Essays on multi-sided assignment markets

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Annem Zeliha ve babam Zeki'ye

"The only way of discovering the limits of the possible is to venture a little way past them into the impossible."

Arthur C. Clarke

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Contents

1 Introduction

This dissertation covers the study of assignment problems in a game theoretical framework, focusing on multi-sided assignment games and especially in stability notions. One of the earliest works on assignment problems within an economic context is Koopmans and Beckmann (1957). The authors study a market situation in which industrial plants had to be assigned to the designated locations. The idea is to match two disjoint sets (plants and locations) by mixed-pairs where each possible mixed-pair has a given value. The problem in this context is to find a matching with the highest total valuation of mixed-pairs. Making use of Birkhoff-von Neumann Theorem (Birkhoff, 1946; von Neumann, 1953), they show that an optimal assignment can be obtained by solving a linear program. Furthermore, they introduce a system of rents (prices) on the locations that sustain the optimal assignment by solving the dual linear program. Related to that, Gale (1960) defines competitive equilibrium prices and shows they exist for any assignment problem.

Shapley and Shubik (1972) introduces the assignment problem in a cooperative game framework. The authors study a two-sided (house) market. In their setting, there are two disjoint sets that consist of m buyers and n sellers respectively. Each buyer wants to buy at most one house and each seller has one house on sale. Utility is identified with money, each buyer has a value (which can be different) for every house, and each seller has a reservation value. The valuation matrix represents the joint profit obtained by each mixed-pair. They define the corresponding coalitional game (*assignment game*) for the market. The question is how to share the profit and, to this end, the authors analyse a solution concept: the core (the set of allocations that cannot be improved upon by any coalition). They show that the core of an assignment game is always non-empty. Furthermore, it coincides with the set of dual solutions to the assignment problem, also with the set of competitive equilibrium payoff vectors, and has a lattice structure. Demange (1982) and Leonard (1983) prove that in the buyers-optimal core allocation each buyer attains his/her marginal contribution and in the sellers optimal core allocation each seller attains his/her marginal contribution.

Nevertheless, we observe several examples of real markets that consist of more than two sides. For instance, production lines consist of different industries in the market where agents from each industry have different roles: e.g. dairies $\rightarrow$ supermarkets $\rightarrow$ customers. A customer implicitly pays for the transaction between dairies and supermarkets when he/she buys a carton of milk from a supermarket. Although two-sided assignment games always have a non-empty core with nice properties, most of these properties depend on its two-sided structure.

In a multi-sided assignment market there exist m finite and pairwise disjoint sets (the sectors or sides of the market) and a non-negative valuation for each m-tuple, that is, an ordered set of m agents from different sectors. In this setting, a matching is a set of m-tuples such that each agent belongs to at most one of them. Then, a coalitional game is defined by maximizing the sum of values of all m-tuples of a matching, the maximum taken over all possible matchings. When there exist more than two sides in the market, the core may be empty (Kaneko and Wooders, 1982). Hence, most relevant properties cannot be extended to the multi-sided markets. Even when the core is non-empty, Tejada (2011) builds several counterexamples to show, among other facts, that the core may not be a lattice and an agent may not reach his/her marginal contribution as a core payoff.

To overcome these, several authors have proposed specific classes of multi-sided assignment games where the core is non-empty and may preserve some appealing structural properties. Sherstyuk (1999) defines "supermodular matching games" where the function that valuates the m-tuples is supermodular. For this subclass of multi-sided assignment games the core is proved to be non-empty and the existence of an optimal core allocation for each side of the market is guaranteed.

Other authors approach differently the problem of multi-sided assignment markets and the possible emptiness of the core. Quint (1991), Stuart (1997), and Tejada (2013) rely on some additivity principle in the definition of the value of an m-tuple when defining a subclass of multi-sided assignment games with non-empty core. In the two first models, the authors consider valuations of those coalitions that contain exactly one agent from each side of the market. Otherwise, a coalition has a value equal to zero. In both generalizations, there exist given weights on each pair of agents of different sectors, that sum up to the value of the coalition. Yet, the value of a pair is equal to zero since only coalitions formed by one agent from each side of the market may have positive value. The difference is that Quint (1991) adds up the weights of all pairs contained in an m-tuple, while Stuart (1997) considers that the sectors are organized in a line, take for instance a supplier-firm-buyer chain, and adds up the weights of pairs of agents in consecutive sectors.

The present dissertation aims to contribute to the study of multi-sided markets in two directions. Chapter 3 and Chapter 4 introduce a more general notion of multi-sided markets where r-tuples, with $r < m$, are assumed to have a reservation value and cooperation may be restricted by a network. Chapter 5 and Chapter 6, return to the classical model where only m-tuples may have a positive value and propose a solution concept, different to the core but also based on a dominance relation, that is always non-empty.

The outline of this dissertation is the following:

In Chapter 2, we provide some preliminaries on assignment markets and assignment games. We give some needed definitions and crucial results with their proof. First, we focus on the two-sided assignment game together with related solution concepts, and the notion of stability. Then, we introduce multi-sided assignment markets and the corresponding coalitional game. For these games we also point out the crucial definitions and results. The last section of this chapter is dedicated to some important classes of the multi-sided case that have a non-empty core.

In Chapter 3, we introduce a generalization of three-sided assignment markets. In this model, we consider three-sided assignment markets where value is obtained by means of (basic) coalitions formed by agents of different sides, that is, either triplets, pairs or individuals. Once the valuations of all these basic coalitions are known, a coalitional game is defined. The worth of an arbitrary coalition is obtained by taking the partition in basic coalitions that attains the maximal worth. Since we allow for a positive worth of two-player coalitions with agents belonging to different sectors, together with individual coalitions and triplets, this generalization is different from the classical class in Kaneko and Wooders (1982). Nonetheless, these games may also have an empty core. Hence, some well-known characterizations of the core of the two-sided assignment games do not extend to this class of generalized three-sided assignment games. Take for instance the pairwise-monotonicity property satisfied by the core of two-sided assignment games: we may have a three-sided market with non-empty core but if we rise the value of one triplet the core may become empty. Hence, some other properties must replace monotonicity for a possible characterization of the core of generalized three-sided assignment games.

In this class of generalized three-sided assignment markets, we introduce a reduced market by extending to the three-sided case the derived market that Owen (1992) defines for the two-sided case. The reduced market can be defined in our setting of generalized three-sided markets because both individuals and mixed-pairs have reservation values that play a role whenever they do not take part of a triplet. We show that *consistency with respect to the derived market* is satisfied by the core and the nucleolus (a single-valued solution concept for coalitional games). Together with two other properties, *singleness best* and *individual anti-monotonicity*, we give an axiomatic characterization of the core on the domain of generalized three-sided assignment markets. Finally, we show that the set of competitive equilibrium payoff vectors coincides with the core, which generalizes the results for the two-sided case of Gale (1960) and also some results of Tejada (2010) for classical multi-sided markets. The paper on which this chapter is based has been published in *TOP* under the title "Generalized three-sided assignment markets: core consistency and competitive prices" (Atay et al, 2016).

In Chapter 4, we consider generalized multi-sided assignment games where cooperation is restricted by an underlying network structure. Recall that a two-sided assignment game can be described by means of an underlying weighted bi-partite graph. The agents correspond to the set of nodes and the value of a basic coalition formed by adjacent nodes is the weight of the corresponding edge. From this point of view, we give a generalization to multi-sided assignment games. First, we consider a graph on the set of sectors that is, a graph which indicates which sectors are linked. This graph induces an m-partite graph on the set of agents: if two sectors are connected, each mixed-pair of agents of these sectors is connected in the m-partite graph. We consider as basic coalitions those connected by the m-partite graph and with no more than one agent from each sector. The weights on the graph define an underlying two-sided assignment market for each pair of connected sectors and by additivity give rise to the value of basic coalitions. Then, we define a coalitional game, the corresponding multi-sided assignment game. Notice that for two sectors we recover two-sided assignment games of Shapley and Shubik (1972). In this new setting, multi-sided assignment games on an m-partite graph, we introduce sufficient conditions on the weights that guarantee the non-emptiness of the core. When we impose that the underlying structure of the graph that connects sectors is cycle-free, we guarantee non-emptiness of the core regardless the system of weights. Moreover, we show that the core of a multi-sided assignment game on an m-partite graph where the quotient graph on the set of sectors is cycle-free, is fully described by the "composition" of the cores of all underlying two-sided markets. As a consequence, we study properties of the core of these multi-sided assignment games by means of the cores of their underlying two-sided games. For instance, we extend the result of Demange (1982) and Leonard (1983) to this multi-sided situation and show that, for each sector, there exists a core allocation where all agents in this sector achieve their marginal contribution. Furthermore, with the previous cycle-free condition, we provide the equivalence between core and competitive equilibria. This result also extends the well-known result for two-sided assignment games.

In Chapter 5 and Chapter 6, we focus on the notion of stability of von Neumann and Morgenstern applied to the classical multi-sided assignment games where a basic coalition contains exactly one agent from each sector. As introduced above, Shapley and Shubik (1972) studies a solution concept, the core, for the two-sided assignment games. The core can also be defined by means of von Neumann-

Morgenstern domination as introduced by Gillies (1959). In fact, the first solution concept introduced for coalitional games was the stable set (von Neumann and Morgenstern, 1944). It is a set of imputations that satisfies internal stability and external stability: it does not exist an imputation in the set that dominates another imputation in the set and each imputation outside the set is dominated by some imputation in the set. The core is the set of undominated imputations whenever it is non-empty, which is always the case for two-sided assignment games. Hence, it always satisfies internal stability. Although there are games with no stable set, see Lucas (1968), if a game has a stable set, it contains the core. Moreover, the core is included in any stable set and, if it satisfies external stability, then it is the unique stable set. This relationship between the two solution concepts implies that the characterization of core stability is an important approach to the study of stable sets. Solymosi and Raghavan (2001) characterizes the core stability for two-sided assignment games by means of the dominant diagonal property (each diagonal element is column and row maximum). Later, Núñez and Rafels (2013) proves the existence of a stable set for any two-sided assignment game. Since the core may be empty when there are more than two sides in an assignment game, it is more appealing in this multi-lateral setting to study the stable sets as a solution concept to replace the empty core.

In Chapter 5, we focus on two-sided assignment games. Solymosi and Raghavan (2001) uses a graph theoretical approach in order to show that the core of an assignment game (with the same number of agents on each side and an optimal matching on the main diagonal of the valuation matrix) is a stable set if and only if its diagonal elements are row and column maxima. In this chapter, we provide an alternative proof of the same characterization of the core stability. This new proof is based on results from Núñez and Rafels (2002), where a lower bound for the core payoff of a mixed-pair is provided. A paper based on this chapter is under revision at *Operations Research Letters*.

In Chapter 6, we study von Neumann-Morgenstern stability for three-sided assignment games. First, we generalize the dominant diagonal property and show that it is a necessary condition for the core to be a von Neumann-Morgenstern stable set. Furthermore, making use of the non-emptiness conditions of Lucas (1995) for the particular case where each side has two agents, we show that the dominant diagonal property is also a sufficient condition for core stability in this $2\times2\times2$ class of markets. Then, we extend the notion of μ-compatible subgame introduced by Núñez and Rafels (2013) to three-sided assignment games and we consider the set formed by the union of the extended cores of all μ-compatible subgames. We show that this set consists of imputations that are undominated by any element of the principal section, which is the set of payoff vectors where each optimally matched triplet shares exactly the worth of the coalition they form whereas unassigned agents get a null payoff. If we consider the principal section as the set of feasible outcomes, then the union of the cores of all μ-compatible subgames is the set of undominated outcomes, that is, the “core” with respect to this set of feasible outcomes. However, we provide a counterexample to show that this set may fail to satisfy external stability. Hence, in general, it is not a von Neumann-Morgenstern stable set.

Finally, Chapter 7 concludes this dissertation with some remarks. In this chapter, we higlight our main contributions and provide some hints about possible future research.

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2 Assignment markets and assignment games

This chapter deals with the preliminaries we need in order to explain the findings and contributions of this dissertation. This is the reason why in this chapter we introduce the multi-sided markets and games, and what is known about them in the literature. Also, we will review the main notions and results known for two-sided assignment games, in particular those we intend to generalize to the multi-sided case. Nevertheless since each chapter corresponds to a potential paper, each of them is self-contained.

2.1 The two-sided assignment game

An assignment game is a model for a two-sided market introduced by Shapley and Shubik (1972). There are two disjoint sets of agents, let us call them buyers and sellers, and denote them by M and M' respectively. In this market, there are m buyers, m' sellers, and a valuation matrix $A = (a_{ij})_{\substack{i\in M \\ j\in M'}}$ that represents the joint profit obtained by a mixed-pair of a buyer and a seller. Formally, we denote this market by $\gamma = (M, M'; A)$. In this market, each buyer $i \in M$ wants to buy at most one good, whereas each seller $j \in M'$ has an indivisible good to sell. Utility is identified with money. We will denote by $x = (u, v) \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$ a payoff vector, where $u_i \in \mathbb{R}_+$ stands for the payoff to buyer $i \in M$ and $v_j \in \mathbb{R}_+$ stands for the payoff to seller $j \in M'$.[1]

Let $\gamma = (M, M'; A)$ be an assignment market. A *matching* μ between M and M' is a subset of the Cartesian product, $M \times M'$, such that each agent belongs to at most one pair. We denote by $\mathcal{M}(M, M')$ the set of all possible matchings. A matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for the market $(M, M'; A)$ if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for all $\mu' \in \mathcal{M}(M, M')$. The set of all optimal matchings for the market $(M, M'; A)$ is denoted by $\mathcal{M}_A(M, M')$. An optimal matching μ can be found by solving the

[1] Throughout this dissertation, $\mathbb{R}_+$ stands for the set of non-negative real numbers. Similarly, $\mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i \in \{1, \ldots, m\}\}$.

so-called linear assignment problem:

$$\begin{aligned} \max \sum_{i\in M}\sum_{j\in M'} a_{ij}x_{ij} \\ \text{s.t.} \sum_{i\in M} x_{ij} \le 1, \text{ for all } j\in M', \\ \sum_{j\in M'} x_{ij} \le 1, \text{ for all } i\in M, \\ x_{ij}\in\{0,1\} \text{ for all } (i,j)\in M\times M'. \end{aligned} \tag{2.1}$$

If $x\in\{0,1\}^{M\times M'}$ is a solution of (2.1), then $\mu=\{(i,j)\mid x_{ij}=1\}$ is an optimal matching.

Shapley and Shubik (1972) defines a coalitional game (N,w_A) to describe the market.[2] The player set is $N=M\cup M'$ and the characteristic function

$$w_A(S)=\max_{\mu\in\mathcal{M}(M\cap S,M'\cap S)}\sum_{(i,j)\in\mu} a_{ij} \text{ for all } S\subseteq N.$$

They show that it is sufficient to take into account mixed-pair coalitions to describe the core.[3] Then, for each optimal matching μ, the core of the corresponding assignment game (N,w_A) is described by

$$C(w_A)=\left\{(u,v)\in\mathbb{R}^M_+\times\mathbb{R}^{M'}_+ \;\middle|\; \begin{array}{l} u_i+v_j=a_{ij} \text{ for all } (i,j)\in\mu \text{ and} \\ u_i+v_j\ge a_{ij} \text{ for all } (i,j)\in M\times M' \end{array}\right\}.$$

Shapley and Shubik prove that the core of an assignment game is always non-empty, that is, assignment games are balanced.[4]

Theorem 2.1 (Shapley and Shubik, 1972)**.** *Let $\gamma=(M,M';A)$ be a two-sided assignment market. Then, its corresponding coalitional game (N,w_A) has a non-*

[2] A *coalitional game* is defined by a pair (N,v) where N is the (finite) player set and the characteristic function v assigns a real number $v(S)$ to each coalition $S\subseteq N$, with $v(\emptyset)=0$.

[3] The main solution concept studied for coalitional games is the *core*. The core of (N,v) is the set of payoff vectors $x\in\mathbb{R}^N$, where x_i stands for the payoff to agent $i\in N$, that satisfy efficiency and coalitional rationality:

$$C(v)=\{x\in\mathbb{R}^N \mid \sum_{i\in N} x_i=v(N) \text{ and } \sum_{i\in S} x_i\ge v(S) \text{ for all } S\subseteq N\}.$$

The core is a subset of imputations, $I(v)$, that is, efficient payoff vectors that are individually rational, $x_i\ge v(\{i\})$ for all $i\in N$.

[4] A game (N,v) is said to be balanced if it has a non-empty core.

empty core. Moreover, the core coincides with the set of dual solutions to the assignment problem.

Proof. Consider the assignment market $\gamma = (M, M'; A)$ and its corresponding game (N, w_A). An optimal matching μ can be found by solving the so-called linear assignment problem:

$$\begin{aligned} \max & \sum_{i \in M} \sum_{j \in M'} a_{ij} x_{ij} \\ \text{s.t.} & \sum_{i \in M} x_{ij} \leq 1, \text{ for all } j \in M', \\ & \sum_{j \in M'} x_{ij} \leq 1, \text{ for all } i \in M, \\ & x_{ij} \in \{0,1\} \text{ for all } (i,j) \in M \times M'. \end{aligned} \tag{2.2}$$

By the Birkhoff-von Neumann Theorem (Birkhoff, 1946; von Neumann, 1953), the solution of the above integer linear program coincides with its LP relaxation, which is the related continuous linear program with $x_{ij} \geq 0$ for all $(i,j) \in M \times M'$. The fundamental duality theorem (Dantzig, 1963) states that every linear program can be transposed into a dual form and, if the primal program has a solution, then the optimal values of both programs coincide. Then, the dual of the LP relaxation of the primal program (2.2) is:

$$\begin{aligned} \min & \sum_{i \in M} u_i + \sum_{j \in M'} v_j \\ \text{s.t. } & u_i + v_j \geq a_{ij} \text{ for all } (i,j) \in M \times M', \\ & u_i \geq 0 \text{ for all } i \in M, \\ & v_j \geq 0 \text{ for all } j \in M'. \end{aligned} \tag{2.3}$$

In our case, the fundamental duality theorem tells that (2.3) has a solution and, over the respective sets of constraints, $\min \sum_{i \in M} u_i + \sum_{j \in M'} v_j = \max \sum_{i \in M} \sum_{j \in M'} a_{ij} x_{ij} = w_A(M \cup M')$. Hence, a payoff vector (u, v) is a solution of the dual program (2.3) if and only if it is an element of the core of (N, w_A). As a consequence, the core is non-empty. □

The set of dual solutions of the assignment problem had already been analysed by Gale (1960) and related to his notion of competitive equilibrium. As in Roth and Sotomayor (1990), let us assume that M' contains as many copies as necessary of a null object $\mathcal{O}$ such that $a_{i\mathcal{O}} = 0$ for all $i \in M$. Then, for any matching μ, all buyers can be assumed to be matched either to a real object or to a null object $\mathcal{O}$.

Definition 2.2 (Gale, 1960)**.** Given a vector of non-negative prices $p \in \mathbb{R}^{M'}$, with $p_{\mathcal{O}} = 0$, the *demand set* of buyer $i \in M$ at prices p is

$$D_p(i) = \{j \in M' | a_{ij} - p_j = \max_{k \in M'} \{a_{ik} - p_k\}\}.$$

Then, a pair (p, μ) formed by a vector of prices and a matching is a *competitive equilibrium* if $\mu(i) \in D_i(p)$ for all $i \in M$ and $p_j = 0$ whenever $j \in M'$ is unassigned by μ. In this case, p is said to be a *competitive equilibrium price vector*. Given a competitive equilibrium (p, μ), the payoff vector (u, v) where $u_i = a_{i\mu(i)} - p_{\mu(i)}$ for all $i \in M$ and $v_j = p_j$ for all $j \in M'$ is a *competitive equilibrium payoff vector*.

Theorem 2.3 (Gale, 1960)**.** *For any assignment game, the set of solutions of the dual program of (2.1) coincides with the set of competitive equilibrium payoff vectors.*

Proof. Given a solution (u, v) of the dual program, define $p = v \in \mathbb{R}^{M'}_{+}$. Take μ an optimal matching. From $\sum_{(i,j) \in \mu} a_{ij} = \sum_{i \in M} u_i + \sum_{j \in M'} v_j$ and $u_i + v_j \geq a_{ij}$ for all $(i, j) \in \mu$ it follows that $p_j = v_j = 0$ for all unassigned object $j \in M'$ and $u_i + v_j = a_{ij}$ if $(i, j) \in \mu$. Moreover, for all $i \in M$,

$$a_{i\mu(i)} - p_{\mu(i)} = u_i \geq a_{ij} - p_j \quad \text{for all } j \in M',$$

where the inequality follows from the dual program constraints. Hence, p is a competitive price vector.

Conversely, if p is a competitive price vector, then there exists $\mu \in \mathcal{M}(M, M')$ such that $p_j = 0$ if j is unassigned by μ and for all $i \in M$

$$\mu(i) \in D_i(p).$$

Define now $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ by $v_j = p_j$ for all $j \in M'$ and $u_i = a_{i\mu(i)} - p_{\mu(i)}$ for all $i \in M$. Notice that if $i \in M$ is assigned to a null object, then $u_i = 0$. Also, $v_j = 0$ if $j \notin \mu(M)$. Let us check that (u, v) is a solution of the dual problem.

We see first that if (p, μ) is a competitive equilibrium, then μ is an optimal matching. Indeed, take another matching $\mu' \in \mathcal{M}(M, M')$. Now, since $a_{i\mu(i)} - p_{\mu(i)} \geq a_{i\mu'(i)} - p_{\mu'(i)}$ for all $i \in M$,

$$\begin{aligned}
\sum_{(i,j) \in \mu} a_{ij} = \sum_{i \in M} a_{i\mu(i)} &\geq \sum_{i \in M} (a_{i\mu'(i)} - p_{\mu'(i)}) + \sum_{i \in M} p_{\mu(i)} \\
&= \sum_{i \in M} a_{i\mu'(i)} - \sum_{j \in \mu'(M)} p_j + \sum_{j \in \mu(M)} p_j \\
&= \sum_{i \in M} a_{i\mu'(i)} - \sum_{j \in \mu'(M) \setminus \mu(M)} p_j + \sum_{j \in \mu(M) \setminus \mu'(M)} p_j \\
&\geq \sum_{i \in M} a_{i\mu'(i)}
\end{aligned}$$

where the last inequality follows from the fact that (p,μ) is a competitive equilibrium and hence $p_j = 0$ for all $j \notin \mu(M)$.

Since μ is an optimal matching and agents assigned to the null object receive zero,

$$w_A(M \cup M') = \sum_{i \in M} a_{i\mu(i)} = \sum_{i \in M} u_i + v_{\mu(i)} = \sum_{i \in M} u_i + \sum_{j \in M'} v_j,$$

which means (u,v) is efficient.

Finally, for all $i \in M$ and for all $j \in M'$,

$$\begin{aligned} u_i + v_j = u_i + p_j &= a_{i\mu(i)} - p_{\mu(i)} + p_j \\ &\geq a_{ij} - p_j + p_j = a_{ij}, \end{aligned}$$

which concludes the proof that (u,v) is a solution of the dual program. □

Furthermore, Shapley and Shubik study the structure of the core and show that it has a lattice structure with respect to the partial order $(u,v) \geq_M (u',v')$ if $u_i \geq u'_i$ for all $i \in M$.

Theorem 2.4 (Shapley and Shubik, 1972)**.** *Let $\gamma = (M,M';A)$ be an assignment market. Given two core elements $(u,v) \in C(w_A)$ and $(u',v') \in C(w_A)$, the join*

$$(u,v) \vee (u',v') = ((\max\{u_i,u'_i\})_{i \in M}, (\min\{v_j,v'_j\})_{j \in M'})$$

and the meet

$$(u,v) \wedge (u',v') = ((\min\{u_i,u'_i\})_{i \in M}, (\max\{v_j,v'_j\})_{j \in M'})$$

belong to the core.

Besides, the lattice structure of the core leads to the existence of two best core allocations, one for each side of the market, namely, *buyers-optimal* core allocation and *sellers-optimal* core allocation.

Remark 2.5 (Shapley and Shubik, 1972)**.** There exist two special extreme core allocations. In the buyers-optimal core allocation, $(\overline{u}^A, \underline{v}^A) \in C(w_A)$, each buyer maximizes his/her payoff in the core, while each seller minimizes his/her, and vice versa in the sellers-optimal core allocation $(\underline{u}^A, \overline{v}^A) \in C(w_A)$.

The assignment market is studied from a strategic point of view by Demange (1982) and Leonard (1983). The authors, independently, show that the maximum core allocation of an agent is equal to his/her marginal contribution to the grand coalition. As a result, the authors prove that there is no incentive for a buyer (seller) to misrepresent his/her true valuations if profits will be shared by means of the buyers-optimal (respectively, sellers-optimal) core allocation.

Theorem 2.6 (Demange, 1982; Leonard, 1983)**.** *Given an assignment game* $(M \cup M', w_A)$*, the maximum core payoff of an agent is his/her marginal contribution to the grand coalition, that is,*

$$\overline{u}_i^A = w_A(M \cup M') - w_A((M \setminus \{i\}) \cup M') \qquad \text{for all } i \in M,$$
$$\overline{v}_j^A = w_A(M \cup M') - w_A((M \cup (M' \setminus \{j\})) \qquad \text{for all } j \in M'.$$

Other cooperative solutions have been studied for the assignment game. Among the single-valued solutions, that are defined for arbitrary coalitional games, the *nucleolus* stands out. We will briefly give its definition applied to the assignment game. Consider all basic coalitions $\mathcal{B}$ (singletons and mixed-pairs) and at each imputation $x \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$, the *excess of* x *at coalition* $S \in \mathcal{B}$, $e(S,x) := w_A(S) - \sum_{i \in S} x_i$. Let us denote by $\theta(x)$ the vector formed by the decreasingly ordered excesses of all basic coalitions at imputation $x \in \mathbb{R}_+^M \times \mathbb{R}_+^{M'}$. Then, the *nucleolus* (Schmeidler, 1969) is the imputation that lexicographically minimizes this vector of excesses: $\theta(\eta) \leq_L \theta(x)$ for all $x \in I(w_A)$. An algorithm to compute the nucleolus of the assignment game is given in Solymosi and Raghavan (1994), a geometric characterization is in Llerena and Núñez (2011), and an axiomatization in Llerena et al (2015).

Another single-valued solution for the assignment game was introduced by Thompson (1981) with the name of *fair division point*, since it is the midpoint of the segment between the buyers-optimal and the sellers-optimal core allocation:

$$\tau(w_A) = \tfrac{1}{2}(\overline{u}^A, \underline{v}^A) + \tfrac{1}{2}(\underline{u}^A, \overline{v}^A).$$

Núñez and Rafels (2002) proves that the fair division point coincides with the τ-value (Tijs, 1981).

Among many set-valued solutions that have been defined for coalitional games, we will consider the von Neumann-Morgenstern stable sets in Chapters 5 and 6 of this dissertation. This was the first notion of solution proposed for coalitional games. It is even previous to the core, although both can be defined by means of the same dominance relation. von Neumann and Morgenstern (1944) introduced the following notion of domination between imputations. Given a coalitional game (N,v) and two imputations $x, y \in I(v)$, we say x *dominates* y if there is a coalition $S \subseteq N$ such that $x_i > y_i$ for all $i \in S$ and $\sum_{i \in S} x_i \leq v(S)$.

Given this dominance relation, stable sets are defined as follows:

Definition 2.7 (von Neumann and Morgenstern, 1944)**.** Given a coalitional game (N,v), a subset V of the set of imputations is a von Neumann-Morgenstern stable set if it satisfies the following conditions:

(i) internal stability: no imputation in the set V is dominated by another imputation in V,

(ii) external stability: each imputation outside the set V is dominated by some imputation of the set V.

The existence of stable sets for arbitrary classes of games is a difficult task. Nonetheless, Lucas (1968) provides a game with ten players that has no stable set. In comparison, as we have seen before, the core can be defined by means of linear inequalities and, for a given instance, it is not difficult to determine whether it is empty or not. However, both solutions are more tightly related than what may seem at first sight. The core can also be defined through the above domination relation of von Neumann and Morgenstern (1944): when the core is non-empty, it consists of all undominated imputations. Thus, internal stability is always satisfied. Furthermore, the core is included in any stable set and, when it also satisfies external stability, it is the unique stable set.

Let us now go back to the two-sided assignment game and recall what is known about core stability and stable sets for these games. Solymosi and Raghavan (2001) shows that the core of a square two-sided assignment game, $|M| = |M'| = m$, where an optimal matching is placed on the diagonal of the valuation matrix, is a von Neumann-Morgenstern stable set if and only if the valuation matrix has a *dominant diagonal*, that is to say, diagonal elements of the matrix are row and column maxima:

$$a_{ii} \geq \max\{a_{ij}, a_{ji}\} \qquad \text{for all } i, j \in \{1, 2, ..., m\}.$$

Theorem 2.8 (Solymosi and Raghavan, 2001)**.** *Let* $(M \cup M', w_A)$ *be a two-sided square assignment game. Then, the following statements are equivalent:*

(i) A *has a dominant diagonal*

(ii) $C(w_A)$ *is a von Neumann-Morgenstern stable set.*

When the core of the assignment game is not a stable set, it is natural to ask whether it can be enlarged in some way as to obtain a stable set. This was suggested by Shapley in some personal notes and in Shubik (1984), and proved by Núñez and Rafels (2013). To construct a stable set for the assignment game, we need to introduce the notion of compatible subgame. For all $I \subseteq M$, $J \subseteq M'$, we will denote by $((M \setminus I) \cup (M' \setminus J), w_{A_{-I \cup J}})$ the subgame with agents $(M \setminus I) \cup (M' \setminus J)$ and where the valuation matrix is restricted to the rows $i \notin I$ and the columns $j \notin J$.

Definition 2.9 (Núñez and Rafels, 2013)**.** Given an assignment game $(M \cup M', w_A)$ and an optimal matching $\mu \in \mathcal{M}_A(M,M')$, the subgame obtained by removing buyers in $I \subseteq M$ and sellers in $J \subseteq M'$, $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$, is said to be *$\mu$-compatible* if

$$w_A(M \cup M') = \sum_{i \in I} a_{i\mu(i)} + \sum_{j \in J} a_{\mu^{-1}(j)j} + w_A((M \setminus I) \cup (M' \setminus J)).$$

If $a_{i\mu(i)} > 0$ for all $i \in M$, this is equivalent to saying that the restriction of μ to $(M \setminus I) \times (M' \setminus J)$ is optimal for the subgame. Notice that the core of a μ-compatible subgame $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ lies in a linear space of lower dimension than the one that contains $C(w_A)$. Because of that, if the subgame $((M \setminus I) \cup (M' \setminus J), w_{A_{-I\cup J}})$ is μ-compatible, we consider its extended core $\hat{C}(w_{A_{-I\cup J}})$, that is,

$$\hat{C}(w_{A_{-I\cup J}}) = \{x \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+ \mid x_{-I\cup J} \in C(w_{A_{-I\cup J}}), x_k = a_{kk} \text{ for all } k \in I \cup J\}.$$

By means of this definition, a stable set for the assignment game can be described.

Theorem 2.10 (Núñez and Rafels, 2013)**.** *Let $(M \cup M', w_A)$ be an assignment game and $\mu \in \mathcal{M}_A(M,M')$ be an optimal matching. The set V, the union of the extended cores of all μ-compatible subgames, is a von Neumann-Morgenstern stable set of $(M \cup M', w_A)$.*

This stable set is included in the μ-principal section of the game $(M \cup M', w_A)$, that is,

$$B^\mu(w_A) = \left\{ x \in \mathbb{R}^M_+ \times \mathbb{R}^{M'}_+ \;\middle|\; \begin{array}{l} x_i + x_j = a_{ij} \text{ for all } (i,j) \in \mu, \\ x_k = 0 \text{ if } k \in M \cup M' \text{ is unassigned by } \mu. \end{array} \right\}.$$

This means that in the aforementioned stable set V, third-party payments are excluded. That is, side-payments only take place between optimally matched pairs. In fact, V is the only stable set with this property. In Chapter 6 we will analyse if this stability property can be extended to the multi-sided case.

2.2 Multi-sided assignment markets and games

The classical generalization of two-sided assignment markets considers the market situations where there are m disjoint sectors N_1, N_2, …, N_m and a non-negative m-dimensional matrix $A = (a_E)_{E \in \prod_{k=1}^m N_k}$. This valuation matrix assigns a value to those coalitions that contain exactly one agent from each sector of the market. With some abuse of notation, such coalitions can be identified with the m-tuples

$E=(i_1,i_2,\ldots,i_m)\in\prod_{k=1}^m N_k$. These coalitions are called *basic coalitions*. Thus, each matrix element, $a_E\geq 0$, represents the joint profit obtained by a basic coalition $E\in\prod_{k=1}^m N_k$. A *matching* for an arbitrary coalition $S\subseteq N$ is a subset of basic coalitions such that each agent belongs to at most one basic coalition. The set of all matchings for a coalition $S\subseteq N$ is denoted by $\mathcal{M}(S\cap N_1,S\cap N_2,\ldots,S\cap N_m)$. A multi-sided ($m$-sided) assignment market is defined by $\gamma=(N_1,N_2,\ldots,N_m;A)$ and the corresponding coalitional game for this market situation is a pair (N,w_A), where $N=\bigcup\limits_{k=1}^{m} N_k$ and w_A is the characteristic function defined below.

Definition 2.11. The *multi-sided assignment game* corresponding to a multi-sided assignment market $\gamma=(N_1,N_2,\ldots,N_m;A)$ is the pair (N,w_A) where $N=\bigcup\limits_{k=1}^{m} N_k$ is the set of players and the characteristic function is defined by

$$w_A(S)=\max_{\mu\in\mathcal{M}(N_1\cap S,N_2\cap S,\ldots,N_m\cap S)}\sum_{E\in\mu}a_E \qquad \text{for all } S\subseteq N, \tag{2.4}$$

with $w_A(\emptyset)=0$.

A matching $\mu\in\mathcal{M}(N_1\cap S,N_2\cap S,\ldots,N_m\cap S)$ that solves the maximization problem in (2.4) is said to be an *optimal matching* and the set of all optimal matchings for coalition $S\subseteq N$ is denoted by $\mathcal{M}_A(N_1\cap S,N_2\cap S,\ldots,N_m\cap S)$.

For multi-sided assignment games we analyse the core of the game. Once selected an arbitrary optimal matching $\mu\in\mathcal{M}_A(N_1,\ldots,N_m)$, it follows from Definition 2.11 that it is sufficient to take into account only basic coalitions in order to define the core. Let (N,w_A) be a multi-sided assignment game and consider an optimal matching $\mu\in\mathcal{M}_A(N_1,N_2,...,N_m)$. Then, the core $C(w_A)$ is described by

$$C(w_A)=\left\{x\in\mathbb{R}^N_+ \mid \sum_{i\in E}x_i=v(E) \text{ for all } E\in\mu,\ \sum_{i\in E}x_i\geq v(E),\ \text{for all } E\in\prod_{k=1}^m N_k\right\}.$$

Different from the two-sided case, Kaneko and Wooders (1982) shows by means of an example that m-sided assignment markets may have an empty core.

Example 2.12 (Kaneko and Wooders, 1982)**.** Let $M_1=\{1,2,3\}$, $M_2=\{1',2',3'\}$, and $M_3=\{1'',2'',3''\}$ be three sectors, and consider a three-sided assignment game (N,v) where $N=M_1\cup M_2\cup M_3$ and the characteristic function is

$$v(S)=\begin{cases}|S| & \text{if } S\in\mathcal{C},\\ 0 & \text{otherwise,}\end{cases}$$

where $\mathcal{C}=\{\{1,1',1''\},\{1,2',3''\},\{2,1',2''\},\{2,3',3''\},\{3,2',1''\},\{3,3',2''\}\}$ is a family of basic coalitions. So, the three-dimensional valuation matrix of this game is the following:

$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{pmatrix} \mathbf{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \\ 1'' \end{array} \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & \mathbf{3} \end{pmatrix} \\ 2'' \end{array} \quad \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{pmatrix} 0 & 3 & 0 \\ 0 & \mathbf{0} & 3 \\ 0 & 0 & 0 \end{pmatrix} \\ 3'' \end{array}.$$

Now, take the optimal matching $\mu = \{\{1,1',1''\},\{3,3',2''\},\{2,2',3''\}\}$. Then, triplets matched under the matching μ exactly share their profit: $u_1+v_1+w_1 = 3$, $u_3+v_3+w_2 = 3$, and $u_2+v_2+w_3 = 0$. Hence, $u_2 = v_2 = w_3 = 0$. On the other hand, by the core constraint $u_1+v_2+w_3 \geq 3$ we have $u_1 = 3$ and as a consequence $v_1 = w_1 = 0$. Then, $u_2+v_1+w_2 \geq 3$ implies $w_2 \geq 3$ and hence $u_3 = v_3 = 0$. But then $u_2+v_3+w_3 \geq 3$ is not satisfied. Thus, the core of the game is empty.

For the particular case of three-sided assignment markets in which there are two agents in each sector, Lucas (1995) provides necessary and sufficient conditions for the non-emptiness of the core.

Proposition 2.13 (Lucas, 1995)**.** *Let $(N_1,N_2,N_3;A)$ be a three-sided assignment market where each side of the market consists of two agents, $N_1 = \{1,2\}$, $N_2 = \{1',2'\}$, $N_3 = \{1'',2''\}$, and let the main diagonal be an optimal matching, that is, $a_{11'1''}+a_{22'2''} = w_A(N)$. Then, the core of the corresponding coalitional game (N,w_A) is non-empty if and only if*

(i) $2a_{11'1''}+a_{22'2''} \geq a_{21'1''}+a_{12'1''}+a_{11'2''}$,

(ii) $a_{11'1''}+2a_{22'2''} \geq a_{12'2''}+a_{21'2''}+a_{22'1''}$.

Hence, although multi-sided assignment games seem to be the natural generalization of the two-sided assignment games of Shapley and Shubik (1972), they behave very differently, basically because the core of two-sided assignment games is always non-empty.

2.3 Some balanced classes of multi-sided assignment games

Since the core may be empty for multi-sided assignment games, conditions for the non-emptiness of the core, that is balancedness conditions, have been studied by some authors. Quint (1991) introduces a class where the worth of an arbitrary basic coalition (formed by exactly one agent from each side of the market) is obtained by addition of the weights that are attributed to each pair of agents in the basic coalition. When the market has more than two sectors, values of two-player coalitions are considered to be null.

Lemma 2.14 (Quint, 1991). *Let $(a_E)_{E\in\prod_{k=1}^m N_k}$ define an m-sided square assignment game with n agents in each side. Let α_1 and α_2 be two non-negative weights satisfying $\alpha_1+\alpha_2=1$. To each $r,s\in\{1,\dots,m\}$ and each pair $(i_r,i_s)\in N_r\times N_s$, a non-negative number $d^{rs}_{i_ri_s}$ is attached. If $a_E=\sum\limits_{1\le r<s<m} d^{rs}_{i_ri_s}$, for each $E=(i_1,i_2,\dots,i_m)$ and*

$$d^{rs}_{i_ri_s}\le \alpha_1 d^{rs}_{i_ri_r}+\alpha_2 d^{rs}_{i_si_s} \tag{2.5}$$

for all $i_r,i_s\in\{1,\dots n\}$ and all $r,s\in\{1,\dots,m\}$ with $r<s$, then the core of the m-sided assignment game is non-empty.

The idea behind the balanced class introduced by Quint (1991) is to consider *"the whole is only as good as the sum of its parts"*. So, the author assigns a non-negative constant $d^{rs}_{i_ri_s}$ to each pair of agents of different sectors and the worth of a basic coalition formed by one agent from each sector is simply the addition of the weights of its pairs. If, moreover, the weights satisfy inequality (2.5), then a balanced multi-sided assignment game is obtained.

In a similar spirit, Stuart (1997) introduces another class with non-empty core. In this class, so-called supplier-buyer-firm model, the sectors of the market are established on a chain. The difference is that only pairs of agents that are from consecutive sectors generate a value. In Stuart's class, coalitions that do not contain exactly one agent from each side also have a worth equal to zero.

Definition 2.15 (Stuart, 1997). An m-sided assignment market $(N_1,N_2,\dots,N_m;A)$ satisfies *local additivity* if there exists a set of matrices $B^k=(b^k_{ij})_{(i,j)\in N_k\times N_{k+1}}$, for $k\in\{1,2,\dots,m-1\}$ such that

$$a_E=\sum_{k=1}^{m-1} b^k_{i_ki_{k+1}} \qquad \text{for all } E=(i_1,i_2,\dots,i_m)\in\prod_{k=1}^m N_k.$$

Proposition 2.16 (Stuart, 1997). *Let $(N_1,N_2,\dots,N_m;A)$ be a locally additive m-sided assignment market. The core of the corresponding m-sided assignment game (N,w_A) is non-empty.*

Another interesting class of multi-sided assignment games is the m-sided Böhm-Bawerk assignment markets.[5] Tejada (2013) studies this class where there are $m-1$

[5]A particular case of assignment market where product differentiation is not present, due to Böhm-Bawerk (1923).

sectors that consist of different types of sellers and there exists a sector of buyers with the requirement that each buyer values in the same way each bundle formed by one seller of each type. This game is the natural generalization of the Böhm-Bawerk horse market to a market with several sectors. The author shows that the core is non-empty.

Definition 2.17 (Tejada, 2013)**.** An m-sided Böhm-Bawerk market is a pair $(c;w)$ where $c=(c_1,c_2,\dots,c_{m-1})\in\mathbb{R}_+^{N_1}\times\dots\times\mathbb{R}_+^{N_{m-1}}$ are the sellers' valuations and $w=(w_1,\dots,w_{n_m})\in\mathbb{R}^{N_m}$ are the buyers' valuations. Given an m-sided Böhm-Bawerk market $(c;w)$, $A(c;w)$ denotes the m-dimensional valuation matrix defined by

$$a_E=\max\{0,w_{i_m}-\sum_{k=1}^{m-1}c_{ki_k}\},\quad \text{for all } E=(i_1,\dots,i_m)\in\prod_{k=1}^{m}N_k.$$

Tejada (2013) proves not only that the core is non-empty, but also that is determined by the core of a convex coalitional game played by the sectors instead of the agents. As a consequence, the dimension of the core of the m-sided Böhm-Bawerk assignment game is bounded above by $m-1$.

For the sake of completeness, and to finish this section, note that there is another type of balanced multi-sided assignment games introduced by Sherstyuk (1999). In that case, balancedness does not follow from an additivity property of the coalitional function but from supermodularity of the function that valuates the m-tuples.

In Chapter 4 of this dissertation, we propose another balanced subclass of m-sided assignment markets also defined in an additive way based on an m-partite network that connects the agents.

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3 Generalized three-sided assignment markets: core consistency and competitive prices[§]

3.1 Introduction

We consider a market with three sectors where value is obtained by means of coalitions formed by agents of different sectors, that is, either triplets, pairs or individuals. Once the valuations of all these basic coalitions are known, a coalitional game is defined, the worth of an arbitrary coalition being the maximum worth that can be obtained by a partition of this coalition into basic ones.

Think, for instance, of one sector formed by firms providing landline telephone and internet service, on the second sector firms providing cable TV and on the third sector firms providing mobile telephone service. A triplet formed by one firm of each sector can achieve a profit by pooling their customers and offering them more services, but also a firm alone or a pair of firms of different sectors can attain some value.

These markets have already been considered in Tejada (2013) to see that agents of different sectors do not need to be complements and agents of the same sector do not need to be substitutes which is different from the two-sided case. Clearly, this class of coalitional games includes the classical three-sided assignment games of Quint (1991b) where value is only generated by triplets of agents belonging to different sectors. Another possible generalization of three-sided assignment games would be just assigning a reservation value to each individual and assuming that whenever an agent does not form part of any triplet this agent can attain his/her reservation value, in the way Owen (1992) generalizes the classical two-sided assignment game of Shapley and Shubik (1972).

The difference between the generalized three-sided markets that we consider and the three-sided assignment markets with individual reservation values is that when

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an agent does not form part of a triplet in the optimal partition (that we will name optimal matching), apart from being alone in an individual coalition, he/she may form part of a two-player coalition with some agent belonging to a different sector and, in that case, the value of this two-player coalition may be larger than the addition of the individual reservation value of the two agents. As a consequence, ours is a wider class since it includes games that are not strategically equivalent to a Quint (1991b) three-sided assignment game. Nevertheless, as in the classical three-sided assignment games, these games may not be balanced, that is, the core may be empty.

However, we present a subclass of balanced generalized three-sided assignment markets. Besides non-negativeness, two additional properties define this subclass: (a) the worth of a triplet is the addition of the worths of the three pairs that can be formed with its members and (b) there is an optimal partition such that, when restricted to each pair of sectors, is also optimal for the related two-sided market. This subclass of generalized three-sided assignment markets is inspired by the balanced subclass introduced by Quint (1991b) and the supplier-firm-buyer market of Stuart (1997), where also the value of a triplet is obtained by the addition of the value of some of the pairs that can be formed with its elements. However, in their classes, such a pair cannot attain its value if not matched with an agent of the remaining sector.

We restrict to the three-sided case to keep notation simpler, but all the arguments and results on the present paper can be extended to the multi-sided case.

In this class of generalized three-sided assignment markets, we introduce a reduced market at a given coalition and payoff vector, which represents the situation in which members outside the coalition leave the game with a predetermined payoff and the agents that remain in the market reevaluate their coalitional worth taking into account the possibility of cooperation with the agents outside. In the case of only two sectors, this reduced market coincides with the derived market defined by Owen (1992) for two-sided assignment markets with agents' reservation values.

Making use of consistency with respect to the derived market and two additional axioms, singleness best and individual anti-monotonicity, we provide an axiomatic characterization of the core on the domain of generalized three-sided assignment markets. Sasaki (1995) and Toda (2005) characterize the core on the domain of two-sided assignment markets by means of some monotonicity property that is not satisfied by the core in the three-sided case. The reason is that when we raise the value of a triplet, a pair or an individual in a three-sided market, the new market may fail to have core elements. This is the reason why the previous characterizations cannot be straightforwardly extended to the three-sided case.

In the last part of the paper we consider that one of the sectors is formed by buyers and the others by sellers of two different types of goods. Each buyer can buy at most one good of each type and valuates all basic coalitions she/he can take part in. From these valuations we introduce the demand of a buyer, given a price for each object on sale. Then, as usual, prices are competitive if there exists a matching such that each buyer takes part in a basic coalition in its demand set, and prices of unsold objects are zero. We show that the set of payoff vectors related to competitive equilibria coincide with the core. This generalizes the result in Gale (1960) for two-sided assignment markets and Tejada (2010) for the classical multi-sided assignment markets where buyers are forced to acquire exactly one item of each type.

The paper is organized as follows. The model is described in Section 2. The derived consistency of the core and the nucleolus is proved in Section 3, and an axiomatic characterization of the core is presented in Section 4. Section 5 focuses on the case with one sector of buyers and two sectors of sellers of different type of goods to prove the coincidence of core elements and competitive equilibria payoff vectors. The Appendix contains some technical proofs.

3.2 The model

In this section, we introduce a generalized three-sided assignment market and its corresponding assignment game.

Let $\mathcal{U}_1, \mathcal{U}_2$ and $\mathcal{U}_3$ be three pairwise disjoint countable sets. A *generalized three-sided assignment market* consists of three different sectors, $M_1 \subseteq \mathcal{U}_1$, $M_2 \subseteq \mathcal{U}_2$, and $M_3 \subseteq \mathcal{U}_3$ with a finite number of agents each, such that $N = M_1 \cup M_2 \cup M_3 \neq \emptyset$, and a valuation fuction v. The *basic coalitions* in this market are the ones formed by exactly one agent of each sector and all their possible subcoalitions. Let us denote by $\mathcal{B}^N$, or just $\mathcal{B}$, this set of basic coalitions,

$$\begin{aligned}\mathcal{B} = &\{\{i,j,k\} \mid i \in M_1, j \in M_2, k \in M_3\}\\ &\cup \{\{i,j\} \mid i \in M_r, j \in M_s,\ r,s \in \{1,2,3\}, r \neq s\} \cup \{\{i\} \mid i \in M_1 \cup M_2 \cup M_3\}.\end{aligned}$$

The valuation function v, from the set $\mathcal{B}$ to the real numbers $\mathbb{R}$ associates to each basic coalition $E \in \mathcal{B}$ its value $v(E)$.

Given a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$, for each non-empty coalition $S \subseteq N = M_1 \cup M_2 \cup M_3$ we can define a *submarket* $\gamma_{|S} = (M_1 \cap S, M_2 \cap S, M_3 \cap S; v_{|S})$ where $(v_{|S})(E) = v(E)$ for all $E \in \mathcal{B}^S = \{R \in \mathcal{B} \mid R \subseteq S\}$. Notice that if one of the sectors is empty, then this generalized three-sided assignment market is a two-sided assignment market with reservation values as introduced in Owen (1992).

Let $\emptyset \neq S \subseteq N$ be a coalition such that $S = S_1 \cup S_2 \cup S_3$ with $S_1 \subseteq M_1$, $S_2 \subseteq M_2$, and $S_3 \subseteq M_3$. Then, a *matching* μ for S is a partition of S in coalitions of $\mathcal{B}^S$. Let $\mathcal{M}(S_1, S_2, S_3)$ be the set of all possible matchings for coalition $S = S_1 \cup S_2 \cup S_3$. A matching $\mu \in \mathcal{M}(S_1, S_2, S_3)$ is *optimal* for the submarket $\gamma_{|S}$ if $\sum_{E \in \mu} v(E) \geq \sum_{E \in \mu'} v(E)$ for any $\mu' \in \mathcal{M}(S_1, S_2, S_3)$. We denote by $\mathcal{M}_\gamma(S_1, S_2, S_3)$ the set of optimal matchings for the market $\gamma_{|S}$.

Given a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$, its corresponding generalized three-sided assignment game[1] is a pair (N, w_γ) where $N = M_1 \cup M_2 \cup M_3$ is the player set and the characteristic function w_γ satisfies $w_\gamma(\emptyset) = 0$ and for all $S \subseteq N$,

$$w_\gamma(S) = \max_{\mu \in \mathcal{M}(S_1, S_2, S_3)} \left\{ \sum_{E \in \mu} v(E) \right\},$$

where $S_1 = S \cap M_1$, $S_2 = S \cap M_2$ and $S_3 = S \cap M_3$. Notice that the game (N, w_γ) is superadditive because it is a special type of partitioning game as introduced by Kaneko and Wooders (1982).

From now on, we denote by Γ_{3-GAM} indistinctly the set of generalized three-sided assignment markets or games.

An outcome for a generalized three-sided assignment market will be a matching and a distribution of the profits of this matching among the agents that take part.

Given $\gamma = (M_1, M_2, M_3; v)$, a *payoff vector* is $x \in \mathbb{R}^N$, where x_i stands for the payoff of player $i \in N$. We write $x_{|S}$ to denote the projection of a payoff vector x to agents in coalition $S \subseteq N$. Moreover, $x(S) = \sum_{i \in S} x_i$ with $x(\emptyset) = 0$. A payoff vector $x \in \mathbb{R}^N$ is *individually rational* for γ if $x_i \geq w_\gamma(\{i\})$ for all $i \in N$, and efficient if $x(N) = w_\gamma(N)$.

The core of a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$ is the core of the associated assignment game (N, w_γ), where $N = M_1 \cup M_2 \cup M_3$. Then, a market γ is *balanced* if its associated game (N, w_γ) has a non-empty core. It is straightforward to see that this core is formed by those efficient payoff vectors that satisfy coalitional rationality for all coalitions in $\mathcal{B}$. Given any optimal matching $\mu \in \mathcal{M}_\gamma(M_1, M_2, M_3)$,

$$C(\gamma) = \left\{ x \in \mathbb{R}^N \;\middle|\; x(N) = \sum_{E \in \mu} v(E) \text{ and } x(E) \geq v(E) \text{ for all } E \in \mathcal{B} \right\}.$$

[1] A game is a pair formed by a finite set of players N and a characteristic function r that assigns a real number $r(S)$ to each coalition $S \subseteq N$, with $r(\emptyset) = 0$. The *core* of a coalitional game (N, r) is $C(r) = \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = r(N), \sum_{i \in S} x_i \geq r(S) \text{ for all } S \subseteq N\}$. A game is *balanced* if it has a non-empty core. A game is said to be *superadditive* if for any two disjoint coalitions $S, T \subseteq N$, $S \cap T = \emptyset$, it holds $r(S \cup T) \geq r(S) + r(T)$.

As a consequence, given any optimal matching μ, if $x \in C(\gamma)$, then $x(E) = v(E)$ for all $E \in \mu$. Since this class is a generalization of the classical three-sided assignment games, the core may be empty.

The following two examples show that this class of generalized three-sided assignment games is indeed different from the class of classical three-sided assignment games. If we give values to some two-player coalitions in a classical three-sided assignment game with empty core (non-empty core), the core of the new generalized three-sided assignment game may become non-empty (empty). Moreover, we show that a generalized three-sided assignment game may not be strategically equivalent to any classical three-sided assignment game.

Example 3.1. Consider $M_1 = \{1,2\}$, $M_2 = \{1',2'\}$ and $M_3 = \{1'',2''\}$ and the three-sided assignment game taken from Quint (1991b) where the value of triplets is given by the following three-dimensional matrix A,

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\underset{1''}{\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array}} \qquad \begin{array}{c} \\ 1 \\ 2 \end{array}\underset{2''}{\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}},$$

and it is straightforward to see that the core is empty.

Define now a generalized three-sided market $\gamma_1 = (M_1, M_2, M_3; v_1)$ where

$$v_1(\{i,j,k\}) = a_{ijk} \text{ for } (i,j,k) \in M_1 \times M_2 \times M_3,$$

$$v_1(\{1,1'\}) = 1 \text{ and } v_1(S) = 0 \text{ for any other } S \in \mathcal{B}.$$

Notice that $w_{\gamma_1}(\{1,1',1''\}) = w_{\gamma_1}(\{1,1',2''\}) = 1$ and $x = (0,0;1,1;0,0) \in C(\gamma_1)$.

Moreover, the game (N, w_{γ_1}), where $N = M_1 \cup M_2 \cup M_3$, is not strategically equivalent to any classical three-sided assignment game. Indeed, if there existed $d \in \mathbb{R}^N$ and a three-dimensional matrix B such that $w_{\gamma_1}(S) = w_B(S) + \sum_{i \in S} d_i$ for all $S \subseteq N$, then

$$1 = w_{\gamma_1}(\{1,1'\}) = w_B(\{1,1'\}) + d_1 + d_{1'} = d_1 + d_{1'}$$

which means either $d_1 > 0$ or $d_{1'} > 0$. If we assume without loss of generality that $d_1 > 0$, then we get a contradiction since $0 = w_{\gamma_1}(\{1\}) = w_B(\{1\}) + d_1 > 0$.

Example 3.2. Consider now a classical three-sided assignment game with a non-empty core given in Quint (1991b). It is defined by $M_1 = \{1,2\}$, $M_2 = \{1',2'\}$,

$M_3 = \{1'', 2''\}$ and the three-dimensional matrix C, where an optimal matching is in boldface:

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} 1' \quad 2' \\ \begin{pmatrix} \mathbf{2} & 0 \\ 1 & 0 \end{pmatrix} \\ 1'' \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} 1' \quad 2' \\ \begin{pmatrix} 3 & 2 \\ 0 & \mathbf{2} \end{pmatrix} \\ 2'' \end{array}.$$

Define now a generalized three-sided market $\gamma_2 = (M_1, M_2, M_3; v_2)$ where

$$v_2(\{i, j, k\}) = c_{ijk} \text{ for } (i, j, k) \in M_1 \times M_2 \times M_3,$$

$$v_2(\{1, 2'\}) = v_2(\{2, 1'\}) = 2 \text{ and } v_2(S) = 0 \text{ for any other } S \in \mathcal{B}.$$

This implies that $w_{\gamma_2}(\{1, 1', 2, 2'\}) = 4$. If $x \in C(\gamma_2)$, we have $x_1 + x_{1'} + x_2 + x_{2'} \geq 4$ and hence $x_{1''} = x_{2''} = 0$. Moreover, from $w_{\gamma_2}(\{1, 1', 1''\}) = 2$ we have $x_1 + x_{1'} = x_1 + x_{1'} + x_{1''} = 2$. On the other side, from $w_{\gamma_2}(\{1, 1', 2''\}) = 3$, $x_1 + x_{1'} = x_1 + x_{1'} + x_{2''} \geq 3$, which leads to a contradiction and implies that $C(\gamma_2) = \emptyset$.

To conclude the discussion of the model, we introduce a subclass of generalized three-sided assignment markets. For the markets in this subclass, core allocations always exist.

3.2.1 A subclass of markets with non-empty core

We say a generalized three-sided assignment market is 2-additive if the three following conditions hold. The first one requires non-negativeness of the valuation function, with null value for single-player coalitions. Secondly, the valuation of each triplet $(i, j, k) \in M_1 \times M_2 \times M_3$ is the sum of the valuations of all pairs of agents in the triplet. Finally, we require the existence of an optimal matching that induces an optimal matching in each two-sided market. The reader will notice that the spirit of this class of 2-additive generalized three-sided assignment markets, that we denote by Γ^{add}_{3-GAM}, is similar to that of the balanced classes of multi-sided assignment games in Quint (1991b) and Stuart (1997). In both cases, the authors impose that the worth of a triplet is the addition of some numbers attached to its pairs. The difference is that in their models a pair cannot attain its worth if not matched with a third agent of the remaining sector, while in our case there is an underlying two-sided market for each pair of sectors.

As in Quint (1991b), we will assume from now on that the market is square, that is $|M_1| = |M_2| = |M_3|$. Let us introduce some notation: given a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$, for all $r, s \in \{1, 2, 3\}$, $r < s$, we consider the two-sided market $\gamma^{rs} = (M_r, M_s; v_{|\mathcal{B}^{M_r \cup M_s}})$. Then, we denote by

$\mathcal{M}_{\gamma^{rs}}(M_r, M_s)$ the set of optimal matchings for the two-sided market γ^{rs}, that is, partitions of $M_r \cup M_s$ in mixed pairs and singletons that maximize the sum of the valuations of the coalitions in the partition. Naturally, $C(\gamma^{rs})$ stands for the core of the underlying two-sided assignment game $(M_r \cup M_s, w_{\gamma^{rs}})$.

Given a matching $\mu \in \mathcal{M}(M_1, M_2, M_3)$ and two different sectors $r, s \in \{1,2,3\}$, $r < s$, the matching μ induces a matching μ^{rs} in the two-sided market γ^{rs} simply by defining $E \in \mu^{rs}$ if there exists a basic coalition $E' \in \mu$ such that $E = E' \cap (M_r \cup M_s)$ and $E \neq \emptyset$.

Definition 3.3. A generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$ with $|M_1| = |M_2| = |M_3|$, belongs to the class Γ^{add}_{3-GAM} if and only if

1. $v \geq 0$ and $v(\{k\}) = 0$ for all $k \in M_1 \cup M_2 \cup M_3$,
2. $v(\{i,j,k\}) = v(\{i,j\}) + v(\{i,k\}) + v(\{j,k\})$ for all $(i,j,k) \in M_1 \times M_2 \times M_3$,
3. there exists $\mu \in \mathcal{M}_\gamma(M_1, M_2, M_3)$ such that $\mu^{rs} \in \mathcal{M}_{\gamma^{rs}}(M_r, M_s)$ for all $r, s \in \{1,2,3\}$, $r < s$.

Conditions (1) and (2) imply that the valuation function v is superadditive. Condition (3) requires that there is an optimal matching $\mu \in \mathcal{M}_\gamma(M_1, M_2, M_3)$ that induces an optimal matching in each bilateral market γ^{rs}, for $r < s$. Next proposition shows that the three conditions together guarantee that the core of any generalized three-sided assignment market in the class Γ^{add}_{3-GAM} is non-empty.

Proposition 3.4. *Each 2-additive generalized three-sided assignment market is balanced.*

Proof. Let $\gamma = (M_1, M_2, M_3; v) \in \Gamma^{add}_{3-GAM}$ and let $\mu = \{E_1, E_2, \ldots, E_p\}$ be an optimal matching, $\mu \in \mathcal{M}_\gamma(M_1, M_2, M_3)$, such that $\mu^{rs} \in \mathcal{M}_{\gamma^{rs}}(M_r, M_s)$ for all $r, s \in \{1,2,3\}$, $r < s$. For all $r, s \in \{1,2,3\}$, $r < s$, and $l \in \{1,2,\ldots,p\}$, define $E^{rs}_l = E_l \cap (M_r \cup M_s)$ and notice that by definition $\mu^{rs} = \{E^{rs}_l \mid 1 \leq l \leq p, E^{rs}_l \neq \emptyset\}$.

From Shapley and Shubik (1972), it is known that each two-sided assignment market is balanced. So, take core allocations $(x^1, y^1) \in C(\gamma^{12})$, $(x^2, z^2) \in C(\gamma^{13})$ and $(y^3, z^3) \in C(\gamma^{23})$. We will see that $(x^1 + x^2, y^1 + y^3, z^2 + z^3) \in C(\gamma)$.

By optimality of μ^{12}, we have that if for some $l \in \{1,2,\ldots,p\}$, $E^{12}_l = \{i,j\}$, then $x^1_i + y^1_j = v(\{i,j\})$. Similarly, if $E^{12}_l = \{i\}$, for $i \in M_1$, then $x^1_i = 0$; and if $E^{12}_l = \{j\}$ for some $j \in M_2$, then $y^1_j = 0$. Analogous equalities are obtained for E^{13}_l and E^{23}_l, for $l \in \{1,2,\ldots,p\}$.

Hence,

$$\begin{aligned}&\textstyle\sum_{i\in M_1}(x_i^1+x_i^2)+\sum_{j\in M_2}(y_j^1+y_j^3)+\sum_{k\in M_3}(z_k^2+z_k^3)=\\&\textstyle\sum_{l=1}^{p}\left[\sum_{\{i,j\}\in E_l^{12}}(x_i^1+y_j^1)+\sum_{\{i,k\}\in E_l^{13}}(x_i^2+z_k^2)+\sum_{\{j,k\}\in E_l^{23}}(y_j^3+z_k^3)\right]=\\&\textstyle\sum_{l=1}^{p}\left[\sum_{\{i,j\}\in E_l^{12}}v(\{i,j\})+\sum_{\{i,k\}\in E_l^{13}}v(\{i,k\})+\sum_{\{j,k\}\in E_l^{23}}v(\{j,k\})\right]=\\&\textstyle\sum_{l=1}^{p}v(E_l)=w_\gamma(N).\end{aligned}$$

Once proved efficiency, it only remains to prove coalitional rationality of the payoff vector $(x^1+x^2, y^1+y^3, z^2+z^3)$. Indeed, take any $\{i,j,k\}\in\mathcal{B}$ and notice that

$$\begin{aligned}x_i^1+x_i^2+y_j^1+y_j^3+z_k^2+z_k^3 &= (x_i^1+y_j^1)+(x_i^2+z_k^2)+(y_j^3+z_k^3)\\&\geq v(\{i,j\})+v(\{i,k\})+v(\{j,k\})=v(\{i,j,k\})\end{aligned}$$

where the inequality follows from the core constraints of (x^1,y^1), (x^2,z^2) and (y^3,z^3) in each two-sided market.

Similarly, if $\{i,j\}\in\mathcal{B}$, we may assume without loss of generality that $i\in M_1$ and $j\in M_2$, and hence, taking into account $x_i^2\geq v(\{i\})=0$ and $y_j^3\geq v(\{j\})=0$, we get

$$x_i^1+x_i^2+y_j^1+y_j^3=(x_i^1+y_j^1)+x_i^2+y_j^3\geq v(\{i,j\}).$$

Finally, if $\{i\}\in\mathcal{B}$, let us assume without loss of generality that $i\in M_1$. Then $x_i^1+x_i^2\geq 0=v(\{i\})$ follows also from the individual rationality of (x^1,y^1) and (x^2,z^2). □

In the above proposition we have deduced the existence of core elements for $\gamma\in\Gamma_{3-GAM}^{add}$ by operating with three core elements of the related two-sided markets. However, as the next example shows, there are 2-additive generalized three-sided markets where not all core elements can be obtained in this way.

Example 3.5. Let us consider a generalized three-sided assignment market γ where $M_1=\{1,2\}$, $M_2=\{1',2'\}$ and $M_3=\{1'',2''\}$. The value of individual coalitions is null, the value of those basic coalitions formed by a pair of agents is given by

$$\begin{array}{c|cc} & 1' & 2' \\ \hline 1 & \textcircled{4} & 6 \\ 2 & 0 & \textcircled{4}\end{array}\qquad\begin{array}{c|cc} & 1'' & 2'' \\ \hline 1 & \textcircled{6} & 9 \\ 2 & 1 & \textcircled{5}\end{array}\qquad\begin{array}{c|cc} & 1'' & 2'' \\ \hline 1' & \textcircled{2} & 0 \\ 2' & 8 & \textcircled{7}\end{array},$$

and the value of triplets is given by the following three-dimensional matrix

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} \mathbf{12} & 20 \\ 3 & 13 \end{pmatrix} \\ 1'' \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 13 & 22 \\ 5 & \mathbf{16} \end{pmatrix} \\ 2'' \end{array} .$$

The reader can check that the above values define a 2-additive generalized three-sided market. Optimal matchings of the underlying two-sided markets are circled and the optimal matching of the three-sided market is shown in boldface.

The payoff vector $u = (6,0;0,8;6,8)$ belongs to the core but cannot be obtained by core allocations of the three underlying two-sided assignment markets. Indeed, if there existed $(x^1, y^1) \in C(\gamma^{12})$, $(x^2, z^2) \in C(\gamma^{13})$ and $(y^3, z^3) \in C(\gamma^{23})$ such that $(x^1 + x^2; y^1 + y^3; z^2 + z^3) = (6,0;0,8;6,8)$, then $0 = x^1_2 + x^2_2$ and $0 = y^1_1 + y^3_1$ imply $x^2_2 = y^3_1 = 0$. Then, from the core constraints in the underlying two-sided markets, $x^2_2 + z^2_2 = 5$ and $y^3_1 + z^3_1 = 2$, we obtain $z^2_2 = 5$ and $z^3_1 = 2$. Now, $6 = z^2_1 + z^3_1$ implies $z^2_1 = 4$, and by substitution in (x^2, z^2) we obtain $(x^2, z^2) = (x^2_1, 0; 4, 5)$. But such a payoff vector is not in the core of γ^{13} since the two core constraints $x^2_1 + z^2_1 = x^2_1 + 4 = 6$ and $x^2_1 + z^2_2 = x^2_1 + 5 \geq 9$ are not compatible.

Once established our model, and shown one subclass with non-empty core, we look for a notion of reduction that makes the core a consistent solution on the class of generalized three-sided assignment markets.

3.3 Consistency of the core and the nucleolus

In this section, we introduce the derived market (and game) for the generalized three-sided assignment market, and the corresponding consistency property.

Given any coalitional game, and given a particular distribution of the worth of the grand coalition, we may ask what happens when some agents leave the market after being paid according to that given distribution. The remaining agents must reevaluate the worth of all the coalitions they can form. The different ways in which this reevaluation is done correspond to the different notions of reduced game that exist in the literature.

Maybe the best known notion of reduced game is that of Davis and Maschler (1965), where the remaining coalitions take into account what they could obtain by joining some agents that have left, with the condition of preserving the amount they have already been paid.

Definition 3.6 (Davis and Maschler, 1965)**.** Given a generalized three-sided assignment game (N,w_γ), a non-empty coalition S and a payoff vector $x\in\mathbb{R}^{N\setminus S}$, the *Davis and Maschler reduced game* for the coalition S at x is the game $(S,w_\gamma^{S,x})$ defined by

$$w_\gamma^{S,x}(T)=\begin{cases}0 & \text{if} \quad T=\emptyset,\\ w_\gamma(N)-x(N\setminus S) & \text{if} \quad T=S,\\ \max\limits_{Q\subseteq N\setminus S}\{w_\gamma(T\cup Q)-x(Q)\} & \text{otherwise.}\end{cases}$$

In general, the reduced game of a generalized three-sided assignment game is not superadditive, and hence it is not a generalized three-sided assignment game. Take for instance coalition $S=\{1,2',1'',2''\}$ and the core element $u=(6,0;0,8;6,8)$ in the market of Example 3.5 and notice that $w_\gamma^{S,u}(\{1''\})+w_\gamma^{S,u}(\{2''\})=3+5>7=w_\gamma^{S,u}(\{1'',2''\})$.

To solve this, we introduce a new and different reduction for the generalized three-sided assignment market (and game) that extends the derived game introduced by Owen (1992) for the two-sided case. We will see that this notion of reduced game is closely related to the Davis and Maschler reduction.

Definition 3.7. Let $\gamma=(M_1,M_2,M_3;v)$ be a three-sided assignment market. For any $\emptyset\neq S=S_1\cup S_2\cup S_3$, $S\neq N$, where $S_1\subseteq M_1$, $S_2\subseteq M_2$, $S_3\subseteq M_3$ and $x\in\mathbb{R}^{N\setminus S}$, the *derived market* at S and x is $\hat{\gamma}^{S,x}=(S_1,S_2,S_3;\hat{v}^{S,x})$ where

$$\hat{v}^{S,x}(E)=\max_{\substack{Q\subseteq N\setminus S\\ E\cup Q\in\mathcal{B}}}\{v(E\cup Q)-x(Q)\} \text{ for all } E\in\mathcal{B}^S. \tag{3.1}$$

Then, the corresponding *derived game* at S and x is $(S,w_{\hat{\gamma}^{S,x}})$ where for all $R\subseteq S$,

$$w_{\hat{\gamma}^{S,x}}(R)=\max_{\mu\in\mathcal{M}(M_1\cap R,M_2\cap R,M_3\cap R)}\Big\{\sum_{E\in\mu}\hat{v}^{S,x}(E)\Big\}. \tag{3.2}$$

To obtain the derived game, we first consider the valuation in the reduced situation of the basic coalitions of the submarket. The valuation of such a basic coalition is obtained by allowing it to cooperate only with agents that have left but with whom it can form a basic coalition of the initial market. In particular, when $E=\{i,j,k\}$ with $i\in S_1$, $j\in S_2$ and $k\in S_3$, then $\hat{v}^{S,x}(\{i,j,k\})=v(\{i,j,k\})$. Thus, the worth $w_{\hat{\gamma}^{S,x}}(R)$ in the derived game for any coalition $R\subseteq S$ is obtained from the valuations $\hat{v}^{S,x}$ of the basic coalitions in $\mathcal{B}^S$ by imposing superadditivity. Hence, the derived assignment game is always a superadditive game.

Notice that in (3.2) different basic coalitions E in the same matching $\mu\in\mathcal{M}(M_1\cap R,M_2\cap R,M_3\cap R)$ can use the same coalition $Q\subseteq N\setminus S$ to establish their value

$\hat{v}^{S,x}(E)$. Thus, $(S, w_{\hat{\gamma}^{S,x}})$ serves only to determine the distribution of $w_{\hat{\gamma}^{S,x}}(S)$ among the members of S. Also in the Davis and Maschler reduced game the expectations of different disjoint subcoalitions may not be compatible with each other, because they may require cooperation of the same subset of $N \setminus S$.

However, it is interesting to remark that the worth of the grand coalition of the derived game (at a core allocation) is indeed attainable. The reason is that there exists an optimal matching of the derived game such that no two basic coalitions of this matching need the cooperation of a same outside agent to attain their worth. We will discuss this fact in Remark 3.13, after the proof of Theorem 3.12.

A market with some empty sector is a two-sided market (with individual reservation values) and the definition of derived game coincides with the one given by Owen (1992) for these markets.

Given a game (N, w), its *superadditive cover* is the minimal superadditive game $(N, \tilde{w})$ such that $\tilde{w} \geq w$. Next proposition extends a result obtained for two-sided assignment games by Owen (1992). We show that for any generalized three-sided assignment game (N, w_γ), its derived game $(S, w_{\hat{\gamma}^{S,x}})$ at any coalition S and core allocation x is the superadditive cover of the corresponding Davis and Maschler reduced game $(S, w_\gamma^{S,x})$. This means that the derived game of a generalized three-sided assignment market is closely related to the Davis and Maschler reduced game. The proof is consigned to the Appendix of this chapter.

Proposition 3.8. *Let $\gamma = (M_1, M_2, M_3; v)$ be a generalized three-sided assignment market, $N = M_1 \cup M_2 \cup M_3$, (N, w_γ) the associated generalized three-sided game and $x \in C(w_\gamma)$. Then for any $\emptyset \neq S \subsetneq N$, the derived game $(S, w_{\hat{\gamma}^{S,x}})$, where $\hat{\gamma}^{S,x} = (M_1 \cap S, M_2 \cap S, M_3 \cap S; \hat{v}^{S,x})$, is the superadditive cover of the Davis and Maschler reduced game $(S, w_\gamma^{S,x})$.*

Our objective now is to introduce a consistency property with respect to the derived market. We name this property *derived consistency*.

Before doing that, we need to introduce the notion of solution in the class of generalized three-sided assignment markets or games. Next definition extends to our setting the notion of feasibility that is usual in two-sided assignment markets.

Definition 3.9. Let $\gamma = (M_1, M_2, M_3; v)$ be a generalized three-sided assignment market. An allocation $x \in \mathbb{R}^{M_1} \times \mathbb{R}^{M_2} \times \mathbb{R}^{M_3}$ is *feasible-by-matching* if there exists a matching $\mu \in \mathcal{M}(M_1, M_2, M_3)$ such that for all $E \in \mu$, $x(E) = v(E)$.

In that case, we say that x and μ are compatible. Notice that a matching μ compatible with x may not be optimal. Moreover, the set of feasible-by-matching allocations is always non-empty since we can take the matching $\mu = \{\{i\}\}_{i \in N}$ and then $x = (v(\{i\}))_{i \in N}$ is feasible with respect to μ.

Definition 3.10. A *solution* on a class $\Gamma \subseteq \Gamma_{3-GAM}$ is a correspondence σ that assigns a subset of feasible-by-matching payoff vectors to each $\gamma \in \Gamma$.

Given $\gamma \in \Gamma$, we write $\sigma(\gamma)$ to denote the subset of feasible-by-matching payoff vectors assigned by solution σ to the assignment market γ. Notice that a solution σ is allowed to be empty. The core correspondence and the mapping that gives to each agent his/her individual value (compatible with the matching formed by all individual coalitions) are examples of solutions on the class of generalized three-sided assignment markets. Similarly, the nucleolus, which will be defined below, is a solution on the subclass of balanced generalized three-sided assignment markets.

Definition 3.11. A solution σ on the class of generalized three-sided assignment markets satisfies *derived consistency* if for all $\gamma = (M_1, M_2, M_3; v)$, all $\emptyset \neq S \subsetneq N$ and all $x \in \sigma(\gamma)$, it holds $x_{|S} \in \sigma(\hat{\gamma}^{S,x})$.

Next theorem shows that the core satisfies derived consistency on the domain of generalized three-sided assignment markets.

Theorem 3.12. *On the domain of generalized three-sided assignment markets, the core satisfies derived consistency.*

Proof. Let $\gamma = (M_1, M_2, M_3; v)$ be a generalized three-sided assignment market, let x be a core allocation and $\emptyset \neq S \subsetneq M_1 \cup M_2 \cup M_3$. To simplify notation, let us write $\hat{v} = \hat{v}^{S,x}$ and $\hat{w} = w_{\hat{\gamma}^{S,x}}$.

Consider all possible basic coalitions in $\mathcal{B}^S$. First, for all $(i,j,k) \in M_1 \cap S \times M_2 \cap S \times M_3 \cap S$, $x_i + x_j + x_k \geq v(\{i,j,k\}) = \hat{v}(\{i,j,k\})$. Secondly, for all $(i,j) \in (M_1 \cap S) \times (M_2 \cap S)$, $x_i + x_j \geq v(\{i,j\})$ and $x_i + x_j \geq v(\{i,j,k\}) - x_k$ for all $k \in M_3 \setminus S$. Hence, $x_i + x_j \geq \hat{v}(\{i,j\})$. Finally, for all $i \in M_1 \cap S$, $x_i \geq v(\{i\})$, and $x_i \geq v(\{i,j\}) - x_j$ for all $j \in M_2 \setminus S$, and $x_i \geq v(\{i,k\}) - x_k$ for all $k \in M_3 \setminus S$, and $x_i \geq v(\{i,j,k\}) - x_j - x_k$ for all $j \in M_2 \setminus S$ and for all $k \in M_3 \setminus S$. Hence, $x_i \geq \hat{v}(\{i\})$. Proceeding similarly for the remaining $E \in \mathcal{B}^S$, we obtain

$$x(E) \geq \hat{v}(E) \text{ for all } E \in \mathcal{B}^S. \tag{3.3}$$

Finally, it remains to show that $x(S) = \hat{w}(S)$. Expression (3.3) implies $x(R) \geq \hat{w}(R)$ for all $R \subseteq S$. Now, appyling Proposition 3.8 we obtain

$$x(S) \geq \hat{w}(S) \geq w_{\gamma}^{S,x}(S) = x(S),$$

where the second inequality follows from Proposition 3.8 and the last equality from the Davis and Maschler reduced game property of the core (Peleg, 1986). Thus, $x(S) = \hat{w}(S)$ and this completes the proof of $x_{|S} \in C(\hat{\gamma}^{S,x})$. □

As a consequence of the proof of the above theorem, we can justify, by means of the following remark, that the worth of the grand coalition of the derived game is indeed attainable.

Remark 3.13. An optimal matching of the derived market at a core allocation is induced by an optimal matching of the initial market. To see that, take a balanced generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$ and μ an optimal matching, $\mu \in \mathcal{M}_\gamma(M_1, M_2, M_3)$. Let $\hat{\gamma}^{S,x} = (M_1 \cap S, M_2 \cap S, M_3 \cap S; \hat{v}^{S,x})$ be the derived market at $S \subseteq M_1 \cup M_2 \cup M_3$ and $x \in C(\gamma)$. It turns out that $\mu_{|S} = \{E \cap S \mid E \in \mu\}$ is optimal for $\hat{\gamma}^{S,x}$. Indeed, given any other $\mu' \in \mathcal{M}_{\hat{\gamma}^{S,x}}(M_1 \cap S, M_2 \cap S, M_3 \cap S)$,

$$\begin{aligned} \sum_{E \in \mu'} \hat{v}^{S,x}(E) &\leq \sum_{E \in \mu'} x(E) = x(S) = \sum_{E \in \mu_{|S}} x(E) \\ &= \sum_{E \in \mu_{|S}} v(D(E)) - x(D(E) \setminus E) \leq \sum_{E \in \mu_{|S}} \hat{v}^{S,x}(E), \end{aligned}$$

where the first inequality follows from (3.3); for all $E \in \mu_{|S}$, $D(E)$ is defined as the unique basic coalition in μ such that $D(E) \cap S = E$; and the last inequality follows from (3.1). Hence, $\mu_{|S}$ is optimal for $\hat{\gamma}^{S,x}$. Because of that, no two basic coalitions of $\mu_{|S}$ need the cooperation of the same outside agent to attain their worth.

To finish this section we show another solution concept that satisfies derived consistency. The *nucleolus* is a well-known single-valued solution for coalitional games introduced by Schmeidler (1969). When the game is balanced, the nucleolus is the unique core allocation that lexicographically minimizes the vector of decreasingly-ordered excesses of coalitions.[2]

The nucleolus of a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v)$ is the nucleolus of the associated assignment game (N, w_γ), and it will be denoted by $\eta(\gamma)$. Next, we show that when a generalized three-sided assignment market is balanced the nucleolus also satisfies derived consistency.

Theorem 3.14. *On the class of balanced generalized three-sided assignment markets, the nucleolus satisfies derived consistency.*

Proof. Let $\gamma = (M_1, M_2, M_3; v)$ be a balanced generalized three-sided assignment market, $\eta(\gamma) = \eta$ be the nucleolus and $\emptyset \neq S \subsetneq M_1 \cup M_2 \cup M_3$. Since the nucleolus satisfies the Davis and Maschler reduced game property (Potters, 1991), $\eta_{|S} = \eta(w_\gamma^{S,\eta})$ which implies $\eta(S) = w_\gamma^{S,\eta}(S)$. On the other hand, since $\eta \in C(\gamma)$, by Theorem 3.12 we know that $\eta_{|S} \in C(w_{\hat{\gamma}^{S,\eta}})$ which implies $\eta(S) = w_{\hat{\gamma}^{S,\eta}}(S)$.

[2] Given a game (N, r), the excess of a coalition $S \subseteq N$ at a payoff vector $x \in \mathbb{R}^N$ is $r(S) - \sum_{i \in S} x_i$.

Hence, taking into account Proposition 3.8, we have that the Davis and Maschler reduced game $(S, w_{\gamma}^{S,\eta})$ and its superadditive cover have the same efficiency level. By Miquel and Núñez (2011) this implies that both games have the same nucleolus. Therefore, $\eta_{|S} = \eta(w_{\hat{\gamma}^{S,\eta}})$. □

In the next section we combine derived consistency with two additional properties in order to characterize the core of generalized three-sided assignment games.

3.4 An axiomatic characterization of the core

In this section, we give an axiomatic characterization of the core on the class of generalized three-sided assignment markets making use of derived consistency and two additional properties, *singleness best* and *individual anti-monotonicity*, that are introduced in the sequel.

Definition 3.15. A solution σ on $\Gamma \subseteq \Gamma_{3-GAM}$ satisfies *singleness best* if for all $\gamma = (M_1, M_2, M_3; v) \in \Gamma$, it holds that whenever the partition in singletons is optimal in γ, then $(v(\{i\}))_{i\in N} \in \sigma(\gamma)$.

Singleness best simply says that if the partition in individual coalitions is optimal, then the vector of individual values should be an outcome of the solution. This axiom has some resemblance with the *zero inessential game property* of Hwang and Sudhölter (2001) in the sense that it is a non-emptiness axiom for generalized three-sided assignment games that are trivial or inessential.

Before introducing the property of individual anti-monotonicity we need to establish how to compare the individual values of all agents across different games.

Given a matching $\mu \in \mathcal{M}(M_1, M_2, M_3)$, two payoff vectors $x = (x_i)_{i\in N}$ and $x' = (x'_i)_{i\in N}$, we write $x' \geq_\mu x$ when $x'_i = x_i$ for all $\{i\} \in \mu$ and $x'_i \geq x_i$ if $\{i\} \notin \mu$. That is, x' is greater than x with respect to μ when agents that are matched with some other partner receive at least as much in x' than in x, while agents that are alone receive the same payoff in both allocations.

Definition 3.16. A solution σ on $\Gamma \subseteq \Gamma_{3-GAM}$ satisfies *individual anti-monotonicity* if for all $\gamma' = (M_1, M_2, M_3; v') \in \Gamma$, all $\gamma = (M_1, M_2, M_3; v) \in \Gamma$, all $u \in \sigma(\gamma')$ and matching μ compatible with u, if $v(E) = v'(E)$ for all $E \in \mathcal{B}$ with $|E| > 1$ and $(v'(\{i\}))_{i\in N} \geq_\mu (v(\{i\}))_{i\in N}$, then it holds $u \in \sigma(\gamma)$.

Individual anti-monotonicity says that if the individual values decrease (in the sense defined above) any payoff vector in the solution of the original market should remain in the solution of the new market. Notice that the value of pairs and triplets coincide in both markets. Individual anti-monotonicity is a weaker version of *anti-monotonicity* introduced by Keiding (1986) and also used by Toda (2003).

Now, we characterize the core on the class of generalized three-sided assignment games by means of derived consistency, singleness best and individual anti-monotonicity.

Theorem 3.17. *On the domain of generalized three-sided assignment markets, the core is the unique solution that satisfies derived consistency, singleness best and individual anti-monotonicity.*

Proof. By Theorem 3.12 we know the core satisfies derived consistency. It is straightforward that the core satisfies singleness best and individual anti-monotonicity. Assume now that σ is a solution on Γ_{3-GAM} also satisfying these axioms. Take any $\gamma = (M_1, M_2, M_3; v) \in \Gamma_{3-GAM}$.

We first show that $\sigma(\gamma) \subseteq C(\gamma)$. Take $x \in \sigma(\gamma)$. We need to show that x satisfies coalitional rationality and efficiency. Notice that if some side of the market is empty, the game is a two-sided assignment market and the statement follows from Proposition 2 in (Llerena et al, 2015). So, we can assume without loss of generality that $M_l \neq \emptyset$ for all $l \in \{1,2,3\}$. Then, for all $i \in M_1 \cup M_2 \cup M_3$ consider the derived market relative to $S = \{i\}$ at x. By derived consistency of σ, $x_i \in \sigma(\hat{\gamma}^{\{i\},x})$. Moreover, feasibility-by-matching of σ implies that $x_i = \hat{v}^{\{i\},x}(\{i\})$. Now, let $E \in \mathcal{B}$ be any basic coalition such that $i \in E$. By definition of derived market at $\{i\}$ and x we have $x_i = \hat{v}^{\{i\},x}(\{i\}) \geq v(E) - \sum_{k \in E \setminus \{i\}} x_k$. Hence, $\sum_{k \in E} x_k \geq v(E)$ which states that x satisfies coalitional rationality.

In order to prove efficiency, let μ be an optimal matching and μ' be a matching compatible with x. Then, $w_\gamma(N) = \sum_{E \in \mu} v(E) \leq \sum_{E \in \mu} (\sum_{i \in E} x_i) = \sum_{E \in \mu'} (\sum_{i \in E} x_i) = \sum_{E \in \mu'} v(E)$, where the last equality follows from the fact that μ' is compatible with x. Since μ is optimal and $w_\gamma(N) \leq \sum_{E \in \mu'} v(E)$, we get that μ' is also optimal and x is efficient. Hence, $x \in C(\gamma)$ and we have proved $\sigma(\gamma) \subseteq C(\gamma)$.

To show that $C(\gamma) \subseteq \sigma(\gamma)$, take $u \in C(\gamma)$ and $\mu \in \mathcal{M}(M_1, M_2, M_3)$ compatible with u. Then, μ is optimal for γ. Now, define a market $\gamma' = (M_1, M_2, M_3; v')$ where $v'(E) = v(E)$ for all $E \in \mathcal{B}$ such that $|E| > 1$ and $v'(E) = u_i$ for all $E = \{i\}$. Notice that $v'(\{i\}) = u_i = v(\{i\})$ for all $\{i\} \in \mu$ and $v'(\{i\}) = u_i \geq v(\{i\})$ for all $\{i\} \notin \mu$. Hence, $(v'(\{i\}))_{i \in N} \geq_\mu (v(\{i\}))_{i \in N}$. Let us see that $\mu' = \{\{i\} \mid i \in N\}$ is optimal for γ'. To this end, take any matching $\mu'' \in \mathcal{M}(M_1, M_2, M_3)$. Then,

$$\sum_{E\in\mu'} v'(E) = \sum_{i\in N} v'(\{i\}) = \sum_{i\in N} u_i = \sum_{\substack{E\in\mu'' \\ |E|>1}} \sum_{i\in E} u_i + \sum_{\substack{E\in\mu'' \\ |E|=1}} \sum_{i\in E} u_i$$
$$\geq \sum_{\substack{E\in\mu'' \\ |E|>1}} v'(E) + \sum_{\substack{E\in\mu'' \\ |E|=1}} v'(E) = \sum_{E\in\mu''} v'(E).$$

The inequality follows from the fact that $u \in C(\gamma)$ and the relationship between v and v'. Thus, μ' is optimal for γ'. By singleness best, $u = (u_i)_{i\in N} = (v'(\{i\}))_{i\in N} \in \sigma(\gamma')$ and then, by individual anti-monotonicity, $u \in \sigma(\gamma)$. Hence, $C(\gamma) \subseteq \sigma(\gamma)$. Together with the reverse inclusion, $\sigma(\gamma) \subseteq C(\gamma)$, we conclude that $C(\gamma) = \sigma(\gamma)$. □

We now show that no axiom in the above characterization is implied by the others. To this end, we introduce different solutions satisfying all axioms but one.

Example 3.18. For all $\gamma = (M_1, M_2, M_3; v) \in \Gamma_{3-GAM}$, let us consider

$$\sigma_1(\gamma) = \emptyset.$$

Clearly, σ_1 satisfies derived consistency and individual anti-monotonicity but not singleness best.

Example 3.19. For all $\gamma = (M_1, M_2, M_3; v) \in \Gamma_{3-GAM}$, write $N = M_1 \cup M_2 \cup M_3$ and let us consider

$$\sigma_2(\gamma) = \left\{ u \in \mathbb{R}^N \;\middle|\; \begin{array}{l} u \text{ is feasible-by-matching for } \gamma, \\ u_i \geq w_\gamma(\{i\}), \text{ for all } i \in N, \\ u(N) = w_\gamma(N) \end{array} \right\}.$$

Notice that if $u \in \sigma_2(\gamma)$, every matching μ that is compatible with u is optimal. It can be easily checked that σ_2 satisfies singleness best and individual anti-monotonicity but, since σ_2 is different from the core, the characterization of the core in Theorem 3.17 implies that σ_2 does not satisfy derived consistency.

Example 3.20. For all $\gamma = (M_1, M_2, M_3; v) \in \Gamma_{3-GAM}$, let $\eta(\gamma)$ be the nucleolus of γ and consider

$$\sigma_3(\gamma) = \begin{cases} \emptyset & \text{if } C(\gamma) = \emptyset, \\ \{\eta(\gamma)\} & \text{if } C(\gamma) \neq \emptyset. \end{cases}$$

The solution σ_3 satisfies singleness best and derived consistency (see Theorem 3.14), but, since σ_3 is different from the core, the characterization of the core in Theorem 3.17 implies that σ_3 does not satisfy individual anti-monotonicity.

These three examples prove that none of the axioms is redundant in the above characterization of the core.

3.5 Core and competitive equilibria

We now focus on the particular case where $M_1 = \{1,...,m\}$ and $M_2 = \{1',...,m'\}$ are two sets of sellers, each selling an indivisible good. Goods of sellers in M_1 are of a different type of those of sellers in M_2. The third sector $M_3 = \{1'',...,m''\}$ is formed by buyers, each interested in buying at most one unit of each type of good. Each seller $r \in M_1 \cup M_2$ has a reservation value $c_r \geq 0$ for his object, meaning he will not sell for a price lower than that. We denote by c the vector of sellers' reservation values.

We denote by $\mathcal{B}^k$, the set of basic coalitions that contain buyer $k \in M_3$, $\mathcal{B}^k = \{E \in \mathcal{B} \mid k \in E\}$. Then, each buyer $k \in M_3$ places a value $w^k(E) \in \mathbb{R}_+$ on each basic coalition $E \in \mathcal{B}^k$ and we denote by $w = (w^k)_{k \in M_3}$ the vector of buyers' valuations.

All these valuations (w,c) give rise to a generalized three-sided assignment market $(M_1, M_2, M_3; v^{w,c})$ where $v^{w,c}(E) = w^k(E) - c(E \setminus \{k\})$ if $E \in \mathcal{B}^k$ for some $k \in M_3$ and $v^{w,c}(E) = 0$ if $E \in \mathcal{B}$ with $E \cap M_3 = \emptyset$. We denote by Γ_{SSB} this subclass of generalized three-sided assignment markets that are defined by some valuations (w,c).

We want to show that each core allocation is the result of trading goods following an optimal matching and according to some prices. To introduce the notion of competitive price vector, some previous definitions are needed.

Given a generalized three-sided assignment market $\gamma = (M_1, M_2, M_3; v^{w,c}) \in \Gamma_{SSB}$, a *feasible price vector* is $p \in \mathbb{R}_+^{M_1 \cup M_2}$ such that $p_r \geq c_r$ for all $r \in M_1 \cup M_2$.

Next, for each feasible price vector $p \in \mathbb{R}_+^{M_1 \cup M_2}$ we introduce the *demand set* of each buyer in sector M_3.

Definition 3.21. Given a market $\gamma = (M_1, M_2, M_3; v^{w,c}) \in \Gamma_{SSB}$, the *demand set* of a buyer $k \in M_3$ at a feasible price vector $p \in \mathbb{R}_+^{M_1 \cup M_2}$ is

$$D_k(p) = \{E \in \mathcal{B}^k \mid w^k(E) - p(E \setminus \{k\}) \geq w^k(E') - p(E' \setminus \{k\}) \text{ for all } E' \in \mathcal{B}^k\}.$$

Note that $D_k(p)$ describes the set of basic coalitions containing buyer k that maximize the net valuation of buyer k at prices p. Notice also that the demand set of a buyer $k \in M_3$ is always non-empty. If $\mu \in \mathcal{M}(M_1, M_2, M_3)$, for all $k \in M_3$ we will write $\mu(k)$ to denote the basic coalition E such that $k \in E$ and $E \in \mu$.

Given a matching $\mu \in \mathcal{M}(M_1, M_2, M_3)$, we say a seller $r \in M_1 \cup M_2$ is *unassigned* (by μ) if there is no $k \in M_3$ such that $r \in \mu(k)$

Now, we can introduce the notion of *competitive equilibrium* for our market.

Definition 3.22. Given a market $\gamma = (M_1, M_2, M_3; v^{w,c}) \in \Gamma_{SSB}$, a pair (p, μ), where $p \in \mathbb{R}_+^{M_1 \cup M_2}$ is a feasible price vector and $\mu \in \mathcal{M}(M_1, M_2, M_3)$, is a *competitive equilibrium* if

i for all buyer $k \in M_3$, $\mu(k) \in D_k(p)$,

ii for all seller $r \in M_1 \cup M_2$, if r is unassigned by μ, then $p_r = c_r$.

If a pair (p, μ) is a competitive equilibrium, then we say that the price vector p is a *competitive equilibrium price vector*. The corresponding payoff vector for a given pair (p, μ) is called *competitive equilibrium payoff vector*. This payoff vector is $(x(p, \mu), y(p, \mu), z(p, \mu)) \in \mathbb{R}^{M_1} \times \mathbb{R}^{M_2} \times \mathbb{R}^{M_3}$, defined by

$$x_i(p, \mu) = p_i - c_i \text{ for all sellers } i \in M_1,$$
$$y_j(p, \mu) = p_j - c_j \text{ for all sellers } j \in M_2,$$
$$z_k(p, \mu) = w^k(\mu(k)) - p(\mu(k) \setminus \{k\}) \text{ for all buyers } k \in M_3.$$

We denote the set of competitive equilibrium payoff vectors of market γ by $\mathcal{CE}(\gamma)$.

We now study the relationship between the core of $\gamma = (M_1, M_2, M_3; v^{w,c}) \in \Gamma_{SSB}$ and the set of competitive equilibrium payoff vectors. First, we point out that if a matching μ constitutes a competitive equilibrium with a feasible price vector p, then μ is an optimal matching. The proof is consigned to the Appendix.

Lemma 3.23. *Given a market $\gamma = (M_1, M_2, M_3; v^{w,c}) \in \Gamma_{SSB}$, if a pair (p, μ) is a competitive equilibrium, then μ is an optimal matching.*

Now, we can give the main result in this section.

Theorem 3.24. *Given a market $\gamma = (M_1, M_2, M_3; v^{w,c}) \in \Gamma_{SSB}$, the core of the market, $C(\gamma)$, coincides with the set of competitive equilibrium payoff vectors, $\mathcal{CE}(\gamma)$.*

Proof. First, we show that if (p, μ) is a competitive equilibrium, then its corresponding competitive equilibrium payoff vector $X = (x(p, \mu), y(p, \mu), z(p, \mu)) \in \mathcal{CE}(\gamma)$ is a core element. Recall from its definition that $x_i(p, \mu) = p_i - c_i$ for all $i \in M_1$, $y_j(p, \mu) = p_j - c_j$ for all $j \in M_2$ and $z_k(p, \mu) = w^k(\mu(k)) - p(\mu(k) \setminus \{k\})$ for all $k \in$

M_3. Let us check that for all basic coalitions $E \in \mathcal{B}$ it holds $X(E) \geq v^{w,c}(E)$. Notice that if E does not contain any buyer $k \in M_3$, then $v^{w,c}(E) = 0$ and hence the core inequality holds. Otherwise, take $E \in \mathcal{B}$ such that $k \in E$ for some $k \in M_3$. Then,

$$\begin{aligned} X(E) &= p(E \setminus \{k\}) - c(E \setminus \{k\}) + w^k(\mu(k)) - p(\mu(k) \setminus \{k\}) \\ &\geq p(E \setminus \{k\}) - c(E \setminus \{k\}) + w^k(E) - p(E \setminus \{k\}) \\ &= w^k(E) - c(E \setminus \{k\}) = v^{w,c}(E), \end{aligned}$$

where the inequality follows from the fact that (p, μ) is a competitive equilibrium. It remains to check that X is efficient. Since μ is a partition of $N = M_1 \cup M_2 \cup M_3$, we get

$$\begin{aligned} X(N) &= \sum_{k \in M_3} \Big[w^k(\mu(k)) - p(\mu(k) \setminus \{k\}) \Big] + p(M_1 \cup M_2) - c(M_1 \cup M_2) \\ &= \sum_{k \in M_3} \Big[w^k(\mu(k)) - p(\mu(k) \setminus \{k\}) + p(\mu(k) \setminus \{k\}) - c(\mu(k) \setminus \{k\}) \Big] \\ &+ \sum_{l \notin \bigcup\limits_{k \in M_3} \mu(k)} (p_l - c_l) \\ &= \sum_{k \in M_3} \Big[w^k(\mu(k)) - c(\mu(k) \setminus \{k\}) \Big] \\ &= \sum_{k \in M_3} v^{w,c}(\mu(k)) = \sum_{E \in \mu} v^{w,c}(E), \end{aligned}$$

where the third equality holds since $p_l = c_l$ for unassigned objects l.

We have shown that if (p, μ) is a competitive equilibrium, then its competitive equilibrium payoff vector X is a core allocation.

Next, we show that the reverse implication holds. That is, if $X \in \mathbb{R}^N$ is a core allocation, then it is the payoff vector related to some competitive equilibrium (p, μ), where μ is any optimal matching and p is a competitive equilibrium price vector.

Let us define $p \in \mathbb{R}^{M_1} \times \mathbb{R}^{M_2}$ by $p_l = X_l + c_l$ for all $l \in M_1 \cup M_2$. Notice first that since $X \in C(\gamma)$, if an object $l \in M_1 \cup M_2$ is unassigned by the matching μ, then $p_l = X_l + c_l = c_l$. Moreover, $X(\mu(k)) = v^{w,c}(\mu(k))$ for all $k \in M_3$ and $X(E') \geq$

$v^{w,c}(E')$ for all $E' \in \mathcal{B}^k$ where $k \in M_3$. Then, for all $k \in M_3$ and all $E' \in \mathcal{B}^k$,

$$\begin{aligned} w^k(\mu(k)) - p(\mu(k) \setminus \{k\}) &= v^{w,c}(\mu(k)) + c(\mu(k) \setminus \{k\}) - p(\mu(k) \setminus \{k\}) \\ &= X(\mu(k)) + c(\mu(k) \setminus \{k\}) - p(\mu(k) \setminus \{k\}) \\ &= X_k \\ &\geq v^{w,c}(E') - X(E' \setminus \{k\}) \\ &= v^{w,c}(E') - \left[p(E' \setminus \{k\}) - c(E' \setminus \{k\})\right] \\ &= w^k(E') - p(E' \setminus \{k\}) \end{aligned}$$

where the inequality follows from the fact that $X \in C(\gamma)$. This shows that $\mu(k) \in D_k(p)$ which concludes the proof. □

Once shown that on the class of generalized three-sided assignment markets the set of competitive equilibrium payoff vectors, $\mathcal{CE}(\gamma)$, coincides with the core of the market, $C(\gamma)$, we have that competitive equilibria exist for this model if and only if the core is non-empty.[3]

Notice to conclude that the class Γ_{SSB} contains all classical three-sided assignment markets as defined in Kaneko and Wooders (1982) or Quint (1991a). Indeed, the class Γ_{SSB} is characterized by two facts: a) individual values are null, $v(\{i\}) = 0$ for all seller $i \in M_1 \cup M_2$ and b) any pair of sellers is also valued at zero, $v(\{i,j\}) = 0$ if $i \in M_1$ and $j \in M_2$. Now, if we have any classical three-sided assignment market defined by a three-dimensional matrix $A = (a_{ijk})_{(i,j,k) \in M_1 \times M_2 \times M_3}$, simply define $c_i = 0$ for all $i \in M_1$, $c_j = 0$ for all $j \in M_2$ and, for all $k \in M_3$, $w^k(\{i,j,k\}) = a_{ijk}$ for all $(i,j) \in M_1 \times M_2$, $w^k(\{i,k\}) = 0$ for all $i \in M_1$, and for all $j \in M_2$, $w^k(\{j,k\}) = 0$. This defines a market in Γ_{SSB}.

Since Γ_{SSB} contains all classical three-sided assignment markets, balancedness is not guaranteed in this class.

3.6 Appendix

Proof of Proposition 3.8: Let us write $\hat{w} = w_{\hat{\gamma}^{S,x}}$. We have to show that $\hat{w}$ is superadditive, $\hat{w} \geq w_\gamma^{S,x}$ and $\hat{w}$ is minimal with these two properties.

By definition, $\hat{w}$ is superadditive. Now, we show that $\hat{w}(T) \geq w_\gamma^{S,x}(T)$ for all $T \subseteq S$. Notice that, for all $T \subseteq S$ there exists $Q \subseteq N \setminus S$ such that

$$w_\gamma^{S,x}(T) = w_\gamma(T \cup Q) - \sum_{l \in Q} x_l. \tag{3.4}$$

[3]See Quint (1991a) for a characterization of the non-emptiness of the core of games in partition form in terms of the solutions of the linear program that provides an optimal matching.

Let μ be a matching on $T \cup Q$ such that $w_\gamma(T \cup Q) = \sum_{E \in \mu} v(E)$. We introduce the following partition of the set of basic coalitions in μ:
$I_1 = \{\{i,j,k\} \in \mu \mid i \in T, j \in T, k \in T\}$
$I_2 = \{\{i,j,k\} \in \mu \mid i \notin T, j \notin T, k \notin T\}$
$I_3 = \{\{i,j,k\} \in \mu \mid i \in T, j \in T, k \notin T\}$
$I_4 = \{\{i,j,k\} \in \mu \mid i \in T, j \notin T, k \notin T\}$
$I_5 = \{\{i,j\} \in \mu \mid i \in T, j \in T\}$
$I_6 = \{\{i,j\} \in \mu \mid i \notin T, j \notin T\}$
$I_7 = \{\{i,j\} \in \mu \mid i \in T, j \notin T\}$
$I_8 = \{\{i\} \in \mu \mid i \in T\}$.
$I_9 = \{\{i\} \in \mu \mid i \notin T\}$.

We write $w_\gamma(T \cup Q)$ in terms of the above partition.

$$\begin{aligned} w_\gamma(T \cup Q) = & \sum_{\{i,j,k\} \in I_1} v(\{i,j,k\}) + \sum_{\{i,j,k\} \in I_2} v(\{i,j,k\}) + \sum_{\{i,j,k\} \in I_3} v(\{i,j,k\}) \\ & + \sum_{\{i,j,k\} \in I_4} v(\{i,j,k\}) + \sum_{\{i,j\} \in I_5} v(\{i,j\}) + \sum_{\{i,j\} \in I_6} v(\{i,j\}) \\ & + \sum_{\{i,j\} \in I_7} v(\{i,j\}) + \sum_{\{i\} \in I_8} v(\{i\}) + \sum_{\{i\} \in I_9} v(\{i\}). \end{aligned} \tag{3.5}$$

Then, substitute (3.5) in equation (3.4) and distribute $\sum\limits_{l \in Q} x_l$ among the sets of the partition.

$$\begin{aligned} w_\gamma^{S,x}(T) &= w_\gamma(T \cup Q) - \sum_{i \in Q} x_i \\ &= \sum_{\{i,j,k\} \in I_1} v(\{i,j,k\}) + \sum_{\{i,j,k\} \in I_2} v(\{i,j,k\}) - x_i - x_j - x_k \\ &+ \sum_{\{i,j,k\} \in I_3} v(\{i,j,k\}) - x_k + \sum_{\{i,j,k\} \in I_4} v(\{i,j,k\}) - x_j - x_k \\ &+ \sum_{\{i,j\} \in I_5} v(\{i,j\}) + \sum_{\{i,j\} \in I_6} v(\{i,j\}) - x_i - x_j + \sum_{\{i,j\} \in I_7} v(\{i,j\}) - x_j \\ &+ \sum_{\{i\} \in I_8} v(\{i\}) + \sum_{\{i\} \in I_9} v(\{i\}) - x_i. \end{aligned}$$

Since $x \in C(\gamma)$, the second, the sixth and the last term are non-positive. Let us consider $\hat{v} = \hat{v}^{S,x}$ (see Definition 3.7). For all $t, r, s \in \{1,2,3\}$ such that $r \neq s$, $r \neq t$, $s \neq t$ and all $i \in M_r \cap T$, $j \in M_s \cap T$,

$$\hat{v}(\{i,j\}) = \max_{k \in Q \cap M_t} \{v(\{i,j,k\}) - x_k, v(\{i,j\})\}.$$

As a consequence, for all $\{i,j,k\} \in I_3$, $v(\{i,j,k\}) - x_k \leq \hat{v}(\{i,j\})$ and for all $\{i,j\} \in I_5$, $v(\{i,j\}) \leq \hat{v}(\{i,j\})$.

Also, for all $t \in \{1,2,3\}$ and $l \in M_t \cap T$, if r,s are such that $r \neq s$, $s \neq t$ and $t \neq r$, then,

$$\hat{v}(\{l\}) = \max_{\substack{i \in M_r \cap Q \\ j \in M_s \cap Q}} \{v(\{i,j,l\}) - x_i - x_j, v(\{i,l\}) - x_i, v(\{j,l\}) - x_j, v(\{l\})\}.$$

As a consequence, for all $\{i,j,k\} \in I_4$, $v(\{i,j,k\}) - x_j - x_k \leq \hat{v}(\{i\})$; for all $\{i,j\} \in I_7$, $v(\{i,j\}) - x_j \leq \hat{v}(\{i\})$ and trivially $v(\{i\}) \leq \hat{v}(\{i\})$ for all $\{i\} \in I_8$.

To sum up, taking into account that $\hat{w}$ is superadditive by definition,

$$w_\gamma^{S,x}(T) \leq \sum_{\{i,j,k\} \in I_1} \hat{v}(\{i,j,k\}) + \sum_{\substack{\{i,j,k\} \in I_3 \\ \{i,j\} \in I_5}} \hat{v}(\{i,j\}) + \sum_{\substack{\{i,j,k\} \in I_4 \\ \{i,j\} \in I_7 \\ \{i\} \in I_8}} \hat{v}(\{i\}) \leq \hat{w}(T).$$

Now, we only need to show that $\hat{w}$ is the minimal superadditive game satisfying the above inequality. First, consider $\{k\} \in \mathcal{B}^S$. Then,

$$\begin{aligned} w_\gamma^{S,x}(\{k\}) &= \max_{Q \subseteq N \setminus S} \{w_\gamma(\{k\} \cup Q) - x(Q)\} \\ &\geq \max_{\substack{Q \subseteq N \setminus S \\ \{k\} \cup Q \in \mathcal{B}}} \{w_\gamma(\{k\} \cup Q) - x(Q)\} \\ &\geq \max_{\substack{Q \subseteq N \setminus S \\ \{k\} \cup Q \in \mathcal{B}}} \{v(\{k\} \cup Q) - x(Q)\} \\ &= \hat{v}(\{k\}). \end{aligned} \tag{3.6}$$

Similarly, we obtain

$$w_\gamma^{S,x}\{i,j\} \quad \geq \quad \hat{v}(\{i,j\}) \text{ for all } \{i,j\} \in \mathcal{B}^S, \tag{3.7}$$

$$w_\gamma^{S,x}(\{i,j,k\}) \quad \geq \quad \hat{v}(\{i,j,k\}) \text{ for all } \{i,j,k\} \in \mathcal{B}^S. \tag{3.8}$$

Assume now (N,w) is superadditive and $w \geq w_\gamma^{S,x}$. For all $T \subseteq S$, let μ be an optimal matching for $\hat{\gamma}_{|T}^{S,x}$. Then,

$$\begin{aligned} w(T) &\geq \sum_{\{i,j,k\} \in \mu} w(\{i,j,k\}) + \sum_{\{i,j\} \in \mu} w(\{i,j\}) + \sum_{\{k\} \in \mu} w(\{k\}) \\ &\geq \sum_{\{i,j,k\} \in \mu} w_\gamma^{S,x}(\{i,j,k\}) + \sum_{\{i,j\} \in \mu} w_\gamma^{S,x}(\{i,j\}) + \sum_{\{k\} \in \mu} w_\gamma^{S,x}(\{k\}) \\ &\geq \sum_{\{i,j,k\} \in \mu} \hat{v}(\{i,j,k\}) + \sum_{\{i,j\} \in \mu} \hat{v}(\{i,j\}) + \sum_{\{k\} \in \mu} \hat{v}(\{k\}) \\ &= \hat{w}(T), \end{aligned}$$

where the last inequality follows from (3.6), (3.7) and (3.8).

This shows that $\hat{w}$ is the minimal superadditive game such that $\hat{w} \geq w_\gamma^{S,x}$, which implies that $\hat{w}$ is the superadditive cover of $w_\gamma^{S,x}$. □

Proof of Lemma 3.23: In order to see this, we need to show that if (p,μ) is a competitive equilibrium, then the matching μ is a partition of maximal value. Consider a competitive equilibrium (p,μ) and another matching $\mu' \in \mathcal{M}(M_1,M_2,M_3)$. Then,

$$\begin{aligned}
&\sum_{E\in\mu} v^{w,c}(E) = \sum_{k\in M_3} w^k(\mu(k)) - c(\mu(k)\setminus\{k\})\\
&\geq \sum_{k\in M_3} w^k(\mu'(k)) - c(\mu(k)\setminus\{k\}) - p(\mu'(k)\setminus\{k\}) + p(\mu(k)\setminus\{k\})\\
&= \sum_{k\in M_3} w^k(\mu'(k)) - c(\mu(k)\setminus\{k\}) - p\left(\bigcup_{k\in M_3}\mu'(k)\setminus M_3\right) + p\left(\bigcup_{k\in M_3}\mu(k)\setminus M_3\right)\\
&= \sum_{k\in M_3} w^k(\mu'(k)) - c\left(\bigcup_{k\in M_3}\mu(k)\setminus M_3\right) - p\left(\left(\bigcup_{k\in M_3}\mu'(k)\setminus\bigcup_{k\in M_3}\mu(k)\right)\setminus M_3\right)\\
&+p\left(\left(\bigcup_{k\in M_3}\mu(k)\setminus\bigcup_{k\in M_3}\mu'(k)\right)\setminus M_3\right)\\
&= \sum_{k\in M_3} w^k(\mu'(k)) - c\left(\bigcup_{k\in M_3}\mu(k)\setminus M_3\right) - c\left(\left(\bigcup_{k\in M_3}\mu'(k)\setminus\bigcup_{k\in M_3}\mu(k)\right)\setminus M_3\right)\\
&+p\left(\left(\bigcup_{k\in M_3}\mu(k)\setminus\bigcup_{k\in M_3}\mu'(k)\right)\setminus M_3\right)\\
&= \sum_{k\in M_3} w^k(\mu'(k)) - c\left(\bigcup_{k\in M_3}\mu'(k)\setminus M_3\right) - c\left(\left(\bigcup_{k\in M_3}\mu(k)\setminus\bigcup_{k\in M_3}\mu'(k)\right)\setminus M_3\right)\\
&+p\left(\left(\bigcup_{k\in M_3}\mu(k)\setminus\bigcup_{k\in M_3}\mu'(k)\right)\setminus M_3\right)\\
&\geq \sum_{k\in M_3} w^k(\mu'(k)) - c(\mu'(k)\setminus\{k\}) = \sum_{E\in\mu'} v^{w,c}(E),
\end{aligned}$$

where the first inequality follows from the definition of the demand set and the fact that (p,μ) is a competitive equilibrium: $w^k(\mu(k)) \geq w^k(\mu'(k)) - p(\mu'(k)\setminus\{k\}) + p(\mu(k)\setminus\{k\})$. The fourth equality follows from the fact that for all $l \in \left(\bigcup_{k\in M_3}\mu'(k)\setminus\bigcup_{k\in M_3}\mu(k)\right)\setminus M_3$, $p_l = c_l$, and the last inequality follows from the feasibility of the price vector p. □

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4 Multi-sided assignment games on m-partite graphs

4.1 Introduction

Two-sided assignment games (Shapley and Shubik, 1972) have been generalized to the multi-sided case. In this case, agents are distributed in m disjoint sectors. Usually it is assumed that these agents are linked by a hypergraph defined by the (basic) coalitions formed by exactly one agent from each sector (see for instance Kaneko and Wooders, 1982; Quint, 1991). A matching for a coalition S is a partition of the set of agents of S in basic coalitions and, since each basic coalition has a value attached, the worth of an arbitrary coalition of agents is obtained by maximizing, over all possible matchings, the addition of values of basic coalitions in a matching.

If we do not require that each basic coalition has exactly one agent of each side but allow for coalitions of smaller size, as long as they do not contain two agents from the same sector, we obtain a larger class of games, see Chapter 3 or Atay et al (2016) for the three-sided case. But in both cases, the classical multi-sided assignment market and this enlarged model, the core of the corresponding coalitional game may be empty, and this is the main difference with the two-sided assignment game of Shapley and Shubik (1972), where the core is always non-empty.

A two-sided assignment game can also be looked at in another way. There is an underlying bi-partite (weighted) graph, where the set of nodes corresponds to the set of agents and the weight of an edge is the value of the basic coalition formed by its adjacent nodes. From this point of view, the generalization to a market with $m > 2$ sectors can be defined by a weighted m-partite graph G. In an m-partite graph the set of nodes N is partitioned in m sets $N_1, N_2, \ldots, N_m$ in such a way that two nodes in a same set of the partition are never connected by an edge. Each node in G corresponds to an agent of our market and each set N_i, for $i \in \{1, 2, \ldots, m\}$, to a different sector. We do not assume that the graph is complete but we do assume that the subgraph determined by any two sectors N_i and N_j, with $i \neq j$, is either empty or complete. Because of that, the graph G determines a quotient graph $\overline{G}$, the nodes of which are the sectors and two sectors are connected in $\overline{G}$ whenever their corresponding subgraph in G is non-empty.

For each pair of sectors N_r and N_s, $r \neq s$, that are connected in $\overline{G}$, we have a bilateral assignment market with valuation matrix $A^{\{r,s\}}$. For each $i \in N_r$ and $j \in N_s$, entry $a_{ij}^{\{r,s\}}$ is the weight in G of the edge $\{i, j\}$, and represents the value created by the cooperation of i and j.

Given the m-partite graph G, a coalition of agents in N is basic if it does not contain two agents from the same sector and its members are connected in G. Then, the worth of a basic coalition is the addition of the weights of the edges in G that are determined by nodes in the coalition. An optimal matching in this market is a partition of N in basic coalitions such that the sum of values is maximum among all possible such partitions.

We show that if there exists an optimal matching for the multi-sided m-partite market that induces an optimal matching in each bilateral market determined by the connected sectors, then the core of the multi-sided market is non-empty. Moreover, a core element can be obtained by the merging of one core element from each of the underlying bilateral markets associated to the connected sectors.

Secondly, if the quotient graph $\overline{G}$ is cycle-free, then the above sufficient condition for a non-empty core always holds and, moreover, the core of the multi-sided assignment game is fully described by the "merging" or "composition" of the cores of the underlying bilateral games. As a consequence, we prove several properties of the core of this multi-sided market. For instance, for each sector there exists a core allocation where all agents in the sector achieve their marginal contribution.

This model of multi-sided assignment market on an m-partite graph G where the quotient graph $\overline{G}$ is cycle-free can be related to the locally-additive multi-sided assignment games of Stuart (1997), where the sectors are organized on a chain and the worth of a basic coalition is also the addition of the worths of pairs of consecutive sectors. However, in Stuart's model all coalitions of size smaller than m have null worth. It can also be related with a model in Quint (1991) in which a value is attached to each pair of agents of different sectors and then the worth of an m-tuple is the addition of the values of its pairs. Again, the difference with our model is that in Quint (1991) the worth of smaller coalitions is zero. In particular, the worth of a two-player coalition is taken to be zero instead of the value of this pair. Notice that in these models the cooperation of one agent from each side is needed to generate some profit. Compared to that, in our model, any set of connected agents from different sectors yields some worth that can be shared.

For arbitrary coalitional games, cooperation restricted by communication graphs was introduced by Myerson (1977) and some examples of more recent studies are van Velzen et al (2008), Khmelnitskaya and Talman (2014), and González–Arangüena et al (2015). The difference with our work is that in the multi-sided assignment game on an m-partite graph there exist well-structured subgames, the two-sided markets between connected sectors, that provide valuable information about the multi-sided market.

Section 4.2 introduces the model. In Section 4.3, for an arbitrary m-partite graph, we provide a sufficient condition for the non-emptiness of the core. Section 4.4 focuses on the case in which the quotient graph is cycle-free. In that case, we completely characterize the non-empty core in terms of the cores of the two-sided markets between connected sectors. From that fact, additional consequences on some particular core elements are derived. In Section 4.5, once selected a spanning tree of the cycle-free graph $\overline{G}$, we characterize the core of the multi-sided assignment game in terms of competitive prices. Finally, Section 4.6 concludes with a remark pointing out a novelty of our generalization comparing with earlier models.

4.2 The model

Let N be the finite set of agents in a market situation. The set N is partitioned in m sets $N_1, N_2, \dots, N_m$, each sector maybe representing a set of agents with a specific role in the market. There is a graph $\overline{G}$ with set of nodes $\{N_1, N_2, \dots, N_m\}$, that we simply denote $\{1, 2, \dots, m\}$ when no confusion arises, and we will identify the graph with its set of edges.[1] The graph $\overline{G}$ induces another graph on the set of agents N such that $\{i, j\} \in G$ if and only if there exist $r, s \in \{1, 2, \dots, m\}$ such that $r \neq s$, $i \in N_r$, $j \in N_s$ and $\{r, s\} \in \overline{G}$. Notice that the graph G is an m-partite graph, that meaning that two agents on the same sector are not connected in G. We say that graph $\overline{G}$ is the quotient graph of G.[2]

For any pair of connected sectors $\{r, s\} \in \overline{G}$, there is a non-negative valuation matrix $A^{\{r,s\}}$ and for all $i \in N_r$ and $j \in N_s$, $v(\{i, j\}) = a_{ij}^{\{r,s\}}$ represents the value obtained by the cooperation of agents i and j. Notice that these valuation matrices, $A = \{A^{\{r,s\}}\}_{\{r,s\} \in \overline{G}}$, determine a system of weights on the graph G, and for each pair of connected sectors $\{r, s\} \in \overline{G}$, $(N_r, N_s, A^{\{r,s\}})$ defines a bilateral assignment market. Sometimes, to simplify notation, we will write A^{rs}, with $r < s$, instead of $A^{\{r,s\}}$.

Then, $\gamma = (N_1, N_2, \dots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\} \in \overline{G}})$ is a *multi-sided assignment mar-*

[1] A graph consists of a (finite) set of nodes and a set of edges, where an edge is a subset formed by two different nodes. If $\{r, s\}$ is an edge of a given graph, we say that the nodes r and s belong to this edge or are adjacent to this edge.

[2] Equivalently, we could introduce the model by first imposing a (weighted) m-partite garph on $N = N_1 \cup N_2 \cup \ldots \cup N_m$ with the condition that its restriction to $N_r \cup N_s$ for all $r, s \in \{1, \dots, m\}$ and different, is either empty or a bi-partite complete graph. Then, the quotient graph $\overline{G}$ is easily defined.

ket on an m-partite graph. When necessary, we will write G^A to denote the weighted graph with the nodes and edges of G and the weights defined by the matrices $\{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}}$. Given any such market γ, a coalition $S \subseteq N$ defines a submarket $\gamma_{|S} = (S\cap N_1,\dots,S\cap N_m; G_{|S}; A_{|S})$ where $G_{|S}$ is the subgraph of G defined by the nodes in S and $A_{|S}$ consists of the values of A that correspond to edges $\{i,j\}$ in the subgraph $G_{|S}$.

We now introduce a coalitional game related to the above market situation. To this end, we first define the worth of some coalitions that we name *basic coalitions* and then the worth of arbitrary coalitions will be obtained just imposing superadditivity. A basic coalition E is a subset of agents belonging to sectors that are connected in the quotient graph $\overline{G}$ and with no two agents of the same sector. That is, $E = \{i_1,i_2,\dots,i_k\} \subseteq N$ is a basic coalition if $(i_1,i_2,\dots,i_k) \in N_{l_1}\times N_{l_2}\times\cdots\times N_{l_k}$ and the sectors $\{l_1,l_2,\dots,l_k\}$ are all different and connected in $\overline{G}$. Sometimes we will identify the basic coalition $E = \{i_1,i_2,\dots,i_k\}$ with the k-tuple $(i_1,i_2,\dots,i_k)$. For the sake of notation, we denote by $\mathcal{B}^N$ the set of basic coalitions of market γ, though we should write $\mathcal{B}^{N_1,\dots,N_m}$, since which coalitions are basic heavily depends on the partition in sectors of the set of agents. Notice that all edges of G belong to $\mathcal{B}^N$. Moreover, if $S\subseteq N$, we denote by $\mathcal{B}^S$ the set of basic coalitions that have all their agents in S: $\mathcal{B}^S = \{E\in\mathcal{B}^N \mid E\subseteq S\}$.

The valuation function, until now defined on the edges of G, is extended to all basic coalitions by additivity: the value of a basic coalition $E\in\mathcal{B}^N$ is the addition of the weights of all edges in G with adjacent nodes in E. For all $E\in\mathcal{B}^N$,

$$v(E) = \sum_{\{i,j\}\in G_{|E}} v(\{i,j\}) = \sum_{\substack{i\in E\cap N_r,\, j\in E\cap N_s\\ \{r,s\}\in\overline{G}}} a_{ij}^{\{r,s\}}. \tag{4.1}$$

A *matching* μ for the market γ is a partition of $N = N_1\cup N_2\cup\cdots\cup N_m$ in basic coalitions in $\mathcal{B}^N$. We denote by $\mathcal{M}(N_1,N_2,\dots,N_m)$ the set of all matchings. Similarly, a matching for a submarket $\gamma_{|S}$ with $S\subseteq N$ is a partition of S in basic coalitions in $\mathcal{B}^S$.

A matching $\mu\in\mathcal{M}(N_1,N_2,\dots,N_m)$ is an *optimal matching* for the market $\gamma = (N_1,N_2,\dots,N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$ if it holds $\sum_{T\in\mu} v(T) \geq \sum_{T\in\mu'} v(T)$ for all other matching $\mu'\in\mathcal{M}(N_1,N_2,\dots,N_m)$. We denote by $\mathcal{M}_\gamma(N_1,N_2,\dots,N_m)$ the set of optimal matchings for market γ.

Then, the *multi-sided assignment game* associated with the market γ is the pair (N, w_γ), where the worth of an arbitrary coalition $S\subseteq N$ is the addition of the values of the basic coalitions in an optimal matching for this coalition S:

$$w_\gamma(S) = \max_{\mu\in\mathcal{M}(S\cap N_1,\dots,S\cap N_m)} \sum_{T\in\mu} v(T), \tag{4.2}$$

with $w_\gamma(\emptyset) = 0$. Notice that if $S \subseteq N$ is a basic coalition, $w_\gamma(S) = v(S)$, since no partition of S in smaller basic coalitions can yield a higher value, because of its definition (4.1) and the non-negativity of weights. Trivially, the game (N, w_γ) is superadditive as it is a special type of partitioning game introduced by Kaneko and Wooders (1982).

Multi-sided assignment games on m-partite graphs combine the idea of cooperation structures based on graphs (Myerson, 1977) and also the notion of (multi-sided) matching that only allows for at most one agent of each sector in a basic coalition. It is clear that for $m = 2$, multi-sided assignment games on bi-partite graphs coincide with the classical Shapley and Shubik (1972) assignment games. Notice also that for $m = 3$, multi-sided assignment games on 3-partite graphs are a particular case of the generalized three-sided assignment games in Chapter 3 or Atay et al (2016), with the constraint that the value of a three-person coalition is the addition of the values of all its pairs.

As for the related quotient graphs, for $m = 2$ the quotient graph $\overline{G}$ consists of only one edge while, for $m = 3$, $\overline{G}$ can be either a complete graph[3] or a chain. Figure 4.1 illustrates both the graph G and its quotient graph $\overline{G}$ for the cases $m = 2$ and $m = 3$.

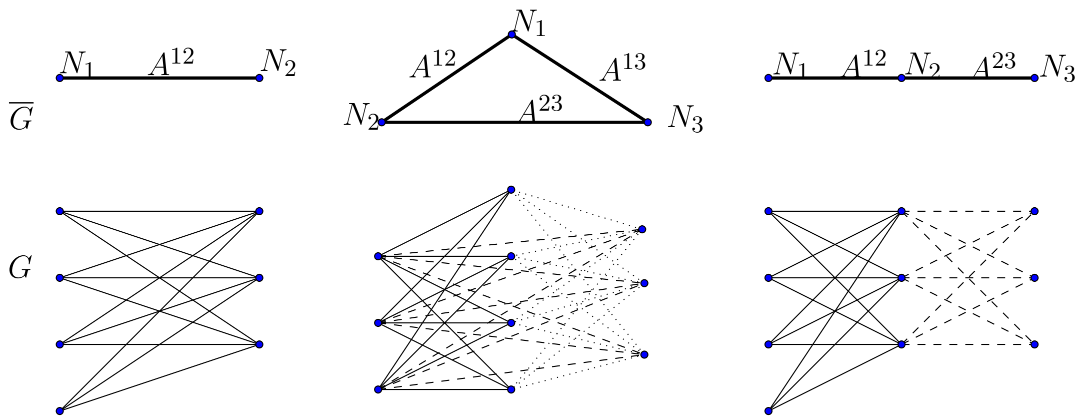

Figure 4.1: 2-partite and 3-partite graphs, and their quotient representation

As in any coalitional game, the aim is to allocate the worth of the grand coalition in such a way that preserves the cooperation among the agents. Given a multi-sided assignment market on an m-partite graph $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\} \in \overline{G}})$, a vector $x \in \mathbb{R}^N$, where $N = N_1 \cup N_2 \cup \cdots \cup N_m$, is a *payoff vector*. An *imputation* is a payoff vector $x \in \mathbb{R}^N$ that is *efficient*, $\sum_{i \in N} x_i = w_\gamma(N)$, and *individually rational*,

[3]A graph is *complete* if any two of its nodes are connected by an edge. Hence, an m-partite graph with more than one node in some of the sectors is never complete in this sense. A *complete m-partite* graph is an m-partite graph such that any two nodes from different sectors are connected by an edge.

$x_i \geq w_\gamma(\{i\}) = 0$ for all $i \in N$. Then, the *core* $C(w_\gamma)$ is the set of imputations that no coalition can object, that is $\sum_{i\in S} x_i \geq w_\gamma(S)$ for all $S \subseteq N$. Because of the definition of the characteristic function w_γ in (4.2), given any optimal matching $\mu \in \mathcal{M}_\gamma(N_1,\ldots,N_m)$, the core is described by

$$C(w_\gamma) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i\in E} x_i = v(E) \text{ for all } E \in \mu, \sum_{i\in E} x_i \geq v(E), \text{ for all } E \in \mathcal{B}^N \right\}.$$

A multi-sided assignment game on an m-partite graph is *balanced* if it has a non-empty core. Moreover, and following Le Breton et al (1992), we will say an m-partite graph $(N_1,N_2,\ldots,N_m;G)$ is *strongly balanced* if for any set of non-negative weights $\{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}}$ the resulting multi-sided assignment game is balanced. Recall from Shapley and Shubik (1972) that bi-partite graphs are strongly balanced. Our aim is to study whether this property extends to m-partite graphs or balancedness depends on properties of the weights or the structure of the graph.

4.3 Balancedness conditions

The first question above is easily answered. For $m \geq 3$, m-partite graphs are not strongly balanced. Take for instance a market with three agents on each sector. Sectors are connected by a complete graph: $N_1 = \{1,2,3\}$, $N_2 = \{1',2',3'\}$, $N_3 = \{1'',2'',3''\}$ and $\overline{G} = \{(N_1,N_2),(N_1,N_3),(N_2,N_3)\}$. From Le Breton et al (1992) we know that a graph is strongly balanced if any balanced collection[4] formed by basic coalitions contains a partition. In our example, the collection

$$\mathcal{C} = \{\{1,1'\},\{1,2''\},\{2',1''\},\{2,3'\},\{3,2''\},\{3',1''\},\{3,3''\},\{2,1'\},\{2',3''\}\}$$

is balanced (notice each agent belongs to exactly two coalitions in $\mathcal{C}$) but we cannot extract any partition. To better understand what causes the core to be empty we complete the above 3-partite graph with a system of weights and analyse some core constraints.

Example 4.1. Let us consider the following valuations on the complete 3-partite

[4] Given a coalitional game (N,v), a collection of coalitions $\mathcal{C} = \{S_1,S_2,\ldots,S_k\}$ with $S_l \subseteq N$ for all $l \in \{1,2,\ldots,k\}$, is balanced if there exist positive numbers $\delta_{S_l} > 0$ such that, for all $i \in N$, it holds $\sum_{i\in S_l\subseteq\mathcal{C}} \delta_{S_l} = 1$.

graph with three agents in each sector:

$$A^{12} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1' & 2' & 3' \end{array} \\ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ \mathbf{9} & 0 & 4 \\ 0 & \mathbf{0} & 0 \end{pmatrix} \end{array} \quad A^{13} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1'' & 2'' & 3'' \end{array} \\ \begin{pmatrix} 0 & \mathbf{5} & 0 \\ \mathbf{0} & 0 & 0 \\ 0 & 2 & \mathbf{4} \end{pmatrix} \end{array} \quad A^{23} = \begin{array}{c} \\ 1' \\ 2' \\ 3' \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1'' & 2'' & 3'' \end{array} \\ \begin{pmatrix} 0 & \mathbf{0} & 0 \\ 4 & 0 & \mathbf{6} \\ \mathbf{2} & 0 & 0 \end{pmatrix} \end{array}.$$

In boldface we show the optimal matching for each two-sided assignment market. Now, applying (4.1), the reader can obtain the worth of all three-player basic coalitions and check that the optimal matching of the three-sided market is

$$\mu = \{(2,1',1''),(1,3',2''),(3,2',3'')\}.$$

Notice that $v(\{2,1',1''\}) = 9+0+0 = 9$, $v(\{1,3',2''\}) = 0+5+0 = 5$ and $v(\{3,2',3''\}) = 0+4+6 = 10$.

Take $x = (u,v,w) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3}$. If $x = (u,v,w) \in C(w_\gamma)$, from core constraints $u_2+v_1+w_1 = 9$ and $u_2+v_1 \geq 9$ we obtain $w_1 = 0$. Then, from $v_3 + w_1 \geq 2$ we deduce $v_3 \geq 2$. Hence, $u_1+v_3+w_2 = 5$ implies $u_1+w_2 \leq 3$, which contradicts the core constraint $u_1+w_2 \geq 5$. Therefore, $C(w_\gamma) = \emptyset$.

We observe that the optimal matching μ in the above example induces a matching $\mu^{23} = \{(1',1''),(3',2''),(2',3'')\}$ for the market $(N_2,N_3,A^{\{2,3\}})$ which is not optimal. Let us relate more formally the matchings in a multi-sided assignment market on an m-partite graph with the matchings of the two-sided markets associated with the edges of the quotient graph.

Definition 4.2. Given $\gamma = (N_1,N_2,\ldots,N_m;G;\{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$, for each matching $\mu \in \mathcal{M}(N_1,\ldots,N_m)$ and each adjacent sectors $\{r,s\} \in \overline{G}$, we define a matching $\mu^{\{r,s\}} \in \mathcal{M}(N_r,N_s)$ by

$$\{i,j\} \in \mu^{\{r,s\}} \text{ if and only if there exists } E \in \mu \text{ such that } \{i,j\} \subseteq E. \tag{4.3}$$

We then say that μ is the composition of $\mu^{\{r,s\}}$ for $\{r,s\} \in \overline{G}$ and write

$$\mu = \bigoplus_{\{r,s\}\in\overline{G}} \mu^{\{r,s\}}.$$

Conversely, the composition of matchings of each underlying two-sided market not always results in a matching of the multi-sided assignment market. Take for instance matchings $\mu^{\{1,2\}} = \{(2,1'),(1,3'),(3,2')\}$, $\mu^{\{1,3\}} = \{(1,2''),(2,1''),(3,3'')\}$ and $\mu^{\{2,3\}} = \{(1',2''),(2',3''),(3',1'')\}$ in Example 4.1. Since $(1',2'') \in \mu^{\{2,3\}}$, $(2,1') \in \mu^{\{1,2\}}$ and $(1,2'') \in \mu^{\{1,3\}}$, both 1 and 2 should be in the same coalition when composing $\mu^{\{1,2\}} \oplus \mu^{\{1,3\}} \oplus \mu^{\{2,3\}}$, but then this coalition would not be basic since it contains two agents from N_1, and the composition would not be a matching of the three-sided market.

Next proposition states that whenever the composition of optimal matchings of the underlying two-sided markets results in a matching of the multi-sided market on an m-partite graph, then that matching is optimal and the core of the multi-sided assignment market is non-empty. To show this second part we need to combine payoff vectors of each underlying two-sided market $(N_r, N_s, A^{\{r,s\}})$, with $\{r,s\} \in \overline{G}$, to produce a payoff vector $x \in \mathbb{R}^N$ for the multi-sided market γ. We write $C(w_{A^{\{r,s\}}})$ to denote the core of these two-sided assignment games.

Definition 4.3. Given $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$, let $x^{\{r,s\}} \in \mathbb{R}^{N_r} \times \mathbb{R}^{N_s}$ for all $\{r,s\} \in \overline{G}$. Then,

$$x = \bigoplus_{\{r,s\}\in\overline{G}} x^{\{r,s\}} \in \mathbb{R}^N \text{ is defined by}$$

$$x_i = \sum_{\{r,s\}\in\overline{G}} x_i^{\{r,s\}}, \text{ for all } i \in N_r, r \in \{1,2,\ldots,m\}.$$

We then say that the payoff vector $x = \bigoplus_{\{r,s\}\in\overline{G}} x^{\{r,s\}} \in \mathbb{R}^N$ is the composition of the payoff vectors $x^{\{r,s\}} \in \mathbb{R}^{N_r} \times \mathbb{R}^{N_s}$. Similarly, we denote the set of payoff vectors in $\mathbb{R}^N$ that result from the composition of core elements of the underlying two-sided assignment markets by $\bigoplus_{\{r,s\}\in\overline{G}} C(w_{A^{\{r,s\}}})$.

Proposition 4.4. *Let $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$ be a multi-sided assignment market on an m-partite graph. If there exists $\mu \in \mathcal{M}(N_1, \ldots, N_m)$ such that $\mu^{\{r,s\}}$ is an optimal matching of $(N_r, N_s, A^{\{r,s\}})$ for all $\{r,s\} \in \overline{G}$, then*

1. *μ is optimal for γ and*
2. *γ is balanced and moreover $\bigoplus_{\{r,s\}\in\overline{G}} C(w_{A^{\{r,s\}}}) \subseteq C(w_\gamma)$.*

Proof. To see that $\mu = \bigoplus_{\{r,s\}\in\overline{G}} \mu^{\{r,s\}}$ is optimal for γ, take any other matching $\tilde{\mu} \in \mathcal{M}(N_1, \ldots, N_m)$ and let $\tilde{\mu}^{\{r,s\}} \in \mathcal{M}(N_r, N_s)$, for $\{r,s\} \in \overline{G}$, be the matching $\tilde{\mu}$ induces in each underlying two-sided market. That is, $\tilde{\mu} = \bigoplus_{\{r,s\}\in\overline{G}} \tilde{\mu}^{\{r,s\}}$. Now, applying (4.1),

$$\begin{aligned}\sum_{E\in\mu} v(E) &= \sum_{E\in\mu} \sum_{\substack{i\in N_r\cap E\\ j\in N_s\cap E\\ \{r,s\}\in\overline{G}}} v(\{i,j\}) = \sum_{\{r,s\}\in\overline{G}} \sum_{\{i,j\}\in\mu^{\{r,s\}}} v(\{i,j\})\\ &\geq \sum_{\{r,s\}\in\overline{G}} \sum_{\{i,j\}\in\tilde{\mu}^{\{r,s\}}} v(\{i,j\}) = \sum_{E\in\tilde{\mu}} v(E),\end{aligned}$$

where the inequality follows from the assumption on the optimality of $\mu^{\{r,s\}}$ in each market $(N_r, N_s, A^{\{r,s\}})$, for $\{r,s\} \in \overline{G}$. Hence, μ is optimal for the multi-sided market γ.

Take now, for each $\{r,s\} \in \overline{G}$, $x^{\{r,s\}} \in C(w_{A^{\{r,s\}}})$. Define the payoff vector $x \in \mathbb{R}^N$ as in Definition 4.3, $x_i = \sum_{\{r,s\}\in\overline{G}} x_i^{\{r,s\}}$, for all $i \in N_r$, $r \in \{1,2,\ldots,m\}$. We will see that $x \in C(w_\gamma)$. Given any basic coalition $E \in \mathcal{B}^N$,

$$\begin{aligned}
\sum_{i\in E} x_i &= \sum_{r=1}^{m} \sum_{i\in E\cap N_r} x_i = \sum_{r=1}^{m} \sum_{i\in E\cap N_r} \sum_{\{r,s\}\in\overline{G}} x_i^{\{r,s\}} \\
&\geq \sum_{r=1}^{m} \sum_{i\in E\cap N_r} \sum_{\substack{\{r,s\}\in\overline{G} \\ E\cap N_s\neq\emptyset}} x_i^{\{r,s\}} = \sum_{\substack{\{r,s\}\in\overline{G} \\ E\cap N_r\neq\emptyset \\ E\cap N_s\neq\emptyset}} \sum_{\substack{i\in E\cap N_r \\ j\in E\cap N_s}} \left(x_i^{\{r,s\}} + x_j^{\{r,s\}}\right) \\
&\geq \sum_{\substack{\{r,s\}\in\overline{G} \\ i\in E\cap N_r \\ j\in E\cap N_s}} v(\{i,j\}) = v(E),
\end{aligned}$$

where both inequalities follow from $x^{\{r,s\}} \in C(w_{A^{\{r,s\}}})$ for all $\{r,s\} \in \overline{G}$. Notice also that if $E \in \mu$ the above inequalities cannot be strict and hence $\sum_{i\in E} x_i = v(E)$. Indeed, if $i \in E \cap N_r$, $\{r,s\} \in \overline{G}$ and $E \cap N_s = \emptyset$, then i is unmatched by $\mu^{\{r,s\}}$ and, because of the optimality of $\mu^{\{r,s\}}$, $x_i^{\{r,s\}} = 0$. Similarly, if $i \in E \cap N_r$ and $j \in E \cap N_s$, then $\{i,j\} \in \mu^{\{r,s\}}$ and hence $x_i^{\{r,s\}} + x_j^{\{r,s\}} = v(\{i,j\})$. □

The above proposition gives a sufficient condition for optimality of a matching and for balancedness of a multi-sided assignment game on an m-partite graph. However, this condition is not necessary. The matching μ in Example 4.1 is optimal while $\mu^{\{2,3\}}$ is not. The core of the market in Example 4.1 is empty, but one can find similar examples with non-empty core.

Finally, even under the assumption of the proposition, that is, when the composition of optimal matchings of the two-sided markets leads to a matching of the multi-sided market, the core may contain more elements than those produced by the composition of the cores of $(N_r, N_s, A^{\{r,s\}})$, for $\{r,s\} \in \overline{G}$. Example 3.5 illustrates this fact in the three-sided case. The inclusion $\bigoplus_{\{r,s\}\in\overline{G}} C(w_{A^{\{r,s\}}}) \subseteq C(w_\gamma)$ will become an equality for some particular graphs.

4.4 When $\overline{G}$ is cycle-free: strong balancedness

In this section we assume that the quotient graph $\overline{G}$ of the m-partite graph G does not contain cycles. We will assume without loss of generality that it is connected, since the results in that case are easily extended to the case of a finite union of disjoint cycle-free graphs.

We select a node of $\overline{G}$ as a source, that is, we select a spanning tree of $\overline{G}$. Define the distance d of any other node as the number of edges in the unique path that connects this node to the source. Then, without loss of generality, we rename the nodes of $\overline{G}$ in such a way that the source has label 1 and, given two other nodes r and s, if $d(1,r) < d(1,s)$ then $r < s$. Notice that the labels of nodes at the same distance to the source are assigned arbitrarily.

A partial order is defined on the set of nodes of a tree in the following way: given two nodes r and s, we say that s *follows* r, and write $s \succeq r$, if given the unique path in the tree that connects s to the source, $\{s_1 = 1, s_2, \ldots, s_q = s\}$, it holds $r = s_p$ for some $p \in \{1, \ldots, q-1\}$. If $r = s_{q-1}$ we say that s is an *immediate follower* of r. We denote by $\mathcal{S}_r^{\overline{G}}$ the set of followers of $r \in \{1, 2, \ldots, m\}$, we write $\hat{\mathcal{S}}_r^{\overline{G}} = \{r\} \cup \mathcal{S}_r^{\overline{G}}$ when we need to include sector r, and we denote by $\mathcal{I}_r^{\overline{G}}$ the set of immediate followers of $r \in \{1, 2, \ldots, m\}$.

Our main result states that an m-partite graph G where the quotient graph $\overline{G}$ is a tree is strongly balanced.

Theorem 4.5. *Let $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\} \in \overline{G}})$ be a multi-sided assignment market on an m-partite graph. If $\overline{G}$ is cycle-free, then (N, w_γ) is balanced and*

$$C(w_\gamma) = \bigoplus_{\{r,s\} \in \overline{G}} C(w_{A^{\{r,s\}}}).$$

Proof. Notice first that when $\overline{G}$ is a tree, the composition of optimal matchings $\mu^{\{r,s\}}$ of each underlying two-sided market $(N_r, N_s, A^{\{r,s\}})$, for $\{r,s\} \in \overline{G}$, leads to a matching in $\mathcal{M}(N_1, N_2, \ldots, N_m)$. To see that, we define a binary relation on the set of agents $N = N_1 \cup N_2 \cup \cdots \cup N_m$. Two agents $i \in N_r$ and $j \in N_s$, with $r \leq s$, are related if either $i = j$ or there exist sectors $\{r = s_1, s_2, \ldots, s_t = s\} \subseteq \{1, 2, \ldots, m\}$ and agents $i_k \in N_{s_k}$ for $k \in \{1, 2, \ldots, t\}$ such that $\{s_k, s_{k+1}\} \in \overline{G}$ and $\{i_k, i_{k+1}\} \in \mu^{\{s_k, s_{k+1}\}}$, for all $k \in \{1, 2, \ldots, t-1\}$. This is an equivalence relation and, because $\overline{G}$ is a tree, in each equivalence class there are no two agents of the same sector. Hence, the set μ of all equivalence classes is a matching and by its definition it is the composition of the matchings $\mu^{\{r,s\}}$ of the two-sided markets: $\mu = \bigoplus_{\{r,s\} \in \overline{G}} \mu^{\{r,s\}}$. Now, by Proposition 4.4, μ is an optimal matching for the multi-sided market γ and $\bigoplus_{\{r,s\} \in \overline{G}} C(w_{A^{\{r,s\}}}) \subseteq C(w_\gamma)$, which guarantees balancedness.

We will now prove that the converse inclusion also holds.

Let it be $u = (u^1, u^2, \ldots, u^m) \in C(w_\gamma)$. We will define, for each $\{r,s\} \in \overline{G}$, a payoff vector $(x^{\{r,s\}}, y^{\{r,s\}}) \in \mathbb{R}^{N_r} \times \mathbb{R}^{N_s}$. Take the optimal matching $\mu = \bigoplus_{\{r,s\} \in \overline{G}} \mu^{\{r,s\}}$ and $E \in \mu$. Let us denote by $\overline{E} = \overline{G}_{|E}$ the subtree in $\overline{G}$ determined by the sectors containing agents in E and take as the source of $\overline{E}$ its sector s_1 with the lowest label. Take any leaf[5] s_r of $\overline{E}$ and let $\{s_1, s_2 \ldots, s_q, s_{q+1}, \ldots, s_{r-1}, s_r\}$ be the unique path in $\overline{E}$ connecting s_r to the source s_1. Let s_q be the sector in this path with the highest label among those that have more than one immediate follower in $\overline{E}$ (let us assume for simplicity that s_q has two immediate followers, s_{q+1} and $s_{q'+1}$). Figure 4.2 depicts such a subtree $\overline{E}$.

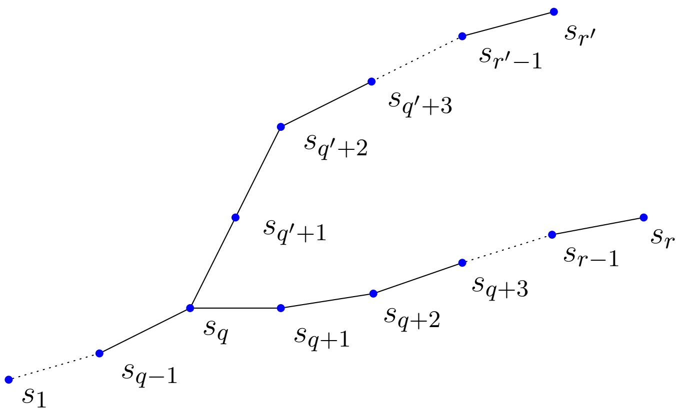

Figure 4.2: A subtree $\overline{E}$ for $E \in \mu$

For each sector s_t with $t \in \{1, 2, \ldots, r\}$ we denote by i_t the unique agent in E that belongs to this sector. Then, we define

$$y_{i_r}^{\{s_{r-1}, s_r\}} = u_{i_r}^{s_r}, \tag{4.4}$$
$$x_{i_{r-1}}^{\{s_{r-1}, s_r\}} = a_{i_{r-1} i_r}^{\{s_{r-1}, s_r\}} - y_{i_r}^{\{s_{r-1} s_r\}}, \text{ and} \tag{4.5}$$
$$y_{i_{r-1}}^{\{s_{r-2}, s_{r-1}\}} = u_{i_{r-1}}^{s_{r-1}} - x_{i_{r-1}}^{\{s_{r-1}, s_r\}}. \tag{4.6}$$

Iteratively, for all $t \in \{q+1, \ldots, r-2\}$, we define

$$x_{i_t}^{\{s_t, s_{t+1}\}} = a_{i_t i_{t+1}}^{\{s_t, s_{t+1}\}} - y_{i_{t+1}}^{\{s_t, s_{t+1}\}}, \text{ and} \tag{4.7}$$
$$y_{i_t}^{\{s_{t-1}, s_t\}} = u_{i_t}^{s_t} - x_{i_t}^{\{s_t, s_{t+1}\}}, \tag{4.8}$$

while for sector s_q we define $x_{i_q}^{\{s_q, s_{q+1}\}} = a_{i_q i_{q+1}}^{\{s_q, s_{q+1}\}} - y_{i_{q+1}}^{\{s_q, s_{q+1}\}}$, and, assuming $x_{i_q}^{\{s_q, s_{q'+1}\}}$ has been defined analogously from the branch $\{s_{q'+1}, s_{q'+2}, \ldots, s_{r'-1}, s_{r'}\}$, we also define $y_{i_q}^{\{s_{q-1}, s_q\}} = u_{i_q}^{s_q} - \left(x_{i_q}^{\{s_q, s_{q+1}\}} + x_{i_q}^{\{s_q, s_{q'+1}\}}\right)$. More generally, if s_q

[5]Given a tree, a leaf is a node with no followers.

has several immediate followers in $\overline{E}$, then

$$y_{i_q}^{\{s_{q-1},s_q\}} = u_{i_q}^{s_q} - \sum_{\substack{\{s_q,s_l\}\in\overline{E}\\ s_q<s_l}} x_{i_q}^{\{s_q,s_l\}}. \tag{4.9}$$

We proceed backwards until we reach $x_{i_1}^{\{s_1,s_l\}}$ for all $\{s_1,s_l\}\in\overline{E}$ with $s_1<s_l$.

In addition, if $i\in N_r$ and for some $\{r,s\}\in\overline{G}$, $r<s$, i is unmatched by $\mu^{\{r,s\}}$, define $x_i^{\{r,s\}}=0$. Similarly, if $i\in N_r$ and for all $\{s,r\}\in\overline{G}$, $s<r$, i is unmatched by $\mu^{\{s,r\}}$, define $y_i^{\{s,r\}}=0$.

We will first check that the payoff vectors $(x^{\{r,s\}},y^{\{r,s\}})$ we have defined are non-negative for all $\{r,s\}\in\overline{G}$. From (4.4) to (4.9) above, it follows that, for all maximal path in $\overline{E}$ starting at s_1, $\{s_1,s_2,\ldots,s_r\}$, and all $t\in\{1,2,\ldots,r-1\}$, we can express $x_{i_t}^{\{s_t,s_{t+1}\}}$ in terms of the payoffs in u to agents in following sectors in $\overline{E}$:

$$\begin{aligned} x_{i_t}^{\{s_t,s_{t+1}\}} &= a_{i_ti_{t+1}}^{\{s_t,s_{t+1}\}} - y_{i_{t+1}}^{\{s_t,s_{t+1}\}} = a_{i_ti_{t+1}}^{\{s_t,s_{t+1}\}} - \Big(u_{i_{t+1}}^{s_{t+1}} - \sum_{\substack{\{s_{t+1},l\}\in\overline{E}\\ l>s_{t+1}}} x_{i_{t+1}}^{\{s_{t+1},l\}}\Big) \\ &= \cdots = a_{i_ti_{t+1}}^{\{s_t,s_{t+1}\}} + \sum_{\substack{i\in N_r\cap E\\ j\in N_s\cap E\\ \{r,s\}\in\overline{E},r,s\in\hat{\mathcal{S}}_{s_{t+1}}^{\overline{E}}}} a_{ij}^{\{r,s\}} - \sum_{\substack{k\in N_r\cap E\\ r\in\hat{\mathcal{S}}_{s_{t+1}}^{\overline{E}}}} u_k^r. \end{aligned} \tag{4.10}$$

Hence, if $T=\{i_t\}\cup\{i\in E \mid i\in N_r,\, r\in\hat{\mathcal{S}}_{s_{t+1}}^{\overline{E}}\}$, we have

$$x_{i_t}^{\{s_t,s_{t+1}\}} = v(T) - u(T\setminus\{i_t\}). \tag{4.11}$$

Notice that for $t=1$, because of efficiency of $u\in C(w_\gamma)$, we obtain

$$\sum_{\{s_1,l\}\in\overline{E}} x_{i_1}^{\{s_1,l\}} = v(E) - \sum_{\substack{k\in E\cap N_r\\ k\neq i_1}} u_k^r = u_{i_1}^{s_1}. \tag{4.12}$$

Equation (4.10), together with (4.9) gives, for all $t\in\{2,\ldots,r\}$,

$$\begin{aligned} y_{i_t}^{\{s_{t-1},s_t\}} &= u_{i_t}^{s_t} - \sum_{\substack{\{s_t,s_l\}\in\overline{E}\\ s_t<s_l}} x_{i_t}^{\{s_t,s_l\}} \\ &= u_{i_t}^{s_t} - \sum_{\substack{\{s_t,s_l\}\in\overline{E}\\ s_t<s_l}} \left(a_{i_ti_l}^{\{s_t,s_l\}} + \sum_{\substack{i\in N_r\cap E\\ j\in N_s\cap E\\ \{r,s\}\in\overline{E},r,s\in\hat{\mathcal{S}}_{s_l}^{\overline{E}}}} a_{ij}^{\{r,s\}} - \sum_{\substack{k\in N_r\cap E\\ r\in\hat{\mathcal{S}}_{s_l}^{\overline{E}}}} u_k^r \right) \geq 0, \end{aligned}$$

where the inequality follows from the core constraint satisfied by $u \in C(w_\gamma)$ for coalition $T = \{i_t\} \cup \{i \in E \mid i \in N_r, r \in \mathcal{S}^{\overline{E}}_{s_t}\}$, that is, $y^{\{s_{t-1},s_t\}}_{i_t} = u(T) - v(T) \geq 0$.

Now, again making use of (4.4) to (4.12), we express $x^{\{s_t,s_{t+1}\}}_{i_t}$ in terms of the payoffs in u to agents in sectors that do not follow s_t in $\overline{E}$:

$$\begin{aligned} x^{\{s_t,s_{t+1}\}}_{i_t} &= u^{s_t}_{i_t} - y^{\{s_{t-1},s_t\}}_{i_t} - \sum_{\substack{\{s_t,l\}\in\overline{E} \\ l>s_t, l\neq s_{t+1}}} x^{\{s_t,l\}}_{i_t} \\ &= u^{s_t}_{i_t} - a^{\{s_{t-1},s_t\}}_{i_{t-1}i_t} + x^{\{s_{t-1},s_t\}}_{i_{t-1}} - \sum_{\substack{\{s_t,l\}\in\overline{E} \\ l>s_t, l\neq s_{t+1}}} x^{\{s_t,l\}}_{i_t} = \cdots \end{aligned}$$

$$= u^{s_t}_{i_t} - a^{\{s_{t-1},s_t\}}_{i_{t-1}i_t} + x^{\{s_{t-1},s_t\}}_{i_{t-1}} - \sum_{\substack{\{s_t,l\}\in\overline{E} \\ s_t<l\neq s_{t+1}}} (v(T_l) - u(T_l \setminus \{i_t\})),$$

where $T_l = \{i_t\} \cup \{i \in E \mid i \in N_r, r \in \hat{\mathcal{S}}^{\overline{E}}_l\}$. Recursively applying the same argument (in first place to $x^{\{s_{t-1},s_t\}}_{i_{t-1}}$), we eventually obtain

$$x^{\{s_t,s_{t+1}\}}_{i_t} = u((T' \setminus T) \cup \{i_t\}) - v((T' \setminus T) \cup \{i_t\}) \geq 0,$$

with $T' = \{i_1\} \cup \{i \in E \mid i \in N_r, r \in \mathcal{S}^{\overline{E}}_{s_1}\}$, T as defined above, and where the inequality also follows from $u \in C(w_\gamma)$.

Once proved that for all $\{r,s\} \in \overline{G}$, $(x^{\{r,s\}}, y^{\{r,s\}})$ is a non-negative payoff vector, let us check it is in $C(w_{A^{\{r,s\}}})$. If $(i,j) \in \mu^{\{r,s\}}$ for some $\{r,s\} \in \overline{G}$, then i and j belong to the same basic coalition E of μ and $x^{\{r,s\}}_i + y^{\{r,s\}}_j = a^{\{r,s\}}_{ij}$ follows by definition from equations (4.5) and (4.7).

It only remains to prove that if $i \in N_r$, $j \in N_s$, with $\{r,s\} \in \overline{G}$, $r < s$, and $(i,j) \notin \mu^{\{r,s\}}$, then $x^{\{r,s\}}_i + y^{\{r,s\}}_j \geq a^{\{r,s\}}_{ij}$. Since i and j are not matched in $(N_r, N_s, A^{\{r,s\}})$, they belong to different basic coalitions in μ. Let E and E' be the basic coalitions containing i and j respectively. Let us consider a maximal path $\{s_1, s_2, \ldots, s_t, \ldots, s_p\}$ in $\overline{E}$ with origin in the node in $\overline{E}$ with the lowest label (that we will name the source of the subtree $\overline{E}$) and such that there exists $t \in \{1,\ldots,q\}$ with $r = s_t$. We write $i_1 \in E \cap N_{s_1}$. Similarly, let $\{s'_1, s'_2, \ldots, s'_l, \ldots, s'_p\}$ be the maximal path in $\overline{E'}$ with origin in the node in $\overline{E'}$ with the lowest label (the source) and such that there exists $l \in \{1,\ldots,p\}$ with $s = s'_l$.

Recall that, $y^{\{r,s\}}_j = u(R) - v(R)$, where $R = \{j\} \cup \{b \in E' \mid b \in N_k, k \in \mathcal{S}^{\overline{E'}}_{s'_l}\}$, and $x^{\{r,s\}}_i = u((T' \setminus T) \cup \{i\}) - v((T' \setminus T) \cup \{i\})$, where $T = \{i\} \cup \{b \in E \mid b \in N_k, k \in \mathcal{S}^{\overline{E}}_{s_t}\}$ and $T' = \{i_1\} \cup \{b \in E \mid b \in N_k, k \in \mathcal{S}^{\overline{E}}_{s_1}\}$. Since $E \cap E' = \emptyset$, $(T' \setminus$

$T) \cup \{i\}$ and R are also disjoint. Then,

$$x_i^{\{r,s\}} + y_j^{\{r,s\}} = u((T' \setminus T) \cup \{i\}) + u(R) - v((T' \setminus T) \cup \{i\}) - v(R) \geq a_{ij}^{\{r,s\}}$$

since $v((T' \setminus T) \cup \{i\} \cup R) = v((T' \setminus T) \cup \{i\}) + v(R) + a_{ij}^{\{r,s\}}$ and $u \in C(w_\gamma)$. This completes the proof of $C(w_\gamma) = \bigoplus_{\{r,s\} \in \overline{G}} C(w_{A^{\{r,s\}}})$. □

The fact that the core of the multi-sided assignment game on an m-partite graph is completely described by the cores of all underlying two-sided markets allows us to deduce some properties of $C(w_\gamma)$ from the known properties of $C(w_{A^{\{r,s\}}})$, with $\{r,s\} \in \overline{G}$.

One of these consequences is that, for each sector $r \in \{1,2,\ldots,m\}$, there is a core element $u \in C(w_\gamma)$ where all agents in sector r simultaneously receive their maximum core payoff, which is their marginal contribution to the grand coalition. This is one property of two-sided assignment markets that does not extend to arbitrary multi-sided markets but it is preserved when sectors are connected by a tree and the value of basic coalitions is defined additively as in (4.1).

Proposition 4.6. *Let $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\} \in \overline{G}})$ be a multi-sided assignment market on an m-partite graph. If $\overline{G}$ is a tree, then for each sector $k \in \{1,2,\ldots,m\}$ there exists $u \in C(w_\gamma)$ such that $u_i = w_\gamma(N) - w_\gamma(N \setminus \{i\})$ for all $i \in N_k$.*

Proof. Take any $k \in \{1,2,\ldots,m\}$. For all $s \in \{1,2,\ldots,m\}$ with $\{k,s\} \in \overline{G}$, take $(x^{\{k,s\}}, y^{\{k,s\}}) = (\overline{x}^{\{k,s\}}, \underline{y}^{\{k,s\}})$ the element of $C(w_{A^{\{k,s\}}})$ that is optimal for all agents in N_k. Similarly, for all $r \in \{1,2,\ldots,m\}$ such that $\{r,k\} \in \overline{G}$, take the element $(x^{\{r,k\}}, y^{\{r,k\}}) = (\underline{x}^{\{r,k\}}, \overline{y}^{\{r,k\}})$ of $C(w_{A^{\{r,k\}}})$ that is optimal for the agents in N_k. These optimal core elements exist in any bilateral assignment market (see Shapley and Shubik, 1972). Moreover, by Demange (1982) and Leonard (1983), it is known that for all $i \in N_k$, $\overline{x}_i^{\{k,s\}} = w_{A^{\{k,s\}}}(N_k \cup N_s) - w_{A^{\{k,s\}}}(N_k \cup N_s \setminus \{i\})$ and $\overline{y}_i^{\{r,k\}} = w_{A^{\{r,k\}}}(N_r \cup N_k) - w_{A^{\{r,k\}}}(N_r \cup N_k \setminus \{i\})$. Finally, for all $\{r,s\} \in \overline{G}$ with $r \neq k$ and $s \neq k$, take an arbitrary core element $(x^{\{r,s\}}, y^{\{r,s\}}) \in C(w_{A^{\{r,s\}}})$.

Now, if we consider the composition of the core elements defined above, we get, given $k \in \{1,2,\ldots,m\}$, $\overline{u}^k = \bigoplus_{\{r,s\} \in \overline{G}} (x^{\{r,s\}}, y^{\{r,s\}})$.

Then, for all $i \in N_k$, if $\{r,k\} \in \overline{G}$ with $r < k$,

$$\overline{u}_i^k = \overline{y}_i^{\{r,k\}} + \sum_{\substack{\{k,s\} \in \overline{G} \\ k < s}} \overline{x}_i^{\{k,s\}} \geq u_i$$

for all other $u \in C(w_\gamma)$, as a consequence of Theorem 4.5.

Moreover, if $k \in \{1,2,\ldots,m\}$ is such that there exists $r \in \{1,2,\ldots,m\}$ with $\{r,k\} \in \overline{G}$ and $r < k$, and there exists $s \in \{1,2,\ldots,m\}$ with $\{k,s\} \in \overline{G}$ and $k < s$, then

$$\begin{aligned} w_\gamma(N) - w_\gamma(N \setminus \{i\}) &= \left[w_{A^{\{r,k\}}}(N_r \cup N_k) - w_{A^{\{r,k\}}}(N_r \cup N_k \setminus \{i\})\right] \\ &+ \sum_{\substack{\{k,s\} \in \overline{G} \\ k<s}} \left[w_{A^{\{k,s\}}}(N_k \cup N_s) - w_{A^{\{k,s\}}}(N_k \cup N_s \setminus \{i\})\right] \\ &= \overline{u}_i^k, \end{aligned}$$

for all $i \in N_k$.

Similarly, if k is a leaf of $\overline{G}$, then

$$w_\gamma(N) - w_\gamma(N \setminus \{i\}) = w_{A^{\{r,k\}}}(N_r \cup N_k) - w_{A^{\{r,k\}}}(N_r \cup N_k \setminus \{i\}) = \overline{u}_i^k$$

for the only $r \in \{1,2,\ldots,m\}$ such that $\{r,k\} \in \overline{G}$ and for all $i \in N_k$. Also, if k is the source of the tree $\overline{G}$, then

$$w_\gamma(N) - w_\gamma(N \setminus \{i\}) = \sum_{\substack{\{k,s\} \in \overline{G} \\ k<s}} \left[w_{A^{\{k,s\}}}(N_k \cup N_s) - w_{A^{\{k,s\}}}(N_k \cup N_s \setminus \{i\})\right] = \overline{u}_i^k,$$

for all $i \in N_k$.

Then, for all $k \in \{1,2,\ldots,m\}$ we have $\overline{u}_i^k = w_\gamma(N) - w_\gamma(N \setminus i)$ for all $i \in N_k$.
□

Once proved in Theorem 4.5 that for an assignment market on an m-partite graph with a cycle-free quotient graph $\overline{G}$ the core can be completely described from the cores of the two-sided markets between connected sectors, the question arises whether some other single valued cooperative solutions of the market can be obtained in the same way.

As a first consequence we obtain that all extreme core allocations of the multi-sided assignment game are obtained as the composition of extreme core allocations of the underlying two-sided markets.

Proposition 4.7. *Let $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\} \in \overline{G}})$ be a multi-sided assignment market on an m-partite graph. If $\overline{G}$ is cycle-free, then any extreme core allocation $x \in Ext(C(w_\gamma))$ is the composition of extreme core allocations of the underlying two-sided markets, $x = \bigoplus_{\{r,s\} \in \overline{G}} x^{\{r,s\}}$, where $x^{\{r,s\}} \in Ext(C(w_{A^{\{r,s\}}}))$ for all $\{r,s\} \in \overline{G}$.*

Proof. From Theorem 4.5, it is straightforward to see that $x \in Ext(C(w_\gamma))$ satisfies $x = \bigoplus_{\{r,s\} \in \overline{G}} x^{\{r,s\}}$ with $x^{\{r,s\}} \in C(w_{A^{\{r,s\}}})$. Assume now that $x^{\{r',s'\}} \notin$

$Ext(C(w_{A^{\{r',s'\}}}))$ for some $\{r',s'\} \in \overline{G}$. Then, there exist two different elements, $y^{\{r',s'\}}$ and $z^{\{r',s'\}}$, in $C(w_{A^{\{r',s'\}}})$ such that $x^{\{r',s'\}} = \frac{1}{2}y^{\{r',s'\}} + \frac{1}{2}z^{\{r',s'\}}$.

We now consider two different elements in $C(w_\gamma)$ by composing $\bigoplus_{\substack{\{r,s\}\in\overline{G}\\ \{r,s\}\neq\{r',s'\}}} x^{\{r,s\}}$ either with $y^{\{r',s'\}}$ or $z^{\{r',s'\}}$,

$$x^y = \left(\bigoplus_{\substack{\{r,s\}\in\overline{G}\\ \{r,s\}\neq\{r',s'\}}} x^{\{r,s\}}\right) \oplus y^{\{r',s'\}} \text{ and } x^z = \left(\bigoplus_{\substack{\{r,s\}\in\overline{G}\\ \{r,s\}\neq\{r',s'\}}} x^{\{r,s\}}\right) \oplus z^{\{r',s'\}}.$$

It is then straightforward to check that $x = \frac{1}{2}x^y + \frac{1}{2}x^z$, which contradicts the assumption $x \in Ext(C(w_\gamma))$. □

However, the converse implication does not hold, that is, the composition of extreme core allocations of the underlying two-sided markets provides an element in $C(w_\gamma)$ which may not be an extreme point (see Example 4.14 in the Appendix 4.7).

We now consider single-valued core selections that are not extreme points but usually interior core points. As a consequence of Theorem 4.5, the composition $\eta^{\oplus}(w_\gamma) = \oplus_{\{r,s\}\in\overline{G}}\eta(w_{A^{\{r,s\}}})$ of the nucleolus[6] of the two-sided markets between connected sectors belongs to $C(w_\gamma)$. Moreover, well-known algorithms to compute the nucleolus of a two-sided assignment game (Solymosi and Raghavan, 1994; Martínez–de–Albéniz et al, 2014) can be used to obtain $\eta^{\oplus}(w_\gamma)$. However, this composition does not coincide with the nucleolus of the initial m-sided market $\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$, as Example 4.13 in the Appendix 4.7 shows.

If we select the τ-value or fair-division point[7] as the cooperative solution concept to distribute the profits in each bilateral market, we can propose the composition of the τ-values of all connected two-sided markets, $\tau^{\oplus}(w_\gamma) = \oplus_{\{r,s\}\in\overline{G}}\tau(w_{A^{\{r,s\}}})$ as an allocation of the profit of the multi-sided assignment market with a tree quotient graph. Because of Theorem 4.5, this composition belongs to $C(w_\gamma)$ and can be considered as a fair division solution for the m-sided market. However, different to the two-sided case, it may not coincide with the τ-value of the initial m-sided market

[6] The nucleolus of a coalitional game (N,v) is the payoff vector $\eta(v) \in \mathbb{R}^N$ that lexicographically minimizes the vector of decreasingly ordered excesses of coalitions among all possible imputations (Schmeidler, 1969). An imputation for the game (N,v) is a payoff vector $x \in \mathbb{R}^N$ that satisfies $\sum_{i\in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$. The excess of coalition $S \subseteq N$ at $x \in \mathbb{R}^N$ is $v(S) - \sum_{i\in S} x_i$.

[7] The fair-division point of a two-sided assignment market is the midpoint of the buyers-optimal and the sellers-optimal core allocations Thompson (1981), and it coincides with the τ-value of the corresponding assignment game (Núñez and Rafels, 2002).

$\gamma = (N_1, N_2, \ldots, N_m; G; \{A^{\{r,s\}}\}_{\{r,s\}\in\overline{G}})$. In fact, the τ-value of a multi-sided assignment market on an m-partite graph may lie outside the core (see Example 4.12 in the Appendix 4.7), even when the quotient graph $\overline{G}$ is cycle-free.

4.5 Core and competitive prices in a market network

The aim of this section is to extend to multi-sided assignment games on an m-partite graph the equivalence between core and competitive equilibria that Shapley and Shubik (1972) prove for two-sided markets. To introduce prices and payments, we need to assign some roles of buyers and/or sellers to the agents in the network.

Consider now m sectors $N_1, N_2, \ldots, N_m$ connected by a tree $\overline{G}$ and assume that the source is at N_1. Let us denote by $L(\overline{G})$ the set of leaves of this tree, and by N_L the agents in these leaves. Each agent i in a sector $r \neq 1$ offers an object on sale and has a reservation value $c_i \geq 0$ for this object, meaning that he/she will not sell below that value. We denote by c the vector of sellers' reservation values. At the same time, each agent $i \in N_r$, with $r \notin L(\overline{G})$, is willing to buy one object from each sector $s > r$ such that $\{s,r\} \in \overline{G}$. Assume this agent $i \in N_r$ places a value of $w^i_j \geq 0$ on the object of agent $j \in N_s$ with $s > r$ and $\{r,s\} \in \overline{G}$, and we denote w^i the vectors of buyer i's valuations and by w the vector of all buyers' valuations. Notice that each agent can sell at most one object and buy several objects but at most one from the same sector.

This situation may represent a market network in which each agent at an intermediate sector acts independently both as a buyer in the downstream (higher labels) direction and as a seller in the upstream direction, and pulls together the payoffs obtained in both transactions. We assume all these transactions are independent, that is, an agent can sell an item even if he/she is unmatched in the markets where he/she acts as a buyer. That is, the basic coalitions are, as before, those coalitions connected by G and with no two agents belonging to the same sector. Recall we do not require all sectors to be present in each basic coalition.

These valuations (w,c) give rise to a multi-sided assignment market $\gamma = (N_1, N_2, ..., N_m; G; \{A^{r,s}\}_{\{r,s\}\in\overline{G}})$ on an m-partite graph G with a tree quotient graph $\overline{G}$, where for all $r,s \in \{1,2,\ldots,m\}$, with $\{r,s\} \in \overline{G}$, $a^{\{r,s\}}_{ij} = \max\{0, w^i_j - c_j\}$ for all $i \in N_r$ and $j \in N_s$. We will then simply denote the market by $\gamma = (N_1, N_2, ..., N_m; G; w, c)$. Then, the value of a basic coalition E is

$$v^{w,c}(E) = \sum_{\substack{i\in E\cap N_r, j\in E\cap N_s \\ r<s, \{r,s\}\in\overline{G}}} \max\{0, w^i_j - c_j\},$$

and from this valuation function the coalitional game (N, w_γ) is defined as in (4.2).

Let us denote by $\mathcal{B}_k$ those basic coalitions containing buyer $k \in N_r$, for some $r \in \{1,2,\ldots,m\}$, and only sellers in sectors that immediatly follow r. We refer to these coalitions as *k-basic coalitions*. That is,

$$\mathcal{B}_k = \{E \in \mathcal{B}^N \mid k \in E \cap N_r \text{ and } (E \setminus \{k\}) \cap N_t = \emptyset, \text{ for all } t \in \{1,2,\ldots,m\} \setminus \mathcal{I}_r^{\overline{G}}\}.$$

Recall that $\mathcal{I}_r^{\overline{G}}$ is the set of the immediate followers of r

$$\mathcal{I}_r^{\overline{G}} = \{s \in \{1,2,\ldots,m\} \mid s > r, \{r,s\} \in \overline{G}\}.$$

We want to show that each core allocation is obtained as the result of trading at competitive prices. Therefore, we need to introduce some previous definitions in order to define the notion of competitive price vector.

Given a multi-sided assignment market $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ where $\overline{G}$ is a tree with source at N_1, a *feasible price vector* is $p \in \mathbb{R}_+^{N_2 \cup N_3 \cup \ldots \cup N_m}$ such that $p_j \geq c_j$ for all $j \in \bigcup_{l=2}^{m} N_l$. Next, for each feasible price vector $p \in \mathbb{R}_+^{N_2 \cup N_3 \cup \ldots \cup N_m}$ we introduce the *demand set* of each buyer $k \in N_r$, with $r \in \{1,2,\ldots,m\} \setminus L(\overline{G})$.

Definition 4.8. Let $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ be a multi-sided assignment market where $\overline{G}$ is a tree with source at N_1. The *demand set* of a buyer $k \in N_r$, $r \in \{1,2,\ldots,m\} \setminus L(\overline{G})$, at a feasible price vector $p \in \mathbb{R}_+^{N_2 \cup N_3 \cup \ldots \cup N_m}$ is

$$\begin{aligned} D_k(p) = \{E \in \mathcal{B}_k \mid & w^k(E \setminus \{k\}) - p(E \setminus \{k\}) \\ & \geq w^k(E' \setminus \{k\}) - p(E' \setminus \{k\}), \forall E' \in \mathcal{B}_k\}, \end{aligned} \tag{4.13}$$

where for all coalition T of sellers, $w^k(T) = \sum_{j \in T} w_j^k$ and $p(T) = \sum_{j \in T} p_j$.

Note that $D_k(p)$ describes the set of k-basic coalitions that maximize the net valuation of buyer k at prices p. Notice also that the demand set of a buyer $k \in N_r$, for some $r \in \{1,2,\ldots,m\} \setminus L(\overline{G})$, is always non-empty since k can always demand $E = \{k\}$ with a net profit of 0. If $\mu \in \mathcal{M}(N_1, N_2, ..., N_m)$, for all $k \in N \setminus N_L$ we write $\mu(k)$ to denote the k-basic coalition E such that $k \in E \subseteq E' \in \mu$, that is, $\mu(k) = \{E \in \mathcal{B}_k \mid \text{ there exists } E' \in \mu \text{ such that } E \subseteq E'\}$. Notice that $\mu(k)$ may consist of only agent k, that meaning that k is not matched to any of his/her immediately follower sellers.

Given a matching $\mu \in \mathcal{M}(N_1, N_2, ..., N_m)$, we say a seller $j \in \bigcup_{l=2}^{m} N_l$ is *unassigned* (by μ) if there is no $k \in N_r$ for some $r \in \{1,2,\ldots,m\} \setminus L(\overline{G})$ such that $j \in \mu(k)$.

Now, we can introduce the notion of *competitive equilibrium* for the market γ on an m-partite graph where $\overline{G}$ is a tree.

Definition 4.9. Let $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ be a multi-sided assignment market on an m-partite graph, where $\overline{G}$ is a tree with source at N_1. A pair (p, μ), where $p \in \mathbb{R}_+^{N_2 \cup N_3 \cup ... \cup N_m}$ is a feasible price vector and $\mu \in \mathcal{M}(N_1, N_2, ..., N_m)$, is a *competitive equilibrium* if

(i) for all buyer $k \in N_r$, $r \in \{1, 2, \ldots, m\} \setminus L(\overline{G})$, it holds $\mu(k) \in D_k(p)$,

(ii) for all seller $j \in \bigcup_{l=2}^{m} N_l$, if j is unassigned by μ, then $p_j = c_j$.

If a pair (p, μ) is a competitive equilibrium, then we say that the price vector p is a *competitive equilibrium price vector*. The corresponding payoff vector for a given pair (p, μ) is called *competitive equilibrium payoff vector*. This payoff vector is $(u^1(p, \mu), u^2(p, \mu), ..., u^m(p, \mu)) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times ... \times \mathbb{R}^{N_m}$, defined by

$$\begin{array}{rcl} u_k^1(p,\mu) & = & w^k(\mu(k) \setminus \{k\}) - p(\mu(k) \setminus \{k\}) \quad \text{for all } k \in N_1, \\ u_k^r(p,\mu) & = & w^k(\mu(k) \setminus \{k\}) - p(\mu(k) \setminus \{k\}) + p_k - c_k \\ & & \text{for all } k \in N_r,\ r \in \{2, \ldots, m\} \setminus L(\overline{G}), \\ u_k^r(p,\mu) & = & p_k - c_k \quad \text{for all } k \in N_r, r \in L(\overline{G}), \end{array}$$

where for all coalition T of sellers, $w^k(T) = \sum_{j \in T} w_j^k$ and $p(T) = \sum_{j \in T} p_j$. We denote the set of competitive equilibrium payoff vectors of market γ by $\mathcal{CE}(\gamma)$.

We now study the relationship between the core of a multi-sided assignment market on an m-partite graph $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ where $\overline{G}$ is a tree, and the set of competitive equilibrium payoff vectors. First, we show that if a matching μ constitutes a competitive equilibrium with a feasible price vector p, then μ is an optimal matching.

Lemma 4.10. *Let $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ be a multi-sided assignment market on an m-partite graph. If $\overline{G}$ is a tree and the pair (p, μ) is a competitive equilibrium, then μ is an optimal matching.*

Proof. Let us assume that $\overline{G}$ has a source at N_1, and hence competitive equilibria are defined as in Definition 4.9. In order to prove the statement, we need to show that if (p, μ) is a competitive equilibrium, then the matching μ is a partition of maximal value. We can assume without loss of generality that for all $E \in \mu$, if $i \in E \cap N_r$, $j \in E \cap N_s$ with $r < s$ and $\{r, s\} \in \overline{G}$, then it holds $w_j^i - c_j \geq 0$ and hence $v^{w,c}(E) = \sum_{\substack{i \in E \cap N_r, j \in E \cap N_s \\ r < s, \{r,s\} \in \overline{G}}} w_j^i - c_j$, since otherwise E could be partitioned in basic coalitions satisfying the above condition to obtain another matching that gives rise to the same value. Consider now another matching $\mu' \in \mathcal{M}(N_1, N_2, ..., N_m)$.

Then,

$$\begin{aligned}
\sum_{E\in\mu} v^{w,c}(E) &= \sum_{k\in N\setminus N_L} w^k(\mu(k)\setminus\{k\}) - c(\mu(k)\setminus\{k\}) \\
&\geq \sum_{k\in N\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c(\mu(k)\setminus\{k\}) - p(\mu'(k)\setminus\{k\}) + p(\mu(k)\setminus\{k\}) \\
&= \sum_{k\in N\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c(\mu(k)\setminus\{k\}) \\
&- p\left(\bigcup_{k\in N\setminus N_L} \mu'(k)\setminus N_1\right) \\
&+ p\left(\bigcup_{k\in N\setminus N_L} \mu(k)\setminus N_1\right) \\
&= \sum_{k\in N\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c\left(\bigcup_{k\in N\setminus N_L} \mu(k)\setminus N_1\right) \\
&- p\left(\left(\bigcup_{k\in N\setminus N_L} \mu'(k)\setminus \bigcup_{k\in N\setminus N_L} \mu(k)\right)\setminus N_1\right) \\
&+ p\left(\left(\bigcup_{k\in N\setminus N_L} \mu(k)\setminus \bigcup_{k\in N\setminus N_L} \mu'(k)\right)\setminus N_1\right) \\
&= \sum_{k\in N\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c\left(\bigcup_{k\in N\setminus N_L} \mu(k)\setminus N_1\right) \\
&- c\left(\left(\bigcup_{k\in N\setminus N_L} \mu'(k)\setminus \bigcup_{k\in N\setminus N_L} \mu(k)\right)\setminus N_1\right) \\
&+ p\left(\left(\bigcup_{k\in N\setminus N_L} \mu(k)\setminus \bigcup_{k\in N\setminus N_L} \mu'(k)\right)\setminus N_1\right) \\
&= \sum_{k\in N\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c\left(\bigcup_{k\in N\setminus N_L} \mu'(k)\setminus N_1\right) \\
&- c\left(\left(\bigcup_{k\in N\setminus N_L} \mu(k)\setminus \bigcup_{k\in N\setminus N_L} \mu'(k)\right)\setminus N_1\right) \\
&+ p\left(\left(\bigcup_{k\in N\setminus N_L} \mu(k)\setminus \bigcup_{k\in N\setminus N_L} \mu'(k)\right)\setminus N_1\right) \\
&\geq \sum_{k\in\setminus N_L} w^k(\mu'(k)\setminus\{k\}) - c(\mu'(k)\setminus\{k\}) = \sum_{E\in\mu'} v^{w,c}(E),
\end{aligned}$$

where the first inequality follows from the definition of the demand set and the fact that (p,μ) is a competitive equilibrium:

$$w^k(\mu(k)\setminus\{k\}) \geq w^k(\mu'(k)\setminus\{k\}) - p(\mu'(k)\setminus\{k\}) + p(\mu(k)\setminus\{k\}).$$

The fourth equality follows from the fact that for all

$$j \in \left(\bigcup_{k\in N\setminus N_L} \mu'(k) \setminus \bigcup_{k\in N\setminus N_L} \mu(k) \right) \setminus N_1, \qquad p_j = c_j,$$

and the last inequality follows from the feasibility of the price vector p. □

Now, we can give the main result in this section.

Theorem 4.11. *Let $\gamma = (N_1, N_2, ..., N_m; G; w, c)$ be a multi-sided assignment market on an m-partite graph, where $\overline{G}$ is a tree. The core of the market, $C(w_\gamma)$, coincides with the set of competitive equilibrium payoff vectors, $\mathcal{CE}(\gamma)$.*

Proof. Assume that $\overline{G}$ has a source at N_1, and hence competitive equilibria are defined as in Definition 4.9. First, we show the implication that states if (p,μ) is a competitive equilibrium, then its corresponding competitive equilibrium payoff vector $x = (u^1(p,\mu), u^2(p,\mu), ..., u^m(p,\mu)) \in \mathcal{CE}(\gamma)$ is a core element. As in the proof of Lemma 4.10, we can assume without loss of generality that $v^{w,c}(E) = \sum_{\substack{i\in E\cap N_r, j\in E\cap N_s\\ r<s,\{r,s\}\in\overline{G}}} (w^i_j - c_j)$.

Recall that by definition $u^r_k(p,\mu) = w^k(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\}) + p_k - c_k$ for all $k \in N_r$ for $r \in \{1,2,\ldots,m\}\setminus L(\overline{G})$. Let us check that for all basic coalitions $E \in \mathcal{B}^N$ it holds $x(E) \geq v^{w,c}(E)$. Notice that if E only contains one agent, then $v^{w,c}(E) = 0$ and hence the core inequality holds. Otherwise, take $E \in \mathcal{B}^N$ such that $k \in E$ for some $k \in N_r$ with $r \in \{1,2,\ldots,m\}\setminus L(\overline{G})$. For each $k \in E\cap N_r$ with $r \in \{1,2,\ldots,m\}\setminus L(\overline{G})$, denote by E_k the union of $\{k\}$ with the set of $j \in E\cap N_s$ for some $r < s$ with $\{r,s\} \in \overline{G}$. Notice that E_k is formed by agent k and those of his immediate followers that belong to E. Then,

$$\begin{aligned}
x(E) &= p(E\setminus N_1) - c(E\setminus N_1) + \sum_{r=1}^{m} \sum_{\substack{k\in E\cap N_r\\ r\notin L(\overline{G})}} w^k(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\}) \\
&\geq p(E\setminus N_1) - c(E\setminus N_1) + \sum_{r=1}^{m} \sum_{\substack{k\in E\cap N_r\\ r\notin L(\overline{G})}} w^k(E_k\setminus\{k\}) - p(E_k\setminus\{k\}) \\
&= \sum_{\substack{k\in E\cap N_r, j\in E\cap N_s\\ r<s,\{r,s\}\in\overline{G}}} w^k_j - c_j = v^{w,c}(E),
\end{aligned}$$

where the inequality follows from the fact that (p,μ) is a competitive equilibrium. It remains to check that x is efficient. Since μ is a partition of $N = N_1 \cup N_2 \cup \ldots \cup N_m$, we get

$$\begin{aligned}
x(N) &= \sum_{r=1}^{m} \sum_{\substack{k\in N_r \\ r\notin L(\overline{G})}} \Big[w^k(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\})\Big] + p(N\setminus N_1) - c(N\setminus N_1) \\
&= \sum_{r=1}^{m} \sum_{\substack{k\in N_r \\ r\notin L(\overline{G})}} \Big[w^k(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\}) + p(\mu(k)\setminus\{k\}) - c(\mu(k)\setminus\{k\})\Big] \\
&+ \sum_{s=2}^{m} \sum_{\substack{q\in N_s, q\notin\mu(k) \\ \forall k\in N\setminus N_L}} (p_q - c_q) \\
&= \sum_{r=1}^{m} \sum_{\substack{k\in N_r \\ r\notin L(\overline{G})}} \Big[w^k(\mu(k)\setminus\{k\}) - c(\mu(k)\setminus\{k\})\Big] \\
&= \sum_{r=1}^{m} \sum_{\substack{k\in N_r \\ r\notin L(\overline{G})}} v^{w,c}(\mu(k)\setminus\{k\}) = \sum_{E\in\mu} v^{w,c}(E),
\end{aligned}$$

where the third equality holds since $p_q = c_q$ for unassigned sellers q.

We have shown that if (p,μ) is a competitive equilibrium, then its competitive equilibrium payoff vector x is a core allocation.

Next, we show that the reverse implication holds. That is, if $x \in \mathbb{R}^N$ is a core allocation, then it is the payoff vector related to a competitive equilibrium (p,μ), where μ is any optimal matching and p is a competitive equilibrium price vector. Recall from Theorem 4.5 that $x = \bigoplus_{\{r,s\}\in\overline{G}} x^{\{r,s\}}$, where $x^{\{r,s\}} \in C(w_{A^{\{r,s\}}})$.

For all $s \in \{2,\ldots,m\}$ and all $j \in N_s$, define $p_j = x_j^{\{r,s\}} + c_j$, where r is the unique sector in $\{1,2,\ldots,m\}$ such that $\{r,s\} \in \overline{G}$. Notice first that since $x \in C(w_{A^{\{r,s\}}})$, if an object $j \in N_s$ is not assigned by the matching μ to any $k \in N_r$, then $p_j = x_j + c_j = c_j$. Moreover, $x^{\{r,s\}}(\mu(k)) = v^{w,c}(\mu(k))$ for all $k \in N \setminus N_L$ and $x^{\{r,s\}}(E') \geq v^{w,c}(E')$ for all $E' \in \mathcal{B}_k$ where $k \in N \setminus N_L$.

Then, for all $k \in N \setminus N_L$ and all $E' \in \mathcal{B}_k$,

$$\begin{aligned}
w^k(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\}) &= v^{w,c}(\mu(k)) + c(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\}) \\
&= x^{\{r,s\}}(\mu(k)) + c(\mu(k)\setminus\{k\}) - p(\mu(k)\setminus\{k\}) \\
&= x_k^{\{r,s\}} \\
&\geq v^{w,c}(E') - x^{\{r,s\}}(E'\setminus\{k\}) \\
&= v^{w,c}(E') - \big[p(E'\setminus\{k\}) - c(E'\setminus\{k\})\big] \\
&= w^k(E'\setminus\{k\}) - p(E'\setminus\{k\})
\end{aligned}$$

where the inequality follows from the fact that $x \in C(w_{A\{r,s\}})$. This shows that $\mu(k) \in D_k(p)$ which concludes the proof. $\square$

Once shown that the set of competitive equilibrium payoff vectors of a multi-sided assignment market on a cycle-free quotient graph $\overline{G}$, $\mathcal{CE}(\gamma)$, coincides with the core of the market, $C(w_\gamma)$, we have that a competitive equilibrium always exists for this model, since we already know that the core is non-empty.

4.6 A concluding remark

We have considered multi-sided markets where agents are on an m-partite graph that induces a cycle-free network among the sectors. Basic coalitions do not need to have agents from all sectors. As in the previous chapter, it is enough not to have two agents from the same sector. Moreover, the worth of a basic coalition is the addition of the worths of all its pairs that are an edge of the m-partite graph.

A similar situation is considered in Stuart (1997), although restricted to the case in which the network that connects the sectors is a chain. There, the worth of a basic coalition is also defined additively, but, as in the classical multi-sided assignment games in Kaneko and Wooders (1982) and Quint (1991), the set of basic coalitions is smaller since it is required that a basic coalition contains exactly one agent of each side. Although the core of Stuart's multi-sided game is also non-empty, it does not contain the composition of all core elements of the underlying two sided markets.

Indeed, take $N_1 = \{1,2,3\}$, $N_2 = \{1',2',3'\}$ and $N_3 = \{1'',2''\}$, and consider the chain $\overline{G} = \{\{N_1,N_2\},\{N_2,N_3\}\}$. Assume also that $a_{ij}^{\{r,s\}} = 1$ for all $(i,j) \in N_r \times N_s$ such that $\{r,s\} \in \overline{G}$, but, unlike the model we present in this chapter, only triplets may have a positive value. It is easy to see that $(0.5,0.5,0.5;0.5,0.5,0.5) \in C(w_{A\{1,2\}})$ and $(0,0,0;1,1) \in C(w_{A\{2,3\}})$. However,

$$z = x \oplus y = (0.5,0.5,0.5;0.5,0.5,0.5;1,1) \notin C(w_\gamma),$$

since an optimal matching consists of two triplets and hence the unassigned agents in sectors N_1 and N_2 can only receive zero payoff in the core. Hence, our generalized multi-sided markets, together with the cycle-free network structure, where the set of basic coalitions has been enlarged, better inherits some properties of the core of the well-known two-sided markets.

4.7 Appendix

We consign to this appendix two examples that show that for a multi-sided assignment game on a cycle-free quotient graph, the composition of the τ-values (or the nucleolus) of each underlying two-sided market may not coincide with the τ-value or the nucleolus of the initial multi-sided market. Similarly, the third example shows that by composition of arbitrary extreme core allocations of each two-sided market we may not obtain an extreme core allocation of the muti-sided market.

Example 4.12. Let us consider an assignment market γ on a 3-partite graph such that the quotient graph is $\overline{G} = \{\{1,2\},\{2,3\}\}$ which is cycle-free. The sectors are $N_1 = \{1,2\}$, $N_2 = \{1',2'\}$, and $N_3 = \{1'',2''\}$. The valuation matrices of the two underlying two-sided markets are

$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ 1' \\ 2' \end{array} \begin{array}{c} \begin{array}{cc} 1'' & 2'' \end{array} \\ \begin{pmatrix} 3 & 4 \\ 0 & 3 \end{pmatrix} \end{array},$$

and the value of triplets is given by the following three-dimensional matrix

$$\begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 5 & 0 \\ 8 & 4 \end{pmatrix} \\ 1'' \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 6 & 3 \\ 9 & 7 \end{pmatrix} \\ 2'' \end{array}.$$

The τ-value of this multi-sided market game is $\tau(\gamma) = (\frac{5}{9}, \frac{24}{9}; \frac{29}{9}, \frac{15}{9}; \frac{15}{9}, \frac{20}{9})$ which is not in the core. Hence, $\tau(\gamma)$ cannot coincide with $\tau(w_{A^{\{1,2\}}}) \oplus \tau(w_{A^{\{2,3\}}})$.

Example 4.13. Let us consider an assignment market γ on the following 4-partite graph related to the the quotient graph $\overline{G} = \{\{1,2\},\{2,3\},\{2,4\}\}$ which is cycle free. The sectors are $N_1 = \{1,2\}$, $N_2 = \{1',2'\}$, $N_3 = \{1'',2''\}$, $N_4 = \{1''',2'''\}$, and the valuation matrices of the two-sided markets are

$$A^{\{1,2\}} = \begin{pmatrix} \mathbf{2} & 3 \\ 0.5 & \mathbf{2} \end{pmatrix}, \quad A^{\{2,3\}} = \begin{pmatrix} \mathbf{3} & 0.8 \\ 4 & \mathbf{2} \end{pmatrix} \text{ and } A^{\{2,4\}} = \begin{pmatrix} \mathbf{2} & 0.6 \\ 2.4 & \mathbf{2} \end{pmatrix}.$$

The nucleolus of the three underlying two-sided markets are

$$\eta^{\{1,2\}} = (1.625, 0.375; 0.375, 1.625), \quad \eta^{\{2,3\}} = (0.45, 1.55; 2.55, 0.45)$$
$$\text{and } \eta^{\{2,4\}} = (0.55, 1.45; 1.45, 0.55)$$

and their composition is

$$\eta^{\oplus} = (1.625, 0.375; 1.375, 4.625; 2.55, 0.45; 1.45, 0.55),$$

while the nucleolus of the six-player game (N, w_γ) can be computed and is

$$\eta = (1.65, 0.4; 1.6, 4.75; 2.55, 0.45; 1.2, 0.4).$$

Example 4.14. Let us consider an assignment market γ on a 4-partite graph related to the quotient graph $\overline{G} = \{\{1,2\},\{2,3\},\{2,4\}\}$ which is cycle-free. The sectors are $N_1 = \{1,2\}$, $N_2 = \{1',2'\}$, $N_3 = \{1'',2''\}$, and $N_4 = \{1''',2'''\}$. The valuation matrices of the three underlying two-sided markets are

$$A^{\{1,2\}} = \begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} \mathbf{2} & 0 \\ 1 & \mathbf{2} \end{pmatrix} \end{array} \qquad A^{\{2,3\}} = \begin{array}{c} \\ 1' \\ 2' \end{array}\begin{array}{c} \begin{array}{cc} 1'' & 2'' \end{array} \\ \begin{pmatrix} \mathbf{2} & 1 \\ 0 & \mathbf{2} \end{pmatrix} \end{array} \qquad A^{\{2,4\}} = \begin{array}{c} \\ 1' \\ 2' \end{array}\begin{array}{c} \begin{array}{cc} 1''' & 2''' \end{array} \\ \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \end{array}.$$

Take respective extreme core allocations of the three underlying two-sided markets $A^{\{1,2\}}$, $A^{\{2,3\}}$, and $A^{\{2,4\}}$: $(2,1;0,1)$, $(2,0;0,2)$, and $(1,0;0,1)$. Then, by composition we get a core allocation for the multi-sided assignment market, $x^{\oplus} = (2,1;3,1;0,2;0,1) \in C(w_\gamma)$. But, there exist two core elements

$$y = (1.8, 0.8; 3.2, 1.2; 0, 2; 0, 1) \in C(w_\gamma)$$

and

$$z = (2.2, 1.2; 2.8, 0.8; 0, 2; 0, 1) \in C(w_\gamma)$$

such that $x^{\oplus} = \frac{1}{2}y + \frac{1}{2}z$. Hence, $x^{\oplus} \notin Ext(C(w_\gamma))$.

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5 An alternative proof of the characterization of core stability for two-sided assignment games[§]

5.1 Introduction and preliminaries

A *two-sided assignment market* $(M, M'; A)$ consists of two different sectors, let us say a finite set of buyers M and a finite set of sellers M' (M and M' disjoint) and a non-negative valuation matrix $A = (a_{ij})_{\substack{i \in M \\ j \in M'}}$ that represents the potential joint profit obtained by each mixed-pair $(i, j) \in M \times M'$. As in Solymosi and Raghavan (2001) and Núñez and Rafels (2002), we assume that the assignment market is *square*, that is $|M| = |M'|$.

A *matching* μ between M and M' is a subset of the Cartesian product, $M \times M'$, such that each agent belongs, at most, to one pair. The set of all possible matchings is denoted by $\mathcal{M}(M, M')$. A matching $\mu \in \mathcal{M}(M, M')$ is *optimal* for the market (M, M', A) if $\sum_{(i,j) \in \mu} a_{ij} \geq \sum_{(i,j) \in \mu'} a_{ij}$ for all $\mu' \in \mathcal{M}(M, M')$. We denote by $\mathcal{M}_A(M, M')$ the set of all optimal matchings for the market (M, M', A). The corresponding *assignment game* $(M \cup M', w_A)$ has a player set $M \cup M'$ and a characteristic function $w_A(S \cup T) = \max_{\mu \in \mathcal{M}(S,T)} \sum_{(i,j) \in \mu} a_{ij}$ for all $S \subseteq M$ and $T \subseteq M'$.

Without loss of generality, we assume throughout the paper that the main diagonal corresponds to an optimal matching, $\mu = \{(i, i) \mid i \in M\}$. We use "$j$" to denote both the j^{th} buyer and the j^{th} seller, since the distinction is always clear from the context.

Once a matching between buyers and sellers that maximizes the total profit in the market has been chosen, we need to determine how this profit can be allocated among the agents. Given an assignment game $(M \cup M', w_A)$, an *allocation* is a payoff vector $(u; v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$, where u_l denotes the payoff to buyer $l \in M$ and v_l denotes the payoff to seller $l \in M'$. An *imputation* is a payoff vector that is efficient,

[§]A working paper based on this chapter is under second revision at *Operations Research Letters*.

$\sum_{i \in M} u_i + \sum_{j \in M'} v_j = w_A(M \cup M')$ and individually rational, $u_i \geq w_A(\{i\}) = 0$ for all $i \in M$ and $v_j \geq w_A(\{j\}) = 0$ for all $j \in M'$. We denote the set of imputations of an assignment game $(M \cup M', w_A)$ by $I(w_A)$.

We define the *principal section* of $(M \cup M', w_A)$ as the set of imputations such that $u_i + v_i = a_{ii}$ for all $i \in M$. We denote it by $B(w_A)$. In the principal section, the only side-payments that take place are those between matched agents. Among the outstanding allocations that we use later are the *sector-optimal allocations*. These are $(a;0) \in \mathbb{R}^M \times \mathbb{R}^{M'}$, defined by $a_k = a_{kk}$ for $k \in M$ and $a_k = 0$ for $k \in M'$, and analogously $(0;a) \in \mathbb{R}^M \times \mathbb{R}^{M'}$.

A binary relation, known as *domination*, is defined on the set of imputations. Given two imputations $(u;v)$ and $(u';v')$, we say that $(u;v)$ *dominates* $(u';v')$ if and only if there exists $(i,j) \in M \times M'$ such that $u_i > u'_i$, $v_j > v'_j$ and $u_i + v_j \leq a_{ij}$. We then write $(u;v)\ \mathrm{dom}\,^A_{\{i,j\}}(u';v')$, and $(u;v)\ \mathrm{dom}^A(u';v')$ to denote that $(u;v)$ dominates $(u';v')$ by means of some pair $(i,j) \in M \times M'$. In assignment games, this dominance relation that only makes use of mixed-pair allocations is equivalent to the usual dominance relation of von Neumann and Morgenstern (1944).

The first solution concept that appears in the literature is the notion of stable set. A subset V of the set of imputations $I(w_A)$ is a *von Neumann-Morgenstern stable set* (a stable set) if it satisfies *internal stability*, that is, for any $(u;v),(u';v') \in V$, $(u;v)\ \mathrm{dom}^A(u';v')$ does not hold, and *external stability*, that is, for any $(u';v') \in I(w_A) \setminus V$, there exists $(u;v) \in V$ such that $(u;v)\ \mathrm{dom}^A(u';v')$.

The *core* $C(w_A)$ is another solution concept that can also be defined, see Gillies (1959), by means of the dominance relation: it is the set of undominated imputations.

Equivalently, an imputation $(u;v) \in I(w_A)$ belongs to the core of the assignment game $C(w_A)$ if for all $(i,j) \in M \times M'$ it holds $u_i + v_j \geq a_{ij}$. It is shown in Shapley and Shubik (1972) that an assignment game $(M \cup M', w_A)$ always has a non-empty core. Notice that the core always satisfies internal stability but may fail to satisfy external stability. This raises the question as to which valuation matrices correspond to assignment games with an externally stable (and hence stable) core.

Solymosi and Raghavan (2001) introduces the *dominant diagonal* property for valuation matrices. A square valuation matrix A has a dominant diagonal if all diagonal elements are row and column maxima: $a_{ii} \geq \max\{a_{ij}, a_{ji}\}$ for all $(i,j) \in M \times M'$. Hence, an optimal matching is placed on the main diagonal. It is straightforward to see that a valuation matrix A has a dominant diagonal if and only if the sector-optimal allocations $(a;0)$ and $(0;a)$ belong to the core. It is then proved in Solymosi and Raghavan (2001) that *"the core of a square assignment game $(M \cup M', w_A)$ is a von Neumann-Morgenstern stable set if and only if the valu-*

ation matrix A has a dominant diagonal". The authors' proof is based on graph-theoretical arguments while here we base ours on the properties of the buyer-seller exact representative of an assignment game proposed in Núñez and Rafels (2002).

Given any assignment game $(M \cup M', w_A)$, there exists a unique valuation matrix A^r such that $C(w_A) = C(w_{A^r})$ and A^r is the maximum with this property. That is, if any entry in A^r is raised, the resulting game has a different core. As a consequence, if the matrix A^r is the buyer-seller exact representative of A, then for all $(i,j) \in M \times M'$ there exists $(u,v) \in C(w_{A^r})$ such that $u_i + v_j = a^r_{ij}$. Notice that for each $(i,j) \in M \times M'$, a^r_{ij} is the lower bound for the joint payoff of agents $i \in M$ and $j \in M'$ in the core.

Based on Núñez and Rafels (2002), we are now able to offer a proof of the characterization of core stability for assignment games alternative to that provided in Solymosi and Raghavan (2001). The advantage of this new approach is that it relies solely on the structure of the assignment game, above all, on the known bounds for the payoff to each mixed-pair in the core. For this reason, it might be possible to apply these ideas to the characterization of core stability for markets with more than two sectors, which, to the best of our knowledge, remains an open question.

5.2 Core stability

In this section, we provide the main result of this paper, an alternative proof of the characterization of core stability for the two-sided assignment game.

To do so, we first adapt a lemma provided by Shapley without a proof in his unpublished notes for the stable sets of the assignment game. Assuming that the valuation matrix has a dominant diagonal, we prove that through each core allocation there is a continuous monotonic curve parameterized by τ that is included in the core and connects the two sector-optimal allocations $(a;0)$ and $(0;a)$. The payoff to any agent in this curve, for a given value of the parameter τ, is computed as the median of three terms.

Lemma 5.1. *Let $(M \cup M', w_A)$ be a square two-sided assignment game such that its valuation matrix A has a dominant diagonal. Given any vector belonging to the core of the game, $(u;v) \in C(w_A)$, and any $\tau \in \mathbb{R}$, the vector $(u(\tau);v(\tau))$ defined by*

$$\begin{aligned} u_i(\tau) &= med\,\{0, u_i - \tau, a_{ii}\} \qquad \text{for all } i \in M, \\ v_i(\tau) &= med\,\{0, v_i + \tau, a_{ii}\} \qquad \text{for all } i \in M', \end{aligned} \tag{5.1}$$

belongs to $C(w_A)$.

Proof. Note first that for $\tau_1 = \max_{i \in M} a_{ii}$, $(u(\tau_1); v(\tau_1)) = (0; a)$ and for $\tau_2 = -\tau_1$, $(u(\tau_2); v(\tau_2)) = (a; 0)$. Notice that since $(u; v) \in C(w_A)$, we have $u_i + v_i = a_{ii}$ for all $i \in M$ and hence, for all $\tau \in \mathbb{R}$ and all $i \in M$,

$$v_i + \tau = a_{ii} - u_i + \tau = a_{ii} - (u_i - \tau). \tag{5.2}$$

It is then straightforward to show that $u_i(\tau) + v_i(\tau) = a_{ii}$ for all $i \in M$.

Take now $i \neq j$ and consider three different cases to check that $(u(\tau); v(\tau))$ satisfies the core constraints:

1. $\tau < -\min\{v_i, v_j\}$, that is either $u_i(\tau) = a_{ii}$ or $v_j(\tau) = 0$. In the first case, $u_i(\tau) = a_{ii}$, we have $u_i(\tau) + v_j(\tau) \geq u_i(\tau) = a_{ii} \geq a_{ij}$ where the last inequality follows from the dominant diagonal assumption. Otherwise, if $u_i(\tau) < a_{ii}$ and $v_j(\tau) = 0$, then since $v_j(\tau) = \text{med}\{0, v_j + \tau, a_{jj}\}$, $v_j + \tau \leq 0$. This implies $\tau \leq 0$ and also $u_i(\tau) + v_j(\tau) = u_i - \tau \geq u_i + v_j \geq a_{ij}$, where the last inequality follows from (u, v) being in the core.

2. $\tau > \min\{u_i, u_j\}$, that is either $v_j(\tau) = a_{jj}$ or $u_i(\tau) = 0$. If $v_j(\tau) = a_{jj}$ then $u_i(\tau) + v_j(\tau) \geq a_{jj} \geq a_{ij}$ because of the dominant diagonal assumption. If $v_j(\tau) < a_{jj}$ but $u_i(\tau) = 0$, since $u_i(\tau) = \text{med}\{0, u_i - \tau, a_{ii}\}$, we have $u_i - \tau \leq 0$. Then, $\tau \geq 0$ and hence $u_i(\tau) + v_j(\tau) = v_j(\tau) = v_j + \tau \geq v_j + u_i \geq a_{ij}$, where the last inequality follows from (u, v) being in the core.

3. $-\min\{v_i, v_j\} \leq \tau \leq \min\{u_i, u_j\}$. This implies $u_i(\tau) = u_i - \tau$ and $v_j(\tau) = v_j + \tau$ and hence, again from (u, v) being in the core, $u_i(\tau) + v_j(\tau) = u_i + v_j \geq a_{ij}$.

□

Next, to show that the core of a square two-sided assignment game is a von Neumann-Morgenstern stable set if and only if its valuation matrix has a dominant diagonal, we need to prove the following lemma that states a property of the principal section.

Lemma 5.2. *Let $(M \cup M', w_A)$ be a square two-sided assignment game with an optimal matching on the main diagonal. Given $(x; y) \in B(w_A) \setminus C(w_A)$, there exists a pair $(i, j) \in M \times M'$ and a core allocation $(u; v) \in C(w_A)$ such that $x_i + y_j < a_{ij} = u_i + v_j$.*

Proof. From Núñez and Rafels (2002), for any assignment game $(M \cup M', w_A)$ there exists another assignment game $(M \cup M', w_{A^r})$ with the same core, $C(w_A) = C(w_{A^r})$, and A^r maximum with this property. Hence, if $(x; y) \notin C(w_A)$, then

$(x;y) \notin C(w_{A^r})$. This means $x_i + y_j < a^r_{ij}$ for some $(i,j) \in M \times M'$ and there exists a core allocation $(u;v)$ such that

$$x_i + y_j < a^r_{ij} = u_i + v_j. \tag{5.3}$$

If $a^r_{ij} = a_{ij}$, the lemma is proved. Otherwise, by the definition of A^r, see page 428 in Núñez and Rafels (2002), $a^r_{ij} = a_{ik_1} + a_{k_1k_2} + a_{k_2k_3} + \ldots + a_{k_rj} - a_{k_1k_1} - \ldots - a_{k_rk_r}$ for some $k_1, \ldots, k_r \in M \setminus \{i,j\}$ and different.

Since $(u;v)$ is a core allocation and the main diagonal is an optimal matching,

$$\begin{aligned} u_i + v_j &= a_{ik_1} + a_{k_1k_2} + \ldots + a_{k_rj} - a_{k_1k_1} - \ldots - a_{k_rk_r} \\ &= a_{ik_1} + a_{k_1k_2} + \ldots + a_{k_rj} - (u_{k_1} + v_{k_1}) - \ldots - (u_{k_r} + v_{k_r}). \end{aligned} \tag{5.4}$$

By rearranging (5.4) we obtain

$$u_i + v_{k_1} + u_{k_1} + v_{k_2} + \ldots + u_{k_r} + v_j = a_{ik_1} + \ldots + a_{k_rj}. \tag{5.5}$$

From $(u;v) \in C(w_A)$, and (5.5), we obtain $u_{l_1} + v_{l_2} = a_{l_1l_2}$ for all $(l_1,l_2) \in \{(i,k_1),(k_1,k_2),\ldots,(k_{r-1},k_r),(k_r,j)\}$.

Since $(x;y) \in B(w_A)$, we know $x_t + y_t = a_{tt} = u_t + v_t$ for all $t \in \{k_1,k_2,\ldots,k_r\}$. Now,

$$\begin{aligned} x_i + y_{k_1} + x_{k_1} + y_{k_2} + \ldots + x_{k_r} + y_j &= x_i + y_j + \sum_{l=1}^{r} x_{k_l} + y_{k_l} \\ &< u_i + v_j + \sum_{l=1}^{r} u_{k_l} + v_{k_l} \\ &= u_i + v_{k_1} + u_{k_1} + v_{k_2} + \ldots + u_{k_r} + v_j \\ &= a_{ik_1} + \ldots + a_{k_rj}, \end{aligned}$$

where the inequality follows from (5.3) and the last equality follows from (5.5).

Then, $x_i + y_{k_1} + x_{k_1} + y_{k_2} + \ldots + x_{k_r} + y_j < a_{ik_1} + a_{k_1k_2} + \ldots + a_{k_rj}$ means that either $x_i + y_{k_1} < a_{ik_1} = u_i + v_{k_1}$ or $x_{k_l} + y_{k_{l+1}} < a_{k_lk_{l+1}} = u_{k_l} + v_{k_{l+1}}$ for some $l \in \{1,\ldots,r-1\}$ or $x_{k_r} + y_j < a_{k_rj} = u_{k_r} + v_j$, which concludes the proof of the lemma. □

We can now state and prove the main result.

Theorem 5.3. *Let $(M \cup M', w_A)$ be a square assignment game with an optimal matching on the main diagonal. Then the following statements are equivalent:*

(i) A has a dominant diagonal,

(ii) $C(w_A)$ *is a von Neumann-Morgenstern stable set.*

Proof. We first consider $(i) \Rightarrow (ii)$. Recall that the core of a game is always internally stable. The fact that every allocation outside the principal section is dominated by some core allocation is proved in Shapley's notes, but we reproduce the proof for the sake of completeness. Assume $(x;y) \in I(w_A) \setminus B(w_A)$. Since $(x;y) \in I(w_A)$ and $\mu = \{(k,k)|k \in M\}$ is an optimal matching, $\sum_{k\in M} x_k + y_k = \sum_{k\in M} a_{kk}$. Moreover, since $(x;y) \notin B(w_A)$, there is some $i \in M$ such that $x_i + y_i \neq a_{ii}$. We can assume $x_i + y_i < a_{ii}$ since if $x_{i'} + y_{i'} > a_{i'i'}$ for some $i' \in M$, because of $\sum_{k\in M} x_k + y_k = \sum_{k\in M} a_{kk}$, there is $i \in M \setminus \{i'\}$ with $x_i + y_i < a_{ii}$. Thus, $x_i < a_{ii} - y_i$, which implies that there exists $0 \leq x_i < \lambda < a_{ii} - y_i \leq a_{ii}$. By Lemma 5.1, there exists $(u;v) \in C(w_A)$ with $u_i = \lambda$. Then, $u_i > x_i$ and $u_i < a_{ii} - y_i$ which implies $y_i < a_{ii} - u_i = v_i$. Moreover, $x_i + y_i < a_{ii} = u_i + v_i$. Hence, $(u;v)\ \mathrm{dom}^A_{\{i,i\}}(x;y)$.

Assume now that $(x;y) \in B(w_A) \setminus C(w_A)$. We know by Lemma 5.2 that there exists a pair $(i,j) \in M \times M'$ and $(u;v) \in C(w_A)$ such that $x_i + y_j < a_{ij} = u_i + v_j$. Now, assume without loss of generality $u_i > x_i$. If also $v_j > y_j$, we obtain $(u;v)\ \mathrm{dom}^A_{\{i,j\}}(x;y)$.

Otherwise, assume $v_j \leq y_j$. Since both $(x;y)$ and $(u;v)$ belong to $B(w_A)$, $x_j + y_j = u_j + v_j = a_{jj}$. Then, $u_j \geq x_j$. Notice that $u_i > x_i + y_j - v_j = x_i + (u_j + v_j - x_j) - v_j = x_i + u_j - x_j$. Hence,

$$u_j - u_i + x_i < x_j. \tag{5.6}$$

We want to show that a core allocation exists that dominates $(x;y)$ via coalition $\{i,j\}$. To this end, we consider some cases:

1. $x_j > 0$. Consider two cases:

   a) $0 \leq x_i < a_{ii}$. Consider the continuous monotonic curve defined as in (5.1) through $(u;v)$, and take the point corresponding to $\tau^\varepsilon = u_i - x_i - \varepsilon$ where $0 < \varepsilon \leq a_{ii} - x_i$. We prove that for some $0 < \varepsilon \leq a_{ii} - x_i$, $(u(\tau^\varepsilon); v(\tau^\varepsilon))$ dominates $(x;y)$ via $\{i,j\}$. Notice that, for all $0 < \varepsilon \leq a_{ii} - x_i$, $u_i(\tau^\varepsilon) = \mathrm{med}\{0, u_i - u_i + x_i + \varepsilon, a_{ii}\} = x_i + \varepsilon > x_i$. Now, since $u_j(\tau^\varepsilon) = \mathrm{med}\{0, u_j - u_i + x_i + \varepsilon, a_{jj}\}$ and by (5.6) $u_j - u_i + x_i < x_j \leq a_{jj}$, there exists $0 < \varepsilon_1 \leq a_{ii} - x_i$ small enough such that $u_j(\tau^{\varepsilon_1}) \neq a_{jj}$. Then, we examine two cases:

      i. $u_j(\tau^{\varepsilon_1}) = u_j - u_i + x_i + \varepsilon_1$. Notice that $u_i(\tau^{\varepsilon_1}) > x_i$, $u_j(\tau^{\varepsilon_1}) < x_j$ or equivalently $v_j(\tau^{\varepsilon_1}) > y_j$ which together with $u_i(\tau^{\varepsilon_1}) + v_j(\tau^{\varepsilon_1}) = u_i + v_j = a_{ij}$ proves $(u(\tau^{\varepsilon_1}); v(\tau^{\varepsilon_1}))\ \mathrm{dom}^A_{\{i,j\}}(x;y)$.

ii. $u_j(\tau^{\varepsilon_1}) = 0 < x_j$. Then, $v_j(\tau^{\varepsilon_1}) = a_{jj} > y_j$. Moreover, $v_j(\tau^{\varepsilon_1}) = a_{jj}$ implies $v_j(\tau^{\varepsilon_1}) \leq v_j + \tau^{\varepsilon_1}$. Since $u_i(\tau^{\varepsilon_1}) = x_i + \varepsilon = u_i - \tau^{\varepsilon_1}$, we have $u_i(\tau^{\varepsilon_1}) + v_j(\tau^{\varepsilon_1}) \leq u_i + v_j = a_{ij}$. Together with $v_j(\tau^{\varepsilon_1}) > y_j$ and $u_i(\tau^{\varepsilon_1}) > x_i$ this implies that $(u(\tau^{\varepsilon_1}); v(\tau^{\varepsilon_1}))\ \mathrm{dom}^{A}_{\{i,j\}}(x;y)$.

b) $x_i = a_{ii}$. Since, by assumption, $u_i > x_i$, we obtain $a_{ii} = x_i < u_i$ which contradicts $(u;v) \in C(w_A)$.

2. $x_j = 0$. Since $(x;y) \in B(w_A)$, $y_j = a_{jj}$. We obtain from $x_i + y_j < a_{ij}$ that $a_{jj} < a_{ij}$, which contradicts the dominant diagonal assumption regarding the valuation matrix.

This shows that any $(x;y) \in B(w_A) \setminus C(w_A)$ is dominated by a core allocation via coalition $\{i,j\}$, which concludes the proof of $(i) \Rightarrow (ii)$.

Next, we prove $(ii) \Rightarrow (i)$. Let us suppose, on the contrary, that the core of a square two-sided assignment game $(M \cup M', w_A)$ is a von Neumann-Morgenstern stable set but that its corresponding valuation matrix A does not have a dominant diagonal. Since A does not have a dominant diagonal, there exists a sector-optimal allocation, let us say $(a;0)$, that does not belong to the core. Since the assumption states that $C(w_A)$ is a von Neumann-Morgenstern stable set, there exists $(u;v) \in C(w_A)$ such that $(u;v)\ \mathrm{dom}^{A}_{\{i,j\}}(a;0)$ for some $(i,j) \in M \times M'$. Then $u_i > a_{ii}$ which contradicts $(u;v) \in C(w_A)$. □

In this chapter, we have provided an alternative proof of the characterization of core stability for the assignment game. Solymosi and Raghavan (2001) show that the core of an assignment game is a von Neumann-Morgenstern stable set if and only if its valuation matrix has a dominant diagonal. While their proof makes use of graph-theoretical tools, the alternative proof presented in this chapter relies on the notion of the buyer-seller exact representative, as introduced by Núñez and Rafels (2002).

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6 Three-sided assignment games: core stability and undominated allocations

6.1 Introduction

In this chapter we consider markets with three different sectors or sides. Coalitions of agents can achieve a non-negative joint profit only by means of triplets if formed by one agent of each side in the market. Then, a three-dimensional valuation matrix represents the joint profit of all these possible triplets. These markets, introduced by Kaneko and Wooders (1982), are a generalization of Shapley and Shubik (1972) two-sided assignment games.

In a two-sided assignment game, each seller has one unit of an indivisible good to sell and each buyer wants to buy at most one unit. Buyers have valuations over goods. The valuation matrix represents the joint profit obtained by each buyer-seller trade. From these valuations a coalitional game is obtained and the total profit under an optimal matching between buyers and sellers yields the worth of the grand coalition.

Several set-solution concepts for coalitional games and hence also for assignment games, are based on a dominance relation between imputations, that is, individually rational payoff vectors that distribute the worth of the grand coalition within agents in the market. A *von Neumann-Morgenstern stable set* (von Neumann and Morgenstern, 1944) is a set of imputations that satisfy internal stability and external stability: *(a)* no imputation in the set is dominated by any other imputation in the set and *(b)* each imputation outside the set is dominated by some imputation in the set. Even if its computation can be difficult, the conjecture was that all games had a stable set. However, Lucas (1968) provided an example of a game with no stable set. Another solution concept introduced by Gillies (1959), the core, has been more widely studied. Gillies (1959) defines the *core* through the von Neumann-Morgenstern domination relation. Whenever it is non-empty, the core is the set of undominated imputations and hence it always satisfies internal stability. Moreover, the core is included in any stable set and if the core is externally stable, then it is the only stable set. Therefore, characterization of the core stability is an important approach to the study of stability. An intermediate form of stability, weaker than stable sets but stronger than the core, is the notion of *subsolution* introduced by Roth (1976). Roughly speaking, a set of imputations is a subsolution if it is internally stable and it is not dominated by the set of allocations it fails to dominate.

In the case of two-sided assignment games, Shapley and Shubik (1972) shows that the core is always non-empty. They also show that it has a lattice structure that leads to an optimal allocation for each side in the market. Solymosi and Raghavan (2001) shows that the core of a two-sided assignment game is a stable set if and only if the valuation matrix has a dominant diagonal. Later, Núñez and Rafels (2013) proves the existence of a stable set for all two-sided assignment games. The stable set they introduce is the only one that excludes third party payments with respect to an optimal matching μ and is defined through certain subgames, which are called μ-compatible subgames.

However, when the market has more than two sides, most results for the two-sided case do not extend to the multi-sided case. Kaneko and Wooders (1982) shows that the core of a three-sided assignment game may be empty. Moreover, when the core is non-empty it fails to have a lattice structure. Lucas (1995) provides necessary and sufficient conditions that yield non-emptiness of the core for the particular case where each side of the market consists of two agents. Nonetheless, there are no results focusing on stable sets which can shed light on stability for multi-sided assignment games.

The fact that the core may be empty makes the notions of subsolution and of stable sets more appealing as a solution concept for multi-sided assignment games. To keep notation as simple as possible, we restrict ourselves to the three-sided case.

First, we generalize the notion of dominant diagonal to the three-sided case and prove that it is a necessary condition for the core to be a stable set. We show that for three-sided markets with only two agents on each side, the dominant diagonal property suffices to guarantee that the core is stable. Furthermore, we extend the notion of μ-compatible subgames introduced by Núñez and Rafels (2013) to the three-sided case. As a consequence, given an optimal matching μ, we consider the set V^{μ} formed by the union of the cores of all μ-compatible subgames. However, different to the two-sided case, we show by means of a counterexample that V^{μ} may not be a stable set, not even a subsolution. Given an optimal matching μ, we show that the set V^{μ}, formed by the union of the cores of all μ-compatible subgames, is the set of allocations that are undominated by any allocation compatible with μ, that is, allocations such that the only side-payments take place within the triplets in μ.

Although the usual definition of the core and the stable sets of a coalitional game takes as the set of feasible outcomes of the game the set of imputations (efficient allocations that are individually rational), a more general setting can be considered. Lucas (1992) defines an abstract game by a set of (feasible) outcomes B and a dominance relation D (irreflexive binary relation) over this set of outcomes. Then, the core of an abstract game is the set of undominated outcomes, $C = B \setminus D(B)$, and a stable set V is a set of outcomes such that $V = B \setminus D(V)$.

In a three-sided assignment game it seems natural to restrict the set of feasible outcomes to those imputations that are compatible with some optimal matching μ. These allocations are known as the principal section B^{μ} of the assignment game and we prove that V^{μ} introduced before is the set of undominated allocations: $V^{\mu} = B^{\mu} \setminus D(B^{\mu})$. In this sense, V^{μ}, which is always non-empty, is the "core" with respect to the principal section. Moreover, V^{μ} coincides with the usual core if and only if the valuation matrix has a dominant diagonal.

The chapter is organized as follows. In Section 6.2 we give preliminaries on assignment games and solution concepts. Section 6.3 is devoted to conditions on the three-sided valuation matrix in order to obtain core stability. In Section 6.4, μ-compatible subgames are introduced and the union of cores of all μ-compatible subgames is shown to coincide with the core if the valuation matrix has a dominant diagonal. In Section 6.5, we show that if the μ-principal section is considered as the set of feasible outcomes, the union of the cores of all μ-compatible subgames, V^{μ}, is the set of undominated outcomes, that is, the "core" with respect to the set of feasible outcomes. Finally, Section 6.6 concludes.

6.2 Preliminaries

Let $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ be pairwise disjoint countable sets. An $m \times m \times m$ *assignment market* $\gamma = (M_1, M_2, M_3; A)$ consists of three different sectors with m agents each: $M_1 = \{1, 2, ..., m\} \subseteq \mathcal{U}_1$, $M_2 = \{1', 2', ..., m'\} \subseteq \mathcal{U}_2$, $M_3 = \{1'', 2'', ..., m''\} \subseteq \mathcal{U}_3$, and a three-dimensional valuation matrix $A = (a_{ijk})_{\substack{i \in M_1 \\ j \in M_2 \\ k \in M_3}}$ that represents the potential joint profit obtained by triplets formed by one agent of each sector. These triplets are the *basic coalitions* of the three-sided assignment game, as defined by Quint (1991).

Given subsets of agents of each sector, $S_1 \subseteq M_1$, $S_2 \subseteq M_2$, and $S_3 \subseteq M_3$, a *matching* μ for the submarket $\gamma_{|S} = (S_1, S_2, S_3; A_{|S_1 \times S_2 \times S_3})$ is a subset of the cartesian product, $\mu \subseteq S_1 \times S_2 \times S_3$, such that each agent belongs to at most one triplet. We denote by $\mathcal{M}(S_1, S_2, S_3)$ the set of all possible matchings. A matching $\mu \in \mathcal{M}(S_1, S_2, S_3)$ is an *optimal matching* for the submarket if

$$\sum_{(i,j,k)\in\mu} a_{ijk} \geq \sum_{(i,j,k)\in\mu'} a_{ijk}$$

for all other $\mu' \in \mathcal{M}(S_1,S_2,S_3)$. We denote by $\mathcal{M}_A(S_1,S_2,S_3)$ the set of all optimal matchings for the submarket $(S_1,S_2,S_3;A_{|S_1\times S_2\times S_3})$.

The $m\times m\times m$ *assignment game*, (N,w_A), related to the above assignment market has player set $N = M_1\cup M_2\cup M_3$ and characteristic function

$$w_A(S) = \max_{\mu\in\mathcal{M}(S\cap M_1,S\cap M_2,S\cap M_3)} \sum_{(i,j,k)\in\mu} a_{ijk}$$

for all $S\subseteq N$. In the sequel, we will need to exclude some agents. Then, if we exclude some agents $I\subseteq M_1$, $J\subseteq M_2$, and $K\subseteq M_3$, we will write $w_{A_{-I\cup J\cup K}}$ instead of $w_{A_{|(M_1\setminus I)\times(M_2\setminus J)\times(M_3\setminus K)}}$. Notice that these subgames need not have the same number of agents in each sector. Nevertheless, the notion of matching and characteristic function is defined analogously as for the $m\times m\times m$ case.

Given an $m\times m\times m$ assignment game, a payoff vector, or allocation, is $(u,v,w)\in\mathbb{R}^m_+\times\mathbb{R}^m_+\times\mathbb{R}^m_+$ where u_l denotes the payoff to agent $l\in M_1$, v_l denotes the payoff to agent $l'\in M_2$ and w_l denotes the payoff to agent $l''\in M_3$. An *imputation* is a non-negative payoff vector that is efficient, $\sum_{i\in M_1} u_i + \sum_{j\in M_2} v_j + \sum_{k\in M_3} w_k = w_A(M_1\cup M_2\cup M_3)$. We denote the set of imputations of the assignment game (N,w_A) by $I(w_A)$.

Given an optimal matching $\mu\in\mathcal{M}_A(M_1,M_2,M_3)$ we define the *μ-principal section* of (N,w_A), as the set of payoff vectors such that $u_i+v_j+w_k = a_{ijk}$ for all $(i,j,k)\in\mu$ and the payoff to agents unassigned by μ is zero. We denote it by $B^\mu(w_A)$. Notice that $B^\mu(w_A)\subseteq I(w_A)$. In the μ-principal section the only side payments that take place are those among agents matched together by μ.

We can assume that the optimal matching is on the main diagonal of the valuation matrix, $\mu = \{(i,i,i)|i\in\{1,2,\ldots,m\}\}$. Notice that the allocation $(a,0,0)$, that is $u_i = a_{iii}$ for all $i\in M_1$, $v_j = w_k = 0$ for all $j\in M_2, k\in M_3$, always belongs to the μ-principal section. The same happens with the allocations $(0,a,0)$ and $(0,0,a)$. These three vertices of the polytope $B^\mu(w_A)$ will be named the *sector-optimal allocations*. The core of a game is the set of imputations (u,v,w) such that no coalition S can improve upon: $u(S\cap M_1)+v(S\cap M_2)+w(S\cap M_3)\geq w_A(S)$. In our case, it is easy to see that it is enough to consider individual and basic coalitions. An imputation (u,v,w) belongs to the core, $(u,v,w)\in C(w_A)$, if for all $(i,j,k)\in M_1\times M_2\times M_3$ it holds $u_i+v_j+w_k\geq a_{ijk}$.

Notice that this means the core is a subset of the μ-principal section for any optimal matching $\mu\in\mathcal{M}_A(M_1,M_2,M_3)$.

It is known (see Example 2.12) that the core of a three-sided assignment game may be empty. For the particular case of $2\times 2\times 2$ assignment games, Lucas (1995) gives necessary and sufficient conditions for non-emptiness of the core (see Proposition 2.13).

Given a three-sided assignment market $\gamma = (M_1, M_2, M_3; A)$, we define a binary relation on the set of imputations. It is called the *dominance relation*. Given two imputations (u,v,w) and (u',v',w'), we say (u,v,w) *dominates* (u',v',w') if and only if there exists $(i,j,k) \in M_1 \times M_2 \times M_3$ such that $u_i > u'_i$, $v_j > v'_j$, $w_k > w'_k$ and $u_i + v_j + w_k \leq a_{ijk}$. We denote it by (u,v,w) dom $^A_{\{i,j,k\}}(u',v',w')$. We write (u,v,w) dom $^A(u',v',w')$ to denote that (u,v,w) dominates (u',v',w') by means of some triplet (i,j,k).[1] Given a set of imputations $V \subseteq I(w_A)$, we denote by $D(V)$ the set of imputations dominated by some element in V and by $U(V)$ those imputations not dominated by any element in V.

Two main set-solution concepts are defined by means of this dominance relation: the core and the stable set. On the one side, the core, whenever it is non-empty, coincides with the set of undominated imputations. That is, $C(w_A) = U(I(w_A))$. The other solution concept defined by means of domination is the von Neumann-Morgenstern stable set.

A subset of the set of imputations, $V \subseteq I(w_A)$, is a *von Neumann-Morgenstern solution* or a *stable set* if it satisfies internal and external stability:

(i) *internal stability:* for all $(u,v,w),(u',v',w') \in V$, (u,v,w) dom$^A(u',v',w')$ does not hold,

(ii) *external stability:* for all $(u',v',w') \in I(w_A) \setminus V$, there exists $(u,v,w) \in V$ such that (u,v,w) dom$^A(u',v',w')$.

Internal stability of a set of imputations V guarantees that no imputation of V is dominated by another imputation of $V : V \subseteq U(V)$. The core is internally stable. External stability imposes that all imputations outside V are dominated by an imputation in $V : I(w_A) \setminus V \subseteq D(V)$. In general, the core fails to satisfy external stability and hence the von Neumann-Morgenstern stability is stronger than the notion of stability satisfied by the core. Both conditions (internal and external stability) can be summarized in $V = U(V)$.

There is an intermediate notion of stability introduced by Roth (1976). A subset of imputations $V \subseteq I(w_A)$ is a *subsolution* if

(i) V is internally stable, that is, $V \subseteq U(V)$,

(ii) $V = U^2(V) = U(U(V))$.

[1] This dominance relation is the usual one introduced by von Neumann and Morgenstern (1944). It is clear that in the case of multi-sided assignment games, we only need to consider domination via basic coalitions. When no confusion regarding the valuation matrix can arise, we will simply write (u,v,w) dom (u',v',w').

Together with the internal stability that it shares with the core and the stable sets, the second condition for a set V to be a subsolution requires that if an imputation $x \in V$ is dominated by some $y \notin V$, then y will be dominated by some other $z \in V$. Notice that this is like an external stability restricted to those external imputations that dominate some element of V. In this sense, this stability notion is weaker than that of stable sets.

For arbitrary coalitional games, Roth (1976) proves a subsolution always exists but the existence of a non-empty subsolution is not guaranteed. Since for three-sided assignment games the core may be empty, a first aim of this paper is to look for some other set-solution concepts that yield a non-empty solution for three-sided assignment games. Before looking for stable sets or subsolutions, we analyze when the core is a stable set.

6.3 Dominant diagonal and core stability

In this section we look for conditions on the multi-sided valuation matrix that guarantee the core satisfies external stability and hence it is a von Neumann-Morgenstern stable set.

We begin by generalizing to the multi-sided case the dominant diagonal property introduced by Solymosi and Raghavan (2001) for two-sided assignment games. They prove that, in the two-sided case, this condition characterizes stability of the core. Therefore, we must define the appropriate generalization. We will assume that the valuation matrix is square, that is, there is the same number of agents on each side. Notice that, whenever necessary, we can assume without loss of generality that an optimal matching is placed on the main diagonal.

Definition 6.1. Let $(M_1, M_2, M_3; A)$ be a square three-sided assignment market where $m = |M_1| = |M_2| = |M_3|$. Matrix A has a *dominant diagonal* if and only if for all $i \in \{1, 2, ..., m\}$ it holds

$$a_{iii} \geq \max\{a_{ijk}, a_{jik}, a_{jki}\} \quad \text{for all } j, k \in \{1, 2, ..., m\}.$$

Clearly, if A has a dominant diagonal, then $\mu = \{(i,i,i) \mid i \in \{1, 2, \ldots, m\}\}$ is an optimal matching.

As in the two-sided case, the dominant diagonal property characterizes those markets where giving the profit of each optimal partnership to the agent on the same sector leads to a core element.

Proposition 6.2. *A three-dimensional square valuation matrix A has a* dominant diagonal *if and only if all sector-optimal allocations belong to the core.*

Proof. First, we prove the "*if*" part. Take the optimal allocation for the first sector: $(u,v,w) = (a_{111},...,a_{mmm};0,...,0;0,...,0)$. If it belongs to the core, then we have $a_{iii} = u_i = u_i + v_j + w_k \geq a_{ijk}$ for all $(i,j,k) \in M_1 \times M_2 \times M_3$. For the rest of optimal allocations the proof is analogous.

To prove the "*only if*" part, let A be a three-dimensional valuation matrix with the dominant diagonal property. By Definition 6.1, for all $i \in \{1,2,...,m\}$ and for all $j,k \in \{1,2,...,m\}$, $a_{iii} \geq \max\{a_{ijk}, a_{jik}, a_{jki}\}$. If we take the sector-optimal allocation $(u,v,w) = (a_{111},...,a_{mmm};0,...,0;0,...,0)$, the above inequality trivially shows that it belongs to the core. Analogously, $(0,a,0)$ and $(0,0,a)$ are also core allocations. □

Next proposition shows that the dominant diagonal property is necessary for the stability of the core.

Proposition 6.3. *If the core of a square three-sided assignment game (N,w_A) with an optimal matching on the main diagonal is a von Neumann-Morgenstern stable set, then its corresponding valuation matrix A has a dominant diagonal.*

Proof. Let us suppose, on the contrary, that the core of a three-sided assignment game (N,w_A) is a von Neumann-Morgenstern stable set but its corresponding three-dimensional valuation matrix A is not dominant diagonal. If A is not dominant diagonal, then there exists one sector-optimal allocation, let us say $(a,0,0)$, that does not belongs to the core. But then, since $C(w_A)$ is assumed to be a von Neumann-Morgenstern stable set, there exists $(u',v',w') \in C(w_A)$ such that

$$(u',v',w') \text{ dom}^A_{\{i,j,k\}} (a,0,0).$$

Then, $u'_i > u_i = a_{iii}$ which contradicts $(u',v',w') \in C(w_A)$. □

Proposition 6.3 arises the question of the equivalence between the von Neumann-Morgenstern stability of core and the dominant diagonal property of the matrix. That is to say, if A has dominant diagonal, does the core of the assignment game, $C(w_A)$, satistfy von Neumann-Morgenstern stable set conditions? We can answer this question affirmatively when the market has only two agents in each sector.

6.3.1 The $2 \times 2 \times 2$ Case

In this subsection, we show that the property of dominant diagonal is also a sufficient condition for core stability in the particular case of three-sided assignment games with two agents in each side. To this end, we need a remark regarding 2×2 assignment games that will be of use in the proof of Proposition 6.5.

Remark 6.4. Let $(M \cup M', w_B)$ be a 2×2 assignment game with $M = \{1,2\}$, $M' = \{1',2'\}$, and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Let us denote by $(\bar{u}, \underline{v})$ the buyers-optimal core allocation and $(\underline{u}, \bar{v})$ the sellers-optimal core allocation, see Remark 2.5. Assume the optimal matching is in the main diagonal, i.e. $b_{11} + b_{22} \geq b_{12} + b_{21}$ and $b_{22} \geq \max\{b_{12}, b_{21}\}$. Then, for each $0 \leq \eta \leq b_{11}$, there exists a core element (u, v) of w_B such that $v_1 = \eta$. Indeed, we know from Demange (1982) and Leonard (1983) that the maximum core-payoff of an agent in a two-sided assignment game is his/her marginal contribution. Then, the reader can check that under the above assumption $\bar{u}_1 = \bar{v}_1 = b_{11}$ and $\underline{u}_1 = b_{11} - \bar{v}_1 = 0$.

Similarly, given a 2×2 assignment game, if it holds $b_{11} + b_{22} \geq b_{12} + b_{21}$ and $b_{11} \geq \max\{b_{12}, b_{21}\}$, then for each $0 \leq \eta \leq b_{22}$ there exists a core element (u, v) of w_B such that $u_2 = \eta$.

Next, we show that, for the particular case of 2×2×2 assignment games, the dominant diagonal property is a necessary and sufficient condition for core stability.

Proposition 6.5. *Given a $2 \times 2 \times 2$ assignment game (N, w_A) with an optimal matching on the main diagonal, the core $C(w_A)$ is a von Neumann-Morgenstern stable set if and only if A has a dominant diagonal.*

Proof. The *"only if"* part is proved in Proposition 6.3. To prove the *"if"* part, assume A has a dominant diagonal and denote by μ the optimal matching on the main diagonal. Take an allocation $\alpha = (x, y, z)$ that is in the μ-principal section but outside the core. Let us see that α is dominated by some core allocation. Since it is in the μ-principal section, it satisfies the following conditions:

$$x_1 + y_1 + z_1 = a_{111}$$
$$x_2 + y_2 + z_2 = a_{222}.$$

Since (x, y, z) does not belong to the core, assume that $x_2 + y_1 + z_1 < a_{211}$. All other cases are treated similarly. We first look for a core allocation $\beta = (u, v, w)$ that satisfies $u_2 + v_1 + w_1 = a_{211}$ such that β dominates α via coalition $\{2, 1', 1''\}$. This equality, together with the core constraint $u_1 + v_1 + w_1 = a_{111}$ leads to $u_1 = u_2 + a_{111} - a_{211}$. Now, if we had such core allocation β, by substitution in the core constraints, we would get:

(i) $u_2 + v_1 + w_1 = a_{211}$

(ii) $u_2 + v_2 + w_2 = a_{222}$

(iii) $u_2+v_2+w_1 \geq a_{121}+a_{211}-a_{111}$

(iv) $u_2+v_1+w_1 \geq a_{211}$

(v) $u_2+v_2+w_1 \geq a_{221}$

(vi) $u_2+v_1+w_2 \geq a_{112}+a_{211}-a_{111}$

(vii) $u_2+v_2+w_2 \geq a_{122}+a_{211}-a_{111}$

(viii) $u_2+v_1+w_2 \geq a_{212}$.

Note that *(i)* implies *(iv)* and since $\{(1,1',1''),(2,2',2'')\}$ is an optimal matching, *(ii)* implies *(vii)*. By *(iii)* and *(v)* we get $v_2+w_1 \geq \max\{a_{221}-u_2, a_{121}+a_{211}-a_{111}-u_2, 0\}$ and by *(vi)* and *(viii)* we get $v_1+w_2 \geq \max\{a_{212}-u_2, a_{112}+a_{211}-a_{111}-u_2, 0\}$. Hence, a core element $\beta=(u,v,w)$ satisfies $u_2+v_1+w_1=a_{211}$ if and only if its projection (v,w) belongs to the core of the 2×2 assignment game defined by matrix B^{u_2}:

$$\begin{pmatrix} a_{211}-u_2 & \max\{a_{212}-u_2, a_{112}+a_{211}-a_{111}-u_2, 0\} \\ \max\{a_{221}-u_2, a_{121}+a_{211}-a_{111}-u_2, 0\} & a_{222}-u_2 \end{pmatrix}.$$

Define $\tilde{u}_2 = x_2+\varepsilon$ with $0<\varepsilon<\min\{a_{222}-x_2, a_{211}-x_2-y_1-z_1\}$. Notice that this is always possible since $x_2+y_1+z_1<a_{211}$ and because of the dominant diagonal assumption $x_2<a_{211}\leq a_{222}$. We now consider the matrix $B^{\tilde{u}_2}$.

By the dominant diagonal property and the fact that $\{(1,1',1''),(2,2',2'')\}$ is optimal, we always have

$$\begin{aligned} b_{22}^{\tilde{u}_2} = a_{222}-\tilde{u}_2 \geq \max\Big\{ &\max\{a_{212}-\tilde{u}_2, a_{112}+a_{211}-a_{111}-\tilde{u}_2, 0\}, \\ &\max\{a_{221}-\tilde{u}_2, a_{121}+a_{211}-a_{111}-\tilde{u}_2, 0\}\Big\} \\ = \max\{b_{12}^{\tilde{u}_2}, b_{21}^{\tilde{u}_2}\}. \end{aligned} \tag{6.1}$$

Case 1: $b_{11}^{\tilde{u}_2}+b_{22}^{\tilde{u}_2} \geq b_{12}^{\tilde{u}_2}+b_{21}^{\tilde{u}_2}$. That is,

$$\begin{aligned} a_{211}-\tilde{u}_2+a_{222}-\tilde{u}_2 \geq &\max\{a_{212}-\tilde{u}_2, a_{112}+a_{211}-a_{111}-\tilde{u}_2, 0\} \\ &+\max\{a_{221}-\tilde{u}_2, a_{121}+a_{211}-a_{111}-\tilde{u}_2, 0\}. \end{aligned}$$

Let us now define

$$\begin{aligned} v_1 &= y_1+\frac{a_{211}-x_2-y_1-z_1-\varepsilon}{2} > y_1 \geq 0, \\ w_1 &= z_1+\frac{a_{211}-x_2-y_1-z_1-\varepsilon}{2} > z_1 \geq 0. \end{aligned}$$

Note that $v_1 + w_1 = a_{211} - \tilde{u}_2$ and $v_1 \geq 0, w_1 \geq 0$.

By Remark 6.4, for all v_1 such that $0 \leq v_1 \leq a_{211} - \tilde{u}_2$ there exists a core allocation $\gamma = (\tilde{v}_1, \tilde{v}_2; \tilde{w}_1, \tilde{w}_2)$ of $B^{\tilde{u}_2}$ with $\tilde{v}_1 = v_1$. Notice that such a core allocation γ satisfies the constraint $\tilde{v}_2 + \tilde{w}_2 = a_{222} - \tilde{u}_2$ since by assumption of Case 1, $\{(1,1'),(2,2')\}$ is optimal for $B^{\tilde{u}_2}$. Then, by completion with $\tilde{u}_1 = \tilde{u}_2 + a_{111} - a_{211}$, we obtain a core allocation, $\beta = (\tilde{u}_1, \tilde{u}_2; \tilde{v}_1, \tilde{v}_2; \tilde{w}_1, \tilde{w}_2)$, of the three-sided assignment game such that $\beta \ \mathrm{dom}^{A}_{\{2,1,1\}} \alpha$.

Case 2: $b^{\tilde{u}_2}_{12} + b^{\tilde{u}_2}_{21} > b^{\tilde{u}_2}_{11} + b^{\tilde{u}_2}_{22}$.
Since $b^{\tilde{u}_2}_{22} \geq \max\{b^{\tilde{u}_2}_{12}, b^{\tilde{u}_2}_{21}\}$, it holds in this case that $b^{\tilde{u}_2}_{11} < b^{\tilde{u}_2}_{12}$ and $b^{\tilde{u}_2}_{11} < b^{\tilde{u}_2}_{21}$.
To sum up,

$$\begin{aligned}
\max\{a_{212} - \tilde{u}_2, a_{112} + a_{211} - a_{111} - \tilde{u}_2, 0\} &> a_{211} - \tilde{u}_2,\\
\max\{a_{221} - \tilde{u}_2, a_{121} + a_{211} - a_{111} - \tilde{u}_2, 0\} &> a_{211} - \tilde{u}_2,\\
a_{211} - \tilde{u}_2 + a_{222} - \tilde{u}_2 < \max\{a_{212} - \tilde{u}_2, a_{112} &+ a_{211} - a_{111} - \tilde{u}_2, 0\}\\
+\max\{a_{221} - \tilde{u}_2, a_{121} &+ a_{211} - a_{111} - \tilde{u}_2, 0\}. \qquad (6.2)
\end{aligned}$$

Note that, taking into account the dominant diagonal property, this implies

$$\begin{aligned}
\max\{a_{212} - \tilde{u}_2, a_{112} + a_{211} - a_{111} - \tilde{u}_2, 0\} &= a_{212} - \tilde{u}_2\\
\max\{a_{221} - \tilde{u}_2, a_{121} + a_{211} - a_{111} - \tilde{u}_2, 0\} &= a_{221} - \tilde{u}_2. \qquad (6.3)
\end{aligned}$$

Then, by (6.2) and (6.3), $a_{211} - \tilde{u}_2 + a_{222} - \tilde{u}_2 < a_{212} - \tilde{u}_2 + a_{221} - \tilde{u}_2$ which is equivalent to $a_{211} + a_{222} < a_{212} + a_{221}$. Hence, $(x_2 + y_1 + z_2) + (x_2 + y_2 + z_1) = (x_2 + y_1 + z_1) + (x_2 + y_2 + z_2) < a_{211} + a_{222} < a_{212} + a_{221}$. This means that either $x_2 + y_1 + z_2 < a_{212}$ or $x_2 + y_2 + z_1 < a_{221}$.

Case 2.1: $x_2 + y_1 + z_2 < a_{212}$.

We now look for a core allocation $\beta = (u, v, w)$ of w_A such that β dominates α via $\{2, 1', 2''\}$, and hence $u_2 + v_1 + w_2 = a_{212}$. Together with the core constraint $u_2 + v_2 + w_2 = a_{222}$, we get $v_2 = v_1 + (a_{222} - a_{212})$.

If we had such core allocation β, by substitution in the core constraints, we would get

(i) $u_1 + v_1 + w_1 = a_{111}$

(ii) $u_2 + v_1 + w_2 = a_{222} + a_{212} - a_{222} = a_{212}$

(iii) $u_1 + v_1 + w_1 \geq a_{121} + a_{212} - a_{222}$

(iv) $u_2 + v_1 + w_1 \geq a_{211}$

(v) $u_2+v_1+w_1 \geq a_{221}+a_{212}-a_{222}$

(vi) $u_1+v_1+w_2 \geq a_{112}$

(vii) $u_1+v_1+w_2 \geq a_{122}+a_{212}-a_{222}$

(viii) $u_2+v_1+w_2 \geq a_{212}$.

Note that from the fact that $\{(1,1',1''),(2,2',2'')\}$ is optimal for A and the dominant diagonal property, *(i)* implies *(iii)* and *(ii)* implies *(viii)*. By *(vi)* and *(vii)* we get $u_1+w_2 \geq \max\{a_{112}-v_1, a_{122}+a_{212}-a_{222}-v_1, 0\}$ and by *(iv)* and *(v)* we get $u_2+w_1 \geq \max\{a_{211}-v_1, a_{221}+a_{212}-a_{222}-v_1, 0\}$. Hence $\beta=(u,v,w) \in C(w_A)$ satisfies $u_2+v_1+w_2 = a_{212}$ if and only if its projection $(u,w)=(u_1,u_2;w_1,w_2)$ belongs to the core of the 2×2 assignment game B^{v_1}

$$\begin{pmatrix} a_{111}-v_1 & \max\{a_{112}-v_1, a_{122}+a_{212}-a_{222}-v_1, 0\} \\ \max\{a_{211}-v_1, a_{221}+a_{212}-a_{222}-v_1, 0\} & a_{212}-v_1 \end{pmatrix}.$$

Let us now take $\tilde{v}_1 = y_1+\varepsilon$ where $0<\varepsilon<\min\{a_{111}-y_1, a_{212}-x_2-y_1-z_2\}$. Notice this is always possible since $0 \leq y_1 < a_{212} \leq a_{111}$. Consider now $B^{\tilde{v}_1}$. Note that

$$\begin{aligned} b_{11}^{\tilde{v}_1} = a_{111}-\tilde{v}_1 \geq \max\Big\{ & \max\{a_{112}-\tilde{v}_1, a_{122}+a_{212}-a_{222}-\tilde{v}_1, 0\}, \\ & \max\{a_{211}-\tilde{v}_1, a_{221}+a_{212}-a_{222}-\tilde{v}_1, 0\}\Big\} \\ = \max\{ & b_{12}^{\tilde{v}_1}, b_{21}^{\tilde{v}_1}\}. \end{aligned} \tag{6.4}$$

From $a_{211}+a_{222} < a_{212}+a_{221}$ and $a_{222} \geq a_{221}$ we know that $a_{211} < a_{212}$. Together with (6.4) this implies that $a_{111}-\tilde{v}_1+a_{212}-\tilde{v}_1 \geq \max\{a_{112}-\tilde{v}_1, a_{122}+a_{212}-a_{222}-\tilde{v}_1, 0\}+\max\{a_{211}-\tilde{v}_1, a_{221}+a_{212}-a_{222}-\tilde{v}_1, 0\}$, that is $b_{11}^{\tilde{v}_1}+b_{22}^{\tilde{v}_1} \geq b_{12}^{\tilde{v}_1}+b_{21}^{\tilde{v}_1}$.

Let us define

$$u_2 = x_2 + \frac{a_{212}-x_2-y_1-z_2-\varepsilon}{2} > x_2 \geq 0,$$
$$w_2 = z_2 + \frac{a_{212}-x_2-y_1-z_2-\varepsilon}{2} > z_2 \geq 0.$$

Note that $u_2+w_2 = a_{212}-\tilde{v}_1$ and $u_2>0, w_2>0$.

By Remark 6.4, there exists a core allocation γ of $B^{\tilde{v}_1}$ with $\tilde{u}_2 = u_2$. Such a core allocation γ satisfies the constraint $\tilde{u}_1+\tilde{w}_1 = a_{111}-\tilde{v}_1$. Then, by completion with $\tilde{v}_2 = \tilde{v}_1+a_{222}-a_{212}$, we obtain a core allocation of the three-sided assignment game, $(\tilde{u}_1,\tilde{u}_2;\tilde{v}_1,\tilde{v}_2;\tilde{w}_1,\tilde{w}_2)$, such that $\beta\ \mathrm{dom}^A_{\{2,1,2\}}\alpha$.

Case 2.2: $x_2+y_2+z_1 < a_{221}$.
We now look for a core allocation $\beta=(u,v,w)$ of w_A such that β dominates α via $\{2,2',1''\}$ and $u_2+v_2+w_1 = a_{221}$. Together with the core constraint $u_2+v_2+w_2 = a_{222}$, we get $w_2 = w_1+(a_{222}-a_{221})$.

If we had such a core allocation β, by substitution in the core constraints we would obtain

(i) $u_1+v_1+w_1=a_{111}$

(ii) $u_2+v_2+w_1=a_{222}+a_{221}-a_{222}=a_{221}$

(iii) $u_1+v_1+w_1\geq a_{112}+a_{221}-a_{222}$

(iv) $u_1+v_2+w_1\geq a_{121}$

(v) $u_1+v_2+w_1\geq a_{122}+a_{221}-a_{222}$

(vi) $u_2+v_1+w_1\geq a_{211}$

(vii) $u_2+v_1+w_1\geq a_{212}+a_{221}-a_{222}$

(viii) $u_2+v_2+w_1\geq a_{221}$.

Note that because of the dominant diagonal property and the fact that the matching $\{(1,1',1''),(2,2',2'')\}$ is optimal for A, we have *(i)* implies*(iii)* and *(ii)* implies *(viii)*. By *(iv)* and *(v)* we get $u_1+v_2\geq\max\{a_{121}-w_1,a_{122}+a_{221}-a_{222}-w_1,0\}$ and by *(vi)* and *(vii)* we get $u_2+v_1\geq\max\{a_{211}-w_1,a_{212}+a_{221}-a_{222}-w_1,0\}$. Hence, a core element $\beta=(u,v,w)$ satisfies $u_2+v_2+w_1=a_{221}$ if and only if its projection $(u,v)=(u_1,u_2;v_1,v_2)$ belongs to the core of the 2×2 assignment game B^{w_1}:

$$\begin{pmatrix} a_{111}-w_1 & \max\{a_{121}-w_1,a_{122}+a_{221}-a_{222}-w_1,0\} \\ \max\{a_{211}-w_1,a_{212}+a_{221}-a_{222}-w_1,0\} & a_{221}-w_1 \end{pmatrix}.$$

Let us now take $\tilde{w}_1=z_1+\varepsilon$ where $0<\varepsilon<\min\{a_{111}-z_1,a_{221}-x_2-y_2-z_1\}$. Notice that this is always possible since $0\leq z_1<a_{221}\leq a_{111}$. Consider now $B^{\tilde{w}_1}$. Then,

$$\begin{aligned} b_{11}^{\tilde{w}_1}=a_{111}-\tilde{w}_1\geq\max\Big\{&\max\{a_{121}-\tilde{w}_1,a_{122}+a_{212}-a_{222}-\tilde{w}_1,0\},\\ &\max\{a_{211}-\tilde{w}_1,a_{212}+a_{221}-a_{222}-\tilde{w}_1,0\}\Big\} \\ =\max\{b_{12}^{\tilde{w}_1},b_{21}^{\tilde{w}_1}\}. \end{aligned} \tag{6.5}$$

Now, from $a_{211}+a_{222}<a_{212}+a_{221}$ and $a_{222}\geq a_{212}$ we get $a_{221}>a_{211}$, and together with (6.5) this implies $a_{111}-\tilde{w}_1+a_{221}-\tilde{w}_2\geq\max\{a_{121}-\tilde{w}_1,a_{122}+a_{212}-a_{222}-\tilde{w}_1,0\}+\max\{a_{211}-\tilde{w}_1,a_{212}-\tilde{w}_1+a_{221}-a_{222},0\}$, that is $b_{11}^{\tilde{w}_1}+b_{22}^{\tilde{w}_1}\geq b_{12}^{\tilde{w}_1}+b_{21}^{\tilde{w}_1}$.

Let us define

$$u_2=x_2+\frac{a_{221}-x_2-y_2-z_1-\varepsilon}{2}>x_2\geq 0,$$
$$v_2=y_2+\frac{a_{221}-x_2-y_2-z_1-\varepsilon}{2}>y_2\geq 0.$$

Note that $u_2+v_2=a_{221}-\tilde{w}_1$ and $u_2>0, v_2>0$.

By Remark 6.4, there exists a core allocation $\gamma=(u_1,u_2;v_1,v_2)$ of $B^{\tilde{w}_1}$ with $\tilde{u}_2=u_2$. Such a core allocation γ satisfies the constraint $\tilde{u}_1+\tilde{v}_1=a_{111}-\tilde{w}_1$. Then, by completion with $\tilde{w}_2=\tilde{w}_1+a_{222}-a_{221}$, we obtain a core allocation of the three-sided assignment game $\beta=(\tilde{u}_1,\tilde{u}_2;\tilde{v}_1,\tilde{v}_2;\tilde{w}_1,\tilde{w}_2)$ such that $\beta \operatorname{dom}^A_{\{2,2,1\}}\alpha$. $\square$

Now, we return to the general case, that is to say, $m\times m\times m$ assignment games, and define μ-compatible subgames in search of a stable set. We give some results related to stability but we do not achieve a characterization or an existence theorem.

6.4 The μ-compatible subgames and some stability notions

In this section, we follow an approach similar to the one in Núñez and Rafels (2013) to construct a stable set for two-sided assignment markets. First, we extend to multi-sided assignment games the notion of the μ-compatible subgame. Then, we introduce a set that consists of the union of the extended cores of all μ-compatible subgames and we look for stability properties of this set. We show that, in general, it fails to satisfy external stability and hence, different from the two-sided case, it does not always result in a von Neumann-Morgenstern stable set.

Definition 6.6. Let $(M_1\cup M_2\cup M_3, w_A)$ be a three-sided assignment game, with $m=|M_1|=|M_2|=|M_3|$, $\mu\in\mathcal{M}_A(M_1,M_2,M_3)$ an optimal matching, and $I\subseteq M_1$, $J\subseteq M_2$ and $K\subseteq M_3$. A subgame

$$(M_1\setminus I, M_2\setminus J, M_3\setminus K, w_{A_{-I\cup J\cup K}})$$

is a *μ-compatible subgame* if and only if

$$\begin{aligned} w_A(M_1\cup M_2\cup M_3) \quad &= w_A((M_1\setminus I)\cup(M_2\setminus J)\cup(M_3\setminus K)) \\ &+ \sum_{\substack{(i,j,k)\in\mu \\ i\in I}} a_{ijk} + \sum_{\substack{(i,j,k)\in\mu \\ j\in J}} a_{ijk} + \sum_{\substack{(i,j,k)\in\mu \\ k\in K}} a_{ijk}. \end{aligned}$$

Without loss of generality, assume that the diagonal matching is an optimal matching for A: $\mu=\{(i,i,i)|i\in\{1,2,\ldots,m\}\}$. Then, given a μ-compatible subgame $w_{A_{-I\cup J\cup K}}$ we define its extended core,

$$\hat{C}(w_{A_{-I\cup J\cup K}})=\left\{(x,z)\in B^{\mu}(w_A)\;\middle|\; \begin{array}{l} x_i=a_{iii} \text{ for all } i\in I\cup J\cup K, \\ z\in C(w_{A_{-I\cup J\cup K}}) \end{array}\right\}.$$

Note that if $C(w_{A_{-I\cup J\cup K}})=\emptyset$, then $\hat{C}(w_{A_{-I\cup J\cup K}})=\emptyset$. The following ones are two straightforward properties of μ-compatible subgames.

If $w_{A_{-I\cup J\cup K}}$ is a μ-compatible subgame, then:

(i) $\mu_{|(M_1\setminus I)\times(M_2\setminus J)\times(M_3\setminus K)}=\{(i,j,k)\in\mu \mid i\in M_1\setminus I, j\in M_2\setminus J, k\in M_3\setminus K\}$ is an optimal matching for $w_{A_{-I\cup J\cup K}}$, which implies that the partners of agents in $I\cup J\cup K$ remain unmatched in the subgame,

(ii) if $i,j\in I\cup J\cup K$, then i and j cannot belong to the same basic coalition in μ except if the value of this triplet is null.

Hence, if $A>0$, that is all entries are positive, all μ-compatible subgames come from the exclusion of a set of agents of only one side of the market. In particular, if we exclude all agents in M_1, then the game $(N\setminus M_1, w_{A_{-M_1}})$ is always a μ-compatible subgame since $w_{A_{-M_1}}(N\setminus M_1)=0$. The core of this μ-compatible subgame is reduced to $\{(0,0)\}\subseteq\mathbb{R}^{M_2}\times\mathbb{R}^{M_3}$ and the corresponding extended core is $\hat{C}(w_{A_{-M_1}})=\{(a,0,0)\}$. Analogous μ-compatible subgames are obtained when we exclude the agents of one of the remaining sides of the market.

Given a three-sided assignment market $\gamma=(M_1,M_2,M_3;A)$, we define the set of all coalitions that give rise to μ-compatible subgames:

$$\mathcal{C}^{\mu}(A)=\{R\subseteq M_1\cup M_2\cup M_3 \mid w_{A_{-R}} \text{ is a } \mu\text{-compatible subgame}\}.$$

Notice that when $R=\emptyset$ we retrieve the core of the initial game (N,w_A).

Now, for any assignment market $\gamma=(M_1,M_2,M_3;A)$, we define the set $V^{\mu}(w_A)$ formed by the union of extended cores of all μ-compatible subgames:

$$V^{\mu}(w_A)=\bigcup_{R\in\mathcal{C}^{\mu}(A)}\hat{C}(w_{A_{-R}}) \tag{6.6}$$

A first immediate consequence of the above definition is that $V^{\mu}(w_A)$ is a subset of the μ-principal section:

$$V^{\mu}(w_A)\subseteq B^{\mu}(w_A).$$

Notice also that differently from the core, the set $V^{\mu}(w_A)$ is always non-empty since it contains at least the three points $(a,0,0)$, $(0,a,0)$, and $(0,0,a)$, which result from the μ-compatible subgames where all agents of one sector have been excluded. In fact the following example shows that $V^{\mu}(w_A)$ can be reduced to only these three points and hence be non-convex and disconnected.

Example 6.7. Consider a three-sided assignment game where each sector has two agents, $M_1 = \{1,2\}$, $M_2 = \{1',2'\}$, and $M_3 = \{1'',2''\}$, and the valuation matrix A is the following

$$A = \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} \mathbf{3} & 1 \\ 2 & 5 \end{pmatrix} \\ 1'' \end{array} \qquad \begin{array}{c} \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1' & 2' \end{array} \\ \begin{pmatrix} 1 & 4 \\ 5 & \mathbf{4} \end{pmatrix} \\ 2'' \end{array}$$

Notice there is a unique optimal matching $\mu = \{(1,1',1''),(2,2',2'')\}$. By Lucas' conditions for balancedness (see Proposition 2.13), we see that the core is empty: $a_{111} + 2a_{222} = 11 < 14 = a_{221} + a_{122} + a_{212}$. We observe that the only μ-compatible subgames are $w_{A_{-\{1,2\}}}$, $w_{A_{-\{1',2'\}}}$ and $w_{A_{-\{1'',2''\}}}$. Hence $V^{\mu}(w_A) = \{(a,0,0),(0,a,0),(0,0,a)\} = \{(3,4;0,0;0,0),(0,0;3,4;0,0),(0,0;0,0;3,4)\}$. Now it is easy to realize that such points do not dominate any imputation in the μ-principal section. Thus, external stability does not hold for the set $V^{\mu}(w_A)$. This implies that the set $V^{\mu}(w_A)$ is not a von Neumann-Morgenstern stable set.

Now, take the imputation $(1,4.5;1,0.25;0.25,0)$. Notice that it is not an element of the set $V^{\mu}(w_A)$ and there is no element of the set $V^{\mu}(w_A)$ that dominates it. Furthermore, it dominates an element, $(3,4;0,0;0,0)$, of the set $V^{\mu}(w_A)$ via coalition $\{2,2',1''\}$. We observe that there exist an imputation that dominates one allocation in $V^{\mu}(w_A)$ and no point in $V^{\mu}(w_A)$ dominates the aforementioned allocation, which contradicts the definition of subsolution. Hence, the set $V^{\mu}(w_A)$ is not a subsolution.

The following proposition provides an equivalent definition of the set $V^{\mu}(w_A)$.

Proposition 6.8. *Let $(M_1,M_2,M_3;A)$ be a three-sided assignment market with $|M_1| = |M_2| = |M_3| = m$, and an optimal matching on the main diagonal. Let (u,v,w) be an allocation of the principal section, that is, $(u,v,w) \in B^{\mu}(w_A)$. Then $(u,v,w) \in V^{\mu}(w_A)$ if and only if for all $(i,j,k) \in M_1 \times M_2 \times M_3$ at least one of the four following statements holds:*

(i) either $u_i = a_{iii}$

(ii) or $v_j = a_{jjj}$

(iii) or $w_k = a_{kkk}$

(iv) or $u_i + v_j + w_k \geq a_{ijk}$.

Proof. First, we prove the "*only if*" part. Assume $(u,v,w) \in \hat{C}(w_{A_{-R}})$ for some $R \subseteq M_1 \cup M_2 \cup M_3$ and take $(i,j,k) \in M_1 \times M_2 \times M_3$. If $i \in R$, then $u_i = a_{iii}$. If $j \in R$, then $v_j = a_{jjj}$. If $k \in R$, then $w_k = a_{kkk}$. Otherwise, $u_i + v_j + w_k \geq a_{ijk}$.

Next, we show the "*if*" implication. Take $(u,v,w) \in B^{\mu}(w_A)$ such that all $(i,j,k) \in M_1 \times M_2 \times M_3$ satisfy either (i), or (ii), or (iii), or (iv). Define $I = \{i \in M_1 \mid u_i = a_{iii}\}$, $J = \{j \in M_2 \mid v_j = a_{jjj}\}$, and $K = \{k \in M_1 \mid w_k = a_{kkk}\}$, and also $R = I \cup J \cup K$. Notice that $z = (u,v,w) \in \hat{C}(w_{A_{-R}})$, since $z_l = a_{lll}$ for all $l \in R$, and for all $(i,j,k) \in (M_1 \setminus R) \times (M_2 \setminus R) \times (M_3 \setminus R)$ it holds $u_i + v_j + w_k \geq a_{ijk}$. Hence, $z = (u,v,w) \in V^{\mu}(w_A)$. □

Making use of the above equivalent expression of the set $V^{\mu}(w_A)$, we can characterize under which condition this set reduces to the core of the three-sided assignment market.

Proposition 6.9. *Let (M_1, M_2, M_3, w_A) be a square three-sided assignment game and μ an optimal matching on the main diagonal, $\mu \in \mathcal{M}_A(M_1, M_2, M_3)$. A has a dominant diagonal if and only if $V^{\mu}(w_A) = C(w_A)$.*

Proof. First, we prove the "*if*" part. Taking $R = M_1$ always gives a μ-compatible subgame and $\hat{C}(w_{A_{-M_1}}) = \{(a,0,0)\}$. Then, by the assumption, $(a,0,0) \in C(w_A)$. Similarly, $(0,a,0) \in C(w_A)$ and $(0,0,a) \in C(w_A)$. By Proposition 6.2, we obtain that A has dominant diagonal.

To prove the "*only if*" part, assume $(u,v,w) \in \hat{C}(w_{A_{-R}})$. Since $\hat{C}(w_{A_{-R}}) \subseteq B^{\mu}(w_A)$, (u,v,w) satisfies the efficiency condition. By the definition of the extended core, we know that, for all $i \in R \cap M_1$, $u_i = a_{iii}$; for all $j \in R \cap M_2$, $v_j = a_{jjj}$; for all $k \in R \cap M_3$, $w_k = a_{kkk}$; and for all $(i,j,k) \in (M_1 \setminus R) \times (M_2 \setminus R) \times (M_3 \setminus R)$ it satisfies $u_i + v_j + w_k \geq a_{ijk}$. Now, if $i \in R$, for all $j \in M_2$ and $k \in M_3$ it holds $u_i + v_j + w_k \geq a_{iii} + v_j + w_k \geq a_{iii} \geq a_{ijk}$, where the last inequality follows from the dominant diagonal property. Similarly, if $j \in R$ and $i \in M_1$, $k \in M_3$ or $k \in R$ and $i \in M_1$, $j \in M_2$ we obtain $u_i + v_j + w_k \geq a_{ijk}$. Together with efficiency this means $(u,v,w) \in C(w_A)$. □

We have seen that in general the set $V^{\mu}(w_A)$ is not a stable set nor a subsolution, but it is always a non-empty set. In the next section, we give an interpretation of the set $V^{\mu}(w_A)$ by means of the dominance relation.

6.5 The core of a three-sided assignment game with respect to the principal section

We have just seen that under the dominant diagonal property the set $V^{\mu}(w_A)$ coincides with the core and hence it is the set of undominated imputations.

In an assignment market, once an optimal matching is agreed on, agents must negotiate on an outcome that distributes the profit of each optimally matched triplet among its members. That is to say, it seems natural to consider payoff vectors that exclude third-party payments, that is, exclude side-payments among agents that are not in the same optimal triplet. These payoff vectors are those in the μ-principal section $B^{\mu}(w_A)$.

Next theorem shows that, if we reduce to the outcomes in the principal section, the set $V^{\mu}(w_A)$ is precisely the set of undominated outcomes, even if the dominant diagonal property does not hold.

Theorem 6.10. *Let* $(M_1,M_2,M_3;A)$ *be a three-sided square assignment market and* $\mu \in \mathcal{M}_A(M_1,M_2,M_3)$. *Then,*

$$V^{\mu}(w_A) = U(B^{\mu}(w_A))$$

where $U(B^{\mu}(w_A))$ *is the set of imputations that are undominated by the* μ*-principal section.*

Proof. Let us write $V = V^{\mu}(w_A)$ and assume μ is on the main diagonal. First, we prove $U(B^{\mu}(w_A)) \subseteq B^{\mu}(w_A)$. Notice that this inclusion is equivalent to $I(w_A) \setminus B^{\mu}(w_A) \subseteq D(B^{\mu}(w_A))$, where $D(B^{\mu}(w_A))$ is the set of imputations that are dominated by some allocation in the μ-principal section.

Take $(x,y,z) \in I(w_A) \setminus B^{\mu}(w_A)$. Then, there exists $i \in \{1,...,m\}$ such that $x_i + y_i + z_i < a_{iii}$. Take $\varepsilon = a_{iii} - x_i - y_i - z_i > 0$, and define λ_1, λ_2 and λ_3 by $\lambda_1 = \frac{x_i+\frac{\varepsilon}{3}}{a_{iii}}$, $\lambda_2 = \frac{y_i+\frac{\varepsilon}{3}}{a_{iii}}$ and $\lambda_3 = \frac{z_i+\frac{\varepsilon}{3}}{a_{iii}}$. Note that $\lambda_1+\lambda_2+\lambda_3 = 1$, and $\lambda_1 a_{iii} = x_i + \frac{\varepsilon}{3} > x_i$, $\lambda_2 a_{iii} = y_i + \frac{\varepsilon}{3} > y_i$ and $\lambda_3 a_{iii} = z_i + \frac{\varepsilon}{3} > z_i$.

Now, recall that $(a,0,0)$, $(0,a,0)$ and $(0,0,a)$ all belong to $B^{\mu}(w_A)$ and take the point $(u,v,w) = \lambda_1(a,0,0) + \lambda_2(0,a,0) + \lambda_3(0,0,a) \in B^{\mu}(w_A)$. Then, for all $i \in \{1,...,m\}$, $u_i + v_i + w_i = (\lambda_1+\lambda_2+\lambda_3)a_{iii} = a_{iii}$. Together with $u_i > x_i$, $v_i > y_i$ and $w_i > z_i$, this implies that (u,v,w) dom $_{\{i,i,i\}}(x,y,z)$ and hence $(x,y,z) \in D(B^{\mu}(w_A))$.

Now, we prove the equality, $V = U(B^{\mu}(w_A))$. First, we prove $V \subseteq U(B^{\mu}(w_A))$. We want to show that no allocation in V is dominated by an allocation in the μ-principal section. Consider two allocations $(u,v,w) \in B^{\mu}(w_A)$ and $(u',v',w') \in V$. We want to show that (u,v,w) cannot dominate (u',v',w') via any triplet $\{i,j,k\}$.

Assume that for some $(i,j,k) \in M_1 \times M_2 \times M_3$, (u,v,w) dom$_{\{i,j,k\}}$ (u',v',w') holds, which means $u_i + v_j + w_k \leq a_{ijk}$ together with $u_i > u'_i$, $v_j > v'_j$ and $w_k > w'_k$. Two cases are considered.

Case 1: $(u',v',w') \in C(w_A)$.

We reach straightforwardly a contradiction, since core elements are undominated.

Case 2: $(u',v',w') \in \hat{C}(w_{A_{-R}})$ for some $R \in \mathcal{C}^{\mu}(A)$.

If $i \in R$, then $u'_i = a_{iii}$. Then $u_i > u'_i = a_{iii}$ which contradicts $(u,v,w) \in B^{\mu}(w_A)$. The same argument leads to contradiction if $j \in R$ or $k \in R$. If $i \notin R$, $j \notin R$ and $k \notin R$, then by Proposition 6.8, $u'_i + v'_j + w'_k \geq a_{ijk} \geq u_i + v_j + w_k$ which contradicts our assumption $u_i > u'_i$, $v_j > v'_j$ and $w_k > w'_k$. This finishes the proof of $(u,v,w) \in U(B^{\mu}(w_A))$.

Now, we move to $U(B^{\mu}(w_A)) \subseteq V$. Assume on the contrary that $(u,v,w) \in U(B^{\mu}(w_A))$ and $(u,v,w) \notin V$. Since $U(B^{\mu}(w_A)) \subseteq B^{\mu}(w_A)$, $(u,v,w) \in B^{\mu}(w_A)$. Then, $(u,v,w) \in B^{\mu}(w_A)$ and $(u,v,w) \notin V$ which implies by Proposition 6.8 there exist $(i,j,k) \in M_1 \times M_2 \times M_3$ such that $u_i < a_{iii}$, $v_j < a_{jjj}$, $w_k < a_{kkk}$ and $u_i + v_j + w_k < a_{ijk}$. Define $\varepsilon_1 = a_{iii} - u_i > 0$, $\varepsilon_2 = a_{jjj} - v_j > 0$, $\varepsilon_3 = a_{kkk} - w_k > 0$ and $\varepsilon_4 = a_{ijk} - u_i - v_j - w_k > 0$. Also, let us define $u'_i = u_i + \min\{\varepsilon_1, \frac{\varepsilon_4}{3}\}$, $v'_j = v_j + \min\{\varepsilon_2, \frac{\varepsilon_4}{3}\}$ and $w'_k = w_k + \min\{\varepsilon_3, \frac{\varepsilon_4}{3}\}$. Note that $u'_1 > u_1$, $v'_j > v_j$, $w'_k > w_k$ and $u'_i + v'_j + w'_k < u_i + v_j + w_k + 3\frac{\varepsilon_4}{3} = a_{ijk}$. Now, we complete the definition of $(u',v,'w')$ in the following way:

Since, by definition, $u'_i \leq a_{iii}$, define $v'_i = a_{iii} - u'_i$ and $w'_i = 0$. Similarly, since $v'_j \leq a_{jjj}$, define $u'_j = a_{jjj} - v'_j$ and $w'_i = 0$. And finally, since $w'_k \leq a_{kkk}$, define $v'_k = a_{kkk} - w'_k$ and $u'_k = 0$. For all $l \in \{1,...,m\} \setminus \{i,j,k\}$ define $u'_l = a_{lll}$, $v'_l = 0$ and $w'_l = 0$. Then $(u',v',w') \in B^{\mu}(w_A)$ and (u',v',w') dom $_{\{i,j,k\}}(u,v,w)$ which contradicts $(u,v,w) \in U(B^{\mu}(w_A))$. Hence, if $(u,v,w) \in U(B^{\mu}(w_A))$, then $(u,v,w) \in V$. □

In Theorem 6.10 we show that there is no allocation in the μ-principal section that dominates any element of $V^{\mu}(w_A)$. This ensures internal stability of $V^{\mu}(w_A)$. But, we already know from Example 6.7 that $V^{\mu}(w_A)$ may not be externally stable. Hence, it may not be a stable set.

As we have seen that the set $V^{\mu}(w_A)$ may not be a stable set, in this section we have provided an interpretation through dominance relation for the set $V^{\mu}(w_A)$. That is, if the set of outcomes is not the whole imputation set but the set of imputations that exclude third-party payments with respect to some optimal matching μ, that is the μ-principal section $B^{\mu}(w_A)$, then the set of undominated outcomes (the “core") is the set $V^{\mu}(w_A)$. Hence, $V^{\mu}(w_A)$ is like a “core" with respect to the μ-principal section.

6.6 A concluding remark

Two important questions remain open regarding the stable sets for three-sided assignment games. One is whether stable sets always exist for these games. Once checked that the μ-compatible subgames do not provide a stable set in this new setting we have no hint related to this existence problem.

The second open question is the conjecture regarding the characterization of the core stability by means of the dominant diagonal property. The proof of Solymosi and Raghavan (2001) for the two-sided case cannot be extended to markets with three sides since it strongly relies on the weighted bi-partite graph structure.

This is the reason why in Chapter 5 of this dissertation we have provided an alternative proof that relies solely on properties of the core of the assignment game: the buyer-seller exact representative and the existence of a continuous monotonic curve through each core element that connects the two sector-optimal allocations.

If these two elements could be built for the three-sided assignment game, it might be shown that a three-sided market with a dominant diagonal valuation matrix has a stable core. This will be the objective of future research.

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7 Concluding remarks

This dissertation undertakes the study of assignment markets with several sectors. The main difficulty when passing from two-sided markets to the multi-sided case lies in the fact that core allocations may not exist. The first part of the thesis introduces a generalization of the classical multi-sided assignment markets that allows each coalition formed by agents of different sectors to attain a non-negative value. This generalized model fits better with the usual notion of reduced game and makes the core a consistent solution. Also, if a network with certain properties is attached to the market data, a closer relationship with the two-sided case is obtained. The second part of the dissertation deals with the classical notion of multi-sided assignment game, where one agent from each sector is needed to yield a profit, and focuses on von Neumann-Morgenstern stability as an alternative to the core. The problem of existence of a stable set for these multi-sided markets remains open but a set-solution that is always non-empty is found. This set-solution contains the core (whenever it exists) and, like the core, satisfies internal stability.

In Chapter 3, we introduce some three-sided assignment markets which have a different structure than earlier work on those markets. That is to say, we consider a generalization of the classical three-sided assignment market, where value is generated by pairs or triplets of agents belonging to different sectors, as well as by individuals. The difference between the classical three-sided model (see Kaneko and Wooders, 1982) and ours is that we allow pairs and individuals to attain non-negative values, whereas in the classical model even though pairs may have a weight, they have null value. As a consequence, our class is wider since it includes games that are not strategically equivalent to a Kaneko and Wooders (1982) three-sided assignment game. These generalized three-sided assignment games that we introduce may also have an empty core as the classical three-sided assignment games. However, we present a subclass of balanced generalized three-sided assignment markets. This subclass is defined by means of three properties: (a) non-negativeness, (b) the worth of a triplet is obtained by summing up the worths of its two-player subcoalitions, and (c) there is an optimal partition that is also optimal for each underlying two-sided market.

For these generalized three-sided assignment markets we represent the situation that arises when some agents leave the market with some payoff by means of a generalization of Owen (1992) derived market. The idea behind is that, in general, the Davis and Maschler (1965) reduced game of a three-sided assignment game is not superadditive, hence not an assignment game. We show that for any generalized three-sided assignment game, its derived game is the superadditive cover of the corresponding Davis and Maschler reduced game, and hence we stay within the class of generalized three-sided assignment markets.

Sasaki (1995) and Toda (2005) give axiomatic characterizations of the core of two-sided assignment games. These characterizations cannot be extended to the three-sided assignment games. Here, making use of a consistency property with respect to the derived market, that is *derived consistency*, together with *singleness best*, and *individual anti-monontonicity*, we give an axiomatic characterization of the core for these generalized three-sided assignment markets. Furthermore, by means of examples, we show that the properties are logically independent.

In Chapter 3, we also extend an important result for two-sided assignment games to the three-sided case. We consider the market situation where one of the sectors is formed by buyers and the others by sellers of two different types of goods. In the two-sided case, it is known that the core coincides with the set of competitive equilibrium payoff vectors. We show that the set of competitive equilibrium payoff vectors also coincides with the core for the generalized three-sided assignment markets. This generalizes the result in Gale (1960) for two-sided assignment markets and Tejada (2010) for the classical three-sided assignment markets where buyers are requested to get exactly one item of each type.

Another extension that we propose is studied in Chapter 4. In this chapter, we study multi-sided assignment games on m-partite graphs. We consider a multi-sided assignment game with the following characteristics: (a) the agents are organized in m sectors that are connected by a graph $\overline{G}$ that induces a weighted m-partite graph G on the set of agents, (b) a basic coalition is formed by agents from different sectors that are connected by $\overline{G}$, and (c) the worth of a basic coalition is the addition of the weights of all its pairs that belong to connected sectors.

We provide a sufficient condition on the weights to guarantee balancedness (non-emptiness of the core) of the related multi-sided assignment game since these multi-sided assignment games may also have an empty core. The idea behind the sufficient condition is due to the relation between an optimal matching of the multi-sided assignment game on an m-partite graph and optimal matchings of the underlying two-sided markets. When this sufficient condition holds, the composition of core elements of all underlying two-sided markets gives a core element of the multi-sided one. However, in general, not all core elements can be obtained in this way. Moreover, this sufficient condition depends on the weights, that is to say, a same graph structure may lead to balanced or unbalanced markets depending on the set of weights. Thus, we assume a specific graph structure on the set of sectors, that is cycle-free graphs, to obtain a balancedness result that does not depend on the weights.

We show that when the quotient graph $\overline{G}$, defined on the sectors of the market is cycle-free, the game is strongly balanced and hence the multi-sided assignment game on an m-partite graph has a non-empty core regardless the set of weights. Moreover, we characterize the core of the multi-sided assignment game on an m-partite graph when the quotient graph $\overline{G}$ is cycle-free. We show that the core is fully described by means of the cores of the underlying two-sided assignment games associated with the edges of $\overline{G}$. The characterization of the core of a multi-sided assignment game by means of the cores of associated two-sided assignment games allows us to deduce some properties of the core of the multi-sided assignment game from the known properties of the core of two-sided assignment games. These results related to the core structure cannot be extended to an arbitrary multi-sided assignment game because they rely on the m-partite structure of the network. First, we show that there exists a core allocation where all agents from a sector attain their maximum core payoff, which is their marginal contribution to the grand coalition. Second, we prove that any extreme core allocation is the composition of extreme core allocations of the underlying two-sided markets, whereas the reverse inclusion does not hold in general, as we provide a counter-example.

In this case where we have the characterization of the core, we also study some single-valued solutions for the multi-sided assignment games. We show that neither the nucleolus nor the τ-value of a multi-sided assignment market coincides with the composition of the corresponding single-valued solutions of the two-sided markets, even when $\overline{G}$ is cycle-free. Nevertheless, since the core is fully described by the cores of those underlying two-sided assignment games, we provide two outstanding core allocations, that are, the composed nucleolus and the composed τ-value.

Furthermore, we extend to multi-sided assignment games on an m-partite graph the equivalence between the core and the competitive equilibria for two-sided markets (Gale, 1960; Shapley and Shubik, 1972).

Another main focus of this dissertation is the notion of stability. In Chapter 5, we provide an alternative proof of the characterization of core stability for the two-sided assignment game. Solymosi and Raghavan (2001) characterizes the stability of the core of the assignment game by means of a property of the valuation matrix. They show that the core of an assignment game is a von Neumann-Morgenstern stable set if and only if its valuation matrix has a dominant diagonal. Their proof makes use of some graph-theoretical tools, while ours relies on the notion of buyer-seller exact representative introduced by Núñez and Rafels (2002). The advantage of our approach is that it is based solely on the known bounds for the payoff to each mixed-pair in the core. This approach might be fruitful to do further research to characterize core stability for multi-sided market situations.

Finally, in Chapter 6, we study other stability notions for three-sided assignment games. Since the core may be empty in this case, we focus on other notions of stability such as the subsolutions and von Neumann-Morgenstern stable sets. We first show that the dominant diagonal property of the valuation matrix is a necessary condition for the core to be a von Neumann-Morgenstern stable set. Furthermore, we see that it is also sufficient in the particular case of where each of the three sectors contains only two agents. Hence, we extend the result of Solymosi and Raghavan (2001) on core stability to this particular case.

Then, we extend the notion of μ-compatible subgames introduced by Núñez and Rafels (2013) to the three-sided case. We consider the set formed by the union of the cores of all μ-compatible subgames. Different to the two-sided case, we show by means of a counterexample, that this set may fail to satisfy external stability. Hence, in general it is not a stable set. Moreover, we provide another counterexample and show that it is neither a subsolution. When we restrict the set of feasible outcomes to those imputations that are compatible with some optimal matching μ (these allocations are known as the μ-principal section of the assignment game) we show that the set formed by the union of the cores of all μ-compatible subgames is the set of undominated allocations. In this sense, the aforementioned set, which is always non-empty, is the “core” with respect to the principal section.

Many questions remain open related to multi-sided assignment markets. We have already mentioned the characterization of core stability and the existence and description of some stable set. Also, now that we have proven that the nucleolus of these games is consistent with respect to the derived game reduction, it would be interesting to find some additional property to characterize the nucleolus, as it is done in Llerena et al (2015) for the two-sided assignment markets.

The network situation studied in Chapter 4 could also be reviewed in the classical way, that is, assuming that exactly one agent from each sector were necessary to achieve a positive value. This would be a generalization of the supply-chain model of Stuart (1997). Then, the close relationship with the underlying two-sided markets would be lost but still consequences on the core and other solutions could be derived.

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