Chapter 2

Coalition Formation Games with Separable Preferences

2.1 Introduction

Many of the models used to study coalition formation explore two related questions: which coalitions will form, and how will each coalition share the benefits generated by cooperation? In order to focus solely on the first of these questions, we restrict our attention to the purely hedonic games of Banerjee, Konishi, and Sönmez (2001) and Bogomolnaia and Jackson (1998).[1] In this setting, individuals' preferences are completely determined by the composition of the coalition they belong to, and so the structure of the cooperative game becomes very simple. A feasible allocation consists of a partition of individuals into coalitions. There is no bargaining over a choice of alternatives available to a given coalition or over a choice of payoff distributions among the members of a given coalition, and there is no transferable good that could be exchanged among the members of the coalition or across different coalitions. In effect, each coalition has exactly one payoff vector as a feasible payoff allocation and there are no spillovers among coalitions.[2] Matching models represent instances of hedonic games, even though they only admit a limited subset of coalitional structures.

Despite the simplicity of the model, the existence of stable coalitional structures proves to be a major problem in hedonic situations. Banerjee, Konishi, and Sönmez (2001) find that there are many natural

[1] The terms hedonic game and hedonic coalition structures have been introduced by Bogomolnaia and Jackson (1998) and were borrowed from Drèze and Greenberg (1981), where the hedonic aspect of coalition formation indicates that the utility that agents derive from belonging to a given coalition might also depend on the membership of the coalition itself.

[2] Hedonic games can be easily thought of as the "reduced form" of more complex coalition formation games where, for every coalition, one can foresee which is the feasible alternative that the coalition will choose and how the gains arising from cooperation will be divided. We direct the reader to Bogomolnaia and Jackson (1998) for additional motivation.

settings that do not guarantee the existence of core-stable partitions, that is of partitions for which there is no group of individuals who can all benefit by moving out of their current coalitions and constitute a new (deviating) coalition. In particular, they consider the assumption of additively separable preferences coupled with symmetry that are generally seen as beneficial in ruling out cyclical coalitional deviations. Additive separability means that each player is endowed with a utility function with which he ranks every other player in society. Then each player's valuation of a given coalition to which he belongs is simply the sum of the individual utilities separately assigned by that player to the other members of the coalition. Moreover, additively separable preferences are symmetric if the utilities can be chosen so that the valuations that each pair of individuals assign to each other are the same. Banerjee et al. (2001) provide counterexamples showing that these properties are not sufficient to guarantee the non-emptiness of the core of a hedonic game. Bogomolnaia and Jackson (1998) investigate stability concepts that only involve individual movements and they show that additively separable and symmetric preferences always guarantee the existence of a Nash-stable coalition partition. Nash-stability requires that no individual would choose to disrupt the partition by abandoning his (possibly singleton) coalition for a different one, whose members need not welcome his addition.[3]

Inspired by these findings, we initially focus our attention on symmetric additively separable preferences. Are there natural additional assumptions guaranteeing that core-stable partitions exist? If so, will the methods of proof suggest ways to relax the already strong assumption of symmetric additive separability? We introduce a decomposition of the utility vectors representing symmetric additively separable preferences into two components, namely the cardinal component and the alternating component. It turns out that these components have different implications for stability. When the alternating component is the only one present, we show (by means of a counterexample involving 14 players) that the core might be empty. But when the cardinal component stands alone, stability is guaranteed. If agents' preferences are purely cardinal, it is as if each agent were assigned a cardinal weight, i.e. a real number, which represents the fixed contribution this agent brings to the relationship with any other agent. Therefore, the sum of two agents' intrinsic weights coincides with the increment (or decrement) each contributes to the utility of the other by their joint membership in a coalition. One of our main results is that, when agents' preferences are restricted to be purely cardinal, there always exists a coalition structure which is both core and Nash-stable. If preferences are not required to be strict, then such a stable coalition structure need not be unique.

Implicit in purely cardinal preferences are individual weights that induce a ranking of agents, which is such that players with higher weights come above those with lower ones. This ranking plays a dual role. First, it may be thought of as a preference ordering over individuals that is common to all agents

[3] If this requirement is weakened so that it only applies when all members of the new coalition welcome the addition of the moving individual, the resulting notion is individual stability. It is implied both by Nash stability and by core stability.

in society, so that the players with the highest weights are unanimously preferred to players with lower weights. This condition is reminiscent of the ones introduced in Farrell and Scotchmer (1988) and in Banerjee, Konishi and Sönmez (2001), where a common ranking of coalitions is imposed in order to get non-emptiness of the core of a coalition structure. Nevertheless, we must draw a contrast here with the assumptions made in the works we have just mentioned. Our common relative ranking of individuals does not extend to a common ranking over coalitions, because individuals typically differ as to which other individuals they like to associate with, in an absolute sense rather than in comparison with others. For instance, consider three players, a, b and c; having individual weights of 10, 1 and -5 respectively. Then, player a prefers $\{a, b, c\}$ to $\{a, b\}$ while player b prefers $\{a, b\}$ to $\{a, b, c\}$. Secondly, as an individual's weight decreases, not only does his desirability to all others decrease correspondingly, but his degree of desire for associating with others decreases as well. An individual's weight thus serves as a sort of congestion parameter that helps set the absolute size of the coalition that such individual most prefers belonging to. Notice that stability is unlikely if there exists some individual who desires to associate with many others but who is not considered desirable by many others. Hence, a degree of agreement between desire and desirability is necessary in some form. Purely cardinal preferences are ones for which this agreement is complete.

We also prove that precisely the same results concerning the existence of core and Nash-stable coalition partitions hold under a weaker set of requirements, that we call Descending Separable (or DS) Preferences. The properties in this set preserve some of the qualitative features of purely cardinal preferences and fall into several types (weakened, ordinal versions of additive separability, of symmetry, of descending desirability of individuals, and of descending desire of individuals). The existence of a common ranking of individuals is still an important feature of this class of individuals' preferences.

Very similar assumptions have already been studied in the literature. Weak forms of additive separability date back at least as far as de Finetti's seminal work in mathematical psychology (1931 and 1937), but also see Kraft, Pratt and Seidenberg (1959) and Fishburn (1986). Such assumptions have played a key role in matching theory, see for example Roth and Sotomayor (1990), or Dutta and Massò (1997) and Martínez, Massò, Neme and Oviedo (2000). Banerjee, Konishi and Sönmez (2001) consider mutuality, an ordinal weakening of symmetry.

Intuitively, there is a simple process by which both purely cardinal preferences and DS preferences lead to stable coalitional structures. Suppose that the top-ranked individual, agent a, is entitled to form his coalition within the partition. Agent a proposes to each of the other agents in turn, in descending order. He starts with the second-ranked agent and asks her whether or not she prefers joining her coalition to remaining alone. If she joins, he proceeds down the ranking, terminating the process once he reaches the first agent who does not wish to be added to the growing coalition. A coalition is then set, which we call the top segment coalition, and all the remaining agents are singletons. If agents' preferences are all strict, this coalition partition then seems to represent the unique subgame perfect equilibrium of the non-cooperative game of group formation that we have just described.

There seems to be a wide variety of economic, social and political situations in which agents can be naturally ordered according to some attribute or characteristic. For instance, consumers might be ranked according to their willingness to pay for a particular good or service, workers are ordered according to some objective measure of their ability, firms that compete on the same output market might be ranked according either to the quality or to the location of the product sold, and so on. The idea has been shown to yield interesting results in a framework somewhat different from the one we use here. Among others, Greenberg and Weber (1986), Farrell and Scotchmer (1988), Demange and Henriet (1991) and Demange (1994) study the process of coalition formation when coalitions have to choose from among a set of feasible alternatives and there is a single attribute that orders agents' preferences. In these models, the existence of core-stable coalitional structures is guaranteed under the following basic assumptions: (i) individuals are ranked according to some parameters that order their preferences; (ii) individuals' preference orderings satisfy the intermediate preference property;[4] (iii) the game satisfies some form of monotonicity.[5] Not only do these assumptions ensure the existence of core-stable coalitional structures, but they also determine the qualitative features of stable structures. All coalitions in a stable partition are consecutive, in the sense that, if two distinct individuals are members of the same coalition, then all the players that lie between them (according to the given ordering of players) are also members of that coalition. Thus, as in our model, stable coalitional structures have the property that individuals with similar attributes tend to cluster together. Monotonicity does not seem to apply at all to our context, and the form of intermediate preference property that holds under purely cardinal preferences may fail under the more relaxed assumptions of Descending Separable preferences.

The structure of the paper is as follows. Section 2.2 describes the model and introduces the notation and the definitions we will be working with. Section 2.3 describes several related decompositions of additively separable preferences, which we think may have their own interest, with attention mainly devoted to the symmetric case. The examples in Section 2.3.1 are quite compelling, allowing the reader who wishes to skip Section 2.3.2 and go directly to Section 2.4, where the stability properties of symmetric additively separable preferences are examined. In Section 2.5 we present the DS set of properties, which is weaker than purely cardinal preferences but yields the same qualitative results. In the last two sections we also examine the connections between our set of conditions and other properties appearing in the literature.

[4] This means that for any three distinct agents, if the two extreme agents (according to the ranking of players) agree as to their preference ordering of two alternatives, then the agent who is in between also does.

[5] When a coalition increases in size it may widen, but never narrow, its set of feasible alternatives. In this sense, a growing coalition can only improve its prospects.

2.2 Notation and Definitions

Consider a finite society composed of $N = \{1, 2, ..., n\}$ members. A coalition C is an element of 2^N, i.e. a subset of individuals that belong to society. A coalition partition or coalition structure is a partition of N and it will be denoted as $\pi = \{C_h\}_{h=1}^{H}$, where H is a positive integer with $N \geq H$. Thus, π consists of an exhaustive collection of non-empty, pairwise disjoint coalitions $C_h \subseteq N$.

Agents' preferences are defined over the set of all coalition structures in N, which is denoted by $\Pi(N)$. Throughout, we will assume that each individual $i \in N$ has preferences that are purely hedonic: each agent's preferences over partitions are completely characterized by his preferences over the coalitions that he belongs to in each partition. Therefore, each agent i is endowed with a preference ordering $\succsim_i$ (a reflexive, complete and transitive preference relation) over the set $C_i(N) \equiv \left\{C \in 2^N \mid i \in C\right\}$ which satisfies that

$$\pi \succsim_i \pi' \Leftrightarrow C_\pi(i) \succsim_i C_{\pi'}(i), \tag{2.1}$$

where $C_\pi(i)$ denotes the coalition in π to which agent i belongs. We denote by $\succ_i$ the strict preference relation.

Definition 1 A hedonic coalition formation game is a pair $\Gamma = [N; \{\succsim_i\}_{i=1}^{n}]$, where N is the finite set of players or members of society and preference profiles $\{\succsim_i\}_{i=1}^{n}$ satisfy condition (2.1).

We consider the solution concepts arising from core-stability and Nash-stability, with formal definitions as follows.

Definition 2 A coalition partition π is **core-stable** (or is in the core of a coalition structure) if $\nexists S \subseteq N$, with $S \notin \pi$, such that $S \succ_i C_\pi(i)$ for all $i \in S$.

Definition 3 A coalition partition π is **Nash-stable** if, for all $i \in N$ and for all $C_h \in \pi$, $C_\pi(i) \succsim_i C_h \cup \{i\}$.

It is straightforward to check that neither type of stability implies the other. A necessary condition that every partition π has to satisfy in order to be either core-stable or Nash-stable is individual rationality.

Definition 4 A coalition partition π is **individually rational for player** i if $C_\pi(i) \succsim_i \{i\}$ and is **individually rational** if, for every agent $i \in N$, it is individually rational for player i.

Let us now turn to some properties that preference profiles might satisfy.

Definition 5 A profile of agents' preferences $(\succsim_1, \succsim_2, ..., \succsim_n)$ satisfies **separability** if, for every $i, j \in N$ and every coalition C such that $C \in C_i(N)$ and $j \notin C$,

$$\{i, j\} \succsim_i \{i\} \Leftrightarrow C \cup \{j\} \succsim_i C \quad \text{and} \quad \{i, j\} \succ_i \{i\} \Leftrightarrow C \cup \{j\} \succ_i C.$$

Preferences are separable if the effect of a given player on another player's preferences is consistently positive, negative, or neutral, regardless of which coalition the latter player is a member of. Hence, there is no complementariness among players belonging to a given coalition. It is rather as if each player divided the remaining players in three disjoint sets: the set of good agents, the set of bad agents and the set of neutral agents. Adding a good agent to a coalition always makes the coalition better, adding a bad agent always makes it worse, whereas adding a neutral agent never changes it.

Definition 6 A profile of agents' preferences $(\succeq_1, \succeq_2, ..., \succeq_n)$ satisfies **additive separability** (or is additively representable) if, for every $i \in N$, there exists a real-valued function $v_i : N \to \mathbb{R}$ such that

$$C \succeq_i C' \Leftrightarrow \sum_{j \in C} v_i(j) \geq \sum_{j \in C'} v_i(j)$$

for all $C, C' \in \mathcal{C}_i(N)$.

Thus, additively separable preferences are such that every individual attaches a value to each other individual in society and the utility that an agent receives from being in a given coalition is simply the sum of the values that the agent assigns to the other members of the coalition. Note that $v_i(j)$ stands for the cardinal utility assigned by player i to player j, which is the contribution that players j makes to the total utility that individual i obtains from membership in any coalition containing both players. The value that player i assigns to himself has no effect on his ranking, and so it is commonly set at $v_i(i) = 0$. Additive separability is a stronger requirement than separability.

A profile of additively separable preferences satisfies *symmetry* if it can be represented by a vector $v = (v_1, v_2, ..., v_n)$ satisfying that agents assign the same reciprocal value to each other, i.e. $v_i(j) = v_j(i)$ for every $i, j \in N$. In the symmetric case, we will use $v(i, j)$ to denote the common value of $v_i(j)$ and $v_j(i)$.

A profile of preferences, satisfies *mutuality* if

$$\{i\} \succeq_i \{i,j\} \Leftrightarrow \{j\} \succeq_j \{i,j\} \quad \text{and} \quad \{i,j\} \succ_i \{i\} \Leftrightarrow \{i,j\} \succ_j \{j\}. \tag{2.2}$$

For additively separable preferences, this is equivalent to saying that the profile can be represented by cardinal utilities satisfying $v_i(j) \geq 0 \Leftrightarrow v_j(i) \geq 0$ and $v_i(j) > 0 \Leftrightarrow v_j(i) > 0$ for all $i, j \in N$.[6]

It is already known that there are basic problems in finding stable coalition partitions in the context of purely hedonic situations, even when domain restrictions on individual preferences are imposed. Banerjee, Konishi and Sönmez (2001) showed that the set of core-stable coalition partitions can be empty when preferences are additively separable and satisfy symmetry. Nonetheless, Bogomolnaia and Jackson (1998) prove that under the same restrictions, i.e. if individuals' preferences are additively separable and symmetric, Nash-stable partitions always exist. In particular, any partition that maximizes the sum of all agents' utilities is Nash-stable.

[6] Therefore, symmetry implies mutuality.

2.3 Vector Decompositions of Additively Separable Preferences

The purpose of this section is to show that any assignment $v = (v_1; v_2; \ldots; v_n)$ of utilities representing additively separable preferences may be thought of as a vector in an appropriate finite-dimensional vector space. We introduce a sequence of decompositions that break this vector v into components. The various components then have different implications for the existence of stable solutions for the corresponding hedonic coalition formation game.

Let agents' preferences be additively separable, but not necessarily symmetric, and let such preferences be represented by the vector $v = (v_1; v_2; \ldots; v_n)$. Given that $v_i(i) = 0$ for all $i \in N$; and that each agent assigns a real-number utility to the other $n - 1$ players in society, the utility profile v may be considered as a vector in the space $\mathbb{R}^{n(n-1)}$: This fact will allow us to associate a certain labeled graph to any vector $v = (v_1; v_2; \ldots; v_n)$ representing additively separable preferences.

Definition 7 A **directed graph** G consists of a finite set N of **vertices** together with a set E of directed **edges**, each of which is an ordered pair $(i; j)$ of vertices satisfying $i \neq j$; depicted as an arrow from vertex i to vertex j, and denoted by $e_{i \to j}$: The **complete directed graph** $K_{\to}(N)$ on the vertex set N is the directed graph that includes all possible directed edges.

In the present context, the vertices of a complete directed graph correspond to the players in N and each edge $e_{i \to j}$ directed from player i to player j has some weight associated to it, which represents the utility $v_j(i)$ that agent i contributes to agent j. The utility $v_i(j)$; which is generically different from $v_j(i)$; can be represented as the weight on the opposite edge $e_{j \to i}$.

In the following Section 2.3.1, we will present an extended example as an introduction to these ideas. In the next Section 2.3.2, we will sketch the history of, and fill in the general theory behind, some of the unproved assertions appearing in Section 2.3.1.

2.3.1 An Extended Example

Let $N = \{1; 2; 3; 4\}$ and let agents' preferences be depicted by the complete graph in Figure 2.1. The 12 edge weights of this figure are the 12 components of the vector of utilities $v = (v_1; v_2; \ldots; v_4)$ representing players' additively separable preferences. For example, the edge $e_{1 \to 2}$ directed from vertex 1 to vertex 2 has weight 7, so $v_2(1) = 7$ is the utility agent 1 contributes to agent 2 whenever 1 is in 2's coalition. The utility $v_1(2) = 5$ is different, and appears as the edge weight on the opposite edge $e_{2 \to 1}$:

Any vector v can be decomposed into a symmetric component v^S; satisfying $v_i^S(j) = v_j^S(i)$ for each i and j, and an antisymmetric component v^A; satisfying $v_i^A(j) = -v_j^A(i)$ for each i and j. It is easy to compute v^S and v^A from any vector of utilities v, as follows

$$v_i^S(j) = \frac{v_i(j) + v_j(i)}{2} \tag{2.3}$$

and

$$v_i^A(j) = \frac{v_i(j) - v_j(i)}{2}. \tag{2.4}$$

Moreover this decomposition is unique, in that there is only one way to write v as the sum $v = v^S + v^A$

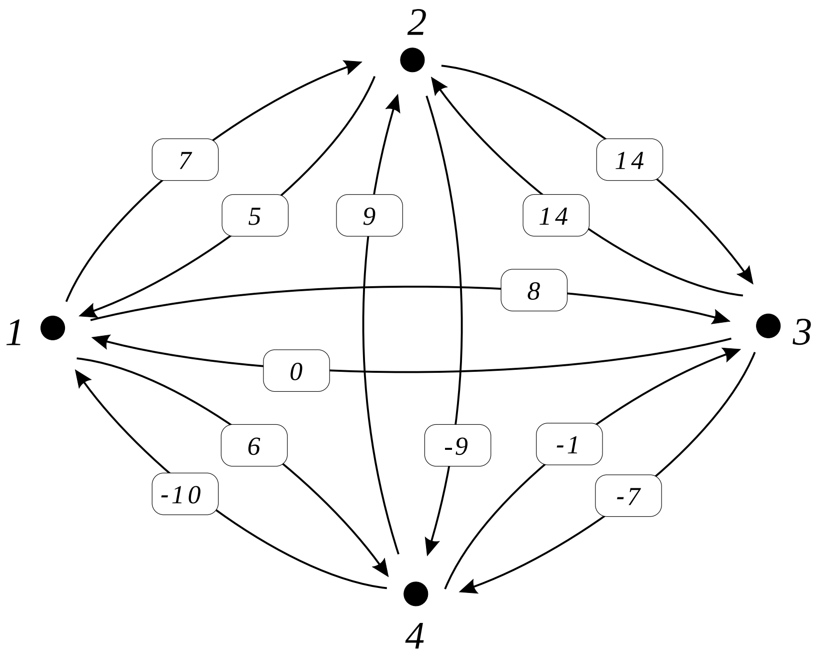

Figure 2.1: An asymmetric vector v of utilities

of a symmetric vector and an asymmetric vector, where two vectors are added (which in R^n is done by adding corresponding components) by adding the weights on corresponding edges. Notice that vector v is itself symmetric if and only if $v = v^S$ (equivalently, if and only if $v^A = 0$) and is antisymmetric if and only if $v = v^A$ (equivalently, if and only if $v^S = 0$).

Consider the two vectors v^S and v^A appearing in Figure 2.2. Notice that v^S is indeed symmetric and that v^A is antisymmetric. Furthermore, the vectors v^S and v^A are orthogonal to each other in the sense that $v^S \cdot v^A = 0$, where the inner or dot product $v \cdot u$ of any two vectors v and u is computed by multiplying weights on corresponding directed edges, and then adding these products[7]

$$v \cdot u = \sum \{v_i(j)\, u_i(j) \mid i, j \in N \text{ and } i \neq j\}. \tag{2.5}$$

If we let $V = V(N)$ be the vector space of all possible utility assignments (of agents in the set N to each other), $S = S(N)$ be the subset of V containing all the symmetric vectors, and $A = A(N)$ be the

[7] To make this inner product agree with the one introduced later for symmetric vectors, we would need to define

$$v \cdot u = \frac{1}{2} \sum \{v_i(j)\, u_i(j) \mid i, j \in N; \text{ with } i \neq j\}$$

but this scale factor of $\frac{1}{2}$ is of little account.

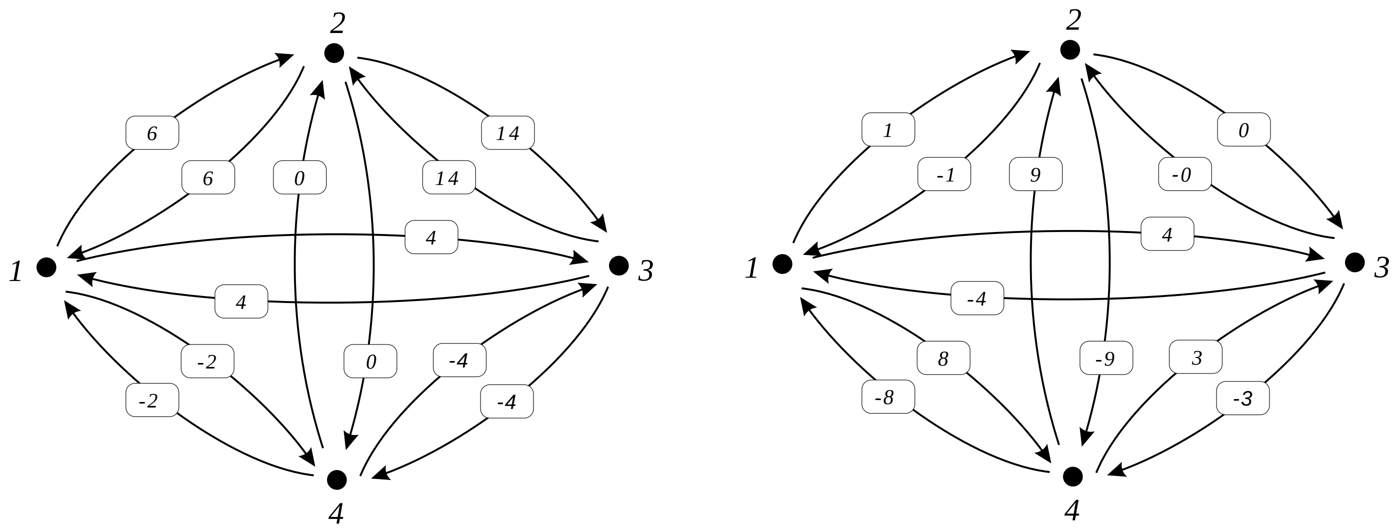

Figure 2.2: Decomposition of v from Figure 2.1 as $v = v^S + v^A$

subset of V containing all the antisymmetric vectors, it is easy to see that A and S are subspaces of V, and in fact are orthogonal complements in V, with V being the direct sum decomposition of A and S

$$V = S \oplus A.$$

Thus, formula (2.3) used to find v^S is actually calculating the orthogonal projection of v onto S, while (2.4) calculates the orthogonal projection v^A of v onto A.

The representations of the components v^S and v^A that appear in Figure 2.2 can be simplified by eliminating the redundant information in the edge weightings. In the case of v^S; the two edges connecting any pair of vertices always receive the same weight, so the v^S information may be conveyed by assigning this common weight to a single, undirected edge between the pair. For v^A these two edges also receive the same weight, but with opposite signs, so we arbitrarily discard one from each pair of directed edges, while saving the other directed edge with its weight. Because our main concern in this paper will be with the symmetric component, we will use the term *edge weighting* to refer to a weighting of edges of a complete undirected graph. Thus an edge weighting is the same as a symmetric vector or assignment of a real number utility $v(i; j)$ to each unordered pair or edge $\{i; j\}$ (with $i \neq j$) in the complete undirected graph $K(N)$.

In Figure 2.3 we consider a further decomposition of the symmetric part v^S of our example into a cardinal component and an alternating component

$$v^S = v^{CARD} + v^{ALT}.$$

This second decomposition is similarly based on a direct sum decomposition of S

$$S = S^{CARD} \oplus S^{ALT}$$

into two orthogonal and complementary subspaces, with the components v^{CARD} and v^{ALT} being the orthogonal projections of v^S onto these subspaces. Here, the appropriate inner product is exactly what one might expect, i.e.

$$v \cdot u = \sum \{ v(i;j)\, u(i;j) \mid \{i;j\} \subseteq N \text{ and } i \neq j \}. \tag{2.6}$$

To explain this decomposition, we begin by exploring some properties of the components.

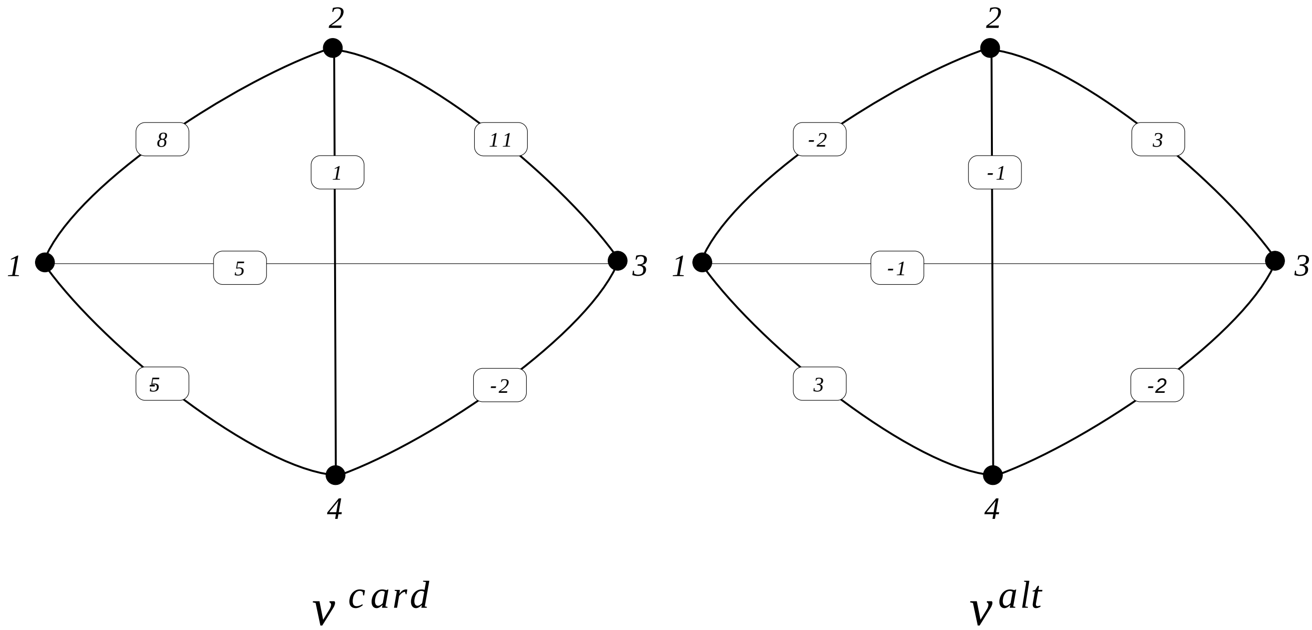

Figure 2.3: Decomposition of v^S into v^{CARD} and v^{ALT}

What characterizes the cardinal component in the decomposition of Figure 2.3? Any assignment w of real number weights to the vertices of a graph induces a corresponding edge weighting, as follows. To find the weight on any edge, sum the two vertex weights on the vertices joined by that edge (we will refer to such a sum as an edge sum). The cardinal component v^{CARD} of a vector is always induced as the edge sums of some vertex weighting, and S^{CARD} is defined to be the linear subspace of all edge weightings induced, via edge sums, by some vertex weighting. But how do we compute the decomposition

$$v = v^{CARD} + v^{ALT}$$

for an arbitrary edge weighting (symmetric utility assignment) v?

Definition 8 For each vertex i of the (undirected and complete) graph $K(N)$, the **load on** i is the sum $\sum \{v_i(j) \mid i \neq j\}$ of all edge weights on edges incident to i, and the **load off** i is the sum $\sum \{v_k(l) \mid k < l$ and $k, l \neq i\}$ of all edge weights on edges not incident to i. The **average load on** i is the average value

$$\frac{1}{(n-1)} \sum \{v_i(j) \mid i \neq j\}$$

of all edge weights on edges incident to i, and the **average load off** i is the average value

$$\frac{1}{(n-1)(n-2)} \sum \{v_k(l) \mid k < l \text{ and } k, l \neq i\}$$

of all edge weights on edges not incident to i.

Let the quantity $w(i)$ be defined by

$$w(i) \equiv [\text{average load on vertex } i] - \frac{1}{2}[\text{average load off vertex } i]; \qquad (2.7)$$

then $w(i)$ gives the vertex weights that induce, via edge sums, the component v^{CARD}. For example, if we take v to be the vector v^S in Figure 2.2, then

$$w(1) = \frac{(6+4-2)}{3} - \frac{1}{2}\frac{(14-4+0)}{3} = 1.$$

Figure 2.4 provides the vertex weighting that induces the v^{CARD} of Figure 2.3.

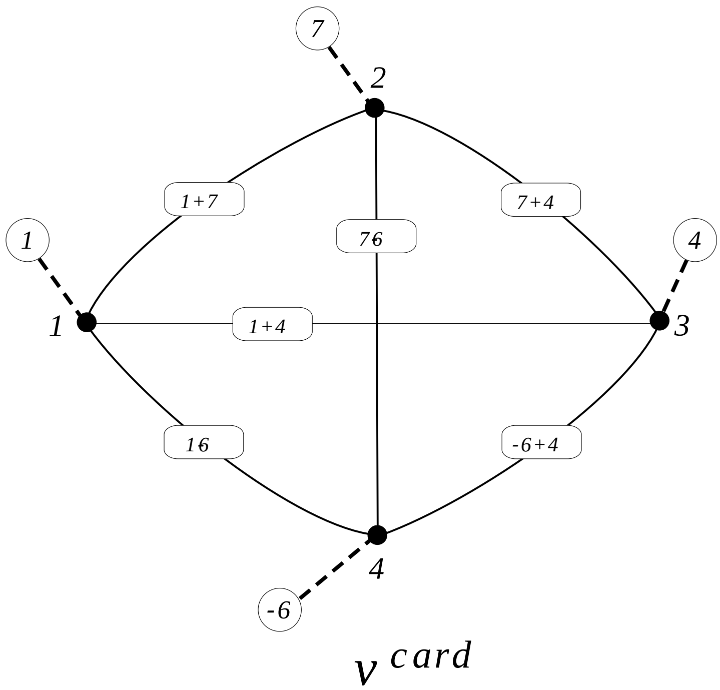

Figure 2.4: v^{CARD} induced as the edge sum of a vertex weighting

As far as a formula for computing the alternating component v^{ALT}, the simplest approach is to use

$$v^{ALT} = v - v^{CARD}.$$

Along the same lines, the subspace S^{ALT} that alternating components belong to could be defined as the orthogonal complement, within the space S, of S^{CARD}. However, we seek a more useful characterization of the part of a utility vector that is not cardinal. In Figure 2.5, we show that v^{ALT} can be expressed as a linear combination of two basic alternating cycles

$$v^{ALT} = \alpha_1 + 2\alpha_2;$$

where α_1 and α_2 are the edge weightings that appear on the right side of Figure 2.5. More generally, S^{ALT} may be characterized as the linear span of all alternating cycles. Before we introduce some definitions in order to make these notions precise, notice one additional feature of v^{ALT} in Figure 2.5: the net load on each vertex is zero. In fact, this property also characterizes membership in S^{ALT}.

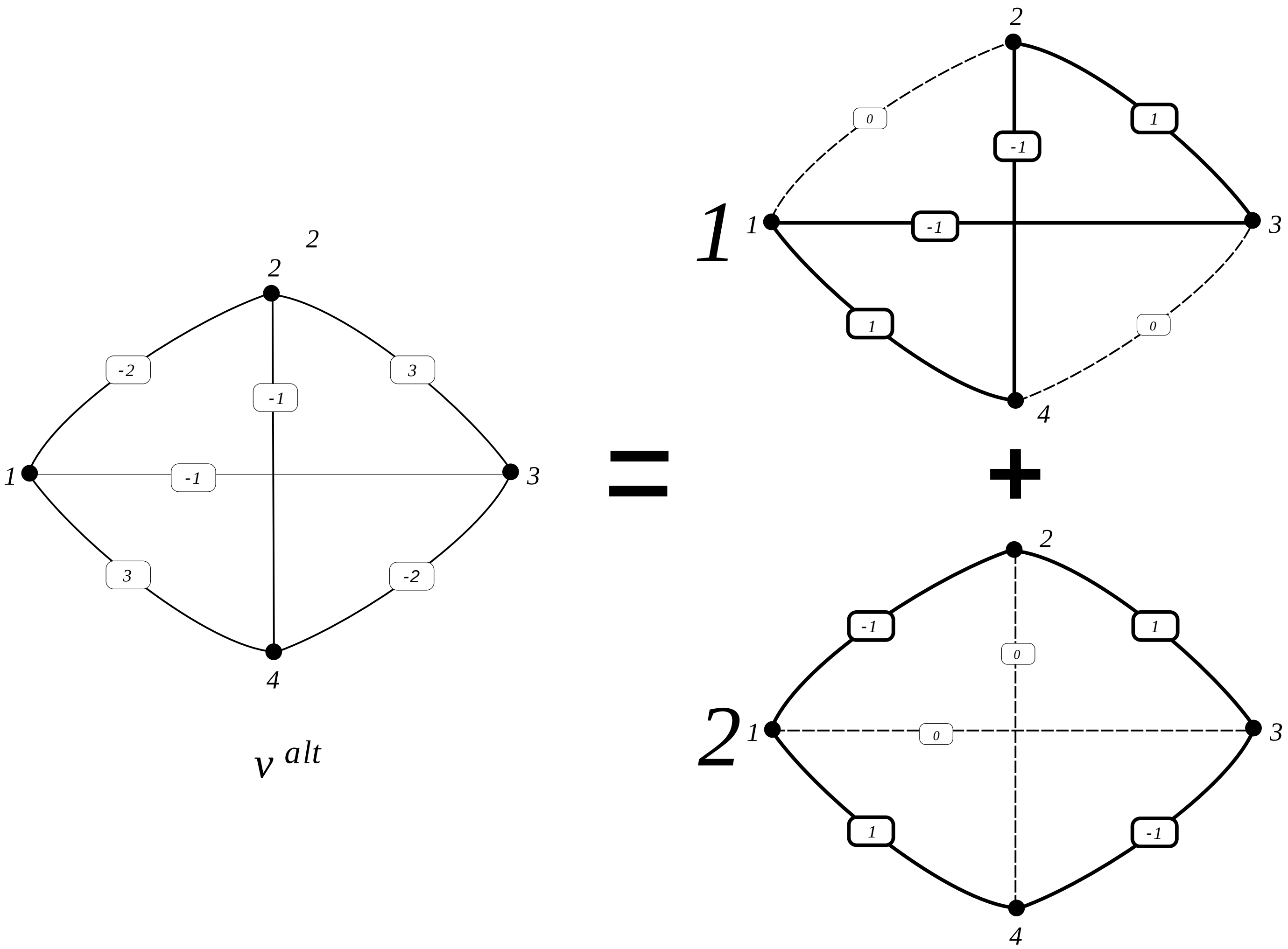

Figure 2.5: v^{ALT} as a linear combination of alternating cycles

Definition 9 A cycle of length m in the (undirected and complete) graph $K(N)$ is a sequence $i_1; i_2; \ldots; i_{m+1}$ of vertices such that there is an edge from i_l to i_{l+1} for each $i = 1; 2; \ldots; m$;[8] where $i_l = i_{m+1}$ and the vertices $i_1; i_2; \ldots; i_m$ are all distinct from each other.

The length of a cycle thus coincides with the number of distinct vertices or edges in the cycle.

[8]Our graphs are complete, so a cycle automatically satisfies the requirement that the graph contains an edge from i_l to i_{l+1} for each l:

Definition 10 An **even cycle** is a cycle of even length m. The **basic alternating cycle** $(i_1, i_2, \ldots, i_m)$ corresponding to the even cycle $C = i_1, i_2, \ldots, i_{m+1}$ is the vector v given by

$$v(a,b) = \begin{cases} (-1)^l & \text{if } \{a,b\} = \{i_l, i_{l+1}\} \text{ is an edge in cycle } C \\ 0 & \text{if } \{a,b\} \text{ is an edge that does not belong to cycle } C \end{cases}.$$

A scalar multiple of a basic alternating cycle is called an alternating cycle. We can now define S^{ALT} to be the linear span of the set of all basic alternating cycles. Equivalently, a vector v is an element of S^{ALT} if and only if v is a sum of alternating cycles.

Notice that in our example there are alternating cycles other than the α_1 and α_2 appearing in Figure 2.5. In fact, any other such cycle is a linear combination of α_1 and α_2. We will show in the next section that when $n = 4$, the dimension of S^{ALT} is equal to 2, so that any two linearly independent alternating cycles, such as α_1 and α_2, serve as a basis for S^{ALT}.

2.3.2 History and General Theory of Vector Decompositions

The two decompositions discussed in the previous section are

$$V = S \oplus A \tag{2.8}$$

and

$$S = S^{CARD} \oplus S^{ALT}. \tag{2.9}$$

In fact, there is a third decomposition that completes the picture, which is

$$A = A^{BORDA} \oplus A^{CYCLE}. \tag{2.10}$$

This means that we may take any vector $v \in V(N)$ and write it uniquely as the sum of the four mutually orthogonal components

$$v = v^S + v^A = (v^{CARD} + v^{ALT}) + (v^{BORDA} + v^{CYCLE}).$$

There is a loose but important analogy between the last two decompositions: decomposition (2.9) is to graphs as decomposition (2.10) is to directed graphs.

Historically, decomposition (2.10) arose first, under a different terminology. Essentially, it is the decomposition induced by the boundary map of homology theory (a branch of algebraic topology) in the one-dimensional case (see, for example, Giblin [1977]). This decomposition has had some interesting applications. In the study of electric circuits, a flow of current in a complex circuit may be decomposed into the part arising from sources and sinks of current, and the part consisting of cycling currents. This analysis serves as the algebraic basis for the well known Kirchoff's Laws in circuit theory. The decomposition has also been applied to social choice theory. Imagine that the vertices of our directed graph are the candidates in a multicandidate election, and that the "flow" in the "wires" indicates net preferences for these candidates among the electorate, given some particular profile of preferences. So, for example, a flow of 14 units on the edge from candidate i to candidate j indicates that the number of voters preferring j to i is greater, by a margin of 14, than the number who prefer i to j. When one applies decomposition (2.10), one of the components corresponds to the sequence of total scores awarded to each candidate in a Borda count election applied to the profile, and the other component represents the underlying tendency for a Condorcet cycle (paradox of voting) for the profile given. This idea has been applied in Zwicker (1991) to find necessary and sufficient conditions for transitive outcomes, and more recently it has been applied in a variety of ways by Saari (2000).

At the moment, this decomposition seems less useful to the study of stability in hedonic coalition formation games, because the entire antisymmetric component has strongly negative implications for stability. Consider, for example, the case of two agents, a and b, with $v_a(b) = 1$ and $v_b(a) = -1$. It is easy to see that the partition $\{\{a, b\}\}$ is not core-stable, while the only alternative partition $\{\{a\}, \{b\}\}$ is not Nash-stable. We have not, however, studied games based on utilities that have both a non-zero symmetric component and also a non-zero antisymmetric component, and it remains possible that decomposition (2.10) will play a role in the analysis of such games.

Decomposition (2.9) is of much more recent vintage. It was used in work of Bolker (1979) on the rigidity of structures, as well as in Zaslavsky (1982), Bouchet (1983), Khelladi (1987), and may have been anticipated by Edmonds, as reported in Lawler (1976). However, it appears most explicitly in Grossman et al. (1994), and this is the reference we recommend to the interested reader.[9]

As far as we know, our work here represents the first application of decomposition (2.9) to the social sciences. As with decomposition (2.10), however, this application is solely concerned with the special case of complete graphs, while the earlier literature focuses on graphs that are not complete. With complete graphs, as we might expect, more can be said and some things can be said more simply. This explains why our Theorem 12 below looks somewhat different in content, as well as in terminology, from what appears in Grossman et al. (1994). One important difference is that for complete graphs, alternating cycles alone suffice to span the orthogonal complement of S^{CARD}, while for graphs in general one needs to add other vectors (called odd handcuffs) to the alternating cycles. For these reasons, our discussion of decomposition (2.9), and the proof of Theorem 12, are self-contained.

Decomposition (2.8) appears to be new, although it is clearly related to familiar ideas with a long

[9] We are indebted to Thomas Zaslavsky for pointing out the parallels between our work on decomposition (2.9) and this earlier literature. Comments on the papers we cite here, and on other related work, can be found on Zaslavsky's excellent "Mathematical bibliography of signed and gain graphs and allied areas", at http://www.math.binghamton.edu/zaslav/Bsg/index.html.

history. It is also a very simple decomposition, and so we outline its basic properties in the following theorem, while leaving the routine proof to the reader.

Theorem 11 Let $N = \{1, 2, \ldots, n\}$ be a finite set and $V = V(N)$ denote the vector space of all assignments v of real number weights to the edges of the complete directed graph $\vec{K}(N)$ on the vertex set N. If V is endowed with the standard inner product[10] then $V(N)$ is a Hilbert space satisfying the following properties. (1) The dimension of V is $(n)(n-1)$. (2) The subset $S = S(N)$ consisting of all symmetric edge-weight assignments, satisfying $v(e_{i\to j}) = v(e_{j\to i})$ for all i and j, forms a subspace of V having dimension $\frac{1}{2}(n)(n-1)$. (3) The subset $A = A(N)$ consisting of all antisymmetric edge-weight assignments satisfying $v(e_{i\to j}) = -v(e_{j\to i})$, for all i and j, forms a subspace of V having dimension $\frac{1}{2}(n)(n-1)$. (4) The subspaces S and A are orthogonal complements in V, from which it follows that $V = S \oplus A$. (5) The symmetric and antisymmetric components of any vector $v \in V(N)$ are given by $v^S(e_{i\to j}) = \frac{1}{2}(v(e_{i\to j}) + v(e_{j\to i}))$ and $v^A(e_{i\to j}) = \frac{1}{2}(v(e_{i\to j}) - v(e_{j\to i}))$.

Let us now move to consider decomposition (2.9). There is another characterization of the cardinal component v^{CARD}, equivalent to the one presented in Section 2.3.1, that will be useful. Let us define the star σ_i on vertex i to be the edge weighting that assigns weight 1 to each edge incident to vertex i and weight 0 to each edge not incident to i. Stating that the cardinal component v^{CARD} of any vector v is induced, via edge sums, from some vertex weighting is equivalent to saying that v^{CARD} is a linear combination of stars, and the coefficients in this linear combination are equal to the vertex weights inducing v^{CARD}. In particular, for the v^{CARD} of Figure 2.3, we have

$$v^{CARD} = 1\sigma_1 + 7\sigma_2 + 4\sigma_3 - 6\sigma_4 .$$

Thus, an equivalent definition of the linear subspace S^{CARD} is that it is the linear span of the stars. Also, the claim that formula (2.7) for $w(i)$ is correct may be rephrased as

$$v^{CARD} = w(1)\sigma_1 + w(2)\sigma_2 + \cdots + w(n)\sigma_n .$$

which is the form we prove it in, in what follows.

Theorem 12 Let $N = \{1, 2, \ldots, n\}$ be a finite set and $S = S(N)$ denote the vector space of all assignments v of real number weights to the edges of the complete (undirected) graph $K(N)$ on the vertex set N. Endow S with the standard inner product[11] so that it becomes a Hilbert space. Let S^{CARD} denote the linear span of $\{\sigma_i \mid i \in N\}$ and S^{ALT} denote the linear span of all the alternating cycles. Then the following properties hold. (1) The dimension of S is $\frac{1}{2}(n)(n-1)$. (2) S^{CARD} is a subspace of S of dimension n. (3) S^{ALT} is a subspace of S of dimension $\frac{1}{2}(n)(n-1) - n = \frac{1}{2}(n)(n-3)$. (4) The subspaces S^{CARD}

[10] See condition (2.5) in Section 2.3.
[11] See condition (2.6) in Section 2.3.

and S^{ALT} are orthogonal complements in S, from which it follows that $S = S^{CARD} \oplus S^{ALT}$. (5) An element v of S is a member of S^{CARD} if and only if every alternating cycle sum of v is zero (to form an alternating cycle sum of v, choose any even cycle $i_1, i_2, \ldots, i_{2k}$ in $K(N)$ and form the sum of all terms of form $(-1)^j v(\{i_j, i_{j+1}\})$ for $i = 1, 2, \ldots, 2k$, where $i_{2k+1} = i_1$). (6) An element v of S is a member of S^{ALT} if and only if the load on each vertex is zero. (7) The cardinal component of any vector $v \in S$ is given by $v^{CARD} = w(1)\Lambda_1 + w(2)\Lambda_2 + \ldots + w(n)\Lambda_n$, where each coefficient $w(i)$ is determined by (2.7).

Proof. It is clear that S has dimension $\frac{1}{2}(n)(n-1)$ and that S^{CARD} is a subspace of S. It is easy to check that $\{\Lambda_i \mid i \in N\}$ is a linearly independent set and thus forms a basis for S^{CARD}, which must therefore have dimension n.[12] It is also routine to check that for every element v of S^{CARD} and element u of S^{ALT}, $v \cdot u = 0$, so that S^{CARD} and S^{ALT} are orthogonal in S. It will follow that S^{CARD} and S^{ALT} are orthogonal complements in S if we can show that the dimension of S^{ALT} is at least $\frac{1}{2}(n)(n-3)$, because then the sum $\frac{1}{2}(n)(n-3) + n = \frac{1}{2}(n)(n-1)$ of the dimensions of the orthogonal subspaces S^{CARD} and S^{ALT} is equal to that of the space S. We accomplish this by producing, for each n, a linearly independent set T_n of $\frac{1}{2}(n)(n-3)$ basic alternating cycles, each of length 4, in $S(\{1, 2, \ldots, n\})$.[13] It follows immediately that the dimension of S^{ALT} is exactly $\frac{1}{2}(n)(n-3)$. The construction of T_n is by induction on $n \geq 3$. The base step for $n = 3$ is immediate, as the quantity $\frac{1}{2}(n)(n-3)$ is equal to zero when $n = 3$. Assume that T_n is a set of $\frac{1}{2}(n)(n-3)$ linearly independent basic alternating cycles, each of length 4, in $S(\{1, 2, \ldots, n\})$. Let $T_{n+1} = T_n \cup S_n$ where

$$S_n = \{(1, n+1, 2, 3), (2, n+1, 3, 4), \ldots, (n-2, n+1, n-1, n), (n-1, n+1, n, 1), (n, n+1, 1, 2)\}.$$

Note that

$$|T_{n+1}| = |T_n| + |S_n| = \frac{1}{2}(n)(n-3) + n = \frac{1}{2}[(n+1)((n+1)-3)],$$

which is what we desire. It is routine to check that T_{n+1} is a set of linearly independent, basic alternating cycles, each of length 4, in $S(\{1, 2, \ldots, n+1\})$. Now that (4) is confirmed, parts (5) and (6) follow as instances of the general fact that a vector lies in the orthogonal complement of a subspace if and only if it is orthogonal to all elements of a spanning set of that subspace, and so these are polished off as well. It remains to prove (7). Let v be a vector in S. As $v = v^{CARD} + v^{ALT}$, and v^{ALT} contributes 0 to each of the loads appearing in the formula for $w(i)$, it suffices to assume $v \in S^{CARD}$ and prove that $v = u$, where

$$u \equiv (w(1)\Lambda_1 + w(2)\Lambda_2 + \ldots + w(n)\Lambda_n) \in S^{CARD}.$$

[12] But note that $\{\Lambda_i \mid i \in N\}$ is not an orthonormal basis or even an orthogonal basis.

[13] Note that the set of all basic alternating cycles does not form a basis for S^{ALT}. Indeed, S^{ALT} appears to lack a natural basis.

As the $¤_i$ span S^{CARD}, it then suffices to prove that $v \cdot ¤_i = u \cdot ¤_i$ for each $i = 1; 2; \ldots; n$.

We begin with the special case $N = \{1; 2; 3; 4\}$ and prove that $v \cdot ¤_1 = u \cdot ¤_1$. Then we sketch the general case. Let v be given by the edge weights appearing in Figure 2.6. Notice that, for each i and j in this special case,

$$¤_i \cdot ¤_j = \begin{cases} 1 & \text{if } i \neq j; \\ 3 & \text{if } i = j: \end{cases}$$

Thus,

$$u \cdot ¤_1 = w(1)\, ¤_1 \cdot ¤_1 + w(2)\, ¤_2 \cdot ¤_1 + w(3)\, ¤_3 \cdot ¤_1 + w(4)\, ¤_4 \cdot ¤_1 = 3w(1) + w(2) + w(3) + w(4):$$

The first term is such that

$$3w(1) = 3\left(\frac{1}{3}(v_1 + v_2 + v_3) - \frac{1}{2}\left(\frac{1}{3}(v_4 + v_5 + v_6)\right)\right):$$

This is a linear combination of the v_i, whose coefficients appear as the first row of the table below; the other terms, $w(2)$, $w(3)$ and $w(4)$, are likewise given by the coefficients of rows 2, 3 and 4 respectively.

	v_1	v_2	v_3	v_4	v_5	v_6
$3w(1) =$	1	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
$w(2) =$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$
$w(3) =$	$-\frac{1}{6}$	$\frac{1}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$
$w(4) =$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{6}$	$\frac{1}{3}$
$u \cdot ¤_1 =$	1	1	1	0	0	0

The coefficients of $u \cdot ¤_1$ (as a linear combination of the v_j) are then obtained by summing the columns of this table, so that $u \cdot ¤_1 = v_1 + v_2 + v_3$, the total load on vertex 1. As $v \cdot ¤_1$ is clearly also such that $v \cdot ¤_1 = v_1 + v_2 + v_3$, we see that $v \cdot ¤_1 = u \cdot ¤_1$, as desired.

Now consider the more general case, wherein $N = \{1; 2; \ldots; n\}$, and we are proving that $v \cdot ¤_i = u \cdot ¤_i$. Note that in this case,

$$¤_i \cdot ¤_j = \begin{cases} 1 & \text{if } i \neq j \\ n - 1 & \text{if } i = j \end{cases};$$

and the table corresponding to that above has n rows. If we consider a typical column in the table, there are two cases to consider: (i) if the column corresponds to an edge joining vertex i to some other vertex j, then it is straightforward to show that the entries are 1 from row i, $1/(n-1)$ from row j and $-1/(n-1)(n-2)$ from each of the $(n-2)$ other rows, which sums to 1; (ii) if the column corresponds to an edge joining vertex s to a different vertex t, with $s \neq i$ and $t \neq i$ then the entries are $-1/(n-2)$

from row i, $1/(n-1)$ from row s and row t and $-1/(n-1)(n-2)$ from each of the $(n-3)$ other rows, which sums to 0. This leads, as above, to

$$u \cdot \chi_i = \text{total load on vertex } i = v \cdot \chi_i;$$

as desired.

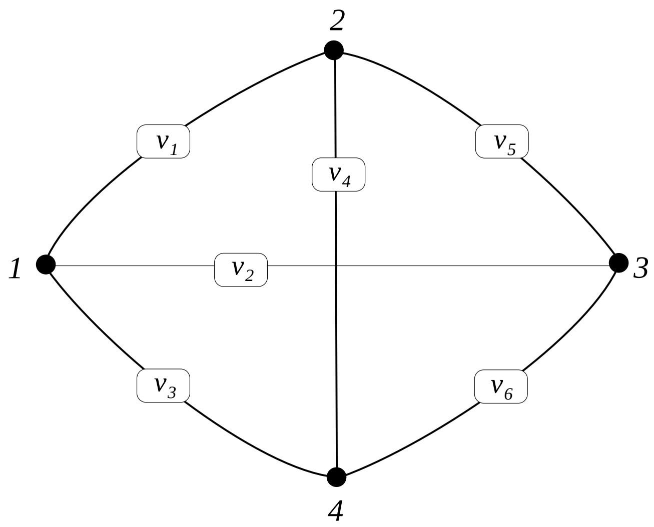

Figure 2.6: The edge weights in the proof of Theorem 12

2.4 Stability under Symmetric Additively Separable Preferences

What does the decomposition $S = S^{CARD} \oplus S^{ALT}$ tell us about hedonic coalition formation games?

Definition 13 Given a profile $(\succsim_1; \succsim_2; \ldots; \succsim_n)$ of purely hedonic preferences, we say that such profile is **purely cardinal** if there exists a utility vector $v \in S^{CARD}$ that represents it. Similarly, a profile $(\succsim_1; \succsim_2; \ldots; \succsim_n)$ of agents' preferences is **purely alternating** if there exists a utility vector $v \in S^{ALT}$ that represents it.

Note that either condition implies that the preference profile $(\succsim_1; \succsim_2; \ldots; \succsim_n)$ is both additively separable and symmetric. We show that when preferences are purely cardinal very stable solutions exist, but when preferences are purely alternating the core of a coalition structure might be empty.

2.4.1 Purely Cardinal Preferences

When agents' preferences are purely cardinal, the common value that two players contribute to each other's utility when they belong to the same coalition is given by the sum of two real numbers. These numbers are the weights that function $w(\cdot)$, as defined in (2.7), assigns to each individual in society. We might interpret each such weight as the fixed worth that an agent brings to the relationship with any other agent.

Theorem 14 Let agents' preferences be purely cardinal. Then, there always exist coalition structures that are both core-stable and Nash-stable.

We claim that the following algorithm provides a coalition partition that is both core-stable and Nash-stable.

Consider the permutation $p : N \longrightarrow N$ which renames individuals from p_1 to p_n in non-decreasing order of their weights. Individuals with the same weight are ordered arbitrarily. Hence, $w(i) > w(j)$ implies $p_i > p_j$ while $w(i) \geq w(j)$ might result in either $p_i > p_j$ or $p_j > p_i$.

Then construct the top segment partition π^* according to the following two steps.

Step 1. Define the top segment coalition T^* with the following iterative procedure. The first player in the ordering p_1 belongs to the top segment coalition. If the next player p_2 strictly prefers being alone to joining p_1; i.e. if $\{p_2\} \succ_{p_2} \{p_1, p_2\}$, then the top segment coalition is completed and $T^* = \{p_1\}$. If however $\{p_1, p_2\} \succeq_{p_2} \{p_2\}$, then add p_2 to the top segment coalition and move on to p_3. Continue to add players from left to right until a player is reached who is denoted as p_{l+1} and who strictly prefers staying alone to joining the growing coalition (or until everyone joins, if such an agent p_{l+1} is never reached). Therefore, the top segment coalition is represented by $T^* = \{p_1, p_2, ..., p_l\}$.

Step 2. Let agents from p_{l+1} until p_n each form a one-member coalition.

The top segment partition contains the top segment coalition and all remaining players as singletons and it is given by $\pi^* = \{T^*, \{p_{l+1}\}, ..., \{p_n\}\}$.

For the proof of Theorem 14 see Corollary 41 to Theorem 35 in Section 2.5.2.

Remark 15 Note that we could have defined the top segment coalition by admitting agents to T^* as long as they strictly prefer joining it to staying alone (as opposed to weak preference). It is easy to construct an example in which a player is indifferent between joining T^* and remaining alone. This alternate definition then yields a slightly different version of the top segment partition π^*, that has all the same qualitative features as the original one. In particular it is both Nash and core-stable. Also, by varying the random ordering of agents who have the same weight, it is clearly possible to generate other stable variants of π^*.

Let us now consider the relationships between purely cardinal preferences and some of the other properties that have been shown to affect the non-emptiness of the core of games in coalition structure.

Definition 16 (Demange [1993]) Agents' preferences satisfy the **intermediate preference property** if it is possible to order the individuals in such a way that for any three individuals i, j, k and any two coalitions S, T with $i > j > k$ and $i, j, k \in S \cap T$ if it is true that $T \succ_i S$ and $T \succ_k S$; then $T \succ_j S$.

Proposition 17 Let agents' preferences be purely cardinal. Then, the intermediate preference property is satisfied.

Proof. Given that preferences are purely cardinal as represented by the weight function $w(\cdot)$, let us order individuals from p_1 to p_n in non-increasing order of their weights. Consider then any three agents $p_i > p_j > p_k$ and two coalitions S and T with $p_i, p_j, p_k \in T \cap S$ that satisfy conditions $T \succ_i S$ and $T \succ_k S$. Such preference relations can be rewritten as

$$(|T| - |S|)\, w(p_i) + \sum_{p_x \in T} w(p_x) - \sum_{p_x \in S} w(p_x) > 0 \tag{2.11}$$

and

$$(|T| - |S|)\, w(p_k) + \sum_{p_x \in T} w(p_x) - \sum_{p_x \in S} w(p_x) > 0 \tag{2.12}$$

respectively. Suppose first that $(|T| - |S|) > 0$. Then inequality (2.12) and the fact that $w(p_j) \geq w(p_k)$ imply that

$$(|T| - |S|)\, w(p_j) + \sum_{p_x \in T} w(p_x) - \sum_{p_x \in S} w(p_x) > 0, \tag{2.13}$$

which in turn yields $T \succ_j S$. In the case in which $(|T| - |S|) < 0$, inequality (2.11) together with $w(p_i) \geq w(p_j)$ lead us back again to condition (4.3) and finally, even when $(|T| - |S|) = 0$, agent p_j prefers coalition T to coalition S, because expression (4.3) reduces to $\sum_{p_x \in T} w(p_x) > \sum_{p_x \in S} w(p_x)$ in this case. ■

In Section 2.5, we will show that the intermediate preference property need not hold when agents' preferences are required to satisfy a weaker version of pure cardinality.[14]

Definition 18 (Banerjee, Konishi, and Sönmez [2001]) (a) A game satisfies the **top coalition property** if for any $V \subseteq N$ with $V \neq \emptyset$ there exists a non-empty subset $S \subseteq V$ such that, for any $i \in S$ and for any $T \in C_i(V)$; $S \succeq_i T$. (b) Given any $V \subseteq N$ with $V \neq \emptyset$; a non-empty subset $S \subseteq V$ is a **weak top coalition** of V if it has an ordered partition $\{S^1, \ldots, S^l\}$ such that: (i) for any $i \in S^1$ and for any $T \subseteq V$ with $i \in T$ we have $S \succeq_i T$; and (ii) for any $k > 1$, any $i \in S^k$ and for any $T \subseteq V$ with $i \in T$ we have $T \succ_i S \Rightarrow T \cap (\cup_{m<k} S^m) \neq \emptyset$. A game satisfies the **weak top coalition property** if for any non-empty set of players $V \subseteq N$ there exists a weak top coalition of V.

[14] Greenberg and Weber (1986) have a different version of the intermediate preference property, which seems less well-suited to our context. Indeed, it might fail to hold when preferences are purely cardinal and some agents have negative weights.

Proposition 19 The assumption of purely cardinal preferences does not imply the weak top coalition property (and so it does not imply the top coalition property).

Proof. Let $N = \{1;2;3;4;5;6;7\}$ and let players' preferences be purely cardinal with weights $w(1) = 6$; $w(i) = 1$ for $i = 2;3$ and $w(j) = -2$ for $j = 4;5;6;7$. For each player, the utility of the remaining players to him is represented by the following table

	1	2	3	4	5	6	7
v_1	0	7	7	4	4	4	4
v_2	7	0	2	-1	-1	-1	-1
v_3	7	2	0	-1	-1	-1	-1
v_4	4	-1	-1	0	-4	-4	-4
v_5	4	-1	-1	-4	0	-4	-4
v_6	4	-1	-1	-4	-4	0	-4
v_7	4	-1	-1	-4	-4	-4	0

We will show that there does not exist a weak top coalition of N: The only candidates are coalition N; which is the coalition on top of player 1's preferences, coalition $\{1;2;3\}$ which is the best coalition for both agents 2 and 3 and finally $\{1;j\}$; the best possible coalition for agent j with $j = 4;5;6;7$. Now, the grand coalition cannot be a weak top coalition of itself as coalition $\{4\}$ would be strictly preferred to N by player 4 and, regardless of the ordered partition of N; coalition $\{4\}$ is clearly disjoint from those sets in the ordered partition that do not contain 4 as a member. Coalition $\{1;2;3\}$ cannot be a weak top coalition of N since player 1 strictly prefers coalition $T = \{1;4;5;6;7\}$ to $\{1;2;3\}$ and thus cannot be put in the necessary ordered partition of $\{1;2;3\}$. By the same token, coalition $\{1;j\}$ cannot be a weak top coalition of N because player 1 strictly prefers $\{1;2;3\}$ to the former and thus cannot be put in the ordered partition of $\{1;j\}$: ■

2.4.2 Purely Alternating Preferences

Proposition 20 Let agents' preferences be purely alternating. Then, the set of core-stable partitions might be empty.

Proof. Let $n = 14$ and let players' preferences be summarized by table 2.14.[15] Notice that the sum of utilities in any single row is zero, implying that players' preferences are indeed purely alternating. Agents' preferences can be described as follows. Imagine that players are located on a circle from 1 to 14 in clockwise order. All agents prefer their neighbours to distant players and players can be divided into two different categories (namely odd players and even players) according to their preferences. Odd players

[15] An example with $n = 14$ is the smallest we could find with an empty core under purely alternating preferences.

like the individual immediately following them the most, they like all players that are at a distance not greater than 2 from them, and they also like the odd players that are opposite to them on the circle; they dislike all remaining agents. Even players, on the contrary, like the individual immediately preceding them the most, better than the individual immediately following them, they like even players that are at distance greater than 2 from them, and dislike all other agents.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
v_1	0	9	3	$-\frac{13}{4}$	-4	$-\frac{13}{4}$	2	-5	2	$-\frac{13}{4}$	-4	$-\frac{13}{4}$	3	7
v_2	9	0	7	-6	$-\frac{13}{4}$	5	$-\frac{13}{4}$	2	-5	2	$-\frac{13}{4}$	5	$-\frac{13}{4}$	-6
v_3	3	7	0	9	3	$-\frac{13}{4}$	-4	$-\frac{13}{4}$	2	-5	2	$-\frac{13}{4}$	-4	$-\frac{13}{4}$
v_4	$-\frac{13}{4}$	-6	9	0	7	-6	$-\frac{13}{4}$	5	$-\frac{13}{4}$	2	-5	2	$-\frac{13}{4}$	5
v_5	-4	$-\frac{13}{4}$	3	7	0	9	3	$-\frac{13}{4}$	-4	$-\frac{13}{4}$	2	-5	2	$-\frac{13}{4}$
v_6	$-\frac{13}{4}$	5	$-\frac{13}{4}$	-6	9	0	7	-6	$-\frac{13}{4}$	5	$-\frac{13}{4}$	2	-5	2
v_7	2	$-\frac{13}{4}$	-4	$-\frac{13}{4}$	3	7	0	9	3	$-\frac{13}{4}$	-4	$-\frac{13}{4}$	2	-5
v_8	-5	2	$-\frac{13}{4}$	5	$-\frac{13}{4}$	-6	9	0	7	-6	$-\frac{13}{4}$	5	$-\frac{13}{4}$	2
v_9	2	-5	2	$-\frac{13}{4}$	-4	$-\frac{13}{4}$	3	7	0	9	3	$-\frac{13}{4}$	-4	$-\frac{13}{4}$
v_{10}	$-\frac{13}{4}$	2	-5	2	$-\frac{13}{4}$	5	$-\frac{13}{4}$	-6	9	0	7	-6	$-\frac{13}{4}$	5
v_{11}	-4	$-\frac{13}{4}$	2	-5	2	$-\frac{13}{4}$	-4	$-\frac{13}{4}$	3	7	0	9	3	$-\frac{13}{4}$
v_{12}	$-\frac{13}{4}$	5	$-\frac{13}{4}$	2	-5	2	$-\frac{13}{4}$	5	$-\frac{13}{4}$	-6	9	0	7	-6
v_{13}	3	$-\frac{13}{4}$	-4	$-\frac{13}{4}$	2	-5	2	$-\frac{13}{4}$	-4	$-\frac{13}{4}$	3	7	0	9
v_{14}	7	-6	$-\frac{13}{4}$	5	$-\frac{13}{4}$	2	-5	2	$-\frac{13}{4}$	5	$-\frac{13}{4}$	-6	9	0

(2.14)

There are over 190 million distinct partitions of a 14-element set. However, candidate core-stable partitions contain only coalitions that are individually rational and have no deviating subcoalitions – the internally stable coalitions. Professor Davide Cervone wrote an algorithm, implemented in TCL/TK, to check the example. His algorithm first reduces the list of 16,383 nonempty coalitions to the 252 that are internally stable, and then uses these to build subpartitions, which are partitions of a subset M of N (because if a subpartition has a deviating coalition S contained in M, then the same S is deviating for any extension of the subpartition to a partition of N). The rotational symmetries of the example allow us to further restrict our attention to those subpartitions for which the largest coalition (or tied for largest) contains agent 1, agent 2, or both, and this brings the number of subpartitions actually checked down to 34,928. Each of these was found to have a deviating coalition (and, in fact, it was enough to consider as potentially deviating only those coalitions that are internally stable). As a check, we ran the algorithm on an earlier, unsuccessful 10-agent example that was found (by hand calculation) to have a core-stable partition; the algorithm identified that partition as the unique (up to symmetry) core-stable partition of the example.

The example was constructed to have a number of cyclical coalitional deviations. For example, consider the coalition structure $\pi_1 = \{1,2,3 \mid 5,6,7 \mid 11,12,13 \mid C_1\}$ where the set $C_1 \equiv \{4,8,9,10,14\}$ might consist of a single coalition, or might be partitioned as $\{8,9 \mid 4,10,14\}$ or $\{9,10 \mid 4,8,14\}$. A profitable deviation is represented by $\{7,8,9\}$, which leads to coalition structure $\pi_2 = \{1,2,3 \mid 7,8,9 \mid 11,12,13 \mid C_2\}$, where $C_2 \equiv \{4,5,6,10,14\}$, which again might consist of a single coalition or might itself be partitioned as $\{4,5 \mid 6,10,14\}$ or $\{5,6 \mid 4,10,14\}$. A profitable deviation is now represented by $\{3,4,5\}$, which leads to $\pi_3 = \{3,4,5 \mid 7,8,9 \mid 11,12,13 \mid C_3\}$, with $C_3 \equiv \{1,2,6,10,14\}$. Coalition $\{13,14,1\}$ blocks partition π_3 and this yields coalition structure $\pi_4 = \{13,14,1 \mid 3,4,5 \mid 7,8,9 \mid C_4\}$ where $C_4 \equiv \{2,6,10,11,12\}$. Now, agents in $\{9,10,11\}$ find it beneficial to deviate and thus partition $\pi_5 = \{13,14,1 \mid 3,4,5 \mid 9,10,11 \mid C_5\}$ is formed, with $C_5 \equiv \{2,6,7,8,12\}$ and $\{5,6,7\}$ being the blocking coalition. This leads to the partition $\pi_6 = \{13,14,1 \mid 5,6,7 \mid 9,10,11 \mid C_6\}$, with $C_6 \equiv \{2,3,4,8,12\}$ and with coalition $\{1,2,3\}$ deviating, which yields partition $\pi_7 = \{1,2,3 \mid 5,6,7 \mid 9,10,11 \mid C_7\}$, where $C_7 \equiv \{4,8,12,13,14\}$. The incentive to deviate comes from coalition $\{11,12,13\}$ and this leads back to partition π_1. ■

It seems plausible that if a profile of symmetric and additively separable preferences $(v_1, \ldots, v_n)$ is such that the purely alternating component is sufficiently small as compared to the purely cardinal component, then core-stable partitions might always be found. By this we mean that the magnitude of all the components of vector v^{ALT} has to be significantly smaller than the magnitude of the corresponding components of v^{CARD} in such a way that a common ordering of individuals is still preserved. It is this consideration that motivates the following section.

2.5 Stability under Descending Separable Preferences

Our goal in this section is to isolate, in the form of a set of properties, termed DS preferences for Descending Separable preferences, those features of purely cardinal preferences that appear to be actually necessary in proving the existence of Nash and core-stable partitions (Theorem 14). The elements of this package fall into roughly four types. The properties of the first two types, descending desirability of players that is common to other players (named CRI below) and descending desire of players for other players, require that we assume the existence of a strict linear reference ranking of individuals,

$$p_1 > p_2 > \ldots > p_n \tag{2.15}$$

When agents' preferences are purely cardinal, this ranking is generated by the descending cardinal weights associated to each player (where players of equal weight are ordered arbitrarily). Properties of the third type are weak forms of additive separability, and thus entail a relationship between the original ranking

(2.15) of players and the preference rankings that players have over coalitions. Finally, the property of the fourth type is a weak form of symmetry.

2.5.1 Descending Separable Preferences

In the list of properties that follows, each player p_i is assumed to have a preference ordering $\succsim_{p_i}$ over $C_{p_i}(N)$; that is over the collection of subsets of N which include p_i. The properties we call conditions in what follows actually enter in the definition of DS preferences while these presented as definitions appear only for comparison purposes.

Condition 21 Common Ranking of Individuals (CRI) For any three distinct players p_i, p_j and p_k, if $p_j > p_k$ then $\{p_i, p_j\} \succsim_{p_i} \{p_i, p_k\}$.

Property CRI states that the relative desirability of various individuals to any fixed player weakly decreases as we move to the right in the reference order; that is, players share a roughly common ranking over the other players. Recall that this condition does not imply the same absolute distinction between good and bad individuals, and hence does not imply a common ranking of coalitions.

Condition 22 Descending Desire (DD). For any pair p_i, p_j of distinct players with $p_i > p_j$ and for any coalition X containing neither player p_i nor p_j, if $\{p_j\} \cup X \succsim_{p_j} \{p_j\}$ then $\{p_i\} \cup X \succsim_{p_i} \{p_i\}$ and if $\{p_j\} \cup X \succ_{p_j} \{p_j\}$ then $\{p_i\} \cup X \succ_{p_i} \{p_i\}$.

Property DD states that the absolute appeal of any fixed coalition to various players weakly decreases as we move to the right in the reference order.

Definition 23 Separable Preferences (SP). For any two distinct players p_i and p_j and any coalition X such that $p_j \notin X$ and $p_i \in X$; we have $\{p_i, p_j\} \succsim_{p_i} \{p_i\}$ if and only if $\{p_j\} \cup X \succsim_{p_i} X$ and $\{p_i, p_j\} \succ_{p_i} \{p_i\}$ if and only if $\{p_j\} \cup X \succ_{p_i} X$.

Property SP states that the effect of an absolutely desirable (respectively, undesirable) individual p_j on a player's preferences remains positive (respectively, negative), regardless of which additional individuals belong to the player's coalition. A useful consequence of separability of preferences is the following.

Definition 24 Iterated Separable Preferences (ISP). Let p_i be any player and let X and Y be two coalitions such that $X \cap Y = \emptyset$ and $p_i \in X$. If $\{p_i, y\} \succsim_{p_i} \{p_i\}$ for every $y \in Y$, then $X \cup Y \succsim_{p_i} X$, and if $\{p_i, y\} \succ_{p_i} \{p_i\}$ for every $y \in Y$, then $X \cup Y \succ_{p_i} X$.

To derive Condition 24 from separable preferences, add the members of coalition Y to X in a step-by-step fashion, applying definition SP at each step.

Condition 25 Group Separable Preferences (GSP). For any player p_i and for any two disjoint coalitions X and Y with $p_i \in X \setminus Y$, if $\{p_i\} \cup Y \succeq_{p_i} \{p_i\}$ then $X \cup Y \succeq_{p_i} X$ and if $\{p_i\} \cup Y \succ_{p_i} \{p_i\}$ then $X \cup Y \succ_{p_i} X$.

Property GSP states that the effect of an absolutely desirable coalition Y on a player's preferences remains positive, regardless of which additional individuals belong to the player's coalition. This particular form of group separability does not imply SP because it does not require that an undesirable coalition have a consistently negative effect in the presence of others.

Condition 26 Responsive Preferences (RESP). For any triple of players $p_i; p_j; p_k$ and any coalition X such that $p_j; p_k \notin X$ and $p_i \in X$; $\{p_i; p_j\} \succeq_{p_i} \{p_i; p_k\}$ if and only if $\{p_j\} \cup X \succeq_{p_i} \{p_k\} \cup X$ and $\{p_i; p_j\} \succ_{p_i} \{p_i; p_k\}$ if and only if $\{p_j\} \cup X \succ_{p_i} \{p_k\} \cup X$.

Property RESP states that the relative appeal of two given players to a third player is the same, regardless of which additional players belong to the coalition containing such a third player. Note that responsiveness does not imply separability.

Conditions CRI and RESP together imply the following property.

Definition 27 Right-Shifted Coalitions. For any two coalitions X and Y both containing player p_i, if $X \geq_{SRS} Y$ then $X \succeq_{p_i} Y$.

Here $X \geq_{SRS} Y$ means that coalition Y is a simple right-shift of X. Either $Y = X$ or coalition Y is obtained from X by replacing one or more members p_j of X with members p_k to their right (i.e. such that $p_j > p_k$) not already in the coalition. Thus $X \geq_{SRS} Y$ implies $|X| = |Y|$.[16] Hence, the condition above says that the appeal of a shifting coalition to a given player weakly decreases as the coalition shifts to the right. In order to derive Definition 27 from properties 21 and 26, replace members of coalition X by those of Y in a step-by-step fashion, working from right to left (that is starting with the player with highest index). For example, let $X = \{p_1; p_3; p_4; p_7\}$ and $Y = \{p_1; p_4; p_8; p_9\}$. Then we obtain that, for agent $p_1 \in X \cap Y$, the following preference ordering holds

$$\{p_1; p_3; p_4; p_7\} \succeq_{p_1} \{p_1; p_3; p_4; p_9\} \succeq_{p_1} \{p_1; p_3; p_8; p_9\} \succeq_{p_1} \{p_1; p_4; p_8; p_9\};$$

thus implying that $X \succeq_{p_i} Y$:

Definition 28 Mutual Preferences: For any pair $p_i; p_j$ of distinct players, $\{p_i; p_j\} \succeq_{p_i} \{p_i\}$ if and only if $\{p_i; p_j\} \succeq_{p_j} \{p_j\}$ and $\{p_i; p_j\} \succ_{p_i} \{p_i\}$ if and only if $\{p_i; p_j\} \succ_{p_j} \{p_j\}$.

One player finds another player desirable if and only if the latter finds the former desirable.

[16] In the standard version of the right-shift order, a right shift Y of X is obtained by replacement together with additions, so that $|X| \leq |Y|$: We use the qualifier "simple" to distinguish the version we intend.

Definition 29 Descending Mutual Preferences. For any pair p_i, p_j of distinct players with $p_i > p_j$, if $\{p_i, p_j\} \succeq_{p_j} \{p_j\}$ then $\{p_i, p_j\} \succeq_{p_i} \{p_i\}$ and if $\{p_i, p_j\} \succ_{p_j} \{p_j\}$ then $\{p_i, p_j\} \succ_{p_i} \{p_i\}$.

The latter condition describes a weaker form of mutuality because possibly p_i likes p_j but p_j dislikes p_i when p_j is to the right of p_i.

Condition 30 Replaceable Preferences (REP): For any pair p_i, p_j of distinct players with $p_i > p_j$ and for any coalition X containing neither player p_i nor p_j, if $\{p_i, p_j\} \cup X \succeq_{p_j} \{p_j\}$ then $\{p_i, p_j\} \cup X \succeq_{p_i} \{p_i\}$ and if $\{p_i, p_j\} \cup X \succ_{p_j} \{p_j\}$ then $\{p_i, p_j\} \cup X \succ_{p_i} \{p_i\}$.

Condition REP implies descending mutual preferences (simply take coalition X to be the empty set) and through this connection is the sole member of our final package containing a measure of symmetry. However, REP also bears a resemblance to DD, leaving some doubt over the extent to which the roles of symmetry and of descending desire can be disentangled in the proof of Theorem 35.

Definition 31 A profile $(\succeq_1; \succeq_2; \dots; \succeq_n)$ of agents' preferences is said to be **descending separable (DS)** if there exists a reference ordering (2.15) under which Conditions 21 (CRI), 22 (DD), 23 (SP), 25 (GSP), 26 (RESP), and 30 (REP) all hold.

Let us provide an example in order to show how the conditions previously stated interact with each other and with other properties appearing in the literature.

Example 32 Let players be $N = \{1, 2, 3, 4, 5\}$ and let each agent's preferences over the remaining players in society be summarized by the following profile, satisfying CRI and descending mutuality.

$$
\begin{aligned}
1 &: \{2\} \succ_1 \{3\} \succ_1 \{4\} \succ_1 \{5\} \succ_1 \emptyset; \\
2 &: \{1\} \succ_2 \{3\} \succ_2 \{4\} \succ_2 \emptyset \succ_2 \{5\}; \\
3 &: \{1\} \succ_3 \{2\} \succ_3 \emptyset \succ_3 \{4\} \succ_3 \{5\}; \\
4 &: \{1\} \succ_4 \emptyset \succ_4 \{2\} \succ_4 \{3\} \succ_4 \{5\}; \\
5 &: \emptyset \succ_5 \{1\} \succ_5 \{2\} \succ_5 \{3\} \succ_5 \{4\}.
\end{aligned}
\tag{2.16}
$$

Thus, the underlying linear ordering of players according to their desirability is $1 > 2 > \dots > 5$. Players' preferences over coalitions to which they belong must then be related to the ordering 2.16 through the conditions SP,GSP, RESP, DD, and REP.

Such a preference profile can be described as in (2.17) below

$$
\begin{array}{ll}
1: & \{1,2,3,4,5\} \succ_1 \{1,2,3,4\} \succ_1 \{1,2,3,5\} \succ_1 \{1,2,3\} \succ_1 \{1,2,4,5\} \succ_1 \\
& \{1,3,4,5\} \succ_1 \{1,2,4\} \succ_1 \{1,2,5\} \succ_1 \{1,3,4\} \succ_1 \{1,2\} \succ_1 \{1,3,5\} \succ_1 \\
& \{1,3\} \succ_1 \{1,4,5\} \succ_1 \{1,4\} \succ_1 \{1,5\} \succ_1 \{1\}; \\
2: & \{1,2,3,4\} \succ_2 \{1,2,3\} \succ_2 \{1,2,4\} \succ_2 \{1,2\} \succ_2 \{2,3,4\} \succ_2 \\
& \{1,2,3,4,5\} \succ_2 \{2,3\} \succ_2 \{2,4\} \succ_2 \{1,2,3,5\} \succ_2 \{1,2,4,5\} \succ_2 \{2\} \succ_2 \\
& \{2,3,4,5\} \succ_2 \{1,2,5\} \succ_2 \{2,3,5\} \succ_2 \{2,4,5\} \succ_2 \{2,5\}; \\
3: & \{1,2,3\} \succ_3 \{1,3\} \succ_3 \{2,3\} \succ_3 \{1,2,3,4\} \succ_3 \{3\} \succ_3 \{1,3,4\} \succ_3 \\
& \{2,3,4\} \succ_3 \{1,2,3,5\} \succ_3 \{1,2,4,3,5\} \succ_3 \{3,4\} \succ_3 \{3,5\} \succ_3 \\
& \{1,3,4,5\} \succ_3 \{1,3,5\} \succ_3 \{2,3,5\} \succ_3 \{2,3,4,5\} \succ_3 \{3,4,5\}; \\
4: & \{1,4\} \succ_4 \{1,2,4\} \succ_4 \{4\} \succ_4 \{1,3,4\} \succ_4 \{1,4,5\} \succ_4 \{2,4\} \succ_4 \\
& \{1,2,3,4\} \succ_4 \{3,4\} \succ_4 \{1,2,4,5\} \succ_4 \{2,3,4\} \succ_4 \{4,5\} \succ_4 \\
& \{2,4,5\} \succ_4 \{1,3,4,5\} \succ_4 \{1,2,3,4,5\} \succ_4 \{3,4,5\} \succ_4 \{2,3,4,5\}; \\
5: & \{5\} \succ_5 \{1,5\} \succ_5 \{2,5\} \succ_5 \{3,5\} \succ_5 \{4,5\} \succ_5 \{1,2,5\} \succ_5 \{1,3,5\} \succ_5 \\
& \{1,4,5\} \succ_5 \{2,3,5\} \succ_5 \{1,2,3,5\} \succ_5 \{2,4,5\} \succ_5 \{3,4,5\} \succ_5 \\
& \{1,2,4,5\} \succ_5 \{1,3,4,5\} \succ_5 \{2,3,4,5\} \succ_5 \{1,2,3,4,5\}.
\end{array} \tag{2.17}
$$

We conclude with two propositions that point to the extent to which the assumption of DS preferences is weaker than that of purely cardinal preferences.

Proposition 33 The descending separable preference property neither implies nor is implied by the property of additive separability.

Proof. It is straightforward to construct a four-agent preference profile satisfying symmetric additive separability with the property that two of the agents differ in their ranking of the other two. Clearly, there is no common ranking of individuals, so such preferences do not satisfy DS.

The profile of preferences (2.17) in Example 32 satisfies all conditions which define DS preferences but agents' preferences are not additively separable. Indeed, let us consider the preference ordering of agent 1: He strictly prefers coalition $\{1,3,4,5\}$ to $\{1,2,4\}$ and at the same time he would be strictly better-off in $\{1,2\}$ rather than in $\{1,3,5\}$. Thus, were player 1's preferences additively representable, we would need both $v_1(\{1,3,4,5\}) > v_1(\{1,2,4\})$, which would imply $v_1(3) + v_1(5) > v_1(2)$, and $v_1(\{1,2\}) > v_1(\{1,3,5\})$, which would instead lead to $v_1(3) + v_1(5) < v_1(2)$. Obviously, there does not exist a real-valued function v_1 satisfying both inequalities at the same time. ■

Recall that purely cardinal preferences always satisfy the intermediate preference property, perhaps suggesting a link between our results and some of the earlier results from the literature that provide sufficient conditions for stable coalitional structures. However, the following proposition suggests that any such link may be weaker than it first appears to be.

Proposition 34 Let agents' preferences satisfy the descending separable preference property. Then, the intermediate preference property need not hold.

Proof. Consider $N = \{1, 2, \ldots, 7\}$ and the profile of purely cardinal preferences generated by the following individual weighs: $w(1) = 3$; $w(2) = 2$; $w(3) = 1$; $w(4) = -200$; $w(5) = -300$; $w(6) = -400$ and $w(7) = -500$. Notice that agents 1, 2 and 3 are each indifferent between coalitions $\{1, 2, 3, 4, 7\}$ and $\{1, 2, 3, 5, 6\}$. Now modify slightly these agents' preferences in such a way that

$$\{1, 2, 3, 5, 6\} \succ_1 \{1, 2, 3, 4, 7\} \quad \text{and} \quad \{1, 2, 3, 5, 6\} \succ_3 \{1, 2, 3, 4, 7\};$$

whereas

$$\{1, 2, 3, 4, 7\} \succ_2 \{1, 2, 3, 5, 6\};$$

The resulting preference profile is no longer purely cardinal, it satisfies the DS preference property but it does not satisfy the intermediate preference property. ■

2.5.2 Stability of the Top Segment Partition

The existence of coalition partitions with desirable properties is guaranteed when agents have DS preferences.

Theorem 35 Consider a hedonic coalition formation game and let agents' preferences be descending separable. Then, there always exists a coalition structure that is both core-stable and Nash-stable.

The proof will proceed by constructing a coalition structure via the same technique used in Section 2.4.1 and showing, through a sequence of lemmata, that it is both core-stable and Nash-stable.

For a given linear ordering of players $p_1 > p_2 > \ldots > p_n$; construct the top segment partition π^* according to the same steps as in Section 2.4.1. Recall that the top segment partition is represented by $\pi^* = \{T^*; \{p_{l+1}\}; \ldots; \{p_n\}\}$; where T^* is called the top segment coalition and is formed by the first l players in the ordering.

The proof that π^* is core-stable follows immediately from Lemmata 36 through 40.

Lemma 36 Every coalition $C \in 2^N$ which is individually rational contains at most l members.

Proof. By construction of the top segment coalition, agent p_{l+1} is such that $\{p_{l+1}\} \succ_{p_{l+1}} T^* \cup \{p_{l+1}\}$. By the right-shifted coalitions property, any right-shift C of coalition T^* omitting player p_{l+1} must satisfy that $T^* \cup \{p_{l+1}\} \succeq_{p_{l+1}} C \cup \{p_{l+1}\}$. Then coalition $C \cup \{p_{l+1}\}$; which contains exactly $l + 1$ members, is not individually rational for agent p_{l+1}. Now, by descending desire, $\{p_{l+s}\} \succ_{p_{l+s}} C \cup \{p_{l+s}\}$ for every individual $p_{l+s} < p_{l+1}$ with $p_{l+s} \notin C$; implying that no coalition of size $l + 1$ is individually rational. The same conclusion holds for coalitions of size greater than $l + 1$. ■

Lemma 37 Partition π^* is individually rational.

Proof. It suffices to prove that coalition T^* is individually rational. Consider agents p_l and any individual $p_i \in T^* \setminus \{p_l\}$. By construction, p_l weakly prefers T^* to $\{p_l\}$, and so by replaceable preferences $T^* \succeq_{p_i} \{p_i\}$, which says that p_i at least as well off being a member of T^* as being alone. ∎

Since for p_l (the last player included in the top segment coalition) $T^* \succeq_{p_l} \{p_l\}$ holds, it must be the case that there exists some player $p_i \in T^* \setminus \{p_l\}$ for whom $\{p_l; p_i\} \succeq_{p_l} \{p_l\}$: Why? Suppose that for every $p_i \in T^* \setminus \{p_l\}$ we have $\{p_l\} \succ_{p_l} \{p_l; p_i\}$: Then iterated separable preferences would imply that $\{p_l\} \succ_{p_l} T^*$, contradicting the individual rationality of the top segment partition. Now, each agent $p_i \in T^* \setminus \{p_l\}$ for whom $\{p_l; p_i\} \succeq_{p_l} \{p_l\}$ satisfies that $\{p_l; p_i\} \succeq_{p_i} \{p_i\}$, by descending mutuality. Hence, the set of all agents $p_i > p_l$ who consider player p_l to be a good player is non-empty.

By DD this set forms an initial segment $T^{**} \equiv \{p_1; p_2; \ldots; p_f\}$ of players contained in T^*, where player $p_f > p_l$ is the member of $T^* \setminus \{p_l\}$ with the highest index for whom $\{p_l; p_f\} \succeq_{p_f} \{p_f\}$ still holds. It is easy to show that members of coalition T^{**} rank the top segment coalition in the same way.

Lemma 38 For each of the players in $T^{**} = \{p_1; p_2; \ldots; p_f\} \subset T^*$; coalition T^* is top ranked among individually rational coalitions (or tied for top). Therefore, if π^* is not in the core of a coalition structure, no deviating coalition C can contain any of the players in T^{**}.

Proof. Let p_i be any player such that $p_i \in T^{**}$ and let D be any individually rational coalition containing p_i. By left-shifting D to the greatest possible extent while keeping p_i as a member, we form a coalition C which is such that $C \succeq_{p_i} D$ (and such that C is a subset of T^* by Lemma 36). Now, any player $p_j \in T^* \setminus \{p_i\}$ is good to agent p_i; (that is, $\{p_i; p_j\} \succeq_{p_i} \{p_i\}$), because $\{p_i; p_l\} \succeq_{p_i} \{p_i\}$ for all members of T^{**} and either $p_j = p_l$ or $p_j > p_l$, in which case $\{p_i; p_j\} \succeq_{p_i} \{p_i; p_l\}$ by CRI. If coalition C omits any such players $p_j \in T^* \setminus \{p_i\}$; then SP implies that $\{p_j\} \cup C \succeq_{p_i} C$ and adding in the remaining members of $T^* \setminus \{p_i\}$ continues to give a coalition that is at least as good as C to p_i. Hence, $T^* \succeq_{p_i} C \succeq_{p_i} D$, as desired. ∎

Lemma 39 Let C be any non-empty coalition such that $C \not\subseteq T^*$: Assume that all agents $p_i \in C \cap T^*$ are such that $C \succ_{p_i} T^*$ and all agents $p_j \in C \setminus T^*$ are such that $C \succ_{p_j} \{p_j\}$. Then $\{p_l\}$ is not the sole member of $C \cap T^*$ and all members of $C \cap T^*$ strictly prefer $C \cap T^*$ to C.

Proof. For each agent $p_i \in C \cap T^*$; coalition C is strictly preferred to T^*, so by Lemma 38, none of the players $p_1; p_2; \ldots; p_f$ forming coalition T^{**} are in C. This implies that each agent $p_i \in C \cap T^* \setminus \{p_l\}$ has preferences such that $\{p_i\} \succ_{p_i} \{p_i; p_l\}$; and by CRI this leads to $\{p_i\} \succ_{p_i} \{p_i; p_s\}$ for all $p_s < p_l$. In particular, we have that $\{p_i\} \succ_{p_i} \{p_i; p_j\}$ for each agent $p_j \in C \setminus T^*$: Therefore expanding $C \cap T^*$ to C; by adding in the members of $C \setminus T^*$ one at a time, strictly decreases the appeal of the coalition to p_i at each step. Hence, p_i strictly prefers $C \cap T^*$ to C. To handle the case $p_i = p_l$, first note that by CRI and iterated separable preferences, our assumption that all agents $p_j \in C \setminus T^*$ are such that $C \succ_{p_j} \{p_j\}$ rules out the possibility that p_l is the sole member of $C \cap T^*$. It cannot be for $p_s \in C \setminus T^*$ that $\{p_l; p_s\} \succeq_{p_l} \{p_l\}$ else choose any $p_i \in C \cap T^* \setminus \{p_l\}$ and it would follow by DD that $\{p_i; p_s\} \succeq_{p_i} \{p_i\}$, contradicting what we just showed above. Hence $\{p_l\} \succ_{p_l} \{p_l; p_s\}$ for all $p_s \in C \setminus T^*$, and we may conclude, as above, that p_l also strictly prefers $C \cap T^*$ to C. ■

Lemma 40 Let C be any coalition containing at least two players such that $C \subset T^*$ and let p_c denote the leftmost member of C. Then agent p_c weakly prefers coalition T^* to C.

Proof. Let coalition A consist of the members of $T^* \setminus C$ who lie to the left of p_c according to the reference ranking of players and let coalition B be formed by those members of $T^* \setminus C$ who lie to the right of p_c. Therefore, coalitions A, B and C are all disjoint and $T^* = A \cup B \cup C$.[17]

There are two possible cases to consider.

Case (1) Assume that $B = \emptyset$: Subcase (1a) Assume that, for each individual p_i such that $p_i > p_c$, $\{p_c; p_i\} \succeq_{p_c} \{p_c\}$ holds. Condition ISP implies then that p_c weakly prefers T^* to C; because coalition $T^* \setminus C$ is formed by players that are all good to p_i. Subcase (1b): Assume that, for some p_i such that $p_i > p_c$, it happens that $\{p_c\} \succ_{p_c} \{p_c; p_i\}$. It then follows that $\{p_c\} \succ_{p_c} \{p_c; p_j\}$ for each $p_j \in C$ as $p_i > p_j$ and CRI applies. Property ISP implies now that p_c strictly prefers being alone to being a member of C. But p_c weakly prefers T^* to $\{p_c\}$ by Lemma 37, hence transitivity of preferences yields that p_c strictly prefers T^* to C.

Case (2) Assume that $B \neq \emptyset$ and let p_b denote the rightmost member of B. Subcase (2a) Assume that p_b weakly prefers $\{p_b; p_c\}$ to $\{p_b\}$. Then, since $p_c > p_b$ by construction, decreasing mutuality leads to $\{p_b; p_c\} \succeq_{p_c} \{p_c\}$, from which it follows that p_c weakly prefers being paired with any individual in $A \cup B$ to staying alone. And by property ISP we obtain once again that p_c weakly prefers T^* to C. Subcase (2b) Assume that p_b is such that $\{p_b\} \succ_{p_b} \{p_b; p_c\}$: Then p_b strictly prefers staying alone to being paired with any member of C. Condition ISP implies that p_b has the following preference ordering

$$A \cup B \cup \{p_c\} \succ_{p_b} A \cup B \cup C = T^* \succeq_{p_b} \{p_b\}:$$

As p_b strictly prefers $A \cup B \cup \{p_c\}$ to being alone, p_c strictly prefers $A \cup B \cup \{p_c\}$ to $\{p_c\}$ because of property REP. Finally, it follows that p_c strictly prefers $A \cup B \cup C = T^*$ to C because $A \cup B$ is a good coalition and condition GSP applies.[18] ■

Let us now go back to the proof of the main theorem.

[17] Either coalition A or coalition B might be empty, but not both, otherwise $C = T^*$, contradicting the premises of the Lemma.

[18] Notice that we used condition GSP in the proof of this lemma, whereas condition SP sufficed in all preceeding Lemmata.

Proof of Theorem 35. Consider the facts established in Lemmata 36 through 40. If there is a coalition C that defects from π^* then: (i) C contains members from both T^* and $N \setminus T^*$, or (ii) C is a proper subset of T^*, or (iii) C is disjoint from T^*. Lemma 39 says that if the deviating coalition is such that (i) holds, then it triggers another deviation for which (ii) holds. But Lemma 40 says that (ii) never happens as no deviating coalition could be a proper subset of the top segment coalition. It is easy to see that DD rules out (iii). Being immune to all possible coalitional deviation, partition π^* is thus in the core of a coalition structure.

It remains to check that DS preferences imply Nash stability of the top segment partition π^*. First, π^* is immune to unilateral deviations of individuals in $N \setminus T^*$. No player outside of T^* would be weakly better off by joining T^* than by staying alone, because we have $\{p_{l+1}\} \succ_{p_{l+1}} T^* \cup \{p_{l+1}\}$ by construction. Descending desire in turn yields that all players that lie to the right of p_{l+1} also have this preference. Moreover, p_{l+1} is such that $\{p_{l+1}\} \succ_{p_{l+1}} \{p_l, p_{l+1}\}$. Properties DD and CRI then imply that for any two distinct individuals p_i and p_j lying to the right of p_l, $\{p_i\} \succ_{p_i} \{p_i, p_j\}$. Therefore, no two players outside of T^* would find it profitable to merge. As for agents in T^*, Lemma 37 says that each individual $p_i \in T^*$ weakly prefers belonging to T^* to being by himself. Lemma 38 then tells us that for all members of coalition $T^{**} = \{p_1, p_2, ..., p_f\}$ the top segment coalition T^* is top-ranked among individually rational coalitions (or tied for top). This means that individuals in T^{**} weakly prefer staying in T^* to joining any of the singletons $\{p_s\}$ with $p_s < p_l$. Players $p_j \in T^* \setminus T^{**}$ such that $p_j < p_f$ strictly prefer $\{p_j\}$ to $\{p_j, p_l\}$ by definition of player p_f, and so by condition CRI we have $\{p_j\} \succ_{p_j} \{p_j, p_s\}$ for any such individual p_j and any $p_s < p_l$. Transitivity of preferences then leads to $T^* \succ_{p_j} \{p_j, p_s\}$ for all $p_j \in T^* \setminus T^{**}$ and all $p_s < p_l$. This completes the proof of Nash stability of the partition π^*. ■

The results established in the lemmata and in Theorem 35 also hold in the case in which agents' preferences are purely cardinal.

Corollary 41 Let individuals' preferences be purely cardinal. Then, there always exist coalition partitions that are both Nash-stable and in the core of a coalition structure.

Proof. It is easy to check that each of the conditions CRI, SP, GSP, RESP, DD and REP that make up DS follows from the assumption of purely cardinal preferences, so that the premises of Theorem 35 hold and its results follow. ■