4.2 The Model

Consider a banking sector with n *risk neutral* banks distributed uniformly on a unit circle. A bank i has equity capital k_i.[1] Banks compete in deposit rates $\mathbf{r} = (r_i, r_{-i})$ in order to mobilise deposits, $D(r_i, r_{-i})$, which they invest in assets together with their own equity capital.

There is a continuum of depositors, also uniformly distributed on the unit circle, with one unit of fund apiece. A depositor can deposit her fund in a bank and earns deposit rate next period. We assume away any sort of deposit insurance. A depositor incurs a per unit transport cost t to travel to a bank.

Banks can choose between a prudent asset and a gambling asset. The prudent asset yields a return α, and the gambling asset, gives a return γ with probability θ and zero with probability $1-\theta$. The prudent asset has higher expected return ($\alpha > \theta\gamma$), but if the gamble is successful it yields higher private return ($\gamma > \alpha$). The bank invests $k + D(r_i, r_{-i})$ in an asset.[2]

If bank i chooses to invest in prudent and the gambling assets then his expected profits, respectively are

$$\begin{aligned}
\pi^P(r_i, r_{-i}, k) &= \alpha k + (\alpha - r_i)D(r_i, r_{-i}),\\
\pi^G(r_i, r_{-i}, k) &= \theta(\gamma k + (\gamma - r_i)D(r_i, r_{-i})).
\end{aligned}$$

The timing of the game is as follows: Banks simultaneously offer deposit rates. Depositors then choose the bank in which to deposit their funds. The deposit mobilisation is followed by the portfolio choice by the banks. Finally, project outputs are realised and the depositors are paid.

4.3 Equilibrium

In this section, we characterise the equilibrium of the economy where banks compete in the deposit market by offering deposit rates and choose a prudent asset or a gambling asset to invest in, and the depositors choose banks to place their funds. We

[2] We assume that the each bank has fixed amount of equity capital.

will focus on two types of equilibria in symmetric strategies. A *prudent* equilibrium, where all banks choose to invest in the prudent asset, and a *gambling* equilibrium, where all banks invest in the gambling asset. The natural solution concept used here is *Subgame Perfect Equilibrium*. Since, the depositors have to incur a per unit transport cost t to travel to a bank, the transport cost relative to the number of banks in the economy $(\frac{t}{n})$ can be used as a proper measure of market concentration.[2] This is because, if the transport cost increases relative to the number of banks, given the total number of depositors, each bank can lower the deposit rate to earn higher rent.

When the transport cost relative to the number of banks is very high, there might be some depositors who, in equilibrium, would not find it profitable to travel to a bank to place their fund (since, deposit rate would not be high enough in order to compensate for the increased travelling cost). In that case, we say that a *Local Monopoly* arises. On the contrary, if concentration is low, all the depositors are willing to participate in equilibrium. Then a *Competitive Market* arises. Prior to characterizing the equilibrium, we first analyse the necessary conditions for existence of prudent and gambling equilibria in both types of market conditions.

Banks will choose to invest in prudent asset if the expected profits from prudent asset exceeds the expected profits from gambling asset ($\pi^P \geq \pi^G$), i.e., if the deposit of a bank satisfies the following *No Gambling Condition*:

$$D_i \leq \frac{mk}{(1-\theta)r_i - m}, \qquad \text{(NG)}$$

where $m = a - \theta\gamma$. We assume that $(1-\theta) - m > 0$ in order that the term in the *right hand side* of the above inequality is positive.

4.3.1 Competitive Equilibrium

A competitive banking sector is a market where all the depositors place their funds in any of the n banks rather than staying home.

First, consider a Competitive Prudent Equilibrium (CPE). We first compute the demand for deposit of bank i when he offers r_i and all the other banks, r. A depositor

[2] See Salop [27].

at a distance x from bank i is indifferent between going to bank i and bank $i+1$ if

$$r_i - tx = r - t\left(\frac{1}{n} - x\right).$$

If the depositors anticipate that all banks are going to choose the prudent asset, then the demand for deposit of bank i is given by:

$$D^{CP}(r_i, r) = 2x(r_i, r) = \frac{r_i - r}{t} + \frac{1}{n}. \tag{4.1}$$

Here, one should take two restrictions into account. First, all the banks must comply with the *No Gambling Condition*. Second, there is no depositor who has incentive not to participate in the market, i.e., for any depositor at a distance x from any bank i, $r_i - 1 \geq tx$. This restriction implies the following *Participation Condition*:

$$D^{CP}(r_i) \leq \frac{2(r_i - 1)}{t} \tag{PC}$$

Hence, bank i's shareholders will solve the following problem:

$$\max_{r_i} \alpha k + (\alpha - r_i)\left(\frac{r_i - r}{t} + \frac{1}{n}\right).$$

s. t. (NG) and (PC).

From Kuhn-Tucker conditions of the above optimization problem, one can find three disjoint regions over which the symmetric deposit rate $r_i = r = r^{CP}$ is a candidate optimum. This is summarised below.

$$r^{CP} = \begin{cases} \bar{r} & \text{if } \frac{t}{n} \leq \alpha - \bar{r} \\ \alpha - \frac{t}{n} & \text{if } \alpha - \bar{r} \leq \frac{t}{n} \leq \frac{2(\alpha-1)}{3} \\ 1 + \frac{t}{2n} & \text{if } \frac{2(\alpha-1)}{3} \leq \frac{t}{n} \leq 2(\bar{r} - 1), \end{cases}$$

where $\bar{r} = \text{[illegible]}$ is the deposit rate which makes the (NG) bind with equal deposit for all banks. In order to interprete the above, first consider the corner solution $\bar{r}$. This deposit rate must satisfy (PC), which implies $\frac{t}{n} \leq 2(\bar{r} - 1)$. Also at $\bar{r}$, the profit function must have a non-negative slope which implies $\frac{t}{n} \leq \alpha - \bar{r}$. Then consider the interior solution $\alpha - \frac{t}{n}$. This must satisfy both (NG) and (PC), which implies $\alpha - \bar{r} \leq \frac{t}{n} \leq \frac{2(\alpha-1)}{3}$. Finally, consider the other corner solution $\left(1 + \frac{t}{2n}\right)$ which must satisfy (NG) and at this point the profit function ust have a ne ative slope. These two to ether i plies that $\frac{2(\gamma-1)}{3} \leq \frac{t}{n} \leq 2(\bar{r}-1)$. Therefore, a s etric CPE exists onl if

$$\frac{t}{n} \leq 2(\bar{r}-1) \equiv \phi^{r}.$$

Next, we anal se the Co petitive Ga blin Equilibriu (CGE). Note that when a bank i offers a eposit rate r_i, a epositor in this bank ets (in expecte ter s) θr_i. The in ifferent epositor x between i an $i+1$ is iven b in ifferent between oin to bank i an bank $i+1$ if

$$\theta r_i - tx = \theta r - t\left(\frac{1}{n} - x\right).$$

If the epositors anticipate that all banks are oin to choose the a blin asset, the eposit of bank i is iven b :

$$D^{CG}(r_i, r) = 2x(r_i, r) = \frac{\theta(r_i - r)}{t} + \frac{1}{n}.$$

Bank i's sharehol ers will solve the followin proble

$$\max_{r_i} \alpha k + (\alpha - r_i)\left(\frac{\theta(r_i - r)}{t} + \frac{1}{n}\right)$$

takin into account the (PC) an that (NG) is reverse .

A ain one can i entif three isjoint re ions on which a s etric eposit rate $r_i = r = r^{CG}$ (iven in the equation bellow) is can i ate for opti u . The can i ate opti a for s etric eposit rate are iven b :

$$r^{CG} = \begin{cases} \gamma - \frac{t}{\theta n} & \text{if } \frac{t}{n} \leq min\left\{\frac{2\theta(\gamma-1)}{3}, \theta(\gamma - \bar{r})\right\} \\ \bar{r} & \text{if } \theta(\gamma - \bar{r}) \leq \frac{t}{n} \leq 2(\theta\bar{r} - 1) \\ \frac{1}{\theta}\left(1 + \frac{t}{2n}\right) & \text{if } \frac{t}{n} \geq 2(\theta\bar{r} - 1) \end{cases}$$

First consi er the interior solution $\gamma - \frac{t}{\theta n}$. This ust satisf both the reverse (NG) an (PC), which i plies $\frac{t}{n} \leq \theta(\gamma - \bar{r})$ an $\frac{t}{n} \leq \frac{2\theta(\gamma-1)}{3}$. Then consi er the corner solution $\bar{r}$. This eposit rate ust satisf (PC), which i plies $\frac{t}{n} \leq 2(\theta\bar{r}-1)$. Also at $\bar{r}$, the profit function ust have a ne ative slope which i plies $\frac{t}{n} \geq \theta(\gamma - \bar{r})$. Finall , consi er the other corner solution $\frac{1}{\theta}\left(1 + \frac{t}{2n}\right)$ which ust satisf the reverse (NG) and at this point the profit function must have a negative slope. These two together imply that $\frac{t}{n} \geq 2(\theta^{*}-1)$.

The above only gives the necessary conditions for all candidates of r^{CG} to exist. In Section 3.3 we will show that the two corner solutions can be ruled out, and only the interior solution constitutes a CGE.

4.3.2 Local Monopoly Equilibrium

A local monopoly banking sector is a market where between any two consecutive banks on the circle there is a non-empty subset of depositors who will not place their funds in either of the banks. Consider a bank i offering deposit rate r_i. A depositor at distance x from i will prefer to stay home if $r_i - 1 < tx$. Hence, bank i will get a maximum deposit of $x \leq \frac{r_i - 1}{t}$ from either side and he will have following deposit demand:

$$D(r_i) = \frac{2(r_i - 1)}{t}.$$

In this section, we focus on two possible kinds of monopoly equilibria: Prudent and Gambling.

First, we look for the conditions under which does a Monopoly Prudent Equilibrium (MPE) exist. In such equilibrium, banks simply maximise profits subject to the *No Gambling Condition*, and the *No Participation Constraint*. Since banks offer deposit rate which is independent of the other banks' decisions, it is sufficient to check that the depositor at $x = \frac{1}{2n}$ does not deposit in either of the banks. Hence, the *No Participation Constraint* boils down to

$$r_i \leq 1 + \frac{t}{2n}. \qquad \text{(NPC)}$$

Therefore, bank i's shareholders will solve the following problem:

$$\max_{r_i} \alpha k + (\alpha - r_i)\left(\frac{2(r_i - 1)}{t}\right).$$

s. t. (NG) and (NPC).

From Kuhn-Tucker conditions of the above maximisation problem, the candidates for $r_i = r = r^{MP}$ are su arise below.

$$r^{MP} = \begin{cases} 1 + \frac{\varepsilon}{3n} & \text{if } \frac{\varepsilon}{n} \le min\{\alpha - 1, 2(\hat{r} - 1)\}, \\ \hat{r} & \text{if } \frac{\varepsilon}{n} \ge 2(\bar{r} - 1) \text{ and } \alpha - 1 \ge 2(\bar{r} - 1), \\ \frac{\alpha+1}{2} & \text{if } \frac{\varepsilon}{n} \ge \alpha - 1 \text{ and } \alpha - 1 \le 2(\bar{r} - 1), \end{cases}$$

where $\hat{r}$ is defined by:

$$\frac{2(\hat{r} - 1)}{\varepsilon} = \frac{mk}{(1-\theta)\bar{r} - m}.$$

Note that when $\frac{\varepsilon}{n} \ge min\{\alpha - 1, 2(\bar{r} - 1)\}$, the deposit rate offered by a bank is $r^{MP} = min\left\{\frac{\alpha+1}{2}, \hat{r}\right\}$. This form of deposit rate depends on the slope of the profit function at $\hat{r}$. If $\hat{r} \ge \frac{\alpha+1}{2}$, then the slope is negative, and hence the deposit rate that maximizes bank's profit is simply $\frac{\alpha+1}{2}$.[4] Also an MPE exists only if: [5]

$$\frac{\varepsilon}{n} \ge min\{\alpha - 1, 2(\hat{r} - 1)\} = \psi^P.$$

In the rest of this section, we analyse the *Monopoly Gambling Equilibrium* (MPE).[6] Bank i, operates on the part of the demand curve $\frac{2(\theta r_i - 1)}{\varepsilon}$ above $\frac{mk}{(1-\theta)r_i - m}$ (i.e., *No Gambling Condition* is reversed). Also, in this case, the *No Participation Constraint* is slightly different from the case of a prudent monopolist:

$$r_i \le \frac{1}{\theta}\left(1 + \frac{\varepsilon}{2n}\right). \qquad \text{(NPC')}$$

Hence, bank i's shareholders will solve the following problem:

$$\max_{r_i} \theta\gamma k + \theta(\gamma - r_i)\left(\frac{2(\theta r_i - 1)}{\varepsilon}\right).$$

[4] Notice that when $r^{MP} = 1 + \frac{\varepsilon}{3n}$, this is same as a CPE with deposit rate $1 + \frac{\varepsilon}{3n}$. Hence, the part of MPE, when $\frac{\varepsilon}{n} \le min\{\alpha - 1, 2(\hat{r} - 1)\}$, will be referred to as a CPE.

[5] Given the timing of the stage game, a bank choosing prudent or gambling asset depends only on the total size of the deposit in the bank (i.e., on the *No Gambling Condition*), not on who are the depositors placing fund with him. In case of local monopoly, we only concentrate on the prudent equilibrium where the total deposit comes from the depositors staying closest to a bank. Readers should also take into account that there might be equilibria of other types, given that in all these equilibria, the total deposit of a bank is of the same size.

[6] While interpreting the above necessary conditions, notice that $\hat{r}$ is a function of $\frac{\varepsilon}{n}$. After several steps of tedious algebra one can show that $\frac{\varepsilon}{n} \ge 2(\hat{r} - 1)$ if and only if $\frac{\varepsilon}{n} \ge 2(\bar{r} - 1)$.

s. t. the reversed (NG) and (NPC^{g}).

The candidates for $r_1 = r = r^{MG}$ are

$$r^{MG} = \begin{cases} \frac{1}{\theta}\left(1+\frac{t}{2n}\right) & \text{if } \frac{t}{n} \le \theta\gamma - 1 \text{ and } \frac{t}{n} \ge 2(\theta r^P - 1)\}, \\ \hat{r} & \text{if } \frac{t}{n} \ge 2(\theta r^P - 1) \text{ and } \theta\gamma - 1 \le 2(\theta r^P - 1), \\ \frac{\theta\gamma+1}{2\theta} & \text{if } \frac{t}{n} \ge \theta\gamma - 1 \text{ and } \theta\gamma - 1 \ge 2(\theta r^P - 1), \end{cases}$$

where $\hat{r}$ is defined by:

$$\frac{2(\theta\hat{r} - 1)}{t} = \frac{mk}{(1-\theta)\hat{r} - m} \ .$$

Also when $\frac{t}{n} \ge max\{\theta\gamma - 1, 2(\theta r^P - 1)\}$, then

$$r^{MG} = max\left\{\frac{\theta\gamma + 1}{2\theta}, \hat{r}\right\}.$$

In the following proposition, we show that none of the candidates for r^{MG} can constitutes an equilibrium.[7]

Proposition 3. *A Monopoly Gambling Equilibrium never exists.*

Proof. We are going to provide one deviation from each of the above candidates, $\hat{r}$ and $\frac{\theta\gamma+1}{2\theta}$. All these deviations have the common feature of generating same deposits as the candidates they deviate from.

First consider the deposit rate $\hat{r}$. A bank would be strictly better off by choosing a prudent asset and offering a deposit rate $\theta\hat{r}$. Hence, the MGE with deposit rate $\hat{r}$ cannot survive as an equilibrium. Next, consider the deposit rate $\frac{\theta\gamma+1}{2\theta}$. In the similar fashion, this is also dominated (in terms of profits) by a deposit rate $\frac{\theta\gamma+1}{2}$ and a bank choosing a prudent asset.

The only thing remains to be checked is that any non-participant who can alter bank's investment decision will stay out. This may happen only when $r^{MP} = \hat{r}$ (when (NG) is binding). Let x be the maximum distance that a depositor travels from. We call x the marginal consumer, for whom $tx = \hat{r} - 1$. She will not participate if $\theta\hat{r} - 1 \le tx$. Hence, it is sufficient to show that $\theta\hat{r} - \hat{r} \le 0$. This is always true, since $m > 0$ and $\theta < 1$. □

[7] In the same vein as the MPE, the MGE with deposit rate $\frac{1}{\theta}\left(1+\frac{t}{2n}\right)$ will be called a CGE.

Till now we have provided only the necessary conditions for the existence of different types of equilibria and showed that an MGE never exists. In the following section, we also provide the sufficient conditions for existence.

4.3.3 Characterisation of Equilibrium

In the following proposition, we characterise the equilibriums. Recall that the term $\frac{k}{n}$ is used as a measure of market concentration.

Proposition 4. *For a given level equity capital of each bank, k,*

(a) there exists a threshold $\tilde{\phi}$ such that if $\frac{k}{n} \leq \tilde{\phi}$ (low market concentration), only the Competitive Gambling Equilibrium exists, with the banks offering deposit rate $\gamma - \frac{k}{3n}$;
(b) if $\frac{k}{n} \in [\tilde{\phi}, \phi^{P}]$ (intermediate levels of market concentration), both a Competitive Gambling Equilibrium and a Competitive Prudent Equilibrium exist, with banks offering $\gamma - \frac{k}{3n}$ and $\bar{r}$ or $\alpha - \frac{k}{n}$, respectively;
(c) if $\frac{k}{n} \in [\phi^{P}, \psi^{P}]$ (moderately high levels of concentration), only Competitive Prudent Equilibrium exists, with banks offering $\alpha - \frac{k}{n}$ or $1 + \frac{k}{3n}$;
(d) if $\frac{k}{n} \geq \psi^{P}$ (very high concentration), only Monopoly Prudent Equilibrium exists, with banks offering $\frac{\alpha+1}{2}$ or $\bar{r}$.

Proof. First we show that in case of CGE, only the interior solution survives. Consider the solution $\frac{1}{2}(1 + \frac{k}{3n})$. Notice that this deposit rate is optimal only if $\frac{k}{n} \geq 2(\theta\bar{r} - 1)$. It is easy to check that $\pi^{MP}(1 + \frac{k}{3n}) > \pi^{CG}(\frac{1}{2}(1 + \frac{k}{3n}))$. Hence, a bank will have incentive to reduce the deposit rate, attracting the same deposit, and becoming a local (prudent) monopolist. Next, when the deposit rate $\bar{r}$ is a candidate optimum, a bank can gain strictly higher profit by reducing the deposit rate to $r^{d} = \frac{\alpha}{2}$, attracting the same demand, and switching to prudent asset. Hence, only the interior optimum survives. In this region, no bank can gain by offering a different deposit rate and choosing a prudent asset. a CGE exists if and only if:

$$\frac{k}{n} \leq min\left\{\frac{2(\theta\gamma - 1)}{3}, \theta(\gamma - \bar{r})\right\} \equiv \phi^{G}.$$

Next, there exists a threshold value $\tilde{\phi} \leq \alpha - \bar{r}$ such that if $\frac{t}{n} \leq \tilde{\phi}$,[8] a bank will find it profitable to switch to a gambling asset by offering a deposit rate which is his best response to $\bar{r}$. Hence, for $\frac{t}{n} \leq \tilde{\phi}$, there is no CPE.

Recall that $\psi^P = min\{\alpha - 1, 2(\bar{r} - 1)\} = min\{\alpha - 1, \phi^P\}$. Therefore, CPE and MPE might co-exist if $\alpha - 1 < \phi^P$. However, this does not occur, since a bank has incentive to choose to become a local monopolist by offering a deposit rate $\frac{\alpha+1}{2}$. In this region, no bank can gain by offering a different deposit rate and choosing a gambling asset. Therefore, a symmetric CPE exists if and only if

$$\tilde{\phi} \leq \frac{t}{n} \leq \psi^P$$

Notice that if $\tilde{\phi} < \phi^G$, then CPE and CGE do not exist together.

Finally, when $\frac{t}{n} \geq \psi^P$, only MPE exists since, by Proposition 1,there is no MGE.

□

The intuition behind the above proposition is fairly straightforward. When the market concentration is very low, competition erodes banks' profit, thus leaving little incentive for them to invest in prudent asset. Also, with fierce competition, banks offer high deposit rate which compensate for the travelling cost of the depositors (although they receive only θ fraction of it). On the other hand, for very high degree of concentration, banks gain monopoly rent, and hence they have incentive to choose prudent asset in order to preserve that. For, even a higher values of $\frac{t}{n}$, the market becomes monopolistic, i.e., banks offer even lower deposit rate which is not conducive to attract the depositors located at a longer distance. The above proposition is summarized in the following figure.

[Insert Figure 1 about here]

Also, for intermediate levels of concentration, banks might invest in the prudent asset by offering a lower deposit rate or in the gambling asset offering a higher rate which compensates for the expected loss for the depositors due to a possible failure in gambling.

---

[8]The threshold is given by $\tilde{\phi} = \left(\sqrt{(1-\theta)\bar{r}} - \sqrt{\theta(\gamma - \bar{r})}\right)^2$.

4.4 Comments on Welfare

In this section, we discuss the connection between market concentration and welfare. In the current set up social welfare is simply the total consumer's surplus, since the deposit rate is transfer from the banks to the depositors.

Proposition 4 shows that for a very low level of market concentration, banks only invest in the gambling asset. For a fixed level of equity capital, as concentration increases prudent behaviour also becomes part of the equilibrium strategy. And ultimately, risk-taking disappears. Depositor benefit from the higher deposit rate offered in the prudent equilibrium. Consequently, market concentration enhances social welfare. For very high concentration, the market becomes monopolistic, and some depositors are left out of the market. In this case, social welfare decreases.

Clearly, welfare is maximum if the market concentration is neither too high nor too low, although efficiency increases (in the sense that, bank choose the prudent asset rather than the gambling asset) with the level of market concentration.

Moreover, a change in bank's equity capital influence our results in an important way. Recall that the threshold values of market concentration that determine different types of equilibria depends on $\bar{r}$, which is increasing in k. When k increases, risk-taking and local monopoly become less likely since the regions over which CGE and MPE emerge shrink. As competition along with prudent behaviour is more likely, social welfare is potentially higher.

4.5 Conclusions

In this chapter, we use a model of baking sector based on spatial competition, and analyse the role of market concentration in influencing the risk-taking behaviour of banks. Using a static model we show that, for a very low level of market concentration, banks invest in the gambling asset. On the other hand, when the market concentration increases, banks invest only in the prudent asset. We assert that, more market concentration works as a device to refrain banks from being involved in high risk activities. We also show non-monotonic relation between concentration and social welfare.

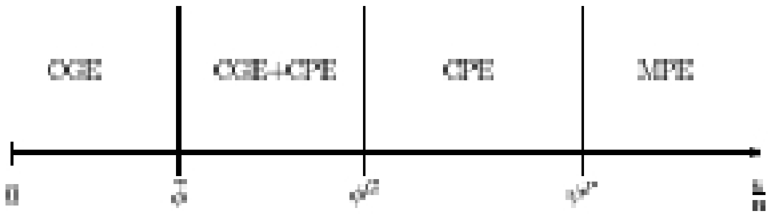

Figure 1: Characterisation of Equilibrium

Appendix

A. Appendix to Chapter 2

The Principal-Agent Contracts

We solve for the optimal principal-agent contract for a pair (P_i, A^j):

$$
\begin{cases}
\underset{\{\theta_S, \theta_F, K\}}{\text{maximise}} & u_{P_i} = \pi_1(K)\theta_S + (1 - \pi_1(K))\theta_F - K \\
\text{subject to} & (PC) \quad \pi_1(K)(y - \theta_S + \theta_F) - \theta_F \geq 1 \\
 & (IC^a) \quad [\pi_1(K) - \pi_0(K)](y - \theta_S + \theta_F) \geq 1 \\
 & (LS) \quad \theta_S \leq y + w^j \\
 & (LF) \quad \theta_F \leq w^j.
\end{cases}
\qquad (P1)
$$

At the optimum, (IC^a) binds, so we write the constraint with equality.[8] Using this, one can replace θ_S in the objective function and the other three constraints. Moreover, if (PC) and (LF) are satisfied, (LS) also holds. Hence, the above programme reduces to the following:

$$
\begin{cases}
\underset{\{\theta_F, K\}}{\text{maximise}} & \pi_1(K)y - \frac{\pi_1(K)}{\pi_1(K) - \pi_0(K)} + \theta_F - K \\
\text{subject to} & (PC^a) \quad \frac{\pi_0(K)}{\pi_1(K) - \pi_0(K)} - \theta_F - 1 \geq 0 \\
 & (LF) \quad w^j - \theta_F \geq 0.
\end{cases}
\qquad (P1')
$$

We denote μ_1 and μ_2 the Lagrangean multipliers of (P1'). Then, the Kuhn-Tucker

[8] To be more precise, (IC^a) does not bind if w^j is very high, that is in the region where the limited liability constraints do not play any role and the first best contract is signed. This corresponds to Case 2 in the analysis that follows.

(first-order) conditions are given by:[10]

$$y\pi_1' - 1 + (1-\mu_1)\frac{\pi_1'\pi_0 - \pi_1\pi_0'}{(\pi_1-\pi_0)^2} = 0 \tag{4.2}$$

$$1 - \mu_1 - \mu_2 = 0 \tag{4.3}$$

$$\mu_1\left(\frac{\pi_1(K)}{\pi_1(K)-\pi_0(K)} - \theta_F - 1\right) = 0 \tag{4.4}$$

$$\mu_2\left(w^J - \theta_F\right) = 0 \tag{4.5}$$

$$\frac{\pi_1(K)}{\pi_1(K)-\pi_0(K)} - \theta_F - 1 \geq 0 \tag{4.6}$$

$$w^J - \theta_F \geq 0 \tag{4.7}$$

$$\mu_1, \mu_2 \geq 0 \tag{4.8}$$

Now we study different regions where the Kuhn-Tucker conditions can be satisfied. For simplicity, we develop the analysis when $\pi_1'\pi_0 - \pi_1\pi_0' < 0$.
CASE 1: $\mu_1 = \mu_2 = 0$ (Both the constraints are non binding)
From (4.3), we can see that this case is not possible.

CASE 2: $\mu_1 > 0$, $\mu_2 = 0$ ((LL^Y) is non-binding and (PC^Y) is binding)
From (4.3), $\mu_1 = 1$. Then from (4.2), we have $y\pi_1'(K^*) = 1$, where K^* is the first-best level of investment. Using (PC^Y) and (LL^Y), one has

$$w^J \geq \frac{\pi_1(K^*)}{\pi_1(K^*)-\pi_0(K^*)} - 1 \equiv w^*.$$

Hence, if $w^J \geq w^*$ a candidate for optimal solution exists involving $K = K^*$. In particular, an optimal payment vector is $(\theta_S = y + w^J - \frac{1+w^J}{\pi_1(K^*)}, \theta_F = w^J)$.

CASE 3: $\mu_1 = 0$, $\mu_2 > 0$ ((LL^Y) is binding and (PC^Y) is non-binding)
From (4.3), $\mu_2 = 1$. Then (4.2) implicitly defines the level of optimum investment $\overline{K}$,

$$y\pi_1'(\overline{K}) - 1 + \frac{\pi_1'(\overline{K})\pi_0(\overline{K}) - \pi_1(\overline{K})\pi_0'(\overline{K})}{(\pi_1(\overline{K}) - \pi_0(\overline{K}))^2} = 0.$$

[10] The hypotheses on $\pi_1(K)$ and y make sure the optimal K must be interior and it satisfies the first-order conditions. The corner solution for θ_F is explicitly taken into account.

From (LF), we also have $\theta_F = w^d$. Moreover, θ_S is determined by (IC^S) as $\theta_S = y + w^d - \frac{1}{\pi_1(\overline{K}) - \pi_0(\overline{K})}$. And from the non-binding (PC^S) we have

$$w^d \leq \frac{\pi_1(\overline{K})}{\pi_1(\overline{K}) - \pi_0(\overline{K})} - 1 \equiv \overline{w}.$$

That is, the previous contract can only be a candidate if $w^d \leq \overline{w}$.

Case 4: $\mu_1 > 0$, $\mu_2 > 0$ (Both the constraints are binding)
From (LF), $\theta_F = w^d$. Then (PC^S) defines the optimal K as an implicit function of w^d. Denote this by $K(w^d)$, which must satisfy the following condition

$$\frac{\pi_1(K(w^d))}{\pi_1(K(w^d)) - \pi_0(K(w^d))} = w^d + 1. \tag{4.9}$$

Finally, θ_S is determined by (IC^S). Previously found θ_F, θ_S and $K(w^d)$ are indeed the candidates for optimum if the Lagrange multiplier, μ_1, implicitly defined by (4.2) lies in the interval [0, 1] (so that constraints (4.3) and (4.8) are satisfied). Given that $\pi_1' \pi_0 - \pi_1 \pi_0' < 0$, $\mu_1 < 1$ if and only if

$$y\pi_1'(K(w^d)) - 1 > 0 \Rightarrow K(w^d) < K^*.$$

Again using $\pi_1' \pi_0 - \pi_1 \pi_0' < 0$, $K(w^d) < K^*$ is optimal if

$$\frac{\pi_1(K^*)}{\pi_1(K^*) - \pi_0(K^*)} < w^d + 1 \Rightarrow w^d < w^*.$$

Similarly, $\mu_1 > 0$ if and only if

$$y\pi_1'(K(w^d)) - 1 + \frac{\pi_1'(K(w^d))\pi_0(K(w^d)) - \pi_1(K(w^d))\pi_0'(K(w^d))}{(\pi_1(K(w^d)) - \pi_0(K(w^d)))^2} < 0.$$

The above inequality implies $K(w^d) > \overline{K} \Rightarrow \frac{\pi_1(\overline{K})}{\pi_1(\overline{K}) - \pi_0(\overline{K})} < 1 + w^d \Rightarrow w^d > \overline{w}$. Hence, the optimal contract corresponds to the solution found in Case 3 when $w^d < \overline{w}$, is the candidate found in Case 4 when $\overline{w} < w^d < w^*$, and it is the first-best contract of Case 2 when $w^* \leq w^d$.

Proof of Proposition 1

From (LF), we also have $\theta_F = w^d$. Moreover, θ_S is determined by (IC^S) as $\theta_S = y + w^d - \frac{1}{\pi_1(\overline{K}) - \pi_0(\overline{K})}$. And from the non-binding (PC^S) we have

$$w^d \leq \frac{\pi_1(\overline{K})}{\pi_1(\overline{K}) - \pi_0(\overline{K})} - 1 \equiv \overline{w}.$$

That is, the previous contract can only be a candidate if $w^d \leq \overline{w}$.

CASE 4: $\mu_1 > 0$, $\mu_2 > 0$ (Both the constraints are binding)
From (LF), $\theta_F = w^d$. Then (PC^S) defines the optimal K as an implicit function of w^d. Denote this by $K(w^d)$, which must satisfy the following condition

$$\frac{\pi_1(K(w^d))}{\pi_1(K(w^d)) - \pi_0(K(w^d))} = w^d + 1. \tag{4.9}$$

Finally, θ_S is determined by (IC^S). Previously found θ_F, θ_S and $K(w^d)$ are indeed the candidates for optimum if the Lagrange multiplier, μ_1, implicitly defined by (4.2) lies in the interval $[0,1]$ (so that constraints (4.3) and (4.8) are satisfied). Given that $\pi_1'\pi_0 - \pi_1\pi_0' < 0$, $\mu_1 < 1$ if and only if

$$y\pi_0'(K(w^d)) - 1 > 0 \Rightarrow K(w^d) < K^*.$$

Again using $\pi_1'\pi_0 - \pi_1\pi_0' < 0$, $K(w^d) < K^*$ is optimal if

$$\frac{\pi_1(K^*)}{\pi_1(K^*) - \pi_0(K^*)} < w^d + 1 \Rightarrow w^d < w^*.$$

Similarly, $\mu_1 > 0$ if and only if

$$y\pi_1'(K(w^d)) - 1 + \frac{\pi_1'(K(w^d))\pi_0(K(w^d)) - \pi_1(K(w^d))\pi_0'(K(w^d))}{(\pi_1(K(w^d)) - \pi_0(K(w^d)))^2} < 0.$$

The above inequality implies $K(w^d) > \overline{K} \Rightarrow \frac{\pi_1(\overline{K})}{\pi_1(\overline{K}) - \pi_0(\overline{K})} < 1 + w^d \Rightarrow w^d > \overline{w}$. Hence, the optimal contract corresponds to the solution found in Case 3 when $w^d < \overline{w}$, is the candidate found in Case 4 when $\overline{w} < w^d < w^*$, and it is the first-best contract of Case 2 when $w^* \leq w^d$.

Proof of Proposition 1

We are to show that if $w^j > w^k$ in the region $w^j < w^B$, then $u_{P2}(\Lambda^j, c^{j*}) > u_{P2}(\Lambda^k, c^{k*})$. From the previous section one can write the value function $v(w^j) = u_{P2}(\Lambda^j, c^{j*})$. Using the Envelope theorem, we get $v'(w^j) = \mu_2 > 0$ and hence the proposition.

Contracts in a Stable Outcome

Let us rewrite (P2):

$$\begin{cases} \underset{(\theta_S, \theta_F, K)}{\text{maximize}} & u_{AP} = \pi_1(K)(y - \theta_S) - (1 - \pi_1(K))\theta_F - 1 \\ \text{subject to} & (PCP) \quad \pi_1(K)\theta_S + (1 - \pi_1(K))\theta_F - K \geq \bar{u} \\ & (IC^a) \quad [\pi_1(K) - \pi_0(K)](y - \theta_S + \theta_F) \geq 1 \\ & (LS) \quad \theta_S \leq y + w^j \\ & (LF) \quad \theta_F \leq w^j. \end{cases} \qquad \text{(P2)}$$

As we have pointed out in the paper, this programme is individually rational for the agent only if $\bar{u} \leq u_{P2}(\Lambda^j, c^{j*})$. Denote by $w^{min}(\bar{u})$ the level of wealth such that $\bar{u}$ is the utility of a principal that hires an agent with this wealth under a principal-agent contract. Programme $(P2)$ is only well defined for $w^j \geq w^{min}(\bar{u})$. At the optimum, (PCP) binds. Hence, one can substitute for θ_S in the objective function and the rest of the constraints. Also, if both (IC^a) and (LF) hold, then (LS) becomes redundant. Then one has the above programme reduced as the following:

$$\begin{cases} \underset{(\theta_F, K)}{\text{maximize}} & \pi_1(K)y - \bar{u} - K - 1 \\ \text{subject to} & (IC^{a\prime}) \quad \pi_1(K)y - \frac{\pi_1(K)}{\pi_1(K) - \pi_0(K)} + \theta_F - K - \bar{u} \geq 0 \\ & (LF) \quad \theta_F \leq w^j. \end{cases} \qquad \text{(P2}'\text{)}$$

Let ν_1 and ν_2 be the Lagrange multipliers for (IC$^{a\prime}$) and (LF), respectively. The

Kuhn-Tucker (first-order) conditions are

$$
\begin{aligned}
y\pi_1' - R + \nu_1\left(y\pi_1'(K) - 1 + \frac{\pi_1'(K)\pi_0(K) - \pi_1(K)\pi_0'(K)}{(\pi_1(K) - \pi_0(K))^2}\right) &= 0 \qquad (4.10)\\
\nu_1\left(\frac{\pi_1(K) - \pi_0(K)}{\pi_0(K)}\right) - \nu_2 &= 0 \qquad (4.11)\\
\nu_1\left((\pi_1(K) - \pi_0(K))\left(y - \frac{B - \theta_F + K}{\pi_1(K)}\right) - 1\right) &= 0 \qquad (4.12)\\
\nu_2(w^d - \theta_F) &= 0 \qquad (4.13)\\
\left((\pi_1(K) - \pi_0(K))\left(y - \frac{B - \theta_F + K}{\pi_1(K)}\right) - 1\right) &\geq 0 \qquad (4.14)\\
w^d - \theta_F &\geq 0 \qquad (4.15)\\
\nu_1,\ \nu_2 &\geq 0 \qquad (4.16)
\end{aligned}
$$

Now we study different regions for the Kuhn-Tucker conditions to be satisfied.

Case 1: $\nu_1 = 0$, $\nu_2 > 0$ ((LF) is binding and (IC^m), non-binding)
Using (4.11), one can see that this case is not possible.

Case 2: $\nu_1 > 0$, $\nu_2 = 0$ ((LF) is non-binding and (IC^m), binding)
From (4.11), it is clear that this case is not possible either.

Case 3: $\nu_1 = \nu_2 = 0$ (Both the constraints are non-binding)
From (4.10), $K = K^{fb}$, the first best level of investment. The payment made to the principal in case of failure, θ_F is calculated from (PCP). For example, $\theta_F = w^d$ and $\theta_S = \frac{B + K^{fb} - (1 - \pi_1(K^{fb}))w^d}{\pi_1(K^{fb})}$ are optimal. From (IC^m) and (LF), the above is only possible if

$$w^d \geq -\pi_1(K^{fb})y + K^{fb} + B + \frac{\pi_1(K^{fb})}{\pi_1(K^{fb}) - \pi_0(K^{fb})} \equiv w(B).$$

Case 4: $\nu_1 > 0$, $\nu_2 > 0$ (Both the constraints are binding)
In this case, $\theta_F = w^d$ and optimal investment is a function of w^d, $\bar{K}(w^d; B)$, that is implicitly defined by the condition

$$-\pi_1(K)y + K + B + \frac{\pi_1(K)}{\pi_1(K) - \pi_0(K)} = w^d. \qquad (4.17)$$

Notice that, from (4.10), for $K \leq K^{fb}$, $y\pi_1'(K) - 1 + \frac{\pi_1'(K)\pi_0(K) - \pi_1(K)\pi_0'(K)}{[\pi_1(K) - \pi_0(K)]^2} \geq 0$. From the above expression, this immediately implies that $\tilde{K}(.)$ is increasing in w^d. The previous values of θ_F, θ_S and K are optimal solutions to the above programme if the multipliers ν_1 and ν_2 defined in equations (4.10) and (4.11) satisfy (4.16), i.e., they are non-negative. Notice that (4.10) implies $\nu_2 > 0$ if and only if $\nu_1 > 0$. To check when $\nu_1 > 0$, notice that if $w^d > w(\bar{s})$, then it is necessary that

$$\begin{aligned} w^d &= -\pi_1(\tilde{K}(w^d;\bar{s}))y + \tilde{K}(w^d;\bar{s}) + \bar{s} + \frac{\pi_1(\tilde{K}(w^d;\bar{s}))}{\pi_1(\tilde{K}(w^d;\bar{s})) - \pi_0(\tilde{K}(w^d;\bar{s}))} \\ &> -\pi_1(K^{fb})y + K^{fb} + \bar{s} + \frac{\pi_1(K^{fb})}{\pi_1(K^{fb}) - y\pi_0'(K^{fb})} \equiv w(\bar{s}). \end{aligned}$$

Now we can characterise the optimal contract as follows.

$$K = \begin{cases} \tilde{K}(w^d;\bar{s}) & \text{if } w^d < w(\bar{s}) \\ K^{fb} & \text{if } w^d \geq w(\bar{s}). \end{cases}$$

$$\theta_S = \begin{cases} \frac{\pi_1\tilde{K}(w^d;\bar{s}) - \bar{s} - \pi_1(\tilde{K}(w^d;\bar{s}))y - w^d}{\pi_0(\tilde{K}(w^d;\bar{s}))} & \text{if } w^d < w(\bar{s}) \\ \frac{\pi_1 + K^{fb} - \bar{s} - \pi_1(K^{fb})y - w^d}{\pi_0(K^{fb})} & \text{if } w^d \geq w(\bar{s}) \end{cases}$$

and $\theta_F = w^d$

Here we also want to prove that for any level of $w^d \geq w^{min}(\bar{s})$, $\tilde{K}(w^d) \geq K(w^d)$. First of all we know that, $\tilde{K}(w^d) > \overline{K}$. Comparing (9) and (17), it is clear that proving $\tilde{K}(w^d) \geq K(w^d)$ is equivalent to showing that $\pi_1(\tilde{K})y - \tilde{K} - \bar{s} \geq 1$. Suppose that $w^{min}(\bar{s}) \leq \overline{w}$. Then $\bar{s}$ is given by

$$\bar{s} = \pi_1(\overline{K})y - \frac{\pi_1(\overline{K})}{\pi_1(\overline{K}) - \pi_0(\overline{K})} + w^{min}(\bar{s}) - \overline{K}.$$

Using the above together with (4.6), it is easy to see that $\pi_1(\tilde{K})y - \tilde{K} - \bar{s} > 1$. This also proves that $w(\bar{s}) \leq w^{fb}$. We now do the same considering $w^{min}(\bar{s}) > \overline{w}$. Notice that, in this case $\bar{s} = \pi_1(K(w^{min}(\bar{s})))y - K(w^{min}(\bar{s}))$. Also , $[\pi_1(\tilde{K})y - \tilde{K}] - [\pi_1(K(w^{min}(\bar{s})))y - K(w^{min}(\bar{s}))] > 0$, since investment is increasing in wealth. These previous two facts imply the above assertion that $\tilde{K}(w^d) \geq K(w^d)$ for all $w^d \geq w^{min}(\bar{s})$.

The Case when $\pi_1(K)\pi'_0(K) < \pi'_1(K)\pi_0(K)$

In the paper we have analysed our model under the assumption that $\pi_1\pi'_0 > \pi'_1\pi_0$. We also asserted that, all the qualitative results of our model would hold good under the assumption that $\pi_1\pi'_0 < \pi'_1\pi_0$. Under this assumption, the findings in Appendix A imply $\overline{K} > K(w^d) > K^*$ and $K(w^d)$ is decreasing for $w^d \in (\overline{w},\ w^*)$. The reason behind this is the following. When $\pi_1(K)$ is increasing relative to $\pi_0(K)$, for a high level of initial investment, giving incentives is much easier. Because of this, for low level of wealth, the principal gives over incentives to the agent by lending more money (equivalently, the optimal investment is higher). Similarly, under this assumption, the findings of Appendix C imply that $\tilde{K}(w^d; \bar{\theta}) > K^*$ for $w^d > w(\bar{\theta})$.

Proof of Theorem 3

Consider $m > n$. First we prove that each SPE outcome is stable. We do that through several claims. **(a)** At any SPE, the contracts accepted are (among) the ones yielding the highest utility to the principals. Otherwise, a principal accepting a contract that yields lower utility would have incentives to switch to a better contract that has not already been taken. **(b)** At any SPE, all the contracts that are accepted provide the same utility to all the principals. Otherwise, on the contrary, consider one of the (at most $n-1$) contracts that gives the maximum utility to the principals. If one of the agents slightly decreases the payments offered at the first stage, his contract will still be accepted at any Nash equilibrium (NE) of the second-stage game for the new set of offers (because of **(a)**). **(c)** At any SPE, exactly n contracts are accepted. To see this, suppose on the contrary that at most $n-1$ contracts are accepted. Then there is a (unmatched) principal with zero utility. This is not possible since **(b)** holds. **(d)** The contracts that are finally accepted are those offered by the wealthiest agents. Suppose $w^k > w^j$ and the contract offered by A^j is accepted, but not the one by A^k. Then A^k can offer a slightly better (for the principals) contract than A^j. Given **(a)**-**(c)**, this new contract will be accepted at any NE of the second-stage game. This is a contradiction. **(e)** All the contracts signed are optimal. Otherwise, an agent offering a non-optimal contract could improve it for both (any principal and himself). This new contract will certainly be among the n best contracts for the principals (since the previous contract was) and hence, will be accepted at any SPE outcome. **(f)** Finally, any SPE outcome is stable. It only remains to prove that the common utility level of the principals at an SPE, denoted by $\overline{u}$, lies in $[u_{P_1}(A^{n+1}, c^{(n+1)*}), u_{P_1}(A^n, c^{n*})]$. First, $\overline{u} \leq u_{P_1}(A^n, c^{n*})$, because otherwise, some agents would be better-off by not offering any contract. Secondly, $\overline{u} \geq u_{P_1}(A^{n+1}, c^{(n+1)*})$ for agent A^{n+1} not to have incentives to propose a contract that would have been accepted.

We now prove that any stable outcome can be supported by an SPE strategy. Let (μ, C) be a stable allocation where each principal gets utility $\overline{u}$. Consider the following strategies of each agent A^j for all j and of each principal P_i for all i:

$$\overline{s}^j = \begin{cases} c_{\mu(A^j)} & \text{if } \mu(A^j) \in \mathcal{P} \\ \overline{c} \quad \text{s.t.} \quad u_{P_i}(A^j, \overline{c}) = \overline{u} \;;\; \text{for any } \; P_i \in \mathcal{P}, & \text{otherwise.} \end{cases}$$

And $\overline{s}_i = \mu(P_i)$ if $\overline{s}$ is played in the first stage. Otherwise, principals select any agent compatible with an NE in pure strategies given any other message s sent in the first period. These strategies constitute an SPE yielding the stable outcome (μ, C). To see this, notice that given any message $s^j \neq \overline{s}^j$, principals play their NE strategies. Given that $\overline{s}$ is played in the first stage, by deviating any principal P_i she cannot gain more than $\overline{u}$. This is true because any contract offered in the first stage yields the same utility $\overline{u}$ to any principal. Now consider deviations by the agents. Given that $\overline{u} \geq u_{P_1}(A^{n+1}, c^{(n+1)*})$, by stability, there does not exist any contract that would be offered by an unmatched agent that guarantees him a positive utility while yielding at least $\overline{u}$ to a principal. Hence, unmatched agents do not have incentives to deviate. Also, given the efficiency of the contracts in a stable allocation, there does not exist a different contract that a matched agent could offer at which he could have strictly improved while still guaranteeing at least $\overline{u}$ to the principals. If there is a plethora of contracts that yields utility $\overline{u}$ to the principals, it is easy to check that there is no NE of the game at which a contract providing utility lower than $\overline{u}$ is accepted by a principal. Hence, the matched agents do not also have any incentive to deviate from $\overline{s}$.

The proof when $m \leq n$ is easier than before and follows similar arguments. To prove that each SPE yields stable outcomes where principals obtain zero profits, it is sufficient to check that the following three claims hold. (a) At any SPE, the contracts accepted are (among) the ones yielding the highest utility to the principals. (b) At any SPE, precisely m contracts are accepted and they provide zero utility to all the principals. (c) All the contracts signed are optimal.

To prove that the stable outcomes (the agents' optimal, if $m \leq n$) can be supported by an SPE strategy, let (μ, C) be a stable allocation where each principal gets utility $\bar{u}_i$. Consider the following strategies of each agent A^j for all j and of each principal P_i for all i:

$$\bar{s}^j = c^j(1) \text{ for any } A^j$$

And $\bar{s}_i = \mu(P_i)$ if $\bar{s}$ is played in the first stage. Otherwise, principals select any agent compatible with an NE in pure strategies given any other message s sent in the first period.

B. Appendix to Chapter 3

Analysis of the Optimal Contract

We solve for the optimal contract for an intermediary-firm pair (z_i, w^j). The contract should solve the following maximisation programme:

$$\max_{(p,R,r)} u_i = pR + (1-p)r \tag{P*}$$

$$\text{s.t.} \quad u_j = p(y - R) - (1-p)r + \frac{1 - p^2}{2z_i} \tag{4.18}$$

$$z_i(y - R + r) = p \tag{4.19}$$

$$R \leq y + w^j \tag{4.20}$$

$$r \leq w^j. \tag{4.21}$$

Since, constraint (4.19) is satisfied with equality we can substitute for R in the objective function and the other constraints in order to obtain the following reduced programme:

$$\max_{(p,r)} py - \frac{p^2}{z_i} + r \tag{P**}$$

$$\text{s.t.} \quad \frac{p^2}{2z_i} - r + \frac{1}{2z_i} \geq \bar{u}^j \tag{4.22}$$

$$r \leq w^j. \tag{4.23}$$

Let ν_1 and ν_2 be the Lagrange multipliers of the above programme. The Kuhn-Tucker (first-order) conditions are given by:

$$
\begin{aligned}
y - \frac{2p}{\theta_1} + \nu_1 \frac{p}{\theta_1} &= 0 && (4.24)\\
1 - \nu_1 - \nu_2 &= 0 && (4.25)\\
\nu_1 \left(\frac{p^2}{2\theta_1} - r + \frac{1}{2\theta_1} - \bar{u}^l \right) &= 0 && (4.26)\\
\nu_2 (u^l - r) &= 0 && (4.27)\\
\frac{p^2}{2\theta_1} - r + \frac{1}{2\theta_1} - \bar{u}^l &\geq 0 && (4.28)\\
u^l - r &\geq 0 && (4.29)\\
\nu_1, \nu_2 &\geq 0. && (4.30)
\end{aligned}
$$

We consider the following cases.

CASE 1: $\nu_1 = \nu_2 = 0$. This is not compatible with equation (4.25).

CASE 2: $\nu_1 > 0$ and $\nu_2 = 0$. Let (p^a, R^a, r^a) be the candidate solution in this case. From (4.25), $\nu_1 = 1$. Then from (4.24) one gets $p = \theta_1 y$. Given $\theta_1 y \geq 1$, $p^a = 1$. From constraint (4.19) in programme (P) and equation (4.26), one gets

$$R^a = r^a = y - \bar{u}^l.$$

The utilities are given by:

$$
\begin{aligned}
u_1^{a} &= y - \bar{u}^l,\\
u^{la} &= \bar{u}^l.
\end{aligned}
$$

Finally, the solution must satisfy (4.29), i.e.,

$$\frac{\theta_1 y^2}{2} + \frac{1}{2\theta_1} - \bar{u}^l < u^l.$$

Hence, for (θ_1, u^l) in the above region, (p^a, R^a, r^a) is candidate for an optimum. In this region, the contract is the *first-best* contract where the provision of incentive does not involve any cost.

Case 3: $\nu_1 = 0$ and $\nu_2 > 0$. Then from equation (4.27), $r^B = w^j$. From equations (4.25) and (4.24) we have $p^B = \frac{s_1 y}{2}$. Then from constraint (4.19) of programme (P') we get $R^B = \frac{y}{2} + w^j$. The utilities are given by:

$$
\begin{aligned}
u_1^B &= \frac{s_1 y^2}{4} + w^j, \\
u^{jB} &= \frac{s_1 y^2}{8} + \frac{1}{2s_1} - w^j.
\end{aligned}
$$

Notice that a firm w^j gets his project financed if $w^j \geq \tilde{w} \equiv \rho(1+s_1) - \frac{s_1 y^2}{4}$, since for $w^j \geq \tilde{w}$, $u_1^B \geq \rho(1+s_1)$. Finally, the solution must satisfy equation (4.28) which implies

$$\frac{s_1 y^2}{8} + \frac{1}{2s_1} - \bar{u}^j > w^j.$$

Hence, for (s_1, w^j) in the above region, (p^B, R^B, r^B) is candidate for an optimum.

Case 4: $\nu_1 > 0$ and $\nu_2 > 0$. Then from (4.27), $\bar{r} = w^j$. Substituting this in equation (4.26) we get,

$$\bar{p} = \sqrt{2s_1(w^j + \bar{u}^j) - 1}.$$

Then from constraint (4.19) of programme (P') we get

$$\bar{R} = y + w^j - \frac{1}{s_1}\sqrt{2s_1(w^j + \bar{u}^j) - 1}.$$

The utilities are given by:

$$
\begin{aligned}
\bar{u}_1 &= y\sqrt{2s_1(w^j + \bar{u}^j) - 1} - 2\bar{u}^j - w^j + \frac{1}{s_1}, \\
\bar{u}^j &= \bar{u}^j.
\end{aligned}
$$

Since, $\nu_2 > 0$ from equation (4.24) we have $s_1 y - 2\bar{p} \leq 0$. This implies

$$\frac{s_1 y^2}{8} + \frac{1}{2s_1} - \bar{u}^j \leq w^j$$

Also $\nu_1 < 1$ implies that $s_1 y - \bar{p} \geq 0$ (equation (4.24)). Hence we get

$$\frac{s_1 y^2}{2} + \frac{1}{2s_1} - \bar{u}^j \geq w^j$$

Hence, for (s_i, w^j) in the above region, $(\hat{p}, \hat{r}, F)$ is candidate for an optimum. Given the previous analysis, we can conclude that the optimal contracts are those described in the text. ■

Proof of Lemma 3

Consider the value function $u(s_i, w^j, \bar{\pi}^j)$ of programme (P). The Lagrange function is given by: Using Envelope Theorem we get,

$$\frac{\partial u(s_i, w^j, \bar{\pi}^j)}{\partial w^j} = \nu_2 > 0.$$

The above implies:

$$u(s_i, w^j, \bar{\pi}^j) > u(s_i, w^{j'}, \bar{\pi}^j) \quad \text{if} \quad w^j > w^{j'} \tag{a}$$

Also

$$\frac{\partial u(w^j, s_i, \bar{\pi}_j)}{\partial \bar{\pi}_j} = -\nu_1 < 0,$$

since, at the (incentive constrained) optimum $\nu_2 > 0$. Hence, we have

$$u(s_i, w^{j'}, \bar{\pi}^j) > u(s_i, w^{j'}, \bar{\pi}^{j'}) \quad \text{if} \quad \bar{\pi}^j < \bar{\pi}^{j'} \tag{b}$$

The above two together imply:

$$u(s_i, w^j, \bar{\pi}^j) > u(s_i, w^{j'}, \bar{\pi}^{j'}) \quad \text{if} \quad w^j > w^{j'} \quad \text{and} \quad \bar{\pi}^j \leq \bar{\pi}^{j'}.$$

This completes the proof of the lemma. ■

Detailed Proof of Theorem 6

First we show that in a stable outcome if the willingness to pay decreases (condition (DWP)) then the matching is negatively assortative. Consider the following condition.

$$\Delta u_1(j, j') \leq \Delta u_2(j, j') \quad \text{for} \quad s_1 > s_2 \quad \text{and} \quad w^j > w^{j'}. \tag{DWP}$$

Then the above and Part (c) of Theorem 1 together imply that in a stable outcome we must have $\mu(w^j) = s_2$ and $\mu(w^{j'}) = s_1$, and hence μ is negatively assortative.

The only thing remains to be shown is that, given stability, the condition (DWP) is always satisfied. As we have discussed earlier that the solution to programme (P') is candidate to be optimal over three disjoint regions of the parameter space. It is easy check that under first-best and when the firm's individual rationality constraint is not binding $\Delta u_l(j, j') = \Delta u_h(j, j')$ for $x_l > x_h$ and $w^j > w^{j'}$. So (DWP) is automatically satisfied.

To see this in the intermediate region, consider the maximum value function $u(x_k, w^j, \bar{w}^j)$ of the maximisation programme (P'). From this we get

$$\frac{\partial^2 u}{\partial w^j \partial x_k} = \frac{\partial^2 u}{\partial \bar{w}^j \partial x_k} = \beta\sqrt{2x_k(w^j + \bar{w}^j) - 1}\,\left[1 - x_k(w^j + \bar{w}^j)(2x_k(w^j + \bar{w}^j) - 1)^{-1}\right] \leq 0,$$

since $\beta \leq 1$. The above equation implies

$$u(x_l, w^j, \bar{w}^j) - u(x_h, w^j, \bar{w}^j) \leq u(x_l, w^{j'}, \bar{w}^j) - u(x_h, w^{j'}, \bar{w}^j) \qquad (4.31)$$

$$u(x_l, w^{j'}, \bar{w}^j) - u(x_h, w^{j'}, \bar{w}^j) \leq u(x_l, w^{j'}, \bar{w}^{j'}) - u(x_h, w^{j'}, \bar{w}^{j'}). \qquad (4.32)$$

The above two together imply

$$u(x_l, w^j, \bar{w}^j) - u(x_l, w^{j'}, \bar{w}^{j'}) \leq u(x_h, w^j, \bar{w}^j) - u(x_h, w^{j'}, \bar{w}^{j'})$$

This is nothing but (DWP), and hence the theorem. ■

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