Universitat Politècnica de Catalunya
Programa de Doctorat de Matemàtica Aplicada

Departament de Matemàtica Aplicada

On the fractional Yamabe problem with isolated singularities and related issues

by

Azahara DelaTorre Pedraza

PhD dissertation

Advisor: María del Mar González Nogueras

Barcelona, December 2016

Felicidades...va por tí!
(Bolillón /Ball-eeh-john/)

"And now let's thank the speaker again"
(The chairman)

Acknowledgements

First, I would like to thank my advisor María del Mar González Nogueras for all her advice and dedication.

I also would like to thank Robin Graham and Xavier Ros-Oton for their useful suggestions, Alice Chang and Juncheng Wei for their support and dedication during my visit to Princeton and British Columbia University respectively.

I would also like to thank Xavier Cabré and Joan Solà-Morales for the help they always bring me.

Moreover I want to thank my office mates for putting up with me in my bad moments and enjoy with me in the good ones. I will never forget, among other things, how many teas Fernando Charro pretended to want just to walk me to the coffee shop when I needed a break or how Albert Mas singed "Loca" to make me laugh in the worse moments. Also Tomás Sanz and Matteo Cozzi thanks for being great office mates. Thanks also for all the mathematical help you all gave me.

In addition, I would like to thank all my math friends around the world for making me enjoy each conference I have attended.

Seria impossible no agrair a tot el departament MA1ri per fer-me sentir com a casa. Per acollir-me i integrar-me com una més d'aquesta gran família de la qual espero seguir formant part. Sempre recordaré les clases de catalán/andaluz, *cotilleos*, bromes, riures i altres moments viscuts a la sala de cafè. Especial agraïment a la Marta, la Tere, l'Amadeu, el Toni, el Jordi, la Maria Rosa, la Gemma, el Tomás, el Rafael, el Marcel, el Jaume, el Jesús, l'Inma, el Carles (sense tu seria una *homeless*), el Juan Ramon... i als que, encara que no pertanyin ja al departament, per mi sempre en formaran part: el Ferran, la Patri, l'Adriá, l'Oriol, el Víctor i l'Anna. No deixeu mai que es perdi el bon rotllo, les festes del departament o els inoblidables teatres de Nadal. Grácies també al Pedro per totes les hores que ha passat solucionant els meus problemes amb l'ordinador.

Gracias también a todos los que me habéis ayudado a llegar hasta aquí en el mundo matemático. Empezando por la "señorita Paquita", siguiendo por Lourdes y Andrea y terminando por Pablo Flores, Cesar Rosales y Francisco Martín. Gracias a todos vosotros decidí dedicarme a la investigación en matemáticas y os estoy enormemente agradecida por ello. De igual manera, gracias a todos mis compañeros de Licenciatura porque ellos me enseñaron a ver el mundo matemático de otra manera y a realmente disfrutar con matemáticos $\pi rad2$.

No puedo dejar atrás a todos los compañeros y amigos que han estado a mi lado durante estos años y, en especial, a todos aquellos que me enseñaron a “salir de tranquis” por Barcelona y cómo no, a quien me hizo disfrutar en un piso compartido como nunca antes y se ha convertido en un amigo inigualable. Gracias a todos vosotros: Ángel, Rojas, Antonio, Girela, Anna, Carme, David, Juan Pablo... espero que el doctorado sólo sea el comienzo de una gran amistad que dure para siempre. Gracias también a Augusto por acogerme y “aguantar” a una compañera de piso en los días que te hace reir a carcajadas pero también en los días malos.

Gracias a Toni, Christophel, Diego, Pedram, Sandra, Roberto y Yessica por estar a mi lado durante estos cuatro años. Gracias a Rubén por prepararme tantos cafés en buena compañia durante todo este tiempo. También gracias a Jessica y Fran porque siempres estáis cuando necesito un amigo a mi lado.

Por último, y no por ello menos importante, quiero agradecer enormemente a mis padres y mi hermana por todo el apoyo y ánimo que me han dado durante este tiempo, por aprender matemáticas junto a mí (ayudandome con la redacción, buscando un 2 perdido, escuchando una y otra vez mis charlas, buscando imágenes, etc.). A ellos le debo todo por lo que he luchado y quiero dedicarles todo mi esfuerzo y trabajo. Sin ellos a mi lado, nada de esto habra sido possible.

The work presented in this dissertation has been developed with the financial support given by the following grants:

- FPI2012 (Research staff training grant), associated to grant MTM2011-27739-C04-01. Ministerio de Economía y competitividad, Gobierno de España.
- MTM2011-27739-C04-01 Ecuaciones en derivadas parciales: problemas de reacción difusión y problemas geométricos (2012-2014, Expanded 2012-2015). Ministerio de Economía y competitividad, Gobierno de España.
- MTM2014-52402- C3-1-P Ecuaciones en derivadas parciales: problemas de reacción-difusión, integro-diferenciales y geomtricos (2015-2017).Ministerio de Economía y competitividad, Gobierno de España.
- 2009SGR345 Equacions diferencials en derivades parcials i aplicacions (2009-2013). Generalitat de Catalunya
- 2014SGR1083 Equacions diferencials en derivades parcials i aplicacions (2014-2016). Generalitat de Catalunya

UNIVERSITAT POLITÈCNICA DE CATALUNYA
BARCELONATECH
Escola de Doctorat

Curs acadèmic: 2016-2017

Acta de qualificació de tesi doctoral

Nom i cognoms
AZAHARA DE LA TORRE PEDRAZA

Programa de doctorat
MATEMÀTICA APLICADA

Unitat estructural responsable del programa
FACULTAT DE MATEMÀTIQUES I ESTADÍSTICA

Resolució del Tribunal

Reunit el Tribunal designat a l'efecte, el doctorand / la doctoranda exposa el tema de la seva tesi doctoral titulada

On the fractional Yamabe problem with isolated singularities and related issues

Acabada la lectura i després de donar resposta a les qüestions formulades pels membres titulars del tribunal, aquest atorga la qualificació:

☐ NO APTE ☐ APROVAT ☐ NOTABLE ☐ EXCEL·LENT

(Nom, cognoms i signatura) President/a		(Nom, cognoms i signatura) Secretari/ària
(Nom, cognoms i signatura) Vocal	(Nom, cognoms i signatura) Vocal	(Nom, cognoms i signatura) Vocal

Barcelona , _______ d'/de _________________ de _______________

El resultat de l'escrutini dels vots emesos pels membres titulars del tribunal, efectuat per la Comissió Permanent de l'Escola de Doctorat, atorga la MENCIÓ CUM LAUDE:

☐ SÍ ☐ NO

(Nom, cognoms i signatura) President/a de la Comissió Permanent de l'Escola de Doctorat	(Nom, cognoms i signatura) Secretari/ària de la Comissió Permanent de l'Escola de Doctorat

Barcelona, _______ d'/de ____________________ de _________

Contents

Chapter 1

Introduction

My research is based on **non-local elliptic semilinear equations in conformal geometry**. The fractional curvature is defined from the conformal fractional Laplacian and it is a non-local version of some of the classical local curvatures such as the scalar curvature, the fourth-order Q-curvature or the mean curvature. This new notion of non-local curvature has good conformal properties that allow to treat classical problems from a more general convexity point of view. Note that the fractional curvature in my research is different from the one defined by Caffarelli, Roquejoffre and Savin in [35].

In particular, I have worked on the fractional singular Yamabe problem and related issues. This problem arises in conformal geometry when we try to find a conformal metric to a given one having constant fractional curvature and prescribed singularities. The precise problem I considered in my thesis was to find solutions for the fractional Yamabe problem in $\mathbb{R}^n$, $n > 2\gamma$,

$$(-\Delta)^\gamma w = c_{n,\gamma} w^{\frac{n+2\gamma}{n-2\gamma}} \text{ for } \gamma \in (0,1),$$

with prescribed isolated singularities: first, I just considered radial solutions when there is an isolated singularity and, later, the problem of removing a finite number of points.

I started my research focusing on the geometric interpretation of the problem for an isolated singularity [62]. This study is based on an **extension problem** for the computation of the conformal fractional Laplacian. This is a Dirichlet-to-Neumann problem for a degenerate elliptic, but local, equation, which gives an example of a boundary reaction problem where the nonlinearity is of power type with the critical Sobolev exponent.

Later, I treated the problem as an integro-differential equation, facing two main difficulties: the lack of compactness and the fact that we are dealing with a non-local ODE [61]. Our study is carried out using **variational methods** and it proves the existence of Delaunay-type solutions for the problem. These are radially symmetric metrics with constant fractional curvature.

Finally, I applied some **gluing methods** together with a **Lyapunov reduction** to construct solutions for the singular fractional Yamabe problem when the singular set consists of a given finite number of points [11].

At the moment, I am working on the fractional Caffarelli-Kohn-Nirenberg inequality, which is an interpolation between the Hardy and Sobolev fractional inequalities. In particular, I am looking at the radial symmetry or symmetry breaking of the minimizers.

A future research plan is underlined in Section 1.6.

I have collaborated with Weiwei Ao (University of British Columbia), María del Mar González (Universitat Politècnica de Catalunya, Universidad Autónoma de Madrid), Manuel del Pino (Universidad de Santiago, Chile) and Juncheng Wei (University of British Columbia).

This thesis consists of nine chapters. First, we give is a brief introduction and summary of the thesis. Next, provide some background, notation and known results. Later, we show the main results, i.e, Chapters 3, 4, 5 and 6. After this, we introduce the research plan to come. The thesis also has two appendixes with useful computations. Below I present an introduction for each one of the main chapters:

1.1 Background

First, in Chapter 2 we will provide some background, notation and known results.

The problem of finding radial solutions for the fractional Yamabe problem in $\mathbb{R}^n$, $n > 2\gamma$, with an isolated singularity at the origin is equivalent to looking for positive, radially symmetric solutions to the semilinear equation

$$(-\Delta_{\mathbb{R}^n})^\gamma w = c_{n,\gamma} w^{\frac{n+2\gamma}{n-2\gamma}} \text{ in } \mathbb{R}^n \setminus \{0\}, \tag{1.1.1}$$

where $c_{n,\gamma}$ is any positive constant and $\gamma \in (0,1)$. In geometric terms, given the Euclidean metric $|dx|^2$ on $\mathbb{R}^n$, we are looking for a conformal metric

$$g_w = w^{\frac{4}{n-2\gamma}}|dx|^2, \; w > 0,$$

with positive constant fractional curvature $Q_\gamma^{g_w} \equiv c_{n,\gamma}$, which is radially symmetric and has a prescribed singularity at the origin.

Caffarelli, Jin, Sire and Xiong in [33] characterized the asymptotic behavior of all nonnegative solutions to (1.1.1), thus we know that we should look for solutions of the form

$$w(r) = r^{-\frac{n-2\gamma}{2}} v(r) \text{ on } \mathbb{R}^n \setminus \{0\}, \tag{1.1.2}$$

for some function $0 < c_1 \le v \le c_2$.

The main difficulty here is to compute the fractional Laplacian in radial coordinates. The fractional Laplacian on $\mathbb{R}^n$ is defined as the pseudo-differential operator with Fourier symbol $|\xi|^{2\gamma}$, or, equivalently for $\gamma \in (0,1)$ and $u \in L^\infty \cap \mathcal{C}^2$ as integro-differential operator

$$(-\Delta)^\gamma w(x) = \kappa_{n,\gamma} \text{P.V.} \int_{\mathbb{R}^n} \frac{w(x)-w(y)}{|x-y|^{n+2\gamma}}\, dy, \tag{1.1.3}$$

where *P.V.* denotes the principal value, and the constant $\kappa_{n,\gamma} > 0$. Caffarelli and Silvestre introduced in [36] a different way to compute the fractional Laplacian in $\mathbb{R}^n$ for $\gamma \in (0,1)$:

let w be any smooth function defined on $\mathbb{R}^n$ and consider the extension $W : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ solution to the following (local) degenerate elliptic equation:

$$\begin{cases} \operatorname{div}(y^{1-2\gamma}\nabla W) = 0, & x \in \mathbb{R}^n,\ y \in \mathbb{R}_+, \\ W(x,0) = w(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.1.4}$$

Then,

$$(-\Delta)^\gamma w = -\tilde{d}_\gamma \lim_{y\to 0^+} y^{1-2\gamma}\partial_y W \tag{1.1.5}$$

for a constant $\tilde{d}_\gamma > 0$.

This construction was generalized to the curved setting by Chang and González in [43]: on a general Riemannian manifold (M^n, g), the fractional curvature Q_γ^g is defined from the conformal fractional Laplacian P_γ^g, and it is a nonlocal version of the scalar curvature (which corresponds to the local case $\gamma = 1$). The conformal fractional Laplacian is constructed from the scattering theory on the conformal infinity M^n of a conformally compact Einstein manifold (X^{n+1}, g^+) and it is a (non-local) pseudo-differential operator of order 2γ. Its principal symbol is the same as the one for the fractional Laplacian, but it presents some curvature terms. In the Euclidean case, these curvature terms equal zero and thus the conformal fractional Laplacian reduces to the standard fractional Laplacian.

Let $\gamma \in (0,1)$ and (X^{n+1}, g^+) be a conformally compact Einstein manifold with conformal infinity $(M^n, [g])$. The conformal fractional Laplacian P_γ^g on the conformal infinity can be computed as a Dirichlet-to-Neumann operator for a generalized extension problem in the spirit of (1.1.4)-(1.1.5). In the particular cases $\gamma = 1$ or $\gamma = 2$ we recover the classical conformal Laplacian and the fourth order Paneitz operator, respectively.

The main property of P_γ^g is its conformal invariance; indeed, for a conformal change of metric $g_w = w^{\frac{4}{n-2\gamma}} g$, we have that

$$P_\gamma^{g_w}(f) = w^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^g(fw), \quad \text{for all } f \text{ smooth},$$

which, in particular when $f = 1$, reduces to the fractional curvature equation

$$P_\gamma^g(w) = Q_\gamma^{g_w} w^{\frac{n+2\gamma}{n-2\gamma}}.$$

This is a very natural generalization of the classical Yamabe equation.

1.2 Non-local ODEs – Geometric interpretation [62]

After establishing the necessary preliminaries in the previous section, it is possible now to introduce the main results in Chapter 3 using the natural geometric interpretation of the problem (1.1.1) and the extension formulation. This work constitutes the first article of my thesis [62].

The main idea is to perform a conformal change in order to find an equivalent, but more tractable, problem. We look for radial solutions for (1.1.1), i.e. in $\mathbb{R}^n \setminus \{0\}$, using the extension (1.1.4)-(1.1.5) for a suitable metric $\bar{g}$. After writing the Euclidean metric in polar coordinates we can use the Emder-Fowler change of variable $r = e^{-t}$ to get

$$|dx|^2 = dr^2 + r^2 g_{\mathbb{S}^{n-1}} = e^{-2t}[dt^2 + g_{\mathbb{S}^{n-1}}] =: e^{-2t} g_0.$$

This conformal change allows to formulate this equivalent problem: let the extension manifold be $X^{n+1} = (0,2) \times \mathbb{R} \times \mathbb{S}^{n-1}$ with the metric given by

$$\bar{g} = d\rho^2 + \left(1 + \tfrac{\rho^2}{4}\right)^2 dt^2 + \left(1 - \tfrac{\rho^2}{4}\right)^2 g_{\mathbb{S}^{n-1}}, \tag{1.2.1}$$

and the conformal infinity $M = \mathbb{R} \times \mathbb{S}^{n-1}$ with the cylindrical metric given by

$$g_0 = \bar{g}|_M = dt^2 + g_{\mathbb{S}^{n-1}},$$

for $\rho \in (0,2)$ and $t \in \mathbb{S}^1(L)$. Recalling (1.1.2), we can write any conformal change of metric on M as

$$g_v := w^{\frac{4}{n-2\gamma}} |dx|^2 = v^{\frac{4}{n-2\gamma}} g_0. \tag{1.2.2}$$

Using g_0 as background metric on M, and writing the conformal change of metric in terms of v as (1.2.2), to find radial (in the variable $|x|$) positive solutions for (1.1.1) with an isolated singularity at the origin is equivalent to looking for positive solutions $v = v(t)$ to

$$P_\gamma^{g_0}(v) = c_{n,\gamma} v^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{on} \quad \mathbb{R} \times \mathbb{S}^{n-1}, \tag{1.2.3}$$

with $0 < c_1 \le v \le c_2$; we hope to find those that are periodic in t, i.e, $v(t+L) = v(t)$. These are known as Delaunay solutions for the fractional curvature, and can be found by solving a fractional order ODE.

The first result in Chapter 3, which is valid for all $\gamma \in (0, \frac{n}{2})$, is a definition of the conformal fractional Laplacian $P_\gamma^{g_0}$ on the cylinder using the Fourier symbol for the operator. Any function on $\mathbb{R} \times \mathbb{S}^{n-1}$ may be decomposed as $\sum_k v_k(t) E_k$, where $\{E_k\}$ is a basis of eigenfunctions associated to the eigenvalues $\{\mu_k\}$ of $\Delta_{\mathbb{S}^{n-1}}$, repeated according to multiplicity. Since the operator $P_\gamma^{g_0}$ diagonalizes under such eigenspace decomposition, if P_γ^k denotes the projection of the operator $P_\gamma^{g_0}$ over each eigenspace $\langle E_k \rangle$, then for $\gamma \in (0, \frac{n}{2})$, taking the Fourier transform in t,

$$\widehat{P_\gamma^k(v_k)} = \Theta_\gamma^k(\xi)\, \widehat{v_k}, \qquad \text{where } \Theta_\gamma^k(\xi) = 2^{2\gamma} \frac{\left|\Gamma\left(\frac{1}{2} + \frac{\gamma}{2} + \frac{\sqrt{(\frac{n}{2}-1)^2 - \mu_k}}{2} + \frac{\xi}{2} i\right)\right|^2}{\left|\Gamma\left(\frac{1}{2} - \frac{\gamma}{2} + \frac{\sqrt{(\frac{n}{2}-1)^2 - \mu_k}}{2} + \frac{\xi}{2} i\right)\right|^2},$$

which is simplified in the case of radial solutions.

Next, we explore how much of the standard ODE study for the scalar curvature (see the lecture notes [159]) can be generalized in our setting. Rewriting the extension (1.1.4)-(1.1.5) for the new metric (1.2.1), we obtain for $\gamma \in (0,1)$ that (1.2.3) is equivalent to the following boundary-reaction PDE

$$\begin{cases} -\operatorname{div}_{\bar{g}}(\rho^a \nabla_{\bar{g}} V) + E(\rho) V = 0 \quad \text{in } (X^{n+1}, \bar{g}), \\ V = v \text{ on } \{\rho = 0\}, \\ -\tilde{d}_\gamma \lim_{\rho \to 0} \rho^a \partial_\rho V = c_{n,\gamma} v^{\frac{n+2\gamma}{n-2\gamma}} \text{ on } \{\rho = 0\}, \end{cases} \tag{1.2.4}$$

where $E(\rho)$ is a lower order term and V is a solution of (1.2.4) only depending on t and ρ.

We have analyzed (1.2.4) in the spirit of a "phase portrait" for a standard second order ODE. In particular, we have studied the linearized problem around the equilibrium point $v_1 \equiv 1$, which corresponds to the cylindrical metric. We explicitly find periodic solutions with period

$$L_0^\gamma = \frac{2\pi}{\sqrt{\lambda_\gamma}}, \tag{1.2.5}$$

where λ_γ is unique and positive. In addition,

$$\lim_{\gamma \to 1} L_0^\gamma = \tfrac{2\pi}{\sqrt{n-2}},$$

so we recover the classical case for the scalar curvature $\gamma = 1$.

Finally, we were able to find a Hamiltonian quantity preserved along the trajectories of (1.2.4). Indeed, the Hamiltonian

$$H_\gamma(t) := \frac{1}{2} \int_0^2 \rho^a \left\{ e_1(\rho)(\partial_t V)^2 - e(\rho)(\partial_\rho V)^2 - e_2(\rho) V^2 \right\} d\rho + C_{n,\gamma} v^{\frac{2n}{n-2\gamma}},$$

is constant with respect to t. Here $e_i(\rho)$, $i = 1,2,3$, denote polynomial expressions and $C_{n,\gamma}$ is a constant.

Summarizing, in Chapter 3 we provide the geometric setting, the "phase portrait"-type study and the linear study for the construction of "Delaunay" solutions of (1.1.1). The existence of such solutions is proved in the following.

1.3 ODEs for Integro-differential equations – Variational - approach [61]

In Chapter 4, which corresponds with a second article of my thesis [61], we considered the same problem (1.1.1), but this time it was treated as an integro-differential equation, i.e., as an equation of the type $\mathscr{L} w = f(w)$ where $\mathscr{L} w$ has the general form

$$\mathscr{L} w(x) = \text{P.V.} \int_{\mathbb{R}^n} (w(x) - w(y)) K(x,y) \, dy,$$

and K represents a singular kernel. Our problem (1.1.1) becomes an integro-differential equation using the definition for the fractional Laplacian given in (1.1.3).

In the classical case $\gamma = 1$, equation (1.1.1) reduces to a standard second order ODE. However, in the fractional case, (1.1.1) becomes a fractional order ODE, so classical methods cannot be directly applied here. As we did in the previous Section 1.2, we reformulate the problem for v using the relation between w and v given in (1.1.2); but here, instead of using the boundary reaction problem (1.2.4) we work directly with the nonlocal operator. The main idea is to use the Emden-Fowler change of variable $r = e^{-t}$ in the singular integral (1.1.3). This yields that equation (1.1.1) can be written as

$$\mathscr{L}_\gamma v = c_{n,\gamma} v^{\frac{n+2\gamma}{n-2\gamma}}, \quad v > 0, \tag{1.3.1}$$

which is a semilinear equation with critical exponent in dimension n and $\mathscr{L}_\gamma$ is the nonlocal operator defined by

$$\mathscr{L}_\gamma v(t) = \kappa_{n,\gamma} P.V. \int_{-\infty}^{\infty} (v(t) - v(\tau)) K(t-\tau)\, d\tau + c_{n,\gamma} v(t),$$

for K a singular kernel which can be precisely computed in terms of hypergeometric functions. The behaviour of K near the origin is the same as the kernel of the fractional Laplacian $(-\triangle)^\gamma$ in $\mathbb{R}$ and near infinity it presents an exponential decay. This kind of kernels corresponds to tempered stable process in probability.

If we just take into account periodic functions $v(t+L) = v(t)$, solutions for problem (1.3.1) may be found by minimizing the following functional:

$$\mathscr{F}_L(v) = \frac{\kappa_{n,\gamma} \int_0^L \int_0^L (v(t) - v(\tau))^2 K_L(t-\tau)\, dt\, d\tau + c_{n,\gamma} \int_0^L v(t)^2 dt}{\left(\int_0^L v(t)^{\frac{2n}{n-2\gamma}} dt\right)^{\frac{n-2\gamma}{n}}},$$

where $K_L := \sum_{j\in\mathbb{Z}} K(t-\tau-jL)$ is a periodic singular kernel. We prove that, for $n > 2+2\gamma$, a minimizer always exists and, if $c(L)$ denotes the minimum value for the functional, $c(L)$ is attained by a nonconstant minimizer v_L when $L > L_0^\gamma$, and when $L \le L_0^\gamma$, $c(L)$ is attained by the constant only; here L_0^γ is the minimal period given in (1.2.5).

We call these solutions "Delaunay"-type manifolds of constant fractional curvature because they can be understood as a generalization of the well known Delaunay surfaces of constant mean curvature. Moreover, these manifolds converge to cylinders when the period L tends to the minimal period L_0^γ, and to spheres when L tends to infinity.

1.4 Gluing methods for the fractional singular Yamabe problem [11]

Finally, in Chapter 5 I present the third article of my thesis [11], where we considered the problem of finding solutions for the fractional Yamabe problem in $\mathbb{R}^n$, $n > 2\gamma$, for $\gamma \in (0,1)$, with isolated singularities at a prescribed finite number of points. This is, to find positive solutions for the equation

$$\begin{cases} (-\Delta_{\mathbb{R}^n})^\gamma w = c_{n,\gamma} w^{\frac{n+2\gamma}{n-2\gamma}} \text{ in } \mathbb{R}^n \backslash \Sigma, \\ w \to +\infty \text{ as } x \to \Sigma, \end{cases} \tag{1.4.1}$$

where $\Sigma = \{p_1, \cdots, p_k\}$.

In the previous work, we showed the existence of "Delaunay"-type solutions for (1.1.1), i.e, solutions of the form

$$w_L(r) = r^{-\frac{n-2\gamma}{2}} v_L(-\log r) \text{ on } \mathbb{R}^n \setminus \{0\},$$

for some smooth function v_L that is periodic in the variable $t = -\log r$, for any period $L > L_0^\gamma$.

In this Chapter 5 we are able to use the gluing method for the non-local problem (1.4.1). This work generalizes the result given by Mazzeo and Pacard ([141]) or Schoen ([158]) for the classical case (i.e, for the scalar curvature), but using similar methods to the ones developed by Malchiodi in [132]. Apart from the obvious difficulty of passing from a local problem to a non-local one, which is handled by careful estimates of the non-local terms, the main obstacle we find is the lack of a second order ODE for radial solutions. Thus we use Delaunay solutions but not directly as a model for an isolated singularity; instead, we construct a bubble tower at each singular point. (Note that by *bubble* we denote the "unique" solution for problem (1.4.1) when $\Sigma = \varnothing$).

This allows to construct a suitable approximate solution with an infinite number of parameters to be chosen. Note that the linearization at this approximate solution is not injective due to the presence of a kernel, so we use a Lyapunov-Schimdt reduction procedure. It is well known that one single bubble is non-degenerate and the kernel for the linearized operator can be explicitly characterized. However, for our problem, we perturb each bubble in the bubble tower separately. Of course, this perturbation will not be independent from one bubble to another; we find an infinite dimensional Toda-type system of compatibility conditions that allows to solve the original problem from the perturbed one.

1.5 Work in progress: Fractional Caffarelli-Kohn-Nirenberg inequality

At this moment I am working on the generalization of the classical Caffarelli-Kohn-Nirenberg inequality [34] to the fractional case $\gamma \in (0, 1)$. In the classical case, the existence or non existence of extremal solutions and their properties have extensively been studied since 1984, and in some particular cases, even before. The starting point is the $p = 2$-case, that reads as follows: for all $\alpha \leq \beta \leq \alpha + 1$ and $\alpha \neq \frac{n-2}{2}$, in space dimension $n > 2$, it holds that

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*}}{|x|^{\beta 2^*}} \, dx \right)^{2/2^*} \leq (\Lambda^n_{\alpha,\beta})^{-1} \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^{2\alpha}} \, dx, \ \forall \, u \in D_{\alpha,\beta},$$

where $2^* = \frac{2n}{n-2+2(\beta-\alpha)}$, $D_{\alpha,\beta} = \{|x|^{-\beta} u \in L^{2^*}(\mathbb{R}^n), |x|^{-\alpha}|\nabla u| \in L^2(\mathbb{R}^n)\}$ and $(\Lambda^n_{\alpha,\beta})^{-1}$ denotes the optimal constant. This inequality represents an interpolation between the usual Sobolev inequality ($\alpha = 0, \beta = 0$) and the Hardy inequality ($\alpha = 0, \beta = 1$) or weighted Hardy inequality ($\beta = \alpha + 1$).

Among the existing results in this direction, we should point out that, even though it was expected that all the minimizers were radially symmetric, Catrina and Wang in

[42] discovered that symmetry for minimizers can be broken. Felli and Schneider in [87] highlighted the symmetry-breaking phenomenon when they found non-radial minimizers for a small perturbation of the problem. They conjectured that the symmetry region and the non-symmetry region are separated by a curve called the Felli-Schneider curve. This fact was proven, in many cases, in a series of papers by Dolbeault, Esteban, del Pino, Filippas, Loss, Tarantello and Tertikas (see the survey [65]).

So far, I have focused my research on the case $p = 2$, for which I conjecture the following: for all functions u regular enough,

$$\Lambda \left(\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*}}{|x|^{\beta p^*}} \right)^{\frac{2}{2^*}} \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\gamma}|x|^{\alpha}|y|^{\alpha}} \, dy \, dx \tag{1.5.1}$$

where Λ is a positive constant independent of u and $2* = \frac{2n}{n-2\gamma+2(\beta-\alpha)}$, whenever $n > 2\gamma$, $\alpha \neq \frac{n-2\gamma}{2}$ and $\beta < \alpha < \beta + \gamma$.

We found no reference (in particular, no proof) of this inequality in the literature. We expect to have a full proof of it soon – recall that both the Hardy and the Sobolev inequality have fractional versions. Assuming the inequality to hold, I have established that extremal functions for (1.5.1) must be radially symmetric if $0 < \alpha < \frac{n-2\gamma}{2}$ and $\beta < \alpha < \beta + \gamma$; and I am working on the symmetry breaking case. Inspired by [42], I expect to cover the case $\alpha < 0$ and to find some region where radial symmetry of the optimizers is broken.

Later on, I expect to find the optimal symmetry range of the parameters using flows methods as Dolbeault, Esteban and Loss did for the classical case in [70]-[71].

1.6 Research plan

In some future works, I plan to consider the following problems:

- On the one hand, I would like to generalize the Caffarelli-Kohn-Nirenberg inequality to the fractional setting without the parameter restrictions I am considering at present, using the recently developed flow method. First, I plan to follow the steps of [70], where Dolbeault, Esteban and Loss solved the conjecture for the optimal symmetry range of the parameters. Since rearrangement inequalities, reflection methods or moving plane cannot be applied in some regions, it was not enough to study only the optimizers in the radial class. The key idea in their work was to rewrite the inequality in terms of a new variable $p = v^{-n}$ and assume that v satisfies a fast diffusion equation. This idea of exhibiting a nonlinear fast diffusion flow under a monotone action (non linear *carré du champ* method) allows to use the fast diffusion flow to drive the functional towards its optimal value. Good notes for this work are written in [71].

  Later on, I would like to complete this work by generalizing to the fractional setting all the symmetry and symmetry breaking results for the most general Caffarelli-Kohn-Nirenbeg inequality,

  $$\big\| |x|^{\gamma} u \big\|_{L^r} \leq C \big\| |x|^{\alpha} |\nabla u| \big\|_{L^p}^{a} \, \big\| |x|^{\beta} u \big\|_{L^q}^{1-a},$$

that holds under suitable parameter conditions. The starting point is the recent work of Dolbeault, Muratori and Nazaret in [75].

- On the other hand, it would be interesting to study how much of the results given in Chapters $3, 4$ and 5 ([62]-[61]-[11]) can be generalized for $\gamma > 1$, in particular $\gamma \in (1,2)$. One of the difficulties here is the lack of a general maximum principle. To deal with this issue, the idea is to follow the work of Gursky and Malchiodi in [104], where they proved that the Paneitz operator (P_2) satisfies a strong maximum principle under the extra hypothesis of nonnegative scalar curvature ($\gamma = 1$).

Chapter 2

Background

2.1 Preliminaries

2.1.1 Introduction to Riemannian Geometry

First of all, we introduce some basic notions in Riemannian Geometry. We will follow the notation and definitions given in the books by Aubin [16, 15]. Note that we are using the Einstein summation convention.

A **connection on a differentiable manifold** M is a mapping D (called the covariant derivative) of $T(M) \times \Gamma(M)$ into $T(M)$ which has the following properties:

- If $X \in T_P(M)$, then $D(X, Y)$ (denoted by D_{XY}) is in $T_P(M)$.
- For any $P \in M$ the restriction of D to $T_P(M) \times \Gamma(M)$ is bilinear.
- If f is a differentiable function, then $D(fY) = X(f)Y + fDY$.
- If X and Y belong to $\Gamma(M)$, X is of class $\mathcal{C}^r$ and Y of class $\mathcal{C}^{r+1}$, then DY is in $\Gamma(M)$ and is of class $\mathcal{C}^r$,

where $\Gamma(M)$ denotes the vector space of vector fields on M. A **Riemannian metric** is a twice-covariant tensor field g such that at each point $P \in M$, g_0 is a positive definite bilinear symmetric form.

In a Riemannian manifold we can also define the **torsion tensor**: It is a $(1, 2)-$tensor which depends on the connection D in the following way $T(X, Y) = D_X Y - D_Y X - [X, Y]$

The **Riemannian connection** is the unique connection with vanishing torsion tensor, for which the covariant derivative of the metric tensor is zero ($\nabla g = 0$).

In the following, M will always be an oriented Riemannian manifold of dimension n unless otherwise stated. Since the Riemannian connection has no torsion we can define the Christoffel symbols in a local coordinate system as

$$\Gamma_{ij}^l = \frac{1}{2}[\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij}]g^{kl},$$

where g^{kl} are, by definition, the components of the inverse matrix of the matrix $(g_{ij})_{ij}$.

A **volume form** on M, given in an oriented coordinate system $\{x^i\}$ is

$$dvol_g := \sqrt{|g|}dx^1 \wedge ... \wedge dx^n,$$

where the dx^i are the 1−forms forming the dual basis to the basis vectors $\partial_i := \frac{\partial}{\partial x^i}$ and $\wedge$ is the wedge product. We denote by $|g|$ the determinant of the metric tensor g_{ij}. Given a 2−tensor E we define its **contraction or trace** as:

$$tr(E_{ij}) = \sum_{i,j} g^{ij} E_{ij}.$$

The **curvature of a connection** D is a 2−form with values in $Hom(\Gamma, \Gamma)$ defined by

$$(X, Y) \to Riem(X, Y) = D_X D_Y - D_Y D_X - D_{[X,Y]}.$$

For the definition we suppose that the vector fields are at least $\mathcal{C}^2$.

The **curvature tensor** is the 4−covariant tensor $Riem(X, Y, Z, T) = g[Riem(X, Y)T, Z]$; its components are $Riem_{ijkl} = g_{im} Riem^m_{jkl}$. It has the properties:

- $Riem_{ijkl} = -Riem_{ijlk}$,
- $Riem_{ijkl} = Riem_{klij}$.

According to the expression of the components of the curvature tensor and considering a normal coordinate system around P,

$$Riem^l_{kij}(P) = (\partial_i \Gamma^l_{jk})_P - (\partial_j \Gamma^l_{ik})_P.$$

The **sectional curvature** of a 2−dimensional subspace of $T(M)$ defined by vectors X and Y, where X is orthonormal to Y (i.e., $g(X, X) = 1$, $g(Y, Y) = 1$, $g(X, Y) = 0$), is

$$\sigma(X, Y) = Riem(X, Y, X, Y).$$

If X, Y are not orthonormal, the definition is

$$\sigma(X, Y) = \frac{Riem(X, Y, X, Y)}{g(X, X)g(Y, Y) - (g(X, Y))^2}.$$

We can obtain, by contraction, the so called **Ricci tensor**, whose components are

$$Ric^g_{ij} = Riem^k_{ikj} = Riem_{iklj} g^{lk}. \tag{2.1.1}$$

(The Ricci tensor is symmetric).

The contraction of the Ricci tensor is called the **scalar curvature**.

$$R_g = Ric^g_{ij} g^{ij}.$$

A **conformal map** is a transformation which preserves angles. Given two metrics g and $\tilde{g}$ on M, they are conformally related if $\tilde{g} = fg$ with $f > 0$.
Note that we will usually write the conformal change as $g_w = w^{\frac{4}{(n-2)}} g$, where w is a $\mathcal{C}^\infty$ and strictly positive function on M; and we will denote $[g]$ the class of all metrics conformal to g.

Once we know the previous definitions we can introduce the Laplace-Beltrami and the conformal Laplacian operators.

2.1.2 Laplacian operators on manifolds and the Yamabe problem

The divergence of a vector field X ($\operatorname{div} X$) on a Riemannian manifold (M^n, g) is defined as the scalar function with the property

$$(\operatorname{div} X)vol_g := \mathfrak{L} vol_g,$$

where $\mathfrak{L}$ is the Lie derivative along the vector field X. In local coordinates we obtain

$$\operatorname{div} X = \frac{1}{\sqrt{|g|}} \partial_i(\sqrt{|g|} X^i).$$

Additionally, the gradient of a scalar function f is the vector field grad f that may be defined through the inner product $< \cdot, \cdot >$ on the manifold, as

$$< \operatorname{grad} f(x), v_x >= df(x)(v_x),$$

for all vectors $v_x \in T_x M$; where df is the exterior derivative of the function f. So in local coordinates, we have

$$(\operatorname{grad})^i f = \partial^i f = g^{ij} \partial_j f.$$

We will denote it by ∇f. We also write

$$< \nabla w, \nabla v >_g = \nabla^i w \nabla_i v \quad \text{and} \quad |\nabla f|_g^2 = \nabla^i \nabla_i f.$$

The **Laplace-Beltrami** operator on a manifold M is defined as:

$$\Delta_g f = \operatorname{div} \ \operatorname{grad} f.$$

Combining the definitions of the gradient and divergence, we can give an explicit formula, in local coordinates, for the Laplace-Beltrami operator Δ_g:

$$\Delta_g f = \frac{1}{\sqrt{|g|}} \partial_i(\sqrt{|g|} g^{ij} \partial_j f).$$

The **conformal Laplacian** operator on (M^n, g) is defined as

$$L_g = -\Delta_g + c_n R_g, \quad \text{where } c_n = \tfrac{(n-2)}{4(n-1)}, \tag{2.1.2}$$

The conformal Laplacian is a conformally covariant operator, this is:

Proposition 2.1.1. *Given $\tilde{g}, g$ two conformally related metrics with $\tilde{g} = w^{\frac{4}{n-2}} g$, $w > 0$, then the operator L_g satisfies*

$$L_{\tilde{g}}(\varphi) = w^{\frac{-(n+2)}{n-2}} L_g(w\varphi),$$

for every $\varphi \in \mathcal{C}^\infty(M)$. In the case $\varphi = 1$ we obtain the classical scalar curvature equation:

$$L_g(w) = c_n R_{\tilde{g}} w^{\frac{n+2}{n-2}}. \tag{2.1.3}$$

Proof. We will only present the proof of (2.1.3). If we denote $\tilde{\Gamma}^l_{ik}$ and Γ^l_{ik} the Christoffel symbols corresponding to $\tilde{g}$ and g, respectively; and we write here the conformal change as $\tilde{g} = e^f g$, we obtain that

$$\begin{aligned}\tilde{\Gamma}^l_{ik} - \Gamma^l_{ik} =& \frac{1}{2}\tilde{g}^{lm}(\frac{\partial \tilde{g}_{mi}}{\partial x^k} + \frac{\partial \tilde{g}_{mk}}{\partial x^i} - \frac{\partial \tilde{g}_{ik}}{\partial x^m}) - \frac{1}{2}g^{lm}(\frac{\partial g_{mi}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^m}) \\ =& \frac{1}{2}e^{-f}g^{lm}(\frac{\partial e^f g_{mi}}{\partial x^k} + \frac{\partial e^f g_{mk}}{\partial x^i} - \frac{\partial e^f g_{ik}}{\partial x^m}) - \frac{1}{2}g^{lm}(\frac{\partial g_{mi}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^m}) \\ =& \frac{1}{2}g^{lm}(\partial_k f g_{mi} + \partial_i f g_{mk} - \partial_m f g_{ik}).\end{aligned}$$

Using (2.1.1),

$$Ric^{\tilde{g}}_{kj} - Ric^g_{kj} = -\frac{n-2}{2}\nabla_k\nabla_j f + \frac{1}{2}\Delta_g f g_{jk} + \frac{n-2}{4}\nabla_k f\nabla_j f - \frac{n-2}{4}\nabla^s f\nabla_s f g_{jk}.$$

Thus if we use $\tilde{g} = e^f g$ and we contract by g^{kj}, we obtain

$$R_{\tilde{g}}e^f - R_g = (n-1)\Delta_g f - \frac{(n-2)(n-1)}{4}\nabla^s f\nabla_s f.$$

Substituting the change $f = \frac{4}{n-2}\log w$, we find that

$$-\Delta_g w + c_n R_g w = c_n R_{\tilde{g}} w^{\frac{n+2}{n-2}}.$$

Recalling the definition of the conformal Laplacian (2.1.2) we have proved the result. □

The Yamabe problem.
A very good reference for the classical Yamabe problem is the survey [122]. The problem proposed by Yamabe is: given a compact Riemannian manifold (M^n, g) of dimension $n \geq 3$, find a new metric g_w conformal to g with constant scalar curvature. Let R_g be the scalar curvature of (M^n, g).

Because of Proposition 2.1.1 we obtain that the Yamabe problem (with a conformal metric) is equivalent to proving that the equation

$$-\Delta_g w + c_n R_g w = R_{g_w} w^{\frac{n+2}{n-2}}, \tag{2.1.4}$$

with R_{g_w} constant, has a $\mathcal{C}^\infty$solution; and that this solution is strictly positive.

Yamabe's original idea was to use the variational method, by minimizing the functional

$$J[w] = \frac{\int_M (|\nabla w|_g^2 + c_n R_g w^2)\, dvol_g}{(\int_M |w|^{2^*}\, dvol_g)^{\frac{2}{2^*}}}. \tag{2.1.5}$$

The Euler-Lagrange equation for functional (2.1.5) is precisely (2.1.4) with R_{g_w} constant.

Here $2^* = \frac{2n}{n-2}$ denotes the critical exponent for Sobolev embedding.

Definition 2.1.2. *Given a manifold (M, g) we define the Yamabe constant as:*

$$\lambda(M) := \lambda(M, g) = \inf\{J[w];\ w \text{ is positive smooth on (M,g)}\}. \tag{2.1.6}$$

Remark 2.1.3. The sign of $\lambda(M)$ is equal to the sign of R_{g_w} (which is constant).

We can restrict to non-negative solutions because if $w \in W^{1,2}$, then $|w| \in W^{1,2}$ and $|\nabla|w\| = |\nabla w|$ almost everywhere, so $J(w) = J(|w|)$. Positivity is a consequence of the maximum principle. Regularity follows from elliptic theory, so it is enough to take the infimum in $W^{1,2}$.

Yamabe problem on the sphere (model case)

The analysis of the Yamabe equation (2.1.4) depends on the case of the sphere $\mathbb{S}^n$ with its standard metric $g_{\mathbb{S}^n}$. So we are going to describe the solution to the Yamabe problem on $\mathbb{S}^n$ and prove that the infimum of the Yamabe functional (2.1.5) in this case is realized by the standard metric on the sphere. We will also show the relation with the sharp form of the Sobolev inequality in $\mathbb{R}^n$.

We call σ the stereographic projection (a conformal diffeomorphism) defined by

$$\sigma : \mathbb{S}^n - \{P\} \to \mathbb{R}^n, \tag{2.1.7}$$

$$\sigma(z_1, ..., z_n, \xi) = (x_1, ...x_n),$$

where $P = (0, ..., 0, 1)$ is the north pole on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$,

$$x_j = \frac{z - j}{(1 - \xi)},\ j \in \{1, ...n\}$$

and $(z, \xi) \in \mathbb{S}^n \setminus \{P\}$.

We will denote $|dx|^2$ the Euclidean metric on $\mathbb{R}^n$ and $\rho = \sigma^{-1}$. Under σ, $g_{\mathbb{S}^n}$ corresponds to

$$\rho^* g_{\mathbb{S}^n} = \frac{4}{(|x|^2 + 1)^2}|dx|^2.$$

Moreover using the stereographic projection we can write down all the conformal diffeomorphisms of the sphere, which are generated by the rotations and maps of the form $\sigma^{-1}\tau_v\sigma$ or $\sigma^{-1}\delta_\mu\sigma$, where τ_v, δ_μ are, respectively, translation by $v \in \mathbb{R}^n$ and dilation by $\mu > 0$:

$$\tau_v, \delta_\mu : \mathbb{R}^n \to \mathbb{R}^n,$$

$$\tau_v(x) = x - v,$$

$$\delta_\mu = \mu^{-1}x.$$

Under dilations, the spherical metric on $\mathbb{R}^n$, $(\rho^* g_{\mathbb{S}^n})$, is transformed into

$$\delta_\mu^* \rho^* g_{\mathbb{S}^n} = 4 w_\mu^{\frac{4}{n-2}} |dx|^2,\ \text{where } w_\mu(x) = \left(\frac{\mu}{|x|^2 + \mu^2}\right)^{\frac{(n-2)}{2}}. \tag{2.1.8}$$

Theorem 2.1.4. *(Obata, [149]). If g is a metric on $\mathbb{S}^n$ that is conformal to the standard metric $g_{\mathbb{S}^n}$ and has constant scalar curvature, then up to a constant factor, g is obtained from $g_{\mathbb{S}^n}$ by a conformal diffeomorphism of the sphere.*

In this way, the Yamabe functional (2.1.5) on $(\mathbb{S}^n, g_{\mathbb{S}^n})$ is minimized by constant multiplies of $g_{\mathbb{S}^n}$ and its images under conformal diffeomorphisms. These are the only metrics conformal to the standard one on $\mathbb{S}^n$ that have constant scalar curvature.

This theorem is closely related to the Sobolev inequality in $\mathbb{R}^n$,

$$\|u\|_{2^*}^2 \leq \sigma_n \int_{\mathbb{R}^n} |\nabla u|^2 dx, \qquad u \in W^{1,2}(\mathbb{R}^n). \tag{2.1.9}$$

Since the infimum of the Yamabe functional on the sphere is conformally invariant, stereographic projection converts the Yamabe problem on $\mathbb{S}^n$ to an equivalent on $\mathbb{R}^n$.

More precisely, for $w \in \mathcal{C}^\infty(\mathbb{S}^n)$, let w_0 denote the weighted push-forward function on $\mathbb{R}^n$ defined by $w_0 = w_1 \rho^* w$ with $w_1(x) = (|x|^2+1)^{(2-n)/2}$ the conformal factor. Then we have

$$\rho^*(w^{\frac{4}{n-2}} g_{\mathbb{S}^n}) = 4 w_0^{\frac{4}{n-2}} |dx|^2.$$

Because of the conformal invariance, $J(\mathbb{R}^n) = J(\mathbb{S}^n)$. Recalling the definition of J from (2.1.5), since $R_{|dx|^2} = 0$, we have

$$\lambda(\mathbb{R}^n) = \inf_{u_0 \in \mathcal{C}^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |\nabla w_0|^2 \, dx}{(\int_{\mathbb{R}^n} |w_0|^{2^*} \, dx)^{2/2^*}}.$$

Because of density we can restrict to smooth compactly supported functions:

$$\lambda(\mathbb{R}^n) = \inf_{w \in \mathcal{C}_0^\infty(\mathbb{R}^n)} \frac{\|\nabla w\|_2^2}{\|w\|_{2^*}^2}.$$

Using the Sobolev inequality (2.1.9), $\lambda(\mathbb{S}^n) > 0$. Therefore, it is equivalent identifying $\lambda(\mathbb{S}^n)$ and the associated extremal functions to identifying the best constant and extremal functions for the Sobolev inequality.

Theorem 2.1.5. ***Sharp Sobolev inequality in the sphere.***

The $n-$dimensional Sobolev constant σ_n is equal to $\frac{c_n}{\Lambda}$, where

$$\Lambda = \lambda(\mathbb{S}^n) = J(g_{\mathbb{S}^n}) = n(n-1) vol(\mathbb{S}^n)^{2/n}.$$

Thus the sharp form of the Sobolev inequality on $\mathbb{R}^n$ is:

$$\|w\|_{2^*}^2 \leq \frac{c_n}{\Lambda} \int_{\mathbb{R}^n} |\nabla w|^2 dx.$$

Equality is attained only by constant multiples and translates of the functions w_μ defined by (2.1.8).

Lemma 2.1.6. *If M is any compact Riemannian manifold of dimension $n \geq 3$, then $\lambda(M) \leq \lambda(\mathbb{S}^n)$.*

With all these ingredients, one can give a solution for the Yamabe problem. Here we just present the main theorems:

Theorem 2.1.7. *(Yamabe, Trudinger and Aubin (1976).) The Yamabe problem can be solved for any compact manifold M such that $\lambda(M) < \lambda(\mathbb{S}^n)$.*

This theorem shifts the focus of the proof from analysis to understanding the geometric meaning of the invariant $\lambda(M)$. The idea of the proof to show that $\lambda(M) < \lambda(\mathbb{S}^n)$ is to find a test function ϕ with $J(\phi) < \lambda(\mathbb{S}^n)$. Then,

Theorem 2.1.8. *[122] If M is any compact Riemannian manifold of dimension $n \geq 3$, then*

$$\lambda(M) < \lambda(\mathbb{S}^n); \tag{2.1.10}$$

unless M is already conformal to the sphere $\mathbb{S}^n$.

This theorem was proved in several steps:

- Aubin (1976)[13]: He proved that if M has dimension $n \geq 6$ and it is not locally conformally flat then (2.1.10) holds.
- Schoen (1984)[157]: Who finally proved that if M has dimension $3, 4$, or 5, or M is locally conformally flat, then (2.1.10) holds, unless M is already conformal to the sphere $\mathbb{S}^n$. Note that his proof uses the positive mass theorem.

2.2 Conformal fractional Laplacian and fractional Q_γ-curvature

2.2.1 Fractional Laplacian in $\mathbb{R}^n$.

We can cite as references, the surveys in [169] and [64], and the book [120].

Let $\gamma \in (0,1)$ and $w \in L^\infty \cap \mathcal{C}^2$ in $\mathbb{R}^n$, the **fractional Laplacian in $\mathbb{R}^n$** is given by

$$(-\Delta)^\gamma w(x) = \kappa_{n,\gamma} \text{P.V.} \int_{\mathbb{R}^n} \frac{w(x+y) - w(x)}{|y|^{(n+2\gamma)}}\, dy,$$

where *P.V.* denotes the principal value, that is defined as

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon} \frac{w(x+y) - w(x)}{|y|^{(n+2\gamma)}}\, dy,$$

and the constant $\kappa_{n,\gamma}$ is given by

$$\kappa_{n,\gamma} = \pi^{-\frac{n}{2}} 2^{2\gamma} \frac{\Gamma\left(\frac{n}{2}+\gamma\right)}{\Gamma(1-\gamma)} \gamma.$$

This it is a good definition because $|y|^{-(n+2\gamma)}$ is integrable at ∞ and $\int_{B_1} \frac{\nabla w(x)\cdot y}{|y|^{n+2\gamma}} = 0$ (since $\frac{y}{|y|^{n+2\gamma}}$ is odd). Moreover using Taylor's expansion at the origin, we have that

$$\frac{|w(x+y) - w(x) - \nabla w(x)\cdot y|}{|y|^{n+2\gamma}} \leq \frac{\|D^2 w\|_{L^\infty}}{|y|^{n+2\gamma-2}},$$

where the right hand side is integrable. Then, we obtain that the integral is convergent, and thus, near zero (2.2.1) can be expressed without the need of $P.V$ as

$$(-\Delta)^\gamma w(x) = \int_{\mathbb{R}^n} \frac{w(x+y) - w(x) - \nabla w(x)\cdot y}{|y|^{n+2\gamma}}\, dy. \tag{2.2.1}$$

Fourier symbols and fractional Laplacian
The fractional Laplacian on $\mathbb{R}^n$ is defined through Fourier transform as

$$\widehat{(-\Delta)^\gamma} w = |\xi|^{2\gamma}\widehat{w}, \quad \forall \gamma \in \mathbb{R}.$$

Note that we use the Fourier transform defined by

$$\widehat{w}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} w(x) e^{-i\xi\cdot x}\, dx.$$

Fractional Laplacian as solution of a degenerate elliptic equation in the extension
We have seen two different ways to define fractional Laplacian, now we are going to introduce another one [36].

Let $w : \mathbb{R}^n \to \mathbb{R}$ be a function such that $\int_{\mathbb{R}^n} \frac{|w(x)|}{(1+|x|)^{n+2\gamma}} < \infty$, and let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}_+$. We consider the extension $W : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ that satisfies the following partial differential equation:

$$\begin{cases} W(x,0) = w(x), \quad x \in \mathbb{R}^n, \\ \Delta_x W + \dfrac{a}{y} W_y + W_{yy} = 0, \quad x \in \mathbb{R}^n, \ y \in \mathbb{R}. \end{cases} \tag{2.2.2}$$

Then, on the one hand, the second equation in 2.2.2 can be written in divergence form as

$$\operatorname{div}(y^a \nabla W) = 0.$$

On the other hand, solutions for this differential equation are critical points for the following functional:

$$J(W) = \int_{\mathbb{R}^{n+1}_+} |\nabla W|^2 y^a\, dxdy.$$

Definition 2.2.1. *Let $\gamma \in (0,1)$, we define the fractional Laplacian on $\mathbb{R}^n$ as*

$$(-\Delta)^\gamma w = -\tilde{d}_\gamma \lim_{y\to 0^+} y^a \partial_y W$$

where $a = 1 - 2\gamma$ and

$$\tilde{d}_\gamma = -\frac{2^{2\gamma-1}\Gamma(\gamma)}{\gamma\Gamma(-\gamma)}. \tag{2.2.3}$$

We will prove that this construction of the fractional Laplacian is equivalent to the two previous definitions. If we take Fourier transform with respect to x in the system (2.2.2), we obtain

$$\begin{cases} \hat{W}(\xi,0) = \hat{w}(\xi), \ \xi \in \mathbb{R}^n, \\ -|\xi|^2\hat{W}(\xi,y) + \frac{a}{y}\hat{W}_y(\xi,y) + \hat{W}_{yy}(\xi,y) = 0, \ \xi \in \mathbb{R}^n, y > 0. \end{cases} \tag{2.2.4}$$

Fixed ξ, we can call $\psi(y) = \hat{W}(\xi,y)$ and we get

$$-|\xi|^2\psi + \frac{a}{y}\psi_y + \psi_{yy} = 0.$$

Then we know that the solution of (2.2.4) is given by

$$\hat{W}(\xi,y) = \hat{w}(\xi)\phi(|\xi|y), \tag{2.2.5}$$

where ϕ is the solution of the following system:

$$\begin{cases} -\phi(y) + \frac{a}{y}\partial_y\phi(y) + \partial_{yy}\phi(y) = 0, \\ \phi(0) = 1, \\ \lim_{y\to\infty}\phi(y) = 0. \end{cases} \tag{2.2.6}$$

Applying the following Lemma 2.2.2 we obtain that the solution of (2.2.6) can be written as $\phi(y) = y^\gamma\left(c_1 I_\gamma(y) + c_2 K_\gamma(y)\right)$; and imposing $\lim_{y\to\infty}\phi(y) = 0$ we obtain $c_1 = 0$. If we impose $\phi(0) = 1$, we get that the constant c_2 must be equal to $\Gamma^{-1}(\gamma)2^{1-\gamma}$.

Differentiating (2.2.5) with respect to y we get

$$\partial_y\hat{W} = \hat{w}(\xi)\phi'(|\xi|y)|\xi|.$$

Letting y tend to zero, after the change of variable $z = |\xi|y$, we obtain

$$\lim_{y\to 0} y^a\partial_y\hat{W} = \hat{w}(\xi)|\xi|\lim_{y\to 0}\phi'(|\xi|y)y^a = \hat{w}(\xi)|\xi|^{2\gamma}\lim_{z\to 0}\phi'(z)z^a = c\hat{w}|\xi|^{2\gamma},$$

where $c = \lim_{z\to 0}\phi'(z)z^a = c_2\lim_{z\to 0} z^\gamma K'(z)z^a = -\tilde{d}_\gamma^{-1}$. This shows (2.2.1).

Lemma 2.2.2. *[4] The solution of the ODE*

$$\partial_{yy}\phi + \frac{a}{y}\,\partial_y\phi - \phi = 0.$$

may be written as $\phi(y) = y^\gamma\psi(y)$, for $a = 1-2\gamma$, where ψ solves the well known Bessel equation

$$y^2\psi'' + y\psi' - (y^2+\gamma^2)\psi = 0. \tag{2.2.7}$$

In addition, (2.2.7) *has two linearly independent solutions,* I_γ, K_γ*, which are the modified Bessel functions; their asymptotic behavior is given precisely by*

$$\begin{aligned} I_\gamma(y) &\sim \frac{1}{\Gamma(\gamma+1)}\left(\frac{y}{2}\right)^\gamma\left(1+\frac{y^2}{4(\gamma+1)}+\frac{y^4}{32(\gamma+1)(\gamma+2)}+\dots\right),\\ K_\gamma(y) &\sim \frac{\Gamma(\gamma)}{2}\left(\frac{2}{y}\right)^\gamma\left(1+\frac{y^2}{4(1-\gamma)}+\frac{y^4}{32(1-\gamma)(2-\gamma)}+\dots\right)\\ &\quad+\frac{\Gamma(-\gamma)}{2}\left(\frac{y}{2}\right)^\gamma\left(1+\frac{y^2}{4(\gamma+1)}+\frac{y^4}{32(\gamma+1)(\gamma+2)}+\dots\right), \end{aligned}$$

for $y \to 0^+$, $\gamma \notin \mathbb{Z}$*. And when* $y \to +\infty$,

$$\begin{aligned} I_\gamma(y) &\sim \frac{1}{\sqrt{2\pi y}}e^y\left(1-\frac{4\gamma^2-1}{8y}+\frac{(4\gamma^2-1)(4\gamma^2-9)}{2!(8y)^2}-\dots\right),\\ K_\gamma(y) &\sim \sqrt{\frac{\pi}{2y}}e^{-y}\left(1+\frac{4\gamma^2-1}{8y}+\frac{(4\gamma^2-1)(4\gamma^2-9)}{2!(8y)^2}+\dots\right). \end{aligned}$$

The Poisson Kernel

Finally we would like to obtain an explicit formula for the solution of (2.2.2). The proof may be found in [36].

Given $\gamma \in (0,1)$, let $a = 1-2\gamma \in (-1,1)$. The function

$$\mathcal{K}_\gamma(x,y) = C_{n,a}\frac{y^{2\gamma}}{(|x|^2+|y|^2)^{\frac{n+2\gamma}{2}}} = C_{n,\gamma}\frac{y^{1-a}}{(|x|^2+y^2)^{\frac{n+1-a}{2}}} \tag{2.2.8}$$

is a solution of

$$\begin{cases} \operatorname{div}(y^a\nabla\mathcal{K}_\gamma) = 0 & \text{in } \mathbb{R}^{n+1}_+,\\ \mathcal{K}_\gamma = \delta_0 & \text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n, \end{cases}$$

where δ_0 is the delta distribution at the origin, and $C_{n,\gamma}$ is a positive constant depending only on n and γ which is chosen such that, for all $y>0$,

$$\int_{\mathbb{R}^n}\mathcal{K}_\gamma(x,y)\,\,dx = 1.$$

Proposition 2.2.3. *[31] For* $w \in \mathcal{C}_c(\mathbb{R}^n)$*, the solution of problem* (2.2.2) *is given by the Poisson formula*

$$W(x,y) = \int_{\mathbb{R}^n}\mathcal{K}_\gamma(x-\xi,y)f(\xi)\,d\xi,$$

where $\mathcal{K}_\gamma$ *is the Poisson Kernel for the problem, that is given in* (2.2.8).

2.2.2 Geometric setting

The conformal fractional Laplacian P_γ^g is constructed from scattering theory on the conformal infinity (M^n, g) of a conformally compact Einstein manifold (X^{n+1}, g^+) as a generalized Dirichlet-to-Neumann operator for the eigenvalue problem

$$-\Delta_{g^+}U - s(n-s)U = 0 \text{ in } X, \quad s = \tfrac{n}{2} + \gamma,$$

and it is a (non-local) pseudo-differential operator of order 2γ. The fractional curvature, which is a generalization of the scalar curvature, is constructed from the conformal fractional Laplacian [102, 43]. This is natural from the point of view of the AdS/CFT correspondence in Physics ([5, 172]). The mathematical definition was given by Graham-Zworski [102] and Mazzeo-Melrose [138]. These were originally based on the work of Newmann, Penrose and Lebrun [121] on four-dimentional gravitational Physics, in the spirit of Maldacena's AdS/CFT correspondence [133].

We give here some necessary definitions to introduce the concept of conformally compact Einstein and asymptotically hyperbolic manifolds. Let X^{n+1} be a smooth manifold of dimension $n+1$ with smooth boundary $\partial X = M^n$. A **defining function** for the boundary M^n in X^{n+1} is a function ρ on $\bar{X}^{n+1}$ which satisfies:

$$\begin{cases} \rho > 0 \text{ in } X, \\ \rho = 0 \text{ on } M, \\ d\rho \neq 0 \text{ on } M. \end{cases}$$

A Riemannian metric g^+ on X^{n+1} is **conformally compact** if $(\bar{X}^{n+1}, \bar{g})$ is a compact Riemannian manifold with boundary M^n for a defining function ρ and

$$\bar{g} = \rho^2 g^+.$$

Any conformally compact manifold (X^{n+1}, g^+) carries a well-defined conformal structure $[g]$ on the boundary M^n; where each g is the restriction of $\bar{g} = \rho^2 g^+$ for a defining function ρ. We call $(M^n, [g])$ **the conformal infinity** of the conformally compact manifold (X^{n+1}, g^+). We usually write these conformal changes on M as $g_w = w^{\frac{4}{n-2\gamma}} g$, for a positive smooth function w. Near the conformal infinity, given a defining function ρ, we have the following asymptotically expansion of the Riemannian tensor

$$Riem^{g^+}_{ijkl} = -|d\rho|^2_{\bar{g}}(g^+_{ik}g^+_{jl} - g^+_{il}g^+_{jk}) + O(\rho^3), \tag{2.2.9}$$

in a coordinate system on $(0, \epsilon) \times M^n \in X^{n+1}$.

A Riemannian metric g^+ is called **asymptotically hyperbolic** if there exists a defining function ρ such that

$$|\nabla\rho|^2_{\bar{g}} = 1 \text{ on } \partial X.$$

Remark 2.2.4. From (2.2.9) one sees that for a conformally compact manifold, if it is asymptotically hyperbolic, then the sectional curvature goes to -1 at infinity.

Lemma 2.2.5. *[100] Given a conformally compact, asymptotically hyperbolic manifold (X^{n+1}, g^+) and a representative g in $[g]$ on the conformal infinity M^n, there is a unique defining function ρ such that, on $M \times (0, \varepsilon)$ in X, g^+ has the normal form*

$$g^+ = \rho^{-2}(d\rho^2 + g_\rho), \tag{2.2.10}$$

where g_ρ is a family on M of metrics depending on the defining function and satisfying $g_\rho|_M = g$.

An **Einstein metric** is a metric for which the Ricci tensor and the metric tensor are proportional:

$$Ric^{g^+}_{ij} = f g^+_{ij}, \tag{2.2.11}$$

for some f smooth on X. Note that for an Einstein metric, $R_{g^+} = (n+1)f$.

Lemma 2.2.6. *[15] Under condition (2.2.11), the function f must be constant, when $n \geq 2$, so, in particular, an Einstein metric has constant scalar curvature $R_{g^+} = -n(n+1)$.*

Thus we may give the definition:

Definition 2.2.7. *A conformally compact manifold (X^{n+1}, g^+) is called conformally compact Einstein manifold if the metric satisfies $Ric_{g^+} = -ng^+$.*

Note that a conformally compact Einstein manifold must be asymptotically hyperbolic. Let us give some examples of conformally compact Einstein manifolds:

i. [20, 151] **Hyperbolic space.** We describe the Upper half space model for the hyperbolic space as

$$\mathbb{H}^{n+1} = \{z = (x, y); x \in \mathbb{R}^n,\ y \in \mathbb{R}_+\},$$

The metric in these coordinates is

$$g^+ = y^{-2}(|dx|^2 + dy^2),$$

and the volume element is

$$dvol_{g^+} = y^{-(n+1)} dxdy.$$

The conformal infinity is $\mathbb{R}^n \cup \{\infty\}$ where $\mathbb{R}^n$ is interpreted as the hyperplane $\{y = 0\}$, and the metric here is precisely the Euclidean one:

$$g = y^2 g^+|_{y=0} = |dx|^2.$$

The Laplace Beltrami operator is given by

$$\Delta_{\mathbb{H}} = y^2(\Delta_x + \partial_{yy}) - (n-1)y\partial_y. \tag{2.2.12}$$

The hyperbolic space can also be represented with the Poincaré Ball model. In this way $\mathbb{H}^{n+1}$ is realized as a set

$$\mathbb{B}^{n+1} = \{x \in \mathbb{R}^{n+1}/|x| < 1\}.$$

We take x as a global coordinate and define a metric:

$$g^{\mathbb{B}} = 4(1-|x|^2)^{-2}(dx_1^2 + ... + dx_{n+1}^2).$$

Here the volume element is

$$dvol_{g^{\mathbb{B}}} = 2^{n+1}(1-|x|^2)^{-(n+1)}dx_1 dx_2 ... dx_{n+1}.$$

We call $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1}; |x| = 1\}$, which represents the conformal infinity $\partial_\infty \mathbb{B}^{n+1}$, where the metric is the standard for $\mathbb{S}^n$.

Comparing with $\mathbb{H}^{n+1}$, we see that whereas the $\partial_\infty \mathbb{H}^{n+1} = \mathbb{R}^n \cup \{\infty\}$ has a "distinguished" point at ∞, this does not happen in ball model because the boundary at infinity $\partial_\infty \mathbb{H}^{n+1}$ is the one point of compactification of $\mathbb{R}^n$.

Remark 2.2.8. The relation between both models is given by:

$$G : \mathbb{B}^{n+1} \longrightarrow \mathbb{H}^{n+1}.$$

$$G(x) = \frac{(x_1, x_2, ..., \frac{1}{2}(1-|x|^2))}{(1+|x|^2 - 2x_1)},$$

and the inverse map,

$$G^{-1} : \mathbb{H}^{n+1} \longrightarrow \mathbb{B}^{n+1}.$$

$$G^{-1}(z) = \frac{(z_1, z_2, ..., z_n, z_{n+1}^2 + \|z'\|^2 - \frac{1}{4})}{((\frac{1}{2} + \|z_{n+1}\|)^2 + \|z'\|^2)}$$

where $z' = (z_1, ..., z_n)$.

ii. **A generalized hyperbolic manifold.** A different realization for hyperbolic space may be given as $\mathbb{R} \times \mathbb{R}^3$, with the metric

$$g^+ = \left((1+R^2)dt^2 + \frac{dR^2}{1+R^2} + R^2 g_{\mathbb{S}^2}\right). \tag{2.2.13}$$

If for $x \in \mathbb{R}^3$ we use the change of variables

$$|x| = \frac{1-\frac{\rho^2}{4}}{\rho} = \sinh \log \frac{2}{\rho},$$

$$d|x| = \left(-\frac{1}{\rho^2} - \frac{1}{4}\right) d\rho = -\frac{1+\frac{\rho^2}{4}}{\rho^2} d\rho$$

and also,

$$1 + R^2 = \left(\frac{1+\frac{\rho^2}{4}}{\rho}\right)^2,$$

we can observe that $(\mathbb{R}\times\mathbb{R}^3, g^+)$ is a conformally compact Einstein manifold, expressed in the normal form (2.2.10):

$$g^+ = \rho^{-2}\left(d\rho^2 + \left(1-\frac{\rho^2}{4}\right)^2 g_{\mathbb{S}^2} + \left(1+\frac{\rho^2}{4}\right)^2 dt^2\right).$$

Thus the conformal infinity is $(\mathbb{R}\times\mathbb{S}^2, g_0)$, where $g_0 := dt^2 + g_{\mathbb{S}^2}$. If now take the quotient in our manifolds with the group generated by the translations, we obtain a generalized hyperbolic manifold:

$$X^4 = \mathbb{S}^1(L)\times\mathbb{R}^3,$$

with conformal infinity

$$M^3 = \mathbb{S}^1(L)\times\mathbb{S}^2,$$

and the same metrics g^+ and g_0.

iii. **Anti de Sitter space.** We will explain it in detail in Section 2.3. This example is important because it gives two different examples of conformally compact Einstein manifolds with the same conformal infinity. This model is well known in cosmology since they provide the simplest background for the study of thermodynamically stable black holes (see [172, 105], for instance, or the survey paper [45]).

The standard examples of static Riemannian AdS-type black holes solutions are manifolds $M = N^{n-2}\times\mathbb{R}^2$ where N^{n-2} is compact and the given metric has the following form $g_M = V^{-1}dr^2 + Vd\theta^2 + r^2 g_N$, where g_N is any Einstein metric and V is a function that will be described later in the particular case we are going to study. Some examples of these Ads-type black holes are [9, 10]:

- AdS-$\mathbb{S}^2$-black holes: $M = \mathbb{R}^2\times\mathbb{S}^{n-1}$.
- AdS toral black holes: $M = \mathbb{R}^2\times\mathbb{T}^{n-1}$. (Note $\mathbb{T}^{n-1}$ represents the $(n-1)$-torus).

2.2.3 Conformal fractional Laplacian

First, we look at the spectrum of the Laplacian on hyperbolic space:

Lemma 2.2.9. *[56] The spectrum of $-\Delta_{\mathbb{H}^{n+1}}$ is equal to $[(\frac{n}{2})^2,\infty)$.*

Proof. Let us prove here that the spectrum of $-\Delta_{\mathbb{H}^{n+1}}$ is contained in $[(n/2)^2,\infty)$. Using (2.2.12),

$$-\Delta_{\mathbb{H}^{n+1}}(y^s) = s(n-s)y^s.$$

The claim follows from Theorem 2.2.10 with $\phi = y^{\frac{n}{2}}$. □

Theorem 2.2.10. *[56] Suppose that H is elliptic on $L^2(\Omega)$ and that there is a positive continuous function ϕ in $W^{1,2}_{loc}(\Omega)$ and a potential V in $L^1_{loc}(\Omega)$, such that*

$$H\phi \geq V\phi.$$

Then the quadratic form inequality

$$H \geq V$$

is valid on $\mathcal{C}_c^\infty(\Omega)$.

We can read about the spectrum of the Laplacian of a general asymptotically hyperbolic metric in [135, 137, 138]. It can be described as

$$\sigma(-\Delta_{g^+}) = [(n/2)^2, \infty) \cup \sigma_{pp}(-\Delta_{g^+}), \text{ where } \sigma_{pp}(-\Delta_{g^+}) \subset (0, (n/2)^2).$$

We note that $\sigma_{pp}(-\Delta_g)$ is the pure point spectrum, i.e, the set of L^2-eigenvalues, and it is finite; and $[(n/2)^2, \infty)$ is the continuous spectrum.

More refined statements follow from the main result of [138], which is the existence of the meromorphic continuation of the resolvent

$$R(s) = (-\Delta_{g^+} - s(n-s))^{-1}.$$

Here $\lambda = s(n-s)$ is symmetric with respect to $Re(s) = \frac{n}{2}$. We will choose $s \in (\frac{n}{2}, n)$ and denote $s = \frac{n}{2} + \gamma$ for $\gamma \in (0, \frac{n}{2})$.

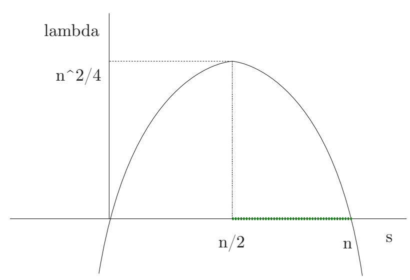

Figure 2.1: Representation of $\lambda(s)$.

Let (X, g^+) be a conformally compact Einstein manifold with conformal infinity $(M, [g])$. As we can check in Graham-Zworski and Mazzeo-Melrose [102, 138], given $w \in \mathcal{C}^\infty(M)$ and $s \in \mathbb{C}$, if $s(n-s)$ does not belong to the pure point spectrum of $-\Delta_{g^+}$ then there exists a unique solution of the form

$$U = W\rho^{n-s} + W_1\rho^s, \quad W, W_1 \in \mathcal{C}^\infty(\overline{X}), \quad W|_{\rho=0} = w. \tag{2.2.14}$$

for the **scattering problem**

$$-\Delta_{g^+}U - s(n-s)U = 0 \text{ in } X. \tag{2.2.15}$$

The same is true for a more general asymptotically hyperbolic manifold, but there may be other additional poles.

Definition 2.2.11. *Taking a representative g of the conformal infinity $(M^n,[g])$ we can define a family of meromorphic pseudo-differential operators $S(s)$, called **scattering operators** as*

$$S(s)w = W_1|_M. \tag{2.2.16}$$

It is defined for $Re(s) > \frac{n}{2}$. As it is explained in the next theorem, the values $s = \frac{n}{2}+k;\ k = 0,1,2...$ are simple poles of finite rank (they are known as trivial poles). It is possible that $S(s)$ has another poles, but we will assume here that our value of s is not one of these exceptional values. We will also assume, for technical reasons that the first eigenvalue for $-\Delta_{g^+}$ is greater than $\frac{n^2}{4} - \left(s-\frac{n}{2}\right)^2$.

Theorem 2.2.12. *[102] Let (X,g^+) be a conformally compact Einstein manifold with conformal infinity $(M,[g])$. Suppose that $k\in\mathbb{N}$ and $k\le\frac{n}{2}$ if n is even, and that $(\frac{n}{2})^2-k^2$ is not an L^2-eigenvalue of $-\Delta_g$. If $S(s)$ is the scattering operator of (X,g^+), and P^g_k the conformally invariant operators on M constructed in [101], then $S(s)$ has a simple pole at $s=\frac{n}{2}+k$ and*

$$c_kP^g_k = -Res_{s=\frac{n}{2+k}}S(s),\ \ c_k = (-1)^k[2^{2k}k!(k-1)!]^{-1},$$

where $Res_{s=s_0}S(s)$ denotes the residue at s_0 of the meromorphic family of operators $S(s)$.

Consequently these are local operators which satisfy

$$P^g_k = (-\Delta_g)^k + l.o.t.$$

In particular, $P^g_k = (-\Delta_g)^k$ if $\bar{g}$ is flat.

- If $k=1$ we have the conformal Laplacian

$$P^g_1 = -\Delta_g + \frac{n-2}{4(n-1)}R_g.$$

- If $k=2$, the Paneitz operator

$$P^g_2 = (-\Delta_g)^2 + \delta(a_nR_g + b_nRic_g)d + \frac{n-4}{2}Q^g_2.$$

Note that up to constant Q_1 is the classical scalar curvature and Q_2 is the so called Q-curvature.
It is also possible define conformally covariant fractional powers of Laplacian in the case $\gamma\notin\mathbb{N}$.

Definition 2.2.13. *For $s=\frac{n}{2}+\gamma;\gamma\in(0,\frac{n}{2}),\gamma\notin\mathbb{N}$, we define the conformally covariant fractional powers of the Laplacian on (M^n,g) as*

$$P_\gamma[g^+,g] = d_\gamma S\left(\frac{n}{2}+\gamma\right);\ \ d_\gamma = 2^{2\gamma}\frac{\Gamma(\gamma)}{\Gamma(-\gamma)}. \tag{2.2.17}$$

As a pseudodifferential operator, its principal symbol is the same as the one of the fractional Laplacian $(-\Delta_g)^\gamma$, so it has order 2γ. Note also that $P_\gamma[g^+, g]$ is a self-adjoint operator on M (see: [102]), and it is non-local.

In the rest of the thesis we will use the simplified notation:

$$P_\gamma^g = P_\gamma[g^+, g].$$

Proposition 2.2.14. *These operators satisfy an important conformal property*

$$P_\gamma^{g_w}\phi = w^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^g(w\phi),\ \forall \phi \in \mathcal{C}^\infty(M), \tag{2.2.18}$$

where

$$g_w := w^{\frac{4}{n-2\gamma}} g.$$

Proof. Given g on M as in Lemma 2.2.5, there exists ρ such that $g^+ = \frac{d\rho + g_\rho}{\rho^2}$ and $g_\rho|_M = g$. Given $g_w = w^{\frac{4}{n-2\gamma}} g$ on M, there exist $\tilde{\rho}$ such that $g^+ = \frac{d\tilde{\rho}^2 + g_{\tilde{\rho}}}{\tilde{\rho}^2}$ and $g_{\tilde{\rho}}|_M = g_w$. From the proof in [100] one gets that

$$\left.\frac{\tilde{\rho}}{\rho}\right|_M = w^{\frac{2}{n-2\gamma}}. \tag{2.2.19}$$

So we can find a solution U for the eigenvalue problem (2.2.15) in the following way:

$$U = F\rho^{n-s} + F_1\rho^s = \tilde{F}\tilde{\rho}^{n-s} + \tilde{F}_1\tilde{\rho}^s.$$

And using (2.2.19) and $s = \frac{n}{2} + \gamma$, up to lower order terms,

$$F = \tilde{F}w \quad \text{and} \quad F_1 = \tilde{F}_1 w^{\frac{n+2\gamma}{n-2\gamma}}.$$

Restricting these equalities to M one gets $\tilde{F}|_{\rho=0} = \tilde{f}$ and $\tilde{F}w|_{\rho=0} = f$, which means $\tilde{f} = fw^{-1}$. Morever $\tilde{F}_1|_{\rho=0} = \tilde{f}_1$ and $\tilde{F}_1 w^{\frac{n+2\gamma}{n-2\gamma}}|_{\rho=0} = f_1$, which means $\tilde{f}_1 = f_1 w^{-\frac{n+2\gamma}{n-2\gamma}}$.

Because the definition of Scattering operator we can assert that $S(s)f = f_1$ and $\tilde{S}(s)\tilde{f} = \tilde{f}_1$, and applying the definition of conformally covariant fractional powers of fractional Laplacian (2.2.17) we get

$$P_\gamma^{g_w}(fw^{-1}) = P_\gamma^{g_w}(\tilde{f}) = \tilde{f}_1 d_\gamma = f_1 w^{-\frac{n+2\gamma}{n-2\gamma}} d_\gamma = P_\gamma^g(f) w^{-\frac{n+2\gamma}{n-2\gamma}}.$$

Taking $f = \phi w$ we get the desired result. □

Definition 2.2.15. *We define the fractional order curvature as:*

$$Q_\gamma^g := P_\gamma^g(1).$$

Note that this fractional curvature is a nonlocal version of the scalar curvature (which corresponds to the local case $\gamma = 1$). Note also that Q_γ^g is different to the one defined by Caffarelli, Roquejoffre and Savin in [35], which is a non local version of the mean curvature (see also the review [170]), and it has also received a lot of attention recently..

Remark 2.2.16. Using the previous definition we can express the conformal property (2.2.18) as

$$P_\gamma^g(w) = w^{\frac{n+2\gamma}{n-2\gamma}} Q_\gamma^{g_w}. \tag{2.2.20}$$

The conformal fractional Laplacian on Euclidean spaces.
In the case of $M = \mathbb{R}^n$ and $X = \mathbb{R}^{n+1}_+$ with coordinates $x \in \mathbb{R}^n$ and $y > 0$, with the hyperbolic metric $g^+ \equiv g_{\mathbb{H}} = \frac{dy^2+|dx|^2}{y^2}$ (where $|dx|^2$ is the Euclidean metric on $\mathbb{R}^n$) the construction of the scattering operator is precisely the Caffareli-Silvestre extension problem for the fractional Laplacian when $\gamma \in (0,1)$. We remark that in this case $\bar{g} = dy^2 + |dx|^2$ is the flat metric in $\mathbb{R}^{n+1}_+$.

Theorem 2.2.17. *[43] Given $\gamma \in (0,1)$, for a smooth function $w : \mathbb{R}^n \to \mathbb{R}$, there exists an unique solution $V = V(x,y) : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$ to the following extension problem*

$$\begin{cases} \Delta_x V + \dfrac{a}{y}\partial_y V + \partial_{yy} V = 0; \quad x \in \mathbb{R}^n, \quad y \in [0,+\infty), \\ \qquad\qquad V(x,0) = w(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{2.2.21}$$

where $a = 1 - 2\gamma$. Moreover, $U = y^{n-s}V$ is a solution of the eigenvalue problem

$$-\Delta_{g_{\mathbb{H}}} U - s(n-s)U = 0, \text{ in } \mathbb{H}^{n+1},$$

for $s = \frac{n}{2} + \gamma$, and

$$P_\gamma^{|dx|^2} w = -\tilde{d}_\gamma \lim_{y\to 0}(y^a \partial_y V) = (-\Delta_x)^\gamma(w), \tag{2.2.22}$$

where $\tilde{d}_\gamma$ is defined in (2.2.3).

Proof. Given w fixed, we know that the solution U of the scattering problem

$$-\Delta_{\mathbb{H}} U - s(n-s)U = 0 \text{ in } \mathbb{H}^{n+1}, \tag{2.2.23}$$

can be written as

$$U = y^{n-s}W + y^s W_1, \tag{2.2.24}$$

where W and W_1 satisfy:

$$\begin{aligned} W|_{y=0} = w \text{ and } W(x,y) = w(x) + w_2(x)y^2 + o(y^2), \\ \text{and } S(s)w = w_1 \text{ for } w_1 = W_1|_{y=0} \text{ and } W_1(x,y) = w_1(x) + \tilde{w}_2(x)y^2 + o(y^2). \end{aligned} \tag{2.2.25}$$

If we recall the definition of conformal Laplacian (2.1.2) and use that in the hyperbolic space we have $R_{g_{\mathbb{H}}} = -n(n+1)$ (because it is an Einstein manifold), we have that

$$-\Delta_{g_{\mathbb{H}}} = L_{g_{\mathbb{H}}} + \frac{n^2-1}{4}.$$

We use the conformal property of the conformal Laplacian given in Proposition 2.1.1 for the change of metric $\bar{g} = y^2 g_{\mathbb{H}}$ (where $\bar{g}$ is the Euclidean metric), getting

$$L_{g_{\mathbb{H}}}\phi = y^{\frac{n+3}{2}} L_{\bar{g}}(y^{-\frac{n-1}{2}}\phi).$$

But we know that $L_{\bar{g}} = -\Delta_{\bar{g}} = -\Delta_x - \partial_{yy}$. So we can do the change of variable

$$U = y^{n-s}V, \tag{2.2.26}$$

sustitute $s = \frac{n}{2} + \gamma$, and use all the previous equivalences in (2.2.23) to get

$$\Delta_x V + \partial_{yy} V + \frac{a}{y}\partial_y V = 0.$$

For the second part, we only need to realize that with the definition of w_1 given in (2.2.25), the following equivalence holds

$$P_\gamma^{|dx|^2} w = d_\gamma S(s) w = d_\gamma w_1.$$

If we substitute the expansion (2.2.24) in the change of variable (2.2.26) we obtain $V = W + y^{2s-n}W_1$. And we can compute

$$\begin{aligned}
\lim_{y\to 0} y^a \partial_y V &= \lim_{y\to 0} y^a \partial_y (W + y^{2s-n}W_1) \\
&= \lim_{y\to 0} y^a \partial_y \left[w(x) + w_2(x)y^2 + o(y^2) + y^{2s-n}\left(w_1(x) + \tilde{w}_2(x)y^2 + o(y^2) \right) \right] \\
&= (2s-n)w_1 = 2\gamma w_1.
\end{aligned}$$

Therefore $w_1 = \frac{1}{2\gamma}\lim_{y\to 0} y^a \partial_y V$, and so that

$$P_\gamma^{|dx|^2} w = \frac{d_\gamma}{2\gamma} \lim_{y\to 0} y^a \partial_y V,$$

as desired. □

The conformal fractional Laplacian on the sphere.
In this section we look at the sphere $\mathbb{S}^n$ with the round metric $g_{\mathbb{S}^n}$, understood as the conformal infinity of the Poincaré ball model for hyperbolic space $\mathbb{H}^{n+1}$. Note that hyperbolic space is the simplest example of a Poincaré-Einstein manifold, and the model for the general development.

On $\mathbb{S}^n$ one explicitly knows ([25], see also the lecture notes [26], for instance) that the conformal fractional Laplacian (or intertwining operator) has the explicit expression

$$P_\gamma^{g_{\mathbb{S}^n}} = \frac{\Gamma\left(A_{1/2} + \gamma + \frac{1}{2}\right)}{\Gamma\left(A_{1/2} - \gamma + \frac{1}{2}\right)}, \quad A_{1/2} = \sqrt{-\Delta_{\mathbb{S}^n} + \left(\tfrac{n-1}{2}\right)^2}, \tag{2.2.27}$$

for all $\gamma \in (0, n/2)$. From here one easily calculates that the fractional curvature of the sphere is a positive constant

$$Q_\gamma^{g_{\mathbb{S}^n}} = P_\gamma^{g_{\mathbb{S}^n}}(1) = \frac{\Gamma\left(\frac{n}{2} + \gamma\right)}{\Gamma\left(\frac{n}{2} - \gamma\right)}. \tag{2.2.28}$$

Formula (2.2.27) may be easily derived from the scattering problem (2.2.14)-(2.2.15). A proof can be found in the book [19], which also makes the link to the representation theory community. Note, however, a different factor of 2, which is always an issue when passing from representation theory to geometry. For convenience of the reader not familiar with this subject we provide a direct proof below [95].

Consider the Poincaré metric for hyperbolic space $\mathbb{H}^{n+1}$, written in normal form (2.2.10) as

$$g^+ = \rho^{-2}\left(d\rho^2 + \left(1-\tfrac{\rho^2}{4}\right)^2 g_{\mathbb{S}^n}\right),$$

for $\rho \in (0,2]$. Remark that $\rho = 2$ corresponds to the origin of the Poincaré ball and thus the apparent singularity is just a consequence of the expression for the metric in polar-like coordinates.

Calculating the Laplace-Beltrami operator with respect to g^+ we obtain, recalling that $s = \frac{n}{2} + \gamma$, that the eigenvalue equation (2.2.15) is equivalent to the following:

$$\rho^{n+1}\left(1-\tfrac{\rho^2}{4}\right)^{-n}\partial_\rho\left[\rho^{-n+1}\left(1-\tfrac{\rho^2}{4}\right)^{n}\partial_\rho U\right] + \rho^2\left(1-\tfrac{\rho^2}{4}\right)^{-2}\Delta_{\mathbb{S}^n}U + \left(\tfrac{n^2}{4}-\gamma^2\right)U = 0. \tag{2.2.29}$$

We will show that the operator $P_\gamma^{g_{\mathbb{S}^n}}$ diagonalizes in the spherical harmonic decomposition for $\mathbb{S}^n$. With some abuse of notation, let $\mu_k = k(k+n-1)$, $k = 0,1,2,...$ be the eigenvalues of $-\Delta_{\mathbb{S}^n}$, repeated according to multiplicity, and $\{E_k\}$ be the corresponding basis of eigenfunctions. The projection of (2.2.29) onto each eigenspace $\langle E_k\rangle$ yields

$$\rho^{n+1}\left(1-\tfrac{\rho^2}{4}\right)^{-n}\partial_\rho\left[\rho^{-n+1}\left(1-\tfrac{\rho^2}{4}\right)^{n}\partial_\rho U_k\right] - \rho^2\left(1-\tfrac{\rho^2}{4}\right)^{-2}\mu_k U_k + \left(\tfrac{n^2}{4}-\gamma^2\right)U_k = 0.$$

This is a hypergeometric ODE with general solution

$$U_k(\rho) = c_1\rho^{\frac{n}{2}-\gamma}\varphi_1(\rho) + c_2\rho^{\frac{n}{2}+\gamma}\varphi_2(\rho), \quad c_1, c_2 \in \mathbb{R}, \tag{2.2.30}$$

for

$$\begin{aligned}\varphi_1(\rho) &:= (\rho^2-4)^{\frac{-n-\beta+1}{2}}\,{}_2F_1\left(\tfrac{-\beta+1}{2}, \tfrac{-\beta+1}{2}-\gamma, 1-\gamma, \tfrac{\rho^2}{4}\right),\\ \varphi_2(\rho) &:= (\rho^2-4)^{\frac{-n-\beta+1}{2}}\,{}_2F_1\left(\tfrac{-\beta+1}{2}, \tfrac{-\beta+1}{2}+\gamma, 1+\gamma, \tfrac{\rho^2}{4}\right),\end{aligned}$$

where we have defined

$$\beta := \sqrt{(n-1)^2 + 4\mu_k}$$

and ${}_2F_1$ is the usual Hypergeometric function (see Appendix 7).

In order to calculate the conformal fractional Laplacian, first one needs to obtain an asymptotic expansion of the form (2.2.14) for W, $\tilde{W}$ smooth up to $\overline{X}$. Since U must be smooth at the central point $\rho = 2$, one should choose the constants c_1, c_2 such that in (2.2.30) the singularities of φ_1 and φ_2 at $\rho = 2$ cancel out. This is,

$$c_1 2^{\frac{n}{2}-\gamma}\,{}_2F_1\left(\tfrac{-\beta+1}{2}, \tfrac{-\beta+1}{2}-\gamma, 1-\gamma, 1\right) + c_2 2^{\frac{n}{2}+\gamma}\,{}_2F_1\left(\tfrac{-\beta+1}{2}, \tfrac{-\beta+1}{2}+\gamma, 1+\gamma, 1\right) = 0. \tag{2.2.31}$$

In order to simplify this expression, recall the property (7.0.7) of the Hypergeometric function given in Lemma 7.0.1 in the Apendix. After some calculation, (2.2.31) yields

$$\frac{c_2}{c_1} = 2^{-2\gamma} \frac{\Gamma\big(\frac{1}{2}+\gamma+\frac{\beta}{2}\big)\Gamma(-\gamma)}{\Gamma\big(\frac{1}{2}-\gamma+\frac{\beta}{2}\big)\Gamma(\gamma)}. \tag{2.2.32}$$

Next, looking at the definition of the conformal fractional Laplacian from (2.2.17), and noting that both φ_1, φ_2 are smooth at $\rho = 0$, we conclude from (2.2.32) that

$$P_\gamma^{g_{\mathbb{S}^n}}|_{\langle E_k\rangle} w_k = d_\gamma \frac{c_2}{c_1} w_k = \frac{\Gamma\big(\frac{1}{2}+\gamma+\frac{\beta}{2}\big)}{\Gamma\big(\frac{1}{2}-\gamma+\frac{\beta}{2}\big)} w_k.$$

This concludes the proof of (2.2.27) when $\gamma \in (0, n/2)$ is not an integer.

From another point of view, on $\mathbb{R}^n$ with the Euclidean metric, the fractional Laplacian for $\gamma \in (0,1)$ can be computed as the principal value of the integral

$$(-\Delta)^\gamma w(x) = C(n,\gamma) \int_{\mathbb{R}^n} \frac{w(x) - w(\xi)}{\mid x - \xi \mid^{n+2\gamma}}\, d\xi. \tag{2.2.33}$$

Our next objective is to give an analogous expression for $P_\gamma^{g_{\mathbb{S}^n}}$ in terms of a singular integral operator, using stereographic projection (given in (2.1.7)) in expression (2.2.33):

Proposition 2.2.1. (see [95]) Let $\gamma \in (0,1)$. Given $w(z)$ in $\mathcal{C}^\infty(\mathbb{S}^n)$, it holds

$$P_\gamma^{g_{\mathbb{S}^n}} w(z) = \int_{\mathbb{S}^n} [w(z) - w(\zeta)]\, K_\gamma(z,\zeta)\, d\zeta + A_{n,\gamma} u(z),$$

where the kernel K_γ is given by

$$K_\gamma(z,\zeta) = 2^{\gamma+n/2} C(n,s) \left(\frac{1 - z_{n+1}}{1 + z_{n+1}}\right)^{\gamma+n/2} \left(\frac{1 - \zeta_{n+1}}{1 + \zeta_{n+1}}\right)^{\gamma+n/2} \frac{1}{(1 - z\cdot\zeta)^{\gamma+n/2}}.$$

and the (positive) constant

$$A_{n,\gamma} = \frac{\Gamma\big(\frac{n}{2}+\gamma\big)}{\Gamma\big(\frac{n}{2}-\gamma\big)}.$$

The conformal fractional Laplacian on conformally compact Einstein manifolds. Now we are going to study the same extension problem (2.2.21) as before but in any conformally compact Einstein manifold (X^{n+1}, g^+).

Theorem 2.2.18. *[43] Let (X, g^+) be any conformally compact Einstein manifold with conformal infinity $(M,[g])$. For any defining function ρ of M satisfying (2.2.10) in X, the problem*

$$-\Delta_{g^+} U - s(n-s) U = 0 \text{ in } (X, g^+),$$

with Dirichlet condition w, is equivalent to

$$\begin{cases} -\operatorname{div}(\rho^a \nabla W) + E(\rho) W = 0 \ \textit{in} \ (X, \bar{g}), \\ \qquad\qquad\qquad\qquad W = w \ \textit{on} \ M. \end{cases} \tag{2.2.34}$$

where

$$\bar{g} = \rho^2 g^+, \quad W = \rho^{s-n} U, \quad s = \frac{n}{2} + \gamma, \quad a = 1 - 2\gamma.$$

and the derivatives in (2.2.34) *are taken respect to the metric* $\bar{g}$*. The lower order term is given by*

$$E(\rho) = -\Delta_{\bar{g}}(\rho^{\frac{a}{2}})\rho^{\frac{a}{2}} + \left(\gamma^2 - \frac{1}{4}\right)\rho^{-2+a} + \frac{n-1}{4n} R_{\bar{g}} \rho^a. \tag{2.2.35}$$

If we write it in the metric g^+ *we have*

$$E(\rho) = \tfrac{n-1-a}{4n} \left[R_{\bar{g}} - \left\{ n(n+1) + R_{g^+} \right\} \rho^{-2} \right] \rho^a. \tag{2.2.36}$$

Moreover we have the following formula for the calculation of the conformal fractional Laplacian

$$P_\gamma^g w = -\tilde{d}_\gamma \lim_{\rho \to 0} \rho^a \partial_\rho W, \tag{2.2.37}$$

where $\tilde{d}_\gamma$ *is defined in* (2.2.3).

The proof is analogous to the one given in Theorem 2.2.17 for the Euclidean case, we only have to take into account that we are working with an Einstein manifold provided with a metric g^+, for which

$$\Delta_{\bar{g}} = \partial_{\rho\rho} + \frac{1}{2}\psi \partial_\rho + \Delta_{g_\rho},$$

where $\bar{g} = \rho^2 g^+$ and $\psi := \partial_\rho(\log \det(g_\rho))$. The second term on the right hand side is the one that generates the lower order term $E(\rho)$.

Remark 2.2.19. For a conformally compact Einstein metric given in normal form as (2.2.10)

$$E(\rho) = \frac{-n+1+a}{4} \psi \rho^{a-1} = \frac{n-1-a}{4n} R_{\bar{g}} \rho^a, \ \text{in } M \times (0, \delta). \tag{2.2.38}$$

Remark 2.2.20. We recall how to compute the Q_γ^g curvature. We set $w \equiv 1$, and we find the solution W for the extension problem (2.2.34). Then,

$$Q_\gamma^g = -\tilde{d}_\gamma \lim_{\rho \to 0} \rho^a \partial_\rho W.$$

Now we are going to choose a **suitable defining function** ρ^*, in order to transform the problem (2.2.10) into one of pure divergence form. We follow the study in [43].

Lemma 2.2.21. *Let (X, g^+) be a conformally compact Einstein manifold with conformal infinity $(M, [g])$. Fixed a metric g on M, and assuming that ρ is a defining function, we can assert that for each $\gamma \in (0,1)$, there exists another (positive) defining function ρ^* on X, satisfying $\rho^* = \rho + O(\rho^{2\gamma+1})$ and such that for the term E defined in (2.2.35) we have*

$$E(\rho^*) = 0.$$

Moreover, the metric $g^* = (\rho^*)^2 g^+$ satisfies $g^*|_{\rho=0} = g$ and has asymptotic expansion

$$g^* = (d\rho^*)^2[1 + O((\rho^*)^{2\gamma})] + g[1 + O((\rho^*)^{2\gamma})].$$

Theorem 2.2.22. *Let $\gamma \in (0,1)$ fixed, and f any smooth function on M. If under the hypothesis and the special defining function ρ^* constructed in the Lemma 2.2.21, W solves the following extension problem*

$$\begin{cases} -\operatorname{div}((\rho^*)^a \nabla W) = 0 \text{ in } (X, g^*), \\ W = w \text{ on } M. \end{cases}$$

(where the derivatives are taken with respect to the metric $g^* = (\rho^*)^2 g^+$); then

$$P_\gamma^g w = -\tilde{d}_\gamma \lim_{\rho^* \to 0} (\rho^*)^a \partial_{\rho^*} W + w Q_\gamma^g. \tag{2.2.39}$$

2.2.4 Fractional Yamabe problem [97]

From now and on, we fix $\gamma \in (0,1)$. The Fractional Yamabe problem is: given a conformally compact Einstein manifold $(X^{n+1}, \bar{g})$ of dimension $n > 2\gamma$ with conformal infinity $(M, [g])$, to find a new metric conformal to g, $g_w = w^{\frac{4}{n-2\gamma}} g$ (where w is a strictly positive $\mathcal{C}^\infty$ function on M) with constant fractional curvature $Q_\gamma^{g_w}$.

Since we impose that the metric g_w has constant fractional curvature, the conformal property (2.2.20) is equivalent to assert that there exists a constant c on M such that

$$P_\gamma^g(w) = c w^{\frac{n+2\gamma}{n-2\gamma}}, \quad w > 0, \tag{2.2.40}$$

which thanks to Theorem 2.2.18 is equivalent to the existence of a strictly positive $\mathcal{C}^\infty$ solution for extension problem:

$$\begin{cases} -\operatorname{div}(\rho^a \nabla W) + E(\rho) W = 0 \text{ in } (X, \bar{g}), \\ -\tilde{d}_\gamma \lim_{\rho \to 0} \rho^a \partial_\rho W = c w^{\frac{n+2\gamma}{n-2\gamma}} \text{ where } W|_M = w. \end{cases} \tag{2.2.41}$$

Remark 2.2.23. Using the special defining function (2.2.21) the fractional Yamabe problem (2.2.41) can be written as

$$\begin{cases} -\operatorname{div}((\rho^*)^a \nabla W) = 0 \text{ in } (X, g^*), \\ -\tilde{d}_\gamma \lim_{\rho^* \to 0} (\rho^*)^a \partial_{\rho^*} W + w Q_\gamma^g = c w^{\frac{n+2\gamma}{n-2\gamma}} \text{ where } W|_M = w. \end{cases}$$

Indeed, we only need to use the equation for the Yamabe problem (2.2.40) and the expression of P_γ^g with ρ^* from (2.2.39).

The fractional Yamabe problem can be seen as a generalization of **Escobar's problem** [79, 78, 80, 134]: this is to find a conformal metric to the given one on a manifold X^{n+1} with constant mean curvature on the boundary $M^n = \partial X^{n+1}$, or equivalently to find a solution for

$$\begin{cases} -\Delta_{\bar{g}} u + \frac{n-1}{4n} R_{\bar{g}} u = 0 & \text{in } (X^{n+1}, \bar{g}), \\ \partial_\nu u + \frac{n-1}{2} H u = c u^{\frac{n+1}{n-1}} & \text{on } M^n. \end{cases} \tag{2.2.42}$$

In the particular case $\gamma = 1/2$, the fractional Yamabe problem and this one are equivalent modulo some lower order terms. The main different between them is that in the first one, we are allowed to take a conformal metric in X^{n+1}, while in the second problem we are restricted to conformal metrics on the boundary M^n.

For the fractional Yamabe problem one may use the variational method as in the classical case $\gamma = 1$. We define the $\gamma-$**Yamabe functional** as

$$I_\gamma[g] = \frac{\int_M Q_\gamma^g \, dvol_g}{(\int_M dvol_g)^{\frac{n-2\gamma}{n}}}. \tag{2.2.43}$$

Now we can ask about the existence of the minimizer of I_γ among metrics in the class $[g]$.

Remark 2.2.24. We will use the notation $2^* = \frac{2n}{n-2\gamma}$.

Definition 2.2.25. *We define the $\gamma-$Yamabe constant as*

$$\lambda_\gamma(M, [g]) = \inf\{I_\gamma[h]; h \in [g]\},$$

which is an invariant of the conformal class $[g]$ when g^+ is fixed.

Taking the conformal metric $g_w = w^{\frac{4}{n-2\gamma}} g$ we ca define the previous functional as a functional on w by:

$$I_\gamma[w] = \frac{\int_M w P_\gamma^g w \, dvol_g}{(\int_M w^{\frac{2n}{n-2\gamma}} \, dvol_g)^{\frac{n-2\gamma}{n}}}.$$

Indeed, using the conformal property (2.2.20) we have:

- $\int_{M^n} Q_\gamma^{g_w} \, dvol_{g_w} = \int_{M^n} w^{1-\frac{2n}{n-2\gamma}} P_\gamma^g w \, dvol_{g_w} = \int_M w P_\gamma^g w \, dvol_g$.
- $(\int_{M^n} dvol_{g_w})^{\frac{n-2\gamma}{n}} = (\int_{M^n} w^{\frac{2n}{n-2\gamma}} \, dvol_g)^{\frac{n-2\gamma}{n}}$.

The functional $I_\gamma[g]$ also can be represented as a functional in the extension:

$$\tilde{I}_\gamma[W] = \frac{\tilde{d}_\gamma \int_{X^{n+1}} (\rho^a |\nabla W|^2 + E(\rho) W^2) \, dvol_{\bar{g}}}{\int_{M^n} (W^{\frac{2n}{n-2\gamma}} \, dvol_g)^{\frac{n-2\gamma}{n}}}, \tag{2.2.44}$$

where $\tilde{d}_\gamma$ is defined in (2.2.3). Indeed if we take the equation (2.2.34), multiply it by W, integrate over X^{n+1} and apply divergence theorem we get the equality

$$\int_M W(\rho^a \nabla W) \, dvol_g = \int_{X^{n+1}} \left(\rho^a |\nabla W|^2 + E(\rho) W^2\right) \, dvol_{\bar{g}}.$$

Using that $w \equiv W|_M$, the definition of P_γ^g given in (2.2.37) and (2.2.34) we get

$$\begin{aligned} I_\gamma[w] &= \frac{\int_{M^n} w P_\gamma^g w \, dvol_g}{(\int_{M^n} w^{\frac{2n}{n-2\gamma}} \, dvol_g)^{\frac{n-2\gamma}{n}}} = \frac{\int_{M^n} w P_\gamma^g w \, dvol_g}{(\int_{M^n} W^{\frac{2n}{n-2\gamma}} \, dvol_g)^{\frac{n-2\gamma}{n}}} \\ &= \frac{\tilde{d}_\gamma \int_{M^n} W(y^a \partial_y W) \, dvol_g}{(\int_{M^n} W^{\frac{2n}{n-2\gamma}} \, dvol_g)^{\frac{n-2\gamma}{n}}} = \frac{\tilde{d}_\gamma \int_{X^{n+1}} (y^a |\nabla W|^2 + E(y) W^2) \, dvol_{\bar{g}}}{(\int_{M^n} W^{\frac{2n}{n-2\gamma}} \, dvol_g)^{\frac{n-2\gamma}{n}}} \\ &= \tilde{I}_\gamma[W]. \end{aligned}$$

Note that the infimum of $\tilde{I}_\gamma[V]$ among $V \in W^{1,2}(X, \rho^a)$ with $TV = w$ is attained at W satisfying (2.2.34).

This equivalence tells us that

$$\lambda_\gamma(M, [g]) = \inf\{\tilde{I}_\gamma[W, \bar{g}]; \ W \in W^{1,2}(X, \rho^a)\}.$$

Remark 2.2.26. If we use the special defining function defined in Lemma 2.2.21, the functional (2.2.43) can be represented as

$$I_\gamma^*[W] = \frac{\tilde{d}_\gamma \int_{X^{n+1}} (\rho^*)^a |\nabla W|^2 \, dvol_{g^*} + \int_{M^n} w^2 Q_\gamma^g \, dvol_g}{(\int_{M^n} W^{\frac{2n}{n-2\gamma}} \, dvol_g)^{\frac{n-2\gamma}{n}}}.$$

Before giving an example of manifold where the fractional Yamabe problem is solved, we will note here that, as in the classical case, the sign of $\lambda_\gamma(M)$ is the same that the sign of the $Q_\gamma^{\tilde{g}}$, where $\tilde{g}$ is the metric which solves the fractional Yamabe problem. Indeed, this results follows from Theorem 4.2 and Corollary 4.3 in [97]. We summarize below both results:

Lemma 2.2.27. *Let $(X^{n+1}; g+)$ be an asymptotically hyperbolic manifold with conformal infinity (M^n, g). For each $\gamma \in (0,1)$, under the assumption of zero mean curvature when $\gamma \in (1/2, 1)$, we have three mutually exclusive possibilities:*

1. *The first eigenvalue of P_γ^g is positive, the γ-Yamabe constant $\lambda_\gamma(M)$ is positive, and M admits a metric $\tilde{g}$ in $[g]$ that has pointwise positive fractional scalar curvature $Q_\gamma^{\tilde{g}}$.*

2. *The first eigenvalue of P_γ^g is negative, the γ-Yamabe constant $\lambda_\gamma(M)$ is negative, and M admits a metric $\tilde{g}$ in $[g]$ that has pointwise negative fractional scalar curvature $Q_\gamma^{\tilde{g}}$.*

3. *The first eigenvalue of P_γ^g is zero, the γ-Yamabe constant $\lambda_\gamma(M)$ is zero, and M admits a metric $\tilde{g}$ in $[g]$ that has vanishing fractional scalar curvature $Q_\gamma^{\tilde{g}}$.*

We consider here the **fractional Yamabe problem on** $\mathbb{S}^n$ (equivalently on $\mathbb{R}^n$). We give some results regarding the trace Sobolev inequality on $\mathbb{R}^n$ and its relation with $\mathbb{S}^n$. This reminds the model example for the classical case given in Section 2.1.2

Theorem 2.2.28. *[124, 54, 48] Let $w \in W^{\gamma,2}(\mathbb{R}^n)$, $\gamma \in (0,1)$, $a = 1-2\gamma$, and $W \in W^{1,2}(\mathbb{R}^{n+1}_+, y^a)$ with trace $TW = w$. Then*

$$\|w\|^2_{L^{2^*}(\mathbb{R}^n)} \leq \bar{S}(n,\gamma) \int_{\mathbb{R}^{n+1}_+} y^a |\nabla W|^2 \, dx \, dy,$$

where, being $g_{\mathbb{S}^n}$ the standard metric of $\mathbb{S}^n$,

$$\bar{S}(n,\gamma) = \frac{\tilde{d}_\gamma}{\lambda_\gamma(\mathbb{S}^n, [g_{\mathbb{S}^n}])}.$$

Moreover the equality holds if and only if

$$w_\mu(x) = c\left(\frac{\mu}{|x-x_0|^2+\mu^2}\right)^{\frac{n-2\gamma}{2}} \; ; \; x \in \mathbb{R}^n,$$

for $c \in \mathbb{R}$, $\mu > 0$ and $x_0 \in \mathbb{R}^n$ fixed.

We remark that if we look at the fractional Yamabe problem in $\mathbb{R}^n$ without singularities, all the entire solutions for

$$(-\Delta)^\gamma w = c_{n,\gamma} w^{\frac{n+2\gamma}{n-2\gamma}} \text{ in } \mathbb{R}^n, \quad w > 0,$$

have been completely classified by Jin, Li and Xiong [111] and Chen, Li and Ou [48], for instance. In particular, they must be the standard "bubbles" given in (2.2.28).

Suppose that (X^{n+1}, g^+) is an asymptotically hyperbolic manifold with a geodesic defining function ρ and set $\bar{g} = \rho^2 g^+$. Let $(M^n, [g])$ be its conformal infinity. One can show that ([97, 39]) the fractional Yamabe constant satisfies

$$-\infty < \lambda_\gamma(M, [g]) \leq \lambda_\gamma(\mathbb{S}^n, [g_{\mathbb{S}^n}]).$$

Theorem 2.2.2 ([97]). In the setting above, if

$$\lambda_\gamma(M, [g]) < \lambda_\gamma(\mathbb{S}^n, [g_{\mathbb{S}^n}]), \tag{2.2.45}$$

then the γ-Yamabe problem is solvable for $\gamma \in (0,1)$.

Therefore, it suffices to find a suitable test function in the functional (2.2.44) that attains this strict inequality. For this, one needs to find suitable conformal normal coordinates on M by conformal change, and then deal with the corresponding extension metric. Hence one needs to make some assumptions on the behavior of the asymptotically hyperbolic manifold g^+ . The underlying idea here is to have g^+ as close as possible as a Poincaré-Einstein manifold. The first one of these assumptions is

$$R_{g^+} + n(n+1) = o(\rho^2) \quad \text{as} \quad \rho \to 0,$$

which looks very reasonable in the light of (2.2.36). In particular, under this condition one has that

$$E(\rho) = \tfrac{n-1+a}{4n} R_{\bar{g}} \rho^a + o(\rho^2) \quad \text{as} \quad \rho \to 0.$$

(compare to (2.2.38)). Another consequence of this expression is that the $1/2$-Yamabe problem coincides to the prescribing constant mean curvature problem (2.2.42), up to a small error. In general one needs a higher order of vanishing for g^+ (see [115] for the precise statements), which is automatically true if g^+ is Poincaré-Einstein and not just asymptotically hyperbolic.

Definition 2.2.29. *We say that a manifold has a non-umbilic point, when there exists any point such that in its neighbourhood the manifold is not as a piece of a sphere.*

The first attempt to prove (2.2.45) was [97] in the non-umbilic case, where the authors use a bubble as a test function. The umbilic, non-locally conformally flat case in high dimensions was considered in [99]. Finally, Kim, Musso and Wei [115] have provided the latest development, covering all the cases that do not need a positive mass theorem for the conformal fractional Laplacian. Their test function is not a "bubble" but instead it has a more complicated geometry. Summarizing, some hypothesis under which the fractional Yamabe problem for $\gamma \in (0,1)$ is solvable (in addition to those on g^+ above) are:

- $n \geq 2$, $\gamma \in (0, 1/2)$, M has a point of negative mean curvature.
- $n \geq 4$, $\gamma \in (0,1)$, M is not umbilic.
- $n > 4 + 2\gamma$, M is umbilic but not locally conformally flat.
- M is locally conformally flat or $n = 2$, and the fractional positive mass theorem holds.

However, we see from this last point that to cover all the cases one still needs to develop a positive mass theorem for the Green's function of the conformal fractional Laplacian, which is at this time a puzzling open question.

Finally, one may look at the lack of compactness phenomenon. In general, Palais-Smale sequences can be decomposed into the solution of the limit equation plus a finite number of bubbles. Moreover, the multi-bubbles are non-interfering even though the operator is non-local (see, for instance, [84, 150, 116, 117]).

2.3 Non uniqueness issues

2.3.1 Two different extensions for the same conformal infinity

As we mentioned in Section 2.2, the Swarzchild-Anti-de-Sitter space is an interesting example because it gives two different examples of conformally compact Einstein manifolds with the same conformal infinity. It is not known yet if the scattering operator for both extensions coincide [55], Important references for this section are [105, 151] and the lectures given by Graham in "Mini-courses and Conference on Nonlinear Elliptic Equations" (May 13-18, 2013, Rutgers University and May 20-22, 2013, Courant Institute).

Motivation from physics (informal)

Definition 2.3.1. *Anti-de Sitter space is the submanifold described by one of the sheets of the hyperboloid of two sheets $x_1^2 + x_2^2 + x_3^2 - x_0^2 = -\alpha^2$, with the pseudometric given by $ds^2 = dx_1^2 + dx_2^2 + dx_3^2 - dx_0^2$, and where α is a nonzero constant with dimensions of length (the radius of curvature). It is related with the cosmological constant by $\alpha = \Lambda^{-\frac{1}{2}}$.*

Remark 2.3.2. The hyperboloid of one sheet $x_1^2+x_2^2+x_3^2-x_0^2 = \alpha^2$ is called de Sitter space.

The Lorentzian metric of the covering space of anti-de Sitter space can be written in the static form

$$g_{AdS} = -V d\tau^2 + V^{-1}dr^2 + r^2 g_{\mathbb{S}^2},$$

where $V = 1 + \frac{r^2}{b^2}$ and $b = (\frac{-3}{\Lambda})^{1/2}$. A positive definite metric may be attained using Wick's rotation $t = i\tau$.

$$g_{AdS+} = V dt^2 + V^{-1}dr^2 + r^2 g_{\mathbb{S}^2}. \tag{2.3.1}$$

This metric (2.3.1) reminds the hyperbolic metric given in (2.2.13)

More generally, we are going to consider of the Schwarzchild-anti-de Sitter metric, which has the form of (2.3.1), for

$$V = 1 - \frac{2m}{m_p^2 r} + \frac{r^2}{b^2}.$$

In this expression we have used the following notation:

- $m > 0$ is the mass of the black hole.
- m_p is a constant called "Planck mass" and given by $m_p = G^{-\frac{1}{2}}$, where G is the gravitational constant
- r_h is the positive root of $1 - \frac{2m}{m_p^2 r} + \frac{r^2}{b^2} = 0$ (because the metric must be positive).
- $r \in [r_h, +\infty)$.
- $t \in \mathbb{S}^1(L)$, L to be chosen.
- $(\theta, \phi) \in \mathbb{S}^2$.

Rigorous definition

We restrict to the case the case $b = 1$, $m_p = 1$ to simplify the computations. Then the AdS-Schwarzchild is defined as the $4-$manifold

$$(\mathbb{R}^2 \times \mathbb{S}^2, g_{+1}^m), \tag{2.3.2}$$

where

$$g_{+1}^m = V dt^2 + V^{-1}dr^2 + r^2 g_{\mathbb{S}^2}. \tag{2.3.3}$$

for

$$V = 1 + r^2 - \frac{2m}{r}. \tag{2.3.4}$$

Definition 2.3.3. *We call r_h the positive root for $1 + r^2 - \frac{2m}{r} = 0$, so $r \in [r_h, +\infty)$, $t \in \mathbb{S}^1(L)$ and $(\theta, \phi) \in \mathbb{S}^2$.*

Even though the metric g^m_{+1} seems singular at r_h, we will prove that this is not the case if we make the t variable periodic.

Since we would like $d\rho^2 = V^{-1}(r)\, dr^2$, we define

$$\rho : (r_h, \infty) \longrightarrow (0, \infty),$$

$$\rho(r) = \int_{r_h}^{r} V^{\frac{-1}{2}}.$$

Taylor's expansion around the point r_h yields

$$V(r) \sim V'(r_h)(r - r_h). \tag{2.3.5}$$

We call $V'_h = V'(r_h)$ and substituting (2.3.5) in the definition of ρ we obtain

$$\rho \sim \int_{r_h}^{r} (V'_h)^{\frac{-1}{2}} (r - r_h)^{\frac{-1}{2}} = (V'_h)^{\frac{-1}{2}} 2(r - r_h)^{\frac{1}{2}}.$$

Isolating $r - r_h$, we obtain

$$r - r_h = \frac{V'_h}{4}\rho^2 + ...$$

And using this approximation in (2.3.5), the metric (2.3.3) can be written as

$$g^m_{+1} = d\rho^2 + \frac{(V'_h)^2}{4}\rho^2 dt^2 + r^2 g_{\mathbb{S}^2},$$

where for periodicity we must impose $0 \leq \frac{V'_h}{2} t \leq 2\pi$, which is $0 \leq t \leq 2\pi L$, for

$$L = \frac{V'_h}{2}. \tag{2.3.6}$$

Remark 2.3.4. The manifold (2.3.2)-(2.3.3) is conformally compact Einstein. Indeed, let $s = \frac{1}{r}$ and using the definition of $V(r)$ from (2.3.4) we obtain

$$V(r) = 1 + \frac{1}{s^2} - 2ms \approx \frac{1}{s^2}, \text{ when } s \to 0 \text{ (and therefore } r \to \infty).$$

Using $dr = \frac{-dt}{t^2}$ yields

$$g^m_{+1} = V^{-1}(r)\, dr^2 + V(r)\, dt^2 + r^2\, g_{\mathbb{S}^2} \sim \frac{1}{s^2}[ds^2 + dt^2 + g_{\mathbb{S}^2}].$$

Uniqueness

We would like to find two different conformally compact Einstein manifolds with the same conformal infinity. We have seen that for each m, g^m_{+1} is a conformally compact Einstein metric with conformal infinity ($r = \infty$) given by $\mathbb{S}^1(L) \times \mathbb{S}^2$ with the metric

$$g_0 = V^{-1} g^m_{+1}|_{r=\infty} = dt^2 + g_{\mathbb{S}^2}.$$

Now we are going to observe how this radius L depends on m (or equivalently, the relation between L and r_h).
The definition of r_h (given in Definition 2.3.3) yields $V(r_h) = 0$, or equivalently $r_h^3 + r_h = 2m$. Then, $V'(r_h) = 2r_h + \frac{2m}{r_h^2} = 3r_h + \frac{1}{r_h}$, and using the definition of L from (2.3.6) we have

$$L = \frac{2r_h}{3r_h^2 + 1}. \tag{2.3.7}$$

Proposition 2.3.5. *Depending of the value of L there exist one, two or zero values of m such that g^m_+ has as conformal infinity $\mathbb{S}^1(L) \times \mathbb{S}^2$:*

- *If $L < L_0$ there exist two different masses m_1 and m_2 with the same $L(m_i)$; and thus they give same conformal infinity.*
- *If $L = L_0$ there exist only one mass m and which gives us $L(m)$.*
- *If $L > L_0$ there does not exist any mass which gives us $L(m)$.*

Proof. Calculate from (2.3.7)

$$L'(r_h) = \frac{-2(3r_h^2 - 1)}{3r_h^2 + 1}.$$

So the unique critical point for $L(r_h)$ is $r_{h_0} = \frac{1}{\sqrt{3}}$, and $L_0 = L(r_{h_0}) = \frac{1}{\sqrt{3}}$. Moreover,

$$L''(r_h) = \frac{-24r_h}{(3r_h^2 + 1)^2} < 0.$$

We can conclude that r_{h_0} is a maximum for $L(r_h)$ and so for each $0 < L < \frac{1}{\sqrt{3}}$ there are two different masses m_1, m_2 which share the same L, as desired $\square$

Because the previous statement, for the same conformal infinity $\mathbb{S}^1(L) \times \mathbb{S}^2$ when $0 < L < \frac{1}{\sqrt{3}}$, there are two non-isometric AdS-Schwarzschild spaces with metrics $g^+_{m_1}$ and $g^+_{m_2}$ on $\mathbb{R}^2 \times \mathbb{S}^2$. In this way we have proved the non-uniqueness for conformally compact Einstein metrics on the topologically same 4-manifold.

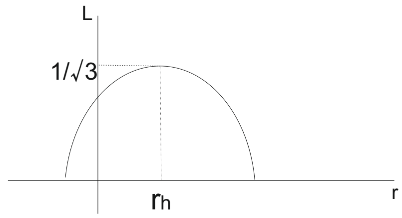

Figure 2.2: Representation of L(r)

2.3.2 Uniqueness of solutions for the fractional Yamabe problem

Classical Case

For an introduction to the Yamabe problem, see Section 2.1.2. We may see that depending on the sign of the minimizer of the classical Yamabe constant $\lambda(M)$ given in (2.1.6), it holds:

i. If $\lambda(M) = 0$ we have uniqueness of solutions (up to multiplicative constant).

ii. If $\lambda(M) < 0$ we also get uniqueness in the solution (up to constant).

iii. However, we have possible non uniqueness of solutions if $\lambda(M) > 0$. It will be explained in the next Section 2.4.

Fractional case

As in the classical case, the uniqueness of solution for the fractional Yamabe problem (explained in Section 2.2.4) also depends on the sign of fractional Yamabe constant $\lambda_\gamma(M)$. Indeed, recalling that the sign of $\lambda_\gamma(M)$ is equal to the sign of the constant $Q_\gamma^{\tilde{g}}$, where $\tilde{g}$ is the $\gamma-$Yamabe metric as we can check in Lemma 2.2.27, we have

i. If $\lambda_\gamma(M) < 0$: Given (X^{n+1}, g^+) with conformal infinity (M, g), suppose that $g_1 = w_1^{\frac{4\gamma}{n-2\gamma}} g$ is a solution with $Q_\gamma^{g_1} = \mu_1$. We also suppose that $g_2 = w_2^{\frac{4\gamma}{n-2\gamma}} g$ is a solution with $Q_\gamma^{g_2} = \mu_2$. Then,

$$g_2 = w_2^{\frac{4}{n-2\gamma}} g = w_2^{\frac{4}{n-2\gamma}} w_1^{\frac{-4}{n-2\gamma}} g_1 = w^{\frac{4}{n-2\gamma}} g_1, \tag{2.3.8}$$

and, using g_1 as a background metric, the conformal factor w is solution of

$$\begin{cases} -\operatorname{div}((\rho^*)^a \nabla_{\bar{g}_1} W) = 0 \text{ in } (X, \bar{g}_1), \\ -\tilde{d}_\gamma \lim\limits_{\rho^* \to 0} (\rho^*)^a \partial_{\rho^*} W + w\mu_1 = \mu_2 w^{\frac{n+2\gamma}{n-2\gamma}} \text{ on } M. \end{cases}$$

We are under the hypothesis $\mu_1, \mu_2 < 0$, so, up to constant, we can assume $\mu_1 = \mu_2 = \mu < 0$. If we take a look at the system, the first equation is an elliptic one, so the maximum (and minimun) of W is attained at the boundary. So we can consider $P \in M$ the point where W reaches the maximum value, and $Q \in M$ the point where it reaches the minimum value:

- At P, the function W is maximum so that the outward normal derivative in this point must be nonnegative, and negative if we take the derivative in the direccion of ∂_{ρ^*}

$$-\mu(w^{\frac{n+2\gamma}{n-2\gamma}} - w)(P) = \tilde{d}_\gamma \lim_{\rho^* \to 0} (\rho^*)^a \partial_{\rho^*} W \leq 0,$$

  which under the assuption $\mu < 0$ implies $w(P) \leq 1$.

- At Q, the function W is minimum so that

$$-(w^{\frac{n+2\gamma}{n-2\gamma}} - w)(Q) = \tilde{d}_\gamma \lim_{\rho^* \to 0} (\rho^*)^a \partial_{\rho^*} W \geq 0,$$

  which under the assuption $\mu < 0$ implies $w(Q) \geq 1$.

Since $\min w \geq 1$ and $\max w \leq 1$, we must have $w \equiv 1$, as desired.

ii. If $\lambda_\gamma(M) = 0$ we also have uniqueness of solutions (up to constant).

Given (X^{n+1}, g^+) with conformal infinity (M, g), suppose that $g_1 = w_1^{\frac{4}{n-2\gamma}} g$ is a solution with $Q_\gamma^{g_1} = \mu_1$. We also suppose that $g_2 = w_2^{\frac{4}{n-2\gamma}} g$ is a solution with $Q_\gamma^{g_2} = \mu_2$. Equality (2.3.8) holds again, and for the w appearing there, there exists a unique W such that $W|_M = w$ and

$$\begin{cases} -\operatorname{div}((\rho^*)^a \nabla_{\bar{g}_1} W) = 0 \text{ in } X, \\ -\tilde{d}_\gamma \lim_{\rho^* \to 0} (\rho^*)^a \partial_{\rho^*} W(x, 0) = 0 \text{ on } M, \end{cases}$$

since we are under the assumption $\mu_1 = \mu = 0$. Using that $W = W(\rho^*)$, then $w = W|_M$ is constant, as desired.

iii. However, we have possible non uniqueness of solutions if $\lambda_\gamma(M) > 0$. This is one of our main contributions and it will be explained in the next Chapter 3.

Example:
We will provide here an example of non uniqueness for the Yamabe metric in the **classical case**: $M = \mathbb{S}^1(L) \times \mathbb{S}^{n-1}$ by Schoen [159].
It follows from a straightforward computation that the non uniqueness for the problem of finding a conformal metric with constant scalar curvature for $M_1 = \mathbb{R} \times \mathbb{S}^{n-1}$ with the metric $g_0 = dt^2 + g_{\mathbb{S}^{n-1}}$ is equivalent to finding solutions with an isolated singularity in $M_2 = \mathbb{R}^n \setminus \{0\}$, with the Euclidean metric in polar coordinates $|dx|^2 = dr^2 + r^2 g_{\mathbb{S}^{n-1}}$.

Indeed, we just need to notice that both manifolds are conformally related, in fact, using the Emden-Fowler change of variable $r = e^{-t}$, we have

$$|dx|^2 = dr^2 + r^2 g_{\mathbb{S}^{n-1}} = e^{-2t}\left(dt^2 + e^{-2t} g_{\mathbb{S}^{n-1}}\right) = e^{-2t} g_0.$$

Thus, having a metric conformal to $|dx|^2$ with constant curvature and an isolated singularity is equivalent to having a metric conformal to g_0 with constant curvature and smooth. This conformal metric is of the form

$$\tilde{g} := w^{\frac{4}{n-2}} |dx|^2 = v^{\frac{4}{n-2}} g_0$$

where v and w are related by

$$w(x) = |x|^{-\frac{n-2}{2}} v(x).$$

We will look for radially symmetric solutions, so the example reduces to solving and ODE. We will give more details about the non uniqueness example, when we consider the Yamabe problem in the Euclidean space with isolated singularities. For the classical case, this is written in the next Section 2.4, for the fractional case, this construction is new and it is written in Chapter 3.

2.4 Isolated singularities

In this section we are going to start considering the Yamabe problem in $\mathbb{R}^n$, $n \geq 3$ with an isolated singularity at the origin. We will focus here on the **classical case** $\gamma = 1$, following the study in [159, 160]. The fractional case will be studied in the Chapter 3, since it is one of the main results in this thesis.

We consider the Yamabe problem

$$-\Delta w = c_{n,1} w^{\frac{n+2}{n-2}}, \quad w > 0, \tag{2.4.1}$$

with an isolated singularity at the origin. Here the constant is given by $c_{n,1} = \frac{n-2}{4}$. equation for the metric g_w. It is well known ([37]) that positive solutions of equation (2.4.1) in $\mathbb{R}^n \backslash \{0\}$ must be radially symmetric and, if the singularity at the origin is not removable, then the solution must behave as

$$w(x) = |x|^{-\frac{n-2}{2}} v(x), \tag{2.4.2}$$

where $0 < v \leq c_1 \leq v \leq c_2 < \infty$.

First, we will give two explicit solutions for (2.4.1).

- Indeed, for the constant solution v_1 we just need to use polar coordinates to compute

$$\Delta w = \partial_{rr} w + \frac{n-1}{r} \partial_r w + \frac{1}{r^2} \Delta_{\mathbb{S}^2}.$$

  Then, we can easily check that the only solution $w_1 = v r^{-\frac{n-2}{2}}$ with v being a constant function is $v \equiv v_1 \equiv 1$. This is the cylindrical metric.

- For the non constant solution, we recall that the spherical metric is

$$g_1 = \frac{4}{(1+|x|^2)^2}|dx|^2, \tag{2.4.3}$$

which represents the so called bubble from (2.1.8). After the Emden Fowler change of variable $r = e^{-t}$, we can observe that, in the new variable, the conformal factor in (2.4.3) is

$$w_\infty = \left(\frac{4}{(1+|x|^2)^2}\right)^{\frac{n-2}{4}} = e^{-\frac{n-2}{2}t}v_\infty, \quad \text{ where } v_\infty(t) = (\cosh t)^{-\frac{n-2}{2}},$$

up to multiplicative constant.

Now we will look for all the radial solutions with an isolated singularity at the origin. Thus we take again

$$w(r) = r^{-\frac{n-2}{2}}v(r).$$

Substituting $r = e^{-t}$ our equation (2.4.1) reduces to a standard second order ODE for a function $v = v(t)$:

$$-\ddot{v} + \tfrac{(n-2)^2}{4}v = \tfrac{(n-2)^2}{4}v^{\frac{n+2}{n-2}}, \quad v > 0. \tag{2.4.4}$$

First we draw the phase portrait, transforming the equation in a first order system. We call $\dot{v}(t) = \partial_t v(t)$ and we get the Hamiltonian system

$$X(v,\dot{v}) = \left(\dot{v}(t), -\tfrac{(n-2)^2}{4}\left(v^{\frac{n+2}{n-2}} - v\right)\right).$$

There exist two critical points for this system: $V_0 = (v_0, 0) = (0,0)$ and $V_1 = (v_1, 0) = (1,0)$. Now we linearize at each critical point,

- At V_0, the Jacobian is

$$J(v_0,0) = \begin{pmatrix} 0 & 1 \\ \frac{(n-2)^2}{4} & 0 \end{pmatrix},$$

which has eigenvalues $\lambda = \pm\frac{(n-2)}{2}$. Therefore, $(v_0,0)$ is an saddle point.

- At V_1,

$$J(v_1,0) = \begin{pmatrix} 0 & 1 \\ -(n-2) & 0 \end{pmatrix},$$

with eigenvalues $\lambda = \pm\sqrt{-(n-2)}$, and therefore, the point $(v_1, 0)$ is a center.

This equation it is easily integrated and the analysis of its phase portrait gives that all the bounded solutions must be periodic.

More precisely, the Hamiltonian

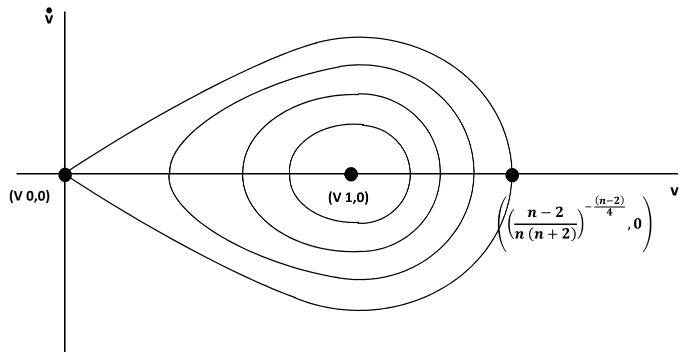

Figure 2.3: Representation of the phase portrait.

$$H_1(v,\dot v) := \tfrac{1}{2}\dot v^2 + \tfrac{(n-2)^2}{4}\left(\tfrac{(n-2)}{2n} v^{\frac{2n}{n-2}} - \tfrac{1}{2}v^2\right) \tag{2.4.5}$$

is preserved along trajectories. Thus looking at its level sets we conclude that there exists a family of periodic solutions $\{v_L\}$ of periods $L \in (L_0^1, \infty)$. Here

$$L_0^1 = \tfrac{2\pi}{\sqrt{n-2}} \tag{2.4.6}$$

is the minimal period and it can be calculated from the linearization at the equilibrium solution $v_1 \equiv 1$. These $\{v_L\}$ are known as the Fowler ([89]) or Delaunay solutions for the scalar curvature.

The metric g_1 is not a complete metric on $\mathbb{R} \times \mathbb{S}^{n-1}$. But taking the metric in $\mathbb{S}^1(L) \times \mathbb{S}^{n-1}$, when $L > L_0^1$, given by $g_v = v^{\frac{4}{n-2}}(dt^2 + g_{\mathbb{S}^{n-1}})$ for $v = v(t)$ any solution of the ODE; we find a complete metric with constant scalar curvature that is different from the standard one.

2.5 The general singular fractional Yamabe problem

Before going further in the study of the isolated singularities case we will provide here a summary of the known results for the general singular fractional Yamabe problem (see also

[110]). From the analysis point of view, one wishes to understand the semilinear problem

$$\begin{cases} (-\Delta)^\gamma w = c\, w^{\frac{n+2s}{n-2s}} \text{ in } \mathbb{R}^n \setminus \Lambda, \\ w(x) \to \infty \text{ as } x \to \Lambda, \end{cases} \tag{2.5.1}$$

where Λ is a closed set of Hausdorff dimension $0 < k < n$ and $c \in \mathbb{R}$. The first difficulty one encounters is precisely how to define the fractional Laplacian $(-\Delta)^\gamma$ on $\Omega := \mathbb{R}^n \setminus \Lambda$ since it is a non-local operator. Nevertheless, this is better understood from the conformal geometry point of view.

In order to put (2.5.1) into a broader context, let us give a brief review of the classical singular Yamabe problem ($\gamma = 1$). Let (M, g) be a compact n-dimensional Riemannian manifold, $n \geq 3$, and $\Lambda \subset M$ is any closed set as above. We are concerned with the existence and geometric properties of complete (non-compact) metrics of the form $g_w = w^{\frac{4}{n-2}} g$ with constant scalar curvature. This corresponds to solving the partial differential equation (recall (2.1.4))

$$-\Delta_g w + \tfrac{n-2}{4(n-1)} R_g w = \tfrac{n-2}{4(n-1)} R\, w^{\frac{n+2}{n-2}}, \quad w > 0,$$

where $R_{g_w} \equiv R$ is constant and with a boundary condition that $w \to \infty$ sufficiently quickly at Λ so that g_w is complete. It is known that solutions with $R < 0$ exist quite generally if Λ is large in a capacitary sense ([130, 119]), whereas for $R > 0$ existence is only known when Λ is a smooth submanifold (possibly with boundary) of dimension $k < (n-2)/2$ ([139, 82]).

There are both analytic and geometric motivations for studying this problem. For example, in the positive case ($R > 0$), solutions to this problem are actually weak solutions across the singular set ([161]), so these results fit into the broader investigation of possible singular sets of weak solutions of semilinear elliptic equations.

On the geometric side, a well-known theorem by Schoen and Yau ([161, 162]) states that if (M, g) is a compact manifold with a locally conformally flat metric g of positive scalar curvature, then the developing map D from the universal cover $\widetilde{M}$ to $\mathbb{S}^n$, which by definition is conformal, is injective, and moreover, $\Lambda := \mathbb{S}^n \setminus D(\widetilde{M})$ has Hausdorff dimension less than or equal to $(n-2)/2$. Regarding the lifted metric $\tilde{g}$ on $\widetilde{M}$ as a metric on Ω, this provides an interesting class of solutions of the singular Yamabe problem which are periodic with respect to a Kleinian group, and for which the singular set Λ is typically nonrectifiable. More generally, they also show that if $g_{\mathbb{S}^n}$ is the canonical metric on the sphere and if $g = w^{\frac{4}{n-2}} g_{\mathbb{S}^n}$ is a complete metric with positive scalar curvature and bounded Ricci curvature on a domain $\Omega = \mathbb{S}^n \setminus \Lambda$, then

$$\dim \Lambda \leq (n-2)/2.$$

Going back to the non-local case, although it is not at all clear how to define $P_\gamma^{\tilde{g}}$ and $Q_\gamma^{\tilde{g}}$ on a general complete (non-compact) manifold $(\Omega, \tilde{g})$, in the paper [96] the authors give a reasonable definition when Ω is an open dense set in a compact manifold M and the metric $\tilde{g}$ is conformally related to a smooth metric g on M. Namely, one can define them by demanding that the conformal property (2.2.18) holds (as usual, we assume that a Poincaré-Einstein filling (X, g^+) has been fixed). Note, however, that this is not as simple as it first appears since, because of the nonlocal character of $P_s^{\tilde{h}}$, we must extend w as a distribution on all of M. There is no difficulty in using the relationship (2.2.18) to define $P_\gamma^{\tilde{g}}\phi$ when $\phi \in \mathcal{C}_0^\infty(\Omega)$. From here one can use an abstract functional analytic argument to extend $P_\gamma^{\tilde{g}}$ to act on any $\phi \in L^2(\Omega, dv_{\tilde{g}})$. Indeed, it is straightforward to check that the operator $P_\gamma^{\tilde{g}}$ defined in this way is essentially self-adjoint on $L^2(\Omega, dv_{\tilde{g}})$ when γ is real. However, observe that $P_\gamma^{\tilde{g}} = (-\Delta_{\tilde{h}})^\gamma + \mathcal{K}$, where $\mathcal{K}$ is a pseudo-differential operator of order $2\gamma - 1$. Furthermore, $(-\Delta_{\tilde{h}})^\gamma$ is self-adjoint. Since $\mathcal{K}$ is a lower order symmetric perturbation, then $P_\gamma^{\tilde{g}}$ is also essentially self-adjoint.

Another interesting development is [103], where they give a sharp spectral characterization of conformally compact Einstein manifolds with conformal infinity of positive Yamabe type.

The singular fractional Yamabe problem on $(M, [g])$ is then formulated as

$$\begin{cases} P_\gamma^g w = c w^{\frac{n+2\gamma}{n-2\gamma}} & \text{in } M \setminus \Lambda, \\ w(x) \to \infty & \text{as } x \to \Lambda, \end{cases} \tag{2.5.2}$$

for $c \equiv Q_\gamma^{\tilde{g}}$ constant. A separate, but also very interesting issue, is whether $c > 0$ implies that the conformal factor w is actually a weak solution of (2.5.2) on all of M.

The first result in [96] partially generalizes Schoen-Yau's theorem:

Theorem 2.5.1 ([96]). Suppose that (M^n, g) is compact and $g_w = w^{\frac{4}{n-2\gamma}} g$ is a complete metric on $\Omega = M \setminus \Lambda$, where Λ is a smooth k-dimensional submanifold in M. Assume furthermore that w is polyhomogeneous along Λ with leading exponent $-n/2+\gamma$. If $\gamma \in \left(0, \frac{n}{2}\right)$, and if $Q_\gamma^g > 0$ everywhere for any choice of asymptotically Poincaré-Einstein extension (X, g^+) then n, k and γ are restricted by the inequality

$$\Gamma\left(\frac{n}{4} - \frac{k}{2} + \frac{\gamma}{2}\right) \Big/ \Gamma\left(\frac{n}{4} - \frac{k}{2} - \frac{\gamma}{2}\right) > 0. \tag{2.5.3}$$

This inequality holds in particular when

$$k < \frac{n - 2\gamma}{2}, \tag{2.5.4}$$

and in this case then there is a unique distributional extension of w on all of M which is still a solution of (2.5.2) on all of M.

Remark that w is called polyhomogeneous along Λ if in terms of any cylindrical coordinate system (r, θ, y) in a tubular neighborhood of Λ, where r and θ are polar coordinates in disks in the normal bundle and y is a local coordinate along Λ, w admits an asymptotic expansion $w \sim \sum a_{jk}(y, \theta) r^{\mu_j} (\log r)^k$, where μ_j is a sequence of complex numbers with real part tending to infinity, for each j, a_{jk} is nonzero for only finitely many nonnegative integers k, and such that every coefficient $a_{jk} \in \mathcal{C}^\infty$.

As we have noted, inequality (2.5.3) is satisfied whenever $k < (n-2\gamma)/2$, and in fact is equivalent to this simpler inequality when $\gamma = 1$. When $\gamma = 2$, i.e. for the standard $Q-$curvature, this result is already known: [44] shows that complete metrics with $Q_2 > 0$ and positive scalar curvature must have singular set with dimension less than $(n-4)/2$, which again agrees with (2.5.3).

Of course, the main open question is to remove the smoothness assumption on the singular set Λ. Recent results of [175] show that, under a positive scalar curvature assumption, if $Q_\gamma > 0$ for $\gamma \in (1,2)$, then (2.5.4) holds for any Λ.

We also remark that a dimension estimate of the type (2.5.3) implies some topological restrictions on M: on the homotopy ([162], chapter VI), on the cohomology ([148]), or even classification results in the low dimensional case ([108]).

Finally, one can also obtain existence of solutions when γ is sufficiently near 1 and Λ is smooth by perturbation theory:

Theorem 2.5.2 ([96]). Let (M^n, g) be compact with nonnegative Yamabe constant and Λ a k-dimensional submanifold with $k < \frac{1}{2}(n-2)$. Then there exists an $\epsilon > 0$ such that if $\gamma \in (1-\epsilon, 1+\epsilon)$, there exists a solution to the fractional singular Yamabe problem (2.5.2) with $c > 0$ which is complete on $M \setminus \Lambda$.

2.6 Integro-differential operators

Linear integro-differential operators are generators of Levy processes. According to the Levy-Kintchine formula, they have the general form

$$\mathbb{L}u(x) = \operatorname{tr}\left(A(x)D^2u\right) + b(x)\cdot\nabla u + c(x)u + d(x) + \int_{\mathbb{R}^n} \left(u(x+y) - u(x) - y\cdot\nabla u(x)\chi_{\mathbb{B}_1}(y)\right) d\mu_x(y),$$

where $A(x)$ is nonnegative matrix and μ_x is non negative measure satisfying

$$\int_{\mathbb{R}^n} \min(y^1, 1)\, d\mu_x < +\infty.$$

In most of the cases, the nonnegative measure μ_x is assumed to be absolutely continuous, and thus, $\mu_x(y) = K(x,y)\,dy$, where K will denote the kernel.

Since we are interested here in purely integro-differential operators we will neglect the local part of the operator and we will study operators with the general form

$$\mathbb{L}u(x) = \int_{\mathbb{R}^n} \left(u(x+y) - u(x) - y\cdot\nabla u(x)\chi_{\mathbb{B}(1)}(y)\right) K(x,y)\,dy.$$

Note that if the kernel is symmetric the previous expression becomes

$$\mathbb{L}u(x) = \int_{\mathbb{R}^n} \left(u(x+y) + u(x-y) - 2u(x)\right) K(x,y)\,dy.$$

The model example for these operators is the fractional Laplacian expressed as in (2.2.1); this operator corresponds to a stable process. We will consider this kind of operators but also these corresponding to a tempered stable process. An example of these kernels is the one defined in Chapter 4 in (4.2.6).

For more details about integro-differential operators see, for instance, some of these references [114],[155],[18],[165],[163].

Chapter 3

Isolated singularities for a semilinear equation for the fractional Laplacian arising in conformal geometry

In this chapter, we introduce the study of isolated singularities for a semilinear equation involving the fractional Laplacian. In conformal geometry, it is equivalent to the study of singular metrics with constant fractional curvature. Our main ideas are: first, to set the problem into a natural geometric framework, and second, to perform some kind of phase portrait study for this non-local ODE.

3.1 Introduction and statement of results

We consider the problem of finding radial solutions for the fractional Yamabe problem in $\mathbb{R}^n$, $n > 2\gamma$, with an isolated singularity at the origin. This means to look for positive, radially symmetric solutions of

$$(-\Delta)^\gamma w = c_{n,\gamma} w^{\frac{n+2\gamma}{n-2\gamma}} \text{ in } \mathbb{R}^n \setminus \{0\}, \tag{3.1.1}$$

where $c_{n,\gamma}$ is any positive constant that, without loss of generality, will be normalized as in Proposition 3.1.1. Unless we state the contrary, $\gamma \in (0, \frac{n}{2})$. In geometric terms, given the Euclidean metric $|dx|^2$ on $\mathbb{R}^n$, we are looking for a conformal metric

$$g_w = w^{\frac{4}{n-2\gamma}} |dx|^2, \ w > 0, \tag{3.1.2}$$

with positive constant fractional curvature $Q_\gamma^{g_w} \equiv c_{n,\gamma}$, that is radially symmetric and has a prescribed singularity at the origin.

Because of the well known extension theorem for the fractional Laplacian [36, 40, 43] we can assert that equation (3.1.1) for the case $\gamma \in (0, 1)$ is equivalent to the boundary reaction problem

$$\begin{cases} -\operatorname{div}(y^a \nabla W) = 0 \text{ in } \mathbb{R}^{n+1}_+, \\ W = w \text{ on } \mathbb{R}^n \setminus \{0\}, \\ -\tilde{d}_\gamma \lim_{y\to 0} y^a \partial_y W = c_{n,\gamma} w^{\frac{n+2\gamma}{n-2\gamma}} \text{ on } \mathbb{R}^n \setminus \{0\}, \end{cases} \tag{3.1.3}$$

where $a = 1 - 2\gamma$ and the constant $\tilde{d}_\gamma$ is defined in Chapter 2 in (2.2.3). We note that it is possible to write $W = \mathcal{K}_\gamma *_x w$, where $\mathcal{K}_\gamma$ is the Poisson kernel (2.2.8) for this extension problem.

It is known that $w_1(r) = r^{-\frac{n-2\gamma}{2}}$, $r = |x|$, together with $W_1 = \mathcal{K}_\gamma *_x w_1$, is an explicit solution for (3.1.3); this fact will be proved in Proposition 3.1.1 and as a consequence we will obtain the normalization of the constant $c_{n,\gamma}$. Therefore, w_1 is the model solution for isolated singularities, and it corresponds to the cylindrical metric.

In the recent paper [33] Caffarelli, Jin, Sire and Xiong characterize all the nonnegative solutions to (3.1.3). Indeed, let W be any nonnegative solution of (3.1.3) in $\mathbb{R}^{n+1}_+$ and suppose that the origin is not a removable singularity. Then, we must have that

$$W = W(r, y) \quad \text{and} \quad \partial_r W(r, y) < 0 \quad \forall\ 0 < r < \infty.$$

In addition, they also provide its asymptotic behavior. More precisely, if $w = W(\cdot, 0)$ denotes the trace of W, then near the origin one must have that

$$c_1 r^{-\frac{n-2\gamma}{2}} \leq w(x) \leq c_2 r^{-\frac{n-2\gamma}{2}}, \tag{3.1.4}$$

where c_1, c_2 are positive constants.

We remark that if the singularity at the origin is removable, all the entire solutions for (3.1.3) have been completely classified by Jin, Li and Xiong [111] and Chen, Li and Ou [48], for instance. In particular, they must be the standard "bubbles"

$$w(x) = c \left(\frac{\lambda}{\lambda^2 + |x - x_0|} \right)^{\frac{n-2\gamma}{2}}, \quad c, \lambda > 0,\ x_0 \in \mathbb{R}^n. \tag{3.1.5}$$

In this chapter we initiate the study of positive radial solutions for (3.1.1). It is clear from the above that we should look for solutions of the form

$$w(r) = r^{-\frac{n-2\gamma}{2}} v(r) \text{ on } \mathbb{R}^n \setminus \{0\}, \tag{3.1.6}$$

for some function $0 < c_1 \leq v \leq c_2$. In the classical case $\gamma = 1$, equation (3.1.1) reduces to a standard second order ODE (2.4.4). However, in the fractional case, (3.1.1) becomes a fractional order ODE, so classical methods cannot be directly applied here.

The objective of this chapter is two-fold: first, to use the natural interpretation of problem (3.1.1) in conformal geometry in order to obtain information about isolated singularities for the operator $(-\Delta_{\mathbb{R}^n})^\gamma$ from the scattering theory definition. And second, to take a dynamical system approach to explore how much of the standard ODE study can be generalized to the PDE (3.1.3).

Before we consider the conformal geometry approach, let us give the necessary background. We present now the natural coordinates for studying isolated singularities of (3.1.1). Let $M = \mathbb{R}^n\backslash\{0\}$ and use the Emden-Fowler change of variable $r = e^{-t}$, $t \in \mathbb{R}$; with some abuse of the notation, we write $v = v(t)$. Then, in radial coordinates, M may be identified with the manifold $\mathbb{R} \times \mathbb{S}^{n-1}$, for which the Euclidean metric is written as

$$|dx|^2 = dr^2 + r^2 g_{\mathbb{S}^{n-1}} = e^{-2t}[dt^2 + g_{\mathbb{S}^{n-1}}] =: e^{-2t} g_0. \tag{3.1.7}$$

Since the metrics $|dx|^2$ and g_0 are conformally related, we prefer to use g_0, the cylindrical metric, as a background metric and thus any conformal change (3.1.2) may be rewritten as

$$g_v = w^{\frac{4}{n-2\gamma}} |dx|^2 = v^{\frac{4}{n-2\gamma}} g_0,$$

where we have used relation (3.1.6). Looking at the conformal transformation property for P_γ^g given in (2.2.18) and relation (3.1.6) again, it is clear that

$$P_\gamma^{g_0}(v) = r^{\frac{n+2\gamma}{2}} P_\gamma^{|dx|^2}(r^{-\frac{n-2\gamma}{2}} v) = r^{\frac{n+2\gamma}{2}} (-\Delta)^\gamma w, \tag{3.1.8}$$

and thus the original problem (3.1.1) is equivalent to the following one: fixed g_0 as a background metric on $\mathbb{R} \times \mathbb{S}^{n-1}$, find a conformal metric $g_v = v^{\frac{4}{n-2\gamma}} g_0$ of positive constant fractional curvature $Q_\gamma^{g_v}$, i.e., find a positive smooth solution v for

$$P_\gamma^{g_0}(v) = c_{n,\gamma} v^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{on} \quad \mathbb{R} \times \mathbb{S}^{n-1}. \tag{3.1.9}$$

Proposition 3.1.1. *The fractional curvature of the cylindrical metric $g_{w_1} = w_1^{\frac{4}{n-2\gamma}} |dx|^2$ for the conformal change*

$$w_1(x) = |x|^{-\frac{n-2\gamma}{2}}, \tag{3.1.10}$$

is the constant

$$c_{n,\gamma} = 2^{2\gamma} \left(\frac{\Gamma(\frac{1}{2}(\frac{n}{2} + \gamma))}{\Gamma(\frac{1}{2}(\frac{n}{2} - \gamma))} \right)^2 > 0.$$

Proof. The value is calculated using the conformal property (2.2.20), as follows:

$$Q_\gamma^{g_{w_1}} = w_1^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^{|dx|^2}(w_1) = w_1^{-\frac{n+2\gamma}{n-2\gamma}} (-\Delta)^\gamma (w_1) =: c_{n,\gamma}.$$

The last equality follows from the calculation of the fractional Laplacian of a power function; it can be found in [96, 153]. □

The point of view of this chapter is to consider problem (3.1.9) instead of (3.1.1), since it allows for a simpler analysis. In our first theorem we compute the principal symbol of the operator $P_\gamma^{g_0}$ on $\mathbb{R} \times \mathbb{S}^{n-1}$ using the spherical harmonic decomposition for $\mathbb{S}^{n-1}$. With some abuse of notation, let $\mu_k = -k(k + n - 2)$, $k = 0, 1, 2, ...$ be the eigenvalues of $\Delta_{\mathbb{S}^{n-1}}$, repeated according to multiplicity. Then any function on $\mathbb{R} \times \mathbb{S}^{n-1}$ may be decomposed as $\sum_k v_k(t) E_k$, where $\{E_k\}$ is a basis of eigenfunctions. We show that the operator $P_\gamma^{g_0}$

diagonalizes under such eigenspace decomposition, and moreover, it is possible to calculate the Fourier symbol of each projection. Let

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi \cdot t} v(t)\, dt \tag{3.1.11}$$

be our normalization for the one-dimensional Fourier transform.

Theorem 3.1.2. *Fix $\gamma \in (0, \frac{n}{2})$ and let P_γ^k be the projection of the operator $P_\gamma^{g_0}$ over each eigenspace $\langle E_k \rangle$. Then*

$$\widehat{P_\gamma^k(v_k)} = \Theta_\gamma^k(\xi)\, \widehat{v_k},$$

and this Fourier symbol is given by

$$\Theta_\gamma^k(\xi) = 2^{2\gamma} \frac{\left| \Gamma\left(\frac{1}{2} + \frac{\gamma}{2} + \frac{\sqrt{(\frac{n}{2}-1)^2 - \mu_k}}{2} + \frac{\xi}{2} i \right) \right|^2}{\left| \Gamma\left(\frac{1}{2} - \frac{\gamma}{2} + \frac{\sqrt{(\frac{n}{2}-1)^2 - \mu_k}}{2} + \frac{\xi}{2} i \right) \right|^2}. \tag{3.1.12}$$

Since we are mainly interested in radial solutions $v = v(t)$, in many computations we will just need to consider the symbol for the first eigenspace $k = 0$ (that corresponds to the constant eigenfunctions):

$$\Theta_\gamma^0(\xi) = 2^{2\gamma} \frac{\left| \Gamma(\frac{n}{4} + \frac{\gamma}{2} + \frac{\xi}{2} i) \right|^2}{\left| \Gamma(\frac{n}{4} - \frac{\gamma}{2} + \frac{\xi}{2} i) \right|^2}.$$

Now we look at the question of finding smooth solutions $v = v(t)$, $0 < c_1 \le v \le c_2$, for equation (3.1.9), and we expect to have periodic solutions. The local case $\gamma = 1$, presented in the previous Chapter 2 in Section 2.4, provides some motivation for this statement and, in the next Chapter 4 we will construct such periodic solutions from the variational point of view. These will be called "Delaunay" solutions for the fractional curvature. Here we look at the geometrical interpretation of such solutions and provide a dynamical system approach for the problem.

Delaunay solutions are, originally, rotationally symmetric surfaces with constant mean curvature and they have been known for a long time ([63, 77]). In addition, let $\Sigma \in \mathbb{R}^3$ be a noncompact embedded constant mean curvature surfaces with k ends. It is known that any of such ends must be asymptotic to one of the Delaunay surfaces ([118, 107]), which is very similar to what happens in the constant scalar curvature setting (see, for instance [112]), where any positive solution of the constant scalar curvature equation (1.3.1) must be asymptotic to a precise deformation of one Delaunay.

Let us comment here that, as we explained in Chapter 2 when we introduced Escobar's problem (2.2.42), the $\gamma = 1/2$ case is very related to the constant mean curvature problem. However, there is a further restriction since in the present chapter we consider only rotationally symmetric metrics on the boundary and thus, not allowing full generality for the original constant mean curvature problem.

Going back to (2.4.4), we would like to understand how much of this picture is preserved in the non-local case, so we look for radial solutions of equation (3.1.1), which becomes a fractional order ODE. On the one hand, we formulate the problem through the extension (3.1.3). This point of view has the advantage that the new equation is local (and degenerate elliptic) but, on the other hand, it is a PDE with a non-linear boundary condition. Note that because we will be using the extension from Theorem 2.2.18 for the calculation of the fractional Laplacian, we need to restrict ourselves to $\gamma \in (0,1)$ at this stage.

The first difficulty we encounter with our approach is how to write the original extension equation (3.1.3) in a natural way after the change of variables $r = e^{-t}$. Looking at the construction of the fractional Laplacian from the scattering equation (2.2.15) on hyperbolic space $(X^{n+1}, g^+) = (\mathbb{H}^{n+1}, g^+_{\mathbb{H}^{n+1}})$ given in Chapter 2, we need to find a parametrization of hyperbolic space in such a way that its conformal infinity $M^n = \{\rho = 0\}$ is precisely $(\mathbb{R} \times \mathbb{S}^{n-1}, g_0)$. The precise metric on the extension is $g^+ = \bar{g}/\rho^2$ for

$$\bar{g} = d\rho^2 + \left(1 + \tfrac{\rho^2}{4}\right)^2 dt^2 + \left(1 - \tfrac{\rho^2}{4}\right)^2 g_{\mathbb{S}^{n-1}}, \tag{3.1.13}$$

where $\rho \in (0,2)$ and $t \in \mathbb{R}$. The motivation for this change of metric will be made clear in Section 3.2.

Rewriting the equations in this new metric, our original equation (3.1.3), written in terms of the change (3.1.6), is equivalent to the extension problem

$$\begin{cases} -\operatorname{div}_{\bar{g}}(\rho^a \nabla_{\bar{g}} V) + E(\rho) V = 0 \;\text{ in } (X^{n+1}, \bar{g}), \\ V = v \text{ on } \{\rho = 0\}, \\ -\tilde{d}_\gamma \lim_{\rho \to 0} \rho^a \partial_\rho V = c_{n,\gamma} v^{\frac{n+2\gamma}{n-2\gamma}} \text{ on } \{\rho = 0\}, \end{cases} \tag{3.1.14}$$

where the expression for the lower order term $E(\rho)$ will be given in (3.4.2). We look for solutions V to (3.1.14) that only depend on ρ and t, and that are bounded between two positive constants.

We show first that equation (3.1.14) exhibits a Hamiltonian quantity that generalizes (2.4.5):

Theorem 3.1.3. *Fix $\gamma \in (0,1)$. Let V be a solution of (3.1.14) only depending on t and ρ. Then the Hamiltonian quantity*

$$H_\gamma(t) := \frac{1}{2} \int_0^2 \rho^a \left\{ e_1(\rho)(\partial_t V)^2 - e(\rho)(\partial_\rho V)^2 - e_2(\rho) V^2 \right\} d\rho + C_{n,\gamma} v^{\frac{2n}{n-2\gamma}}, \tag{3.1.15}$$

is constant with respect to t. Here we write

$$\begin{aligned} e_1(\rho) &= \left(1 + \tfrac{\rho^2}{4}\right)^{-1} \left(1 - \tfrac{\rho^2}{4}\right)^{n-1}, \\ e_2(\rho) &= \tfrac{n-1+a}{4} \left(1 - \tfrac{\rho^2}{4}\right)^{n-2} \left(n - 2 + n\tfrac{\rho^2}{4}\right), \\ e(\rho) &= \left(1 + \tfrac{\rho^2}{4}\right) \left(1 - \tfrac{\rho^2}{4}\right)^{n-1}, \end{aligned} \tag{3.1.16}$$

and the constant is given by

$$C_{n,\gamma} = \frac{n-2\gamma}{2n} \frac{c_{n,\gamma}}{\tilde{d}_\gamma}. \tag{3.1.17}$$

Hamiltonian quantities for fractional problems have been recently developed in the setting of one-dimensional solutions for fractional semilinear equations $(-\Delta)^\gamma w + f(w) = 0$. The first reference we find a conserved Hamiltonian quantity for this type of non-local equations is the paper by Cabré and Solá-Morales [32] for the particular case $\gamma = 1/2$. The general case $\gamma \in (0,1)$ was carried out by Cabré and Sire in [31]. On the one hand, in these two papers [31, 32], the authors impose a nonlinearity coming from a double-well potential and look for layers (i.e., solutions that are monotone and have prescribed limits at infinity), and they are able to write a Hamiltonian quantity that is preserved. In addition, if one considers the same problem but on hyperbolic space, one finds that the geometry at infinity plays a role and the analogous Hamiltonian is only monotone (see [98]).

On the other hand, if, instead, one looks for radial solutions for semilinear equations, then Cabré and Sire in [31] and Frank, Lenzman and Silvestre in [90] have developed a monotonicity formula for the associated Hamiltonian. In the setting of radial solutions with an isolated singularity for the fractional Yamabe problem, our Theorem 3.1.3 states that, if one uses the metric (3.1.13) to rewrite the problem, then the associated Hamiltonian (3.1.15) is constant along trajectories.

If one insists on performing an ODE-type study for the PDE problem (3.1.14), a possibility is to look for some kind of phase portrait of the boundary values (at $\rho = 0$), while keeping in mind that the equation is defined on the whole extension. From this point of view, one can prove the existence of two critical points: the constant solutions $v_0 \equiv 0$ and $v_1 \equiv 1$. Moreover, there exists an explicit homoclinic solution v_∞, whose precise expression will be given in (3.5.1); it corresponds to the n-sphere (3.1.5).

The next step is to linearize the equation. As we will observe in Section 3.6, the classical Hardy inequality, rewritten in terms of the background metric g_0, decides the stability of the explicit solutions v_0, v_1 and v_∞. Stability issues for semilinear fractional Laplacian equations have received a lot of attention recently. Some references are: [28] for the half-Laplacian, [154] for extremal solutions with exponential nonlinearity, [83] for semilinear equations with Hardy potential. In the particular case of the fractional Lane-Emden equation, stability was considered in [59, 86], for $\gamma \in (0,1)$ and [85] for $\gamma \in (1,2)$. We believe that our methods, although still at their initial stage, would provide tools for a unified approach for all $\gamma \in \left(0, \frac{n}{2}\right)$.

Finally, we consider the linearization of equation (3.1.9) around the equilibrium $v_1 \equiv 1$:

$$P_\gamma^{g_0} v = \tfrac{n+2\gamma}{n-2\gamma} v \quad \text{on} \quad \mathbb{R} \times \mathbb{S}^{n-1},$$

and look at the projection over each eigenspace $\langle E_k \rangle$, $k = 0, 1, \ldots$,

$$P_\gamma^k v_k = \tfrac{n+2\gamma}{n-2\gamma} v_k. \tag{3.1.18}$$

Although we will not provide a complete calculation of the spectrum, we can say the following:

Theorem 3.1.4. *For the projection $k = 0$, equation (3.1.18) has periodic solutions $v(t)$ with period $L_0^\gamma = \frac{2\pi}{\sqrt{\lambda_\gamma}}$, where λ_γ is the unique positive solution of (3.6.3). In addition,*

$$\lim_{\gamma \to 1} L_0^\gamma = L_0^1,$$

so we recover the classical case (2.4.6).

Remark 3.1.5. We also give some motivation to show that the projection on the k-eigenspace of (3.1.18) does not have periodic solutions if $k = 1, 2, \ldots$.

Theorem 3.1.4 gives the existence of periodic radial solutions for the linear problem. In addition, the existence of a conserved Hamiltonian hints that the original non-linear problem has periodic solutions too. Based on the results presented here, we will show in Chapter 4 that for every period $L > L_0^\gamma$, there exists a non trivial periodic solution v_L (called Delaunay solution) for the non-nonlinear problem (3.1.9).

The construction of Delaunay solutions allows for many further studies. For instance, as a consequence of their construction one obtains the non-uniqueness of the solutions for the fractional Yamabe problem in the positive curvature case, since it gives different conformal metrics on $\mathbb{S}^1(L) \times \mathbb{S}^{n-1}$ that have constant fractional curvature. This is well known in the scalar curvature case as we explained in the previous Chapter 2 in Section 2.3. In addition, this gives some motivation to define a total fractional scalar curvature functional, which maximizes the standard fractional Yamabe quotient ([97]) across conformal classes. We hope to return to this problem elsewhere.

From another point of view, Delaunay solutions can be used in gluing problems. Classical references are, for instance, [140, 143] for the scalar curvature, and [141, 142] for the construction of constant mean curvature surfaces with Delaunay ends. In Chapter 5 we use Delaunay solutions to construct metrics of constant fractional curvature with isolated singularities at a prescribed number of points.

Recently it has been introduced a related notion of nonlocal mean curvature H_γ for the boundary of a set in $\mathbb{R}^n$ (see [35, 170]). Finding Delaunay-type surfaces with constant nonlocal mean curvature has just been accomplished in [29]. For related nonlocal equations with periodic solutions see also [57, 30].

Other non-local problems that present periodic solutions can be found in [7, 8, 6, 164].

We finally comment that the negative fractional curvature case has not been explored yet, except for the works [46, 47]. They consider singular solutions for the problem $(-\Delta)^\gamma w + |w|^{p-1}w = 0$ in a domain Ω with zero Dirichlet condition on $\partial\Omega$. This setting is very different from the positive curvature case because the maximum principle is valid here. We also cite the work [152], where they consider singular solutions of $\Delta W = 0$ in a domain Ω with a nonlinear Neumann boundary condition $\partial_\nu W = f(x, W) - W$ on $\partial\Omega$.

This chapter will be structured as follows: in Section 3.2 we will give a geometric interpretation of the problem. Next, in Section 3.3 we will analyze the scattering equation to give a proof for theorem (3.1.2). That is, we will compute the Fourier symbol for the conformal fractional Laplacian. In Section 3.4 we face the problem from an ODE-type point of view. This kind of study over the extension problem (3.1.14) gives us two equilibria and the existence of a Hamiltonian quantity conserved along the trajectories. Moreover we will find in Section 3.5 an explicit homoclinic solution, which corresponds to the $n-$sphere. Finally, in Section 3.6 we will perform a linear analysis close to the constant solution which corresponds to the n-cylinder.

3.2 Geometric setting

We give now the natural geometric interpretation of problem (3.1.1) and the extension formulation (3.1.3). Thanks to Theorem 2.2.17 given in the previous Chapter 2, the initial extension problem (3.1.3) can be transformed into the scattering equation (2.2.15) in hyperbolic space, denoted by $X_1 = \mathbb{H}^{n+1}$, with the metric $g^+ = \frac{dy^2+|dx|^2}{y^2}$. Our point of view is to use the metric g_0 from (3.1.7) as the representative of the conformal infinity $\mathbb{R}^n \setminus \{0\}$, instead of the Euclidean one $|dx|^2$. Let us introduce some notation now. The conformal infinity (with an isolated singularity) is $M_1 = \mathbb{R}^n \backslash \{0\}$, which in polar coordinates can be represented as $M_1 = \mathbb{R}^+ \times \mathbb{S}^{n-1}$ and $|dx|^2 = dr^2 + r^2 g_{\mathbb{S}^{n-1}}$, or using the change of variable $r = e^{-t}$, the Euclidean metric may be written as

$$|dx|^2 = e^{-2t}[dt^2 + g_{\mathbb{S}^{n-1}}] =: e^{-2t} g_0. \tag{3.2.1}$$

Thus we need to rewrite the hyperbolic metric in a different normal form

$$g^+ = \frac{d\rho^2 + g_\rho}{\rho^2} \quad \text{with} \quad g_\rho|_{\rho=0} = g_0, \tag{3.2.2}$$

for a suitable defining function ρ. We consider now several models for hyperbolic space, identified with the Riemannian version of AdS space-time. Inspired by cosmology studies, as we explained in Chapter 2, when we provide the example of the Anti-de Sitter space in Section 2.2.2 and also in Section 2.3.1, we write the hyperbolic metric as

$$g^+ = d\sigma^2 + \cosh^2\sigma\, dt^2 + \sinh^2\sigma\, g_{\mathbb{S}^{n-1}}, \tag{3.2.3}$$

where $t \in \mathbb{R}$, $\sigma \in (0, \infty)$ $\theta \in \mathbb{S}^{n-1}$. Using the change of variable $R = \sinh\sigma$,

$$g^+ = \frac{1}{1+R^2}\, dR^2 + (1+R^2)\, dt^2 + R^2 g_{\mathbb{S}^{n-1}}.$$

This metric can be written in the normal form (3.2.2) as

$$g^+ = \rho^{-2}\left[d\rho^2 + \left(1 + \tfrac{\rho^2}{4}\right)^2 dt^2 + \left(1 - \tfrac{\rho^2}{4}\right)^2 g_{\mathbb{S}^{n-1}}\right], \tag{3.2.4}$$

for $\rho \in (0,2)$, $t \in \mathbb{R}$, $\theta \in \mathbb{S}^{n-1}$. Here we have used the relations

$$\rho = 2e^{-\sigma} \quad \text{and} \quad 1 + R^2 = \left(\tfrac{4-\rho^2}{4\rho}\right)^2 \tag{3.2.5}$$

Let $\bar{g} = \rho^2 g^+$ be a compactification of g^+. Note that the apparent singularity at $\rho = 2$ in the metric (3.2.4) is just a consequence of the polar coordinate parametrization and thus the metric is smooth across this point.

We define now $X_2 = (0,2) \times \mathbb{S}^1(L) \times \mathbb{S}^{n-1}$, with coordinates $\rho \in (0,2)$, $t \in \mathbb{S}^1(L)$, $\theta \in \mathbb{S}^{n-1}$, and the same metric given by (3.2.4). The conformal infinity $\{\rho = 0\}$ is $M_2 = \mathbb{S}^1(L) \times \mathbb{S}^{n-1}$, with the metric given by $g_0 = dt^2 + g_{\mathbb{S}^{n-1}}$.

Note that $(X_1, g^+_{\mathbb{H}^{n+1}})$ is a covering of (X_2, g^+). Indeed, X_2 is the quotient $X_2 = \mathbb{H}^{n+1}/\mathbb{Z} \approx \mathbb{R}^n \times \mathbb{S}^1(L)$ with $\mathbb{Z}$ the group generated by the translations, if we make the t variable periodic. As a consequence, also $(M_1, |dx|^2)$ is a covering of (M_2, g_0) after the conformal change (3.2.1).

Summarizing, we denote $X = (0,2) \times \mathbb{R} \times \mathbb{S}^{n-1}$ and $M = \mathbb{R} \times \mathbb{S}^{n-1}$ and recall that the metric $\bar{g} = \rho^2 g^+$ is given by

$$\bar{g} = d\rho^2 + \left(1 + \tfrac{\rho^2}{4}\right)^2 dt^2 + \left(1 - \tfrac{\rho^2}{4}\right)^2 g_{\mathbb{S}^{n-1}}, \quad \text{and} \quad g_0 = \bar{g}|_M = dt^2 + g_{\mathbb{S}^{n-1}}. \tag{3.2.6}$$

Equality (3.2.1) shows that the metrics $|dx|^2$ and g_0 are conformally related and therefore using (3.1.6), we can write any conformal change of metric on M as

$$g_v := w^{\frac{4}{n-2\gamma}} |dx|^2 = v^{\frac{4}{n-2\gamma}} g_0. \tag{3.2.7}$$

Our aim is to to find radial (in the variable $|x|$), positive solutions for (3.1.3) with an isolated singularity at the origin. Using g_0 as background metric on M, and writing the conformal change of metric in terms of v as (3.2.7), this is equivalent to look for positive solutions $v = v(t)$ to (3.1.9) with $0 < c_1 \leq v \leq c_2$, and we hope to find those that are periodic in t.

Finally, we check that the background metric g_0 given in (3.2.6) has constant fractional curvature $Q^{g_0}_\gamma \equiv c_{n,\gamma}$. This is true because of the definition of $c_{n,\gamma}$ given in Proposition 3.1.1, and the conformal equivalence given in (3.2.1). Thus, by construction, the trivial change $v_1 \equiv 1$ is a solution to (3.1.9).

3.3 The conformal fractional Laplacian on $\mathbb{R} \times \mathbb{S}^{n-1}$.

In this section we present the proof of Theorem 3.1.2, i.e, the calculation of the Fourier symbol for the conformal fractional Laplacian on $\mathbb{R} \times \mathbb{S}^{n-1}$. This computation is based on the analysis of the scattering equation given in (2.2.15)-(2.2.14) for the extension metric (3.2.4). We recall that the scattering operator is defined as

$$P^g_\gamma w = S(s)w = W_1|_{\rho=0}, \tag{3.3.1}$$

and $s = \frac{n}{2} + \gamma$.

We also remark that the proof of formula (3.1.12) is inspired in the calculation of the Fourier symbol for the conformal fractional Laplacian on the sphere $\mathbb{S}^n$ as we explained in Section 2.2.3. The method in both cases is, using spherical harmonics, to reduce the scattering equation (2.2.15) to an ODE that can be explicitly solved. Note that this idea of studying the scattering problem on certain Lorentzian models has been long used in Physics papers, but in general it is very hard to obtain explicit expressions for the solution and the majority of the existing results are numeric (see, for example, [55]).

For the calculations below it is better to use the hyperbolic metric given in the coordinates (3.2.3). Then the conformal infinity corresponds to the value $\{\sigma = +\infty\}$. The scattering equation (2.2.15) can be written in terms of the variables $\sigma \in (0, \infty)$, $t \in \mathbb{R}$ and $\theta \in \mathbb{S}^{n-1}$ as

$$\partial_{\sigma\sigma} U + Q(\sigma)\partial_\sigma U + \cosh^{-2}(\sigma)\partial_{tt} U + \sinh^{-2}(\sigma)\Delta_{\mathbb{S}^{n-1}} U + \left(\tfrac{n^2}{4} - \gamma^2\right) U = 0, \tag{3.3.2}$$

where $U = U(\sigma, t, \theta)$, and

$$Q(\sigma) = \frac{\partial_\sigma(\cosh\sigma \sinh^{n-1}\sigma)}{\cosh\sigma \sinh^{n-1}\sigma}.$$

After the change of variable

$$z = \tanh(\sigma), \tag{3.3.3}$$

equation (3.3.2) reads:

$$\begin{aligned}(1 - z^2)^2 \partial_{zz} U + \left(\tfrac{n-1}{z} - z\right)(1 - z^2)\partial_z U + (1 - z^2)\partial_{tt} U \\ + \left(\tfrac{1}{z^2} - 1\right)\Delta_{\mathbb{S}^{n-1}} U + \left(\tfrac{n^2}{4} - \gamma^2\right) U = 0.\end{aligned} \tag{3.3.4}$$

We compute the projection of equation (3.3.4) over each eigenspace of $\Delta_{\mathbb{S}^{n-1}}$. Given $k \in \mathbb{N}$, let $U_k(z, t)$ be the projection of U over the eigenspace $\langle E_k \rangle$ associated to the eigenvalue $\mu_k = -k(k + n - 2)$. Each U_k satisfies the following equation:

$$(1 - z^2)\partial_{zz} U_k + \left(\tfrac{n-1}{z} - z\right)\partial_z U_k + \partial_{tt} U_k + \mu_k \tfrac{1}{z^2} U_k + \tfrac{\frac{n^2}{4} - \gamma^2}{1 - z^2} U_k = 0. \tag{3.3.5}$$

Taking the Fourier transform (3.1.11) in the variable t we obtain

$$(1 - z^2)\partial_{zz} \widehat{U_k} + \left(\tfrac{n-1}{z} - z\right)\partial_z \widehat{U_k} + \left[\mu_k \tfrac{1}{z^2} + \tfrac{\frac{n^2}{4} - \gamma^2}{1 - z^2} - \xi^2\right] \widehat{U_k} = 0. \tag{3.3.6}$$

Fixed k and ξ, we know that

$$\widehat{U_k} = \widehat{w_k}(\xi)\varphi_k^\xi(z), \tag{3.3.7}$$

where $\varphi := \varphi_k^\xi(z)$ is the solution of the following ODE problem:

$$\begin{cases} (1 - z^2)\partial_{zz}\varphi + \left(\tfrac{n-1}{z} - z\right)\partial_z \varphi + \left(\tfrac{\mu_k}{z^2} + \tfrac{\frac{n^2}{4} - \gamma^2}{1 - z^2} - \xi^2\right)\varphi = 0, \\ \text{has the expansion (2.2.14) with } w \equiv 1 \text{ near the conformal infinity } z = 1, \\ \varphi \text{ is regular at } z = 0. \end{cases} \tag{3.3.8}$$

This ODE has only regular singular points z. The first equation in (3.3.8) can be explicitly solved,

$$\begin{aligned}\varphi(z) =&A(1-z^2)^{\frac{n}{4}-\frac{\gamma}{2}} z^{1-\frac{n}{2}+\sqrt{(\frac{n}{2}-1)^2-\mu_k}}\ {}_2\mathrm{F}_1(a,b;c;z^2)\\ &+B(1-z^2)^{\frac{n}{4}-\frac{\gamma}{2}} z^{1-\frac{n}{2}-\sqrt{(\frac{n}{2}-1)^2-\mu_k}}\ {}_2\mathrm{F}_1(\tilde{a},\tilde{b};\tilde{c};,z^2),\end{aligned} \tag{3.3.9}$$

for any real constants A, B, where

- $a = \frac{-\gamma}{2} + \frac{1}{2} + \frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2} + i\frac{\xi}{2}$,
- $b = \frac{-\gamma}{2} + \frac{1}{2} + \frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2} - i\frac{\xi}{2}$,
- $c = 1 + \sqrt{(\frac{n}{2}-1)^2-\mu_k}$,
- $\tilde{a} = \frac{-\gamma}{2} + \frac{1}{2} - \frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2} + i\frac{\xi}{2}$,
- $\tilde{b} = \frac{-\gamma}{2} + \frac{1}{2} - \frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2} - i\frac{\xi}{2}$,
- $\tilde{c} = 1 - \sqrt{(\frac{n}{2}-1)^2-\mu_k}$,

and ${}_2\mathrm{F}_1$ denotes the standard hypergeometric function introduced in Lemma 7.0.1 (Appendix 7). Note that we can write ξ instead of $|\xi|$ in the arguments of the hypergeometric functions because $a = \bar{b}$, $\tilde{a} = \bar{\tilde{b}}$ and property (7.0.6) given in the same Lemma 7.0.1.

The regularity at the origin $z = 0$ implies $B = 0$ in (3.3.9). Moreover, property (7.0.5) from Lemma 7.0.1 in the Appendix 7 makes it possible to rewrite φ as

$$\begin{aligned}\varphi(z) =&A\Big[\alpha(1+z)^{\frac{n}{4}-\frac{\gamma}{2}}(1-z)^{\frac{n}{4}-\frac{\gamma}{2}} z^{1-\frac{n}{2}+\sqrt{(\frac{n}{2}-1)^2-\mu_k}} \cdot {}_2\mathrm{F}_1(a,b;a+b-c+1;1-z^2)\\ &+\beta(1+z)^{\frac{n}{4}+\frac{\gamma}{2}}(1-z)^{\frac{n}{4}+\frac{\gamma}{2}} z^{1-\frac{n}{2}+\sqrt{(\frac{n}{2}-1)^2-\mu_k}} \cdot {}_2\mathrm{F}_1(c-a,c-b;c-a-b+1;1-z^2)\Big],\end{aligned} \tag{3.3.10}$$

where

$$\begin{aligned}\alpha &= \frac{\Gamma\left(1+\sqrt{(\frac{n}{2}-1)^2-\mu_k}\right)\Gamma(\gamma)}{\Gamma\left(\frac{1}{2}+\frac{\gamma}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}-i\frac{\xi}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\gamma}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}+i\frac{\xi}{2}\right)},\\ \beta &= \frac{\Gamma\left(1+\sqrt{(\frac{n}{2}-1)^2-\mu_k}\right)\Gamma(-\gamma)}{\Gamma\left(\frac{-\gamma}{2}+\frac{1}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}+i\frac{\xi}{2}\right)\Gamma\left(\frac{-\gamma}{2}+\frac{1}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}-i\frac{\xi}{2}\right)}.\end{aligned} \tag{3.3.11}$$

The constant coefficient A will be fixed from the second statement in (3.3.8). From the definition of the scattering operator in (3.3.1), φ must have the asymptotic expansion near $\rho = 0$

$$\varphi(\rho) = \rho^{n-s}(1 + ...) + \rho^{s}(\widehat{S^k}(s)1 + ...), \tag{3.3.12}$$

where $S^k(s)$ is the projection of the scattering operator $S(s)$ over the eigenspace $\langle E_k \rangle$.

We now use the changes of variable (3.3.3) and (3.2.5), obtaining

$$z = \tanh(\sigma) = \frac{4-\rho^2}{4+\rho^2} = 1 - \frac{1}{2}\rho^2 + \cdots . \tag{3.3.13}$$

Therefore, substituting (3.3.13) into (3.3.10) we can express φ as a function on ρ as follows

$$\begin{aligned}\varphi(\rho) \sim A \Big[& \alpha\rho^{\frac{n}{2}-\gamma} \;\; {}_2\mathrm{F}_1(a, b; a+b-c+1; \rho^2) \\ & + \beta\rho^{\frac{n}{2}+\gamma} \;\; {}_2\mathrm{F}_1(c-a, c-b; c-a-b+1; \rho^2) \Big], \quad \text{as } \rho \to 0.\end{aligned}$$

Using property (7.0.1) from Lemma 7.0.1 in the Appendix 7, we have that near the conformal infinity,

$$\varphi(\rho) \simeq A \left[\alpha\rho^{\frac{n}{2}-\gamma} + \beta\rho^{\frac{n}{2}+\gamma} + \ldots \right]. \tag{3.3.14}$$

Comparing (3.3.14) with the expansion of φ given in (3.3.12), we have

$$A = \alpha^{-1}, \tag{3.3.15}$$

and

$$\widehat{S^k}(s) = \beta\alpha^{-1}. \tag{3.3.16}$$

Recalling the definition of the conformal fractional Laplacian given in (2.2.17), and taking into account (3.3.7), we can assert that the Fourier symbol $\Theta^k_\gamma(\xi)$ for the projection P^k_γ of the conformal fractional Laplacian $P^{g_0}_\gamma$ satisfies

$$\Theta^k_\gamma(\xi) = \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} 2^{2\gamma} \widehat{S^k}(s).$$

From here we can calculate the value of this symbol and obtain (3.1.12); just take (3.3.16) into account and property (7.0.8) from Lemma 7.0.2. This completes the proof of Theorem 3.1.2.

Remark 3.3.1. When $\gamma = m$, an integer, we recover the principal symbol for the GJMS operators $P^{g_0}_m$. Indeed, from Theorem 3.1.2 we have that for any dimension $n > 2m$, the

Fourier symbol of $P_m^{g_0}$ is given by

$$\begin{aligned}
\Theta_m^k(\xi) &= 2^{2m} \frac{|\Gamma(\frac{1}{2}+\frac{m}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}+\frac{\xi}{2}i)|^2}{|\Gamma(\frac{1}{2}-\frac{m}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}+\frac{\xi}{2}i)|^2} \\
&= 2^{2m} \prod_{j=1}^{m} \left(\frac{[4(m-j)-m+1+\sqrt{(\frac{n}{2}-1)^2+k(k+n-1)}]^2}{4} + \frac{\xi^2}{4} \right) \\
&= \Psi(m,n,k,\xi,\xi^2,...,\xi^{2m-1}) + \xi^{2m},
\end{aligned}$$

where we have used the property (7.0.9) of the Gamma function given in Lemma 7.0.2. Note that Ψ is a polynomial function on ξ of degree less than $2m$.

For instance, for the classical case $m=1$,

$$\Theta_1^k(\xi) = \xi^2 + \tfrac{(n-2)^2}{4} - \mu_k, \quad k=0,1,\ldots,$$

so we recover the usual conformal Laplacian $P_1^{g_0}$ given, in Fourier decomposition, by

$$P_1^k(v) = -\ddot{v} + \left[\tfrac{(n-2)^2}{4} - \mu_k\right] v, \quad k=0,1,\ldots.$$

Note that P_1^0 is precisely the operator appearing in (2.4.4) for radial functions $v=v(t)$.

This proof also allows us to explicitly calculate the special defining function ρ^* from Theorem 2.2.21:

Corolary 3.3.2. *We have*

$$(\rho^*)^{n-s} = \alpha^{-1} \left(\tfrac{4\rho}{4+\rho^2}\right)^{\frac{n}{2}-\gamma} {}_2\mathrm{F}_1\left(\tfrac{n}{4}-\tfrac{\gamma}{2}, \tfrac{n}{4}-\tfrac{\gamma}{2}; \tfrac{n}{2}, \left(\tfrac{4-\rho^2}{4+\rho^2}\right)^2\right),$$

where α is the constant from (3.3.11). As a consequence, $\rho^* \in (0,\rho_0^*)$ where we have defined $(\rho_0^*)^{n-s} = \alpha^{-1}$.

Proof. From the proof of Theorem 2.2.21, which corresponds to Lemma 4.5 in [43], we know that

$$\rho^* = (\varphi_0^0)^{\frac{1}{n-s}}(z),$$

where φ is the solution of (3.3.8). Thus from formula (3.3.9) for $B=0$ and the relation between z and ρ from (3.3.13) we arrive at the desired conclusion. The behavior when $\rho \to 2$ can be calculated directly from (3.3.9) and, as a consequence, $(\rho_0^*)^{n-s} = \varphi(0) = \alpha^{-1}$. □

We end this section with a remark on the classical fractional Hardy inequality. On Euclidean space $(\mathbb{R}^n, |dx|^2)$, it is well known that, for all $w \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ and $\gamma \in \left(0, \frac{n}{2}\right)$,

$$\begin{aligned}
c_H \int_{\mathbb{R}^n} \frac{|w|^2}{|x|^{2\gamma}}\,dx &\leq \int_{\mathbb{R}^n} |\xi|^{2\gamma} |\hat{w}(\xi)|^2\,d\xi \\
&= \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\gamma}{2}} w|^2\,dx = \int_{\mathbb{R}^n} w(-\Delta)^\gamma w\,dx.
\end{aligned} \tag{3.3.17}$$

Moreover, the constant c_H is sharp (although it is not achieved) and its value is given by

$$c_H = c_{n,\gamma},$$

which is the constant in Proposition 3.1.1. This is not a coincidence, since the functions that are used in the proof of the sharpness statement are suitable approximations of (3.1.10). This constant was first calculated in [106], but there have been many references [173, 21, 92], for instance.

A natural geometric context for the fractional Hardy inequality is obtained by taking g_0 as a background metric, and using the changes (3.1.6) and (3.1.7). Indeed, using the conformal relation given by expression (3.1.8), we conclude that (3.3.17) is equivalent to the following:

$$c_{n,\gamma}\int_{\mathbb{R}\times\mathbb{S}^{n-1}} v^2\, dvol_{g_0} \leq \int_{\mathbb{R}\times\mathbb{S}^{n-1}} v(P_\gamma^{g_0}v)\, dvol_{g_0}, \tag{3.3.18}$$

for every $v \in \mathcal{C}_0^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$.

3.4 ODE-type analysis

In this section we fix $\gamma \in (0,1)$. As we have explained, the fractional Yamabe problem with an isolated singularity at the origin is equivalent to the extension problem (3.1.3). We look for radial solutions of the form (3.1.6). Based on our previous study, it is equivalent to consider solutions $V = V(t,\rho)$ of the extension problem (3.1.14), for the metric (3.1.13). In this section we perform an ODE-type analysis for the PDE problem (3.1.14).

Firstly we calculate

$$\begin{aligned}\operatorname{div}_{\bar g}(\rho^a\nabla_{\bar g}V) =& \sum_{i,j}\frac{1}{\sqrt{|\bar g|}}\partial_i(\bar g^{ij}\rho^a\sqrt{|\bar g|}\partial_j V)\\ =& \tfrac{1}{e(\rho)}\partial_\rho\left(\rho^a e(\rho)\partial_\rho V\right) + \tfrac{\rho^a}{(1+\frac{\rho^2}{4})^2}\partial_{tt}V + \tfrac{\rho^a}{(1-\frac{\rho^2}{4})^2}\Delta_{\mathbb{S}^{n-1}}V,\end{aligned} \tag{3.4.1}$$

where

$$e(\rho) = \left(1+\tfrac{\rho^2}{4}\right)\left(1-\tfrac{\rho^2}{4}\right)^{n-1}.$$

Using the expression given in (2.2.35),

$$E(\rho) = \tfrac{n-1+a}{4}\rho^a\frac{n-2+n\frac{\rho^2}{4}}{\left(1+\frac{\rho^2}{4}\right)\left(1-\frac{\rho^2}{4}\right)}. \tag{3.4.2}$$

Remark 3.4.1. Let V be the (unique) solution of (3.1.14). If v does not depend on the spherical variable $\theta \in \mathbb{S}^{n-1}$, then V does not either. Analogously, if v is independent on t and θ, then V is just a function of ρ. The proof is a straightforward computation using that the variables in (3.4.1) are separated.

As a consequence of the previous remark, it is natural to look for solutions V of (3.1.14) that only depend on ρ and t, i.e. solutions of

$$\begin{cases} -\dfrac{1}{e(\rho)}\partial_\rho\left(\rho^a(e(\rho)\partial_\rho V\right) - \dfrac{\rho^a}{(1+\frac{\rho^2}{4})^2}\partial_{tt}V + E(\rho)V = 0 \ \text{ for } \rho\in(0,2), t\in\mathbb{R}, \\ V = v \ \text{ on } \{\rho=0\}, \\ -\tilde{d}_\gamma \lim_{\rho\to 0}\rho^a\partial_\rho V = c_{n,\gamma}v^{\frac{n+2\gamma}{n-2\gamma}} \ \text{ on } \{\rho=0\}. \end{cases} \tag{3.4.3}$$

Now we take the special defining function ρ^* given in Theorem 2.2.21, whose explicit expression is given in Corollary 3.3.2. Then we can rewrite the original problem (3.1.14) in g^*, defined on the extension $X^*=\{\rho\in(0,\rho_0^*), t\in\mathbb{R}, \theta\in\mathbb{S}^{n-1}\}$, as

$$\begin{cases} -\operatorname{div}_{g^*}((\rho^*)^a\nabla_{g^*}V) = 0 \ \text{ in } (X^*,g^*), \\ V = v \ \text{ on } \{\rho^*=0\}, \\ -\tilde{d}_\gamma \lim_{\rho^*\to 0}(\rho^*)^a\partial_{\rho^*}V + c_{n,\gamma}v = c_{n,\gamma}v^{\frac{n+2\gamma}{n-2\gamma}} \ \text{ on } \{\rho^*=0\}, \end{cases} \tag{3.4.4}$$

where $g^*=\frac{(\rho^*)^2}{\rho^2}\bar{g}$, for $\rho^*=\rho^*(\rho)$.

Note that Proposition 3.1.1 calculates the value $Q_\gamma^{g_0}\equiv c_{n,\gamma}$. The advantage of (3.4.4) over the original (3.1.14) is that it is a pure divergence elliptic problem and has nicer analytical properties.

Next, if we look for radial solutions (that depend only on t and ρ^*), then the extension problem (3.4.4) reduces to:

$$\begin{cases} \dfrac{1}{e^*(\rho)}\partial_{\rho^*}\left((\rho^*)^a e^*(\rho)\partial_{\rho^*}V\right) + \dfrac{(\rho^*)^a}{\left(1+\frac{\rho^2}{4}\right)^2}\partial_{tt}V = 0 \text{ for } t\in\mathbb{R}, \rho^*\in(0,\rho_0^*), \\ v = V \text{ on } \{\rho^*=0\}, \\ -\tilde{d}_\gamma(\rho^*)^a\partial_{\rho^*}V + c_{n,\gamma}v = c_{n,\gamma}v^{\frac{n+2\gamma}{n-2\gamma}} \text{ on } \{\rho^*=0\}, \end{cases} \tag{3.4.5}$$

where

$$e^*(\rho) = \left(\frac{\rho^*}{\rho}\right)^2 e(\rho).$$

Summarizing, we will concentrate in problems (3.4.3) and (3.4.5). In some sense (3.4.5) is closer to the local equation (2.4.4) and shares many of its properties. For instance, it has two critical points: $v_0\equiv 0$ and $v_1\equiv 1$, since these are the only constant solutions of the boundary condition $v=v^{\frac{n+2\gamma}{n-2\gamma}}$ on $\rho^*=0$. Moreover, by uniqueness of the solution and Remark 3.4.1, the only critical points in the extension are simply $V_0\equiv 0$ and $V_1\equiv 1$.

Remark 3.4.2. The calculation of the critical points $v_0\equiv 0$ and $v_1\equiv 1$ also holds for any $\gamma\in\left(0,\frac{n}{2}\right)$, since the corresponding extension problem shares many similarities with (3.4.5) (c.f. [43, 40, 174]).

The linearization at $V_1\equiv 1$ will be considered in Section 3.6.

3.4.1 A conserved Hamiltonian

Here we give the proof of Theorem 3.1.3. The idea comes from [31], where they consider layer solutions for semilinear equations with fractional Laplacian and a double-well potential. Multiply the first equation in (3.4.3) by $e(\rho)\partial_t V$, and integrate with respect to $\rho \in (0,2)$, obtaining

$$-\int_0^2 \partial_\rho\left(\rho^a e(\rho)\partial_\rho V\right)\partial_t V\,d\rho - \int_0^2 \rho^a e_1(\rho)\partial_{tt}V\partial_t V\,d\rho + \int_0^2 \rho^a e_2(\rho)V\partial_t V\,d\rho = 0,$$

where we have defined e, e_1, e_2 as in (3.1.16). We realize that $\partial_{tt}V\partial_t V = \frac{1}{2}\partial_t((\partial_t V)^2)$ and $V\partial_t V = \frac{1}{2}\partial_t(V^2)$, thus integrating by parts in the first term above we get

$$\begin{aligned}&\int_0^2 \rho^a e(\rho)\partial_\rho V\partial_{t\rho}V\,d\rho + \left(\rho^a e(\rho)\partial_\rho V\partial_t V\right)|_{\rho=0}\\ &-\partial_t\left(\tfrac{1}{2}\int_0^2 \rho^a e_1(\rho)(\partial_t V)^2\,d\rho\right) + \partial_t\left(\tfrac{1}{2}\int_0^2 \rho^a e_2(\rho)V^2\,d\rho\right) = 0.\end{aligned}$$

Here we have used the regularity of V at $\rho=2$. Again we note that $\partial_\rho V\partial_{t\rho}V = \frac{1}{2}\partial_t((\partial_\rho V)^2)$ and using the boundary condition, i.e., the third equation in (3.4.3), we have

$$\begin{aligned}\tfrac{1}{2}\partial_t\left(\int_0^2 \rho^a e(\rho)(\partial_\rho V)^2\,d\rho\right) - \tfrac{1}{2}\partial_t\left(\int_0^2 \rho^a e_1(\rho)(\partial_t V)^2\,d\rho\right)&\\ + \tfrac{1}{2}\partial_t\left(\int_0^2 \rho^a e_2(\rho)V^2\,d\rho\right) = \frac{c_{n,\gamma}}{\tilde{d}\gamma}v^{\frac{n+2\gamma}{n-2\gamma}}\partial_t v.&\end{aligned} \tag{3.4.6}$$

Define

$$G(v) = C_{n,\gamma}v^{\frac{2n}{n-2\gamma}},$$

where the constant is defined in (3.1.17). In this way, we have from (3.4.6) that

$$\tfrac{1}{2}\partial_t\int_0^2 \left\{\rho^a e(\rho)(\partial_\rho V)^2 - \rho^a e_1(\rho)(\partial_t V)^2 + \rho^a e_2(\rho)V^2\right\}\,d\rho - \partial_t(G(v)) = 0.$$

So we can conclude that the Hamiltonian

$$-H_\gamma(t) := \tfrac{1}{2}\int_0^2 \rho^a\left\{e(\rho)(\partial_\rho V)^2 - e_1(\rho)(\partial_t V)^2 + e_2(\rho)V^2\right\}\,d\rho - G(v),$$

is constant respect to t. This concludes the proof of Theorem 3.1.3.

Remark 3.4.3. One can rewrite the Hamiltonian in terms of the defining function ρ^*. For this, we may follow similar computations as above but starting with equation (3.4.5). Indeed, let V be a solution of (3.4.4), then the new Hamiltonian quantity

$$H^*_\gamma(t) := \frac{c_{n,\gamma}}{\tilde{d}_\gamma}\left(\tfrac{n-2\gamma}{2n}v^{\frac{2n}{n-2\gamma}} - \tfrac{1}{2}v^2\right) + \tfrac{1}{2}\int_0^{\rho_0^*}(\rho^*)^a\left\{e_1^*(\rho)(\partial_t V)^2 - e^*(\rho)(\partial_{\rho^*}V)^2\right\}\,d\rho^* \tag{3.4.7}$$

is constant respect to t. Here

$$e^*(\rho) = \left(\frac{\rho^*}{\rho}\right)^2, \quad e_1^*(\rho) = \left(\frac{\rho^*}{\rho}\right)^2 e_1(\rho).$$

This quantity H_γ^* is the natural generalization of (2.4.5).

Now we observe that in the local case, the Hamiltonian (2.4.5) is a convex function in the domain we are interested, thus its level sets are well defined closed trajectories around the equilibrium $v_1 \equiv 1$. We would like to have the analogous result for the Hamiltonian quantity H_γ^* from (3.4.7). This is a very interesting open question that we conjecture to be true. In any case, the second variation for H_γ^* near this equilibrium is:

$$\frac{d^2}{d\epsilon^2}\bigg|_{\epsilon=0} H_\gamma^*(V_1+\epsilon V) = \tfrac{c_{n,\gamma}}{\tilde{d}_\gamma}\tfrac{4\gamma}{n-2\gamma}v^2 + \tfrac{1}{2}\int_0^{\rho_0^*} (\rho^*)^a \tfrac{(\rho^*)^2}{\rho^2}\left\{e_1(\rho)(\partial_t V)^2 - e(\rho)(\partial_{\rho^*} V)^2\right\} d\rho^*.$$

3.5 The homoclinic solution

For this section we will take $\gamma \in \left(0, \frac{n}{2}\right)$, since it does not depend on the extension problem (3.1.3). It is clear that the standard bubble (3.1.5) is a solution of equation (3.1.1) that has a removable singularity at the origin. Note that, because of our choice of the constant $c_{n,\gamma}$, we need to normalize it by a positive multiplicative constant. We prove here that, on the boundary phase portrait, the equilibrium $v_1 \equiv 1$ stays always bounded by this homoclinic solution at the boundary. More precisely:

Proposition 3.5.1. *The positive function*

$$v_\infty(t) = C(\cosh t)^{-\frac{n-2\gamma}{2}}, \qquad \text{with} \quad C = \left(c_{n,\gamma}\frac{\Gamma(\frac{n}{2}-\gamma)}{\Gamma(\frac{n}{2}+\gamma)}\right)^{-\frac{n-2\gamma}{4\gamma}} > 1 \equiv v_1, \tag{3.5.1}$$

is a smooth solution of the fractional Yamabe problem (3.1.9). The value of $c_{n,\gamma}$ is given in Proposition 3.1.1.

Proof. The canonical metric on the sphere, rescaled by a constant, maybe written as

$$g_C = C^{\frac{4}{n-2\gamma}} g_{\mathbb{S}^n} = [C(\cosh t)^{-\frac{n-2\gamma}{2}}]^{\frac{4}{n-2\gamma}} g_0.$$

We choose C such that the fractional curvature of the standard sphere is normalized to

$$Q_\gamma^{g_C} \equiv c_{n,\gamma}. \tag{3.5.2}$$

Now we use the conformal property (2.2.20) for the operator $P_\gamma^{g_{\mathbb{S}^n}}$:

$$P_\gamma^{g_{\mathbb{S}^n}}(C) = C^{\frac{n+2\gamma}{n-2\gamma}} Q_\gamma^{g_C}. \tag{3.5.3}$$

One checks that the fractional curvature is homogeneous of order γ under rescaling of the metric. Indeed, because of (3.5.3) and the linearity of the operator P_γ

$$Q_\gamma^{g_C} = C^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^{g_{\mathbb{S}^n}}(C) = C^{-\frac{(n+2\gamma)}{n-2\gamma}+1} P_\gamma^{g_{\mathbb{S}^n}}(1) = C^{-\frac{4\gamma}{n-2\gamma}} Q_\gamma^{g_{\mathbb{S}^n}}. \tag{3.5.4}$$

Comparing equalities (3.5.2) and (3.5.4), together with the value of the curvature on the standard sphere (2.2.28) given in the previous Chapter 2, we find the precise value of C as claimed in (3.5.1).

Next, let us check that the value of the constant C is larger than one. Because of Proposition 3.1.1 we have to test that

$$2^{2\gamma}\left(\frac{\Gamma(\frac{1}{2}(\frac{n}{2}+\gamma))}{\Gamma(\frac{1}{2}(\frac{n}{2}-\gamma))}\right)^2\frac{\Gamma(\frac{n}{2}-\gamma)}{\Gamma(\frac{n}{2}+\gamma)}<1.$$

Using the property (7.0.10) of the Gamma function, given in Lemma 7.0.2, we only need to verify that

$$X(n,\gamma):=\frac{\Gamma(\frac{1}{2}(\frac{n}{2}+\gamma))}{\Gamma(\frac{1}{2}(\frac{n}{2}-\gamma))}\frac{\Gamma(\frac{1}{2}(\frac{n}{2}-\gamma)+\frac{1}{2})}{\Gamma(\frac{1}{2}(\frac{n}{2}+\gamma)+\frac{1}{2})}<1.$$

Thanks to Lemma 3.5.2 below, it is enough to see that

$$X(n,1)=1-\tfrac{2}{n}\leq 1\quad \forall n,$$

which holds trivially. □

Lemma 3.5.2. *The function $X(n,\gamma)$ defined as follows*

$$X(n,\gamma):=\frac{\Gamma(\frac{1}{2}(\frac{n}{2}+\gamma))}{\Gamma(\frac{1}{2}(\frac{n}{2}-\gamma))}\frac{\Gamma(\frac{1}{2}(\frac{n}{2}-\gamma)+\frac{1}{2})}{\Gamma(\frac{1}{2}(\frac{n}{2}+\gamma)+\frac{1}{2})},$$

is (strictly) increasing in n, and decreasing in γ.

Proof. If we denote $\psi(z)$ the Digamma function from Lemma 7.0.2, we can use the expansion (7.0.11) to study the growth of the function $X(n,\gamma)$ with respect to n and γ. First,

$$\begin{aligned}\frac{\partial}{\partial n}(\log X(n,\gamma))&=\tfrac{1}{4}\left(\psi(\tfrac{n}{4}+\tfrac{\gamma}{2})+\psi(\tfrac{n}{4}-\tfrac{\gamma}{2}+\tfrac{1}{2})-\psi(\tfrac{n}{4}-\tfrac{\gamma}{2})-\psi(\tfrac{n}{4}+\tfrac{\gamma}{2}+\tfrac{1}{2})\right)\\&=\tfrac{\gamma}{4}\sum_{m=0}^{\infty}\frac{m+\frac{n}{4}+\frac{1}{4}}{\left[(m+\frac{n}{4})^2-\frac{\gamma^2}{4}\right]\left[(m+\frac{n}{4}+\frac{1}{2})^2-\frac{\gamma^2}{4}\right]}>0.\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial \gamma}(\log X(n,\gamma))&=\tfrac{1}{2}\left(\psi(\tfrac{n}{4}+\tfrac{\gamma}{2})-\psi(\tfrac{n}{4}-\tfrac{\gamma}{2}+\tfrac{1}{2})+\psi(\tfrac{n}{4}-\tfrac{\gamma}{2})-\psi(\tfrac{n}{4}+\tfrac{\gamma}{2}+\tfrac{1}{2})\right)\\&=-\tfrac{1}{2}\sum_{m=0}^{\infty}\left[\frac{\left(m+\frac{n}{4}+\frac{1}{2}\right)(m+\frac{n}{4})+\frac{\gamma^2}{4}}{\left((m+\frac{n}{4}+\frac{1}{2})^2-\frac{\gamma^2}{4}\right)\left((m+\frac{n}{4})^2-\frac{\gamma^2}{4}\right)}\right]<0.\end{aligned}$$

□

3.6 Linear analysis

Let us say a few words about stability. Let v_* be a solution of (3.1.9). The corresponding linearized equation is

$$P_\gamma^{g_0} v = c_{n,\gamma} \tfrac{n+2\gamma}{n-2\gamma}\, v_*^{\frac{4\gamma}{n-2\gamma}} v.$$

We say that v_* is a stable solution of (3.1.9) if

$$\int_M v(P_\gamma^{g_0} v)\, dvol_{g_0} - c_{n,\gamma} \tfrac{n+2\gamma}{n-2\gamma} \int_M v_*^{\frac{4\gamma}{n-2\gamma}} v^2\, dvol_{g_0} \geq 0, \quad \text{for all} \quad v \in \mathcal{C}_0^\infty(M). \tag{3.6.1}$$

We observe here that the equilibrium $v_1 \equiv 1$ is not a stable solution for (3.1.9) just by comparing the constant appearing in (3.6.1) and in the Hardy inequality (3.3.18). In addition, one easily checks that the equilibrium solution $v_0 \equiv 0$ is stable.

But it is more interesting to look at the explicit solution v_∞ given in (3.5.1). It follows from the Hardy inequality (3.3.18) that this explicit solution is not stable. The kernel of the linearization at v_∞ is calculated in [58], where they show that, although non-trivial, is non-degenerate, i.e., is generated by translations and dilations of the standard bubble.

Let us look more closely at the spectrum of the operator $P_\gamma^{g_0}$. It is well known that $P_\gamma^{g_0}$ is self-adjoint ([102]), and then we can compute its first eigenvalue through the Rayleigh quotient. Thus we minimize

$$\inf_{v \in \mathcal{C}_0^\infty(M)} \frac{\int_M v P_\gamma^{g_0} v\, dvol_{g_0}}{\int_M v^2\, dvol_{g_0}},$$

where $M = \mathbb{R} \times \mathbb{S}^{n-1}$. We can apply Theorem 4.2 and Corollary 4.3 in [97] (or the Hardy inequality (3.3.18)) to conclude that $P_\gamma^{g_0}$ is positive-definite. Moreover, the first eigenspace is of dimension one.

Now we consider the linear analysis around the equilibrium solution $v_1 \equiv 1$. In order to motivate our results, let us explain what happens in the local case $\gamma = 1$, explained in the previous Chapter 2 in Section 2.4, for the linearization (see [140, 144, 112]). In these papers the authors actually characterize the spectrum for the linearization of the equation

$$P_1^{g_0} v = \tfrac{(n-2)^2}{4} v^{\frac{n+2}{n-2}},$$

given by (after projection over each eigenspace $\langle E_k \rangle$, $k = 0, 1, \ldots$)

$$-\ddot{v} - [n - 2 + \mu_k] v = 0.$$

Note that this equation has periodic solutions only for $k = 0$, of period $L_0^1 = \frac{2\pi}{\sqrt{\lambda^0}}$ for $\lambda^0 = n - 2$. Thus we recover (2.4.6). For the rest of $k = 1, \ldots,$ the corresponding $\lambda^k = n - 2 + \mu_k < 0$, so we do not get periodic solutions.

The linearization of equation (3.1.9) around the equilibrium $v_1 \equiv 1$ is given by

$$P_\gamma^{g_0} v = c_{n,\gamma} \tfrac{n+2\gamma}{n-2\gamma} v \tag{3.6.2}$$

Here we will calculate the period of solutions for this linearized problem (for the projection $k=0$), as stated in Theorem 3.1.4, by the method of separation of variables. We also conjecture that there are not periodic solutions for the linearized problem (3.1.18) for the rest of $k = 1, ...,$ as it happens in the classical clase.

Therefore, we consider the projection of equation (3.3.2) over each eigenspace $\langle E_k \rangle$, $k = 0, 1, \ldots$. Let

$$U_k(z,t) = T(t)Z(z),$$

be a solution of (3.3.5). Then

$$(1-z^2)\frac{Z''(z)}{Z(z)} + \left(\tfrac{n-1}{z} - z\right)\frac{Z'(z)}{Z(z)} + \frac{\frac{n^2}{4} - \gamma^2}{1-z^2} + \frac{\mu_k}{z^2} = -\frac{T''(t)}{T(t)} = \lambda^k,$$

for a constant $\lambda^k := \lambda^k(\gamma) \in \mathbb{R}$. We are only interested in the case $\lambda > 0$, which is the one that leads to periodic solutions in the variable t. The period would be calculated from $L_0^{\gamma,k} := \frac{2\pi}{\sqrt{\lambda^k}}$.

Note that the equation for $Z(z)$ is simply (3.3.6) with ξ^2 replaced by λ^k. From the discussion in Section 3.3, in particular (3.3.10), (3.3.15) and (3.3.16) we have that

$$\begin{aligned} Z(z) =& (1+z)^{\frac{n}{4}-\frac{\gamma}{2}}(1-z)^{\frac{n}{4}-\frac{\gamma}{2}} z^{1-\frac{n}{2}+\sqrt{(\frac{n}{2}-1)^2-\mu_k}} \;{}_2\mathrm{F}_1(a,b;a+b-c+1;1-z^2) \\ &+ \kappa(1+z)^{\frac{n}{4}+\frac{\gamma}{2}}(1-z)^{\frac{n}{4}+\frac{\gamma}{2}} z^{1-\frac{n}{2}-\sqrt{(\frac{n}{2}-1)^2-\mu_k}} \;{}_2\mathrm{F}_1(c-a,c-b;c-a-b+1;1-z^2), \end{aligned}$$

where

$$\begin{aligned} a &= -\tfrac{\gamma}{2} + \tfrac{1}{2} + \frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2} + i\frac{\sqrt{\lambda^k}}{2} \\ b &= -\tfrac{\gamma}{2} + \tfrac{1}{2} + \frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2} - i\frac{\sqrt{\lambda^k}}{2}, \\ c &= 1 + \sqrt{(\tfrac{n}{2}-1)^2 - \mu_k}, \\ \kappa &= \frac{\Gamma(-\gamma)\left|\Gamma\left(\frac{1}{2}+\frac{\gamma}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}+i\frac{\sqrt{\lambda^k}}{2}\right)\right|^2}{\Gamma(\gamma)\left|\Gamma\left(\frac{1}{2}-\frac{\gamma}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}+i\frac{\sqrt{\lambda^k}}{2}\right)\right|^2}. \end{aligned}$$

We use the change of variable (3.3.13) to analyze the asymptotic behavior of Z near the conformal infinity $\rho = 0$

$$Z \sim \rho^{\frac{n}{2}-\gamma} + \kappa\rho^{\frac{n}{2}+\gamma}.$$

From the definition of the scattering operator (2.2.14), (2.2.16), and the definition of the conformal fractional Laplacian we have that

$$P_\gamma^k v_k = 2^{2\gamma} \frac{\left|\Gamma(\frac{1}{2}+\frac{\gamma}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}+\frac{\sqrt{\lambda^k}}{2}i)\right|^2}{\left|\Gamma(\frac{1}{2}-\frac{\gamma}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}+\frac{\sqrt{\lambda^k}}{2}i)\right|^2} v_k.$$

Imposing the boundary condition (3.6.2) and the value of $c_{n,\gamma}$ given in (3.1.1), the unknown λ^k must be a solution of

$$\frac{\left|\Gamma(\frac{1}{2}+\frac{\gamma}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}+\frac{\sqrt{\lambda^k}}{2}i)\right|^2}{\left|\Gamma(\frac{1}{2}-\frac{\gamma}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}+\frac{\sqrt{\lambda^k}}{2}i)\right|^2} = \frac{n+2\gamma}{n-2\gamma} \frac{\left|\Gamma\left(\frac{1}{2}\left(\frac{n}{2}+\gamma\right)\right)\right|^2}{\left|\Gamma\left(\frac{1}{2}\left(\frac{n}{2}-\gamma\right)\right)\right|^2}. \tag{3.6.3}$$

Note that for the canonical projection $k=0$, equality (3.6.3) simplifies to

$$\frac{\left|\Gamma(\frac{n}{4}+\frac{\gamma}{2}+\frac{\sqrt{\lambda^0}}{2}i)\right|^2}{\left|\Gamma(\frac{n}{4}-\frac{\gamma}{2}+\frac{\sqrt{\lambda^0}}{2}i)\right|^2} = \frac{n+2\gamma}{n-2\gamma} \frac{\left|\Gamma\left(\frac{1}{2}\left(\frac{n}{2}+\gamma\right)\right)\right|^2}{\left|\Gamma\left(\frac{1}{2}\left(\frac{n}{2}-\gamma\right)\right)\right|^2}. \tag{3.6.4}$$

This equation (3.6.4) lets us recover the value of λ^0 for the classical case $\gamma=1$. Indeed, using property (7.0.9) we get $\lambda^0 = n-2$ and we recover (2.4.6).

Going back to equation (3.6.3) we can assert that the value of λ^k can not be zero and it is unique for each k. Indeed if $\lambda = 0$ we get a contradiction, and if $\lambda > 0$ we may proceed as follows. Define

$$F(\beta) = \frac{\frac{|\Gamma(\alpha_k+\beta i)|^2}{|\Gamma(\tilde{\alpha}_k+\beta i)|^2}}{\frac{n+2\gamma}{n-2\gamma}\frac{\left|\Gamma\left(\frac{1}{2}\left(\frac{n}{2}+\gamma\right)\right)\right|^2}{\left|\Gamma\left(\frac{1}{2}\left(\frac{n}{2}-\gamma\right)\right)\right|^2}},$$

where

$$\alpha_k = \frac{1}{2}+\frac{\gamma}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}, \quad \tilde{\alpha}_k = \frac{1}{2}-\frac{\gamma}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2} \quad \text{and} \quad \beta = \frac{\sqrt{\lambda^k}}{2}.$$

Note that equation (3.6.3) is written as $F(\beta)=1$, for some $\beta>0$. We derive this expression with respect to β,

$$(\log F(\beta))' = 2\,\Im[\psi(\tilde{\alpha}_k+\beta i) - \psi(\alpha_k+\beta i)].$$

Here $\Im$ represents the imaginary part of a complex number and $\psi(z)$ the Digamma function from Lemma 7.0.2. We can use the expansion (7.0.11) to arrive at

$$(\log F(\beta))' = c\sum_{m=0}^{\infty} \frac{\gamma\beta\left(2m+1+\sqrt{(\frac{n}{2}-1)^2-\mu_k}\right)}{\left[\left(m+\frac{1}{2}+\frac{\sqrt{(\frac{n}{2}-1)^2-\mu_k}}{2}\right)^2-\beta^2-\frac{\gamma^2}{4}\right]^2+\left[\left(2m+1+\sqrt{(\frac{n}{2}-1)^2-\mu_k}\right)\beta\right]^2},$$

for some positive constant c. Therefore $F(\beta)$ is an strictly increasing function of β.

Next, note that

$$\lim_{\beta \to +\infty} F(\beta) = +\infty,$$

for all $k = 0, 1, \ldots$. This follows easily writing

$$\frac{n+2\gamma}{n-2\gamma} F(\beta) = \frac{B\left(\alpha_k + \beta i, \frac{n}{4} - \frac{\gamma}{2}\right)}{B\left(\tilde{\alpha}_k + \beta i, \frac{n}{4} + \frac{\gamma}{2}\right)},$$

and the asymptotic behavior for the Beta function (7.0.12) from Lemma 7.0.2.

Now we look at the projection $k = 0$. One immediately calculates

$$F(0) = \frac{n-2\gamma}{n+2\gamma} < 1,$$

so there exists (and it is unique) a solution $\lambda^0 = \lambda^0(\gamma) > 0$ for the equation $F(\beta) = 1$. From the proof one also gets that

$$\lim_{\gamma \to 1} \lambda^0(\gamma) = n - 2.$$

This concludes the proof of Theorem 3.1.4.

We believe that, as in the classical case $F(\beta) = 1$ does not have any positive solution for $k = 1, 2, \ldots$. This is a well supported conjecture that only depends on making more rigorous some numerical analysis. In order to motivate this conjecture, let us try to show that $f_k > 1$ for $k = 1, 2, \ldots$, where we have defined

$$F(0) = \frac{(n-2\gamma)|\Gamma(\alpha_k)|^2 \left|\Gamma\left(\frac{1}{2}\left(\frac{n}{2} - \gamma\right)\right)\right|^2}{(n+2\gamma)|\Gamma(\tilde{\alpha}_k)|^2 \left|\Gamma\left(\frac{1}{2}\left(\frac{n}{2} + \gamma\right)\right)\right|^2} =: f_k.$$

Using the same ideas as above, one checks that f_k is an increasing function of k, and it is enough to show that

$$f_1 = \frac{(n-2\gamma)\left|\Gamma\left(\frac{1}{2} + \frac{\gamma}{2} + \frac{n}{4}\right)\Gamma\left(\frac{n}{4} - \frac{\gamma}{2}\right)\right|^2}{(n+2\gamma)\left|\Gamma\left(\frac{1}{2} - \frac{\gamma}{2} + \frac{n}{4}\right)\Gamma\left(\frac{n}{4} + \frac{\gamma}{2}\right)\right|^2} = \frac{n-2\gamma}{n+2\gamma} X(n,\gamma)^{-2} > 1,$$

where $X(n,\gamma)$ is defined in Lemma 3.5.2. We have numerically observed that $f_1 = f_1(\gamma)$ is an increasing function in γ. Since for $\gamma = 0$ we already have that $f_1(0) = 1$, we would conclude that $f_k > f_1 \geq 1$, as desired.

Chapter 4

Delaunay-type singular solutions for the fractional Yamabe problem

Here we construct Delaunay-type solutions for the fractional Yamabe problem with an isolated singularity

$$(-\Delta)^\gamma w = c_{n,\gamma} w^{\frac{n+2\gamma}{n-2\gamma}}, w > 0 \text{ in } \mathbb{R}^n \backslash \{0\}.$$

We follow a variational approach, in which the key is the computation of the fractional Laplacian in polar coordinates.

4.1 Introduction and statement of the main result

As in the previous Chapter 3, we also consider here the problem of finding radial solutions for the fractional Yamabe problem in $\mathbb{R}^n$ with an isolated singularity at the origin (3.1.1). Fix $\gamma \in (0,1)$ and $n > 2\gamma$. We reformulate the problem into a variational one for the the periodic function v. The main difficulty is to compute the fractional Laplacian in polar coordinates.

Our approach does not use the extension problem (3.1.3). Instead we work directly with the nonlocal operator, after suitable Emden-Fowler transformation. For $\gamma \in (0,1)$ we know that the fractional Laplacian can be defined as a singular kernel as

$$(-\Delta)^\gamma w(x) = \kappa_{n,\gamma} \text{P.V.} \int_{\mathbb{R}^n} \frac{w(x) - w(x+y)}{|y|^{n+2\gamma}}\, dy,$$

where *P.V.* denotes the principal value, and the constant $\kappa_{n,\gamma}$ (see [120]) is given by

$$\kappa_{n,\gamma} = \pi^{-\frac{n}{2}} 2^{2\gamma} \frac{\Gamma\left(\frac{n}{2}+\gamma\right)}{\Gamma(1-\gamma)} \gamma.$$

After some more changes of variable, equation (3.1.1) will be written as

$$\mathscr{L}_\gamma v = c_{n,\gamma} v^\beta, v > 0, \tag{4.1.1}$$

where

$$\beta = \tfrac{n+2\gamma}{n-2\gamma}$$

is the critical exponent in dimension n and $\mathscr{L}_\gamma$ is the linear operator defined by

$$\mathscr{L}_\gamma v(t) = \kappa_{n,\gamma} P.V. \int_{-\infty}^{\infty} (v(t) - v(\tau)) K(t-\tau)\, d\tau + c_{n,\gamma} v(t),$$

for K a singular kernel which is precisely written in (4.2.6). The behaviour of K near the origin is the same as the kernel of the fractional Laplacian $(-\triangle)^\gamma$ in $\mathbb{R}$ and near infinity it presents an exponential decay. This kind of kernels corresponds to tempered stable process and they have been studied in [113] and [165], for instance.

If we take into account just periodic functions $v(t+L) = v(t)$, the operator $\mathscr{L}_\gamma$ can be rewritten as

$$\mathscr{L}_\gamma^L v(t) = \kappa_{n,\gamma} P.V. \int_0^L (v(t) - v(\tau)) K_L(t-\tau)\, d\tau + c_{n,\gamma} v(t), \tag{4.1.2}$$

where K_L is a periodic singular kernel that will be defined in (4.2.12). For periodic solutions, problem (4.1.1) is equivalent to finding a minimizer for the functional

$$\mathscr{F}_L(v) = \frac{\kappa_{n,\gamma} \int_0^L \int_0^L (v(t) - v(\tau))^2 K_L(t-\tau)\, dt\, d\tau + c_{n,\gamma} \int_0^L v(t)^2 dt}{\left(\int_0^L v(t)^{\beta+1} dt\right)^{\frac{2}{\beta+1}}}.$$

Note that a minimizer always exists as we can check in Lemma 4.4.1. The minimum value for the functional will be denoted by $c(L)$.

Our main result is the following:

Theorem 4.1.1. Let $n > 2 + 2\gamma$. There is a unique $L_0^\gamma > 0$ such that $c(L)$ is attained by a nonconstant minimizer when $L > L_0^\gamma$ and when $L \le L_0^\gamma$, $c(L)$ is attained by the constant only.

In the previous Chapter 3 we studied this fractional problem (3.1.1) from two different points of view. We carried out an ODE-type study and explain the geometrical interpretation of the problem. In addition, we gave some results towards the description of some kind of generalized phase portrait. For instance, we proved the existence of periodic radial solutions for the linearized equation around the equilibrium $v_1 \equiv 1$, with period L_0^γ. For the original non-linear problem we showed the existence of a Hamiltonian quantity conserved along trajectories, which suggests that the non-linear problem has periodic solutions too, for every period larger than this minimal period L_0^γ. Theorem 4.1.1 proves this conjecture.

The construction of Delaunay solutions allows for many further studies. For instance, as a consequence of our construction one obtains the non-uniqueness of the solutions for the fractional Yamabe equation in the positive curvature case, since it gives different conformal metrics on $\mathbb{S}^1(L) \times \mathbb{S}^{n-1}$ that have constant fractional curvature; as we announced in Section 2.3.2 in Chapter 2. This is well known in the scalar curvature case $\gamma = 1$, which is explained in Section 2.4 in Chapter 2. In addition, this fact gives some examples for the calculation of the total fractional scalar curvature functional, which maximizes the standard fractional Yamabe quotient across conformal classes.

From another point of view, Delaunay solutions can be used in gluing problems. Classical references are, for instance, [140, 143] for the scalar curvature, and [141, 142] for the construction of constant mean curvature surfaces with Delaunay ends. In the non-local case, we use Delaunay-type singularities to deal with the problem of constructing metrics of constant fractional curvature with prescribed isolated singularities (see Chapter 5 for more details).

This chapter will be structured as follows: in Section 2 we will introduce the problem. In particular we will recall some known results for the classical case and we will present the formulation of the problem through some properties of the singular kernel. In Section 3 we will show some technical results that we will need in the last Section; where we will use the variational method to prove the main result in this chapter, this is, Theorem 4.1.1.

4.2 Set up of the problem

4.2.1 Formulation of the problem.

We now consider the singular Yamabe problem

$$(-\Delta)^\gamma w = c_{n,\gamma} w^\beta \quad \text{in } \mathbb{R}^n\backslash\{0\},\ w>0 \tag{4.2.1}$$

for $\gamma \in (0,1)$, $n > 2\gamma$, β the critical exponent given by

$$\beta = \frac{n+2\gamma}{n-2\gamma}$$

and

$$(-\Delta)^\gamma w(x) = \kappa_{n,\gamma} \text{P.V.} \int_{\mathbb{R}^n} \frac{w(x)-w(x+y)}{|y|^{n+2\gamma}}\, dy,$$

where $P.V.$ denotes the principal value, and the constant $\kappa_{n,\gamma}$ (see [120]) is given by

$$\kappa_{n,\gamma} = \pi^{-\frac{n}{2}} 2^{2\gamma} \frac{\Gamma\left(\frac{n}{2}+\gamma\right)}{\Gamma(1-\gamma)} \gamma.$$

Because of (3.1.4) we only consider radially symmetric solutions of the form

$$w(x) = |x|^{-\frac{n-2\gamma}{2}} v(|x|),$$

where v is some function $0 < c_1 \leq v \leq c_2$. In radial coordinates ($r = |x|, \theta \in \mathbb{S}^{n-1}$ and $s = |y|, \sigma \in \mathbb{S}^{n-1}$), we can express the fractional Laplacian as

$$(-\Delta)^\gamma u = \kappa_{n,\gamma} P.V. \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{r^{-\frac{n-2\gamma}{2}} v(r) - s^{-\frac{n-2\gamma}{2}} v(s)}{|r^2+s^2-2rs\langle\theta,\sigma\rangle|^{\frac{n+2\gamma}{2}}} s^{n-1} d\sigma ds.$$

Inspired by the computations by Ferrari and Verbitsky in [88], we write $s = r\bar{s}$, so the radial function v can be expressed as

$$v(r) = (1 - \bar{s}^{-\frac{n-2\gamma}{2}})v(r) + \bar{s}^{-\frac{n-2\gamma}{2}}v(r).$$

Thus the equation (4.2.1) for v becomes

$$\kappa_{n,\gamma} P.V. \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{\bar{s}^{n-1-\frac{n-2\gamma}{2}}(v(r) - v(r\bar{s}))}{|1 + \bar{s}^2 - 2\bar{s}\langle\theta, \sigma\rangle|^{\frac{n+2\gamma}{2}}} d\sigma d\bar{s} + Av = c_{n,\gamma} v^\beta(r), \tag{4.2.2}$$

where

$$A = \kappa_{n,\gamma} P.V. \int_0^\infty \int_{\mathbb{S}^{n-1}} \frac{(1 - \bar{s}^{-\frac{n-2\gamma}{2}})\bar{s}^{n-1}}{|1 + \bar{s}^2 - 2\bar{s}\langle\theta, \sigma\rangle|^{\frac{n+2\gamma}{2}}} d\sigma d\bar{s}.$$

Remark 4.2.1. The constant A is strictly positive. Indeed, from (4.2.2) we have

$$A = c_{n,\gamma} > 0,$$

since $c_{n,\gamma}$ is normalized such that $v_1 \equiv 1$ is a solution for the singular Yamabe problem (see Proposition 3.1.1 in the previous Chapter 3).

Finally we do the Emden-Fowler changes of variable $r = e^t$ and $s = e^\tau$ in (4.2.2) to obtain

$$\mathscr{L}_\gamma v = c_{n,\gamma} v^\beta, \tag{4.2.3}$$

where the operator $\mathscr{L}_\gamma$ is defined as

$$\mathscr{L}_\gamma v = \kappa_{n,\gamma} P.V. \int_{-\infty}^\infty (v(t) - v(\tau))K(t - \tau)\, d\tau + c_{n,\gamma} v, \tag{4.2.4}$$

for a function $v = v(t)$ and the kernel K is given by

$$K(\xi) = 2^{-\frac{n+2\gamma}{2}} \int_{\mathbb{S}^{n-1}} \frac{1}{|\cosh(\xi) - \langle\theta, \sigma\rangle|^{\frac{n+2\gamma}{2}}} d\sigma = \int_{\mathbb{S}^{n-1}} \frac{e^{-\frac{n+2\gamma}{2}\xi}}{(1 + e^{-2\xi} - 2e^{-\xi}\langle\theta, \sigma\rangle)^{\frac{n+2\gamma}{2}}}\, d\sigma. \tag{4.2.5}$$

Remark 4.2.2. K is rotationally invariant in the variable θ, thus we drop the dependence on θ in the argument of K. Indeed if we identify $e_1 = (1, 0, \ldots, 0)$ with a fixed point in $\mathbb{S}^{n-1}$ via the usual embedding $\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$ and we define

$$J(\theta) := \int_{\mathbb{S}^{n-1}} \frac{e^{-\frac{n+2\gamma}{2}\xi}}{|1 + e^{-2\xi} - 2e^{-\xi}\langle\theta, \sigma\rangle|^{\frac{n+2\gamma}{2}}}\, d\sigma,$$

it is easy to check that $J(\theta) = J(e_1)$. The proof is trivial using equality (4.2.5) and the change of variable $\tilde{\sigma} = R^\top \sigma$, where R is any rotation such that $R(e_1) = \theta$.

The kernel also can be written using spherical coordinates as

$$\begin{aligned} K(\xi) &= \bar{c}_n e^{-\frac{n+2\gamma}{2}\xi} \int_0^\pi \frac{(\sin\phi_1)^{n-2}}{(1+e^{-2\xi}-2e^{-\xi}\cos\phi_1)^{\frac{n+2\gamma}{2}}}\, d\phi_1 \\ &= \bar{c}_n 2^{-\frac{n+2\gamma}{2}} \int_0^\pi \frac{(\sin\phi_1)^{n-2}}{(\cosh(\xi)-\cos(\phi_1))^{\frac{n+2\gamma}{2}}}\, d\phi_1, \end{aligned} \tag{4.2.6}$$

where ϕ_1 is the angle between θ and σ, and $\bar{c}_n$ is a positive dimensional constant that only depends on the integral in the rest of the spherical coordinates.

Remark 4.2.3. The expression (4.2.6) implies that $K(\xi)$ is an even function. Moreover, since $\phi_1 \in (0,\pi)$ and $\cosh(x) \geq 1$, $\forall x \in \mathbb{R}$, K is strictly positive.

In the next paragraphs we will find a more explicit formula for K that will help us calculate its asymptotic behavior.

Lemma 4.2.4. The kernel K can be expressed in terms of a hypergeometric function as

$$K(\xi) = c_n(\sinh\xi)^{-1-2\gamma}(\cosh\xi)^{\frac{2-n+2\gamma}{2}}\ {}_2F_1\left(\tfrac{a+1}{2}-b, \tfrac{a}{2}-b+1; a-b+1; (\operatorname{sech}\,\xi)^2\right), \tag{4.2.7}$$

where $c_n = \bar{c}_n 2^{-\frac{n+2\gamma}{2}} \frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}$, and ${}_2F_1$ is the hypergeometric function defined in Lemma 7.0.1 in the Appedix 7.

Proof. Because of the parity of the kernel K it is enough to study its behavior for $\xi > 0$. Using property (7.0.2) given in Lemma 7.0.1 in the Appendix 7, we can assert that, if $\xi > 0$,

$$K(\xi) = \bar{c}_n \frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} e^{-\frac{n+2\gamma}{2}\xi}\ {}_2F_1(a,b;c;e^{-2\xi}),$$

where

$$a = \tfrac{n+2\gamma}{2}, \quad b = 1+\gamma, \quad c = \tfrac{n}{2}. \tag{4.2.8}$$

An important observation is that

$$a-b+1 = c,$$

which, from property (7.0.3) in Lemma 7.0.1 in the Appendix 7, yields (4.2.7). □

Lemma 4.2.5. The asymptotic expansion of the kernel K is given by

- $K(\xi) \sim |\xi|^{-1-2\gamma}$ if $|\xi| \to 0$,
- $K(\xi) \sim e^{-|\xi|\frac{n+2\gamma}{2}}$ if $|\xi| \to \infty$.

Proof. Note that K is an even function. Using property (7.0.7) to estimate expression (4.2.7) for $K(\xi)$, we obtain that, for $|\xi|$ small enough,

$$K(\xi) \sim |\sinh\xi|^{-1-2\gamma} \sim |\xi|^{-1-2\gamma}. \tag{4.2.9}$$

Moreover, this expression (4.2.7), the behaviour of the hyperbolic secant function at infinity and the hypergeometric function property (7.0.1) given in Lemma 7.0.1 in the Appendix 7 show the exponential decay of the kernel at infinity:

$$K(\xi) \sim c_n (\sinh \xi)^{-1-2\gamma} (\cosh \xi)^{\frac{2-n+2\gamma}{2}} \sim c e^{-|\xi| \frac{n+2\gamma}{2}}. \tag{4.2.10}$$

where c is a positive constant. □

Remark 4.2.6. The asymptotic behaviour of this kernel near the origin and near infinity given in Lemma 4.2.5 correspond to a tempered stable process.

We recall here that, as we have seen in the previous Chapter 3, the problem (3.1.14) can be rewritten on the extension $X^* = M \times (0, \rho_0^*)$, as

$$\begin{cases} -\operatorname{div}_{g^*}((\rho^*)^{1-2\gamma} \nabla_{g^*} V) = 0 \text{ in } (X^*, g^*), \\ V = v \text{ on } \{\rho^* = 0\}, \\ -\tilde{d}_\gamma \lim_{\rho^* \to 0} (\rho^*)^{1-2\gamma} \partial_{\rho^*} V + c_{n,\gamma} v = c_{n,\gamma} v^\beta \text{ on } \{\rho^* = 0\}, \end{cases} \tag{4.2.11}$$

where $g^* = \frac{(\rho^*)^2}{\rho^2} \bar{g}$. We look for radially symmetric solutions $v = v(t)$, $V = V(t, \rho)$ of (4.2.11). For such solutions we have that $\mathscr{L}_\gamma$ is the Dirichlet-to-Neumann operator for this problem, i.e.,

$$\mathscr{L}_\gamma(v) = -\tilde{d}_\gamma \lim_{\rho^* \to 0} (\rho^*)^{1-2\gamma} \partial_{\rho^*} V + c_{n,\gamma} v.$$

4.2.2 Periodic solutions

We are looking for periodic solutions of (4.2.3). Assume that $v(t + L) = v(t)$: in this case equation (4.2.3) becomes

$$\mathscr{L}_\gamma^L v = \kappa_{n,\gamma} P.V. \int_0^L (v(t) - v(\tau)) K_L(t - \tau) d\tau + c_{n,\gamma} v = c_{n,\gamma} v^\beta, \quad \text{where } \beta = \frac{n+2\gamma}{n-2\gamma},$$

and

$$K_L(t - \tau) = \sum_{j \in \mathbb{Z}} K(t - \tau - jL), \tag{4.2.12}$$

for K the kernel given in (4.2.6). Note that the argument in the integral above has a finite number of poles, but K_L is still well defined.

Lemma 4.2.7. The periodic kernel K_L satisfies the following inequality:

$$\frac{L}{L_1} K_L \left(\frac{L}{L_1} (t - \tau) \right) < K_{L_1}(t - \tau), \quad \forall L > L_1 > 0. \tag{4.2.13}$$

Proof. By evenness we just need to show that the function $\xi K(\xi)$ is decreasing for $\xi > 0$. By (4.2.7), up to positive constant,

$$\begin{aligned} \xi K(\xi) =& \xi(\sinh\xi)^{-1-2\gamma}(\cosh\xi)^{\frac{2-n+2\gamma}{2}} \\ & \cdot {}_2F_1(\tfrac{a+1}{2}-b, \tfrac{a}{2}-b+1; a-b+1; (\operatorname{sech}\xi)^2), \end{aligned} \tag{4.2.14}$$

where a, b, c are given in (4.2.8).

Observe that

$$ {}_2F_1(\tfrac{a+1}{2}-b, \tfrac{a}{2}-b+1; a-b+1; (\operatorname{sech}\xi)^2) > 0, \tag{4.2.15}$$

and

$$\tfrac{a+1}{2}-b > 0, \quad a-b+1 = c > 0,$$

since $n > 2+2\gamma$. Property (7.0.4) yields that (4.2.15) is decreasing. Indeed,

$$\begin{aligned} & \frac{d}{d\xi}\left[{}_2F_1(\tfrac{a+1}{2}-b,\ \tfrac{a}{2}-b+1; a-b+1; (\operatorname{sech}\xi)^2)\right] \\ & \qquad = -2\frac{(\frac{a+1}{2}-b)(\frac{a}{2}-b+1)}{c}(\operatorname{sech}\xi)^2\tanh\xi \\ & \qquad\quad \cdot {}_2F_1(\tfrac{a+1}{2}-b+1, \tfrac{a}{2}-b+2; a-b+2; (\operatorname{sech}\xi)^2) < 0. \end{aligned}$$

Thus we just need to show that the function $\xi(\sinh\xi)^{-1-2\gamma}(\cosh\xi)^{\frac{2-n+2\gamma}{2}}$ in (4.2.14) is decreasing in ξ. In fact by writing

$$\xi(\sinh\xi)^{-1-2\gamma}(\cosh\xi)^{\frac{2-n+2\gamma}{2}} = \tfrac{\xi}{\sinh\xi}(\tanh\xi)^{-\gamma}(\sinh\xi)^{-\gamma}(\cosh\xi)^{\frac{2-n}{2}},$$

we have that $\xi K(\xi)$ is a product of positive decreasing functions.

Finally, inequality (4.2.13) follows from the definition of $K_L(\xi)$ given in (4.2.12):

$$\frac{L}{L_1}K_L\left(\frac{L}{L_1}(t-\tau)\right) = \sum_{j=-\infty}^{+\infty}\frac{L}{L_1}K\left(\frac{L}{L_1}(t-\tau-jL_1)\right) < \sum_{j=-\infty}^{+\infty}K(t-\tau-jL_1) = K_{L_1}(t-\tau).$$

□

4.3 Technical results

4.3.1 Functional Spaces

Definition 4.3.1. We shall work with the following function space

$$\begin{aligned} H_L^\gamma = \{ & v : \mathbb{R} \to \mathbb{R};\ v(t+L) = v(t) \text{ and} \\ & \int_0^L\int_0^L (v(t)-v(\tau))^2 K_L(t-\tau)\,d\tau dt + \int_0^L v(t)^2 dt < +\infty \Big\} \end{aligned}$$

with the norm given by

$$\|v\|_{H_L^\gamma} = \left(\int_0^L v(t)^2\,dt + \int_0^L \int_0^L |v(t)-v(\tau)|^2 K_L(t-\tau)\,dt\,d\tau\right)^{1/2}.$$

Note that we will denote

$$\begin{aligned} W_L^{\gamma,p} = \{v:\mathbb{R}\to\mathbb{R};\ v(t+L)=v(t) \text{ and} \\ \|v\|^p_{L^p(0,L)} + \int_0^L\int_0^L \frac{|v(t)-v(\tau)|^p}{|t-\tau|^{1+\gamma p}}\,dt\,d\tau < \infty\}, \end{aligned}$$

with the norm given by

$$\|v\|_{W_L^{\gamma,p}} = \left(\|v\|^p_{L^p(0,L)} + \int_0^L\int_0^L \frac{|v(t)-v(\tau)|^p}{|t-\tau|^{1+\gamma p}}\,dt\,d\tau\right)^{1/p},$$

which is equivalent to the norm

$$\|v\|_{\tilde{W}_L^{\gamma,p}} = \left(\|v\|^p_{L^p(0,L)} + \int_0^L\int_0^L |v(t)-v(\tau)|^p K(t-\tau)\,dt\,d\tau\right)^{1/p},$$

for the kernel K given in (4.2.6).

Now we are going to introduce some fractional inequalities, continuity and compactness results whose proofs for an extension domain can be found in [64]. Here we are working with periodic functions, which avoids the technicalities of extension domains but the same proofs as in [64] are valid.

Proposition 4.3.2. (Fractional Sobolev inequalities.) (Theorems 6.7 and 6.10, [64]) Let $\gamma \in (0,1)$, $p \in [1,+\infty)$ such that $\gamma p \le 1$ and $p^* = \frac{np}{n-\gamma p}$. Then there exists a positive constant $C = C(p,\gamma)$ such that, for any $v \in W_L^{\gamma,p}$, we have

$$\|v\|_{L^q(0,L)} \le C\|v\|_{W_L^{\gamma,p}},$$

for any $q \in [1,p^*)$; i.e., the space $W_L^{\gamma,p}$ is continuously embedded in $L^q(0,L)$ for any $q \in [1,p^*)$.

Proposition 4.3.3. (Compact embeddings) (Theorem 7.1 and Corollary 7.2 in [64].) Let $\gamma \in (0,1)$ and $p \in [1,+\infty)$, $q \in [1,p]$, and J be a bounded subset of $L^p(0,L)$. Suppose

$$\sup_{f\in J}\int_{[0,L]}\int_{[0,L]} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\gamma p}}\,dx\,dy < +\infty.$$

Then J is pre-compact in $L^q(0,L)$. Moreover, if $\gamma p < 1$, then J is pre-compact in $L^q(0,L)$, for all $q \in [1,p^*)$.

Remark 4.3.4. If $\gamma = 1/2$, we have the compact embedding

$$W_L^{1/2,2} \subset\subset L^q(0,L), \quad \text{for } q \in (1,\infty).$$

Indeed, a consequence of Proposition 4.3.2 is $W_L^{1/2,2} \subset W_L^{\gamma,2}$, $\forall \gamma < 1/2$, thus Proposition 4.3.3 provides

$$W_L^{1/2,2} \subset W_L^{\gamma,2} \subset\subset L^q(0,L), \quad \forall q \in (1, \tfrac{2}{1-2\gamma}), \gamma < 1/2.$$

We conclude by letting $\gamma \to 1/2$.

Proposition 4.3.5. (Hölder fractional regularity.) (Theorem 8.2 in [64].) Let $p \in [1,+\infty)$, $\gamma \in (0,1)$ such that $\gamma p > 1$. Then there exists $C > 0$, depending on γ and p, such that

$$\|v\|_{\mathcal{C}^{0,\alpha}([0,L])} \leq C \left(\|v\|_{L^p(0,L)}^p + \int_0^L \int_0^L \tfrac{|v(t)-v(\tau)|^p}{|t-\tau|^{1+\gamma p}} \, dt \, d\tau \right)^{1/p}$$

for any L-periodic function $v \in L^p(0,L)$, with $\alpha = \gamma - 1/p$.

Note that with the equi-continuity given in Proposition 4.3.5 we can apply Arzelà-Ascoli to show the compactness

$$W_L^{\gamma,2} \subset\subset L^q(0,L) \ \forall q \in (1,\infty) \text{ with } \gamma > 1/2.$$

Remark 4.3.6. We have the compact embedding

$$H_L^\gamma \subset\subset L^q(0,L), \ \forall \gamma \in (0,1),$$

where

$$q \in (1, \tfrac{2}{1-2\gamma}) \text{ if } \gamma \leq \tfrac{1}{2} \quad \text{and} \quad q \geq 1 \text{ if } \gamma > \tfrac{1}{2}. \tag{4.3.1}$$

Indeed, Proposition 4.3.3, Remark 4.3.4 and Proposition 4.3.5 with provide $W_L^{\gamma,2} \subset\subset L^q(0,L)$ for all $\gamma \in (0,1)$ and q as in (4.3.1). But from the definition of K_L given in (4.2.12) and the positivity of the function K, we have the following inequality between norms

$$\|v\|_{W_L^{\gamma,2}} \leq \|v\|_{H_L^\gamma}.$$

Proposition 4.3.7. (Poincare's fractional inequality.) Let $v \in H_L^\gamma$ with zero average (i.e. $\int_0^L v(t)\, dt = 0$), then there exists $c > 0$ such that

$$\|v\|_{L^2(0,L)}^2 \leq c \int_0^L \int_0^L \frac{(v(t)-v(\tau))^2}{|t-\tau|^{1+2\gamma}} \, dt \, d\tau. \tag{4.3.2}$$

Proof. Inspired on the proof of the classical Poincare's inequality given in Theorem 7.16 in [156], we prove (4.3.2). By contradiction assume that, $\forall j \geq 1$, there exists $v_j \in H_L^\gamma$ satisfying

$$\|v_j\|_{L^2(0,L)}^2 > j \int_0^L \int_0^L \frac{(v_j(t)-v_j(\tau))^2}{|t-\tau|^{1+2\gamma}} \, dt \, d\tau. \tag{4.3.3}$$

On the one hand, we normalize v_j in $L^2(0,L)$ by $w_j := \frac{v_j}{\|v_j\|_{L^2(0,L)}}$, so $\|w_j\|_{L^2(0,L)} = 1$. Because of (4.3.3) it follows that

$$\int_0^L \int_0^L \frac{(w_j(t) - w_j(\tau))^2}{|t-\tau|^{1+2\gamma}}\, dt\, d\tau < \tfrac{1}{j} \leq 1, \tag{4.3.4}$$

that is, $\{w_j\}$ is bounded in the H_L^γ norm. By the compactness from Remark 4.3.6, we obtain a subsequence $\{w_i\}$ that converges strongly in $L^2(0,L)$, i.e, there exists $w \in L^2(0,L)$ such that $w_i \to w$ in $L^2(0,L)$. Thus,

$$\|w\|_{L^2(0,L)} = \lim_{j\to\infty} \|w_j\|_{L^2(0,L)} = 1.$$

On the other hand, also by the compactness given in Remark 4.3.6, we have weak semiconvergence in H_L^γ. Thus the following inequality follows

$$\int_0^L \int_0^L \frac{(w(t) - w(\tau))^2}{|t-\tau|^{1+2\gamma}}\, dt\, d\tau \leq \liminf_{j\to\infty} \int_0^L \int_0^L \frac{(w_j(t) - w_j(\tau))^2}{|t-\tau|^{1+2\gamma}}\, dt\, d\tau.$$

Thanks to (4.3.4), this gives

$$\int_0^L \int_0^L \frac{(w(t) - w(\tau))^2}{|t-\tau|^{1+2\gamma}}\, dt\, d\tau = 0,$$

that is, w must be constant and, since it has zero average, it has to be the zero function. □

4.3.2 Maximum principles

Proposition 4.3.8. (Strong maximum principle). Let $v \in H_L^{\gamma,2} \cap \mathcal{C}^0(\mathbb{R})$ with $v \geq 0$ be a solution of

$$\mathcal{L}_\gamma v = f(v), \quad \text{in } \mathbb{R},$$

where f satisfies $f(v) \geq 0$ if $v \geq 0$. Then $v > 0$ or $v \equiv 0$.

Proof. Since $v \geq 0$, we have that

$$\mathscr{L}_\gamma v = f(v) \geq 0. \tag{4.3.5}$$

Suppose that there exists a point $t_0 \in \mathbb{R}$ with $v(t_0) = 0$, then

$$\begin{aligned}\mathscr{L}_\gamma v(t_0) &= \kappa_{n,\gamma} \mathrm{P.V} \int_{-\infty}^{+\infty} (v(t_0) - v(\tau)) K(t_0 - \tau)\, d\tau + c_{n,\gamma} v(t_0)\\ &= \kappa_{n,\gamma} \mathrm{P.V} \int_{-\infty}^{+\infty} (-v(\tau)) K(t_0 - \tau)\, d\tau \leq 0\end{aligned}$$

satisfies (4.3.5) only in the case $v \equiv 0$. □

4.3.3 Regularity

In the following Proposition 4.3.9 we concentrate on the local regularity, using the equivalent characterization for $\mathscr{L}_\gamma$ as a Dirichlet-to-Neumann operator for problem (4.2.11). First, we fix some notation that we will use here. Let $0 < R < \rho_0^*$, we denote

$$\begin{aligned} B_R^+ &= \{(t,\rho^*) \in \mathbb{R}^2 \,:\, \rho^* > 0, |(t,\rho^*)| < R\}, \\ \Gamma_R^0 &= \{(t,0) \in \partial\mathbb{R}_+^2 \,:\, |t| < R\}. \end{aligned}$$

Proposition 4.3.9. Fix $\gamma < 1/2$ and let $V = V(t,\rho^*)$ be a solution of the extension problem

$$\begin{cases} -\operatorname{div}_{g^*}((\rho^*)^{1-2\gamma}\nabla_{g^*}V) = 0 \;\text{ in } (B_{2R}^+, g^*), \\ -\tilde{d}_\gamma \lim\limits_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\partial_{\rho^*}V + c_{n,\gamma}v = c_{n,\gamma}v^\beta \;\text{ on } \Gamma_{2R}^0. \end{cases} \tag{4.3.6}$$

If

$$\int_{\Gamma_{2R}^0} |v|^{\frac{2}{1-2\gamma}}\,dt =: \zeta < \infty,$$

then for each $p > 1$, there exists a constant $C_p = C(p,\zeta) > 0$ such that

$$\sup_{B_R^+}|V| + \sup_{\Gamma_R^0}|v| \le C_p\left[\left(\tfrac{1}{R^{n+1+a}}\right)^{1/p}\|V\|_{L^p(B_{2R}^+)} + \left(\tfrac{1}{R^n}\right)^{1/p}\|v\|_{L^p(\Gamma_{2R}^0)}\right].$$

Proof. This L^∞ bound is proven for linear right hand side in Theorem 2.3.1 in [81]. A generalization for the nonlinear subcritical case is given in Theorem 3.4 in [97]. Here we can follow the same proof as in [97] because we have reduced our problem to one-dimensional problem for $t \in \mathbb{R}$ and thus, $\beta = \frac{n+2\gamma}{n-2\gamma}$ is a subcritical exponent. $\square$

The following two propositions could be also proved using the extension problem (4.3.6). However, they can be phrased in terms of a general convolution kernel, as we explain here. Thus we fix $K : \mathbb{R} \to [0,\infty)$ a measurable kernel satisfying:

a) $\nu \le K(t)|t|^{1+\frac{\gamma}{2}} \le \nu^{-1}$ a.e $t \in \mathbb{R}$ with $|t| \le 1$,

b) $K(t) \le M|t|^{-n-\eta}$ a.e. $t \in \mathbb{R}$ with $|t| > 1$,

for some $\gamma \in (0,1)$, $\nu \in (0,1)$, $\eta > 0$, $M \ge 1$. Consider the functional defined in (4.2.4) by

$$(\mathscr{L}_\gamma v)(t) = \kappa_{n,\gamma}\text{P.V}\int_{-\infty}^{+\infty}(v(t)-v(\tau))K(t-\tau)\,d\tau + c_{n,\gamma}v,$$

for $v \in L^p(\mathbb{R})$. We study the regularity of solutions to

$$\mathscr{L}_\gamma v = f. \tag{4.3.7}$$

Proposition 4.3.10. Let $f \in L^q$ for some $q > n$ and v solution of (4.3.7) in $B_R(x_0)$, then there exist constants $c > 0$ and $\alpha \in (0,1)$ which depend on n, ν, M, η, γ, q and A, and remain positive as $\gamma \to 1$, such that for any $R \in (0,1)$,

$$|v(t) - v(\tau)| \leq c|t - \tau|^{\alpha} \left(R^{-\alpha} \|v\|_{L^\infty} + \|f\|_{L^q} \right).$$

Proof. Since our kernel corresponds to a tempered stable process, this regularity was given by Kassmann in his article [113] (see Theorem 1.1 and Extension 5). We could also follow the same steps as for Theorem 5.1 in [165] since Lemma 4.1 and Remark 4.3 in this paper [165] hold for our K (note the expansion in Lemma 4.2.5). □

Proposition 4.3.11. Let $\alpha \in (0,1)$. Assume $f \in \mathcal{C}^\alpha(\mathbb{R})$, and let $v \in L^\infty(\mathbb{R})$ be a solution of (4.3.7) in $\mathbb{R}^n$. Then there exists $c > 0$ depending on n, α, γ such that

$$\|v\|_{\mathcal{C}^{\alpha+2\gamma}} \leq c \left(\|v\|_{\mathcal{C}^\alpha} + \|f\|_{\mathcal{C}^\alpha} \right).$$

Proof. Under our assumptions, on the one hand, Dong and Kim proved in Theorem 1.2 from [76] that $(-\Delta)^\gamma v \in \mathcal{C}^\alpha$ and moreover the following estimate holds:

$$\|(-\Delta)^\gamma v\|_{\mathcal{C}^\alpha} \leq c \left(\|v\|_{\mathcal{C}^\alpha} + \|f\|_{\mathcal{C}^\alpha} \right). \tag{4.3.8}$$

On the other hand, Silvestre in Proposition 2.8 in [166], showed that

- If $\alpha + 2\gamma \leq 1$, then $v \in \mathcal{C}^{\alpha+2\gamma}$ and

$$\|v\|_{\mathcal{C}^{\alpha+2\gamma}(\mathbb{R})} \leq c(\|v\|_{L^\infty} + \|(-\Delta)^\gamma v\|_{\mathcal{C}^\alpha}). \tag{4.3.9}$$

- If $\alpha + 2\gamma > 1$, then $v \in \mathcal{C}^{1,\alpha+2\gamma-1}$ and

$$\|v\|_{\mathcal{C}^{1,\alpha+2\gamma-1}(\mathbb{R})} \leq c(\|v\|_{L^\infty} + \|(-\Delta)^\gamma v\|_{\mathcal{C}^\alpha}). \tag{4.3.10}$$

Thus, combining (4.3.8) with (4.3.9) and (4.3.10) we have the claimed regularity. □

Remark 4.3.12. The previous Propositions 4.3.9, 4.3.10, 4.3.11 imply that for $\gamma < 1/2$ any $v \in L^{\beta+1}$ solution of equation (4.2.3) satisfies $v \in \mathcal{C}^\infty$. A standard argument yields the same conclusion for $\gamma = 1/2$ too. Finally, if $\gamma > 1/2$ Proposition 4.3.5 automatically implies that any function $v \in H^\gamma_L$ also satisfies $v \in \mathcal{C}^\infty$.

4.3.4 Subcritical case.

Note that the following Lemma 4.3.13 has been studied by different authors if $N > 2\gamma$, even for $1 < p < \frac{N+2\gamma}{N-2\gamma}$ (see [49, 48, 123]), but here we need this result also for $2\gamma \geq N$ since we have reduced our problem to dimension $N = 1$ for any $\gamma \in (0,1)$. We will use it for $p = \frac{n+2\gamma}{n-2\gamma}$.

Lemma 4.3.13. Let w be solution for

$$(-\Delta)^{\gamma} w = w^p, \quad 0 \le w \le 1, \quad p > 1, \quad (N-2\gamma)p < N. \tag{4.3.11}$$

Then $w \equiv 0$.

Proof. Let η be a smooth function. In fact we may choose

$$\eta = (1+|x|)^{-m}, \quad \text{where } m = N + 2\gamma. \tag{4.3.12}$$

Then multiplying (4.3.11) by the test function η, integrating over $\mathbb{R}^N$ and using integration by parts in the right hand side of (4.3.11) we obtain the following inequality

$$\begin{aligned}
\left| \int_{\mathbb{R}^N} w^p \eta \, dx \right| &= \left| \int_{\mathbb{R}^N} \left(w(x) \int_{\mathbb{R}^N} \frac{\eta(x)-\eta(y)}{|x-y|^{N+2\gamma}} \, dy \right) \, dx \right| \\
&\le \left| \int_{\mathbb{R}^N} \left((w(x)\eta^{1/p}(x))\eta(x)^{-1/p} \int_{\mathbb{R}^N} \frac{\eta(x)-\eta(y)}{|x-y|^{N+2\gamma}} \, dy \right) \, dx \right| \\
&\le \left| \int_{\mathbb{R}^N} w^p(x)\eta(x) \, dx \right|^{1/p} \left(\int_{\mathbb{R}^N} \left| (\eta(x)^{-1/p}(-\Delta)^{\gamma}\eta(x))^{p/(p-1)} \right| \, dx \right)^{(p-1)/p}.
\end{aligned} \tag{4.3.13}$$

We just need to compute the second term in the right hand side. Firstly we can check that it is bounded. Since

$$\eta(x)^{-\frac{1}{p-1}} |(-\Delta)^{\gamma}\eta(x)|^{\frac{p}{p-1}} \le c(1+|x|)^{(N+2\gamma)\frac{1}{p-1}} (1+|x|)^{-\frac{p}{p-1}(N+2\gamma)} \le (1+|x|)^{-(N+2\gamma)}, \tag{4.3.14}$$

we have

$$\int_{\mathbb{R}^N} \eta(x)^{-\frac{1}{p-1}} |(-\Delta)^{\gamma}\eta(x)|^{\frac{p}{p-1}} \, dx < \infty.$$

Note that for inequality (4.3.14) we have used the definition of the test function given in (4.3.12) and the following bound

$$|(-\Delta)^{\gamma}\eta| \le c(1+|x|)^{-(N+2\gamma)}, \quad \text{for } x \text{ large enough;} \tag{4.3.15}$$

which is proven at the end of the proof of this Lemma. Now we chose

$$\eta_R(x) = \eta(x/R).$$

Performing a similar analysis to that of (4.3.13), we obtain

$$\int_{\mathbb{R}^N} w^p(x)\eta_R(x) \le \int_{\mathbb{R}^N} \eta_R(x)^{-1/(p-1)} \left| \int_{\mathbb{R}^N} \frac{\eta(x)-\eta(y)}{|x-y|^{N+2\gamma}} \, dy \right|^{p/(p-1)} dx.$$

Then, by scaling,

$$\int_{|x|\le R} w^p(x) \le cR^{N-\frac{2p\gamma}{p-1}} \int_{\mathbb{R}^N} \eta(x)^{-1/(p-1)} \left| \int_{\mathbb{R}^N} \frac{\eta(x)-\eta(y)}{|x-y|^{N+2\gamma}} \, dy \right|^{p/(p-1)} dx.$$

Note that $N - \frac{2p\gamma}{p-1} < 0$ by hypothesis. Then, letting R tend to infinity, we obtain

$$\int_{|x|\leq R} w^p(x)\,dx \to 0 \text{ as } R \to +\infty.$$

Therefore, we have $w \equiv 0$.

In order to conclude we just need to check inequality (4.3.15) before. It follows from standard potential analysis. In fact, for $|x| \geq 1$ we have that

$$|(-\Delta)^\gamma \eta(x)| = \left| \text{P.V.} \int_{\mathbb{R}^N} \frac{\eta(x) - \eta(y)}{|x-y|^{N+2\gamma}}\,dy \right| \leq |I_1| + |I_2| + |I_3| + |I_4|,$$

where these integrals can be bounded as follows: for the first integral we use that $|x - y|$ is small enough to check that

$$\begin{aligned}|I_1| &= \left| \text{P.V.} \int_{|x-y|<1} \frac{\eta(x) - \eta(y)}{|x-y|^{N+2\gamma}}\,dy \right| = \left| \int_{|x-y|<1} \frac{\eta(x) - \eta(y) - \eta'(x)|x-y|}{|x-y|^{N+2\gamma}}\,dy \right| \\ &\leq C \int_{|x-y|<1} \frac{|\eta''(x)||x-y|^2}{|x-y|^{N+2\gamma}}\,dy \leq \frac{C}{(1+|x|)^{N+2\gamma}}.\end{aligned}$$

For the second one, we have that $|x - y| < \frac{|x|}{2}$, then, we can use that

$$|\eta(x) - \eta(y)| \leq |\eta'(\xi)||x-y| \leq C(1+|x|)^{-(N/2+2\gamma-1)}|x-y|,$$

and bound the integral as follows

$$\begin{aligned}|I_2| &= \left| \int_{1<|x-y|<\frac{|x|}{2}} \frac{\eta(x) - \eta(y)}{|x-y|^{N+2\gamma}}\,dy \right| \\ &\leq C|x|^{1-2\gamma}(1+|x|)^{-(N/2+2\gamma-1)} \leq \frac{C}{(1+|x|)^{N+2\gamma}},\end{aligned}$$

since x is large enough and $|x| \sim |y|$, indeed $|y| \geq |x| - |x-y| \geq \frac{|x|}{2}$ and $|y| \leq \frac{3}{2}|x|$.

The third one is directly bounded,

$$\begin{aligned}|I_3| &= \left| \int_{\frac{|x|}{2}<|x-y|<2|x|} \frac{\eta(x) - \eta(y)}{|x-y|^{N+2\gamma}}\,dy \right| \leq \frac{2^{N+2\gamma}}{|x|^{N+2\gamma}} \left| \int_{\frac{|x|}{2}<|x-y|<2|x|} (\eta(x) - \eta(y))\,dy \right| \\ &\leq \frac{2^{N+2\gamma}}{|x|^{N+2\gamma}} \left| \eta(x)|x|^{-N} - \int_{\frac{|x|}{2}<|x-y|<2|x|} \eta(y)\,dy \right| \leq \frac{C}{|x|^{N+2\gamma}} \sim \frac{C}{(1+|x|)^{N+2\gamma}},\end{aligned}$$

using that $|x|$ is large enough.

For the fourth and last one, we use that $|y| \geq |x-y| - |x| \geq |x|$, then

$$|I_4| = \left| \int_{|x-y|>2|x|} \left(\frac{\eta(x)-\eta(y)}{|x-y|^{N+2\gamma}} \right) dy \right| \leq C \left(\int_{|x-y|>2|x|} \frac{1}{|x-y|^{N+2\gamma}} \, dy \right) (1+|x|)^{-(N+2\gamma)}$$
$$\leq \frac{C}{(1+|x|)^{(N+2\gamma)}}.$$

□

4.4 Proof of Theorem 4.1.1

4.4.1 Variational Formulation

We consider the following minimization problem

$$c(L) = \inf_{v \in H^\gamma_L, v \not\equiv 0} \mathscr{F}_L(v), \tag{4.4.1}$$

where

$$\mathscr{F}_L(v) = \frac{\kappa_{n,\gamma} \int_0^L \int_0^L (v(t)-v(\tau))^2 K_L(t-\tau) \, dt \, d\tau + c_{n,\gamma} \int_0^L v(t)^2 \, dt}{(\int_0^L v(t)^{\beta+1} dt)^{\frac{2}{\beta+1}}}. \tag{4.4.2}$$

Our first lemma shows that

Lemma 4.4.1. For any $L > 0$, $c(L)$ is achieved by a positive function $v_L \in \mathcal{C}^\infty$ which solves

$$\mathscr{L}^L_\gamma v = \kappa_{n,\gamma} P.V. \int_0^L (v(t)-v(\tau)) K_L(t-\tau) d\tau + c_{n,\gamma} v = c_{n,\gamma} v^\beta, \quad \text{where } \beta = \tfrac{n+2\gamma}{n-2\gamma}. \tag{4.4.3}$$

Proof. Considering that the value of multiplicative constants does not affect this proof, we may assume that $c_{n,\gamma} = 1$ and $\kappa_{n,\gamma} = 1$. Since $c(L)$ is invariant by rescaling we can assume that

$$\int_0^L v^{\beta+1} \, dt = 1; \tag{4.4.4}$$

thus $\mathscr{F}_L[v] = \|v\|^2_{H^\gamma_L}$. First note that if $c(L)$ is achieved by a function v_L, then this function solves (4.4.3) because this is the Euler-Lagrange equation for the functional (4.4.2).

By construction, the functional $\mathscr{F}_L(v)$ is non-negative and therefore it is bounded from below, so the infimum is finite. Next we show that a minimizer exists. Let $\{v_i\}$ be a minimizing sequence normalized to satisfy (4.4.4), such that $\mathscr{F}_L(v_i) \leq c(L) + 1$. Because of Remark 4.3.6, for all $\gamma \in (0,1)$ we have the compact embedding of H^γ_L in L^q, with $q \in (1, \frac{2}{1-2\gamma})$ if $\gamma \leq \frac{1}{2}$ and $q \geq 1$ if $\gamma > \frac{1}{2}$ so, in particular, for $q = \beta + 1$. Moreover, there exists $v_L \in H^\gamma_L$ such that $v_i \rightharpoonup v_L$. This implies

$$\|v_L\|_{H^\gamma_L} \leq \liminf_j \|v_j\|_{H^\gamma_L}. \tag{4.4.5}$$

Since $\{v_i\}$ is a minimizing sequence, $\liminf \|v_j\|_{H_L^\gamma} = c(L)$, and (4.4.5) implies that we have a minimizer $v_L \in H_L^\gamma$. The compact embedding assures that convergence is strong in $L^{\beta+1}$, i.e.,

$$1 = \lim_j \|v_j\|_{L^{\beta+1}} = \|v_L\|_{L^{\beta+1}}.$$

Now we apply Remark 4.3.12 to obtain $v_L \in \mathcal{C}^\infty$.

Finally we observe that the minimizer $v_L \in H_L^\gamma$ must be positive. If v_L is not non-negative we take $w = |v_L| \in H_L^\gamma$ and the following inequality holds

$$\mathscr{F}_L(w) \leq \mathscr{F}_L(v_L), \tag{4.4.6}$$

obtaining a contradiction. Indeed if $\operatorname{sign}(v(t)) = \operatorname{sign}(v(\tau))$, equality holds in (4.4.6) and if $\operatorname{sign}(v(t)) \neq \operatorname{sign}(v(\tau))$, (4.4.6) is also true because

$$\begin{aligned}(w(t) - w(\tau))^2 = (v_L(t) + v_L(\tau))^2 &\leq \max\{(v_L(t))^2, (v_L(\tau))^2\} \\ &\leq (|v_L(t)| + |v_L(\tau)|)^2 = (v_L(t) - v_L(\tau))^2.\end{aligned}$$

Once we have the non-negativity of the minimizer, since $\|v_L\|_{L^\beta} = 1$, the maximum principle given in Proposition 4.3.8 applied to equation (4.4.3) assures that $v_L > 0$. Therefore we conclude the proof of the Lemma 4.4.1. □

We now introduce the weak formulation of the problem. We will say that $v \in H_L^\gamma$ is weak solution of (4.4.3) if it satisfies

$$\langle \mathscr{L}_\gamma^L v, \phi \rangle = c_{n,\gamma} \int_0^L v^\beta(t)\phi(t)\, dt, \quad \forall \phi \in H_L^\gamma \tag{4.4.7}$$

where $\langle \, , \, \rangle$ is defined by

$$\langle \mathscr{L}_\gamma^L v, \phi \rangle = \frac{\kappa_{n,\gamma}}{2} P.V. \int_0^L \int_0^L (v(t) - v(\tau))(\phi(t) - \phi(\tau)) K_L(t-\tau)\, dt\, d\tau + c_{n,\gamma} \int_0^L v(t)\phi(t)\, dt.$$

4.4.2 Proof of Theorem 4.1.1:

At this moment it is unclear if the minimizer v_L for (4.4.2) is the constant solution. Let

$$c^*(L) = c_{n,\gamma} L^{\frac{\beta-1}{\beta+1}}$$

be the energy of the constant solution. The next key lemma provides a criteria:

Lemma 4.4.2. Assume that $c(L_1)$ is attained by a nonconstant function v_{L_1}. Then $c(L) < c^*(L)$ for all $L > L_1$.

Proof. Let v_{L_1} be the minimizer for L_1, then v_{L_1} is the solution to

$$\mathscr{L}_\gamma^{L_1}(v_{L_1}) := \kappa_{n,\gamma} \int_0^{L_1} (v_{L_1}(t) - v_{L_1}(\tau)) K_{L_1}(t-\tau) d\tau + c_{n,\gamma} v_{L_1} = c_{n,\gamma} v_{L_1}^\beta.$$

By assumption $v_{L_1} \not\equiv 1$. Now let

$$t = \frac{L_1}{L}\bar{t} \quad \text{and} \quad v(\bar{t}) = v_{L_1}\left(\frac{L_1}{L}\bar{t}\right),$$

which is an $L-$periodic function. By definition it is clear that

$$\begin{aligned}
c(L) &\leq \frac{\kappa_{n,\gamma}\int_0^L\int_0^L (v(\bar{t}) - v(\bar{\tau}))^2 K_L(\bar{t}-\bar{\tau})\, d\bar{t}\, d\bar{\tau} + c_{n,\gamma}\int_0^L v^2(\bar{t})\, d\bar{t}}{(\int_0^L v^{\beta+1}(\bar{t})\, d\bar{t})^{\frac{2}{\beta+1}}} \\
&= \left(\tfrac{L}{L_1}\right)^{1-\frac{2}{\beta+1}} \frac{\kappa_{n,\gamma}\int_0^{L_1}\int_0^{L_1}(v_{L_1}(t) - v_{L_1}(\tau))^2 \frac{L}{L_1} K_L(\frac{L}{L_1}(t-\tau))\, dt\, d\tau + c_{n,\gamma}\int_0^{L_1} v_{L_1}^2(t)\, dt}{(\int_0^{L_1} v_{L_1}^{\beta+1}(t)\, dt)^{\frac{2}{\beta+1}}} \\
&< \left(\tfrac{L}{L_1}\right)^{1-\frac{2}{\beta+1}} \frac{\kappa_{n,\gamma}\int_0^{L_1}\int_0^{L_1}(v_{L_1}(t) - v_{L_1}(\tau))^2 (K_{L_1}(t-\tau))\, dt\, d\tau + c_{n,\gamma}\int_0^{L_1} v_{L_1}^2(t)\, dt}{(\int_0^{L_1} v_{L_1}^{\beta+1}(t)\, dt)^{\frac{2}{\beta+1}}} \\
&\leq \left(\tfrac{L}{L_1}\right)^{1-\frac{2}{\beta+1}} c(L_1) \leq \left(\tfrac{L}{L_1}\right)^{1-\frac{2}{\beta+1}} c^*(L_1) = c^*(L).
\end{aligned}$$

The second inequality above follows from Lemma 4.2.7.

Thus we conclude that $c(L) < c^*(L)$ for all $L > L_1$ and hence $c(L)$ is attained by a nonconstant minimizer. $\square$

Lemma 4.4.3. If the period L is small enough, then $c(L)$ is attained by the constant only.

Proof. First, we claim that, for $L \leq 1$, the minimizer v_L is uniformly bounded. This follows from a standard Gidas-Spruck type blow-up argument. In fact, suppose not, we may assume that there exist sequences $\{L_i\}$, $\{v_{L_i}\}$ and $\{t_i\}$ with $t_i \in [0, L_i]$ such that

$$\max_{0\leq t\leq L_i} v_{L_i}(t) = \max_{t\in\mathbb{R}} v_{L_i}(t) = v_{L_i}(t_i) = M_i \to +\infty.$$

Note that v_{L_i} satisfies (4.4.3). Now rescale

$$\tilde{t} = \epsilon_i^{-1}(t - t_i), \quad \tilde{v}_{L_i}(\tilde{t}) = \epsilon_i^{\frac{2\gamma}{\beta-1}} v_{L_i}(\epsilon_i\tilde{t}),$$

where

$$M_i = \epsilon_i^{\frac{-2\gamma}{\beta-1}}.$$

With this change of variable, (4.4.3) reads

$$\kappa_{n,\gamma}\int_{\mathbb{R}} \epsilon_i(\tilde{v}_{L_i}(\tilde{t}) - \tilde{v}_{L_i}(\tilde{\tau}))K(\epsilon_i(\tilde{t}-\tilde{\tau}))\, d\tilde{\tau} + c_{n,\gamma}\tilde{v}_{L_i}(\tilde{t}) = \epsilon_i^{-2\gamma} c_{n,\gamma} v_{L_i}^{\beta}(\tilde{t}).$$

Because of (4.2.9)

$$\int_{\mathbb{R}} \epsilon_i(\tilde{v}_{L_i}(\tilde{t}) - \tilde{v}_{L_i}(\tilde{\tau}))K(\epsilon_i(\tilde{t}-\tilde{\tau}))\, d\tilde{\tau} \sim \frac{1}{\epsilon_i^{2\gamma}}\int_{\mathbb{R}} \frac{\tilde{v}_{L_i}(\tilde{t}) - \tilde{v}_{L_i}(\tilde{\tau})}{|\tilde{t}-\tilde{\tau}|^{1+2\gamma}} d\tilde{\tau} \sim \frac{1}{\epsilon_i^{2\gamma}\kappa_{n,\gamma}}(-\Delta)^{\gamma}\tilde{v}_{L_i}.$$

Therefore $\tilde{v}_{L_i}$ satisfies

$$(-\Delta)^\gamma \tilde{v}_{L_i} + c_{n,\gamma}\epsilon^{2\gamma}\tilde{v}_{L_i}(\tilde{t}) = c_{n,\gamma}\tilde{v}_{L_i}^{\beta}(\tilde{t}) + o(1) \text{ as } i \to \infty.$$

Remark 4.3.12 assures that all the derivatives of v_{L_i} are equi-continuous functions, thus we can apply Ascoli-Arzelá theorem to find $v_\infty \in \mathcal{C}^\infty$ such that $\tilde{v}_{L_i} \to v_\infty$ as $i \to +\infty$ and which satisfies

$$(-\Delta)^\gamma v_\infty = c_{n,\gamma} v_\infty^\beta \text{ in } \mathbb{R}.$$

Note that v_∞ is positive. By the result given in Lemma 4.3.13 we derive that $v_\infty \equiv 0$, which contradicts with the assumption that $v_\infty(0) = 1$.

Secondly, we use Poincare's inequality given in (4.3.2) to show that $v_L \equiv Constant$. In fact we observe that $\phi = \frac{\partial v_L}{\partial t}$ satisfies

$$\mathscr{L}_\gamma^L \phi - c_{n,\gamma}\beta v_L^{\beta-1}\phi = 0, \tag{4.4.8}$$

where $\mathscr{L}_\gamma^L$ is defined as in (4.1.2). The weak formulation for the problem from (4.4.7), the fact that v_L is bounded and equation (4.4.8) give

$$\int_0^L \int_0^L (\phi(t) - \phi(\tau))^2 K_L(t-\tau)\,dt\,d\tau \le C \int_0^L \phi^2.$$

Rescaling $t = L\tilde{t}, \tilde{\phi} = \phi(L\tilde{t})$ and using (4.2.9), since L is small enough, we obtain that

$$\int_0^1 \int_0^1 \frac{(\tilde{\phi}(\tilde{t}) - \tilde{\phi}(\tilde{\tau}))^2}{|\tilde{t} - \tilde{\tau}|^{1+2\gamma}} d\tilde{t}\,d\tilde{\tau} \le CL^{2\gamma} \int_0^1 \tilde{\phi}^2.$$

By Poincare's inequality (4.3.2) (since ϕ has average zero) there exists $C_0 > 0$ for which

$$C_0 \int_0^1 \tilde{\phi}^2 \le \int_0^1 \int_0^1 \frac{(\tilde{\phi}(\tilde{t}) - \tilde{\phi}(\tilde{\tau}))^2}{|\tilde{t} - \tilde{\tau}|^{1+2\gamma}} d\tilde{t}\,d\tilde{\tau} \le CL^{2\gamma} \int_0^1 \tilde{\phi}^2,$$

which yields that

$$\int_0^1 \tilde{\phi}^2 = 0$$

for L small. □

Lemma 4.4.4. If the period L is large enough, then

$$c(L) < c^*(L), \tag{4.4.9}$$

and therefore, we have a non constant positive solution for (4.1.1).

Proof. Let

$$b(t) := \left(\frac{e^t}{e^{2t} + 1} \right)^{\frac{n-2\gamma}{2}}, \tag{4.4.10}$$

which is a ground state solution for (4.1.1). This follows because the "bubble"

$$w(x) = \left(\frac{1}{|x|^2+1}\right)^{\frac{n-2\gamma}{2}}, \tag{4.4.11}$$

is a solution of (3.1.1) that is regular at the origin. Note that $b(t) > 0$ and $b(\pm\infty) = 0$.

Now we take a cut-off function η_L which is identically 1 in the ball of radius $L/4$ and null outside the ball of radius $L/2$. We define a new function

$$v_L(t) = b(t)\eta_L(t).$$

We will denote $\tilde{v}_L(t) \in H_L^\gamma$ the $L-$periodic extension of v_L. The definitions of $c(L)$ and K_L, given in (4.4.1) and (4.2.12) respectively, give us the following equality:

$$\begin{aligned} c(L) &= \inf_{v\in H_L^\gamma, v\not\equiv 0} \frac{\kappa_{n,\gamma}\int_0^L\int_{\mathbb{R}}(v(s+\tau)-v(\tau))^2K(s)\,dsd\tau + c_{n,\gamma}\int_0^L v(t)^2dt}{(\int_0^L v(t)^{\beta+1}\,dt)^{\frac{2}{\beta+1}}} \\ &= \inf_{v\in H_L^\gamma, v\not\equiv 0} \frac{\kappa_{n,\gamma}\int_{-L/2}^{L/2}\int_{\mathbb{R}}(v(s+\tau)-v(\tau))^2K(s)\,dsd\tau + c_{n,\gamma}\int_{-L/2}^{L/2} v(t)^2\,dt}{(\int_{-L/2}^{L/2} v(t)^{\beta+1}\,dt)^{\frac{2}{\beta+1}}}, \end{aligned} \tag{4.4.12}$$

where $s := t-\tau$ and we have used the $L-$periodicity of any $v \in H_L^\gamma$. We use $\tilde{v}_L$ as a test function in the functional (4.4.12). Taking the limit $L \to \infty$,

$$\begin{aligned} \lim_{L\to\infty} c(L) &\leq \lim_{L\to\infty} \frac{\kappa_{n,\gamma}\int_{-L/2}^{L/2}\int_{\mathbb{R}}(\tilde{v}_L(s+\tau)-\tilde{v}_L(\tau))^2K(s)\,dsd\tau + c_{n,\gamma}\int_{-L/2}^{L/2}\tilde{v}_L(t)^2\,dt}{(\int_{-L/2}^{L/2}\tilde{v}_L(t)^{\beta+1}\,dt)^{\frac{2}{\beta+1}}} \\ &= \frac{\kappa_{n,\gamma}\int_{\mathbb{R}}\int_{\mathbb{R}}(b(t)-b(\tau))^2K(t-\tau)\,dtd\tau + c_{n,\gamma}\int_{\mathbb{R}} b(t)^2\,dt}{(\int_{\mathbb{R}} b(t)^{\beta+1}\,dt)^{\frac{2}{\beta+1}}} < \infty, \end{aligned}$$

since the "bubble" (4.4.11) has finite energy. Let us check that all the integrals above are uniformly bounded in order to use the Dominated Convergence Theorem. First, both integrals $\int_{-L/2}^{L/2}\tilde{v}_L^2(t)\,dt$ and $\int_{-L/2}^{L/2}\tilde{v}_L^{\beta+1}(t)\,dt$ are uniformly bounded since $b(t) \sim e^{-\frac{n-2\gamma}{2}|t|}$. Finally, recalling that $b(t), \eta_L \in L^\infty$ and the behaviour of the kernel (4.2.10),

$$\int_{-L/2}^{L/2}\int_{\mathbb{R}}(\tilde{v}_L(s+\tau)-\tilde{v}_L(\tau))^2K(s)\,dsd\tau = I_1 + I_2,$$

where

$$\begin{aligned} I_1 &\sim \int_{-L/2}^{L/2}\int_{\mathbb{R}\setminus[-\epsilon,\epsilon]}(\tilde{v}_L(s+\tau)-\tilde{v}_L(\tau))^2e^{-|s|\frac{n+2\gamma}{2}}\,dsd\tau \\ &\sim \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} e^{-|s|\frac{n+2\gamma}{2}}\int_{-L/2}^{L/2}\tilde{v}_L(s+\tau)^2\,d\tau ds + \int_{-L/2}^{L/2}\tilde{v}_L(\tau)^2\,d\tau\int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} e^{-|s|\frac{n+2\gamma}{2}}\,ds < \infty. \\ I_2 &\sim \int_{-L/2}^{L/2}\int_{-\epsilon}^{\epsilon}\frac{(\tilde{v}_L(s+\tau)-\tilde{v}_L(\tau))^2}{|s|^{1+2\gamma}}\,dsd\tau \sim \int_{-L/2}^{L/2}\int_{-\epsilon}^{\epsilon}\tilde{v}_L'(\tau)^2|s|^{1-2\gamma}\,dsd\tau < \infty. \end{aligned}$$

In this second integral, we have used the Taylor expansion of $\tilde{v}_L$.

On the other hand, $c^*(L) = c_{n,\gamma} L^{\frac{\beta-1}{\beta+1}} \to +\infty$ as $L \to +\infty$. This proves (4.4.9).

$\square$

Remark 4.4.5. When $L \to \infty$, the minimizer v_L for the functional given in (4.4.2) satisfies that

$$v_L \to v_\infty \equiv b,$$

where $b(t)$ is defined as in (4.4.10) up to multiplicative constant. The proof of this fact will be postponed to the next Chapter 5.

Let v be a L-periodic solution of equation (4.1.1), i.e.,

$$\kappa_{n,\gamma} P.V. \int_0^L (v(t) - v(\tau)) K_L(t-\tau)\, d\tau + c_{n,\gamma} v(t) = c_{n,\gamma} v(t)^\beta. \tag{4.4.13}$$

The linearization of this equation around the constant solution $v_1 \equiv 1$ is:

$$\kappa_{n,\gamma} \int_0^L (v(t) - v(\tau)) K_L(t-\tau)\, d\tau - c_{n,\gamma}(\beta - 1) v(t) = 0. \tag{4.4.14}$$

We consider the eigenvalue problem for this linearized operator:

$$\kappa_{n,\gamma} \int_0^L (v(t) - v(\tau)) K_L(t-\tau)\, d\tau - c_{n,\gamma}(\beta - 1) v(t) = \delta_L v(t). \tag{4.4.15}$$

Lemma 4.4.6. There exists $\tilde{L}_0^\gamma > 0$ such that

$$\delta_L < 0 \text{ if } L > \tilde{L}_0^\gamma, \quad \delta_L > 0 \text{ if } L < \tilde{L}_0^\gamma, \text{ and } \delta_{\tilde{L}_0^\gamma} = 0.$$

Proof. Following the computations in the previous Chapter 3 we get that the first eigenvalue δ_L is given by the implicit expression

$$\frac{\left|\Gamma(\frac{n}{4} + \frac{\gamma}{2} + \frac{\sqrt{\lambda}}{2} i)\right|^2}{\left|\Gamma(\frac{n}{4} - \frac{\gamma}{2} + \frac{\sqrt{\lambda}}{2} i)\right|^2} = \frac{n + 2\gamma}{n - 2\gamma} \frac{\left|\Gamma\left(\frac{1}{2}\left(\frac{n}{2} + \gamma\right)\right)\right|^2}{\left|\Gamma\left(\frac{1}{2}\left(\frac{n}{2} - \gamma\right)\right)\right|^2} + \delta_L.$$

Here λ is univocally related with the period by $L = \frac{2\pi}{\sqrt{\lambda}}$. δ_L is a strictly decreasing function of L. We now define $\tilde{L}_0^\gamma$ as the period corresponding to the zero eigenvalue. $\square$

We are now ready to conclude the proof of Theorem 4.1.1. Let

$$L_0^\gamma = \sup\{L \mid c(l) = c^*(l) \text{ for } l \in (0, L)\}. \tag{4.4.16}$$

By Lemma 4.4.3 we see that $L_0^\gamma > 0$. By Lemma 4.4.4, also $L_0^\gamma < +\infty$. Then we are left to check that if $L = L_0^\gamma$ we just have the constant solution.

Proposition 4.4.7. If $L = \tilde{L}_0^\gamma$ the unique solution for (4.4.13) is the constant solution $v_1 \equiv 1$.

Proof. Let $v > 0$ and $v_1 \equiv 1$ be $\tilde{L}_0^\gamma$-periodic solutions of (4.4.13). We define

$$w = v - 1. \tag{4.4.17}$$

On the one hand, using the weak formulation for the problem (4.4.13) given in (4.4.7), we have

$$\langle \mathscr{L}_\gamma^L v, \phi \rangle = \langle \tilde{\mathscr{L}}_\gamma^L v, \phi \rangle + c_{n,\gamma} \int_0^L v(t)\phi(t)\, dt = c_{n,\gamma} \int_0^L v^\beta(t)\phi(t)\, dt, \quad \forall \phi \in H_L^\gamma, \tag{4.4.18}$$

where we have defined

$$\langle \tilde{\mathscr{L}}_\gamma^L v, \phi \rangle = \kappa_{n,\gamma} P.V. \int_0^L \int_0^L (v(t) - v(\tau))(\phi(t) - \phi(\tau)) K_L(t - \tau)\, dt\, d\tau.$$

Thus, in particular for $v = w + 1$, equation (4.4.18) reads

$$\langle \tilde{\mathscr{L}}_\gamma^L(w), \phi \rangle + c_{n,\gamma} \int_0^L (w(t) + 1)\phi(t)\, dt = c_{n,\gamma} \int_0^L (w(t) + 1)^\beta \phi(t)\, dt, \quad \forall \phi \in H_L^\gamma,$$

which, interchanging ϕ and w in the first term, is equivalent to

$$\langle \tilde{\mathscr{L}}_\gamma^L(\phi), w \rangle + c_{n,\gamma} \int_0^L \Big((w(t) + 1) - (w(t) + 1)^\beta \Big) \phi(t)\, dt = 0, \quad \forall \phi \in H_L^\gamma. \tag{4.4.19}$$

On the other hand, if φ_1 denotes the first eigenfunction for the linearized problem around $v \equiv 1$, given in (4.4.15), for the period $\tilde{L}_0^\gamma$ (i.e. the corresponding to the zero eigenvalue $\delta_{\tilde{L}_0^\gamma} = 0$), the following holds

$$\langle \tilde{\mathscr{L}}_\gamma^L \varphi_1, \phi \rangle + c_{n,\gamma} \int_0^L \varphi_1(t)\phi(t)\, dt = \beta c_{n,\gamma} \int_0^L \varphi_1(t)\phi(t)\, dt, \quad \forall \phi \in H_L^\gamma.$$

Now we choose the test function here to be $\phi = w$, the function defined in (4.4.17), and the equality above becomes

$$\langle \tilde{\mathscr{L}}_\gamma^L \varphi_1, w \rangle = (\beta - 1) c_{n,\gamma} \int_0^L \varphi_1(t) w(t)\, dt. \tag{4.4.20}$$

Coming back to equation (4.4.19) for the test function $\phi = \varphi_1$, then we have

$$\langle \tilde{\mathscr{L}}_\gamma^L(\varphi_1), w \rangle + c_{n,\gamma} \int_0^L \Big((w(t) + 1) - (w(t) + 1)^\beta \Big) \varphi_1(t)\, dt = 0,$$

which using equality (4.4.20) reads

$$\int_0^L \Big(\beta w(t) + 1 - (w(t) + 1)^\beta \Big) \varphi_1(t)\, dt = 0. \tag{4.4.21}$$

The positivity of the first eigenfunction φ_1 and the convexity of the function $f(w) = \beta w(t) + 1 - (w(t) + 1)^\beta$ assure that the only possible solution for (4.4.21) is $w \equiv 0$. $\square$

Let $v \in H_L^\gamma$ and E_L, $\tilde{E}_L$ be the energy functionals for the non-linear and the linear problems (4.4.13) and (4.4.14) defined by

$$E_L(v) := \tfrac{\kappa_{n,\gamma}}{2} \int_0^L \int_0^L (v(t) - v(\tau))^2 K_L(t-\tau)\, d\tau\, dt + \tfrac{c_{n,\gamma}}{2} \int_0^L v^2(t) - \tfrac{c_{n,\gamma}}{\beta+1} \int_0^L v^{\beta+1}(t)\, dt, \tag{4.4.22}$$

and

$$\tilde{E}_L(v) := \tfrac{\kappa_{n,\gamma}}{2} \int_0^L \int_0^L (v(t) - v(\tau))^2 K_L(t-\tau)\, d\tau\, dt - \tfrac{c_{n,\gamma}}{2}(\beta - 1) \int_0^L v^2(t)\, dt, \tag{4.4.23}$$

respectively. The variational formulation of the first eigenvalue δ_L (Rayleygh quotient) for (4.4.23) implies the following Poincaré inequality

$$\kappa_{n,\gamma} \int_0^L \int_0^L (v(t) - v(\tau))^2 K_L(t-\tau)\, d\tau\, dt - c_{n,\gamma}(\beta-1) \int_0^L v^2(t)\, dt \geq \delta_L \int_0^L v^2(t)\, dt, \quad \forall v \in H_L^\gamma.$$

In particular, if φ_1 denotes, as before, the first eigenfunction for the linearized problem around $v \equiv 1$ at the period $\tilde{L}_0^\gamma$, we have the equality

$$\kappa_{n,\gamma} \int_0^{\tilde{L}_0^\gamma} \int_0^{\tilde{L}_0^\gamma} (\varphi_1(t) - \varphi_1(\tau))^2 K_{\tilde{L}_0^\gamma}(t-\tau)\, d\tau\, dt - c_{n,\gamma}(\beta - 1) \int_0^{\tilde{L}_0^\gamma} \varphi_1^2(t)\, dt = 0. \tag{4.4.24}$$

Proposition 4.4.8. The period L_0^γ defined in (4.4.16) coincides with the period $\tilde{L}_0^\gamma$ given by the zero eigenvalue in equation (4.4.15).

Proof. First, because of the definition of L_0^γ given in (4.4.16) we can easily check that $L_0^\gamma \geq \tilde{L}_0^\gamma$. Indeed, Proposition 4.4.7 asserts that $c(\tilde{L}_0^\gamma) = c^*(\tilde{L}_0^\gamma)$ and Lemma 4.4.2 assures that this is not possible if $\tilde{L}_0^\gamma > L_0^\gamma$.

We now are going to check the opposite inequality. We have defined $\tilde{L}_0^\gamma$ as the period where the constant solution $v_1 \equiv 1$ loses stability. This is, if we define

$$L_\epsilon^\gamma = \tilde{L}_0^\gamma + \epsilon \tag{4.4.25}$$

with $\epsilon > 0$, we have instability for the constant solution and thus $c(L_\epsilon^\gamma) < c^*(L_\epsilon^\gamma)$. To prove this, we compute the energy (4.4.22) for the function $1 + \sigma\phi_{L_\epsilon^\gamma}$, where $\sigma > 0$ small enough and $\phi_{L_\epsilon^\gamma} \in H_{L_\epsilon^\gamma}^\gamma$. We have

$$\begin{aligned} E_{L_\epsilon^\gamma}(1 + \sigma\phi_{L_\epsilon^\gamma}) = & E_{L_\epsilon^\gamma}(1) \\ & + \sigma^2 \Bigg[\kappa_{n,\gamma} \int_0^{L_\epsilon^\gamma} \int_0^{L_\epsilon^\gamma} (\phi_{L_\epsilon^\gamma}(t) - \phi_{L_\epsilon^\gamma}(\tau))^2 K_{L_\epsilon^\gamma}(t-\tau)\, d\tau\, dt \\ & \qquad - c_{n,\gamma}(\beta - 1) \int_0^{L_\epsilon^\gamma} \phi_{L_\epsilon^\gamma}^2(t)\, dt \Bigg] \\ & + \text{h.o.t.} \end{aligned}$$

Therefore, if we find $\phi_{L_\epsilon^\gamma}$ such that

$$\kappa_{n,\gamma}\int_0^{L_\epsilon^\gamma}\int_0^{L_\epsilon^\gamma}(\phi_{L_\epsilon^\gamma}(t)-\phi_{L_\epsilon^\gamma}(\tau))^2 K_{L_\epsilon^\gamma}(t-\tau)\,d\tau\,dt - c_{n,\gamma}(\beta-1)\int_0^{L_\epsilon^\gamma}\phi^2_{L_\epsilon^\gamma}(t)\,dt<0, \quad (4.4.26)$$

the instability of $v_1 \in H^\gamma_{L_\epsilon^\gamma}$ is proved. Let $\phi_{L_\epsilon^\gamma}(t)=\varphi_1(\frac{\tilde L_0^\gamma}{L_\epsilon^\gamma}t)$, where φ_1 is the first eigenfunction defined in (4.4.24). Under the changes of variable $\bar t = \frac{L_\epsilon^\gamma}{\tilde L_0^\gamma}t$ and $\bar\tau=\frac{L_\epsilon^\gamma}{\tilde L_0^\gamma}\tau$, equality (4.4.24) and Lemma 4.2.7 imply (4.4.26). Here we have also used that $\beta=\frac{n+2\gamma}{n-2\gamma}>1$, and $L_\epsilon^\gamma>\tilde L_0^\gamma$ (4.4.25).

The definition of L_0^γ (4.4.16) and Lemma 4.4.2 imply $L_0^\gamma \le \tilde L_0^\gamma+\epsilon$. Taking limit as ϵ goes to zero we have the claimed equality $L_0^\gamma=\tilde L_0^\gamma$. □

This completes the proof of Theorem 4.1.1. □

Chapter 5

A gluing approach for the fractional Yamabe problem with isolated singularities

In this chapter, we construct solutions for the fractional Yamabe problem that are singular at a prescribed number of isolated points. This seems to be the first time that a gluing method is successfully applied to a non-local problem. The main step is an infinite-dimensional Lyapunov-Schmidt reduction method, that reduces the problem to an (infinite dimensional) Toda type system.

5.1 Introduction

We consider the problem of finding solutions for the fractional Yamabe problem in $\mathbb{R}^n$, $n > 2\gamma$ for $\gamma \in (0,1)$ with isolated singularities at a prescribed finite number of points. This is, to find positive solutions for the equation

$$\begin{cases} (-\Delta_{\mathbb{R}^n})^\gamma u = c_{n,\gamma} u^\beta & \text{in } \mathbb{R}^n \backslash \Sigma, \\ u \to +\infty & \text{as } x \to \Sigma, \end{cases} \tag{5.1.1}$$

where $\Sigma = \{p_1, \cdots, p_k\}$ and

$$\beta = \frac{n+2\gamma}{n-2\gamma}$$

is the critical exponent in dimension n. Remark that we are using the notation t^β to denote the power nonlinearity $|t|^{\beta-1}t$, but this does not constitute any abuse of notation since any solution must be positive thanks to the maximum principle. $c_{n,\gamma} > 0$ is a normalization constant and can be chosen arbitrarily.

Instead, one could look at the singular version. Here the sign of Q_γ is related to the size of the singular set Σ. For instance, when Σ is a smooth submanifold, [96] shows that the positivity of fractional curvature imposes some geometric and topological restrictions, while [175] considers very general singular sets in the case $\gamma \in (1,2)$, with the additional assumption of positive fractional curvature. See also [109] for some capacitary arguments on the local behavior of singularities.

But all these results give necessary conditions for the existence of such metrics. On the contrary, the question of sufficiency is expected to have only partial answers, requiring that Σ has a very particular structure. Here we initiate the study of this issue, looking at the singular Yamabe problem with prescribed isolated singularities at the points $\{p_1, \ldots, p_k\}$.

Thus our main theorem is:

Theorem 5.1.1. *Fixed any configuration $\Sigma = \{p_1, \cdots, p_k\}$ of k different points in $\mathbb{R}^n$, there exists a smooth, positive solution to* (5.1.1).

As a corollary, we also obtain existence of conformal metrics on the unit sphere $\mathbb{S}^n$ of constant fractional curvature with a finite number of isolated singularities. Note that our results will imply that this metric is complete.

We know from the previous chapters that a non-removable isolated singularities for the problem

$$\begin{cases} (-\Delta_{\mathbb{R}^n})^\gamma u = u^\beta \text{ in } \mathbb{R}^n \backslash \{0\}, \\ u \to +\infty \text{ as } x \to 0, \quad u > 0, \end{cases} \tag{5.1.2}$$

must satisfy the asymptotic behavior

$$c_1 r^{-\frac{n-2\gamma}{2}} \leq u(x) \leq c_2 r^{-\frac{n-2\gamma}{2}}, \quad r \to 0,$$

where c_1, c_2 are positive constants and $r = |x|$ (see [33] for more details).

In Chapter 3, we considered the geometric interpretation of (5.1.2) which provides the equivalence of this problem with the fractional Yamabe problem in a cylinder and motivates the change (5.1.3) below. In Chapter 4, using a variational approach, we showed the existence of "Delaunay"-type solutions for (5.1.2), i.e, solutions of the form

$$u_L(r) = r^{-\frac{n-2\gamma}{2}} v_L(-\log r) \text{ on } \mathbb{R}^n \setminus \{0\}, \tag{5.1.3}$$

for some smooth function v_L that is periodic in the variable $t = -\log r$, for any period $L \geq L_0$. L_0 is known as the minimal period and has been completely characterized.

Delaunay-type of solutions are useful in gluing problems, since they model isolated singularities: we cite, for instance, [141, 142, 158] for the construction of constant mean curvature surfaces with Delaunay ends, or [140, 143] for solutions to (5.1.1) in the local case $\gamma = 1$. However, these classical constructions exploit the local nature of the problem and, above all, the fact that (5.1.2) reduces to a standard second ODE in the radial case. There the space of solutions of this ODE can be explicitly written in terms of two given parameters, which is not the case for a non-local equation.

Here we are able to use the gluing method for the non-local problem (5.1.1). This seems to be the first time where this construction is successfully applied in a non-local setting. The first difficulty is obvious: one needs to make sure that the errors created by the cut-and-glue procedure are not propagated by the non-locality of the problem but, instead, they can be handled through careful estimates.

Nevertheless, the main obstacle we find is the lack of a standard ODE for the calculation of model radial solutions with an isolated singularity, as one does in the classical cases. As we have mentioned this is the starting point of [140] or [141]. Thus, even though a Delaunay solution is our basic model for an isolated singularity, we construct bubble towers at each singular point that consist of perturbed half-Delaunay solutions (also known as half-Dancer solutions).

Our source of inspiration for this approach is [132], where the author constructs new entire solutions for a semilinear equation with subcritical exponent, different from the spike solutions that were known for a long time. Malchiodi's new solutions do not tend to zero at infinity, but decay to zero away from three half lines; the method is to construct a half Dancer solution along each half-line.

The idea of gluing bubble towers allows to construct a suitable approximate solution for (5.1.1) with an infinite number of parameters to be chosen. Note that the linearization at this approximate solution is not injective due to the presence of an infinite dimensional kernel, so we use a Lyapunov-Schimdt reduction procedure. It is well known that one single bubble is non-degenerate [58], and the kernel can be explicitly characterized. However, for our problem we perturb each bubble in the bubble tower separately; we find an infinite dimensional system of compatibility conditions, of Toda type, that allows to solve the original problem from the perturbed one.

These compatibility conditions do not impose any restrictions on the location of the singularity points $p_1, \dots, p_k$, but only on the Delaunay parameter (the neck size) at each point. We also remark that the first compatibility condition is analogous to that of the local case $\gamma = 1$ of [140], this is due to the strong influence of the underlying geometry, while the rest of the configuration depends on the Toda type system. On the other hand, in the local setting a similar procedure to remove the resonances of the linearized problem was considered in [12] and the references therein. However, in their case the Toda type system is finite dimensional.

We remark here that in all our results we do not use the well known extension problem for the fractional Laplacian [36]. Instead we are inspired to the previous Chapter 4 to rewrite the fractional Laplacian in radial coordinates in terms of a new integro-differential operator in the variable t. In any case, if we write our problem in the extension, at least for the linear theory, it provides an example of an edge boundary value problem of the type considered in [136, 145].

There are still many open problems. For instance, to find radially symmetric solutions for the fourth order Q-curvature equation. Here the difficulty is the lack of maximum principle, which one may be able to handle using [104]. We hope to return to this problem elsewhere.

The next natural question is to look at problem (5.5.7) when the singular set Σ has larger Hausdorff dimension N. In this case, in order to have a solution one needs to impose some necessary conditions (see [96, 175]). The existence of singular solutions with larger Hausdorff dimension singular set will be studied separately.

The chapter will be structured as follows: in Section 5.2 we recall some results about Delaunay solutions for (5.1.2) from the previous Chapter 4, while in Section 5.3 we use those as models to construct a suitable approximate solution for our problem. Sections 5.4 and 5.5 are of technical nature. Finally, the proof of Theorem 5.1.1 is contained in Section 5.6.

5.2 Delaunay-type solutions

In this section we recall some results explained in the previous Chapters 3 and 4 on the Delaunay solutions of

$$(-\Delta_{\mathbb{R}^n})^\gamma u = c_{n,\gamma} u^\beta \text{ in } \mathbb{R}^n \backslash \{0\}. \tag{5.2.1}$$

We may reduce (5.2.1) by writing

$$u(x) = r^{-\frac{n-2\gamma}{2}} v(-\log|x|)$$

and using $t = -\log|x|$.

There are two distinguished solutions to (5.2.1):

i. The cylinder, which is $v(t) \equiv C$, that corresponds to the singular solution $u(x) = Cr^{-\frac{n-2\gamma}{2}}$.

ii. The standard sphere (also known as "bubble")

$$v(t) = (\cosh(t-t_0))^{-\frac{n-2\gamma}{2}},$$

for any $t_0 \in \mathbb{R}$, which is regular at the origin.

Moreover, it is well-known that all the smooth solutions to problem (5.1.1) are of the form

$$w(x) = \Big(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\Big)^{\frac{n-2\gamma}{2}}.$$

For the standard bubble solution we have the following non-degeneracy result (Theorem 1 in [58]):

Lemma 5.2.1. *The solution $w(x) = (\frac{1}{1+|x|^2})^{\frac{n-2\gamma}{2}}$ of (5.1.1) is non-degenerate in the sense that all bounded solutions of equation*

$$(-\Delta)^\gamma \psi - c_{n,\gamma}\beta w^{\beta-1}\psi = 0 \textit{ in } \mathbb{R}^n$$

are linear combinations of the functions

$$\frac{n-2\gamma}{2} w + x\cdot\nabla w, \quad \textit{and} \quad \partial_{x_i} w,\ 1 \le i \le n.$$

Note that we normalize the constant $c_{n,\gamma}$ in (5.2.1) such that the standard bubble is a solution. The exact value of the constants may be found in Chapter 3 but in this chapter this is not important.

In the previous Chapter 4, we considered the existence of solutions $v(t)$ which are periodic in t. Using the change of variable $t = -\log|x|$, the equation (5.2.1) can be written as

$$\mathcal{L}_\gamma v = c_{n,\gamma} v^\beta, \quad t \in \mathbb{R},\ v > 0, \tag{5.2.2}$$

where $\mathcal{L}_\gamma$ is the linear operator defined by

$$\mathcal{L}_\gamma v = \kappa_{n,\gamma} P.V. \int_{-\infty}^{+\infty} (v(t) - v(\tau)) K(t-\tau)\, d\tau + c_{n,\gamma} v(t)$$

for K a singular kernel given in (4.2.7) and

$$\kappa_{n,\gamma} = \pi^{-\frac{n}{2}} 2^{2\gamma} \frac{\Gamma(\frac{n}{2}+\gamma)}{\Gamma(1-\gamma)} \gamma.$$

The asymptotic behaviour for K is given the previous Chapter 4 in Lemma 4.2.5. Since we are looking for periodic solutions of (5.2.2), we assume that $v(t+L) = v(t)$; in this case, equation (5.2.2) becomes

$$\mathcal{L}_\gamma^L(v) = c_{n,\gamma} v^\beta, \quad v > 0,$$

where

$$\mathcal{L}_\gamma^L(v) = \kappa_{n,\gamma} P.V. \int_{-\frac{L}{2}}^{\frac{L}{2}} (v(t) - v(\tau)) K_L(t-\tau)\, d\tau + c_{n,\gamma} v$$

for the singular kernel

$$K_L(t-\tau) = \sum_{j \in Z} K(t - \tau - jL).$$

We are going to consider the problem

$$\begin{cases} \mathcal{L}_\gamma^L(v) = c_{n,\gamma} v^\beta & \text{in } \ (-\frac{L}{2}, \frac{L}{2}), \\ v'(-\frac{L}{2}) = v'(\frac{L}{2}) = 0. \end{cases} \tag{5.2.3}$$

For this we shall work with the norm given by

$$\|v\|_{H_L^\gamma} = \Big(\int_{-L/2}^{L/2} \int_{-L/2}^{L/2} (v(t) - v(\tau))^2 K_L(t-\tau)\, d\tau dt + \int_{-L/2}^{L/2} v^2\, dt \Big)^{1/2},$$

and the following functional space

$$H_L^\gamma = \Big\{ v : (-\tfrac{L}{2}, \tfrac{L}{2}) \to \mathbb{R}\ ;\ v'(-\tfrac{L}{2}) = v'(\tfrac{L}{2}) = 0 \text{ and } \|v\|_{H_L^\gamma} < \infty \Big\}.$$

Proposition 5.2.2. *Consider problem (5.2.3). Then for L large there exists a unique positive solution v_L in H_L^γ with the following properties* *(a)* v_L *is even in* t*;*

(b) $v_L = \sum_{j\in Z} v(t-jL) + \psi_L$*, where* $\|\psi_L\|_{H^L_\gamma} \to 0$ *as* $L\to\infty$*,*

where

$$v(t) := (\cosh t)^{-\frac{n-2\gamma}{2}}$$

corresponds to the standard bubble solution.

More precisely, for $\gamma\in(0,1)$*, and for* L *large we have the following Holder estimates on* ψ_L*:*

$$\|\psi_L\|_{\mathcal{C}^{2\gamma+\alpha}(-L/2,L/2)} \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}$$

for some $\alpha\in(0,1)$*, and* $\xi>0$ *independent of* L *large.*

As an immediate consequence of Proposition 5.2.2 we obtain periodic solutions for the original equation (5.2.2):

Corolary 5.2.3. *For* L *large there exists a unique positive solution* v_L *of* (5.2.2) *with the following properties*

(a) v_L *is periodic and even in* t*;*

(b) $v_L = \sum_{j\in\mathbb{Z}} v(t-jL) + \psi_L$*, where* $\|\psi_L\|_{H^\gamma_L} \to 0$ *as* $L\to\infty$ *in* $(-L/2, L/2)$*.*

More precisely, for $\gamma\in(0,1)$*, and for* L *large we have the following Holder estimates on* ψ_L*:*

$$\|\psi_L\|_{\mathcal{C}^{2\gamma+\alpha}} \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}$$

for some $\alpha\in(0,1)$*, and* $\xi>0$ *independent of* L*.*

Proof of Proposition 5.2.2. We denote the function

$$v_{0,L}(t) = \sum_{j=-\infty}^{\infty} v_j(t), \tag{5.2.4}$$

where $v_j(t) = v(t-jL) = (\cosh(t-jL))^{-\frac{n-2\gamma}{2}}$. By symmetry, this function satisfies the boundary condition at $t=\pm\frac{L}{2}$. We consider next the functional

$$\begin{aligned} F_L(v) =& \frac{\kappa_{n,\gamma}}{4}\int_{-L/2}^{L/2}\int_{-L/2}^{L/2} (v(t)-v(\tau))^2 K_L(t-\tau)\,d\tau dt \\ &+ \frac{c_{n,\gamma}}{2}\int_{-L/2}^{L/2} v^2\,dt - \frac{c_{n,\gamma}}{\beta+1}\int_{-L/2}^{L/2} v^{\beta+1}\,dt, \end{aligned}$$

in the space

$$v\in H^\gamma_*, \quad H^\gamma_* = \{v\in H^\gamma_L,\ v(t)=v(-t)\}.$$

Solutions of equation (5.2.3) are critical points of F_L. Moreover, we have

$$F'_L(v_{0,L})[\varphi] = \langle \mathcal{L}_\gamma^L(v_{0,L}), \varphi\rangle - c_{n,\gamma}\int_{-L/2}^{L/2} v_{0,L}^\beta \varphi\, dt = \langle S(v_{0,L}), \varphi\rangle$$

for every test function φ, where $\langle\, ,\, \rangle$ is defined by

$$\begin{aligned}\langle \mathcal{L}_\gamma^L(v_{0,L}), \varphi\rangle =& \frac{\kappa_{n,\gamma}}{2} P.V. \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} (v(t)-v(\tau))(\varphi(t)-\varphi(\tau))K_L(t-\tau)\, dt d\tau \\ &+ c_{n,\gamma}\int_{-L/2}^{L/2} v_{0,L}\varphi(t)\, dt\end{aligned}$$

and

$$S(v_{0,L}) := \mathcal{L}_\gamma^L(v_{0,L}) - c_{n,\gamma} v_{0,L}^\beta.$$

Therefore, using Hölder's inequality, we easily get

$$\|F'_L\|_{H^\gamma} \le C\|S(v_{0,L})\|_{L^2},$$

where C is independent of L large. Hence, we need to estimate the L^2 norm of $S(v_{0,L})$ in $(-L/2, L/2)$. Recalling (5.2.4) and the definition of v_j, we have

$$S(v_{0,L}) = c_{n,\gamma}\big[\sum_{j=-\infty}^{\infty} v_j^\beta - (\sum_{j=-\infty}^{\infty} v_j)^\beta\big] \quad \text{in } (-L/2, L/2).$$

For $t \ge 0$, since $v_{-j} \le v_j$, by symmetry, for L large,

$$|S(v_{0,L})| \le C v_0^{\beta-1}\sum_{j\ne 0} v_j + \sum_{j\ne 0} v_j^\beta.$$

As a consequence, we have

$$\int_{-L/2}^{L/2} S(v_{0,L})^2 dt \le C\int_{-L/2}^{L/2} v_0^{2(\beta-1)}(\sum_{j\ne 0} v_j)^2 + \int_{-L/2}^{L/2}(\sum_{j\ne 0} v_j^\beta)^2.$$

In order to estimate the first term, we divide the domain into two subsets, $\{|t| \le \frac{\alpha L}{2}\}$ and $\{|t| \ge \frac{\alpha L}{2}\}$ for $\alpha \in (0,1)$. In these two sets we have the estimates $\sum_{j\ne 0} v_j \le Ce^{-\frac{(n-2\gamma)L}{4}(2-\alpha)}$ and $\sum_{j\ne 0} v_j \le Ce^{-\frac{(n-2\gamma)L}{4}}$, respectively, by the exponential decay of v_0. Hence one easily finds

$$\begin{aligned}\int_{-L/2}^{L/2} v_0^{2(\beta-1)}(\sum_{j\ne 0} v_j)^2 dt &\le Ce^{-\frac{(n-2\gamma)L}{2}(2-\alpha)} + Ce^{-\frac{(n-2\gamma)L}{2}} e^{-2(\beta-1)\frac{(n-2\gamma)}{2}\frac{\alpha L}{2}} \\ &= Ce^{-\frac{(n-2\gamma)L}{2}(2-\alpha)} + Ce^{-\frac{(n-2\gamma)L}{2}(1+\frac{4\alpha\gamma}{n-2\gamma})}\end{aligned}$$

and

$$\int_{-L/2}^{L/2}(\sum_{j\neq 0} v_j^{\beta})^2 dt \leq Ce^{-\frac{(n-2\gamma)L\beta}{2}}.$$

In conclusion, we have

$$\|S(v_{0,L})\|_{L^2(-L/2,L/2)} \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}$$

for some $\xi > 0$ independent of L large.

Next we claim that the operator $F_L''(v_{0,L})$ is invertible in the space $H_*^{\gamma}(-L/2, L/2)$. This follows from the non-degeneracy of the standard bubble and the fact that we are working in the subspace of even functions in t. This allows us to solve the problem via local inversion. In fact, we write $v_L = v_{0,L} + \psi$ and we have $F_L'(v_{0,L} + \psi) = 0$ if and only if $\psi \in H_*^{\gamma}(-L/2, L/2)$ satisfies

$$\psi = -(F_L(v_{0,L}))''[F_L'(v_{0,L}) + N(\psi)],$$

where $N(\psi) = c_{n,\gamma}[(v_{0,L} + \psi)^{\beta} - v_{0,L}^{\beta} - \beta v_{0,L}^{\beta-1}\psi]$ is superlinear in ψ. We can apply the contraction mapping theorem, obtaining a solution ψ which satisfies

$$\|\psi\|_{H_L^{\gamma}} \leq C\|F_L'(v_{0,L})\|_{H_L^{\gamma}} \leq C\|S(v_{0,L})\|_{L^2} \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}.$$

For $\gamma \in (0, 1)$, by the regularity estimates given in Chapter 4 and summarized in Remark 4.3.12 in the same chapter (see also [64]), it follows that ψ is smooth and we have the following estimate:

$$\|\psi\|_{\mathcal{C}^{2\gamma+\alpha}} \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}.$$

The maximum principle (Proposition 4.3.8) of Chapter 4 concludes the proof of the proposition.

□

Remark 5.2.4. Since the equation for v is translational invariant, if $v(t)$ is a solution of (5.2.2), then $v(t - t_0)$ is also a solution. In the following, we will use the periodic solution v_L with period L which attains its minimum at the points $t = jL$, $j \in \mathbb{Z}$. By Corollary 5.2.3, this periodic solution can be expressed as a perturbation of a bubble tower (or Dancer solution)

$$v_L(t) = \sum_{j=-\infty}^{\infty} v\big(t - \tfrac{L}{2} - jL\big) + \psi_L(t),$$

where $\|\psi_L\|_{\mathcal{C}^{2\gamma+\alpha}} \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}$ for some $\xi > 0$ independent of L. For the rest of the chapter we write

$$t_j = \tfrac{L}{2} + jL, \quad j \in \mathbb{Z},$$

and

$$v_j(t) := v(t - t_j) = \cosh(t - t_j)^{-\frac{n-2\gamma}{2}},\ t \in \mathbb{R}.$$

Now we consider only half a bubble tower; this is needed in order to have fast decay far from the singularity ($t \to -\infty$). We define

$$\tilde{v}_L(t) = \sum_{j=0}^{\infty} v\big(t - \tfrac{L}{2} - jL\big) + \psi_L(t),$$

then one has the following asymptotic behaviour of $\tilde{v}_L$:

$$\tilde{v}_L(0) = e^{-\frac{(n-2\gamma)L}{4}}(1 + o(1)),$$

(this is the *neck size*). And for $t \le 0$, i.e. $|x| \ge 1$, using the fact that v is exponential decaying,

$$\tilde{v}_L(t) = v_0(t)(1 + o(1)) = \big(\cosh(t - \tfrac{L}{2})\big)^{-\frac{n-2\gamma}{2}}(1 + o(1)) = |x|^{-\frac{n-2\gamma}{2}} e^{-\frac{(n-2\gamma)L}{4}}(1 + o(1)),$$

and the corresponding solution $\tilde{u}_L = |x|^{-\frac{n-2\gamma}{2}} \tilde{v}_L$ satisfies

$$\tilde{u}_L(x) = |x|^{-\frac{n-2\gamma}{2}} \tilde{v}_L = |x|^{-(n-2\gamma)} e^{-\frac{(n-2\gamma)L}{4}}(1 + o(1)). \tag{5.2.5}$$

5.3 Construction of the approximate solutions

We now proceed to define a family of approximate solutions to the problem using the Delaunay solutions from the previous section. We know that the Delaunay solution with period L has the form of a bubble tower, i.e,

$$\begin{aligned} u_L(x) &= |x|^{-\frac{n-2\gamma}{2}} \Big(\sum_{j=-\infty}^{\infty} v\big(-\log|x| - \tfrac{L}{2} - jL\big) + \psi_L(-\log|x|)\Big) \\ &=: \sum_{j=-\infty}^{\infty} \Big(\frac{\lambda_j}{\lambda_j^2 + |x|^2}\Big)^{\frac{n-2\gamma}{2}} + \phi_L(x), \end{aligned} \tag{5.3.1}$$

where

$$\lambda_j = e^{-\frac{1+2j}{2}L} \quad \text{and} \quad \phi_L(x) = |x|^{-\frac{n-2\gamma}{2}} \psi_L(-\log|x|), \tag{5.3.2}$$

for ψ_L the perturbation function constructed in Corollary 5.2.3.

As we have mentioned, one of the main ideas is that, although we would like the approximate solution to have Delaunay-type singularities around each point of Σ, it should have a fast decay once we are away from Σ in order to glue to the background manifold $\mathbb{R}^n$. To this end, we will only take half a Delaunay solution (this is, only values $j = 0, 1, \ldots$).

In addition, we would like to introduce some perturbation parameters $R \in \mathbb{R}$, $a \in \mathbb{R}^n$, since each standard bubble has $n + 1$ free parameters which correspond to scaling and translations. This is done for each bubble in the bubble tower independently, thus we will have an infinite dimensional set of perturbations.

Keeping both aspects in mind, let us give the precise construction of our approximate solution $\bar{u}$. First, one can always assume that all the balls $B(p_i, 2)$ are disjoint, since we may dilate the problem by some factor $\kappa > 0$ that will change the set Σ into $\kappa\Sigma$ and a function u defined in $\mathbb{R}^n \backslash \Sigma$ into $\kappa^{-\frac{n-2\gamma}{2}} u(x/\kappa)$ defined in $\mathbb{R}^n \backslash \kappa\Sigma$.

Let χ be a cut-off function such that

$$\chi(x) = \begin{cases} 1, & \text{if } |x| \leq \frac{1}{2}, \\ 0, & \text{if } |x| \geq 1, \\ \chi \in [0,1], & \text{if } \frac{1}{2} \leq |x| \leq 1, \end{cases}$$

and set $\chi_i(x) = \chi(x - p_i)$.

Given $L > 0$ large enough, we will fix

$$\bar{L} = (L_1, \cdots, L_k)$$

to be the Delaunay parameters, which also are related to the neck sizes of each Delaunay solution. They will be chosen (large enough) in the proof. They will satisfy the following conditions:

$$|L_i - L| \leq C.$$

More precisely, they will be related by the following:

$$q_i e^{-\frac{(n-2\gamma)L}{4}} = e^{-\frac{(n-2\gamma)L_i}{4}}, \quad i = 1, \ldots, k \tag{5.3.3}$$

Also, for $i = 1, \ldots, k$, $j = 0, 1, \ldots$, set $a_j^i \in \mathbb{R}^n$ and $R_j^i = R^i(1 + r_j^i) \in \mathbb{R}$ to be the perturbation parameters. Define the approximate solution $\bar{u}$ as

$$\begin{aligned} \bar{u}(x) &= \sum_{i=1}^{k} \Big[\sum_{j=0}^{\infty} \big[|x - p_i - a_j^i|^{-\frac{n-2\gamma}{2}} v(-\log|x - p_i - a_j^i| - \tfrac{L_i}{2} - jL_i + \log R_j^i) \big] \\ &\qquad + \chi_i(x)|x - p_j|^{-\frac{n-2\gamma}{2}} \psi_i(-\log|x - p_i| + \log R^i) \Big] \\ &= \sum_{i=1}^{k} \Big[\sum_{j=0}^{\infty} \Big(\frac{\lambda_j^i}{|\lambda_j^i|^2 + |x - p_i - a_j^i|^2} \Big)^{\frac{n-2\gamma}{2}} + \chi_i(x)\phi_i(x - p_i) \Big] \\ &=: \sum_{i=1}^{k} \Big[\sum_{j=0}^{\infty} w_j^i + \chi_i(x)\phi_i(x - p_i) \Big], \end{aligned} \tag{5.3.4}$$

where we have set

$$\lambda_j^i = R_j^i e^{-\frac{(1+2j)L_i}{2}}.$$

Next we will explain in detail the perturbation parameters q_i, a_j^i, R_j^i. First fix a set of positive numbers $q_1^b, \cdots, q_k^b$, and let $a_0^{i,b}, R^{i,b}$ be determined by the following balancing conditions:

$$q_i^b = A_2 \sum_{l \neq i} q_{i'}^b (R^{i,b} R^{i',b})^{\frac{n-2\gamma}{2}} |p_i - p_{i'}|^{-(n-2\gamma)}, \quad i = 1, \ldots, k, \tag{5.3.5}$$

and

$$\frac{a_0^{i,b}}{(\lambda_0^{i,b})^2} = -\frac{A_3}{A_0}\sum_{i'\neq i}\frac{p_{i'}-p_i}{|p_{i'}-p_i|^{n-2\gamma+2}}\frac{q_{i'}^b}{q_i^b}(R^{i,b}R^{i',b})^{\frac{n-2\gamma}{2}}, \quad i=1,\dots,k, \tag{5.3.6}$$

where $\lambda_0^{i,b} = R^{i,b}e^{-\frac{(1+2j)L_i^b}{2}}$, and the L_i^b are defined from the q_i^b, $i=1,\dots,k$ and the constants A_0,$A_2>0$, $A_3<0$ are defined in Appendix 8.

Remark 5.3.1. It has been shown in Remark 3 of [140] that for $\bar{q} := (q_1^b,\dots,q_k^b)$ in the positive octant, there exists a solution $R^{i,b}$ to equation (5.3.5). Once $R^{i,b}$ is chosen, then we can use equation (5.3.6) to determine $a_0^{i,b}$.

Although the meaning of these compatibility conditions will become clear in the next sections, we have just seen that they are the analogous to those of [140] for the local case. The idea is that, at the *base* level, perturbations should be very close to those for a single bubble. This also shows, in particular, that although our problem is non-local, very near the singularity it presents a local behavior due to the strong influence of the underlying geometry.

However, for the rest of the parameters a_j^i, R_j^i, $i=1,\dots k$, $j=0,1,\dots$, we will have to solve an infinite dimensional system of equations. First let R^i, q_i be $2k$ parameters which satisfy

$$|R^i - R^{i,b}|\le C, \quad |q_i - q_i^b|\le C. \tag{5.3.7}$$

and let $\lambda_0^{i,0} = R^i e^{-\frac{(1+2j)L_i}{2}}$.

Set also $\hat{a}_0^i$ given by $\frac{a_0^{i,0}}{(\lambda_0^{i,0})^2} = \hat{a}_0^i$ be k parameters satisfying

$$|\hat{a}_0^i - \hat{a}_0^{i,b}|\le C, \tag{5.3.8}$$

where $\hat{a}_0^{i,b} = \frac{a_0^{i,b}}{(\lambda_0^{i,b})^2}$.

Last we define the parameters R_j^i, a_j^i by

$$R_j^i = R^i(1+r_j^i), \quad \frac{a_j^i}{(\lambda_j^i)^2} = \bar{a}_j^i = \hat{a}_0^i + \tilde{a}_j^i, \quad i=1,\dots,k, \quad j=0,\dots,\infty, \tag{5.3.9}$$

where $r_j^i, \tilde{a}_j^i$ satisfy

$$|r_j^i|\le Ce^{-\tau t_j^i},\ |\tilde{a}_j^i|\le Ce^{-\tau t_j^i}, \tag{5.3.10}$$

for some $\tau>0$, where $t_j^i = (\frac{1}{2}+j)L_i$. The exact value of the parameters will be determined in Section 5.6.

Let us give some explanation about the choice of parameters. Given the $k(n+2)$ balancing parameters $q_i^b, R^{i,b}, \hat{a}_0^{i,b}$ satisfying the balancing conditions (5.3.5)-(5.3.6), we first choose $k(n+2)$ initial perturbation parameters $q_i, R^i, \hat{a}_0^i$ which are close to the balancing parameters, i.e (5.3.7)-(5.3.8). After that, we introduce infinitely many other perturbation parameters $\tilde{a}_j^i, r_j^i$ which are exponential decaying in t_j^i, i.e. (5.3.9)-(5.3.10).

We will prove next some quantitative estimates on the function $\bar{u}$, and in particular on its behaviour near the singular points. Before that, we need to introduce the function spaces we will work with.

Definition 5.3.2. *We set the weighted norm*

$$\|u\|_{\mathcal{C}^{\alpha}_{\gamma_1,\gamma_2}} = \|\mathrm{dist}\,(x,\Sigma)^{-\gamma_1}u\|_{\mathcal{C}^{\alpha}(B_1(\Sigma))} + \||x|^{-\gamma_2}u\|_{\mathcal{C}^{\alpha}(\mathbb{R}^n\setminus B_1(\Sigma))}.$$

In other words, to check if u is an element of some $\mathcal{C}^{\alpha}_{\gamma_1,\gamma_2}$, it is sufficient to check that u is bounded by a constant times $|x-p_i|^{\gamma_1}$ and has its ℓ-th order partial derivatives bounded by a constant times $|x-p_i|^{\gamma_1-\ell}$ for $\ell\leq\alpha$ near each singular point p_i. Away from the singular set Σ, u is bounded by $|x|^{\gamma_2}$ and has its ℓ-th order partial derivatives bounded by a constant times $|x|^{\gamma_2-\ell}$ for $\ell\leq\alpha$ (note that here we are implicitly assuming that $0\in\Sigma$, in order to simplify the notation).

First, we define $Z^i_{j,l}$ to be the (normalized) approximate kernels

$$Z^i_{j,0}=\frac{\partial}{\partial r^i_j}w^i_j,\quad Z^i_{j,l}=\lambda^i_j\frac{\partial}{\partial a^i_{j,l}}w^i_j=-\lambda^i_j\frac{\partial}{\partial x_l}w^i_j,\quad l=1,\cdots,n.$$

Without loss of generality, assume in the following that $p_i=0$. For $l=0$ we will repeatedly use the following estimates

$$|Z^i_{j,0}|\leq C\begin{cases}|x|^{-\frac{n-2\gamma}{2}}|v^i_j|, & |x|\leq 1,\\ |x|^{-(n-2\gamma)}(\lambda^i_j)^{\frac{n-2\gamma}{2}}, & |x|\geq 1.\end{cases}\tag{5.3.11}$$

In addition, for $l=1,\ldots,n$, we have

$$Z^i_{j,l}=(n-2\gamma)(v^i_j)^{1+\frac{2}{n-2\gamma}}|x-a^i_j-p_i|^{-\frac{n-2\gamma}{2}-1}(x-a^i_j-p_i)_l,\tag{5.3.12}$$

where we have used the obvious notation $w^i_j=|x-p_i-a^i_j|^{-\frac{n-2\gamma}{2}}v^i_j$. Then one has the following *orthogonality* conditions (recentering at $p_i=0$):

$$\begin{aligned}&\int_{\mathbb{R}^n}(w^i_j)^{\beta-1}Z^i_{j,l}Z^i_{j',l'}\,dx\\
&\quad=\frac{4(n-2\gamma)^2}{n}\delta_{l,l'}\int_{\mathbb{R}^n}|x|^{-2\gamma}(v_j)^{\beta-1}|x|^{-\frac{n-2\gamma}{2}}(v_j)^{\frac{n-2\gamma+2}{n-2\gamma}}|x|^{-\frac{n-2\gamma}{2}}v_{j'}^{\frac{n-2\gamma+2}{n-2\gamma}}\,dx+o(1)\\
&\quad=\frac{4(n-2\gamma)^2}{n}\delta_{l,l'}\int_{\mathbb{R}^n}|x|^{-n}(v_j)^{\beta+\frac{2}{n-2\gamma}}(v_{j'})^{1+\frac{2}{n-2\gamma}}\,dx+o(1)\\
&\quad=\frac{4(n-2\gamma)^2}{n}(\delta_{l,l'}+o(1))\,e^{-\frac{n-2\gamma+2}{2}|t^i_j-t^i_{j'}|}.\end{aligned}\tag{5.3.13}$$

Similar estimates also hold true for $l = 0$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^n} (w_j^i)^{\beta-1} Z_{j,0}^i Z_{j',0}^i \, dx &= (1 + o(1)) \int_{\mathbb{R}^n} |x - p_i|^{-n} v_j^{\beta-1} v_j' v_{j'}' \, dx \\ &= C_0(1 + o(1)) e^{-\frac{n-2\gamma}{2}|t_j^i - t_{j'}^i|}\end{aligned} \tag{5.3.14}$$

for some $C_0 > 0$.

From now on, we choose $-\frac{n-2\gamma}{2} < \gamma_1 < \min\{-\frac{n-2\gamma}{2} + 2\gamma, 0\}$. Define also

$$\|u\|_* = \|u\|_{\mathcal{C}^{2\gamma+\alpha}_{\min\{\gamma_1, -\frac{n-2\gamma}{2}+\tau\}, -(n-2\gamma)}}, \quad \|h\|_{**} = \|h\|_{\mathcal{C}^{\alpha}_{\min\{\gamma_1, -\frac{n-2\gamma}{2}+\tau\}-2\gamma, -(n+2\gamma)}},$$

and denote by $\mathcal{C}_*$ and $\mathcal{C}_{**}$ the corresponding weighted Hölder spaces. Here τ (small enough) is given in the definition of the perturbation parameters (5.3.9)-(5.3.10). Remark that, to simplify the notation, many times we will ignore the small τ perturbation and just the weight near the singular set as $\operatorname{dist}(x, \Sigma)^{-\gamma_1}$, $\operatorname{dist}(x, \Sigma)^{-(\gamma_1 - 2\gamma)}$, respectively.

Our main result in this section is the following proposition:

Proposition 5.3.3. *Suppose the parameters satisfy (5.3.7)-(5.3.10), and let $\bar{u}$ be as in (5.3.4). Then for L large enough, there exists a function ϕ and a sequence $\{c_{j,l}^i\}$ which satisfies the following properties:*

$$\begin{cases} (-\Delta)^\gamma (\bar{u} + \phi) - c_{n,\gamma} (\bar{u} + \phi)^\beta = \displaystyle\sum_{i=1}^{k} \sum_{j=0}^{\infty} \sum_{l=0}^{n} c_{j,l}^i (w_j^i)^{\beta-1} Z_{j,l}^i, \\ \int_{\mathbb{R}^n} \phi (w_j^i)^{\beta-1} Z_{j,l}^i \, dx = 0 \quad \text{for } i = 1, \cdots, k, \ j = 0, \cdots, \infty, \ l = 0, \cdots, n. \end{cases} \tag{5.3.15}$$

Moreover, one has

$$\|\phi\|_* \leq C e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}, \tag{5.3.16}$$

for some $-\frac{n-2\gamma}{2} < \gamma_1 < \min\{-\frac{n-2\gamma}{2} + 2\gamma, 0\}$ and $\xi > 0$ independent of L large.

The proof is technically involved, so we prove some preliminary lemmas. We first show a result involving the auxiliary linear equation

$$\begin{cases} (-\Delta)^\gamma \phi - c_{n,\gamma} \beta \bar{u}^{\beta-1} \phi = h + \displaystyle\sum_{i=1}^{k} \sum_{j=0}^{\infty} \sum_{l=0}^{n} c_{j,l}^i (w_j^i)^{\beta-1} Z_{j,l}^i, \\ \int_{\mathbb{R}^n} \phi (w_j^i)^{\beta-1} Z_{j,l}^i \, dx = 0 \quad \text{for } i = 1, \cdots, k, \ j = 0, \cdots, \infty, \ l = 0, \cdots, n. \end{cases} \tag{5.3.17}$$

Lemma 5.3.4. *Suppose the parameters satisfy (5.3.7)-(5.3.10). Then there exists a weight γ_1 satisfying $-\frac{n-2\gamma}{2} < \gamma_1 < \min\{-\frac{n-2\gamma}{2} + 2\gamma, 0\}$ such that, given h with $\|h\|_{**} < \infty$, equation (5.3.17) has a unique solution ϕ in the space $\mathcal{C}_*$. Moreover, there exists a constant C independent of L such that*

$$\|\phi\|_* \leq C \|h\|_{**}. \tag{5.3.18}$$

Note that Fredholm properties for the problem (5.3.17) in weighted spaces have been shown in [136, 145], since it is an example of an edge boundary value problem when we look at the usual extension formulation for the fractional Laplacian from [36]. However, in Lemma 5.3.4 we show, in addition, that the estimates are independent of the choice of Delaunay parameters $(L_1, \dots, L_k)$.

We will postpone the proof of this lemma, instead we will show first some quantitative estimates on the function $\bar{u}$ and in particular its behaviour near the singular set Σ and at infinity.

Lemma 5.3.5. *Let $S(\bar{u}) = (-\Delta)^\gamma \bar{u} - c_{n,\gamma}\bar{u}^\beta$. Then if the parameters satisfy (5.3.7)-(5.3.10), we have the following estimate on $S(\bar{u})$:*

$$\|S(\bar{u})\|_{**} \le Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)} \tag{5.3.19}$$

for some $\xi > 0$ independent of L large.

Proof. As usual, for simplicity, we prove the estimates in (5.3.19) for the L^∞ norm, namely, we prove the following estimates:

$$|S(\bar{u})(x)| \le C|x - p_i|^{\min\{\gamma_1, -\frac{n-2\gamma}{2}+\tau\}-2\gamma} e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}, \tag{5.3.20}$$

near each singular point p_i and

$$|S(\bar{u})(x)| \le C|x|^{-(n+2\gamma)} e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}. \tag{5.3.21}$$

for dist $(x, \Sigma) \ge 1$.

First we show the estimates for the particular case that all the parameters a_j^i, r_j^i are zero. Let $\bar{u}_0$ be the approximate solution from (5.3.4) in this case. Without loss of generality, assume $p_1 = 0$ and we consider in the region dist $(x, \Sigma) \ge 1$. In this region, $\chi_i = 0$ for all i, one has

$$\begin{aligned} S(\bar{u}_0) = (-\Delta)^\gamma \bar{u}_0 - c_{n,\gamma}\bar{u}_0^\beta &= (-\Delta)^\gamma (\sum_{i=1}^k \sum_{j=0}^\infty w_j^i + \chi_i\phi_i) - c_{n,\gamma}(\sum_{i=1}^k \sum_{j=0}^\infty w_j^i)^\beta \\ &= -c_{n,\gamma}[(\sum_{i,j} w_j^i)^\beta - \sum_{i,j}(w_j^i)^\beta] + (-\Delta)^\gamma(\sum_{i=1}^k \chi_i\phi_i) =: I_1 + I_2. \end{aligned}$$

First, using the fact that

$$|w_j^i| \sim (\lambda_j^i)^{\frac{n-2\gamma}{2}} |x|^{-(n-2\gamma)},$$

and recalling the relation between L and L_i from (5.3.3) we have

$$I_1 \le C(e^{-\frac{(n-2\gamma)L}{4}}|x|^{-(n-2\gamma)})^\beta \le Ce^{-\frac{(n+2\gamma)L}{4}}|x|^{-(n+2\gamma)}.$$

For I_2, recall that by Corollary 5.2.3

$$\phi_i = |x - p_i|^{-\frac{n-2\gamma}{2}}\psi_i = |x - p_i|^{-\frac{n-2\gamma}{2}} O(e^{-\frac{(n-2\gamma)L_i}{4}(1+\xi)}),$$

we have for $|x|$ large,

$$(-\Delta)^\gamma(\chi_i\phi_i)(x) = P.V.\int_{\mathbb{R}^n} \frac{\chi_i(x)\phi_i(x)-\chi_i(y)\phi_i(y)}{|x-y|^{n+2\gamma}}\,dy = P.V.\int_{B_1} \frac{-\chi_i(y)\phi_i(y)}{|x-y|^{n+2\gamma}}\,dy$$
$$= |x|^{-(n+2\gamma)}O(e^{-\frac{(n-2\gamma)L_i}{4}(1+\xi)}).$$

Thus one has for $\operatorname{dist}(x,\Sigma)\geq 1$,

$$|S(\bar{u}_0)| \leq C|x|^{-(n+2\gamma)}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}.$$

Next, we consider the region $\frac{1}{2}\leq |x-p_i|\leq 1$. In this case, it is easy to check that

$$|S(\bar{u}_0)| \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}.$$

Last we consider the region $|x|\leq\frac{1}{2}$. In this region we have $\chi_1=1$ and $\chi_i=0$ for $i\neq 1$, so

$$\bar{u}_0 = u_{L_1} - (1-\chi_1)\phi_1 + \sum_{i\neq 1}\left(\sum_{j=0}^{\infty} w_j^i + \chi_i\phi_i\right) - \sum_{j=-\infty}^{-1} w_j^1.$$

Hence

$$\begin{aligned}
S(\bar{u}_0) &= (-\Delta)^\gamma u_{L_1} - c_{n,\gamma}\Big(\sum_{j=0}^{\infty} w_j^1 + \phi_1 + O(e^{-\frac{(n-2\gamma)L}{4}})\Big)^\beta + O(e^{-\frac{(n-2\gamma)L}{4}(1+\xi)})\\
&= -c_{n,\gamma}\big[(u_{L_1}+O(e^{-\frac{(n-2\gamma)L}{4}}))^\beta - u_{L_1}^\beta\big] + O(e^{-\frac{(n-2\gamma)L}{4}(1+\xi)})\\
&\leq Cu_{L_1}^{\beta-1}e^{-\frac{(n-2\gamma)L}{4}} + O(e^{-\frac{(n-2\gamma)L}{4}(1+\xi)})\\
&\leq C|x|^{-2\gamma}\Big(\sum_{j=-\infty}^{\infty} v_j(-\log|x|)\Big)^{\beta-1} e^{-\frac{(n-2\gamma)L}{4}} + O(e^{-\frac{(n-2\gamma)L}{4}(1+\xi)})\\
&\leq C|x|^{\gamma_1-2\gamma}|x|^{-\gamma_1}\Big(\sum_{j=-\infty}^{\infty} v_j(-\log|x|)\Big)^{\beta-1} e^{-\frac{(n-2\gamma)L}{4}} + O(e^{-\frac{(n-2\gamma)L}{4}(1+\xi)})\\
&\leq C|x|^{\gamma_1-2\gamma}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)},
\end{aligned}$$

where for the last inequality we have used (5.3.24) below in the region $|x|\leq\frac{1}{2}$. We have also denoted

$$v_j(t) := v\big(t-\tfrac{L_1}{2}-jL_1\big), \quad \text{for} \quad t=-\log|x|.$$

In any case, for $t=-\log|x|<\frac{L_1}{4}$, we have

$$\sum_{j=-\infty}^{\infty} v_j(-\log|x|) \leq Ce^{-\frac{(n-2\gamma)L_1}{8}}, \quad |x|\leq C, \tag{5.3.22}$$

and for $t \geq \frac{L_1}{4}$, we have

$$|x| \leq Ce^{-\frac{L_1}{4}}, \quad \sum_{j=-\infty}^{\infty} v_j(-\log|x|) \leq C. \tag{5.3.23}$$

Combining the above two estimates, we have for $\gamma_1 < 0$,

$$|x|^{-\gamma_1}\Big(\sum_{j=-\infty}^{\infty} v_j(-\log|x|)\Big)^{\beta-1} \leq Ce^{-\xi L_1}. \tag{5.3.24}$$

So for $|x| \leq \frac{1}{2}$, one has

$$|S(\bar{u}_0)| \leq C|x|^{\gamma_1-2\gamma}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}.$$

Thus we get estimates (5.3.20) and (5.3.21) in this particular case.

Now we consider the case of a general configuration r_j^i, a_j^i. First we differentiate $S(\bar{u}_0)$ with respect to these parameters. Since the variation is linear in the displacements of the parameters, we vary the parameter of one point at one time. Varying r_j^i, we obtain

$$\frac{\partial}{\partial r_j^i}S(\bar{u}_0) = (-\Delta)^\gamma \frac{\partial w_j^i}{\partial r_j^i} - c_{n,\gamma}\beta\bar{u}_0^{\beta-1}\frac{\partial w_j^i}{\partial r_j^i} = \beta c_{n,\gamma}\big[(w_j^i)^{\beta-1} - \bar{u}_0^{\beta-1}\big]\frac{\partial w_j^i}{\partial r_j^i}.$$

From the estimate on ϕ_i and the condition on r_j^i, we have the following estimates:

For $\operatorname{dist}(x, \Sigma) \geq 1$,

$$\begin{aligned}
[(w_j^i)^{\beta-1} - \bar{u}_0^{\beta-1}]\frac{\partial w_j^i}{\partial r_j^i} &\leq C(e^{-\frac{(n-2\gamma)L}{4}}|x|^{-(n-2\gamma)})^{\beta-1}e^{-\frac{(n-2\gamma)}{4}L(2j+1)}|x|^{-(n-2\gamma)} \\
&\leq C|x|^{-(n+2\gamma)}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\sigma t_j^i},
\end{aligned}$$

for a suitable choice of $\sigma > 0$. Next, when $|x - p_i| \leq 1$ for $i \neq 1$, for instance, similar to the estimates (5.3.22) and (5.3.23), one has

$$\begin{aligned}
[(w_j^i)^{\beta-1} - \bar{u}_0^{\beta-1}]\frac{\partial w_j^i}{\partial r_j^i} &\leq C|x-p_i|^{-2\gamma}(\sum_{j=0}^{\infty} v_j(-\log|x-p_i|))^{\beta-1}e^{-\frac{(n-2\gamma)L}{4}(2j+1)} \\
&\leq C|x-p_i|^{\gamma_1-2\gamma}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\sigma t_j^i},
\end{aligned}$$

while for $|x-p_1|\le 1$, if $|t-t^i_j|\le \frac{L_1}{2}$ it is true that

$$\begin{aligned}
[(w^i_j)^{\beta-1}-\bar u_0^{\beta-1}]\frac{\partial w^i_j}{\partial r^i_j} &\le C(w^i_j)^{\beta-1}\big[\sum_{l\neq j} w^i_l + e^{-\frac{(n-2\gamma)L}{4}}\big]\\
&\le C\big[|x|^{-\frac{n+2\gamma}{2}}\sum_{l\neq j} v_j^{\beta-1}v_l + |x|^{-2\gamma}v_j^{\beta-1}e^{-\frac{(n-2\gamma)L}{4}}\big]\\
&\le C\big[|x|^{-\frac{n+2\gamma}{2}}e^{-\eta|t-t_j|}\sum_{l\neq j} e^{-(2\gamma-\eta)|t-t^i_j|}e^{-\frac{n-2\gamma}{2}|t-t^i_l|}\\
&\qquad + |x|^{\gamma_1-2\gamma}|x|^{\gamma_1}e^{-2\gamma|t-t^i_j|}e^{-\frac{(n-2\gamma)L}{4}}\big]\\
&\le C\big[|x|^{-\frac{n+2\gamma}{2}}e^{-\eta|t-t^i_j|}+|x|^{\gamma_1-2\gamma}e^{-\sigma t^i_j}\big]e^{-\frac{(n-2\gamma)L}{4}(1+\xi)},
\end{aligned}$$

if we choose $0<\eta<2\gamma$. On the other hand, if $|t-t^i_l|\le\frac{L_1}{2}$ for some $l\neq j$, one has

$$\begin{aligned}
[(w^i_j)^{\beta-1}-\bar u_0^{\beta-1}]\frac{\partial w^i_j}{\partial r^i_j} &\le C(w^i_l)^{\beta-1}\frac{\partial w^i_j}{\partial r^i_j}\\
&\le C|x|^{-\frac{n+2\gamma}{2}}v_l^{\beta-1}v_j\\
&\le C|x|^{-\frac{n+2\gamma}{2}}e^{-\eta|t-t^i_j|}e^{\eta|t-t^i_j|}e^{-\frac{n-2\gamma}{2}|t-t_j|}e^{-2\gamma|t-t^i_l|}\\
&\le C|x|^{-\frac{n+2\gamma}{2}}e^{-\eta|t-t^i_j|}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)},
\end{aligned}$$

if $\eta<\frac{n-2\gamma}{2}$ which is chosen small enough. Combining the above two estimates yields, for $|x|\le 1$,

$$[(w^i_j)^{\beta-1}-\bar u_0^{\beta-1}]\frac{\partial w^i_j}{\partial r^i_j}|r^i_j| \le C|x|^{-\frac{n+2\gamma}{2}}e^{-\tau t}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}+|x|^{\gamma_1-2\gamma}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}.$$

Moreover, recalling (5.3.9), one can get that for $\operatorname{dist}(x,\Sigma)\ge 1$,

$$[(w^i_j)^{\beta-1}-\bar u_0^{\beta-1}]\frac{\partial w^i_j}{\partial r^i_j}|r^i_j| \le C|x|^{-(n+2\gamma)}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\sigma t^i_j},$$

and for $\operatorname{dist}(x,\Sigma)\le 1$,

$$\begin{aligned}
&[(w^i_j)^{\beta-1}-\bar u_0^{\beta-1}]\frac{\partial w^i_j}{\partial r^i_j}|r^i_j|\\
&\quad\le C[\operatorname{dist}(x,\Sigma)^{-\frac{n+2\gamma}{2}}\operatorname{dist}(x,\Sigma)^{\tau}+\operatorname{dist}(x,\Sigma)^{\gamma_1-2\gamma}]e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}\\
&\quad\le C\operatorname{dist}(x,\Sigma)^{\min\{-\frac{n-2\gamma}{2}+\tau,\gamma_1\}-2\gamma}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}
\end{aligned}$$

for some $-\frac{n-2\gamma}{2}<\gamma_1<\min\{0,-\frac{n-2\gamma}{2}+2\gamma\}$ and τ small enough.

Similar estimates hold for $\frac{\partial}{\partial a^i_{j,l}}S(\bar{u})$. We conclude from the above that

$$
\begin{aligned}
|S(\bar{u})-S(\bar{u}_0)| &\leq \sum_{i=1}^{k}\sum_{j=0}^{\infty}\sum_{l=1}^{n}\Big|[(w^i_j)^{\beta-1}-\bar{u}_0^{\beta-1}]\frac{\partial w^i_j}{\partial r^i_j}|r^i_j|+[(w^i_j)^{\beta-1}-\bar{u}_0^{\beta-1}]\frac{\partial w^i_j}{\partial a^i_{j,l}}|a^i_{j,l}|\Big| \\
&\leq \begin{cases} C|x|^{-(n+2\gamma)}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}, & \text{if } \operatorname{dist}(x,\Sigma)\geq 1, \\ C\operatorname{dist}(x,\Sigma)^{\min\{\gamma_1,-\frac{n-2\gamma}{2}+\tau\}-2\gamma}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}, & \text{if } \operatorname{dist}(x,\Sigma)<1. \end{cases}
\end{aligned}
$$

Thus we have

$$\|S(\bar{u})\|_{**}\leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)},$$

as desired.

□

Proof of Lemma 5.3.4. The proof relies on a standard finite-dimensional Lyapunov-Schmidt reduction.

Step 1: Preliminary calculations. Multiply equation (5.3.17) by $Z^{i'}_{j',l'}$ and integrate over $\mathbb{R}^n$; we have

$$\int_{\mathbb{R}^n}[(-\Delta)^\gamma\phi-c_{n,\gamma}\beta\bar{u}^{\beta-1}\phi]Z^{i'}_{j',l'}\,dx=\int_{\mathbb{R}^n}hZ^{i'}_{j',l'}\,dx+\sum_{i,j,l}c^i_{j,l}\int_{\mathbb{R}^n}(w^i_j)^{\beta-1}Z^i_{j,l}Z^{i'}_{j',l'}\,dx. \tag{5.3.25}$$

By the orthogonality condition satisfied by ϕ, we have that the left hand side of (5.3.25) is

$$
\begin{aligned}
\int_{\mathbb{R}^n}[(-\Delta)^\gamma\phi-c_{n,\gamma}\beta\bar{u}^{\beta-1}\phi]Z^{i'}_{j',l'}\,dx&=c_{n,\gamma}\beta\int_{\mathbb{R}^n}[(w^{i'}_{j'})^{\beta-1}-\bar{u}^{\beta-1}]\phi Z^{i'}_{j',l'}\,dx \\
&=\Big[\int_{B(p_{i'},1)}+\sum_{i\neq i'}\int_{B(p_i,1)}+\int_{\mathbb{R}^n\setminus\,\cup_{i=1}^k B(p_i,1)}\Big]=:I_1+I_2+I_3.
\end{aligned} \tag{5.3.26}
$$

Without loss of generality, assume that $i'=1$ and $p_1=0$. First we consider the case $l'=0$. Recalling the estimates for $Z^{i'}_{j',0}$ from (5.3.11),

$$
\begin{aligned}
I_1&=\int_{B_1}[(w^{i'}_{j'})^{\beta-1}-\bar{u}^{\beta-1}]\phi Z^{i'}_{j',l'}\,dx\leq\|\phi\|_*\int_{B_1}\big|(w^{i'}_{j'})^{\beta-1}-\bar{u}^{\beta-1}\big|\,|x|^{\gamma_1}Z^{i'}_{j',l'}\,dx \\
&\leq\|\phi\|_*\int_{B_1}|x|^{\gamma_1-\frac{n+2\gamma}{2}}v_{j'}^{\beta-1}\sum_{j\neq j'}v_j\,dx\leq C\|\phi\|_*\int_0^\infty e^{-(\gamma_1+\frac{n-2\gamma}{2})t}v_{j'}^{\beta-1}\sum_{j\neq j'}v_j\,dt \\
&\leq C\|\phi\|_*\,e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-(\gamma_1+\frac{n-2\gamma}{2})t^i_{j'}},
\end{aligned} \tag{5.3.27}
$$

and notice that $\gamma_1 > -\frac{n-2\gamma}{2}$. Next,

$$\begin{aligned}
I_2 &= \sum_{i\neq 1}\int_{B(p_i,1)} [(w_{j'}^{i'})^{\beta-1} - \bar{u}^{\beta-1}]\phi Z_{j',l'}^{i'}\,dx \\
&\leq C\|\phi\|_* \sum_{i\neq 1}\int_{B(p_i,1)} \left|(w_{j'}^{i'})^{\beta-1} - \bar{u}^{\beta-1}\right| Z_{j',l'}^{i'}|x-p_i|^{\gamma_1}\,dx \\
&\leq C\|\phi\|_* \sum_{i\neq 1}\int_{B(p_i,1)} |x-p_i|^{\gamma_1-2\gamma}(\lambda_{j'}^{i'})^{\frac{n-2\gamma}{2}}(\sum_j v_j^i)\,dx \\
&\leq C\|\phi\|_*\,(\lambda_{j'}^{i'})^{\frac{n-2\gamma}{2}} e^{-(n+\gamma_1-2\gamma)\frac{L}{2}} \\
&\leq C\|\phi\|_*\, e^{\gamma_1 t_{j'} - \frac{n-2\gamma}{2}L - \frac{\gamma_1}{2}L} e^{-(\gamma_1+\frac{n-2\gamma}{2})t_{j'}^i} \\
&\leq C\|\phi\|_*\, e^{-\frac{(n-2\gamma)L}{4}(1+\xi)} e^{-(\gamma_1+\frac{n-2\gamma}{2})t_{j'}^i}
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \int_{\mathbb{R}^n/\cup_i B(p_i,1)} [(w_{j'}^{i'})^{\beta-1} - \bar{u}^{\beta-1}]\phi Z_{j',l'}^{i'}\,dx \\
&\leq C\|\phi\|_* \int_{\mathbb{R}^n\setminus\cup_i B(p_i,1)} |x|^{-(n-2\gamma)}|x|^{-(n+2\gamma)}(\lambda_{j'}^{i'})^{\frac{n-2\gamma}{2}} e^{-\gamma L}\,dx \\
&\leq C\|\phi\|_*\, e^{\gamma_1 t_{j'} - \gamma L} e^{-(\gamma_1+\frac{n-2\gamma}{2})t_{j'}^i} \\
&\leq C\|\phi\|_*\, e^{-\frac{(n-2\gamma)L}{4}(1+\xi)} e^{-(\gamma_1+\frac{n-2\gamma}{2})t_{j'}^i},
\end{aligned}$$

where we have used $-\frac{n-2\gamma}{2} < \gamma_1 < -\frac{n-2\gamma}{2} + 2\gamma$.

On the other hand, for $l' = 1, \cdots, n$, recalling from (5.3.12) that $Z_{j',l'}^{i'} = O(|x - p_i|^{-\frac{n-2\gamma}{2}}(v_{j'}^{i'})^{1+\frac{2}{n-2\gamma}})$, then one can get similar estimates as above. In conclusion, one has

$$\int_{\mathbb{R}^n} [(-\Delta)^\gamma \phi - c_{n,\gamma}\beta\bar{u}^{\beta-1}\phi] Z_{j',l'}^{i'}\,dx \leq C\|\phi\|_* e^{-\frac{(n-2\gamma)L}{4}(1+\xi)} e^{-(\gamma_1+\frac{n-2\gamma}{2})t_{j'}^i},$$

for every $l' = 0, \ldots, n$, which gives a good control of the left hand side of (5.3.25).

Now, for the first term in the right hand side of (5.3.25),

$$\begin{aligned}
\int_{\mathbb{R}^n} hZ_{j',l'}^{i'}\,dx \leq C &\int_{\mathbb{R}^n\setminus\cup_i B(p_i)} \|h\|_{**}\,|x|^{-(n+2\gamma)}|x|^{-(n-2\gamma)}e^{-\frac{(n-2\gamma)}{2}t_{j'}^i}\,dx \\
&+ \int_{B(p_{i'})} \|h\|_{**}\,|x-p_{i'}|^{\gamma_1-2\gamma}|x-p_{i'}|^{-\frac{n-2\gamma}{2}}[e^{-\frac{(n-2\gamma)}{2}t_{j'}^i} + e^{-(\frac{n-2\gamma}{2}+1)t_{j'}^i}]\,dx \\
&+ \sum_{i\neq i'}\int_{B(p_i)} \|h\|_{**}\,|x-p_i|^{\gamma_1-2\gamma}e^{-\frac{n-2\gamma}{2}t_{j'}^i}\,dx \\
&\leq C\|h\|_{**}\, e^{-(\gamma_1+\frac{n-2\gamma}{2})t_{j'}^i}.
\end{aligned}$$

The next step is to isolate the term $c^i_{j,l}$ in (5.3.25), by inverting the matrix $\int_{\mathbb{R}^n}(w^i_j)^{\beta-1}Z^i_{j,l}Z^{i'}_{j',l'}\,dx$. For this, recall the orthogonality estimates from (5.3.13)-(5.3.14), which yield, for all $l=0,\ldots,n$,

$$\int_{\mathbb{R}^n}(w^i_j)^{\beta-1}Z^i_{j,l}Z^i_{j,l'}\,dx = C_0\delta_{l,l'} \quad \text{and} \quad \int_{\mathbb{R}^n}(w^i_j)^{\beta-1}Z^i_{j,l}Z^i_{j',l'}\,dx = O(e^{-\frac{n-2\gamma}{2}|t^i_j-t^i_{j'}|}) \text{ if } j\neq j',$$

plus a tiny error. Then using Lemma A.6 in [132] for the inversion of a Toepliz-type operator, one has from (5.3.25) that

$$\begin{aligned}
|c^i_{j,l}| &\leq C[e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}\|\phi\|_* + \|h\|_{**}]e^{-(\gamma_1+\frac{n-2\gamma}{2})t^i_j} \\
&\quad + C\sum_{j'\neq j}[e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}\|\phi\|_* + \|h\|_{**}]e^{-\frac{n-2\gamma}{2}(1+o(1))|t_j-t_{j'}|}e^{-(\gamma_1+\frac{n-2\gamma}{2})t^i_{j'}} \\
&\leq C[e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}\|\phi\|_* + \|h\|_{**}]e^{-(\gamma_1+\frac{n-2\gamma}{2})t^i_j}.
\end{aligned}$$

From the estimates for $Z^i_{j,l}$ from (5.3.11)-(5.3.12) and the previous bound for $c^i_{j,l}$ one can check that in $B_1(p_i)$,

$$\begin{aligned}
|c^i_{j,l}(w^i_j)^{\beta-1}Z^i_{j,l}| &\leq C|c^i_{j,l}|\cdot|x-p_i|^{-\frac{n+2\gamma}{2}}e^{-\frac{n+2\gamma}{2}|t^i-t^i_j|} \\
&\leq C|x-p_i|^{\gamma_1-2\gamma}[e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}\|\phi\|_* + \|h\|_{**}] \\
&\quad \cdot\, e^{-(\gamma_1+\frac{n-2\gamma}{2})t^i_j}e^{(\gamma_1+\frac{n-2\gamma}{2})t^i}e^{-\frac{n+2\gamma}{2}|t^i-t^i_j|} \\
&\leq C|x-p_i|^{\gamma_1-2\gamma}[e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}\|\phi\|_* + \|h\|_{**}]e^{-\sigma|t^i-t^i_j|}
\end{aligned}$$

for some $\sigma>0$.

For $x\in\mathbb{R}^n\setminus\cup_i B_1(p_i)$, one has

$$\begin{aligned}
|c^i_{j,l}(w^i_j)^{\beta-1}Z^i_{j,l}| &\leq C(\lambda^i_j)^{\frac{n-2\gamma}{2}}|x|^{-(n+2\gamma)}|c^i_{j,l}| \\
&\leq C|x|^{-(n+2\gamma)}[e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}\|\phi\|_* + \|h\|_{**}]e^{-\sigma t^i_j}.
\end{aligned}$$

Combining the above two estimates yields

$$\big\|\sum c^i_{j,l}(w^i_j)^{\beta-1}Z^i_{j,l}\big\|_{**} \leq C\big[e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}\|\phi\|_* + \|h\|_{**}\big]. \tag{5.3.28}$$

Step 2: A priori estimates. We are going to prove the a priori estimate (5.3.18) by a contradiction argument. First let us recall the problem we are going to consider:

$$\begin{cases} (-\Delta)^\gamma\phi - c_{n,\gamma}\beta\bar{u}^{\beta-1}\phi = \bar{h}, \\ \displaystyle\int_{\mathbb{R}^n}\phi(w^i_j)^{\beta-1}Z^i_{j,l}\,dx = 0, \quad i=1,\ldots,k,\ j=0,1,\ldots,\ l=0,\ldots,n, \end{cases} \tag{5.3.29}$$

where have we denoted $\bar{h} := h + \sum c^i_{j,l}(w^i_j)^{\beta-1}Z^i_{j,l}$, and which satisfies, by (5.3.28), that

$$\|\bar{h}\|_{**} \leq C(\|h\|_{**} + o(1)\|\phi\|_*).$$

We are going to prove that

$$\|\phi\|_* \le C\|\bar{h}\|_{**}, \tag{5.3.30}$$

for where (5.3.18) follows immediately.

Assume that there exist sequences $\{L_i^{(n)}\}$ with $L_i^{(n)} \to \infty$, $\{r_j^{i,(n)},\ a_j^{i,(n)}\}$, $\{h^{(n)}\}$, $\{c_{j,l}^{i,(n)}\}$ and the corresponding solution $\{\phi^{(n)}\}$ such that

$$\|h^{(n)}\|_{**} \to 0, \quad \|\phi^{(n)}\|_* = 1. \tag{5.3.31}$$

In the following we will drop the index n if needed. First by the Green's representation formula for the first equation in (5.3.29) we have

$$\phi(x) = \int_{\mathbb{R}^n} (c_{n,\gamma}\bar{u}^{\beta-1}\phi + \bar{h})(y)G(x,y)\,dy =: I_1 + I_2, \tag{5.3.32}$$

where G is the Green's function for the fractional Laplacian $(-\Delta)^\gamma$, given by ([36])

$$G(x,y) = C|x-y|^{-(n-2\gamma)}.$$

First we consider the region $\{\mathrm{dist}\,(x,\Sigma) \ge 1\}$. Here, for I_2,

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}^n} \bar{h}(y)G(x,y)\,dy \\
&\le \Bigg[\int_{\{\mathrm{dist}\,(y,\Sigma)\le 1\}} + \int_{\{1<\mathrm{dist}\,(y,\Sigma)<\frac{|x|}{2}\}} \\
&\quad + \int_{\{\frac{|x|}{2}<\mathrm{dist}\,(y,\Sigma)<2|x|\}} + \int_{\{\mathrm{dist}\,(y,\Sigma)\ge 2|x|\}}\Bigg]\bar{h}(y)G(x,y)\,dy \\
&=: I_{21} + I_{22} + I_{23} + I_{24},
\end{aligned}$$

one has

$$\begin{aligned}
I_{21} &\le \int_{\{\mathrm{dist}\,(y,\Sigma)<1\}} \frac{1}{|x-y|^{n-2\gamma}}\|\bar{h}\|_{**}|y|^{\gamma_1-2\gamma}\,dy \le C\|\bar{h}\|_{**}|x|^{-(n-2\gamma)}, \\
I_{22} &\le \int_{\{1<\mathrm{dist}\,(y,\Sigma)<\frac{|x|}{2}\}} \frac{1}{|x-y|^{n-2\gamma}}\|\bar{h}\|_{**}|y|^{-(n+2\gamma)}\,dy \\
&\le C\|\bar{h}\|_{**}|x|^{-(n-2\gamma)}\int_{\{1<\mathrm{dist}\,(y,\Sigma)<\frac{|x|}{2}\}} |y|^{-(n+2\gamma)}\,dy \le C\|\bar{h}\|_{**}|x|^{-(n-2\gamma)}, \\
I_{23} &\le \int_{\frac{|x|}{2}<\mathrm{dist}\,(y,\Sigma)<2|x|} \frac{1}{|x-y|^{n-2\gamma}}\|\bar{h}\|_{**}|y|^{-(n+2\gamma)}\,dy \\
&\le \|\bar{h}\|_{**}|x|^{-(n+2\gamma)}\int_{\{x-y|<\frac{5|x|}{2}\}} \frac{1}{|x-y|^{n-2\gamma}}\,dy \le C\|\bar{h}\|_{**}|x|^{-(n-2\gamma)}, \\
I_{24} &\le \int_{\{\mathrm{dist}\,(y,\Sigma)\ge 2|x|\}} \frac{1}{|x-y|^{n-2\gamma}}\|\bar{h}\|_{**}|y|^{-(n+2\gamma)}\,dy \le C\|\bar{h}\|_{**}|x|^{-(n-2\gamma)}.
\end{aligned}$$

Putting all together,
$$I_2 \le C\|\bar{h}\|_{**}|x|^{-(n-2\gamma)}. \tag{5.3.33}$$
Next for I_1,
$$I_1 = \left[\int_{\{\text{dist}\,(y,\Sigma)\le 1\}} + \int_{\{\text{dist}\,(y,\Sigma)\ge 1\}}\right] c_{n,\gamma}\beta\bar{u}^{\beta-1}\phi\, G(x,y)\,dy =: I_{11} + I_{12}.$$
Since for $\text{dist}\,(y,\Sigma)\ge 1$ it holds that $\bar{u} = O(e^{-\frac{n-2\gamma}{4}L})|y|^{-(n-2\gamma)}$ (recall (5.2.5)), then
$$I_{12} \le Ce^{-\gamma L}\|\phi\|_* \int_{\{\text{dist}\,(y,\Sigma)\ge 1\}} |y|^{-(n+2\gamma)}G(x,y)\,dy,$$
and similar to the estimate above we get that
$$I_{12} \le o(1)\|\phi\|_*|x|^{-(n-2\gamma)}.$$
Moreover,
$$\begin{aligned}
I_{11} &\le \sum_{i=1}^k \int_{\{|y-p_i|\le 1\}} |y-p_i|^{-2\gamma}(\sum_{j=0}^\infty v_j^i)^{\beta-1}\|\phi\|_*|y-p_i|^{\gamma_1}|x-y|^{-(n-2\gamma)}\,dy\\
&\le C\|\phi\|_*|x|^{-(n-2\gamma)}\int_{\{|y-p_i|<1\}} |y-p_i|^{\gamma_1-2\gamma}(\sum_{j=0}^\infty v_j^i)^{\beta-1}\,dy\\
&\le C\|\phi\|_*|x|^{-(n-2\gamma)}\int_0^\infty e^{-(n+\gamma_1-2\gamma)t}(\sum_{j=0}^\infty v_j^i)^{\beta-1}\,dt\\
&\le Ce^{-(n+\gamma_1-2\gamma)\frac{L}{2}}\|\phi\|_*|x|^{-(n-2\gamma)}.
\end{aligned}$$
Since $\gamma_1 > -\frac{n-2\gamma}{2}$, by the above estimates one has
$$I_1 \le o(1)\|\phi\|_*|x|^{-(n-2\gamma)}. \tag{5.3.34}$$
Summarizing, from (5.3.33) and (5.3.34) we obtain that, for $\text{dist}\,(x,\Sigma)\ge 1$,
$$\sup_{\text{dist}\,(x,\Sigma)\ge 1}\left\{|x|^{n-2\gamma}|\phi(x)|\right\} \le C(\|\bar{h}\|_{**} + o(1)\|\phi\|_*) \to 0 \quad \text{as} \quad L\to\infty,$$
by our initial hypothesis (5.3.31). Moreover, because of the same reason, we know that there exists p_i such that
$$\sup_{|x-p_i|\le 1}\left\{|x-p_i|^{-\gamma_1}\phi(x)|\right\} \ge \tfrac{1}{2}. \tag{5.3.35}$$

The next step is to consider the region $\{|x-p_i|\le 1\}$. In order to simplify the notation, we assume that $p_i = 0$, $|x|<1$. Again, we use Green's representation formula (5.3.32), and we estimate both integrals I_1, I_2. On the one hand,
$$\begin{aligned}
I_2 &= \left[\int_{\{|y|\ge 1\}} + \int_{\{|y|<\frac{|x|}{2}\}} + \int_{\{\frac{|x|}{2}<|y|<2|x|\}} + \int_{\{2|x|<|y|<1\}}\right] G(x,y)\bar{h}\,dy\\
&=: I_{21} + I_{22} + I_{23} + I_{24},
\end{aligned}$$

where

$$\begin{aligned}
I_{21} &\le C\int_{\{|y|\ge 1\}} \frac{1}{|x-y|^{n-2\gamma}}\|\bar h\|_{**}|y|^{-(n+2\gamma)}\,dy \le C\|\bar h\|_{**}|x|^{\gamma_1},\\
I_{22} &\le \int_{\{|y|<\frac{|x|}{2}\}} \frac{1}{|x-y|^{n-2\gamma}}\|\bar h\|_{**}|y|^{\gamma_1-2\gamma}\,dy\\
&\le C\|\bar h\|_{**}\int_{\{|y|<\frac{|x|}{2}\}} \frac{1}{|x|^{n-2\gamma}}|y|^{\gamma_1-2\gamma}\,dy \le C\|\bar h\|_{**}|x|^{\gamma_1},\\
I_{23} &\le \int_{\{\frac{|x|}{2}<|y|<2|x|\}} \frac{1}{|x-y|^{n-2\gamma}}\|\bar h\|_{**}|y|^{\gamma_1-2\gamma}\,dy\\
&\le C\|\bar h\|_{**}|x|^{\gamma_1-2\gamma}\int_{\{|x-y|<\frac{5|x|}{2}\}} \frac{1}{|x-y|^{n-2\gamma}}\,dy \le C\|\bar h\|_{**}|x|^{\gamma_1},\\
I_{24} &\le \int_{\{2|x|<|y|<2\}} \frac{1}{|x-y|^{n-2\gamma}}\|\bar h\|_{**}|y|^{\gamma_1-2\gamma}\,dy\\
&\le C\|\bar h\|_{**}\int_{\{2|x|<|y|<2\}} |y|^{\gamma_1-2\gamma-n+2\gamma}\,dy \le C\|\bar h\|_{**}|x|^{\gamma_1}.
\end{aligned}$$

Thus one has

$$I_2 \le C\|\bar h\|_{**}|x|^{\gamma_1}.$$

On the other hand, for I_1,

$$I_1 = \left[\int_{\{|y|>1\}} + \int_{\{|y|<1\}}\right] c_{n,\gamma}\beta\bar u^{\beta-1}\phi\, G(x,y)\,dy =: I_{11}+I_{12}.$$

Similar to the estimates for $\bar h$,

$$I_{11} \le C\int_{\{\mathrm{dist}\,(y,\Sigma)\ge 1\}} e^{-\gamma L}|y|^{-4\gamma}\frac{1}{|x-y|^{n-2\gamma}}\|\phi\|_*|y|^{-(n-2\gamma)}\,dy \le o(1)\|\phi\|_*|x|^{\gamma_1}.$$

The final step is to estimate I_{12}. For this we consider ϕ in the region $A_j := \sqrt{\lambda^i_{j+1}\lambda^i_j} < |x| < \sqrt{\lambda^i_j\lambda^i_{j-1}}$, and define a scaled function $\tilde\phi_j(\tilde x) = (\lambda^i_j)^{-\gamma_1}\phi(\lambda^i_j\tilde x)$ defined in the region $\tilde A_j = \frac{A_j}{\lambda^i_j} \to (0,\infty)$ as $n\to\infty$. Then $\tilde\phi_j$ will satisfy the following equation

$$\begin{cases}
(-\Delta)^\gamma\tilde\phi_j - c_{n,\gamma}\beta\Big(\dfrac{1}{1+|\tilde x|^2}\Big)^{2\gamma}(1+o(1))\tilde\phi_j = (\lambda^i_j)^{2\gamma-\gamma_1}\bar h(\lambda^i_j\tilde x) \quad \text{in } \tilde A_j,\\
\displaystyle\int_{\mathbb{R}^n}\tilde\phi_j(w^i_j)^{\beta-1}(\lambda^i_j\tilde x)Z^i_{j,l}(\lambda^i_j\tilde x)\,d\tilde x = 0.
\end{cases}$$

Since $|\bar h| \le C\|\bar h\|_{**}|\lambda^i_j\tilde x|^{\gamma_1-2\gamma}$ as $n\to\infty$, $\tilde\phi_j\to\bar\phi$ in any compact set $\frac{1}{R}\le|\tilde x|\le R$ for R large enough (to be determined later), where $\bar\phi$ is a solution of the following equation

$$\begin{cases}
(-\Delta)^\gamma\bar\phi - c_{n,\gamma}\beta w^{\beta-1}\bar\phi_j = 0,\\
\displaystyle\int_{\mathbb{R}^n}\bar\phi w^{\beta-1}Z_l\,dx = 0,
\end{cases}$$

where w is the standard bubble solution and $Z_l,\ l=0,\cdots,n$, are the corresponding kernels mentioned in Lemma 5.2.1. By the non-degeneracy of the bubble, one has $\bar{\phi}=0$, i.e. $\tilde{\phi}_j \to 0$ in $\frac{1}{R}<|\tilde{x}|<R$. If we consider the original ϕ, this is equivalent to that $|x|^{-\gamma_1}\phi(x)\to 0$ in $\cup_j\{\frac{\lambda_j^i}{R}<|x|<R\lambda_j^i\}$ as $n\to\infty$. Using this result, we now consider I_{12}:

$$
\begin{aligned}
I_{12} &= \int_{\{|y|<1\}} c_{n,\gamma}\beta\bar{u}^{\beta-1}\phi\, G(x,y)\,dy \\
&\leq \sum_j \left[\int_{\{\frac{\lambda_j^i}{R}<|x|<R\lambda_j^i\}} + \int_{\{|y|<1\}\setminus\cup_j\{\frac{\lambda_j^i}{R}<|x|<\lambda_j^i R\}}\right]\bar{u}^{\beta-1}\phi\, G(x,y)\,dy =: I_{121}+I_{122}.
\end{aligned}
$$

Recalling (5.3.4), we have that in $\{|y|<1\}$, $\bar{u}=|y|^{-\frac{n-2\gamma}{2}}(\sum_{j=0}^{\infty}v_j^i)(1+o(1))$. Then in the region $\{|y|<1\}\setminus\cup_j\{\frac{\lambda_j^i}{R}<|x|<\lambda_j^i R\}$, one has $\sum_j v_j^i\leq Ce^{-\frac{(n-2\gamma)}{2}R}$ which can be small enough choosing R large enough but independent of n. Using this estimate we can assert that

$$
I_{122}\leq Ce^{-2R}\int_{|y|<1}|y|^{-2\gamma}\|\phi\|_*|y|^{\gamma_1}\frac{1}{|x-y|^{n-2\gamma}}\,dy\leq Ce^{-2R}|x|^{\gamma_1}.
$$

In addition, by the previous argument we know that $|x|^{-\gamma_1}\phi(x)\to 0$ in $\cup_j\{\frac{\lambda_j^i}{R}<|x|<R\lambda_j^i\}$, and one has

$$
\begin{aligned}
I_{121} &\leq C\sum_j\int_{\{\frac{\lambda_j^i}{R}<|y|<R\lambda_j^i\}}|\phi||y|^{-\gamma_1}\frac{|y|^{\gamma_1-2\gamma}(\sum_j v_j^i)^{\beta-1}}{|x-y|^{n-2\gamma}}\,dy \\
&\leq o(1)\int_{\{|y|\leq 1\}}\frac{|y|^{\gamma_1-2\gamma}}{|x-y|^{n-2\gamma}}\,dy\leq o(1)|x|^{\gamma_1}.
\end{aligned}
$$

Combining all the above estimates yields that in the set $\{|x|<1\}$ we must have $|x|^{-\gamma_1}\phi(x)=o(1)$ as $n\to\infty$, which is a contradiction to (5.3.35). This completes the proof of the a priori estimate (5.3.30), as desired.

Step 3: Existence and uniqueness. Consider the space

$$
\mathcal{H}=\Big\{u\in H^{\gamma}(\mathbb{R}^n),\ \int_{\mathbb{R}^n}u(w_j^i)^{\beta-1}Z_{j,l}^i\,dx=0\quad\text{for all } i,j,l\Big\}.
$$

Notice that the problem (5.3.17) in ϕ gets rewritten as

$$
\phi+K(\phi)=\bar{h}\text{ in }\mathcal{H}, \tag{5.3.36}
$$

where $\bar{h}$ is defined by duality and $K:\mathcal{H}\to\mathcal{H}$ is a linear compact operator. Using Fredholm's alternative, showing that equation (5.3.36) has a unique solution for each $\bar{h}$ is equivalent to showing that the equation has a unique solution for $\bar{h}=0$, which in turn follows from the previous a priori estimate. This concludes the proof of Lemma 5.3.4.

$\square$

Proof of Proposition 5.3.3. The proof relies on the contraction mapping in the above weighted norms. We set

$$S(\bar{u}) = (-\Delta)^\gamma \bar{u} - c_{n,\gamma}\bar{u}^\beta,$$

and also define the linear operator

$$\mathbb{L}(\phi) = (-\Delta)^\gamma \phi - c_{n,\gamma}\beta\bar{u}^{\beta-1}\phi.$$

We have that $\bar{u}+\phi$, $\phi \in \mathcal{C}_*$ solves equation (5.3.15) if and only if ϕ satisfies

$$\phi = G(\phi) \tag{5.3.37}$$

where

$$G(\phi) := \mathbb{L}^{-1}(S(\bar{u})) + c_{n,\gamma}\mathbb{L}^{-1}(N(\phi)).$$

Here we have defined

$$N(\phi) := (\bar{u}+\phi)^\beta - \bar{u}^\beta - \beta\bar{u}^{\beta-1}\phi.$$

Also, by $\mathbb{L}^{-1}$, we are denoting the linear operator which, according to Lemma 5.3.4, associates with $h \in \mathcal{C}_{**}$ the function $\phi \in \mathcal{C}_*$ solving (5.3.17).

We find a solution for (5.3.37) by a standard contraction mapping argument. First by the definition of G, one has

$$\|G(\phi)\|_* \leq C\left(\|S(\bar{u})\|_{**} + \|N(\phi)\|_{**}\right).$$

Fixing a large $C_1 > 0$, we define the set

$$B_{C_1} = \big\{\phi \in \mathcal{C}_* \ : \ \|\phi\|_* \leq C_1 e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}, \int_{\mathbb{R}^n} \phi(w_j^i)^{\beta-1}Z_{j,l}^i\,dx \ = 0, \forall i,j,l\big\}.$$

Note that

$$|(\bar{u}+\phi)^\beta - \bar{u}^\beta - \beta\bar{u}^{\beta-1}\phi| \leq C\left\{\begin{array}{ll} \bar{u}^{\beta-2}\phi^2, & \text{if } |\bar{u}| \geq \frac{1}{4}\phi, \\ \phi^\beta, & \text{if } |\bar{u}| \leq \frac{1}{4}\phi. \end{array}\right.$$

Now, let $\phi \in B_{C_1}$. By our construction, we have that if $\operatorname{dist}(x,\Sigma) < 1$,

$$\begin{aligned}
|N(\phi)| &\leq C(\bar{u}^{\beta-2}\phi^2 + \phi^\beta) \\
&\leq C\sum_{i=1}^k \left[\|\phi\|_*^2\bar{u}^{\beta-2}|x-p_i|^{2\gamma_1} + \|\phi\|_*^\beta|x-p_i|^{\beta\gamma_1}\right] \\
&\leq C\sum_{i=1}^k \left[\|\phi\|_*^2|x-p_i|^{\gamma_1-2\gamma}|x-p_i|^{\gamma_1+\frac{n-2\gamma}{2}} + \|\phi\|_*^\beta|x-p_i|^{\gamma_1-2\gamma}|x-p_i|^{\beta\gamma_1-\gamma_1+2\gamma}\right] \\
&\leq C\sum_{i=1}^k \left[\|\phi\|_*^2 + \|\phi\|_*^\beta\right]|x-p_i|^{\gamma_1-2\gamma},
\end{aligned}$$

and for dist $(x,\Sigma)\geq 1$,

$$
\begin{aligned}
|N(\phi)| &\leq C[\|\phi\|_*^2 \bar{u}^{\beta-2}|x|^{-2(n-2\gamma)} + \|\phi\|_*^{\beta}|x|^{-\beta(n-2\gamma)}] \\
&\leq C|x|^{-(n+2\gamma)}[e^{-(\beta-2)\frac{(n-2\gamma)L}{4}}\|\phi\|_*^2 + \|\phi\|_*^{\beta}].
\end{aligned}
$$

Combining the above two estimates, one has

$$
\|N(\phi)\|_{**} \leq C[e^{-(\beta-2)\frac{(n-2\gamma)L}{4}}\|\phi\|_*^2 + \|\phi\|_*^{\beta}] \leq o(1)\|\phi\|_*.
$$

Now we consider two functions $\phi_1,\phi_2\in B_{C_1}$, it is easy to see that for L large,

$$
\|N(\phi_1)-N(\phi_2)\|_{**} \leq o(1)\|\phi_1-\phi_2\|_*.
$$

Therefore, by the above estimates for $N(\phi)$ and (5.3.19), G is a contraction mapping in B_{C_1}, thus it has a fixed point in this set. This completes the proof of the Proposition. □

5.4 Estimates on the coefficients $c_{j,l}^i$

In this section we prove some estimates related to the coefficients $c_{j,l}^i$ obtained in the last section, first in the special case of the configuration $(r_j^i,a_j^i)=(0,0)$ and then for a general configuration of parameters satisfying (5.3.7)-(5.3.10). These are studied in subsections 5.4.1 and 5.4.2 respectively. Later on, in the next section, we study also the derivative with respect to a variation of the parameters.

5.4.1 Estimates on the $c_{j,l}^i$ for $(a_j^i,r_j^i)=(0,0)$

In this subsection, we prove the decay of $c_{j,l}^i$ when the parameters $(a_j^i,r_j^i)=(0,0)$. We denote $\bar{u}_0$ to be the approximate solution and ϕ the perturbation function found in Proposition 5.3.3 for this particular case. Define the numbers $\bar{\beta}_{j,l}^i$ as

$$
\bar{\beta}_{j,l}^i = \int_{\mathbb{R}^n}[(-\Delta)^{\gamma}(\bar{u}_0+\phi) - c_{n,\gamma}(\bar{u}_0+\phi)^{\beta}]Z_{j,l}^i\,dx.
$$

Then we have the following estimates on $\bar{\beta}_{j,l}^i$:

Lemma 5.4.1. *Given $\{R^i\}$ satisfying (5.3.7) and let $\bar{u}_0$ be the function defined in (5.3.4) for the parameters $(r_j^i,a_j^i)=(0,0)$. Let ϕ and $(c_{j,l}^i)$ be given in Proposition 5.3.3. Then the coefficients $\bar{\beta}_{j,l}^i$ satisfy*

$$
\begin{aligned}
\bar{\beta}_{0,0}^i =& -c_{n,\gamma}q_i\Big[A_2\sum_{i'\neq i}|p_{i'}-p_i|^{-(n-2\gamma)}(R^iR^{i'})^{\frac{n-2\gamma}{2}}q_{i'}-q_i\Big]e^{-\frac{(n-2\gamma)L}{2}}(1+o(1)) \\
&+O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}), \\
\bar{\beta}_{0,l}^i =& c_{n,\gamma}\lambda_0^i\Big[A_3\sum_{i'\neq i}\frac{(p_{i'}-p_i)_l}{|p_{i'}-p_i|^{n-2\gamma+2}}(R^iR^{i'})^{\frac{n-2\gamma}{2}}q_{i'}q_ie^{-\frac{n-2\gamma}{2}L}+O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)})\Big]
\end{aligned}
$$

for $l=1,\cdots,n$.

For $j \geq 1$, we have

$$\bar{\beta}^i_{j,0} = O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\sigma t^i_j}),$$
$$\bar{\beta}^i_{j,l} = O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-(1+\sigma)t^i_j}),$$

where $A_2 > 0, A_3 < 0$ are two constants independent of L and $\sigma = \min\{\gamma_1 + \frac{n-2\gamma}{2}, \frac{n-2\gamma}{4}\}$ independent of L large.

Proof. With some manipulation and the orthogonality condition satisfied by ϕ, we find that

$$\bar{\beta}^i_{j,l} = \bar{\beta}^i_{j,l,1} + \bar{\beta}^i_{j,l,2} + \bar{\beta}^i_{j,l,3},$$

for

$$\begin{aligned}
\bar{\beta}^i_{j,l,1} &= \int_{\mathbb{R}^n} [(-\Delta)^\gamma \bar{u}_0 - c_{n,\gamma}(\bar{u}_0)^\beta] Z^i_{j,l}\, dx,\\
\bar{\beta}_{j,l,2} &= \int_{\mathbb{R}^n} \mathbb{L}_0(\phi) Z^i_{j,l}\, dx = -c_{n,\gamma}\beta \int_{\mathbb{R}^n} [(\bar{u}_0)^{\beta-1} - (w^i_j)^{\beta-1}] Z^i_{j,l}\phi\, dx,\\
\bar{\beta}^i_{j,l,3} &= -c_{n,\gamma}\int_{\mathbb{R}^n} [(\bar{u}_0+\phi)^\beta - (\bar{u}_0)^\beta - \beta(\bar{u}_0)^{\beta-1}\phi] Z^i_{j,l}\, dx,
\end{aligned}$$

where we have defined $\mathbb{L}_0(\phi) = (-\Delta)^\gamma\phi - c_{n,\gamma}\beta(\bar{u}_0)^{\beta-1}\phi$.

Step 1: Estimate for $\bar{\beta}^i_{j,l,2}$ and $\bar{\beta}^i_{j,l,3}$. By the estimates in the proof of Lemma 5.3.4 and the bounds satisfied by ϕ, one has

$$|\bar{\beta}^i_{j,l,2}| \leq Ce^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-(\gamma_1+\frac{n-2\gamma}{2})t_j}.$$

In addition,

$$\begin{aligned}
-\bar{\beta}^i_{j,l,3} &= \int_{\mathbb{R}^n} N(\phi) Z^i_{j,l}\, dx\\
&= \int_{B(p_i,1)} + \sum_{i'\neq i}\int_{B(p_{i'},1)} + \int_{\mathbb{R}^n\setminus\cup_i B(p_i,1)} =: J_1 + J_2 + J_3.
\end{aligned}$$

We estimate this expression term by term. For $l = 0$, one has

$$\begin{aligned}
J_1 &= \int_{B(p_i,1)} N(\phi) Z^i_{j,l}\, dx \leq C\int_{B(p_i,1)} \|\phi\|^2_* |x-p_i|^{2\gamma_1}\bar{u}^{\beta-2} Z^i_{j,l}\, dx\\
&\leq C\|\phi\|^2_* \int_{B(p_i,1)} |x-p_i|^{2\gamma_1-\frac{6\gamma-n}{2}}|x-p_i|^{-\frac{n-2\gamma}{2}}|v'_j|\, dx\\
&\leq Ce^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\min\{2(\gamma_1+\frac{n-2\gamma}{2}),\frac{n-2\gamma}{2}\}t^i_j},
\end{aligned}$$

recalling (5.3.16). Similarly,

$$\begin{aligned}J_2 &= \sum_{i'\neq i}\int_{B(p_{i'},1)} N(\phi) Z^i_{j,l}\,dx \le C\|\phi\|_*^2 \sum_{i'\neq i}\int_{B(p_{i'},1)} |x-p_{i'}|^{2\gamma_1}|x-p_{i'}|^{\frac{6\gamma-n}{2}} Z^i_{j,l}\,dx\\ &\le C\|\phi\|_*^2 \,(\lambda^i_j)^{\frac{n-2\gamma}{2}} \le Ce^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\frac{n-2\gamma}{2}t^i_j}\end{aligned}$$

and

$$\begin{aligned}J_3 &= \int_{\mathbb{R}^n\setminus\cup_i B(p_i,1)} N(\phi) Z^i_{j,l}\,dx\\ &\le C\|\phi\|_*^2 \int_{\mathbb{R}^n\setminus\cup_i B(p_i,1)} |x|^{-2(n-2\gamma)} \bar u^{\beta-2} Z^i_{j,l}\,dx\\ &\le C\|\phi\|_*^2 \int_{\mathbb{R}^n\setminus\cup_i B(p_i,1)} |x|^{-2(n-2\gamma)} e^{-(\beta-2)\frac{(n-2\gamma)L}{4}} |x|^{-(n-2\gamma)(\beta-2)} (\lambda^i_j)^{\frac{n-2\gamma}{2}} |x|^{-(n-2\gamma)}\,dx\\ &\le Ce^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\frac{n-2\gamma}{2}t^i_j}.\end{aligned}$$

Combining the above estimates, we have for $l=0$,

$$|\bar\beta^i_{j,0,2}| + |\bar\beta^i_{j,0,3}| \le Ce^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-(\gamma_1+\frac{n-2\gamma}{2})t^i_j}.$$

For $l=1,\cdots,n$, by the estimate for ϕ given in (5.3.16) and the bounds for the term I_1 from (5.3.26) in Section 5.3 one obtains a similar estimate as above. But this is not enough for our analysis; one needs to be more precise. In order to do this, first recall the definition of $\bar u_0$ from (5.3.4),

$$\bar u_0 = \sum_{i=1}^k U_i(|x-p_i|),$$

where U_i are radial functions in $|x-p_i|$.

Near each singular point p_i, we can decompose

$$\bar u_0 = U_i(|x-p_i|) + D_i + O(\lambda^i_0|x-p_i|),$$

where D_i depends on $p_i-p_{i'}$ for $i'\neq i$. And similarly, we can decompose $S(\bar u_0)$ into two parts,

$$S(\bar u_0) = \mathcal{E}(|x-p_i|) + \mathcal{E}_1(x),$$

where $\mathcal{E}$ is radial function in $|x-p_i|$ and can be controlled by $Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}|x-p_i|^{\gamma_1-2\gamma}$; the second term can be controlled by $Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}|x-p_i|^{\gamma_1-2\gamma+1}$. We now proceed as follows. Let $\varphi_i=\varphi_i(|x-p_i|)$ be the solution to

$$\begin{cases}(-\Delta)^\gamma\varphi_i - c_{n,\gamma}\beta[U_i+D_i]^{\beta-1}\varphi_i\chi_i = \mathcal{E}(|x-p_i|)\chi_i + \displaystyle\sum_{j=0}^\infty c^i_{j,0}(w^i_j)^{\beta-1}Z^i_{j,0},\\ \displaystyle\int_{\mathbb{R}^n}(w^i_j)^{\beta-1}Z^i_{j,0}\varphi_i\,dx = 0 \quad \text{for } j=0,\cdots,\infty.\end{cases}$$

Note that the existence of such a φ_i can be proved similarly to the arguments in Section 5.3. Moreover, as in (5.3.16) one has

$$\|\varphi_i\|_* \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}.$$

Then we decompose $\phi = \sum_{i=1}^k \varphi_i\chi_i + \tilde{\varphi}$. In this case, since we have cancelled the radial part in the error near each singular point p_i by φ_i, then the extra error will have an extra factor $|x-p_i|$ and $\tilde{\varphi}$ will satisfy

$$|\tilde{\varphi}| \leq C\begin{cases} e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}|x-p_i|^{\gamma_1+1} \text{ in } B(p_i,1), \\ e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}|x|^{-(n-2\gamma)} \text{ in } \mathbb{R}^n\setminus\cup_i B(p_i,1). \end{cases}$$

Therefore, by the above decomposition of ϕ into radial and nonradial parts one deduces

$$\begin{aligned} -\bar{\beta}^i_{j,l,2} &= -\int_{\mathbb{R}^n}\mathbb{L}_0(\phi)Z^i_{j,l}\,dx \\ &= \int_{\mathbb{R}^n}[(\bar{u}_0)^{\beta-1}-(w^i_j)^{\beta-1}]Z^i_{j,l}\phi\,dx \\ &= \int_{\mathbb{R}^n}[(U_i+D_i)^{\beta-1}-(w^i_j)^{\beta-1}]Z^i_{j,l}\tilde{\varphi}\,dx + \int_{\mathbb{R}^n}[(\bar{u}_0)^{\beta-1}-(U_i+D_i)^{\beta-1}]Z^i_{j,l}\phi\,dx. \end{aligned}$$

Similar to the estimate of I_1 in the proof of Lemma 5.3.4, recalling the asymptotic behaviour of $\tilde{\varphi}$ near p_i, we can get that the first term can be controlled by

$$\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-(\gamma_1+\frac{n-2\gamma}{2})t^i_j}$$

in a ball $B(p_i,1)$ (see (5.3.27) and notice the extra factor λ^i_j). For the second term,

$$\begin{aligned} &\int_{B(p_i,1)}[(\bar{u}_0)^{\beta-1}-(U_i+D_i)^{\beta-1}]Z^i_{j,l}\phi\,dx \\ &\qquad\leq Ce^{-\frac{(n-2\gamma)L}{4}}\|\phi\|_*\int_{B(p_i,1)}|x|^{\gamma_1-2\gamma+1}(\sum_{j'}v_{j'})^{\beta-2}v_j^{1+\frac{2}{n-2\gamma}}\,dx \\ &\qquad\leq Ce^{-\frac{(n-2\gamma)L}{2}(1+\xi)}\int_{B(p_i,1)}|x|^{\gamma_1-2\gamma+1}v_j^{\beta-1+\frac{2}{n-2\gamma}}\,dx \\ &\qquad\leq C\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-(\gamma_1+\frac{n-2\gamma}{2})t^i_j}. \end{aligned}$$

Combining the above two estimates,

$$\int_{B(p_i,1)}\mathbb{L}_0(\phi)Z^i_{j,l}\,dx = O(\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-(\gamma_1+\frac{n-2\gamma}{2})t^i_j}).$$

Next, the asymptotic behaviour of $Z^i_{j,l}$ at infinity, given by

$$|Z^i_{j,l}| = (\lambda^i_j)^{\frac{n-2\gamma}{2}+1}|x|^{-(n-2\gamma+1)} \quad \text{if } |x-p_i|\geq 1,$$

yields that

$$\sum_{i'\neq i}\int_{B(p_{i'},1)}\mathbb{L}_0(\phi)Z_{j,l}^i\,dx+\int_{\mathbb{R}^n\setminus\cup_{i'}B(p_{i'},1)}\mathbb{L}_0(\phi)Z_{j,l}^i\,dx\le C\lambda_j^i e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-(\gamma_1+\frac{n-2\gamma}{2})t_j^i}.$$

Using similar argument, we obtain an analogous estimate for $\bar{\beta}_{j,l,3}^i$. Thus for $l=1,\cdots,n$,

$$|\bar{\beta}_{j,l,2}^i|+|\bar{\beta}_{j,l,3}^i|\le C\lambda_j^i e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-(\gamma_1+\frac{n-2\gamma}{2})t_j^i},$$

which completes the proof of Step 1.

Step 2: Estimate for $\bar{\beta}_{j,l,1}^i$. Denote by $E:=S(\bar{u}_0)=(-\Delta)^\gamma\bar{u}_0-c_{n,\gamma}\bar{u}_0^\beta$. We compute

$$\begin{aligned}\bar{\beta}_{j,l,1}^i&=\int_{\mathbb{R}^n}EZ_{j,l}^i\,dx=\Big[\int_{B(p_i,1)}+\sum_{i'\neq i}\int_{B(p_{i'},1)}+\int_{\mathbb{R}^n\setminus\cup_{i'}B(p_{i'},1)}\Big]\,dx\\&=I_{1,j,l}+I_{2,j,l}+I_{3,j,l}.\end{aligned}$$

Recalling the estimate for E from (5.3.19) one has

$$I_{2,j,0}\le Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}\sum_{i'\neq i}\int_{B(p_{i'},1)}|x-p_{i'}|^{\gamma_1-2\gamma}(\lambda_j^i)^{\frac{n-2\gamma}{2}}\,dx\le Ce^{\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\frac{n-2\gamma}{4}t_j^i}$$

and

$$\begin{aligned}I_{3,j,0}&\le Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}\int_{\mathbb{R}^n/\cup_{i'}B(p_{i'},1)}|x|^{-(n+2\gamma)}(\lambda_j^i)^{\frac{n-2\gamma}{2}}|x|^{-(n-2\gamma)}\,dx\\&\le Ce^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\frac{n-2\gamma}{4}t_j^i}.\end{aligned}$$

For $l=1,\cdots,n$, we know that $|Z_{j,l}^i|=O(\lambda_j^i)Z_{j,0}^i$, which yields easily that

$$|I_{2,j,l}|+|I_{3,j,l}|\le Ce^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-(1+\frac{n-2\gamma}{4})t_j^i}.$$

Next, for $l=0,\dots,n$, we consider $I_{1,j,l}$: fixed i,j, substitute the expression for $\bar{u}_0$ from

(5.3.4),

$$
\begin{aligned}
E &= (-\Delta)^\gamma \bar u_0 - c_{n,\gamma}\bar u_0^\beta \\
&= (-\Delta)^\gamma(\sum_{j'=0}^\infty w^i_{j'}) + (-\Delta)^\gamma\phi_i - \beta c_{n,\gamma}(w^i_j)^{\beta-1}\phi_i + (-\Delta)^\gamma(1-\chi_i)\phi_i \\
&\quad - c_{n,\gamma}[(\sum_{j'=0}^\infty w^i_{j'} + \phi_i + \sum_{i'\neq i} w^{i'}_j)^\beta - \sum_{i'\neq i}(w^{i'}_j)^\beta - \beta(w^i_j)^{\beta-1}\phi_i] \\
&= -c_{n,\gamma}[(\sum_{j'=0}^\infty w^i_{j'} + \phi_i + \sum_{i'\neq i} w^{i'}_j)^\beta - \sum_{j=0}^\infty (w^i_j)^\beta - \sum_{i'\neq i}(w^{i'}_j)^\beta - \beta(w^i_j)^{\beta-1}\phi_i] \\
&\quad + \mathbb{L}^i_j\phi_i + (-\Delta)^\gamma(1-\chi_i)\phi_i \\
&= -c_{n,\gamma}\big[\bar u^\beta - \sum_{j'=0}^\infty (w^i_{j'})^\beta - \sum_{i'\neq i}(w^{i'}_j)^\beta - \beta(w^i_j)^{\beta-1}\big(\phi_i + \sum_{i'\neq i} w^{i'}_j + \sum_{j'\neq j} w^i_{j'}\big) \\
&\quad + \beta(w^i_j)^{\beta-1}\big(\sum_{i'\neq i} w^{i'}_j + \sum_{j'\neq j} w^i_{j'}\big)\big] + \mathbb{L}^i_j\phi_i + (-\Delta)^\gamma(1-\chi_i)\phi_i.
\end{aligned}
$$

Here $\mathbb{L}^i_j$ denotes the linearized operator around w^i_j. Looking at the equation that ϕ_i satisfies and its bounds (see formula (5.3.2) and Corollary 5.2.3), we have in general the following estimates:

On the one hand, for $l=0$, since $Z^i_{j,0}$ is odd in the variable $t^i - t^i_j$, where we have defined $t^i = -\log|x-p_i|$, by the above expansion for E,

$$
\begin{aligned}
I_{1,j,0} &\leq C\int_{B(p_i,1)} (w^i_j)^{\beta-1} Z^i_{j,0}\big(\sum_{i'\neq i} w^{i'}_0 + \sum_{j'\neq j} w^i_{j'}\big)\,dx + O(e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}(\lambda^i_j)^{\frac{n-2\gamma}{2}}) \\
&\leq C\sum_{i'\neq i}(\lambda^{i'}_0)^{\frac{n-2\gamma}{2}}\int_{B_1}|x|^{-2\gamma}(v^i_j)^{\beta-1}|x|^{-\frac{n-2\gamma}{2}}|(v^i_j)'|\,dx + \int_{B_1}\sum_{j'\neq j}|x|^{-n}(v^i_j)^{\beta-1}(v^i_j)'v^i_{j'}\,dx \\
&\leq Ce^{-\frac{n-2\gamma}{4}L}\int_0^\infty e^{-\frac{n-2\gamma}{2}t}(v^i_j)^\beta\,dt + \int_0^\infty (v^i_j)^{\beta-1}(v^i_j)'\sum_{j'\neq j} v^i_{j'}\,dt \\
&\quad + O(e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}(\lambda^i_j)^{\frac{n-2\gamma}{2}}) \\
&\leq Ce^{-\frac{n-2\gamma}{4}L}e^{-\frac{n-2\gamma}{2}t^i_j} + \sum_{j'\leq 2j}\int_0^\infty (v^i_j)^{\beta-1}(v^i_j)'v^i_{j'}\,dt + \sum_{j'\geq 2j+1}\int_0^\infty (v^i_j)^{\beta-1}(v^i_j)'v^i_{j'}\,dt.
\end{aligned}
$$

Let us bound the two terms in this expression:

$$\begin{aligned}
&\int_0^\infty (v_j^i)^{\beta-1}(v_j^i)' \sum_{j'\le 2j} v_{j'}^i \, dt \\
&= \int_0^{t_{2j}^i+\frac{L_i}{2}} (v_j^i)^{\beta-1}(v_j^i)' \sum_{j'\le 2j} v_{j'}^i \, dt + \int_{t_{2j}^i+\frac{L_i}{2}}^\infty (v_j^i)^{\beta-1}(v_j^i)' \sum_{j'\le 2j} v_{j'}^i \, dt \\
&(t' = t - t_j^i) \\
&= \int_{-t_j^i}^{t_j^i} v^{\beta-1} v'(t') \sum_{0\le j'\le 2j} v_{j'}^i \, dt' + \int_{t\ge t_{2j}^i+\frac{L_i}{2}} (v_j^i)^{\beta-1}(v_j^i)' \sum_{j'\le 2j} v_{j'}^i \, dt
\end{aligned}$$

For the first integral, since $v'(t')$ is odd in t' and $\sum_{j'\le 2j} v_{j'}^i$ is an even function of t', this integral is 0. In the meantime, thanks to the exponential decaying of v, the second integral is bounded by $e^{-\frac{n+2\gamma}{2}t_j^i}$, and we may conclude that

$$\int_0^\infty (v_j^i)^{\beta-1}(v_j^i)' \sum_{j'\le 2j} v_{j'}^i \, dt \le C e^{-\frac{n+2\gamma}{2}t_j^i}.$$

In addition,

$$\begin{aligned}
&\sum_{j'\ge 2j+1} \int_0^\infty (v_j^i)^{\beta-1}(v_j^i)' v_{j'}^i \, dt \\
&\le C e^{-\frac{(n-2\gamma)}{2}|t_{2j+1}^i - t_j^i|} \le C e^{-\frac{n-2\gamma}{4}L} e^{-\frac{n-2\gamma}{2}t_j^i}.
\end{aligned}$$

In conclusion, one has

$$I_{1,j,0} \le C e^{-\frac{n-2\gamma}{4}L} e^{-\frac{n-2\gamma}{2}t_j^i} + e^{-\frac{n+2\gamma}{2}t_j^i}.$$

On the other hand, for $l = 1, \cdots, n$, since $Z_{j,l}^i$ is odd in $x - p_i$, and both w_j^i, ϕ_i are even in $x - p_i$, one has

$$\begin{aligned}
I_{1,j,l} &\le C \int_{B(p_i,1)} \sum_{i'\ne i} (w_j^i)^{\beta-1} Z_{j,l}^i w_0^{i'} \, dx \\
&\le C \int_{B(p_i,1)} \sum_{i'\ne i} (w_j^i)^{\beta-1} |x-p_i|^{-\frac{n-2\gamma}{2}+1} v_j^{1+\frac{2}{n-2\gamma}} (\lambda_0^{i'})^{\frac{n-2\gamma}{2}} \, dx \\
&\le C e^{-\frac{n-2\gamma}{4}L} \int_{B_1} |x|^{-2\gamma} v_j^{\beta-1} |x|^{-\frac{n-2\gamma}{2}+1} v_j^{1+\frac{2}{n-2\gamma}} \, dx \\
&\le C e^{-\frac{n-2\gamma}{4}L} e^{-(1+\frac{n-2\gamma}{2})t_j^i}.
\end{aligned}$$

From the above two estimates, when $j \ge 1$,

$$I_{1,j,l} \le \begin{cases} e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma t_j^i} & \text{if } l = 0, \\ e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-(1+\sigma)t_j^i} & \text{if } l = 1, \cdots, n, \end{cases}$$

for some $\xi, \sigma > 0$.

On the contrary, for $j = 0$ one has

$$I_{1,0,0} = O(e^{-\frac{n-2\gamma}{2}L}), \quad I_{1,0,l} = O(\lambda_0^i)e^{-\frac{n-2\gamma}{2}L},$$

but can obtain more accurate estimates in this case. This is going to be the crucial step in the proof of the Lemma since it gives the formula for the compatibility conditions.

First, if $l = 0$,

$$\begin{aligned} I_{1,0,0} &= -c_{n,\gamma}\beta \int_{\mathbb{R}^n} (w_0^i)^{\beta-1}(w_1^i + \sum_{i' \neq i} w_0^{i'}) Z_{0,0}^i \, dx + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}) \\ &=: T_{1,0} + T_{2,0} + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}). \end{aligned}$$

In this case, by expression (8.0.1),

$$T_{1,0} = -c_{n,\gamma} F(|\log \tfrac{\lambda_1^i}{\lambda_0^i}|) \frac{\log \frac{\lambda_1^i}{\lambda_0^i}}{|\log \frac{\lambda_1^i}{\lambda_0^i}|},$$

and by formula (8.0.2) and the relation that $Z_{0,0}^i = \frac{\partial w_0^i}{\partial \lambda_0^i} \lambda_0^i R^i$,

$$T_{2,0} = -c_{n,\gamma} \sum_{i' \neq i} A_2 |p_{i'} - p_i|^{-(n-2\gamma)} (\lambda_0^i \lambda_0^{i'})^{\frac{n-2\gamma}{2}} [1 + O(\lambda_0^i)^2].$$

Combining the above two estimates yields

$$\begin{aligned} I_{1,0,0} &= c_{n,\gamma}\Big[F(L_1) - \sum_{i' \neq i} A_2 |p_{i'} - p_i|^{-(n-2\gamma)} (\lambda_0^i \lambda_0^{i'})^{\frac{n-2\gamma}{2}}\Big] + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}) \\ &= c_{n,\gamma} q_i \Big[q_i - A_2 \sum_{i' \neq i} q_{i'} (R^i R^{i'})^{\frac{n-2\gamma}{2}} |p_i - p_{i'}|^{-(n-2\gamma)}\Big] e^{-\frac{(n-2\gamma)L}{2}} (1 + o(1)) \\ &+ O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}). \end{aligned}$$

On the other hand, for $l = 1, \cdots, n$

$$I_{1,0,l} = -c_{n,\gamma}\beta \int_{\mathbb{R}^n} (w_0^i)^{\beta-1} (\sum_{i' \neq i} w_0^{i'}) Z_{0,0}^i \, dx + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} \lambda_0^i),$$

and recall that $Z_{j,l}^i = -\frac{\partial w_j^i}{\partial x_l} \lambda_j^i$, by the estimate (8.0.3), one has

$$I_{1,0,l} = c_{n,\gamma} A_3 \lambda_0^i [\sum_{i' \neq i} \frac{(p_{i'} - p_i)_l}{|p_{i'} - p_i|^{n-2\gamma+2}} (\lambda_0^i \lambda_0^{i'})^{\frac{n-2\gamma}{2}}] + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} \lambda_0^i).$$

Then combining the estimates for $\bar{\beta}_{j,l,1}^i, \bar{\beta}_{j,l,2}^i, \bar{\beta}_{j,l,3}^i$, we achieve the proof of the Lemma. □

Remark 5.4.2. Fixed i, if we consider the approximate solution $\bar{u}_i^0$ with $a_j^i = 0, r_j^i = 0$ for $j = 0, \cdots, \infty$, then the same estimates for $\bar{\beta}_{j,l}^i$ in the above lemma hold.

5.4.2 Estimates for general parameters

Next we study the coefficients $c^i_{j,l}$ for a general configuration space satisfying (5.3.9) and (5.3.10). Most of the estimates of the previous subsection will continue to hold, but we need to be especially careful when considering $\beta^i_{j,l,1}$. First, from Remark 5.4.2, one can see that only the perturbations of a^i_j, r^i_j will affect the numbers $\beta^i_{j,l}$, i.e, we can get the same estimates for $\beta^i_{j,l}$ for general parameters $(a^{i'}_j, r^{i'}_j)$ satisfying (5.3.9) and (5.3.10) if $(a^i_j, r^i_j) = (0,0)$. So fixing $i = I$, we would like to study the estimates for $\beta^i_{j,l}$ when we vary the parameters a^i_j, r^i_j. First we have the following estimates:

Lemma 5.4.3. *Suppose that the parameters R^i satisfy (5.3.7). Let e^I_J be a vector in $\mathbb{R}^n$ and r^I_J be a real number in $\mathbb{R}$. We let $X(t)$ be the configuration for which all the parameters (a^i_j, r^i_j) are fixed to be $(0,0)$ if $j \neq J$ and where $(a^I_J, R^I_J) = (te^I_J, R^I(1+tr^I_J))$. Assume that $|e^I_J| \leq C(\lambda^I_J)^2$ and $|r^I_J| \leq Ce^{-\tau t^I_J}$. We also let $w^I_{J,t} = w_{R^I(1+tr^I_J)}(x - te^I_J)$. Then we have the following:*

- *If $i = I$, $J \neq j$,*

$$\frac{\partial}{\partial t}\Big|_{t=0} \int_{\mathbb{R}^n} S(\bar{u}_t) Z^i_{j,l}\, dx$$
$$= \begin{cases} -\dfrac{\partial}{\partial t}\Big[c_{n,\gamma} F(|\log \frac{\lambda^i_J}{\lambda^i_j}|) \dfrac{\log \frac{\lambda^i_J}{\lambda^i_j}}{|\log \frac{\lambda^i_J}{\lambda^i_j}|}\Big] + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\tau t^I_J} e^{-\sigma|t^I_j - t^I_J|}) \text{ if } l = 0, \\ -\dfrac{\partial}{\partial t}\Big[c_{n,\gamma} A_0 \lambda^i_j \min\{\frac{\lambda^i_J}{\lambda^i_j}, \frac{\lambda^i_j}{\lambda^i_J}\}^{\frac{n-2\gamma}{2}} \dfrac{te_l}{|\max\{\lambda^i_j, \lambda^i_J\}|^2}\Big] \\ \qquad + O(\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\tau t^I_J} e^{-\sigma|t^I_j - t^I_J|}) \text{ if } l = 1, \cdots, n, \end{cases}$$

  for some $\sigma > 0$ independent of τ and L.

- *If $i = I$, $J = j = 0$,*

$$\frac{\partial}{\partial t}\Big|_{t=0} \int_{\mathbb{R}^n} S(\bar{u}_t) Z^i_{j,l}\, dx$$
$$= \begin{cases} -c_{n,\gamma} \dfrac{\partial}{\partial t}\Big[A_2 \sum_{i' \neq i} |p_{i'} - p_i|^{-(n-2\gamma)} (\lambda^i_0 \lambda^{i'}_0)^{\frac{n-2\gamma}{2}} + F(|\log \frac{\lambda^i_0}{\lambda^i_1}|) \dfrac{\log \frac{\lambda^i_1}{\lambda^i_0}}{|\log \frac{\lambda^i_1}{\lambda^i_0}|}\Big] \\ \qquad + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}) \text{ if } l = 0, \\ c_{n,\gamma} \lambda^i_0 \dfrac{\partial}{\partial t}\Big[A_3 \sum_{i' \neq i} \dfrac{p_{i'} - p_i}{|p_{i'} - p_i|^{n-2\gamma+2}} (\lambda^i_0 \lambda^{i'}_0)^{\frac{n-2\gamma}{2}} \\ \quad + A_0 \min\{\frac{\lambda^i_0}{\lambda^i_1}, \frac{\lambda^i_1}{\lambda^i_0}\}^{\frac{n-2\gamma}{2}} \dfrac{te_l}{|\max\{\lambda^i_{j'}, \lambda^i_J\}|^2}\Big] + O(\lambda^i_0 e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}) \text{ if } l = 1, \cdots, n. \end{cases}$$

- *If* $i = I,\ J = j \geq 1$,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\Big|_{t=0} \int_{\mathbb{R}^n} S(\bar{u}_t) Z^i_{j,l}\, dx \\
&= \begin{cases}
-c_{n,\gamma} \sum_{j' \neq j} \frac{\partial}{\partial t}\big[F(|\log \frac{\lambda^i_j}{\lambda^i_{j'}}) \frac{\log \frac{\lambda^i_{j'}}{\lambda^i_j}}{|\log \frac{\lambda^i_{j'}}{\lambda^i_j}|}\big] + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\tau t_j}) & \textit{if } l = 0, \\
c_{n,\gamma} \lambda^i_j \frac{\partial}{\partial t}\big[A_0 \sum_{j' \neq J} \min\{\frac{\lambda^i_{j'}}{\lambda^i_J}, \frac{\lambda^i_J}{\lambda^i_{j'}}\}^{\frac{n-2\gamma}{2}} \frac{te_l}{|\max\{\lambda^i_{j'}, \lambda^i_J\}|^2}\big] & \\
\qquad\qquad + O(\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\tau t_J}) & \textit{if } l = 1, \cdots, n.
\end{cases}
\end{aligned}
$$

Proof. Fix $i = I$. We first consider the case in which $J \neq j$. We have

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \int_{\mathbb{R}^n} S(\bar{u}_t) Z^i_{j,l}\, dx &= c_{n,\gamma}\beta \int_{\mathbb{R}^n} [(w^i_{J,t})^{\beta-1} - \bar{u}_t^{\beta-1}] \frac{\partial w^i_{J,t}}{\partial t} Z^i_{j,l}\, dx \\
&= -c_{n,\gamma}\beta \int_{\mathbb{R}^n} (w^i_j)^{\beta-1} \frac{\partial w^i_{J,t}}{\partial t} Z^i_{j,l}\, dx \\
&\quad + c_{n,\gamma}\beta \int_{\mathbb{R}^n} [(w^i_j)^{\beta-1} + (w^i_{J,t})^{\beta-1} - \bar{u}_t^{\beta-1}] \frac{\partial w^i_{J,t}}{\partial t} Z^i_{j,l}\, dx \\
&=: M_1 + M_2.
\end{aligned}
$$

From the proof of Appendix 8, more precisely, (8.0.1) for $l = 0$ and (8.0.5) for $l = 1, \ldots, n$, one can find that

$$
\begin{aligned}
M_1 &= -c_{n,\gamma}\beta \int_{\mathbb{R}^n} (w^i_j)^{\beta-1} \frac{\partial w^i_{J,t}}{\partial t} Z^i_{j,l}\, dx = -c_{n,\gamma}\beta \frac{\partial}{\partial t} \int_{\mathbb{R}^n} (w^i_j)^{\beta-1} Z^i_{j,l} w^i_{J,t}\, dx \\
&= \begin{cases}
-\frac{\partial}{\partial t}\Big[c_{n,\gamma} F(|\log \frac{\lambda^i_J}{\lambda^i_j}|) \frac{\log \frac{\lambda^i_J}{\lambda^i_j}}{|\log \frac{\lambda^i_J}{\lambda^i_j}|}\Big] & \text{if } l = 0, \\
-\frac{\partial}{\partial t}\Big[c_{n,\gamma} A_0 \lambda^i_j \min\{\frac{\lambda^i_J}{\lambda^i_j}, \frac{\lambda^i_j}{\lambda^i_J}\}^{\frac{n-2\gamma}{2}} \frac{te_l}{|\max\{\lambda^i_j, \lambda^i_J\}|^2}\Big]
\end{cases}
\end{aligned}
$$

for a constant $A_0 > 0$. Moreover,

$$
\begin{aligned}
M_2 &= c_{n,\gamma}\beta \int_{\mathbb{R}^n} [(w^i_j)^{\beta-1} + (w^i_{J,t})^{\beta-1} - \bar{u}_t^{\beta-1}] \frac{\partial w^i_{J,t}}{\partial t} Z^i_{j,l}\, dx \\
&= \begin{cases}
O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}) e^{-\tau t^I_J} e^{-\sigma|t^I_j - t^I_J|} & \text{for } l = 0, \\
O(\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}) e^{-\tau t^I_J} e^{-\sigma|t^I_j - t^I_J|} & \text{for } l = 1, \cdots, n,
\end{cases}
\end{aligned}
$$

which proves the assertion when $J \neq j$.

Now we consider the case $J = j$, for which we have

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\int_{\mathbb{R}^n} S(\bar{u}_t)Z^i_{j,t,l}\,dx &= \int_{\mathbb{R}^n}\frac{\partial}{\partial t}S(\bar{u}_t)Z^i_{j,t,l}\,dx + \int_{\mathbb{R}^n} S(\bar{u}_t)\frac{\partial}{\partial t}Z^i_{j,t,l}\,dx \\
&= \int_{\mathbb{R}^n}\big[(-\Delta)^\gamma - c_{n,\gamma}\beta(w^i_{J,t})^{\beta-1}\big]\big[\sum_{j'\neq J} w^i_{j'} + \sum_{i'\neq i} w^{i'}_j + \phi_{L_i}\big]\frac{\partial}{\partial t}Z^i_{j,t,l}\,dx \\
&\quad - \beta(\beta-1)c_{n,\gamma}\int_{\mathbb{R}^n}(w^i_{J,t})^{\beta-2}\big[\sum_{j'\neq J} w^i_{j'} + \sum_{i'\neq i} w^{i'}_j + \phi_{L_i}\big]\frac{\partial w^i_{J,t}}{\partial t}Z^i_{j,t,l}\,dx \\
&\quad + \begin{cases} O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\tau t^I_j}) & \text{if } l = 0, \\ O(\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\tau t^I_j}) & \text{if } l \geq 1. \end{cases}
\end{aligned}
$$

From the equation satisfied by $Z_{t,l,i}$, and taking derivative with respect to t, one can cancel the terms containing ϕ_{L_i}, which yields

$$
\begin{aligned}
\frac{d}{dt}\Big|_{t=0}\int_{\mathbb{R}^n} S(\bar{u}_t)Z^i_{j,t,l}\,dx &= \int_{\mathbb{R}^n} -c_{n,\gamma}\beta(w^i_{J,t})^{\beta-1}\big[\sum_{j'\neq J} w^i_{j'} + \sum_{i'\neq i} w^{i'}_j\big]\frac{\partial}{\partial t}Z^i_{j,t,l}\,dx \\
&\quad - \beta(\beta-1)c_{n,\gamma}\int_{\mathbb{R}^n}(w^i_{J,t})^{\beta-2}\big[\sum_{j'\neq J} w^i_{j'} + \sum_{i'\neq i} w^{i'}_j\big]\frac{\partial w^i_{J,t}}{\partial t}Z^i_{j,t,l}\,dx \\
&\quad + \begin{cases} O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\tau t^I_j}) & \text{if } l = 0, \\ O(\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\tau t^I_j}) & \text{if } l \geq 1, \end{cases} \\
&= -c_{n,\gamma}\frac{\partial}{\partial t}\int_{\mathbb{R}^n}\beta(w^i_{J,t})^{\beta-1}Z^i_{j,t,l}\big[\sum_{j'\neq J} w^i_{j'} + \sum_{i'\neq i} w^{i'}_j\big]\,dx \\
&\quad + \begin{cases} O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\tau t^I_j}) & \text{if } l = 0, \\ O(\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\tau t^I_j}) & \text{if } l \geq 1, \end{cases} \\
&=: N_1 + N_2 + O(\cdots).
\end{aligned}
$$

Similar to the estimates before, one can get that for $l = 1, \cdots, n$, by estimate (8.0.5) in the Appendix 8,

$$
\begin{aligned}
N_1 &= \int_{\mathbb{R}^n}\beta(w^i_{J,t})^{\beta-1}Z^i_{j,t,l}\sum_{j'\neq J} w^i_{j'}\,dx \\
&= -A_0\lambda^i_j\sum_{j'\neq J}\min\Big\{\frac{\lambda^i_{j'}}{\lambda^i_J}, \frac{\lambda^i_J}{\lambda^i_{j'}}\Big\}^{\frac{n-2\gamma}{2}}\frac{te_l}{|\max\{\lambda^i_{j'}, \lambda^i_J\}|^2} + O(\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}}(1+\xi)e^{-\tau t^I_J}),
\end{aligned}
$$

and from (8.0.3),

$$N_2 = \int_{\mathbb{R}^n} \beta(w^i_{J,t})^{\beta-1} Z^i_{j,t,l} \sum_{i'\neq i} w^{i'}_j \, dx$$
$$= \begin{cases} -\lambda^i_0 \Big[A_3 \sum_{i'\neq i} \frac{(p_{i'}-p_i)_l}{|p_{i'}-p_i|^{n-2\gamma+2}} (\lambda^i_0 \lambda^{i'}_0)^{\frac{n-2\gamma}{2}} + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}) \Big] & \text{if } J = 0, \\ O(\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\tau t^I_J}) & \text{if } J \geq 1. \end{cases}$$

On the other hand, for $l = 0$, by (8.0.1),

$$N_1 = \int_{\mathbb{R}^n} \beta(w^i_{J,t})^{\beta-1} Z^i_{j,t,l} \sum_{j'\neq J} w^i_{j'} \, dx = \sum_{j'\neq j} F\big(\big|\log \frac{\lambda^i_j}{\lambda^i_{j'}}\big|\big) \frac{\log \frac{\lambda^i_{j'}}{\lambda^i_j}}{|\log \frac{\lambda^i_{j'}}{\lambda^i_j}|},$$

and using (8.0.2),

$$N_2 = \int_{\mathbb{R}^n} \beta(w^i_{J,t})^{\beta-1} Z^i_{j,t,l} \sum_{i'\neq i} w^{i'}_j \, dx$$
$$= \begin{cases} A_2 \sum_{i'\neq i} |p_{i'}-p_i|^{-(n-2\gamma)} (\lambda^i_0 \lambda^{i'}_0)^{\frac{n-2\gamma}{2}} + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}) & \text{if } J = 0, \\ e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\tau t^I_J} & \text{if } J \geq 1. \end{cases}$$

Combining all the above estimates, the proof of the Lemma is completed.

□

From Lemma 5.4.1 and Lemma 5.4.3, we find the decay estimate for the $\beta^i_{j,l}$ for general parameters a^i_j, r^i_j satisfying conditions (5.3.9) and (5.3.10):

Lemma 5.4.4. *For the parameters (a^i_j, R^i_j) satisfying (5.3.9) and (5.3.10), we have the following estimates:*

$$\beta^i_{0,0} = -c_{n,\gamma} q_i \Big[A_2 \sum_{i'\neq i} |p_{i'}-p_i|^{-(n-2\gamma)} (R^i_0 R^{i'}_0)^{\frac{n-2\gamma}{2}} q_{i'} - \Big(\frac{R^i_1}{R^i_0}\Big)^{\frac{n-2\gamma}{2}} q_i \Big] e^{-\frac{(n-2\gamma)L}{2}} (1+o(1))$$
$$+ O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}),$$
$$\beta^i_{0,l} = c_{n,\gamma} \lambda^i_0 \Big[A_3 \sum_{i'\neq i} \frac{(p_{i'}-p_i)_l}{|p_{i'}-p_i|^{n-2\gamma+2}} (R^i_0 R^{i'}_0)^{\frac{n-2\gamma}{2}} q_{i'} + A_0 \Big(\frac{R^i_1}{R^i_0}\Big)^{\frac{n-2\gamma}{2}} \frac{a^i_0 - a^i_1}{(\lambda^i_0)^2} q_i \Big] q_i e^{-\frac{n-2\gamma}{2}L}$$
$$+ O(\lambda^i_0 e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}), \ \textit{for } l \geq 1,$$
$$\beta^i_{j,l} = \begin{cases} O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma t^i_j} + e^{-\frac{(n-2\gamma)L}{2}} e^{-\tau t^i_{j-1}}), & l = 0, \\ O(\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma t^i_j} + \lambda^i_j e^{-\frac{(n-2\gamma)L}{2}} e^{-\tau t^i_{j-1}}), & l \geq 1, \end{cases} \quad \textit{for } j \geq 1.$$

where σ is obtained in Lemma 5.4.1.

Proof. Using the notation in the previous subsection, we first estimate $\beta^i_{j,l,1}$. Using Lemma 5.4.1 and Lemma 5.4.3, and integrating in t from 0 to 1, varying the parameters $(a, R) = (0, R^i)$ to (a^i_j, R^i_j), and using the estimates satisfied by the parameters. The integration yields

$$\begin{aligned}
\beta^i_{0,0,1} &= -c_{n,\gamma} q_i \Big[A_2 \sum_{i' \neq i} |p_{i'} - p_i|^{-(n-2\gamma)} (R^i_0 R^{i'}_0)^{\frac{n-2\gamma}{2}} q_{i'} - \Big(\frac{R^i_1}{R^i_0}\Big)^{\frac{n-2\gamma}{2}} q_i \Big] \\
&\quad \cdot e^{-\frac{(n-2\gamma)L}{2}} (1 + o(1)) + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}), \\
\beta^i_{0,l,1} &= c_{n,\gamma} \lambda^i_0 \Big[A_3 \sum_{i' \neq i} \frac{(p_{i'} - p_i)_l}{|p_{i'} - p_i|^{n-2\gamma+2}} (R^i_0 R^{i'}_0)^{\frac{n-2\gamma}{2}} q_{i'} + A_0 \Big(\frac{R^i_1}{R^i_0}\Big)^{\frac{n-2\gamma}{2}} \frac{a^i_0 - a^i_1}{(\lambda^i_0)^2} q_i \Big] q_i e^{-\frac{n-2\gamma}{2}L} \\
&\quad + O(\lambda^i_0 e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}), \ \text{for } l \geq 1, \\
\beta^i_{j,l,1} &= \begin{cases} O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma t^i_j} + e^{-\frac{(n-2\gamma)L}{2}} e^{-\tau t^i_{j-1}}) & \text{if } l = 0, \\ O(\lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma t^i_j} + \lambda^i_j e^{-\frac{(n-2\gamma)L}{2}} e^{-\tau t^i_{j-1}}) & \text{if } l \geq 1, \end{cases} \quad \text{for } j \geq 1.
\end{aligned}$$

Similarly to the estimates in subsection 5.4.1, $\beta^i_{j,l,2}$ and $\beta^i_{j,l,3}$ can be bounded by

$$|\beta^i_{j,l,2}| + |\beta^i_{j,l,3}| \leq \begin{cases} C e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma t^i_j}, \\ C \lambda^i_j e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma t^i_j}. \end{cases}$$

Hence we get the desired bounds on $\beta^i_{j,l}$. □

5.5 Derivatives of the coefficients $\beta^i_{j,l}$ with respect to the variation of $\{a^i_{j,l}\}$ and $\{r^i_{j,l}\}$

Here study the derivatives of the coefficients $\beta^i_{j,l}$ with respect to the parameters $\{a^i_{j,l}\}$ and $\{r^i_{j,l}\}$. As in the previous remark, we only need to care about the perturbation of a^i_j, r^i_j. Thus we first consider the derivatives of $\beta^i_{j,l}$ with respect to a^i_j, r^i_j for the special configuration space that $a^i_j, r^i_j = 0$ for fixed i. For this, we need to consider the variation of ϕ with respect to these parameters.

5.5.1 Derivatives of $\beta^i_{j,l}$ for a^i_j, r^i_j all equal to zero

In this section, we fix i and let $\bar{u}^0_i$ to be the approximate solution with $a^i_j, r^i_j = 0$. Given ϕ as in Proposition 5.3.3 for the approximate solution $\bar{u} = \bar{u}^i_0$, we introduce the operator

$$\tilde{\mathbb{L}} = (-\Delta)^\gamma - c_{n,\gamma} \beta (\bar{u}^0_i + \phi)^{\beta - 1}. \tag{5.5.1}$$

Denote by $\xi^i_{j,0} = r^i_j$, and $\xi^i_{j,l} = a^i_{j,l}$ for $l = 1, \cdots, n$.

Lemma 5.5.1. *For L large, let $\bar{u}_i^0$ and ϕ be as above. Then we have the following estimates on $\frac{\partial\phi}{\partial\xi_{j,l}^i}$ near p_i:*

$$\Big|\frac{\partial\phi}{\partial\xi_{j,l}^i}\Big| \leq \left\{ \begin{array}{ll} Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}|x-p_i|^{-\frac{n-2\gamma}{2}}e^{-\sigma|t^i-t_j^i|} & \text{for } l=0, \\ \frac{C}{\lambda_j^i}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}|x-p_i|^{-\frac{n-2\gamma}{2}}e^{-\sigma|t^i-t_j^i|} & \text{for } l\geq 1, \end{array} \right. \quad \text{in } B(p_i,1). \tag{5.5.2}$$

Proof. We first consider the case $l=0$. If we differentiate the first equation in Proposition 5.3.3 with respect to r_j^i, after some manipulation, we obtain that

$$\tilde{\mathbb{L}}\Big(\frac{\partial\phi}{\partial r_j^i}\Big) = \tilde{h} + \sum_{j',l,i'} \frac{\partial c_{j',l}^{i'}}{\partial r_j^i}(w_{j'}^{i'})^{\beta-1}Z_{j',l}^{i'},$$

where

$$\tilde{h} = -\frac{\partial S(\bar{u}_i^0)}{\partial r_j^i} + c_{n,\gamma}\beta[(\bar{u}_i^0+\phi)^{\beta-1}-(\bar{u}_i^0)^{\beta-1}]\frac{\partial\bar{u}_i^0}{\partial r_j^i} + \sum_l c_{j,l}^i\frac{\partial}{\partial r_j^i}[(w_j^i)^{\beta-1}Z_{j,l}^i]. \tag{5.5.3}$$

We now introduce two new norms:

$$\begin{aligned}\|\phi\|_{*_\sigma} &= \big\|\,|x-p_i|^{\frac{n-2\gamma}{2}}e^{\sigma|t^i-t_j^i|}\phi\big\|_{\mathcal{C}^{2\gamma+\alpha}(B(p_i,1))} + \sum_{i'\neq i}\big\|\,|x-p_{i'}|^{\frac{n-2\gamma}{2}}\phi\|_{\mathcal{C}^{2\gamma+\alpha}(B(p_{i'},1))} \\ &\quad + \big\|\,|x|^{n-2\gamma}\phi\,\big\|_{\mathcal{C}^{2\gamma+\alpha}(\mathbb{R}^n\setminus\cup_{i'}B(p_{i'},1))}\end{aligned}$$

and

$$\begin{aligned}\|\phi\|_{**_\sigma} &= \big\|\,|x-p_i|^{\frac{n+2\gamma}{2}}e^{\sigma|t^i-t_j^i|}\phi\big\|_{\mathcal{C}^{2\gamma+\alpha}(B(p_i,1))} + \sum_{i'\neq i}\big\|\,|x-p_{i'}|^{\frac{n+2\gamma}{2}}\phi\big\|_{\mathcal{C}^{2\gamma+\alpha}(B(p_{i'},1))} \\ &\quad + \big\|\,|x|^{n+2\gamma}\phi\big\|_{\mathcal{C}^{2\gamma+\alpha}(\mathbb{R}^n\setminus\cup_{i'}B(p_{i'},1))},\end{aligned}$$

where $t^i=-\log|x-p_i|$ and $\sigma>0$ is a small positive constant to be determined later.

Similarly to the proof of Lemma 5.3.4, if we work in the above weighted norm spaces, one can check that given $\|\tilde{h}\|_{**_\sigma}<+\infty$, the following problem is solvable:

$$\left\{ \begin{array}{l} \tilde{\mathbb{L}}v = \tilde{h} + \sum_{j,l,i} c_{j,l}^i(w_j^i)^{\beta-1}Z_{j,l}^i, \\ \int_{\mathbb{R}^n} v(w_j^i)^{\beta-1}Z_{j,l}^i\,dx = 0, \quad j=0,1,\ldots,\ l=0,\ldots,n, \end{array} \right.$$

and the solution v satisfies $\|v\|_{*_\sigma}\leq C\|\tilde{h}\|_{**_\sigma}$ where C only depends on σ. We would like to apply this estimate to (5.5.1), but we do not have the orthogonality condition on $\frac{\partial\phi}{\partial r_j^i}$. This can be recovered by adding some corrections.

For this, the L^2-product of $\frac{\partial\phi}{\partial r_j^i}$ and $(w_j^i)^{\beta-1}Z_{j,l}^i$ can be estimated as follows: differentiating the second equation in Proposition 5.3.3 with respect to r_j^i, we obtain by the estimate satisfied by ϕ given in (5.3.16) that

$$\Big|\int_{\mathbb{R}^n}\frac{\partial\phi}{\partial r_j^i}(w_{j'}^i)^{\beta-1}Z_{j',l}^i\,dx\Big| = \Big|-\int_{\mathbb{R}^n}\phi\frac{\partial}{\partial r_j^i}[(w_{j'}^i)^{\beta-1}Z_{j',l}^i]\,dx\Big| \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}\delta_{jj'}e^{-\sigma t_j^i}, \tag{5.5.4}$$

for some $\sigma > 0$ independent of L large.

Since for $i' \neq i$, the orthogonality is satisfied, we set

$$\hat{\phi} = \frac{\partial\phi}{\partial r_j^i} + \sum_{j,l}\alpha_{j,l}^i Z_{j,l}^i, \tag{5.5.5}$$

for some $\alpha_{j,l}^i \in \mathbb{R}$. We would like to choose the numbers $\alpha_{j,l}^i$ so that the new function $\hat{\phi}$ will satisfy the orthogonality condition. In order to have this, we need

$$\int_{\mathbb{R}^n}\frac{\partial\phi}{\partial r_j^i}(w_{j'}^{i'})^{\beta-1}Z_{j',l'}^{i'}\,dx + \sum_{j,l}\int_{\mathbb{R}^n}\alpha_{j,l}^i Z_{j,l}^i(w_{j'}^{i'})^{\beta-1}Z_{j',l'}^{i'}\,dx = 0.$$

From estimate (5.5.4), one has

$$|\alpha_{j,l}^i| \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\sigma t_j^i}. \tag{5.5.6}$$

Then $\hat{\phi}$ will satisfy the following equation

$$\begin{cases} \tilde{\mathbb{L}}(\hat{\phi}) = \hat{h} + \sum_{j',l',i'}\frac{\partial c_{j',l'}^{i'}}{\partial r_j^i}(w_{j'}^{i'})^{\beta-1}Z_{j',l'}^{i'}, \\ \int_{\mathbb{R}^n}\hat{\phi}(w_{j'}^{i'})^{\beta-1}Z_{j',l'}^{i'}\,dx = 0, \end{cases}$$

where

$$\hat{h} = \tilde{h} + c_{n,\gamma}\beta\sum_{j,l,i}\alpha_{j,l}^i Z_{j,l}^i[(\bar{u}_i^0)^{\beta-1} - (\bar{u}_i^0+\phi)^{\beta-1}].$$

In conclusion, to estimate $\hat{\phi}$ and hence $\frac{\partial\phi}{\partial r_j^i}$, it suffices to estimate $\hat{h}$. So we now bound $\hat{h}$ term by term.

Concerning $\frac{\partial S(\bar{u}_i^0)}{\partial r_j^i}$, we have from the arguments in Section 5.3 that

$$\Big\|\frac{\partial S(\bar{u}_i^0)}{\partial r_j^i}\Big\|_{**_\sigma} \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}. \tag{5.5.7}$$

From the estimates satisfied by ϕ, the same estimate holds for the second term in (5.5.3) if $\sigma < \gamma_1 + \frac{n-2\gamma}{2}$. For the third term, it contains the symbol $\delta_{jj'}$, so the estimate follows by the bounds for $c_{j,l}^i$ in the proof of Lemma 5.3.4. Moreover, from (5.5.6), one can get the same estimate for the fourth term. In conclusion, one has

$$\|\hat{h}\|_{**_\sigma} \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}.$$

Hence by the above reasoning, $\|\hat{\phi}\|_{*_\sigma} \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}$. Formulas (5.5.5) and (5.5.6) yield

$$\Big\|\frac{\partial \phi}{\partial r_j^i}\Big\|_{*_\sigma} \leq Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}.$$

Finally, by the definition of $\|\cdot\|_{*_\sigma}$ norm, we obtain the first assertion in (5.5.2).

Similarly, differentiating the first equation in Proposition 5.3.3 with respect to a_j^i and arguing as above, always keeping in mind that $\frac{\partial w_j^i}{\partial a_j^i} \sim \frac{1}{\lambda_j^i}|x-p_i|^{-\frac{n-2\gamma}{2}}(v_j^i)^{\frac{2}{n-2\gamma}+1}$, one can get that

$$\Big\|\frac{\partial \phi}{\partial a_j^i}\Big\|_{*_\sigma} \leq \frac{C}{\lambda_j^i}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)},$$

which yields the second estimate in (5.5.2). □

We now describe the asymptotic profile of the function $\frac{\partial \phi}{\partial \xi_{j,l}^i}$. First of all, we consider the ideal case when there is only one point singularity at $p=0$ and $u=u_L$, i.e., the exact Delaunay solution from (5.3.1). By definition, u_L is a solution of (5.3.15) with $\phi=0$ and vanishing right hand side. For $j\geq 1$, we assume that we are varying w_j by a_j, r_j, and denote the corresponding approximate solution by $\bar{u}_L$. We are still able to perform the reduction in Proposition 5.3.3 to find a solution of the form $\bar{u}_L+\bar{\phi}$ of the following equation

$$\begin{cases} (-\Delta)^\gamma(\bar{u}_L+\bar{\phi})-c_{n,\gamma}(\bar{u}_L+\bar{\phi})^\beta = \sum_{j,l} c_{j,l}w_j^{\beta-1}Z_{j,l}, \\ \int_{\mathbb{R}^n}\bar{\phi}w_j^{\beta-1}Z_{j,l}\,dx=0. \end{cases} \tag{5.5.8}$$

Note that an estimate similar to that of Lemma 5.5.1 will hold true for the corresponding $\bar{\phi}$. But we also need to control the derivative of $c_{j,l}$ with respect to the perturbations. In order to do so, we first introduce some notation. Define

$$\beta_{j,l} := \int_{\mathbb{R}^n}[(-\Delta)^\gamma(\bar{u}_L+\bar{\phi})-c_{n,\gamma}(\bar{u}_L+\bar{\phi})^\beta]Z_{j,l}\,dx.$$

We are interested in the derivatives of $\beta_{j',l'}$ with respect to $\{\xi_{j,l}\}=\{r_j, a_{j,1},\cdots,a_{j,n}\}$ for $l=0,\cdots,n$.

Lemma 5.5.2. *For L large, the following estimates hold:*

$$\begin{aligned}
&\frac{\partial \beta_{j,0}}{\partial r_j} = -2c_{n,\gamma}F'(L)+O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}),\\
&\frac{\partial \beta_{j-1,0}}{\partial r_j} = c_{n,\gamma}F'(L)+O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}),\\
&\frac{\partial \beta_{j+1,0}}{\partial r_j} = c_{n,\gamma}F'(L)+O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}),\\
&\frac{\partial \beta_{J,0}}{\partial r_j} = O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\sigma|t_J-t_j|}),\ \textit{for } |J-j|\geq 2.
\end{aligned}$$

And for $l = 1, \dots, n$,

$$\frac{\partial \beta_{j,l}}{\partial a_{j,l}} = c_{n,\gamma} \lambda_j \sum_{j' \neq j} \min\big\{ \frac{\lambda_{j'}}{\lambda_j}, \frac{\lambda_j}{\lambda_{j'}} \big\}^{\frac{n-2\gamma}{2}} \frac{1}{\max\{\lambda_{j'}^2, \lambda_j^2\}} + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}),$$

$$\frac{\partial \beta_{J,l}}{\partial a_{j,l}} = -c_{n,\gamma} \lambda_J \Big[\min\big\{ \frac{\lambda_J}{\lambda_j}, \frac{\lambda_j}{\lambda_J} \big\}^{\frac{n-2\gamma}{2}} \frac{1}{\max\{\lambda_J^2, \lambda_j^2\}} + O\big(\frac{1}{\lambda_i} e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma|t_J - t_j|}\big) \Big], \ \textit{if } j \neq J,$$

In addition,

$$\frac{\partial \beta_{J,l}}{\partial \xi_{j,l'}} = 0, \ \textit{if } l \neq l'.$$

Here the derivatives are evaluated at $a_j, r_j = 0$.

Proof. Differentiating the expression for $\beta_{J,l}$ with respect to r_j, and recalling equation (5.5.8) one has

$$\frac{\partial \beta_{J,l}}{\partial r_j} = \int_{\mathbb{R}^n} \Big[\bar{\mathbb{L}}\big(\frac{\partial \bar{\phi}}{\partial r_j}\big) + \frac{\partial S(\bar{u}_L)}{\partial r_j} \Big] Z_{J,l} \, dx + \sum_{j'',l'} c_{j'',l'} \int_{\mathbb{R}^n} w_{j''}^{\beta-1} Z_{j'',l'} \frac{\partial Z_{J,l}}{\partial r_j} \, dx, \tag{5.5.9}$$

where we have defined

$$\bar{\mathbb{L}} = (-\Delta)^\gamma - c_{n,\gamma} \beta u_L^{\beta-1},$$

since $\bar{\phi} = 0$ when $\bar{u}_L = u_L$. We write

$$\begin{aligned}
\int_{\mathbb{R}^n} \bar{\mathbb{L}}\big(\frac{\partial \bar{\phi}}{\partial r_j}\big) Z_{J,l} \, dx = & \int_{\mathbb{R}^n} \mathbb{L}_J(Z_{J,l}) \frac{\partial \bar{\phi}}{\partial r_j} \, dx + \int_{\mathbb{R}^n} (\bar{\mathbb{L}} - \mathbb{L}_J)(Z_{J,l}) \frac{\partial \bar{\phi}}{\partial r_j} \, dx \\
& + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma|t_J - t_j|}) \\
= & c_{n,\gamma} \beta \int_{\mathbb{R}^n} [w_J^{\beta-1} - u_L^{\beta-1}] \frac{\partial \bar{\phi}}{\partial r_j} Z_{J,l} \, dx + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma|t_J - t_j|}) \\
= & O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma|t_J - t_j|}),
\end{aligned}$$

where $\mathbb{L}_J = (-\Delta)^\gamma - c_{n,\gamma} \beta w_J^{\beta-1}$.

Since u_L is the exact solution, the corresponding $c_{j'',l} = 0$. Thus for the last term in (5.5.9) we have

$$\sum_{j'',l} c_{j'',l} \int_{\mathbb{R}^n} w_{j''}^{\beta-1} Z_{j'',l'} \frac{\partial Z_{J,l}}{\partial r_j} \, dx = 0.$$

In conclusion, one has

$$\frac{\partial \beta_{J,l}}{\partial r_j} = \frac{\partial}{\partial r_j} \int_{\mathbb{R}^n} S(\bar{u}_L) Z_{J,l} \, dx + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma|t_J - t_j|}),$$

where we have used the fact that $S(\bar{u}_L) = 0$ for $\bar{u}_L = u_L$.

Similarly, recalling the definition of $Z_{J,l} = \lambda_j \frac{\partial w_J}{\partial a_{J,l}}$, and the estimates for $\frac{\partial \bar\phi}{\partial a_j}$ from the previous paragraphs,

$$\frac{\partial \beta_{J,l}}{\partial a_j} = \frac{\partial}{\partial a_j} \int_{\mathbb{R}^n} S(\bar u_L) Z_{J,l}\, dx + O\big(\frac{\lambda_J}{\lambda_j} e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma|t_J - t_j|}\big).$$

Both variations above can be calculated from Lemma 5.4.3, with the obvious modifications as we just have one singular point so there is no summation in i. Thus one has

$$\frac{\partial \beta_{J,0}}{\partial r_j} = c_{n,\gamma} \sum_{j' \neq j'} \frac{\partial}{\partial r_j} \Big[F(|\log \frac{\lambda_{j'}}{\lambda_J}|) \frac{\log \frac{\lambda_{j'}}{\lambda_J}}{|\log \frac{\lambda_{j'}}{\lambda_J}|} \Big] + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma|t_J - t_j|}).$$

The first four conclusions in Lemma 5.5.2 follow by taking different values of J and from the definition of λ_j.

Very similarly, for $l = 1, \ldots, n$, applying Lemma 5.4.3 we obtain

$$\begin{aligned}\frac{\partial \beta_{J,l}}{\partial a_{j,l}} &= O\big(\frac{\lambda_J}{\lambda_j} e^{-\frac{(n-2\gamma)L}{2}(1+\xi)} e^{-\sigma|t_J - t_j|}\big) \\ &+ \begin{cases} c_{n,\gamma} \lambda_j \dfrac{\partial}{\partial a_{j,l}} \displaystyle\sum_{j' \neq j} \Big[\min\big\{ \frac{\lambda_{j'}}{\lambda_j}, \frac{\lambda_j}{\lambda_{j'}} \big\}^{\frac{n-2\gamma}{2}} \frac{a_{j,l}}{\max\{\lambda_{j'}^2, \lambda_j^2\}} \Big] & \text{if } J = j, \\ -c_{n,\gamma} \lambda_J \dfrac{\partial}{\partial a_{j,l}} \Big[\min\big\{ \frac{\lambda_J}{\lambda_j}, \frac{\lambda_j}{\lambda_J} \big\}^{\frac{n-2\gamma}{2}} \frac{a_{j,l}}{\max\{\lambda_J^2, \lambda_j^2\}} \Big] & \text{if } J \neq j. \end{cases}\end{aligned}$$

In addition, by the symmetry of the problem, we have $\frac{\partial \beta_{J,l'}}{\partial \xi_{j,l}} = 0$ if $l \neq l'$. This completes the proof of the Lemma.

□

The reason we have studied the special configuration u_L is that we will identify the quantities $\frac{\partial \beta^i_{j,l}}{\partial \xi^i_{j,l}}$ as the limits of the derivatives of $\beta_{j,l}$ with respect to $\xi_{j,l}$ as $j \to \infty$. We fix a point $p = p_i$ and a Delaunay parameter $L = L_i$. We denote $\bar u_{L_i}$, $\bar\phi_i$ the pair that gives the solution to (5.5.8). Before we state the result, we first need to compare the functions $\frac{\partial \phi}{\partial \xi^i_{j,l}}$ and $\frac{\partial \bar\phi_i}{\partial \xi_{j,l}}$ for i fixed, as in the lemma below:

Lemma 5.5.3. *Take $\bar u_i^0$ and ϕ as in Lemma 5.5.1. For i fixed and $j \geq 1$ we have the following estimate:*

$$\Big\| \frac{\partial \phi}{\partial \xi^i_{j,l}} - \frac{\partial \bar\phi_i}{\partial \xi^i_{j,l}} \Big\|_{*\sigma} \leq \begin{cases} C e^{-\frac{(n-2\gamma)L}{4}(1+\xi)} e^{-\sigma t^i_j} & \textit{for } l = 0, \\ \frac{C}{\lambda^i_j} e^{-\frac{(n-2\gamma)L}{4}(1+\xi)} e^{-\sigma t^i_j} & \textit{for } l \geq 1. \end{cases}$$

Proof. As before, we write down the equations satisfied by ϕ and $\bar\phi_i$,

$$(-\Delta)^\gamma (\bar u_i^0 + \phi) - c_{n,\gamma} (\bar u_i^0 + \phi)^\beta = \sum_{i',j,l} c^{i'}_{j,l} (w^{i'}_j)^{\beta-1} Z^{i'}_{j,l}$$

and

$$(-\Delta)^{\gamma}(\bar{u}_{L_i}+\bar{\phi}_i)-c_{n,\gamma}(\bar{u}_{L_i}+\bar{\phi}_i)^{\beta}=\sum_{j,l}c_{j,l}(w_j^i)^{\beta-1}Z_{j,l}^i.$$

We will differentiate both expressions with respect to $\xi^i_{j,l}$; one has from the first equation that

$$\begin{aligned}(-\Delta)^{\gamma}\Big(\frac{\partial \bar{u}_i^0}{\partial \xi^i_{j,l}}+\frac{\partial \phi}{\partial \xi^i_{j,l}}\Big)-c_{n,\gamma}\beta(\bar{u}_i^0+\phi)^{\beta-1}\Big(\frac{\partial \bar{u}_i^0}{\partial \xi^i_{j,l}}+\frac{\partial \phi}{\partial \xi^i_{j,l}}\Big)\\ =\sum_{i'}\sum_{j=0}^{\infty}c_{j,l}^{i'}\frac{\partial}{\partial \xi^i_{j,l}}[(w_j^{i'})^{\beta-1}Z_{j,l}^{i'}]+\sum_{i'}\sum_{j'=0}^{\infty}\frac{\partial c^{i'}_{j',l}}{\partial \xi^i_{j,l}}(w_{j'}^{i'})^{\beta-1}Z^{i'}_{j',l}\end{aligned}$$

Here, by the definition of the approximate solution $\bar{u}_i^0$, one knows that $\frac{\partial \bar{u}_i^0}{\partial \xi^i_{j,l}}=\frac{\partial w_j^i}{\partial \xi^i_{j,l}}$.

Next, differentiating the second equation,

$$\begin{aligned}(-\Delta)^{\gamma}\Big(\frac{\partial \bar{u}_{L_i}}{\partial \xi_{j,l}}+\frac{\partial \bar{\phi}_i}{\partial \xi_{j,l}}\Big)-c_{n,\gamma}\beta(\bar{u}_{L_i}+\bar{\phi}_i)^{\beta-1}\Big(\frac{\partial \bar{u}_{L_i}}{\partial \xi_{j,l}}+\frac{\partial \bar{\phi}_i}{\partial \xi_{j,l}}\Big)\\ =\sum_{j'=-\infty}^{\infty}\frac{\partial c_{j',l'}}{\partial \xi_{j,l}}(w^i_{j'})^{\beta-1}Z_{j,l}+\sum_{j=-\infty}^{\infty}c_{j,l}\frac{\partial}{\partial \xi_{j,l}}[w_j^{\beta-1}Z_{j,l}].\end{aligned}$$

To simplify this expression, recall that when $\bar{u}_{L_i}=u_{L_i}$ is a exact solution, one has $c_{j,l}=0,\ \bar{\phi}_i=0$. So when evaluating at $\xi_{j,l}=0$, this equation becomes

$$(-\Delta)^{\gamma}\frac{\partial \bar{\phi}_i}{\partial \xi_{j,l}}-c_{n,\gamma}u_{L_i}^{\beta-1}\frac{\partial \bar{\phi}_i}{\partial \xi_{j,l}}+(-\Delta)^{\gamma}\frac{\partial w_j}{\partial \xi_{j,l}}-c_{n,\gamma}\beta u_{L_i}^{\beta-1}\frac{\partial w_j}{\partial \xi_{j,l}}=\sum_{j'=-\infty}^{\infty}\frac{\partial c_{j',l'}}{\partial \xi_{j,l}}(w^i_{j'})^{\beta-1}Z_{j,l}$$

Denote $\mathbb{L}_{\bar{u}_i^0}=(-\Delta)^{\gamma}-c_{n,\gamma}\beta(\bar{u}_i^0)^{\beta-1}$. Taking the difference of the above two expressions we obtain an equation for $\frac{\partial \phi}{\partial \xi^i_{j,l}}-\frac{\partial \bar{\phi}}{\partial \xi_{j,l}}$:

$$
\begin{aligned}
&\mathbb{L}_{\bar{u}_i^0}\Big(\frac{\partial \phi}{\partial \xi_{j,l}^i}-\frac{\partial \bar{\phi}_i}{\partial \xi_{j,l}}\Big)\\
&=c_{n,\gamma}\beta\Big[\big((\bar{u}_i^0+\phi)^{\beta-1}-u_{L_i}^{\beta-1}\big)\frac{\partial w_j^i}{\partial \xi_{j,l}^i}+\big((\bar{u}_i^0)^{\beta-1}-u_{L_i}^{\beta-1}\big)\frac{\partial \bar{\phi}_i}{\partial \xi_{j,l}}\Big]\\
&\quad+c_{n,\gamma}\beta[(\bar{u}_i^0+\phi)^{\beta-1}-(\bar{u}_i^0)^{\beta-1}]\frac{\partial \phi}{\partial \xi_{j,l}^i}\\
&\quad+\sum_{i'}\sum_{j=0}^{\infty}c_{j,l}^{i'}\frac{\partial}{\partial \xi_{j,l}^i}[(w_j^{i'})^{\beta-1}Z_{j,l}^{i'}]+\sum_{i'}\sum_{j'=0}^{\infty}\frac{\partial c_{j',l}^{i'}}{\partial \xi_{j,l}^i}(w_{j'}^{i'})^{\beta-1}Z_{j',l}^{i'}-\sum_{j'=-\infty}^{\infty}\frac{\partial c_{j',l'}}{\partial \xi_{j,l}}(w_{j'}^i)^{\beta-1}Z_{j,l}\\
&=c_{n,\gamma}\beta\Big[((\bar{u}_i^0+\phi)^{\beta-1}-u_{L_i}^{\beta-1})\frac{\partial w_j^i}{\partial \xi_{j,l}^i}+((\bar{u}_i^0)^{\beta-1}-u_{L_i}^{\beta-1})\frac{\partial \bar{\phi}_i}{\partial \xi_{j,l}}\Big]\\
&\quad+c_{n,\gamma}\beta[(\bar{u}_i^0+\phi)^{\beta-1}-(\bar{u}_i^0)^{\beta-1}]\frac{\partial \phi}{\partial \xi_{j,l}^i}\\
&\quad+\sum_{j=0}^{\infty}c_{j,l}^{i}\frac{\partial}{\partial \xi_{j,l}^i}[(w_j^{i})^{\beta-1}Z_{j,l}^{i}]+\sum_{j'<0}\frac{\partial c_{j',l'}}{\partial \xi_{j,l}}(w_{j'}^i)^{\beta-1}Z_{j',l'}\\
&\quad+\sum_{j'=0}^{\infty}\big(\frac{\partial c_{j',l'}^{i}}{\partial \xi_{j,l}^i}-\frac{\partial c_{j',l'}}{\partial \xi_{j,l}}\big)(w_{j'}^i)^{\beta-1}Z_{j',l'}^i+\sum_{i'\neq i}\frac{\partial c_{j',l'}^{i'}}{\partial \xi_{j,l}^i}(w_{j'}^{i'})^{\beta-1}Z_{j',l'}^{i'}.
\end{aligned}
$$

Neglecting the terms in the last line, taking into account the estimates in Section 5.3, the estimates for $\frac{\partial \phi}{\partial r_j^i}$ in Lemma 5.5.1 and the estimates in Lemma 5.5.2, one can find that the right hand side of the above equation can be bounded by $e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\sigma t_j^i}$ in $\|\cdot\|_{**_\sigma}$ norm.

We first consider the case $l=0$, i.e. $\xi_{j,l}^i=r_j^i$. In this case, to have control on $\frac{\partial \phi}{\partial r_j^i}-\frac{\partial \bar{\phi}_i}{\partial r_j^i}$, we can reason similarly as in Lemma 5.5.1. More precisely, we first set

$$
\hat{\phi}=\frac{\partial \phi}{\partial r_j^i}-\frac{\partial \bar{\phi}_i}{\partial r_j^i}+\sum_{i',j',l'}\alpha_{j',l'}^{i'}Z_{j',l'}^{i'}.
$$

In order to get the orthogonality condition $\int_{\mathbb{R}^n}\hat{\phi}\,(w_J^I)^{\beta-1}Z_{J,L}^I\,dx=0$ for every I,J,L. we need

$$
\int_{\mathbb{R}^n}\Big[\frac{\partial \phi}{\partial r_j^i}-\frac{\partial \bar{\phi}_i}{\partial r_j^i}\Big](w_J^I)^{\beta-1}Z_{J,L}^I\,dx=\sum_{i',j',l'}\alpha_{j',l'}^{i'}\int_{\mathbb{R}^n}(w_J^I)^{\beta-1}Z_{j',l'}^{i'}Z_{J,L}^I\,dx. \tag{5.5.10}
$$

Differentiating the orthogonality condition of ϕ and $\bar{\phi}_i$ w.r.t r_j^i, in analogy with (5.5.4), we get

$$\Big|\int_{\mathbb{R}^n}\Big[\frac{\partial\phi}{\partial r^i_j}-\frac{\partial\bar{\phi}_i}{\partial r^i_j}\Big](w^I_J)^{\beta-1}Z^I_{J,L}\,dx\Big| = \sum_{I,J,L}\Big|\int_{\mathbb{R}^n}(\phi-\bar{\phi}_i)\frac{\partial}{\partial r^i_j}[(w^I_J)^{\beta-1}Z^I_{J,L}]\,dx\Big|$$
$$\le \begin{cases} Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\sigma t^i_j} & \text{if } j=J,\\ 0 \ \text{ if } j\neq J,\end{cases}$$

where we have used the fact that $\bar{\phi}_i = 0$ for u_{L_i}. Therefore, from (5.5.10) we have the following estimates:

$$|\alpha^{i'}_{j',l'}| \le \begin{cases} Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\sigma t^i_j} & \text{if } i'=i,\ j'=j,\\ Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\sigma|t^i_j-t^i_{j'}|} & \text{if } i=i',\ j'\neq j,\\ 0 \text{ if } i\neq i'.\end{cases} \tag{5.5.11}$$

Moreover, $\hat{\phi}$ solves

$$\begin{aligned}\mathbb{L}_{\bar{u}^0_i}(\hat{\phi}) &= \mathbb{L}_{\bar{u}^0_i}\big(\frac{\partial\phi}{\partial r^i_j}-\frac{\partial\bar{\phi}_i}{\partial r^i_j}\big)+\mathbb{L}_{\bar{u}^0_i}\big(\sum_{j',l'}\alpha^i_{j',l'}Z^i_{j',l'}\big)\\ &= \mathbb{L}_{\bar{u}^0_i}\big(\frac{\partial\phi}{\partial r^i_j}-\frac{\partial\bar{\phi}_i}{\partial r^i_j}\big)-c_{n,\gamma}\beta\sum_{j',l'}\alpha^i_{j',l'}((\bar{u}^0_i)^{\beta-1}-(w^i_{j'})^{\beta-1})Z^i_{j',l'}.\end{aligned}$$

By estimate (5.5.11), we know that the second term is bounded by $e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\sigma t^i_j}$ in $\|\cdot\|_{**_\sigma}$ norm. So the right hand side of the above equation can be controlled by $e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\sigma t^i_j}$ in $\|\cdot\|_{**_\sigma}$ norm, and thus, applying Proposition 5.3.3 to $\hat{\phi}$,

$$\|\hat{\phi}\|_{*_\sigma} \le Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\sigma t^i_j}.$$

Looking back at (5.5.5), from the estimates for $\alpha^i_{j,l}$ we have

$$\Big\|\frac{\partial\phi}{\partial r^i_j}-\frac{\partial\bar{\phi}_i}{\partial r^i_j}\Big\|_{*_\sigma} \le Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\sigma t^i_j}.$$

A similar argument yields

$$\Big\|\frac{\partial\phi}{\partial a^i_j}-\frac{\partial\bar{\phi}_i}{\partial a^i_j}\Big\|_{*_\sigma} \le \frac{C}{\lambda^i_j}e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\sigma t^i_j}.$$

The proof of the Lemma is completed. □

From the previous lemma we can obtain estimates on derivatives of $\beta^i_{j,l}$ with respect to $\xi^i_{j,l}$.

Lemma 5.5.4. *In the previous setting, we have the following estimates*

$$\Big|\frac{\partial\beta^i_{j',l'}}{\partial\xi^i_{j,l}}-\frac{\partial\beta_{j',l'}}{\partial\xi_{j,l}}\Big|\le\begin{cases}Ce^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\sigma t^i_j}e^{-\sigma|t^i_j-t^i_{j'}|} & \text{if } l=0,\\ \frac{C}{\lambda^i_j}e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\sigma t^i_j}e^{-\sigma|t^i_j-t^i_{j'}|} & \text{if } l\ge 1,\end{cases}\quad \text{for } j\ge 1.$$

Proof. Recall that by the definition of $\beta^i_{j,l}$ and $\beta_{j,l}$,

$$\begin{aligned}\beta^i_{j,l}&=\int_{\mathbb{R}^n}[(-\Delta)^\gamma(\bar u^0_i+\phi)-c_{n,\gamma}(\bar u^0_i+\phi)^\beta]Z^i_{j,l}\,dx,\\ \beta_{j,l}&=\int_{\mathbb{R}^n}[(-\Delta)^\gamma(\bar u_{L_i}+\bar\phi_i)-c_{n,\gamma}(\bar u_{L_i}+\bar\phi_i)^\beta]Z^i_{j,l}\,dx.\end{aligned}$$

Differentiating the above equations w.r.t $\xi^i_{j,l}$ and taking the difference, one has

$$\begin{aligned}&\frac{\partial}{\partial\xi^i_{j,l}}(\beta^i_{j',l'}-\beta_{j',l'})\\ &\quad=\int_{\mathbb{R}^n}\mathbb{L}_{\bar u^0_i}\Big(\frac{\partial\phi}{\partial\xi^i_{j,l}}-\frac{\partial\bar\phi_i}{\partial\xi^i_{j,l}}\Big)Z^i_{j',l'}\,dx-c_{n,\gamma}\beta\int_{\mathbb{R}^n}((\bar u^0_i)^{\beta-1}-u^{\beta-1}_{L_i})\frac{\partial\bar\phi_i}{\partial\xi^i_{j,l}}Z^i_{j',l'}\,dx\\ &\quad-c_{n,\gamma}\beta\int_{\mathbb{R}^n}(u^{\beta-1}_{L_i}-(\bar u^0_i+\phi)^{\beta-1})\frac{\partial w^i_j}{\partial\xi^i_{j,l}}Z^i_{j',l'}\,dx\\ &\quad-c_{n,\gamma}\beta\int_{\mathbb{R}^n}[(\bar u^0_i+\phi)^{\beta-1}-(\bar u^0_i)^{\beta-1}]\frac{\partial\phi}{\partial\xi^i_{j,l}}Z^i_{j',l'}\,dx\\ &\quad+c^i_{j,l}\int_{\mathbb{R}^n}(w^i_j)^{\beta-1}Z^i_{j,l}\frac{\partial}{\partial\xi^i_{j,l}}Z^i_{j',l'}\,dx.\end{aligned}$$

By oddness, one can first get that the term in the last line vanishes. Moreover, by the estimates in Lemma 5.5.1 and 5.5.3, one can get that the first two lines can be controlled by $e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\sigma t^i_j}e^{-\sigma|t^i_j-t^i_{j'}|}$ when $l=0$ and $\frac{1}{\lambda^i_j}e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\sigma t^i_j}e^{-\sigma|t^i_j-t^i_{j'}|}$ when $l\ge 1$. Thus one has

$$\Big|\frac{\partial}{\partial\xi^i_{j,l}}(\beta^i_{j',l'}-\beta_{j',l'})\Big|\le\begin{cases}Ce^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\sigma t^i_j}e^{-\sigma|t^i_j-t^i_{j'}|} & \text{if } l=0,\\ \frac{C}{\lambda^i_j}e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\sigma t^i_j}e^{-\sigma|t^i_j-t^i_{j'}|} & \text{if } l\ge 1,\end{cases}\quad \text{for } j\ge 1.$$

The proof is completed. □

5.5.2 Derivatives of the numbers $\beta^i_{j,l}$ for the general parameters a^i_j, r^i_j

In this subsection, we consider the derivatives of the $\beta^i_{j,l}$'s for the general parameters a^i_j, r^i_j satisfying (5.3.7)-(5.3.10). We write the counterpart of Lemmas 5.5.3 and 5.5.4, but we do not prove them since the methods are quite similar. In the following, we assume that $\tau<\sigma$.

Lemma 5.5.5. *Suppose a_j^i, r_j^i satisfy (5.3.7)-(5.3.10). For L large, let $\bar{u}$ and ϕ be as in Proposition 5.3.3, then we have the following estimates for $\frac{\partial \phi}{\partial \xi_{j,l}^i} - \frac{\partial \bar{\phi}_i}{\partial \xi_{j,l}}$:*

$$\Big\|\frac{\partial \phi}{\partial \xi_{j,l}^i} - \frac{\partial \bar{\phi}_i}{\partial \xi_{j,l}}\Big\|_{*_\sigma} \leq C \begin{cases} Ce^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\tau t_j^i} & \text{if } l = 0, \\ \frac{C}{\lambda_j^i} e^{-\frac{(n-2\gamma)L}{4}(1+\xi)}e^{-\tau t_j^i} & \text{if } l \geq 1, \end{cases}$$

for $j \geq 1$ if we choose $\tau < \sigma$.

Lemma 5.5.6. *Suppose a_j^i, r_j^i satisfy (5.3.7) - (5.3.10). For L large, let $\bar{u}$ and ϕ be as in Proposition 5.3.3, then we have the following estimates*

$$\Big|\frac{\partial}{\partial \xi_{j,l}^i}(\beta_{j',l'}^i - \beta_{j',l'})\Big| \leq \begin{cases} Ce^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\tau t_j^i}e^{-\sigma|t_j^i - t_{j'}^i|} & \text{if } l = 0, \\ \frac{C}{\lambda_j^i} e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}e^{-\tau t_j^i}e^{-\sigma|t_j^i - t_{j'}^i|} & \text{if } l \geq 1, \end{cases}$$

for $j \geq 1$ where $\xi > 0$ is a positive constant independent of L large.

5.6 Proof of the main theorem

In this section we prove our main results. We keep the notation and assumptions in the previous sections. Before we start, we define some notation:

$$\tilde{\mathbf{a}}^i = (\tilde{a}_0^i, \cdots, \tilde{a}_j^i, \cdots)^t, \quad \mathbf{r}^i = (r_0^i, r_1^i, \cdots, r_j^i, \cdots)^t,$$

$$T_{\tilde{a}}^i(\tilde{\mathbf{a}}^i) = T_{\tilde{a}}^i \tilde{\mathbf{a}}^i, \quad T_r^i(\mathbf{r}^i) = T_r^i \mathbf{r}^i,$$

where

$$T_{\tilde{a}}^i = \begin{pmatrix} -1 & 1+e^{-2L_i} & -e^{-2L_i} & 0 & \cdots & \cdots & \vdots \\ 0 & -1 & 1+e^{-2L_i} & -e^{-2L_i} & 0 & \cdots & \vdots \\ 0 & 0 & -1 & 1+e^{-2L_i} & -e^{-2L_i} & 0 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and

$$T_r^i = \begin{pmatrix} -1 & 2 & -1 & 0 & \cdots & \cdots & \vdots \\ 0 & -1 & 2 & -1 & 0 & \cdots & \vdots \\ 0 & 0 & -1 & 2 & -1 & 0 & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

For $\tau > 0$, let us also introduce the weighted norm and space

$$|(x_j)|_\tau = \sup_j e^{(2j+1)\tau}|x_j|$$

and

$$\ell_\tau = \{(\tilde{a}^i_j, r^i_j) : |(\tilde{a}^i_j)|_\tau + |(r^i_j)|_\tau < +\infty\}.$$

At first glance, these infinite dimensional matrices are not invertible, since they have the trivial kernel $(1, 1, \cdots)^t$, but they are indeed invertible in some suitable weighted norm, which is given in the following:

Lemma 5.6.1. *The operators $T^i_{\tilde{a}}$, T^i_r have inverse, whose norm can be bounded by $Ce^{-2\tau}$.*

Proof. Given $(f_i)_{i\geq 1}$ with $|(f_j)_j|_\tau < \infty$, our goal is to solve $T^i_{\tilde{a}}(\tilde{\mathbf{a}}^i) = (f_i)_i$. Defining

$$\tilde{a}^i_j = \sum_{l=j+1}^{\infty} \Big(\sum_{s=0}^{l-j-1} e^{-2L_i s} \Big) f_l,$$

one can easily check that the solution $\tilde{a}^i_j$ satisfies the required conditions and that the operator is an inverse of $T^i_{\tilde{a}}$ both from the left and from the right (here the index for f starts from 1, while the index for a starts from 0). Moreover, one has

$$|\tilde{a}^i_j| \leq C|f_j|_\tau \sum_{l=j+1}^{\infty} \Big(\sum_{s=0}^{l-j-1} e^{-2L_i s} \Big) e^{-(2l+1)\tau} \leq Ce^{-(2j+3)\tau}|f_j|_\tau,$$

which proves the result for $T^i_{\tilde{a}}$. The proof for the inverse for T^i_r has been given in Lemma 7.3 of [132].

The lemma is proved.

∎

Recall that in Proposition 5.3.3 one has found a solution $u = \bar{u} + \phi$ for

$$(-\Delta)^\gamma u - c_{n,\gamma} u^\beta = \sum_{i,j,l} c^i_{j,l} (w^i_j)^{\beta-1} Z^i_{j,l}.$$

The solvability of the original problem (5.1.1) is reduced to the following system of equations:

$$\beta^i_{j,l} = \int_{\mathbb{R}^n} [(-\Delta)^\gamma u - c_{n,\gamma} u^\beta] Z^i_{j,l}\, dx = 0,$$

for all $i = 1, \cdots, k$, $j = 0, \cdots, +\infty$, and $l = 0, \cdots, n$.

Using the above lemma and a perturbation argument, we can prove the following result:

Proposition 5.6.2. *Given $\{R^i, \hat{a}^i_0, q_i\}$ satisfying (5.3.7) and (5.3.8) with L sufficiently large, if we choose $\tau < \min\{\xi, \sigma\}$, there exist $(\tilde{a}^i_j)_{i,j}$ and $(r^i_j)_{i,j}$ such that (5.3.10) holds true with $\beta^i_{j,l} = 0$ for $j \geq 1$ and all $l = 0, \cdots, n$, $i = 1, \cdots, k$.*

Proof. For $l = 0$, consider

$$G_0^i = \begin{pmatrix} \vdots \\ \frac{1}{F(L_i)}[\beta_{j,0}^i - \bar{\beta}_{j,0}^i] \\ \vdots \end{pmatrix} - T_r^i(\mathbf{r}^i)$$

and for $l = 1, \cdots, n$,

$$G_l^i = \begin{pmatrix} \vdots \\ \frac{e^{\frac{n-2\gamma}{2}L_i}}{\lambda_j^i}[\beta_{j,l}^i - \bar{\beta}_{j,l}^i] \\ \vdots \end{pmatrix} - T_{\tilde{a}}^i(\tilde{\mathbf{a}}^i),$$

where $\bar{\beta}_{j,l}^i$ correspond to the numbers $\beta_{j,l}^i$ for the approximate solution $\bar{u}_i^0$, i.e., the solution when $\bar{a}_j^i$, r_j^i are all zero.

One can easily see that $\beta_{j,l}^i = 0$ for $j \geq 1$ if

$$\tilde{\mathbf{a}}^i = -T_{\tilde{a}}^{-1}\Big\{ \begin{pmatrix} \vdots \\ \frac{e^{\frac{n-2\gamma}{2}L_i}}{\lambda_j^i}\bar{\beta}_{j,l}^i \\ \vdots \end{pmatrix} + G_l^i \Big\} \tag{5.6.1}$$

and

$$\mathbf{r}^i = -T_r^{-1}\Big\{ \begin{pmatrix} \vdots \\ \frac{1}{F(L_i)}\bar{\beta}_{j,0}^i \\ \vdots \end{pmatrix} + G_0^i \Big\}. \tag{5.6.2}$$

Next we show that the terms on the right hand sides of (5.6.1)-(5.6.2) are contractions in an appropriate sense. First, by Lemma 5.4.1, one has

$$|\bar{\beta}_{j,l}^i| \leq \begin{cases} Ce^{-\frac{(n-2\gamma)L_i}{2}(1+\xi)}e^{-\sigma t_j^i} & \text{if } l = 0, \\ C\lambda_j^i e^{-\frac{(n-2\gamma)L_i}{2}(1+\xi)}e^{-\sigma t_j^i} & \text{if } l \geq 1, \end{cases} \quad \text{for } j \geq 1.$$

We write the $j-$th component of $\big[\frac{e^{\frac{n-2\gamma}{2}L_i}}{\lambda_j^i}(\beta_{j,l}^i - \bar{\beta}_{j,l}^i)\big] - T_{\tilde{a}}^i(\tilde{\mathbf{a}}^i)$ as $G_{1,l,j} + G_{2,l,j}$, where

$$G_{1,l,j} = \int_0^1 \Big[\frac{e^{\frac{n-2\gamma}{2}L_i}}{\lambda_j^i}\frac{\partial \beta_{j,l}^i(t(\bar{\mathbf{a}}^i, \mathbf{r}^i))}{\partial t} - \bar{A}^i\Big](\bar{\mathbf{a}}^i, \mathbf{r}^i)\, dt$$

and

$$G_{2,l,j} = \bar{A}^i - T_a^i(\tilde{\mathbf{a}}^i)$$

for

$$\bar{A}_j^i(\bar{\mathbf{a}}^i, \mathbf{r}^i) = \sum_{j'=0}^{\infty} \frac{e^{\frac{n-2\gamma}{2}L}}{\lambda_j^i}\frac{\partial \beta_{j,l}}{\partial \bar{a}_{j'}^i} \cdot [\bar{a}_{j'}^i],$$

where $\beta_{j,l}$ is given before Lemma 5.5.2 and $\bar{a}^i_j$ corresponds to the translation perturbation of the $j-$th bubble in the Delaunay solution, see Lemma 5.5.2. Also observe that

$$T_{\tilde{a}}(\bar{\mathbf{a}}^i) = T_{\tilde{a}}(\tilde{\mathbf{a}}^i).$$

Let us begin by estimating $G_{1,l,j}$: using Lemma 5.5.4 for $l = 1, \ldots, n$, one finds

$$\begin{aligned}
|G_{1,l,j}| &\leq \sum_{j'=0}^{\infty} \frac{e^{\frac{n-2\gamma}{2}L}}{\lambda^i_j} \Big| \frac{\partial \beta^i_{j,l}}{\partial \bar{a}^i_{j'}} - \frac{\partial \beta_{j,l}}{\partial \bar{a}^i_{j'}} \Big| \, |\bar{a}^i_{j'}| + O(e^{-\frac{(n-2\gamma)L_i}{2}\xi} e^{-\min\{\sigma,\tau\} t^i_j}) \\
&\leq C e^{-\frac{(n-2\gamma)L_i}{2}\xi} \sum_{j'} e^{-\sigma t_{j'}} e^{-\sigma |t_j - t_{j'}|} |\bar{a}^i_{j'}| + O(e^{-\frac{(n-2\gamma)L_i}{2}\xi} e^{-\min\{\sigma,\tau\} t^i_j}) \\
&\leq C (e^{-\frac{(n-2\gamma)L}{2}\xi} e^{-\min\{\sigma,\tau\} t^i_j}).
\end{aligned}$$

To estimate $G_{2,l,j}$, we apply Lemma 5.5.2 which gives

$$|G_{2,l,j}| \leq C e^{-\frac{(n-2\gamma)L_i}{2}\xi} \Big[e^{-\sigma L_i} \big(|\tilde{a}^i_{j-1}| + |\tilde{a}^i_{j+1}| \big) + \sum_{j' \neq j \pm 1} e^{-\sigma |t^i_{j'} - t^i_j|} |\tilde{a}^i_{j'}| \Big].$$

Combining the above two estimates, one has for $\tau < \sigma$,

$$\|G^i_l\|_{\frac{\tau L_i}{2}} \leq C e^{\tau L} e^{-\frac{(n-2\gamma)L_i}{2}\xi} \|\tilde{\mathbf{a}}^i\|_{\frac{\tau L_i}{2}} + O(e^{-\frac{(n-2\gamma)L_i}{2}\xi}), \quad \text{for } l = 1, \ldots, n.$$

Similarly, for $l = 0$, one can get that

$$\|G^i_0\|_{\frac{\tau L_i}{2}} \leq C e^{\tau L} e^{-\frac{(n-2\gamma)L_i}{2}\xi} \|\mathbf{r}^i\|_{\frac{\tau L_i}{2}} + O(e^{-\frac{(n-2\gamma)L_i}{2}\xi}).$$

Next, with some abuse of notation, equations (5.6.1) and (5.6.2) are equivalent to

$$\tilde{\mathbf{a}}^i = T_a^{-1} [e^{\tau L} e^{-\frac{(n-2\gamma)L_i}{2}\xi} \|\tilde{\mathbf{a}}^i\|_{\tau L_i} + O(e^{-\frac{(n-2\gamma)L_i}{2}\xi})] =: G_{\mathbf{a}}(\tilde{\mathbf{a}}^i)$$

and

$$\mathbf{r}^i = T_r^{-1} [e^{\tau L} e^{-\frac{(n-2\gamma)L_i}{2}\xi} \|\mathbf{r}^i\|_{\tau L_i} + O(e^{-\frac{(n-2\gamma)L_i}{2}\xi})] =: G_{\mathbf{r}}(\mathbf{r}^i),$$

where the terms on the right hand sides of the above two equations are estimated in $\|\cdot\|_{\frac{\tau L_i}{2}}$ norm. We now consider the set

$$\mathcal{B} = \{(\tilde{\mathbf{a}}^i_j, \mathbf{r}^i_j) \, : \, \|\tilde{\mathbf{a}}^i\|_{\frac{\tau L_i}{2}} + \|\mathbf{r}^i\|_{\frac{\tau L_i}{2}} \leq C e^{-\tau L}\}. \tag{5.6.3}$$

For $\tau < \xi$ small enough, it follows that $(G_{\mathbf{a}}, G_{\mathbf{r}})$ maps $\mathcal{B}$ into itself for L large. Furthermore, it is a contraction mapping. So by fixed point theory, there exists a fixed point in set $\mathcal{B}$. Thus we have found $\tilde{a}^i_j, r^i_j$ such that $\beta^i_{j,l} = 0$ for all $j \geq 1$, as desired.

$\square$

We are now in the position to prove our existence result.

Proof of Theorem 5.1.1. By Proposition 5.6.2, we are reduced to find $R^i, \hat{a}_0^i$ and q_i for which $\beta_{0,l}^i = 0$.

For $j = 0$, from Lemma 5.4.4, one has that equation $\beta_{0,0}^i = 0$ is reduced to

$$\beta_{0,0}^i = -c_{n,\gamma} q_i \Big[A_2 \sum_{i' \neq i} |p_{i'} - p_i|^{-(n-2\gamma)} (R_0^i R_0^{i'})^{\frac{n-2\gamma}{2}} q_{i'} - \Big(\frac{R_1^i}{R_0^i}\Big)^{\frac{n-2\gamma}{2}} q_i \Big] e^{-\frac{(n-2\gamma)L}{2}} (1 + o(1))$$
$$+ O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}) = 0.$$

Recall that by the definition of R_j^i, i,e., $R_0^i = R^i(1 + r_0^i)$, and the estimate for r_j^i (5.6.3), then the above equation can be rewritten as

$$A_2 \sum_{i' \neq i} |p_{i'} - p_i|^{-(n-2\gamma)} (R^i R^{i'})^{\frac{n-2\gamma}{2}} q_{i'} - q_i = o(1). \tag{5.6.4}$$

On the other hand, the equations $\beta_{0,l}^i = 0$ for $l = 1, \cdots, n$ are reduced to

$$\beta_{0,l}^i = c_{n,\gamma} \Big[A_3 \sum_{i' \neq i} \frac{(p_{i'} - p_i)_l}{|p_{i'} - p_i|^{n-2\gamma+2}} (R_0^i R_0^{i'})^{\frac{n-2\gamma}{2}} q_{i'}$$
$$+ A_0 \Big(\frac{R_1^i}{R_0^i}\Big)^{\frac{n-2\gamma}{2}} \frac{a_0^i - a_1^i}{(\lambda_0^i)^2} q_i \Big] q_i e^{-\frac{n-2\gamma}{2}L} + O(e^{-\frac{(n-2\gamma)L}{2}(1+\xi)}) = 0.$$

By the definition of a_j^i, i.e. $a_j^i = (\lambda_j^i)^2 \bar{a}_j^i$ and $\bar{a}_j^i = \hat{a}_0^i + \tilde{a}_j^i$, and the estimates satisfied by $\tilde{a}_j^i$ (5.6.3), the above equation can be rewritten as

$$A_3 \sum_{i' \neq i} \frac{(p_{i'} - p_i)_l}{|p_{i'} - p_i|^{n-2\gamma+2}} (R^i R^{i'})^{\frac{n-2\gamma}{2}} q_{i'} + A_0 \hat{a}_0^i q_i = o(1). \tag{5.6.5}$$

Our last step is to choose suitable $\hat{a}_0^i, R^i, q_i$ such that equations (5.6.4) and (5.6.5) are solvable. Recalling the balancing conditions (5.3.5)-(5.3.6) satisfied by $\hat{a}_0^{i,b}, R^{i,b}, q_i^b$, the solvability of (5.6.4)-(5.6.5) depends on the following invertibility property of the linearized operator of the above equations around $\hat{a}_0^{i,b}, R^{i,b}, q_i^b$:

Lemma 5.6.3. *If we denote by*

$$\mathcal{F}(R^i, q_i) = A_2 \sum_{i' \neq i} |p_{i'} - p_i|^{-(n-2\gamma)} (R^i R^{i'})^{\frac{n-2\gamma}{2}} q_{i'} - q_i,$$

then the linearized operator of $\mathcal{F}$ around $(R^{i,b}, q_i^b)$ is invertible.

Proof. From the definition of $\mathcal{F}$, one has the following expression for the linearized operator

$$\mathcal{F}_{R^i, q_i}|_{(R^{i,b}, q_i^b)} : \mathbb{R}^{2k} \to \mathbb{R}^k,$$

where

$$\mathcal{F}_{q_i} = (\mathbf{q}_{ij})$$

for

$$\mathbf{q}_{ii} = -1, \quad \mathbf{q}_{ij} = A_2 |p_i - p_j|^{-(n-2\gamma)} (R^{i,b} R^{j,b})^{\frac{n-2\gamma}{2}}, \, i \neq j,$$

and

$$\mathcal{F}_{R^i} = (\mathbf{R}_{ij})$$

for

$$\begin{aligned}
\mathbf{R}_{ii} &= \frac{n-2\gamma}{2} \frac{1}{R^{i,b}} \sum_{i' \neq i} A_2 |p_{i'} - p_i|^{-(n-2\gamma)} (R^{i,b} R^{i',b})^{\frac{n-2\gamma}{2}} q_i^b, \\
\mathbf{R}_{ij} &= \frac{n-2\gamma}{2} \frac{1}{R^{j,b}} A_2 |p_j - p_i|^{-(n-2\gamma)} (R^{i,b} R^{j,b})^{\frac{n-2\gamma}{2}} q_j^b, \, i \neq j.
\end{aligned}$$

From the balancing condition (5.3.5) we know that

$$\mathcal{F}(R^{i,b}, q_i^b) = 0.$$

One can easily see that the matrix $\mathcal{F}_{q_i}$ is symmetric and has only one-dimensional kernel, which is given by

$$\mathrm{Ker}(\mathcal{F}_{q_i}) = Span\{(q_1^b, \cdots, q_k^b)\}.$$

The balancing condition (5.3.5) also implies that

$$\mathcal{F}_{R^i} \begin{pmatrix} R^{1,b} \\ \vdots \\ R^{k,b} \end{pmatrix} = \frac{(n-2\gamma)}{2} \begin{pmatrix} q_1^b \\ \vdots \\ q_k^b \end{pmatrix}.$$

Thus we conclude that the operator $\mathcal{F}_{R^{i,b}, q_i^b}$ is surjective.

□

From Lemma 5.6.3 and the balancing condition (5.3.5), one can easily find (R^i, q_i) which solves (5.6.4) by perturbing near $(R^{i,b}, q_i^b)$. Looking at the second balancing condition (5.3.6), once (R^i, q_i) are known, one can find $\hat{a}_0^i$ around $\hat{a}_0^{i,b}$ which solves (5.6.5).

In conclusion, we have chosen $R^i, q_i, \hat{a}_0^i$ such that (5.6.4)-(5.6.5) are solved, i.e. $\beta_{0,l}^i = 0$. The last step in our argument is to use the maximum principle (Proposition 4.3.8) in the previous Chapter 4 to show that $u > 0$. This concludes the proof of the main Theorem.

□

Chapter 6

Fractional Caffarelli-Kohn-Nirenberg inequality

6.1 Background

The Caffarelli-Kohn-Nirenberg inequality was introduced by Caffarelli, Kohn and Nirenberg in 1984 (see [34]). The existence or non existence of extremal solutions and the properties of these solutions have extensively studied since them. The classical inequality was stated as follows: let $p, q, r; \alpha, \beta, \sigma; a \in \mathbb{R}$ be fixed numbers satisfying

$$p, q \geq 1, \quad r > 0, \quad 0 \leq a \leq 1, \quad \frac{1}{p} + \frac{\alpha}{n},\ \frac{1}{q} + \frac{\beta}{n},\ \frac{1}{r} + \frac{\gamma}{n} > 0; \quad \gamma := a\sigma + (1-a)\beta,$$

then, for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, there exists a constant C such that

$$\big\| |x|^\gamma u \big\|_{L^r} \leq C \big\| |x|^\alpha |\nabla u| \big\|_{L^p}^a \ \big\| |x|^\beta u \big\|_{L^q}^{1-a}, \tag{6.1.1}$$

if and only if

$$\frac{1}{r} + \frac{\gamma}{n} = a\left(\frac{1}{p} + \frac{\alpha - 1}{n}\right) + (1-a)\left(\frac{1}{q} + \frac{\beta}{n}\right)$$

and

$$0 \leq \alpha - \sigma \text{ if } a > 0, \quad \text{or} \quad \alpha - \sigma \leq 1 \text{ if } a > 0 \text{ and } \frac{1}{p} + \frac{\alpha - 1}{n} = \frac{1}{r} + \frac{\gamma}{n}.$$

In 1986, Lin clarified under which necessary and sufficient conditions the previous inequality holds (see: [126]).

If we restrict the study to the case $p = 2$, $a = 1$, the Caffarelli-Kohn-Nirenberg inequality (6.1.1) establishes that for all $\alpha \leq \beta \leq \alpha + 1$ and $\alpha \neq \frac{n-2}{2}$, in space dimension $n > 2$, it holds that

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{2^*}}{|x|^{\beta 2^*}}\, dx\right)^{2/2^*} \leq (\Lambda^n_{\alpha,\beta})^{-1} \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^{2\alpha}}\, dx,\ \forall\ u \in D_{\alpha,\beta} \tag{6.1.2}$$

where

$$2^* = \frac{2n}{n-2+2(\beta-\alpha)},$$

$$D_{\alpha,\beta} = \{|x|^{-\beta}u \in L^{2^*}(\mathbb{R}^n), |x|^{-\alpha}|\nabla u| \in L^2(\mathbb{R}^n)\}$$

and $(\Lambda^n_{\alpha,\beta})^{-1}$ denotes the optimal constant. This inequality represents an interpolation between the usual Sobolev inequality ($\alpha = 0, \beta = 0$) and the Hardy inequality ($\alpha = 0, \beta = 1$) or weighted Hardy inequality ($\beta = \alpha + 1$). A strategy to find extremal solutions (or minimizers) for inequality (6.1.2) is to look for solutions to the following Euler-Lagrange equation:

$$-\operatorname{div}(|x|^{-2\alpha}\nabla u) = \frac{u^{2^*-1}}{|x|^{2^*\beta}}. \tag{6.1.3}$$

One could also consider the more general equation

$$-\operatorname{div}(|x|^{-2\alpha}\nabla u) = \frac{\lambda}{|x|^{2(1+\alpha)}}u + \frac{u^{2^*-1}}{|x|^{2^*\beta}}, \tag{6.1.4}$$

where $-\infty < \lambda < \left(\frac{n-2\alpha-2}{2}\right)^2$.
Some particular cases were studied before the general inequality (6.1.1) was published. For example, the best constant and the minimizers for the Sobolev inequality ($\alpha = 0,\ \beta = 0$) were given by Talent in [168] and Aubin in [14]; or the particular case $\alpha = 0,\ 0 < \beta < 1$ was studied by Lieb in [124], where he found the best constant and explicit minimizers.

The inequality for the nonnegative range $0 \leq \alpha < \frac{n-2}{2}$ with $\alpha \leq \beta \leq \alpha + 1$ has been studied in different works (see [52]-[128]-[129]-[171]) and the symmetry of the minimizers in this region has been studied in depth.

The negative region for α, which is more delicate because the symmetrization methods were not applicable, was studied in [42], where Catrina and Wang provided results for the best constant, the existence (or nonexistence) of minimizers and their symmetry properties.

The symmetry of extremal functions for inequality (6.1.2) for whole range $-\infty \leq \alpha < \frac{n-2}{2}$ was considered by Lin and Wang in [127], where they proved using moving plane method that all non-radial extremal functions are axially symmetric with respect to a line passing through the origin. Some years later, Costa gave in [53] a new and short proof for inequality (6.1.2) in some particular cases; this proof was based on some definitions of weighted Sobolev spaces and their embbeding into the weighted L^2-spaces. Bouchez and Willen also gave in [24] a simpler proof for the result of Lin and Wang; their proof relies on the use of polarizations.

The fact that symmetry for minimizers can be broken was discovered by Catrina and Wang in [41]-[42]. In [87] Felli and Schneider highlighted the symmetry-breaking phenomenon when they found non-radial minimizers for a small perturbation of equation (6.1.4) This work conjectures that the symmetry region and the non-symmetry region are separated by a curve that we will call Felli-Schneider curve. (See also: [67] for numerical results or [68] for a formal expansion). This fact was proven, in many cases, in a series of papers by Dolbeault, Esteban, Filippas, Loss, del Pino, Tarantello and Tertikas. A good summary for all this work was done by Dolbeault and Esteban in [65]. Firstly, in collaboration with Esteban and Tarantello, they studied in [73] the symmetry breaking phenomena in two space dimensions in a suitable sets of parameters by a blow up argument and an analysis of the convergence to a solution of a Liouville equation; later, in collaboration also with Loss [72], they proved some new results for the extremals of inequality (6.1.2) in any dimension larger or equal than 2. In [60]-[74] they focused on the study of a logarithmic Hardy inequality. Later, following a similar analysis to the one in [60], Dolbeault and Esteban studied in [66] a more general inequality than (6.1.2): they kept the value of $p = 2$ in (6.1.1) but they let $a \in (0, 1)$. They showed the positivity of the minimizers, but they could prove that the infimum is attained only for a certain range of α.

In the particular case $\alpha = \beta$, the problem was already studied by Musso and Wei in [147]; where they provided, under different assumptions, nonradial solution to the C-K-N equation (6.1.3) in dimension $n \geq 5$.

A similar problem to equation (6.1.3) (or (6.1.4)) has also been studied for a bounded domain, where it becomes a boundary problem, as we can check, for instance, in the work by Abdelaoui, Colorado and Peral in [3]-[2] for mixed Dirichlet-Neumann boundary conditions; or the case of a general smooth bounded domain Ω, which was considered by Ghoussoub and Robert ([94]) when $0 \in \Omega$, and generalized by Chern and Lin ([50]) when $0 \in \partial\Omega$.

More general cases than $p = 2$, i.e., for a general $p \in \mathbb{R}$, have been studied from 2002 to nowadays. For example, in [27], Byen and Wang with the symmetry property of the extremal functions for the L^p version of (6.1.1) for $p > 1$. The existence of extremals for inequality (6.1.1) for a general $p \in (1, n)$ was studied by Musina in [146]. The symmetry breaking of extremals for inequality (6.1.1) for $1 < p < q < p^*$, where $p^* = \frac{np}{n-p}$ if $p < n$, and $p^* = \infty$ if $p \geq n$, was introduced in [38], where Caldiroli and Musina provided an explicit necessary condition to have that no extremal for the best constant in inequality (6.1.1) is radially symmetric. The case $p = 1$ has recently studied by Chiba and Horiuchi in [51], where they proved that the symmetry breaking of the best constant occurs under some assumptions.

Finally the conjecture that the Felli-Schneider curve (given in [87]) is the threshold between the symmetry and the symmetry breaking region for the minimizers of inequality (6.1.1) (for a general p but $a = 1$) has been resolved by Dolbeault, Esteban and Loss using non linear flows in the recent work [70]. Since rearrangement inequalities, reflection methods or moving plane can not be applied in some regions, it was not enough to study only the optimizers in the radial class. The key idea in their work was to rewrite the inequality in terms of a new variable $p = v^{-n}$ and assume that v satisfies a fast diffusion equation. This idea of exhibiting a nonlinear fast diffusion flow under a monotone action (non linear *carré du champ* method) allows to use the fast diffusion flow to drive the functional towards its optimal value. In their notes [71], they gave a simpler explanation and reformulated the result in [70]. A more general case, was studied by Dolbeault in collaboration with Muratori and Nazaret who recently focused the problem from another point of view; in [75] they followed a concentration-compactness analysis and on a perturbation method which uses a spectral gap and let them establish the existence of optimal functions, study their properties and prove that they are radial when the power in the weight is small enough.

Later on Dolbeault, Esteban and Loss also used non linear flows for studying the CKN inequality in the sphere ([69]) They built a counter-example which shows why heat flow methods definitely cannot cover the whole range of the exponents p up to the critical exponent 2^*, while nonlinear flows, with a proper choice of the nonlinearity, do it. Before this work, it was not known whether the limitation was of technical nature, or if there was a deep reason for it.

6.2 Introduction

The main aim in this work is to generalize the Caffarelli-Kohn-Nirenberg inequality (6.1.2) to the fractional setting with $\gamma \in (0,1)$.

Conjecture: Let $n > 2\gamma$, this generalization can be written as

$$\Lambda \left(\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*_\gamma}}{|x|^{\beta 2^*_\gamma}}\, dx \right)^{\frac{2}{2^*_\gamma}} \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\gamma}|x|^\alpha |y|^\alpha}\, dy\, dx, \tag{6.2.1}$$

for all $\alpha \leq \beta \leq \alpha + \gamma$, $\alpha \neq \frac{n-2\gamma}{2}$ and $u \in D^\gamma_{\alpha,\beta}$. The value of the positive constant Λ (independent of u) is given in (6.3.2), the value of

$$2^*_\gamma = \frac{2n}{n-2\gamma+2(\beta-\alpha)} \tag{6.2.2}$$

is computed by scaling argument and our functional space is

$$D^\gamma_{\alpha,\beta} = \{|x|^{-\beta} u \in L^{2^*}(\mathbb{R}^n), \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\gamma}|x|^\alpha |y|^\alpha}\, dy\, dx, < \infty\},$$

provided with the following norm:

$$\|u\|_{\gamma,\alpha,\beta} = \left(\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*_\gamma}}{|x|^{\beta 2^*_\gamma}}\, dx \right)^{\frac{1}{2^*_\gamma}} + \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^2}{|x-y|^{n+2\gamma}|x|^\alpha |y|^\alpha}\, dy\, dx \right)^{\frac{1}{2}}.$$

We found no reference (in particular, no proof) of this inequality in the literature. We expect to have a full proof of it soon. As in the classical case (6.1.2), this inequality (6.2.1) is an interpolation between the fractional Sobolev inequality ($\alpha = 0, \beta = 0$; see: [91]) and the weighted fractional Hardy inequality ($\beta = \alpha + \gamma$) (see [1]).

The goal of this work is also to show some results for the symmetry and symmetry breaking region for the minimizers. Following the framework of [72] we expect, first, to give a range for the parameters α and β where the extremal solutions for (6.2.1) are radially symmetric, and later, to provide a region were none of the extremal solutions are radially symmetric. In order to get this result we reformulate the fractional Caffarelli-Kohn-Nirenberg inequality in cylindrical variables and we provide a non-local ODE to find the radially symmetric extremals.

Assuming the inequality to hold, we start the study for all $\alpha \leq \beta \leq \alpha+1$, $0 \leq \alpha < \frac{n-2\gamma}{2}$ and $u \in D^\gamma_{\alpha,\beta}$.

6.3 Preliminaries

The extremal functions for (6.2.1) are the minimizers for the functional

$$F(u) = \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+2\gamma}|x|^\alpha|y|^\alpha}\,dy\,dx}{\left(\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\gamma}}{|x|^{\beta 2^*_\gamma}}\,dx\right)^{\frac{2}{2^*_\gamma}}}.$$

The Euler-lagrange equation associated to this minimization problem is

$$\kappa^n_{\alpha,\gamma}\frac{|u(x)|^{2^*_\gamma-2}u(x)}{|x|^{\beta 2^*_\gamma}} = \int_{\mathbb{R}^n} \frac{(u(x)-u(y))}{|x-y|^{n+2\gamma}|x|^\alpha|y|^\alpha}\,dy, \tag{6.3.1}$$

where the constant $\kappa^n_{\alpha,\gamma}$ is normalized as in (6.3.9). Indeed, let $\phi \in C^\infty_c$. The Euler-Lagrange equation for inequality (6.2.1) is

$$C\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*_\gamma-2}u(x)\phi(x)}{|x|^{\beta 2^*_\gamma}}\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2\gamma}|x|^\alpha|y|^\alpha}\,dy,$$

which is the weak formulation of equation (6.3.1). Here $C = 2\kappa^n_{\alpha,\gamma}$.
The value of the constant Λ in (6.2.1) is given by

$$\Lambda = 2\kappa^n_{\alpha,\gamma}\left(\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*_\gamma}}{|x|^{\beta 2^*_\gamma}}\,dx\right)^{1-\frac{2}{2^*_\gamma}}. \tag{6.3.2}$$

Lemma 6.3.1. *Let $\alpha,\ \bar\alpha \in \mathbb{R}$, the integral function given by*

$$I(x) = \int_{\mathbb{R}^n} \frac{(|x|^{-\bar\alpha}-|y|^{-\bar\alpha})}{|x-y|^{n+2\gamma}|y|^\alpha}\,dy$$

is radially symmetric and can be expressed as

$$I(x) = \frac{1}{|x|^{2\gamma}}\kappa^{n,\bar\alpha}_{\alpha,\gamma},$$

where $\kappa^{n,\bar\alpha}_{\alpha,\gamma}$ is the constant defined as

$$\kappa^{n,\bar\alpha}_{\alpha,\gamma} := P.V.\int_{\mathbb{R}^n} \frac{(1-|\zeta|^{-\bar\alpha})}{|e_1-\zeta|^{n+2\gamma}|\zeta|^\alpha}\,d\zeta. \tag{6.3.3}$$

Note that the value of this constant (6.3.3) is finite for all $\gamma \in (0,1)$ when

$$\begin{cases} -\bar\alpha-\alpha < 2\gamma \text{ and } n > \alpha \text{ if } \bar\alpha < 0, \\ \alpha > -2\gamma \text{ and } n > \alpha+\bar\alpha \text{ if } \bar\alpha > 0. \end{cases}$$

Proof. First, we can check that I is radially symmetric, i.e, if R denotes any rotation, then $I(x)=I(Rx)$. Since $R^{-1}=R^T$, where R^T denotes the transpose matrix and $|R|=1$, then

$$\begin{aligned}
I(Rx) &= \int_{\mathbb{R}^n} \tfrac{(|Rx|^{-\bar{\alpha}}-|y|^{-\bar{\alpha}})}{|Rx-y|^{n+2\gamma}|y|^{\alpha}}\,dy = \int_{\mathbb{R}^n} \tfrac{(|x|^{-\bar{\alpha}}-|y|^{-\bar{\alpha}})}{(|x|^2+|y|^2-2<Rx,y>)^{\frac{N+2\gamma}{2}}|y|^{\alpha}}\,dy \\
&= \int_{\mathbb{R}^n} \tfrac{(|x|^{-\bar{\alpha}}-|R^Ty|^{-\bar{\alpha}})}{(|x|^2+|R^Ty|^2-2<x,R^Ty>)^{\frac{n+2\gamma}{2}}|R^Ty|^{\alpha}}\,dy = \int_{\mathbb{R}^n} \tfrac{(|x|^{-\bar{\alpha}}-|\tilde{y}|^{-\bar{\alpha}})}{(|x|^2+|\tilde{y}|^2-2<x,\tilde{y}>)^{\frac{n+2\gamma}{2}}|\tilde{y}|^{\alpha}}|R^T|^{-n}\,dy \\
&= \int_{\mathbb{R}^n} \tfrac{(|x|^{-\bar{\alpha}}-|\tilde{y}|^{-\bar{\alpha}})}{(|x|^2+|\tilde{y}|^2-2<x,\tilde{y}>)^{\frac{n+2\gamma}{2}}|\tilde{y}|^{\alpha}}\,d\tilde{y} = I(x),
\end{aligned}$$

where we have used the change of variable $\tilde{y}=R^Ty$.

Secondly, after the change of variable $y=|x|\zeta$, since I is rotationally invariant, if we denote $e_1=(1,0,\cdots,0)$ we observe that $I(x)$ reads

$$I(x)=\tfrac{1}{|x|^{2\gamma}}\int_{\mathbb{R}^n}\tfrac{(1-|\zeta|^{-\bar{\alpha}})}{|\frac{x}{|x|}-\zeta|^{n+2\gamma}|\zeta|^{\alpha}}\,d\zeta=\tfrac{1}{|x|^{2\gamma}}\int_{\mathbb{R}^n}\tfrac{(1-|\zeta|^{-\bar{\alpha}})}{|e_1-\zeta|^{n+2\gamma}|\zeta|^{\alpha}}\,d\zeta=\tfrac{1}{|x|^{2\gamma}}\kappa^{n,\bar{\alpha}}_{\alpha,\gamma}.$$

□

We will see in Section 6.4.1 that these integrals can be studied using hypergeometric functions.

Corolary 6.3.2. *Let $\bar{\alpha}<0$, the constant $\kappa^{n,\bar{\alpha}}_{\alpha,\gamma}$ is positive for all $\alpha>\frac{n-2\gamma}{2}-\frac{\bar{\alpha}}{2}$, negative for all $\alpha<\frac{n-2\gamma}{2}-\frac{\bar{\alpha}}{2}$ and zero if $\alpha=\frac{n-2\gamma}{2}-\frac{\bar{\alpha}}{2}$. What's more, if $\bar{\alpha}>0$, the constant $\kappa^{n,\bar{\alpha}}_{\alpha,\gamma}$ is positive for all $\alpha<\frac{n-2\gamma}{2}-\frac{\bar{\alpha}}{2}$, negative for all $\alpha>\frac{n-2\gamma}{2}-\frac{\bar{\alpha}}{2}$ and zero if $\alpha=\frac{n-2\gamma}{2}-\frac{\bar{\alpha}}{2}$.*

Proof. We first use the polar coordinates for the variable ζ: $\varrho=|\zeta|$, $\theta\in\mathbb{S}^{n-1}$, and represent e_1 by $\sigma\in\mathbb{S}^{n-1}$, then we have

$$\kappa^{n,\bar{\alpha}}_{\alpha,\gamma}=\text{P.V.}\int_{\mathbb{R}^n}\tfrac{(1-|\zeta|^{-\bar{\alpha}})}{|e_1-\zeta|^{n+2\gamma}|\zeta|^{\alpha}}\,d\zeta=\int_{\mathbb{S}^{n-1}}J(\theta)\,d\theta, \tag{6.3.4}$$

where

$$J(\theta)=\text{P.V.}\int_0^\infty \frac{(1-\varrho^{-\bar{\alpha}})\varrho^{n-1-\alpha}}{(1+\varrho^2-2\varrho<\sigma,\theta>)^{\frac{n+2\gamma}{2}}}\,d\varrho.$$

We can write this function $J(\theta)$, using the change of variable $\tilde{\varrho}=1/\varrho$ in the first integral of the two integrals in the second line as

$$\begin{aligned}
J(\theta) &= \lim_{\epsilon\to0}\int_0^{1-\epsilon}\frac{(1-\varrho^{-\bar{\alpha}})\varrho^{n-1-\alpha}}{(1+\varrho^2-2\varrho<\sigma,\theta>)^{\frac{n+2\gamma}{2}}}\,d\varrho+\int_{1+\epsilon}^\infty\frac{(1-\varrho^{-\bar{\alpha}})\varrho^{n-1-\alpha}}{(1+\varrho^2-2\varrho<\sigma,\theta>)^{\frac{n+2\gamma}{2}}}\,d\varrho \\
&= \lim_{\epsilon\to0}\int_{1+\epsilon}^\infty\frac{-(1-\varrho^{-\bar{\alpha}})\varrho^{2\gamma-1+\alpha+\bar{\alpha}}}{(1+\varrho^2-2\varrho<\sigma,\theta>)^{\frac{n+2\gamma}{2}}}\,d\varrho+\int_{1+\epsilon}^\infty\frac{(1-\varrho^{-\bar{\alpha}})\varrho^{n-1-\alpha}}{(1+\varrho^2-2\varrho<\sigma,\theta>)^{\frac{n+2\gamma}{2}}}\,d\varrho \\
&= \lim_{\epsilon\to0}\int_{1+\epsilon}^\infty\frac{(1-\varrho^{-\bar{\alpha}})\varrho^{-1}}{(1+\varrho^2-2\varrho<\sigma,\theta>)^{\frac{n+2\gamma}{2}}}(\varrho^{n-\alpha}-\varrho^{2\gamma+\alpha+\bar{\alpha}})\,d\varrho.
\end{aligned}$$

Since $\varrho \in (1,\infty)$, we can assert that for all $\bar{\alpha} < 0$,

$$J(\theta) \begin{cases} > 0 \ \text{ iff } \ \alpha > \dfrac{n-2\gamma}{2} - \dfrac{\bar{\alpha}}{2}, \\ = 0 \ \text{ iff } \ \alpha = \dfrac{n-2\gamma}{2} - \dfrac{\bar{\alpha}}{2}, \\ < 0 \ \text{ iff } \ \alpha < \dfrac{n-2\gamma}{2} - \dfrac{\bar{\alpha}}{2}; \end{cases} \tag{6.3.5}$$

the same as for all $\bar{\alpha} > 0$,

$$J(\theta) \begin{cases} > 0 \ \text{ iff } \ \alpha < \dfrac{n-2\gamma}{2} - \dfrac{\bar{\alpha}}{2}, \\ = 0 \ \text{ iff } \ \alpha = \dfrac{n-2\gamma}{2} - \dfrac{\bar{\alpha}}{2}, \\ < 0 \ \text{ iff } \ \alpha > \dfrac{n-2\gamma}{2} - \dfrac{\bar{\alpha}}{2}. \end{cases} \tag{6.3.6}$$

Then, going back to the expression for $\kappa^{n,\bar{\alpha}}_{\alpha,\gamma}$ given in (6.3.4), we complete the proof of the corollary. $\square$

Remark 6.3.3. As a direct consequence of Corollary 6.3.2 we have that the constant $\kappa^{n,-\alpha}_{\alpha,\gamma}$ defined as in (6.3.3) with $\bar{\alpha} = -\alpha$ is negative for all $0 < \alpha < \frac{n-2\gamma}{2}$.

Lemma 6.3.4. *The function $u(x) = |x|^{-\nu}$ attains the equality for* (6.2.1)*, where*

$$\nu := \frac{n-2\gamma}{2} - \alpha. \tag{6.3.7}$$

Proof. Imposing that $u(x) = |x|^{-\nu}$ is a solution for (6.3.1), it yields

$$\begin{aligned} \kappa^n_{\alpha,\gamma} |x|^{-\nu(2^*_\gamma - 1) - \beta 2^*_\gamma + \alpha} &= \int_{\mathbb{R}^n} \tfrac{|x|^{-\nu} - |y|^{-\nu}}{|x-y|^{n+2\gamma}|y|^\alpha}\, dy \\ &= |x|^{-\nu - 2\gamma - \alpha} \int_{\mathbb{S}^n} \int_0^\infty \tfrac{(1-\varrho^{-\nu})\varrho^{n-1-\alpha}}{(1+\varrho^2 - 2\varrho <\sigma,\theta>)^{\frac{n+2\gamma}{2}}}\, d\varrho\, d\theta, \end{aligned} \tag{6.3.8}$$

where we have used polar coordinates with $\varrho = \frac{|y|}{|x|}$ and $\theta, \sigma \in \mathbb{S}^{n-1}$ for x, y, respectively. Equality (6.3.8) is possible if and only if the constant $\kappa^n_{\alpha,\gamma}$ is normalized to be

$$\kappa^n_{\alpha,\gamma} := \kappa^{n,\nu}_{\alpha,\gamma} \quad \text{and} \quad \nu = \frac{n-2\gamma}{2} - \alpha, \tag{6.3.9}$$

using (6.2.2). $\square$

Remark 6.3.5. Recalling the definition of the constant $\kappa^{n,\bar{\alpha}}_{\alpha,\gamma}$ given in (6.3.3) we can assure, thanks to Lemma 6.3.1 and Corollary 6.3.2, that $0 < \kappa^n_{\alpha,\gamma} < \infty$ for all $-2\gamma < \alpha < \frac{n-2\gamma}{2}$.

6.4 Results

Theorem 6.4.1. *Let $n > 2\gamma$. Extremal functions for (6.2.1) are radially symmetric for all $0 \leq \alpha < \frac{n-2\gamma}{2}$ and $\alpha \leq \beta < \alpha + 1$.*

Proof. Let $u \in \mathcal{C}_c^\infty(\mathbb{R}^N \setminus \{0\})$ and consider

$$v(x) = |x|^{-\alpha} u(x), \tag{6.4.1}$$

for any $x \in \mathbb{R}^n$. Inequality (6.2.1) for a function v as in (6.4.1) reads

$$\begin{aligned}
\Lambda \left(\int_{\mathbb{R}^n} \frac{|v(x)|^{2_\gamma^*}}{|x|^{(\beta-\alpha)2_\gamma^*}}\, dx \right)^{\frac{2}{2_\gamma^*}} &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(|x|^\alpha v(x) - |y|^\alpha v(y))^2}{|x-y|^{n+2\gamma} |x|^\alpha |y|^\alpha}\, dy\, dx \\
&= 2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(|x|^\alpha - |y|^\alpha) v^2(x)}{|x-y|^{n+2\gamma} |y|^\alpha}\, dy\, dx \\
&\quad + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x)-v(y))^2}{|x-y|^{n+2\gamma}}\, dy\, dx,
\end{aligned} \tag{6.4.2}$$

where we have used that

$$v^2(x) = \left(1 - \left|\frac{x}{y}\right|^{-\alpha}\right) v^2(x) + \left|\frac{x}{y}\right|^{-\alpha} v^2(x) \quad \text{and} \quad v^2(y) = \left(1 - \left|\frac{y}{x}\right|^{-\alpha}\right) v^2(y) + \left|\frac{y}{x}\right|^{-\alpha} v^2(y).$$

The first term in the right hand side of (6.4.2) can be written as $2 \int_{\mathbb{R}^N} v^2(x) I(x)\, dx$, where I is the integral studied in Lemma 6.3.1 when we take $\bar{\alpha} = -\alpha$. Thus inequality (6.4.2) is equivalent to

$$\Lambda \left(\int_{\mathbb{R}^n} \frac{|v(x)|^{2_\gamma^*}}{|x|^{(\beta-\alpha)2_\gamma^*}}\, dx \right)^{\frac{2}{2_\gamma^*}} - 2\kappa_{\alpha,\gamma}^{n,-\alpha} \int_{\mathbb{R}^n} \frac{v^2(x)}{|x|^{2\gamma}}\, dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x)-v(y))^2}{|x-y|^{n+2\gamma}}\, dy\, dx. \tag{6.4.3}$$

Finally, we will rewrite this inequality for $\tilde{v}$, the decreasing rearrangement of v. For the left hand side, we can apply Theorem 3.4 in Chapter 3 of [125] to assure that

$$\int_{\mathbb{R}^n} \frac{|\tilde{v}(x)|^{2_\gamma^*}}{|x|^{(\beta-\alpha)2_\gamma^*}}\, dx \geq \int_{\mathbb{R}^n} \frac{|v(x)|^{2_\gamma^*}}{|x|^{(\beta-\alpha)2_\gamma^*}}\, dx \quad \text{and} \quad \int_{\mathbb{R}^n} \frac{|\tilde{v}(x)|^2}{|x|^{2\gamma}}\, dx \geq \int_{\mathbb{R}^n} \frac{|v(x)|^2}{|x|^{2\gamma}}\, dx.$$

Because of Remark 6.3.3 we have that $\kappa_{\alpha,\gamma}^{n,-\alpha} < 0$ for $\alpha \in [0, \frac{n-2\gamma}{2})$.

For the right hand side, we can apply theorem $I.1$ in [93] which assures that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\tilde{v}(x)-\tilde{v}(y))^2}{|x-y|^{n+2\gamma}}\, dy\, dx \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x)-v(y))^2}{|x-y|^{n+2\gamma}}\, dy\, dx.$$

Then, since the symmetrization gives a better approximation in (6.4.3), we can assure that if the equality is attained by a function v, this function is going to be a radially symmetric one, which implies $\tilde{v} = v$. Indeed, suppose u to be a function which reaches the equality in (6.2.1), then v given as in (6.4.1) reaches the equality in (6.4.2) and Theorem 3.4 in [125] completes the proof.

□

6.4.1 The inequality in cylindrical variables

Inspired by the transformation we did in Chapter 3 to obtain the Hardy inequality (3.3.18), we would like to reformulate the fractional Caffarelli-Kohn-Nirenberg inequality in cylindrical variables. First, in the light of Lemma 6.3.4, we can write any function $u \in D^{\gamma}_{\alpha,\beta}$ as

$$u(x) = |x|^{-\nu} v(x), \tag{6.4.4}$$

where $v \in \tilde{D}^{\gamma}_{\alpha,\beta}$, which will be defined in (6.4.8), and ν is given in (6.3.7).

Proposition 6.4.2. *For a function v as in (6.4.4), the Euler Lagrange equation (6.3.1) in cylindrical coordinates ($r = e^t$, $s = e^{\tau}$) reads*

$$\kappa^n_{\alpha,\gamma} |v(t,\theta)|^{2^*_\gamma - 2} v(t,\theta) = \kappa^n_{\alpha,\gamma} v(t,\theta) + \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \frac{e^{-\frac{n+2\gamma}{2}(t-\tau)}(v(t,\theta) - v(\tau,\sigma))}{(1 + e^{-2(t-\tau)} - 2e^{-(t-\tau)} < \theta, \sigma >)^{\frac{n+2\gamma}{2}}} \, d\sigma \, d\tau, \tag{6.4.5}$$

and inequality (6.2.1) becomes

$$\begin{aligned} \Lambda \left(\int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} |v(t,\theta)|^{2^*_\gamma} \right)^{\frac{2}{2^*_\gamma}} \leq 2\kappa^{n,\nu}_{\alpha,\gamma} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} v^2(t,\theta) \, dt \, d\theta \\ + \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\frac{n+2\gamma}{2}(t-\tau)}(v(t,\theta) - v(\tau,\sigma))^2}{(1 + e^{-2(t-\tau)} - 2e^{-(t-\tau)} < \theta, \sigma >)^{\frac{n+2\gamma}{2}}} \, d\tau \, dt \, d\theta \, d\sigma. \end{aligned} \tag{6.4.6}$$

Proof. In polar coordinates ($r = |x|$, $\theta \in \mathbb{S}^{n-1}$ and $s = |y|$, $\sigma \in \mathbb{S}^{n-1}$), the Euler-Lagrange equation (6.3.1) reads

$$\kappa^n_{\alpha,\gamma} |v(r,\theta)|^{2^*_\gamma - 2} v(r,\theta) r^{-\nu(2^*_\gamma - 1) - \beta 2^*_\gamma} = \int_{\mathbb{S}^{n-1}} \int_0^{\infty} \frac{(r^{-\nu} v(r,\theta) - s^{-\nu} v(s,\sigma)) s^{n-1-\alpha} r^{-\alpha}}{(s^2 + r^2 - 2sr < \theta, \sigma >)^{\frac{n+2\gamma}{2}}} \, ds \, d\sigma,$$

which after the change of variable $\bar{s} = \frac{s}{r}$ is equivalent to

$$\kappa^n_{\alpha,\gamma} |v(r,\theta)|^{2^*_\gamma - 2} v(r,\theta) r^{-2^*_\gamma(\nu+\beta) + 2\nu + 2\alpha + 2\gamma} = \int_{\mathbb{S}^{n-1}} \int_0^{\infty} \frac{(v(r,\theta) - \bar{s}^{-\nu} v(r\bar{s},\sigma)) \bar{s}^{n-1-\alpha}}{(1 + \bar{s}^2 - 2\bar{s} < \theta, \sigma >)^{\frac{n+2\gamma}{2}}} \, d\bar{s} \, d\sigma.$$

Moreover, using the trivial equality $v(r,\theta) = (1 - \bar{s}^{-\nu} v(r,\theta)) + \bar{s}^{-\nu} v(r,\theta)$, we have

$$\begin{aligned} \kappa^n_{\alpha,\gamma} |v(r,\theta)|^{2^*_\gamma - 2} v(r,\theta) r^{-2^*_\gamma(\nu+\beta) + 2(\nu+\alpha+\gamma)} = & \kappa^n_{\alpha,\gamma} v(r,\theta) \\ & + \int_0^{\infty} \int_{\mathbb{S}^{n-1}} \frac{(v(r,\theta) - v(r\bar{s},\sigma)) \bar{s}^{n-1-\alpha-\nu}}{(1 + \bar{s}^2 - 2\bar{s} < \theta, \sigma >)^{\frac{n+2\gamma}{2}}} \, d\sigma \, d\bar{s}. \end{aligned} \tag{6.4.7}$$

Now, we can rewrite (6.4.7) in cylindrical coordinates using the Emden-Fowler change of variable ($r = e^t$, $s = e^{\tau}$ and thus, $\bar{s} = e^{-(t-\tau)}$):

$$\kappa^n_{\alpha,\gamma}|v(t,\theta)|^{2^*_\gamma-2}v(t,\theta)e^{\left(-2^*_\gamma(\nu+\beta)+2(\nu+\alpha+\gamma)\right)t} = \kappa^n_{\alpha,\gamma}v(t,\theta)+\int_{-\infty}^{\infty}\int_{\mathbb{S}^{n-1}}\frac{e^{(\nu-n+\alpha)(t-\tau)}(v(t,\theta)-v(\tau,\sigma))}{(1+e^{-2(t-\tau)}-2e^{-(t-\tau)}<\theta,\sigma>)^{\frac{n+2\gamma}{2}}}\,d\sigma\,d\tau,$$

which, for our choice of ν as in (6.3.7) reduces to (6.4.5).

Once we have the Euler Lagrange equation in cylindrical coordinates, we follow the same steps to rewrite the inequality (6.2.1). First, in polar coordinates it reads

$$\Lambda\left(\int_{\mathbb{S}^{n-1}}\int_0^\infty |v(r,\theta)|^{2^*_\gamma}r^{-2^*_\gamma(\nu+\beta)+n-1}\,dr\,d\theta\right)^{2/2^*_\gamma}\leq \int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_0^\infty r^{n-1-2(\alpha+\gamma+\nu)}\int_0^\infty\frac{(v(r,\theta)-\bar{s}^{-\nu}v(r\bar{s},\sigma))^2\bar{s}^{n-1-\alpha}}{(1+\bar{s}^2-2\bar{s}<\theta,\sigma>)^{\frac{n+2\gamma}{2}}}\,d\bar{s}\,dr\,d\sigma\,d\theta,$$

which using the trivial equalities

$$v^2(r,\theta)=(1-\bar{s}^{-\nu})v^2(r,\theta)+\bar{s}^{-\nu}v^2(r,\theta)$$

and

$$v^2(r\bar{s},\sigma)=(1-\bar{s}^{\nu})v^2(r\bar{s},\sigma)+\bar{s}^{\nu}v^2(r\bar{s},\sigma)$$

is equivalent to

$$\begin{aligned}\Lambda&\left(\int_{\mathbb{S}^{n-1}}\int_0^\infty |v(r,\theta)|^{2^*_\gamma}r^{-2^*_\gamma(\nu+\beta)+n-1}\,dr\,d\theta\right)^{2/2^*_\gamma}\leq\\ &\kappa^{n,\nu}_{\alpha,\gamma}\int_{\mathbb{S}^{n-1}}\int_0^\infty r^{n-1-2(\alpha+\gamma+\nu)}v^2(r,\theta)\,dr\,d\theta\\ &+\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_0^\infty\int_0^\infty\frac{r^{n-1-2(\alpha+\gamma+\nu)}v^2(r\bar{s},\sigma)\bar{s}^{n-1-\alpha-2\nu}(1-\bar{s})}{(1+\bar{s}^2-2\bar{s}<\theta,\sigma>)^{\frac{n+2\gamma}{2}}}\,dr\,d\bar{s}\,d\theta\,d\sigma\\ &+\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_0^\infty r^{n-1-2(\alpha+\gamma+\nu)}\int_0^\infty\frac{(v(r,\theta)-v(r\bar{s},\sigma))^2\bar{s}^{n-1-\alpha-\nu}}{(1+\bar{s}^2-2\bar{s}<\theta,\sigma>)^{\frac{n+2\gamma}{2}}}\,d\bar{s}\,dr\,d\theta\,d\sigma,\end{aligned}$$

which after the changes of variable $s=\bar{s}r$ and $\bar{r}=rs^{-1}$ in the second integral in the right hand side reads

$$\begin{aligned}\Lambda&\left(\int_{\mathbb{S}^{n-1}}\int_0^\infty |v(r,\theta)|^{2^*_\gamma}r^{-2^*_\gamma(\nu+\beta)+n-1}\,dr\,d\theta\right)^{2/2^*_\gamma}\leq\\ &2\kappa^{n,\nu}_{\alpha,\gamma}\int_{\mathbb{S}^{n-1}}\int_0^\infty r^{n-1-2(\alpha+\gamma+\nu)}v^2(r,\theta)\,dr\,d\theta\\ &+\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_0^\infty r^{n-1-2(\alpha+\gamma+\nu)}\int_0^\infty\frac{(v(r,\theta)-v(r\bar{s},\sigma))^2\bar{s}^{n-1-\alpha-\nu}}{(1+\bar{s}^2-2\bar{s}<\theta,\sigma>)^{\frac{n+2\gamma}{2}}}\,d\bar{s}\,dr\,d\theta\,d\sigma,\end{aligned}$$

and in cylindrical coordinates ($r = e^t$, $s = e^\tau$ and $\bar{s} = e^{-(t-\tau)}$) it becomes

$$\Lambda \left(\int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} |v(t,\theta)|^{2^*_\gamma} e^{(-2^*_\gamma(\nu+\beta)+n)t} \, dt \, d\theta \right)^{2/2^*_\gamma} \leq$$
$$2\kappa^{n,\nu}_{\alpha,\gamma} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} e^{(n-2(\alpha+\gamma+\nu))t} v^2(t,\theta) \, dt \, d\theta$$
$$+ \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} e^{(n-2(\alpha+\gamma+\nu))t} \int_{-\infty}^{\infty} \frac{e^{-(n-\alpha-\nu)(t-\tau)}(v(t,\theta) - v(\tau,\sigma))^2}{(1 + e^{-2(t-\tau)} - 2e^{-(t-\tau)} < \theta, \sigma >)^{\frac{n+2\gamma}{2}}} \, d\tau \, dt \, d\theta \, d\sigma.$$

To sum up, for our choice of ν in (6.3.7) we have that, in cylindrical coordinates, inequality (6.2.1) reduces to (6.4.6).

□

Definition 6.4.3. *Here we will define the new functional space*

$$\tilde{D}^\gamma_{\alpha,\beta} = \{ v \in L^2(\mathbb{R} \times \mathbb{S}^{n-1}) : \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-\frac{n+2\gamma}{2}(t-\tau)}(v(t,\theta)-v(\tau,\sigma))^2}{(1+e^{-2(t-\tau)}-2e^{-(t-\tau)}<\theta,\sigma>)^{\frac{n+2\gamma}{2}}} \, d\tau \, dt \, d\theta \, d\sigma < \infty \}. \tag{6.4.8}$$

Now we rewrite the Euler-Lagrange equation (6.3.1) (or equivalently, (6.4.7)) in radial coordinates for a radially symmetric function

$$u(|x|) = |x|^{-\nu} v(|x|),$$

where $u \in D^\gamma_{\alpha,\beta}$, $v \in \tilde{D}^\gamma_{\alpha,\beta}$ and ν is given in (6.3.7):

$$\kappa^n_{\alpha,\gamma} |v(t)|^{2^*_\gamma - 2} v(t) = \kappa^n_{\alpha,\gamma} v(t) + \int_{-\infty}^{\infty} (v(t) - v(\tau)) K(t-\tau) \, d\tau,$$

where the kernel is

$$K(\xi) = c_n e^{-\frac{n+2\gamma}{2}\xi} {}_2F_1(\tilde{a}, \tilde{b}; \tilde{c}; e^{-2\xi}),$$

the constant $\kappa^n_{\alpha,\gamma}$ is given in (6.3.9) and $\tilde{a} = \frac{n+2\gamma}{2}$, $\tilde{b} = 1+\gamma$, $\tilde{c} = \frac{n}{2}$. The asymptotic behaviour of the kernel can be studied using some properties of hypergeometric functions (see Lemma 4.2.5 in Chapter 4) and it satisfies:

- $K(\xi) \sim |\xi|^{-1-2\gamma}$ if $|\xi| \to 0$,
- $K(\xi) \sim e^{-\frac{n+2\gamma}{2}|\xi|}$ if $|\xi| \to \infty$.

6.5 Research plan

In some future works, I plan to generalize the Caffarelli-Kohn-Nirenberg inequality to the fractional setting without the parameter restrictions I am considering at present, using the recently developed flow method. **Conjecture:**

Let $p \in \mathbb{R}$ and $\gamma \in (0,1)$, the generalization to the fractional setting of the Caffarelli-Kohn-Nirenberg inequality in space dimension $n > 2\gamma$ can be written as

$$\Lambda \left(\int_{\mathbb{R}^n} \frac{|u(x)|^{p^*_\gamma}}{|x|^{2\beta \frac{p^*_\gamma}{p}}} \, dx \right)^{\frac{p}{p^*_\gamma}} \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^p}{|x-y|^{n+p\gamma} |x|^\alpha |y|^\alpha} \, dy \, dx, \tag{6.5.1}$$

for all $\alpha \leq \beta \leq \alpha + 1$ and $\alpha \neq \frac{n-2\gamma}{2}$. The value of Λ is given in (6.3.2),

$$D^\gamma_{\alpha,\beta} = \{ |x|^{-\frac{2\beta}{p}} u \in L^{p^*}(\mathbb{R}^n), \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)-u(y))^p}{|x-y|^{n+p\gamma} |x|^\alpha |y|^\alpha} \, dy \, dx, < \infty \} \tag{6.5.2}$$

and

$$p^*_\gamma = \frac{pn}{n - p\gamma + p(\beta - \alpha)} \tag{6.5.3}$$

is computed by scaling argument.

First, I plan to follow the steps of [70], where Dolbeault, Esteban and Loss solved the conjecture for the optimal symmetry range of the parameters. Since rearrangement inequalities, reflection methods or moving plane cannot be applied in some regions, it was not enough to study only the optimizers in the radial class. The key idea in their work was to rewrite the inequality in terms of a new variable $p = v^{-n}$ and assume that v satisfies a fast diffusion equation. This idea of exhibiting a nonlinear fast diffusion flow under a monotone action (non linear *carré du champ* method) allows to use the fast diffusion flow to drive the functional towards its optimal value. Good notes for this work are written in [71].

Later on, I would like to complete this work by generalizing to the fractional setting all the symmetry and symmetry breaking results for the most general Caffarelli-Kohn-Nirenbeg inequality,

$$\big\| |x|^\gamma u \big\|_{L^r} \leq C \big\| |x|^\alpha |\nabla u| \big\|^a_{L^p} \, \big\| |x|^\beta u \big\|^{1-a}_{L^q},$$

that holds under suitable parameter conditions. The starting point is the recent work of Dolbeault, Muratori and Nazaret in [75].

Another point of view to confront problems related with the general inequality (6.1.1) has been developed in the two recent works [22] and [23], where they related the inequality with the weighted fast diffusion equation.

Chapter 7

Appendix: Hypergeometric functions

In this appendix we will show the definition and some important properties of the Hypergeometric and related functions.

Lemma 7.0.1. [4, 167, 131] *Let $z \in \mathbb{C}$. The hypergeometric function is defined for $|z| < 1$ by the power series*

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}. $$

It is undefined (or infinite) if c equals a non-positive integer. Some properties of this function are

1. *The hypergeometric function evaluated at $z = 0$ satisfies*

$$ {}_2F_1(a+j, b-j; c; 0) = 1; \;\; j = \pm 1, \pm 2, ... \tag{7.0.1} $$

2. *If Re $b > 0, |z| < 1$,*

$$ {}_2F_1(a, a-b+\tfrac{1}{2}; b+\tfrac{1}{2}; z^2) = \frac{\Gamma(b+\frac{1}{2})}{\sqrt{\pi}\Gamma(b)} \int_0^{\pi} \frac{(\sin t)^{2b-1}}{(1+2z\cos t + z^2)^a} \, dt. \tag{7.0.2} $$

3. *If $a - b + 1 = c$, the following identity holds*

$$ \begin{aligned} &{}_2F_1(a, b; a-b+1; z) \\ &\quad = (1-z)^{1-2b}(1+z)^{2b-a-1} \; {}_2F_1\left(\tfrac{a+1}{2} - b, \tfrac{a}{2} - b + 1; a-b+1; \tfrac{4z}{(z+1)^2}\right). \end{aligned} \tag{7.0.3} $$

4. *The derivative of the hypergeometric function with respect to the last argument is*

$$ \frac{d}{dz} \; {}_2F_1(a, b; c; z) = \frac{ab}{c} \; {}_2F_1(a+1, b+1; c+1; z). \tag{7.0.4} $$

5. *If* $|arg(1-z)| < \pi$, *then*

$$
\begin{aligned}
{}_2\mathrm{F}_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ {}_2\mathrm{F}_1\left(a,b;a+b-c+1;1-z\right) \\
&+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\ {}_2\mathrm{F}_1(c-a,c-b;c-a-b+1;1-z).
\end{aligned}
\tag{7.0.5}
$$

6. *The hypergeometric function is symmetric with respect to first and second arguments, i.e*

$$ {}_2\mathrm{F}_1(a,b;c;z) = {}_2\mathrm{F}_1(b,a;c;z). \tag{7.0.6} $$

7. *From* (7.0.5) *and* (7.0.1), *if* $a+b<c$, *the following expansion holds*

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{7.0.7} $$

Lemma 7.0.2. [4, 167] *Let* $z \in \mathbb{C}$. *Some properties of the Gamma function* $\Gamma(z)$ *are*

$$ \Gamma(\bar{z}) = \overline{\Gamma(z)}, \tag{7.0.8} $$

$$ \Gamma(z+1) = z\Gamma(z), \tag{7.0.9} $$

$$ \Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z). \tag{7.0.10} $$

Let $\psi(z)$ *denote the Digamma function defined by*

$$ \psi(z) = \frac{d\ln\Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}. $$

This function has the expansion

$$ \psi(z) = \psi(1) + \sum_{m=0}^{\infty}\left(\tfrac{1}{m+1} - \tfrac{1}{m+z}\right). \tag{7.0.11} $$

Let $B(z_1,z_2)$ *denote the Beta function defined by*

$$ B(z_1,z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}. $$

If z_2 *is a fixed number and* $z_1 > 0$ *is big enough, then this function behaves*

$$ B(z_1,z_2) \sim \Gamma(z_2)(z_1)^{-z_2}. \tag{7.0.12} $$

Chapter 8

Appendix: Integral computations

In this appendix we will derive some useful integrals which are important for Chapter 5. All of the following expressions may be found in A. Bahri's book [17] for the special case $\gamma = 1$. Below we derive the estimates for general γ.

We define

$$w_1 = \Big(\frac{\lambda_1}{\lambda_1^2 + |x|^2}\Big)^{\frac{n-2\gamma}{2}}, \quad w_2 = \Big(\frac{\lambda_2}{\lambda_2^2 + |x|^2}\Big)^{\frac{n-2\gamma}{2}}, \quad w_3 = \Big(\frac{\lambda_3}{\lambda_3^2 + |x-p|^2}\Big)^{\frac{n-2\gamma}{2}}.$$

Lemma 8.0.3. *It holds*

$$\beta \int_{\mathbb{R}^n} w_1^{\beta-1} w_2 \frac{\partial w_1}{\partial \lambda_1}\, dx = \frac{1}{\lambda_1} F\big(\big|\log \tfrac{\lambda_2}{\lambda_1}\big|\big) \frac{\log \frac{\lambda_2}{\lambda_1}}{|\log \frac{\lambda_2}{\lambda_1}|}, \tag{8.0.1}$$

where

$$F(\ell) := \beta \int_{\mathbb{R}} v(t)^{\beta-1} v(t+\ell) v'(t)\, dt = e^{-\frac{n-2\gamma}{2}\ell}(1 + o(1)), \quad \ell \to \infty.$$

Proof. By the relation between w and v, one has

$$w_1 = |x|^{-\frac{n-2\gamma}{2}} v(-\log|x| + \log\lambda_1),\ w_2 = |x|^{-\frac{n-2\gamma}{2}} v(-\log|x| + \log\lambda_2).$$

Thus

$$\begin{aligned}
\beta \int_{\mathbb{R}^n} w_1^{\beta-1} w_2 \frac{\partial w_1}{\partial \lambda_1}\, dx &= \beta \int_{\mathbb{R}^n} |x|^{-2\gamma} v_1^{\beta-1} |x|^{-\frac{n-2\gamma}{2}} v_1' \frac{1}{\lambda_1} |x|^{-\frac{n-2\gamma}{2}} v_2\, dx \\
&= \beta \frac{1}{\lambda_1} \int_{\mathbb{R}} v^{\beta-1}(t + \log\lambda_1) v'(t+\log\lambda_1) v(t+\log\lambda_2)\, dt \\
&= \beta \frac{1}{\lambda_1} \int_{\mathbb{R}} v^{\beta-1}(t) v'(t) v(t + \log \tfrac{\lambda_2}{\lambda_1})\, dt \\
&= \frac{1}{\lambda_1} F(|\log \frac{\lambda_2}{\lambda_1}|) \frac{\log\frac{\lambda_2}{\lambda_1}}{|\log \frac{\lambda_2}{\lambda_1}|}.
\end{aligned}$$

□

Lemma 8.0.4. *If $\lambda_3 = O(\lambda_1)$ then the following estimates hold:*

$$\beta \int_{\mathbb{R}^n} w_1^{\beta-1} w_3 \frac{\partial w_1}{\partial \lambda_1}\, dx = A_2 \frac{|p|^{-(n-2\gamma)}}{\lambda_1} (\lambda_1\lambda_3)^{\frac{n-2\gamma}{2}} [1 + O(\lambda_1)^2], \tag{8.0.2}$$

$$\beta \int_{\mathbb{R}^n} w_1^{\beta-1} w_3 \frac{\partial w_1}{\partial x_l}\, dx = A_3 \frac{p_l}{|p|^{n-2\gamma+2}} (\lambda_1\lambda_3)^{\frac{n-2\gamma}{2}} (1 + O(\lambda_1^2)), \quad l = 1, \dots, n, \tag{8.0.3}$$

and the constants are given by

$$A_2 = \frac{n+2\gamma}{2} \int_{\mathbb{R}^n} \frac{|x|^2 - 1}{(1+|x|^2)^{\frac{n+2\gamma+2}{2}}}\, dx > 0,$$

$$A_3 = -\frac{(n-2\gamma)^2}{n} \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^{\frac{n+2\gamma+2}{2}}}\, dx < 0.$$

Proof. We calculate

$$\begin{aligned}
\beta \int_{\mathbb{R}^n} w_1^{\beta-1} w_3 \frac{\partial w_1}{\partial \lambda_1}\, dx &= \frac{n+2\gamma}{2} \int_{\mathbb{R}^n} \frac{\lambda_1^{\frac{n+2\gamma}{2}-1}(|x|^2 - \lambda_1^2)}{(|x|^2+\lambda_1^2)^{\frac{n+2\gamma}{2}+1}} \Big(\frac{\lambda_3}{|x-p|^2 + \lambda_3^2} \Big)^{\frac{n-2\gamma}{2}} dx \\
&= \frac{n+2\gamma}{2} \lambda_1^{\frac{n-2\gamma}{2}-1} \lambda_3^{\frac{n-2\gamma}{2}} \int_{\mathbb{R}^n} \frac{|x|^2-1}{(1+|x|^2)^{\frac{n+2\gamma}{2}+1}} \frac{1}{(|\lambda_1 x - p|^2 + \lambda_3^2)^{\frac{n-2\gamma}{2}}}\, dx \\
&= \frac{n+2\gamma}{2} \lambda_1^{\frac{n-2\gamma}{2}-1} \lambda_3^{\frac{n-2\gamma}{2}} |p|^{-(n-2\gamma)} \int_{\mathbb{R}^n} \frac{|x|^2-1}{(1+|x|^2)^{\frac{n+2\gamma}{2}+1}}\, dx\, (1 + O(\lambda_1^2)),
\end{aligned}$$

where we have used the expansion

$$(\lambda_3^2 + |\lambda_1 x - p|^2)^{-\frac{n-2\gamma}{2}} = |p|^{-(n-2\gamma)} + (n-2\gamma)\frac{\lambda_1 p \cdot x}{|p|^{n-2\gamma+2}} + O(\lambda_1^2). \tag{8.0.4}$$

Moreover, rescaling λ_1 in the second step,

$$\begin{aligned}
\frac{n+2\gamma}{2} \int_{\mathbb{R}^n} \frac{|x|^2-1}{(1+|x|^2)^{\frac{n+2\gamma}{2}+1}}\, dx &= \frac{\partial}{\partial \lambda_1}\Big|_{\lambda_1=1} \int_{\mathbb{R}^n} w_1^{\beta}\, dx \\
&= \frac{\partial}{\partial \lambda_1}\Big|_{\lambda_1=1} \int_{\mathbb{R}^n} \frac{\lambda_1^{\frac{n-2\gamma}{2}}}{(1+|x|^2)^{\frac{n+2\gamma}{2}}}\, dx \\
&= \frac{n-2\gamma}{2} \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{\frac{n+2\gamma}{2}}}\, dx > 0.
\end{aligned}$$

Next, by (8.0.4) again,

$$\begin{aligned}
\int_{\mathbb{R}^n} \beta w_1^{\beta-1} w_3 \frac{\partial w_1}{\partial x_l}\, dx &= -(n-2\gamma) \int_{\mathbb{R}^n} \frac{\lambda_1^{\frac{n+2\gamma}{2}} x}{(\lambda_1^2+|x|^2)^{\frac{n+2\gamma}{2}+1}} \Big(\frac{\lambda_3}{\lambda_3^2 + |x-p|^2} \Big)^{\frac{n-2\gamma}{2}} dx \\
&= -\frac{(n-2\gamma)^2}{n} (\lambda_1\lambda_3)^{\frac{n-2\gamma}{2}} \frac{p_l}{|p|^{n-2\gamma+2}} \int_{\mathbb{R}^n} \frac{|x|^2}{(1+|x|^2)^{\frac{n+2\gamma}{2}+1}}\, dx\, (1 + O(\lambda_1^2)).
\end{aligned}$$

□

Lemma 8.0.5. *For* $|a| \le \max\{\lambda_1^2, \lambda_2^2\} << 1$ *and* $\min\{\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}\} << 1$*, the following estimates hold:*

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{\partial}{\partial a}\Big(\frac{\lambda_1}{\lambda_1^2+|x-a|^2}\Big)^{\frac{n+2\gamma}{2}}\Big(\frac{\lambda_2}{\lambda_2^2+|x|^2}\Big)^{\frac{n-2\gamma}{2}}dx \\ &= -A_0 \min\Big\{\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{n-2\gamma}{2}}, \Big(\frac{\lambda_2}{\lambda_1}\Big)^{\frac{n-2\gamma}{2}}\Big\}\frac{a}{\max\{\lambda_1^2,\lambda_2^2\}} \\ &+ O\Big(\Big(\frac{a}{\max\{\lambda_1,\lambda_2\}}\Big)^2 + \min\Big\{\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{n-2\gamma}{2}}, \Big(\frac{\lambda_2}{\lambda_1}\Big)^{\frac{n-2\gamma}{2}}\Big\}\frac{a}{\max\{\lambda_1,\lambda_2\}}\Big) \\ &\cdot \min\Big\{\Big(\frac{\lambda_1}{\lambda_2}\Big)^{\frac{n-2\gamma}{2}}, \Big(\frac{\lambda_2}{\lambda_1}\Big)^{\frac{n-2\gamma}{2}}\Big\}, \end{aligned} \tag{8.0.5}$$

where

$$A_0 = \frac{(n+2\gamma)(n-2\gamma)}{n}\int_{\mathbb{R}^n}\frac{1}{|x|^{n-2\gamma}(1+|x|^2)^{\frac{n+2\gamma}{2}+1}}\,dx > 0.$$

Proof. We consider the case $\lambda_2 << \lambda_1$.

$$\begin{aligned} &\frac{1}{n+2\gamma}\int_{\mathbb{R}^n}\frac{\partial}{\partial a}\Big(\frac{\lambda_1}{\lambda_1^2+|x-a|^2}\Big)^{\frac{n+2\gamma}{2}}\Big(\frac{\lambda_2}{\lambda_2^2+|x|^2}\Big)^{\frac{n-2\gamma}{2}}dx \\ &= \int_{\mathbb{R}^n}\frac{\lambda_1^{\frac{n+2\gamma}{2}}(x-a)}{(\lambda_1^2+|x-a|^2)^{\frac{n+2\gamma}{2}+1}}\Big(\frac{\lambda_2}{\lambda_2^2+|x|^2}\Big)^{\frac{n-2\gamma}{2}}dx \\ &= \lambda_1^{\frac{n+2\gamma}{2}}\lambda_2^{\frac{n-2\gamma}{2}}\int_{\mathbb{R}^n}\frac{\lambda_1 x}{\lambda_1^{n+2\gamma+2}(1+|x|^2)^{\frac{n+2\gamma}{2}+1}}\frac{1}{(\lambda_2^2+|\lambda_1 x+a|^2)^{\frac{n-2\gamma}{2}}}\lambda_1^n\,dx \\ &= \lambda_1^{-\frac{n-2\gamma}{2}-1}\lambda_2^{\frac{n-2\gamma}{2}}\int_{\mathbb{R}^n}\frac{x}{(1+|x|^2)^{\frac{n+2\gamma}{2}+1}}\frac{1}{((\frac{\lambda_2}{\lambda_1})^2+|x|^2+\frac{2a\cdot x}{\lambda_1}+|\frac{a}{\lambda_1}|^2)^{\frac{n-2\gamma}{2}}}\,dx. \end{aligned}$$

Using the assumption that $|a| \le C\lambda_1^2$ and $\lambda_2 << \lambda_1$, by Taylor's expansion for the second term in the integral, the above integral is

$$\begin{aligned} &-(n-2\gamma)\Big(\frac{\lambda_2}{\lambda_1}\Big)^{\frac{n-2\gamma}{2}}\lambda_1^{-1}\int_{\mathbb{R}^n}\frac{\lambda_1^{-1}(a\cdot x)x|x|^{-(n-2\gamma)-2}}{(1+|x|^2)^{\frac{n+2\gamma}{2}+1}}\Big[1+O\Big(\frac{a}{\lambda_1}\Big)^2+O\Big(\frac{\lambda_2}{\lambda_1}\Big)^2\frac{a}{\lambda_1}\Big]dx \\ &= -\frac{n-2\gamma}{n}\Big(\frac{\lambda_2}{\lambda_1}\Big)^{\frac{n-2\gamma}{2}}\frac{a}{\lambda_1^2}\int_{\mathbb{R}^n}\frac{1}{|x|^{n-2\gamma}(1+|x|^2)^{\frac{n+2\gamma}{2}+1}}\,dx\Big[1+O\Big(\frac{a}{\lambda_1}\Big)^2+O\Big(\frac{\lambda_2}{\lambda_1}\Big)^2\frac{a}{\lambda_1}\Big] \\ &= -\frac{A_0}{n+2\gamma}\Big(\frac{\lambda_2}{\lambda_1}\Big)^{\frac{n-2\gamma}{2}}\frac{a}{\lambda_1^2}+O\Big(\frac{\lambda_2}{\lambda_1}\Big)^{\frac{n-2\gamma}{2}}\Big[\Big(\frac{a}{\lambda_1}\Big)^2+\Big(\frac{\lambda_2}{\lambda_1}\Big)^2\frac{a}{\lambda_1}\Big]. \end{aligned}$$

One can deal similarly with the case $\lambda_1 << \lambda_2$; we leave this proof to the reader. □

Bibliography

[1] B. Abdellaoui and R. Bentifour. Caffarelli-Kohn-Nirenberg type inequalities of fractional order with applications. Preprint.

[2] B. Abdellaoui, E. Colorado, and I. Peral. Some remarks on elliptic equations with singular potentials and mixed boundary conditions. *Adv. Nonlinear Stud.*, 4(4):503–533, 2004.

[3] B. Abdellaoui, E. Colorado, and I. Peral. Effect of the boundary conditions in the behavior of the optimal constant of some Caffarelli-Kohn-Nirenberg inequalities. Application to some doubly critical nonlinear elliptic problems. *Adv. Differential Equations*, 11(6):667–720, 2006.

[4] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[5] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz. Large N field theories, string theory and gravity. *Phys. Rep.*, 323(3-4):183–386, 2000.

[6] V. Ambrosio. Periodic solutions for a superlinear fractional problem without the Ambrosetti-Rabinowitz condition. Preprint arXiv:1601.06282.

[7] V. Ambrosio. Periodic solutions for the non-local operator $(-\delta + m^2)^s - m^{2s}$ with $m \geq 0$. Preprint arXiv:1510.05808.

[8] V. Ambrosio. Periodic solutions for a pseudo-relativistic Schrödinger equation. *Nonlinear Anal.*, 120:262–284, 2015.

[9] M. T. Anderson. Geometric aspects of the AdS/CFT correspondence. In *AdS/CFT correspondence: Einstein metrics and their conformal boundaries*, volume 8 of *IRMA Lect. Math. Theor. Phys.*, pages 1–31. Eur. Math. Soc., Zürich, 2005.

[10] M. T. Anderson, P. T. Chruściel, and E. Delay. Non-trivial, static, geodesically complete space-times with a negative cosmological constant. II. $n \geq 5$. In *AdS/CFT correspondence: Einstein metrics and their conformal boundaries*, volume 8 of *IRMA Lect. Math. Theor. Phys.*, pages 165–204. Eur. Math. Soc., Zürich, 2005.

[11] W. Ao, A. DelaTorre, M. González, and J. Wei. A gluing approach for the fractional yamabe problem with prescribed isolated singularities. Preprint. Avalaible at arXiv:1609.08903.

[12] W. Ao, M. Musso, and J. Wei. On spikes concentrating on line-segments to a semilinear Neumann problem. *J. Differential Equations*, 251(4-5):881–901, 2011.

[13] T. Aubin. Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. *J. Math. Pures Appl. (9)*, 55(3):269–296, 1976.

[14] T. Aubin. Problèmes isopérimétriques et espaces de Sobolev. *J. Differential Geometry*, 11(4):573–598, 1976.

[15] T. Aubin. *Some nonlinear problems in Riemannian geometry.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[16] T. Aubin. *A course in differential geometry*, volume 27 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001.

[17] A. Bahri. *Critical points at infinity in some variational problems*, volume 182 of *Pitman Research Notes in Mathematics Series.* Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989.

[18] R. F. Bass. Regularity results for stable-like operators. *J. Funct. Anal.*, 257(8):2693–2722, 2009.

[19] H. Baum and A. Juhl. *Conformal differential geometry*, volume 40 of *Oberwolfach Seminars.* Birkhäuser Verlag, Basel, 2010. Q-curvature and conformal holonomy.

[20] A. F. Beardon. *The geometry of discrete groups*, volume 91 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995. Corrected reprint of the 1983 original.

[21] W. Beckner. Pitt's inequality and the uncertainty principle. *Proc. Amer. Math. Soc.*, 123(6):1897–1905, 1995.

[22] M. Bonforte, J. Dolbeault, M. Muratori, and B. Nazaret. Weighted fast diffusion equations (part i): Sharp asymptotyc rates without symmetry and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities. arXiv:1602.08319, 2016.

[23] M. Bonforte, J. Dolbeault, M. Muratori, and B. Nazaret. Weighted fast diffusion equations (part ii): Sharp asymptotic rates of convergence in relative error by entropy methods. arXiv:1602.08315, 2016.

[24] V. Bouchez and M. Willem. Extremal functions for the Caffarelli-Kohn-Nirenberg inequalities: a simple proof of the symmetry. *J. Math. Anal. Appl.*, 352(1):293–300, 2009.

[25] T. P. Branson. Sharp inequalities, the functional determinant, and the complementary series. *Trans. Amer. Math. Soc.*, 347(10):3671–3742, 1995.

[26] T. P. Branson. Spectral theory of invariant operators, sharp inequalities, and representation theory. In *The Proceedings of the 16th Winter School "Geometry and Physics" (Srní, 1996)*, number 46, pages 29–54, 1997.

[27] J. Byeon and Z.-Q. Wang. Symmetry breaking of extremal functions for the Caffarelli-Kohn-Nirenberg inequalities. *Commun. Contemp. Math.*, 4(3):457–465, 2002.

[28] X. Cabré and E. Cinti. Energy estimates and 1-D symmetry for nonlinear equations involving the half-Laplacian. *Discrete Contin. Dyn. Syst.*, 28(3):1179–1206, 2010.

[29] X. Cabré, M. Fall, J. Solà-Morales, and T. Weth. Curves and surfaces with constant mean curvature: meeting Alexandrov and Delaunay. arXiv:1503.00469.

[30] X. Cabré, A. Mas, and J. Solá-Morales. Periodic stationary solutions of fractional parabolic and dispersive evolution equations. In preparation.

[31] X. Cabré and Y. Sire. Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 31(1):23–53, 2014.

[32] X. Cabré and J. Solà-Morales. Layer solutions in a half-space for boundary reactions. *Comm. Pure Appl. Math.*, 58(12):1678–1732, 2005.

[33] L. Caffarelli, T. Jin, Y. Sire, and J. Xiong. Local analysis of solutions of fractional semi-linear elliptic equations with isolated singularities. *Arch. Ration. Mech. Anal.*, 213(1):245–268, 2014.

[34] L. Caffarelli, R. Kohn, and L. Nirenberg. First order interpolation inequalities with weights. *Compositio Math.*, 53(3):259–275, 1984.

[35] L. Caffarelli, J.-M. Roquejoffre, and O. Savin. Nonlocal minimal surfaces. *Comm. Pure Appl. Math.*, 63(9):1111–1144, 2010.

[36] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[37] L. A. Caffarelli, B. Gidas, and J. Spruck. Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth. *Comm. Pure Appl. Math.*, 42(3):271–297, 1989.

[38] P. Caldiroli and R. Musina. Symmetry breaking of extremals for the Caffarelli-Kohn-Nirenberg inequalities in a non-Hilbertian setting. *Milan J. Math.*, 81(2):421–430, 2013.

[39] J. Case. Some energy inequalities involving fractional GJMS operators. Preprint arXiv:1509.08347.

[40] J. Case and S. Chang. On fractional GJMS operators. To appear in Communications on Pure and Applied Mathematics.

[41] F. Catrina and Z.-Q. Wang. On the Caffarelli-Kohn-Nirenberg inequalities. *C. R. Acad. Sci. Paris Sér. I Math.*, 330(6):437–442, 2000.

[42] F. Catrina and Z.-Q. Wang. On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Comm. Pure Appl. Math.*, 54(2):229–258, 2001.

[43] S. Y. A. Chang and M. d. M. González. Fractional Laplacian in conformal geometry. *Adv. Math.*, 226(2):1410–1432, 2011.

[44] S. Y. A. Chang, F. Hang, and P. Yang. On a class of locally conformally flat manifolds. *Int. Math. Res. Not.*, (4):185–209, 2004.

[45] S. Y. A. Chang, J. Qing, and P. Yang. Some progress in conformal geometry. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 3:Paper 122, 17, 2007.

[46] H. Chen and L. Véron. Singular solutions of fractional elliptic equations with absorption. Preprint, arXiv:1302.1427.

[47] H. Chen and L. Véron. Weakly and strongly singular solutions of semilinear fractional elliptic equations. *Asymptot. Anal.*, 88(3):165–184, 2014.

[48] W. Chen, C. Li, and B. Ou. Classification of solutions for an integral equation. *Comm. Pure Appl. Math.*, 59(3):330–343, 2006.

[49] W. Chen, Y. Li, and R. Zhang. Direct method of moving spheres on fractional order equations. arXiv:1509.03785.

[50] J. L. Chern and C. S. Lin. Minimizers of Caffarelli-Kohn-Nirenberg inequalities with the singularity on the boundary. *Arch. Ration. Mech. Anal.*, 197(2):401–432, 2010.

[51] N. Chiba and T. Horiuchi. Radial symmetry and its breaking in the Caffarelli-Kohn-Nirenberg type inequalities for $p = 1$. *Proc. Japan Acad. Ser. A Math. Sci.*, 92(4):51–55, 2016.

[52] K. S. Chou and C. W. Chu. On the best constant for a weighted Sobolev-Hardy inequality. *J. London Math. Soc. (2)*, 48(1):137–151, 1993.

[53] D. G. Costa. Some new and short proofs for a class of Caffarelli-Kohn-Nirenberg type inequalities. *J. Math. Anal. Appl.*, 337(1):311–317, 2008.

[54] A. Cotsiolis and N. K. Tavoularis. Best constants for Sobolev inequalities for higher order fractional derivatives. *J. Math. Anal. Appl.*, 295(1):225–236, 2004.

[55] C. Csáki, H. Ooguri, Y. Oz, and J. Terning. Glueball mass spectrum from supergravity. *J. High Energy Phys.*, (1):Paper 17, 21 pp. (electronic), 1999.

[56] E. B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[57] J. Dávila, M. del Pino, S. Dipierro, and E. Valdinoci. Nonlocal Delaunay surfaces. *Nonlinear Anal.*, 137:357–380, 2016.

[58] J. Dávila, M. del Pino, and Y. Sire. Nondegeneracy of the bubble in the critical case for nonlocal equations. *Proc. Amer. Math. Soc.*, 141(11):3865–3870, 2013.

[59] J. Dávila, L. Dupaigne, and J. Wei. On the fractional Lane-Emden equation. Preprint 2014. Avalaible at arXiv:1404.3694.

[60] M. del Pino, J. Dolbeault, S. Filippas, and A. Tertikas. A logarithmic Hardy inequality. *J. Funct. Anal.*, 259(8):2045–2072, 2010.

[61] A. DelaTorre, M. del Pino, M. d. M. González, and J. Wei. Radial solutions with an isolated singularity for a semilinear equation with fractional laplacian. To appear in Mathematische Annalen, DOI 10.1007/s00208-016-1483-1.

[62] A. DelaTorre and M. d. M. González. Isolated singularities for a semilinear equation for the fractional Laplacian arising in conformal geometry. Preprint. Avalaible at arXiv:1504.03493.

[63] C. Delaunay. Sur la surface de revolution dont la courbure moyenne est constante. *J. de Math. Pure et Appl.*, 6, 1841.

[64] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[65] J. Dolbeault and M. J. Esteban. Extremal functions in some interpolation inequalities: symmetry, symmetry breaking and estimates of the best constants. In *Mathematical results in quantum physics*, pages 178–182. World Sci. Publ., Hackensack, NJ, 2011.

[66] J. Dolbeault and M. J. Esteban. Extremal functions for Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities. *Proc. Roy. Soc. Edinburgh Sect. A*, 142(4):745–767, 2012.

[67] J. Dolbeault and M. J. Esteban. A scenario for symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities. *J. Numer. Math.*, 20(3-4):233–249, 2012.

[68] J. Dolbeault and M. J. Esteban. Branches of non-symmetric critical points and symmetry breaking in nonlinear elliptic partial differential equations. *Nonlinearity*, 27(3):435–465, 2014.

[69] J. Dolbeault, M. J. Esteban, and M. Loss. Interpolation inequalities on the sphere: linear vs. nonlinear flows. arXiv:1509.09099.

[70] J. Dolbeault, M. J. Esteban, and M. Loss. Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces. To appear in Inventiones Mathematicae, 2015.

[71] J. Dolbeault, M. J. Esteban, and M. Loss. Symmetry of optimizers of the Caffarelli-Kohn-Nirenberg inequalities. arXiv:1603.03574, 2016.

[72] J. Dolbeault, M. J. Esteban, M. Loss, and G. Tarantello. On the symmetry of extremals for the Caffarelli-Kohn-Nirenberg inequalities. *Adv. Nonlinear Stud.*, 9(4):713–726, 2009.

[73] J. Dolbeault, M. J. Esteban, and G. Tarantello. The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 7(2):313–341, 2008.

[74] J. Dolbeault, M. J. Esteban, G. Tarantello, and A. Tertikas. Radial symmetry and symmetry breaking for some interpolation inequalities. *Calc. Var. Partial Differential Equations*, 42(3-4):461–485, 2011.

[75] J. Dolbeault, M. Muratori, and B. Nazaret. Weighted interpolation inequalities: a perturbation approach. arXiv:1509.09127, 2015.

[76] H. Dong and D. Kim. Schauder estimates for a class of non-local elliptic equations. *Discrete Contin. Dyn. Syst.*, 33(6):2319–2347, 2013.

[77] J. Eells. The surfaces of Delaunay. *Math. Intelligencer*, 9(1):53–57, 1987.

[78] J. F. Escobar. Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary. *Ann. of Math. (2)*, 136(1):1–50, 1992.

[79] J. F. Escobar. The Yamabe problem on manifolds with boundary. *J. Differential Geom.*, 35(1):21–84, 1992.

[80] J. F. Escobar. Addendum: "Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary" [Ann. of Math. (2) **136** (1992), no. 1, 1–50; MR1173925 (93e:53046)]. *Ann. of Math. (2)*, 139(3):749–750, 1994.

[81] E. Fabes, C. Kenig, and R. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 7(1):77–116, 1982.

[82] S. Fakhi. Positive solutions of $\Delta u + u^p = 0$ whose singular set is a manifold with boundary. *Calc. Var. Partial Differential Equations*, 17(2):179–197, 2003.

[83] M. Fall. Semilinear elliptic equations for the fractional laplacian with hardy potential. Preprint 2014. Avalaible at arXiv:1109.5530.

[84] Y. Fang and M. d. M. González. Asymptotic behavior of Palais-Smale sequences associated with fractional Yamabe-type equations. *Pacific J. Math.*, 278(2):369–405, 2015.

[85] M. Fazly and J. Wei. On finite Morse index solutions of higher order fractional Lane-Emden equation. Preprint 2014.

[86] M. Fazly and J. Wei. On stable solutions of the fractional Hénon–Lane–Emden equation. *Commun. Contemp. Math.*, 18(5):1650005, 24, 2016.

[87] V. Felli and M. Schneider. Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type. *J. Differential Equations*, 191(1):121–142, 2003.

[88] F. Ferrari and I. E. Verbitsky. Radial fractional Laplace operators and Hessian inequalities. *J. Differential Equations*, 253(1):244–272, 2012.

[89] R. Fowler. Furter studies of Emden's and similar differential equations. *Quarterly Journal of Mathematics*, 1931.

[90] R. L. Frank, E. Lenzmann, and L. Silvestre. Uniqueness of radial solutions for the fractional Laplacian. *Comm. Pure Appl. Math.*, 69(9):1671–1726, 2016.

[91] R. L. Frank and E. H. Lieb. A new, rearrangement-free proof of the sharp Hardy-Littlewood-Sobolev inequality. In *Spectral theory, function spaces and inequalities*, volume 219 of *Oper. Theory Adv. Appl.*, pages 55–67. Birkhäuser/Springer Basel AG, Basel, 2012.

[92] R. L. Frank, E. H. Lieb, and R. Seiringer. Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators. *J. Amer. Math. Soc.*, 21(4):925–950, 2008.

[93] A. M. Garsia and E. Rodemich. Monotonicity of certain functionals under rearrangement. *Ann. Inst. Fourier (Grenoble)*, 24(2):vi, 67–116, 1974. Colloque International sur les Processus Gaussiens et les Distributions Aléatoires (Colloque Internat. du CNRS, No. 222, Strasbourg, 1973).

[94] N. Ghoussoub and F. Robert. The effect of curvature on the best constant in the Hardy-Sobolev inequalities. *Geom. Funct. Anal.*, 16(6):1201–1245, 2006.

[95] M. d. M. González. Recent progress on the fractional laplacian in conformal geometry. Preprint.

[96] M. d. M. González, R. Mazzeo, and Y. Sire. Singular solutions of fractional order conformal Laplacians. *J. Geom. Anal.*, 22(3):845–863, 2012.

[97] M. d. M. González and J. Qing. Fractional conformal Laplacians and fractional Yamabe problems. *Anal. PDE*, 6(7):1535–1576, 2013.

[98] M. d. M. González, M. Sáez, and Y. Sire. Layer solutions for the fractional laplacian on hyperbolic space.

[99] M. d. M. González and M. Wang. Further results on the fractional yamabe problem: the umbilic case. Preprint.

[100] C. R. Graham. Volume and area renormalizations for conformally compact Einstein metrics. In *The Proceedings of the 19th Winter School "Geometry and Physics" (Srní, 1999)*, number 63, pages 31–42, 2000.

[101] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling. Conformally invariant powers of the Laplacian. I. Existence. *J. London Math. Soc. (2)*, 46(3):557–565, 1992.

[102] C. R. Graham and M. Zworski. Scattering matrix in conformal geometry. *Invent. Math.*, 152(1):89–118, 2003.

[103] C. Guillarmou and J. Qing. Spectral characterization of Poincaré-Einstein manifolds with infinity of positive Yamabe type. *Int. Math. Res. Not. IMRN*, (9):1720–1740, 2010.

[104] M. J. Gursky and A. Malchiodi. A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature. *J. Eur. Math. Soc. (JEMS)*, 17(9):2137–2173, 2015.

[105] S. W. Hawking and D. N. Page. Thermodynamics of black holes in anti-de Sitter space. *Comm. Math. Phys.*, 87(4):577–588, 1982/83.

[106] I. W. Herbst. Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$. *Comm. Math. Phys.*, 53(3):285–294, 1977.

[107] W. H. III Meeks. The topology and geometry of embedded surfaces of constant mean curvature. *J. Differential Geom.*, 27(3):539–552, 1988.

[108] H. Izeki. Limit sets of Kleinian groups and conformally flat Riemannian manifolds. *Invent. Math.*, 122(3):603–625, 1995.

[109] T. Jin, O. de Queiroz, Y. Sire, and J. Xiong. On local behavior of singular positive solutions to nonlocal elliptic equations. Preprint arXiv:1604.00448.

[110] T. Jin, O. de Queiroz, Y. Sire, and J. Xiong. On local behavior of singular positive solutions to nonlocal elliptic equations. Preprint arXiv:1604.00448, 2016.

[111] T. Jin, Y. Li, and J. Xiong. On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions. *J. Eur. Math. Soc. (JEMS)*, 16(6):1111–1171, 2014.

[112] N. J.Korevaar, R. Mazzeo, F. Pacard, and R. M. Schoen. Refined asymptotics for constant scalar curvature metrics with isolated singularities. *Invent. Math.*, 135(2):233–272, 1999.

[113] M. Kassmann. A priori estimates for integro-differential operators with measurable kernels. *Calc. Var. Partial Differential Equations*, 34(1):1–21, 2009.

[114] M. Kassmann, M. Rang, and R. Schwab. Hölder regularity for integro-differential equations with nonlinear directional dependence. Indiana Univ. Math. J.(2016),in press.

[115] S. Kim, M. Musso, and J. Wei. Existence theorems of the fractional yamabe problem. Preprint arXiv:1603.06617.

[116] S. Kim, M. Musso, and J. Wei. A non-compactness result on the fractional yamabe problem in large dimensions. Preprint arXiv:1505.06183.

[117] S. Kim, M. Musso, and J. Wei. A compactness theorem of the fractional yamabe problem. part i: the non-umbilic conformal infinity. Preprint., 2016.

[118] N. J. Korevaar, R. Kusner, and B. Solomon. The structure of complete embedded surfaces with constant mean curvature. *J. Differential Geom.*, 30(2):465–503, 1989.

[119] D. A. Labutin. Wiener regularity for large solutions of nonlinear equations. *Ark. Mat.*, 41(2):307–339, 2003.

[120] N. Landkof. *Foundations of modern potential theory.* Springer-Verlag, New York-Heidelberg, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[121] C. R. LeBrun. $\mathcal{H}$-space with a cosmological constant. *Proc. Roy. Soc. London Ser. A*, 380(1778):171–185, 1982.

[122] J. M. Lee and T. H. Parker. The Yamabe problem. *Bull. Amer. Math. Soc. (N.S.)*, 17(1):37–91, 1987.

[123] Y. Y. Li. Remark on some conformally invariant integral equations: the method of moving spheres. *J. Eur. Math. Soc. (JEMS)*, 6(2):153–180, 2004.

[124] E. H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. *Ann. of Math. (2)*, 118(2):349–374, 1983.

[125] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, second edition, 2001.

[126] C. S. Lin. Interpolation inequalities with weights. *Comm. Partial Differential Equations*, 11(14):1515–1538, 1986.

[127] C. S. Lin and Z.-Q. Wang. Symmetry of extremal functions for the Caffarrelli-Kohn-Nirenberg inequalities. *Proc. Amer. Math. Soc.*, 132(6):1685–1691, 2004.

[128] P. L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.

[129] P. L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. II. *Rev. Mat. Iberoamericana*, 1(2):45–121, 1985.

[130] C. Loewner and L. Nirenberg. Partial differential equations invariant under conformal or projective transformations. In *Contributions to analysis (a collection of papers dedicated to Lipman Bers)*, pages 245–272. Academic Press, New York, 1974.

[131] W. Magnus, F. Oberhettinger, and R. P. Soni. *Formulas and theorems for the special functions of mathematical physics.* Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52. Springer-Verlag New York, Inc., New York, 1966.

[132] A. Malchiodi. Some new entire solutions of semilinear elliptic equations on $\mathbb{R}^n$. *Adv. Math.*, 221(6):1843–1909, 2009.

[133] J. Maldacena. The large N limit of superconformal field theories and supergravity [MR1633016 (99e:81204a)]. In *Trends in theoretical physics, II (Buenos Aires, 1998)*, volume 484 of *AIP Conf. Proc.*, pages 51–63. Amer. Inst. Phys., Woodbury, NY, 1999.

[134] F. C. Marques. Conformal deformations to scalar-flat metrics with constant mean curvature on the boundary. *Comm. Anal. Geom.*, 15(2):381–405, 2007.

[135] R. Mazzeo. The Hodge cohomology of a conformally compact metric. *J. Differential Geom.*, 28(2):309–339, 1988.

[136] R. Mazzeo. Elliptic theory of differential edge operators. I. *Comm. Partial Differential Equations*, 16(10):1615–1664, 1991.

[137] R. Mazzeo. Unique continuation at infinity and embedded eigenvalues for asymptotically hyperbolic manifolds. *Amer. J. Math.*, 113(1):25–45, 1991.

[138] R. Mazzeo and R. B. Melrose. Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *J. Funct. Anal.*, 75(2):260–310, 1987.

[139] R. Mazzeo and F. Pacard. A construction of singular solutions for a semilinear elliptic equation using asymptotic analysis. *J. Differential Geom.*, 44(2):331–370, 1996.

[140] R. Mazzeo and F. Pacard. Constant scalar curvature metrics with isolated singularities. *Duke Math. J.*, 99(3):353–418, 1999.

[141] R. Mazzeo and F. Pacard. Constant mean curvature surfaces with Delaunay ends. *Comm. Anal. Geom.*, 9(1):169–237, 2001.

[142] R. Mazzeo, F. Pacard, and D. Pollack. Connected sums of constant mean curvature surfaces in Euclidean 3 space. *J. Reine Angew. Math.*, 536:115–165, 2001.

[143] R. Mazzeo, D. Pollack, and K. Uhlenbeck. Connected sum constructions for constant scalar curvature metrics. *Topol. Methods Nonlinear Anal.*, 6(2):207–233, 1995.

[144] R. Mazzeo, D. Pollack, and K. Uhlenbeck. Moduli spaces of singular Yamabe metrics. *J. Amer. Math. Soc.*, 9(2):303–344, 1996.

[145] R. Mazzeo and B. Vertman. Elliptic theory of differential edge operators, II: Boundary value problems. *Indiana Univ. Math. J.*, 63(6):1911–1955, 2014.

[146] R. Musina. Existence of extremals for the Maz'ya and for the Caffarelli-Kohn-Nirenberg inequalities. *Nonlinear Anal.*, 70(8):3002–3007, 2009.

[147] M. Musso and J. Wei. Nonradial solutions to critical elliptic equations of Caffarelli-Kohn-Nirenberg type. *Int. Math. Res. Not. IMRN*, (18):4120–4162, 2012.

[148] S. Nayatani. Patterson-Sullivan measure and conformally flat metrics. *Math. Z.*, 225(1):115–131, 1997.

[149] M. Obata. The conjectures on conformal transformations of Riemannian manifolds. *J. Differential Geometry*, 6:247–258, 1971/72.

[150] G. Palatucci and A. Pisante. A global compactness type result for Palais-Smale sequences in fractional Sobolev spaces. *Nonlinear Anal.*, 117:1–7, 2015.

[151] J. Qing. On the rigidity for conformally compact Einstein manifolds. *Int. Math. Res. Not.*, (21):1141–1153, 2003.

[152] P. Quittner and W. Reichel. Very weak solutions to elliptic equations with nonlinear Neumann boundary conditions. *Calc. Var. Partial Differential Equations*, 32(4):429–452, 2008.

[153] X. Ros-Oton. Regularity for the fractional Gelfand problem up to dimension 7. *J. Math. Anal. Appl.*, 419(1):10–19, 2014.

[154] X. Ros-Oton and J. Serra. The extremal solution for the fractional Laplacian. *Calc. Var. Partial Differential Equations*, 50(3-4):723–750, 2014.

[155] X. Ros-Oton and J. Serra. Regularity theory for general stable operators. *J. Differential Equations*, 260(12):8675–8715, 2016.

[156] S. Salsa. *Partial differential equations in action*. Universitext. Springer-Verlag Italia, Milan, 2008. From modelling to theory.

[157] R. M. Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature. *J. Differential Geom.*, 20(2):479–495, 1984.

[158] R. M. Schoen. The existence of weak solutions with prescribed singular behavior for a conformally invariant scalar equation. *Comm. Pure Appl. Math.*, 41(3):317–392, 1988.

[159] R. M. Schoen. Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. In *Topics in calculus of variations (Montecatini Terme, 1987)*, volume 1365 of *Lecture Notes in Math.*, pages 120–154. Springer, Berlin, 1989.

[160] R. M. Schoen. On the number of constant scalar curvature metrics in a conformal class. In *Differential geometry*, volume 52 of *Pitman Monogr. Surveys Pure Appl. Math.*, pages 311–320. Longman Sci. Tech., Harlow, 1991.

[161] R. M. Schoen and S. T. Yau. Conformally flat manifolds, Kleinian groups and scalar curvature. *Invent. Math.*, 92(1):47–71, 1988.

[162] R. M. Schoen and S. T. Yau. *Lectures on harmonic maps.* Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997.

[163] R. Schwab and L. Silvestre. Regularity for parabolic integro-differential equations with very irregular kernels. Anal. PDE, in press., (2016).

[164] P. Sicbaldi. New extremal domains for the first eigenvalue of the Laplacian in flat tori. *Calc. Var. Partial Differential Equations*, 37(3-4):329–344, 2010.

[165] L. Silvestre. Hölder estimates for solutions of integro-differential equations like the fractional Laplace. *Indiana Univ. Math. J.*, 55(3):1155–1174, 2006.

[166] L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, 60(1):67–112, 2007.

[167] S. Slavyanov and W. Lay. *Special functions.* Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000. A unified theory based on singularities, With a foreword by Alfred Seeger, Oxford Science Publications.

[168] G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372, 1976.

[169] E. Valdinoci. From the long jump random walk to the fractional Laplacian. *Bol. Soc. Esp. Mat. Apl. S$\vec{\rm e}$MA*, (49):33–44, 2009.

[170] E. Valdinoci. A fractional framework for perimeters and phase transitions. *Milan J. Math.*, 81(1):1–23, 2013.

[171] Z. Q. Wang and M. Willem. Singular minimization problems. *J. Differential Equations*, 161(2):307–320, 2000.

[172] E. Witten. Anti de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2(2):253–291, 1998.

[173] D. Yafaev. Sharp constants in the Hardy-Rellich inequalities. *J. Funct. Anal.*, 168(1):121–144, 1999.

[174] R. Yang. On higher order extensions for the fractional Laplacian. Preprint.

[175] R. Zhang. Non-local curvature and topology of locally confomally flat manifolds. arXiv:1510.00957.