Chapter 4

Matching, Search, and Intermediation with Two-Sided Heterogeneity

4.1 Introduction

This paper analyzes the role played by intermediation in the context of a decentralized market where trade is carried out through bilateral bargaining, and where the bargaining outcome is interrelated with the process of search for a suitable trading partner. To this purpose, first a model of random-matching and two-sided search with heterogeneous agents is developed. The characterization of all possible search equilibria is provided and it is shown that inefficiencies might emerge as a result of sorting externalities. Secondly, the incentive for intermediation to arise endogenously within this environment is considered. Intermediation is viable because of the agents' heterogeneity and it might serve as an efficiency-enhancing mechanism in that it induces separation of the agents' types.

The basic structure of the random-matching and bargaining model considered here can be traced back to the framework introduced by Diamond and Maskin (1979), Diamond (1982) and Mortensen (1982), and further developed by Rubinstein and Wolinsky (1985). It is as follows.

There are two continuous groups of agents, buyers and sellers. Each buyer is endowed with a unit of a perfectly divisible commodity (which plays the role of money), and each seller has a unit of an indivisible good which he wants to exchange for money. As in Jackson and Palfrey (1998), it is assumed that both buyers and sellers are heterogeneous with respect to their reservation values for the indivisible good. In particular, there are only two types (high and low valuation) of agents on each side of the market.

In every period, each individual is randomly matched with an agent of the opposite endowment type. Once a match is formed, the pair bargain over the terms of the transactions. It is assumed that upon getting matched the parties observe each other's types, so that bargaining proceeds under complete information. The bargaining outcome will be either an agreement on a price that divides the net surplus associated to the given match according to a predetermined sharing rule (for instance the symmetric Nash Bargaining solution), or disagreement. In the event agreement is reached, a transaction will be carried out and the pair will leave the market forever. Upon disagreement, the pair return to the pool of the unmatched agents and they search for a better trading partner during the next period. Being matched precludes further search and thus each individual faces a trade-off between the opportunity cost of abstaining from search, and the benefits associated with concluding a transaction immediately. Such trade-off is the main source of delay in trade and it thus represents the implicit cost of search.

Clearly the types of agents that trade with each other, and the agents' expected utility of being unmatched are endogenous in this environment. Each individual faces a decision problem that consists in determining which is the set of partners with whom he is willing to be matched and ultimately conclude a transaction, or else which is the set of acceptable agents of the opposite type. Only mutually acceptable matches will be consummated. The agents' decisions to consummate matches involve widespread externalities: such choices affect the distribution of unmatched agents, this in turn alters the probability that a particular match will be formed and thus the traders' expected unmatched utilities.

We are concerned with identifying the set of Stationary Search Equilibria (SSE) that emerge in this context. An equilibrium is such that the agents' acceptance decisions maximize their discounted expected unmatched values, and such that the distributions of unmatched types (and hence the prices at which trade occurs) are constant through time.

In this respect, the present analysis is related to the recent strand of literature on decentralized two-sided search (see Lu and Mac Afee [1996], Burdett and Coles [1997], Bloch and Ryder [2000] and Shimer and Smith [2000]), but it departs from it in several aspects. The main contribution of these models consists in showing that stationary equilibria in the context of search with heterogeneous agents exist, which can be characterized by the endogenous formation of clusters of agents who adopt the same search strategy. It is proven that under certain conditions positive assortative matching, i.e. a positive association between the characteristics of partners, arises at equilibrium. Burdett and Coles (1997) and Bloch and Ryder (2000) obtain such results in the context of a marriage market where agents' utilities from a match are non-transferable (each individual's utility only depends on the type of the agent on the other side of the market). Conversely, Lu and Mac Afee (1996) and Shimer and Smith (2000) consider transferable utilities in a model with search frictions and where meeting other agents is time-consuming and haphazard. They assume that when a match is formed it generates some divisible output according to a production function which depends on the agents' types. Shimer and Smith (2000) show that a search equilibrium exists if the production function is strictly supermodular in the agents' types and if it satisfies some other regularity conditions. Additional restrictive requirements on the match output need to be imposed in order to guarantee assortative matching as well.

In this paper, agents do transfer the utility generated in a match between each other. But, since the aim of the present analysis is to describe a pure exchange economy, it seems natural to suppose that the gross surplus "produced" by a pair of matched agents simply consists in the difference between reservation values. This assumption does not meet the above mentioned sufficient conditions for the existence of search equilibria, and consequently the properties of equilibria described do not hold in the present context. The existence of sorting externalities might cause inexistence or multiplicity of equilibria to arise. It can be shown that, for some parameter ranges, some matched pairs of agents systematically disagree and remain on the market. Since waiting implies a loss in overall utility (because of discounting), some equilibria are inefficient from a utilitarian perspective. The nature of results obtained here bears many similarities with the conclusions reached by Sattinger (1995) and Bose (1996), although they use a different model.

In the second part of the paper, we consider how this picture is altered when the agents have the additional option of trading through a monopolistic intermediary. The objective is to show that intermediation is a profitable activity in search markets with two-sided heterogeneity, and that there is an incentive for intermediation to arise endogenously.

According to Yavaş (1994), there are two possible types of middlemen, marketmakers and matchmakers. A marketmaker sets a bid and an ask price at which he sells and purchases the products for his own account. This would be the case for specialists in the stock market, retailers and wholesalers. A matchmaker does not trade and proposes to match potential partners instead. Employment and matrimonial agencies, and real estate brokers belong to this class of middlemen.

The present model focuses on the first type of intermediary and assumes that the middleman does not possess a technology to distinguish between the agents' types. Hence, the middleman is not able to price discriminate and is forced to quote fixed bid and ask prices instead. Such prices are publicly observable by the whole market. Unlike buyers and sellers who can trade only if matched, the middleman can conclude large volumes of trade at each period. Moreover, exclusive brokerage is considered, in that trading through the marketmaker or searching on the decentralized market are mutually exclusive activities, at least within a single time period.

Therefore, it is as if the economy consisted of two trading posts, the search and the intermediated market, which are governed by different rules. At the beginning of each period, buyers and sellers simultaneously decide which post to visit, conditional on the observed bid and ask prices. Again, these participation decisions are the source of externalities, because each agent's participation decision affects other traders' expected discounted utility of searching in the decentralized market and their value of trading through the middleman. In this context, we identify the possible Stationary Equilibria with Intermediation, namely situations where buyers and sellers adopt optimal participation decisions given the participation policy of all other agents, where the middleman sets bid and ask prices in such a way as to maximize profits, and where the market is in a stationary state.

The conditions which guarantee that both the search and the intermediated markets coexist are provided. The presence of a middleman in the market narrows the set of agents' types who search. In particular, it is shown that the existence of such an alternative trading mechanism as intermediation might induce separation between agents' types. Intuitively, traders who generate small gains from trade prefer to trade in the search market rather than sustain the transaction costs implicit in the bid-ask spread. Conversely, agents who generate high gains from trade prefer intermediation because it eliminates the likelihood of disagreement in bargaining and the risk of matching unsuitable partners. In this circumstance, the presence of the middleman is beneficial because it enhances the total volume of trade and eliminates delay.

The analysis of the middlemen's activity in search markets was initiated by Rubinstein and Wolinsky (1987), who consider the matching framework for examining the endogenous determination of the extent of intermediation. Their model features a continuum of middlemen facing a continuum of homogeneous buyers and sellers. Intermediaries have access to the same matching technology as buyers and sellers but they need to be at least as efficient as buyers and sellers in making contacts in order for their activity to be viable at equilibrium. Gehrig (1993) and Yavaş (1994) analyze the role of the middleman in a market with heterogeneous agents and two-sided search. They show that the marketmakers provide a service of immediacy by reducing the costs of search and the possibility of delay. When the intermediary posts bid and ask prices, the equilibrium is such that only agents with high gains from trade transact through the middleman. The present analysis departs substantially from Gehrig (1993) and Yavaş (1994) in that these models consider a market which operates for a single period in isolation and which is not embedded in a dynamic environment. The model presented by Wooders (1997) deals with a market that remains in place for infinitely many rounds but traders on the two sides of this market are perfectly homogeneous. When there are unequal measures of buyers and sellers and when agents are impatient, it is shown that the intermediary is able to capture all gains arising from the elimination of delay.

The rest of the paper is organized as follows. Section 4.2 presents the basic model of search with two-sided heterogeneity and without the middleman. Some properties of this market are derived in Section 4.2.1 and then the characterization of search equilibria is provided in Section 4.2.2. Intermediation is introduced in Section 4.3, and in Section 4.4 the stationary equilibria with intermediation are identified and compared with the search equilibria.

4.2 The Search Market

Consider an economy with two goods, a homogeneous indivisible good, and a perfectly divisible commodity that plays the role of numeraire (money). Both commodities are supposed to be perfectly storable. There are two kinds of agents in the market, buyers and sellers. Each seller is endowed with one unit of the indivisible good and zero units of money, while each buyer is endowed with one unit of the perfectly divisible good and zero units of the indivisible good.

Time is discrete and there is an infinite number of dates, indexed by $t = 1, 2, \ldots,$ at which trade can take place. All agents are expected utility maximizers and they are perfectly patient.[1] Nonetheless there does exist a source of discounting in the model: agents that conclude trade in a given period t are assumed to leave the market forever, whereas agents that do not transact face an exogenous and constant probability $\theta \in (0, 1)$ of being terminated. Buyers' and sellers' preferences are characterized by the reservation value of the indivisible good. A seller with valuation s who sells the indivisible good for P units of the numeraire good after t rounds of trade receives a net utility equal to $(1-\theta)^t (P - s)$, and a buyer with reservation value b who buys the indivisible good at period t in exchange for P units of the numeraire receives a payoff of $(1-\theta)^t (b - P)$. If agents never trade, their utility is equal to zero.

At each period t there is a continuum of buyers and sellers, with measures B_t and S_t respectively, but there is only a finite number of types of buyers and sellers. Namely, the set of buyers B_t can be partitioned into the set of high valuation buyers $B_{H,t}$, whose members have valuation b_H, and the set of low valuation buyers $B_{L,t}$, whose members have valuation b_L. Similarly the set of sellers consists in the union of the set of low valuation sellers $S_{L,t}$, whose members have valuation s_L, with the set of high valuation sellers $S_{H,t}$, whose members have valuation s_H. It is assumed that the agents' reservation values are all non-negative, with $b_H > b_L \geq s_L$ and $b_H \geq s_H > s_L$, and such that the differences in reservation values are the same for both buyers and sellers, i.e. $b_H - b_L = s_H - s_L$ or equivalently $b_H - s_H = b_L - s_L \geq 0$. The latter assumption amounts to requiring that the types' distributions have symmetric supports about the point $\frac{|b_L - s_H|}{2}$, where $|\cdot|$ denotes the absolute value. Let $F_{B,t}(b)$ denote the distribution of buyers that are in the market at time t, where

$$F_t(b) = \begin{cases} 0 & \text{if } b < b_L \\ 1 - \eta_{B,t} & \text{if } b_L \leq b < b_H \\ 1 & \text{if } b \geq b_H \end{cases}.$$

Therefore $\eta_{B,t}$ is the proportion of high valuation buyers at time t where $\eta_{B,t} = \frac{B_{H,t}}{B_t}$. Likewise, let $F_t(s)$ denote the distribution of sellers, where $\eta_{S,t} = \frac{S_{L,t}}{S_t}$ is the proportion of low valuation sellers that are in

[1] Letting agents be (equally) impatient would not alter the qualitative features of the results that will be obtained. This follows from the fact that discounting is already present in the model, so the introduction of a pure time preference is redundant.

the market at time t.

It is assumed that there is an exogenous and constant flow of entry at each period. Buyers and sellers enter the market at the same rate E. Moreover, the flow-in distribution of buyers $F_E(b)$ is supposed to be equal to the flow-in distribution of sellers $F_E(s)$, where $\gamma \in (0,1)$ represents the proportion of both high valuation buyers and low valuation sellers that enter the market at each period.

All agents know the distribution functions $F_E(\cdot)$ which govern entries, and have rational expectations about the market distributions $F_t(b)$ and $F_t(s)$ which stem from the agents' matching decisions.

The search market is modeled as a random matching market. At each period, buyers and sellers search for each other and they will either be matched in pairs or remain unmatched. The probability that a buyer of any type meets a seller depends only on the total measures of agents that are active in the market, and it is denoted by $\alpha_{B,t}(B_t, S_t)$. Likewise, the probability that a seller is matched with a buyer is given by $\alpha_{S,t}(B_t, S_t)$. It is assumed that there are no search frictions and thus the matching probabilities α_t have the following simple form

$$\alpha_{B,t}(B_t, S_t) = \begin{cases} 1 & \text{if } B_t < S_t, \\ \frac{S_t}{B_t} & \text{if } B_t \geq S_t \end{cases} \quad \text{and} \quad \alpha_{S,t}(B_t, S_t) = \begin{cases} 1 & \text{if } B_t > S_t, \\ \frac{B_t}{S_t} & \text{if } B_t \leq S_t \end{cases}. \tag{4.1}$$

The agents in the short side of the market find a trading partner with certainty, while some agents in the long side of the market get rationed.[2]

If a buyer and a seller get matched, they immediately observe each other's valuations and they decide whether or not to trade and at which conditions. The gross benefit associated with a match between a type b buyer and a type s seller is simply $b - s$. The agents' expected discounted value of continued search at time t are denoted by $(1-\theta)V_{t+1}(b)$ and $(1-\theta)V_{t+1}(s)$. Then the net gain from trade associated with the match (b, s) at time t is given by

$$G_t(b, s) = b - s - (1-\theta)V_{t+1}(b) - (1-\theta)V_{t+1}(s).$$

If the net gain from trade is negative, the parties will not transact because at least one of them prefers to wait for the next period. Conversely, if the match's net surplus is non-negative, then the pair bargain over the division of such surplus. Denote by $(\pi_{b,t}(b, s), \pi_{s,t}(b, s))$ the vector of shares representing the outcome of the bargaining between buyer b and seller s. Only divisions that are feasible, efficient and both individually and jointly rational can be solutions to the bilateral bargaining problem faced by a pair of agents that get matched. Formally, the agreement schedule $(\pi_{b,t}(b, s), \pi_{s,t}(b, s))$ must satisfy that

$$\pi_{b,t}(b, s) + \pi_{s,t}(b, s) = b - s \tag{4.2}$$

[2] The results obtained in the sequel would still hold in the presence of search frictions, i.e. for matching probabilities of the kind $\alpha_{B,t}(B_t, S_t) = \alpha$ if $B_t < S_t$, with $\alpha < 1$. The cost of search implied by agents' heterogeneity is the main force that drives the results.

and that

$$\pi_{b,t}(b,s) \geq (1-\theta)V_{t+1}(b) \text{ and } \pi_{s,t}(b,s) \geq (1-\theta)V_{t+1}(s);$$

whereby $G_t(b,s) \geq 0$: The strategic approach to bargaining is discarded here because it would complicate the analysis without probably adding relevant insights. Instead, an exogenous bargaining rule is considered: the net gains from trade are supposed to be shared symmetrically and efficiently, for instance according to the Nash Bargaining Solution. Moreover, it is assumed that all agents in the market have identical expectations on the division of the gains from trade in bargaining. Therefore, the price at which the transaction takes place is

$$P_t(b,s) = \left[(1-\theta)V_{t+1}(s) + \tfrac{1}{2}G_t(b,s)\right] = \pi_{s,t}(b,s); \tag{4.3}$$

which is seller s's payoff, and thus buyer b is left with utility

$$\pi_{b,t}(b,s) = b - s - P_t(b,s) = \left[(1-\theta)V_{t+1}(b) + \tfrac{1}{2}G_t(b,s)\right]. \tag{4.4}$$

Throughout the paper, the analysis focuses on steady state situations in which the flow of agents who complete their transactions and leave the market is exactly balanced by the flow of new agents entering the market. When the inflows of both buyers and sellers are constant, the distribution of the types of the buyers and the sellers in the market will converge to a steady state distribution. Therefore, the matching probabilities, the prices and the expected utilities will also converge. Time indices will henceforth be omitted.

A stationary pure strategy for every agent consists in consummating any match which yields at least his expected present value of being unmatched. Define thus buyer b's matching set as the set of sellers' types such that the pair (b,s) generates non-negative gains from trade or else

$$M(b) \equiv \{s \mid G(b,s) \geq 0\}; \tag{4.5}$$

Seller s's matching set $M(s)$ is defined similarly. Observe that matching sets are symmetric by construction (because function $G(b,s)$ is symmetric), hence $b \in M(s)$ if and only if $s \in M(b)$. Indeed, if buyer b is willing to trade with sellers s it is because the match (b,s) generates non-negative net surplus and therefore seller s must also find it beneficial to trade with buyer b: Then the pair (b,s) represents a mutually agreeable match.[3]

Remark 58 Note that the individuals' strategy space has been restricted in two respects. First, all agents of the same type are required to adopt the same strategy, and secondly a tie-breaking rule is implicitly introduced which requires that, upon indifference between accepting or rejecting a match, both parties choose to consummate the given match.

[3] This terminology follows Shimer and Smith (2000).

Given the probabilities α_B and α_B of finding a match, and the distributions $F(b)$ and $F(s)$ of unmatched buyers and sellers, the expected utilities for buyer b and seller s can be defined implicitly as

$$V(b) = (1-\alpha_B)(1-\theta)V(b) + \alpha_B(1-\Pr(s \in M(b)))(1-\theta)V(b) + \alpha_B E[\pi_b(b,s) \mid s \in M(b)] \tag{4.6}$$

and

$$V(s) = (1-\alpha_S)(1-\theta)V(s) + \alpha_S(1-\Pr(b \in M(s)))(1-\theta)V(s) + \alpha_S E[\pi_s(b,s) \mid b \in M(s)] \tag{4.7}$$

respectively. The first term on the right hand side of both equations above corresponds to the case in which no match is found. The second represents the fact that a match is found but it is not mutually agreeable and the last term is the expected surplus associated to a mutually agreeable match, where the expectation is computed with respect to the distribution functions $F(b)$ and $F(s)$.

Substituting (4.4) into (4.6) and (4.3) into (4.7) and rearranging yields

$$V(b) = \tfrac{\alpha_B}{2\theta} E[\max\{0, G(b,s)\}] \tag{4.8}$$

and

$$V(s) = \tfrac{\alpha_S}{2\theta} E[\max\{0, G(b,s)\}] . \tag{4.9}$$

4.2.1 Properties of Value Functions and Matching Sets

The value functions and the matching sets satisfy some useful properties.

Define the functions $\nu(b) \equiv (b-(1-\theta)V(b))$ and $\nu(s) \equiv (s+(1-\theta)V(s))$.

Lemma 59 The value $V(b)$ is always non-decreasing in b and the function $\nu(b)$ is strictly increasing in b. Symmetrically, the value $V(s)$ is always non-increasing in s and the function $\nu(s)$ is strictly increasing in s.

Proof. Attention will be focused on $V(b)$ and $\nu(b)$. Symmetric results hold for sellers.

Buyer b's expected unmatched utility $V(b)$ takes different values depending on the magnitude of the expression $G(\cdot, s)$. Suppose that $\min\{G(b,s_L), G(b,s_H)\} \geq 0$, in which case (4.8) writes extensively as

$$\tfrac{2\theta}{\alpha_B}V(b) = \eta_S(b-s_L-(1-\theta)(V(b)+V(s_L))) + (1-\eta_S)(b-s_H-(1-\theta)(V(b)+V(s_H))) ,$$

which yields $\frac{\partial V(b)}{\partial b} = \frac{\alpha_B}{2\theta+\alpha_B(1-\theta)} > 0$, and $\frac{\partial}{\partial b}\nu(b) = \frac{2\theta}{2\theta+\alpha_B(1-\theta)} > 0$. Suppose instead that $G(b,s_L) \geq 0 > G(b,s_H)$, in which case (4.8) has expression

$$\tfrac{2\theta}{\alpha_B}V(b) = \eta_S(b-s_L-(1-\theta)(V(b)+V(s_L))) ,$$

and consequently $\frac{\partial V(b)}{\partial b} = \frac{\eta_S \alpha_B}{2\theta + \eta_S \alpha_B (1-\theta)} > 0$, and $\frac{\partial}{\partial b}\psi(b) = \frac{2\theta}{2\theta + \eta_S \alpha_B (1-\theta)} > 0$. Conversely, if $G(b, s_H) \geq 0 > G(b, s_L)$, then

$$\frac{2\theta}{\alpha_B} V(b) = (1-\eta_S)(b - s_H - (1-\theta)(V(b) + V(s_H)))$$

which yields $\frac{\partial V(b)}{\partial b} = \frac{(1-\eta_S)\alpha_B}{2\theta + \alpha_B(1-\eta_S)(1-\theta)} > 0$ and $\frac{\partial}{\partial b}\psi(b) = \frac{2\theta}{2\theta + \alpha_B(1-\eta_S)(1-\theta)} > 0$. Finally, in the case in which $\max\{G(b, s_L), G(b, s_H)\} < 0$, one has $\frac{\partial V(b)}{\partial b} = 0$ and $\frac{\partial}{\partial b}\psi(b) = 1$. ■

Lemma 59 has the following implication. Whenever buyer b's matching set $M(b)$ includes high valuation sellers, then it will also include low valuation sellers. Likewise, if $b_L \in M(s)$ then $b_H \in M(s)$ as well. This is stated formally in the proposition that follows.

Proposition 60 For any seller s, the net surplus from a match is strictly increasing in the buyer's type, i.e. $G(b_H, s) > G(b_L, s)$. For any buyer b, the net surplus from a match is strictly decreasing in the seller's type, i.e. $G(b, s_L) > G(b, s_H)$.

Proof. Note that $G(b_H, s) > G(b_L, s)$ is equivalent to

$$b_H - s - (1-\theta)V(b_H) - (1-\theta)V(s) > b_L - s - (1-\theta)V(b_L) - (1-\theta)V(s)$$

which simplifies as $b_H - (1-\theta)V(b_H) > b_L - (1-\theta)V(b_L)$. And the latter inequality is always satisfied because, by Lemma 59, function $\psi(b)$ is strictly increasing in b. A similar reasoning can be used to show that $G(b, s_L) > G(b, s_H)$. ■

Define the match (b_H, s_L) as a high surplus match and the match (b_L, s_H) as a low surplus match. Accordingly, types b_H and s_L will be called high surplus agents, whereas types b_L and s_H will be denoted as low surplus individuals. It is straightforward to check that the following is true.

Corollary 61 If low surplus matches are consummated, then all other matches are consummated as well.

Proof. Observe that by Proposition 60 the chain of inequalities

$$G(b_H, s_L) > \max\{G(b_H, s_H), G(b_L, s_L)\} > \min\{G(b_H, s_H), G(b_L, s_L)\} > G(b_L, s_H)$$

always holds. Low surplus matches are consummated if and only if the right-most term is non-negative, in which case all other terms are strictly positive. ■

4.2.2 Stationary Search Equilibria

It is of interest to identify the set of equilibria in the search market. In a Stationary Search Equilibrium (SSE) all individuals maximize their expected payoffs taking the strategies of all the other agents as given, matches are consummated if and only if the associated gains from trade are non-negative and the distributions of types of buyers and sellers (and therefore the matching probabilities, the prices, and the expected unmatched utilities) are constant over time. The definition below states formally which conditions have to be met for a SSE to arise in the present context.

Definition 62 A Stationary Search Equilibrium (SSE) is represented by a vector $(V(\cdot), M(\cdot), F(\cdot))$ such that: (i) the value functions $V(\cdot)$ solve the system formed by (4.8) and (4.9); (ii) the matching sets $M(\cdot)$ obey condition (4.5), given the values $V(\cdot)$; (iii) the distributions $F(\cdot)$ are constant over time, given agents' strategies $M(\cdot)$.

Do steady state search equilibria exist in the first place? Depending upon the relative magnitude of buyers' and sellers' reservation values and on the agents' matching decisions (which in turn influence the expected unmatched values and the equilibrium distribution of types), different possible stationary situations arise that will be considered in the sequel.

Observe that a positive death rate θ is necessary for the existence of SSE where some types of agents never trade, otherwise their measure would grow indefinitely and stationarity would not be attained.[4]

It can be shown that in any stationary search equilibrium, the total measures of active traders are related with each other in a peculiar way.

Lemma 63 In any Stationary Search Equilibrium the total measures of buyers and sellers are such that $B = S$, whereby $\alpha_S = \alpha_B = 1$.

Proof. Let $p(b_\tau, s_\rho)$ denote the probability that match between a buyer with reservation value $\tau = H, L$ and a seller with valuation $\rho = H, L$ be consummated. Since matching sets are symmetric, the above probabilities are the same for both buyers and the sellers, and such that $p(b_\tau, s_\rho) = \{0, 1\}$.[5] The stationary total measure of buyers must satisfy that

$$\begin{aligned} B = \; & E + (1-\alpha_B)(1-\theta)B + \alpha_B(1-\theta)((\eta_S(1-p(b_L,s_L)) + (1-\eta_S)(1-p(b_L,s_H)))B_L) \\ & + \alpha_B(1-\theta)(\eta_S(1-p(b_H,s_L))B_H + (1-\eta_S)(1-p(b_H,s_H))B_H) \end{aligned} . \quad (4.10)$$

The first term on the right-hand side of the equality represents the new buyers entering the market, the others represent the measure of buyers that remain on the market from one period to the next. In particular, the second term features the surviving buyers who have not found a match and the last ones represent the surviving buyers that did find a match but have not consummated it. Likewise, at a stationary equilibrium the total measure of sellers must be such that

$$\begin{aligned} S = \; & E + (1-\alpha_S)(1-\theta)S + \alpha_S(1-\theta)(\eta_B(1-p(b_H,s_L)) + (1-\eta_B)(1-p(b_L,s_L)))S_L \\ & + \alpha_S(1-\theta)(\eta_B(1-p(b_H,s_H)) + (1-\eta_B)(1-p(b_L,s_H)))S_H \end{aligned} . \quad (4.11)$$

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[4] An exogenous probability of match dissolution, coupled with infinitely lived agents and no inflow of new individuals, serves the same purpose.

[5] See Remark 58.

Solving (4.10) and (4.11) explicitly and rearranging yields

$$B = \frac{E + \alpha_B (1-\theta)(B - (((1-\eta_S)p(b_H, s_H) + \eta_S p(b_H, s_L))B_H + ((1-\eta_S)p(b_L, s_H) + \eta_S p(b_L, s_L))B_L))}{(\theta + \alpha_B (1-\theta))} \tag{4.12}$$

and

$$S = \frac{E + \alpha_S (1-\theta)(S - (((1-\eta_B)p(b_L, s_L) + \eta_B p(b_H, s_L))S_L + ((1-\eta_B)p(b_L, s_H) + \eta_B p(b_H, s_H))S_H))}{(\theta + \alpha_S (1-\theta))}. \tag{4.13}$$

Since agents leave the market in pairs, the following equalities always have to be satisfied

$$\begin{aligned} \alpha_S \eta_B S_L &= \alpha_B \eta_S B_H \\ \alpha_S \eta_B S_H &= \alpha_B (1-\eta_S) B_H \\ \alpha_S (1-\eta_B) S_L &= \alpha_B \eta_S B_L \\ \alpha_S (1-\eta_B) S_H &= \alpha_B (1-\eta_S) B_L \end{aligned} . \tag{4.14}$$

Suppose now, contrary to the assertion in the lemma that at a stationary equilibrium $B > S$, which implies that $\alpha_B = \frac{S}{B}$ and $\alpha_S = 1$. Substituting these facts into (4.12) and (4.13) and taking (4.14) into account yields

$$S = \frac{E - (1-\theta)K}{\theta} = B,$$

where $K \equiv \alpha_B ((\eta_S p(b_H, s_L) + (1-\eta_S) p(b_H, s_H)) B_H + (\eta_S p(b_L, s_L) + (1-\eta_S) p(b_L, s_H)) B_L)$, which contradicts the premises. The same argument holds for $B < S$, and hence $S = B$ is true at a Stationary Search Equilibrium. ■

It turns out that there exist three candidate SSE, which will be examined in turn.

Non-Elitist Equilibrium

A Non-Elitist (NE) Stationary Search Equilibrium is such that all possible matches end up in trade. By Corollary 61, this occurs if and only if $G(b_L, s_H) \geq 0$. Define

$$b_L^{NE} \equiv \frac{(1+\theta)s_H + 2\gamma(1-\theta)(s_H - s_L)}{(1+\theta)} \quad \text{and} \quad \tau \equiv \frac{(1+\theta)(b_L - s_H)}{2(1-\theta)(s_H - s_L)}.$$

Then the given Stationary Search Equilibrium exists under the following conditions.

Proposition 64 A Non-Elitist SSE exists if and only if $b_L \geq b_L^{NE}$, where $s_H < b_L^{NE} < b_H$, or equivalently if and only if $(b_L - s_H) > 0$ and $\gamma \leq \tau$.

Proof. Consider first the steady state conditions that characterize the candidate equilibrium. At a steady state where all matches are consummated, the total measures of buyers and sellers are such that $B^{NE} = S^{NE} = E$, the measures of high surplus buyers and sellers are $B_H^{NE} = S_L^{NE} = \gamma E$, whereby the steady state fraction of high surplus individuals is

$$\eta_B^{NE} = \eta_S^{NE} = \eta_{NE} = \gamma.$$

This situation represents a SSE if and only if low surplus matches are actually consummated, or else if and only if $G(b_L; s_H) \geq 0$ holds. In order to check this requirement, the value functions need to be computed for all agents' types. When all matches are consummated, the value functions are given by

$$\begin{aligned} V(b_H) = \; & \gamma \tfrac{1}{2}\left(b_H - s_L + (1-\theta)V(b_H) - (1-\theta)V(s_L)\right) \\ & + (1-\gamma)\tfrac{1}{2}\left(b_H - s_H + (1-\theta)V(b_H) - (1-\theta)V(s_H)\right) \end{aligned} \tag{4.15}$$

for high valuation buyers,

$$\begin{aligned} V(b_L) = \; & \gamma \tfrac{1}{2}\left(b_L - s_L + (1-\theta)V(b_L) - (1-\theta)V(s_L)\right) \\ & + (1-\gamma)\tfrac{1}{2}\left(b_L - s_H + (1-\theta)V(b_L) - (1-\theta)V(s_H)\right) \end{aligned} \tag{4.16}$$

for low valuation buyers and by

$$\begin{aligned} V(s_L) = \; & \gamma \tfrac{1}{2}\left(b_H - s_L + (1-\theta)V(s_L) - (1-\theta)V(b_H)\right) \\ & + (1-\gamma)\tfrac{1}{2}\left(b_L - s_L + (1-\theta)V(s_L) - (1-\theta)V(b_L)\right) \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} V(s_H) = \; & \gamma \tfrac{1}{2}\left(b_H - s_H + (1-\theta)V(s_H) - (1-\theta)V(b_H)\right) \\ & + (1-\gamma)\tfrac{1}{2}\left(b_L - s_H + (1-\theta)V(s_H) - (1-\theta)V(b_L)\right) \end{aligned} \tag{4.18}$$

for low and high valuation sellers respectively. It is easy to check that the system formed by equations (4.15), (4.16), (4.17), and (4.18) has a unique solution which, under the assumption that $b_H - b_L = s_H - s_L$, is symmetric and consists in

$$V_{NE}(b_H) = V_{NE}(s_L) = \frac{(1+\theta)(b_H - s_H) + (1-\theta(1-2\gamma))(s_H - s_L)}{2(1+\theta)}$$

and

$$V_{NE}(b_L) = V_{NE}(s_H) = \frac{(1+\theta)(b_L - s_H) + 2\gamma\theta(s_H - s_L)}{2(1+\theta)}.$$

Consequently, the condition $G(b_L; s_H) \geq 0$ is satisfied at a NE equilibrium if and only if

$$b_L \geq \frac{(1+\theta)s_H + 2\gamma(1-\theta)(s_H - s_L)}{(1+\theta)} \equiv b_L^{NE}, \tag{4.19}$$

where $s_H < b_L^{NE} < b_H$, or equivalently if and only if

$$\gamma \leq \frac{(1+\theta)(b_L - s_H)}{2(1-\theta)(s_H - s_L)} \equiv \bar{\eta}, \tag{4.20}$$

and this completes the proof. ■

A Non-Elitist SSE exists provided that the gross benefit generated by a low surplus match, namely the quantity $b_L - s_H$, be not only positive but sufficiently high. Alternatively, such an equilibrium exists when the fraction $\eta_{NE} = \gamma$ of unmatched high surplus individuals on both sides of the market is sufficiently low. Otherwise, low surplus agents might be induced not to consummate matches between each other and wait in order to meet high surplus agents, with whom trade is concluded at more favorable terms. This is precisely what happens in the candidate SSE that will be examined next.

Partially Elitist Equilibrium

At a Partially Elitist (SE) Stationary Search Equilibrium all matches are consummated except for low surplus matches $(b_L; s_H)$: This occurs when low valuation buyers and high valuation sellers do not belong to each others' matching sets, or else provided that $G(b_L; s_H) < 0$ but $\min\{G(b_H; s_H); G(b_L; s_L)\} \geq 0$: Let

$$\bar{b}_L^{PE} \equiv \frac{(1+\theta)s_H + 2\eta_{PE}(1-\theta)(s_H - s_L)}{(1+\theta)} \quad \text{and} \quad \underline{b}_L^{PE} \equiv \frac{(2\theta+\eta_{PE}(1-\theta))s_L + \eta_{PE}(1-\theta)(s_H - s_L)}{(2\theta+\eta_{PE}(1-\theta))}$$

and also

$$\underline{\eta} \equiv \frac{2\theta(b_L - s_L)}{(1-\theta)(s_H - b_L)} \quad \text{and} \quad \eta_{PE} \equiv \frac{2\gamma(1-\theta) - 1 + \sqrt{1 - 4\gamma(1-\gamma)(1-\theta)}}{2\gamma(1-\theta)}:$$

Then the existence of such a steady state equilibrium requires the following conditions to be met.

Proposition 65 A Partially Elitist SSE exists if and only if $\underline{b}_L^{PE} \leq b_L < \bar{b}_L^{PE}$; where $s_L < \underline{b}_L^{PE} < s_H < \bar{b}_L^{PE} < b_H$; or equivalently if and only if either $(b_L - s_H) > 0$ and $\eta_{PE} > \underline{\eta}_{PE}(b_L)$; or $(b_L - s_H) < 0$ and $\eta_{PE} \leq \underline{\eta}$:

Proof. At a steady state in which all matches except low surplus ones are consummated, the total number of buyers is $B^{PE} = E + (1-\theta)(1-\eta_S)B_L^{PE}$, the measure of high valuation buyers is equal to $B_H^{PE} = \gamma E$, and the measure of low valuation buyers is $B_L^{PE} = \frac{(1-\gamma)E}{\theta+\eta_S(1-\theta)}$: Therefore, the proportion of high valuation buyers is

$$\eta_B = \frac{\gamma(\theta+\eta_S(1-\theta))}{1-\gamma(1-\theta)(1-\eta_S)}: \tag{4.21}$$

Likewise, the total measure of sellers satisfies that $S^{PE} = E + (1-\theta)(1-\eta_B)S_H^{PE}$; the measure of low valuation sellers is equal to $S_L^{PE} = B_H^{PE} = \gamma E$ and consequently

$$\eta_S = \frac{\gamma(\theta+\eta_B(1-\theta))}{1-\gamma(1-\theta)(1-\eta_B)} \tag{4.22}$$

is the fraction of unmatched low valuation sellers. Solving the system formed by (4.21) and (4.22) one obtains

$$\eta_B^{PE} = \eta_S^{PE} = \eta_{PE} \equiv \frac{2\gamma(1-\theta) - 1 + \sqrt{1 - 4\gamma(1-\gamma)(1-\theta)}}{2\gamma(1-\theta)}; \tag{4.23}$$

where $0 < \eta_{PE} < \eta_{NE} = \gamma$.

Consider now the expected present value of being unmatched in a stationary market where low surplus matches always end in disagreement. The value functions for high valuation buyers and low valuation sellers are as in (4.15) and (4.17) respectively (except that now η_{PE} replaces $\eta_{NE} = \gamma$); while the expected utility for low valuation buyers has expression

$$V(b_L) = \eta_{PE}\tfrac{1}{2}(b_L - s_L + (1-\theta)(V(b_L) - V(s_L))) + (1-\theta)(1-\eta_{PE})V(b_L); \tag{4.24}$$

and finally

$$V(s_H) = \eta_{PE}\tfrac{1}{2}\left(b_H - s_H + (1-\theta)\left(V(s_H) - V(b_H)\right)\right) + (1-\theta)(1-\eta_{PE})V(s_H) \tag{4.25}$$

holds for high valuation sellers. The system formed by equations (4.24), (4.25) and by the analogous to (4.15) and (4.17) has the unique symmetric solution

$$V_{PE}(b_H) = V_{PE}(s_L) = \frac{2\left(\theta(1+\eta_{PE})+\eta_{PE}^2(1-\theta)\right)(s_H-s_L)+\left(2\theta+\eta_{PE}^2(1-\theta)\right)(b_L-s_H)}{2\left(\theta+(\theta+\eta_{PE}(1-\theta))^2\right)},$$

and

$$V_{PE}(b_L) = V_{PE}(s_H) = \eta_{PE}\frac{2\theta(s_H-s_L)+(2\theta+\eta_{PE}(1-\theta))(b_L-s_H)}{2\left(\theta+(\theta+\eta_{PE}(1-\theta))^2\right)}.$$

Note that this result implies that $G(b_H, s_H) = G(b_L, s_L)$ at a PE equilibrium. Then the present situation describes a stationary search equilibrium if and only if the requirements $G(b_L, s_H) < 0$ and $G(b_H, s_H) = G(b_L, s_L) \geq 0$ are both satisfied. The former inequality is satisfied for

$$b_L < \frac{(1+\theta)s_H+2\eta_{PE}(1-\theta)(s_H-s_L)}{(1+\theta)} = \bar{b}_L^{PE} \tag{4.26}$$

and the latter inequality holds if and only if

$$b_L \geq \frac{(2\theta+\eta_{PE}(1-\theta))s_L+\eta_{PE}(1-\theta)(s_H-s_L)}{(2\theta+\eta_{PE}(1-\theta))} = \underline{b}_L^{PE}, \tag{4.27}$$

where $\underline{b}_L^{PE} < \bar{b}_L^{PE}$ always holds and where $s_L < \underline{b}_L^{PE} < s_H < \bar{b}_L^{PE} < b_H$. Alternatively, $G(b_L, s_H) < 0$ always holds at a PE equilibrium if $(b_L - s_H) \leq 0$ and otherwise it is true if and only if $\eta_{PE} > \bar{\eta}$; whereas $G(b_H, s_H) = G(b_L, s_L) \geq 0$ is always satisfied for $(b_L - s_H) \geq 0$ and otherwise it holds if and only if

$$\eta_{PE} \leq \frac{2\theta(b_L-s_L)}{(1-\theta)(s_H-b_L)} = \underline{\eta}.$$

In the particular case in which $b_L = s_H$, this equilibrium is always attained. ■

One can compare the Non-Elitist and the Partially Elitist SSE in terms of total welfare. Then the following result holds.

Proposition 66 A Non-Elitist SSE always dominates a Partially Elitist SSE in terms of total welfare.

Proof. The total welfare is conventionally measured by the sum of the agents' payoffs. By equation (4.2), the joint payoff of a match (b, s) simply coincides with the difference $(b - s)$. Then total welfare can be straightforwardly computed as the weighted sum of the terms $(b - s)$, with weights represented by the number of agents who consummate matches (b, s) at each period. Consider the Non-Elitist equilibrium first. The total measure of both buyers and sellers is E and the proportion of high surplus agents on both sides of the market is γ. Then one obtains

$$W_{NE} = \gamma^2 E(b_H - s_L) + 2\gamma(1-\gamma)E(b_L - s_L) + (1-\gamma)^2 E(b_L - s_H) = E\left(b_L - s_H + 2\gamma(s_H - s_L)\right) \tag{4.28}$$

At the Partially Elitist SSE, the total measure of agents is $B^{PE} = S^{SE} > E$ and the proportion of high surplus traders on both sides of the market is $\eta_{PE} < \gamma$: Together, these conditions yield that the flow-out of high surplus agents is the same as in the Non-Elitist equilibrium, i.e. $\eta_{PE} B^{PE} = \gamma E$. Total welfare has then expression

$$W_{PE} = \eta_{PE}^2 B^{PE} (b_H - s_L) + 2\eta_{PE} (1 - \eta_{PE}) B^{PE} (b_L - s_L) = \gamma E (2(s_H - s_L) + (2 - \eta)(b_L - s_H)): \tag{4.29}$$

Thus

$$W_{NE} > W_{PE} \iff \gamma < \frac{1}{2 - \eta_{PE}};$$

and considering the equilibrium value of η_{PE} given by (4.23) one obtains that

$$W_{NE} > W_{PE} \iff \gamma < \frac{2\gamma(1-\theta)}{2\gamma(1-\theta)+1-\sqrt{1-4\gamma(1-\gamma)(1-\theta)}}$$

where the above inequality is always satisfied. ■

Therefore, a utilitarian social planner would prefer the Non-Elitist to the Partially Elitist SSE.

There remains one more candidate equilibrium to be considered.

Elitist Equilibrium

At an Elitist (E) Stationary Search Equilibrium, the only active agents in the market are high valuation buyers and low valuation sellers. No other agents find it beneficial to trade when they meet each other, in which case it must be that $G(b_H; s_L) \geq 0$ whereas $\max\{G(b_H; s_H); G(b_L; s_L)\} < 0$.[6] Define

$$b_L^E \equiv \frac{(2\theta + \eta_E(1-\theta))s_L + \eta_E(1-\theta)(s_H - s_L)}{(2\theta + \eta_E(1-\theta))}$$

and

$$\eta_E \equiv \frac{-\theta + \sqrt{\theta(\theta + 4\gamma(1-\gamma)(1-\theta))}}{2(1-\theta)(1-\gamma)};$$

then the next proposition provides the conditions that guarantee the existence of an equilibrium.

Proposition 67 An Elitist SSE exists if and only if $b_L < b_L^E$; where $s_L < b_L^E < s_H$; or alternatively if and only if $(b_H - s_L) < 0$ and $\eta_E > \underline{\eta}$.

Proof. When only high surplus matches are consummated, the total measure of buyers satisfies that $B^E = \frac{E - \eta_S(1-\theta)B_H^E}{\theta}$ and the measures of high and low valuation buyers are given by $B_H^E = \frac{\gamma E}{\theta + \eta_S(1-\theta)}$ and $B_L^E = \frac{(1-\gamma)E}{\theta}$ respectively. As a consequence, the steady state fraction of high valuation buyers is

$$\eta_B = \frac{\gamma\theta}{\theta + \eta_S(1-\theta)(1-\gamma)}: \tag{4.30}$$

[6] No steady state equilibrium exists for the cases in which $G(b_H; s_H) \geq 0 > G(b_L; s_L)$ and vice-versa.

Similarly, the total measure of sellers is $S^E = \frac{E - \eta_B(1-\theta)S_L^E}{\theta}$; the measure of low valuation sellers satisfies $S_L^E = \frac{\gamma E}{\theta + \eta_B(1-\theta)}$ and the stationary measure of high valuation sellers is $S_H^E = B_L^E = \frac{(1-\gamma)E}{\theta}$. Thus the stationary proportion of low valuation sellers is

$$\eta_S = \frac{\gamma\theta}{\theta + \eta_B(1-\theta)(1-\gamma)} . \tag{4.31}$$

The expressions (4.30) and (4.31) must be equal and they give

$$\eta_B^E = \eta_S^E = \eta_E = \frac{-\theta + \sqrt{\theta(\theta + 4\gamma(1-\gamma)(1-\theta))}}{2(1-\theta)(1-\gamma)} \tag{4.32}$$

where $0 < \eta_E < \gamma$ always holds.

The value functions in this case are $V_E(b_L) = V_E(s_H) = 0$, since low valuation buyers and high valuation sellers never trade, whereas

$$V_E(b_H) = V_E(s_L) = \frac{\eta_E(b_H - s_L)}{2(\theta + \eta_E(1-\theta))}$$

for high surplus agents. Therefore at an Elitist equilibrium, the requirement $G(b_H; s_L) \geq 0$ holds if and only if $(b_H - s_L) \geq 0$, which is always the case, and the condition $G(b_H; s_H) = G(b_L; s_L) < 0$ is satisfied if and only if

$$b_L < \frac{(2\theta + \eta_E(1-\theta))s_L + \eta_E(1-\theta)(s_H - s_L)}{(2\theta + \eta_E(1-\theta))} = b_L^E ;$$

where $s_L < b_L^E < s_H$; or else if and only if $\eta_E > \underline{\eta}$; which is the case only if $b_L - s_H < 0$. ■

An Elitist equilibrium exists provided that the gross benefit generated by low surplus agents, i.e. the difference $b_L - s_H$, be not only negative but sufficiently low. Alternatively, existence is guaranteed when the fraction of high surplus agents who are active on both sides of the market is sufficiently high. Thus high surplus agents do not to consummate matches with low surplus counterparts because it is more beneficial to wait in order to find a match with other high surplus agents. If this is not the case then an intermediate situation, represented by the Partially Elitist equilibrium, might be relevant.

The theorem below summarizes the results obtained so far and provides a characterization of Stationary Search Equilibria.

Theorem 68 The Non-Elitist SSE, when it exists, is unique. The Partially Elitist and the Elitist SSE might coexist if and only if $\gamma < \frac{1}{2}$ and $\underline{b}_L^{PE} \leq b_L < b_L^E < s_H$. No SSE exists for $s_H < \bar{b}_L^{PE} \leq b_L < b_L^{NE}$ and for $\gamma > \frac{1}{2}$ and $b_L^E \leq b_L < \underline{b}_L^{PE} < s_H$.

Proof. A NE equilibrium can not coexist with a PE equilibrium because $b_L^{NE} > \bar{b}_L^{PE}$ if and only if $\gamma > \eta_{PE}$ which is always the case. The Elitist equilibrium can not coexists with the PE equilibrium if $b_L^E \leq \underline{b}_L^{PE}$. But $b_L^E < \underline{b}_L^{PE}$ is satisfied if and only if $\eta_E < \eta_{PE}$ or equivalently if and only if $\gamma > \frac{1}{2}$;

otherwise $b_L^E = \underline{b}_L^{PE}$ for either $\eta_E = \eta_{PE}$ or $\gamma = \frac{1}{2}$; and $b_L^E > \underline{b}_L^{PE}$ holds if and only if $\eta_E > \eta_{PE}$ or else if $\gamma < \frac{1}{2}$. Therefore, when $\gamma \geq \frac{1}{2}$ the Elitist equilibrium does not coexist with the PE equilibrium, but for $\gamma < \frac{1}{2}$ both steady states exist for

$$\underline{b}_L^{PE} \leq b_L < b_L^E < s_H.$$

There are some parameter ranges that are covered by none of the above Propositions 64, 65 or 67. In these cases a SSE does not exist. ■

Consider the issue of multiplicity of equilibria. The same remarks as in Burdett and Coles (1997) are relevant in the present context.[7]

Observe that for $\gamma < \frac{1}{2}$ one has that $\eta_E > \eta_{PE}$. Hence the proportion of high surplus agents on both sides of the market is greater at the Elitist than at the PE equilibrium. This fact induces high surplus agents to be more selective and to reject matches with low surplus counterparts. When instead $\gamma \geq \frac{1}{2}$ and $\eta_E \leq \eta_{PE}$, the likelihood of meeting a high surplus agent is higher at a PE than at an Elitist equilibrium: In this circumstance η_E, the fraction of high surplus agents at an Elitist equilibrium, cannot support a PE equilibrium as well.

Also notice that $B^E = S^E > B^{PE} = S^{PE}$ which implies that there are more unmatched agents at an Elitist equilibrium than at a PE one. High surplus agents are more selective at an Elitist equilibrium and this depresses the total measure of agents leaving the market at each period. It is then necessary that the measure of unmatched agents be higher in order for the flow-out of consummated matches to balance the exogenous inflow of individuals.

Finally it can be shown that, when both Partially Elitist and Elitist SSE exist, low surplus agents prefer the SE to the Elitist equilibrium, being $V_{PE}(b_L) = V_{PE}(s_H) > V_E(b_L) = V_E(s_H) = 0$. Conversely, high surplus agents are better-off at the Elitist equilibrium since $V_E(b_H) = V_E(s_L) > V_{PE}(b_H) = V_{PE}(s_L)$. Therefore, the two equilibria cannot be ranked in terms of Pareto efficiency. Nevertheless a utilitarian social planner would prefer the Partially Elitist equilibrium because it involves less delay and a higher volume of trade. This is the consequence of the presence of sorting externalities. Indeed, the decision of high surplus agents to consummate matches in the PE equilibrium reduces the expected unmatched value of the other high surplus types, while it increases the expected unmatched value for low surplus agents. The converse is true at the Elitist equilibrium where, "elitist behavior makes the élite better-off at the cost of the lower (...) types".[8]

The lack of existence of a SSE could depend mainly on the restrictions imposed on agents' strategies. It seems plausible that letting individuals, who are precisely indifferent between accepting or rejecting a given match, adopt mixed strategies might avoid the problem of inexistence of a SSE. Observe that,

[7] See in particular the Example 3 therein, pp. 155-158.
[8] Burdett and Coles (1997), p. 158.

when agents are allowed to randomize, stationarity requires that $p_B(b;s) = p_S(b;s)$; i.e. the probability that a type b buyer consummates a match with a type s seller must be the same for type b buyers and type s sellers.

4.3 Intermediation and Search

Consider now the role that intermediation plays in the context of a search market like the one analyzed in the previous section. Again, attention will be restricted to steady state situations, whereby all variables introduced in the sequel are supposed to be constant through time.

It is assumed that one and only one middleman enters the market at time $t = 1$ and remains active at each period thereafter.[9] The intermediary is not endowed with a stock of neither the indivisible good nor money, and it is assumed that he cannot accumulate one either. His reservation value for the indivisible good is zero.

At the beginning of each period, the middleman o¤ers a common bid price P_b for purchasing the indivisible good (from any seller) and demands a common ask price P_a for selling the indivisible good (to any buyer). Thus it is assumed that the middleman is not able to price discriminate and quote di¤erent prices for di¤erent types of buyers or sellers. The posted prices are immediately observed by all buyers and sellers.

On the basis of this information, buyers and sellers simultaneously decide how to conduct their transactions. A buyer chooses whether to search for a seller and trade directly or to trade with the middleman. Similarly a seller chooses whether to trade directly or to trade through the monopolistic intermediary. These participation decisions determine the emergence of two di¤erent markets, the search or direct market and the intermediated market, that might coexist at each period. The measure of buyers (respectively, sellers) that decide to enter the search market at each period is denoted by B_D (S_D) and the measure of buyers (sellers) that decide to enter the intermediated market is B_M (S_M); where $B_D + B_M = B$ (and $S_D + S_M = S$).[10]

Within each period, the intermediary can conclude a large volume of transactions, and not just one as buyers and sellers. But, since the middleman is not allowed to accumulate a stock of the indivisible good, he can only cross trades and transfer each unit of the indivisible good from a seller to a buyer. Therefore when the same measure of buyers and sellers enter the intermediated market at a given period, the middleman is able to match all agents. However, if unequal measures of buyers and sellers enter the mediated market at period t, the intermediary randomly rations the agents on the long side of the

[9] Unlike buyers and sellers, the intermediary is not a¤ected by the probability μ of termination.

[10] In the rest of the paper, the subscript M will be used to denote the mediated market, as opposed to the subscript D which represents the direct or search market.

market. The probabilities that a buyer and a seller trade through the intermediary at each t are the thus same as the matching probabilities (4.1) in the search market, and are denoted by

$$\mu_B(B_M; S_M) = \begin{cases} 1 & \text{if } B_M < S_M; \\ \frac{S_M}{B_M} & \text{if } B_M \geq S_M \end{cases} \quad \text{and} \quad \mu_S(B_M; S_M) = \begin{cases} 1 & \text{if } B_M > S_M; \\ \frac{B_M}{S_M} & \text{if } B_M \leq S_M \end{cases};$$

respectively.

As for players' payoffs, the expected value of participating in the intermediated market is

$$V_M(b) = \mu_B(b - P_a) + (1 - \theta)(1 - \mu_B)V_M(b)$$

for a buyer with reservation value b and

$$V_M(s) = \mu_S(P_b - s) + (1 - \theta)(1 - \mu_S)V_M(s)$$

for a type s seller. The middleman's profit at each period is

$$\pi_M = q_M(P_a - P_b);$$

where q_M is the total quantity traded by the intermediary, that is $q_M = \min\{\mu_B B_M; \mu_S S_M\}$: It is assumed that the middleman stays in business if and only if $\pi_M > 0$:

In the search market, the matching probabilities remain the same as those in (4.1) with the only difference that the relevant measures on which α_S and α_B depend are B_D and S_D instead of B and S: Thus $\alpha_S = \min\left\{1; \frac{B_D}{S_D}\right\}$ is the probability that sellers find a match in the search market and $\alpha_B = \min\left\{1; \frac{S_D}{B_D}\right\}$ is the corresponding matching probability for buyers. The payoff functions remain the same as (4.8) and (4.9).

The rest of analysis is devoted to examining the equilibria that arise in a search market with intermediation. Such equilibria are termed Stationary Equilibria with Intermediation (SEI) and are defined in what follows.

Definition 69 A Stationary Equilibrium with Intermediation (SEI) is such that: (i) the middleman sets profit-maximizing bid and ask prices; (ii) buyers' and sellers' participation decisions are optimal given the bid and ask prices posted by the middleman, the prices negotiated in the private trading market, and the participation policy of all other agents; (iii) the market is in a stationary state.

The implications of the above listed requirements will be considered in turn, starting from the last one.

4.3.1 Steady States

Start with requirement (iii) above, and examine the conditions under which stationarity is attained when both the intermediated and the search market coexist. It is easy to show that the market must be balanced at a steady state with intermediation, as in the case of pure search.

Lemma 70 At any Stationary Equilibrium with Intermediation the total measures of buyers and sellers are such that $B = S$.

Proof. The total measure of buyers in the market is time invariant if and only if

$$B = E + (1 - \theta)((1 - \mu_B) B_M + (1 - \alpha_B) B_D) , \tag{4.33}$$

and similarly the total measure of sellers in the market is constant at each period if

$$S = E + (1 - \theta)((1 - \mu_S) S_M + (1 - \alpha_S) S_D) . \tag{4.34}$$

Rearranging the above expressions, and taking into account that $B_D = B - B_M$, one obtains $B = E + (1 - \theta)(B_M - \mu_B B_M + B_D - \alpha_B B_D)$ and $S = E + (1 - \theta)(S_M - \mu_S S_M + S_D - \alpha_S S_D)$ respectively. Since agents leave the market in pairs, the conditions $\alpha_B B_D = \alpha_S S_D$ and $\mu_B B_M = \mu_S S_M$ must be satisfied at each period. Moreover, $B_M + B_D = B$ and similarly $S_M + S_D = S$. Taking these requirements into account yields

$$B = \frac{E - (1-\theta)(\mu_B B_M + \alpha_B B_D)}{\theta} = S ,$$

which completes the proof. ■

Denote by β_τ the proportion of buyers of type $\tau = H, L$ who enter the intermediated market at each period. Similarly, denote by σ_ρ the proportion of sellers or type $\rho = H, L$ who trade through the intermediary at each period. Thus $(1 - \beta_\tau)$ and $(1 - \sigma_\rho)$ represent the fractions of type τ buyers and type ρ sellers, respectively, who enter the private trading market at each period. Moreover, the proportion of buyers that enter the intermediated market at each t is given by $\gamma\beta_H + (1 - \gamma)\beta_L$, whereas the fraction of buyers that enter the search market at t is $\gamma(1 - \beta_H) + (1 - \gamma)(1 - \beta_L) = 1 - (\gamma\beta_H + (1 - \gamma)\beta_L)$. Symmetric expressions hold for sellers. It turns out that the quantities β_τ and σ_ρ take specific values at a steady state equilibrium.

Proposition 71 At any Stationary Equilibrium with Intermediation the admissible fractions of high and low valuation buyers entering the intermediated market, i.e. β_H and β_L, are represented by the vectors

$$(\beta_H, \beta_L) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

Similarly the only possible combinations of proportions of low and high valuation sellers entering the intermediated market are

$$(\sigma_L, \sigma_H) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

Proof. Attention will be limited to one side of the market, namely the buyers' side. Similar arguments and symmetric expressions are true for sellers. Consider first the stationary measure of low valuation buyers, which must satisfy that

$$B_L = (1-\gamma)E + (1-\theta)((1-\mu_B)\beta_L + (1-\alpha_B)(1-\beta_L))B_L. \quad (4.35)$$

Likewise the steady state measure of high valuation buyers is such that

$$B_H = \gamma E + (1-\theta)((1-\mu_B)\beta_H + (1-\alpha_B)(1-\beta_H))B_H. \quad (4.36)$$

Solving expressions (4.35) and (4.36) explicitly and adding them yields the expression for the total measure of buyers

$$B = B_H + B_L = E\frac{\theta+\alpha_B(1-\theta)+(\gamma\beta_L+(1-\gamma)\beta_H)(\mu_B-\alpha_B)(1-\theta)}{(\theta+\alpha_B(1-\theta)+\beta_H(\mu_B-\alpha_B)(1-\theta))(\theta+\alpha_B(1-\theta)+\beta_L(\mu_B-\alpha_B)(1-\theta))}. \quad (4.37)$$

Moreover, since at a steady state the measure of buyers exiting each market must equal the measure of new agents entering each market, one obtains that the measure of buyers participating in the intermediated market is

$$B_M = E\frac{\gamma\beta_H+(1-\gamma)\beta_L}{(\theta+\mu_B(1-\theta))},$$

whereas

$$B_D = E\frac{1-(\gamma\beta_H+(1-\gamma)\beta_L)}{(\theta+\alpha_B(1-\theta))}$$

is the measure of buyers entering the search market. Hence, an alternate way to compute the stationary total measure of buyers in the market is the following

$$B = B_M + B_D = E\frac{(\theta+\mu_B(1-\theta))-(\mu_B-\alpha_B)(\gamma\beta_H+(1-\gamma)\beta_L)(1-\theta)}{(\theta+\mu_B(1-\theta))(\theta+\alpha_B(1-\theta))}. \quad (4.38)$$

Now, expressions (4.37) and (4.38) are compatible if and only if the relationship between β_H and β_L is such that

$$\beta_L = \frac{b(\beta_H)-\sqrt{b(\beta_H)^2+4\gamma\beta_H(1-\beta_H)(1-\gamma)(\theta+\alpha_B(1-\theta))(\theta+\alpha_B(1-\theta)(1-\beta_H)+\mu_B\beta_H(1-\theta))}}{2(1-\gamma)(\theta+\alpha_B(1-\beta_H)(1-\theta)+\mu_B\beta_H(1-\theta))} \equiv \beta_{L,1}$$

and

$$\beta_L = \frac{K(\beta_H)+\sqrt{K(\beta_H)^2+4\gamma\beta_H(1-\beta_H)(1-\gamma)(\theta+\alpha_B(1-\theta))(\theta+\alpha_B(1-\theta)(1-\beta_H)+\mu_B\beta_H(1-\theta))}}{2(1-\gamma)(\theta+\alpha_B(1-\beta_H)(1-\theta)+\mu_B\beta_H(1-\theta))} \equiv \beta_{L,2},$$

where $K(\beta_H) \equiv ((\theta+\alpha_B(1-\theta))(1-\gamma)+\beta_H(\mu_B-\alpha_B)(1-\gamma\beta_H)(1-\theta))$. But observe that $\beta_{L,1} < 0$ always holds, except when either $\beta_H = 0$ or $\beta_H = 1$, in which case $\beta_{L,1} = 0$. Moreover, $\beta_{L,2} > 1$ is always true except for either $\beta_H = 0$ or $\beta_H = 1$, in which case $\beta_{L,2} = 1$. ■

This result has relevant implications on the nature of SEI, because it rules out situations where agents of the same type adopt di¤erent strategies. Instead, all agents of the same type have to abide by the same participation strategy. Moreover, there is no stationary state where agents of a given type use mixed strategies upon indi¤erence between entering the search or the intermediated market.

Proposition 71 identi…es the stationary con…gurations that emerge under all possible combinations of proportions (¯$_H$; ¯$_L$; ¾$_L$; ¾$_H$). There are sixteen possible con…gurations to consider. Actually, because of the symmetry of the model, six cases are redundant and can be ignored at the outset, and this leaves ten cases to be analyzed. Furthermore, the interest is concentrated on stationary states where the middleman is active and concludes a strictly positive amount of transactions. This further reduces the number of relevant cases to six.

The next step of the analysis consists in examining the implications of requirement (ii) in De…nition 69 in terms of the relevance of the remaining stationary con…gurations.

4.3.2 Quasi-Equilibria with Intermediation

A situation where the middleman is active and where conditions (ii) and (iii) in De…nition 69 are satis…ed will be termed (stationary) Quasi-Equilibrium with Intermediation (QEI) as opposed to a full-‡edged SEI which also meets requirement (i): The characterization of all possible QEI is provided next.

In order to identify a Quasi-Equilibrium with Intermediation the following procedure is adopted. First a stationary con…guration is considered as described by the vector (¯$_H$; ¯$_L$; ¾$_L$; ¾$_H$). Underlying this con…guration are the individuals' participation decisions, which are initially taken as given. Then, the bid and ask prices that support the con…guration (¯$_H$; ¯$_L$; ¾$_L$; ¾$_H$) are determined. One has to check whether there exist prices for which neither buyers nor sellers have incentive to deviate unilaterally from the given participation decisions. In particular the agents' entry choices represented by the vector (¯$_H$; ¯$_L$; ¾$_L$; ¾$_H$) are optimal if and only if: (a) for any buyer and any seller who participate in the search market, the expected value of entering the direct market is at least as high as the value of entering the intermediated market; (b) for any buyer and any seller who enter the intermediated market, the expected value of trading trough the middleman is at least as high as the value of direct search.

Remark 72 By virtue of Proposition 71, deviations by a measurable set of individuals of a given type are not of concern, and only unilateral deviations matter.

The above requirements further pin down the number of stationary con…gurations that are of interest, in that there might not exist bid and ask prices that support a given vector (¯$_H$; ¯$_L$; ¾$_L$; ¾$_H$). The only admissible QEI are analyzed in what follows.

Sorting Quasi-Equilibrium with Intermediation

A Sorting Quasi-Equilibrium with Intermediation is characterized by the configuration $(\beta_H; \beta_L; \sigma_L; \sigma_H) = (1; 0; 1; 0)$; where agents with higher expected gains from trade in pairwise bargaining prefer to enter the intermediated market, while low surplus agents participate in the search market. Then, one observes $B_M = B_H = \eta_B B$ and $B_D = B_L$; and similarly $S_M = S_L = \eta_S S$ and $S_D = S_H$: The features of a Sorting QEI depend on the relative magnitude of low surplus agents' reservation values. Indeed, depending on the sign of $b_L - s_H$; it might or it might not be feasible for low surplus individuals to transact with each other, and this in turn affects all agents' incentives to participate in either the intermediated or the search market. Therefore at a Sorting QEI the following two cases have to be considered: (i) if $b_L - s_H \geq 0$ then low surplus agents actively trade in the search market; (ii) if $b_L - s_H < 0$ then low surplus individuals do not conclude any transaction.

Examine first case (i) where low surplus agents actively trade in the search market. The matching probabilities in both the search and the intermediated market depend on the relative magnitude of η_B and η_S: Suppose then that $\eta_B = \eta_S$; whereby all individuals find a match in all markets and $\alpha_B = \alpha_S = 1$ and $\mu_B = \mu_S = 1$. As a consequence the measures of high and low surplus individuals are equal to $B_H = S_L = \gamma E$ and $B_L = S_H = (1 - \gamma) E$ respectively. Moreover $B = S = E$; and the condition $\eta_B = \eta_S$ is indeed satisfied being $\eta_B = \eta_S = \gamma$. In the private trading market, the value functions for low valuation buyers high valuation sellers are

$$V_D(b_L) = V_D(s_H) = \frac{(b_L - s_H)}{2} : \tag{4.39}$$

and the search market is active provided $G(b_L; s_H) \geq 0$ which amounts precisely to $b_L - s_H \geq 0$: In the intermediated market, the value functions coincide with the agents' per period payoffs and are given by

$$V_M(b_H) = \pi_{b_H} = b_H - P_a$$

for high valuation buyers and by

$$V_M(s_L) = \pi_{s_L} = P_b - s_L$$

for low valuation sellers. Examine the incentives of both low valuation buyers and high valuation sellers to enter the intermediated market. Note that a deviation on the part of a single individual has no effect on the magnitude of the matching probabilities because a continuum of non-atomic agents is considered. Then if a low valuation buyer deviates and participates instead in the intermediated market his payoff is $V_M(b_L) = (b_L - P_a)$, and if a high valuation seller deviates he receives a payoff of $V_M(s_H) = (P_b - s_H)$: No defection from the search market is profitable for low valuation buyers if $V_D(b_L) \geq V_M(b_L)$ holds, or else if

$$P_a \geq \frac{(b_L + s_H)}{2} \equiv \hat{P}_a ;$$

and for high valuation seller if $V_D(s_H) \geq V_M(s_H)$ holds, or else if

$$P_b \leq \frac{(b_L + s_H)}{2} \equiv \hat{P}_b \, .$$

Since $s_H < \hat{P}_a = \hat{P}_b < b_L$, any positive bid and ask spread is immune to unilateral deviations on the part of low surplus agents. As for defections from the intermediated market, observe that if a high valuation buyer deviates and enters the search market his value function becomes

$$V_D(b_H) = \frac{1}{2}\left(b_H - s_H + (1-\theta)V_D(b_H) - (1-\theta)V_D(s_H)\right) , \tag{4.40}$$

where $V_D(s_H)$ is given by (4.39).[11] The requirement $V_M(b_H) \geq V_D(b_H)$ is satisfied if and only if

$$P_a \leq \frac{2(1+\theta)b_H - 2(s_H - s_L) - (1+\theta)(b_L - s_H)}{2(1+\theta)} \equiv \overline{P}_a \, ,$$

where $\overline{P}_a > \hat{P}_a$ always holds under the assumption that $s_H - s_L = b_H - b_L$. For a low valuation seller, deviating and entering the search market entails an expected value of

$$V_D(s_L) = \frac{1}{2}\left(b_L - s_L + (1-\theta)(V_D(s_L) - V_D(b_L))\right) ,$$

where again $V_D(b_L)$ is given by (4.39). Consequently, a defection from the intermediated market is not profitable if $V_M(s_L) \geq V_D(s_L)$ or equivalently if

$$P_b \geq \frac{2(b_L - s_L) - (1-\theta)(b_L - s_H) + 2(1+\theta)s_L}{2(1+\theta)} \equiv \underline{P}_b \, ,$$

with $\underline{P}_b < \hat{P}_b$. Therefore at a Sorting QEI with $b_L - s_H \geq 0$, the middleman sets bid and ask prices equal to $P_b^S \equiv \underline{P}_b$ and $P_a^S \equiv \overline{P}_a$ respectively and obtains a payoff

$$\pi_M^S = \gamma E \frac{2\theta(s_H - s_L)}{(1+\theta)} > 0 \, , \tag{4.41}$$

low surplus agents enter the search market and strictly prefer to transact directly rather than through the middleman, while high surplus agents are exactly indifferent between intermediation and direct trade but choose the former. For further reference, observe that both $P_a^S > b_L$ and $P_b^S < s_H$ hold if and only if

$$b_L < \frac{2\theta(s_H - s_L) + (1+\theta)s_H}{(1+\theta)} \equiv \hat{b}_L^S \, , \tag{4.42}$$

where $s_H < \hat{b}_L^S < b_H$.

If case (ii) is relevant and $b_L < s_H$, or equivalently if transactions are not carried out in the search market, then the stationary measures of high surplus agents are still equal to $B_H = \gamma E = S_L$, while the measures of low surplus types are $B_L = \frac{(1-\gamma)E}{\theta} = S_H$. Thus $B = S = E\frac{1-\gamma(1-\theta)}{\theta} > E$ and $\eta_B = \eta_S = \frac{\gamma\theta}{1-\gamma(1-\theta)} < \gamma$. Low surplus agents have expected values $V_D(b_L) = V_D(s_H) = 0$ in the search market and any positive bid-ask spread is sufficient to prevent low surplus individuals from entering the intermediated

[11] The probability of meeting a deviating high valuation buyer is so insignificant that the sellers' value is not affected.

market. Conversely, the expected utility accruing to either high valuation buyers or low valuation sellers when (unilaterally) defecting from the intermediated market is

$$V_D(b_H) = V_D(s_L) = \frac{(b_H - s_H)}{(1+\theta)}:$$

In order for a high valuation buyer to prefer to enter the intermediated rather than the search market, it must be the case that $V_M(b_H) \geq V_D(b_H)$ or else that

$$P_a \leq \frac{(1+\theta)b_H - (b_H - s_H)}{(1+\theta)} \equiv \underline{P}_a;$$

with $b_L < \underline{P}_a < b_H$: Similarly, a low valuation seller does not find it profitable to deviate and enter the search market if $V_M(s_L) \geq V_D(s_L)$ or equivalently if

$$P_b \geq \frac{(1+\theta)s_L + (b_L - s_L)}{(1+\theta)} \equiv \overline{P}_b;$$

with $s_L < \overline{P}_b < s_H$: Thus at a Sorting QEI with $b_L - s_H < 0$; the bid and ask prices posted by the middleman are $\underline{P}_b^S \equiv \overline{P}_b$ and $\underline{P}_a^S \equiv \underline{P}_a$ respectively and the profit accruing to the middleman is equal to

$$\underline{\pi}_M^S = \gamma E \frac{2\theta(s_H - s_L) - (1+\theta)(s_H - b_L)}{(1+\theta)}; \tag{4.43}$$

where

$$\underline{\pi}_M^S > 0 \iff b_L > \frac{2\theta s_L + (1-\theta)s_H}{(1+\theta)} \equiv b_L^S;$$

with $s_L < b_L^S < s_H$: High surplus individuals participate in the intermediated market but are exactly indifferent between remaining in such market and deviating to the alternative one. Low surplus agents enter the search market but remain inactive because matches are not acceptable.

The proposition below summarized the results obtained so far.

Proposition 73 (i) If $b_L \geq s_H$; a Sorting QEI always exists, which is characterized by prices P_a^S and P_b^S: (ii) If $b_L < s_H$; a Sorting QEI, characterized by prices $\underline{P}_a^S$ and $\underline{P}_b^S$; exists if and only if $b_L^S < b_L < s_H$: No Sorting QEI exists for $s_L < b_L \leq b_L^S$.

The proof is omitted because it follows from the preceding discussion.

One-Sided Pooling Quasi-Equilibrium with Intermediation

Consider next the vector $(\beta_H; \beta_L; \sigma_L; \sigma_H) = (1; 1; 1; 0)$ which represents a market configuration where all agents except high valuation sellers decide to participate in the intermediated market. This situation will be referred to as One-Sided Pooling (OSP) Quasi-Equilibrium with Intermediation and it is characterized by $B_M = B = E\frac{1-\gamma+\gamma\theta}{\theta}$ and $B_D = 0$; and $S_M = S_L = \gamma E$ and $S_D = S_H$: The matching probabilities are $\alpha_B = 1$ and $\alpha_S = 0$ in the search market, and $\mu_B = \eta_S$ and $\mu_S = 1$ in the intermediated market.

Thus, the stationary proportions of high valuation buyers and low valuation sellers are $\eta_B = \gamma$ and $\eta_S = \frac{\gamma\theta}{(1-\gamma(1-\theta))} < \gamma$ respectively, and the total quantity traded in each period coincides with amount of transactions carried out by the middleman, namely $q_M = \gamma E$.

In the search market, high valuation sellers have zero expected value because the probability of meeting a trading partner is zero. In the intermediated market, the expected value for low valuation sellers is

$$V(s_L) = (P_b - s_L);$$

and sellers of type s_L do not have any incentive to deviate and enter the search market, since $\alpha_S = 0$. Hence the middleman can force low valuation sellers to accept a price as low as $P_b = s_L$ without triggering sellers' deviations. As for buyers, their expected value in the intermediated market is

$$V_M(b) = \mu_B(b - P_a) + (1 - \mu_B)(1 - \theta)V_M(b);$$

where $b = b_H; b_L$; which substituting for the steady state value of μ_B yields

$$V_M(b) = \gamma(b - P_a).$$

If a valuation b buyer defects from the intermediated market and searches directly, his expected value is

$$V_D(b) = \frac{b - s_H}{(1+\theta)};$$

with $V_D(b) \geq 0$ if and only if $b \geq s_H$. The sign of the term $b_L - s_H$ influences buyers' incentives to remain in the intermediated market, and thus the same two cases as in Section 4.3.2 have to be considered.

(i) Suppose that $b_L - s_H \geq 0$ which implies that both types of buyers would have positive gains from trade when entering the search market, i.e. in the event of a deviation. Then no low valuation buyer has incentive to deviate unilaterally if

$$V_M(b_L) \geq V_D(b_L) \iff P_a \leq \frac{\gamma(1+\theta)b_L - (b_L - s_H)}{\gamma(1+\theta)} \equiv P_a(b_L);$$

and likewise no high valuation buyer has incentive to deviate if

$$V_M(b_H) \geq V_D(b_H) \iff P_a \leq \frac{\gamma(1+\theta)b_H - (b_H - s_H)}{\gamma(1+\theta)} \equiv P_a(b_H).$$

Observe that the above prices satisfy the following conditions: $s_H < P_a(b_L) < b_L$ always holds, both $P_a(b_H)$ and $P_a(b_L)$ are always positive if

$$\gamma \geq \frac{1}{(1+\theta)} \equiv \gamma_0; \tag{4.44}$$

where $\frac{1}{2} < \gamma_0 < 1$; and $P_a(b_H) \geq P_a(b_L)$ holds if and only if (4.44) is satisfied. Assume this is indeed the case, so that the bid and ask price posted by the middleman are $P_b^{OSP} \equiv s_L$ and $P_a^{OSP} \equiv P_a(b_L)$ respectively, and the corresponding payoff is

$$\pi_M^{OSP} = E\frac{\gamma(1+\theta)(s_H - s_L) + (\gamma(1+\theta) - 1)(b_L - s_H)}{(1+\theta)} > 0. \tag{4.45}$$

When instead $\gamma < \gamma_0$, then the ask price is $P_a^{OSP} = P_a(b_H)$, where $P_a(b_H) > P_b^{OSP} = s_L$ only if

$$\gamma > \frac{1}{2(1+\theta)} \equiv \gamma_1 , \tag{4.46}$$

with $\gamma_1 < \frac{1}{2} < \gamma_0$. Thus $P_a(b_H) > P_b^{OSP}$ if and only if both $\gamma_1 < \gamma < \gamma_0$ and $s_H \leq b_L < b_L^{OSP}$ hold, where

$$b_L^{OSP} \equiv \frac{(2\gamma(1+\theta)-1)(s_H - s_L) + (1-\gamma(1+\theta))s_H}{(1-\gamma(1+\theta))} .$$

The middleman's profit then takes expression

$$\underline{\pi}_M^{OSP} = E\frac{(2\gamma(1+\theta)-1)(s_H - s_L) - (1-\gamma(1+\theta))(b_L - s_H)}{(1+\theta)} , \tag{4.47}$$

and obviously $\underline{\pi}_M^{OSP} > 0$ is satisfied under the same conditions for which $P_a^{OSP} = P_a(b_H) > P_b^{OSP}$ is true. Therefore when $b_L - s_H \geq 0$, a OSP Quasi-Equilibrium does not exist if either $\gamma \leq \gamma_1$, or if $\gamma_1 < \gamma < \gamma_0$ and $b_L \geq b_L^{OSP}$.

(ii) Suppose now that $b_L - s_H < 0$, in which case low valuation buyers do not have incentive to deviate unilaterally and enter the search market (as long as $P_a \leq b_L$), and high valuation buyers do not find it profitable to defect from the intermediated market, provided that $P_a \leq P_a(b_H)$. The middleman's ask price must satisfy that $P_a^{OSP} = \min\{P_a(b_H); b_L\}$, and $P_a(b_H) > b_L$ always holds when $\gamma \geq \gamma_0$, while for $\gamma < \gamma_0$ the inequality $P_a(b_H) \geq b_L$ is true if and only if

$$b_L \leq \gamma(1+\theta)(s_H - s_L) + s_L \equiv \underline{b}_L^{OSP} < s_H .$$

Thus, for either $\gamma \geq \gamma_0$ or for $\gamma < \gamma_0$ and $b_L \leq \underline{b}_L^{OSP}$, one has that $P_a^{OSP} = b_L$ giving the middleman a payoff

$$\underline{\underline{\pi}}_M^{OSP} = \gamma E(b_L - s_L) , \tag{4.48}$$

and otherwise $P_a^{OSP} = P_a(b_H)$ yielding profits $\underline{\pi}_M^{OSP}$, which are positive if and only if $\gamma_1 < \gamma < \gamma_0$.

The next proposition outlines the results obtained relative to the One-Sided Pooling QEI.

Proposition 74 (i) Let $b_L \geq s_H$. If $\gamma \geq \gamma_0$ a One-Sided Pooling QEI always exists which is characterized by prices $P_a^{OSP} = P_a(b_L)$ and $P_b^{OSP} = s_L$; if $\gamma < \gamma_0$ an equilibrium exists if and only if $\gamma_1 < \gamma < \gamma_0$ and $s_H \leq b_L < b_L^{OSP}$, and it is characterized by $P_a^{OSP} = P_a(b_H)$. (ii) Let $b_L < s_H$. A One-Sided Pooling QEI with $P_a^{OSP} = b_L$ always exists if either $\gamma \geq \gamma_0$ or $\gamma < \gamma_0$ and $b_L \leq \underline{b}_L^{OSP}$; otherwise for $b_L > \underline{b}_L^{OSP}$ an equilibrium exists if and only if $\gamma_1 < \gamma < \gamma_0$, and it is characterized by $\underline{P}_a^{OSP} = P_a(b_H)$.

The preceding discussion serves to prove the proposition.

Pooling Quasi-Equilibrium with Intermediation

In a Pooling (P) Quasi-Equilibrium with Intermediation, no agent chooses to trade directly and this is represented by configuration $(\beta_H, \beta_L, \sigma_L, \sigma_H) = (1, 1, 1, 1)$. In this case, one has that $B_M = S_M = B$ and $B_D = S_D = 0$, and consequently the matching probabilities are $\alpha_B = \alpha_S = 0$ in the search market, and $\mu_B = \mu_S = 1$ in the intermediated market. The measure of high valuation buyers is the same as low valuation sellers' and equal to $B_H = \gamma E = S_L$, and the measures of low valuation buyers and high valuation sellers are $B_L = S_H = (1 - \gamma) E$. Thus the total measure of buyers and sellers is $B = S = E$ and the fraction of high surplus agents is $\eta_B = \eta_S = \gamma$.

Proposition 75 *A Pooling QEI exists if and only if* $b_L > s_H$.

Proof. In the intermediated market, buyers' value function takes the form

$$V_M(b) = b - P_a,$$

where $b = b_H, b_L$, and sellers' value function is

$$V_M(s) = P_b - s,$$

being $s = s_H, s_L$. An equilibrium in which all buyers and sellers enter the intermediated market and trade immediately can only be supported by bid and ask prices such that $P_a \leq b_L$ and $P_b \geq s_H$ respectively. Letting $P_a^P = b_L$ and $P_b^P = s_H$ the middleman maximizes his profits

$$\pi_M^P = E(b_L - s_H), \tag{4.49}$$

without inducing deviations. ■

4.4 Stationary Equilibria with Intermediation

The final step of the analysis consists in comparing the Quasi-Equilibria with Intermediation in terms of the middleman's payoffs. This will allow to identify the set of Stationary Equilibria with Intermediation, which are QEI where the intermediary's profits are maximal.

It turns out that all Quasi-Equilibria are relevant, in that the Stationary Equilibrium with Intermediation might coincide either with the Sorting, or with the One-Sided Pooling or else with the Pooling QEI, according to the magnitude of γ and to the relationship among the traders' reservation values. This result stands in contrast with the conclusions obtained in Gehrig (1993) and Yavaş (1994), where it is shown that a sorting kind of equilibrium is always attained when intermediation coexists with search.

Moreover, it might be the case that the set of Stationary Equilibria with Intermediation be empty. Existence of a SEI is only at stake if the gross gain from trade in low surplus matches is negative, i.e.

when $b_L - s_H < 0$; and if the fraction of high surplus traders is low. In such cases, it is evident that the features of the market prevent intermediation from being viable. But when the existence of a Stationary Equilibrium with Intermediation is not a problem, then the equilibrium is generically unique.

The full characterization of the set of SEI in not particularly instructive, because the analysis is complicated by the fact that many different cases have to be taken into account. Therefore, the theorem that comes next is meant not to be exhaustive, but rather to single out the most interesting features of equilibria. Some piece of notation must be introduced first. Define $b_L^{S\succ P}$ and $b_L^{OSP\succ P}$ as

$$\pi_M^S > \pi_M^P \iff b_L < b_L^{S\succ P}$$

and

$$\pi_M^{OSP} > \pi_M^P \iff b_L < b_L^{OSP\succ P};$$

respectively. Finally let $\underline{b}_L^{OSP\succ S}$ be such that

$$\underline{\pi}_M^{OSP} > \underline{\pi}^S \iff b_L < \underline{b}_L^{OSP\succ S}:$$

Theorem 76 (i) If $b_L - s_H \geq 0$; a Stationary Equilibrium with Intermediation always exists and is generically unique. In particular when $\gamma \leq \frac{1}{2}$; the equilibrium cannot be of the OSP type and it coincides with the Sorting QEI if and only if $b_L < b_L^{S\succ P}$; otherwise with the Pooling QEI; when $\gamma \geq \gamma_0 > \frac{1}{2}$; the equilibrium cannot be of the sorting type and it coincides with the One-Sided Pooling QEI if and only if $b_L < b_L^{OSP\succ P}$; otherwise with the Pooling QEI. (ii) If $b_L - s_H < 0$; a SEI does not exist for sufficiently low values of γ; and otherwise it is represented by the OSP quasi-equilibrium for $b_L < \underline{b}_L^{OSP\succ S}$ and by the Sorting QEI for $b_L \geq \underline{b}_L^{OSP\succ S}$ and $\gamma < \gamma_0$:

Proof. See Appendix A.2 ■

The welfare implications of the Stationary Equilibria with Intermediation are interesting. It is possible to show that the presence of a monopolistic intermediary in the context of a search market does not have a clear-cut effect on the economy's aggregate welfare.

Proposition 77 Intermediation and pure search cannot be ranked in terms of aggregate welfare.

Before proving the proposition it is useful to prove the following preliminary result.

Lemma 78 When $b_L - s_H \geq 0$; the aggregate welfare at a Non-Elitist Stationary Search Equilibrium is the same as the total welfare at a Sorting Stationary Equilibrium with Intermediation.

Proof. At a Sorting Stationary Equilibrium with Intermediation with $b_L - s_H \geq 0$; the middleman is active and serves high surplus agents. Low surplus traders prefer to carry out their transactions in the direct market. On each side of the market, the total measure of unmatched individuals is E; and the fraction of high surplus ones is γ. The profit accruing to middleman is thus π_M^S as given by (4.41), the payoff to each high surplus agent is

$$\pi_{b_H,s_L} = \left(b_H - P_a^S\right) = \left(P_b^S - s_L\right) = \frac{(1+\theta)(b_L - s_H) + 2(s_H - s_L)}{2(1+\theta)};$$

and finally the joint payoff of a low surplus match is $(b_L - s_H)$. Thus aggregate welfare takes value

$$W_S = \gamma E \frac{2\theta(s_H - s_L)}{(1+\theta)} + \gamma E \frac{s_H(1-\theta) - 2s_L + b_L(1+\theta)}{(1+\theta)} + (1-\gamma) E (b_L - s_H) = E\left(2\gamma(s_H - s_L) + b_L - s_H\right)$$

which is precisely the same expression as W_{NE} in (4.28). ■

It is now easy to prove Proposition 77.

Proof of Proposition 77. It is sufficient to show that, when the parameters of the model allow for both a Stationary Search Equilibrium and a Stationary Equilibrium with Intermediation to exist, there are situations in which the former equilibrium yields a lower aggregate welfare than the latter and there are other cases in which the reverse is true.

Start considering the Partially Elitist SSE and the Sorting SEI. By Theorem 76, a SEI of the sorting type always attains for $\gamma \le \frac{1}{2}$ and $s_H \le b_L < b_L^{S*P}$. Also recall that a Partially Elitist SSE always exists for $s_H \le b_L < \bar{b}_L^{PE}$, therefore both equilibria coexists if $\gamma \le \frac{1}{2}$ and $s_H \le b_L < \min\left\{b_L^{S*P}; \bar{b}_L^{PE}\right\}$. By Lemma 78, the total welfare at a Sorting SEI coincides with the total welfare at a Non-Elitist SSE, that is $W_S = W_{NE}$. Moreover by Proposition 66, $W_{NE} > W_{PE}$ is always true. Thus intermediation is welfare enhancing because a Sorting Stationary Equilibrium with Intermediation always dominates a Partially Elitist Search Equilibrium in terms of aggregate welfare.

Compare now the Non-Elitist SSE and the One-Sided Pooling SEI. The former always exists for $b_L > b_L^{NE}$ and the latter attains when $\gamma \ge \gamma_0$ and $s_H \le b_L < b_L^{OSP*P}$. Setting $\theta = \frac{1}{3}$ and letting $\gamma \ge \frac{3}{4} = \gamma_0$, one obtains that $b_L^{OSP*P} > b_L^{NE}$ holds and then both equilibria coexist for $b_L^{NE} < b_L < b_L^{OSP*P}$. At a One-Sided Pooling SEI, sellers' utility is zero independently of the type, buyers have individual utility $\left(b - P_b^{OSP}\right)$ and the middleman's profits are given by π_M^{OSP}. The measure of buyers finding a match at each period is γE and thus the total welfare is given by

$$\begin{aligned} W_{OSP} &= \gamma E (1-\gamma)\left(\frac{(b_L - s_H)}{\gamma(1+\theta)}\right) + \gamma^2 E \left(\frac{\gamma(1+\theta)(b_H - b_L) + (b_L - s_H)}{\gamma(1+\theta)}\right) + E\frac{\gamma(1+\theta)(s_H - s_L) + (\gamma(1+\theta) - 1)(b_L - s_H)}{(1+\theta)}. \\ &= \left(\gamma(b_H - b_L) + b_L - s_L\right) E\gamma \end{aligned}$$

It is immediate to check that $W_{OSP} < W_{NE}$, whereby intermediation reduces welfare in this circumstance. ■

The intuition for these results is the following. The existence of an intermediated market together with the search market has two opposing effects. On the one hand, the possibility of trading at fixed prices in the intermediated market allows buyers and sellers to spare the cost of search inherent in the ex ante uncertainty about each other's valuations. The fact that the middleman captures most of the gains from the elimination of sorting externalities has no effect on total welfare (because it simply represents a transfer of resources).

On the other hand, the presence of an additional market introduces search frictions that did not exist in the search market alone, and this might impair efficiency. Consider for instance the most extreme situation where all buyers or sellers of a given type enter the search market and do not find anyone to trade with. The intermediary can therefore take advantage of the coordination problems that arise. Finally observe this would not be a drawback with a continuum of types on each side of the market.