega), \\ \Delta W^2(\theta) &= \sum_{p=1}^{\infty} (A_{2,2})^{-p} g^2(\theta + p\omega). \end{aligned}$$

Since $\|A_{1,1}\| < 1$, $\|A_{2,2}^{-1}\| < 1$ the previous series are absolutely convergent. Moreover, if $g \equiv 0$, the only solution is $\Delta W \equiv 0$.

The convergence of the series of the derivatives is easy to check since $A_{1,1}$ and $A_{2,2}$ do not depend on θ. They are also absolutely convergent if $g(\theta)$ is in $C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$. This means we can exchange the order of derivation and infinite sum and then the solution ΔW is in $C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$.

By the Implicit Function Theorem in Banach spaces (see [Nir01], for instance) we now know that there exists a unique function $\mathcal{G}$ from a neighbourhood $\mathcal{U}$ of F_0 in $C^{t,r}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ to $C^r(\mathbb{T}^d, U)$ such that $\mathcal{G}$ is C^1 and

$$\mathcal{F}(F, \mathcal{G}(F))(\theta) = \mathcal{G}(F)(\theta + \omega).$$

Moreover we have the following bound

$$\|\mathcal{G}(F)\|_{C^r} \le C\|F - F_0\|_{C^{t,r}}, \quad F \in \mathcal{U}$$

as a consequence of the fact that $\mathcal{G}(F)$ depends C^1 with respect to F, where C is a constant depending only on $\|D_x F_0(0)\|$.

To prove Part (iii) we need to work a little more. Recall from Remark 3.4 that bounds involving uncoupled linear maps with Γ-decay have an additional constant, $\Gamma(0)^{-1}$. Therefore we cannot bound iterates directly, since it results in factors $\Gamma(0)^{-n}$ making fixed point arguments divergent.

To avoid this difficulty we consider the invariance of this torus for a high enough iterate of F. Starting with

$$F(W(\theta-\omega), \theta-\omega) = W(\theta),$$

we can write

$$F(F(W(\theta-2\omega),\theta-2\omega),\theta-\omega)=F(W(\theta-\omega),\theta-\omega)=W(\theta)$$

and iterate this procedure. To simplify the notation we introduce

$$\begin{aligned}F^{[1]}(x,\theta)&=F(x,\theta-\omega),\\ F^{[2]}(x,\theta)&=F(F^{[1]}(x,\theta-\omega),\theta-\omega)=F(F(\theta-2\omega),\theta-2\omega),\theta-\omega)\end{aligned}$$

and in general,

$$F^{[n]}(x,\theta)=F(F^{[n-1]}(x,\theta-\omega),\theta-\omega).$$

With this notation the invariant torus parametrised by W has to satisfy

$$F^{[n]}(W(\theta-n\omega),\theta)=W(\theta),$$

and hence we define the operator

$$\mathcal{F}^{[n]}(F,W)(\theta)=F^{[n]}(W(\theta-n\omega),\theta)-W(\theta).$$

Moreover we define $M^{[n]}$ as the linear part of $F^{[n]}$, i.e.

$$M^{[n]}(\theta)=M(\theta-\omega)\cdots M(\theta-n\omega)=M_0^n+\mathcal{O}(\|M_1\|).$$

Observe that M_0^n can be written as

$$\begin{pmatrix}A_{1,1}^n & 0\\ 0 & A_{2,2}^n\end{pmatrix}.$$

By the properties of Γ-norms, $\|A_{1,1}^n\|_\Gamma=\|A_{1,1}^n\|\Gamma(0)^{-1}$, $\|A_{2,2}^{-n}\|_\Gamma=\|A_{2,2}^{-n}\|\Gamma(0)^{-1}$. Let N be the minimum natural number such that both these norms are smaller than 1. This number has to exist since $\|A_{1,1}\|$ and $\|A_{2,2}^{-1}\|$ are both smaller than 1.

Now the proof proceeds analogously to the proof of Part (ii) working with the operator $\widetilde{\mathcal{F}}=\mathcal{F}^{[N]}$ instead of $\mathcal{F}$.

By Lemma 7.5, the operator $\widetilde{\mathcal{F}}(F,W)(\theta)=F^{[N]}(W(\theta-N\omega),\theta)-W(\theta)$ can be differentiated with respect to W and

$$D_W\widetilde{\mathcal{F}}(F,W)\Delta W(\theta)=D_xF^{[N]}(W(\theta-N\omega),\theta)\Delta W(\theta-N\omega)-\Delta W(\theta).$$

To prove the existence of the invariant torus we apply the Implicit Function Theorem in Banach Spaces to $\widetilde{\mathcal{F}}(F,W)=0$. We clearly have that $\widetilde{\mathcal{F}}(F_0,0)=0$. We only have to check the invertibility of

$$D_W\widetilde{\mathcal{F}}(F_0,0)(\theta)=D_xF_0^{[N]}(0,\theta)\Delta W(\theta-N\omega)-\Delta W(\theta)=M_0^N\Delta W(\theta-N\omega)-\Delta W(\theta).$$

As before, given $g\in S^r_{j,\Gamma}(\mathbb{T}^d,\ \ell^\infty(\mathbb{R}^n))$ and projecting onto $\mathcal{E}^1$ and $\mathcal{E}^2$ respectively we are led to consider the equations

$$\begin{aligned}A_{1,1}^N\Delta W^s(\theta-N\omega)-\Delta W^s(\theta)&=g^s(\theta),\\ A_{2,2}^N\Delta W^u(\theta-N\omega)-\Delta W^u(\theta)&=g^u(\theta),\end{aligned}$$

which have the solutions

$$\Delta W^s(\theta) = -\sum_{p=0}^{\infty} A_{1,1}^{Np} g^s(\theta - Np\omega),$$
$$\Delta W^u(\theta) = \sum_{p=1}^{\infty} A_{2,2}^{-Np} g^u(\theta + Np\omega),$$

respectively. By Proposition 4.14 we have that ΔW^s and ΔW^u belong to $S^r_{j,\Gamma}$ and

$$\|\Delta W^s\|_{S^r_{j,\Gamma}} \leq \sum_{p=0}^{\infty} \|A_{1,1}^{Np}\|_\Gamma \|g^s\|_{S^r_{j,\Gamma}} \leq \sum_{p=0}^{\infty} \|A_{1,1}^{N}\|_\Gamma^p \|g^s\|_{S^r_{j,\Gamma}},$$
$$\|\Delta W^u\|_{S^r_{j,\Gamma}} \leq \sum_{p=1}^{\infty} \|A_{2,2}^{-Np}\|_\Gamma \|g^u\|_{S^r_{j,\Gamma}} \leq \sum_{p=1}^{\infty} \|A_{2,2}^{-N}\|_\Gamma^p \|g^u\|_{S^r_{j,\Gamma}},$$

the series on the right-hand sides being convergent by the choice of N.

Then there exists a (locally) unique implicit function $\mathcal{G}^{[N]}$ such that

$$\mathcal{F}^{[N]}(F, \mathcal{G}^{[N]}(F))(\theta) = \mathcal{G}^{[N]}(F)(\theta).$$

Observe that $\mathcal{G}^{[N]}(F)$ is a solution in the $S^r_{j,\Gamma}$-norm and also a C^r solution of $\mathcal{F}^{[N]}(F,W) = 0$. The same argument applied to $\mathcal{F}^{[N]}$ from $\mathcal{V}$ to $C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ proves that $\mathcal{F}^{[N]}(F,W) = 0$ has a unique C^r solution $\widetilde{\mathcal{G}}^{[N]}(F)$.

On the other hand, $\mathcal{F}(F,W) = 0$ has a unique C^r solution $\mathcal{G}(F)$. These two solutions must be the same, since $\mathcal{G}(F)$ is also a solution of $\mathcal{F}^{[N]}(F,W) = 0$ and both solutions are locally unique. Therefore $\mathcal{G}^{[N]}(F)$ is the (locally) unique $S^r_{j,\Gamma} \subset C^r$ solution of $\mathcal{F}(F,W) = 0$ and the result is proved.

As before, we can use the C^1-regularity of the implicit function $\mathcal{G}^{[N]}$ to get the following bound

$$\|\mathcal{G}^{[N]}(F)\|_{S^r_{j,\Gamma}} \leq C\|F - F_0\|_{C^{t,r}_{j,\Gamma}},$$

which will be used in later sections.

□

Chapter 8

Linearisation around the torus

In this chapter we will determine the linearisation around the invariant torus and find the linear term in the parametrisation of *strong stable invariant manifolds* and *non-resonant stable invariant manifolds*. Recall that strong stable invariant manifolds of fixed points of maps F are associated to a subset σ_1 of the spectrum having the following property: if M is the linear part of the map at the fixed point, $\mathrm{Spec}(M) = \sigma_1 \cup \sigma_2$ and there exists a modulus $\rho < 1$ such that

$$\sup_{\lambda\in\sigma_1} |\lambda| < \rho < \inf_{\lambda\in\sigma_2} |\lambda|.$$

We introduce now the annulus generated by a set. Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. Given a set $S \subset \mathbb{C}$, we will define the annulus generated by S as the set

$$\mathcal{A}S = \{e^{i\theta}s \,|\, \theta \in [0, 2\pi),\, s \in S\}.$$

Non-resonant stable invariant manifolds on the other hand are associated to an invariant subspace by the linear part M such that the spectrum of M restricted to this subspace is contained in the unit disc. Some non-resonance conditions are needed so that these manifolds do exist and have the required regularity. In this Chapter our maps F will be perturbations of an uncoupled map f with

$$Df_0(0) = \begin{pmatrix} \mathfrak{a}_{1,1} & 0 \\ 0 & \mathfrak{a}_{2,2} \end{pmatrix}$$

We will require an additional (with respect to the case of manifolds of fixed points) non-resonance condition to determine non-resonant invariant manifolds,

- $\mathcal{A}\,\mathrm{Spec}\,\mathfrak{a}_{1,1} \cdot \mathcal{A}\,\mathrm{Spec}\,\mathfrak{a}_{2,2}^{-1} \cap \{z \in \mathbb{C} \,|\, |z| = 1\} = \emptyset$.

This condition is a technical requirement, used when the extended mapping $D_xF : \ell^\infty(\mathbb{R}^n) \to \ell^\infty(\mathbb{R}^n)$ is not in block triangular form in the decomposition defined by the projections $\Pi_{\mathcal{E}^1}$ and $\Pi_{\mathcal{E}^2}$ of the induced decomposition $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$. When the linear part of the dynamical system is in block triangular form, the linearisation has a trivial solution, and with this spectral condition we can find a linear transformation that puts D_xF in block triangular form, i.e. where $\Pi_{\mathcal{E}^2} D_x\widetilde{F}\mathrm{emb}_{\mathcal{E}^1}$ is 0. This is done by splitting the restrictions $\Pi_{\mathcal{E}^1} D_xF_0 = A_{1,1}$ and $\Pi_{\mathcal{E}^2} D_xF_0 = A_{2,2}$ according to the spectral decomposition.

Actually, strong stable manifolds are particular cases of non-resonant manifolds. We treat both cases separately because we can prove better regularity results for strong stable manifolds.

We want to determine the linear part of the function describing the local dynamics on the invariant manifold close to the torus, traditionally denoted by $R(s,\theta)$ in the parametrisation method ([CFdlL03a]). This linear part of $R(s,\theta)$ is denoted by $R_1(\theta)$, and we write the expansion of $R(s,\theta)$ as

$$R(s,\theta) = R_1(\theta)s + R_2(\theta)s^{\otimes 2} + \dots .$$

Following the same convention in the parametrisation method, the parametrising function $W(x,\theta)$ has a linear part, denoted by $W_1(\theta)$, which we want to determine. The condition R_1 and W_1 satisfy under the action of the dynamical system (up to first order) is

$$F\big(W_0(\theta) + W_1(\theta)s, \theta\big) = W_0(\theta+\omega) + W_1(\theta+\omega)R_1(\theta)s + o(\|s\|). \tag{8.1}$$

For convenience we will translate the torus $W_0(\theta)$ to $0 \in \ell^\infty(\mathbb{R}^n)$. Let $T_{W_0(\theta)}(x,\theta) = (W_0(\theta)+x,\theta)$, which clearly has the same regularity as W_0 with respect to θ. The transformed mapping is $\widetilde{F}(x,\theta) = T^{-1}_{W_0(\theta+\omega)} \circ F \circ T_{W_0(\theta)}(x,\theta)$ and satisfies $\widetilde{F}(0,\theta) = 0$.

Notice that

$$\begin{aligned}\widetilde{F}(x,\theta) &= F_0(x + W_0(\theta)) + F_1(x+W_0(\theta),\theta) - W_0(\theta+\omega)\\ &= F_0(x) + [F_0(x+W_0(\theta)) - F_0(x) + F_1(x+W_0(\theta),\theta) - W_0(\theta+\omega)]\end{aligned}$$

which can be written as

$$\widetilde{F}(x,\theta) = F_0(x) + \widetilde{F}_1(x,\theta),$$

where

$$\widetilde{F}_1 = F_0(x+W_0(\theta)) - F_0(x) + F_1(x+W_0(\theta),\theta) - W_0(\theta+\omega).$$

Then $\widetilde{F}_1$ has the same regularity and smallness properties as F_1.

Note also that

$$D_x\widetilde{F}(0,\theta) = D_xF_0(0) + D_xF_0(W_0(\theta)) - D_xF_0(0) + D_xF_1(W_0(\theta),\theta).$$

From now on we will write F again for $\widetilde{F}$.

With this new notation we also write $F(x,\theta) = D_xF(0,\theta)x + N(x,\theta)$, with $N(0,\theta)=0$ and $DN(0,\theta)=0$.

With the torus translated to 0, Equation (8.1) has the form

$$F(W_1(\theta)s,\theta) = W_1(\theta+\omega)R_1(\theta)s + o(\|s\|). \tag{8.2}$$

8.1 Linear part for a strong stable invariant manifold

Recall from Section 2.1 that the splitting of the spectrum of $D_xF_0(0)$ into a strong stable part and its complement induces a splitting of $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$, where $\mathcal{E}^1$ and $\mathcal{E}^2$ are invariant by $D_xF_0(0)$. We write

$$D_xF_0(0) = \begin{pmatrix} A_{1,1} & 0 \\ 0 & A_{2,2}\end{pmatrix},$$

and

$$D_xF_1(0,\theta) = \begin{pmatrix} B_{1,1}(\theta) & B_{1,2}(\theta) \\ B_{2,1}(\theta) & B_{2,2}(\theta) \end{pmatrix}.$$

Let

$$D_xF(0,\theta) = D_xF_0(0) + D_xF_1(0,\theta) = \begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ M_{2,1}(\theta) & M_{2,2}(\theta) \end{pmatrix}.$$

Recall also from Section 2.1 that we use an adapted norm in $\ell^\infty(\mathbb{R}^n)$ such that

$$\|A_{1,1}\|\|A_{2,2}^{-1}\| < 1,$$

induced by a norm in $\mathbb{R}^n$. From now on we will use this underlying norm unless explicitly stated.

It is straightforward to determine a linear part of the parametrisation $W_1(\theta)$ in the strong stable (also in the non-resonant) case if we assume $D_xF(0,\theta)$ is block triangular in the decomposition $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$ (see for instance [CFdlL03a], [HdlL06b]). In this subsection we will put $D_xF(0,\theta)$ in block triangular form to determine this linear part, assuming $A_{1,1}$ is contracting and $\|A_{1,1}\|\|A_{2,2}^{-1}\| < 1$.

Observe that if $D_xF(0,\theta)$ is block triangular, i.e.

$$D_xF(0,\theta) = \begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ 0 & M_{2,2}(\theta) \end{pmatrix}$$

we can choose $R_1(\theta) = M_{1,1}(\theta)$ and $W_1(\theta) = (\mathrm{Id}_{\mathcal{E}^1}, 0_{\mathcal{E}^2})$ where W_1 is expressed according to the splitting $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$, $\mathrm{Id}_{\mathcal{E}^1}$ denotes the mapping $g(\theta) \equiv \mathrm{Id}_{\mathcal{E}^1}$ for all θ and $0_{\mathcal{E}^2}$ denotes the mapping $h(\theta) \equiv 0_{\mathcal{E}^2}$ for all θ.

We will use the fact that our system is close to a constant coefficients system and the contraction conditions stated above to find a linear change of coordinates transforming $D_xF(0,\theta)$ into a block triangular linear mapping as shown, thus finding the linear terms in the parametrisation.

We look for $W_1(\theta)$ in the form $(\mathrm{Id}, v(\theta))$, where Id denotes the identity in $\mathcal{E}^1$ and $v(\theta) \in \mathcal{E}^2$, $\forall \theta \in \mathbb{T}^d$. Equation (8.2) cut to first order can be written as

$$\begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ M_{2,1}(\theta) & M_{2,2}(\theta) \end{pmatrix} \begin{pmatrix} \mathrm{Id} \\ v(\theta) \end{pmatrix} = \begin{pmatrix} \mathrm{Id} \\ v(\theta+\omega) \end{pmatrix} R_1(\theta). \tag{8.3}$$

Proposition 8.1. *Let F be defined as in Section 2.1 and $D_xF_0(0)$ written as above under the splitting of its spectrum in a strong stable part and its complementary. Assume*

(H1) $\|A_{1,1}\| < 1$ *in some norm,*

(H2) $A_{2,2}$ *is invertible, and* $\|A_{1,1}\|\|A_{2,2}^{-1}\| < 1$ *in the same norm as the previous hypothesis,*

(H3) $\|F_1\|_{C_\Gamma^{\Sigma_{t,r}}}$ *is small enough,* $t \geq r+1$, $r \geq 0$.

Then we can find $R_1 \in C_\Gamma^r(\mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^1))$ *and* $W_1 \in C_\Gamma^r(\mathbb{T}^d, L(\mathcal{E}^1, \ell^\infty(\mathbb{R}^n)))$ *such that*

$$F(W_1(\theta)s, \theta) = W_1(\theta+\omega)R_1(\theta)s + o(\|s\|). \tag{8.4}$$

Moreover

$$R_1 = A_{1,1} + \mathcal{O}(\|F_1\|).$$

Remark 8.2. *In Hypothesis (H3) we ask for $t \geq r+1$ because this condition is needed to determine the invariant torus of class C^r. However, if we start with $F(x,\theta) = F_0(x) + F_1(x,\theta)$ such that*

$$F_0(0) = 0, \quad F_1(0,\theta) = 0, \qquad \forall \theta \in \mathbb{T}^d,$$

to prove the results in this chapter it would be enough to require $F_1 \in C_\Gamma^{\Sigma_{1,r}}$.

As stated in the previous introductory text, this proposition is a direct consequence of the following lemma, which determines a linear transformation converting $D_xF(0,\theta)$ into a block triangular map. The proof of this lemma is very similar to the proof needed in the next section, but the spectral condition is simpler, making the proof easier. We include it for completeness.

Lemma 8.3. *Under the hypotheses of Proposition 8.1 we can find $v \in C_\Gamma^r(\mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2))$ such that the linear transformation*

$$S(\theta) = \begin{pmatrix} \mathrm{Id} & 0 \\ v(\theta) & \mathrm{Id} \end{pmatrix}$$

transforms $D_xF(0,\theta)$ into a block upper triangular matrix. More precisely, there exists a splitting $\ell^\infty(\mathbb{R}^n) = \widetilde{\mathcal{E}_\theta^1} \oplus \widetilde{\mathcal{E}_\theta^2}$ such that the projections $\Pi_{\widetilde{\mathcal{E}_\theta^1}}, \Pi_{\widetilde{\mathcal{E}_\theta^2}} \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and that in this splitting we have

$$D_xF(0,\theta) = \begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ 0 & M_{2,2}(\theta) \end{pmatrix}.$$

Proof of Proposition 8.1. Expanding F by Taylor at 0 and keeping the first order terms in Equation (8.4) we get the following invariance equation

$$D_xF(0,\theta)W_1(\theta) = W_1(\theta+\omega)R_1(\theta).$$

By Lemma 8.3 we can find a linear transformation $S(\theta)$ transforming $D_xF(0,\theta)$ into $D_x\widetilde{F}(0,\theta)$ having a block triangular form, i.e.

$$S(\theta)^{-1}D_xF(0,\theta)S(\theta) = D_x\widetilde{F}(0,\theta) = \begin{pmatrix} \widetilde{M}_{1,1}(\theta) & \widetilde{M}_{1,2}(\theta) \\ 0 & \widetilde{M}_{2,2}(\theta) \end{pmatrix}$$

with $\widetilde{M}_{i,j} \in C_\Gamma^r(\mathbb{T}^d, L(\mathcal{E}^j, \mathcal{E}^i))$.

After this change of coordinates, we can choose $W_1(\theta) = (\mathrm{Id}_{\mathcal{E}^1}, 0_{\mathcal{E}^2})$ and

$$R_1(\theta) = \widetilde{M}_{1,1}(\theta) = M_{1,1}(\theta) + M_{1,2}(\theta)v(\theta) = A_{1,1} + \mathcal{O}(\|F_1\|)$$

and the statement is proved.

□

Proof of Lemma 8.3. Converting a 2×2 block matrix into triangular form amounts to making one of the non-diagonal terms $M_{1,2}(\theta)$ or $M_{2,1}(\theta)$ zero. This problem in this particular setting is equivalent to solving an invariance equation for a linear function $w(\theta) : \mathcal{E}^2 \to \mathcal{E}^1$

or $v(\theta) : \mathcal{E}^1 \to \mathcal{E}^2$. The invariance condition for the graph of a function $v(\theta) : \mathcal{E}^1 \to \mathcal{E}^2$ under $D_x F(0, \theta)$ can be expressed as

$$D_x F(0,\theta) \begin{pmatrix} \mathrm{Id} \\ v(\theta) \end{pmatrix} = \begin{pmatrix} \alpha(\theta) \\ v(\theta+\omega)\alpha(\theta) \end{pmatrix} \tag{8.5}$$

for some suitable function α.

Equation (8.5) is equivalent to the fixed point equation

$$\mathcal{S}(W)(\theta) = W(\theta),$$

where $\mathcal{S} : C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)) \to C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2))$ is the operator defined as

$$\mathcal{S}(v)(\theta) = M_{2,2}^{-1}(\theta)\left[v(\theta+\omega)\left(M_{1,1}(\theta) + M_{1,2}(\theta)v(\theta)\right) - M_{2,1}(\theta)\right].$$

We write this operator as $\mathcal{S} = \mathcal{S}_0 + \mathcal{S}_1$ with

$$\mathcal{S}_0(v)(\theta) = A_{2,2}^{-1} v(\theta+\omega) A_{1,1},$$
$$\mathcal{S}_1(v)(\theta) = \mathcal{S}(v)(\theta) - \mathcal{S}_0(v)(\theta).$$

By hypothesis, $\|A_{1,1}\|\|A_{2,2}^{-1}\| \le \rho < 1$ in the norm of $\ell^\infty(\mathbb{R}^n)$. Thus we can iterate S_0 to a high enough order and find

$$\mathcal{S}_0^N(v)(\theta) = \mathcal{S}_0(\mathcal{S}_0(\dots(v))) = A_{2,2}^{-N} v(\theta + N\omega) A_{1,1}^N$$

which we can bound by

$$\|\mathcal{S}_0^N(v)\|_{C^r_\Gamma} \le \|A_{2,2}^{-N}\|_\Gamma \|A_{1,1}^N\|_\Gamma \|v\|_{C^r_\Gamma} \le \rho^N \Gamma(0)^{-2} \|v\|_{C^r_\Gamma}.$$

Using the triangle inequality we can show that a high iterate of the operator $\mathcal{S}$ is a contraction in C^r_Γ norm. Indeed,

$$\|\mathcal{S}^N\| \le \|\mathcal{S}_0^N\| + \|\mathcal{S}_0^N - \mathcal{S}^N\|,$$

where

$$\mathcal{S}_0^N - \mathcal{S}^N = \sum_{k=0}^{N-1} \mathcal{S}_0^k(\mathcal{S}_0 - \mathcal{S}) \circ \mathcal{S}^{N-k-1}.$$

We can bound this expression by

$$\|\mathcal{S}_0^N - \mathcal{S}^N\| = \|\mathcal{S}_0 - \mathcal{S}\| \sum_{k=0}^{N-1} \|S_0^k\| \le \frac{\Gamma(0)^{-2}}{1-\rho}\|\mathcal{S}_0 - \mathcal{S}\|. \tag{8.6}$$

Let N be such that $\rho^N \Gamma(0)^{-2} < 1 - \delta$ for some $\delta > 0$, and $\|F_1\|_{C_\Gamma^{\Sigma_{t,r}}}$ small enough such that the right-hand side of (8.6) is smaller than δ. Then $\mathcal{S}^N$ sends $C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2))$ into itself and the operator $\mathcal{S}^N$ is a contraction on it.

Since $\mathcal{S}^N$ is a contraction it has a unique fixed point on it giving $W_1(\theta)$ in the form $(\theta, v(\theta))$. Since the fixed point of S_0 is zero and $S - S_0 = \mathcal{O}(\|F_1\|)$, the fixed point of $\mathcal{S}$ is also $\mathcal{O}(\|F_1\|)$. Moreover,

$$R_1(\theta) = M_{1,1}(\theta) + M_{1,2}(\theta)v(\theta) = A_{1,1} + \mathcal{O}(\|F_1\|)$$

as stated.

It is clear that $\widetilde{\mathcal{E}^1_\theta} = \operatorname{graph} v(\theta)$ and $\widetilde{\mathcal{E}^2_\theta} = \mathcal{E}^2$. Then from $x + y = x + v(\theta)x + y - v(\theta)x$ we see that $\Pi_{\widetilde{\mathcal{E}^1_\theta}}(x, y) = x + v(\theta)x$ and $\Pi_{\widetilde{\mathcal{E}^2_\theta}}(x, y) = y - v(\theta)x$ and since $v(\theta) \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ for all $\theta \in \mathbb{T}^d$, we also have $\Pi_{\widetilde{\mathcal{E}^i_\theta}} \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $i = 1, 2$, for all $\theta \in \mathbb{T}^d$.

□

8.2 Linear part of a non-resonant stable invariant manifold

As in the previous section, it is straightforward to determine the linear part $W_1(\theta)$ of the parametrisation $W(x, \theta)$ in the non-resonant case if we assume $D_xF(0, \theta)$ is block triangular in the decomposition $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$. In this subsection we will put $D_xF(0, \theta)$ in block triangular form to determine this linear part, assuming different spectral properties of the non-perturbed system than in the previous section, which make the proof slightly different.

Again, observe that if $D_xF(0, \theta)$ is block triangular, i.e.

$$D_xF(0, \theta) = \begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ 0 & M_{2,2}(\theta) \end{pmatrix}$$

we can choose $R_1(\theta) = M_{1,1}(\theta)$ and $W_1(\theta) = (\mathrm{Id}_{\mathcal{E}^1}, 0_{\mathcal{E}^2})$ where W_1 is expressed according to the splitting $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$, $\mathrm{Id}_{\mathcal{E}^1}$ denotes the mapping $g(\theta) \equiv \mathrm{Id}_{\mathcal{E}^1}$ for all θ and $0_{\mathcal{E}^2}$ denotes the mapping $h(\theta) \equiv 0_{\mathcal{E}^2}$ for all θ.

As before, assume we have a splitting of $\operatorname{Spec} D_xF_0(0)$ and a corresponding splitting of $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$ associated to the spectral decomposition. We write

$$D_xF_0(0) = \begin{pmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{pmatrix}, \quad D_xF_1(0, \theta) = \begin{pmatrix} B_{1,1}(\theta) & B_{1,2}(\theta) \\ B_{2,1}(\theta) & B_{2,2}(\theta) \end{pmatrix} \text{ and}$$
$$D_xF(0, \theta) = \begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ M_{2,1}(\theta) & M_{2,2}(\theta) \end{pmatrix}.$$

Proposition 8.4. *Let F be defined as in Section 2.1 and $D_xF_0(0)$ written as above. Assume*

(H1) $\|A_{1,1}\| < 1$ in some norm,

(H2) $A_{2,2}$ is invertible,

(H3) $\mathcal{A} \operatorname{Spec} A_{1,1} \cap \mathcal{A} \operatorname{Spec} A_{2,2} = \emptyset$,

(H4) $\|F_1\|_{C^{\Sigma_{t,r}}_\Gamma}$ is small enough (with $t \geq r + 1$, $r \geq 0$).

Then we can find $R_1 \in C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E})^1)$ and $W_1 \in C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^1, \ell^\infty(\mathbb{R}^n)))$ such that

$$F(W_1(\theta)s, \theta) = W_1(\theta + \omega)R_1(\theta)s + o(\|s\|). \tag{8.7}$$

Moreover $R_1 = A_{1,1} + \mathcal{O}(\|M_1\|)$.

As in the previous case, this proposition is a direct consequence of the following lemma, which determines a linear transformation converting $D_xF(0, \theta)$ into a block upper triangular map.

Lemma 8.5. *Under the hypotheses of Proposition 8.4 we can find* $v \in C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2))$ *such that the linear transformation*

$$T(\theta) = \begin{pmatrix} \mathrm{Id} & 0 \\ v(\theta) & \mathrm{Id} \end{pmatrix},$$

transforms $D_x F(0, \theta)$ *into a block upper triangular matrix.*

Once we have proved Lemma 8.5, the proof of Proposition 8.4 is exactly the same as the proof of Proposition 8.1.

Proof of Lemma 8.5. The components $B_{i,j}$, $M_{i,j}$ satisfy

$$B_{i,j},\, M_{i,j} \in C^r_\Gamma\big(\mathbb{T}^d, L(\mathcal{E}^j, \mathcal{E}^i)\big),$$

and we have the following bounds

$$\|B_{i,j}\|_{C^r_\Gamma} \le C \|F_1\|_{C^r_\Gamma},$$

where $\|\cdot\|_{C^r_\Gamma}$ applied to $B_{i,j}$ is the norm of the corresponding space $C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^j, \mathcal{E}^i))$.

Now we proceed to prove the existence of the change of variables claimed in the statement of this lemma.

As in the proof of Lemma 8.5 we start by studying an invariance equation for the graph of a function $v(\theta) : \mathcal{E}^2 \to \mathcal{E}^1$.

This invariance condition under $D_x F(0, \theta)$ can be expressed as

$$D_x F(0, \theta) \begin{pmatrix} \mathrm{Id} \\ v(\theta) \end{pmatrix} = \begin{pmatrix} \alpha(\theta) \\ v(\theta + \omega)\alpha(\theta) \end{pmatrix} \tag{8.8}$$

for some function α. Thus we need to find a bundle linear map $v \in C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2))$ satisfying this condition to determine the change of variables.

If we write (8.8) in components, we get the pair of equations

$$M_{1,1}(\theta) + M_{1,2}(\theta)v(\theta) = \alpha(\theta), \tag{8.9}$$

$$M_{2,1}(\theta) + M_{2,2}(\theta)v(\theta) = v(\theta + \omega)\alpha(\theta). \tag{8.10}$$

We can substitute α from Equation (8.9) into Equation (8.10), obtaining

$$M_{2,1}(\theta) + M_{2,2}(\theta)v(\theta) = v(\theta + \omega)\big(M_{1,1}(\theta) + M_{1,2}(\theta)v(\theta)\big). \tag{8.11}$$

We introduce

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}.$$

Note that $B \in C^r_\Gamma(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$.

Now we can determine $v(\theta)$ satisfying Equation (8.11) as a zero of the following operator:

$$\mathcal{T}(v, B)(\theta) = M_{2,2}^{-1}(\theta)\left[v(\theta + \omega)(M_{1,1}(\theta) + M_{1,2}(\theta)v(\theta)) - M_{2,1}(\theta)\right] - v(\theta). \tag{8.12}$$

This operator maps $C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)) \times C^r_\Gamma(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$ into $C^r_\Gamma(\mathbb{T}^d L(\mathcal{E}^1, \mathcal{E}^2))$ if $\|F_1\|_{C^{\Sigma_{t,r}}_\Gamma}$ is small enough, since under this condition, $M_{2,2}^{-1}$ exists by Lemma 4.18.

We will find a zero of $\mathcal{T}$ for B small using the Implicit Function Theorem in Banach spaces.

Lemma 8.6. *The operator* $\mathcal{T}$ *is* C^1*,* $\mathcal{T}(0,0) = 0$ *and*

$$\begin{aligned} D_v\mathcal{T}(v,B)\Delta v(\theta) =& M_{2,2}^{-1}(\theta)\Delta v(\theta+\omega)M_{1,1}(\theta) - \Delta v(\theta) \\ &+ M_{2,2}^{-1}(\theta)\Delta v(\theta+\omega)M_{1,2}v(\theta) + M_{2,2}^{-1}(\theta)v(\theta+\omega)M_{1,2}\Delta v(\theta). \end{aligned}$$

To prove this result we need an auxiliary lemma.

Lemma 8.7. *Under the regularity assumptions and notation introduced above, the mapping* $B_{2,2} \mapsto (A_{2,2} + B_{2,2})^{-1}$ *is* C^1 *from* $C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}_2, \mathcal{E}_2))$ *to* $C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}_2, \mathcal{E}_2))$*.*

Proof. It is sufficient to study the differentiability of the map Inv, that is, $A \mapsto A^{-1}$. In this case we can formally write

$$(A+\Delta)^{-1} = (I + A^{-1}\Delta)^{-1}A^{-1} = A^{-1} - A^{-1}\Delta A^{-1} + \sum_{n=2}^{\infty}(-1)^n(A^{-1}\Delta)^n A^{-1}.$$

Recall that to prove differentiability we actually need to bound

$$\|\mathrm{Inv}(A+\Delta) - \mathrm{Inv}(A) - D\mathrm{Inv}(A)\Delta\|.$$

For $D\mathrm{Inv}(A)$ we will use the formal derivative (cf. [AMR88]).

$$\begin{aligned} \|(A+\Delta)^{-1} - A^{-1} + A^{-1}\Delta A^{-1}\|_{C^r_\Gamma} &\le \sum_{n=2}^{\infty} \|(A^{-1}\Delta)\|^n_{C^r_\Gamma}\|A^{-1}\|_{C^r_\Gamma} \\ &= \|A^{-1}\|_{C^r_\Gamma}\|A^{-1}\Delta\|^2_{C^r_\Gamma}\sum_{n=0}^{\infty}\|(A^{-1})\Delta\|^n_{C^r_\Gamma}. \end{aligned}$$

The n-th term of the infinite sum can be bounded, by using Proposition 4.15, as

$$\left(2^r\|A^{-1}\|_{C^r_\Gamma}\|\Delta\|_{C^r_\Gamma}\right)^n.$$

Thus, the infinite sum is convergent when $\|\Delta\|_{C^r_\Gamma} \le \frac{1}{2^r\|A^{-1}\|_{C^r_\Gamma}}$ and the differentiability is proved. □

Proof of Lemma 8.6. Since $\|(A+\Delta)^{-1} - A^{-1} + A^{-1}\Delta A^{-1}\|_{C^r_\Gamma} = \mathcal{O}(\|\Delta\|^2_{C^r_\Gamma})$ to prove the differentiability of $\mathcal{T}$ first we will prove the differentiability of some auxiliary operators. First, note that the map $B \mapsto B_{i,j}$ is linear and continuous, then it is C^∞.

Let

$$\mathcal{T}_1(v,B)(\theta) = v(\theta+\omega)B_{1,2}(\theta)v(\theta). \tag{8.13}$$

This operator is C^1 with respect to v (since it is quadratic in v) and linear in B. Hence, from Theorem 7.3 it is jointly C^1.

Let

$$\mathcal{T}_2(v,B)(\theta) = v(\theta+\omega)M_{1,1}(\theta) - M_{2,1}(\theta) + \mathcal{T}_1(v,B)(\theta),$$

which is clearly C^1 because $\mathcal{T}_1$ is. Since Lemma 8.7 proves $B_{2,2} \mapsto (A_{2,2}+B_{2,2})^{-1}$ is C^1, the operator

$$\mathcal{T}_3(w,B) = (A_{2,2}+B_{2,2})^{-1}(\theta)w(\theta)$$

is C^1 and the differentiability of the operator $\mathcal{T}$ follows from

$$\mathcal{T}(v,B)(\theta) = \mathcal{T}_3(\mathcal{T}_2(v,B),B)(\theta) - v(\theta).$$

$\square$

To finish the proof of Lemma 8.5 we need to prove the invertibility of $D_v\mathcal{T}(0,0)$. First, we decompose the spectrum of $\mathfrak{a}_{1,1}$ and $\mathfrak{a}_{2,2}$ as

$$\begin{aligned}\operatorname{Spec}\mathfrak{a}_{1,1} &= \Lambda_1 \cup \ldots \cup \Lambda_p,\\ \operatorname{Spec}\mathfrak{a}_{2,2} &= \Lambda_1' \cup \ldots \cup \Lambda_q', \qquad |q-p| \le 1,\end{aligned}$$

where $\lambda \in \Lambda_i$ implies $\alpha_i \le |\lambda| \le \beta_i$ and $\mu \in \Lambda_j'$ implies $\alpha_j' \le |\mu| \le \beta_j'$ so that all intervals $[\alpha_i, \beta_i]$, $[\alpha_j', \beta_j']$ are disjoint. Hence, if $\eta \in \Lambda_i$ and $\zeta \in \Lambda_j'$ then either $\zeta^{-1}\eta < 1$ or $\zeta\eta^{-1} < 1$. Observe that $A_{1,1}$, $A_{2,2}$ can be put in a block diagonal form in this decomposition. Now write

$$\begin{aligned}A_{1,1} &= \operatorname{diag}(A_{1,1}^1, \ldots, A_{1,1}^p),\\ A_{2,2} &= \operatorname{diag}(A_{2,2}^1, \ldots, A_{2,2}^q).\end{aligned}$$

We can also write v in this decomposition using $q \times p$ blocks of linear mappings. We take a norm in $\mathbb{R}^n$ such that

$$\begin{aligned}\|A_{1,1}^i\| &\le \beta_i + \varepsilon, \qquad \|(A_{1,1}^i)^{-1}\| \le (\alpha_i + \varepsilon)^{-1},\\ \|A_{2,2}^j\| &\le \beta_j' + \varepsilon, \qquad \|(A_{2,2}^j)^{-1}\| \le (\alpha_j' + \varepsilon)^{-1},\end{aligned}$$

with $\varepsilon > 0$ so small that the intervals $[\alpha_i - \varepsilon, \beta_i + \varepsilon]$, $[\alpha_j' - \varepsilon, \beta_j' + \varepsilon]$ are disjoint.

Thus, either

$$\|A_{1,1}^i\|\|(A_{2,2}^j)^{-1}\| < 1$$

or

$$\|(A_{1,1}^i)^{-1}\|\|A_{2,2}^j\| < 1.$$

Finally, to prove the invertibility of $D_v\mathcal{T}(0,0)$, let $w \in C_\Gamma^r(\mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2))$ and assume it is also expressed in $q\times p$ blocks according to the same decomposition. To solve $D_v\mathcal{T}(0,0)\Delta v = w$, we have to solve $q \times p$ equations, expressed in matrix form. They are

$$A_{2,2}^{-1}\begin{pmatrix}\Delta v_{1,1}(\theta+\omega) & \cdots & \Delta v_{1,p}(\theta+\omega)\\ \vdots & \vdots & \vdots\\ \Delta v_{q,1}(\theta+\omega) & \cdots & \Delta v_{q,p}(\theta+\omega)\end{pmatrix} A_{1,1} - \begin{pmatrix}\Delta v_{1,1}(\theta) & \cdots & \Delta v_{1,p}(\theta)\\ \vdots & \vdots & \vdots\\ \Delta v_{q,1}(\theta) & \cdots & \Delta v_{q,p}(\theta)\end{pmatrix}$$
$$= \begin{pmatrix} w_{1,1}(\theta) & \cdots & w_{1,p}(\theta)\\ \vdots & \vdots & \vdots\\ w_{q,1}(\theta) & \cdots & w_{q,p}(\theta)\end{pmatrix}.$$

Thus we have to solve equations of the type

$$\big(D_v\mathcal{T}(0,0)\Delta v\big)_{i,j}(\theta) = w_{i,j}(\theta),$$

which can be written as

$$\left(A^i_{2,2}\right)^{-1}\Delta v_{i,j}(\theta+\omega)A^j_{1,1} - v_{i,j}(\theta) = w_{i,j}(\theta).$$

We can solve this type of equation by either setting

$$\Delta v_{i,j}(\theta) = -\sum_{n=0}^{\infty}\left(A^i_{2,2}\right)^{-n} w_{i,j}(\theta+n\omega)\left(A^j_{1,1}\right)^{n} \tag{8.14}$$

or

$$\Delta v_{i,j}(\theta) = \sum_{n=1}^{\infty}\left(A^i_{2,2}\right)^{n} w_{i,j}(\theta-n\omega)\left(A^j_{1,1}\right)^{-n} \tag{8.15}$$

depending on the relationship between eigenvalues.

By Proposition 4.15, the series in 8.14 converges in C^r_Γ because

$$\begin{aligned}\|\Delta v_{i,j}\|_{C^r_\Gamma} &\le \sum_{n=0}^{\infty}\|(A^i_{2,2})^n\|_\Gamma\|(A^j_{1,1})^{-n}\|_\Gamma\|w_{i,j}\|_{C^r_\Gamma}\\ &\le \Gamma(0)^{-2}\sum_{n=0}^{\infty}\left(\|A^i_{2,2}\|\|(A^j_{1,1})^{-1}\|\right)^n\|w_{i,j}\|_{C^r_\Gamma}.\end{aligned}$$

An analogous result holds for (8.15).

This proves $D_v\mathcal{T}(0,0)$ is invertible. By the Implicit Function Theorem, if B is small, from $\mathcal{T}(v,B)=0$ we get $v=v(B)$ such that

$$T_B = \begin{pmatrix}\mathrm{Id} & 0\\ v(B) & \mathrm{Id}\end{pmatrix}$$

transforms $D_xF(0,0)$ into block upper triangular form.

□

8.3 Transforming $D_x F(0,\theta)$ into block diagonal form

Actually we can transform the linear part of M to diagonal form by a C^r_Γ transform. In a completely analogous way to the linearisation in Chapter 8 we can find $w \in C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^2, \mathcal{E}^1))$ such that

$$\begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ M_{2,1}(\theta) & M_{2,2}(\theta) \end{pmatrix} \begin{pmatrix} w(\theta) \\ \mathrm{Id} \end{pmatrix} = \begin{pmatrix} w(\theta+\omega) \\ \mathrm{Id} \end{pmatrix} R^u_1(\theta) \tag{8.16}$$

for some $R^u_1(\theta) \in C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^2, \mathcal{E}^2))$ close to $A_{2,2}$.

Indeed $w(\theta)$ has to satisfy

$$M_{1,1}(\theta)w(\theta) + M_{1,2}(\theta) = w(\theta+\omega)\left[M_{2,1}(\theta)w(\theta) + M_{2,2}(\theta)\right],$$

which is equivalent to

$$w(\theta+\omega) = \left[M_{1,1}(\theta)w(\theta) + M_{1,2}(\theta)\right]\left[M_{2,2}(\theta) + M_{2,1}(\theta)w(\theta)\right]^{-1}.$$

That is, w has to be a fixed point of the operator $\mathcal{S}^u : C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^2, \mathcal{E}^2)) \to C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^2, \mathcal{E}^2))$ defined as

$$\mathcal{S}^u(w)(\theta) = \left[M_{1,1}(\theta-\omega)w(\theta-\omega) + M_{1,2}(\theta-\omega)\right]\left[M_{2,2}(\theta-\omega) + M_{2,1}(\theta-\omega)w(\theta-\omega)\right]^{-1}.$$

Note that $\mathcal{S}^u$ can be written as $\mathcal{S}^u = \mathcal{S}^u_0 + \mathcal{S}^u_1$ with

$$\begin{aligned} \mathcal{S}^u_0(w)(\theta) &= A_{1,1}w(\theta-\omega)A^{-1}_{2,2} \\ \mathcal{S}^u_1(w)(\theta) &= \mathcal{S}^u(w)(\theta) - S^u_0(w)(\theta). \end{aligned}$$

The hypothesis $\|A^{-1}_{2,2}\|\|A_{1,1}\| \le \rho < 1$ used in the previous section allows, in the same way, to prove the existence of a unique fixed point $w \in C^r_\Gamma$ of the operator $\mathcal{S}^u$. As a consequence $\|w\| = \mathcal{O}(\|M_1\|)$ and

$$R^u_1(\theta) = M_{2,1}(\theta)w(\theta) + M_{2,2}(\theta) = A_{2,2} + \mathcal{O}(\|M_1\|).$$

Moreover $R^u_1 \in C^r_\Gamma(\mathbb{T}^d, L(\mathcal{E}^2, \mathcal{E}^2))$.

Now consider the linear map (depending on θ)

$$C(\theta) = \begin{pmatrix} \mathrm{Id} & w(\theta) \\ v(\theta) & \mathrm{Id} \end{pmatrix}$$

where v has been obtained in Lemma 8.3 of Chapter 8. Since v and w are small, $C(\theta)$ is invertible. We claim that

$$C^{-1}(\theta+\omega)M(\theta)C(\theta) \tag{8.17}$$

is the block diagonal matrix

$$\begin{pmatrix} R_1(\theta) & 0 \\ 0 & R^u_1(\theta) \end{pmatrix}.$$

Indeed, the expression (8.17) is equivalent to

$$\begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ M_{2,1}(\theta) & M_{2,2}(\theta) \end{pmatrix} \begin{pmatrix} \mathrm{Id} & w(\theta) \\ v(\theta) & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & w(\theta+\omega) \\ v(\theta+\omega) & \mathrm{Id} \end{pmatrix} \begin{pmatrix} R_1(\theta) & 0 \\ 0 & R^u_1(\theta) \end{pmatrix}$$

which in turn is equivalent to both (8.3) and (8.16).

8.4 Scaling procedure

In this section we will use a "scaling trick" that moves the smallness requirements on the space to smallness requirements on a parameter, allowing us to work always in the unit ball $B(0,1)$ of the corresponding space.

Remember that after translating $W_0(\theta)$ to 0 in the previous section, we write $F(x,\theta) = D_xF(0,\theta)x + N(x,\theta)$, with $N(0,\theta) = 0$ and $DN(0,\theta) = 0$. Using the results of Section 8.3 we know that we can find a linear change of variables C^r-smooth (if the system is C^r with respect to θ, and C^r_Γ if the system is C^r_Γ) such that it transforms

$$D_xF(0,\theta) = \begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ M_{2,1}(\theta) & M_{2,2}(\theta) \end{pmatrix}$$

into

$$D_x\widetilde{F}(0,\theta) = \begin{pmatrix} \widetilde{M}_{1,1}(\theta) & 0 \\ 0 & \widetilde{M}_{2,2}(\theta) \end{pmatrix}.$$

For convenience of notation we will denote $\widetilde{M}_{1,1}$ and $\widetilde{M}_{2,2}$ by $M_{1,1}$ and $M_{2,2}$. We should observe that $M_{1,1}(\theta) = A_{1,1} + B_{1,1}(\theta)$, $M_{2,2} = A_{2,2} + B_{2,2}(\theta)$ and $\|B_{i,i}\|_{C^r_\Gamma} \leq \varepsilon$, $1 \leq i \leq 2$, by the smallness properties of $\|F_1\|$ (and hence of $\|M_1\|$, in $C^{1,0}$, $C^{1,0}_\Gamma$ or $C^{1,0}_{j,\Gamma}$ norms as needed).

We will write F according to the splitting in Section 8.3 as

$$F(x,y,\theta) = \begin{pmatrix} M_{1,1}(\theta) & 0 \\ 0 & M_{2,2}(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} N_1(x,y,\theta) \\ N_2(x,y,\theta) \end{pmatrix}.$$

We will denote by DN the derivative of N with respect to (x,y) and D_xN, D_yN the derivatives of N with respect to x and y respectively. Finally, $D_\theta N$ will denote the derivative of N with respect to θ.

We apply a re-scaling $T(x,y,\theta) = (\delta x, \delta y, \theta)$ to F to get $\overline{F}(x,y,\theta) = T^{-1} \circ F \circ T(x,y,\theta)$ and write $\bar{N}(x,y,\theta) = T^{-1} \circ N \circ T(x,y,\theta)$. Observe that this re-scaling makes $\bar{N}$ and $D\bar{N}$ as small as needed in a ball of radius 1:

$$\begin{aligned} \bar{N}(x,y,\theta) &= \frac{1}{\delta}N(\delta x, \delta y, \theta) = \int_0^1 DN(\mu\delta x, \mu\delta y, \theta)d\mu = o(\delta^0), \\ D\bar{N}(x,y,\theta) &= DN(\delta x, \delta y, \theta) = o(\delta^0). \end{aligned}$$

We will write N instead of $\bar{N}$ for convenience.

Chapter 9

Regularity of local strong stable invariant manifolds of the torus $W_0(\theta)$

This section is devoted to the study of the strong stable manifolds of the invariant torus W_0 determined in Chapter 7 for a system $F = F_0 + F_1$ as defined in Section 2.1, assuming $C^{\Sigma_{t,r}}$ regularity for F as defined in Chapter 5. We will use the same notation for the invariant spaces and projections as the one used in Chapter 8. More precisely, recall from Section 2.1 that we will assume we have a splitting $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$ induced by a splitting of the spectrum of $DF_0(0)$ of the form $\mathrm{Spec}(DF_0(0)) = \sigma_1 \cup \sigma_2$ with

$$\sigma_1 = \{\lambda \in \mathrm{Spec}(DF_0(0)) \,|\, |\lambda| < \alpha\},$$
$$\sigma_2 = \{\lambda \in \mathrm{Spec}(DF_0(0)) \,|\, |\lambda| > \alpha\}$$

for some $\alpha < 1$. We have that $DF_0(0)$ leaves $\mathcal{E}^1$ and $\mathcal{E}^2$ invariant. We will denote the projections $\Pi_1 DF_0(0)\mathrm{emb}_1 = A_{1,1}$, $\Pi_2 DF_0(0)\mathrm{emb}_2 = A_{2,2}$ as introduced in Section 2.1, where $\Pi_i : \ell^\infty(\mathbb{R}^n) \to \mathcal{E}^i$ and $\mathrm{Emb}_i : \mathcal{E}^i \to \ell^\infty(\mathbb{R}^n)$ are the corresponding projections and embeddings, respectively. Notice that, by the spectral conditions, $A_{2,2}$ is invertible. Recall also from Section 2.1 that additionally, they satisfy the following bounds for some adapted norm in $\ell^\infty(\mathbb{R}^n)$:

$$\|A_{1,1}\| < 1, \qquad \|A_{1,1}\|\|A_{2,2}^{-1}\| < 1,$$

a bound we will be using throughout this section.

A strong stable invariant manifold will be determined as an invariant graph under the action of the dynamical system. More precisely, we will look for a function $\varphi(x, \theta)$ such that the local strong invariant manifold of the invariant torus $W_0(\theta)$ found in Chapter 7 can be expressed locally as the graph $(x, \varphi(x, \theta), \theta)$[1].

The regularity result is split into 4 parts. In the first one, we will find the $C^{0,0}$ function parametrising the invariant manifold using a fixed point argument. Then, in the second part we will deal with determining the $C^{1,0}$ parametrisation. The third part is just a sketch of the $C^{1,1}$ case in preparation for the fourth part, which is an inductive proof of the existence of a $C^{\Sigma_{t,r}}$ parametrisation. In each part we will also find Γ and (j, Γ) parametrisations

[1] We write the graph in this form to keep the notation uniform, since this is how it will be used later

when the conditions on the system are adequate to determine them with the techniques we use.

9.1 Regularity in the $C^{1,0}$, $C^{1,0}_{\Gamma}$, $C^{1,0}_{j,\Gamma}$ cases

In this section we deal with the first regularity results, which are the basis for the inductive proof in the general case. This proof is very similar to the regularity proofs found in [CFdlL03a] and [CFdlL03b], but here we deal with a nonlinear operator acting on $C^{1,0}$ functions to find sharp regularity. In this way we obtain the sharp regularity of the manifold when F is just $C^{1,0}$, a case not considered in [CFdlL03a]. The main result is the following.

Theorem 9.1. *Given a dynamical system $F(x,\theta)$ as defined in Section 2.1 and a splitting of $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$ as in the introduction of this chapter, we can determine the unique local strong stable manifold of W_0 tangent to $\mathcal{E}^1$ at 0 as the graph of $\varphi : \mathcal{E}^1 \times \mathbb{T}^d \to \mathcal{E}^2$ in the following regularity cases:*

(i) If $F(x,\theta) \in C^{1,0}\big(\ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)\big)$ and $\|F_1\|_{C^{1,0}}$ is small enough then $\varphi \in C^{1,0}(B(0,1) \times \mathbb{T}^d, \mathcal{E}^2)$.

(ii) If $F(x,\theta) \in C^{1,0}_{\Gamma}\big(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)\big)$ and $\|F_1\|_{C^{1,0}_{\Gamma}}$ is small enough then $\varphi \in C^{1,0}_{\Gamma}(B(0,1)\times \mathbb{T}^d, \mathcal{E}^2)$.

(iii) If $F(x,\theta) \in C^{1,0}_{j,\Gamma}\big(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)\big)$ and $\|F_1\|_{C^{1,0}_{j,\Gamma}}$ is small enough then $\varphi \in C^{1,0}_{j,\Gamma}(B(0,1)\times \mathbb{T}^d, \mathcal{E}^2)$.

Remark 9.2. *The standard local stable manifold of the torus is obtained as a particular case of Theorem 9.1. Also strong unstable manifolds as well as unstable manifolds are obtained applying Theorem 9.1 to the map F^{-1}.*

We will determine first the existence of a $C^{0,0}$ parametrisation under the hypotheses of the theorem and then prove Part (i). We will then prove parts (ii) and (iii) by adding the required conditions and using results from Chapters 4 and 5.

9.2 Lipschitz regularity of the parametrisation of strong stable manifolds

Since we want to determine the strong stable invariant manifolds as graphs of functions $\varphi : \mathcal{E}^1 \times \mathbb{T}^d \to \mathcal{E}^2$, the invariance equation we need to solve has a different expression from the invariance equation we have used in the results of Chapter 7 and Chapter 8. We impose that the graph of φ is invariant, thus

$$F\big(x, \varphi(x,\theta), \theta\big) = \big(\bar{x}, \varphi(\bar{x}, \theta+\omega)\big). \tag{9.1}$$

We assume that we have done the linear change $S(\theta)$ introduced in Section 8.3 to F so that we can write $F(x,y,\theta)$ as

$$F(x,y,\theta) = \begin{pmatrix} M_{1,1}(\theta) & 0 \\ 0 & M_{2,2}(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} N_1(x,y,\theta) \\ N_2(x,y,\theta) \end{pmatrix}. \tag{9.2}$$

If we denote by F^1 and F^2 the projections of the image of F on $\mathcal{E}^1$ and $\mathcal{E}^2$ respectively we can rewrite Equation (9.1) in components as

$$\begin{pmatrix} F^1(x,\varphi(x,\theta),\theta) \\ F^2(x,\varphi(x,\theta),\theta) \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \varphi(\bar{x},\theta+\omega) \end{pmatrix}. \tag{9.3}$$

Substituting $\bar{x}$ by $F^1(x,\varphi(x,\theta),\theta)$ in the projection over $\mathcal{E}^2$ in Equation (9.3) we end up with the following equation:

$$F^2(x,\varphi(x,\theta),\theta) = \varphi(F^1(x,\varphi(x,\theta),\theta),\theta+\omega).$$

By using Expression (9.2) we can write this as

$$M_{2,2}(\theta)\varphi(x,\theta) + N_2(x,\varphi(x,\theta),\theta) = \varphi\big(M_{1,1}(\theta)x + N_1(x,\varphi(x,\theta),\theta),\theta+\omega\big). \tag{9.4}$$

Any solution of (9.4) is a fixed point of the following operator

$$\mathcal{T}(\varphi)(x,\theta) = M_{2,2}(\theta)^{-1}\Big(\varphi\left(\psi_\varphi(x,\theta),\theta+\omega\right) - N_2(x,\varphi(x,\theta),\theta)\Big), \tag{9.5}$$

where

$$\psi_\varphi(x,\theta) = M_{1,1}(\theta)x + N_1(x,\varphi(x,\theta),\theta). \tag{9.6}$$

Now we introduce the spaces we will use to prove the existence of a C^0 fixed point for the operator $\mathcal{T}$. Recall from Chapter 7 that given $f \in C^0(\ell^\infty(\mathbb{R}^n)\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ we have defined

$$\mathrm{Lip}_x(f) := \sup_{\theta\in\mathbb{T}^d} \mathrm{Lip}(f(\cdot,\theta)),$$

and observe that by the definition of $M_{1,1}$, $M_{2,2}$ and the smallness of the perturbating function F_1:

$$\begin{aligned} \|M_{1,1}\| &= \sup_{\theta\in\mathbb{T}^d} \|M_{1,1}(\theta)\| \le \|A_{1,1}\| + \mathcal{O}(\varepsilon), \\ \|M_{2,2}^{-1}\| &= \sup_{\theta\in\mathbb{T}^d} \|M_{2,2}(\theta)^{-1}\| \le \|A_{2,2}^{-1}\| + \mathcal{O}(\varepsilon). \end{aligned}$$

To prove the result for the strong stable manifold, we will use the following space. Consider U a bounded open set in $\mathcal{E}^1$ such that $0\in U$ and

$$\begin{aligned} \mathcal{Y} = \big\{\varphi \in C^{0,0}(U\times\mathbb{T}^d,\mathcal{E}^2) \,|\, & \varphi(0,\theta) = 0, \\ & \varphi \text{ Lipschitz with respect to the first variable}, \\ & \|\varphi\|_{\mathcal{Y}} < \infty,\ \mathrm{Lip}_x(\varphi) \le 1\big\} \end{aligned}$$

with the norm

$$\|\varphi\|_{\mathcal{Y}} = \sup_{\substack{x\in U\setminus\{0\} \\ \theta\in\mathbb{T}^d}} \frac{\|\varphi(x,\theta)\|}{\|x\|}.$$

Remark 9.3. *Observe that $\mathcal{Y}$ is a Banach space.*

Remark 9.4. *Observe that a necessary condition for*

$$\sup_{\substack{x\in U\setminus\{0\}\\ \theta\in\mathbb{T}^d}} \frac{\|\varphi(x,\theta)\|}{\|x\|} < \infty$$

is that

$$\varphi(0,\theta) = 0.$$

A property we will use several times when working with the $\|\cdot\|_{\mathcal{Y}}$ norm and compositions of functions φ, $g \in \mathcal{Y}$ (assuming the composition makes sense) is the following:

$$\|\varphi \circ g\|_{\mathcal{Y}} \leq \|\varphi\|_{\mathcal{Y}} \|g\|_{\mathcal{Y}}, \tag{9.7}$$

where we have made the abuse of notation

$$\varphi \circ g(x,\theta) = \varphi(g(x,\theta),\theta).$$

Indeed, first consider the case where x and θ are such that $g(x,\theta) \neq 0$:

$$\begin{aligned}
\frac{\|\varphi(g(x,\theta),\theta)\|}{\|x\|} &\leq \sup_{\substack{x\in U\setminus\{0\},\,\theta\in\mathbb{T}^d\\ g(x,\theta)\neq 0}} \frac{\|\varphi(g(x,\theta),\theta)\|}{\|x\|}\frac{\|g(x,\theta)\|}{\|g(x,\theta)\|} \\
&= \sup_{\substack{x\in U\setminus\{0\},\,\theta\in\mathbb{T}^d\\ g(x,\theta)\neq 0}} \frac{\|\varphi(g(x,\theta),\theta)\|}{\|g(x,\theta)\|}\frac{\|g(x,\theta)\|}{\|x\|} \\
&\leq \|\varphi\|_{\mathcal{Y}}\|g\|_{\mathcal{Y}}.
\end{aligned}$$

When $g(x,\theta) = 0$,

$$\frac{\|\varphi(g(x,\theta),\theta)\|}{\|x\|} = 0 \leq \|\varphi\|_{\mathcal{Y}}\|g\|_{\mathcal{Y}}$$

and taking supremum and using these two bounds (9.7) follows.

Having defined $\mathcal{Y}$ and having established Property (9.7) we can prove the existence of a local strong stable manifold when $F_1 \in \mathcal{Y}$ is small enough.

In the following we will use $U = B(0,1)$ in the definitions of these function spaces.

Lemma 9.5. *The operator $\mathcal{T}$ defined in (9.5) is well-defined from $\overline{B(0,1)} \subset \mathcal{Y}$ into itself and is a contraction in the $\mathcal{Y}$ norm if $\|F_1\|_{\mathcal{Y}}$ and the scaling parameter δ are small enough.*

Proof. We have to show that $\mathcal{T}$ sends $\mathcal{Y}$ into itself and bound the Lipschitz constant of $\mathcal{T}$ in the norm of $\mathcal{Y}$.

If $\varphi \in \overline{B(0,1)} \subset \mathcal{Y}$, we can use Property (9.7) to bound

$$\begin{aligned}
\sup_{\substack{x\in U\setminus\{0\}\\ \theta\in\mathbb{T}^d}} \frac{\|\mathcal{T}(\varphi)(x,\theta)\|}{\|x\|} \leq \sup_{\substack{x\in U\setminus\{0\}\\ \theta\in\mathbb{T}^d}} &\|M_{2,2}(\theta)^{-1}\|\big(\|M_{1,1}(\theta)\| + \mathrm{Lip}(N)\big) \\
&+ \|M_{2,2}(\theta)^{-1}\|\,\mathrm{Lip}(N) < 1,
\end{aligned}$$

where we have used that $\|\varphi\|_{\mathcal{Y}} \leq 1$ and the fact that given $f \in \mathcal{Y}$ and $g \in C^{0,0}(\mathcal{E}^1 \times \mathcal{E}^2 \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ such that $\|g\|_{C^{0,0}} < 1$, we have $f(0,\theta) = 0$ for all θ and

$$\|f \circ g\|_{C^{0,0}} = \sup_{\substack{x \in \ell^\infty(\mathbb{R}^n) \\ \theta \in \mathbb{T}^d}} \big\| f(g(x,\theta),\theta) - f(0,\theta) \|_{C^{0,0}} \leq \big(\operatorname{Lip}_x f\big) \|g\|_{C^{0,0}}$$

with $f = N$, $g = (\mathrm{Id}, \varphi)$. Observe that $\operatorname{Lip}_x N$ is as small as needed after the scaling procedure in Section 8.4, i.e. if δ is small enough. To bound the Lipschitz constant of $\mathcal{T}(\varphi)$, remember that if φ and g are Lipschitz functions with respect to x, $\operatorname{Lip}_x(\varphi \circ g) \leq \operatorname{Lip}_x(\varphi) \operatorname{Lip}_x(g)$ thus

$$\operatorname{Lip}_x(\mathcal{T}(\varphi)) \leq \|M_{2,2}^{-1}\| \Bigg(\operatorname{Lip}_x(\varphi) \big[\|M_{1,1}\| + \operatorname{Lip}_x(N) + \operatorname{Lip}_x(N) \operatorname{Lip}_x(\varphi) \big] \Bigg),$$

which is smaller than 1 if $\operatorname{Lip}_x(N)$ is small enough, since

$$\begin{aligned} \|M_{1,1}\| &= \sup_{\theta \in \mathbb{T}^d} \|M_{1,1}(\theta)\| \leq \|A_{1,1}\| + \mathcal{O}(\varepsilon), \\ \|M_{2,2}^{-1}\| &= \sup_{\theta \in \mathbb{T}^d} \|M_{2,2}(\theta)^{-1}\| \leq \|A_{2,2}^{-1}\| + \mathcal{O}(\varepsilon). \end{aligned}$$

Also observe that

$$\begin{aligned} \|\mathcal{T}(\varphi)\|_{C^{0,0}} &= \sup_{\substack{x \in U \\ \theta \in \mathbb{T}^d}} \left\| M_{2,2}(\theta)^{-1} \Big(\varphi\left(\psi_\varphi(x,\theta), \theta+\omega\right) - N_2(x, \varphi(x,\theta), \theta) \Big) \right\| \\ &= \sup_{\substack{x \in U \\ \theta \in \mathbb{T}^d}} \Bigg\| M_{2,2}(\theta)^{-1} \Big(\varphi\left(\psi_\varphi(x,\theta), \theta+\omega\right) - N_2(x, \varphi(x,\theta), \theta) \Big) \\ &\qquad - M_{2,2}(\theta)^{-1} \Big(\varphi(0, \theta+\omega) - N_2(0,0,\theta) \Big) \Bigg\| \\ &\leq \sup_{\substack{x \in U \\ \theta \in \mathbb{T}^d}} \Bigg[\|M_{2,2}(\theta)^{-1}\| \big(\|M_{1,1}(\theta)\| + \operatorname{Lip}_x(N) \cdot \operatorname{Lip}_x(\varphi) \big) \\ &\qquad + \|M_{2,2}(\theta)^{-1}\| \cdot \operatorname{Lip}_x(N) \cdot \|\varphi(x,\theta)\| \Bigg] < 1 \end{aligned}$$

by the conditions on the norms of the linear part and the fact that $\operatorname{Lip}_x \varphi < 1$ for $\varphi \in \mathcal{Y}$, ensuring we can compose $\mathcal{T}$ with itself.

Finally observe that

$$\begin{aligned} \sup_{\substack{x \in U\setminus\{0\} \\ \theta \in \mathbb{T}^d}} \frac{\|\mathcal{T}(\varphi)(x,\theta)}{\|x\|}\| &= \sup_{\substack{x \in U\setminus\{0\} \\ \theta \in \mathbb{T}^d}} \frac{\left\| M_{2,2}(\theta)^{-1} \Big(\varphi\left(\psi_\varphi(x,\theta), \theta+\omega\right) - N_2(x, \varphi(x,\theta), \theta) \Big) \right\|}{\|x\|} \\ &= \|M_{2,2}^{-1}\| \Bigg(\operatorname{Lip}_x(N_2) + \sup_{\substack{x \in U\setminus\{0\} \\ \theta \in \mathbb{T}^d}} \frac{\|\varphi\big(\psi_\varphi(x,\theta), \theta+\omega\big)\| \|\psi_\varphi(x,\theta)\|}{\|x\| \|\psi_\varphi(x,\theta)\|} \Bigg) \\ &\leq \|M_{2,2}^{-1}\| \Bigg(\operatorname{Lip}_x(N_2) + \|\varphi\|_{\mathcal{Y}} \big(\|M_{1,1}\| + \operatorname{Lip}_x(N_1) \big) \Bigg) < 1 \end{aligned}$$

if ε is small enough.

Now we will prove the contractivity of the operator. Let $\varphi_1, \varphi_2 \in \overline{B(0,1)} \subset \mathcal{Y}$ and write

$$\big(\mathcal{T}(\varphi_1) - \mathcal{T}(\varphi_2)\big)(x,\theta) = M_{2,2}(\theta)^{-1}\big(T^1(x,\theta) + T^2(x,\theta)\big)$$

with

$$\begin{aligned} T^1(x,\theta) &= \varphi_1(\psi_{\varphi_1}(x,\theta), \theta+\omega) - \varphi_2(\psi_{\varphi_2}(x,\theta), \theta+\omega), \\ T^2(x,\theta) &= N_2(x, \varphi_2(x,\theta), \theta) - N_2(x, \varphi_1(x,\theta), \theta). \end{aligned}$$

Let $T^1(x,\theta) = Z^1(x,\theta) + Z^2(x,\theta)$, with

$$\begin{aligned} Z^1(x,\theta) &= \varphi_1(\psi_{\varphi_1}(x,\theta), \theta+\omega) - \varphi_1(\psi_{\varphi_2}(x,\theta), \theta+\omega), \\ Z^2(x,\theta) &= \varphi_1(\psi_{\varphi_2}(x,\theta), \theta+\omega) - \varphi_2(\psi_{\varphi_2}(x,\theta), \theta+\omega). \end{aligned}$$

We can bound

$$\begin{aligned} \|Z^1\|_{\mathcal{Y}} = \sup_{\substack{x\in U\setminus\{0\} \\ \theta\in\mathbb{T}^d}} \frac{\|Z^1(x,\theta)\|}{\|x\|} &\le \sup_{\substack{x\in U\setminus\{0\} \\ \theta\in\mathbb{T}^d}} \frac{\|\varphi_1(\psi_{\varphi_1}(x,\theta),\theta) - \varphi_1(\psi_{\varphi_2}(x,\theta),\theta)\|}{\|x\|} \\ &\le \operatorname{Lip}_x(\varphi_1) \sup_{\substack{x\in U\setminus\{0\} \\ \theta\in\mathbb{T}^d}} \frac{\|\psi_{\varphi_1}(x,\theta) - \psi_{\varphi_2}(x,\theta)\|}{\|x\|} \\ &\le \operatorname{Lip}_x(\varphi_1)\operatorname{Lip}_x(N)\|\varphi_1 - \varphi_2\|_{\mathcal{Y}} \end{aligned}$$

and

$$\|Z^2\|_{\mathcal{Y}} = \sup_{\substack{x\in U\setminus\{0\} \\ \theta\in\mathbb{T}^d}} \frac{\|Z^2(x,\theta)\|}{\|x\|} \le \|\varphi_1 - \varphi_2\|_{\mathcal{Y}} \|\psi_{\varphi_2}\|_{\mathcal{Y}} \le \Big(\|M_{1,1}\| + \operatorname{Lip}_x(N)\Big)\|\varphi_1 - \varphi_2\|_{\mathcal{Y}}$$

by using Property (9.7) and the bound for $\operatorname{Lip}_x(N_2)$. We can bound $\sup \|T^2(x,\theta)\|/\|x\|$ by $\operatorname{Lip}_x(N)\|\varphi_1 - \varphi_2\|_{\mathcal{Y}}$, and by combining these bounds and assuming the perturbation size ε and the scaling parameter δ are small enough we get

$$\|\mathcal{T}(\varphi_1) - \mathcal{T}(\varphi_2)\|_{\mathcal{Y}} \le \rho \|\varphi_1 - \varphi_2\|_{\mathcal{Y}}$$

for some ρ such that $\|A_{1,1}\|\|A_{2,2}^{-1}\| < \rho < 1$.

□

An immediate consequence of the lemma is that the sequence

$$\{\varphi_n^0 := \mathcal{T}^n(0)\}_{n\in\mathbb{N}}$$

converges in $\mathcal{Y}$ to some function φ_∞^0.

We can summarise this claim in the following proposition.

Proposition 9.6. *Under the assumptions of Theorem 9.1, the sequence $\varphi_n^0 = \{\mathcal{T}^n(0)\}_{n\in\mathbb{N}}$ converges to the unique fixed point φ_∞^0 of the operator $\mathcal{T}$ in $\mathcal{Y}$ and this fixed point gives a parametrisation of the strong stable manifold of $W(\theta)$ as the graph $\big(x, \varphi(x,\theta), \theta\big)$.*

9.3 $C^{1,0}$ regularity of the parametrisation of strong stable manifolds

To prove $C^{1,0}$ regularity, we will assume that the operator $\mathcal{T}$ defined by Equation (9.5) acts over $C^{1,0}(B(0,1)\times\mathbb{T}^d,\mathcal{E}^2)$ where $B(0,1)$ is the unit ball in $\mathcal{E}^1$. With this assumption Proposition 9.6 still applies, thus $\mathcal{T}$ has a fixed point φ^0_∞ in $C^{0,0}$ as before and we only need to prove this fixed point is in $C^{1,0}$. We will prove this by determining its derivative with respect to x as a limit. This proof mimics the proofs in [CFdlL03a] and [CFdlL03b], but extends the result to the $C^{1,0}$ case, a case which was not needed for the results therein. Observe that the regularity proofs in [Irw70] and [HP70] follow a different construction to prove regularity of the fixed point. In the first, the main tool is the implicit function theorem applied to a space of sequences, in the second the graph transform and the fixed point theorem (applied to the graph transform operator) are used.

Observe that for an arbitrary $\varphi\in C^{1,0}(B(0,1)\times\mathbb{T}^d,\mathcal{E}^2)$, $\mathcal{T}(\varphi)\in C^{1,0}(B(0,1)\times\mathbb{T}^d,\mathcal{E}^2)$ and the derivative $D_x\mathcal{T}$ can be computed and has the form

$$\begin{aligned}D_x\mathcal{T}(\varphi)(x,\theta) =& M_{2,2}(\theta)^{-1}\Big(D_x\varphi(\psi_\varphi(x,\theta),\theta+\omega)D_x\psi_\varphi(x,\theta)\\ &- D_xN_2(x,\varphi(x,\theta),\theta)\\ &- D_yN_2(x,\varphi(x,\theta),\theta)D_x\varphi(x,\theta)\Big),\end{aligned} \tag{9.8}$$

where ψ_φ is defined as in (9.6), that is

$$\psi_\varphi(x,\theta)=M_{1,1}(\theta)x+N_1(x,\varphi(x,\theta),\theta).$$

Moreover, if $\|D_x\varphi\|_{C^{0,0}}\le 1$ (consequence of $\|\varphi\|_{C^{1,0}}\le 1$) then the right-hand side of (9.8) has $C^{0,0}$-norm smaller than 1 if $\|N_1\|_{C^{1,0}}$, $\|N_2\|_{C^{1,0}}$ are small enough by the conditions on the norms of $A_{1,1}$ and $A_{2,2}$, since $D_x\psi_\varphi$ is close to $A_{1,1}$. Hence $\|\mathcal{T}(\varphi)\|_{C^{1,0}}\le 1$.

Our goal is now to prove that the limit function φ^0_∞ introduced at the end of the previous section is in $C^{1,0}$. To do so, we will study the convergence of the sequence

$$\{\varphi^1_n(x,\theta):=D_x\varphi^0_n(x,\theta)\}_{n\in\mathbb{N}}.$$

Observe that by differentiating $\varphi^0_{n+1}=\mathcal{T}(\varphi^0_n)(x,\theta)$ we can write the sequence as

$$\varphi^1_{n+1}(x,\theta)=\mathcal{A}_n\varphi^1_n(x,\theta)+\mathcal{C}_n(\varphi^1_n)(x,\theta)+\mathcal{B}_n(x,\theta), \tag{9.9}$$

where

$$\mathcal{A}_n,\mathcal{C}_n: C^{0,0}(\overline{B(0,1)}\times\mathbb{T}^d\subset\mathcal{E}^1\times\mathbb{T}^d,L(\mathcal{E}^1,\mathcal{E}^2))\to C^{0,0}(\overline{B(0,1)}\times\mathbb{T}^d\subset\mathcal{E}^1\times\mathbb{T}^d,L(\mathcal{E}^1,\mathcal{E}^2))$$

and

$$\mathcal{B}_n\in C^{0,0}(\overline{B(0,1)}\times\mathbb{T}^d\subset\mathcal{E}^1\times\mathbb{T}^d,L(\mathcal{E}^1,\mathcal{E}^2))$$

are defined as

$$\begin{aligned}\mathcal{A}_nJ(x,\theta) &= M_{2,2}(\theta)^{-1}J(\psi_n(x,\theta),\theta+\omega)\big(M_{1,1}(\theta)+D_xN_1(x,\varphi^0_n(x,\theta),\theta)\big)\\ &\quad - M_{2,2}(\theta)^{-1}D_yN_2(x,\varphi^0_n(x,\theta),\theta)J(x,\theta),\\ \mathcal{C}_n(J)(x,\theta) &= M_{2,2}(\theta)^{-1}J(\psi_n(x,\theta),\theta+\omega)D_yN_1(x,\varphi^0_n(x,\theta),\theta)J(x,\theta),\\ \mathcal{B}_n(x,\theta) &= -M_{2,2}(\theta)^{-1}D_xN_2(x,\varphi^0_n(x,\theta),\theta),\end{aligned} \tag{9.10}$$

where we have used the notation $\psi_n(x,\theta) = \psi_{\varphi_n^0}(x,\theta)$. We also introduce the objects $\mathcal{A}_\infty$, $\mathcal{C}_\infty$, $\mathcal{B}_\infty$ and ψ_∞ with the same expressions as $\mathcal{A}_n$, $\mathcal{C}_n$, $\mathcal{B}_n$, ψ_n but changing φ_n^0 by the limit function φ_∞^0 whose existence is proved in Proposition 9.6.

Remark 9.7. *Observe that the operator $\mathcal{C}_n$ is nonlinear. This is a difference with the regularity proofs found in [CFdlL03a] and [CFdlL03b], since the operator $\mathcal{T}$ is linear when the regularity required is higher than 1, as happens in those papers.*

We want to show that $\{\varphi_n^1\}_{n\in\mathbb{N}}$ has as limit a continuous function φ_∞^1 and that the limit is uniform in compact sets. First we will prove that this limit exists and it is continuous. Then we will show that the convergence is uniform in compact sets. To end the proof of Part (i) in Theorem 9.1 we will check that the limit function coincides with the derivative of φ_∞^0, thus φ_∞^0 is a $C^{1,0}$ function, proving the differentiability of the function parametrising the strong stable manifold of $W_0(\theta)$.

The following lemma shows that given $J \in C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)\big)$, the sequences $\{\mathcal{A}_n J\}_{n\in\mathbb{N}}$, $\{\mathcal{C}_n(J)\}_{n\in\mathbb{N}}$ and $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ converge in compact sets to $\mathcal{A}_\infty J$, $\mathcal{C}_\infty(J)$ and $\mathcal{B}_\infty$ respectively.

Lemma 9.8. *Under the previous definitions and assumptions,*

(i) The operator $\mathcal{A}_n$ is well defined from $\overline{B(0,1)} \subset C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)\big)$ to itself and is a contraction in $\|\cdot\|_{C^{0,0}}$ for all $n \in \mathbb{N} \cup \{\infty\}$. Furthermore, there exists $\rho_{\mathcal{A}} < 1$ such that $\operatorname{Lip} \mathcal{A}_n < \rho_{\mathcal{A}}$ for all $n \in \mathbb{N} \cup \{\infty\}$.

(ii) Given $J \in C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)\big)$, the sequence $\{\mathcal{A}_n J\}_{n\in\mathbb{N}}$ converges uniformly over compact sets to $\mathcal{A}_\infty J$.

(iii) The operator $\mathcal{C}_n$ is well defined from $\overline{B(0,1)} \subset C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)\big)$ to itself and is a contraction in $\|\cdot\|_{C^{0,0}}$ for all $n \in \mathbb{N} \cup \{\infty\}$. Furthermore, $\sup_n \operatorname{Lip} \mathcal{C}_n \leq \rho_{\mathcal{C}}$ with $\rho_{\mathcal{C}}$ as small as needed by taking the scaling parameter sufficiently small.

(iv) Given $J \in C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)\big)$, the sequence $\{\mathcal{C}_n(J)\}_{n\in\mathbb{N}}$ converges uniformly over compact sets to $\mathcal{C}_\infty(J)$.

(v) The sequence $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ converges uniformly over compact sets to $\mathcal{B}_\infty$ and $\|\mathcal{B}_n\|_{C^{0,0}} \leq \rho_{\mathcal{B}}$ for all $n \in \mathbb{N} \cup \{\infty\}$, with $\rho_{\mathcal{B}}$ as small as needed by taking the scaling parameter sufficiently small.

Before proving this lemma, we need the following auxiliary result.

Lemma 9.9. *Let $\Omega \subset \mathcal{E}^1$ be an open set and $G \subset \Omega$ a compact set. Let $\{\xi_n\}_{n\in\mathbb{N}}$, be a sequence of continuous functions $\xi_n : \Omega \times \mathbb{T}^d \to \mathcal{E}^2$, $n \in \mathbb{N}$, converging uniformly to a continuous function $\xi_\infty(x,\theta)$ in $\Omega \times \mathbb{T}^d$. Then*

$$\mathcal{G} := \left\{ \left(\bigcup_{n=0}^{\infty} \xi_n(x,\theta) \right) \cup \{\xi_\infty(x,\theta)\},\ x \in G,\ \theta \in \mathbb{T}^d \right\}$$

is a compact set. Moreover, let Λ be an open set such that $\mathcal{G} \subset \Lambda$ and $f \in C^{0,0}(G \times \Lambda \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ then

$$\mathcal{G}_f := \left\{ \left(\bigcup_{n=0}^{\infty} f(x, \xi_n(x,\theta), \theta) \right) \cup \{f(x, \xi_\infty(x,\theta), \theta)\},\ x \in G,\ \theta \in \mathbb{T}^d \right\}$$

is a compact set.

Proof. We will only prove the result for the set $\mathcal{G}$, the proof for the set $\mathcal{G}_f$ being analogous. Consider an open covering of $\mathcal{G}$ and denote it by $\mathcal{U}$. Let

$$\mathcal{U}_n = \{U \in \mathcal{U} \,|\, \xi_n(x,\theta) \in U \text{ for some } x \in G,\, \theta \in \mathbb{T}^d\},$$
$$\mathcal{U}_\infty = \{U \in \mathcal{U} \,|\, \xi_\infty(x,\theta) \in U \text{ for some } x \in G,\, \theta \in \mathbb{T}^d\}.$$

Observe that $\bigcup_n \mathcal{U}_n \cup \mathcal{U}_\infty$ is a sub-covering of $\mathcal{U}$ which also covers $\mathcal{G}$. Since for a fixed $n \in \mathbb{N} \cup \{\infty\}$

$$\{\xi_n(x,\theta) \,|\, x \in G,\, \theta \in \mathbb{T}^d\}$$

is a compact set and $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a covering of it, for each $n \in \mathbb{N} \cup \{\infty\}$ there exists a finite sub-covering of $\mathcal{U}_n$, which we will denote by $\mathcal{V}_n$ that covers $\{\xi_n(x,\theta) \,|\, x \in G,\, \theta \in \mathbb{T}^d\}$. The sub-covering $\mathcal{V}_\infty$ has a Lebesgue number $\delta > 0$ and by the convergence of $\{\xi_n\}_{n\in\mathbb{N}}$ to ξ_∞ we can find a $N(\delta) < \infty$ such that for all $n > N(\delta)$,

$$\|\xi_n(x,\theta) - \xi_\infty(x,\theta)\| < \delta, \qquad \forall x \in G,\, \theta \in \mathbb{T}^d.$$

Therefore $\mathcal{V}_\infty$ covers $\{\xi_n(x,\theta),\, x \in G,\, \theta \in \mathbb{T}^d\}$ for all $n > N(\delta)$. Thus

$$\left(\bigcup_{n=0}^{N(\delta)} \mathcal{V}_n\right) \cup \mathcal{V}_\infty$$

is a finite sub-covering of $\mathcal{U}$ covering $\mathcal{G}$, which is then a compact set.

□

After having proved this auxiliary lemma we come back to the proof of Lemma 9.8.

Proof of Lemma 9.8: Recall that $\varphi_n^0 \in \overline{B(0,1)} \subset C^{0,0}(\mathcal{E}^1 \times \mathbb{T}^d, \mathcal{E}^2)$ and $\psi_n(x,\theta) = M_{1,1}(\theta)x + N_1(x, \varphi_n^0(x,\theta), \theta)$.

Part (i) *($\mathcal{A}_n$ is well defined and a contraction):* Given $J \in \overline{B(0,1)} \subset C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)\big)$, $\mathcal{A}_n J$ is clearly continuous. We only need to bound the norm of the image to check that $\mathcal{A}_n J \in \overline{B(0,1)}$:

$$\begin{aligned}
\|\mathcal{A}_n J\|_{C^{0,0}} &\leq \|M_{2,2}^{-1}\|_{C^0} \|J\|_{C^{0,0}} \big(\|M_{1,1}\|_{C^0} + \|D_x N_1\|_{C^{0,0}}\big) \\
&\quad + \|M_{2,2}^{-1}\|_{C^0} \|D_y N_2\|_{C^{0,0}} \|J\|_{C^{0,0}} \\
&\leq \|M_{2,2}^{-1}\|_{C^0} \|J\|_{C^{0,0}} \Bigg(\|M_{1,1}\|_{C^0} + o(\delta^0)\Bigg) \\
&\quad + o(\delta^0) \|M_{2,2}^{-1}\|_{C^0} \|J\|_{C^{0,0}} \\
&\leq \widetilde{\rho} \|J\|_{C^{0,0}},
\end{aligned}$$

with $\widetilde{\rho} = \|M_{2,2}^{-1}\|_{C^0}\big(\|M_{1,1}\|_{C^0} + o(\delta^0)\big)$ which is smaller than some $\rho < 1$ if ε and δ are small enough. Observe that this bound is independent of $n \in \mathbb{N} \cup \{\infty\}$.

To prove the contraction property, let $J_1,\, J_2 \in \overline{B(0,1)} \subset C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)\big)$. We can write

$$\big(\mathcal{A}_n J_1 - \mathcal{A}_n J_2\big)(x,\theta) = T_n^1(x,\theta) + T_n^2(x,\theta) + T_n^3(x,\theta),$$

where

$$\begin{aligned} T_n^1(x,\theta) &= M_{2,2}(\theta)^{-1} J_1(\psi_n(x,\theta), \theta+\omega) M_{1,1}(\theta) \\ &\quad - M_{2,2}(\theta)^{-1} J_2(\psi_n(x,\theta), \theta+\omega) M_{1,1}(\theta), \qquad (9.11)\\ T_n^2(x,\theta) &= M_{2,2}(\theta)^{-1} J_1(\psi_n(x,\theta), \theta+\omega) D_x N_1(x, \varphi_n^0(x,\theta), \theta) \\ &\quad - M_{2,2}(\theta)^{-1} J_2(\psi_n(x,\theta), \theta+\omega) D_x N_1(x, \varphi_n^0(x,\theta), \theta), \qquad (9.12)\\ T_n^3(x,\theta) &= -M_{2,2}(\theta)^{-1} D_y N_2(x, \varphi_n^0(x,\theta), \theta) J_1(x,\theta) \\ &\quad + M_{2,2}(\theta)^{-1} D_y N_2(x, \varphi_n^0(x,\theta), \theta) J_2(x,\theta). \qquad (9.13) \end{aligned}$$

We now bound these three terms separately. We can bound T_n^1 in (9.11) by

$$\|M_{2,2}^{-1}\|_{C^0} \|M_{1,1}\|_{C^0} \|J_1 - J_2\|_{C^{0,0}} < \widetilde{\rho} \|J_1 - J_2\|_{C^{0,0}}$$

with $\widetilde{\rho} = \|M_{2,2}^{-1}\|_{C^0}\|M_{1,1}\|_{C^0} < 1$ if the perturbation of the original system is small enough. Similarly we can bound T_n^2 in (9.12) by $\|M_{2,2}^{-1}\|_{C^0}\|J_1 - J_2\|_{C^{0,0}}\|DN\|_{C^{0,0}}$ and a similar bound follows for T_n^3 in (9.13). All these bounds together result in

$$\big\|\big(\mathcal{A}_n J_1 - \mathcal{A}_n J_2\big)\big\|_{C^{0,0}} \le \mu \|J_1 - J_2\|_{C^{0,0}}$$

with

$$\mu = \|M_{2,2}^{-1}\|_{C^0}\|M_{1,1}\|_{C^0} + 2\|M_{2,2}^{-1}\|_{C^0}\|DN\|_{C^{0,0}}$$

which is smaller than some $\rho < 1$ if the scaling parameter δ and the size of the perturbation term ε are small enough.

Part (ii) *(convergence of* $\{\mathcal{A}_n(J)\}_{n\in\mathbb{N}}$*):* To prove the convergence of the sequence $\{\mathcal{A}_n J\}_{n\in\mathbb{N}}$, observe that we can write

$$\Big[\mathcal{A}_n - \mathcal{A}_\infty\Big](J)(x,\theta) = T_n^1(x,\theta) + T_n^2(x,\theta) + T_n^3(x,\theta),$$

where

$$\begin{aligned} T_n^1(x,\theta) &= M_{2,2}(\theta)^{-1} J(\psi_n(x,\theta), \theta+\omega) M_{1,1}(\theta) \\ &\quad - M_{2,2}(\theta)^{-1} J(\psi_\infty(x,\theta), \theta+\omega) M_{1,1}(\theta),\\ T_n^2(x,\theta) &= M_{2,2}(\theta)^{-1} J(\psi_n(x,\theta), \theta+\omega) D_x N_1(x, \varphi_n^0(x,\theta), \theta) \\ &\quad - M_{2,2}(\theta)^{-1} J(\psi_\infty(x,\theta), \theta+\omega) D_x N_1(x, \varphi_\infty^0(x,\theta), \theta),\\ T_n^3(x,\theta) &= -M_{2,2}(\theta)^{-1} D_y N_2(x, \varphi_n^0(x,\theta), \theta) J(x,\theta) \\ &\quad + M_{2,2}(\theta)^{-1} D_y N_2(x, \varphi_\infty^0(x,\theta), \theta) J(x,\theta). \end{aligned}$$

Given a compact set $G \in \mathcal{E}^1$, by Lemma 9.9 applied to the sequence $\{\psi_n\}_{n\in\mathbb{N}}$, the sets

$$\mathcal{G}_1 = \left\{ \left(\bigcup_{n=0}^{\infty} \psi_n(x,\theta) \right) \cup \{\psi_\infty(x,\theta)\},\ x \in G,\ \theta \in \mathbb{T}^d \right\},$$
$$\mathcal{G}_2 = \left\{ \left(\bigcup_{n=0}^{\infty} \varphi_n(x,\theta) \right) \cup \{\varphi_\infty(x,\theta)\},\ x \in G,\ \theta \in \mathbb{T}^d \right\}, \tag{9.14}$$

$\mathcal{G}_1 \times \mathbb{T}^d$, $\mathcal{G}_2 \times \mathbb{T}^d$ and $G \times \mathcal{G}_2 \times \mathbb{T}^d$ are compact sets. Since J and $D_y N_2$ are continuous, they are uniformly continuous on $\mathcal{G}_1 \times \mathbb{T}^d$ and $G \times \mathcal{G}_2 \times \mathbb{T}^d$ respectively. Thus for each $\varepsilon > 0$ there is some $\delta > 0$ such that if $\|\psi_n - \psi_\infty\|_{C^{0,0}} < \delta$ then $\|T_n^1\| < \varepsilon/3$, where ε and δ are the constants used in the uniform continuity bounds for J in $\mathcal{G}_1 \times \mathbb{T}^d$. The same argument applies to T_n^3, with the compact set $G \times \mathcal{G}_2 \times \mathbb{T}^d$. Finally, to bound $\|T_n^2\|$ we add and subtract the term $M_{2,2}(\theta)^{-1} J(\psi_n(x,\theta), \theta+\omega) D_x N_1(x, \varphi_\infty^0(x,\theta), \theta)$ and use the same argument as above to get analogous bounds. To prove the convergence of $\left\{ \left[\mathcal{A}_n - \mathcal{A}_\infty\right](J) \right\}_{n\in\mathbb{N}}$ to 0 we just have to take n_0 such that if $n > n_0$ then $\|\psi_n - \psi_\infty\|_{C^{0,0}} < \delta$ and $\|\varphi_n^0 - \varphi_\infty^0\|_{C^{0,0}} < \delta$, where δ is related to the definition of uniform continuity of the above mentioned functions.

Part (iii) *($\mathcal{C}_n$ is well defined and a contraction):* It is straightforward to see that $\mathcal{C}_n(J)$, $n \in \mathbb{N} \cup \{\infty\}$, is continuous. To prove the contraction property, let $J_1, J_2 \in \overline{B(0,1)} \subset C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)\big)$. We can bound

$$\|\big(\mathcal{C}_n(J_1) - \mathcal{C}_n(J_2)\big)\|_{C^{0,0}} \leq 2\mathcal{O}(\|D_y N_1\|_{C^{0,0}}) \|J_1 - J_2\|_{C^{0,0}}$$

by adding and subtracting

$$M_{2,2}^{-1}(\theta) J_1(\psi_n(x,\theta), \theta+\omega) D_y N_1(x, \varphi_n^0(x,\theta), \theta) J_2(x,\theta)$$

and using the fact that $\|J_1\|_{C^{0,0}} < 1$, $\|J_2\|_{C^{0,0}} < 1$.

Part (iv) *(convergence of $\{\mathcal{C}_n(J)\}_{n\in\mathbb{N}}$):* We can write

$$\mathcal{C}_n(J) - \mathcal{C}_\infty(J) = T_n^1(x,\theta) + T_n^2(x,\theta),$$

where

$$\begin{aligned} T_n^1(x,\theta) = & M_{2,2}^{-1}(\theta) J(\psi_n(x,\theta), \theta+\omega) D_y N_1(x, \varphi_n^0(x,\theta), \theta) J(x,\theta) \\ & - M_{2,2}^{-1}(\theta) J(\psi_\infty(x,\theta), \theta+\omega) D_y N_1(x, \varphi_n^0(x,\theta), \theta) J(x,\theta) \\ T_n^2(x,\theta) = & M_{2,2}^{-1}(\theta) J(\psi_\infty(x,\theta), \theta+\omega) D_y N_1(x, \varphi_n^0(x,\theta), \theta) J(x,\theta) \\ & - M_{2,2}^{-1}(\theta) J(\psi_\infty(x,\theta), \theta+\omega) D_y N_1(x, \varphi_\infty^0(x,\theta), \theta) J(x,\theta). \end{aligned}$$

Now given a compact set $G \subset \mathcal{E}^1$ we can use the same uniform continuity argument as in the proof of Part (ii) to prove the convergence on compact sets of $\{\mathcal{C}_n(J)\}_{n\in\mathbb{N}}$ to $\mathcal{C}_\infty(J)$.

Part (v) *(convergence of* $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$*):* Given a compact set $G \subset \mathcal{E}^1$, we can use the same argument as before to show the uniform convergence of $\{\mathcal{B}_n - \mathcal{B}_\infty\}_{n\in\mathbb{N}}$ to 0 in $G \times \mathbb{T}^d$. As before, the key ingredient is the uniform convergence of $D_x N_2(x, \varphi_n^0(x,\theta), \theta)$ in the same set as in the proof of Part (ii). The bound $\|\mathcal{B}_n\| < \rho_\mathcal{B} < 1$ is a consequence of the smallness of $\|DN\|_{C^{0,0}}$ and the smallness of the scaling parameter. There is a common bound for all terms, independent of n, as small as needed.

□

Consider the operator

$$\mathcal{T}^1_\infty : C^{0,0}\big(B(0,1) \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)\big) \to C^{0,0}\big(B(0,1) \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^2)\big)$$

defined as

$$\mathcal{T}^1_\infty(J) := \mathcal{A}_\infty J + \mathcal{C}_\infty(J) + \mathcal{B}_\infty.$$

According to the results in Lemma 9.8 this operator is well defined and is a contraction in $\overline{B(0,1)}$ if the perturbation is small enough. By Lemma 9.8 we can have $\rho_\mathcal{A} + \rho_\mathcal{B} + \rho_\mathcal{C} < 1$, then $\mathcal{T}^1_\infty$ sends $\overline{B(0,1)}$ into itself. Thus it has a fixed point $\widetilde{\varphi}^1_\infty \in \overline{B(0,1)}$. Obviously, the fixed point condition holds

$$\widetilde{\varphi}^1_\infty = \mathcal{A}_\infty \widetilde{\varphi}^1_\infty + \mathcal{C}_\infty(\widetilde{\varphi}^1_\infty) + \mathcal{B}_\infty. \tag{9.15}$$

We will prove that the iteration defined as

$$\varphi^1_{n+1} = \mathcal{A}_n \varphi^1_n + \mathcal{C}_n(\varphi^1_n) + \mathcal{B}_n$$

converges to this fixed point uniformly on compact sets. Note that, again by Lemma 9.8, if $\varphi^1_n \in \overline{B(0,1)}$ then $\varphi^1_{n+1} \in \overline{B(0,1)}$.

Let $G \subset \mathcal{E}^1$ be a compact set. We can write

$$\varphi^1_{n+1} - \widetilde{\varphi}^1_\infty = Z^1_n + Z^2_n + Z^3_n,$$

where

$$\begin{aligned} Z^1_n &:= \mathcal{A}_n \varphi^1_n - \mathcal{A}_n \widetilde{\varphi}^1_\infty + \mathcal{A}_n \widetilde{\varphi}^1_\infty - \mathcal{A}_\infty \widetilde{\varphi}^1_\infty, \\ Z^2_n &:= \mathcal{C}_n(\varphi^1_n) - \mathcal{C}_n(\widetilde{\varphi}^1_\infty) + \mathcal{C}_n(\widetilde{\varphi}^1_\infty) - \mathcal{C}_\infty(\widetilde{\varphi}^1_\infty), \\ Z^3_n &:= \mathcal{B}_n - \mathcal{B}_\infty, \end{aligned}$$

where we have used (9.15) and added and subtracted suitable terms to make bounding easier. We can now bound each term separately.

The term Z^1_n can be bounded by $\alpha_n + \|\mathcal{A}_n\|\|\varphi^1_n - \widetilde{\varphi}^1_\infty\|$, where $\alpha_n = \|(\mathcal{A}_n - \mathcal{A}_\infty)\widetilde{\varphi}^1_\infty\|$ on $G \times \mathbb{T}^d$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ converges to 0 by Part (ii) of Lemma 9.8. Similarly for Z^2_n we get the bound $\gamma_n + \operatorname{Lip}\mathcal{C}_n\|\varphi^1_n - \widetilde{\varphi}^1_\infty\|$ where $\gamma_n = \|(\mathcal{C}_n - \mathcal{C}_\infty)(\widetilde{\varphi}^1_\infty)\|$ on $G \times \mathbb{T}^d$ and $\{\gamma_n\}_{n\in\mathbb{N}}$ converges to 0 by Part (iv) of Lemma 9.8. Finally, $\|Z^3_n\| \le \beta_n$ where $\beta_n = \|\mathcal{B}_n - \mathcal{B}_\infty\|$ on $G \times \mathbb{T}^d$ and $\{\beta_n\}_{n\in\mathbb{N}}$ converges to 0 by Part (v) of the same lemma.

With these bounds we can write

$$\|\varphi^1_{n+1} - \widetilde{\varphi}^1_\infty\| \le \rho\|\varphi^1_n - \widetilde{\varphi}^1_\infty\| + \nu_n \le \sum_{i=0}^{n} \rho^{n-i}\nu_i,$$

where $\rho = \sup_{n\in\mathbb{N}}(\|\mathcal{A}_n\| + \operatorname{Lip}\mathcal{C}_n) < 1$ in $G \times \mathbb{T}^d$ and $\nu_i = \alpha_i + \beta_i + \gamma_i$. This sequence of bounds converges to 0 as the following lemma shows.

Lemma 9.10. *Given $\lambda \in \mathbb{R}$ such that $|\lambda| < 1$ and a sequence $\{\nu_n\}_{n\in\mathbb{N}}$ of elements in a Banach space E converging to 0, the sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ defined by*

$$\sigma_n := \sum_{i=0}^{n} \lambda^{n-i}\nu_i,$$

converges to 0 in E (we can use $\lambda \in \mathbb{C}$ if E is a complex Banach space.)

Proof. Let $A := \max_i \|\nu_i\|$. Given any $\varepsilon > 0$ let $N \in \mathbb{N}$ be such that $\|\nu_n\| < \varepsilon(1-|\lambda|)/2$ for $n > N$. For $n > N$, we can write

$$\|\sigma_n\| \le \left\| \sum_{i=0}^{N} \lambda^{n-i}\nu_i \right\| + \left\| \sum_{i=N+1}^{n} \lambda^{n-i}\nu_i \right\|.$$

The first term can be bounded by

$$A\frac{|\lambda|^{n-N} - |\lambda|^{n+1}}{1-|\lambda|} < A\frac{|\lambda|^{n-N}}{1-|\lambda|},$$

which is smaller than $\varepsilon/2$ if $n > K$ for some $K > N$ depending on N, A, ε and λ since $\lambda < 1$. The second term can be bounded by $\frac{1}{1-|\lambda|}\frac{\varepsilon(1-|\lambda|)}{2}$ and the result follows.

□

Let $\varphi^1_\infty = \lim_{n\to\infty} \varphi^1_n$. We have just proved that $\varphi^1_\infty = \widetilde{\varphi}^1_\infty$. Therefore φ^1_∞ is continuous. To finish the proof, we need to check that $\varphi^1_\infty(x,\theta)$ is the derivative with respect to x of $\varphi^0_\infty(x,\theta)$.

Since by definition $\varphi^1_n = D_x\varphi^0_n$, for all $n \in \mathbb{N}$ we can write the following

$$\varphi^0_n(x_2,\theta) - \varphi^0_n(x_1,\theta) = \int_0^1 \varphi^1_n(x_1 + \mu(x_2 - x_1),\theta)(x_2 - x_1)d\mu.$$

As we have uniform convergence over compact sets of $\{\varphi^1_n(x,\theta)\}_{n\in\mathbb{N}}$ to $\varphi^1_\infty(x,\theta)$ and $\{\varphi^0_n(x,\theta)\}_{n\in\mathbb{N}}$ to $\varphi^0_\infty(x,\theta)$ respectively, and the integral is over a compact set (φ^1_n is evaluated on the segment which joins x_1 with x_2,) we can take limit $n \to \infty$ and exchange the integral and the limit in the previous expression to get

$$\begin{aligned}\varphi^0_\infty(x_2,\theta) - \varphi^0_\infty(x_1,\theta) &= \int_0^1 \varphi^1_\infty(x_1 + \mu(x_2 - x_1),\theta)(x_2 - x_1)d\mu,\\ &= \varphi^1_\infty(x_1)(x_2 - x_1)\\ &\quad + \int_0^1 \left[\varphi^1_\infty(x_1 + \mu(x_2 - x_1),\theta) - \varphi^1_\infty(x_1,\theta)\right](x_2 - x_1)d\mu\end{aligned}$$

which implies that $D_x\varphi^0_\infty(x_1,\theta) = \varphi^1_\infty(x_1,\theta)$, since by the continuity of φ^1_∞, the integral can be made smaller than $\varepsilon\|x_2 - x_1\|$ if $\|x_2 - x_1\|$ is small enough.

9.4 $C^{1,0}_\Gamma$ regularity of the parametrisation of strong stable manifolds

To prove Part (ii) of Theorem 9.1 we cannot use the same proof as in the $C^{0,0}$ case, since although for some block diagonal (uncoupled) matrix A_0, $\|A_0\| < 1$, $\|A_0\|_{L_\Gamma} = \|A_0\|\Gamma(0)^{-1}$ which is then not necessarily smaller than 1. To overcome this problem, if $F(x,\theta) = A(\theta)x + N(x,\theta)$ with $A(\theta)$ close to A_0 we will consider $\widetilde{F}(x,\theta) = F^p(x,\theta)$ with $p \in \mathbb{N}$ which we can write as $A(\theta)^p x + \widetilde{N}(x,\theta)$.

The following lemma establishes the contraction property of a high enough iterate of a perturbed coupled linear map.

Lemma 9.11. *Given $A \in C^r_\Gamma(\mathbb{T}^d, L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$ such that*

$$A(\theta) = A_0 + \widetilde{A}(\theta)$$

with $A_0 \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, uncoupled and $\|\widetilde{A}\|_{C^r_\Gamma}$ small, the following bound holds

$$\|A^k\|_{C^r_\Gamma} \leq k^r \Gamma(0)^{-1} \left(\|A_0\| + \Gamma(0)^{-1} \|\widetilde{A}\|_{C^r_\Gamma} \right)^k, \quad k \geq 1.$$

Proof. We will prove first the C^0_Γ bound.

$$A(\theta)^k = \sum_{m=0}^{k} \text{Products of } m \text{ factors } A_0 \text{ and } (k-m) \text{ factors } \widetilde{A}(\theta).$$

Since the product of matrices is not commutative, the order of the factors matters. When we bound this expression in L_Γ norm, we get as many $\Gamma(0)^{-1}$ factors in each summand as groups of consecutive factors A_0, but this is at most $k-m$ for each summand, therefore we can bound the previous expression by:

$$\Gamma(0)^{-1} \left(\|A_0\|^k + \sum_{m=0}^{k-1} C_{m,k} \Gamma(0)^{-k+m} \|A_0\|^m \|\widetilde{A}\|^{k-m}_{C^0_\Gamma} \right),$$

where $C_{m,k}$ is a binomial coefficient. This expression can be bounded by

$$\Gamma(0)^{-1} \left(\|A_0\| + \Gamma(0)^{-1} \|\tilde{A}\|_{C^0_\Gamma} \right)^k.$$

For the general C^r_Γ case, observe that $\partial^i_\theta A(\theta) = \partial^i_\theta \tilde{A}(\theta)$ for all $1 \leq i \leq r$, which is a term of order $\|\widetilde{A}\|_{C^r_\Gamma}$, thus for each derivative of $A(\theta)^k$ with respect to θ we get k new terms which can be bounded by the previous expression, hence

$$\|A(\theta)^k\|_{C^r_\Gamma} \leq k^r \Gamma(0)^{-1} \left(\|A_0\| + \Gamma(0)^{-1} \|\widetilde{A}\|_{C^r_\Gamma} \right)^k.$$

□

By using Lemma 9.11 we can find $p \in \mathbb{N}$ such that $\|A^p_{1,1}\|_{C^0_\Gamma} < 1$ and $\|A^p_{1,1}\|_{C^0_\Gamma} \|A^{-p}_{2,2}\|_{C^0_\Gamma} < 1$. We will fix this p for the rest of this section. For convenience we will write again F and N instead of $\widetilde{F} = F^p$ and $\widetilde{N}$. Now we can prove

Proposition 9.12. *Let $p \in \mathbb{N}$ such that $\|A_{1,1}^p\|_{C_\Gamma^0} < 1$ as defined above, then we can determine the local strong stable manifold of $W_0(\theta)$ for F^p as the graph of $\varphi : \mathcal{E}^1 \times \mathbb{T}^d \to \mathcal{E}^2$ with $\varphi \in C_\Gamma^{1,0}\left(B(0,1) \times \mathbb{T}^d, \mathcal{E}^2\right)$.*

The proof of this proposition follows the same formal outline as the proof of Part (i) of Theorem 9.1. We need to prove analogous auxiliary lemmas and proposition to the ones used in that proof but now in the decay setting.

The formal structure of the proofs of the mentioned lemmas is also similar, thus we will only point out the differences in the bounds, for a shorter exposition.

The operators $\mathcal{A}_n$, $\mathcal{C}_n$ and $\mathcal{B}_n$ defined in (9.10) are now defined in Γ-linear spaces:

$$\mathcal{A}_n, \mathcal{C}_n : C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L_\Gamma(\mathcal{E}^1, \mathcal{E}^2)\big) \to C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L_\Gamma(\mathcal{E}^1, \mathcal{E}^2)\big)$$

and

$$\mathcal{B}_n \in C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L_\Gamma(\mathcal{E}^1, \mathcal{E}^2)\big).$$

Lemma 9.13. *Under the previous definitions and assumptions,*

(i) *$\mathcal{A}_n$ is well defined from $\overline{B(0,1)} \subset C^{0,0}(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L_\Gamma(\mathcal{E}^1, \mathcal{E}^2))$ to itself and is a contraction in $\|\cdot\|_{C^{0,0}}$ for all $n \in \mathbb{N} \cup \{\infty\}$. Furthermore, there exists $\rho_\mathcal{A} < 1$ such that $\operatorname{Lip} \mathcal{A}_n < \rho_\mathcal{A}$ for all $n \in \mathbb{N} \cup \{\infty\}$.*

(ii) *Given $J \in C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L_\Gamma(\mathcal{E}^1, \mathcal{E}^2)\big)$, $\{\mathcal{A}_n J\}_{n\in\mathbb{N}}$ converges uniformly over compact sets to $\mathcal{A}_\infty J$.*

(iii) *$\mathcal{C}_n$ is well defined from $\overline{B(0,1)} \subset C^{0,0}(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L_\Gamma(\mathcal{E}^1, \mathcal{E}^2))$ to itself and is a contraction in $\|\cdot\|_{C^{0,0}}$ for all $n \in \mathbb{N} \cup \{\infty\}$. Furthermore, $\sup_n \operatorname{Lip} \mathcal{C}_n < \rho_\mathcal{C}$ and $\rho_\mathcal{C}$ can be made as small as needed by taking the scaling parameter sufficiently small.*

(iv) *Given $J \in C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L_\Gamma(\mathcal{E}^1, \mathcal{E}^2)\big)$, $\{\mathcal{C}_n(J)\}_{n\in\mathbb{N}}$ converges uniformly over compact sets to $\mathcal{C}_\infty(J)$.*

(v) *$\mathcal{B}_n$ converges uniformly over compact sets to $\mathcal{B}_\infty$ and $\|\mathcal{B}_n\|_{C^{0,0}} < \rho_\mathcal{B}$ for all $n \in \mathbb{N} \cup \{\infty\}$, and $\rho_\mathcal{B}$ can be made as small as needed by taking the scaling parameter sufficiently small.*

Proof of Lemma 9.13: Recall that $\varphi_n^0 \in \overline{B(0,1)} \subset C^{0,0}(\mathcal{E}^1 \times \mathbb{T}^d, \mathcal{E}^2)$ and $\psi_n(x,\theta) = M_{1,1}(\theta)x + N_1(x, \varphi_n^0(x,\theta), \theta)$.

Part (i) *($\mathcal{A}_n$ is well defined and a contraction):* Given $J \in C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L_\Gamma(\mathcal{E}^1, \mathcal{E}^2)\big)$, $\mathcal{A}_n J$ is clearly continuous. Note that we compute $\|J\|_{C^{0,0}}$, although $J(x,\theta) \in$

$L_\Gamma(\mathcal{E}^1, \mathcal{E}^2)$. We only need to bound the norm of the image to check that $\mathcal{A}_n J \in \overline{B(0,1)}$:

$$\begin{aligned}
\|\mathcal{A}_n J\|_{C^{0,0}} &\leq \|M_{2,2}^{-1}\|_{C^0_\Gamma} \|J\|_{C^{0,0}} \big(\|M_{1,1}\|_{C^0_\Gamma} + \|D_x N_1\|_{C^{0,0}}\big) \\
&\quad + \|M_{2,2}^{-1}\|_{C^0_\Gamma} \|D_y N_2\|_{C^{0,0}} \|J\|_{C^{0,0}} \\
&\leq \|M_{2,2}^{-1}\|_{C^0_\Gamma} \|J\|_{C^{0,0}} \left(\|M_{1,1}\|_{C^0_\Gamma} + o(\delta^0)\right) \\
&\quad + o(\delta^0) \|M_{2,2}^{-1}\|_{C^0_\Gamma} \|J\|_{C^{0,0}} \\
&\leq \widetilde{\rho} \|J\|_{C^{0,0}},
\end{aligned}$$

with $\widetilde{\rho} = \|M_{2,2}^{-1}\|_{C^0_\Gamma}\big(\|M_{1,1}\|_{C^0_\Gamma} + o(\delta^0)\big)$. Observe that this bound is independent of $n \in \mathbb{N} \cup \{\infty\}$, and by Lemma 9.11 is smaller than 1 if the perturbation parameter ε is small enough.

To prove the contraction property, let $J_1,\ J_2 \in \overline{B(0,1)} \subset C^{0,0}\big(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L_\Gamma(\mathcal{E}^1, \mathcal{E}^2)\big)$, we can write

$$\big(\mathcal{A}_n J_1 - \mathcal{A}_n J_2\big)(x,\theta) = T_n^1(x,\theta) + T_n^2(x,\theta) + T_n^3(x,\theta),$$

where

$$\begin{aligned}
T_n^1(x,\theta) &= M_{2,2}(\theta)^{-1} J_1(\psi_n(x,\theta), \theta+\omega) M_{1,1}(\theta) \\
&\quad - M_{2,2}(\theta)^{-1} J_2(\psi_n(x,\theta), \theta+\omega) M_{1,1}(\theta), && (9.16)\\
T_n^2(x,\theta) &= M_{2,2}(\theta)^{-1} J_1(\psi_n(x,\theta), \theta+\omega) D_x N_1(x, \varphi_n^0(x,\theta), \theta) \\
&\quad - M_{2,2}(\theta)^{-1} J_2(\psi_n(x,\theta), \theta+\omega) D_x N_1(x, \varphi_n^0(x,\theta), \theta), && (9.17)\\
T_n^3(x,\theta) &= -M_{2,2}(\theta)^{-1} D_y N_2(x, \varphi_n^0(x,\theta), \theta) J_1(x,\theta) \\
&\quad + M_{2,2}(\theta)^{-1} D_y N_2(x, \varphi_n^0(x,\theta), \theta) J_2(x,\theta). && (9.18)
\end{aligned}$$

Now we proceed to bound all these terms separately. We can bound T_n^1 in (9.16) by

$$\|M_{2,2}^{-1}\|_{C^0_\Gamma} \|M_{1,1}\|_{C^0_\Gamma} \|J_1 - J_2\|_{C^{0,0}} < \widetilde{\rho} \|J_1 - J_2\|_{C^{0,0}}$$

with $\widetilde{\rho} = \|M_{2,2}^{-1}\|_{C^0_\Gamma} \|M_{1,1}\|_{C^0_\Gamma} < 1$ if the perturbation of the original system $\|F_1\|$ is small enough. Similarly we can bound T_n^2 in (9.17) by $\|M_{2,2}^{-1}\|_{C^0_\Gamma} \|J_1 - J_2\|_{C^{0,0}} \|DN\|_{C^{0,0}}$ and a similar bound follows for T_n^3 in (9.18). All these bounds together result in

$$\big\|\big(\mathcal{A}_n J_1 - \mathcal{A}_n J_2\big)\big\|_{C^{0,0}} \leq \mu \|J_1 - J_2\|_{C^{0,0}}$$

with

$$\mu = \|M_{2,2}^{-1}\|_{C^0_\Gamma} \|M_{1,1}\|_{C^0_\Gamma} + 2\|M_{2,2}^{-1}\|_{C^0_\Gamma} \|DN\|_{C^{0,0}}$$

which is smaller than 1 if the scaling parameter δ and the size of the perturbation term ε are small enough.

Part (ii) *(convergence of* $\{\mathcal{A}_n(J)\}_{n\in\mathbb{N}}$*):* To prove the convergence of $\mathcal{A}_n J$, observe that we can write

$$\Big[\mathcal{A}_n - \mathcal{A}_\infty\Big](J)(x,\theta) = T_n^1(x,\theta) + T_n^2(x,\theta) + T_n^3(x,\theta),$$

where

$$\begin{aligned}
T_n^1(x,\theta) &= M_{2,2}(\theta)^{-1} J(\psi_n(x,\theta),\theta+\omega) M_{1,1}(\theta) \\
&\quad - M_{2,2}(\theta)^{-1} J(\psi_\infty(x,\theta),\theta+\omega) M_{1,1}(\theta), \\
T_n^2(x,\theta) &= M_{2,2}(\theta)^{-1} J(\psi_n(x,\theta),\theta+\omega) D_x N_1(x,\varphi_n^0(x,\theta),\theta) \\
&\quad - M_{2,2}(\theta)^{-1} J(\psi_\infty(x,\theta),\theta+\omega) D_x N_1(x,\varphi_\infty^0(x,\theta),\theta), \\
T_n^3(x,\theta) &= -M_{2,2}(\theta)^{-1} D_y N_2(x,\varphi_n^0(x,\theta),\theta) J(x,\theta) \\
&\quad + M_{2,2}(\theta)^{-1} D_y N_2(x,\varphi_\infty^0(x,\theta),\theta) J(x,\theta).
\end{aligned}$$

Given a compact set $G \in \mathcal{E}^1$, by Lemma 9.9 applied to the sequence $\{\psi_n\}_{n\in\mathbb{N}}$, the sets

$$\begin{aligned}
\mathcal{G}_1 &= \left\{ \left(\bigcup_{n=0}^{\infty} \psi_n(x,\theta) \right) \cup \{\psi_\infty(x,\theta)\},\ x \in G,\ \theta \in \mathbb{T}^d \right\}, \\
\mathcal{G}_2 &= \left\{ \left(\bigcup_{n=0}^{\infty} \varphi_n(x,\theta) \right) \cup \{\varphi_\infty(x,\theta)\},\ x \in G,\ \theta \in \mathbb{T}^d \right\},
\end{aligned}$$

$\mathcal{G}_1 \times \mathbb{T}^d$, $\mathcal{G}_2 \times \mathbb{T}^d$ and $G \times \mathcal{G} \times \mathbb{T}^d$ are compact sets. Since J and $D_y N_2$ are continuous, they are uniformly continuous on $\mathcal{G}_1 \times \mathbb{T}^d$ and $G \times \mathcal{G}_2 \times \mathbb{T}^d$ respectively. Thus for each $\varepsilon > 0$ there is some $\delta > 0$ such that if $\|\psi_n - \psi_\infty\| < \delta$ then $\|T_n^1\| < \varepsilon/3$, where ε and δ are the constants used in the uniform continuity bounds for J in $\mathcal{G}_1 \times \mathbb{T}^d$. The same argument applies to T_n^3, with the compact set $G \times \mathcal{G}_2 \times \mathbb{T}^d$. Finally, to bound $\|T_n^2\|$ we add and subtract the term $M_{2,2}(\theta)^{-1} J(\psi_n(x,\theta),\theta+\omega) D_x N_1(x,\varphi_\infty^0(x,\theta),\theta)$ and use the same argument as above to get analogous bounds. To prove the convergence of $\big\{\big[\mathcal{A}_n - \mathcal{A}_\infty\big](J)\big\}_{n\in\mathbb{N}}$ to 0 we just have to take n_0 such that if $n > n_0$ $\|\psi_n - \psi_\infty\|_{C^{0,0}} < \delta$ and $\|\varphi_n^0 - \varphi_\infty^0\|_{C^{0,0}} < \delta$ where δ is related with the definition of uniform continuity of the above mentioned functions.

Part (iii) *($\mathcal{C}_n$ is well defined and a contraction):* It is straightforward to see that $\mathcal{C}_n(J)$, $n \in \mathbb{N} \cup \{\infty\}$, is continuous. To prove the contraction property, let J_1, $J_2 \in B(0,1) \subset C^{0,0}\big(B(0,1) \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1,\mathcal{E}^2)\big)$. We can bound

$$\|\left(\mathcal{C}_n(J_1) - \mathcal{C}_n(J_2)\right)\|_{C^{0,0}} \le 2\mathcal{O}(\|D_y N_1\|_{C^{0,0}}) \|J_1 - J_2\|_{C^{0,0}}$$

by adding and subtracting

$$M_{2,2}^{-1}(\theta) J_1(\psi_n(x,\theta),\theta+\omega) D_y N_1(x,\varphi_n^0(x,\theta),\theta) J_2(x,\theta)$$

and using that $\|J_1\|_{C^{0,0}} < 1$, $\|J_2\|_{C^{0,0}} < 1$.

Part (iv) *(convergence of* $\{\mathcal{C}_n(J)\}_{n\in\mathbb{N}}$*):* We can write

$$\mathcal{C}_n(J) - \mathcal{C}_\infty(J) = T_n^1(x,\theta) + T_n^2(x,\theta),$$

where

$$\begin{aligned} T_n^1(x,\theta) =& M_{2,2}^{-1}(\theta) J(\psi_n(x,\theta),\theta+\omega) D_y N_1(x,\varphi_n^0(x,\theta),\theta) J(x,\theta) \\ &- M_{2,2}^{-1}(\theta) J(\psi_\infty(x,\theta),\theta+\omega) D_y N_1(x,\varphi_n^0(x,\theta),\theta) J(x,\theta) \\ T_n^2(x,\theta) =& M_{2,2}^{-1}(\theta) J(\psi_\infty(x,\theta),\theta+\omega) D_y N_1(x,\varphi_n^0(x,\theta),\theta) J(x,\theta) \\ &- M_{2,2}^{-1}(\theta) J(\psi_\infty(x,\theta),\theta+\omega) D_y N_1(x,\varphi_\infty^0(x,\theta),\theta) J(x,\theta). \end{aligned}$$

Now given a compact set $G \subset \mathcal{E}^1$ we can use the same uniform continuity argument as in the proof of Part (ii) to prove the convergence on compact sets of $\{\mathcal{C}_n(J)\}_{n\in\mathbb{N}}$ to $\mathcal{C}_\infty(J)$.

Part (v) *(convergence of* $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$*):* Given a compact set $G \subset \mathcal{E}^1$, we can use the same argument as before to show the uniform convergence of $\{\mathcal{B}_n - \mathcal{B}_\infty\}_{n\in\mathbb{N}}$ to 0 in $G \times \mathcal{G}_2 \times \mathbb{T}^d$ with $\mathcal{G}_2$ the same set as in the proof of Part (ii). The bound $\|\mathcal{B}_n\| < \rho_{\mathcal{B}}$ is a consequence of the smallness of $\|DN\|$ and the smallness of the scaling parameter. There is a common bound for all terms $\mathcal{B}_n$, independent of n, and as small as needed.

□

The proof of Proposition 9.12 now finishes like the proof of the $C^{1,0}$ case.

Observe that the local strong stable invariant manifold of F, denoted for clarity as W_F^s, is a subset of the invariant manifold of $F^p(\theta)$, denoted by $W_{F^p}^s$. Due to the uniqueness of the local invariant graphs tangent to the same linear strong stable space, they are the same local parametrisation proving the result.

9.5 $C^{1,0}_{j,\Gamma}$ regularity of the parametrisation of strong stable invariant manifolds

The proof when $F \in C^{1,0}_{j,\Gamma}$ is completely analogous to the proof in the $C^{1,0}_{\Gamma}$, changing $C^{0,0}$ bounds by S^0_j bounds where appropriate.

9.6 The $C^{\Sigma_{0,1}}$ case

We will only give a rough sketch of the construction of this case, to give an idea of how the proof in the $C^{\Sigma_{r,s}}$ case works. Consider the sequence $\varphi^0_n = \mathcal{T}^n(0)$ introduced at the end of Section 9.2. Taking derivatives with respecto to θ we can write

$$\partial_\theta \varphi^0_{n+1} = \widetilde{\mathcal{A}}_n \partial_\theta \varphi^0_n + \widetilde{\mathcal{B}}_n,$$

where $\widetilde{\mathcal{A}}_n$, $\widetilde{\mathcal{B}}_n$ and ψ_n are defined for $n \in \mathbb{N} \cup \{\infty\}$ as follows

$$\begin{aligned}
\widetilde{\mathcal{A}}_n J(x,\theta) =& M_{2,2}(\theta)^{-1}\big[\varphi^1_n(\psi_n(x,\theta),\theta+\omega) D_y N_1(x,\varphi^0_n(x,\theta)) J(x,\theta) \\
&+ J(\psi_n(x,\theta),\theta+\omega) - D_y N_2(x,\varphi^0_n(x,\theta),\theta) J(x,\theta)\big], \\
\widetilde{\mathcal{B}}_n(x,\theta) =& \partial_\theta M_{2,2}(\theta)^{-1}\left(\varphi^0_n(\psi_n(x,\theta),\theta+\omega) - N_2(x,\varphi^0_n(x,\theta,\theta))\right) \\
&+ M_{2,2}(\theta)^{-1}\big[\varphi^1_n(\psi_n(x,\theta),\theta+\omega)\left(\partial_\theta M_{1,1}(\theta)x + \partial_\theta N_1(x,\varphi^0_n(x,\theta),\theta)\right) \\
&- \partial_\theta N_2(x,\varphi^0_n(x,\theta),\theta)\big], \\
\psi_n(x,\theta) =& M_{1,1}(\theta)x + N_1(x,\varphi^0_n(x,\theta),\theta),
\end{aligned}$$

where we denote $\varphi^1_n = D_x \varphi^0_n$ for $n \in \mathbb{N} \cup \{\infty\}$. Observe that the proof of convergence is now done in a step-wise fashion. From Part (i) of Theorem 9.1 we already know that φ^0_n and φ^1_n converge over compact sets to φ^0_∞ and φ^1_∞ respectively. This allows us to prove a lemma analogous to Lemma 9.8 showing that the sequences $\{\widetilde{\mathcal{A}}_n\}_{n\in\mathbb{N}}$ and $\{\widetilde{\mathcal{B}}_n\}_{n\in\mathbb{N}}$ converge to $\widetilde{\mathcal{A}}_\infty$ and $\widetilde{\mathcal{B}}_\infty$, and that $\widetilde{\mathcal{A}}_\infty$ is a contraction. Observe that in this case, $\widetilde{\mathcal{A}}_n$ is a linear operator and we do not need an operator analogous to $\mathcal{C}_n$. This allows us to prove that the sequence $\{\partial_\theta \varphi^0_n\}_{n\in\mathbb{N}}$ converges uniformly over compact subsets to $\partial_\theta \varphi^0_\infty$.

The proof of the $C^{\Sigma_{s,r}}$ case follows the same lines, covers the $C^{\Sigma_{0,1}}$ sketched here and is done in full detail in the next section.

9.7 Sharp regularity in the $C^{\Sigma_{s,r}}$ case

The proof of this result follows the lines of the proof of sharp regularity in [CFdlL03a]. We use an induction argument that allows us to find higher order derivatives, based on the proof of $C^{1,0}$ regularity.

We will need a few preliminary lemmas describing the structure of derivatives of compositions of functions in $C^{\Sigma_{s,r}}$ and related spaces. The structure of the inductive step then follows closely the regularity proofs in the previous sections.

Theorem 9.14. *Given a dynamical system $F(x,\theta)$ as defined in Section 2.1, we can determine the unique local strong stable manifold of W_0 tangent to $\mathcal{E}^1$ at 0 as a graph $\varphi : \mathcal{E}^1 \times \mathbb{T}^d \to \mathcal{E}^2$ under the following regularity and decay assumptions:*

(i) *If* $F(x,\theta) \in C^{\Sigma_{s,r}}\big(\ell^\infty(\mathbb{R}^n)\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n)\big)$ *and* $\|F_1\|_{C^{\Sigma_{s,r}}}$ *is small enough then* $\varphi \in C^{\Sigma_{s,r}}(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}^1\times\mathbb{T}^d, \mathcal{E}^2)$.

(ii) *If* $F(x,\theta) \in C_\Gamma^{\Sigma_{s,r}}\big(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)\big)$ *and* $\|F_1\|_{C_\Gamma^{\Sigma_{s,r}}}$ *is small enough then* $\varphi \in C_\Gamma^{\Sigma_{s,r}}(B(0,1)\times\mathbb{T}^d, \mathcal{E}^2)$.

(iii) *If* $F(x,\theta) \in C_{j,\Gamma}^{\Sigma_{s,r}}\big(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)\big)$ *and* $\|F_1\|_{C_{j,\Gamma}^{\Sigma_{s,r}}}$ *is small enough then* $\varphi \in C_{j,\Gamma}^{\Sigma_{s,r}}(B(0,1)\times\mathbb{T}^d, \mathcal{E}^2)$.

For the remaining part of this section, let $p(x,\theta)=\theta$ be the projection on the torus $\mathbb{T}^d$.

Lemma 9.15. *Let* s, $r\in\mathbb{Z}^+$ *with* $s+r\geq 1$. *Given a pair of functions* $f\in C^{\Sigma_{s,r}}(\ell^\infty(\mathbb{R}^n)\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ *and* $g\in C^{s,r}(\ell^\infty(\mathbb{R}^n)\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$, *for* i, $j\in\Sigma_{s,r}$ *we can write*

$$D_x^i D_\theta^j (f\circ(g,p))(x,\theta) = \sum_{\substack{(a,b)\in\Sigma^*_{i,j}\\ i_1+\ldots+i_a=i\\ j_1+\ldots+j_a=j-b}} C D_\theta^b D_x^a f\circ(g,p) D_x^{i_1} D_\theta^{j_1} g\cdots D_x^{i_a} D_\theta^{j_a} g,$$

where

$$\Sigma^*_{0,j} = \{(a,b)\in\left(\mathbb{Z}^+\right)^2 \mid a+b\leq j,\ a\geq 1\}\cup\{(0,j)\},\quad j\geq 1$$
$$\Sigma^*_{i,j} = \{(a,b)\in\left(\mathbb{Z}^+\right)^2 \mid a+b\leq i+j,\ a\geq 1,\ b\leq j\}\cup\{(0,j)\},\quad \text{if } i\geq 1,\quad j\geq 0$$

and C *is a combinatorial constant depending on* $a,b,i_1,\ldots,i_a,j_1,\ldots,j_a$.

Proof. We will prove this result by a double induction. First we will prove

$$D_\theta^j\big[f\circ(g,p)\big] = \sum_{\substack{(a,b)\in\Sigma^*_{0,j}\\ j_1+\ldots+j_a=j-b}} C D_\theta^b D_x^a f\circ(g,p) D_\theta^{j_1} g\cdots D_\theta^{j_a} g. \tag{9.19}$$

Indeed, when $j=1$, the summation indices are in $\Sigma^*_{0,1}=\big\{(1,0)\big\}\cup\big\{(0,1)\big\}$ and by Leibniz's rule

$$D_\theta\big[f\circ(g,p)\big] = D_x f(g,p) D_\theta g + D_\theta f(g,p),$$

which coincides with the required expression. Assuming that (9.19) holds for the j-th derivative we will show the expression also holds for the $(j+1)$-th derivative. If we differentiate the general term in the sum in the right-hand side of Equation (9.19) with respect to θ, we get

$$C D_\theta^{b+1} D_x^a f\circ(g,p) D_\theta^{j_1} g\cdots D_\theta^{j_a} g, \tag{9.20}$$
$$+\, C D_\theta^b D_x^{a+1} f\circ(g,p)\big(D_\theta g\big) D_\theta^{j_1} g\cdots D_\theta^{j_a} g, \tag{9.21}$$
$$+\sum_{k=1}^a C D_\theta^b D_x^a f\circ(g,p) D_\theta^{j_1} g\cdots D_\theta^{j_k+1} g\cdots D_\theta^{j_a} g. \tag{9.22}$$

Consider the range of indices $(a,\ b)$ and j_k in the summation of Equation (9.19), which are $(a,\ b)\in\Sigma^*_{0,j}$ and $j_1+\ldots+j_a=j-b$. Then

- in (9.20) the indices span

$$(a,\,b)\in\Sigma^*_{0,j+1}\backslash\{(c,0)\,|\,0\le c\le j+1\},\qquad j_1+\ldots+j_a=j-b,$$

- in (9.21) the indices span

$$(a,b)\in\Sigma^*_{0,j+1}\backslash\Big(\{(0,c)\,|\,0\le c\le j{+}1\}\cup\{(1,c)\,|\,0\le c\le j{-}1\}\Big),\qquad j_1{+}\ldots{+}j_a=j{-}b{+}1,$$

- in (9.22) the indices span

$$(a,\,b)\in\Sigma^*_{0,j},\qquad j_1+\ldots+j_a=j-b+1.$$

Combining these three sets, we get the desired result,

$$D^{j+1}_\theta\left[f\circ(g,p)\right]=\sum_{\substack{(a,b)\in\Sigma^*_{0,j+1}\\ j_1+\ldots+j_a=j+1-b}} CD^a_xD^b_\theta f\circ(g,p)D^{j_1}_\theta g\cdots D^{j_a}_\theta g,$$

since differentiating passes exactly from one set of indices to the next.

Now to prove the general formula, we have derivatives of $D^j_\theta\big[f\circ(g,p)\big]$ with respect to x. When $i=1$,

$$\begin{aligned}D_xD^j_\theta\big[f\circ(g,p)\big]=&\sum_{\substack{(a,b)\in\Sigma^*_{0,j}\\ j_1+\ldots+j_a=j-b}}\Bigg(CD^{a+1}_xD^b_\theta f\circ(g,p)D_xgD^{j_1}_\theta g\cdots D^{j_a}_\theta g\\ &+\sum_{s=1}^{a}CD^a_xD^b_\theta f\circ gD^{j_1}_\theta g\cdots D^{j_s}_\theta D_xg\cdots D^{j_a}_\theta g\Bigg).\end{aligned}\tag{9.23}$$

Since $\Sigma^*_{1,j}=\{(a,b)\in(\mathbb{Z}^+)^2\,|\,a+b\le j+1,\,b\le j,\,a\ge 1\}$, we can re-write the previous expression as

$$D_xD^j_\theta\left[f\circ(g,p)\right]=\sum_{\substack{(a,b)\in\Sigma^*_{1,j}\\ j_1+\ldots+j_a=j-b\\ i_1+\ldots+i_a=1}} CD^a_xD^b_\theta f\circ(g,p)D^{i_1}_xD^{j_1}_\theta g\cdots D^{i_a}_xD^{j_a}_\theta g,\tag{9.24}$$

since the two terms in Equation (9.23) combine to fill the set of indices $\Sigma^*_{1,j}$. Note that there is no term having $D^j_\theta f\circ(g,p)$ in (9.24).

We assume now that the formula holds up to the i-th derivative with respect to x, thus

$$D^i_xD^j_\theta(f\circ(g,p))(x,\theta)=\sum_{\substack{(a,b)\in\Sigma^*_{i,j}\\ i_1+\ldots+i_a=i\\ j_1+\ldots+j_a=j-b}} CD^a_xD^b_\theta f\circ(g,p)D^{i_1}_xD^{j_1}_\theta g\cdots D^{i_a}_xD^{j_a}_\theta g,\tag{9.25}$$

and we differentiate the general term of the sum in the right-hand side of (9.25) with respect to x. We get

$$\begin{aligned}&CD^{a+1}_xD^b_\theta f\circ(g,p)D_xgD^{i_1}_xD^{j_1}_\theta g\cdots D^{i_a}_xD^{j_a}_\theta g\\ +&\sum_{s=1}^{a}CD^a_xD^b_\theta f\circ(g,p)D^{i_1}_xD^{j_1}_\theta g\cdots D^{i_s+1}_xD^{j_s}_\theta g\cdots D^{i_a}_xD^{j_a}_\theta g.\end{aligned}\tag{9.26}$$

Observe that passing from $\Sigma^*_{i,j}$ to $\Sigma^*_{i+1,j}$ is realised by the two terms in (9.26). The first term covers the increment in the a direction, and the second term fills the gaps caused by the shift $a \mapsto a+1$.

□

To denote derivatives of functions like $N(x, y, \theta)$, depending on two space variables and one angle variable we introduce the following sets of indices.

$$\Sigma^{**}_{0,j} = \{(a,b,0) \in (\mathbb{Z}^+)^3 \,|\, a+b \leq j,\, a \geq 1\} \cup \{(0,j,0)\}, \quad j \geq 1$$
$$\Sigma^{**}_{i,j} = \{(a,b,c) \in (\mathbb{Z}^+)^3 \,|\, a+b+c \leq i+j,\, b \leq j,\, c \leq i,\, a \geq 1,\, j \geq 1\} \cup \{(0,j,i)\},\ i \geq 1,\ j \geq 0$$

An analogue of the previous lemma holds for this type of functions.

Lemma 9.16. *Let $f \in C^{\Sigma_{s,r}}(\ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$, $g \in C^{s,r}(\mathcal{E}^1 \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $f = f(x, y, \theta)$. Then we can write*

$$D^i_x D^j_\theta \left[f \circ (\mathrm{Id}, g, p)\right] = \sum_{\substack{(a,b,c)\in\Sigma^{**}_{i,j} \\ i_1+\ldots+i_a=i-c \\ j_1+\ldots+j_a=j-b}} C D^b_\theta D^c_x D^a_y f \circ (\mathrm{Id}, g, p) D^{i_1}_x D^{j_1}_\theta g \cdots D^{i_a}_x D^{j_a}_\theta g.$$

Proof. We will prove this result by induction. The first step in the inductive proof consists in showing that the formula holds for $\Sigma^{**}_{0,j}$, but this is already covered by Lemma 9.15. Therefore the result holds for $i = 0$.

Assume the formula holds up to the i-th derivative,

$$D^i_x D^j_\theta \big[f \circ (\mathrm{Id}, g, p)\big] = \sum_{\substack{(a,b,c)\in\Sigma^{**}_{i,j} \\ i_1+\ldots+i_a=i-c \\ j_1+\ldots+j_a=j-b}} C D^b_\theta D^c_x D^a_y f \circ (\mathrm{Id}, g, p) D^{i_1}_x D^{j_1}_\theta g \cdots D^{i_a}_x D^{j_a}_\theta g. \qquad (9.27)$$

If we differentiate a generic term of the right hand side of Equation (9.27) with respect to x, we get the following expression

$$\begin{aligned} &D^b_\theta D^{c+1}_x D^a_y f \circ (\mathrm{Id}, g, p) D^{i_1}_x D^{j_1}_\theta g \cdots D^{i_a}_x D^{j_a}_\theta g \\ &+ D^b_\theta D^c_x D^{a+1}_y f \circ (\mathrm{Id}, g, p) D_x g D^{i_1}_x D^{j_1}_\theta g \cdots D^{i_a}_x D^{j_a}_\theta g \\ &+ \sum_{s=1}^{a} D^b_\theta D^c_x D^a_y f \circ (\mathrm{Id}, g, p) D^{i_1}_x D^{j_1}_\theta g \cdots D^{i_s+1}_x D^{j_s}_\theta g \cdots D^{i_a}_x D^{j_a}_\theta g. \end{aligned}$$

Note that a term with indices (a, b, c) generates new terms, with indices (a, b, c), $(a+1, b, c)$, $(a, b, c+1)$. Let

$$\widetilde{\Sigma}^m_{i,j} = \{(a, m, c) \,|\, a + c \leq i + j - m,\, c \leq i,\, a \geq 1\}, \quad i \geq 0,\ j \geq 0,\ 0 \leq m \leq i + j.$$

Clearly $\Sigma^{**}_{i,j} = \cup^j_{m=0} \widetilde{\Sigma}^m_{i,j}$. The operation over indices represented by

$$(a, b, c) \mapsto (a, b, c) \cup (a + 1, b, c) \cup (a, b, c + 1)$$

is exhaustive from $\widetilde{\Sigma}^m_{i,j}$ to $\widetilde{\Sigma}^m_{i+1,j}$, since the increments in indices fill up the increment in i. When $i = 0$, $\Sigma^{**}_{0,j} = \cup^j_{m=0} \widetilde{\Sigma}^m_{0,j}$, observe that in the induction procedure the initial case corresponds to $\widetilde{\Sigma}^j_{0,j} = \{(0,j,0)\}$ and this term $(0,j,0)$ generates $(1,j,0)$ and $(0,j,1)$ satisfying

$$\widetilde{\Sigma}^j_{1,j} = \{(1,j,0),(0,j,1)\}.$$

Observe also that the term in $D^i_x D^j_\theta\big[f\circ(\mathrm{Id},g,p)\big]$ with the derivative of highest order in g is $D^i_x D^j_\theta g$ and that this term is multiplied by a factor of the form $D_y f\circ(\mathrm{Id},g,p)$.

□

To end these technical lemmas about derivatives, we need a formula for the derivatives of products of linear mappings by functions of the types we have considered.

Lemma 9.17. *Let $f\in C^{\Sigma_{s,r}}(\ell^\infty(\mathbb{R}^n)\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))$, $g\in C^{s,r}(\ell^\infty(\mathbb{R}^n)\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))$ and $h\in C^r\big(\mathbb{T}^d,L(\ell^\infty(\mathbb{R}^n),\ell^\infty(\mathbb{R}^n))\big)$. Then if $(i,j)\in\Sigma_{s,r}$ the following formula holds*

$$D^i_x D^j_\theta\big[h(\theta)f\circ(g,p)\big] = \sum_{p=0}^{j}\sum_{\substack{(a,b)\in\Sigma^*_{i,p}\\ i_1+\ldots+i_a=i\\ j_1+\ldots+j_a=p-b}} C D^{j-p}_\theta h(\theta)\big(D^b_\theta D^a_x f\circ(g,p)D^{i_1}_x D^{j_1}_\theta g\cdots D^{i_a}_x D^{j_a}_\theta g\big),$$

where C is a combinatorial coefficient depending on all indices.

Proof. The proof is a direct application of Leibniz's rule and Lemma 9.15.

□

Remark 9.18. *Observe that the previous lemmas also hold changing $C^{\Sigma_{s,r}}$ for $C^{\Sigma_{s,r}}_\Gamma$ or $C^{\Sigma_{s,r}}_{j,\Gamma}$ and $C^{s,r}$ for $C^{s,r}_\Gamma$ or $C^{s,r}_{j,\Gamma}$, since they deal with the differentiability of compositions, not with the decay properties of the functions.*

The following theorem establishes the inductive step in the proof of $C^{\Sigma_{s,r}}$ regularity of the strong stable manifold.

Theorem 9.19. *Let $F\in C^{\Sigma_{s,r}}\big(\ell^\infty(\mathbb{R}^n)\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n)\big)$ and let $\Sigma_{i,j}\subset\Sigma_{s,r}$, $(i,j)\neq(1,0)$. Then if we define $\Sigma'_{i,j}=\Sigma_{i,j}\backslash\{(i,j)\}$, $\varphi^{0,0}_n=\mathcal{T}^n(0)$ and assume*

1) $\varphi^{0,0}_n\in C^{\Sigma_{i,j}}(\mathcal{E}^1,\mathcal{E}^2)$ for all $n\in\mathbb{N}$,

2) $\{\varphi^{a,b}_n\}_{n\in\mathbb{N}}$ converges uniformly over compact sets to $\varphi^{a,b}_\infty$ for $(a,\,b)\in\Sigma'_{i,j}$,

then

a) $\varphi^{0,0}_\infty\in C^{\Sigma_{i,j}}$,

b) $\{\varphi^{i,j}_n\}_{n\in\mathbb{N}}$ converges uniformly over compact sets to $\varphi^{i,j}_\infty$, which is a continuous function which coincides with the derivative of $\varphi^{i-1,j}_\infty$ with respect to x and with the derivative of $\varphi^{i,j-1}_\infty$ with respect to θ.

The proof of this result is based on the next lemmas.

Lemma 9.20. *The operator $\mathcal{T}$ defined in (9.5) is well-defined from $\overline{B(0,1)}\subset C^{\Sigma_{i,j}}\big(B(0,1)\times\mathbb{T}^d\subset\mathcal{E}^1\times\mathbb{T}^d,\mathcal{E}^2\big)$ into itself and is a contraction in $\overline{B(0,1)}$ in the C^0, C^0_Γ and $C^0_{j,\Gamma}$ norms.*

Proof. This proof is completely analogous to the proof of Lemma 9.5.

□

Observe that if $\mathcal{T}^n(0) = \varphi_n^{0,0}$ by differentiating $\varphi_{n+1}^{0,0} = \mathcal{T}\varphi_n^{0,0}$, we can write, for $(a,b) \neq (1,0)$,

$$D^{a,b}\varphi_{n+1}^{0,0}(x,\theta) := \varphi_{n+1}^{a,b}(x,\theta) = \mathcal{A}_n\varphi_n^{a,b}(x,\theta) + \mathcal{B}_n(x,\theta), \tag{9.28}$$

with $\mathcal{A}_n : \overline{B(0,1)}^d \subset C^{0,0}(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, \mathcal{L}^{a,b}(\mathcal{E}^1,\mathcal{E}^2))$, $\mathcal{B}_n \in C^{0,0}(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, \mathcal{L}^{a,b}(\mathcal{E}^1,\mathcal{E}^2))$ defined as

$$\begin{aligned}
\mathcal{A}_n J(x,\theta) &= M_{2,2}(\theta)^{-1} J(x,\theta)(\psi_n(x,\theta), \theta+\omega)\,(D_x\psi_n)^{\otimes a} \\
&\quad + M_{2,2}(\theta)^{-1}\varphi_n^{1,0}(\psi_n(x,\theta),\theta+\omega) D_y N_1(x,\varphi_n^{0,0}(x,\theta),\theta) J(x,\theta) \\
&\quad - M_{2,2}(\theta)^{-1} D_y N_2(x,\varphi_n^{0,0}(x,\theta),\theta) J(x,\theta), \\
\mathcal{B}_n(x,\theta) &= D_x^a D_\theta^b \mathcal{T}(\varphi_n^{0,0})(x,\theta) - \mathcal{A}_n D_x^a D_\theta^b \varphi_n^{0,0}(x,\theta),
\end{aligned}$$

where

$$\psi_n(x,\theta) = M_{1,1}(\theta)x + N_1(x,\varphi_n^{0,0}(x,\theta),\theta)$$

for $n \in \mathbb{N} \cup \{\infty\}$ and

$$\mathcal{L}^{a,b}(\mathcal{E}^1,\mathcal{E}^2) = L^a(\mathcal{E}^1, L^b(\mathbb{R}^d,\mathcal{E}^2)).$$

We have dropped the dependence on the differentiation order a, b of $\mathcal{A}_n$ and $\mathcal{B}_n$ for simplicity. Recall that $\varphi_n^{1,0} = D_x\mathcal{T}^n(0)$ which converges to $\varphi_\infty^{1,0}$ as proved in Proposition 9.6.

Remark 9.21. *Observe that $\mathcal{B}_n$ will have multilinear terms in $D_x^{a'} D_\theta^{b'} \varphi_n^{0,0}$ with $a', b' \in \Sigma'_{i,j}$ by the properties of Lemmas 9.15 and 9.16. By the hypotheses of Theorem 9.19, these terms converge uniformly over compact sets to $\varphi_\infty^{a',b'}$, thus $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ also converges.*

Remark 9.22. *Observe that $\|\mathcal{B}_n\|_{C^{0,0}}$ is of order ε if $\|\varphi_n^{a,b}\|_{C^{0,0}} \leq 1$ and δ is small enough. This is a consequence of Lemma 9.15, since all terms in $\mathcal{B}_n$ have either a factor $N(x,\theta)$ or one of its first derivatives, which are smaller than δ, or derivatives of either $M_{1,1}(\theta)$ or $M_{2,2}^{-1}(\theta)$ with respect to θ, which are smaller than ε.*

Observe that the operator $\mathcal{A}_n$ as defined above is very similar to the operator in Lemma 9.8, having similar properties. We can use similar arguments as those used in Lemma 9.8 to prove the following result.

Lemma 9.23. *Under the previous definitions and assumptions,*

(i) *$\mathcal{A}_n$ is well defined from $\overline{B(0,1)} \subset C^{0,0}(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, \mathcal{L}^{a,b}(\mathcal{E}^1,\mathcal{E}^2))$ into itself and is a contraction in $\|\cdot\|_{C^{0,0}}$ for all $n \in \mathbb{N} \cup \{\infty\}$. Furthermore, there exists $\rho_{\mathcal{A}} < 1$ such that $\operatorname{Lip}\mathcal{A}_n < \rho_{\mathcal{A}}$ for all $n \in \mathbb{N} \cup \{\infty\}$.*

(ii) *Given $J \in C^{0,0}(\overline{B(0,1)} \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, L(\mathcal{E}^1,\mathcal{E}^2))$, $\{\mathcal{A}_n J\}_{n\in\mathbb{N}}$ converges uniformly over compact sets to $\mathcal{A}_\infty J$*

(iii) *$\mathcal{B}_n$ converges uniformly over compact sets to $\mathcal{B}_\infty$ and $\|\mathcal{B}_n\|_{C^{0,0}} < \rho_{\mathcal{B}}$ for all $n \in \mathbb{N} \cup \{\infty\}$, with $\rho_{\mathcal{B}}$ as small as needed.*

In the remaining part of this section we will add again the a, b indices to the objects for clarity. Consider the operator

$$\mathcal{T}_\infty^{a,b} : C^{0,0}\big(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}^1\times\mathbb{T}^d, \mathcal{L}^{a,b}(\mathcal{E}^1,\mathcal{E}^2)\big) \to C^{0,0}\big(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}^1\times\mathbb{T}^d, \mathcal{L}^{a,b}(\mathcal{E}^1,\mathcal{E}^2)\big)$$

defined as

$$\mathcal{T}_\infty^{a,b}(J) := \mathcal{A}_\infty^{a,b} J + \mathcal{B}_\infty^{a,b}.$$

According to the results in Lemma 9.23 this operator is well defined and a contraction if the perturbation F_1 of the dynamical system is small enough. Thus it has a fixed point $\widetilde{\varphi}_\infty^{a,b} \in \overline{B(0,1)}$. Obviously

$$\widetilde{\varphi}_\infty^{a,b} = \mathcal{A}_\infty^{a,b}\widetilde{\varphi}_\infty^{a,b} + \mathcal{B}_\infty^{a,b}. \tag{9.29}$$

We will prove that the iteration

$$\varphi_{n+1}^{a,b} = \mathcal{A}_n^{a,b}\varphi_n^{a,b} + \mathcal{B}_n^{a,b}$$

converges to this fixed point uniformly on compact sets.

Let $G \subset \mathcal{E}^1$ be a compact set. We can write

$$\varphi_{n+1}^{a,b} - \widetilde{\varphi}_\infty^{a,b} = Z_n^1 + Z_n^2,$$

where

$$\begin{aligned} Z_n^1 &:= \mathcal{A}_n^{a,b}\varphi_n^{a,b} - \mathcal{A}_n^{a,b}\widetilde{\varphi}_\infty^{a,b} + \mathcal{A}_n^{a,b}\widetilde{\varphi}_\infty^{a,b} - \mathcal{A}_\infty^{a,b}\widetilde{\varphi}_\infty^{a,b}, \\ Z_n^2 &:= \mathcal{B}_n^{a,b} - \mathcal{B}_\infty^{a,b}, \end{aligned}$$

where we have used (9.29) and added and subtracted suitable terms to make bounding easier. We can now bound each term separately as we have done before and prove the convergence of $\varphi_n^{a,b}$ to $\varphi_\infty^{a,b}$.

To prove $\varphi_\infty^{a,b}$ is the derivative with respect to x (or with respect to θ) of $\varphi^{a-1,b}$ (respectively of $\varphi^{a,b-1}$) we use the same argument as in the previous results.

To prove Theorem 9.14, we follow the scheme of Figure 9.7. We prove the $C^{0,0}$ and $C^{1,0}$ regularities via Theorem 9.1. Then advance along the a axis by proving higher regularity with respect to θ, given by the previous inductive step. Then, we prove a higher derivative with respect to x, also given by the previous inductive step (which only fails when the derivative with respect to x is of order 1).

9.8 Sharp regularity in the $C_\Gamma^{\Sigma_{s,r}}$ case

This case is completely analogous, using F^α as introduced in Section 9.4 and the same inductive proof as in the previous section. The equality of the parametrisations is shown using the uniqueness argument already used at the end of Section 9.4.

Chapter 10

Non-resonant manifolds I: Formal expansion

In this chapter and the next we consider non-resonant manifolds of invariant tori. In this chapter we will determine parametrisations $W(s, \theta)$ of the manifolds and a normal form $R(s, \theta)$ for the dynamics restricted to them. In this chapter we will determine the terms of a formal parametrisation up to a predefined degree L, while finding the terms of the restricted dynamics over the invariant manifold at the same time. After this formal part is given, in the next chapter we will proceed to determine the tail of the parametrisation by using a fixed point argument.

We assume that we have translated the torus to the origin as in Chapter 7. After this change is done, $W_0(\theta) = 0$ and hence $M(\theta) = D_x F(0, \theta)$. Assuming we are in the setting of Proposition 8.4, we change variables so that $M(\theta)$ is in block upper triangular form. To simplify the domains of definition of the functions involved in the following proofs, we will introduce a scaling procedure as in Section 8.4 which allows us to work in the unit ball of $\ell^\infty(\mathbb{R}^n)$ and shift the smallness requirements to the size of the perturbing function.

We recall that given $\delta > 0$, we consider the scaling defined by

$$T(x, \theta) = (\delta x, \theta),$$

which we apply to $F(x, \theta)$ to get

$$(T^{-1} \circ F \circ T)(x, \theta) = M(\theta)x + \frac{N(\delta x, \theta)}{\delta},$$

where $M(\theta) = D_x F(0, \theta)$. If the invariance equation

$$F\big(W_0(\theta) + W_1(\theta)s + \dots, \theta\big) = W_0(\theta + \omega) + W_1(\theta + \omega)R_1(\theta)s + \dots$$

is satisfied for W and R defined on $B(0, \delta) \times \mathbb{T}^d \subset \ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d$, after applying this scaling it holds for W and R defined on $B(0, 1) \times \mathbb{T}^d \subset \ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d$. Moreover, assuming $F \in C_\Gamma^{\Sigma_{t,r}}$, the $C_\Gamma^{\Sigma_{t,r}}$ (or the $C_{j,\Gamma}^{\Sigma_{t,r}}$ norm if $F \in C_{j,\Gamma}^{\Sigma_{t,r}}$) norms of $\delta^{-1} N(\delta x, \theta)$ are as small as desired in $B(0, 2) \times \mathbb{T}^d$ (this fact is important later on) by taking δ small, because $N(x, \theta)$ is of order 2 (assuming the system is at least C^2) with respect to x.

Recall that we denote the annulus generated by a set S by $\mathcal{A}S$, as introduced in Chapter 8. The main result of this chapter and the next is the following theorem.

Theorem 10.1. *Let U be an open set of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$ and consider a dynamical system $F : U \times \mathbb{T}^d \subseteq \ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n)$, $F(x, \theta) = M(\theta)x + N_1(x, \theta)$ with $M(\theta) = M_0 + \widetilde{M}(\theta)$ and*

$$M_0 = \begin{pmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{pmatrix}, \; \widetilde{M}(\theta) = \begin{pmatrix} B_{1,1}(\theta) & B_{1,2}(\theta) \\ 0 & B_{2,2}(\theta) \end{pmatrix}, \; M(\theta) = \begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ 0 & M_{2,2}(\theta) \end{pmatrix}.$$

Assume the following hypotheses,

(H1) $F \in C^{\Sigma_{t,r}}_\Gamma\big(\ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)\big)$, *with* $t \geq r + 1$, M_0, $\widetilde{M}(\theta) \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $\sup_{\theta \in \mathbb{T}^d} \|\widetilde{M}(\theta)\|_\Gamma = \mathcal{O}(\varepsilon)$ *and the scaling parameter* δ *are sufficiently small*

(H2) $\mathcal{A}\,\mathrm{Spec}(A_{1,1}) \subset \mathbb{D}\backslash\{0\}$,

(H3) $0 \notin \mathrm{Spec}(A_{2,2})$,

(H4) $\mathcal{A}\,\mathrm{Spec}(A_{1,1})^{L+1} \cdot \mathcal{A}\,\mathrm{Spec}(M_0^{-1}) \subset \mathbb{D}$,

(H5) $\mathcal{A}\,\mathrm{Spec}(A_{1,1})^i \cap \mathcal{A}\,\mathrm{Spec}(A_{2,2}) = \emptyset$ *for* $2 \leq i \leq L$,

(H6) $L + 1 \leq t$.

Then

(a) We can determine a polynomial bundle map $R : \mathcal{E}^1 \times \mathbb{T}^d \to \mathcal{E}^1$ *of degree not larger than* L *in* $C^{\infty,r}_\Gamma(\mathcal{E}_1 \times \mathbb{T}^d, \mathcal{E}_1)$ *such that* $R(0, \theta) = 0$, $D_s R(0, \theta) = M_{1,1}(\theta)$ *and a bundle map* $W : B(0, 1) \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n)$ *in* $C^{\Sigma_{t,r}}_\Gamma(B(0, 1) \times \mathbb{T}^d \subset \mathcal{E}_1 \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ *such that*

$$F(W(s, \theta), \theta) = W(R(s, \theta), \theta + \omega),$$

where $W(0, \theta) = 0$, $\Pi_{\mathcal{E}^1} D_s W(0, \theta) = \mathrm{Id}_{\mathcal{E}^1}$ *and* $\Pi_{\mathcal{E}^2} D_s W(0, \theta) = 0$.

(b) Furthermore, if there is $l \geq 2$ *such that*

$$\mathcal{A}\,\mathrm{Spec}(A_{1,1})^i \cap \mathcal{A}\,\mathrm{Spec}(A_{1,1}) = \emptyset, \quad l \leq i \leq L,$$

then we can choose R *to be a polynomial bundle map of degree not larger than* $l - 1$.

Remark 10.2. *Given a map F as in Chapter 2.1, in general $D_x F(0, \theta)$ is not in triangular form as is required in the theorem above. In such case one has first to apply Proposition 8.4 for which the additional hypothesis $\mathcal{A}\,\mathrm{Spec}\,A_{1,1} \cap \mathcal{A}\,\mathrm{Spec}\,A_{2,2} = \emptyset$ is needed. Under such condition, the results in Section 8.2 give a C^r_Γ linear transformation turning $M(\theta)$ into block triangular form. Hence, if the linear map is already in block triangular form this condition can be skipped.*

Hypotheses (H2) and (H3) have already been used in Section 8.2 to determine the linear part of the parametrisation. Hypothesis (H5) is a non-resonance hypothesis for the spectral decomposition, used to invert the homological equations at each order. Hypothesis (H6) ensures we can still differentiate the remaining terms. The condition in (b) is a stronger non-resonance condition that ensures we can solve all homological equations by setting as zero some higher order terms in the normal form of the restricted dynamics. Hypothesis (H4) is used in controlling the tail of the parametrisation.

10.1 Formal solution up to degree L

The proof of Theorem 10.1 begins with determining a formal solution of the homological equations of the problem under consideration.

Proposition 10.3. *Under the assumptions of Theorem 10.1, we can determine a polynomial bundle map of degree not larger than L,* $R(s,\theta) : \mathcal{E}^1 \times \mathbb{T}^d \to \mathcal{E}^1$ *such that* $R(0,\theta) = 0$, $D_sR(0,\theta) = M_{1,1}(\theta)$ *and a polynomial bundle map of degree not larger than L,* $W : \mathcal{E}^1 \times \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n)$ *such that* $W(0,\theta) = 0$, $\Pi_{\mathcal{E}^1} D_sW(0,\theta) = \mathrm{Id}_{\mathcal{E}^1}$ *and* $\Pi_{\mathcal{E}^2} D_sW(0,\theta) = 0$ *satisfying*

$$F(W(s,\theta),\theta) = W(R(s,\theta),\theta+\omega) + \mathcal{O}(\|s\|^{L+1}).$$

Moreover $W \in C^{\infty,r}_\Gamma(\mathcal{E}^1 \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$, $R \in C^{\infty,r}_\Gamma(\mathcal{E}^1 \times \mathbb{T}^d, \mathcal{E}^1)$ *and*

$$\|F(W(s,\theta),\theta) - W(R(s,\theta),\theta+\omega)\|_{C^{\Sigma_{t,r,L}}_\Gamma} = \mathcal{O}(\varepsilon) + \mathcal{O}(\delta).$$

Furthermore, if

$$\mathcal{A}\,\mathrm{Spec}(A_{1,1})^i \cap \mathcal{A}\,\mathrm{Spec}(A_{1,1}) = \emptyset, \tag{10.1}$$

for $2 \le i \le L$*, then R is linear and equal to* $M_{1,1}(\theta)$*. More generally, if* (10.1) *holds for all i such that* $l \le i \le L$*, we can take R to be a polynomial bundle map of degree not larger than* $l-1$*.*

Proof. Since we have already translated the torus to the origin $x = 0$, we want to determine $W(s,\theta)$ and $R(s,\theta)$ of the form

$$W(s,\theta) = 0 + W_1(\theta)s + \ldots + W_L(\theta)s^{\otimes L},$$
$$R(s,\theta) = R_1(\theta)s + \ldots + R_L(\theta)s^{\otimes L}$$

and satisfying

$$F(W(s,\theta)) = W(R(s,\theta),\theta+\omega) + \mathcal{O}(\|s\|^{L+1}). \tag{10.2}$$

Using that $F(x,\theta) = M(\theta)x + F_2(\theta)x^{\otimes 2} + \ldots + F_L(\theta)x^{\otimes L} + \ldots$, we can expand formally Equation (10.2) in powers of s and find homological equations for each order $k \le L$. For $k = 1$ the condition is

$$M(\theta)W_1(\theta)s = W_1(\theta+\omega)R_1(\theta)s + \mathcal{O}(\|s\|^2). \tag{10.3}$$

As in Section 8.2 once we have $M(\theta)$ in block triangular form we can choose (and it is not the only possibility) $W_1(\theta) = (\mathrm{Id}_{\mathcal{E}^1}, 0)^\top$, that is, the canonical embedding from $\mathcal{E}^1$ to $\ell^\infty(\mathbb{R}^n)$ and $R_1(\theta) = M_{1,1}(\theta)$. This choice solves (10.3). For $k \ge 2$ the homological equations are

$$M(\theta)W_k(\theta) = W_1(\theta+\omega)R_k(\theta) + W_k(\theta+\omega)R_1^{\otimes k}(\theta) + \hat{Q}_k(\theta), \quad k \ge 2,$$

where $\hat{Q}_k(\theta)$ comes inductively and depends on $F_j(\theta)$, $j \le L$, and $W_j(\theta)$ and $R_j(\theta)$ for $j < k$.

To solve these equations, we start by re-parametrising s by $M_{1,1}^{-1}(\theta)$, that is writing $s = M_{1,1}^{-1}(\theta)\widetilde{s}$, obtaining

$$M(\theta)W_k(\theta)M_{1,1}^{-1}(\theta)^{\otimes k} = W_1(\theta+\omega)R_k(\theta)\big(M_{1,1}^{-1}(\theta)\big)^{\otimes k} + W_k(\theta+\omega) + Q_k(\theta).$$

Now we project this equation over $\mathcal{E}^1$ and $\mathcal{E}^2$ and rearrange terms to get a triangular system of equations. We write

$$W_k^j = \Pi_{\mathcal{E}^j} W_k,$$
$$Q_k^j = \Pi_{\mathcal{E}^j} Q_k,$$

for $j = 1, 2$, where $\Pi_{\mathcal{E}^j}$ is the projection $\ell^\infty(\mathbb{R}^n) \to \mathcal{E}^j$. The first equation is

$$\begin{aligned} M_{1,1}(\theta)W_k^1(\theta)\big(M_{1,1}^{-1}(\theta)\big)^{\otimes k} &- W_k^1(\theta+\omega) \\ &= R_k(\theta)M_{1,1}^{-1}(\theta)^{\otimes k} - M_{1,2}(\theta)W_k^2(\theta)M_{1,1}^{-1}(\theta)^{\otimes k} + Q_k^1(\theta), \end{aligned} \tag{10.4}$$

and the second one is

$$M_{2,2}(\theta)W_k^2(\theta)M_{1,1}^{-1}(\theta)^{\otimes k} - W_k^2(\theta+\omega) = Q_k^2(\theta). \tag{10.5}$$

Expanding these equations using the fact that $M_{1,1}(\theta) = A_{1,1} + B_{1,1}(\theta)$, $M_{2,2}(\theta) = A_{2,2} + B_{2,2}(\theta)$ and $M_{1,2}(\theta) = B_{1,2}(\theta)$, we get

$$\begin{aligned} A_{1,1}W_k^1(\theta)\big(A_{1,1}^{-1}\big)^{\otimes k} &+ \widetilde{\mathcal{T}}_1(W_k^1)(\theta) - W_k^1(\theta+\omega) \\ &= R_k(\theta)\big(M_{1,1}^{-1}(\theta)\big)^{\otimes k} - B_{1,2}(\theta)W_k^2(\theta)M_{1,1}^{-1}(\theta)^{\otimes k} + Q_k^1(\theta), \end{aligned} \tag{10.6}$$

and

$$A_{2,2}W_k^2(\theta)\big(A_{1,1}^{-1}\big)^{\otimes k} + \widetilde{\mathcal{T}}_2(W_k^2)(\theta) - W_k^2(\theta+\omega) = Q_k^2(\theta), \tag{10.7}$$

where

$$\widetilde{\mathcal{T}}_1(W_k^1)(\theta) = M_{1,1}(\theta)W_k^1(\theta)M_{1,1}^{-1}(\theta)^{\otimes k} - A_{1,1}W_k^1(\theta)\big(A_{1,1}^{-1}\big)^{\otimes k},$$
$$\widetilde{\mathcal{T}}_2(W_k^2)(\theta) = M_{2,2}(\theta)W_k^2(\theta)M_{1,1}^{-1}(\theta)^{\otimes k} - A_{2,2}W_k^2(\theta)\big(A_{1,1}^{-1}\big)^{\otimes k}.$$

Observe that

$$\|M_{1,1}(\theta) - A_{1,1}\|_{C^r_{L_\Gamma}} \le C\|F_1\|_{C^{t,r}_\Gamma},$$

and $M_{1,1}^{-1}(\theta) = A_{1,1}^{-1} + \hat{B}_{1,1}(\theta)$, where by Lemma 4.18, $\|\hat{B}_{1,1}\|_{C^r_\Gamma} = \mathcal{O}(\|B_{1,1}\|_{C^r_\Gamma})$. From these facts, a telescopic decomposition and Proposition 4.12 we get the bound

$$\|\widetilde{\mathcal{T}}_i(W)(\theta)\|_{C^r_\Gamma} \le \widetilde{C}\|F_1\|_{C^{\Sigma_{t,r}}_\Gamma}, \quad i = 1, 2.$$

To solve the triangular system for W_k^1, W_k^2, we start by solving Equation (10.7). Given $\mathcal{E}^1$, $\mathcal{E}^2$ arbitrary subspaces of $\ell^\infty(\mathbb{R}^n)$ and $A \in L(\mathcal{E}^1, \mathcal{E}^1)$, $B \in L(\mathcal{E}^2, \mathcal{E}^2)$, we define the Sylvester operator $\mathcal{S}_{B,A}(C)$ for $C \in C^r_{L^k_\Gamma}(\mathbb{T}^d, L^k(\mathcal{E}^1, \mathcal{E}^2))$ as

$$\mathcal{S}_{B,A}(C)(\theta) = BC(\theta-\omega)A^{\otimes k}.$$

With this definition, we can rewrite Equation (10.7) as

$$\mathcal{S}_{A_{2,2},A_{1,1}^{-1}}(W_k^2)(\theta+\omega) - W_k^2(\theta+\omega) + \widetilde{\mathcal{T}}_2(W_k^2)(\theta) = Q_k^2(\theta).$$

Since $\|\widetilde{\mathcal{T}}_2(W_k^2)\|_{C_\Gamma^{k,r}}$ is small and all operators involved in the r.h.s. of this equation are linear, we can solve it if we can solve $(\mathcal{S}_{A_{2,2},A_{1,1}^{-1}} - \mathrm{Id})(W_k^2)(\theta+\omega) = Q_k^2(\theta)$, in turn, we can solve this equation if 1 is not in the spectrum of the Sylvester operator $\mathcal{S}_{A_{2,2},A_{1,1}^{-1}}$. Notice that Q_k^2 is determined by simple operations on C_Γ^r functions which maintain the C_Γ^r character. Then $Q_k^2 \in C^r_{L^2_\Gamma}$ (below we show an explicit expression for Q_2.)

Proposition 6.29 in Section 6.8 states that

$$\operatorname{Spec}_\Gamma \mathcal{S}_{A_{2,2},A_{1,1}^{-1}} \subseteq \mathcal{A}\operatorname{Spec}_\Gamma(A_{2,2}) \cdot \mathcal{A}\operatorname{Spec}_\Gamma\left(A_{1,1}^{-1}\right)^k,$$

and by Hypothesis (H5), the r.h.s of this expression does not contain values in the unit circle for $2 \le k \le L$. Therefore we can solve Equation (10.7) and find $W_k^2 \in C_\Gamma^r(\mathbb{T}^d, L_\Gamma^2(\mathcal{E}^1, \ell^\infty(\mathbb{R}^n)))$.

We have two options to solve (10.6). If we can solve the following equation

$$\begin{aligned}(\mathcal{S}_{A_{1,1},A_{1,1}^{-1}} - \mathrm{Id})(W_k^1)(\theta+\omega) &+ \widetilde{\mathcal{T}}_1(W_k^1)(\theta+\omega)\\ &= -B_{1,2}(\theta)W_k^2(\theta)M_{1,1}^{-1}(\theta)^{\otimes k} + Q_k^1(\theta),\end{aligned} \tag{10.8}$$

i.e. if 1 is not in the spectrum of $(\mathcal{S}_{A_{1,1},A_{1,1}^{-1}}, C^r_{L^k_\Gamma})$, we set $W_k^1(\theta)$ equal to the solution of (10.8), and $R_k \equiv 0$. Observe that this spectral condition holds for k large enough by Proposition 6.29. Observe that to solve (10.8) we use the fact that $\mathcal{S}_{A_{1,1},A_{1,1}^{-1}} - \mathrm{Id}$ can be inverted and the smallness and linearity of $\widetilde{\mathcal{T}}_1(W_k^1)$ as before.

If this is not the case and we can not solve equation (10.8), we can solve equation (10.6) by setting $W_k^1 \equiv 0$ and R_k as

$$R_k(\theta) = B_{1,2}(\theta)W_k^2(\theta) - Q_k^1(\theta)(M_{1,1}(\theta))^{\otimes k}.$$

Obviously there are many other possibilities to solve equations (10.6) and (10.7). The remaining part of the proof is devoted to prove that if we use the choices indicated above to solve the equations, the terms W_k, R_k for $k \ge 2$ obtained are small if δ is small. Indeed, this can be shown inductively while proving Q_k are small. When $k = 2$, the homological equation is

$$\begin{aligned}M(\theta)W_2(\theta)s^{\otimes 2} &+ \left[N(W_1(\theta)s + W_2(\theta)s^{\otimes 2})\right]_{\mathcal{O}(2)}\\ &= W_1(\theta+\omega)R_2(\theta)s^{\otimes 2} + W_2(\theta+\omega)\left(M_{1,1}(\theta)s\right)^{\otimes 2}\end{aligned}$$

where the subscript $\mathcal{O}(2)$ means we only keep the order 2 terms. From this equation we find

$$Q_2(\theta) = -\left[N(W_1(\theta)s + W_2(\theta)s^{\otimes 2})\right]_{\mathcal{O}(2)},$$

which is of order δ since N is of order δ. This implies R_2 and W_2 are also small by the following lemma.

Lemma 10.4. *Let A, B be linear operators defined on a Banach space E, A invertible and $f \in E$. If $\|B\|$ is small enough then the equation*

$$Ax + Bx = f$$

has a unique solution, $g \in E$. Furthermore,

$$\|g\| \leq \frac{1}{1 - \|A^{-1}B\|} \|A^{-1}f\|.$$

The proof is straightforward. The smallness of W_2 and R_2 follows considering the Sylvester operator $\mathcal{S}_{A_{2,2}^{-1}, A_{2,2}}$ as A and Q_2 as f for the second equation and $\mathcal{S}_{A_{1,1}^{-1}, A_{1,1}}$ for the first equation. Observe that the case when the Sylvester operator is not invertible also implies W_2 and R_2 are small, since in this case $W_2 = 0$ and R_2 is a sum of small terms. When $k > 2$, $B = \mathcal{T}_1$ or $B = \mathcal{T}_2$ depending on which equation we are solving. As before, if we can not solve Equation (10.8), W_k and R_k are small since in this case $W_k = 0$ and R_k is inductively a small term.

□

Chapter 11

Non-resonant manifolds II: Regularity

11.1 Bounds for shifted iterated maps

In this chapter we will introduce a notation for skew products and skew product-like composition of anisotropic functions. This kind of expressions will appear in Section 11.3 when defining certain inverses of operators via formal series. To prove the convergence of these series we will need sharp bounds of these expressions.

Given $M \in C^r(\mathbb{T}^d, L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)))$ and $R \in C_\Gamma^{\Sigma_{t,r}}(\ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ we define

$$\begin{aligned} M^{[k]}(\theta) &= M(\theta) \cdots M(\theta + (k-1)\omega), \qquad && k \geq 1, \quad M^{[0]}(\theta) = \mathrm{Id}, \\ R^{[k]}(s,\theta) &= R(R^{[k-1]}(s,\theta), \theta + (k-1)\omega), \qquad && k \geq 1, \quad R^{[0]}(s,\theta) = s. \end{aligned} \tag{11.1}$$

These expressions are also called co-cycles over rotations in the literature. The bounds for $M^{[k]}(\theta)$ when $M = M_0 + \widetilde{M}(\theta)$ and M_0 is uncoupled are already established in Lemma 4.19.

The sharp bounds on $R^{[k]}$, based on the ideas in [dlLMM86] (page 574), are obtained by determining bounds for the number of terms and factors when differentiating expressions similar to $R^{[k]}$.

Let $N > 0$ be an arbitrary natural number. For technical reasons we will be interested in grouping compositions of R in $R^{[k]}$ in groups of N maps. For that, given k let

$$k = pN + q$$

with p, $q \in \mathbb{N}$, $q < N$, and define

$$\begin{aligned} \widetilde{R}(s,\theta) &:= R^{[N]}(s,\theta), \\ \widetilde{R}^{[p]} &:= \widetilde{R}\left(\widetilde{R}^{[p-1]}(s,\theta), \theta + (p-1)N\omega\right). \end{aligned} \tag{11.2}$$

Observe that the definition of $\widetilde{R}^{[p]}$ is different from the definition of $R^{[k]}$, as there is a different shift in the angles. This makes the definitions compatible.

Note that with this definition of $\widetilde{R}^{[p]}$, we can show inductively that

$$R^{[pN]} = \widetilde{R}^{[p]}. \tag{11.3}$$

Finally define (as an abuse of notation):

$$\widetilde{R}^{[a]} \circ R^{[b]} := \widetilde{R}^{[a]}(R^{[b]}(s,\theta), \theta + (b-1)\omega), \quad \text{for any } b < N. \tag{11.4}$$

Since $k = pN + q$, statements (11.3) and (11.4) imply the identity

$$R^{[k]} = \widetilde{R}^{[p]} \circ R^{[q]}.$$

Lemma 11.1. *Let $f(x,\theta)$ and $g(x,\theta)$ be such that $f(g^{[k]}(x,\theta),\theta)$ is well defined for any $k \geq 1$ and assume that they are differentiable enough so that the j-th derivative with respect to θ of the above expression makes sense. Let $A(k,j)$ be the maximum number of terms and $B(k,j)$ be the number of factors in each term in the expansion of $D^j_\theta(f(g^{[k]}(x,\theta),\theta))$ expressed in terms of derivatives of g and f. Then*

$$\begin{aligned}
A(k,j) &\leq (k+1)^{(j-1)j/2}(k+2)^{j-1}, && j > 1,\\
B(k,j) &\leq (k+1)^j, && j > 1,\\
A(k,j) &\leq k+1, && j = 1,\\
B(k,j) &\leq k+1, && j = 1.
\end{aligned}$$

Proof. We will prove this result by analysing how the number of terms and factors increase after differentiation with respect to θ. We will start by counting the terms and factors of the first derivative, to get the initial conditions of a recurrence for the number of terms and factors.

When we differentiate for the first time, we get

$$D_\theta f(g^{[k]}) = D_x f(g^{[k]}(x,\theta),\theta) D_\theta(g^{[k]}(x,\theta)) + D_\theta f(g^{[k]}(x,\theta),\theta). \tag{11.5}$$

If we expand the term $D_\theta(g^{[k]}(x,\theta))$, we can use a similar identity k times. Observe that each time we differentiate $g^{[k]}(x,\theta)$ with respect to θ to get the complete expansion we get the terms

$$\begin{aligned}
&D_x g(g^{[k-1]}(x,\theta), \theta + (k-1)\omega) D_\theta g^{[k-1]}(x,\theta),\\
&D_\theta g(g^{[k-1]}(x,\theta), \theta + (k-1)\omega).
\end{aligned}$$

The second term has no more derivatives left to expand and the first term still has one derivative left. Thus when we expand $D_\theta g^{[k-1]}(x,\theta)$ we get two new terms, one with pending expansions and one with no expansions left. We can repeat this process until we have no derivatives left, i.e. when we have differentiated k times and the last product is then $D_\theta g(x,\theta)$. Thus after expanding all derivatives in (11.5), we have $k+1$ terms, each term having a decreasing number of factors. The term with the least number of factors has 1 factor and the term with the most number of factors has $k+1$ factors. Therefore the first derivative has $k+1$ terms and the term with the most number of factors has $k+1$ factors. This proves the bound for $j = 1$.

When we have a term with m factors from the differentiation of $f(g^{[k]}(x,\theta),\theta)$ a certain number of times with respect to θ, we can assume each factor generates at most the same number of factors and terms as $f(g^{[k]}(x,\theta),\theta)$ does under differentiation. This is an upper bound, since a generic factor in this expression will have one of the following forms:

$$\begin{aligned} &D_x^a D_\theta^b g(g^{[c-1]}(x,\theta),\theta+(c-1)\omega), \\ &D_x^a D_\theta^b f(g^{[c]}(x,\theta),\theta+c\omega), \qquad c\le k. \end{aligned}$$

Differentiating such a factor generates at most the same number of factors and terms as differentiating $f(g^{[k]}(x,\theta),\theta)$. Therefore if we have m factors, differentiating this expression with respect to θ generates m terms (the same as the number of factors), each factor then generates at most $k+1$ new terms (following the rule for differentiating (11.5)) and each new term gets an addition of at most $k+1$ new factors. Thus we will have $m(k+2)$ terms arising from each factor in a generic term of m factors of $D_\theta^j f(g^{[k]}(x,\theta),\theta)$, and each term will have at most $m(k+1)$ factors.

Using this argument, we can write the following recurrences for the number of terms (A) and factors (B):

$$\begin{aligned} &A(k,j+1)\le A(k,j)B(k,j)(k+2), && j>1, \quad A(k,1)=k+1, \\ &B(k,j+1)\le B(k,j)(k+1), && j>1, \quad B(k,1)=k+1. \end{aligned}$$

Since this recurrence is of positive terms, we can bound it by the solution of the recurrence with an equality:

$$a(k,j+1)=a(k,j)b(k,j)(k+2), \qquad j>1, \quad a(k,1)=k+1, \tag{11.6}$$

$$b(k,j+1)=b(k,j)(k+1), \qquad j>1, \quad b(k,1)=k+1. \tag{11.7}$$

Expanding Equation (11.7) we get

$$b(k,j+1)=b(k,j)(k+1)=b(k,j-1)(k+1)^2=\ldots=(k+1)^{j+1}.$$

We can now substitute this expansion into Equation (11.6),

$$a(k,j+1)=a(k,j)(k+1)^j(k+2),$$

and expand

$$\begin{aligned} a(k,j+1)&=a(k,j)(k+1)^j(k+2)=a(k,j-1)(k+1)^{j-1}(k+1)^j(k+2)^2 \\ &=\ldots=(k+1)^{(j+1)j/2}(k+2)^j. \end{aligned}$$

This proves that

$$\begin{aligned} &A(k,j)\le a(k,j), \quad j>1, \quad A(k,1)\le(k+1) \\ &B(k,j)\le b(k,j), \end{aligned}$$

as we wanted.

□

The next lemma obtains analogous bounds as Lemma 11.1 for the derivative $D_x^i D_\theta^j(f(g^{[k]}(x,\theta),\theta))$.

Lemma 11.2. *Let $f(x,\theta)$ and $g(x,\theta)$ be such that $f(g^{[k]}(x,\theta),\theta)$ is well defined for any $k \geq 1$ and assume they are differentiable enough so that the (k,i)-th derivatives of the above expression make sense. Let $\overline{A}(k,i)$ be the maximum number of terms and $\overline{B}(k,i)$ be the maximum number of factors in each term in the expansion of $D_x^i D_\theta^j(f(g^{[k]}(x,\theta),\theta))$ expressed in terms of derivatives of f and g. Then*

$$\begin{aligned}
\overline{A}(k,i,j) &\leq (k+1)^{i(i-1)/2}(k+1)^{j(i-1)}(k+1)^{j(j-1)/2}(k+2)^{j-1}, && j>1,\\
\overline{B}(k,i,j) &\leq (k+1)^{i}(k+1)^{j}, && j>1,
\end{aligned}$$

and

$$\begin{aligned}
\overline{A}(k,i) &\leq (k+1)^{i(i-1)/2}, && j=0,\\
\overline{B}(k,i) &\leq (k+1)^{i}, && j=0,\\
\overline{A}(k,i) &\leq (k+1)^{i(i-1)/2}(k+1)^{i+1}, && j=1,\\
\overline{B}(k,i) &\leq (k+1)^{i+1}, && j=1.
\end{aligned}$$

Proof. We will follow the same method as in the proof of Lemma 11.1. We will differentiate with respect to x an expression coming from $D_\theta^j f(g^{[k]}(x,\theta),\theta)$ and count the number of terms appearing in it. Differentiating $f(g^{[k]}(x,\theta),\theta)$ with respect to x generates one term with $k+1$ factors. The number of factors depends on the iteration depth, $k+1$ in this case, and the number of terms is the same as the number of factors, 1 in this case. A generic term of $D_x^i D_\theta^j f(g^{[k]}(x,\theta),\theta)$ will obviously be the product of terms, of one of the following two types:

$$\begin{aligned}
&D_x^a D_\theta^b g(g^{[c]}(x,\theta),\theta), && c\leq k-1,\\
&D_x^a D_\theta^b f(g^{[c]}(x,\theta),\theta), && c\leq k.
\end{aligned}$$

This means that when differentiating with respect to x, each factor generates $k+1$ new factors and one new term. Thus the following recurrences hold, assuming $j>1$.

$$\begin{aligned}
\overline{A}(k,i+1) &\leq \overline{A}(k,i)\overline{B}(k,i),\\
\overline{A}(k,0) &= (k+1)^{j(j-1)/2}(k+2)^{j-1}, \qquad (11.8)\\
\overline{B}(k,i+1) &\leq (k+1)\overline{B}(k,i),\\
\overline{B}(k,0) &= (k+1)^{j}. \qquad (11.9)
\end{aligned}$$

We can follow the same bounding procedure as in Lemma 11.1. First consider the recurrence with an equality,

$$\begin{aligned}
\overline{a}(k,i+1) &= \overline{a}(k,i)\overline{b}(k,i),\\
\overline{a}(k,0) &= (k+1)^{j(j-1)/2}(k+2)^{j-1}, \qquad (11.10)\\
\overline{b}(k,i+1) &= \overline{b}(k,i)(k+1),\\
\overline{b}(k,0) &= (k+1)^{j},, \qquad (11.11)
\end{aligned}$$

and now solve Equation (11.11):

$$\overline{b}(k,i+1) = \overline{b}(k,i)(k+1) = \overline{b}(k,i-1)(k+1)^2 = \ldots = (k+1)^{i+1}(k+1)^{j}.$$

Replace this expression in Equation (11.10)

$$\begin{aligned}\overline{a}(k,i+1) &= \overline{a}(k,i)(k+1)^{i+1}(k+1)^{j} = \ldots = \overline{a}(k,0)(k+1)^{i(i+1)/2}(k+1)^{j(i+1)} \\ &= (k+1)^{(i+1)(i+2)/2}(k+1)^{j(i+1)}(k+1)^{j(j-1)/2}(k+2)^{j-1}.\end{aligned}$$

Finally,

$$\begin{aligned}\overline{A}(k,i) &\le \overline{a}(k,i), \\ \overline{B}(k,i) &\le \overline{b}(k,i),\end{aligned}$$

for the $j > 1$ case, as we wanted.

When $j = 0$, the initial conditions in the recurrences (11.8), (11.9) are $\overline{A}(k,0) = 1$, $\overline{B}(k,0) = 1$. We can obtain teh bound as before

$$\begin{aligned}\overline{A}(k,i) &\le (k+1)^{i(i-1)/2}, \\ \overline{B}(k,i) &\le (k+1)^{i}.\end{aligned}$$

When $j = 1$, the initial conditions in the recurrences (11.8), (11.9) are $\overline{A}(k,0) = k+1$, $\overline{B}(k,0) = k+1$. We can obtain the bound as before

$$\begin{aligned}\overline{A}(k,i) &\le (k+1)^{i(i-1)/2}(k+1)^{i+1}, \\ \overline{B}(k,i) &\le (k+1)^{i+1}.\end{aligned}$$

□

In the case where $f = \widetilde{R}^{[p]}$, $g = \widetilde{R}^{[q]}$ the proof would be (formally) slightly different due to the addition of shifts in the angles. Since shifting angles does not add differentiation terms or factors, the number of terms can be bound exactly in the same way using p instead of k in the formulas.

11.2 Bounds for the iterated local dynamics in Γ-norms

In this section we want to determine the tail terms in the parametrisation of the non-resonant invariant manifold. To do so, we want to bound $\|D_x^i D_\theta^j R^{[k]}\|_{C^0_\Gamma}$ in such a way that a particular series to be defined later is absolutely convergent in Γ-norm. Since we are bounding in Γ-norm, to get contraction properties we have to work with a high enough iterate to turn uncoupled C^0 linear contractions into Γ linear contractions. To find the minimal iterate we need for this procedure, let N be such that $\Gamma(0)^{-4}\|A^{-1}\|^N\|A_{1,1}\|^{LN} < \tilde{\rho} < 1$, for some $\widetilde{\rho}$.

Recall that we define the restricted dynamics on the invariant manifold as

$$R = \sum_{j=1}^{L} R_k(\theta) s^{\otimes j},$$

which is a contraction in C^0 if the perturbation ε and scaling parameter δ are small enough as the following lemma shows. The following lemma is the first step in the proof of existence of a C^0 tail.

Lemma 11.3. *If ε and δ are small enough, we can bound*

$$\|R^{[k]}\|_{C^0} \leq (\|A_{1,1}\| + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta))^k$$

in $B(0,1) \times \mathbb{T}^d \subset \ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d$.

Proof. Observe that we can write $R(s,\theta) = M_{1,1}(\theta)s + \widetilde{Q}(s,\theta)$, with $\widetilde{Q}(s,\theta) = \sum_{j=2}^{L} R_j(\theta)s^{\otimes j}$ where R_j are the multilinear terms determined in Chapter 10. We have the bounds $\|M_{1,1}(\theta)\|_{C^0} \leq \|A_{1,1}\| + \mathcal{O}(\varepsilon)$ and $\|\widetilde{Q}\|_{C^0} = \mathcal{O}(\delta)\|s\|^2$ by the results in Chapter 10. It is then clear that if δ and ε are small enough,

$$\begin{aligned}\|R\|_{C^0} &\leq \sup_{\|s\|\leq 1} (\|A_{1,1}\| + \mathcal{O}(\varepsilon))\|s\| + \mathcal{O}(\delta)\|s\| \\ &\leq \sup_{\|s\|\leq 1} ((\|A_{1,1}\| + \mathcal{O}(\varepsilon))\|s\| + \mathcal{O}(\delta)\|s\|) < \rho,\end{aligned}$$

with $\rho = \|A_{1,1}\| + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta)$, proving the case $k=1$. Thus the map $(s,\theta) \mapsto R^{[2]}(s,\theta)$ is well-defined. Now we can bound $\|R^{[j]}\|_{C^0}$ by induction:

$$\begin{aligned}\|R^{[j+1]}\|_{C^0} &= \|R(R^{[j]}(s,\theta), \theta + j\omega)\|_{C^0} \\ &\leq (\|A_{1,1}\| + \mathcal{O}(\varepsilon))\|R^{[j]}\|_{C^0} + \mathcal{O}(\delta)\|R^{[j]}\|_{C^0}\end{aligned}$$

which by induction hypothesis satisfies

$$\|R^{[j+1]}\|_{C^0} \leq \rho^{j+1}\|s\|.$$

as desired.

□

Of course we also need bounds for higher regularity functions. Thus we need to bound their derivatives by following a similar scheme of proof. In this case the bound will be in Γ norms, since it is the bound needed when differentiating.

Lemma 11.4. *If ε and δ are small enough and for $(i,j) \in \Sigma_{t,r}$ we can bound*

$$\begin{aligned}&\|D_x^i D_\theta^j R^{[k]}(x,\theta)\|_{C^0_\Gamma} \\ &\qquad \leq (p+1)^{i(i-1)/2}(p+1)^{j(i-1)}(p+1)^{j(j-1)/2}(p+2)^{j-1}\Gamma(0)^{-2p}(p+1)^{jp} \\ &\qquad\quad \times (\|A_{1,1}\| + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta))^{pN},\end{aligned}$$

where $k = pN + q$ and C is independent of p.

Proof. Recall that we write $R^{[k]}(x,\theta) = R^{[pN+q]}(x,\theta) = R^{[pN]}(R^{[q]}(x,\theta), \theta + (q-1)\omega)$ and denote $\widetilde{R}(x,\theta) = R^{[N]}(x,\theta)$. Thus we can write $R^{[k]}(x,\theta) = \widetilde{R}^{[p]}(R^{[q]}(x,\theta), \theta + (q-1)\omega)$. Remember that N and L are fixed and observe that if ε and δ are small enough, $\|R\|_{C^0} \leq \rho\|x\|$ as a particular case of Lemma 11.3 which implies all compositions are well-defined. Finally observe that

$$\|M_{1,1}^{[N]}\|_{C^0_\Gamma} \leq \Gamma(0)^{-1}\left(\|M_0\| + \Gamma(0)^{-1}\|\widetilde{M}\|_{C^0_{L_\Gamma}}\right)^N$$

by Lemma 4.19.

The number of terms and factors appearing in the derivative $D_x^i D_\theta^j R^{[k]}$ (in terms of derivatives of $\widetilde{R}$ and $R^{[q]}$) can be bounded by Lemma 11.2, hence the number of terms can be bounded by $(p+1)^{i(i-1)/2}(p+1)^{j(i-1)}(p+1)^{j(j-1)/2}(p+2)^{j-1}$ and the number of factors by $(p+1)^{i+j+1}$. Each factor in a generic term of $D_x^i D_\theta^j[\widetilde{R}^{[p]}(R^{[q]})]$ is of one of the following three types:

1. $D_\theta^b \widetilde{R}^{[p]}(R^{[q]})$, $\qquad b \le j$,
2. $D_x^a D_\theta^b \widetilde{R}(\widetilde{R}^{[c]}(R^{[q]}))$, $\qquad a \le i+j$, $b \le j$, $c < p$, $a \ge 1$, $b \ge 1$,
3. $D_x^a D_\theta^b R^{[q]}$, $\qquad a \le i$, $b \le j$.

We can bound factors of the first type by either a small constant related to the scaling parametre when $a \ge 1$ or using Lemma 4.19 when $a = 0$. Recall that Lemma 4.19 bounds expressions of the form $M^{[k]}(\theta)$ where $M(\theta) = M_0 + \widetilde{M}(\theta)$, and this is essentially equivalent to bounding expressions of the type $R^{[p]}$, since we can write $\widetilde{R} = M_{1,1}^{[N]}(\theta)x + Q(x,\theta)$, where $Q(x,\theta)$ has order larger than 2 and has small coefficients (of order of the scaling parameter δ) in all the considered norms and $M_{1,1}^{[N]}$ is a contraction in L_Γ-norm, since we have chosen N satisfying this propperty.

We can bound factors of the second type likewise. When $a \ge 1$ or $b \ge 1$ the derivative of R is of the order of the scaling parameter or the perturbation term respectively.

Finally, factors of type 3 can be bounded directly by a constant C_q. Observe that this type of factor appears only once in each term.

Let μ be the worst bound of a factor in the previous discussion (which is smaller than 1 with the possible exception of the constant bounding type 3, which we will consider this separately). Such a bound comes necessarily from the factor having no derivatives of R and hence comes from bounding directly by using Lemma 4.19, since all other factors are bound by the scaling or perturbation parameters and are thus as small as needed. Since the factor (3) only appears in the term having the largest number of factors, we can bound the whole expression by bounding all terms with the term having the least amount of factors (which is p factors) and using the largest bound (the one not involving any derivatives, since the others introduce scaling factors which decrease the bound).

Therefore we can bound $D_x^i D_\theta^j(\widetilde{R}^{[p]}(R^{[q]}(x,\theta), \theta + (q-1)\omega))$ by

$$
\begin{aligned}
&(p+1)^{i(i-1)/2}(p+1)^{j(i-1)}(p+1)^{j(j-1)/2}(p+2)^{j-1}\left(\|M_{1,1}^{[N]}\|_{C^0_\Gamma} + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta)\right)^p C_q \\
&\qquad \le (p+1)^{i(i-1)/2}(p+1)^{j(i-1)}(p+1)^{j(j-1)/2}(p+2)^{j-1} \\
&\qquad\quad \times \left(\left(\|A_{1,1}\| + \mathcal{O}(\|B_{1,1}\|_{C^r_{L_\Gamma}})\right)^N + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta)\right)^p C_q \\
&\qquad \le (p+1)^{i(i-1)/2}(p+1)^{j(i-1)}(p+1)^{j(j-1)/2}(p+2)^{j-1}\left(\|A_{1,1}\| + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta)\right)^{pN} C_q
\end{aligned}
$$

with $\widetilde{\rho} = \rho + \mathcal{O}(\delta) < 1$ when ε and δ are small enough, where we have used Lemma 11.2 to bound the number of terms in this expression.

□

11.3 Determining the tail of the parametrisation in Theorem 10.1

The results in Chapters 7, 8 and 10 determine terms W_j and R_j, $0 \leq j \leq L$, such that $W^{\leq}$ and R defined as

$$\begin{aligned} W^{\leq}(s,\theta) &= \sum_{j=0}^{L} W_j(\theta) s^{\otimes j}, \\ R(s,\theta) &= \sum_{j=1} R_j(\theta) s^{\otimes j} \end{aligned}$$

satisfy the invariance equation for a non-resonant manifold up to order L. Note that in this and the next section we denote by s the variable in $\mathcal{E}^1$.

Now assume the parametrisation we are determining is written as

$$W = W^{\leq} + W^{>}, \tag{11.12}$$

with $W^{>} \in C_{\Gamma}^{\Sigma_{t,r},L}\big(B(0,1) \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)\big)$

The term $W^{>}$ is the unknown tail of the parametrisation, $W^{\leq}$ is the formal solution already established and the goal of this section is finding $W^{>}$ and determining its regularity.

We can write

$$F(x,\theta) = M(\theta)x + N(x,\theta),$$

and write the invariance equation for W as

$$\begin{aligned} M(\theta)\left(W^{\leq}(s,\theta) + W^{>}(s,\theta)\right) &+ N\left(W^{\leq}(s,\theta) + W^{>}(s,\theta), \theta\right) \\ &= W^{\leq}\left(R(s,\theta), \theta+\omega\right) + W^{>}(R(s,\theta), \theta+\omega). \end{aligned} \tag{11.13}$$

Since $M(\theta)$ is invertible by combining Hypotheses (H2) and (H3) in Theorem 10.1, we can write Equation (11.13) as

$$\begin{aligned} W^{>}(s,\theta) &- M(\theta)^{-1} W^{>}(R(s,\theta), \theta+\omega) \\ &= M(\theta)^{-1} W^{\leq}(R(s,\theta), \theta+\omega) - W^{\leq}(s,\theta) \\ &\quad - M(\theta)^{-1} N(W^{\leq}(s,\theta) + W^{<}(s,\theta), \theta). \end{aligned} \tag{11.14}$$

Given $W^{\leq}$ and R given by Proposition 10.3 we define

$$\mathcal{G}, \mathcal{H} : C_{\Gamma}^{\Sigma_{t,r},L}(B(0,1) \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) \to C_{\Gamma}^{\Sigma_{t,r},L}(B(0,1) \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)),$$

as

$$\begin{aligned} \mathcal{G}(W)(s,\theta) &= W(s,\theta) - M(\theta)^{-1} W(R(s,\theta), \theta+\omega). \qquad (11.15) \\ \mathcal{H}(W)(s,\theta) &= M(\theta)^{-1} W^{\leq}(R(s,\theta), \theta+\omega) \\ &\quad - W^{\leq}(s,\theta) - M(\theta)^{-1} N((W^{\leq} + W)(s,\theta), \theta). \end{aligned}$$

We can write (11.14) as

$$\mathcal{G}(W^>)(s,\theta) = \mathcal{H}(W^>)(s,\theta), \tag{11.16}$$

or equivalently, assuming $\mathcal{G}$ is invertible,

$$W^>(s,\theta) = \mathcal{G}^{-1}\mathcal{H}(W^>)(s,\theta).$$

If we can prove $\mathcal{G}$ is an invertible operator and the Lipschitz constant of the composition $\mathcal{G}^{-1}\mathcal{H}$ is smaller than 1, we will find $W^>$ satisfying (11.16) using a fixed point argument. In the fixed point process we will lose some regularity of $W^>$ with respect to the regularity of F, but we will recover the lost derivative in the next section.

We need the following bound.

Lemma 11.5. *Given $\eta \in C_\Gamma^{\Sigma_{t,r},L}(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}^1\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $t \geq 2$ we can bound*

$$\|D_\theta^a D_x^b \eta(x,\theta)\|_\Gamma \leq \frac{1}{(L-b)_+!}\|\eta\|_{C_\Gamma^{\Sigma_{t,r},L}}\|x\|^{(L-b+1)_+}, \quad (b,a)\in\Sigma_{t,r},$$

where $(a)_+ = \max(0,a)$.

Proof. Since $D_x^b\eta(0,\theta) = 0$, $\forall\theta\in\mathbb{T}^d$ for $b \leq L$, we can use Taylor's expansion around $x = 0$ to write

$$D_\theta^a D_x^b \eta(x,\theta) = \frac{1}{(L-b-1)!}\int_0^1 (1-t)^{L-b-1} D_\theta^a D_x^L \eta(tx,\theta) x^{\otimes(L-b)}\, dt.$$

We can bound

$$\|D_\theta^a D_x^L \eta(tx,\theta)\|_\Gamma \leq \|\eta\|_{C_\Gamma^{\Sigma_{t,r},L}}\|x\|,$$

since

$$\left\|\frac{D_\theta^a D_x^L \eta(tx,\theta)}{tx} tx\right\|_\Gamma \leq \|\eta\|_{C_\Gamma^{\Sigma_{t,r},L}}\|x\|$$

by the definition of the $C_\Gamma^{\Sigma_{t,r},L}$ norm and the result follows.

□

Proposition 11.6. *The operator*

$$\mathcal{G} : C_\Gamma^{\Sigma_{t,r},L}(B(0,1)\subset\mathcal{E}^1\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) \to C_\Gamma^{\Sigma_{t,r},L}(B(0,1)\subset\mathcal{E}^1\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$$

defined as

$$\mathcal{G}(W)(s,\theta) = W(s,\theta) - M(\theta)^{-1}W(R(s,\theta),\theta+\omega)$$

is a well-defined injective linear operator.

Proof. First observe that $t-1 \geq L$ and that $(x,\theta) \mapsto W^{>}(R(x,\theta),\theta+\omega)$ is well-defined and stays in the domain of definition since $\|R\|_{C^0} < 1$. Also, we already established that $M(\theta)$ is invertible for all θ. Observe also that the image of an L-flat function under the operator $\mathcal{G}$ is also L-flat. Hence, to prove the operator is well-defined we need to check the norm of the image in the target space is finite.

The only term in the norm which requires attention is

$$\frac{\|D_\theta^i D_x^L \left(M(\theta)^{-1} W(R(s,\theta),\theta+\omega)\right)\|}{\|s\|}.$$

Expanding the derivative we have terms with $D_\theta^j D_x^k W(R(s,\theta),\theta+\omega)$ with $j \leq i$ and $k < L$ which can be dealt with the Mean Value Theorem because $D_\theta^j D_x^k W(0,\theta+\omega) = 0$ and $\|R(s,\theta)\|/\|s\|$ is bounded and terms with $D_\theta^j D_s^L W(R(s,\theta),\theta+\omega)$ which can be bounded by

$$\frac{\|D_\theta^j D_s^L W(R(s,\theta),\theta+\omega)\|}{\|R(s,\theta)\|}\|R(s,\theta)\| \leq \|W\|_{C_\Gamma^{\Sigma_{t,r},L}} \left(\|A_{1,1}\| + \mathcal{O}(\delta) + \mathcal{O}(\varepsilon)\right)\|\|s\|,$$

for $s \neq 0$. Recall that $R(s,\theta) = A_{1,1}s + \text{h.o.t.}$ and $A_{1,1}$ is invertible.

To prove the operator is injective we will determine its kernel. Consider the projections in stable and unstable bundles. Write $W = (W^1, W^2)$ where $W^1 = \Pi_{\mathcal{E}^1} W$, $W^2 = \Pi_{\mathcal{E}^2} W$. Since $M(\theta)^{-1}$ is triangular in this decomposition (since $M(\theta)$ is) we can write $\mathcal{G}$ in components as

$$\begin{aligned}
\mathcal{G}^1(W)(s,\theta) &= W^1(s,\theta) - M_{1,1}^{-1} W^1(R(s,\theta),\theta+\omega) + (M_{1,1}^{-1} M_{1,2} M_{2,2}^{-1})(\theta) W^2(R(s,\theta),\theta+\omega) \\
\mathcal{G}^2(W)(s,\theta) &= W^2(s,\theta) - M_{2,2}^{-1}(\theta) W^2\left(R(s,\theta),\theta+\omega\right),
\end{aligned}$$

We can solve $\mathcal{G}^2(W) = 0$ iteratively by setting

$$W^2(s,\theta) = \lim_{k\to\infty} M_{2,2}^{-1}(\theta) \cdots M_{2,2}^{-1}(\theta+(k-1)\omega) W^2(R^{[k]}(s,\theta),\theta+(k-1)\omega).$$

Observe that the limit is actually 0, which can be shown using a Taylor expansion like above and the bound $\|M_{2,2}^{-1}\|\|M_{1,1}\|^{L+1} < \|M_0^{-1}\|\|M_{1,1}\|^{L+1} < 1$. Indeed,

$$\begin{aligned}
&D_\theta^p D_x^q M_{2,2}^{[k]}(\theta) W^2(R^{[k]}(s,\theta),\theta+k\omega) \\
&\quad = \sum_j C D_\theta^{p-m} M_{2,2}^{[k]}(\theta) D_x^q D_\theta^j (W^2 \circ R^{[k]}) \\
&\quad = \sum_{\substack{0\leq m\leq p \\ (\beta,\alpha)\in\Sigma_{q,m}^* \\ i_1\ldots+i_\beta = m-\alpha \\ j_1+\ldots+j_\beta = q-\beta}} C D_\theta^{p_1} M_{2,2}^{-1} \cdots D_\theta^{p_k} M_{2,2}^{-1} C D_\theta^\alpha D_x^\beta W^2 \circ R^{[k]} D_\theta^{i_1} D_x^{j_1} R^{[k]} \cdots D_\theta^{i_\beta} D_x^{j_\beta} R^{[k]}.
\end{aligned}$$

Since for $\eta \in C_\Gamma^{\Sigma_{t,r},L}(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}^1\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ we can bound $\|D_\theta^\alpha D_x^\beta \eta(x,\theta)\|_{C_\Gamma^{\Sigma_{t,r}}} \leq \frac{1}{(L-\beta-1)_+!}\|\eta\|_{C_\Gamma^{\Sigma_{t,r},L}} |x|^{(L-\beta+1)_+}$ by Lemma 11.5 and $\|D_\theta^{i_p} D_x^{j_p} R^{[k]}\|_{C_\Gamma} \leq (\widetilde{p}+1)^{i(i-1)/2}(\widetilde{p}+$

$1)^{j(i-1)}(\widetilde{p}+1)^{j(j-1)/2}(\widetilde{p}+2)^{j-1}\,(\|A_{1,1}\|+\mathcal{O}(\varepsilon)+\mathcal{O}(\delta))^{\widetilde{p}N}\,C_{\widetilde{q}}$ with $k=\widetilde{p}N+\widetilde{q}$. Taking limit when $k\to\infty$ we get that $W^2=0$. In an analogous way we obtain $W^1=0$. the result follows.

□

To prove $\mathcal{G}$ is invertible we need the following auxiliary lemma.

Lemma 11.7. *The definition of* $R^{[k]}(s,\theta)$ *given in* (11.1) *is equivalent to*

$$R^{<k>}(s,\theta)=R^{<k-1>}(R(s,\theta),\theta+\omega)$$

Proof. The proof is immediate by unwrapping the compositions:

$$\begin{aligned}R^{[k]}(s,\theta)&=R(R(R(\dots R(s,\theta),\theta+\omega),\dots),\theta+(k-1)\cdot\omega),\\R^{<k>}(s,\theta)&=R(R(R(\dots R(s,\theta),\theta+\omega),\dots),\theta+(k-1)\cdot\omega),\end{aligned}$$

and observe that both coincide.

□

Proposition 11.8. *The operator* $\mathcal{G}$ *is invertible.*

Proof. Given $\eta\in C_\Gamma^{\Sigma_{t,r},L}(B(0,1)\times\mathbb{T}^d\subset\mathcal{E}^1\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))$, the operator $\mathcal{G}$ has a formal inverse given by

$$\mathcal{G}^{-1}(\eta)(s,\theta)=\sum_{k=0}^{\infty}(M^{-1})^{[k]}(\theta)\eta(R^{[k]}(s,\theta),\theta+k\omega). \tag{11.17}$$

Observe that since R is a contraction in C^0, the composition $\eta(\cdot,\theta+k\omega)\circ R^{[k]}$ is always well-defined.

A straightforward calculation using the definitions of $R^{[k]}$ and $R^{<k>}$ proves this expression is indeed a formal inverse. For completeness, we prove one of the compositions.

$$\begin{aligned}\mathcal{G}^{-1}(\mathcal{G}(\eta))(s,\theta)&=\sum_{k=0}^{\infty}(M^{-1})^{[k]}(\theta)\eta(R^{[k]}(s,\theta),\theta+k\omega)\\&\quad-\sum_{k=0}^{\infty}(M^{-1})^{[k]}(\theta)M^{-1}(\theta+k\omega)\eta(R(R^{[k]}(s,\theta),\theta+(k+1)\omega))\\&=\sum_{k=0}^{\infty}(M^{-1})^{[k]}(\theta)\eta(R^{[k]}(s,\theta),\theta+k\omega)\\&\quad-\sum_{k=0}^{\infty}(M^{-1})^{[k+1]}(\theta)\eta(R^{[k+1]}(s,\theta+(k+1)\omega))=\eta(s,\theta).\end{aligned}$$

The introduction of $R^{<k>}$ is necessary to check $\mathcal{G}(\mathcal{G}^{-1}\eta)=\eta$. To prove the inverse indeed exists we need to show the series in (11.17) is absolutely convergent in the $C_\Gamma^{\Sigma_{t,r},L}$ norm.

This is equivalent to checking the convergence of the series

$$\sum_{k=0}^{\infty} D_\theta^a D_x^b \left((M^{-1})^{[k]}(\theta)\eta(R^{[k]}(s,\theta),\theta+k\omega)\right)$$

for $(b,a) \in \widetilde{\Sigma}_{t,r}$.

According to the results of the previous section, every term in the sum is bounded by a sum of terms of the general form

$$\begin{aligned} C\|D_\theta^{a-m}(M^{-1})^{[k]}(\theta)\|_\Gamma \|D_\theta^\alpha D_x^\beta \eta(R^{[k]}(s,\theta),\theta+k\omega)\|_\Gamma \\ \times \|D_\theta^{i_1} D_x^{j_1} R^{[k]}(s,\theta)\|_\Gamma \cdots \|D_\theta^{i_\beta} D_x^{j_\beta} R^{[k]}(s,\theta)\|_\Gamma, \end{aligned}$$

with $0 \le m \le a$, $(\beta,\alpha) \in \widetilde{\Sigma}_{b,m}$, $i_1 + \ldots + i_\beta = m - \alpha$ and $j_1 + \ldots + j_\beta = b$. This can be bounded by

$$\begin{aligned} & Cn^m \Gamma(0)^{-1}(\|M_0^{-1}\| + \mathcal{O}(\|M_1\|_{C^r_{L_\Gamma}}))^k \|\eta\|_{C_\Gamma^{\Sigma_{t,r},L}} \|R^{[k]}\|^{(L-\beta+1)_+} \kappa(p,i_1,j_1)\cdots\kappa(p,i_\beta,j_\beta) \\ & \quad \times (\|A_{1,1}\| + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta))^{pN\beta} \end{aligned}$$

where

$$\kappa(p,i,j) = (p+1)^{i(i-1)/2}(p+1)^{j(i-1)}(p+1)^{j(j-1)/2}(p+2)^{j-1}$$

comes from Lemma 11.4. Hence each term in the sum can be bounded by

$$C(\|M_0^{-1}\| + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta))^{pN+q}(\|A_{1,1}\| + \mathcal{O}(\varepsilon) + \mathcal{O}(\delta))^{pN(L+1)}\|\eta\|_{C_\Gamma^{\Sigma_{t,r},L}},$$

where C depends on n, a, b polynomially, thus the series is convergent. Moreover we check that the series of the $D_\theta^i D_x^L$ derivatives of the k-th terms divided by $\|s\|$ converges. Hence we obtain norms,

$$\|\mathcal{G}^{-1}(\eta)\|_{C_\Gamma^{\Sigma_{t,r},L}} \le C\|\eta\|_{C_\Gamma^{\Sigma_{t,r},L}}$$

with C independent of k, δ and ε.

□

Once we have proved that $\mathcal{G}$ is invertible, we want to solve the fixed point equation

$$\begin{aligned} W^>(s,\theta) =& \mathcal{G}^{-1}\left(M(\theta)^{-1}W^\le(R(s,\theta),\theta+\omega)\right. \\ & \left. - W^\le(s,\theta) - M(\theta)^{-1}N(W^\le(s,\theta) + W^>(s,\theta))\right). \end{aligned}$$

We will show this problem has a unique solution in $C_\Gamma^{\Sigma_{t-1,r},L}$, and then prove this solution has a derivative with respect to s, a concept similar to the situation in Section 9.1 but with different tools.

We can decompose the operator $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$, with

$$\begin{aligned} \mathcal{H}_1(W)(s,\theta) &= -W^\le(s,\theta) + M(\theta)^{-1}W^\le(R(s,\theta),\theta+\omega) - M(\theta)^{-1}N(W^\le(s,\theta),\theta), \\ \mathcal{H}_2(W)(s,\theta) &= M(\theta)^{-1}N(W^\le(s,\theta),\theta) - M(\theta)^{-1}N((W^\le + W)(s,\theta),\theta). \end{aligned}$$

We rewrite $\mathcal{H}_1(W)$ as

$$\mathcal{H}_1(W)(s,\theta =)M(\theta)^{-1}\left[-M(\theta)W^{\leq}(s,\theta) - N(W^{\leq}(s,\theta),\theta) - W^{\leq}(R(s,\theta),\theta+\omega)\right]$$

and from this expression we readily get that $\mathcal{H}_1(W)$ is flat and $\|H_1(W)\|_{C_\Gamma^{\Sigma_{t,r},L}}$ is as small as needed.

We can write $\mathcal{H}_2(W)(x,\theta)$ as

$$\mathcal{H}_2(W)(x,\theta) = \int_0^1 M(\theta)^{-1} D_x N((W^{\leq} + \mu W)(x,\theta),\theta)W(x,\theta)\,d\mu,$$

which is also as small as needed in the $C_\Gamma^{\Sigma_{t,r},L}$ norm because $\|N\|_{C_\Gamma^{\Sigma_{t,r}}}$ is small and $W \in C_\Gamma^{\Sigma_{t,r},L}$.

We can write the fixed point equation for the tail of the parametrisation $W^>$ as

$$W^>(s,\theta) = \mathcal{G}^{-1}\mathcal{H}(W^>)(s,\theta)$$

which is well defined, since $\|\mathcal{H}\|$ is as small as needed. If we define $\mathcal{T} = \mathcal{G}^{-1}\mathcal{H}$, $\mathcal{T}$ maps the unit ball of $C_\Gamma^{\Sigma_{t,r},L}$ to itself and the only thing left to check is to determine the Lipschitz constant of $\mathcal{T}$ restricted to $C_\Gamma^{\Sigma_{t-1,r},L}$.

Let $\Delta \in C_\Gamma^{\Sigma_{t-1,r},L}$, we can write

$$\begin{aligned}(\mathcal{T}(W^> + \Delta) - \mathcal{T}(W^>))(x,\theta)& \\ = -\int_0^1 \mathcal{G}^{-1} D_x N\big((W^{\leq} + W^> + \lambda\Delta)(x,\theta),\theta\big)\Delta(x,\theta)\,d\lambda.&\end{aligned} \tag{11.18}$$

We can take the $D_\theta^a D_s^b$ derivative of this expression for $W^> \in C_\Gamma^{t,r,L}$ to see that

$$\|D_\theta^a D_s^b \left[\mathcal{T}(W^> + \Delta) - \mathcal{T}(W^>)\right]\|_{C_\Gamma^0} \leq C\|\mathcal{G}^{-1}\|\|D_x N\|_{C_\Gamma^{\Sigma_{t-1,r}}}\|\Delta\|_{C_\Gamma^{\Sigma_{t,r},L}}.$$

Observe that when $b = L$, each term in the sum forming $D_\theta^a D_x^L N \circ (W^{\leq} + W^> + \lambda\Delta)\Delta$ has at least one derivative $D_\theta^\alpha D_s^\beta \Delta$, with $\beta \leq L$ which is then bounded by $C\|x\|^{L-\beta+1}$. Thus

$$\sup_{\|x\|\leq 1,\,\theta\in\mathbb{T}^d} \frac{\|\left(D_\theta^a D_x^b D_x N \circ (W^{\leq} + W^> + \lambda\Delta)\Delta\right)(x,\theta)\|_{C_\Gamma^{0,0}}}{\|x\|} \leq C\|D_x N\|_{C_\Gamma^{\Sigma_{t-1,r}}}.$$

Therefore the operator $\mathcal{T}$ is a contraction in $B(0,1) \subset C_\Gamma^{\Sigma_{t,r},L}(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ and has a fixed point in it satisfying Equation (11.16).

11.4 Recovering the last derivative

In the previous result we lose a derivative with respect to x in the parametrisation W when proving the contractivity of the operator $\mathcal{T}$. Recovering this last derivative is the subject of this section.

Since $W^{>}$ satisfies the fixed point equation

$$\begin{aligned} W^{>}(s,\theta) - M(\theta)^{-1}W^{>}(R(s,\theta),\theta+\omega) \\ = M(\theta)^{-1}W^{\leq}(R(s,\theta),\theta+\omega) - W^{\leq}(s,\theta) \\ - M(\theta)^{-1}N(W^{\leq}(s,\theta)+W^{<}(s,\theta),\theta), \end{aligned} \tag{11.19}$$

its derivative with respect to x, D_sW has to satisfy

$$\begin{aligned} D_sW^{>}(s,\theta) - M(\theta)^{-1}D_sW^{>}(R(s,\theta),\theta+\omega)D_sR(s,\theta) = \\ M(\theta)^{-1}D_sW^{\leq}(R(s,\theta),\theta+\omega)D_sR(s,\theta) - D_sW^{\leq}(s,\theta) \\ - M(\theta)^{-1}D_xN((W^{\leq}+W^{>})(s,\theta),\theta)(D_sW^{\leq}+D_sW^{>})(s,\theta). \end{aligned} \tag{11.20}$$

Let

$$\begin{aligned} U(s,\theta) = & - D_sW^{\leq}(s,\theta) + M(\theta)^{-1}D_sW^{\leq}(R(s,\theta),\theta+\omega)D_sR(s,\theta) \\ & - M(\theta)^{-1}D_xN((W^{\leq}+W^{>})(s,\theta),\theta)D_sW^{\leq}(s,\theta). \end{aligned} \tag{11.21}$$

Observe that now we can write Equation (11.20) as

$$\begin{aligned} D_sW^{>}(s,\theta) - M(\theta)^{-1}D_sW^{>}(R(s,\theta),\theta+\omega)D_sR(s,\theta) = \\ - M(\theta)^{-1}D_xN((W^{\leq}+W^{>})(s,\theta),\theta)D_sW^{>}(s,\theta) + U(s,\theta). \end{aligned} \tag{11.22}$$

Define the operators

$$\widetilde{\mathcal{G}},\widetilde{\mathcal{H}} : C_{\Gamma}^{\Sigma_{t-1},r,L-1}(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}^1\times\mathbb{T}^d,\ell^{\infty}(\mathbb{R}^n)) \to C_{\Gamma}^{\Sigma_{t-1},r,L-1}(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}^1\times\mathbb{T}^d,\ell^{\infty}(\mathbb{R}^n))$$

by

$$\begin{aligned} \widetilde{\mathcal{G}}(K)(s,\theta) &= K(,\theta) - M(\theta)^{-1}K(R(s,\theta),\theta+\omega)D_sR(s,\theta), \\ \widetilde{\mathcal{H}}(K)(s,\theta) &= -M(\theta)^{-1}D_xN((W^{\leq}+W^{>})(s,\theta),\theta)K(s,\theta). \end{aligned}$$

Now the fixed point equation (11.22) can be written as

$$\widetilde{\mathcal{G}}(D_sW^{>})(s,\theta) = \widetilde{\mathcal{H}}(D_sW^{>})(s,\theta) + U(s,\theta).$$

Lemma 11.9. *The operators*

$$\widetilde{\mathcal{G}},\widetilde{\mathcal{H}} : C_{\Gamma}^{\Sigma_{t-1},r,L-1}(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}^1\times\mathbb{T}^d,\ell^{\infty}(\mathbb{R}^n)) \to C_{\Gamma}^{\Sigma_{t-1},r,L-1}(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}^1\times\mathbb{T}^d,\ell^{\infty}(\mathbb{R}^n))$$

are well-defined.

Proof. It is straightforward to prove the well-definedness of $\widetilde{\mathcal{G}}$. Since $M(\theta)^{-1}$ exists, and if K is in $C_{\Gamma}^{\Sigma_{t-1},r}$, so is $\widetilde{\mathcal{G}}(K)$. Likewise, if K is $L-1$-flat, so is $\widetilde{\mathcal{G}}(K)$ since R has no term of order 0.

Similarly for $\widetilde{\mathcal{H}}$, the only thing to check is whether $(W^{\leq}+W^{>})(s,\theta)$ is in the domain of definition of D_xN for all $(s,\theta) \in B(0,1)\times\mathbb{T}^d$, but they are because $W^{\leq}$ contracts and $\|W^{\leq}\|_{C^0} \leq 1$.

□

The main result to recover the lost derivative is the following lemma.

Lemma 11.10. *$\widetilde{\mathcal{G}}, \widetilde{\mathcal{H}} : C_\Gamma^{\Sigma_{t-2},r,L-1}(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}^1\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) \to C_\Gamma^{\Sigma_{t-2},r,L-1}(B(0,1)\times\mathbb{T}^d \subset \mathcal{E}^1\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ are bounded linear operators. Moreover, if $\|N\|_{C_\Gamma^{\Sigma_t,r}}$ is small enough, then $\widetilde{\mathcal{G}}$ is invertible and $\|\widetilde{\mathcal{G}}^{-1}\|\|\widetilde{\mathcal{H}}\| < 1$.*

Remark 11.11. *Observe that the operators are well defined in $C_\Gamma^{\Sigma_{t-1},r,L-1}$, but to prove contractivity we need $C_\Gamma^{\Sigma_{t-2},r,L-1}$. We recover the missing regularity later.*

Proof. They are clearly linear operators.

The proof of the invertibility of $\widetilde{\mathcal{G}}$ is very similar to proving the invertibility for $\mathcal{G}$ (Proposition 11.8). Thus the operator $\widetilde{\mathcal{G}}$ has a formal inverse, given by the series

$$\sum_{k=0}^{\infty}(M^{-1})^{[k]}(\theta)G(R^{[k]}(s,\theta),\theta+k\omega)(D_sR)^{[k]}(s,\theta).$$

Proving its convergence is totally analogous to the proof of $\mathcal{G}$ referenced above. Observe that the missing R factor in the bounds (since $G \in C^{t-1,r,L-1}$) comes from the additional factor D_sR.

The operator $\widetilde{\mathcal{H}}$ is as small as needed, since its size can be adjusted via the scaling as for the operator $\mathcal{H}$, hence the composition is well defined.

Thus the bound for the operator norms follows.

□

To finish the proof we need to prove $U(s,\theta) \in C_\Gamma^{t-1,r,L-1}(B(0,1)\times\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$. Indeed, we can write

$$U(s,\theta) = -D_s(U_1+U_2),$$

where

$$\begin{aligned} U_1 &= W^{\le}(s,\theta) - M(\theta)^{-1}W^{\le}(R(s,\theta),\theta+\omega) - M(\theta)^{-1}N(W \le (s,\theta),\theta), \\ U_2 &= M(\theta)^{-1}N(W^{\le}(s,\theta),\theta) - M(\theta)^{-1}N((W^{\le}+W^{>})(s,\theta),\theta) \end{aligned}$$

where we have added and removed $M(\theta)^{-1}N(W^{\le}(s,\theta),\theta)$.

Observe that U_1 and U_2 are the expressions appearing in (11.22) and hence they are in $C_\Gamma^{\Sigma_t,r,L}$ and then $U \in C_\Gamma^{\Sigma_t,r,L}$.

Finally, we can solve

$$\widetilde{\mathcal{G}}(K)(s,\theta) = \widetilde{\mathcal{H}}(K)(s,\theta) + U(s,\theta) \tag{11.23}$$

by writing

$$K = \left(\widetilde{\mathcal{G}} + \widetilde{\mathcal{H}}\right)^{-1}(U) = \widetilde{\mathcal{G}}^{-1}(\mathrm{Id} + \widetilde{\mathcal{G}}^{-1}\widetilde{\mathcal{H}})^{-1}U.$$

Since Lemma 11.10 states $\|\widetilde{\mathcal{G}}^{-1}\|\|\widetilde{\mathcal{H}}\| < 1$ and $\widetilde{\mathcal{G}}^{-1}$ is well defined by the same result, we can find a unique solution of Equation (11.23), which is $D_sW^{>}$ and therefore $W^{>} \in C_\Gamma^{\Sigma_t,r,L}$, proving Theorem 10.1.

Chapter 12

Normal forms and Sternberg's conjugation theorems

12.1 Normal forms of maps in lattices

In this Section we consider the computation of normal forms around a fixed point of a map in a lattice, assuming the map has decay properties. We estimate the decay properties of the normal form and the transformation to it.

We consider an open set U of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$ and

$$F : U \to \ell^\infty(\mathbb{R}^n)$$

a map such that $F(0) = 0$ and $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$. Let $A = DF(0)$, with $A \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and consider its Γ-spectrum $\mathrm{Spec}_\Gamma(A)$.

Theorem 12.1. *In the previously described setting there exist polynomials $H \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and $R \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ of degree at most r such that $H(0) = 0$, $DH(0) = \mathrm{Id}$ and*

$$F \circ H(x) - H \circ R(x) = o(\|x\|^r)$$

and $R(x) = Ax + \sum_{j \in J} R_j x^{\otimes j}$ where

$$J = \{2 \le j \le r \mid \mathrm{Spec}_\Gamma(A)^j \cap \mathrm{Spec}_\Gamma(A) \neq \emptyset\}$$

and $R_j \in L^j_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$.

Corollary 12.2. *Under the conditions of the previous theorem, if*

$$\mathrm{Spec}_\Gamma(A)^j \cap \mathrm{Spec}_\Gamma(A) = \emptyset, \quad 2 \le j \le r,$$

then there exists a polynomial $H \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that

$$F \circ H(x) - H \circ Ax = o(\|x\|^r).$$

Proof. We look for H and R in the form

$$H(x) = \sum_{j=1}^{r} H_j x^{\otimes j},$$
$$R(x) = \sum_{j=1}^{r} R_j x^{\otimes j},$$

where $H_j, R_j \in L^j_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. Taking derivatives on both sides of

$$F \circ H = H \circ R$$

and evaluating at 0 we have

$$A\,H_1 = H_1\,R_1.$$

This equation has the obvious solution $H_1 = \mathrm{Id}$, $R_1 = A$, although other solutions are possible, for instance H_1 any linear map which commutes with A, as $H_1 = \alpha\,\mathrm{Id}$, $\alpha \in \mathbb{R}$ and $R_1 = A$.

Taking k-th order derivatives using the Faà di Bruno formula,

$$\sum_{j=1}^{k} \sum_{\substack{i_1,\dots,i_j \geq 1 \\ i_1+\dots+i_j = k}} C D^j F \circ H(D^{i_1} H \cdots D^{i_j} H) = \sum_{j=1}^{k} \sum_{\substack{i_1,\dots,i_j \geq 1 \\ i_1+\dots+i_j = k}} C D^j H \circ R(D^{i_1} R \cdots D^{i_j} R)$$

(where for the sake of simplicity we have not written the dependence of C on the indices,) and evaluating the derivatives at 0 we can write

$$AH_k + G^1_k = R_k + H_k A^{\otimes r} + G^2_k, \tag{12.1}$$

where G^1_k, G^2_k are k-linear maps which depend on $D^j F(0)$, $1 \leq j \leq r$, and H_i, R_i, $1 \leq i \leq r-1$.

Let $G_k = G^1_k - G^2_k$. Note that G_k is sum and contraction of multilinear operators.

Using Sylvester operators (introduced in Chapter 6) we rewrite Equation (12.1) as

$$\left(\mathcal{S}_{A^{-1},A} \quad - \mathrm{Id}\right) H_k = A^{-1}(-R_k + G_k). \tag{12.2}$$

Now we proceed inductively from $k = 2$ up to $k = r$. Assume that for j up to the $(k-1)$-th step we have obtained H_j and R_j in $L^j_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ solving Equation (12.2). From the way G_k is defined, from Proposition 3.16 we get that $G_k \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and hence $A^{-1}(-R_k + G_k) \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. Now, if

$$\mathrm{Spec}_\Gamma(A) \cap \mathrm{Spec}_\Gamma(A)^k = \emptyset,$$

then $1 \notin \mathrm{Spec}\,\mathcal{S}_{A^{-1},A}$ thus $\left(\mathcal{S}_{A^{-1},A} - \mathrm{Id}\right) : L^k_\Gamma \to L^k_\Gamma$ is invertible by Proposition 6.21. This implies we can choose $R_k = 0$ and $H_k = (\mathcal{S}_{A^{-1},A})^{-1} G_k$.

Obviously with this choice $R_k, H_k \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. On the other hand, if

$$\mathrm{Spec}_\Gamma(A) \cap \mathrm{Spec}_\Gamma(A)^k \neq \emptyset$$

the operator $\mathcal{S}_{A^{-1},A}$ may not be invertible and we set $R_k = G_k$ and $H_k = 0$. This is not the only possible choice, it is only the simplest one and we also have $H_k, R_k \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$.

Another standard possibility is to decompose $L^k_\Gamma(E,E) = \operatorname{Im} \mathcal{S}_{A^{-1},A} \oplus V$, where Im stands for the range of $\mathcal{S}_{A^{-1},A}$ and V is a complementary subspace in $L^k_\Gamma(E,E)$. Then one decomposes G_k according to this splitting of the space as $G^{\mathrm{I}}_k + G^V_k$ and choose H_k such that $S_{A^{-1},A}\ H_k = G^{\mathrm{I}}_k$ and $R_k = G^v_k$. Of course this choice also depends on the choice of the complementary space V.

By the choice of H_k, R_k, $1 \le k \le r$,

$$D^k \left[F \circ H - H \circ R\right] = 0, \quad 1 \le k \le r$$

and hence, by Taylor's theorem $F \circ H(x) - H \circ R(x) = o(\|x\|^r)$.

□

12.2 Sternberg theorems in lattices

In this section we will prove several Sternberg conjugation theorems for contractions under several non-resonance hypotheses. All are adaptations to our setting of the classical proof in [Ste57], using the normal form theory developed in the previous section.

Although we do not explore it here, another possibility is to follow the ideas in [CC97], based on the fact that *a function φ conjugates h_0 and h_1 if and only if the graph of φ is invariant under $h_0 \times h_1$*. Using this insight and the constructions of invariant manifolds up to a certain order in Chapters 8 and 10 under non-resonance we might also prove Sternberg theorems in lattices with decay.

Theorem 12.3. *Let U be an open set of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$. Let $F : U \to \ell^\infty(\mathbb{R}^n)$ be a C^r_Γ map of the form $F = F_0 + F_1$ where F_0 is an uncoupled map and $F_0(0) = F_1(0) = 0$. Let $A = DF_0(0)$, $B = DF_1(0)$ and $M = A+B$. Assume that $A_{ij} = \mathfrak{a}\delta_{ij}$ with $\mathfrak{a} \in L(\mathbb{R}^n, \mathbb{R}^n)$.*

Let $\operatorname{Spec}(\mathfrak{a}) = \{\lambda_1, \dots, \lambda_n\}$. Assume furthermore

(H1) $0 < |\lambda_i| < 1, \quad 1 \le i \le n,$

(H2) $\lambda_i \ne \lambda^k, \quad k \in \mathbb{Z}^n, \quad |k| \ge 2,\ 1 \le i \le n.$

Let $\alpha = \min_i |\lambda_i|$, $\beta = \max_i |\lambda_i|$, $\nu = \frac{\log \alpha}{\log \beta}$ and $r_0 = [\nu] + 1$. Then if $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ with $r \ge r_0$ and $\|B\|_\Gamma$ is small enough, there exists $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that $R(0) = 0$, $DR(0) = \mathrm{Id}$ and

$$R \circ F = MR$$

in some neighborhood $U_1 \subseteq U$ of 0 in $\ell^\infty(\mathbb{R}^n)$.

Remark 12.4. *Note that we do not require $\|F_1\|$ to be small but only $B = DF_1(0)$ to be small.*

Remark 12.5. *Since $\mathfrak{a}$ is a contraction, assumption (H2) involves only a finite set of conditions.*

Before starting the proof we perform a re-scaling of F as in Section 8.4 to move the smallness conditions on the domains of definition to the smallness of an auxiliary parameter.

Let $\delta \in \mathbb{R}$, $\delta > 0$, and the re-escaling map $T_\delta x = \delta x$. We define

$$F_\delta(x) = T_\delta^{-1} \circ F \circ T_\delta(x) = Mx + N_\delta(x) = Mx + \delta^{-1} N(\delta x).$$

From now on we will not write the dependence on δ of $F_\delta(x)$ and $N_\delta(x)$ and we will assume that F is defined on $B(0,1) \subset \ell^\infty(\mathbb{R}^n)$ and δ is as small as needed. In particular, if F is at least of class C^2, we have that

$$\|N\|_{C^0} = \mathcal{O}(\delta), \quad \|DN\|_{C^0} = \mathcal{O}(\delta) \text{ and } \|D^j N\|_{C^0} = \mathcal{O}(\delta^{j-1}),\ j \geq 2,$$

and moreover

$$\|N\|_{C^r_\Gamma} = \mathcal{O}(\delta).$$

Given $r, r_0 \in \mathbb{N}$, $r \geq r_0$ we introduce the spaces

$$\begin{aligned} \chi^{r,r_0} &= \left\{ g \in C^r(B(0,1), \ell^\infty(\mathbb{R}^n)) \mid D^j g(0) = 0,\ 0 \leq j \leq r_0,\ \|g\|_{C^r} < \infty \right\}, \\ \chi^{r,r_0}_\Gamma &= \left\{ g \in C^r_\Gamma(B(0,1), \ell^\infty(\mathbb{R}^n)) \cap \chi^{r,r_0} \mid \|g\|_{C^r_\Gamma} < \infty \right\}. \end{aligned}$$

Observe that χ^{r,r_0}_Γ is a closed subspace of C^r_Γ.

Lemma 12.6. *There exists $m \in \mathbb{N}$ such that*

$$\Gamma(0)^{-2} \left(\alpha^{-1} \beta^{r_0}\right)^m < 1.$$

Then if $r \geq r_0$ with $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ and $\|B\|_\Gamma$ and the re-scaling parameter are small enough, the operator $\mathcal{G}_m : \chi^{r,r_0}_\Gamma \to \chi^{r,r_0}_\Gamma$ defined by

$$\mathcal{G}_m(g) = M^{-m} g \circ F^m$$

is well defined and a contraction in the C^r_Γ-norm.

Proof. First we fix some quantities to be used throughout the proof. From the definition of r_0 and the fact that $\beta < 1$ we have

$$\alpha^{-1} \beta^{r_0} < 1.$$

Then there exists $m \in \mathbb{N}$ such that

$$\Gamma(0)^{-2} \left(\alpha^{-1} \beta^{r_0}\right)^m < 1$$

and also there exists positive numbers ε_1, ε_2 and ε_3 such that

$$\Gamma(0)^{-1} (\alpha^{-1} + \varepsilon_1)^m \left[\Gamma(0)^{-1} \left((\beta + \varepsilon_1)^m + \varepsilon_2 \right)^{r_0} + \varepsilon_3 \right] < 1. \tag{12.3}$$

On the other hand there exists a norm in $\mathbb{R}^n$ such that

$$\|\mathfrak{a}\| < \beta + \frac{\varepsilon_1}{2}, \quad \|\mathfrak{a}^{-1}\| < \alpha^{-1} + \frac{\varepsilon_1}{2},$$

where $\|\cdot\|$ is the associated operator norm. Clearly,

$$\|\mathfrak{a}^m\| < \left(\beta + \frac{\varepsilon_1}{2}\right)^m, \quad \|\mathfrak{a}^m\| < \left(\alpha^{-1} + \frac{\varepsilon_1}{2}\right)^m$$

and

$$\|A^m\|_\Gamma < \Gamma(0)^{-1}\left(\beta + \frac{\varepsilon_1}{2}\right)^m, \quad \|A^{-m}\|_\Gamma \leq \Gamma(0)^{-1}\left(\alpha^{-1} + \frac{\varepsilon_1}{2}\right)^m.$$

Moreover,

$$\begin{aligned}\|M^m\|_\Gamma &= \|(A+B)^m\|_\Gamma \\ &\leq \|A^m\|_\Gamma + \mathcal{O}(\|B\|_\Gamma) \leq \Gamma(0)^{-1}(\beta + \frac{\varepsilon_1}{2})^m + \Gamma(0)^{-1}\frac{\varepsilon_1}{2}\beta^{m-1} \\ &< \Gamma(0)^{-1}(\beta+\varepsilon_1)^m\end{aligned}$$

if $\|B\|_\Gamma$ is small enough.

In the same way, now using Proposition 3.11,

$$\begin{aligned}\|M^{-m}\|_\Gamma &\leq \|A^{-m}\|_\Gamma + \mathcal{O}(\|B\|_\Gamma) \\ &\leq \Gamma(0)^{-1}(\alpha^{-1} + \frac{\varepsilon_1}{2})^m + \Gamma(0)^{-1}\frac{\varepsilon_1}{2}\alpha^{-(m-1)} < \Gamma(0)^{-1}(\alpha^{-1}+\varepsilon_1)^m,\end{aligned}$$

and finally

$$\|M^m\| \leq \|M\|^m \leq (\beta+\varepsilon_1)^m.$$

Let $g \in \chi_\Gamma^{r,r_0}$. By Remark 4.4 we have

$$\|\mathcal{G}_m(g)\|_{C_\Gamma^r} \leq \|M^{-m}\|_\Gamma \|g \circ F^m\|_{C_\Gamma^r}.$$

To estimate $\|g \circ F^m\|_{C_\Gamma^r}$ we will use again the Faà di Bruno formula for the derivative, $1 \leq p \leq r$,

$$\begin{aligned}D^p(g\circ F^m)(x) =& D^p g(F^m(x))(DF^m(x))^{\otimes p} \\ &+ \sum_{j=1}^{p-1} \sum_{\substack{i_1,\dots,i_j \geq 1 \\ i_1+\dots+i_j = p}} C D^j g(F^m(x)) D^{i_1}F^m(x)\cdots D^{i_j}F^m(x),\end{aligned} \tag{12.4}$$

where C is a combinatorial coefficient which depends on all the indices. From (12.4) it is clear that $D^p(g \circ F^m)(0) = 0$ for $1 \leq p \leq r_0$, since $F^m(0) = 0$.

Since $g \in \chi^{r,r_0}$, by Taylor's theorem in integral form (see [AMR88])

$$g(x) = \frac{1}{(r_0-1)!}\int_0^1 (1-t)^{r_0-1} D^{r_0} g(tx) x^{\otimes r_0}\, dt$$

and also

$$D^j g(x) = \frac{1}{(r_0-j-1)!}\int_0^1 (1-t)^{r_0-j-1} D^{r_0} g(tx) x^{\otimes (r_0-j)}\, dt, \quad 0 \leq j \leq r_0 - 1.$$

Using the previous formulas, the results in Section 6.3 and Proposition 3.14 we have

$$\begin{aligned}\|D^j g(F^m(x))\|_\Gamma &\le \frac{1}{(r_0-j-1)!}\left\|\int_0^1 (1-t)^{r_0-j-1} D^{r_0} g(tF^m(x))\big(F^m(x)\big)^{\otimes(r_0-j)}\,dt\right\|_\Gamma \\ &\le \frac{1}{(r_0-j-1)!}\int_0^1 (1-t)^{r_0-j-1}\|D^{r_0} g(tF^m(x))\|_\Gamma \|F^m(x)\|^{r_0-j}\,dt \\ &\le \frac{1}{(r_0-j)!}\|g\|_{C^r_\Gamma}\|F^m(x)\|^{r_0-j}, \quad 0\le j\le r_0-1\end{aligned}$$

and

$$\|D^j g(F^m(x))\|_\Gamma \le \|g\|_{C^r_\Gamma}, \quad r_0\le j\le r.$$

We can write both bounds in the more compact form

$$\|D^j g(F^m(x))\|_\Gamma \le \|g\|_{C^r_\Gamma}\|F^m(x)\|^{(r_0-j)_+},$$

where $(t)_+ = \max(t,0)$.

Then

$$\begin{aligned}&\|D^p(g\circ F^m)(x)\|_\Gamma \\ &\qquad\le \|g\|_{C^r_\Gamma}\|F^m(x)\|^{(r_0-p)_+}\|DF^m(x)\|_\Gamma\|DF^m(x)\|^{p-1} \\ &\qquad\quad+\sum_{j=1}^{p-1}\sum_{\substack{i_1,\dots,i_j\ge 1\\ i_1+\dots+i_j=p}} C\|g\|_{C^r_\Gamma}\|F^m(x)\|^{(r_0-j)_+}\|D^{i_1}F^m(x)\|_\Gamma\cdots\|D^{i_j}F^m(x)\|_\Gamma.\end{aligned}$$

By the scaling,

$$\|F^m(x)\| \le \|M^m x\| + \mathcal{O}(\delta) \le \|M^m\| + \mathcal{O}(\delta),$$
$$\|DF^m(x)\| \le \|M^m\| + \mathcal{O}(\delta), \quad \|DF^m(x)\|_\Gamma \le \|M^m\|_\Gamma + \mathcal{O}(\delta),$$

and

$$\|D^j F^m(x)\|_\Gamma = \mathcal{O}(\delta), \quad j\ge 2.$$

Also note that for $p\ge 0$, $(r_0-p)_+ + p \ge r_0$.

Then

$$\|D^p(g\circ F^m)(x)\|_\Gamma \le \|g\|_{C^r_\Gamma}\Big[\left(\|M^m\|_\Gamma + \mathcal{O}(\delta)\right)\left(\|M^m\| + \mathcal{O}(\delta)\right)^{r_0-1} + \mathcal{O}(\delta)\Big], \quad 1\le p\le r$$

and finally

$$\begin{aligned}\|\mathcal{G}_m(g)\|_{C^r_\Gamma} &\le \|M^{-m}\|_\Gamma\Big[\left(\|M^m\|_\Gamma + \mathcal{O}(\delta)\right)\left(\|M^m\| + \mathcal{O}(\delta)\right)^{r_0-1} + \mathcal{O}(\delta)\Big]\|g\|_{C^r_\Gamma} \\ &\le \Gamma(0)^{-1}(\alpha^{-1}+\varepsilon_1)^m \\ &\qquad\times\Big[\big(\Gamma(0)^{-1}(\beta+\varepsilon_1)^m + \mathcal{O}(\delta)\big)\left((\beta+\varepsilon_1)^m + \mathcal{O}(\delta)\right)^{r_0-1} + \mathcal{O}(\delta)\Big]\|g\|_{C^r_\Gamma}.\end{aligned}$$

Then if δ is small enough, by (12.3) the factor in front of $\|g\|_{C^r_\Gamma}$ is strictly less than 1 and hence $\mathcal{G}$ is a contraction in χ^{r,r_0}_Γ.

□

Now we use the normal form theory in the previous section to find a decay map which linearises our map F up to order r_0. The form of A implies that $\operatorname{Spec} A = \{\lambda_1, \ldots, \lambda_n\}$ and $\operatorname{Spec} A^m = \{\lambda_1, \ldots, \lambda_n^m\}$. Since A^m is uncoupled, $\operatorname{Spec}_\Gamma(A^m) = \operatorname{Spec}(A^m)$. Moreover the non-resonance condition (H2) implies that

$$\lambda_i^m \neq \lambda_1^{nk_1} \cdots \lambda_n^{mk_n}, \quad k \in \mathbb{Z}^n, \quad |k| \geq 2$$

and therefore

$$(\operatorname{Spec}_\Gamma(A^m))^j \cap \operatorname{Spec}_\Gamma(A^m) = \emptyset, \quad j \geq 2.$$

Taking $\|B\|_\Gamma$ sufficiently small, by Proposition 6.15 we have

$$(\operatorname{Spec}_\Gamma M^m)^j \cap \operatorname{Spec}_\Gamma M^m = \emptyset, \quad 2 \leq j \leq r_0,$$

since we are only asking for a finite set of conditions.

Hence Corollary 12.2 gives us that there exists a polynomial $H \in C_\Gamma^\infty(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ of degree (at most) r_0 such that $H(0) = 0$, $DH(0) = \mathrm{Id}$ and

$$F^m \circ H(x) - H \circ M^m(x) = o(\|x\|^{r_0}).$$

Let $S_0 = H^{-1}$. By Theorem 4.5, $S_0 \in C_\Gamma^r$ and satisfies

$$S_0(0) = 0, \qquad DS_0(0) = \mathrm{Id},$$
$$M^{-m} \circ S_0 \circ F^m - S_0 = o(\|x\|^{r_0}).$$

Starting with this approximate conjugation we define the sequence

$$S_n = M^{-m} S_{n-1} \circ F^m, \quad n \geq 1.$$

The next lemma proves that S_n converges to a well-defined conjugation in the space C_Γ^k.

Lemma 12.7. *The sequence $\{S_n\}_{n\in\mathbb{N}}$ defined above converges to a function $S \in C_\Gamma^k$ satisfying $S(0) = 0$, $DS(0) = \mathrm{Id}$ and*

$$S \circ F^m = M^m S.$$

Proof. Since m is fixed we do not write the dependence of $\mathcal{G}_m$ on m. First we will prove the following relation

$$S_n = S_0 + \sum_{j=0}^{n-1} \mathcal{G}^j (M^{-m} S_0 \circ F^m - S_0), \tag{12.5}$$

where $\mathcal{G}^0 = \mathrm{Id}$ and $\mathcal{G}^j = \mathcal{G} \circ \mathcal{G}^{j-1}$, $j \geq 1$. Observe that $M^{-m} S_0 \circ F^m - S_0 \in \chi^{r,r_0}$, since S_0 solves the conjugation equation formally up to order r_0. Moreover $M^{-m} S_0 \circ F^m - S_0 \in C_\Gamma^k$ since $S_0, F \in C_\Gamma^k$. We will prove (12.5) by induction. When $n = 1$, we can use the definition $S_1 = \mathcal{G}(S_0)$:

$$S_1 = \mathcal{G}(S_0) = M^{-m} S_0 \circ F^m = S_0 + \mathcal{G}^0(M^{-m} S_0 \circ F^m - S_0).$$

Now assume Equation (12.5) is true up to the index n, then

$$\begin{aligned}
S_{n+1} &= M^{m(-n-1)}S_0 \circ F^{m(n+1)} \\
&= M^{-mn}S_0 \circ F^{mn} + M^{m(-n-1)}S_0 \circ F^{m(n+1)} - M^{mn}S_0 \circ F^{mn} \\
&= S_n + M^{-mn}\big(M^{-m}S_0 \circ F^m - S_0\big) \circ F^{mn} \\
&= S_n + \mathcal{G}^n(M^{-m}S_0 \circ F^m - S_0) \\
&= S_0 + \sum_{j=0}^{n} \mathcal{G}^j\big(M^{-m}S_0 \circ F^m - S_0\big).
\end{aligned}$$

Now by Lemma 12.6, $\mathcal{G}$ is a contraction in χ_Γ^{r,r_0} and therefore the series determining $\lim_{n\to\infty} S_n$ is convergent.

Finally, we have to check the conjugacy properties. We can write

$$S \circ F^m = \lim_{n\to\infty} S_n \circ F^m = \lim_{n\to\infty} M^{-mn}S_0 \circ F^{mn+m} = \lim_{n\to\infty} M^m M^{-mn-m}S_0 F^{mn+m} = M^m S,$$

proving the conjugacy part. Finally,

$$S(0) = \lim_{n\to\infty} M^{-mn}S_0 \circ F^{mn}(0) = 0,$$

and since $DF^{mn}(0) = DF^{mn}(F^{mn-1}(0)) \cdots DF(0) = M^{mn}$ and $DS_0(0) = \mathrm{Id}$,

$$DS(0) = \lim_{n\to\infty} M^{-mn}DS_0(F^{mn}(0))DF^{mn}(0) = \lim_{n\to\infty} M^{-mn}\,\mathrm{Id}\,M^{mn} = \mathrm{Id}\,.$$

□

Thus, S conjugates F^m to M^m. The final step is showing that S also conjugates F to M.

By the spectral properties, and also applying Corollary 12.2, there exists a polynomial $\widetilde{H} \in C^\infty(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that

$$\widetilde{H}(0) = 0, \quad D\widetilde{H}(0) = \mathrm{Id},$$

and

$$F \circ \widetilde{H}(x) = \widetilde{H} \circ M(x) = o(\|x\|^{r_0}).$$

Let $R_0 = \widetilde{H}^{-1}$. Thus

$$M^{-1}R_0 \circ F(x) = R_0(x) + o(\|x\|^{r_0})$$

and as a consequence

$$M^{-m}R_0 \circ F^m(x) = R_0(x) + o(\|x\|^{r_0}). \tag{12.6}$$

Lemma 12.8. *Under the hypotheses of Theorem 12.3, if $\|B\|$ is small enough the operator*

$$\widetilde{\mathcal{G}} : C^r(B(0,1), \ell^\infty(\mathbb{R}^n)) \to C^r(B(0,1), \ell^\infty(\mathbb{R}^n))$$

defined by

$$\widetilde{\mathcal{G}}(g) = M^{-1}g \circ F$$

is well defined and a contraction in the C^r-norm.

The proof of this lemma is analogous to the proof of Lemma 12.6 but the estimates are much simpler, since they do not involve decay functions.

Define the sequence

$$R_n = M^{-1} R_{n-1} \circ F, \quad n \geq 1.$$

The same arguments to the ones used in the proof of Lemma 12.7 but now in the space C^r instead of C^r_Γ give that there exists $R = \lim_{n\to\infty} R_n$ and $R \in C^r(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that $R \circ F = MR$.

Now consider the iteration S_n considered in Lemma 12.7 with $S_0 = R_0 \in C^r_\Gamma$. Since R_0 satisfies (12.6), we see that S_n is then a sub-sequence of R_n, being both convergent in the larger space C^r. Then

$$R = \lim_{n\to\infty} R_n = \lim_{n\to\infty} S_n = S \in C^r_\Gamma$$

and S also conjugates F with M, proving Theorem 12.3.

An improvement of Theorem 12.3 consist of not assuming the non-resonant condition (H2). In such case we obtain a C^r_Γ local conjugation to a normal form of F instead to a conjugation to its linear part.

Theorem 12.9. *Under the conditions and notation of Theorem 12.3 except hypothesis (H2), if $F \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ with $r \geq r_0$ and $\|B\|_\Gamma$ is small enough there exists a polynomial $H \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ of degree not larger than r_0 and $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that*

$$R(0) = 0, \quad DR(0) = \mathrm{Id}$$

and

$$R \circ F = H \circ R$$

in some neighborhood $U_1 \subset U$ of 0.

Proof. We will only comment on the differences of this proof with the proof of Theorem 12.3. After re-scaling and defining χ^{r,r_0} and χ^{r,r_0}_Γ, we use the same integer m. We determine a normal form H provided by Theorem 12.1 and define an operator

$$\overline{\mathcal{G}}_m(g) = H^{-m} \circ g \circ F^m.$$

Now the estimates on $\overline{\mathcal{G}}_m$ become more involved because in this case H is not linear. This implies that $\overline{\mathcal{G}}_m$ is not linear anymore. However, because of the re-scaling, H^{-1} is very close to M^{-1} in C^r_Γ (and C^r) norm, a fact which gives similar estimates and thus proves $\mathrm{Lip}\,\overline{\mathcal{G}}_m < 1$. The remaining part of the proof is analogous.

□

The previous theorems assume that the linear part of the maps is close to an uncoupled map and that all restrictions to nodes are equal. This gives sufficient conditions for the conjugation in terms of the eigenvalues of the restriction to the nodes.

A statement using conditions on the Γ-spectrum of the linear part is the following.

Theorem 12.10. *Let U be an open set of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$. Let $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ with $F(0) = 0$. Let $A = DF(0)$. Assume*

(H1) $0 \notin \mathrm{Spec}_\Gamma(A)$ *and* $\mathrm{Spec}_\Gamma(A) \subset \mathbb{D}(0,1)$,

(H2) $\mathrm{Spec}_\Gamma(A) \cap \mathrm{Spec}_\Gamma(A)^j = \emptyset$, $j \geq 2$.

Let $\alpha = \inf\{|\lambda| \mid \lambda \in \mathrm{Spec}_\Gamma(A)\}$, $\beta = \sup\{|\lambda| \mid \lambda \in \mathrm{Spec}_\Gamma(A)\}$, $\nu = \frac{\log\alpha}{\log\beta}$ *and* $r_0 = [\nu] + 1$. *Then if* $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ *with* $r \geq r_0$ *there exists* $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *such that* $R(0) = 0$, $DR(0) = \mathrm{Id}$ *and*

$$R \circ F = AR$$

in a neighborhood $U_1 \subset U$ *of 0.*

Remark 12.11. *Since* $\mathrm{Spec}_\Gamma(A)$ *is compact, (H1) implies that* $0 < \alpha \leq \beta < 1$. *Therefore (H2) needs to be checked only for* $2 \leq j \leq \frac{\log\alpha}{\log\beta}$.

Proof. The proof has a structure very similar to the proof of Theorem 12.3 but has some technical differences. Let r_0 and r be as in the statement of the theorem. Note that $\beta = r_\Gamma(A)$ and $\alpha^{-1} = r_\Gamma(A^{-1})$. Since $r_0 > \nu$, $\alpha^{-1}\beta^{r_0} < 1$ and there exists $\varepsilon_1 > 0$ such that

$$(\alpha^{-1} + \varepsilon_1)(\beta + \varepsilon_1)^{r_0} < 1.$$

From Proposition 6.13 there exists m such that

$$\|A^n\|_\Gamma \leq (r_\Gamma(A) + \varepsilon_1)^n, \quad n \geq m$$

and

$$\|A^{-n}\|_\Gamma \leq (r_\Gamma(A^{-1}) + \varepsilon_1)^n, \quad n \geq m.$$

Obviously

$$(\alpha^{-1} + \varepsilon_1)^m (\beta + \varepsilon_1)^{mr_0} < 1$$

and there exists $\varepsilon_2, \varepsilon_3 > 0$ such that

$$(\alpha^{-1} + \varepsilon_1)^m \left[((\beta + \varepsilon_1)^m + \varepsilon_2)^{r_0} + \varepsilon_3\right] < 1.$$

Now we introduce the operator $\mathcal{G}_m : \chi^{r,r_0}_\Gamma \to \chi^{r,r_0}_\Gamma$ defined by

$$\mathcal{G}_m(g) = A^{-m} g \circ F^m$$

where the space χ^{r,r_0}_Γ is the same as in the proof of Theorem 12.3.

Analogous estimates as in Lemma 12.6 lead us to prove that if the re-scaling parameter is small enough $\mathcal{G}_m$ is well defined in χ^{r,r_0}_Γ and a contraction. Then the proof follows the same lines as the proof of Theorem 12.3.

For the uniqueness arguments needed at the end of the proof, we consider the operator $\widetilde{\mathcal{G}}(g) = A^{-1} g \circ F$ in $C^r(B(0,1), \ell^\infty(\mathbb{R}^n))$. Since $\mathrm{Spec}(A) \subset \mathrm{Spec}_\Gamma(A)$, condition (H2) implies that there are also no resonances among the elements of $\mathrm{Spec}(A)$ and that $r(A) < \beta$ and $r(A^{-1}) > \alpha^{-1}$. Hence we can find a norm in the space, equivalent to the original one such that

$$\|A^{-1}\| \|A\|^{r_0} < 1, \tag{12.7}$$

where in the previous expression $\|\cdot\|$ stands for the associated operator norm. The bound (12.7) allows us to prove the estimates needed to show that $\widetilde{\mathcal{G}}$ is a contraction. With these ingredients we can finish the proof in this setting.

$\square$

The analogous version of Theorem 12.9 in this setting is the following.

Theorem 12.12. *Under the conditions of Theorem 12.10, except condition (H2), if* $F \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *with* $r \geq r_0$ *there exists a polynomial* $H \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *of degree not larger than* r_0 *and* $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *such that*

$$R(0) = 0, \quad DR(0) = \mathrm{Id}$$

and

$$R \circ F = H \circ R$$

in some neighborhood $U_1 \subset U$ *of 0.*

Part II

An Entire Transcendental Family with a Persistent Siegel Disk

Introduction

In the second part we deal with a holomorphic dynamics problem, studying a family of entire transcendental functions with a persistent Siegel disc.

Holomorphic dynamics started with the work of Fatou ([Fat20]) and Julia ([Jul18]) at the beginning of the 20th century, motivated by the complex Newton method. They studied general properties shared between rational and transcendental functions (those having an essential singularity at ∞) alike. They gave a dynamically natural partition of the phase space named after them, the *Fatou set* $\mathcal{F}(f)$ where the iterates of f form a normal family and the *Julia set* $\mathcal{J}(f) = \hat{\mathbb{C}} - \mathcal{F}(f)$ its complement. Informally, the Fatou set is formed by those points that behave under iteration "like their neighbors do", which is why it is sometimes called the *stable set*. Its complement, the Julia set, is also called the *chaotic set* and, indeed the dynamics on $\mathcal{J}(f)$ exhibit high amounts of unpredictability. Both sets are completely invariant, that is $f(\mathcal{J}) = f^{-1}(\mathcal{J}) = \mathcal{J}$, and the same holds for $\mathcal{F}$. Naturally from its definition, the Fatou set is open and the Julia set is closed.

Given a connected component U of the Fatou set, it is always the case that $f(U)$ is also a connected component of the Fatou set. At a first level, a connected component $U \in \mathcal{F}(f)$ may be *periodic* if $f^p(U) = U$ (and we call the minimum p its *period*), *pre-periodic* if there is some n such that $f^n(U)$ is periodic and finally it may be a *wandering domain* if all $\{f^n(U)\}_n$ are distinct.

Fatou also gave a description of all possible periodic connected components of $U \subseteq \mathcal{F}(f)$. They can be classified as:

- *Attracting domain*: U contains an attracting periodic point z_0 of period n and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$. If z_0 is superattracting we also call U a *Böttcher domain*, otherwise a *Schröder domain*.

- *Parabolic domain*: ∂U contains a fixed point z_0 of f^p and $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$ and also $(f^p)'(z_0) = e^{2\pi i\alpha}$, $\alpha \in \mathbb{Q}$. We also call U a *Leau domain* at z_0

- *Siegel disk*: There exists an analytic homeomorphism $\phi : U \to \mathbb{D}$ where $\mathbb{D}$ is the unit disk and such that $\phi(f^n(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

- *Herman ring*: There exists an analytic homeomorphism $\phi : U \to A$ where A is an annulus $A = \{z \,|\, 1 < |z| < r,\, r > 1\}$ such that $\phi(f^n(\phi^{-1}(z))) = e^{2\pi i\alpha}z$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$.

- *Baker domain*: $f^{np}(z) \to \infty$ for $z \in U$ as $n \to \infty$, and ∞ is an essential singularity.

Examples of functions with attracting or indifferent periodic basins are easy to find. The existence of Siegel disks, however, was first shown by C.L. Siegel ([Sie42]), proving the following theorem.

Theorem (Siegel's linearization)**.** *Let $f(z) = \lambda z + \mathcal{O}(z^2)$ be an analytic function with 0 as a fixed point with multiplier λ. If $1/|\lambda^q - 1|$ is less than some polynomial function of q, then f is locally linearisable.*

Being locally linearizable means that there is a conformal homeomorphism conjugating the function to an irrational rotation, hence it is equivalent to the existence of a Siegel disk around $z = 0$. As a corollary, if $\lambda = e^{2\pi i \xi}$ and ξ a Diophantine number, every function with a fixed point of multiplier λ is locally linearisable.

The main difference between real dynamics and holomorphic dynamics is the role that singularities play, since holomorphic mappings are local homeomorphisms except in a discrete set of points, the set of its singular values (i.e. points where some branch of f^{-1} fails to be well defined).

There are two types of singular values, *critical values* and *asymptotic values*. A point z_0 is said to be a *critical point* of $f(z)$ if $f(z)$ is not a local homeomorphism in z_0. If $z_0 \neq \infty$, this is equivalent to $f'(z_0) = 0$. Then $f(z_0)$ is a critical value. For rational functions this is the only type of singularity of f^{-1}. We say v is an asymptotic value if there is a curve $\gamma : [0,1] \to \hat{\mathbb{C}}$ such that $\gamma(t) \to \infty$ as $t \to 1$ and $f(\gamma(t)) \to v$ as $t \to 1$. From now on v will denote an asymptotic value.

The key point with singular values is that every connected component of the Fatou set is 'related' in some way to one of them. Indeed, attracting and parabolic domains must all contain a singular value, while Herman rings and Siegel disks must have a *singular orbit* (the forward orbit of a singular value) accumulating at their boundary. Likewise, Baker domains and wandering domains require the existence of infinitely many singular values (see [Sul85] and [EL92]), satisfying some conditions related to their distance to the boundary of the domains. Hence the asymptotic behavior of the singular orbits contains plenty of information about the possible stable domains in the dynamical plane.

Holomorphic dynamics was stagnating for almost 50 years until D. Sullivan proved his non-wandering domain theorem (which states that a rational map has no wandering domains, ending the classification of components started by Fatou) using as a tool the theory of quasiconformal maps. Then came a new interest, focused mainly in the study of polynomial dynamics, not only because of their simplicity but also because they are "universal", as A. Douady and J.H Hubbard showed with their theory of polynomial-like maps (see [DH85]).

Most of the work was centred around the *quadratic family* $P_c(z) = z^2 + c$, which has only one critical value. Its parameter plane contains the well-known Mandelbrot set, which has been a subject of study for many years and still is the center of several important conjectures.

There is an increasing interest in entire transcendental functions. The first one to be systematically studied was the *exponential family* $f_\lambda(z) = \lambda e^z$. This family has one asymptotic value as its only singularity and is often considered the transcendental analogue of the quadratic family.

S. Zakeri studied a family of cubic polynomials with a fixed Siegel disc. He investigated the interplay of the 2 critical values with the boundary of the Siegel disk.

The dynamics of rational maps are fairly well understood, given the fact that they possess a finite number of critical points and hence of singular values. This motivated the definition and study of special classes of entire transcendental functions like, for example, the class $\mathcal{S}$ of functions of *finite type* which are those with a finite number of singular values. A larger class is $\mathcal{B}$ the class of functions with a bounded set of singularities. These functions share many properties with rational maps, one of the most important is the fact that every connected component of the Fatou set is eventually periodic (see e.g. [EL92] or [GK86]). There is a classification of all possible periodic connected components of the Fatou set for rational maps or for entire transcendental maps in class $\mathcal{S}$. Such a component can only be part of a cycle of rotation domains (Siegel discs) or part of the basin of attraction of an attracting, super-attracting or parabolic periodic orbit. This rules out the existence of Baker or wandering domains, which are naturally unbounded, as well as the existence of Herman rings.

In this work, we are specially interested in the case of rotation domains. We say that Δ is an invariant Siegel disc if there exists a conformal isomorphism $\varphi : \Delta \to \mathbb{D}$ which conjugates f to $\mathcal{R}_\theta(z) = e^{2\pi i\theta}z$ (and φ can not be extended further), with $\theta \in \mathbb{R} \setminus \mathbb{Q} \cap (0, 1)$ called the *rotation number* of Δ. Therefore a Siegel disc is foliated by invariant closed simple curves, where orbits are dense. The existence of such Fatou components was first settled by Siegel [Sie42] who showed that if z_0 is a fixed point of multiplier $\rho = f'(z_0) = e^{2\pi i\theta}$ and θ satisfies a Diophantine condition, then z_0 is *analytically linearisable* in a neighbourhood or, equivalently, z_0 is the centre of a Siegel disc. As already mentioned, the Diophantine condition was relaxed later by Brjuno and Rüssman (for an account of these proofs see e.g. [Mil06] or the original articles [Bry69], [Rüs67]), who showed that the same is true if θ belonged to the set of Brjuno numbers $\mathcal{B}$. The relation of Siegel discs with singular orbits is as follows. Clearly Δ cannot contain critical points since the map is univalent in the disc. Instead, the boundary of Δ must be contained in the *post-critical set* $\cup_{c\in \mathrm{Sing}(f^{-1})}\overline{\mathcal{O}^+(c)}$ i.e., the accumulation set of all singular orbits. In fact something stronger is true, namely that $\partial\Delta$ is contained in the accumulation set of the orbit of at least *one* singular value (see [Mañ93]).

Our goal in this part is to describe the dynamics of the one parameter family of entire transcendental maps

$$f_a(z) = \lambda a(e^{z/a}(z + 1 - a) - 1 + a), \tag{13.1}$$

where $a \in \mathbb{C} \setminus \{0\} = \mathbb{C}^*$ and $\lambda = e^{2\pi i\theta}$ with θ being a fixed irrational Brjuno number. Observe that 0 is a fixed point of multiplier λ and therefore, for all values of the parameter a, there is a persistent Siegel disc Δ_a around $z = 0$. The functions f_a have two singular values: the image of the only critical point $w = -1$ and an asymptotic value at $v_a = \lambda a(a-1)$ which has one and only one finite pre-image at the point $p_a = a - 1$.

The motivation for studying this family of maps is manyfold. On one hand this is the simplest family of entire transcendental maps having one simple critical point and one asymptotic value with a finite pre-image (see Theorem 15.1 for the actual characterisation of f_a). The persistent Siegel disc makes it into a one-parameter family, since one of the two singular orbits must be accumulating on the boundary of Δ_a. We will see that the situation is very different, depending on which of the two singular values is doing that. Therefore, these maps could be viewed as the transcendental version of cubic polynomials with a persistent invariant Siegel disc, studied by Zakeri in [Zak99]. In our case, many new phenomena are possible with respect to the cubic situation, like unbounded Siegel discs for example; but still the two parameter planes share many features like the existence of capture components or semi-hyperbolic ones.

There is a second motivation for studying the maps f_a, namely that this one parameter family includes in some sense three emblematic examples. For $a = 1$ we have the function $f_1(z) = \lambda z e^z$; for large values of a we will see that f_a is polynomial-like of degree 2 in a neighbourhood of the origin (see Theorem 15.2); finally when $a \to 0$, the dynamics of f_a are approaching those of the exponential map $u \mapsto \lambda(e^u - 1)$, as it can be seen changing variables to $u = z/a$. Thus the parameter plane of f_a can be thought of as containing the polynomial $\lambda(z+\frac{z^2}{2})$ at infinity, its transcendental analogue f_1 at $a = 1$, and the exponential map at $a = 0$. The maps $z \mapsto \lambda z e^z$ have been widely studied (see [Gey01] and [Fag95]), among other reasons, because they share many properties with quadratic polynomials: in particular it is known that when θ is of constant type, the boundary of the Siegel disc is a quasi-circle that contains the critical point. It is not known however whether there exist values of θ for which the Siegel disc of f_1 is unbounded. In the long term we hope that this family f_a can throw some light into this and other problems about f_1.

For the maps at hand we prove the following.

Theorem A. *Let f_a as in* (13.1) *and denote by Δ_a the Siegel disc of f_a for all $a \in \mathbb{C}^*$.*

a) There exists $R, M > 0$ such that if θ is of constant type and $|a| > M$ then the boundary of Δ_a is a quasi-circle which contains the critical point. Moreover $\Delta_a \subset D(0, R)$.

b) If θ is Diophantine and the orbit of $c = -1$ belongs to a periodic basin or is eventually captured by the Siegel disc, then either the Siegel disc Δ_a is unbounded or its boundary is an indecomposable continuum.

c) If θ is Diophantine and $f_a^n(-1) \xrightarrow{n\to\infty} \infty$ the Siegel disc Δ_a is unbounded, and the boundary contains the asymptotic value.

Part a) follows from Theorem 15.2 (see Corollary 15.3 below it). The remaining parts (see Theorem 15.4) are based on Herman's proof [Her85] of the fact that Siegel discs of the exponential map are unbounded, if the rotation number is Diophantine. The proof in this case, however, presents some extra difficulties given by the free critical point and the finite pre-image of the asymptotic value.

In this part we are also interested in studying the parameter plane of f_a, which is $\mathbb{C}^*$, and in particular the connected components of its *stable* set, i.e., the parameter values for which the iterates of both singular values form a normal family in some neighbourhood. We denote this set as $\mathcal{S}$ (not to be confused with the class of finite type functions). These connected components are either *capture components*, where an iterate of the free singular value falls into the Siegel disc; or *semi-hyperbolic*, when there exists an attracting periodic orbit (which must then attract the free singular value); otherwise they are called *queer*.

The following theorem summarises the properties of semi-hyperbolic components, and is proved in Chapter 16 (see Proposition 16.4, Theorems 16.7, 16.9 and Proposition 16.10 therein). By a *component* of a set we mean a connected component.

Theorem B. *Let* f_a *be as in* (13.1) *and denote by* $v_a = \lambda(a-1)$ *its asymptotic value. Define*

$$H^c = \{a \in \mathbb{C} | \mathcal{O}^+(-1) \text{ is attracted to an attracting periodic orbit}\},$$
$$H^v = \{a \in \mathbb{C} | \mathcal{O}^+(v_a) \text{ is attracted to an attracting periodic orbit}\}.$$

a) Every component of $H^v \cup H^c$ *is simply connected.*

b) If W *is a component of* H^v *then* W *is unbounded and the multiplier map* $\chi : W \to \mathbb{D}^*$ *is the universal covering map.*

c) There is one component H_1^v *of* H^v *for which* $\mathcal{O}^+(v_a)$ *tends to an attracting fixed point.* H_1^v *contains the segment* $[r, \infty)$ *for* r *large enough.*

d) If W *is a component of* H^c*, then* W *is bounded and the multiplier map* $\chi : W \to \mathbb{D}$ *is a conformal isomorphism.*

Indeed, when the critical point is attracted by a cycle, we naturally see copies of the Mandelbrot set in parameter space. Instead, when it is the asymptotic value that acts in a hyperbolic fashion, we find unbounded exponential-like components, which can be parametrised using quasiconformal surgery.

A dichotomy also occurs with capture components. Numerically we can observe copies of quadratic Siegel discs in parameter space, which correspond to components for which the asymptotic value is being captured. There is in fact a main capture component C_0^v, the one containing $a = 1$ (see Figure 13.1), which corresponds to parameters for which the asymptotic value v_a, belongs itself to the Siegel disc. This is possible because of the existence of a finite pre-image of v_a. The centre of C_0^v is the semi-standard map $f_1(z) = \lambda z e^z$, for which zero itself is the asymptotic value.

The properties we show for capture components are summarised in the following theorem (see Chapter 17: Theorem 17.3 and Proposition 17.5 for the proof).

Theorem C. *Let* f_a *and* v_a *be as in Theorem B and* Δ_a *be the Siegel disc of* f_a *for all* $a \in \mathbb{C}^*$*. Let us define*

$$C^c = \{a \in \mathbb{C} | f_a^n(-1) \in \Delta_a \text{ for some } n \geq 1\},$$
$$C^v = \{a \in \mathbb{C} | f_a^n(v_a) \in \Delta_a \text{ for some } n \geq 0\}.$$

Then

a) C^c *and* C^v *are open sets.*

b) Every component W *of* $C^c \cup C^v$ *is simply connected.*

c) Every component W *of* C^c *is bounded.*

d) There is only one component of $C_0^v = \{a \in \mathbb{C} | v_a \in \Delta_a\}$ *and it is bounded.*

Numerical experiments show that if θ is of constant type, the boundary of C_0^v is a Jordan curve, corresponding to those parameter values for which both singular values lie on the boundary of the Siegel disc (see Figure 13.1). This is true for the slice of cubic polynomials

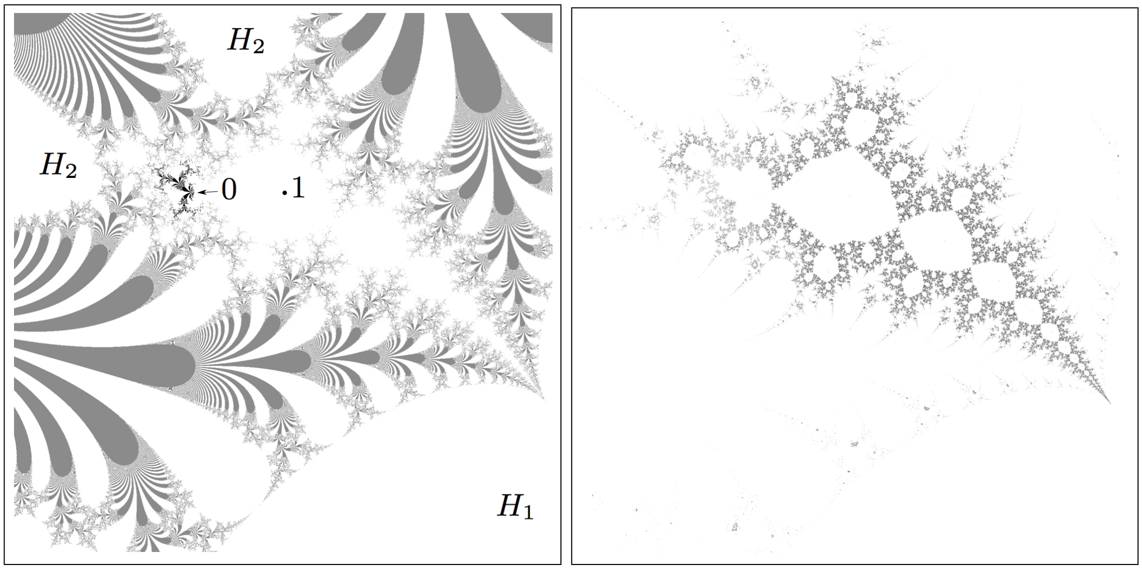

Figure 13.1: **Left:** Simple escape time plot of the parameter plane. Light grey: asymptotic orbit escapes, dark grey critical orbit escapes, white neither escapes. Regions labelled H_1 and H_2 correspond to parameters for which the asymptotic value is attracted to an attracting cycle. **Right:** The same plot, using the algorithm grounded on the results in Chapter 19, which emphasises the capture components. Upper left: $(-2, 2)$, Lower right: $(4, -4)$.

having a Siegel disc of rotation number θ, as shown by Zakeri in [Zak99], but his techniques do not apply to this transcendental case.

As we already mentioned, we are also interested in parameter values for which f_a is $\mathcal{J}$-stable (see [McM94] or [MSS83]), i.e. where both families of iterates $\{f_a^n(-1)\}_{n\in\mathbb{N}}$ and $\{f_a^n(v_a)\}_{n\in\mathbb{N}}$ are normal in a neighbourhood of a (see Chapter 18). We first show that any parameter in a capture component or a semi-hyperbolic component is $\mathcal{J}$-stable.

Proposition D. *Let f_a, H^c, H^v and C^c, C^v be as in Theorems B and C. If $a \in H \cup C$ then f_a is $\mathcal{J}$-stable, where $H = H^c \cup H^v$ and $C = C^c \cup C^v$.*

By using holomorphic motions and the proposition above, it is enough to have certain properties for one parameter value a_0, to be able to "extend" them to all parameters belonging to the same stable component. More precisely we obtain the following corollaries (see Proposition 17.6 and Corollary 18.3).

Proposition E. *Let f_a, H^c, H^v, C^c, C^v, Δ_a be as in Theorems B and C.*

a) If θ is of constant type and $a \in C_0^v$ (i.e. the asymptotic value lies inside the Siegel disc) then $\partial\Delta_a$ is a quasi-circle that contains the critical point.

b) Let $W \subset H^v \cup C^v$ be a component intersecting $\{|z| > M\}$ where M is as in Theorem A. Then, *i) if θ is of constant type, for all $a \in W$ the boundary $\partial\Delta_a$ is a quasi-circle containing the critical point.*

ii) There exist values of $\theta \in \mathbb{R}\backslash\mathbb{Q} \cap (0,1)$ such that if a component $W \subset C^v \cup H^v$ intersects $\{|z| > M\}$, then for all $a \in W$, the boundary of Δ_a is a quasi-circle not *containing the critical point.*

13.0.1 Numerical approximations

During the exploration of this family of functions we used an empirical numerical algorithm to plot capture components. Later on we formalised the method and generalised it to arbitrary holomorphic functions of degree larger than 2.

In his seminal work, Fatou [Fat20] described the set of non-normality for a family of iterates for a rational function, a set he denoted by $\mathcal{F}$ (probably for *fermé*, French for *closed*). This set would be later known as the Julia set of a holomorphic dynamical system. Among the properties he proved for this set is the fact that every point is a limit point of pre-images of almost any point in $\mathbb{C}$. Naturally, a pre-image of a point a for a function f is any point b such that $f^n(b) = a$ for some $n \in \mathbb{N}$.

Later H. Brolin re-wrote and added new results to Fatou's papers in [Bro65], giving a shorter proof of this result and its reciprocal (under suitable conditions any limit point of pre-images is in the Julia set). An interesting consequence of this result was the construction of a measure of weight 1 over the Julia set of a polynomial, based on the proof of this result.

Our interest though is in the parameter plane, not the dynamical plane. Given a dynamical system depending on one parameter, $\{f_c\}_{c\in\mathbb{C}}$, a natural object of study is the *bifurcation locus* of the family: the set of points $c \in \mathbb{C}$ where the dynamical behavior changes significantly in any neighborhood of c. This set can be described in terms of the normality of a certain sequence of functions. The best known example of a bifurcation locus is the *Mandelbrot set*, the bifurcation locus for the quadratic family $P_c(z) = z^2 + c$.

The Mandelbrot set was represented for the first time by R. Brooks and J.P. Matelski in [BM80] while they were studying the dynamics of $PSL(2,\mathbb{C})$. This set is one of the most fruitful objects of study in holomorphic dynamics during the last 30 years for its simple definition, complex behavior and universality among a wide range of dynamical systems. A remarkable property of the Mandelbrot set $\mathcal{M}$ is that any point in its boundary is a limit point of centres of hyperbolic components, centres are parameters for which the attracting orbit of the corresponding quadratic polynomial is superattracting. These components are open regions where the dynamics of the family are governed by an attracting periodic orbit, and its centres are parameters where the critical value is part of this attracting orbit. Centres are easy to compute, and by this property the resulting picture is an accurate representation of $\partial\mathcal{M}$. In Figure 13.2 you can see a plot of this set.

We prove a generalisation of this result for uni-parametric families of functions of degree at least 2 with a finite number of singular values. An additional requirement is for these singular values to be analytic functions with respect to the parameter. We prove that any point in a set we name *bifurcation set*, denoted by $\mathcal{B}$, of an arbitrary uni-parametric family of functions of degree larger than 2 is the limit of certain "pre-image parameters" (to be defined later,) which we denote by C, of almost any point in the plane. We have to exclude some points from a set which we denote by K. This fact can be considered an analogous result to Fatou's pre-image result in parameter plane, and can be used numerically to

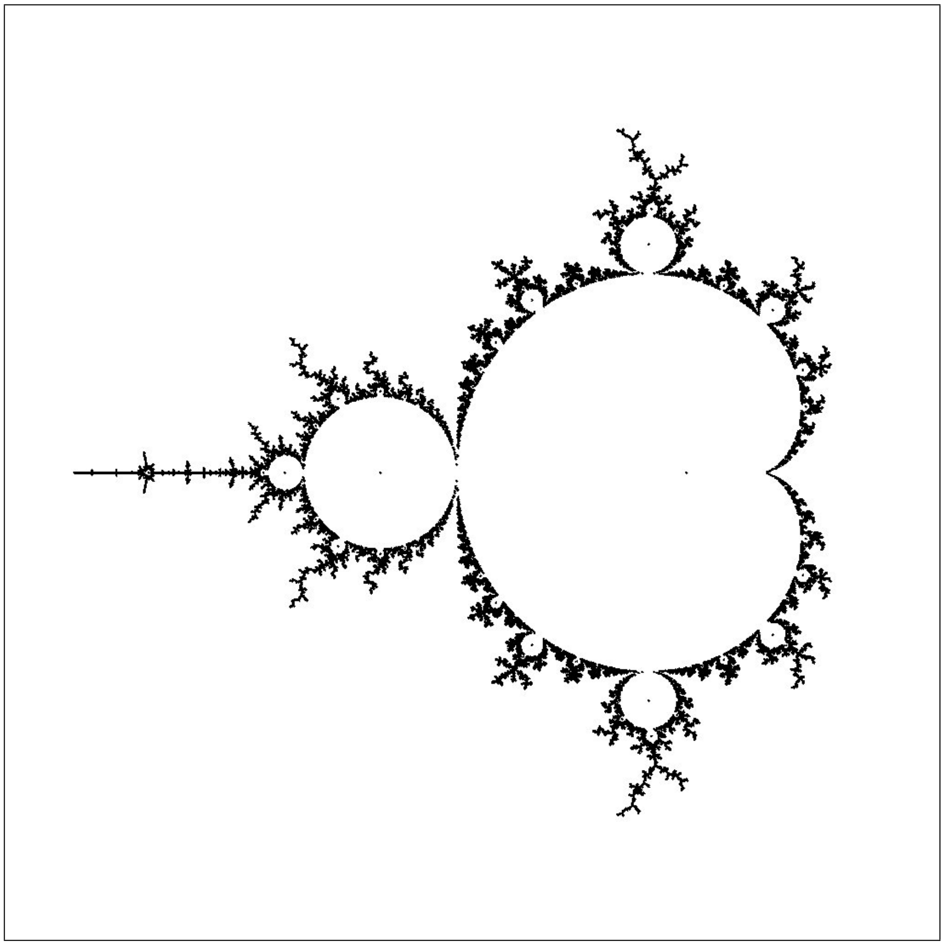

Figure 13.2: **Upper left:** (-2.2,1.6), **lower right:** (1,-1.6) To generate this image, the centres are determined numerically, solving $P_c^n(c) = 0$ for different values of n, using a Newton method

generate images of bifurcation loci, as can be seen in Figure 13.1, right and 13.2. The result by Fatou and Brolin also appears as a relatively easy corollary of this result.

Theorem F. *Let f_c be a one-parameter family of entire functions of degree at least 2 depending analytically on the parameter c. If $v_j(c)$, $1 \leq j \leq N$ denoting the singular and asymptotic values of f_c as functions of c are analytic for all j and all $c \in \mathbb{C}$, then*

$$\mathcal{B} \setminus K_\varepsilon \subseteq C'.$$

Under certain conditions on the dynamics of the family we can also prove the reverse inclusion, see Propositions 19.15 and 19.17. One of this conditions is when the family f_c has a persistent Fatou component for all values of the parameter. Observe that the family $f_a(z) = \lambda a(e^{z/a}(z+1-a) - 1 + a)$, has a non-vanishing, persistent Siegel disk as shown in Part (a) in Theorem C.

The second part is organised as follows. Chapter 14 contains statements and references of some of the results used throughout this part. Chapter 15 contains the characterisation of the family f_a, together with descriptions and images of the possible scenarios in dynamical plane. It also contains the proof of Theorem A. Chapter 16 deals with semi-hyperbolic components and contains the proof of Theorem B, split in several parts, and not necessarily in order. In the same fashion, capture components and Theorem C are treated in Chapter 17. In Chapter 18 we investigate Julia stability and contains the proofs of Propositions D and E. These results have been published in [BF10]. Finally in Chapter 19 we prove Theorem F and related results.

Chapter 14

Preliminary results in holomorphic dynamics

In this section we state results and definitions which will be useful in the sections to follow.

14.1 Quasiconformal mappings and holomorphic motions

First we introduce the concept of *quasiconformal mapping*. Quasiconformal mappings are a very useful tool in holomorphic dynamical systems as they provide a bridge between a geometric construction for a system and its analytic information. They are also a fundamental pillar for the framework of *quasiconformal surgery*, the other one being the *measurable Riemann mapping theorem*. For the groundwork on quasiconformal mappings see for example [Ahl06], and for an exhaustive account on quasiconformal surgery, see [BF14].

Definition 14.1. *Let $\mu : U \subseteq \mathbb{C} \to \mathbb{C}$ be a measurable function. Then it is a k-Beltrami form (or Beltrami coefficient, or complex dilatation) of U if $\|\mu(z)\|_\infty \leq k < 1$.*

Definition 14.2. *Let $f : U \subseteq \mathbb{C} \to V \subseteq \mathbb{C}$ be a homeomorphism. We call it k-quasiconformal if locally it has distributional derivatives in $\mathcal{L}^2$ and*

$$\mu_f(z) = \frac{\frac{\partial f}{\partial \bar{z}}(z)}{\frac{\partial f}{\partial z}(z)}$$

is a k-Beltrami coefficient. Then μ_f is called the complex dilatation of $f(z)$ (or the Beltrami coefficient of $f(z)$).

Given $f(z)$ satisfying all above except being an homeomorphism, we call it k-quasiregular.

The following technical theorem will be used when we have compositions of quasiconformal mappings and finite order mappings.

Theorem 14.3 ([FSV04, p. 750])**.** *A k-quasiconformal mapping in a domain $U \subset \mathbb{C}$ is uniformly Hölder continuous with exponent $(1-k)/(1+k)$ in every compact subset of U.*

Theorem 14.4 ((Measurable Riemann Mapping, MRMT))**.** *Let μ be a Beltrami form over $\mathbb{C}$. Then there exists a quasiconformal homeomorphism f integrating μ (i.e. the Beltrami coefficient of f is μ), unique up to composition with an affine transformation.*

Theorem 14.5 ((MRMT with dependence of parameters))**.** *Let Λ be an open set of $\mathbb{C}$ and let $\{\mu_\lambda\}_{\lambda\in\Lambda}$ be a family of Beltrami forms on $\hat{\mathbb{C}}$. Suppose $\lambda \to \mu_\lambda(z)$ is holomorphic for each fixed $z \in \mathbb{C}$ and $\|\mu_\lambda\|_\infty \leq k < 1$ for all λ. Let f_λ be the unique quasiconformal homeomorphism which integrates μ_λ and fixes three given points in $\hat{\mathbb{C}}$. Then for each $z \in \hat{\mathbb{C}}$ the map $\lambda \to f_\lambda(z)$ is holomorphic.*

The concept of holomorphic motion was in [MSS83] introduced along with the (first) λ-lemma.

Definition 14.6. *Let $h : \Lambda \times X_0 \to \hat{\mathbb{C}}$, where Λ is a complex manifold and X_0 an arbitrary subset of $\hat{\mathbb{C}}$, such that*

- *$h(0, z) = z$,*
- *$h(\lambda, \cdot)$ is an injection from X_0 to $\hat{\mathbb{C}}$,*
- *For all $z \in X_0$, $z \mapsto h(\lambda, z)$ is holomorphic.*

Then $h_\lambda(z) = h(\lambda, z)$ is called a holomorphic motion *of X.*

The following two fundamental results can be found in [MSS83] and [Slo91] respectively.

Lemma 14.7 ((First λ-lemma))**.** *A holomorphic motion h_λ of any set $X \subset \hat{\mathbb{C}}$ extends to a jointly continuous holomorphic motion of $\bar{X}$.*

Lemma 14.8 ((Second λ-lemma))**.** *Let $U \subset \mathbb{C}$ be a set and h_λ a holomorphic motion of U. This motion extends to a holomorphic motion of $\mathbb{C}$.*

14.2 Hadamard's factorisation theorem

We will need the notion of *rank* and *order* to be able to state Hadamard's factorisation theorem, which we will use in the proof of Theorem 15.1. All these results can be found in [Con78].

Definition 14.9. *Given $f : \mathbb{C} \to \mathbb{C}$ an entire function we say it is of* finite order *if there are positive constants $a > 0$, $r_0 > 0$ such that*

$$|f(z)| < e^{|z|^a}, \quad \textit{for } |z| > r_0.$$

Otherwise, we say $f(z)$ is of infinite order. We define

$$\lambda = \inf\{a \,|\, |f(z)| < \exp(|z|^a) \textit{ for } |z| \textit{ large enough}\}$$

as the order of $f(z)$.

Definition 14.10. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function with zeros $\{a_1, a_2, \ldots\}$ counted according to multiplicity. We say f is of* finite rank *if there is an integer p such that*

$$\sum_{n=1}^{\infty} |a_n|^{p+1} < \infty. \tag{14.1}$$

We say it is of rank p if p is the smallest integer verifying (14.1). If f has a finite number of zeros then it has rank 0 by definition.

Definition 14.11. *An entire function $f : \mathbb{C} \to \mathbb{C}$ is said to be of* finite genus *if it has finite rank p and it factorises as:*

$$f(z) = z^m e^{g(z)} \cdot \prod_{n=1}^{\infty} E_p(z/a_n), \tag{14.2}$$

where $g(z)$ is a polynomial, a_n are the zeros of $f(z)$ as in the previous definition and

$$E_p(z) = (1-z)e^{z+\frac{z^2}{2}+\ldots+\frac{z^p}{p}}.$$

We define the genus of $f(z)$ as $\mu = \max\{\deg g, \operatorname{rank} f\}$

Theorem 14.12. *If f is an entire function of finite genus μ then f is of finite order $\lambda < \mu + 1$.*

The converse of this theorem is also true, as we see below.

Theorem 14.13 ((Hadamard's factorisation))**.** *Let f be an entire function of finite order λ. Then f is of finite genus $\mu \leq \lambda$.*

Observe that Hadamard's factorisation theorem implies that every entire function of finite order can be factorised as in (14.2).

14.3 Siegel discs

The following theorem (which is an extension of the original theorem by C.L. Siegel) gives arithmetic conditions on the rotation number of a fixed point to ensure the existence of a Siegel disc around it. J-C. Yoccoz proved that this condition is sharp in the quadratic family. The proof of this theorem can be found in [Mil06].

Theorem 14.14 ((Brjuno-Rüssmann))**.** *Let $f(z) = \lambda z + \mathcal{O}(z^2)$. If $\frac{p_n}{q_n} = [a_1; a_2, \ldots, a_n]$ is the n-th convergent of the continued fraction expansion of θ, where $\lambda = e^{2\pi i\theta}$, and*

$$\sum_{n=0}^{\infty} \frac{\log(q_{n+1})}{q_n} < \infty, \tag{14.3}$$

then f is locally linearisable.

Irrational numbers with this property are called of *Brjuno type.*

We define the notion of *conformal capacity* as a measure of the "size" of Siegel discs.

Definition 14.15. *Consider the Siegel disc Δ and the unique linearising map $h : \mathbb{D}(0,r) \xrightarrow{\sim} \Delta$, with $h(0)$ and $h'(0) = 1$. The radius $r > 0$ of the domain of h is called the* conformal capacity *of Δ and is denoted by $\kappa(\Delta)$.*

A Siegel disc of capacity r contains a disc of radius $\frac{r}{4}$ by Koebe 1/4 Theorem.

The following theorem (see [Yoc95] for a proof) shows that Siegel discs can not shrink indefinitely.

Theorem 14.16. *Let $0 < \theta < 1$ be an irrational number of Brjuno type, and let $\Phi(\theta) = \sum_{n=1}^{\infty}(\log q_{n+1}/q_n) < \infty$ be the Brjuno function. Let $S(\theta)$ be the space of all univalent functions $f : \mathbb{D} \to \mathbb{C}$ with $f(0) = 0$ and $f'(0) = e^{2\pi i\theta}$. Finally, define $\kappa(\theta) = \inf_{f\in S(\theta)} \kappa(\Delta_f)$, where $\kappa(\Delta)$ is the conformal capacity of Δ. Then, there is a universal constant $C > 0$ such that $|\log(\kappa(\theta)) + \Phi(\theta)| < C$.*

We will also need a well-known theorem about the regularity of the boundary of Siegel discs of quadratic polynomials. Its proof can be found in [Dou87].

Theorem 14.17 ((Douady-Ghys))**.** *Let θ be of bounded type, and $p(z) = e^{2\pi i\theta}z + z^2$. Then the boundary of the Siegel disc around 0 is a quasi-circle containing the critical point.*

The following is a theorem by M. Herman concerning critical points on the boundary of Siegel discs. Its proof can be found in [Her85, p. 601]

Theorem 14.18 ((Herman))**.** *Let $g(z)$ be an entire function such that $g(0) = 0$ and $g'(0) = e^{2\pi i\alpha}$ with α Diophantine. Let Δ be the Siegel disc around $z = 0$. If Δ has compact closure in $\mathbb{C}$ and $g|_{\bar{\Delta}}$ is injective then $g(z)$ has a critical point in $\partial\Delta$.*

In fact, the set of Diophantine numbers could be replaced by the set $\mathcal{H}$ of Herman numbers, where $\mathcal{D} \subsetneq \mathcal{H} \subsetneq \mathcal{B}$, as shown in [Yoc02].

Finally, we state a result which is a combination of Theorems 1 and 2 in [Rem08].

Definition 14.19. *We define the class $\mathcal{B}$ as the class of entire functions with a bounded set of singular values.*

Theorem 14.20 ((Rempe))**.** *Let $f \in \mathcal{B}$ with $S(f) \subset \mathcal{J}(f)$, where $S(f)$ denotes the set of singular values of f. If Δ is a Siegel disc of $f(z)$ which is unbounded, then $S(f) \cap \partial\Delta \neq \emptyset$.*

14.4 Topological results

To prove Theorem 15.4 we need to extend a result of Rogers in [Jr.92] to a larger class of functions, namely functions of finite order with no wandering domains.

The result we need follows some preliminary definitions.

Definition 14.21. *A* continuum *is a compact connected non-void metric space.*

Definition 14.22. *A pair (g, Δ) is a local Siegel disc if g is conformally conjugate to an irrational rotation on Δ and g extends continuously to $\bar{\Delta}$.*

Definition 14.23. *We say a bounded local Siegel disc $(f|_{\Delta}, \Delta)$ is* irreducible *if the boundary of Δ separates the centre of the disc from ∞, but no proper closed subset of the boundary of Δ has this property.*

Theorem 14.24. *Suppose Δ is a Siegel disc of a function f in the class $\mathcal{B}$, and $\partial\Delta$ is a decomposable continuum. Then $\partial\Delta$ separates $\mathbb{C}$ into exactly two complementary domains.*

For the proof of this theorem we will need the following ingredients which will be only used in this proof. The topological results can be found in any standard reference on algebraic topology.

Theorem 14.25. *If (Δ, f_θ) is a bounded irreducible local Siegel disc, then the following are equivalent:*

- *$\partial\Delta$ is a decomposable continuum,*
- *each pair of impressions is disjoint, and*
- *the inverse of the map $\varphi : \mathbb{D} \to \Delta$ extends continuously to a map $\Psi : \partial\,\Delta \to S^1$ such that for each $\eta \in S^1$, the fibre $\Psi^{-1}(\eta)$ is the impression $I(\eta)$.*

Proof. See [Jr.92].

□

Theorem 14.26 ((Vietoris-Begle))**.** *Let X and Y be compact metric spaces and $f : X \to Y$ continuous and surjective and suppose that the fibres are acyclic, i.e.*

$$\tilde{H}^r(f^{-1}(y)) = 0, 0 \leq r \leq n-1, \quad \forall y \in Y,$$

where $\tilde{H}^r$ denotes the r-th reduced co-homology group. Then, the induced homomorphism

$$f^* : \tilde{H}^r(Y) \to \tilde{H}^r(X)$$

is an isomorphism for $r \leq n-1$ and is a surjection for $r = n$.

Theorem 14.27 ((Alexander's duality))**.** *Let X be a compact sub-space of the Euclidean space E of dimension n, and Y its complement in E. Then,*

$$\tilde{H}_q(X) \cong \tilde{H}^{n-q-1}(Y)$$

where $\tilde{H}_*$, $\tilde{H}^*$ stands for Čech reduced homology and reduced co-homology respectively.

Remark 14.28. *The case $E = S^2$, $X = S^1$ (or $H^1(X) = \mathbb{Z}$) is Jordan's Curve Theorem.*

Definition 14.29. *If X is a compact subset of $\mathbb{C}$, then the three following conditions are equivalent:*

- *X is cellular,*
- *X is a continuum that does not separate $\mathbb{C}$,*
- *$H^1(X) = 0 = \tilde{H}^0(X)$,*

where $\tilde{H}^r(X)$ stands for reduced Čech co-homology and $H^r(X)$ for Čech co-homology.

Definition 14.30. *We say a map $f : X \to Y$ is cellular if each fibre $f^{-1}(y)$ is a cellular set.*

Remark 14.31. *Recall that $\tilde{H}^1(X) \cong H^1(X)$.*

Remark 14.32. *By definition and in view of the Vietoris-Begle Theorem, cellular maps induce isomorphisms between first reduced co-homology groups.*

Proof of Theorem 14.24. We first show that any Siegel disc Δ for $f \in \mathcal{B}$ is a bounded irreducible local Siegel disc. Recall that we define the escaping set of a function $f : \mathbb{C} \to \mathbb{C}$ as:

$$I(f) = \{z |\, f^n(z) \to \infty \text{ as } n \to \infty\}.$$

Clearly $(f|_{\Delta}, \Delta)$ is a local Siegel disc. It is also bounded by assumption. The only thing left to prove is it is irreducible. If X is a proper closed subset of $\partial\Delta$ and if x is a point of $\partial\Delta \backslash X$, then there is a small disc B containing x and missing X. Since $x \in \partial\Delta$, the disc B contains a point of Δ. As $x \in \partial\Delta \subset \mathcal{J}(f)$, the disc B contains a point $y \in I(f)$. Now, Theorem 3.1.1 in [Rot05] states that for $f \in \mathcal{B}$ the set $I(f) \cup \{\infty\}$ is arc-connected, and thus y can be arc-connected to ∞ through points in $I(f)$. It follows that the centre of the Siegel disc and infinity are in the same complementary domain of $\mathbb{C}\backslash X$.

Clearly $\Psi(\eta)$ for $\eta \in S^1$ is a continuum, which is called the impression of η and denoted $\mathrm{Imp}(\eta)$. Furthermore, $\mathrm{Imp}(\eta)$ does not separate $\mathbb{C}$. Indeed, if U is a bounded complementary domain of $\mathrm{Imp}(\eta)$, then either $f^n(U) \cap U = \emptyset$ for all n or there are intersection points. Clearly $f^n(U) \cap U = \emptyset$, as if $f^n(U) \cap U \neq \emptyset$ for some n, then $f^n(\partial U) \cap \partial U \neq \emptyset$, but this implies $\mathrm{Imp}(\eta) = F^n(\mathrm{Imp}(\eta)) = \mathrm{Imp}(\eta + n\theta)$ and as $\partial\Delta$ is a decomposable continuum, each pair of impressions is disjoint by Theorem 14.25 and this intersection must be empty. Hence, $f^n(U) \cap U = \emptyset$ for all $n \in \mathbb{N}$ which implies U is a wandering domain, and for functions in $\mathcal{B}$ it is known there are no wandering domains (see [EL92]).

Therefore $\mathrm{Imp}(\eta)$ is a cellular set and thus Ψ is a cellular map. The Vietoris-Begle theorem implies that the induced homomorphism $\Psi^* : \tilde{H}^1(S^1) \to \tilde{H}^1(\partial\Delta)$ is an isomorphism (see Remark 14.32). Then $\tilde{H}^1(\partial\Delta) = \mathbb{Z}$ and by Alexander's duality $\partial\Delta$ separates $\mathbb{C}$ into exactly two complementary domains (see Remark 14.28).

□

Chapter 15

The (entire transcendental) family f_a

In this chapter we describe the dynamical plane of the family of entire transcendental maps

$$f_a(z) = \lambda a(e^{z/a}(z + 1 - a) - 1 + a),$$

for different values of $a \in \mathbb{C}^*$, and for $\lambda = e^{2\pi i\theta}$, with θ being a fixed irrational Brjuno number (unless otherwise specified). For these values of λ, in view of Theorem 14.14 there exists an invariant Siegel disc around $z = 0$, for any value of $a \in \mathbb{C}^*$.

We start by showing that this family contains all possible entire transcendental maps with the properties we require.

Theorem 15.1. *Let $g(z)$ be an entire transcendental function having the following properties*

1. *finite order,*
2. *one asymptotic value v, with exactly one finite pre-image p of v,*
3. *a fixed point (normalised to be at 0) of multiplier $\lambda \in \mathbb{C}$,*
4. *a simple critical point (normalised to be at $z = -1$) and no other critical points.*

Then $g(z) = f_a(z)$ for some $a \in \mathbb{C}$ with $v = \lambda a(a - 1)$ and $p = a - 1$. Moreover no two members of this family are conformally conjugate.

Proof. As $g(z) - v = 0$ has one solution at $z = p$, we can write:

$$g(z) = (z - p)^m e^{h(z)} + v,$$

where, by Hadamard's factorisation theorem (Theorem 14.13), $h(z)$ must be a polynomial, as $g(z)$ has finite order. The derivative of this function is

$$g'(z) = e^{h(z)}(z - p)^{m-1}(m + (z - p)h'(z)),$$

whose zeros are the solutions of $z-p=0$ (if $m>1$) and the solutions of $m+(z-p)h'(z)=0$. But as the critical point must be simple and unique, $m=1$ and $\deg h'(z)=0$. Therefore

$$g(z)=(z-p)e^{\alpha z+\beta}+v,$$

and from the expression for the critical points,

$$\alpha=\frac{1}{p+1}.$$

Moreover from the fact that $g(0)=0$ we can deduce that $v=pe^{\beta}$, and from condition 3, i.e. $g'(0)=\lambda$, we obtain $e^{\beta}=\lambda(1+p)$. All together yields

$$g(z)=\lambda(z-p)(1+p)e^{z/(1+p)}+\lambda p(1+p).$$

Writing $a=p+1$ we arrive to

$$g(z)=\lambda a(z-a+1)e^{z/a}+\lambda a(a-1)=f_a(z),$$

as we wanted.

Finally, if $f_a(z)$ and $f_{a'}(z)$ are conformally conjugate, the conjugacy must fix 0,-1 and ∞ and therefore is the identity map.

□

15.1 Dynamical planes

For any parameter value $a\in\mathbb{C}^*$, the Fatou set always contains the Siegel disc Δ_a and all its pre-images. Moreover, one of the singular orbits must be accumulating on the boundary of Δ_a. The other singular orbit may then either eventually fall in Δ_a, or accumulate in $\partial\Delta_a$, or have some independent behaviour. In the first case we say that the singular value is captured by the Siegel disc. More precisely we define the *capture parameters* as

$$\begin{aligned}C=\{a\in\mathbb{C}^*|f_a^n(-1)\in\Delta_a \text{ for some } n\geq 1 \text{ or}\\ f_a^n(v_a)\in\Delta_a \text{ for some } n\geq 0\}\end{aligned}$$

Naturally C splits into two sets $C=C^c\cup C^v$ depending on whether the captured orbit is the critical orbit (C^c) or the orbit of the asymptotic value (C^v). We will follow this convention, superscript c for critical and superscript v for asymptotic, throughout this part.

In the second case, that is, when the free singular value has an independent behaviour, it may happen that it is attracted to an attracting periodic orbit. We define the *semi-hyperbolic parameters* H as

$$H=\{a\in\mathbb{C}^*|f_a \text{ has an attracting periodic orbit}\}.$$

Again this set splits into two sets, $H=H^c\cup H^v$ depending on whether the basin contains the critical point or the asymptotic value.

Notice that these four sets C^c, C^v, H^c, H^v are pairwise disjoint, since a singular value must always belong to the Julia set, as its orbit has to accumulate on the boundary of the Siegel disc.

In the following sections we will describe in detail these regions of parameter space, but let us first show some numerical experiments. For all figures we have chosen $\theta = \frac{1+\sqrt{5}}{2}$, the golden mean number.

Figure 13.1 (in the Introduction) shows the parameter plane, where the left side is made with a simple escaping algorithm. The component containing $a = 1$ is the main capture component for which v_a itself belongs to the Siegel disc. On the right side we see the same parameters, drawn with a different algorithm. Also in Figure 13.1, we can partially see the sets H_1^v and H_2^v (and infinitely many others), where the sub-indices denote the period of the attracting orbit.

In Figure 15.1 (left) we can see the dynamical plane for a chosen in one of the semi-hyperbolic components of Figure 13.1, where the Siegel disc and the attracting orbit and corresponding basin are shown in different colours.

Figure 15.1 (right) shows the dynamical plane of $f_1(z) = \lambda z e^z$, the semi-standard map. In this case the asymptotic value $v_1 = 0$ is actually the centre of the Siegel disc. It is still an open question whether, for some exotic rotation number, this Siegel disc can be unbounded. For bounded type rotation numbers, as the one in the figure, the boundary is a quasi-circle and contains the critical point [Gey01].

Figure 15.2, left side, shows a close-up view of the parameter region around $a = 0$, and in the right side, we can see a closer view of a random spot, in particular a region in H^c, that is, parameters for which the critical orbit is attracted to a cycle.

One of these dynamical planes is shown in Figure 15.4. Observe that the orbit of the asymptotic value is now accumulating on $\partial\Delta_a$ and we may have unbounded Siegel discs.

Finally Figure 15.3 shows some components of C^v, where the orbit of the asymptotic value is captured by the Siegel disc.

We start by considering large values of $a \in \mathbb{C}^*$. By expanding $f_a(z)$ into a power series it is easy to check that as $a \to \infty$ the function approaches the quadratic polynomial $\lambda z(1 + z/2)$. It is therefore not surprising that we have the following theorem, which we shall prove at the end of this section.

Theorem 15.2. *There exists $M > 0$ such that the entire transcendental family $f_a(z)$ is polynomial-like of degree two for $|a| > M$. Moreover, the Siegel disc Δ_a (and in fact, the full small filled Julia set) is contained in a disc of radius R where R is a constant independent of a.*

Figure 15.5 shows the dynamical plane for $a = 15+15i$, $\lambda = e^{2\pi(\frac{1+\sqrt{5}}{2})i}$ where we clearly see the Julia set of the quadratic polynomial $\lambda z(1 + z/2)$, shown on the right side.

An immediate consequence of Theorem 15.2 above follows from Theorem 14.17. This is Part a) of Theorem A in the Introduction.

Corollary 15.3. *For $|a| > M$, and θ of constant type the boundary of Δ_a is a quasi-circle that contains the critical point.*

In fact we will prove in Section 17 (Proposition 17.6) that the same occurs in many other situations like, for example, when the asymptotic value lies itself inside the Siegel disc or when it is attracted to an attracting periodic orbit. See Figures 15.1 (Left) and 15.5.

In fact we believe that this family provides examples of Siegel discs with an asymptotic value on the boundary, but such that the boundary is a quasi-circle containing also the

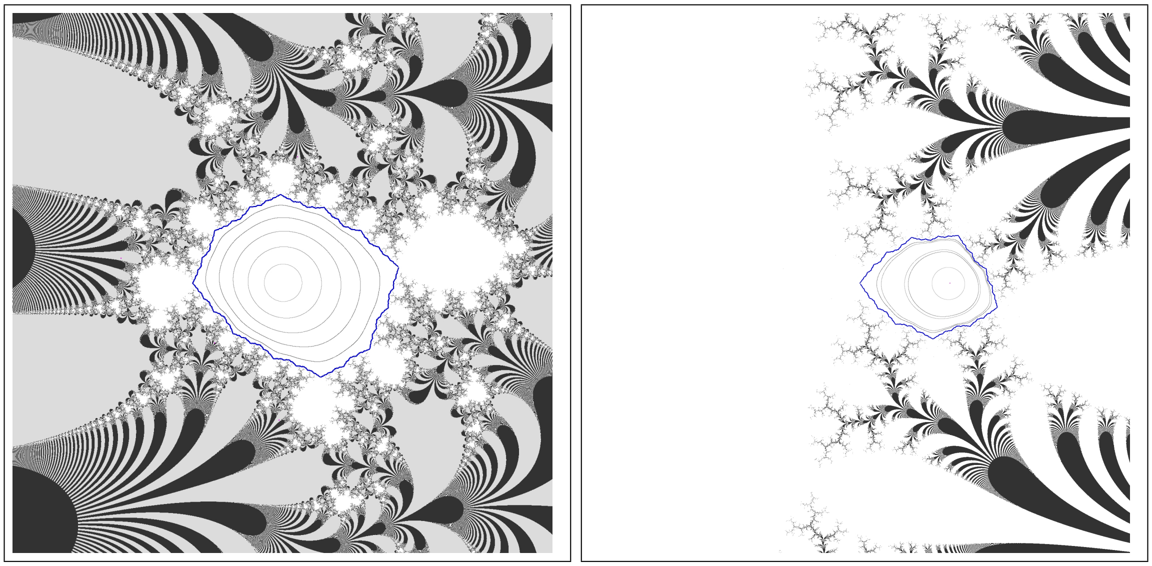

Figure 15.1: **Left:** Julia set for a parameter in a semi-hyperbolic component (for the asymptotic value). Details: $a = (-0.62099, 0.0100973)$, upper left: $(-4, 3)$, lower right: $(2, -3)$. In light grey we see the attracting basin of the attracting cycle, and in white the Siegel disc and its pre-images. **Right:** Julia set of the semi-standard map, corresponding to $f_1(z) = \lambda z e^z$. Upper left: $(-3, 3)$, lower right: $(3, -3)$. The boundary of the Siegel disc around 0 is shown, together with some of the invariant curves. The Fatou set consists exclusively of the Siegel disc and its pre-images.

critical point. A parameter value with this property could be given by $a_0 \approx 1.544913893 + 0.32322773i \in \partial C_0^v$, $\lambda = e^{2\pi(\frac{1+\sqrt{5}}{2})i}$ (see Figure 15.6) where the asymptotic value and the critical point coincide.

The opposite case, that is, the Siegel disc being unbounded and its boundary non-locally connected also takes place for certain values of the parameter a, as we show in the following theorem, which covers parts b) and c) of Theorem A.

Theorem 15.4. *Let θ be Diophantine*[1]*, then:*

a) If $f_a^n(-1) \to \infty$ then Δ_a is unbounded and $v_a \in \partial\Delta_a$,

b) if $a \in H^c \cup C^c$ either Δ_a is unbounded or $\partial\Delta_a$ is an indecomposable continuum.

Proof. The proof of the first part is a slight modification of Herman's proof for the exponential map (see [Her85]). The difference is given by the fact that the asymptotic value of $f_a(z)$ is not an omitted value, and by the existence of a second singular value. For both parts we need the following definitions. Suppose that $\Delta := \Delta_a$ is bounded and let Δ_i denote the bounded components of $\mathbb{C}\backslash\partial\Delta$. Let Δ_∞ be the unbounded component. Since

[1]Diophantine numbers can actually be replaced by the larger class of irrational numbers $\mathcal{H}$ (see [Yoc02], [PM97])

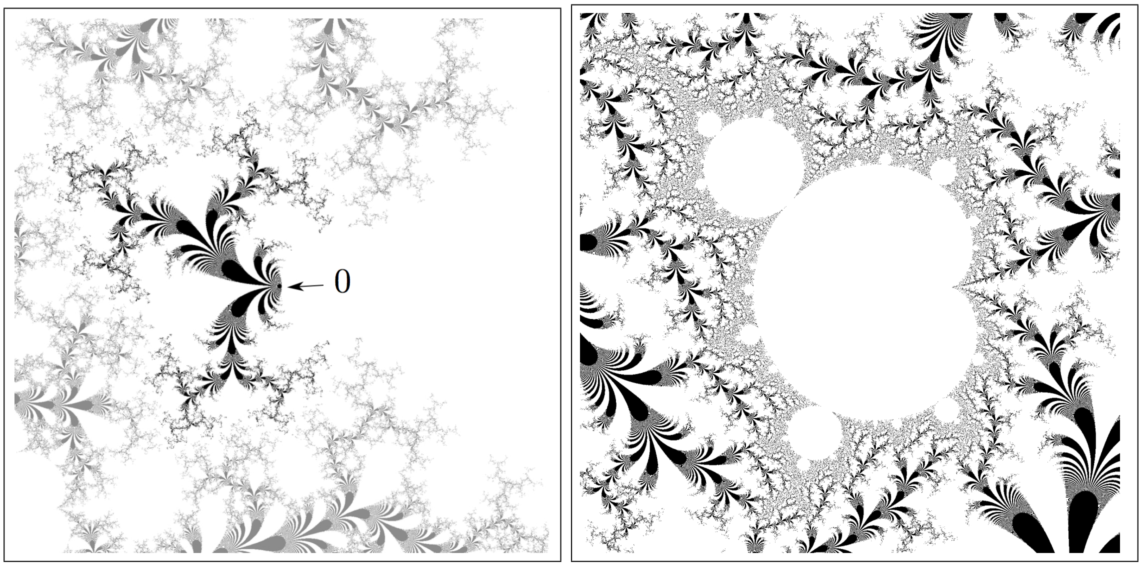

Figure 15.2: **Left:** "Crab"-like structure corresponding to escaping critical orbits (dark grey). Upper left: $(-0.6, 0.6)$, lower right: $(0.6, -0.6)$. In light grey we see parameters for which the orbit of v_a escapes. **Right:** Baby Mandelbrot set from a close-up in the "crab like" structure. Upper left: $(-0.336933, 0.1128)$, lower right: $(-0.322933, 0.08828)$.

Δ and Δ_i are simply connected, then $\hat{\Delta} := \mathbb{C}\backslash\Delta_\infty$ is compact and simply connected. By the Maximum Modulus Principle and Montel's theorem, $\{f_a^n|_{\Delta_i}\}_{n\in\mathbb{N}}$ form a normal family and hence Δ_i is a Fatou component. We also have that $\partial\Delta = \partial\Delta_\infty$, although this does not imply a priori that $\Delta_i = \emptyset$ (see Wada lakes and similar examples [Rog98]).

Proof of Part a). Now suppose the critical orbit is unbounded. Then $c \in \mathcal{J}(f_a)$, but $\hat{\Delta} \cap \mathcal{J}(f_a)$ is bounded and invariant. Hence $c \notin \hat{\Delta}$.

We claim that there exists U a simply connected neighbourhood of $\hat{\Delta}$ such that U contains no singular values. Indeed, suppose that the asymptotic value v_a belongs to $\hat{\Delta}$. Since $v_a \in \mathcal{J}(f)$, then $v_a \in \partial\Delta$. But Δ is bounded, and $f|_{\partial\Delta}$ is surjective, hence the only finite pre-image of v_a, namely $a-1$, also belongs to $\partial\Delta$. This means that v_a is not acting as an asymptotic value but as a regular point, since $f(z)$ is a local homeomorphism from $a-1$ to v_a.

Hence there are no singular values in U. It follows that

$$f|_{f^{-1}(U)} : f^{-1}(U) \to U$$

is a covering and $f^{-1} : \Delta \to \Delta$ extends to a continuous map $h(z)$ from $\bar{\Delta}$ to $\bar{\Delta}$. Since $hf = fh = id$, it follows that $f|_{\partial\Delta}$ is injective. As this mapping is always surjective, it is a homeomorphism. We now apply Herman's main theorem in [Her85] (see Theorem 14.18) to conclude that $\partial\Delta$ must have a critical point, which contradicts our assumptions. It follows that Δ is unbounded. Finally Theorem 14.20 implies that $v_a \in \partial\Delta_a$.

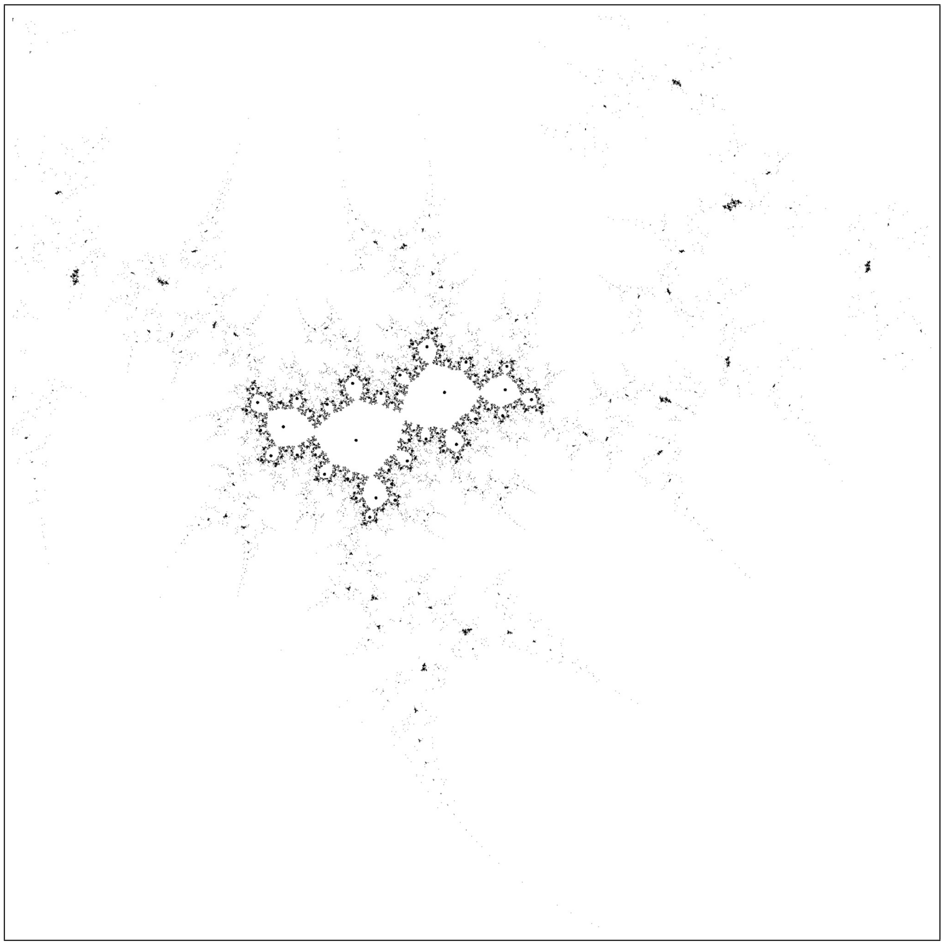

Figure 15.3: A close up of Figure 13.1, Right. A quadratic Siegel disc in parameter space, corresponding to a capture zone for the asymptotic value. Upper left: $(7.477, 4.098)$, Lower right: $(7.777, 3.798)$.

Proof of part b). The work was done already when proving Theorem 14.24. Since f_a has 2 singular values, it belongs to the Eremenko-Lyubich class $\mathcal{B}$. Hence, if we assume that Δ_a is bounded, it follows from Theorem 14.24 that $\partial\Delta_a$ is either an indecomposable continuum or $\partial\Delta_a$ separates $\hat{\mathbb{C}}$ in exactly two complementary domains. This would imply that $\hat{\Delta} = \bar{\Delta}$ and by hypothesis $-1 \notin \bar{\Delta}$. The same arguments as in Part a concludes the proof. $\square$

Remark 15.5. *In part a) it is not strictly necessary that the critical orbit tends to infinity. In fact we only use that the critical point is in $\mathcal{J}(f_a)$ and some element of its orbit belongs to Δ_∞.*

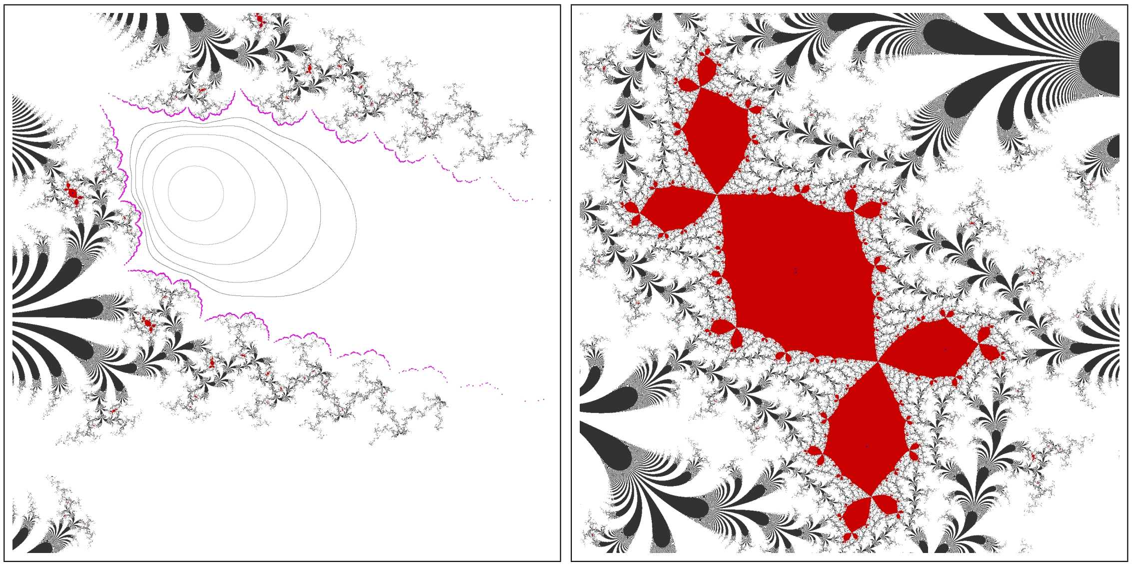

Figure 15.4: **Left:** Julia set for a parameter in a semi-hyperbolic component for the critical value. By Theorem 15.4 this Siegel disc is unbounded. Details: $a = (-0.330897, 0.101867)$, upper left: $(-1.5, 1.5)$., lower right: $[3, -3]$. **Right:** Close-up of a basin of attraction of the attracting periodic orbit. Upper left: $(-1.1, 0.12)$, lower right: $(-0.85, -0.13)$.

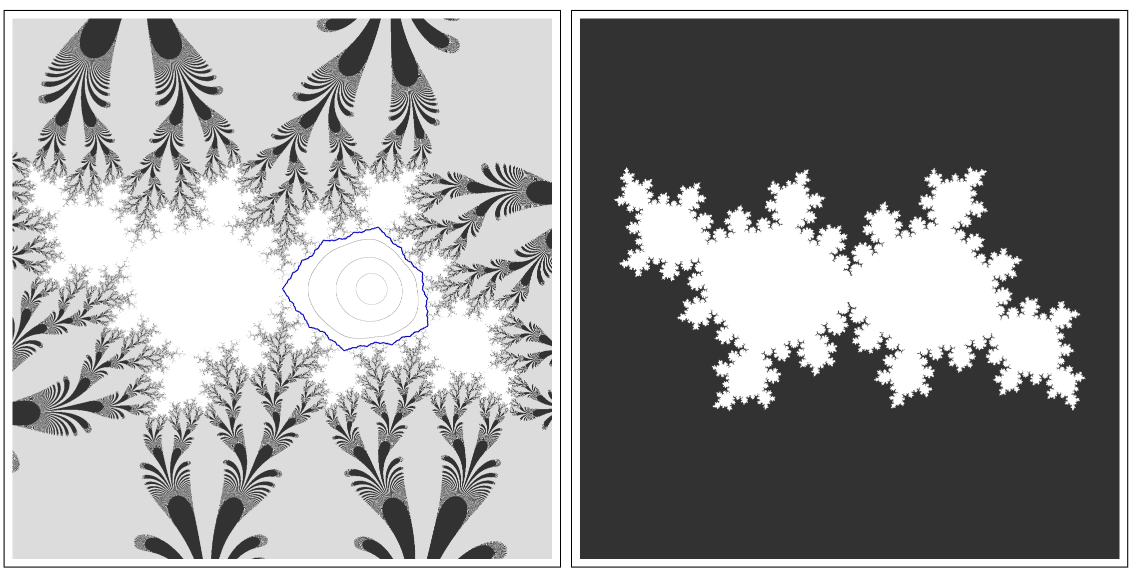

Figure 15.5: **Left:** Julia set corresponding to a polynomial-like mapping. Details: $a = (15, -15)$, upper left: $(-4, 3)$, lower right: $(2, -3)$. **Right:** Julia set corresponding to the related polynomial. Upper left: $(-4, 3)$, lower right: $(-2, 3)$

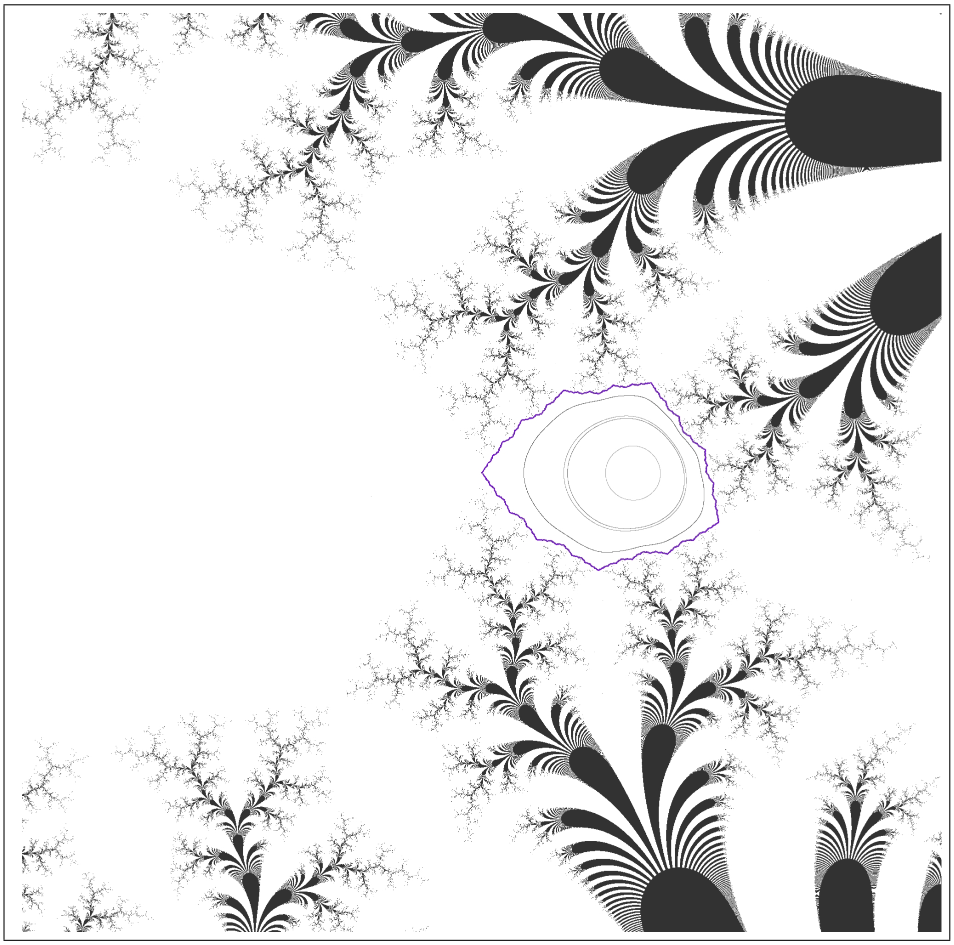

Figure 15.6: Julia set for the parameter $a \approx 1.544913893 + 0.32322773i$. The parameter is chosen so that the critical point and the asymptotic value are at the same point, hence both singular orbits accumulate on the boundary. Upper left: $(-1.5, 1.5)$, lower right: $(3, -3)$.

15.2 Large values of $|a|$: Proof of theorem 15.2

Let $D := \{w \in \mathbb{C} | |w| < R\}$, $\gamma = \partial D$, $g(z) = \lambda z(z/2 + 1)$. If we are able to find some R and S such that

$$|g(z) - w|_{\substack{z\in\gamma \\ w\in D}} \geq S,$$
$$|f(z) - g(z)|_{z\in\gamma} < S,$$

then we will have proved that $D \subset f(D)$ and $\deg f = \deg g = 2$ by Rouché's theorem. Indeed, given $w \in D$ $f(z) - w = 0$ has the same number of solutions as $g(z) - w = 0$, which is exactly 2 counted according with multiplicity. Clearly,

$$|g(z) - w|_{\substack{z\in\gamma \\ w\in D}} \geq |g(z)|_{z\in\gamma} - |w|_{w\in D} \geq (R^2/2 - R) - R.$$

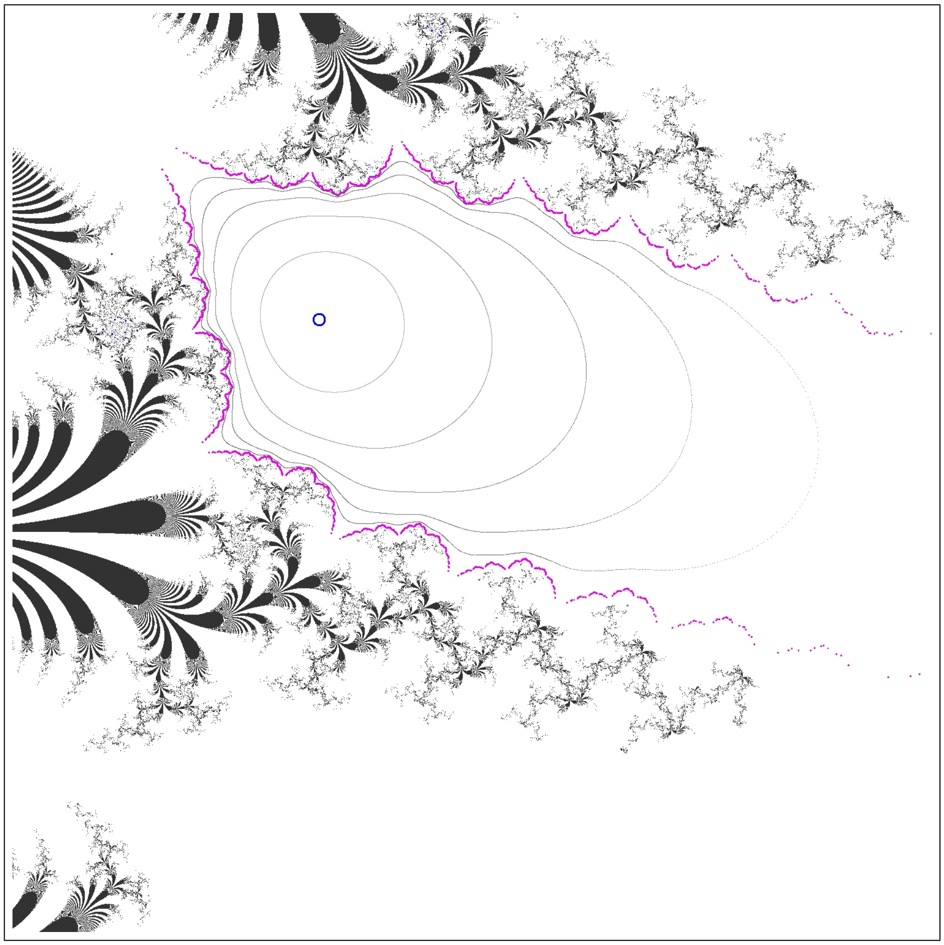

Figure 15.7: Point in a capture component for the critical value, so that the Siegel disc is either unbounded or an indecomposable continuum. Details: $a = (-0.33258, 0.10324)$, upper left: $(-1.5, 1.5)$, lower right: $(-3, -3)$.

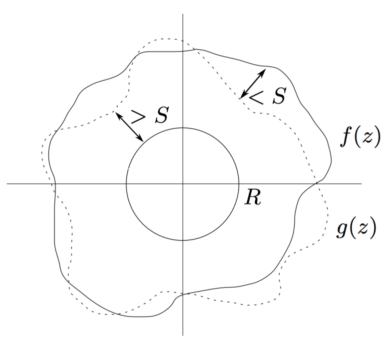

Figure 15.8: Sketch of inequalities

Define $S := R^2/2 - 2\,R$. Since we want $S > R > 0$, we require that $R > 4$. Now expand $\exp(z/a)$ as a power series and let $|a| = b > R$. Then

$$|f(z) - g(z)| = \left| \frac{z^3}{2a} + \frac{z^2}{2a} - a(z+1-a)\sum_{j=3}^{\infty} \frac{z^j}{j!a^j} \right| \leq$$

$$\leq \frac{R^3}{2b} + \frac{R^2}{2b} + \frac{R^3}{6b^3}(3b^2 e^{R/b}) = \frac{R^2}{2b}(1 + (1 + e^{R/b})R).$$

This last expression can be bounded by $\frac{R^2}{2b}(1 + 4R)$ as $b > R$. Now we would like to find some R such that for $b > R$, $\frac{R^2}{2b}(1 + 4R) < S$. It follows that

$$\frac{R + 4R^2}{R - 4} < b,$$

and this function of R has a local minimum at $R \approx 8.12311$. We then conclude that given $R = 8.12311$ b must be larger than 65.9848.

This way the triple $(f_a, D(0,R), f(D(0,R)))$ is polynomial-like of degree two for $|a| \geq 66$.

Remark 15.6. *Numerical experiments suggest that* $|a| > 10$ *would be enough.*

Chapter 16

Semi-hyperbolic components: Proof of Theorem B

In this chapter we deal with the set of parameters a such that the free singular value is attracted to a periodic orbit. We denote this set by H and it naturally splits into the pairwise disjoint subsets

$$H_p^v = \{a \in \mathbb{C} | \mathcal{O}^+(v_a) \text{ is attracted to a periodic orbit of period } p\}$$
$$H_p^c = \{a \in \mathbb{C} | \mathcal{O}^+(-1) \text{ is attracted to a periodic orbit of period } p\}.$$

where $p \geq 1$. We will call these sets *semi-hyperbolic components.*

It is immediate from the definition that semi-hyperbolic components are open. Also connecting with the definition in the previous section we have $H^c = \cup_{p\geq 1} H_p^c$ and $H^v = \cup_{p\geq 1} H_p^v$.

As a first observation note that, by Theorem 15.2, every connected component of H_p^c for every $p \geq 1$ is bounded. Indeed, for large values of a the function $f_a(z)$ is polynomial-like and hence the critical orbit cannot be converging to any periodic cycle, which partially proves Theorem B, Part d). We shall see that, opposite to this fact, all components of H_p^v are unbounded. We start by showing that no semi-hyperbolic component in H_p^c can surround $a = 0$, by showing the existence of continuous curves of parameter values, leading to $a = 0$, for which the critical orbit tends to ∞. These curves can be observed numerically in Figure 15.2 in the previous section.

Proposition 16.1. *If γ is a closed curve contained in a component W of $H^c \cup C^c$, then* $\mathrm{ind}(\gamma, 0) = 0$.

Proof. We shall show that there exists a continuous curve $a(t)$ such that $f_{a(t)}^n(-1) \overset{n\to\infty}{\longrightarrow} \infty$ for all t. It then follows that $a(t)$ would intersect any curve γ surrounding $a = 0$. But if $\gamma \subset H^c \cup C^c$, this is impossible. For $a \neq 0$ we conjugate f_a by $u = z/a$ and obtain the family $g_a(u) = \lambda(e^u(au + 1 - a) - 1 + a)$. Observe that $g_0(u) = \lambda(e^u - 1)$. The idea of the proof is the following. As a approaches 0, the dynamics of g_a converge to those of g_0. In particular we find continuous invariant curves $\{\Gamma_k^a(t), k \in \mathbb{Z}\}_{t\in(0,\infty)}$ (Devaney hairs or dynamic rays) such that $\mathrm{Re}\,\Gamma_k^a(t) \overset{t\to\infty}{\longrightarrow} \infty$ and if $z \in \Gamma_k^a(t)$ then $\mathrm{Re}\, g_a^n(z) \to \infty$. These invariant curves move continuously with respect to the parameter a, and they change less and less as a approaches 0, since g_a converges uniformly to g_0.

On the other hand, the critical point of g_a is now located at $c_a = -1/a$. Hence, when a runs along a half circle around 0, say $\eta_t = \{te^{i\alpha}, \pi/2 \leq \alpha \leq 3\pi/2\}$, c_a runs along a half circle with positive real part, of modulus $|c_a| = 1/t$.

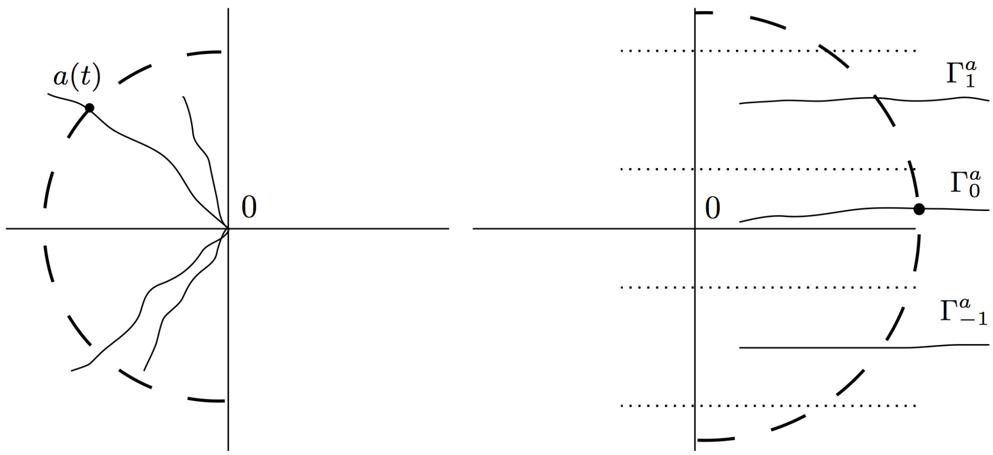

Figure 16.1: **Right:** Parameter plane **Left:** Dynamical plane of $g_a(z)$.

If t is small enough, this circle must intersect, say, Γ_0^a in at least one point. This means that there exists at least one $a(t) \in \eta_t$ such that $g^n_{a(t)}(c_{a(t)} \overset{n\to\infty}{\longrightarrow} \infty)$. Using standard arguments (see for example [Fag95]) it is easy to see that we can choose $a(t)$ in a continuous way so that $a(t) \overset{t\to 0}{\longrightarrow} 0$. Undoing the change of variables, the conclusion follows.

$\square$

Remark 16.2. *It is worth noting that for functions in class $\hat{\mathcal{B}}$, that is, bounded singular set with finite order (f_a belongs to this class), all* periodic *dynamic rays land, and landing points are either repelling or parabolic periodic points (see [Den14]).*

We would like to show now that all semi-hyperbolic components are simply connected. We first prove a preliminary lemma.

Lemma 16.3. *Let $U \subseteq H_p^v$ with $\bar{U}$ compact. Then there is a constant $C > 0$ such that for all $a \in U$ the elements of the attracting hyperbolic orbit, $z_j(a)$, satisfy $|z_j(a)| \leq C$, $j = 1, \ldots, p$.*

Proof. If this is not the case, then for some $1 \leq j \leq p$, $z_j(a) \to \infty$ as $a \to a_0 \in \partial U$ with $a \in U$. But as long as $a \in U$, $z_j(a)$ is well defined, and its multiplier bounded (by 1). Therefore,

$$\prod_{j=1}^{p} |f_a'(z_j(a))| = \prod_{j=1}^{p} |\lambda e^{z_j(a)/a}||z_j(a)+1| < 1.$$

Now, we claim that $z_j(a)+1$ does not converge to 0 for any $1 \leq j \leq p$ as a goes to a_0. Indeed, if this was the case, $z_j(a)$ would converge to -1, which has a dense orbit around the Siegel disc, but as the period of the periodic orbit is fixed, this contradicts the assumption. Hence $\prod_{j=1}^{p} |z_j(a)+1| \to \infty$ and necessarily $\prod_{j=1}^{p} |e^{z_j(a)/a}| \to 0$ as a goes to a_0. This implies that at least one of these elements goes to 0, say $|e^{z_j(a)/a}| \to 0$. But this means that $z_{j+1}(a) \to \lambda a_0(a_0 - 1) = v_{a_0}$ as $a \to a_0$. Now the first $p-1$ iterates of the orbit of v_{a_0} by f_{a_0} are finite. Since f_a is continuous with respect to a in $\bar{U}$, these elements cannot be the limit of a periodic orbit, with one of its points going to infinity. In particular we would have $f_a^{p-1}(z_{j+1}(a)) = z_j(a) \to f_{a_0}^{p-1}(v_{a_0})$ which contradicts the assumption.

□

With these preliminaries, the proof of simple connectedness is standard (see [BR84] or [BDH$^+$99]).

Proposition 16.4. *(Theorem B, Part a) For all $p \geq 1$ every connected component W of H_p^v or H_p^c is simply connected.*

Proof. Let $\gamma \subset W$ a simple curve bounding a domain D. We will show that $D \subset W$. Let $g_n(a) = f_a^{np}(v_a)$ (resp. $f_a^{np}(-1)$). We claim that $\{g_n\}_{n\in\mathbb{N}}$ is a family of entire functions for $a \in D$. Indeed, $f_a(v_a)$ has no essential singularity at $a = 0$ (resp. $f_a(-1)$ has no essential singularity as $0 \notin D$), neither do $f_a^n(f_a(v_a))$, $n \geq 1$ (resp. $f_a^n(f_a(-1))$, $n \geq 1$) as the denominator of the exponential term simplifies.

By definition W is an open set, therefore there is a neighbourhood $\gamma \subset U \subset W$. By Lemma 16.3 $|z_j(a)| < C$, $j = 1, \ldots, p$ and it follows that $\{g_n(a)\}_{n\in\mathbb{N}}$ is uniformly bounded in U, since it must converge to one point of the attracting cycle as n goes to ∞. So by Montel's theorem and the Maximum Modulus Principle, this family is normal, and it has a sub-sequence convergent in D. If we denote by $G(a)$ the limit function, $G(a)$ is analytic and the mapping $H(a) = f_a^p(G(a)) - G(a)$ is also analytic. By definition of H_p, $H(a)$ is identically zero in U, and by analytic continuation it is also identically zero in D. Therefore $G(a) = z(a)$ is a periodic point of period p.

Now let $\chi(a)$ be the multiplier of this periodic point of period p. This multiplier is an analytic function which satisfies $|\chi(a)| < 1$ in U, and by the Maximum Modulus Principle the same holds in D. Hence $D \subset H_p^v$ (resp. $D \subset H_p^c$).

□

The following lemma shows that the asymptotic value itself can not be part of an attracting orbit.

Lemma 16.5. *There are neither a nor p such that $f^p(v_a) = v_a$ and the cycle is attracting.*

Proof. It cannot be a super-attracting cycle since such orbit must contain the critical point and its forward orbit, but the critical orbit is accumulating on the boundary of the Siegel disc and hence its orbit cannot be periodic.

It cannot be attracting either, as the attracting basin must contain a singular value different from the attracting periodic point itself, and this could only be the critical point. But, as before, the critical point cannot be there. The conclusion then follows.

□

We can now show that all components in H_p^v are unbounded, which is part of Part b) of Theorem B. The proof is also analogous to the exponential case (see [BR84] or [BDH$^+$99]).

Theorem 16.6. *Every connected component W of H_p^v is unbounded for $p \geq 1$.*

Proof. From Lemma 16.3 above, the attracting periodic orbit $z(a)$ of Proposition 16.4 above is not only analytic in W but as $\limsup|\chi(a)| \leq 1$ for $a \in W$, $z(a)$ has only algebraic singularities at $b \in \partial W$. These singularities are in fact points where $\chi(b) = 1$ by the implicit function theorem. This entails that the boundary of W is comprised of arcs of curves such that $|\chi(a)| = 1$.

The multiplier in W is never 0 by Lemma 16.5, thus if W is bounded, it is a compact simply-connected domain bounded by arcs $|\chi(a)| = 1$. Now $\partial\chi(W) \subset \chi(\partial W) \subset \{\chi||\chi| = 1\}$ but by the minimum principle this implies $0 \in \chi(W)$ against assumption.

□

To end this section we show the existence of the largest semi-hyperbolic component, the one containing a segment $[r, \infty)$ for r large, which is Theorem B, Part c).

Theorem 16.7. *The parameter plane of $f_a(z)$ has a semi-hyperbolic component H_1^v of period 1 which is unbounded and contains an infinite segment.*

Proof. The idea of the proof is to show that for $a = r > 0$ large enough there is a region $\mathcal{R}$ in dynamical plane such that $\overline{f_a(\mathcal{R})} \subset \mathcal{R}$. By Schwartz's lemma it follows that $\mathcal{R}$ contains an attracting fixed point. By Theorem 15.2 the orbit of v_a must converge to it. Not to break the flow of exposition, the detailed estimates of this proof can be found in the Appendix.

□

Remark 16.8. *The proof can be adapted to the case $\lambda = \pm i$ showing that H_1^v contains an infinite segment in $i\mathbb{R}$. Observe that this case is not in the assumptions of this part since $z = 0$ would be a parabolic point.*

16.1 Parametrisation of H_p^v: Proof of Theorem B, Part b

In this section we will parametrise connected components $W \subset H_p^v$ by means of quasi-conformal surgery. In particular we will prove that the multiplier map $\chi : W \to \mathbb{D}^*$ is a universal covering map by constructing a local inverse of χ. The proof is standard.

Theorem 16.9. *Let $W \subset H_p^v$ be a connected component of H_p^v and $\mathbb{D}^*$ be the punctured disc. Then $\chi : W \to \mathbb{D}^*$ is the universal covering map.*

Proof. For simplicity we will consider $W \subset H_1^v$ in the proof. Take $a_0 \in W$, and observe that $f_a^n(v_a)$ converges to $z(a)$ as n goes to ∞, where $z(a)$ is an attracting fixed point of multiplier $\rho_0 < 1$. By Königs theorem there is a holomorphic change of variables

$$\varphi_{a_0} : U_{a_0} \to \mathbb{D}$$

conjugating $f_{a_0}(z)$ to $m_{\rho_0}(z) = \rho_0 z$ where U_{a_0} is a neighbourhood of $z(a_0)$.

Now choose an open, simply connected neighbourhood Ω of ρ_0, such that $\bar{\Omega} \subset \mathbb{D}^*$, and for $\rho \in \Omega$ consider the map

$$\begin{aligned} \psi_\rho : A_{\rho_0} &\longrightarrow A_\rho \\ re^{i\zeta} &\longmapsto r^\alpha e^{i(\zeta + \beta \log r)}, \end{aligned}$$

where A_r denotes the standard straight annulus $A_r = \{z | r < |z| < 1\}$ and

$$\alpha = \frac{\log|\rho|}{\log|\rho_0|}, \quad \beta = \frac{\arg\rho - \arg\rho_0}{\log|\rho_0|}.$$

This mapping verifies $\psi_\rho(m_{\rho_0}(z)) = m_\rho(\psi_\rho(z)) = \rho\psi_\rho(z)$. With this equation we can extend ψ_ρ to $m_\rho(A_\rho), m_\rho^2(A_\rho), \dots$ and then to the whole disc $\mathbb{D}$ by setting $\psi(0) = 0$. Therefore, the mapping ψ_ρ maps the annuli $m_\rho^k(A_\rho)$ homeomorphically onto the annuli $\{z||\rho^{k+1}| \leq |z| \leq \rho^k\}$.

This mapping has bounded dilatation, as its Beltrami coefficient is

$$\mu_{\psi_\rho} = \frac{\alpha + i\beta - 1}{\alpha + i\beta + 1}e^{2i\zeta}.$$

Now define $\Psi_\rho = \psi_\rho\varphi_{a_0}$, which is a function conjugating f_{a_0} quasiconformally to ρz in $\mathbb{D}$.

Let $\sigma_\rho = \Psi_\rho^*(\sigma_0)$ be the pull-back by Ψ_ρ of the standard complex structure σ_0 in $\mathbb{D}$. We extend this complex structure over U_{a_0} to $f_{a_0}^{-n}(U_{a_0})$ pulling back by f_{a_0}, and prolong it to $\mathbb{C}$ by setting the standard complex structure on those points whose orbit never falls in U_{a_0}. This complex structure has bounded dilatation, as it has the same dilatation as ψ_ρ. Observe that the resulting complex structure is the standard complex structure around 0, because no pre-image of U_{a_0} can intersect the Siegel disc.

Now apply the Measurable Riemann Mapping Theorem (with dependence upon parameters, in particular with respect to ρ) so we have a quasiconformal integrating map h_ρ (which is conformal where the structure was the standard one) so that $h_\rho^*\sigma_0 = \sigma_\rho$. Then the mapping $g_\rho = h \circ f \circ h^{-1}$ is holomorphic as shown in the following diagram:

$$\begin{array}{ccc} (\mathbb{C}, \sigma_{\rho'}) & \xrightarrow{\psi f_a \psi^{-1}} & (\mathbb{C}, \sigma_{\rho'}) \\ \downarrow{\scriptstyle h_{\rho'}} & & \downarrow{\scriptstyle h_{\rho'}} \\ (\mathbb{C}, \sigma_0) & \xrightarrow{g_{\rho'}} & (\mathbb{C}, \sigma_0) \end{array}$$

Moreover, the map $\rho \mapsto h_\rho(z)$ is holomorphic for any given $z \in \mathbb{C}$ since the almost complex structure σ_ρ depends holomorphically on ρ. We normalise the solution given by the Measurable Riemann Mapping Theorem requiring that -1, 0 and ∞ are mapped to themselves. This guarantees that $g_\rho(z)$ satisfies the following properties:

- $g_\rho(z)$ has 0 as a fixed point with rotation number λ, so it has a Siegel disc around it,
- $g_\rho(z)$ has only one critical point, at -1 which is a simple critical point,
- $g_\rho(z)$ has an essential singularity at ∞,
- $g_\rho(z)$ has only one asymptotic value with one finite pre-image.

Moreover $g_\rho(z)$ has finite order by Theorem 14.3. Then Theorem 15.1 implies that $g_\rho(z) = f_b(z)$ for some $b \in \mathbb{C}^*$. Now let's summarise what we have done.

Given ρ in $\Omega \subset \mathbb{D}^*$ we have a $b(\rho) \in W \subset H_1^v$ such that $f_{b(\rho)}(z)$ has a periodic point with multiplier ρ. We claim that the dependence of $b(\rho)$ with respect to ρ is holomorphic. Indeed, recall that v_a has one finite pre-image, $a - 1$. Hence $h_\rho(a-1) = b(\rho) - 1$ which implies a holomorphic dependence on ρ.

We have then constructed a holomorphic local inverse for the multiplier. As a consequence, $\chi : H \to \mathbb{D}^*$ is a covering map and as W is simply connected by Proposition 16.4 and unbounded by Theorem 16.6, χ is the universal covering map.

□

16.2 Parametrisation of H_p^c: Proof of Theorem B, Part d

Let W be a connected component of H_p^c which is bounded and simply connected by Theorem 15.2. The proof of the following proposition is analogous to the case of the quadratic family but we sketch it for completeness.

Proposition 16.10. *The multiplier $\chi : W \to \mathbb{D}$ is a conformal isomorphism.*

Proof. Let $W^* = W \backslash \chi^{-1}(0)$. Using the same surgery construction of the previous section we see that there exists a holomorphic local inverse of χ around any point $\rho = \chi(z(a)) \in \mathbb{D}^*$, $a \in W^*$. It then follows that χ is a branched covering, ramified at most over one point. This shows that $\chi^{-1}(0)$ consists of at most one point by Hurwitz's formula.

To show that the degree of χ is exactly one, we may perform a different surgery construction to obtain a local inverse around $\rho = 0$. This surgery uses an auxiliary family of Blaschke products. For details see [BF14] or [Dou87]. □

Chapter 17

Capture components: Proof of Theorem C

A different scenario for the dynamical plane is the situation where one of the singular orbits is eventually *captured* by the Siegel disc. The parameters for which this occurs are called capture parameters and, as it was the case with semi-hyperbolic parameters, they are naturally classified into two disjoint sets depending whether it is the critical or the asymptotic orbit the one which eventually falls in Δ_a. More precisely, for each $p \geq 0$ we define

$$C = \bigcup_{p \geq 0} C_p^v \cup \bigcup_{p \geq 0} C_p^c,$$

where

$$C_p^v = \{a \in \mathbb{C} | f_a^p(v_a) \in \Delta_a,\, p \geq 0 \text{ minimal}\},$$
$$C_p^c = \{a \in \mathbb{C} | f_a^p(-1) \in \Delta_a,\, p \geq 0 \text{ minimal}\},$$

Observe that the asymptotic value may belong itself to Δ_a since it has a finite pre-image, but the critical point cannot. Hence C_0^c is empty.

We now show that being a capture parameter is an open condition. The argument is standard, but we first need to estimate the minimum size of the Siegel disc in terms of the parameter a. We do so in the following lemma.

Lemma 17.1. *For all $a_0 \neq 0$ exists a neighbourhood V of a_0 such that $f_a(z)$ is univalent in $D(0, R)$.*

Proof. The existence of a Siegel disc around $z = 0$ implies that there is a radius R' such that $f_{a_0}(z)$ is univalent in $D(0, R')$. By continuity of the family $f_a(z)$ with respect to the parameter a, there are $R > 0, \varepsilon > 0$ such that $f_a(z)$ is univalent in $D(0, R)$ for all a in the set $\{a| \, |a - a_0| < \varepsilon\}$.

□

Corollary 17.2. *For all $a_0 \neq 0$ exists a neighbourhood $a_0 \in V$ such that Δ_a contains a disc of radius*

$$\frac{C}{4R}$$

where C is a constant that only depends on θ and R only depends on a_0.

Proof. For any value of a the maps $f_a(z)$ and $\tilde{f}_a(z) = \frac{1}{R}\lambda a(e^{Rz/a}(Rz+1-a)-1+a)$ are affine conjugate through $h(z) = R\cdot z$. For $|a-a_0|<\varepsilon$, $\tilde{f}_a(z)$ is univalent on $\mathbb{D}$, thus we can apply Theorem 14.16 to deduce that the conformal capacity $\tilde{\kappa}_a$ of the Siegel disc $\tilde{\Delta}_a$ is bounded from below by a constant $C = C(\theta)$. Undoing the change of variables we obtain

$$R\kappa = \tilde{\kappa}_a \geq C(\theta)$$

and therefore, by Koebe's 1/4 Theorem, Δ_a contains a disc of radius $\frac{C(\theta)}{4R}$. □

Theorem 17.3 ((Theorem C, Part a))**.** *Let $a \in C_p^v$ (resp. $a \in C_p^c$) for some $p \geq 0$ (resp. $p \geq 1$) which is minimal. Then there exists $\delta > 0$ such that $D(a,\delta) \subset C_p^v$ (resp. C_p^c)*

Proof. Let $b = f_a^p(v_a) \in \Delta_a$ (resp. $b = f_a^p(-1) \in \Delta_a$). Assume $b \neq 0$, (the case $b=0$ is easier and will be done afterwards). Define the annulus A as the region comprised between $\overline{\mathcal{O}(b)}$ and $\partial\Delta_a$ as shown in Figure 17.1.

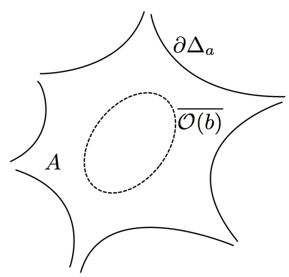

Figure 17.1: The annulus A.

Define $\tilde{\psi}$ as the restriction of the linearising coordinates conjugating $f_a(z)$ to the rotation $\mathcal{R}_\theta$ in Δ_a, taking A to the straight annulus $A(1,\varepsilon)$, where ε is determined by the modulus of A. Also define a quasiconformal mapping $\tilde{\phi} : A(1,\varepsilon) \to A(1,\varepsilon^2)$ conjugating the rotation $\mathcal{R}_\theta$ to itself. Let ϕ be the composition $\tilde{\phi}\circ\tilde{\psi}$.

Let μ be the f_a invariant Beltrami form defined as the pull-back $\mu = \tilde{\phi}^*\mu_0$ in A and spread this structure to $\cup_n f_a^{-n}(A)$ by the dynamics of $f_a(z)$. Finally define $\mu = \mu_0$ in $\mathbb{C}\setminus \cup_n f^{-n}(A)$. Observe that $\mu = \mu_0$ in a neighbourhood of 0. Also ϕ has bounded dilatation, say $k<1$, which is also the dilatation of μ.

Now let $\mu_t = t\cdot\mu$ be a family of Beltrami forms with $t \in \mathbb{D}(0,1/k)$. These new Beltrami forms are integrable, since $\|\mu_t\|_\infty = t\|\mu\| < \frac{1}{k}k = 1$. Thus by the Measurable Riemann Mapping Theorem we get an integrating map ϕ_t fixing 0,-1 and ∞, such that $\phi_t^*\mu_0 = \mu_t$. Let $f^t = \phi_t \circ f_a \circ \phi_t^{-1}$,

$$\begin{array}{ccc}(\mathbb{C},\mu_t) & \xrightarrow{f_a} & (\mathbb{C},\mu_t)\\ \downarrow \phi_t & & \downarrow \phi_t\\ (\mathbb{C},\mu_0) & \overset{f^t}{\dashrightarrow} & (\mathbb{C},\mu_0)\end{array}$$

Since μ_t is f_a-invariant, it follows that $f^t(z)$ preserves the standard complex structure and hence it is holomorphic by Weyl's lemma.

Notice also that by Theorem 14.3 in Section 14 $f^t(z)$ has finite order. Furthermore by the properties of the integrating map and topological considerations, it has an essential singularity at ∞, a fixed point 0 with multiplier λ and a simple critical point in -1. Finally, it has one asymptotic value $\phi_t(a)$ with one finite pre-image, $\phi_t(a-1)$. Hence by Theorem 15.1 $f^t(z) = f_{a(t)}(z)$ for some $a(t)$. Now we want to prove that $a(t)$ is analytic. First observe that for any fixed $z \in \mathbb{C}$, the almost complex structure μ_t is analytic with respect to t. Hence, by the MRMT, it follows that $t \mapsto \phi_t(z)$ is analytic with respect to t. Now, $a-1$ is the finite pre-image of v_a, so $\phi_t(a-1) = a(t)-1$, and this implies $a(t) = 1+\phi_t(a-1)$, which implies that $a(t)$ is also analytic.

It follows that $a(t)$ is either open or constant. But $f_{a(0)} = f_a$ and f_1 are different mappings since the annuli $\phi_0(A) = A$ and $\phi_1(A)$ have different moduli. Then $a(t)$ is open and therefore $\{a(t),\, t \in D(0, 1/k)\}$ is an open neighbourhood of a which belongs to C_p^v (resp. C_p^c).

If $f_{a_0}^p(v_{a_0}) = 0$ (resp. $f_{a_0}^p(-1) = 0$), by Lemma 17.1 and Corollary 17.2 there exists an $\varepsilon > 0$ such that for all a close to a_0, $\Delta_{a_0} \supset D(0, \varepsilon)$. Hence a small perturbation of f_{a_0} will still capture the orbit of v_{a_0} (resp. -1) as we wanted.

□

The theorem above shows that capture parameters form an open set. We call the connected components of this set, *capture components*, which may be *asymptotic* or *critical* depending on whether it is the asymptotic or the critical orbit which falls into Δ_a.

As in the case of semi-hyperbolic components, capture components are simply connected. Before showing that, we also need to prove that no critical capture component may surround $a = 0$. We just state this fact, since the proof is a reproduction of the proof of Proposition 16.1 above.

Proposition 17.4. *Let γ be a closed curve in $W \subset C^v$. Then* $\operatorname{ind}(\gamma, 0) = 0$.

Proposition 17.5. *(Theorem C, Part b) All connected components W of C^v or C^c are simply connected.*

Proof. Let W be a connected component of C^v or C^c and $\gamma \subset W$ a simple closed curve. Let D be the bounded component of $\mathbb{C}\backslash\gamma$. Let U be a neighbourhood of γ such that $U \subset W$. Then, for all $a \in U$, $f_a^n(v_a)$ (resp. $f_a^n(-1)$) belongs to Δ_a for $n \geq n_0$, and even more it remains on an invariant curve. It follows that $G_n^v(a) = f_a^n(v_a)$ (resp. $G_n^c(a) = f_a^n(-1)$) is bounded in U for all $n \geq n_0$.

Since $G_n^v(a)$ is holomorphic in all of $\mathbb{C}$ (resp. in $\mathbb{C}^*$), we have that $G_n^v(a)$ (resp. $G_n^c(a)$) is holomorphic and bounded on D, and hence it is a normal family in D. By analytic continuation the partial limit functions must coincide, so there are no bifurcation parameters in D. Hence $D \subset W$.

□

As it was the case with semi-hyperbolic components, it follows from Theorem 15.2 that all critical capture components must be bounded, since for $|a|$ large, the critical orbit must accumulate on $\partial\Delta_a$. This proves Part c) if Theorem C. Among all asymptotic capture components, there is one that stands out in all computer drawings, precisely the main component in C_0^v. That is, the set of parameters for which v_a itself belongs to the Siegel disc.

We first observe that this component must also be bounded. Indeed, if $v_a \in \Delta_a$ then its finite pre-image $a-1$ must also be contained in the Siegel disc. But for $|a|$ large enough, the disc is contained in $D(0,R)$, with R independent of a (see Theorem 15.2). Clearly C_0^v has a unique component, since $v_a = 0$ only for $a = 0$ or $a = 1$. This proves Part d) of Theorem C.

The "centre" of C_0^v is $a = 1$, or the map $f_a(z) = \lambda z e^z$, for which the asymptotic value $v_1 = 0$ is the centre of the Siegel disc. This map is quite well-known, as it is, in many aspects, the transcendental analogue of the quadratic family. It is known, for example that if θ is of constant type then $\partial\Delta_a$ is a quasi-circle and contains the critical point. This type of properties can be extended to the whole component C_0^v as shown by the following proposition.

Proposition 17.6. *(Proposition E, Part a) If θ is of constant type then for every $a \in C_0^v$ the boundary of the Siegel disc is a quasi-circle that contains the critical point.*

Proof. For $a = 1$, $f_1(z) = \lambda z e^z$ and we know that $\partial\Delta_a$ is a quasi-circle that contains the critical point (see [Gey01]). Define $c_n = f_1^n(-1)$, denote by $\mathcal{O}_a(-1)$ the orbit of -1 by $f_a(z)$ and

$$\begin{aligned} H : \{c_n\}_{n\geq 0} \times C_0^v &\longrightarrow \mathbb{C} \\ (c_n \quad , \quad a) &\longrightarrow f_a^n(-1) \end{aligned}$$

Then this mapping is a holomorphic motion, as it verifies

- $H(c_n, 1) = c_n$,
- it is injective for every a, as if $v_a \in C_0^v$, then $\mathcal{O}_a(-1)$ must accumulate on $\partial\Delta_a$. Hence $f_a^n(-1) \neq f_a^m(-1)$ for all $n \neq m$.
- It is holomorphic with respect to a for all c_n, an obvious assertion as long as $0 \notin C_0^v$ which is always true.

Now by the second λ-lemma (Lemma 14.8 in Section 14), it extends quasiconformally to the closure of $\{c_n\}_{n\in\mathbb{N}}$, which contains $\partial\Delta_a$. It follows that for all $a \in C_0^v$, the boundary of Δ_a satisfies $\partial\Delta_a = H_a(\partial\Delta_a)$ with H_a quasiconformal, and hence $\partial\Delta_a$ is a quasi-circle. Since $-1 \in \partial\Delta_1$, we have that $-1 \in \partial\Delta_a$.

□

We shall see in the next section that this same argument can be generalised to other regions of parameter space.

Chapter 18

Julia stability

The maps in our family are of finite type, hence $f_{a_0}(z)$ is $\mathcal{J}$-stable if both sequences $\{f_a^n(-1)\}_{n\in\mathbb{Z}}$ and $\{f_a^n(v_a)\}_{n\in\mathbb{Z}}$ are normal for a in a neighbourhood of a_0 (see [McM94] or [EL92]).

We define the critical and asymptotic stable components as

$$\mathcal{S}^c = \{a \in \mathbb{C} | G_n^c(a) = f_a^n(-1) \text{ is normal in a neighbourhood of } a\},$$
$$\mathcal{S}^v = \{a \in \mathbb{C} | G_n^v(a) = f_a^n(v_a) \text{ is normal in a neighbourhood of } a\},$$

respectively. Accordingly we define critical and asymptotic unstable components $\mathcal{U}^c$, $\mathcal{U}^v$ as their complements, respectively. These stable components are by definition open, its complements closed. With this notation the set of $\mathcal{J}$-stable parameters is then $\mathcal{S} = \mathcal{S}^c \cap \mathcal{S}^v$.

Capture parameters and semi-hyperbolic parameters clearly belong to $\mathcal{S}^c$ or $\mathcal{S}^v$. Next, we show that, because of the persistent Siegel disc, they actually belong to both sets.

Proposition 18.1. $H^{c,v}, C^{c,v} \subset \mathcal{S}$

Proof. Suppose, say, that $a_0 \in H^v$. The orbit of v_{a_0} tends to an attracting cycle, and hence $a_0 \in \mathcal{S}^v$. In fact, since H^v is open, we have that $a \in \mathcal{S}^v$ for all a in a neighbourhood U of a_0. For all these values of a, the critical orbit is forced to accumulate on $\partial\Delta_a$, hence $\{f_a^n(-1)\}_{n\in\mathbb{N}}$ avoids, for example, all points in Δ_a. It follows that $\{f_a^n(-1)\}_{n\in\mathbb{N}}$ is also normal on U and therefore $a_0 \in \mathcal{S}^c$. The three remaining cases are analogous.

□

Any other component of $\mathcal{S}$ not in H or C will be called a *queer component*, in analogy to the terminology used for the Mandelbrot set. We denote by Q the set of queer components, so that $\mathcal{S} = H \cup C \cup Q$.

At this point we want to return to the proof of Proposition 17.6, where we showed that, for parameters inside C_0^v, the boundary of the Siegel disc was moving holomorphically with the parameter. In fact, this is a general fact for parameters in any non-queer component of the $\mathcal{J}$-stable set.

Proposition 18.2. *Let W be a non-queer component of $\mathcal{S} = \mathcal{S}^c \cap \mathcal{S}^v$, and $a_0 \in W$. Then there exists a function $H : W \times \partial\Delta_{a_0} \to \partial\Delta_a$ which is a holomorphic motion of $\partial\Delta_{a_0}$.*

Proof. Since W is not queer, we have that $W \subset H \cup C$. Let s_a denote the singular value whose orbits accumulates on $\partial\Delta_a$ for $a \in W$, so that $s_a \in \{-1, v_a\}$. Let $s_a^n = f_a^n(s_a)$, and denote the orbit of s_a by $\mathcal{O}_a(s_a)$. Then the function

$$\begin{aligned} H : \mathcal{O}_{a_0}(s_{a_0}) \times W &\longrightarrow \mathbb{C} \\ (s_{a_0}^n \quad , \quad a) &\longrightarrow s_a^n \end{aligned}$$

is a holomorphic motion, since $\mathcal{O}_a(s_a)$ must be infinite for all n, and $f_a^n(s_a)$ is holomorphic on a, because $0 \notin W$. By the second λ-lemma, H extends to the closure of $\mathcal{O}_{a_0}(s_{a_0})$ which contains $\partial\Delta_0$.

□

Combined with the fact that $f_a(z)$ is a polynomial-like map of degree 2 for $|a| > R$ (see Theorem 15.2) we have the following immediate corollary.

Corollary 18.3. *(Proposition E, Part b) Let $W \subset H^v \cup C^v$ be a component intersecting $\{|z| > R\}$ where R is given by Theorem 15.2 (in particular this is satisfied by any component of H^v). Then,*

a) if θ is of constant type, for all $a \in W$, the boundary $\partial\Delta_a$ is a quasi-circle containing the critical point.

b) Depending on $\theta \in \mathbb{R}\backslash\mathbb{Q}$, other possibilities may occur: $\partial\Delta_a$ might be a quasi-circle not containing the critical point, or a $\mathscr{C}^n$, $n \in \mathbb{N}$ Jordan curve not being a quasi-circle containing the critical point, or a $\mathscr{C}^n$, $n \in \mathbb{N}$ Jordan curve not containing the critical point and not being a quasi-circle. In general, any possibility realised by a quadratic polynomial for some rotation number and which persists under quasiconformal conjugacy, is realised for some $f_a = e^{2\pi\theta i}a(e^{z/a}(z + 1 - a) + a - 1)$.

Remark 18.4. *In general, for any $W \subset H^v \cup C^v$ we only need one parameter $a_0 \in W$ for which one of such properties is satisfied, to have it for all $a \in W$.*

Chapter 19

Approximating sets of instability

In this chapter we will prove Theorem F and related results, hence proving a parameter plane analogue of a result by Fatou and Brolin. The chapter is divided in 5 sections. In Section 19.1 we will see two specific examples from the family of quadratic polynomials with the purpose of motivating and illustrating our results. In Section 19.2 we will introduce the notation used in the rest of the chapter. In Section 19.3, we will prove the main result in a general setting. In Section 19.4 we will prove additional results extending the main theorem.

19.1 Two examples for the quadratic family

We will illustrate the main result for the quadratic family, to give a concrete example where the construction is clearer than in the general setting.

19.1.1 Centres of hyperbolic components

Let $P_c(z) = z^2 + c$ and let $\mathcal{M}$ denote the Mandelbrot set, that is, the set

$$\mathcal{M} = \{c \in \mathbb{C} \mid P_c^n(z) \nrightarrow \infty\},$$

or equivalently, the set of parameters for which the filled Julia set of $P_c(z)$ is connected.

The boundary of $\mathcal{M}$, denoted by $\partial\mathcal{M}$, is exactly the set of parameters for which P_c is not $\mathcal{J}$-stable. It can also be characterised as

$$\partial\mathcal{M} = \left\{c \in \mathbb{C} \mid \{g_n(c) := P_c^n(0)\}_{n\in\mathbb{N}} \text{ is not normal in any neighborhood of } c\right\}, \tag{19.1}$$

which is also known as the *bifurcation locus* of the quadratic family.

Recall that we call an open set in the parameter plane of P_c a *hyperbolic component* if all parameters in it have an attracting cycle. This definition generalises for all families with one singular value. In the Mandelbrot set, each hyperbolic component has a distinguished point called the centre, which is the unique parameter for which the critical point is itself periodic, and hence superattracting. Recall that

$$\partial\mathcal{M} = \{\text{centres of hyperbolic components}\}'.$$

This property is the idea for an algorithm suggested by Christian Henriksen, which produces very accurate pictures of $\partial\mathcal{M}$. We want to find points arbitrarily close to $\partial\mathcal{M}$, characterised as in (19.1).

It is easy to see that $P_c(z)$ has *two different* branches of the inverse function, except at the critical point $w = 0$, for every $c \in \mathbb{C}$. Moreover, these two branches are analytic with respect to c and we can denote them by $\varphi_w(c)^+$ and $\varphi_w(c)^-$ as functions depending on c.

Define the sets

$$C_n = \{c \in \mathbb{C} \mid P_c^n(0) = 0 \text{ for } n \text{ in } \mathbb{N}\},$$

$$C = \bigcup_{n=0}^{\infty} C_n,$$

which is the set of centres of hyperbolic components. Observe that the critical point 0 is in C, since $P_0^n(0) = 0$.

Theorem 19.1. *With $\partial\mathcal{M}$ and C as above,*

$$\partial\mathcal{M} \subset \overline{C}.$$

Proof. We prove that $\partial\mathcal{M} \subset \overline{C}$ by contradiction. Assume there is a $d \in \partial\mathcal{M}$ and a neighborhood $d \in U$ such that $U \cap C = \emptyset$. This implies $P_c^n(0) \neq 0$ for all $c \in U$ and all $n \in \mathbb{N}$. Shrink this neighborhood as needed so that $0 \notin U$. We can do this because $0 \notin \partial\mathcal{M}$. Now consider an auxiliary family of functions defined as

$$G_n(c) = \frac{P_c^n(0) - \varphi_0^+(c)}{P_c^n(0) - \varphi_0^-(c)}, \quad c \in U,\, n \in \mathbb{N},$$

which is well-defined for $c \in U$ because $0 \notin U$ and the two branches of the inverse φ_0^+ and φ_0^- are well defined. This family avoids 0 and ∞, as $P_c^n(0)$ can not be equal to a pre-image of 0 for $c \in U$, because $U \cap C = \emptyset$. Since $0 \notin U$, this family also avoids 1 because the branches $\varphi_0(c)^+$ and $\varphi_0(c)^-$ are different. Now by Montel's normality theorem $G_n(c)$ is normal in U as it avoids 3 points and then $g_n(c)$ is also normal, contradicting the fact that $d \in \partial\mathcal{M}$.

□

Observe that in the proof the role of the critical point could be played by an arbitrary analytic function $w(c)$ as long as we defined the set of centres C_n accordingly:

$$\widetilde{C}_n = \{c \in \mathbb{C} \mid P_c^n(w(c)) = 0 \text{ for } n \text{ in } \mathbb{N}\},$$

$$\widetilde{C} = \bigcup_{n=0}^{\infty} \widetilde{C}_n.$$

19.1.2 Misiurewicz points in the quadratic family

Remember that a parameter $c \in \mathbb{C}$ is called a Misiurewicz point (or Misiurewicz parameter) if the critical point of $P_c(z)$ is strictly pre-periodic.

It is well known that (see [DH+84])

$$\partial M = \overline{\{\text{Misiurewicz points}\}},$$

and with slight variations in the construction of the previous section, we can prove a more general result implying this.

Define the following set of (k, q)-Misiurewicz points

$$Mis_{k,q} = \left\{ c \in \mathbb{C} \,|\, P_c^n(P_c^k(0)) = P_c^q(0) \text{ for some } n \geq 0 \right\},$$

for a fixed choice of $k \geq 2$, $k \in \mathbb{N}$ and $1 < q < k$, $q \in \mathbb{N}$. The set $Mis_{k,q}$ is the set of parameters $c \in \mathbb{C}$ such that the critical point is pre-periodic of pre-period q and period $n + k$ for any $n \in \mathbb{N}$. We can simplify the notation used here and write $w(c) = P_c^k(0)$ and $\beta(c) = P_c^q(0)$,

$$Mis_{k,q} = \left\{ c \in \mathbb{C} \,|\, P_c^n(w(c)) = \beta(c) \text{ for some } n \geq 0 \right\}.$$

As in the previous section, we have two different branches for the inverse function. These two branches are different in $\mathbb{C}$, except when $c = 0$ and thus the branches coincide in the following set

$$K = \left\{ c \in \mathbb{C} \,|\, P_c(0) = \beta(c) \right\}.$$

Observe that this set is either discrete or the whole plane, since this is the set of zeros of an analytic function. Now the result is the following.

Theorem 19.2. *With $\partial\mathcal{M}$, $Mis_{k,q}$ and K as above,*

$$\partial\mathcal{M} \subset \overline{Mis_{k,q} \setminus K} = \overline{Mis_{k,q}}.$$

Proof. We will prove this result by contradiction. Let $d \in \partial\mathcal{M}$ and consider a neighborhood $d \in U$ such that $U \cap Mis_{k,q} = \emptyset$ and shrink it such that $U \cap K = \emptyset$. This implies that $P_c^n(w(c)) \neq \beta(c)$ for all $c \in U$ and any $n \in \mathbb{N}$.

As before, we have two branches of the inverse function depending on c, for the point $z = \beta(c)$, written as $\varphi^{\pm}_{\beta(c)}(c)$. These two branches are different in U, since $P_c(0) \neq \beta(c)$ in U (also, $\beta(c) \neq 0$), as $U \cap K = \emptyset$. As in the proof of Theorem 19.1 we can now construct an auxiliary family of functions avoiding 3 points in U, meaning $g_n(c)$ is normal in U in contradiction of our assumption, proving

$$\partial\mathcal{M} = \overline{Mis_{k,q}} \subset \overline{\text{Misiurewicz parameters}}.$$

□

Observe that this setting is a generalisation of the setting in Section 19.1.1, since the sets we are studying now depend on two analytic functions, $w(c)$ and $\beta(c)$.

The main result of this chapter is a generalisation of these examples.

19.2 Definitions

In this section we will give a general framework, valid for general families of functions based on the construction for the Mandelbrot set we have illustrated with centres of hyperbolic components and Misiurewicz points.

Consider a one-parameter family of entire functions of degree at least 2 depending analytically on the parameter c, $f_c : \mathbb{C} \to \mathbb{C}$ with a discrete set of singular (critical and asymptotic) values.

Definition 19.3. *Given $f_c(z)$ as above and $w(c)$ an analytic function we define the sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ as*

$$\left\{ g_n(c) = f_c^n(w(c)) \right\}_{n\in\mathbb{N}}.$$

Observe that in Section 19.1.1 $w(c) = 0$, in Section 19.1.2 $w(c) = P_c^k(0)$ for some $k \in \mathbb{N}$.

Definition 19.4. *We define the bifurcation set associated to $w(c)$*

$$\mathcal{B} = \mathcal{B}_{w(c)} = \left\{ c \in \mathbb{C} \,|\, \{g_n(c)\}_{n\in\mathbb{N}} \textit{ is not normal in any neighborhood of } c \right\}$$

This set has dynamical relevance for specific choices of $w(c)$. For example, $\mathcal{B}$ is the bifurcation locus if $w(c)$ is the unique singular value of f_c, as it was the case in Section 19.1.1 (or a pre-image of the singular value, as in Section 19.1.2). For other choices, $\mathcal{B}$ may be empty. We will omit the dependence of the set $\mathcal{B}$ and of the sequence $\{g_n\}_{n\in\mathbb{N}}$ on the choices of w or f.

Definition 19.5. *We denote by $v_j(c)$, $j = 1, \cdots, N$, the set of analytic functions with respect to $c \in \mathbb{C}$ which correspond to the singular values (asymptotic or critical) of f.*

Remark 19.6. *The functions $v_j(c)$ are not necessarily analytic functions in the whole plane $\mathbb{C}$, in general. A simple example would be $f_c(z) = cz^2 + e^c z + \sin c$. See [Erë06] for a general result on analyticity of asymptotic values with respect to parameters.*

Definition 19.7. *Given $\beta(c)$ an analytic function and $g_n(c)$ defined above, we define the* set of n-centres *as*

$$C_n = \left\{ c \in \mathbb{C} \,|\, g_n(c) = \beta(c), \quad \textit{with minimal } n \in \mathbb{N} \right\}$$

and the set of centres *as*

$$C = \bigcup_{n=1}^{\infty} C_n.$$

These "centres" correspond to centres of hyperbolic components in Section 19.1.1 and to (k,q)-Misiurewicz points in Section 19.1.2.

Definition 19.8. *We define the* period *of a point $c_n \in C$ as the minimal p_n such that*

$$f^{p_n}(w(c_n)) = \beta(c_n).$$

We now define the set of parameters for which the inverse function of f_c does not exist (or fails to have different branches.) These sets have to be excluded from our results.

Definition 19.9. *Given $\beta(c)$ an analytic function and $v_j(c)$ as above, we define the j-th* critical set *as*

$$K_j = \Big\{ c \in \mathbb{C} \,|\, v_j(c) = \beta(c) \Big\}, \quad j = 1, 2, \ldots, N$$

and the set

$$K = \bigcup_{j=1}^{N} K_j.$$

Let K_ε be an ε-neighbourhood of K.

The function $f_c(z)$ has (at least) two different and well-defined branches of the inverse function in a neighborhood of $\beta(c)$ for $c \in \mathbb{C} \backslash K$.

These three sets follow the construction in Sections 19.1.1 and 19.1.2 for the Mandelbrot set. Observe that K_j is the set of zeros of an analytic function and thus it is discrete for each j. Although all these sets depend on $w(c)$ and $\beta(c)$, we will omit this dependence for the sake of a clearer notation.

19.3 First theorem

This is the main result, proving that we can approximate $\mathcal{B}$ by some set C which is numerically approximable.

Theorem 19.10. *Let f_c be a one-parameter family of entire functions of degree at least 2 depending analytically on the parameter c. If $v_j(c)$, $1 \leq j \leq N$ denoting the singular and asymptotic values of f_c as functions of c are analytic for all j and all $c \in \mathbb{C}$, then*

$$\mathcal{B} \setminus K_\varepsilon \subseteq C'.$$

In other words, the set of not *normality is contained in the limit set of the zeros of $\{f_c^n(w(c)) - \beta(c)\}_{n \in \mathbb{N}}$, except at a neighborhood of K.*

Proof. We will prove this result by contradiction. Assume there is some $c_0 \in \mathcal{B} \backslash (\mathcal{B} \cap K_\varepsilon)$ such that there is a neighborhood $c_0 \in U$, $U \cap K_\varepsilon = \emptyset$ and such that $U \cap C = \emptyset$. This implies $g_{n+1}(c) \neq \beta(c)$ for all $c \in U$ and all $n \in \mathbb{N}$, which gets expanded into $f_c(f_c^n(w(c))) \neq \beta(c)$ and thus $f_c^n(w(c)) \notin \big\{ f_c^{-1}(\beta(c)) \big\}$. Let $\varphi_w^{\pm}(c)$ be two branches of the inverse function such that $f_c(\varphi_{\beta(c)}^{\pm}(c)) = \beta(c)$. These two branches exist and are different since $U \cap K_\varepsilon = \emptyset$. Consider the following auxiliary family:

$$G_n(c) = \frac{f_c^n(w(c)) - \varphi_{\beta(c)}^{+}(c)}{f_c^n(w(c)) - \varphi_{\beta(c)}^{-}(c)}, \quad c \in U, \quad n \in \mathbb{N}.$$

This family avoids both 0 and infinity, because $f_c^n(w(c)) \neq \varphi_{\beta(c)}^{\pm}(c)$ for $c \in U$. It avoids 1, as it would imply $\varphi_{\beta(c)}^{+}(c) = \varphi_{\beta(c)}^{-}(c)$ and this condition holds only for $c \in K$ but $\mathrm{dist}(U, K) > \varepsilon$. Thus $G_n(c)$ is a normal family in U and hence $g_n(c)$ is also a normal family in U, yielding a contradiction.

□

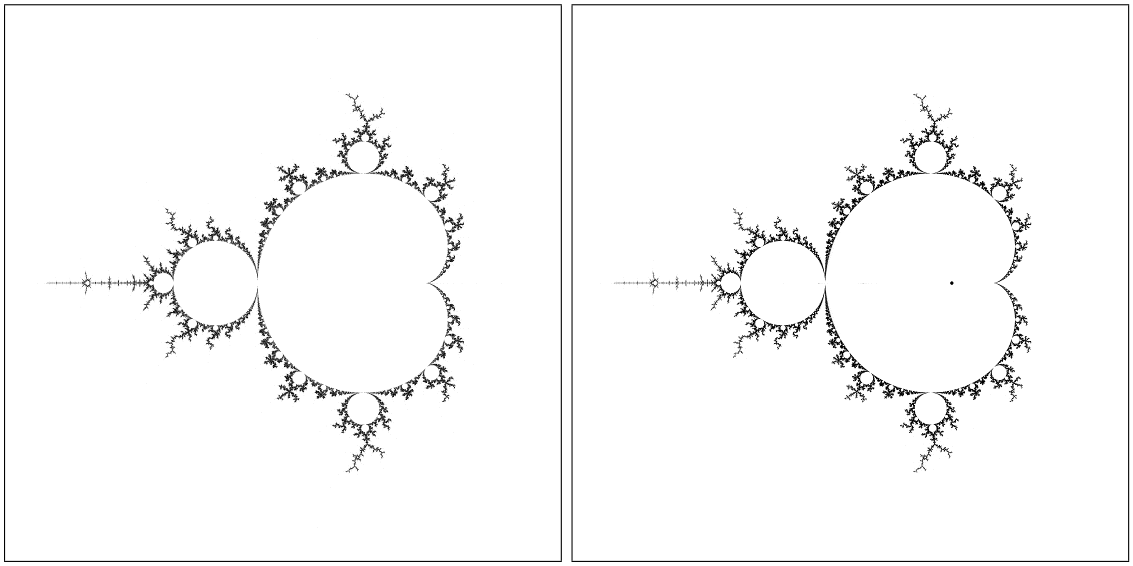

Figure 19.1: **Upper left:** (-2.2,1.6), **lower right:** (1,-1.6). Left shows $c \in \mathbb{C}$ such that $P_c^n(0) = 25$, right shows $c \in \mathbb{C}$ such that $P_c^n(0) = c^3 - c$.

This theorem clearly covers Theorems 19.1.1 and 19.2, in addition to extending to other uni-parametric families like f_a studied in the previous chapters.

The techniques used in the proof can be extended to dynamical planes (in a sense, by exchanging z and c and then fixing c). More concretely, let f be an entire function of degree at least 2, let $w(z)$ and $\beta(z)$ be two arbitrary analytic functions and denote by $v_j(z)$, $j = 1, \cdots, N$ the set of singular values of f. Assume $v_j(z)$ are analytic with respect to z and define the sequence $\{g_n(z) = f^n(w(z))\}_{n\in\mathbb{N}}$ and the sets

$$\begin{aligned}
\widetilde{\mathcal{B}} &= \{z \in \mathbb{C} \,|\, \{g_n(z)\}_{n\in\mathbb{N}} \text{ is not normal in any neighborhood of } z\} \\
\widetilde{C} &= \bigcup_{n\geq 1} \{z \in \mathbb{C} \,|\, g_n(z) = \beta(z) \text{ with minimal } n\}, \\
\widetilde{K} &= \bigcup_{j=1}^{N} \{z \in \mathbb{C} \,|\, v_j(z) = \beta(z)\}, \\
\widetilde{K_\varepsilon} &= \cup_{z\in\widetilde{K}} D(z,\varepsilon), \quad \varepsilon > 0.
\end{aligned}$$

Observe that for $w(z) = z$, the set denoted $\widetilde{\mathcal{B}}$ above is the Julia set of f. Clearly we can prove an equivalent result to Theorem 19.10 but about dynamical planes.

Theorem 19.11. *Let f be an entire function of degree at least 2. If $v_j(z)$, $1 \leq j \leq N$ denoting the singular and asymptotic values of f as functions of z are analytic for all j and all $z \in \mathbb{C}$, then*

$$\widetilde{\mathcal{B}} \setminus \widetilde{K_\varepsilon} \subseteq \widetilde{C}'.$$

In other words, the set of not *normality is contained in the limit set of the zeros of $\{f^n(w(z)) - \beta(z)\}_{n\in\mathbb{N}}$, except at a neighborhood of K.*

Proof. The proof is totally equivalent to the proof of Theorem 19.10. □

This is just a generalisation of a classical result by Fatou and Brolin regarding Julia sets.

Corollary 19.12. *Every point in the Julia set of a function f satisfying the conditions of Theorem 19.11 and with the notation as above is a limit point of pre-images of almost any point in $\mathbb{C}$.*

19.4 Reverse inclusion

We have previously proved that $\mathcal{B} \setminus K_\varepsilon \subset C'$. In this section we want to show the reverse inclusion, $C' \subset \mathcal{B}$, which proves $\mathcal{B} = C'$. This will be done in less generality than Section 19.3, but the result will include several interesting cases. Observe that this result is trivial when $C \subset \mathcal{B}$, as is the case with Misiurewicz points in Section 19.1.2. In this section we will use the same objects that we used in the previous section.

Definition 19.13. *Let f_c be a family of analytic functions. We will say f_c has a persistent Fatou component Λ_c if for any $c \in \mathbb{C}$:*

- *Λ_c is a Fatou component,*
- *There exists $\alpha(c) \in \Lambda_c$ and $r > r_0 > 0$, with r_0 independent of c such that the disc $D(\alpha(c), r) \subset \Lambda_c$.*

Let $\beta(c)$ be a function such that $\beta(c) \in D(\alpha(c), r)$ for all $c \in \mathbb{C}$ and $w(c)$ an arbitrary analytic function.

Definition 19.14. *We say c is a β-capture point of order p for specific choices of $\beta(c)$ and $w(c)$, if $f_c^p(w(c)) = \beta(c)$ and $f_c^k(w(c)) \neq \beta(c)$ for all $0 \leq k < p$.*

This setting includes persistent Fatou components of any type (attracting basins, parabolic components, Siegel disks...)

Proposition 19.15. *If f_c has a persistent Fatou component Λ_c and $\beta(c) \in D(\alpha(c), r/2)$ for all $c \in \mathbb{C}$, then $\mathcal{B} = C'$.*

Proof. Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence of points $c_n \in C$ with limit point $\widetilde{c}$ and let p_n be the order of the point c_n. We claim that the sequence $\{p_n\}_{n\in\mathbb{N}}$ is an unbounded sequence. Indeed, if $p_n \leq P \forall n \in \mathbb{N}$, all points in $\{c_n\}_{n\in\mathbb{N}}$ would be zeros of $g_k(c) - \beta(c)$ for some $1 \leq k \leq P$, where $g_k(c) = f_c^k(w(c))$. But the zeros of an analytic function form a discrete set, and a finite number of discrete sets cannot have accumulation points in the plane.

Let

$$G_r = \{c \in \mathbb{C} \,|\, g_n(c) = f_c^n(w(c)) \text{ lands in } D(\alpha(c), r), \text{ for some } n \geq 0\},$$
$$G = \{c \in \mathbb{C} \,|\, g_n(c) = f_c^n(w(c)) \text{ lands in the Fatou component } \Lambda_c, \text{ for some } n \geq 0\}$$

Any point $c_0 \in G_{r/2}$ has an open neighborhood $U \subset G_{r/2} \subset G$. Indeed, by definition of $G_{r/2}$, there is some minimal n such that $g_n(c_0) = f_{c_0}^n(w(c_0)) \in D(\alpha, r/2)$. Since g_n is continuous with respect to c and $g_n(c_0) \in D(\alpha, r/2)$, there is an open neighborhood $c_0 \in U$ such that $g_n : U \to D(\alpha, r/2)$.

Therefore c_n is in an open component $D_n \subset G_r$ and $\lim_n c_n \in G_{r/2}$ or $\lim_n c_n \in \partial G_{r/2}$. If we denote the limit point as $\widetilde{c}$, then $\widetilde{c} \in \overline{G_{r/2}} \subset G_r \subset G$, thus there is some minimal P such that $g_P(\widetilde{c}) \in \Lambda_c$ and we can find an open neighborhood $\widetilde{D}$ of $\widetilde{c}$ such that $g_P(c) \in \Lambda_c$ for all $c \in \widetilde{C}$. But in any neighborhood of $\widetilde{c}$ we have open neighborhoods D_p associated with points c_p with increasing *minimal* order p_n, which is a contradiction with the fact that in $\widetilde{D}$ the minimal order was P.

□

This proposition includes the family f_a with the function $\beta(a) = v_(a)$ studied in the previous chapters since it has a persistent and non-vanishing Siegel disc (see Corollary 17.2).

Another interesting case is motivated by the components studied in Chapter 16.

Definition 19.16. *Assume $v(c)$ is a critical point of a family f_c under the assumptions of Theorem 19.10. We define the hyperbolic components of f_c for the critical point $v(c)$ as*

$$\mathcal{H}_v = \{c \in \mathbb{C} \,|\, f_c^n(v(c)) \text{ converges to a periodic orbit}\}.$$

Let $\beta(c) = w(c) = v(c)$. In this case, the set $C = \cup_n C_n$ is the set of centres of hyperbolic components (for the critical point $v(c)$), in other words, parameters where $f_c(z)$ has a superattracting periodic orbit of period n and $v(c)$ is part of the cycle.

We restrict the case of *hyperbolic components* to critical values because hyperbolic components associated to asymptotic values do not necessarily have centres, as is the case in the family of functions studied in the preceding chapters (see Lemma 16.5).

Proposition 19.17. *If $\beta(c) = w(c) = v(c)$ then $\mathcal{B} = C'$.*

Proof. Let $\{c_n\}_{n\in\mathbb{N}}$ be a sequence of points in C with limit point $\widetilde{c}$. We claim that the sequence of periods $\{p_n\}_{n\in\mathbb{N}}$ of the points in the sequence $\{c_n\}_{n\in\mathbb{N}}$ form an unbounded sequence. Indeed, let P be a bound for the sequence $\{p_n\}_{n\in\mathbb{N}}$. The points c_n with period p_n are zeros of $h_{p_n}(c) = f_c^{p_n}(v(c)) - v(c)$, which is an analytic function. The set Z_{p_n} of zeros of h_{p_n} is thus a discrete set[1], therefore the set $\bigcup_{p=0}^{P} Z_p$ is also discrete, in contradiction with the fact that $\widetilde{c}$ is an accumulation point.

By the implicit function theorem, there is a neighborhood of c_n, D_n and an analytic function ξ such that $f_{\xi(c)}^{p_n}(\xi(c)) = \xi(c)$ for all $c \in D_n$. In other words, there is a neighborhood of c_n formed by p_n-periodic orbits. These orbits are necessarily attracting, since the implicit function theorem fails when the orbit is indifferent.

This implies that the limit point $\widetilde{c}$ is either in $\mathcal{H}_v$ or in $\partial\mathcal{H}_v$. If $\widetilde{c} \in \mathcal{H}_v$, there is an open neighborhood of $\widetilde{c}$ with periodic orbits of constant period, but in any neighborhood of $\widetilde{c}$ we have open sets D_n with minimal period p_n, which is a contradiction.

If $\widetilde{c} \in \partial\mathcal{H}_v$ the family is not normal and $\widetilde{c} \in \mathcal{B}$.

□

This proposition includes centres of hyperbolic components in many cases, for these specific choices of β and w. A particular case for the family f_a is shown in Figure 19.2.

[1]Unless $f^{p_n}(v(c)) = v(c)$ for all $c \in \mathbb{C}$, which implies $v(c)$ is *always* a periodic point of period p_n.

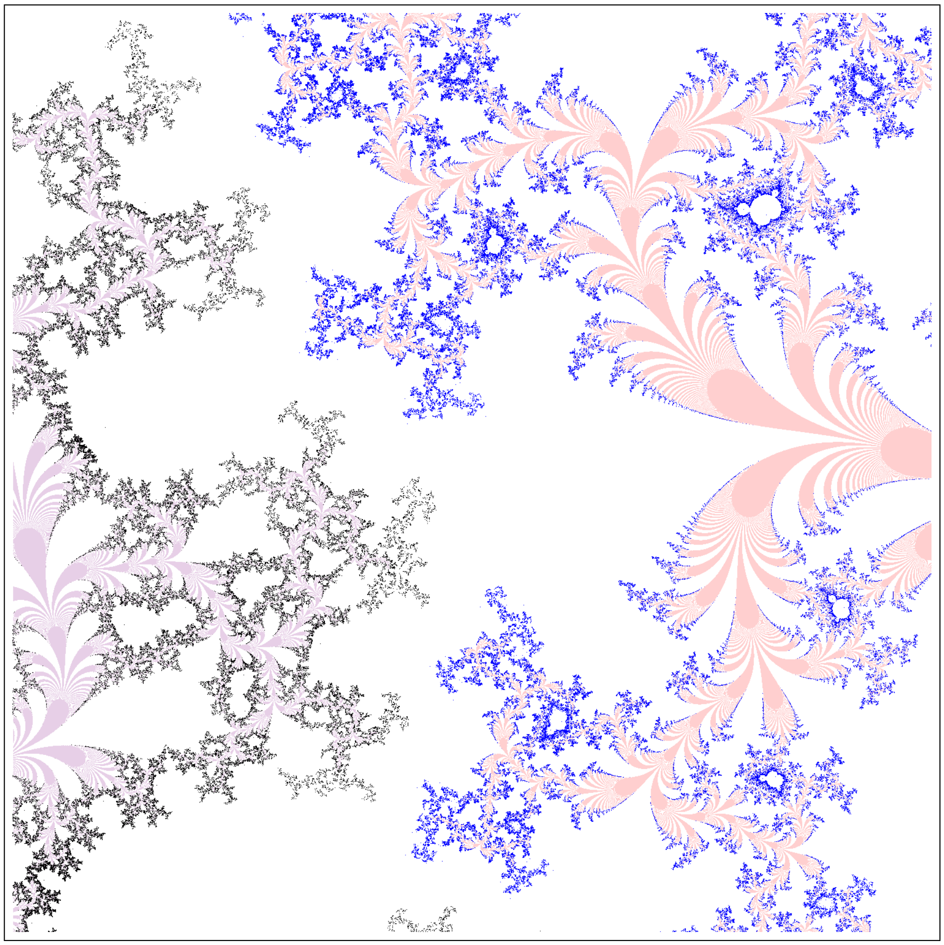

Figure 19.2: Close up around the unit circle of the parameter plane of the family f_a. In red and purple we can see exponential hairs for the asymptotic and critical values. In blue, centres of semi-hyperbolic components for the critical value, H_c, in black, centres of capture components for the asymptotic value, C_c

Appendix A

Proof of Theorem 16.7 and numerical bounds

We may suppose $\lambda \neq \pm i$ since $\theta \neq \pm 1/2$. Let $\lambda = \lambda_1 + i\lambda_2$, $\sigma = \text{Sign}\,(\lambda_1)$ and $\rho = \text{Sign}\,(\lambda_2)$. We define:

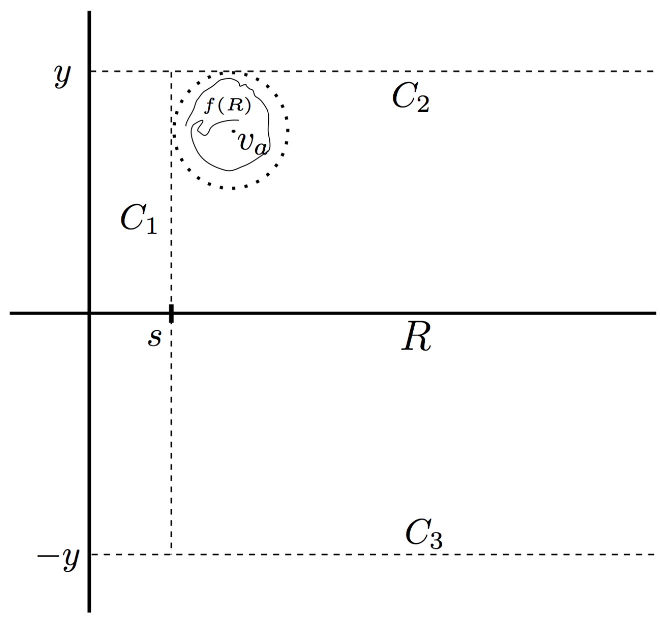

Figure A.1: Sketch of the construction in Thm. 16.7 for the case $\lambda_1, \lambda_2 > 0$.

$$\begin{aligned} C_1 &:= \{\sigma s + ti \,||t| \leq y\} \\ C_2 &:= \{\sigma t + i\rho y | t \geq s\} \\ C_3 &:= \{\sigma t - i\rho y | t \geq s\} \end{aligned}$$

with $y > 0$, $s > 0$, see Figure A.1 for a sketch of this curves. Let R be the region bounded by C_1, C_2, C_3. Recall that $v_a = \lambda(a^2 - a)$ is the asymptotic value. Note that we will consider a real, furthermore following Figure A.1, we will set $a := -\sigma b$ with $b > 0$, as hinted by numerical experiments. Defined this way, the curves that are closer to v_a are C_1 and C_2. We choose y and s in such a way that $d(v_a, C_1) = d(v_a, C_2)$, as in Figure A.1. More precisely,

$$d(v_a, C_{1,2}) = |\lambda_1| \left(b^2 + \sigma b\right) - s = |\lambda_2|(b^2 + \sigma b) - y$$

and hence

$$y = (|\lambda_1| + |\lambda_2|) \left(b^2 + \sigma b\right) - s.$$

To ease notation, define $L = (|\lambda_1| + |\lambda_2|)$. We would like some conditions over s assuring that if $b > b^*$, $d(v_a, f(\partial R)) \leq d(v_a, \partial R)$, as this would imply $f(R) \subset R$ and thus the existence of an attracting fixed point. We write $f_a(z) = v_a + g_a(z)$ where $g_a(z) = a \cdot \lambda e^{z/a} \cdot (z + 1 - a)$. Then

$$d(v_a, f(\partial R)) = d(0, g_a(\partial R)) = |g_a(\partial R)|.$$

Therefore we need to find values such that the following three inequalities hold

$$|g_a(C_1)| < |\lambda_1| \left(b^2 + \sigma b\right) - s, \tag{A.1}$$
$$|g_a(C_2)| < |\lambda_1| \left(b^2 + \sigma b\right) - s, \tag{A.2}$$
$$|g_a(C_3)| < |\lambda_1| \left(b^2 + \sigma b\right) - s. \tag{A.3}$$

For (A.1) to hold the following inequality needs to be satisfied

$$b \cdot e^{-s/b} \sqrt{((\sigma s + \sigma b + 1) + t^2)} \overset{?}{\leq} |\lambda_1| \left(b^2 + \sigma b\right) - s.$$

Observe that

$$\begin{aligned} b \cdot e^{-s/b} \sqrt{(\sigma s + \sigma b + 1)^2 + t^2} &\leq b \cdot e^{-s/b} \left(|\sigma(s + b) + 1| + y\right) = \\ &= b \cdot e^{-s/b} \left(s + b + \sigma + y\right) = \\ &= b \cdot e^{-s/b} \left(b + \sigma + L(b^2 + \sigma b)\right), \end{aligned}$$

so we define the following function

$$h(s) = b \cdot e^{-s/b} \left(b + \sigma + L(b^2 + \sigma b)\right) - |\lambda_1| \left(b^2 + \sigma b\right) + s,$$

and we will find an argument which makes it negative. We need to find s such that $h(s) < 0$ and $0 < s < |\lambda_1|(b^2 + \sigma b)|$. It is easy to check that $h(s)$ has a local minimum at $s^* := b \log \left(b + \sigma + L(b^2 + \sigma b)\right)$ and furthermore

$$h(s^*) = b + b \log \left(b + \sigma + L(b^2 + \sigma b)\right) - |\lambda_1| \left(b^2 + \sigma b\right),$$

which is negative for some b^* big enough (in Appendix A we will give some estimates on how big this b^* must be as a function of λ). This s^* is again in our target interval, for a big enough b (note that if $h(s^*) < 0$ then $s^* < |\lambda_1|(b^2 + \sigma b)|$).

From now on, let $s = s^*$, and check if (A.2) holds, where we will put $s = s^*$ at the end of the calculations.

$$b \cdot e^{-\sigma t/\sigma b} \sqrt{((\sigma t + \sigma b + 1) + y^2)} \overset{?}{\leq} |\lambda_1| \left(b^2 + \sigma b\right) - s.$$

As we have done before, expand

$$\begin{aligned} b\cdot e^{-\sigma t/\sigma b}\sqrt{((\sigma t+\sigma b+1)+y^2)} &\leq b\cdot e^{-t/b}\cdot(|\sigma t+\sigma b+1|+y)= \\ &= b\cdot e^{-t/b}\cdot(t+b+\sigma+y)= \\ &= b\cdot e^{-t/b}\cdot\left(t+b+\sigma+L\left(b^2+\sigma b\right)-s^*\right). \end{aligned}$$

It is easy to check that $b\cdot e^{-t/b}\cdot(b+\sigma+y)$ is a decreasing function in t, and $b\cdot e^{-t/b}t$ has a local maximum at $t=b$ and is a decreasing function for $t>b$. Then, we can bound both terms by setting $t=s^*$, as $s^*\geq b$ whenever $b+\sigma+L(b^2+\sigma b)$ is bigger than e, but this inequality holds if all other conditions are fulfilled. Now we must only check if

$$\begin{aligned} |\lambda_1|\left(b^2+\sigma b\right)-s^* &\overset{?}{\geq} b\cdot e^{-s^*/b}\cdot\left(s^*+b+\sigma|+L\left(b^2+\sigma b\right)-s^*\right)= \\ &= b\cdot\frac{b+\sigma+L\left(b^2+\sigma b\right)}{b+\sigma+L\left(b^2+\sigma b\right)}=b, \end{aligned}$$

which is the same inequality we have for $h(s)$, thus it is also satisfied. Inequality (A.3) is equivalent to (A.1), hence the result follows.

Now we give numerical bounds for how big b must be in Theorem 16.7. We will consider only the general case $\lambda_1\neq 0$, as the other is equivalent.

Consider the inequality

$$b\log\left(b+\sigma+L(b^2+\sigma b)\right)\leq -b+|\lambda_1|\left(b^2+\sigma b\right)$$

If this inequality holds and $b+\sigma+L(b^2+\sigma b)>0$, we have the required estimates to guarantee that all required inequalities in Theorem 16.7 hold. The second inequality is clearly trivial, as it holds when $b>1$. Now, we must find a suitable b for the first.

Simplifying a b factor and taking exponentials in both sides, we must check which b verify

$$b+\sigma+L(b^2+\sigma b)\leq e^{-1+|\lambda_1|\sigma}e^{|\lambda_1|b}. \tag{A.4}$$

We can get a lower bound of e^x:

$$e^{|\lambda_1|b}\geq 1+|\lambda_1|b+\frac{|\lambda_1|^2b^2}{2}+\frac{|\lambda_1|^3b^3}{6}.$$

And this way if

$$b+\sigma+L(b^2+\sigma b)\leq e^{-1+|\lambda_1|\sigma}\left(1+|\lambda_1|b+\frac{|\lambda_1|^2b^2}{2}+\frac{|\lambda_1|^3b^3}{6}\right),$$

then is also true (A.4). Now we must check when a degree 3 polynomial with negative dominant term has negative values. This will be true as long as $b>0$ is greater than the root with bigger modulus. It is well-known (see [HM97]) that a monic polynomial $z^n+\sum_i^{n-1}a_iz^i$ has its roots in a disc of radius $\max(1,\sum_i^{n-1}|a_i|)$, so every $b>1$ and bigger than

$$\frac{6}{e^{\sigma|\lambda_1|-1}|\lambda_1|^3}\cdot\left(|L-e^{\sigma|\lambda_1|-1}\frac{|\lambda_1|^2}{2}|+|1-e^{\sigma|\lambda_1|-1}|\lambda_1|b+L\sigma b|+|b+\sigma-1|\right)$$

satisfies our claims.

Finer estimates for b depending on λ can be obtained with a more careful splitting of λ space, for instance

$$\begin{aligned}\{\lambda | \lambda \in S^1\} = \{\lambda \in [7\pi/4, \pi/4]\} \cup \{\lambda \in [\pi/4, 3\pi/4]\} \cup \{\lambda \in [3\pi/4, 5\pi/4]\} \\ \cup \{\lambda \in [5\pi/4, 7\pi/4]\} = B_1 \cup B_2 \cup B_3 \cup B_4.\end{aligned}$$

The proof can be adapted with very minor changes to this partition, although the exposition and calculations are more cumbersome.

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