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Completion and Decomposition of Hypergraphs by Domination Hypergraphs

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INTRODUCTION

A *graph G* is a finite non-empty set of objects called *vertices* together with a set of unordered pairs of vertices of G called *edges*. A *vertex dominating set* of a graph is a set of vertices D such that every vertex of G is either in D or joined by an edge to some vertex in D.

Domination in graphs is a widely researched branch of graph theory, both from a theoretical and algorithmic point of view (see [3, 11, 21, 23, 24, 25, 38]). In part, it is due to its applications to several fields where graphs are used to model the relationships between a finite number of objects. In this way, for instance, some concepts from domination in graphs appear in problems involving finding sets of representatives, as well as in facility location problems or in problems in monitoring communication, in electrical networks or in network routing (see [32]).

A great many theoretical issues, properties and algorithms have been studied regarding domination in graphs. The wide variety of papers that deal with parameters is remarkable; they range from the standard domination number of a graph to other domination-related parameters (see [13, 27, 40]). There are also many algorithmic papers addressing classes of graphs for which these parameters can be computed in polynomial time, as well as papers providing combinatorial bounds and algorithms for enumerating all the vertex dominating sets of certain graphs (see [7, 14, 15, 18, 19, 26, 29, 31]).

This thesis fits within this context and stems from a theoretical question related with domination in graphs whose solution will lead us to consider a new domination-related parameter, as well as to several issues regarding its computation. The starting point of this work is a question concerning the design of networks on a finite set of nodes Ω whose dominating sets satisfy specific properties. Thus, in this thesis we focus our attention on the collection $\mathscr{D}(G)$ of all the *inclusionwise minimal vertex dominating sets* of a graph G. Specifically, we are looking for graphs G whose collection of vertex dominating sets $\mathscr{D}(G)$ is equal or close to a given collection $\{A_1, \ldots, A_r\}$ of subsets of nodes $A_i \subseteq \Omega$.

Hypergraphs become the natural framework of this problem. A *hypergraph* $\mathscr{H}$ on a finite set Ω is a collection of subsets of Ω, none of which is a proper subset of another. The *domination hypergraph of a graph G* is the collection $\mathscr{D}(G)$ of all the inclusion-minimal vertex dominating sets of a graph G. A hypergraph $\mathcal{H}$ is said to be a *domination hypergraph* if $\mathcal{H}$ is the domination hypergraph of a graph; that is, if $\mathcal{H} = \mathcal{D}(G)$ for some graph G.

In general, a hypergraph $\mathcal{H}$ is far from being the domination hypergraph of a graph. Therefore, a natural question that arises at this point is to determine how it can be transformed into a domination hypergraph.

This kind of questions is frequently considered in other areas of mathematics. We are given a family Σ of mathematical objects of a certain type, and we consider a subset Σ' of Σ consisting of *special* objects that satisfy additional properties. Then we can raise the following questions:

- Characterization: given an element $\sigma \in \Sigma$, we search for necessary and sufficient conditions for σ to be a *special* object; that is, $\sigma \in \Sigma'$.
- Approximation: if $\sigma \notin \Sigma'$, we want to find the optimal approximations of σ by objects of Σ'. In order to deal with this issue we need some kind of notion of approximation in Σ. This notion can be given by a topology arising from a metric or from a partial order, for example.
- Recovering: if $\sigma \notin \Sigma'$, we want to recover σ from its optimal approximations. The recovering of an object from its optimal approximations has to be done by means of some kind of operation we perform with those approximations.

Examples of approximation are the orthogonal projection in linear algebra and, in analysis, the theories concerning the approximation of functions of some kind by simpler functions. Integer factorization into primes fits into the decomposition problem, as do the decomposition of a non-zero non-unit element as a product of prime elements in a unique factorization domain, the primary decomposition of ideals in noetherian rings and the decomposition of algebraic varieties into irreducible ones.

Let us see an elementary example.

Example 0.1 Let $\Sigma = \mathbb{N}$ and $\Sigma' = \{p^k : p \text{ prime}, k \geq 1\}$. We consider on $\mathbb{N}$ the partial order given by divisibility; that is, if $a, b \in \mathbb{N}$, then we set $a \leqslant b$ if and only if $a \mid b$. Then the questions raised above reduce themselves to the following considerations:

- Characterization: to characterize when $\sigma \in \mathbb{N}$ is a power of a prime. For exemple, $\sigma = 12 \notin \Sigma'$.
- Approximation: to approximate $\sigma \in \mathbb{N}$ by powers of primes. For example, $2 \leqslant 2^2 \leqslant 12$, $3 \leqslant 12$. In this case, the optimal approximations of $\sigma = 12$ are 2^2 and 3.
- Decomposition: to recover $\sigma \in \mathbb{N}$ from prime powers. For example, $12 = 2^2 \cdot 3$.

In this thesis we consider these kind of problems in a situation involving discrete objects. That is, we have:

- Ω: a finite set; for instance the set of vertices of a graph.
- Σ: set of families of subsets of Ω; for example, the set of hypergraphs on Ω.
- $\Sigma' \subseteq \Sigma$: a set of special families with additional properties.

The central theme of this thesis deals with the above situation where the special families are hypergraphs whose elements are the dominant sets of vertices of a graph G such that $V(G) \subseteq \Omega$.

In this combinatorial framework, we mention another situation where the special objects considered are hypergraphs arising from matroids [33, 34, 35].

Outline of the Thesis

This Thesis is organized in four chapters and two appendices.

Throughout the rest of this introduction, Ω is a non-empty finite set.

In Chapter 1 we define the sets $\mathrm{Hyp}(\Omega)$ and $\mathrm{Hyp}_0(\Omega)$ of hypergraphs *on* Ω and *with ground set* Ω, respectively. By relating the set $\mathrm{Hyp}(\Omega)$ with the sets $\mathrm{Inc}(\Omega)$, of monotone increasing families of subsets of Ω, and $\mathrm{Dec}(\Omega)$, of monotone decreasing families of subsets of Ω, we introduce two structures of distributive lattice on $\mathrm{Hyp}(\Omega)$. This means that if $\mathcal{H}_1, \mathcal{H}_2 \in \mathrm{Hyp}(\Omega)$, then we can define two partial orders:

$$\mathcal{H}_1 \leqslant^+ \mathcal{H}_2, \quad \mathcal{H}_1 \leqslant^- \mathcal{H}_2$$

and, moreover, four operations:

$$\mathcal{H}_1 \overset{+}{\sqcap} \mathcal{H}_2, \quad \mathcal{H}_1 \overset{+}{\sqcup} \mathcal{H}_2, \quad \mathcal{H}_1 \overset{-}{\sqcap} \mathcal{H}_2, \quad \mathcal{H}_1 \overset{-}{\sqcup} \mathcal{H}_2,$$

such that:

$$\left(\mathrm{Hyp}(\Omega), \leqslant^+, \overset{+}{\sqcup}, \overset{+}{\sqcap}\right) \quad \text{and} \quad \left(\mathrm{Hyp}(\Omega), \leqslant^-, \overset{-}{\sqcup}, \overset{-}{\sqcap}\right)$$

are distributive lattices. We also define some operations on $\mathrm{Inc}(\Omega)$, $\mathrm{Dec}(\Omega)$ and $\mathrm{Hyp}(\Omega)$, for instance the transversal of a hypergraph, and study the behaviour of these operations with respect to the partial orders and the lattice structure. These structures turn out to be fundamental in the rest of this thesis.

The aims of Chapter 2 are twofold. First, we introduce several hypergraphs associated with a graph G; namely:

- $\mathscr{D}(G)$, the domination hypergraph, whose elements are the minimal dominating sets of the graph G;
- $\mathscr{D}_{ind}(G)$, the independence-domination hypergraph, composed by the maximal independent sets of the graph G;
- $\mathscr{N}[G]$, the hypergraph of minimal closed neighborhoods of the graph G;
- $\mathscr{C}(G)$, the hypergraph whose elements are the minimal vertex covers of the graph G;
- $\mathscr{P}(G)$, the hypergraph formed by the maximal pendant sets of vertices of G.

Then we establish several relationships among them, the most important ones being that $\mathscr{D}(G)$ is the tranversal hypergraph of $\mathscr{N}[G]$ and that $\mathscr{D}_{ind}(G)$ is the complementary of the tranversal of $E(G)^c$, where $E(G)$ is the edge set of the graph G considered as a hypergraph. After that, we focus on the study domination hypergraphs of a graph. We compute the domination hypergraph of all graphs, modulo isomorphism, up to order 4 (in Appendix B we tabulate the domination hypergraphs of all graphs of order 5, up to isomorphism.)

The second objective of Chapter 2 is to investigate when a given hypergraph $\mathscr{H}$ is a domination hypergraph; that is, a hypergraph of the form $\mathscr{D}(G)$ for some graph G. When the hypergraph $\mathscr{H}$ is a domination hypergraph, we shed some light on the set of graph realizations of the hypergraph $\mathscr{H}$; that is, the graphs G such that $\mathscr{D}(G)$ coincides with the hypergraph $\mathscr{H}$. We give a necessary condition for a graph to be a domination hypergraph and we set some computational techniques. Finally, we end the chapter by finding all domination hypergraphs in the following cases: order up to 4; sizes 1 or 2; and uniform hypergraphs.

Chapter 3 is the main chapter of this thesis. As we will see in Chapter 2, not every hypergraph is the domination hypergraph of a graph. In Chapter 3 we discuss the problem of how to approximate a given hypergraph by domination hypergraphs. We start by setting a general framework for the approximation problem of hypergraphs by hypergraphs of a given family of hypergraphs. So if Σ is a family of hypergraphs on a finite set Ω and $\mathscr{H}$ is a hypergraph on Ω, we introduce four sets of approximations of $\mathscr{H}$ by elements of Σ, which we call *completions*. Namely, we define the sets $\Sigma^{*_1}_{*_2}(\mathscr{H})$, where $*_1 \in \{+,-\}$ and $*_2 \in \{u,\ell\}$, depending on whether we use the partial order $\leqslant^+$ or $\leqslant^-$ and we approximate the hypergraph from above, u, or from below, ℓ. We then set some sufficient conditions for the existence of completions; introduce the sets of minimal or maximal completions of a hypergraph, depending on the order and side considered; and study the concept of *decomposition*, that corresponds to the idea of *recovering* a hypergraph from its optimal completions. We prove that the existence of decompositions in a general framework is equivalent to the satisfiability of a technical property of hypergraphs which we call the *avoidance property* with respect to a given family of hypergraphs. Once we have presented the general theory, we introduce the corresponding concepts in the domination context. Avoidance properties with respect to the family of domination hypergraphs turn out to be an essential ingredient of the theory. We also define a new parameter related to a hypergraph, the *domination index* of a hypergraph $\mathscr{H}$, that tells us the minimum number of *special* hypergraphs in a decomposition of the hypergraph $\mathscr{H}$. We study its properties and compute it in some cases in the domination context.

Finally, in Chapter 4 we apply the results obtained in the preceding chapters to compute the upper minimal domination completions and the decomposition indices of a hypergraph. We do this only for the partial order $\leqslant^+$, but similar results can be obtained in other cases. We begin by presenting two computational devices that allow us to reduce the computation of the upper minimal domination completions to the case where the intersection set of a hypergraph is empty (that is, there is not any common element to all hyperedges) and where the hyperedges have cardinality at least two.

Next we compute the upper minimal domination completions and the decomposition index of all hypergraphs with order at most four or size equal to one. Some of these computations have been done using a dedicated SAGE program developed to this purpose. In Appendix A we can find the code of this program. And in Appendix B we list the results obtained with this program to calculate all the domination hypergraphs or all graphs of order 5 up to isomorphism. We also compute and list *all* the graphs of order 5 that share the same domination hypergraph; that is, we list the different graph realizations of the domination hypergraph when there is more than one.

CHAPTER 1

POSETS OF HYPERGRAPHS

In this chapter we present the definitions and basic facts concerning hypergraphs. First we introduce the concepts of monotone families of subsets of a finite set and then we relate these families with hypergraphs. From this relationship we define a structure of partially ordered set (*poset*) in the set of hypergraphs and we show that this poset is indeed a distributive lattice. This structure turns out to be fundamental in the rest of the thesis.

Throughout this thesis Ω is a finite set. The power set of Ω is denoted $\mathscr{P}(\Omega)$.

We rely on [2, 5, 16, 22, 37, 39] for general concepts on ordered sets and lattices and on [4, 9, 17, 43] for general background on hypergraphs.

1.1 Families of subsets of Ω

A *monotone increasing* family of subsets Γ of Ω is a collection of subsets of Ω such that any superset of a set in Γ must be in Γ; that is, if $A \in \Gamma$ and $A \subseteq A' \subseteq \Omega$, then $A' \in \Gamma$. We denote by $\mathrm{Inc}(\Omega)$ the collection of monotone increasing families of subsets of Ω. We have $\mathrm{Inc}(\Omega) \subseteq \mathscr{P}(\mathscr{P}(\Omega))$.

A *monotone decreasing* family of subsets Γ of Ω is a collection of subsets of Ω such that any subset of a set in Γ must be in Γ; that is, if $A \in \Gamma$ and $A' \subseteq A \subseteq \Omega$, then $A' \in \Gamma$. We denote by $\mathrm{Dec}(\Omega)$ the collection of monotone decreasing families of subsets of Ω. We have $\mathrm{Dec}(\Omega) \subseteq \mathscr{P}(\mathscr{P}(\Omega))$.

The relationship between both kind of families is stated in Lemma 1.2 and is based in the following two operations on families of subsets of Ω, related to taking complements. For $\Upsilon \subseteq \mathscr{P}(\Omega)$, let $\overline{\Upsilon}$ and Υ^c be:

$$\overline{\Upsilon} = \{B \subseteq \Omega : B \notin \Upsilon\} \subseteq \mathscr{P}(\Omega),$$
$$\Upsilon^c = \{B \subseteq \Omega : B^c \in \Upsilon\} \subseteq \mathscr{P}(\Omega),$$

where $B^c = \Omega \setminus B$. We refer to $\overline{\Upsilon}$ as the *complement* of Υ and to Υ^c as its *complementary* family. The following lemmas states some immediate properties of these two operations.

Lemma 1.1 *If* $\Upsilon_1, \Upsilon_2 \subseteq \mathscr{P}(\Omega)$, *then:*

$$\Upsilon_1 \subseteq \Upsilon_2 \iff \overline{\Upsilon_2} \subseteq \overline{\Upsilon_1} \iff {\Upsilon_1}^c \subseteq {\Upsilon_2}^c.$$

Proof. These equivalences follow from the definitions. □

Lemma 1.2 *If* $\Upsilon \subseteq \mathscr{P}(\Omega)$, *then the following statements hold.*

1) $(\Upsilon^c)^c = \Upsilon$, $\overline{\overline{\Upsilon}} = \Upsilon$, *and* $\overline{(\Upsilon^c)} = (\overline{\Upsilon})^c$.

2) $\Upsilon \in \mathrm{Inc}(\Omega) \iff \Upsilon^c \in \mathrm{Dec}(\Omega) \iff \overline{\Upsilon} \in \mathrm{Dec}(\Omega)$.

3) $\Upsilon \in \mathrm{Dec}(\Omega) \iff \Upsilon^c \in \mathrm{Inc}(\Omega) \iff \overline{\Upsilon} \in \mathrm{Inc}(\Omega)$.

Proof.

1) The first two equalities follow easily from the definitions. Let us prove the third one. We have:

$$B \in \overline{\Upsilon^c} \iff B \notin \Upsilon^c \iff B^c \notin \Upsilon \iff B^c \in \overline{\Upsilon} \iff B \in (\overline{\Upsilon})^c.$$

2) Assume that Υ is an increasing family. Let $A \in \Upsilon^c$ and let $B \subseteq A$. Let us prove that $B \in \Upsilon^c$. We have $A^c \in \Upsilon$ and $A^c \subseteq B^c$. Hence $B^c \in \Upsilon$, by hypothesis. So we have $B \in \Upsilon^c$. Thus we have proved that Υ^c is a decreasing family.

   Assume now that Υ^c is a decreasing family. Let $A \in \overline{\Upsilon}$ and let $B \subseteq A$. We have to show that $B \in \overline{\Upsilon}$. On the contrary, assume that $B \notin \overline{\Upsilon}$. Then $B \in \Upsilon$ and $A \notin \Upsilon$. Hence $B^c \in \Upsilon^c$ and $A^c \notin \Upsilon^c$. As Υ^c is a decreasing family, we cannot have $A^c \subseteq B^c$, and this is a contradiction.

   Assume finally that $\overline{\Upsilon}$ is a decreasing family. Let $A \in \Upsilon$ and let $A \subseteq B$. We have to show that $B \in \Upsilon$. Assume that $B \notin \Upsilon$. Then $B \in \overline{\Upsilon}$ and, as $\overline{\Upsilon}$ is decreasing, we deduce that $A \in \overline{\Upsilon}$, which contradicts the fact that $A \in \Upsilon$.

3) Using the equivalences already proved above, we have:

$$\Upsilon = (\Upsilon^c)^c \in \mathrm{Dec}(\Omega) \iff \Upsilon^c \in \mathrm{Inc}(\Omega),$$
$$\Upsilon = \overline{\overline{\Upsilon}} \in \mathrm{Dec}(\Omega) \iff \overline{\Upsilon} \in \mathrm{Inc}(\Omega).$$

□

1.2 Hypergraphs

1.2.1 Definitions

Since there are different definitions of hypergraphs in the literature, in this section we introduce the one that will be used throughout this thesis.

A *hypergraph* $\mathscr{H}$ *on* Ω is a collection $\mathscr{H}$ of subsets of Ω, none of which is a proper subset of another; that is, if $A, A' \in \mathscr{H}$ and $A \subseteq A'$ then $A = A'$. A hypergraph is also called a *clutter*, a *Sperner system*, an *antichain* or a *simple hypergraph*.

The set of all hypergraphs on Ω is denoted $\mathrm{Hyp}(\Omega)$.

Let $\mathscr{H}$ be a hypergraph on Ω. The elements of Ω are called the *vertices* of $\mathscr{H}$ and the elements of $\mathscr{H}$ are called the *hyperedges* of $\mathscr{H}$. The *order* of $\mathscr{H}$ is the cardinality of Ω and the *size* of $\mathscr{H}$ is the cardinality of $\mathscr{H}$.

Remark 1.3 Observe that a graph can be considered as a special type of hypergraph. Namely, graphs are those hypergraphs whose hyperedges have size two. This fact will be recalled in Section 2.1.

Remark 1.4 Note that $\mathscr{H}$ being a hypergraph on Ω does not imply that every element of Ω belongs to some set in $\mathscr{H}$. For example, $\mathscr{H} = \{\{1,2\},\{1,3\}\}$ is a hypergraph on $\Omega = \{1,2,\dots,n\}$, if $n \geqslant 3$.

Remark 1.5 Notice that both $\emptyset$ and $\{\emptyset\}$ are hypergraphs on every finite set Ω. Observe that if $\mathscr{H}$ is a hypergraph and $\emptyset \in \mathscr{H}$, then $\mathscr{H} = \{\emptyset\}$.

If $\mathscr{H}$ is a hypergrah, we define its *ground set* as:

$$\mathrm{Gr}(\mathscr{H}) = \bigcup_{A \in \mathscr{H}} A.$$

Thus a hypergraph $\mathscr{H}$ with ground set Ω is a hypergraph *on* Ω such that $\mathrm{Gr}(\mathscr{H}) = \Omega$. The set of all hypergraphs with ground set Ω is denoted $\mathrm{Hyp}_0(\Omega)$. Observe that a hypergraph on Ω is a hypergraph with ground set Ω', for some subset Ω' of Ω; namely, if $\mathscr{H} \in \mathrm{Hyp}(\Omega)$, then $\mathscr{H} \in \mathrm{Hyp}_0(\mathrm{Gr}(\mathscr{H}))$. Hence, we have:

$$\mathrm{Hyp}(\Omega) = \bigcup_{\Omega' \subseteq \Omega} \mathrm{Hyp}_0(\Omega').$$

We define the *intersection set* of a hypergraph $\mathscr{H}$ as the set of elements of Ω common to all elements of $\mathscr{H}$; that is:

$$\mathrm{Int}(\mathscr{H}) = \bigcap_{A \in \mathscr{H}} A.$$

Remark 1.6 If $\mathscr{H} = \emptyset$ or $\mathscr{H} = \{\emptyset\}$, then $\mathrm{Gr}(\mathscr{H}) = \mathrm{Int}(\mathscr{H}) = \emptyset$.

We define the *rank*, respectively the *corank*, of $\mathscr{H}$ as the maximum, respectively the minimum, of all the cardinalities of the hyperedges of $\mathscr{H}$; that is:

$$\begin{aligned} \mathrm{rank}(\mathscr{H}) &= \max\{|A| : A \in \mathscr{H}\}, \\ \mathrm{corank}(\mathscr{H}) &= \min\{|A| : A \in \mathscr{H}\}. \end{aligned}$$

Remark 1.7 If $\mathscr{H} = \emptyset$, then $\mathrm{rank}(\mathscr{H})$ and $\mathrm{corank}(\mathscr{H})$ are not defined. If $\mathscr{H} = \{\emptyset\}$, then $\mathrm{rank}(\mathscr{H}) = \mathrm{corank}(\mathscr{H}) = 0$.

A hypergraph $\mathscr{H}$ is called *uniform* if $\mathrm{rank}(\mathscr{H}) = \mathrm{corank}(\mathscr{H})$; that is, if there is an integer $k \geqslant 1$, such that $|A| = k$, for all $A \in \mathscr{H}$.

We define the hypergraph $\mathscr{U}_{k,\Omega}$ as the hypergraph with ground set Ω:

$$\mathscr{U}_{k,\Omega} = \{A \subseteq \Omega : |A| = k\}.$$

We conclude this section with the definition of hypergraph isomorphism. The definition is similar to that of graph isomorphism. We mention that most of the properties on hypergraphs and graphs we will study in this work are invariant under isomorphism; that is, they depends only on the isomorphism class of the hypergraph or graph considered.

A hypergraph $\mathscr{H}' = \{A'_1, \ldots, A'_s\}$ on a finite set Ω' is said to be isomorphic to a hypergraph $\mathscr{H} = \{A_1, \ldots, A_s\}$ on a finite set Ω if there exists a bijection $\phi : \Omega' \to \Omega$ and a permutation σ of $\{1, \ldots, s\}$ such that $\phi(A'_i) = A_{\sigma(i)}$ for all i.

Remark 1.8 The non-isomorphic hypergraphs with ground set Ω, with $|\Omega| \leq 4$, are listed on Tables 2.6, 2.7 and 2.8. We mention at this point that the number of distinct hypergraphs on a set of cardinality n is called the *n-th Dedekind number* and coincides with the number of monotone Boolean functions in n variables (see [30]).

1.2.2 Hypergraphs and monotone families of subsets

A hypergraph $\mathscr{H} \in \mathrm{Hyp}(\Omega)$ on Ω determines both a monotone increasing family $\mathscr{H}^+ \in \mathrm{Inc}(\Omega)$ and a monotone decreasing family $\mathscr{H}^- \in \mathrm{Dec}(\Omega)$ of subsets of Ω as follows:

$$\begin{aligned} \mathscr{H}^+ &= \{A \subseteq \Omega : A_0 \subseteq A \text{ for some } A_0 \in \mathscr{H}\}, \\ \mathscr{H}^- &= \{A \subseteq \Omega : A \subseteq A_0 \text{ for some } A_0 \in \mathscr{H}\}. \end{aligned}$$

Remark 1.9 We observe that the definitions of $\mathscr{H}^+$ and $\mathscr{H}^-$ depend on the set Ω.

Conversely, if Γ is a monotone increasing family of subsets of Ω, then the collection $\min(\Gamma)$ of its inclusion-minimal elements is a hypergraph on Ω. Similarly, if now Γ is a monotone decreasing family of subsets of Ω, then the collection $\max(\Gamma)$ of its inclusion-maximal elements is also a hypergraph on Ω. We summarize in the following proposition the relationships among all these operations.

Proposition 1.10 *Let Ω be a finite set.*

1) *The mappings:*

$$\begin{aligned} \mathrm{Hyp}(\Omega) &\to \mathrm{Inc}(\Omega) \\ \mathcal{H} &\mapsto \mathcal{H}^+ \end{aligned} \qquad\qquad \begin{aligned} \mathrm{Inc}(\Omega) &\to \mathrm{Hyp}(\Omega) \\ \Gamma &\mapsto \min(\Gamma) \end{aligned}$$

are bijections and inverse of each other; that is, if $\mathcal{H}$ is a hypergraph on Ω, then $\min(\mathcal{H}^+) = \mathcal{H}$*; and if Γ is monotone increasing family of subsets of Ω, then* $\min(\Gamma)$ *is a hypergraph on Ω and* $(\min(\Gamma))^+ = \Gamma$.

2) *The mappings:*

$$\begin{aligned} \mathrm{Hyp}(\Omega) &\to \mathrm{Dec}(\Omega) \\ \mathcal{H} &\mapsto \mathcal{H}^- \end{aligned} \qquad\qquad \begin{aligned} \mathrm{Dec}(\Omega) &\to \mathrm{Hyp}(\Omega) \\ \Gamma &\mapsto \max(\Gamma) \end{aligned}$$

are bijections and inverse of each other; that is, if $\mathcal{H}$ is a hypergraph on Ω, then $\max(\mathcal{H}^-) = \mathcal{H}$*; and if Γ is monotone decreasing family of subsets of Ω, then* $\max(\Gamma)$ *is a hypergraph on Ω and* $(\max(\Gamma))^- = \Gamma$.

So a monotone increasing (decreasing) family of subsets Γ is uniquely determined by the hypergraph $\min(\Gamma)$ (respectively, $\max(\Gamma)$), while a hypergraph $\mathcal{H}$ is uniquely determined by either of the families $\mathcal{H}^+$ and $\mathcal{H}^-$.

Remark 1.11 Note that $\emptyset \subseteq \mathcal{P}(\Omega)$ is a hypergraph as well as an increasing and a decreasing family. We have: $\emptyset^+ = \emptyset^- = \emptyset$, $\max(\emptyset) = \min(\emptyset) = \emptyset$. However we observe that $\mathcal{P}(\Omega)$ is not a hypergraph (unless $\Omega = \emptyset$), but it is an increasing and a decreasing family. In this case, we have $\min(\mathcal{P}(\Omega)) = \{\emptyset\}$ and $\max(\mathcal{P}(\Omega)) = \{\Omega\}$. Analogously, we have that $\{\emptyset\}^+ = \mathcal{P}(\Omega)$ and $\{\emptyset\}^- = \{\emptyset\}$. For the hypergraph $\{\Omega\}$ we have $\{\Omega\}^+ = \{\Omega\}$ and $\{\Omega\}^- = \mathcal{P}(\Omega)$. We summarize these properties in Table 1.1.

	$\emptyset$	$\{\emptyset\}$	$\{\Omega\}$
$(\cdot)^+$	$\emptyset$	$\mathcal{P}(\Omega)$	$\{\Omega\}$
$(\cdot)^-$	$\emptyset$	$\{\emptyset\}$	$\mathcal{P}(\Omega)$

Table 1.1 The families $\mathcal{H}^+$ and $\mathcal{H}^-$ for $\mathcal{H} = \emptyset, \{\emptyset\}, \{\Omega\}$.

1.3 Posets of hypergraphs

1.3.1 The partial orders $\leqslant^+$ and $\leqslant^-$

For a finite set Ω, the sets $\mathrm{Inc}(\Omega)$ and $\mathrm{Dec}(\Omega)$ inherit a structure of poset, $(\mathrm{Inc}(\Omega),\subseteq)$ and $(\mathrm{Dec}(\Omega),\subseteq)$, from $(\mathscr{P}(\mathscr{P}(\Omega)),\subseteq)$. Therefore, we can define two partial orders on $\mathrm{Hyp}(\Omega)$ by transport of structure using the bijections of Proposition 1.10 as follows.

If $\mathscr{H}_1$ and $\mathscr{H}_2$ are two hypergraphs on Ω, then we define:

$$\mathscr{H}_1 \leqslant^+ \mathscr{H}_2 \text{ if and only if } \mathscr{H}_1^+ \subseteq \mathscr{H}_2^+;$$

and we define:

$$\mathscr{H}_1 \leqslant^- \mathscr{H}_2 \text{ if and only if } \mathscr{H}_1^- \subseteq \mathscr{H}_2^-.$$

Proposition 1.12 *The binary relations $\leqslant^+$ and $\leqslant^-$ are partial orders on* $\mathrm{Hyp}(\Omega)$ *and on* $\mathrm{Hyp}_0(\Omega)$.

Proof. This is a direct consequence of the fact that both $\mathrm{Inc}(\Omega)$ and $\mathrm{Dec}(\Omega)$ are posets with respect to the order $\subseteq$ and that the mappings in Proposition 1.10 are bijections. □

Therefore, we can restate Proposition 1.10 by saying that the mappings:

$$(\mathrm{Hyp}(\Omega),\leqslant^+) \to (\mathrm{Inc}(\Omega),\subseteq), \qquad \mathscr{H} \mapsto \mathscr{H}^+$$
$$(\mathrm{Hyp}(\Omega),\leqslant^-) \to (\mathrm{Dec}(\Omega),\subseteq), \qquad \mathscr{H} \mapsto \mathscr{H}^-$$

are isomorphisms of posets.

Remark 1.13 It is clear that if $\mathscr{H}_1 \subseteq \mathscr{H}_2$, then $\mathscr{H}_1^+ \subseteq \mathscr{H}_2^+$ and $\mathscr{H}_1^- \subseteq \mathscr{H}_2^-$; that is, $\mathscr{H}_1 \leqslant^+ \mathscr{H}_2$ and $\mathscr{H}_1 \leqslant^- \mathscr{H}_2$. However, the converses are not true, as it is shown in the following example.

Example 1.14 The hypergraphs $\mathscr{H}_1 = \{\{1,2\},\{2,3\}\}$ and $\mathscr{H}_2 = \{\{1\},\{2,3\}\}$ on $\Omega = \{1,2,3\}$ satisfy $\mathscr{H}_1^+ \subseteq \mathscr{H}_2^+$ and $\mathscr{H}_2^- \subseteq \mathscr{H}_1^-$; however $\mathscr{H}_1 \not\subseteq \mathscr{H}_2$ and $\mathscr{H}_2 \not\subseteq \mathscr{H}_1$.

The following lemma rephrases the relations $\leqslant^+$ and $\leqslant^-$ solely in terms of $\mathscr{H}_1$, $\mathscr{H}_2$ and their elements; it will be used repeatedly throughout hereafter.

Lemma 1.15 *Let $\mathscr{H}_1,\mathscr{H}_2$ be two hypergraphs on Ω.*

1) The following statements are equivalent:

a) $\mathscr{H}_1 \leqslant^+ \mathscr{H}_2$.

b) $\mathscr{H}_1 \subseteq \mathscr{H}_2^+$.

   c) For all $A_1 \in \mathcal{H}_1$, there exists $A_2 \in \mathcal{H}_2$ such that $A_2 \subseteq A_1$.

2) The following statements are equivalent:

   a) $\mathcal{H}_1 \leqslant^- \mathcal{H}_2$.

   b) $\mathcal{H}_1 \subseteq \mathcal{H}_2^-$.

   c) For all $A_1 \in \mathcal{H}_1$, there exists $A_2 \in \mathcal{H}_2$ such that $A_1 \subseteq A_2$.

Proof.

1) Assume that $\mathcal{H}_1 \leqslant^+ \mathcal{H}_2$. Then $\mathcal{H}_1 \subseteq \mathcal{H}_1^+ \subseteq \mathcal{H}_2^+$. Next, if $\mathcal{H}_1 \subseteq \mathcal{H}_2^+$ and if $A_1 \in \mathcal{H}_1$, then $A_1 \in \mathcal{H}_2^+$ and, by definition, there exists $A_2 \in \mathcal{H}_2$ such that $A_2 \subseteq A_1$. Assume now that for all $A_1 \in \mathcal{H}_1$, there exists $A_2 \in \mathcal{H}_2$ such that $A_2 \subseteq A_1$. Let us show that $\mathcal{H}_1 \leqslant^+ \mathcal{H}_2$; that is, $\mathcal{H}_1^+ \subseteq \mathcal{H}_2^+$. Let $B \in \mathcal{H}_1^+$. Then there exists $A_1 \in \mathcal{H}_1$ such that $A_1 \subseteq B$. By hypothesis, there exists $A_2 \in \mathcal{H}_2$ such that $A_2 \subseteq A_1$, and hence, $A_2 \subseteq B$. Consequently, $B \in \mathcal{H}_2^+$.

2) Assume that $\mathcal{H}_1 \leqslant^- \mathcal{H}_2$. Then $\mathcal{H}_1 \subseteq \mathcal{H}_1^- \subseteq \mathcal{H}_2^-$. Next if $\mathcal{H}_1 \subseteq \mathcal{H}_2^-$ and if $A_1 \in \mathcal{H}_1$, then $A_1 \in \mathcal{H}_2^-$ and, by definition, there exists $A_2 \in \mathcal{H}_2$ such that $A_1 \subseteq A_2$. Assume now that for all $A_1 \in \mathcal{H}_1$, there exists $A_2 \in \mathcal{H}_2$ such that $A_1 \subseteq A_2$. Let us show that $\mathcal{H}_1 \leqslant^- \mathcal{H}_2$; that is, $\mathcal{H}_1^- \subseteq \mathcal{H}_2^-$. Let $B \in \mathcal{H}_1^-$. Then there exists $A_1 \in \mathcal{H}_1$ such that $B \subseteq A_1$. By hypothesis, there exists $A_2 \in \mathcal{H}_2$ such that $A_1 \subseteq A_2$, and hence, $B \subseteq A_2$. Consequently, $B \in \mathcal{H}_2^-$. □

Lemma 1.16 *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are hypergraphs on Ω, then:*

1) if $\mathcal{H}_1 \leqslant^+ \mathcal{H}_2$, then $\mathrm{Int}(\mathcal{H}_2) \subseteq \mathrm{Int}(\mathcal{H}_1)$;

2) if $\mathcal{H}_1 \leqslant^- \mathcal{H}_2$, then $\mathrm{Gr}(\mathcal{H}_1) \subseteq \mathrm{Gr}(\mathcal{H}_2)$.

Proof. Let us prove the first statement. If $x \in \mathrm{Int}(\mathcal{H}_2)$ and $A_1 \in \mathcal{H}_1$, then there exists $A_2 \in \mathcal{H}_2$ such that $x \in A_2 \subseteq A_1$. So $x \in \mathrm{Int}(\mathcal{H}_1)$. Let us see now the second statement. If $x \in \mathrm{Gr}(\mathcal{H}_1)$, then there exists $A_1 \in \mathcal{H}_1$ such that $x \in A_1$. But, by hypothesis, there exists $A_2 \in \mathcal{H}_2$ such that $A_1 \subseteq A_2$. Thus $x \in A_2$ and then $x \in \mathrm{Gr}(\mathcal{H}_2)$. □

Remark 1.17 Besides those found in Lemma 1.16, there are no more relationships, as the following examples show. All the hypergraphs considered are on the set $\Omega = \{1,2,3,4\}$.

- If $\mathcal{H}_1 = \{\{1\},\{2,3\}\}$, $\mathcal{H}_2 = \{\{1\},\{2\}\}$, then $\mathcal{H}_1 \leqslant^+ \mathcal{H}_2$, but:

$$\mathrm{Gr}(\mathcal{H}_1) \nsubseteq \mathrm{Gr}(\mathcal{H}_2).$$

- If $\mathcal{H}_1 = \{\{1\},\{2,3\}\}$, $\mathcal{H}_2 = \{\{1\},\{2\},\{4\}\}$, then $\mathcal{H}_1 \leqslant^+ \mathcal{H}_2$, but:

$$\mathrm{Gr}(\mathcal{H}_2) \nsubseteq \mathrm{Gr}(\mathcal{H}_1).$$

- If $\mathscr{H}_1 = \{\{1,2\},\{1,3\}\}$, $\mathscr{H}_2 = \{\{2\},\{1,3\}\}$, then $\mathscr{H}_1 \leqslant^+ \mathscr{H}_2$, but:
$$\mathrm{Int}(\mathscr{H}_1) \not\subseteq \mathrm{Int}(\mathscr{H}_2).$$
- If $\mathscr{H}_1 = \{\{1\},\{2,3\}\}$, $\mathscr{H}_2 = \{\{1,4\},\{2,3\}\}$, then $\mathscr{H}_1 \leqslant^- \mathscr{H}_2$, but:
$$\mathrm{Gr}(\mathscr{H}_2) \not\subseteq \mathrm{Gr}(\mathscr{H}_1).$$
- If $\mathscr{H}_1 = \{\{1,2\},\{1,3\}\}$, $\mathscr{H}_2 = \{\{1,2\},\{1,3\},\{2,3\}\}$, then $\mathscr{H}_1 \leqslant^- \mathscr{H}_2$, but:
$$\mathrm{Int}(\mathscr{H}_1) \not\subseteq \mathrm{Int}(\mathscr{H}_2).$$
- If $\mathscr{H}_1 = \{\{1\},\{2,3\}\}$, $\mathscr{H}_2 = \{\{1,4\},\{2,3,4\}\}$, then $\mathscr{H}_1 \leqslant^- \mathscr{H}_2$, but:
$$\mathrm{Int}(\mathscr{H}_2) \not\subseteq \mathrm{Int}(\mathscr{H}_1).$$

1.3.2 The lattice structures on $\mathrm{Hyp}(\Omega)$

If $\Gamma_1, \Gamma_2 \in \mathrm{Inc}(\Omega)$, then it is easily seen that $\Gamma_1 \cup \Gamma_2 \in \mathrm{Inc}(\Omega)$ and $\Gamma_1 \cap \Gamma_2 \in \mathrm{Inc}(\Omega)$. And similarly for $\mathrm{Dec}(\Omega)$. Therefore $(\mathrm{Inc}(\Omega), \subseteq, \cup, \cap)$ and $(\mathrm{Dec}(\Omega), \subseteq, \cup, \cap)$ are distributive lattices. We next investigate the lattice operations on the posets $(\mathrm{Hyp}(\Omega), \leqslant^+)$ and $(\mathrm{Hyp}(\Omega), \leqslant^-)$ inherited from the lattices above, respectively.

Let $\mathscr{H}_1, \mathscr{H}_2 \in \mathrm{Hyp}(\Omega)$. By transport of structure by means of the bijections of Proposition 1.10, we define:

$$\mathscr{H}_1 \overset{+}{\sqcap} \mathscr{H}_2 = \min(\mathscr{H}_1^+ \cap \mathscr{H}_2^+),$$
$$\mathscr{H}_1 \overset{+}{\sqcup} \mathscr{H}_2 = \min(\mathscr{H}_1^+ \cup \mathscr{H}_2^+),$$
$$\mathscr{H}_1 \overset{-}{\sqcap} \mathscr{H}_2 = \max(\mathscr{H}_1^- \cap \mathscr{H}_2^-),$$
$$\mathscr{H}_1 \overset{-}{\sqcup} \mathscr{H}_2 = \max(\mathscr{H}_1^- \cup \mathscr{H}_2^-).$$

Proposition 1.18 *Let Ω be a finite set.*

1) The poset $(\mathrm{Hyp}(\Omega), \leqslant^+)$ *is a distributive lattice with the operations:*
$$\inf(\mathscr{H}_1, \mathscr{H}_2) = \mathscr{H}_1 \overset{+}{\sqcap} \mathscr{H}_2, \qquad \sup(\mathscr{H}_1, \mathscr{H}_2) = \mathscr{H}_1 \overset{+}{\sqcup} \mathscr{H}_2.$$

2) The poset $(\mathrm{Hyp}(\Omega), \leqslant^-)$ *is a distributive lattice with the operations:*
$$\inf(\mathscr{H}_1, \mathscr{H}_2) = \mathscr{H}_1 \overset{-}{\sqcap} \mathscr{H}_2, \qquad \sup(\mathscr{H}_1, \mathscr{H}_2) = \mathscr{H}_1 \overset{-}{\sqcup} \mathscr{H}_2.$$

Remark 1.19 Let us write down explicitly what the first statement of the previous proposition means. If $\mathscr{H}_1, \mathscr{H}_2 \in \mathrm{Hyp}(\Omega)$, then:

1) $\mathcal{H}_1 \overset{+}{\sqcap} \mathcal{H}_2$ is the unique hypergraph $\mathcal{H}_0$ on Ω such that:

   a) $\mathcal{H}_0 \leqslant^+ \mathcal{H}_i$ for $i = 1,2$, and
   b) if $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ satisfies $\mathcal{H}' \leqslant^+ \mathcal{H}_i$ for $i = 1,2$, then $\mathcal{H}' \leqslant^+ \mathcal{H}_0$.

2) $\mathcal{H}_1 \overset{+}{\sqcup} \mathcal{H}_2$ is the unique hypergraph $\mathcal{H}_0$ on Ω such that:

   a) $\mathcal{H}_i \leqslant^+ \mathcal{H}_0$ for $i = 1,2$, and
   b) if $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ satisfies $\mathcal{H}_i \leqslant^+ \mathcal{H}'$ for $i = 1,2$, then $\mathcal{H}_0 \leqslant^+ \mathcal{H}'$.

Analogously, that the poset $(\mathrm{Hyp}(\Omega), \leqslant^-)$ is a lattice with respect to the operations $\overset{-}{\sqcup}$ and $\overset{-}{\sqcap}$ means the following. If $\mathcal{H}_1, \mathcal{H}_2 \in \mathrm{Hyp}(\Omega)$, then:

1) $\mathcal{H}_1 \overset{-}{\sqcap} \mathcal{H}_2$ is the unique hypergraph $\mathcal{H}_0$ on Ω such that:

   a) $\mathcal{H}_0 \leqslant^- \mathcal{H}_i$ for $i = 1,2$, and
   b) if $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ satisfies $\mathcal{H}' \leqslant^- \mathcal{H}_i$ for $i = 1,2$, then $\mathcal{H}' \leqslant^- \mathcal{H}_0$.

2) $\mathcal{H}_1 \overset{-}{\sqcup} \mathcal{H}_2$ is the unique hypergraph $\mathcal{H}_0$ on Ω such that:

   a) $\mathcal{H}_i \leqslant^- \mathcal{H}_0$ for $i = 1,2$, and
   b) if $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ satisfies $\mathcal{H}_i \leqslant^- \mathcal{H}'$ for $i = 1,2$, then $\mathcal{H}_0 \leqslant^- \mathcal{H}'$.

The following lemma provides alternative descriptions of these operations that will be useful in calculations.

Lemma 1.20 *If $\mathcal{H}_1, \mathcal{H}_2 \in \mathrm{Hyp}(\Omega)$, then the following statements hold.*

1) $\mathcal{H}_1 \overset{+}{\sqcap} \mathcal{H}_2 = \min(\{A_1 \cup A_2 : A_i \in \mathcal{H}_i\})$.

2) $\mathcal{H}_1 \overset{+}{\sqcup} \mathcal{H}_2 = \min(\mathcal{H}_1 \cup \mathcal{H}_2)$.

3) $\mathcal{H}_1 \overset{-}{\sqcap} \mathcal{H}_2 = \max(\{A_1 \cap A_2 : A_i \in \mathcal{H}_i\})$.

4) $\mathcal{H}_1 \overset{-}{\sqcup} \mathcal{H}_2 = \max(\mathcal{H}_1 \cup \mathcal{H}_2)$.

Proof.

1) Let $\mathcal{H} = \mathcal{H}_1 \overset{+}{\sqcap} \mathcal{H}_2$ and $\mathcal{H}_0 = \min(\{A_1 \cup A_2 : A_i \in \mathcal{H}_i\})$. As $\leqslant^+$ is a partial order, to proof that $\mathcal{H} = \mathcal{H}_0$, it is enough to prove that $\mathcal{H} \leqslant^+ \mathcal{H}_0$ and that $\mathcal{H}_0 \leqslant^+ \mathcal{H}$. If $A \in \mathcal{H}$, then $A \in \mathcal{H}_i^+$, for $i = 1,2$; so for all i, there exists $A_i \in \mathcal{H}_i$ such that $A_i \subseteq A$. This implies that $A_1 \cup A_2 \subseteq A$, and therefore there exists $B \in \mathcal{H}_0$ such that $B \subseteq A$. By Lemma 1.15, we have $\mathcal{H} \leqslant^+ \mathcal{H}_0$. Now let $B \in \mathcal{H}_0$; that is, $B = A_1 \cup A_2$, with $A_i \in \mathcal{H}_i$, for $i = 1,2$. But then $A_i \subseteq B$, for $i = 1,2$, so we have $B \in \mathcal{H}_i^+$, for $i = 1,2$. Hence $B \in \mathcal{H}_1^+ \cap \mathcal{H}_2^+$. Then by definition of $\mathcal{H}$, there exists $B' \in \mathcal{H}$ such that $B' \subseteq B$. By Lemma 1.15, we have $\mathcal{H}_0 \leqslant^+ \mathcal{H}$.

2) We have:

$$\mathcal{H}_1 \overset{+}{\sqcup} \mathcal{H}_2 = \min(\mathcal{H}_1^+ \cup \mathcal{H}_2^+) = \min(\mathcal{H}_1 \cup \mathcal{H}_2)$$

because $\mathcal{H}_1^+ \cup \mathcal{H}_2^+ = (\mathcal{H}_1 \cup \mathcal{H}_2)^+$ and if Γ is a family of subsets of Ω, then $\min(\Gamma^+) = \min(\Gamma)$.

3) Let $\mathcal{H} = \mathcal{H}_1 \overset{-}{\sqcap} \mathcal{H}_2$ and $\mathcal{H}_0 = \max(\{A_1 \cap A_2 : A_i \in \mathcal{H}_i\})$. Let us prove that $\mathcal{H} \leqslant^- \mathcal{H}_0$ and that $\mathcal{H}_0 \leqslant^- \mathcal{H}$. Let $A \in \mathcal{H}$. By definition of $\mathcal{H}$, for all i, there exists $A_i \in \mathcal{H}_i$ such that $A \subseteq A_i$; but then $A \subseteq A_1 \cap A_2$, so there exists $A' \in \mathcal{H}_0$ such that $A \subseteq A_1 \cap A_2 \subseteq A'$, by definition of $\mathcal{H}_0$. Thus by Lemma 1.15, we have $\mathcal{H} \leqslant^- \mathcal{H}_0$. Now if $B \in \mathcal{H}_0$, then $B = A_1 \cap A_2$, where $A_i \in \mathcal{H}_i$, for all i. As $B \subseteq A_i$, for all i, we have that $B \in \mathcal{H}_i^-$, for all i. Hence $B \in \mathcal{H}_1^- \cap \mathcal{H}_2^-$. So, by definition of $\mathcal{H}$, there exists $B' \in \mathcal{H}$ such that $B \subseteq B'$. By Lemma 1.15, we have $\mathcal{H}_0 \leqslant^- \mathcal{H}$.

4) We have:

$$\mathcal{H}_1 \overset{-}{\sqcup} \mathcal{H}_2 = \max(\mathcal{H}_1^- \cup \mathcal{H}_2^-) = \max(\mathcal{H}_1 \cup \mathcal{H}_2),$$

because $\mathcal{H}_1^- \cup \mathcal{H}_2^- = (\mathcal{H}_1 \cup \mathcal{H}_2)^-$ and if Γ is a family of subsets of Ω, then $\max(\Gamma^-) = \max(\Gamma)$. □

Remark 1.21 The operations $\overset{+}{\sqcap}$ and $\overset{+}{\sqcup}$ are not defined on $\mathrm{Hyp}_0(\Omega)$, as Example 1.22 shows. However, it is easy to check that both $\overset{-}{\sqcup}$ and $\overset{-}{\sqcap}$ define operations on $\mathrm{Hyp}_0(\Omega)$; that is, if $\mathcal{H}_1, \mathcal{H}_2 \in \mathrm{Hyp}_0(\Omega)$, then $\mathcal{H}_1 \overset{-}{\sqcup} \mathcal{H}_2 \in \mathrm{Hyp}_0(\Omega)$ and $\mathcal{H}_1 \overset{-}{\sqcap} \mathcal{H}_2 \in \mathrm{Hyp}_0(\Omega)$. Indeed, if $\mathcal{H}_1, \mathcal{H}_2 \in \mathrm{Hyp}_0(\Omega)$ and $w \in \Omega$, then for $i = 1, 2$ there exist $A_i \in \mathcal{H}_i$ such that $w \in A_i$. By Lemma 1.20, there are $B \in \mathcal{H}_1 \overset{-}{\sqcup} \mathcal{H}_2$ and $C \in \mathcal{H}_1 \overset{-}{\sqcap} \mathcal{H}_2$ such that $A_1 \subseteq B$ and $A_1 \cap A_2 \subseteq C$. Hence we conclude that $w \in B$ and $w \in C$. So, $\mathrm{Gr}(\mathcal{H}_1 \overset{-}{\sqcup} \mathcal{H}_2) = \Omega$ and $\mathrm{Gr}(\mathcal{H}_1 \overset{-}{\sqcap} \mathcal{H}_2) = \Omega$.

Example 1.22 Let $\Omega = \{1, 2, 3, 4\}$. Let us consider the following hypergraphs with ground set Ω:

$$\mathcal{H}_1 = \{\{1\}, \{2, 3\}, \{3, 4\}\}, \quad \mathcal{H}_2 = \{\{1\}, \{2, 3\}, \{2, 4\}\}.$$

We have that :

$$\mathcal{H}_1 \overset{+}{\sqcap} \mathcal{H}_2 = \{\{1\}, \{2, 3\}\} \notin \mathrm{Hyp}_0(\Omega).$$

Now let us consider the following hypergraphs, also with ground set Ω:

$$\mathcal{H}_1 = \{\{1\}, \{2\}, \{3, 4\}\}, \quad \mathcal{H}_2 = \{\{1\}, \{3\}, \{2, 4\}\}.$$

We have that:

$$\mathcal{H}_1 \overset{+}{\sqcup} \mathcal{H}_2 = \{\{1\}, \{2\}, \{3\}\} \notin \mathrm{Hyp}_0(\Omega).$$

Remark 1.23 Let Ω be a finite set. The hypergraphs $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$ on Ω will play an exceptional role throughout this thesis, specially in Chapter 3. Indeed, the hypergraphs $\{\emptyset\}$ and $\emptyset$ are the neutral elements of the distributive lattice $(\mathrm{Hyp}(\Omega), \leqslant^+, \overset{+}{\sqcap}, \overset{+}{\sqcup})$; that is, $\mathscr{H} \overset{+}{\sqcap} \{\emptyset\} = \mathscr{H}$ and $\mathscr{H} \overset{+}{\sqcup} \emptyset = \mathscr{H}$ for all hypergraph $\mathscr{H}$ on Ω. Analogously, the hypergraphs $\{\Omega\}$ and $\emptyset$ are the neutral elements of the distributive lattice $(\mathrm{Hyp}(\Omega), \leqslant^-, \overset{-}{\sqcap}, \overset{-}{\sqcup})$; that is, $\mathscr{H} \overset{-}{\sqcap} \{\Omega\} = \mathscr{H}$ and $\mathscr{H} \overset{-}{\sqcup} \emptyset = \mathscr{H}$ for all hypergraph $\mathscr{H}$ on Ω. We summarize these properties in Table 1.2.

	$* = +$	$* = -$
$\sqcap^*$	$\{\emptyset\}$	$\{\Omega\}$
$\sqcup^*$	$\emptyset$	$\emptyset$

Table 1.2 The neutral elements of the distributive lattices $(\mathrm{Hyp}(\Omega), \leqslant^+, \overset{+}{\sqcap}, \overset{+}{\sqcup})$ and $(\mathrm{Hyp}(\Omega), \leqslant^-, \overset{-}{\sqcap}, \overset{-}{\sqcup})$.

1.4 Operations between hypergraphs

1.4.1 Complementary and complement of a hypergraph

Notice that if $\mathscr{H}$ is a hypergraph on Ω and $|\Omega| \geq 2$, then $\overline{\mathscr{H}}$ is not a hypergraph on Ω. However, if $\mathscr{H} \in \mathrm{Hyp}(\Omega)$, then $\mathscr{H}^c \in \mathrm{Hyp}(\Omega)$.

In the following remarks we list some basic properties of the hypergraph $\mathscr{H}^c$ that will be used hereafter.

Remark 1.24 Observe that if $\mathscr{H} \in \mathrm{Hyp}_0(\Omega)$, then $\mathscr{H}^c \notin \mathrm{Hyp}_0(\Omega)$, in general.

Remark 1.25 If $\mathscr{H}$ is a hypergraph on Ω, then:

1) $(\mathscr{H}^c)^+ = (\mathscr{H}^-)^c$.

2) $(\mathscr{H}^c)^- = (\mathscr{H}^+)^c$.

Remark 1.26 If $\mathscr{H}$ is a hypergraph on Ω, then:

1) $\mathrm{Gr}(\mathscr{H}^c) = \Omega \setminus \mathrm{Int}(\mathscr{H})$ and $\mathrm{Int}(\mathscr{H}^c) = \Omega \setminus \mathrm{Gr}(\mathscr{H})$.

2) $\mathrm{rank}(\mathscr{H}^c) + \mathrm{corank}(\mathscr{H}) = \mathrm{rank}(\mathscr{H}) + \mathrm{corank}(\mathscr{H}^c) = |\Omega|$.

1.4.2 The transversal of a hypergraph

Let $\mathscr{H}$ be a hypergraph on Ω. A subset $T \subseteq \Omega$ is called a *transversal* of $\mathscr{H}$ if $T \cap A \neq \emptyset$, for all $A \in \mathscr{H}$. The family of transversals of a hypergraph is a monotone increasing family of subsets of Ω. Hence the family of inclusion-minimal transversals of $\mathscr{H}$ is another hypergraph on Ω called the *transversal* of $\mathscr{H}$, and denoted by $\mathrm{Tr}(\mathscr{H})$; that is:

$$\mathrm{Tr}(\mathscr{H}) = \min\{T \subseteq \Omega : T \cap A \neq \emptyset, \text{ for all } A \in \mathscr{H}\}.$$

Notice that the transversal of a hypergraph does not depend on whether Ω is the ground set of the hypergraph or not, because we only consider the inclusion-minimal transversals of the hypergraph.

Example 1.27 We have $\mathrm{Tr}(\emptyset) = \min(\mathscr{P}(\Omega)) = \{\emptyset\}$ and $\mathrm{Tr}(\{\emptyset\}) = \emptyset$. Moreover, $\mathrm{Tr}(\{\Omega\}) = \{\{x\} : x \in \Omega\} = \mathscr{U}_{1,\Omega}$. In general, the transversal of the uniform hypergraph $\mathscr{U}_{k,\Omega}$ is another uniform hypergraph, namely:

$$\mathrm{Tr}(\mathscr{U}_{k,\Omega}) = \mathscr{U}_{|\Omega|-k+1,\Omega}.$$

The following proposition establishes the well-known involutive property of the transversal of a hypergraph, but we give here a simple proof that uses the partial order $\leqslant^+$.

Proposition 1.28 *If $\mathscr{H}$ is a hypergraph on Ω, then we have:*

$$\mathrm{Tr}(\mathrm{Tr}(\mathscr{H})) = \mathscr{H}.$$

Proof. By Proposition 1.12, it is enough to demostrate that $\mathscr{H} \leqslant^+ \mathrm{Tr}(\mathrm{Tr}(\mathscr{H}))$ and that $\mathrm{Tr}(\mathrm{Tr}(\mathscr{H})) \leqslant^+ \mathscr{H}$. If $A \in \mathscr{H}$, then for all $B \in \mathrm{Tr}(\mathscr{H})$ we have $B \cap A \neq \emptyset$. Hence A is a tranversal of $\mathrm{Tr}(\mathscr{H})$, and therefore there exists a minimal transversal $B' \in \mathrm{Tr}(\mathrm{Tr}(\mathscr{H}))$ such that $B' \subseteq A$. Thus we have $\mathscr{H} \leqslant^+ \mathrm{Tr}(\mathrm{Tr}(\mathscr{H}))$. Now let $C \in \mathrm{Tr}(\mathrm{Tr}(\mathscr{H}))$. We want to prove that there exists $A \in \mathscr{H}$ such that $A \subseteq C$. Assume on the contrary that $A \not\subseteq C$, for all $A \in \mathscr{H}$. Let $x_A \in A$ be an element such that $x_A \notin C$, and let $B = \{x_A : A \in \mathscr{H}\}$. On one hand we have $B \cap C = \emptyset$, by construction. But on the other hand $B \cap A \neq \emptyset$, for all $A \in \mathscr{H}$; so B is a transversal of $\mathscr{H}$, and therefore there exists $B' \in \mathrm{Tr}(\mathscr{H})$ such that $B' \subseteq B$. But then $B' \cap C \neq \emptyset$. This is a contradiction because $B' \cap C \subseteq B \cap C = \emptyset$. □

In the next lemma, we establish the relationship between the ground sets of a hypergraph and its transversal hypergraph.

Lemma 1.29 *If Ω is a finite set and $\mathscr{H} \in \mathrm{Hyp}(\Omega)$, then* $\mathrm{Gr}(\mathrm{Tr}(\mathscr{H})) = \mathrm{Gr}(\mathscr{H})$. *In particular, $\mathscr{H} \in \mathrm{Hyp}_0(\Omega)$ if and only if* $\mathrm{Tr}(\mathscr{H}) \in \mathrm{Hyp}_0(\Omega)$.

Proof. Let $\mathscr{H} \in \mathrm{Hyp}(\Omega)$ and $v \in \mathrm{Gr}(\mathscr{H})$. We want to demonstrate that there exists $B \in \mathrm{Tr}(\mathscr{H})$ such that $v \in B$. By definition of ground set of a hypergraph, there exists $A \in \mathscr{H}$ such that $v \in A$. Then $A \setminus \{v\} \notin \mathscr{H}$. As $\mathscr{H} = \mathrm{Tr}(\mathrm{Tr}(\mathscr{H}))$, there exists $B \in \mathrm{Tr}(\mathscr{H})$

such that $B \cap (A \setminus \{v\}) = \emptyset$. But, by definition of transversal, we also have $B \cap A \neq \emptyset$. Therefore $v \in B$. Thus $v \in \mathrm{Gr}(\mathrm{Tr}(\mathscr{H}))$. Therefore we have proved that $\mathrm{Gr}(\mathscr{H}) \subseteq \mathrm{Gr}(\mathrm{Tr}(\mathscr{H}))$. Consequently we have the equality because taking the transversal of a hypergraph is an involutive operation. □

The previous lemma deals with the ground set of the transversal hypergraph. The following lemma is about the intersection.

Lemma 1.30 *If* $\mathscr{H} \in \mathrm{Hyp}(\Omega)$, *then we have that* $\mathrm{Int}(\mathrm{Tr}(\mathscr{H})) = \{x \in \Omega : \{x\} \in \mathscr{H}\}$ *and* $\mathrm{Int}(\mathscr{H}) = \{x \in \Omega : \{x\} \in \mathrm{Tr}(\mathscr{H})\}$.

Proof. Let $x \in \Omega$. From the definition it is clear that $\{x\} \in \mathrm{Tr}(\mathscr{H})$ if and only if $x \in \mathrm{Int}(\mathscr{H})$. Therefore we get that $\mathrm{Int}(\mathscr{H}) = \{x \in \Omega : \{x\} \in \mathrm{Tr}(\mathscr{H})\}$, and hence, $\mathrm{Int}(\mathrm{Tr}(\mathscr{H})) = \{x \in \Omega : \{x\} \in \mathrm{Tr}(Tr(\mathscr{H}))\} = \{x \in \Omega : \{x\} \in \mathscr{H}\}$ because of the involutive property of the transversal. □

Remark 1.31 As far as we know, there is no relationship between the rank and the corank of a hypergraph and its transversal hypergraph.

The following lemma provides a description of the transversal hypergraph in terms of the complement and the complementary operations. We mention that this result is not suitable for calculations. However, it will be used frequently in proofs.

Lemma 1.32 *If* $\mathscr{H}$ *is a hypergraph on* Ω, *then:*

$$\mathrm{Tr}(\mathscr{H}) = \min\left(\overline{(\mathscr{H}^{+})^{c}}\right).$$

Proof. We have:

$$\begin{aligned}
\left(\mathscr{H}^{+}\right)^{c} &= \{A \subseteq \Omega : \Omega \setminus A \in \mathscr{H}^{+}\} \\
&= \{A \subseteq \Omega : \exists A_0 \in \mathscr{H} \text{ s.t. } A_0 \subseteq \Omega \setminus A\} \\
&= \{A \subseteq \Omega : \exists A_0 \in \mathscr{H} \text{ s.t. } A \cap A_0 = \emptyset\}
\end{aligned}$$

Thus, we have:

$$\overline{(\mathscr{H}^{+})^{c}} = \{B \subseteq \Omega : B \notin \left(\mathscr{H}^{+}\right)^{c}\} = \{B \subseteq \Omega : B \cap A_0 \neq \emptyset, \text{ for all } A_0 \in \mathscr{H}\};$$

that is, $\overline{(\mathscr{H}^{+})^{c}}$ is the set of all the transversals of $\mathscr{H}$. So, by definition of $\mathrm{Tr}(\mathscr{H})$, we have $\mathrm{Tr}(\mathscr{H}) = \min\left(\overline{(\mathscr{H}^{+})^{c}}\right)$. □

Remark 1.33 Observe that if $\mathscr{H}$ is a hypergraph on Ω, then $\overline{(\mathscr{H}^{+})^{c}}$ is a monotone increasing family of subsets of Ω which is determined by the hypergraph $\min\left(\overline{(\mathscr{H}^{+})^{c}}\right)$ (see Proposition 1.10). From the previous lemma we get that:

$$\mathrm{Tr}(\mathscr{H}) = \min\left(\overline{(\mathscr{H}^{+})^{c}}\right) = \min\{T \subseteq \Omega : T \cap A \neq \emptyset, \text{ for all } A \in \mathscr{H}\}.$$

In view of this, we can also consider the monotone decreasing family of subsets $\overline{(\mathcal{H}^-)^c}$ and the hypergraph $\max\left(\overline{(\mathcal{H}^-)^c}\right)$ associated with any hypergraph $\mathcal{H}$. It can be easily checked that:

$$\max\left(\overline{(\mathcal{H}^-)^c}\right) = \max\{B \subseteq \Omega : A \cup B \neq \Omega, \text{ for all } A \in \mathcal{H}\}.$$

Observe that this equality depends on the set Ω. Le us denote this hypergraph as $\mathrm{Tr}^-(\mathcal{H})$, so that $\mathrm{Tr}(\mathcal{H})$ could also be denoted as $\mathrm{Tr}^+(\mathcal{H})$. The following lemma deals with the relationship between these transversals and their corresponding involutive properties.

Lemma 1.34 *If $\mathcal{H}$ is a hypergraph on Ω, then:*

1) $\mathrm{Tr}^+(\mathcal{H}) = (\mathrm{Tr}^-(\mathcal{H}^c))^c$ *and* $\mathrm{Tr}^-(\mathcal{H}) = (\mathrm{Tr}^+(\mathcal{H}^c))^c$.
2) $\mathrm{Tr}^+(\mathrm{Tr}^+(\mathcal{H})) = \mathcal{H}$ *and* $\mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H})) = \mathcal{H}$.

Proof. Let us begin by proving the second equality of the first statement. From the definitions and by applying Remark 1.25 and Lemma 1.32, we get that:

$$\begin{aligned}
\left(\mathrm{Tr}^+(\mathcal{H}^c)\right)^c &= \left(\min\left(\overline{((\mathcal{H}^c)^+)^c}\right)\right)^c \\
&= \left(\min\left(\overline{(\mathcal{H}^-)^{cc}}\right)\right)^c \\
&= \left(\min\left(\overline{(\mathcal{H}^-)}\right)\right)^c \\
&= \max\left(\left(\overline{(\mathcal{H}^-)}\right)^c\right) \\
&= \max\left(\left(\overline{(\mathcal{H}^-)^c}\right)\right) \\
&= \mathrm{Tr}^-(\mathcal{H}).
\end{aligned}$$

Hence, $(\mathrm{Tr}^-(\mathcal{H}))^c = \mathrm{Tr}^+(\mathcal{H}^c)$. The proof of the first equality of statement 1) is similar.

The equality $\mathrm{Tr}^+(\mathrm{Tr}^+(\mathcal{H})) = \mathcal{H}$ is exactly the one in Proposition 1.28. Therefore, we must prove $\mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H})) = \mathcal{H}$. Next we present two different proofs.

The first proof of this equality is similar to the one of Proposition 1.28. Namely, in order to prove this equality, we use the partial order $\leqslant^-$; that is, we are going to prove that $\mathcal{H} \leqslant^- \mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H}))$ and $\mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H})) \leqslant^- \mathcal{H}$. If $A \in \mathcal{H}$, then for all $B \in \mathrm{Tr}^-(\mathcal{H})$, $A \cup B \neq \Omega$. So, by definition of $\mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H}))$, there exists $C \in \mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H}))$ such that $A \subseteq C$. Hence $\mathcal{H} \leqslant^- \mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H}))$. Now let $X \in \mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H}))$. Assume that $X \not\subseteq A$, for all $A \in \mathcal{H}$. Then, for all $A \in \mathcal{H}$, there exists $x_A \in X$ such that $x_A \notin A$. Consider the set $C = \Omega \setminus \{x_A : A \in \mathcal{H}\}$. As $C \cup A \neq \Omega$, for all $A \in \mathcal{H}$, there exists $C' \in \mathrm{Tr}^-(\mathcal{H})$ such that $C \subseteq C'$. Thus, $X \cup C' \neq \Omega$, because $X \in \mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H}))$. Consequently, we also have $X \cup C \neq \Omega$, but this is a contradiction, because, by construction, $C^c \subseteq X$. Therefore we have proved that $\mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H})) \leqslant^- \mathcal{H}$. So, the equality $\mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H})) = \mathcal{H}$ holds.

To finish the proof of the lemma, let us prove the equality $\mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H})) = \mathcal{H}$ in a different way. From the first statement and applying the involutive property of the transversal (Proposition 1.28) we get that:

$$\begin{aligned}\mathrm{Tr}^-(\mathrm{Tr}^-(\mathcal{H})) &= (\mathrm{Tr}^+(\mathrm{Tr}^-(\mathcal{H})^c))^c\\ &= (\mathrm{Tr}^+((\mathrm{Tr}^+(\mathcal{H}^c))^{cc}))^c\\ &= (\mathrm{Tr}^+(\mathrm{Tr}^+(\mathcal{H}^c)))^c\\ &= (\mathcal{H}^c)^c = \mathcal{H}.\end{aligned}$$

□

1.4.3 Orders and operations

In this subsection we investigate the behaviour between the partial orders and the complementary and transversal operations.

Lemma 1.35 *Let $\mathcal{H}_1, \mathcal{H}_2$ be two hypergraphs on Ω. The following statements hold.*

1) *Order and complementary.*
   a) $\mathcal{H}_1 \leqslant^+ \mathcal{H}_2$ *if and only if* $\mathcal{H}_1^c \leqslant^- \mathcal{H}_2^c$.
   b) $\mathcal{H}_1 \leqslant^- \mathcal{H}_2$ *if and only if* $\mathcal{H}_1^c \leqslant^+ \mathcal{H}_2^c$.
2) *Order and complement.*
   a) $\mathcal{H}_1 \leqslant^+ \mathcal{H}_2$ *if and only if* $\max\left(\overline{\mathcal{H}_2^+}\right) \leqslant^- \max\left(\overline{\mathcal{H}_1^+}\right)$.
   b) $\mathcal{H}_1 \leqslant^- \mathcal{H}_2$ *if and only if* $\min\left(\overline{\mathcal{H}_2^-}\right) \leqslant^+ \min\left(\overline{\mathcal{H}_1^-}\right)$.
3) *Order and transversals.*
   a) $\mathcal{H}_1 \leqslant^+ \mathcal{H}_2$ *if and only if* $\mathrm{Tr}(\mathcal{H}_2) \leqslant^+ \mathrm{Tr}(\mathcal{H}_1)$.
   b) $\mathcal{H}_1 \leqslant^- \mathcal{H}_2$ *if and only if* $\left(\mathrm{Tr}(\mathcal{H}_2^c)\right)^c \leqslant^- \left(\mathrm{Tr}(\mathcal{H}_1^c)\right)^c$.

Proof.

1) Order and complementary.
   a) Assume that $\mathcal{H}_1 \leqslant^+ \mathcal{H}_2$. If $B_1 \in \mathcal{H}_1^c$, then $B_1^c \in \mathcal{H}_1$. Thus there exists $A_2 \in \mathcal{H}_2$ such that $A_2 \subseteq B_1^c$; that is, there exists $B_2 \in \mathcal{H}_2^c$, namely $B_2 = A_2^c$, such that $B_1 \subseteq B_2$. Hence we conclude that $\mathcal{H}_1^c \leqslant^- \mathcal{H}_2^c$.
   Conversely, assume that $\mathcal{H}_1^c \leqslant^- \mathcal{H}_2^c$ and let $B_1 \in \mathcal{H}_1$. Then $B_1^c \in \mathcal{H}_1^c$. So there exists $A_2 \in \mathcal{H}_2^c$ such that $B_1^c \subseteq A_2$; that is, we have proved that there exists $B_2 \in \mathcal{H}_2$, namely $B_2 = A_2^c$, such that $B_2 \subseteq B_1$. Consequently, $\mathcal{H}_1 \leqslant^+ \mathcal{H}_2$.

b) The proof follows by applying the previous statement to $\mathcal{H}_1^c$ and $\mathcal{H}_2^c$.

2) Order and complement.

a) By Lemma 1.2, $\overline{\mathcal{H}_i^+}$ is a decreasing family of subsets of Ω. So by Proposition 1.10:

$$\overline{\mathcal{H}_i^+} = \left(\max\left(\overline{\mathcal{H}_i^+}\right)\right)^- .$$

Then we have:

$$\begin{aligned}
\mathcal{H}_1 \leqslant^+ \mathcal{H}_2 &\iff \mathcal{H}_1^+ \subseteq \mathcal{H}_2^+ \\
&\iff \overline{\mathcal{H}_2^+} \subseteq \overline{\mathcal{H}_1^+} \\
&\iff \left(\max\left(\overline{\mathcal{H}_2^+}\right)\right)^- \subseteq \left(\max\left(\overline{\mathcal{H}_1^+}\right)\right)^- \\
&\iff \max\left(\overline{\mathcal{H}_2^+}\right) \leqslant^- \max\left(\overline{\mathcal{H}_1^+}\right).
\end{aligned}$$

b) By Lemma 1.2, $\overline{\mathcal{H}_i^-}$ is an increasing family of subsets of Ω. So by Proposition 1.10:

$$\overline{\mathcal{H}_i^-} = \left(\min\left(\overline{\mathcal{H}_i^-}\right)\right)^+ .$$

Then we have:

$$\begin{aligned}
\mathcal{H}_1 \leqslant^- \mathcal{H}_2 &\iff \mathcal{H}_1^- \subseteq \mathcal{H}_2^- \\
&\iff \overline{\mathcal{H}_2^-} \subseteq \overline{\mathcal{H}_1^-} \\
&\iff \left(\min\left(\overline{\mathcal{H}_2^-}\right)\right)^+ \subseteq \left(\min\left(\overline{\mathcal{H}_1^-}\right)\right)^+ \\
&\iff \min\left(\overline{\mathcal{H}_2^-}\right) \leqslant^+ \min\left(\overline{\mathcal{H}_1^-}\right).
\end{aligned}$$

3) Order and transversals.

a) By Lemma 1.2, $\overline{(\mathcal{H}_i^+)^c}$ is an increasing family of subsets of Ω. So by Proposition 1.10:

$$\overline{(\mathcal{H}_i^+)^c} = \left(\min\left(\overline{(\mathcal{H}_i^+)^c}\right)\right)^+ .$$

From this equality and Lemma 1.32, we have:

$$\begin{aligned}
\mathrm{Tr}(\mathcal{H}_2) \leqslant^+ \mathrm{Tr}(\mathcal{H}_1) &\iff \min\left(\overline{(\mathcal{H}_2^+)^c}\right) \leqslant^+ \min\left(\overline{(\mathcal{H}_1^+)^c}\right) \\
&\iff \left(\min\left(\overline{(\mathcal{H}_2^+)^c}\right)\right)^+ \subseteq \left(\min\left(\overline{(\mathcal{H}_1^+)^c}\right)\right)^+ \\
&\iff \overline{(\mathcal{H}_2^+)^c} \subseteq \overline{(\mathcal{H}_1^+)^c} \\
&\iff (\mathcal{H}_1^+)^c \subseteq (\mathcal{H}_2^+)^c \\
&\iff \mathcal{H}_1^+ \subseteq \mathcal{H}_2^+ \\
&\iff \mathcal{H}_1 \leqslant^+ \mathcal{H}_2.
\end{aligned}$$

b) From statements 1) and 3a), we have:

$$\begin{aligned} \mathscr{H}_1 \leqslant^- \mathscr{H}_2 &\iff \mathscr{H}_1^c \leqslant^+ \mathscr{H}_2^c \\ &\iff \mathrm{Tr}(\mathscr{H}_2^c) \leqslant^+ \mathrm{Tr}(\mathscr{H}_1^c) \\ &\iff \big(\mathrm{Tr}(\mathscr{H}_2^c)\big)^c \leqslant^- \big(\mathrm{Tr}(\mathscr{H}_1^c)\big)^c . \end{aligned}$$

□

Remark 1.36 The first part of the previous lemma can be restated by saying that the mapping:

$$(\mathrm{Hyp}(\Omega), \leqslant^+) \to (\mathrm{Hyp}(\Omega), \leqslant^-), \qquad \mathscr{H} \mapsto \mathscr{H}^c$$

is an isomorphism of posets. Analogously, the statement 3a) of the lemma says that the mapping:

$$(\mathrm{Hyp}(\Omega), \leqslant^+) \to (\mathrm{Hyp}(\Omega), \leqslant^+), \qquad \mathscr{H} \mapsto \mathrm{Tr}(\mathscr{H})$$

is an anti-automorphism of posets.

Remark 1.37 The third part of Lemma 1.35 can also be written in the following way, using the notations of Remark 1.33:

a) $\mathscr{H}_1 \leqslant^+ \mathscr{H}_2$ if and only if $\mathrm{Tr}^+(\mathscr{H}_2) \leqslant^+ \mathrm{Tr}^+(\mathscr{H}_1)$.

b) $\mathscr{H}_1 \leqslant^- \mathscr{H}_2$ if and only if $\mathrm{Tr}^-(\mathscr{H}_2) \leqslant^- \mathrm{Tr}^-(\mathscr{H}_1)$.

So we have that the mappings:

$$\begin{aligned} (\mathrm{Hyp}(\Omega), \leqslant^+) \to (\mathrm{Hyp}(\Omega), \leqslant^+), &\qquad \mathscr{H} \mapsto \mathrm{Tr}^+(\mathscr{H}) \\ (\mathrm{Hyp}(\Omega), \leqslant^-) \to (\mathrm{Hyp}(\Omega), \leqslant^-), &\qquad \mathscr{H} \mapsto \mathrm{Tr}^-(\mathscr{H}) \end{aligned}$$

are anti-automorphisms of posets.

1.4.4 Lattice structure and operations

Now we study the behaviour of the complementary and transversal operations with respect to the lattice operations.

Lemma 1.38 *The mapping:*

$$(\mathrm{Hyp}(\Omega), \leqslant^+, \overset{+}{\sqcup}, \overset{+}{\sqcap}) \to (\mathrm{Hyp}(\Omega), \leqslant^-, \overset{-}{\sqcup}, \overset{-}{\sqcap}), \qquad \mathscr{H} \mapsto \mathscr{H}^c$$

is an involutive isomorphism of lattices; that is, if $\mathscr{H}_1, \mathscr{H}_2$ are hypergraphs on Ω, then the following statements hold.

1) $(\mathcal{H}_1 \overset{+}{\sqcap} \mathcal{H}_2)^c = \mathcal{H}_1^c \overset{-}{\sqcap} \mathcal{H}_2^c$.

2) $(\mathcal{H}_1 \overset{-}{\sqcap} \mathcal{H}_2)^c = \mathcal{H}_1^c \overset{+}{\sqcap} \mathcal{H}_2^c$.

3) $(\mathcal{H}_1 \overset{+}{\sqcup} \mathcal{H}_2)^c = \mathcal{H}_1^c \overset{-}{\sqcup} \mathcal{H}_2^c$.

4) $(\mathcal{H}_1 \overset{-}{\sqcup} \mathcal{H}_2)^c = \mathcal{H}_1^c \overset{+}{\sqcup} \mathcal{H}_2^c$.

Proof.

1) Let $\mathcal{H} = (\mathcal{H}_1^c \overset{-}{\sqcap} \mathcal{H}_2^c)^c$. As taking complementaries is involutive, by Lemma 1.2, we have to show that $\mathcal{H} = \mathcal{H}_1 \overset{+}{\sqcap} \mathcal{H}_2$. By Remark 1.19, we must prove that $\mathcal{H} \leqslant^+ \mathcal{H}_i$, for $i = 1, 2$, and that if $\mathcal{H}' \leqslant^+ \mathcal{H}_i$, for $i = 1, 2$, then $\mathcal{H}' \leqslant^+ \mathcal{H}$.
By Remark 1.19, we have $\mathcal{H}_1^c \overset{-}{\sqcap} \mathcal{H}_2^c \leqslant^- \mathcal{H}_i^c$, for $i = 1, 2$. So, by Lemma 1.35:

$$\mathcal{H} = (\mathcal{H}_1^c \overset{-}{\sqcap} \mathcal{H}_2^c)^c \leqslant^+ \mathcal{H}_i^{cc} = \mathcal{H}_i,$$

for $i = 1, 2$. Assume now that $\mathcal{H}' \leqslant^+ \mathcal{H}_i$ for $i = 1, 2$. Then, by Lemma 1.35, $(\mathcal{H}')^c \leqslant^- \mathcal{H}_i^c$, for $i = 1, 2$, and, by Remark 1.19, $(\mathcal{H}')^c \leqslant^- \mathcal{H}_1^c \overset{-}{\sqcap} \mathcal{H}_2^c$, and, again by Lemma 1.35:

$$\mathcal{H}' = (\mathcal{H}')^{cc} \leqslant^+ (\mathcal{H}_1^c \overset{-}{\sqcap} \mathcal{H}_2^c)^c = \mathcal{H}.$$

2) It is a direct consequence of the previous item. Indeed, we have:

$$(\mathcal{H}_1 \overset{-}{\sqcap} \mathcal{H}_2)^c = (\mathcal{H}_1^{cc} \overset{-}{\sqcap} \mathcal{H}_2^{cc})^c = (\mathcal{H}_1^c \overset{+}{\sqcap} \mathcal{H}_2^c)^{cc} = \mathcal{H}_1^c \overset{+}{\sqcap} \mathcal{H}_2^c.$$

3) Let $\mathcal{H} = (\mathcal{H}_1^c \overset{-}{\sqcup} \mathcal{H}_2^c)^c$. As taking complementaries is involutive, by Lemma 1.2, we have to show that $\mathcal{H} = \mathcal{H}_1 \overset{+}{\sqcup} \mathcal{H}_2$. By Remark 1.19, we must prove that $\mathcal{H}_i \leqslant^+ \mathcal{H}$, for $i = 1, 2$, and that if $\mathcal{H}_i \leqslant^+ \mathcal{H}'$, for $i = 1, 2$, then $\mathcal{H} \leqslant^+ \mathcal{H}'$.
By Remark 1.19, we have $\mathcal{H}_i^c \leqslant^- \mathcal{H}_1^c \overset{-}{\sqcup} \mathcal{H}_2^c$, for $i = 1, 2$. Hence, by Lemma 1.35:

$$\mathcal{H}_i = \mathcal{H}_i^{cc} \leqslant^+ (\mathcal{H}_1^c \overset{-}{\sqcup} \mathcal{H}_2^c)^c = \mathcal{H},$$

for $i = 1, 2$. Assume now that $\mathcal{H}_i \leqslant^+ \mathcal{H}'$, for $i = 1, 2$. Then, by Lemma 1.35, $\mathcal{H}_i^c \leqslant^- (\mathcal{H}')^c$, for $i = 1, 2$. By Remark 1.19, $\mathcal{H}_1^c \overset{-}{\sqcup} \mathcal{H}_2^c \leqslant^- (\mathcal{H}')^c$, and, again by Lemma 1.35:

$$\mathcal{H} = (\mathcal{H}_1^c \overset{-}{\sqcup} \mathcal{H}_2^c)^c \leqslant^+ (\mathcal{H}')^{cc} = \mathcal{H}'.$$

4) It is a direct consequence of the previous item. Indeed, we have:

$$(\mathcal{H}_1 \overset{-}{\sqcup} \mathcal{H}_2)^c = (\mathcal{H}_1^{cc} \overset{-}{\sqcup} \mathcal{H}_2^{cc})^c = (\mathcal{H}_1^c \overset{+}{\sqcup} \mathcal{H}_2^c)^{cc} = \mathcal{H}_1^c \overset{+}{\sqcup} \mathcal{H}_2^c.$$

□

Lemma 1.39 *The mapping:*

$$(\mathrm{Hyp}(\Omega), \leqslant^+, \overset{+}{\sqcup}, \overset{+}{\sqcap}) \to (\mathrm{Hyp}(\Omega), \leqslant^+, \overset{+}{\sqcup}, \overset{+}{\sqcap}), \qquad \mathscr{H} \mapsto \mathrm{Tr}(\mathscr{H})$$

is an involutive anti-automorphism of lattices; that is, if $\mathscr{H}_1, \mathscr{H}_2$ are hypergraphs on Ω, then the following statements hold.

1) $\mathrm{Tr}(\mathscr{H}_1 \overset{+}{\sqcap} \mathscr{H}_2) = \mathrm{Tr}(\mathscr{H}_1) \overset{+}{\sqcup} \mathrm{Tr}(\mathscr{H}_2)$.

2) $\mathrm{Tr}(\mathscr{H}_1 \overset{+}{\sqcup} \mathscr{H}_2) = \mathrm{Tr}(\mathscr{H}_1) \overset{+}{\sqcap} \mathrm{Tr}(\mathscr{H}_2)$.

Proof.

1) Let $\mathscr{H} = \mathrm{Tr}(\mathscr{H}_1) \overset{+}{\sqcup} \mathrm{Tr}(\mathscr{H}_2)$. We want to prove that $\mathrm{Tr}(\mathscr{H}) = \mathscr{H}_1 \overset{+}{\sqcap} \mathscr{H}_2$. By definition of $\mathscr{H}$ and by Remark 1.19, we get that $\mathrm{Tr}(\mathscr{H}_i) \leqslant^+ \mathscr{H}$, for $i = 1, 2$. Thus, by Lemma 1.35, $\mathrm{Tr}(\mathscr{H}) \leqslant^+ \mathrm{Tr}(\mathrm{Tr}(\mathscr{H}_i)) = \mathscr{H}_i$, for $i = 1, 2$. Again, by Remark 1.19, we get that $\mathrm{Tr}(\mathscr{H}) \leqslant^+ \mathscr{H}_1 \overset{+}{\sqcap} \mathscr{H}_2$. Assume now that $\mathscr{H}'$ is a hypergraph on Ω such that $\mathscr{H}' \leqslant^+ \mathscr{H}_i$, for $i = 1, 2$. Then $\mathrm{Tr}(\mathscr{H}_i) \leqslant^+ \mathrm{Tr}(\mathscr{H}')$, by Lemma 1.35. Hence $\mathscr{H} \leqslant^+ \mathrm{Tr}(\mathscr{H}')$, by Remark 1.19. So by Lemma 1.35 $\mathscr{H}' = \mathrm{Tr}(\mathrm{Tr}(\mathscr{H}')) \leqslant^+ \mathrm{Tr}(\mathscr{H})$. By Remark 1.19, we conclude that $\mathscr{H}_1 \overset{+}{\sqcap} \mathscr{H}_2 = \mathrm{Tr}(\mathscr{H})$.

2) We have:

$$\begin{aligned}\mathrm{Tr}(\mathscr{H}_1 \overset{+}{\sqcup} \mathscr{H}_2) &= \mathrm{Tr}(\mathrm{Tr}(\mathrm{Tr}(\mathscr{H}_1)) \overset{+}{\sqcup} \mathrm{Tr}(\mathrm{Tr}(\mathscr{H}_2))) \\ &= \mathrm{Tr}(\mathrm{Tr}(\mathrm{Tr}(\mathscr{H}_1) \overset{+}{\sqcap} \mathrm{Tr}(\mathscr{H}_2))) \\ &= \mathrm{Tr}(\mathscr{H}_1) \overset{+}{\sqcap} \mathrm{Tr}(\mathscr{H}_2).\end{aligned}$$

□

Remark 1.40 In general, there is no equality between $\mathrm{Tr}(\mathscr{H}_1 \overset{-}{\sqcap} \mathscr{H}_2)$ and the hypergraph $\mathrm{Tr}(\mathscr{H}_1) \overset{-}{\sqcup} \mathrm{Tr}(\mathscr{H}_2)$, or between $\mathrm{Tr}(\mathscr{H}_1 \overset{-}{\sqcup} \mathscr{H}_2)$ and $\mathrm{Tr}(\mathscr{H}_1) \overset{-}{\sqcap} \mathrm{Tr}(\mathscr{H}_2)$, as Examples 1.41 and 1.42 show. This is so because we use $\mathrm{Tr} = \mathrm{Tr}^+$ instead of Tr^-. If we use Tr^- together with the operations $\overset{-}{\sqcup}$ and $\overset{-}{\sqcap}$, then we get a result analogous to the previous lemma (see Lemma 1.43).

Example 1.41 Let us consider the hypergraphs $\mathscr{H}_1 = \{\{1\}, \{2, 3\}\}$, $\mathscr{H}_2 = \{\{1, 2\}, \{3\}\}$ on the finite set $\Omega = \{1, 2, 3\}$. Then $\mathscr{H}_1 \overset{-}{\sqcap} \mathscr{H}_2 = \{\{1\}, \{2\}, \{3\}\}$, and so $\mathrm{Tr}(\mathscr{H}_1 \overset{-}{\sqcap} \mathscr{H}_2) = \{\Omega\}$. However, we have $\mathrm{Tr}(\mathscr{H}_1) = \{\{1, 2\}, \{1, 3\}\}$, and $\mathrm{Tr}(\mathscr{H}_2) = \{\{1, 3\}, \{2, 3\}\}$; hence $\mathrm{Tr}(\mathscr{H}_1) \overset{-}{\sqcup} \mathrm{Tr}(\mathscr{H}_2) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.

Example 1.42 Let us now consider any non-empty hypergraph $\mathscr{H}_1$ and the hypergraph $\mathscr{H}_2 = \{\emptyset\}$.

1) We have $\mathcal{H}_1 \bar{\sqcup} \mathcal{H}_2 = \mathcal{H}_1$, so $\mathrm{Tr}(\mathcal{H}_1 \bar{\sqcup} \mathcal{H}_2) = \mathrm{Tr}(\mathcal{H}_1) \neq \emptyset$. However, we have that $\mathrm{Tr}(\mathcal{H}_1) \bar{\sqcap} \mathrm{Tr}(\mathcal{H}_2) = \mathrm{Tr}(\mathcal{H}_1) \bar{\sqcap} \emptyset = \emptyset$.

2) Analogously, we have that $\mathcal{H}_1 \bar{\sqcap} \mathcal{H}_2 = \{\emptyset\}$ and so $\mathrm{Tr}(\mathcal{H}_1 \bar{\sqcap} \mathcal{H}_2) = \emptyset$. However, we have that $\mathrm{Tr}(\mathcal{H}_1) \bar{\sqcup} \mathrm{Tr}(\mathcal{H}_2) = \mathrm{Tr}(\mathcal{H}_1) \bar{\sqcup} \emptyset = \mathrm{Tr}(\mathcal{H}_1) \neq \emptyset$.

Lemma 1.43 *The mapping:*

$$(\mathrm{Hyp}(\Omega), \leqslant^-, \bar{\sqcup}, \bar{\sqcap}) \to (\mathrm{Hyp}(\Omega), \leqslant^-, \bar{\sqcup}, \bar{\sqcap}), \qquad \mathcal{H} \mapsto \mathrm{Tr}^-(\mathcal{H})$$

is an involutive anti-automorphism of lattices; that is, if $\mathcal{H}_1, \mathcal{H}_2$ are hypergraphs on Ω, then the following statements hold.

1) $\mathrm{Tr}^-(\mathcal{H}_1 \bar{\sqcap} \mathcal{H}_2) = \mathrm{Tr}^-(\mathcal{H}_1) \bar{\sqcup} \mathrm{Tr}^-(\mathcal{H}_2)$.

2) $\mathrm{Tr}^-(\mathcal{H}_1 \bar{\sqcup} \mathcal{H}_2) = \mathrm{Tr}^-(\mathcal{H}_1) \bar{\sqcap} \mathrm{Tr}^-(\mathcal{H}_2)$.

Proof. For the first equality, we have:

$$\begin{aligned}
\mathrm{Tr}^-(\mathcal{H}_1 \bar{\sqcap} \mathcal{H}_2) &= \left(\mathrm{Tr}^+\left((\mathcal{H}_1 \bar{\sqcap} \mathcal{H}_2)^c\right)\right)^c \\
&= \left(\mathrm{Tr}^+(\mathcal{H}_1^c \overset{+}{\sqcap} \mathcal{H}_2^c)\right)^c \\
&= \left(\mathrm{Tr}^+(\mathcal{H}_1^c) \overset{+}{\sqcup} \mathrm{Tr}^+(\mathcal{H}_2^c)\right)^c \\
&= \left(\mathrm{Tr}^+(\mathcal{H}_1^c)\right)^c \bar{\sqcup} \left(\mathrm{Tr}^+(\mathcal{H}_2^c)\right)^c \\
&= \mathrm{Tr}^-(\mathcal{H}_1) \bar{\sqcup} \mathrm{Tr}^-(\mathcal{H}_2).
\end{aligned}$$

The second statement follow by applying the first equality to the hypergraphs $\mathrm{Tr}^-(\mathcal{H}_1)$ and $\mathrm{Tr}^-(\mathcal{H}_2)$ and taking into account the involutive property of the operation Tr^-. □

Remark 1.44 The complementary operation is an isomorphism of lattices between the lattices $(\mathrm{Hyp}(\Omega), \leqslant^+, \overset{+}{\sqcup}, \overset{+}{\sqcap})$ and $(\mathrm{Hyp}(\Omega), \leqslant^-, \bar{\sqcup}, \bar{\sqcap})$, while the transversal operations Tr^+ and Tr^- are anti-isomorphism (see Lemma 1.35). In Table 1.3 we establish the behaviour of the neutral elements of these lattices (see Remark 1.23 and Table 1.2) with respect to these operations.

We end this chapter stating and proving a result that let us to compute the transversal of a hypergraph recurrently using the operation $\overset{+}{\sqcap}$. This result can be found in [4], although with a different notation.

Proposition 1.45 *Let Ω be a finite set. If $\mathcal{H} = \{A_1, \ldots, A_m\} \in \mathrm{Hyp}(\Omega)$, then:*

$$\mathrm{Tr}(\mathcal{H}) = \mathcal{U}_{1,A_1} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{U}_{1,A_m}.$$

$\{\emptyset\}$	$\overset{(\cdot)^c}{\longleftrightarrow}$	$\{\Omega\}$
$\mathrm{Tr}^+(\cdot) \updownarrow$		$\updownarrow \mathrm{Tr}^-(\cdot)$
$\emptyset$	$\overset{(\cdot)^c}{\longleftrightarrow}$	$\emptyset$

Table 1.3 Behaviour of the hypergraphs $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$ with respect to the complementary and transversal operations.

Proof. Let $A \in \mathscr{H}$. As $\mathscr{H}$ is a hypergraph, it is straightforward to check that:

$$\mathscr{H} = (\mathscr{H} \setminus \{A\}) \cup \{A\} = (\mathscr{H} \setminus \{A\}) \overset{+}{\sqcup} \{A\}.$$

Therefore, by Lemma 1.39, we have:

$$\begin{aligned} \mathrm{Tr}(\mathscr{H}) &= \mathrm{Tr}((\mathscr{H} \setminus \{A\}) \overset{+}{\sqcup} \{A\}) \\ &= \mathrm{Tr}(\mathscr{H} \setminus \{A\}) \overset{+}{\sqcap} \mathrm{Tr}(\{A\}) \\ &= \mathrm{Tr}(\mathscr{H} \setminus \{A\}) \overset{+}{\sqcap} \mathscr{U}_{1,A}. \end{aligned}$$

By recurrence, we get the result. □

CHAPTER 2

DOMINATION HYPERGRAPHS

In this chapter we introduce several hypergraphs associated with a graph, with special emphasis on the hypergraph whose elements are the minimal dominating sets of vertices of the graph (the *domination hypergraph* of the graph), and study the relationships among them. Then we investigate the problem of when a given hypergraph is a domination hypergraph and give complete answers in the case that the hypergraph has small order or size and in the case the hypergraph is a uniform hypergraph.

2.1 Graphs

In this section we introduce the standard notations and definitions on graphs that we will use throughout the thesis (for general references on graph theory see [10, 44]).

A *graph* G is an ordered pair $(V(G), E(G))$ comprising a *non-empty* finite set $V(G)$ of *vertices* together with a *possibly empty* set $E(G)$ of *edges* which are two-element subsets of $V(G)$. If $\{x, y\} \in E(G)$ is an edge of G, then x and y are said to be *adjacent vertices*. An *isolated vertex* is a vertex of the graph that is not adjacent to any other vertices; that is, a vertex that does not belong to any edge of the graph. We denote by $V_0(G)$ the set of isolated vertices of G. The *degree* of a vertex x, denoted by $\deg(x)$, is the number of vertices adjacent to it. Thus:

$$V_0(G) = \{x \in V(G) : \deg(x) = 0\}.$$

Remark 2.1 We can consider the set of edges $E(G)$ of G as a uniform hypergraph *on* $V(G)$ of rank 2. The order and size of the hypergraph $E(G)$ coincide with the order and size of the graph G. Moreover, we have $\mathrm{Gr}(E(G)) = V(G) \setminus V_0(G)$ and $\mathrm{Int}(E(G))$ is the set of vertices of degree $|V(G)| - 1$. We also observe that a graph G is isomorphic to a graph G' if and only if the hypergraph $E(G)$ is isomorphic to the hypergraph $E(G')$.

The *open neighborhood* of a vertex x of the graph G is the set of vertices adjacent to it:

$$N(x) = \{y \in V(G) : \{x, y\} \in E(G)\}.$$

The *closed neighborhood* of a vertex x of the graph G is the set of vertices:

$$N[x] = \{x\} \cup N(x).$$

We denote by $N[G]$ the family of closed neighborhoods of vertices of G; that is:

$$N[G] = \{N[x] : x \in V(G)\}.$$

The family $N[G]$ is also known as the *star system* of the graph G. In general, $N[G]$ could be a multiset. The set of minimal closed neighborhoods of G is a hypergraph on $V(G)$, which we will denote by $\mathcal{N}[G]$; that is:

$$\mathcal{N}[G] = \min(N[G]).$$

Lemma 2.2 *The ground set of the hypergraph $\mathcal{N}[G]$ is $V(G)$ and the intersection set of $\mathcal{N}[G]$ is the set of vertices of degree $|V(G)| - 1$.*

Proof. If $x \in V(G)$, then there exists a minimal element $N[y] \in \mathcal{N}[G]$ such that $N[y] \subseteq N[x]$. Then:

$$y \in N[y] \subseteq N[x] \Rightarrow y \in N[x] \Rightarrow x \in N[y].$$

Thus $\mathrm{Gr}(\mathcal{N}[G]) = V(G)$. The second statement follows from the equality $\mathrm{Int}(\mathcal{N}[G]) = \mathrm{Int}(N[G])$. □

Remark 2.3 From the previous lemma, $\mathcal{N}[G]$ is a hypergraph with ground set $V(G)$ and therefore has order $|V(G)|$. Its size is less or equal than $|V(G)|$. Moreover, we have: $\mathrm{corank}(\mathcal{N}[G]) = \delta(G) + 1$ and $\mathrm{rank}(\mathcal{N}[G]) \leqslant \Delta(G) + 1$, where $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degree of G, respectively.

2.2 The domination hypergraph of a graph

2.2.1 Dominating sets

A *dominating set* for a graph $G = (V(G), E(G))$ is a subset D of $V(G)$ such that every vertex not in D is adjacent to at least one member of D.

From the definition it is clear that a subset D of vertices is a dominating set of the graph G if and only if $D \cap N[x] \neq \emptyset$, for every vertex $x \in V(G)$. Therefore, a subset of vertices D is a dominating set of G if and only if the closed neighborhoods of the vertices in D constitute a covering of the set of vertices of the graph; that is, D is a dominating set if and only if $\bigcup_{x \in D} N[x] = V(G)$.

Since any superset of a dominating set of G is also a dominating set of G, the collection $D(G)$ of the dominating sets of a graph G is a monotone increasing family of subsets of the set of vertices $V(G)$; that is:

$$D(G) \in \mathrm{Inc}(V(G)).$$

Therefore, $D(G)$ is uniquely determined by the family $\min(D(G))$ of its inclusion-minimal elements. If we denote by $\mathscr{D}(G) = \min(D(G))$ the family of the *inclusion-minimal dominating sets* of the graph G, then $\mathscr{D}(G)$ is a hypergraph on $V(G)$ and we have:

$$\mathscr{D}(G) = \min(D(G)) \quad \text{and} \quad D(G) = \mathscr{D}(G)^+.$$

We say that $\mathscr{D}(G)$ is *the domination hypergraph of the graph* G.

Remark 2.4 If G_1 and G_2 are isomorphic graphs, then $\mathscr{D}(G_1)$ and $\mathscr{D}(G_2)$ are isomorphic hypergraphs. However the converse statement is no true, as the following example shows. Let G_1 and G_2 be the graphs with vertex set $V(G_1) = V(G_2) = \{1,2,3,4\}$ and edge sets given by:

$$E(G_1) = \{\{1,2\},\{3,4\}\} \quad \text{and} \quad E(G_2) = \{\{1,2\},\{2,3\},\{3,4\}\}.$$

It is clear that G_1 and G_2 are not isomorphic graphs, but:

$$\mathscr{D}(G_1) = \mathscr{D}(G_2) = \{\{1,4\},\{1,3\},\{2,4\},\{2,3\}\}.$$

Now we are going to prove that $\mathrm{Tr}(\mathscr{D}(G)) = \mathscr{N}[G]$ (see Corollary 2.6).

Proposition 2.5 *If G is a graph and $\mathscr{D}(G) = \{D_1,\dots,D_s\}$, then:*

$$\mathscr{N}[G] = \min\left\{\{a_1,\dots,a_s\} : a_i \in D_i \textit{ for } 1 \leqslant i \leqslant s\right\}.$$

Proof. Let us denote $H = \{\{a_1,\dots,a_s\} : a_i \in D_i \text{ for } 1 \leqslant i \leqslant s\}$, and let:

$$H' = \{X \subseteq V(G) : X \cap D \neq \emptyset \text{ for all } D \in \mathscr{D}(G)\}.$$

On one hand, observe that $\min(H) = \min(H')$ (however H and H' are not necessarily equal). On the other hand, from the definition of dominating set we get that $N[x] \in H'$ for all $x \in V(G)$, and hence it follows that $\mathscr{N}[G] \subseteq H'$. Therefore, to prove the equality $\mathscr{N}[G] = \min(H)$, it is enough to show that if $X \in H'$, then there exists $x \in V(G)$ such that $N[x] \subseteq X$.

Let $X \subseteq V(G)$ be a subset of vertices of G such that $X \cap D \neq \emptyset$ for all $D \in \mathscr{D}(G)$. Let us assume that $N[x] \not\subseteq X$ for all $x \in V(G)$. In such a case let $b_x \in N[x] \setminus X$, and let $B = \{b_x : x \in V(G)\}$. Observe that $B \cap N[x] \neq \emptyset$ for all $x \in V(G)$. So B is a dominating set of the graph G. Therefore there exists an inclusion-minimal dominating set $D_0 \in \mathscr{D}(G)$ such that $D_0 \subseteq B$. Since $D_0 \in \mathscr{D}(G)$, the intersection $X \cap D_0$ is non-empty. Thus $X \cap B \neq \emptyset$, and this leads us to a contradiction because $B \subseteq V(G) \setminus X$. This completes the proof. □

Corollary 2.6 *If G is a graph, then:*

$$\mathrm{Tr}(\mathscr{D}(G)) = \mathscr{N}[G] \quad \textit{and} \quad \mathrm{Tr}(\mathscr{N}[G]) = \mathscr{D}(G).$$

Proof. Both properties follow easily from the previous proposition. However, they can be proved directly from the definition. Indeed, recall a subset X of vertices is a dominating set of G if and only if $X \cap N[x] \neq \emptyset$, for every vertex $x \in V(G)$. Hence:

$$\begin{aligned}\mathrm{Tr}(\mathscr{N}[G]) &= \min\{X : X \cap A \neq \emptyset, \text{for all } A \in \mathscr{N}[G]\}\\ &= \min\{X : X \cap A \neq \emptyset, \text{for all } A \in N[G]\}\\ &= \min(D(G)) = \mathscr{D}(G).\end{aligned}$$

□

Corollary 2.7 *If G_1 and G_2 are graphs with the same vertex set, then $\mathscr{D}(G_1) = \mathscr{D}(G_2)$ if and only if $\mathscr{N}[G_1] = \mathscr{N}[G_2]$.*

Proof. It is a direct consequence of the definitions and Corollary 2.6. □

It is easy to check that for graphs G_1 and G_2 of order $n \leqslant 3$, the condition $\mathscr{N}[G_1] = \mathscr{N}[G_2]$ implies $G_1 = G_2$. However, for every $n \geqslant 4$ there exist non-isomorphic graphs of order n with the same inclusion-minimal closed neighborhoods. Let us show an example.

Example 2.8 For $n = 4$, the non-isomorphic graphs G_1 and G_2 with set of vertices $V(G_1) = V(G_2) = \{1,2,3,4\}$ and edges:

$$E(G_1) = \{\{1,2\},\{3,4\}\}, \quad E(G_2) = E(G_1) \cup \{\{2,3\}\}$$

satisfy:

$$\mathscr{N}[G_1] = \mathscr{N}[G_2] = \{\{1,2\},\{3,4\}\};$$

while for every $n \geqslant 5$, the non-isomorphic graphs G_1 and G_2 with set of vertices $V(G_1) = V(G_2) = \{1,2,\ldots,n\}$ and edges:

$$\begin{aligned}&E(G_1) = \{\{1,2\},\{2,3\},\{3,4\},\ldots,\{n-1,n\}\}\\ &E(G_2) = E(G_1) \cup \{\{2,n-1\}\}\end{aligned}$$

verify that:

$$\mathscr{N}[G_1] = \mathscr{N}[G_2] = \{\{1,2\},\{2,3,4\},\ldots,\{n-3,n-2,n-1\},\{n-1,n\}\}.$$

Proposition 2.9 *Let G be a graph.*

1) *The hypergraph* $\mathscr{D}(G)$ *has ground set* $V(G)$.

2) $\mathrm{Int}(\mathscr{D}(G)) = V_0(G)$, *the set of isolated vertices of* G.

3) $\mathscr{D}(G) = \{V(G)\}$ *if and only if* $E(G) = \emptyset$.

Proof.

1) By Lemma 1.29, both a hypergraph and its transversal hypergraph have the same ground set. By Corollary 2.6, we know that $\mathrm{Tr}(\mathscr{D}(G)) = \mathscr{N}[G]$; so by Lemma 2.2, $\mathscr{D}(G)$ has also ground set $V(G)$.

2) If $x \in V_0(G)$, then $N[x] = \{x\}$ and thus $N[x] \in \mathscr{N}[G]$. By Corollary 2.6, $N[x] \cap D \neq \emptyset$, for all $D \in \mathscr{D}(G)$; that is, $x \in D$, for all $D \in \mathscr{D}(G)$. Conversely, assume that $x \in D$, for all $D \in \mathscr{D}(G)$. Then $\{x\} \in \mathrm{Tr}(\mathscr{D}(G)) = \mathscr{N}[G]$; hence $x \in V_0(G)$.

3) It is a direct consequence of the definitions. □

We summarize the main parameters of the hypergraph $\mathscr{D}(G)$:

- The ground set is $V(G)$; the size, that is the number of hyperedges, can be arbitrary. Consequently, the order of $\mathscr{D}(G)$ is $|V(G)|$.
- The corank, the minimum cardinality of a dominating set of G, is called the *domination number* of G and is denoted by $\gamma(G)$.
- The rank of $\mathscr{D}(G)$, the maximum cardinality of a minimal dominating set of G, is called the *upper domination number* of G and is denoted by $\Gamma(G)$.

Both parameters have been extensively studied (see [23, 24]).

We end this section proving a interesting result about minimal dominating sets. Similar results can be found in [38].

Proposition 2.10 *If* G *is a graph with no isolated vertices, then for every* $D_1 \in \mathscr{D}(G)$, *there exists* $D_2 \in \mathscr{D}(G)$ *such that* $D_1 \cap D_2 = \emptyset$.

Proof. Assume that $D_1 \cap D \neq \emptyset$, for all $D \in \mathscr{D}(G) \setminus \{D_1\}$. Then $D_1 \in \mathrm{Tr}(\mathscr{D}(G)) = \mathscr{N}[G]$. Hence, there exists $x_0 \in V(G)$ such that $N[x_0] \subseteq D_1$ and $N[x_0] \in \mathscr{N}[G]$. But then the set $D_1 \setminus \{x_0\}$ is a dominating set of G, if x_0 is not an isolated vertex of G, contradicting the fact that D_1 is a *minimal* dominating set. □

2.2.2 Dominating and independent sets

Dominating sets of a graph are closely related to independent sets. An *independent set* of a graph G is a set of vertices in G such that no two of them are adjacent. We denote by $I(G)$ the set of independent sets of G. It is clear that if $I_1 \in I(G)$ and $I_2 \subseteq I_1$, then $I_2 \in I(G)$. Therefore the collection $I(G)$ of independent sets of G is a monotone decreasing family of subsets of $V(G)$; that is:

$$I(G) \in \mathrm{Dec}(V(G))$$

and hence $I(G)$ is uniquely determined by the set $\max(I(G))$ of inclusion-maximal independent sets. Thus $\mathscr{I}(G) = \max(I(G))$ is a hypergraph on $V(G)$ and we have:

$$\mathscr{I}(G) = \max(I(G)) \quad \text{and} \quad I(G) = \mathscr{I}(G)^{-}.$$

It is clear that an independent set is also a dominating set if and only if it is an inclusion-maximal independent set. Therefore, any inclusion-maximal independent set of a graph is necessarily also an inclusion-minimal dominating set (see [10]). Hence, we have that:

$$\emptyset \neq \mathscr{I}(G) \subseteq \mathscr{D}(G) \subseteq D(G).$$

The following proposition summarizes the relationships between dominating sets and independent sets.

Proposition 2.11 *If G is a graph, then:*

$$D(G) \cap I(G) = \mathscr{D}(G) \cap I(G) = D(G) \cap \mathscr{I}(G) = \mathscr{D}(G) \cap \mathscr{I}(G) = \mathscr{I}(G).$$

Proof. Recall that an independent set is also a dominating set if and only if it is an inclusion-maximal independent set, that is $D(G) \cap I(G) = \mathscr{I}(G)$. Clearly this equality, together with the fact that $\mathscr{I}(G) \subseteq \mathscr{D}(G)$, implies all the others. □

As the main object of study of this thesis is $\mathscr{D}(G)$, we will also denote by $\mathscr{D}_{ind}(G)$ the hypergraph of independent sets of G; that is:

$$\mathscr{D}_{ind}(G) = \{D \in \mathscr{D}(G) : D \text{ is an independent set}\} = \mathscr{I}(G).$$

The hypergraph $\mathscr{D}_{ind}(G)$ is called *the domination-independence hypergraph of the graph G*.

Remark 2.12 We observe that $\mathscr{D}_{ind}(G) \subseteq \mathscr{D}(G)$, although, in general, the inclusion is not an equality; that is, there are graphs satisfying $\mathscr{D}_{ind}(G) = \mathscr{D}(G)$ and graphs satisfying $\mathscr{D}_{ind}(G) \subsetneq \mathscr{D}(G)$. See Example 2.13 below.

Example 2.13 Let us show that the complete graphs K_n of order $n \geq 1$ and the stars $K_{1,n-1}$ of order $n \geqslant 3$ are graphs satisfying $\mathscr{D}_{ind}(G) = \mathscr{D}(G)$, while paths P_n of order

$n \geqslant 6$ satisfy $\mathscr{D}_{ind}(G) \subsetneq \mathscr{D}(G)$. Indeed, the inclusion-minimal dominating sets of K_n are the sets with exactly one vertex and these sets are all inclusion-maximal independent sets; whereas the graph $K_{1,n-1}$ has two inclusion-minimal dominating sets, the set of vertices of degree 1 and the set with the vertex of degree $n-1$, and both are inclusion-maximal independent sets. Hence, $\mathscr{D}_{ind}(K_n) = \mathscr{D}(K_n)$ and $\mathscr{D}_{ind}(K_{1,n-1}) = \mathscr{D}(K_{1,n-1})$. Now let us show that this equality does not hold whenever the graph is a path of order at least 6. Let P_n be the path of order $n \geqslant 6$ with set of vertices $V(P_n) = \{1,\dots,n\}$ and edges $E(P_n) = \{\{1,2\},\{2,3\},\dots,\{n-1,n\}\}$. By Proposition 2.9, there exists an inclusion-minimal dominating set D' of the graph $P_n - \{1,2,3,4,5\}$ containing the vertex 6. It is clear that the set $\{2,3\} \cup D'$ is an inclusion-minimal dominating set of P_n, but it is not an inclusion-maximal independent set of P_n. Therefore, $\mathscr{D}_{ind}(P_n) \subsetneq \mathscr{D}(P_n)$ if $n \geqslant 6$.

Lemma 2.14 *If G is a graph, then:*

1) *for every vertex $x \in V(G)$, there exists $D \in \mathscr{D}_{ind}(G)$ such that $x \in D$;*

2) *for every non-isolated vertex x of G, there exists $D \in \mathscr{D}_{ind}(G)$ such that $x \notin D$.*

Proof. Consider a vertex $x \in V(G)$ of the graph G. Observe that if D is an inclusion-maximal independent set of the subgraph $G - N[x]$ of G induced by $V(G) \setminus N[x]$, then $D \cup \{x\}$ is an inclusion-maximal independent set of G. Hence, $x \in D \cup \{x\} \in \mathscr{D}_{ind}(G)$; that is, there exists an inclusion-maximal independent set containing x.

Moreover, if x is a non-isolated vertex of the graph G, then there exists a vertex $y \in V(G)$ such that $x \in N[y]$. Hence, an inclusion-maximal independent set D of the graph $G - N[y]$ together with the vertex y is an inclusion-maximal independent set of G not containing x; that is, $x \notin D \cup \{y\} \in \mathscr{D}_{ind}(G)$. □

Proposition 2.15 *Let G be a graph.*

1) *The hypergraph $\mathscr{D}_{ind}(G)$ has ground set $V(G)$.*

2) $\mathrm{Int}(\mathscr{D}_{ind}(G)) = V_0(G)$*, the set of isolated vertices of G.*

3) $\mathscr{D}_{ind}(G) = \{V(G)\}$ *if and only if $E(G) = \emptyset$.*

Proof. The first statement follow from Lemma 2.14. The other two are immediate consequences of the definitions. □

Next we are going to demostrate that the hypergraph $\mathscr{D}_{ind}(G)$ uniquely determines the graph G.

Proposition 2.16 *If G is a graph and $\mathscr{D}_{ind}(G) = \{D_1,\ldots,D_t\}$, then:*

$$E(G) = \min\big\{\{a_1,\ldots,a_t\} : a_i \notin D_i \text{ for } 1 \leqslant i \leqslant t\big\}.$$

Proof. Let us denote $H = \{\{a_1,\ldots,a_t\} : a_i \notin D_i$ for $1 \leqslant i \leqslant t\}$.

By Proposition 2.15, if G has no edges, then $\mathscr{D}_{ind}(G) = \{V(G)\}$. Therefore, if G has no edges then $H = \emptyset$, and so $\min(H) = \emptyset = E(G)$.

Now assume that $E(G) \neq \emptyset$. In such a case, in order to prove the equality $E(G) = \min(H)$, it is enough to show that $E(G) \subseteq H$ and that for all $A \in H$, there exists $e \in E(G)$ such that $e \subseteq A$.

First we prove that $E(G) \subseteq H$. Let $\{x,y\} \in E(G)$. Hence, for every independent set D_i, $1 \leqslant i \leqslant t$, we have that either $x \notin D_i$ or $y \notin D_i$. Since x is a non-isolated vertex, without loss of generality we may assume that there exists $\ell \in \{1,\ldots,t-1\}$ such that $x \notin D_i$ if and only if $1 \leqslant i \leqslant \ell$ (observe that, since $E(G) \neq \emptyset$, from Lemma 2.14 it follows that $t \geqslant 2$). Hence, for $\ell < i \leqslant t$ we get that $x \in D_i$ and thus $y \notin D_i$. Set $a_1 = \cdots = a_\ell = x$ and set $a_{\ell+1} = \cdots = a_t = y$. Then $\{x,y\} = \{a_1,\ldots,a_\ell,a_{\ell+1},\ldots,a_t\} \in H$, and so we conclude that $E(G) \subseteq H$.

To finish, we must prove that if $A \in H$ then there exists $e \in E(G)$ such that $e \subseteq A$. Let $A \in H$. Assume, on the contrary, that A is an independent set of vertices. Thus, there exists an inclusion-maximal independent set A' such that $A \subseteq A'$. Since inclusion-maximal independent sets are inclusion-minimal dominating sets, there exists $i_0 \in \{1,\ldots,t\}$ such that $A' = D_{i_0}$. Therefore, $a \in D_{i_0}$ for any $a \in A$, leading to a contradiction because $A = \{a_1,\ldots,a_t\}$ with $a_i \notin D_i$. This completes the proof of the proposition. □

As a consequence of the previous proposition and the definitions we get the following corollary.

Corollary 2.17 *If G is a graph, then:*

$$\mathrm{Tr}(\mathscr{D}_{ind}(G)^c) = E(G) \quad \text{and} \quad \mathscr{D}_{ind}(G) = (\mathrm{Tr}(E(G)))^c.$$

Proof. The first equality is an immediate consequence of Proposition 2.16 and the second follows easily from the first one by applying the transversal and the complementary. □

Corollary 2.18 *Let G_1 and G_2 be graphs with same vertex set. Then $\mathscr{D}_{ind}(G_1) = \mathscr{D}_{ind}(G_2)$ if and only if $G_1 = G_2$.*

Proof. It is a direct consequence of the definitions and Corollary 2.17. □

Remark 2.19 Let G be a graph. It is clear that if G' is a graph such that $\mathscr{D}(G') = \mathscr{D}(G)$, then $\mathscr{D}_{ind}(G') \subseteq \mathscr{D}(G)$. However, the converse is not true, as Example 2.20 shows. So, determining all graphs G' such that $\mathscr{D}(G') = \mathscr{D}(G)$ is not equivalent to determining all possible graphs G' such that $\mathscr{D}_{ind}(G') \subseteq \mathscr{D}(G)$. In Subsection 2.4.2 we deal with graphs with the same domination hypergraphs.

Example 2.20 If G is the graph with vertex set $V(G) = \{1,2,3,4\}$ and edge set $E(G) = \{\{1,2\},\{2,3\},\{3,4\},\{1,4\}\}$, and G' is the graph with vertex set $V(G') = V(G)$ and edge set $E(G') = \{\{1,2\},\{2,4\},\{3,4\}\}$, then we have that:

$$\mathscr{D}_{ind}(G') = \{\{1,3\},\{1,4\},\{2,3\}\}$$

and that:

$$\mathscr{D}(G) = \{A \subseteq V(G) : |A| = 2\}.$$

So $\mathscr{D}_{ind}(G') \subsetneq \mathscr{D}(G)$, while $\mathscr{D}(G') \neq \mathscr{D}(G)$, because $\{1,2\} \notin \mathscr{D}(G')$.

In the following proposition we give an upper bound of the size of the hypergraph of independent dominanting sets of a graph G.

If $n \geqslant 1$ is an integer, we denote by $\iota(n)$ the maximum cardinality of the hypergraphs $\mathscr{D}_{ind}(G)$, where G is a graph of order n.

Proposition 2.21 *If $n \geqslant 1$, then $\iota(n) \leqslant F_n$, where F_n is the n-the term of the Fibonacci sequence $F_0 = F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geqslant 2$.*

Proof. We use complete induction on the order n. Set $V(G) = \{1,\dots,n\}$.

If $n = 1$, then $V(G) = \{1\}$ and $E(G) = \emptyset$. In this case, we have $\mathscr{D}_{ind}(G) = \{\{1\}\}$, so $\iota(1) = 1 = F_1$. If $n = 2$, then $V(G) = \{1,2\}$ and we have two cases: either $E(G) = \emptyset$ or $E(G) = \{\{1,2\}\}$. In the first case, we have $\mathscr{D}_{ind}(G) = \{\{1,2\}\}$ and in the second case, we have $\mathscr{D}_{ind}(G) = \{\{1\},\{2\}\}$. Hence $\iota(2) = 2 = F_2$.

Assume that $\iota(k) \leqslant F_k$, for all integers k such that $2 \leqslant k \leqslant n-1$. Let G be a graph of order n and let us consider a vertex $x \in V(G)$. Let $N[x] = \{x, x_1,\dots,x_r\}$ the closed neighborhood of x. We partition $\mathscr{D}_{ind}(G)$ into two sets $\mathscr{D}_{ind}(G) = \mathscr{D}_1 \cup \mathscr{D}_2$, where $D \in \mathscr{D}_1$ if and only if $x \notin D$, and $D \in \mathscr{D}_2$ if and only if $x \in D$. Observe that $\mathscr{D}_1 = \emptyset$ if and only if $x \in V_0(G)$. If $D \in \mathscr{D}_1$, then D is a maximal independent set of the graph $G - x$, that has order $n-1$; that is $\mathscr{D}_1 \subseteq \mathscr{D}_{ind}(G-x)$. By hypothesis of induction we have $|\mathscr{D}_1| \leqslant \iota(n-1) \leqslant F_{n-1}$. Now let $D \in \mathscr{D}_2$. As D is an independent set of vertices of G, we have $x_1,\dots,x_r \notin D$. Hence $D - \{x\}$ is a maximal independent set of the graph $G - N[x]$. By hypothesis of induction, we have $|\mathscr{D}_2| \leqslant F_{n-r-1} \leqslant F_{n-2}$, this last inequality because $x \notin V_0(G)$ and so $r \geqslant 1$. Therefore we have proved that $\iota(n) \leqslant F_{n-1} + F_{n-2} = F_n$. □

We summarize the main parameters of the hypergraph $\mathscr{D}_{ind}(G)$:

- The order is n, because the ground set is $V(G)$. Proposition 2.21 gives us some information about the maximum value that the size of $\mathscr{D}_{ind}(G)$ can have.
- The corank, that is the minimum cardinality of an independent dominating set of G, is called the *independent domination number* of G and is denoted by $i(G)$.
- The rank of $\mathscr{D}_{ind}(G)$, that is the maximum cardinality of a independent dominating set of G, is called the *independence number* of G and is denoted by $\alpha(G)$.

Both parameters $i(G)$ and $\alpha(G)$ have been extensively studied (see [23, 24, 21]).

2.2.3 Domination hypergraphs of the disjoint union and the join of graphs

Let $G_1,\dots,G_r$ be $r \geqslant 2$ graphs with pairwise disjoint sets of vertices. The *disjoint union* $G_1+\cdots+G_r$ of $G_1,\dots,G_r$ is the graph with:

$$V(G_1+\cdots+G_r) = V(G_1)\cup\cdots\cup V(G_r), \quad \text{and}$$
$$E(G_1+\cdots+G_r) = E(G_1)\cup\cdots\cup E(G_r);$$

while the *join* $G_1\vee\cdots\vee G_r$ of $G_1,\dots,G_r$ is the graph with:

$$V(G_1\vee\cdots\vee G_r) = V(G_1)\cup\cdots\cup V(G_r), \quad \text{and}$$
$$E(G_1\vee\cdots\vee G_r) = E(G_1)\cup\cdots\cup E(G_r)\cup$$
$$\cup\left\{\{x_{i_1},x_{i_2}\} : x_{i_1}\in V(G_{i_1}), x_{i_2}\in V(G_{i_2}), i_1\neq i_2\right\}.$$

The following lemma deals with the minimal dominating sets of these graphs. Its proof is a straightforward consequence of the definitions. Observe that because of this lemma we can reduce ourselves to connected graph when computing the domination hypergraph.

Lemma 2.22 *If $G_1,\dots,G_r$ are $r\geqslant 2$ graphs with pairwise disjoint sets of vertices, then the domination hypergraphs of the graphs $G_1+\cdots+G_r$ and $G_1\vee\cdots\vee G_r$ are:*

$$\mathscr{D}(G_1+\cdots+G_r) = \left\{D_1\cup\cdots\cup D_r : D_i\in\mathscr{D}(G_i)\right\},$$
$$\mathscr{D}(G_1\vee\cdots\vee G_r) = \mathscr{D}(G_1)\cup\cdots\cup\mathscr{D}(G_r)\cup$$
$$\cup\left\{\{x_{i_1},x_{i_2}\} : x_{i_j}\in V(G_{i_j}), N[x_{i_j}]\neq V(G_{i_j}), \textit{for } i_1\neq i_2\right\};$$

and the domination-independence hypergraphs of these graphs are:

$$\mathscr{D}_{ind}(G_1+\cdots+G_r) = \left\{D_1\cup\cdots\cup D_r : D_i\in\mathscr{D}_{ind}(G_i)\right\},$$
$$\mathscr{D}_{ind}(G_1\vee\cdots\vee G_r) = \mathscr{D}_{ind}(G_1)\cup\cdots\cup\mathscr{D}_{ind}(G_r).$$

Remark 2.23 The domination and independence-domination hypergraphs of the complete graph K_Ω on the set of vertices Ω and the null graph $\overline{K_\Omega}$ are, respectively:

$$\mathscr{D}(K_\Omega) = \mathscr{D}_{ind}(K_\Omega) = \{\{x\} : x \in \Omega\},$$

and:

$$\mathscr{D}(\overline{K_\Omega}) = \mathscr{D}_{ind}(\overline{K_\Omega}) = \{\Omega\}.$$

This computations can be done directly or by applying the previous lemma, since K_Ω is the join graph of the graphs $K_{\{x\}}$, where $x \in \Omega$, and $\overline{K_\Omega}$ is the sum of $K_{\{x\}}$, where $x \in \Omega$.

Remark 2.24 In particular, if $G = K_{\Omega_1,\ldots,\Omega_r}$ is the complete multipartite graph with stable sets $\Omega_1,\ldots,\Omega_r$, then $G = \overline{K_{\Omega_1}} \vee \cdots \vee \overline{K_{\Omega_r}}$, so:

$$\mathscr{D}(G) = \{\Omega_1,\ldots,\Omega_r\} \cup \big\{\{a_i, a_j\} : a_i \in \Omega_i, a_j \in \Omega_j, 1 \leqslant i,j \leqslant r, i \neq j\big\},$$

and:

$$\mathscr{D}_{ind}(G) = \{\Omega_1,\ldots,\Omega_r\}.$$

Remark 2.25 As a special case of the previous remark, if G is a star with center x, then $G = K_{\{x\},V(G)\setminus\{x\}}$ and thus:

$$\mathscr{D}(G) = \mathscr{D}_{ind}(G) = \{\{x\}, V(G)\setminus\{x\}\}.$$

2.2.4 Domination hypergraphs of graphs of order at most four

Now we are going to compute the domination hypergraphs and the independent domination hypergraphs of all graphs or order at most 4, up to isomorphism. The results either follow easily from a direct computation or from the previous two lemmas and remark.

It is easy to check that the number of non-isomorphic graphs G with 2, 3 or 4 vertices is 2, 4 and 11, respectively.

The domination hypergraphs $\mathscr{D}(G)$ of the two non-isomorphic graphs G with vertex set $V(G) = \{a, b\}$ are given in Table 2.1.

G	$E(G)$	$\mathscr{D}(G)$
$G_{2,1}$	$\emptyset$	$\{\{a,b\}\}$
$G_{2,2}$	$\{\{a,b\}\}$	$\{\{a\},\{b\}\}$

Table 2.1 Dominating sets of graphs or order 2.

G	$E(G)$	$\mathscr{D}(G)$
$G_{3,1}$	$\emptyset$	$\{\{a,b,c\}\}$
$G_{3,2}$	$\{\{a,b\}\}$	$\{\{a,c\},\{b,c\}\}$
$G_{3,3}$	$\{\{a,b\},\{a,c\}\}$	$\{\{a\},\{b,c\}\}$
$G_{3,4}$	$\{\{a,b\},\{a,c\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\}\}$

Table 2.2 Dominating sets of graphs of order 3.

G	$E(G)$	$\mathscr{D}(G)$
$G_{4,1}$	$\emptyset$	$\{\{a,b,c,d\}\}$
$G_{4,2}$	$\{\{a,b\}\}$	$\{\{a,c,d\},\{b,c,d\}\}$
$G_{4,3}$	$\{\{a,b\},\{b,c\}\}$	$\{\{a,c,d\},\{b,d\}\}$
$G_{4,4}$	$\{\{a,b\},\{c,d\}\}$	$\{\{a,c\},\{a,d\},\{b,c\},\{b,d\}\}$
$G_{4,5}$	$\{\{a,b\},\{a,c\},\{b,c\}\}$	$\{\{a,d\},\{b,d\},\{c,d\}\}$
$G_{4,6}$	$\{\{a,b\},\{b,c\},\{c,d\}\}$	$\{\{a,c\},\{a,d\},\{b,c\},\{b,d\}\}$
$G_{4,7}$	$\{\{a,b\},\{a,c\},\{a,d\}\}$	$\{\{a\},\{b,c,d\}\}$
$G_{4,8}$	$\{\{a,b\},\{a,d\},\{b,c\},\{c,d\}\}$	$\{\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}\}$
$G_{4,9}$	$\{\{a,b\},\{a,c\},\{a,d\},\{b,c\}\}$	$\{\{a\},\{b,d\},\{c,d\}\}$
$G_{4,10}$	$\{\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\}\}$	$\{\{a\},\{b\},\{c,d\}\}$
$G_{4,11}$	$\{\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\}\}$	$\{\{a\},\{b\},\{c\},\{d\}\}$

Table 2.3 Dominating sets of graphs of order 4.

In Table 2.2 we give the domination hypergraphs of the four non-isomorphic graphs G with vertex set $V(G)=\{a,b,c\}$.

The domination hypergraphs of G with set of vertices $V(G)=\{a,b,c,d\}$ are given in Table 2.3.

Remark 2.26 Observe that the six non-isomorphic graphs G with $n \leqslant 3$ vertices satisfy the equality $\mathscr{D}_{ind}(G)=\mathscr{D}(G)$. However, $\mathscr{D}_{ind}(G_{4,k})=\mathscr{D}(G_{4,k})$ for $k \neq 6,8$ while $\mathscr{D}_{ind}(G_{4,k}) \subsetneq \mathscr{D}(G_{4,k})$ for $k=6,8$. Specifically, we have:

$$\begin{aligned}
\mathscr{D}_{ind}(G_{4,6}) &= \{\{a,c\},\{a,d\},\{b,d\}\},\\
\mathscr{D}(G_{4,6}) &= \{\{a,c\},\{a,d\},\{b,c\},\{b,d\}\},\\
\mathscr{D}_{ind}(G_{4,8}) &= \{\{a,c\},\{b,d\}\},\\
\mathscr{D}(G_{4,8}) &= \{\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,c\},\{c,d\}\}.
\end{aligned}$$

Remark 2.27 In Appendix B we list all graphs of order 5, up to isomorphism, together with their respective domination and domination-independence hypergraphs. Such a list has been computed using the SAGE program of Appendix A.

2.3 Other hypergraphs associated with a graph

2.3.1 Vertex covers

A set of vertices C of a graph G is a *vertex cover* of G if for every edge $\{x, y\} \in E(G)$, either $x \in C$ or $y \in C$. We say that C *covers* the edges of G. It is clear that if $C_1 \subseteq C_2 \subseteq V(G)$ and C_1 is a vertex cover of G, then C_2 is also a vertex cover of G. Thus the collection $C(G)$ of the vertex covers of the graph G is a monotone increasing family of subsets of $V(G)$; that is:

$$C(G) \in \mathrm{Inc}(V(G)).$$

Hence, $C(G)$ is uniquely determined by the family $\min(C(G))$ of its inclusion-minimal elements. We denote by $\mathscr{C}(G) = \min(C(G))$. Then $\mathscr{C}(G)$ is a hypergraph *on* $V(G)$ and we have:

$$\mathscr{C}(G) = \min(C(G)) \quad \text{and} \quad C(G) = \mathscr{C}(G)^+.$$

Proposition 2.28 *A set C of vertices of a graph G is a vertex cover if, and only if, its complement $V(G) \setminus C$ is an independent set. Thus we have:*

$$\mathscr{C}(G)^c = \mathscr{D}_{ind}(G) \quad \textit{and} \quad \mathscr{D}_{ind}(G)^c = \mathscr{C}(G).$$

Proof. It is a direct consequence of the definitions. □

Observe that, by using the preceding proposition as well as the results of Subsection 1.4.1 and Proposition 2.15, as a hypergraph on $V(G)$, the hypergraph $\mathscr{C}(G)$ has ground set:

$$\mathrm{Gr}(\mathscr{C}(G)) = V(G) \setminus \mathrm{Int}(\mathscr{D}_{ind}(G)) = V(G) \setminus V_0(G)$$

and intersection set:

$$\mathrm{Int}(\mathscr{C}(G)) = V(G) \setminus \mathrm{Gr}(\mathscr{D}_{ind}(G)) = V(G) \setminus V(G) = \emptyset.$$

The main parameters of the $\mathscr{C}(G)$ as a hypergraph *on* $V(G)$ are the following:

- the order is n;
- the rank is $\Lambda(G)$, the *upper vertex covering number* of G;
- the corank is $\beta(G)$, the *vertex covering number* of G.

Remark 2.29 If G is a graph, then, by Remark 1.26 and the previous proposition, we get that:

$$\mathrm{rank}(\mathscr{D}_{ind}(G)) + \mathrm{corank}(\mathscr{C}(G)) = \alpha(G) + \beta(G) = |V(G)|.$$

This is precisely Gallai's Theorem. Analogously, we have:

$$\mathrm{corank}(\mathscr{D}_{ind}(G)) + \mathrm{rank}(\mathscr{C}(G)) = i(G) + \Lambda(G) = |V(G)|.$$

For this and similar results, see [12, 23].

2.3.2 Pendant vertices

Let S be a set of vertices of a graph $G = (V, E)$. We say that S is a *pendant set* of vertices of G if there exists a spanning forest F of G such that S is formed by the set of leaves of every connected component of F with at least 3 vertices and exactly one vertex for each connected component of F with exactly two vertices.

Let us denote by $P(G)$ the family of pendant sets of vertices of G.

Lemma 2.30 *The collection $P(G)$ of pendant sets of vertices of a graph G is a monotone decreasing family of subsets of the set of vertices of the graph G.*

Proof. Let S be a pendant set of vertices, $x \in S$ and $S' = S \setminus \{x\}$. Let F be a spanning forest of $G = (V, E)$ such that S is the set of leaves of every connected component of F with at least 3 vertices and exactly one vertex for each connected component of F with exactly two vertices. Let $y \in V$ be such that $\{x, y\} \in E(F)$. We distinguish the following cases. If $\deg_F(y) \geqslant 3$ or $\deg_F(y) = 1$, then consider the forest $F' = F - \{\{x, y\}\}$ obtained from F by removing the edge $\{x, y\}$. If $\deg_F(y) = 2$ and the connected component containing x and y is not a path, then consider the forest F' obtained by removing all the edges in the path from x to a vertex of degree at least 3 in F. If $\deg_F(y) = 2$ and the connected component containing x and y is a path, then consider the forest F' obtained by removing all the edges of the path, except the edge incident to the other leaf. It is easy to check that S' is a pendant set of vertices of G with F' as associated forest.

By applying repeatedly this reasoning, we have that every subset of S is a pendant set of vertices of G. □

Thus, for any graph G we have:

$$P(G) \in \mathrm{Dec}(V(G)).$$

Hence $P(G)$ is univocally determined by its set $\mathscr{P}(G) = \max(P(G))$ of inclusion-maximal elements. Then $\mathscr{P}(G)$ is a hypergraph on $V(G)$ and we have:

$$P(G) = \mathscr{P}(G)^- \quad \text{and} \quad \mathscr{P}(G) = \max(P(G)).$$

Proposition 2.31 *If $G = (V, E)$ is a graph and $S \subseteq V$, then, S is a pendant set of vertices of G if and only if $V \setminus S$ is a dominating set of G. Therefore we have:*

$$\mathscr{P}(G)^c = \mathscr{D}(G) \quad \text{and} \quad \mathscr{D}(G)^c = \mathscr{P}(G).$$

Proof. Let S be a pendant set of vertices of G. By definition, every vertex in S is a leaf of a spanning forest adjacent to a vertex in $V \setminus S$. Therefore, $V \setminus S$ is a dominating set of G. Now suppose that $V \setminus S$ is a dominating set. For each vertex $x \in S$, choose one edge $e(x) \in E$ joining x with a vertex in $V \setminus S$ (we know that it exists, because $V \setminus S$ is a dominating set of G). Let $E' = \{e(x) : x \in V \setminus S\}$. By construction, $F = (V, E')$ is a spanning forest of G. □

Observe that, by using the preceding proposition as well as the results of Subsection 1.4.1 and Proposition 2.15, as a hypergraph on $V(G)$, the hypergraph $\mathscr{P}(G)$ has ground set:

$$\mathrm{Gr}(\mathscr{P}(G)) = V(G) \setminus \mathrm{Int}(\mathscr{D}(G)) = V(G) \setminus V_0(G)$$

and intersection set:

$$\mathrm{Int}(\mathscr{P}(G)) = V(G) \setminus \mathrm{Gr}(\mathscr{D}(G)) = V(G) \setminus V(G) = \emptyset.$$

The main parameters of the $\mathscr{P}(G)$ as a hypergraph *on* $V(G)$ are the following:

- the order is n.
- the rank is the parameter $\varepsilon_F(G)$, the *maximum number of pendant edges* in a spanning forest (see [23]).

Remark 2.32 If G is a graph, then, by the previous proposition and Remark 1.26, we get that:

$$\mathrm{corank}(\mathscr{D}(G)) + \mathrm{rank}(\mathscr{P}(G)) = \gamma(G) + \varepsilon_F(G) = |V(G)|.$$

This is Nieminen's Theorem (see [12, 23]).

2.3.3 Relationships among the hypergraphs associated with a graph

In Table 2.4 we summarize all the relationships among the hypergraphs associated with a graph we have studied so far.

Remark 2.33 To our knowledge, empty slots of Table 2.4 are unknown. However, some of them might be related to the operation Tr^- instead of $\mathrm{Tr} = \mathrm{Tr}^+$ (see Subsection 1.4.2). Namely, hypergraphs coming from monotone-increasing families are related to the partial order $\leqslant^+$, and then we have to consider Tr^+; while those hyperhraphs coming from monotone-decreasing families are related to $\leqslant^-$, so we have to take into account Tr^-.

For example, as $\mathscr{H} = \mathscr{D}(G)$ arises from a monotone-increasing family, namely $D(G)$, we have to consider the operation Tr^+; while as $\mathscr{H} = \mathscr{D}_{ind}(G)$ arises from a monotone-decreasing family, namely $I(G)$, we have to consider the operation Tr^-. Hence, by applying Lemma 1.34, we have:

$$\mathrm{Tr}^-(\mathscr{D}_{ind}(G)) = \left(\mathrm{Tr}^+(\mathscr{D}_{ind}(G)^c)\right)^c = E(G)^c,$$

the second equality because of Lemma 2.17.

$\mathscr{H}$	$\mathscr{H}^c$	$\mathrm{Tr}(\mathscr{H})$
$\mathscr{D}(G)$	$\mathscr{P}(G)$	$\mathscr{N}[G]$
$\mathscr{D}_{ind}(G)$	$\mathscr{C}(G)$	
$\mathscr{N}[G]$		$\mathscr{D}(G)$
$E(G)$		$\mathscr{C}(G)$
$\mathscr{C}(G)$	$\mathscr{D}_{ind}(G)$	$E(G)$
$\mathscr{P}(G)$	$\mathscr{D}(G)$	

Table 2.4 The hypergraphs associated with a graph G, their complementaries and their transversal hypergraphs.

The main objects of study of this thesis are the domination hypergraphs. The other hypergraphs are given in terms of domination hypergraphs as it is shown in Table 2.5. However, we will deal mainly with $\mathscr{D}(G)$ and we stress that almost everything can done with hypergraphs of the form $\mathscr{D}_{ind}(G)$.

$E(G)$	$\mathscr{C}(G)$	$\mathscr{N}[G]$	$\mathscr{P}(G)$
		$\mathrm{Tr}(\mathscr{D}(G))$	$\mathscr{D}(G)^c$
$\mathrm{Tr}(\mathscr{D}_{ind}(G)^c)^c$	$\mathscr{D}_{ind}(G)^c$		

Table 2.5 The hypergraphs associated with a graph G in terms of $\mathscr{D}(G)$ and $\mathscr{D}_{ind}(G)$.

2.4 Domination hypergraphs

From now on we will focus only on dominating sets of vertices of graphs. We observe that everywhere we consider a dominating set we could also consider an independent-dominating set and develop a similar theory.

2.4.1 Definitions and notation

Let Ω be a finite set. We say that a hypergraph $\mathscr{H}$ on Ω is a *domination hypergraph* if there exists a graph G with vertex set $V(G) = \mathrm{Gr}(\mathscr{H}) \subseteq \Omega$ and minimal dominating sets

$\mathscr{D}(G) = \mathscr{H}$. Moreover, in such a case we say that the graph G is a *graph realization* of the hypergraph $\mathscr{H}$.

Remark 2.34 Observe that there exist domination hypergraphs $\mathscr{H}$ with more than one graph realization; that is, there exist hypergraphs $\mathscr{H}$ such that $\mathscr{H} = \mathscr{D}(G) = \mathscr{D}(G')$ with G, G' two different graphs (see Remark 2.4). So, a domination hypergraph $\mathscr{H}$ does not necessarily stem from a unique graph G. We can even have two non-isomorphic graphs with the same domination hypergraph. However, from Proposition 2.9 it follows that, if G and G' are two different graphs with $\mathscr{H} = \mathscr{D}(G) = \mathscr{D}(G')$, then $V(G) = V(G')$. In Section 2.4.2 we study this issue in more detail.

Remark 2.35 Observe that if $\mathscr{H} = \mathscr{D}(G)$ is a domination hypergraph, then $\mathscr{H}$ belongs to $\mathrm{Hyp}_0(V(G))$; that is, $\mathrm{Gr}(\mathscr{H}) = V(G)$.

Let us denote by $\mathrm{DomHyp}_0(\Omega)$ the set whose elements are the *domination hypergraphs with ground set* Ω, that is, the hypergraphs $\mathscr{H}$ with ground set Ω such that $\mathscr{H} = \mathscr{D}(G)$ for some graph G with vertex set $V(G) = \Omega$; and let us denote by $\mathrm{DomHyp}(\Omega)$ the set whose elements are the *domination hypergraphs on* Ω, that is, the hypergraphs $\mathscr{H}$ with ground set a subset Ω' of Ω such that $\mathscr{H} = \mathscr{D}(G)$ for some graph G with vertex set $V(G) = \Omega'$. Observe that:

$$\mathrm{DomHyp}_0(\Omega) \subseteq \mathrm{Hyp}_0(\Omega),$$

while:

$$\mathrm{DomHyp}(\Omega) = \bigcup_{\Omega' \subseteq \Omega} \mathrm{DomHyp}_0(\Omega') \subseteq \bigcup_{\Omega' \subseteq \Omega} \mathrm{Hyp}_0(\Omega') = \mathrm{Hyp}(\Omega).$$

It is clear that:

$$\mathrm{DomHyp}_0(\Omega) \subseteq \mathrm{DomHyp}(\Omega).$$

However, in general, the previous inclusion is not an equality. Let us show an example.

Example 2.36 Let $\Omega = \{1,2,3,4\}$. From the description of the domination hypergraphs of the non-isomorphic graphs of order at most four given in Subsection 2.2.4, we know that there are ten non-isomorphic domination hypergraphs with ground set Ω, that is, belonging to $\mathrm{DomHyp}_0(\Omega)$. Whereas there are seventeen non-isomorphic domination hypergraphs on Ω, that is, in $\mathrm{DomHyp}(\Omega)$. For example, $\mathscr{H} = \{\{1\},\{2,3\}\} \in \mathrm{DomHyp}(\Omega)$, but $\mathscr{H} \notin \mathrm{DomHyp}_0(\Omega)$.

Not all hypergraphs are domination hypergraphs, as Example 2.37 shows.

Example 2.37 Let $\Omega = \{1,2,3,4\}$. Let us consider the hypergraph $\mathscr{H} = \{\{1,2\},\{3,4\}\}$ on Ω. Then there does not exist any graph G such that $V(G) \subseteq \Omega$ and $\mathscr{H} = \mathscr{D}(G)$. Indeed, we only have to check that the hypergraph $\mathscr{H}$ does not coincide with any of the domination hypergraphs of Table 2.3 (that is, the hypergraph $\mathscr{H}$ is not isomorphic to any of the domination hypergraphs appearing on the table).

To conclude this section, we state the following proposition that will be used often.

Proposition 2.38 *If $\mathcal{H}$ is a hypergraph on Ω, then $\mathcal{H}$ is a domination hypergraph on Ω if and only if there exists a graph G such that $\mathcal{H} = \mathrm{Tr}(\mathcal{N}[G])$.*

Proof. It is an easy consequence of the definition of domination hypergraph and Corollary 2.6. □

2.4.2 Realizations. Codominating graphs

We say that the graphs G_1 and G_2 are *codominating* graphs if and only if both graphs have the same vertex set and $\mathcal{D}(G_1) = \mathcal{D}(G_2)$; that is, if they are graph realizations of the same domination hypergraph.

In Remark 2.19 we have explored the relationship between codomination and independence. In this subsection we deal with this issue from a different point of view. Namely, in the following proposition we study a way of adding edges to a graph while keeping invariant its domination hypergraph; that is, both the original graph and the resulting graph are codominating graphs. Iterating the construction of this proposition we end up in a *saturated* graph; that is, adding one more edge to that graph changes its domination hypergraph. This result can be found in [29]; we present here a different proof.

Proposition 2.39 *Let $G = (V, E)$ be a graph and set $\mathcal{N}[G] = \{N[x_1], \ldots, N[x_r]\}$. Assume that $r \leq n-1$ and that there exist non-adjacent vertices y_1, y_2 of G such that $y_1, y_2 \notin \{x_1, \ldots, x_r\}$. Then the graphs $G' = (V, E \cup \{\{y_1, y_2\}\})$ and G are codominating graphs; that is, $\mathcal{D}(G') = \mathcal{D}(G)$.*

Proof. Let us denote by $N_G[x]$ the closed neighborhood of the vertex x in G and by $N_{G'}[x]$ the corresponding set in G'. By definition of G', we have:

- $N_{G'}[x] = N_G[x]$, for all $x \in V \setminus \{y_1, y_2\}$,
- $N_{G'}[y_1] = N_G[y_1] \cup \{y_2\}$, and
- $N_{G'}[y_2] = N_G[y_2] \cup \{y_1\}$.

Therefore, $N_{G'}[y_1]$ and $N_{G'}[y_2]$ are not inclusion-minimal elements of $N[G']$, because $N_G[y_1]$ and $N_G[y_2]$ are neither inclusion-minimal elements of $N[G]$. Hence we deduce that $\mathcal{N}[G] = \mathcal{N}[G']$. Thus, $Tr(\mathcal{N}[G]) = Tr(\mathcal{N}[G'])$; that is, $\mathcal{D}(G) = \mathcal{D}(G')$. □

We observe that Proposition 2.39 allows us to build a sequence of graphs with the same domination hypergraph as a given graph.

Example 2.40 Let G, G' and G'' the graphs with vertex set $V(G) = V(G') = V(G'') = \{1, 2, 3, 4\}$ and edge set $E(G) = \{\{1, 2\}, \{3, 4\}\}$, $E(G') = E(G) \cup \{\{2, 3\}\}$ and $E(G'') = E(G) \cup \{\{1, 4\}\}$. Then $\mathcal{D}(G) = \mathcal{D}(G') = \mathcal{D}(G'')$.

However, given two graphs with the same vertex set it is not always possible to get one of them from the other one by applying this process, as the following example shows.

Example 2.41 Let us consider the bipartite complete graph $G = K_{3,3}$, with stable sets $\{1,2,3\}$ and $\{a,b,c\}$, and the cartesian product $G' = K_3 \square K_2$ with the same vertex set $V = \{1,2,3,a,b,c\}$ and cliques of order three generated by the sets $\{1,2,3\}$ and $\{a,b,c\}$ (see Figure 2.1). Then it is easily seen that $\mathscr{D}(G) = \mathscr{D}(G')$ and that adding or deleting any edge to any of these graphs changes the domination hypergraph. Hence it is not possible to get one these graphs from the other one by applying the construction of Proposition 2.39.

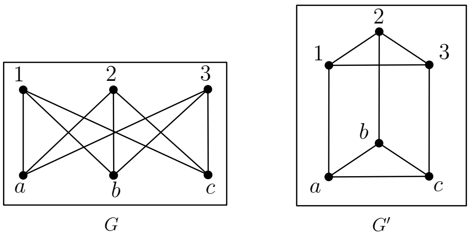


Figure 2.1 Two graphs with the same domination hypergraph.

Remark 2.42 In Appendix B we give a list of all the codominating graphs of order 5.

2.4.3 Necessary conditions and reductions

As mentioned before, not all hypergraphs are domination hypergraphs. The following proposition provides a necessary condition for a hypergraph to be a domination hypergraph.

Proposition 2.43 *If $\mathscr{H}$ is a domination hypergraph on Ω, then:*

$$|\operatorname{Tr}(\mathscr{H})| \leqslant |\operatorname{Gr}(\mathscr{H})|.$$

Proof. If $\mathscr{H} \in \operatorname{DomHyp}(\Omega)$, then, by applying Proposition 2.38, there exists a graph G such that $\mathscr{H} = \operatorname{Tr}(\mathscr{N}[G])$. Hence, applying the involutive property of the transversal and Lemma 1.29, we get that:

$$|\operatorname{Tr}(\mathscr{H})| = |\mathscr{N}[G]| \leqslant |\operatorname{Gr}(\mathscr{N}[G])| = |\operatorname{Gr}(\operatorname{Tr}(\mathscr{N}[G]))| = |\operatorname{Gr}(\mathscr{H})|. \qquad \square$$

Let us see some examples.

Example 2.44 Let $\Omega = \{1,2,3,\ldots,n\}$ with $n \geq 5$. Let us consider the hypergraph $\mathscr{H} = \{\{1,2\},\{3,\ldots,n\}\}$ on Ω. We claim that $\mathscr{H}$ is not a domination hypergraph. Assume that it is so. Then on one hand, by Proposition 2.43, we have $|\mathrm{Tr}(\mathscr{H})| \leqslant |\mathrm{Gr}(\mathscr{H})| = n$. On the other hand, we have:

$$\mathrm{Tr}(\mathscr{H}) = \{\{1,i\} : 3 \leqslant i \leqslant n\} \cup \{\{2,i\} : 3 \leqslant i \leqslant n\},$$

so $|\mathrm{Tr}(\mathscr{H})| = 2(n-2) > n$, if $n \geqslant 5$. So we get a contradiction.

Example 2.45 Let us consider the uniform hypergraph $\mathscr{U}_{r,\Omega}$, with $|\Omega| = n \geqslant 1$ and $3 \leqslant r \leqslant n-1$. Then $\mathscr{U}_{r,\Omega}$ is not a domination hypergraph. On the contrary, assume that it is so. Then, by Proposition 2.43, we have $|\mathrm{Tr}(\mathscr{U}_{r,\Omega})| \leqslant |\mathrm{Gr}(\mathscr{U}_{r,\Omega})| = n$. We know that $\mathrm{Tr}(\mathscr{U}_{r,\Omega}) = \mathscr{U}_{n-r+1,\Omega}$, that has cardinality $\binom{n}{n-r+1}$. Hence we have:

$$\binom{n}{n-r+1} \leqslant n$$

which implies that $r \leqslant 2$ or $r = n$ and we get a contradiction.

Example 2.46 Let Ω be a non-empty finite set. Let us consider $a, b \in \Omega$ such that $a \neq b$ and let $\Omega' = \Omega \setminus \{a,b\}$. We claim that then the hypergraph with ground set Ω:

$$\mathscr{H} = \big\{\{a,b\}, \Omega' \cup \{a\}, \Omega' \cup \{b\}\big\}$$

is not a domination hypergraph. Let us prove our claim. On one hand, we have that $\{x,a\},\{x,b\} \in \mathrm{Tr}(\mathscr{H})$, for all $x \in \Omega'$. Hence $|\mathrm{Tr}(\mathscr{H})| \geqslant 2|\Omega'|$. On the other hand, if $\mathscr{H}$ is a domination hypergraph, then, by Proposition 2.43, we have $|\mathrm{Tr}(\mathscr{H})| \leqslant |\Omega|$. Therefore, $2|\Omega'| \leqslant |\Omega| = |\Omega'| + 2$. So $|\Omega'| \leqslant 2$. If $|\Omega'| = 1$, then $|\Omega| = 3$, so $\Omega = \{a,b,c\}$ for instance, and therefore $\mathscr{H} = \mathscr{U}_{2,\Omega}$, that is not a domination hypergraph because it does not appear on Table 2.2. If $|\Omega'| = 2$, then $|\Omega| = 4$, so $\Omega = \{a,b,c,d\}$ for instance, and thus $\mathscr{H} = \{\{a,b\},\{a,c,d\},\{b,c,d\}\}$. In this case we can check that, up to isomorphism, this hypergraph does not appear on Table 2.3, so it is not a domination hypergraph.

Remark 2.47 The problem of deciding whether a hypergraph is a domination hypergraph is related to the question of realizing a sequence as the degree sequence of a graph. Indeed, if $\mathscr{H}$ is a domination hypergraph on a finite set Ω, say $\mathscr{H} = \mathscr{D}(G)$, then $\mathrm{Tr}(\mathscr{H}) = \mathscr{N}[G]$. If $\mathscr{N}[G] = \{N_1, N_2, \ldots, N_r\}$ with $|N_1| \geqslant |N_2| \geqslant \cdots \geqslant |N_r|$, then the sequence of integers $(|N_1|-1, |N_2|-1, \ldots, |N_r|-1)$ can be completed to a degree sequence, actually the degree sequence of the graph G. This problem is related with the star system problem, see [42].

Now we demonstrate two *reduction* results. The first one, Proposition 2.48, reduces the study of whether a hypergraph is a domination hypergraph to the case when its intersection set is empty. The second result, Proposition 2.49, reduces this study to the case where the corank is at least 2.

Proposition 2.48 *Let Ω be a finite set. Let $\mathcal{H}$ be a hypegraph on Ω such that $A = \mathrm{Int}(\mathcal{H}) \neq \emptyset$. Let $\mathcal{H}^{\langle A\rangle} = \{B \setminus A : B \in \mathcal{H}\}$. Then the following properties hold.*

1) *If G is a graph such that $\mathcal{H} = \mathcal{D}(G)$, then $\mathcal{H}^{\langle A\rangle} = \mathcal{D}(G - A)$.*

2) *If G is a graph such that $\mathcal{H}^{\langle A\rangle} = \mathcal{D}(G)$, then $\mathcal{H} = \mathcal{D}(G + \overline{K_A})$.*

In particular, $\mathcal{H} \in \mathrm{DomHyp}(\Omega)$ if and only if $\mathcal{H}^{\langle A\rangle} \in \mathrm{DomHyp}(\Omega)$.

Proof. Assume that $A \neq \emptyset$ and that $\mathcal{H} = \mathcal{D}(G)$. Then $A = V_0(G)$, so $\mathcal{H}^{\langle A\rangle} = \mathcal{D}(G - A)$. Assume now that $\mathcal{H}^{\langle A\rangle} = \mathcal{D}(G)$. Then $\mathcal{H} = \{B \cup A : B \in \mathcal{H}^{\langle A\rangle}\}$ and hence $\mathcal{H}$ is a domination hypergraph on Ω, namely $\mathcal{H} = \mathcal{D}(G + \overline{K_A})$. □

Proposition 2.49 *Let Ω be a finite set. Let $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ and assume that $\{x\} \in \mathcal{H}$, for some $x \in \Omega$. Then the following properties hold.*

1) *If G is a graph such that $\mathcal{H} = \mathcal{D}(G)$, then $\mathcal{H} \setminus \{\{x\}\} = \mathcal{D}(G - x)$.*

2) *If G is a graph such that $\mathcal{H} \setminus \{\{x\}\} = \mathcal{D}(G)$, then $\mathcal{H} = \mathcal{D}(G \vee K_{\{x\}})$.*

In particular, $\mathcal{H} \in \mathrm{DomHyp}(\Omega)$ if and only if $\mathcal{H} \setminus \{\{x\}\} \in \mathrm{DomHyp}(\Omega)$.

Proof. If $\mathcal{H} = \mathcal{D}(G)$ and $\{x\} \in \mathcal{H}$, then $\deg(x) = |V(G)| - 1$. Therefore $\mathcal{H} \setminus \{\{x\}\} = \mathcal{D}(G - x)$. The second statement is clear. □

Remark 2.50 We observe that there is no reduction result concerning connected components. For example, let $\Omega = \{1,2,3,4\} = \{1,3\} \cup \{2,4\}$ and let us consider the hypergraph $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, where $\mathcal{H}_1 = \{\{1,3\}\}$ and $\mathcal{H}_2 = \{\{2,4\}\}$. Then the hypergraphs $\mathcal{H}_i$ are domination hypergraphs, for $i = 1,2$, but $\mathcal{H}$ is not the domination hypergraph of any graph, as it is shown in Example 2.37.

2.4.4 Domination hypergraphs of small order or size

In this subsection we study hypergraphs of small order or size. Concretely, we identify those hypergraphs with ground set of cardinality at most four and those hypergraphs with one or two hyperedges which are domination hypergraphs. These results have to be understood modulo isomorphism.

Domination hypergraphs of order 1

If $\Omega = \{1\}$, then the only hypergraph with ground set Ω is $\mathcal{H} = \{\{1\}\}$ which is a domination hypergraph, namely $\mathcal{H} = \mathcal{D}(K_\Omega)$.

Domination hypergraphs of order 2

If $\Omega = \{1,2\}$, then there are only two hypergraphs with ground set Ω: $\mathcal{H}_1 = \{\{1,2\}\}$ and $\mathcal{H}_2 = \{\{1\},\{2\}\}$. Both hypergraphs are domination hypergraphs. Indeed: $\mathcal{H}_1 = \mathcal{D}(\overline{K_\Omega})$ and $\mathcal{H}_2 = \mathcal{D}(K_\Omega)$. See Table 2.6.

$\mathcal{H}$	G s.t. $\mathcal{D}(G) = \mathcal{H}$	$G_{i,j}$
$\mathcal{H}_1 = \{\{1,2\}\}$	$\overline{K_\Omega}$	$G_{2,1}$
$\mathcal{H}_2 = \{\{1\},\{2\}\}$	K_Ω	$G_{2,2}$

Table 2.6 Hypergraphs with ground set $\Omega = \{1,2\}$. The third column relates the graph G of the second column with the graphs $G_{i,j}$ of Table 2.1.

Domination hypergraphs of order 3

If $\Omega = \{1,2,3\}$, then, up to isomorphism, there exist five hypergraphs with ground set Ω; four of them are domination hypergraphs.

We summarize in Table 2.7 all these five hypergraphs as well as the graph realizations of the corresponding hypergraph whenever it is a domination hypergraph. We observe that each of the hypergraphs with ground set Ω which is a domination hypergraph has only one graph realization. This can be seen by simple inspection of the domination hypergraphs of graphs of order 3, found in Table 2.2. The actual graph realizations of the second column can be computed by applying Proposition 2.48 for $\mathcal{H}_3$, and Proposition 2.49 for $\mathcal{H}_4$. The uniform hypergraph $\mathcal{H}_2 = \mathcal{U}_{2,\Omega}$ is the only hypergraph of order 3 that is not a domination hypergraph, because it does not appear on Table 2.2.

$\mathcal{H}$	G s.t. $\mathcal{D}(G) = \mathcal{H}$	$G_{i,j}$
$\mathcal{H}_1 = \{\{1,2,3\}\}$	$\overline{K_\Omega}$	$G_{3,1}$
$\mathcal{H}_2 = \{\{1,2\},\{1,3\},\{2,3\}\}$		
$\mathcal{H}_3 = \{\{1,2\},\{1,3\}\}$	$K_{\{2,3\}} + K_{\{1\}}$	$G_{3,2}$
$\mathcal{H}_4 = \{\{1\},\{2,3\}\}$	$K_{\{2,3\}} \vee K_{\{1\}}$	$G_{3,3}$
$\mathcal{H}_5 = \{\{1\},\{2\},\{3\}\}$	K_Ω	$G_{3,3}$

Table 2.7 Hypergraphs with ground set $\Omega = \{1,2,3\}$. The third column relates the graph G of the second column with the graphs $G_{i,j}$ of Table 2.2. For the hypergraph $\mathcal{H}_2$, we have computed its domination completions (see Section 3.2) in Table 4.3.

Domination hypergraphs of order 4

If $\Omega = \{1,2,3,4\}$, then, up to isomorphism, there exist twenty hypergraphs with ground set Ω; ten of them are domination hypergraphs.

We summarize in Table 2.8 all these twenty hypergraphs as well as the graph realizations of the corresponding hypergraph whenever this is a domination hypergraph. We observe that each of the hypergraphs with ground set Ω which is a domination hypergraph has only one graph realization, except for $\mathcal{H}_{10}$ (we will prove this in Proposition 2.52) and for $\mathcal{H}_{13}$ that has nine non-isomorphic graph realizations (eight of them isomorphic to the path graph P_4). This can be seen by simple inspection of the domination hypergraphs of graphs of order 4, found in Table 2.3. The actual graph realizations of the second column can be computed by applying Proposition 2.48 and Proposition 2.49, except for $\mathcal{H}_{13}$.

Moreover, Propositions 2.48 and 2.49 have also been applied in several cases to prove that the corresponding hypergraph is not a domination hypergraph. For example, $\mathcal{H}_3$ is the hypergraph $(\mathcal{U}_{2,\Omega'})^{\langle\{1\}\rangle}$, where $\Omega' = \{2,3,4\}$ and using the notation of Proposition 2.48. By this proposition, $\mathcal{H}_3$ is a domination hypergraph if and only if $\mathcal{U}_{2,\Omega'}$ is so; but, by a direct inspection of Table 2.2, we can check that this hypergraph is not a domination hypergraph. Other cases have been treated using the fact that the transversal hypergraph can not be equal to $\mathcal{N}[G]$ for any graph G (see Proposition 2.38).

Domination hypergraph with one hyperedge

If Ω be a finite set, then every hypergraph with one hyperedge on Ω is a domination hypergraph; namely, if $\mathcal{H} = \{A\} \in \mathrm{Hyp}(\Omega)$, then $\mathcal{H} = \mathcal{D}(\overline{K_A})$.

Domination hypergraph with two hyperedges

Proposition 2.51 *Let Ω be a finite set. If $\mathcal{H} = \{A,B\} \in \mathrm{Hyp}(\Omega)$, then $\mathcal{H} \in \mathrm{DomHyp}(\Omega)$ if and only if either $|A \setminus B| = 1$ or $|B \setminus A| = 1$.*

Proof. Assume first that $\mathcal{H} = \{A,B\}$ is a domination hypergraph; that is, $\mathcal{H} = \mathcal{D}(G)$ for some graph G. Then $V(G) = A \cup B$ and $|A \setminus B| \geq 1$ or $|B \setminus A| \geq 1$, because $\mathcal{H}$ is a hypergraph. The transversal of $\mathcal{H}$ is:

$$\mathcal{N}[G] = \mathrm{Tr}(\mathcal{H}) = \{\{x\} : x \in A \cap B\} \cup \{\{a,b\} : a \in A \setminus B, b \in B \setminus A\}.$$

If $|A \setminus B| \geq 2$ and $|B \setminus A| \geq 2$, the sets $\{a_1,b_1\}$, $\{a_1,b_2\}$, $\{a_2,b_1\}$ and $\{a_2,b_2\}$, where $a_1,a_2 \in A \setminus B$ and $b_1,b_2 \in B \setminus A$ are distinct elements, are closed neighborhoods of G. But if $\{a_1,b_1\} = N[a_1]$, for example, then $\{a_1,b_2\} = N[b_2]$, which is not possible. Hence we have arrived at a contradiction.

$\mathscr{H}$	G s.t. $\mathscr{D}(G) = \mathscr{H}$	$G_{i,j}$
$\mathscr{H}_1 = \{\{1,2,3,4\}\}$	$\overline{K_\Omega}$	$G_{4,1}$
$\mathscr{H}_2 = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$		
$\mathscr{H}_3 = \{\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$		
$\mathscr{H}_4 = \{\{1,2,3\},\{1,2,4\}\}$	$K_{\{3,4\}} + \overline{K_{\{1,2\}}}$	$G_{4,2}$
$\mathscr{H}_5 = \{\{1,2,3\},\{1,2,4\},\{3,4\}\}$		
$\mathscr{H}_6 = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$		
$\mathscr{H}_7 = \{\{1,2\},\{1,3\},\{2,3,4\}\}$		
$\mathscr{H}_8 = \{\{1,2\},\{2,3,4\}\}$	$\left(\overline{K_{\{3,4\}}} \vee K_{\{1\}}\right) + K_{\{2\}}$	$G_{4,3}$
$\mathscr{H}_9 = \{\{1\},\{2,3,4\}\}$	$\overline{K_{\{2,3,4\}}} \vee K_{\{1\}}$	$G_{4,7}$
$\mathscr{H}_{10} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$	see Prop. 2.52	$G_{4,8}$
$\mathscr{H}_{11} = \{\{1,2\},\{1,3\},\{1,4\},\{2,4\},\{3,4\}\}$		
$\mathscr{H}_{12} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\}\}$		
$\mathscr{H}_{13} = \{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$	$K_{\{1,4\}} + K_{\{2,3\}}, P_4$	$G_{4,6}, G_{4,4}$
$\mathscr{H}_{14} = \{\{1,2\},\{1,3\},\{1,4\}\}$	$K_{\{2,3,4\}} + K_{\{1\}}$	$G_{4,5}$
$\mathscr{H}_{15} = \{\{1,2\},\{1,3\},\{2,4\}\}$		
$\mathscr{H}_{16} = \{\{1,2\},\{3,4\}\}$		
$\mathscr{H}_{17} = \{\{1,2\},\{1,3\},\{2,3\},\{4\}\}$		
$\mathscr{H}_{18} = \{\{1,2\},\{1,3\},\{4\}\}$	$(K_{\{2,3\}} + K_{\{1\}}) \vee K_{\{4\}}$	$G_{4,9}$
$\mathscr{H}_{19} = \{\{1,2\},\{3\},\{4\}\}$	$K_{\{1,2\}} \vee \overline{K_{\{3,4\}}}$	$G_{4,10}$
$\mathscr{H}_{20} = \{\{1\},\{2\},\{3\},\{4\}\}$	K_Ω	

Table 2.8 Hypergraphs with ground set $\Omega = \{1,2,3,4\}$. The third column relates the graph G of the second column with the graphs $G_{i,j}$ of Table 2.3. For the hypergraphs that are not domination hypergraphs, we have computed their domination completions (see Section 3.2) in Table 4.6.

Now we prove the converse statement. Assume now that $|A \setminus B| = 1$. Then $\mathcal{H} = \{\{a\} \cup C, B\}$, where $a \notin B$ and $C \subsetneq B$. So the transversal hypergraph is:

$$\mathrm{Tr}(\mathcal{H}) = \{\{c\} : c \in C\} \cup \{\{a,b\} : b \in B \setminus C\}.$$

It is easy to check that this hypergraph is $\mathcal{N}[\overline{K_C} + (\overline{K_{B\setminus C}} \vee K_{\{a\}})]$. Therefore $\mathcal{H} = \mathcal{D}(\overline{K_C} + (\overline{K_{B\setminus C}} \vee K_{\{a\}}))$ and so $\mathcal{H} \in \mathrm{DomHyp}(\Omega)$. □

2.4.5 Uniform domination hypergraphs

In this subsection we characterize the uniform hypergraphs that are domination hypergraphs.

Let Ω be a finite set of size $|\Omega| = n$ and let $1 \leqslant r \leqslant n$. Recall that a hypergraph $\mathcal{H}$ on Ω is called r-uniform if $|A| = r$ for all $A \in \mathcal{H}$. Recall also that we denote by $\mathcal{U}_{r,\Omega}$ the r-uniform hypergraph on Ω whose elements are all the subsets of Ω of size r; that is, $\mathcal{U}_{r,\Omega} = \{A \subseteq \Omega : |A| = r\}$.

The following proposition provides a characterization of the domination hypergraphs of the form $\mathcal{U}_{r,\Omega}$, as well as the description of their graph realizations. This proposition is partially stated in [28].

Proposition 2.52 *Let Ω be a finite set of size $|\Omega| = n$. Let $1 \leqslant r \leqslant n$. Then $\mathcal{U}_{r,\Omega}$ is a domination hypergraph if and only if $r = 1$, or $r = n$, or $r = 2$ and n is even. Moreover, the following statements hold.*

1) *The complete graph K_Ω is the unique graph G such that $\mathcal{U}_{1,\Omega} = \mathcal{D}(G)$.*
2) *The empty graph $\overline{K_\Omega}$ is the unique graph G such that $\mathcal{U}_{n,\Omega} = \mathcal{D}(G)$.*
3) *If $n = 2m$, then there are $(2m)!/(2^m m!)$ graphs G such that $\mathcal{U}_{2,\Omega} = \mathcal{D}(G)$. Namely, G is any graph of the form $G = \overline{K_{\Omega_1}} \vee \cdots \vee \overline{K_{\Omega_m}}$, where $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$ and $|\Omega_i| = 2$ for $1 \leqslant i \leqslant m$; that is, G is any graph obtained from the complete graph K_Ω by deleting the edges of a perfect matching (the vertices of the sets Ω_i are the endpoints of each one of the edges of the perfect matching).*

Proof. First, observe that statements 1) and 2) are a straightforward consequence of the definitions. Next let us prove the third statement.

Assume that $n = 2m$ is even. From the description of the minimal domination sets of the join graph, Lemma 2.22, it follows that $\mathcal{D}(\overline{K_{\Omega_1}} \vee \cdots \vee \overline{K_{\Omega_m}}) = \mathcal{U}_{2,\Omega}$ if $\Omega = \Omega_1 \cup \cdots \cup \Omega_m$ and $|\Omega_i| = 2$ for $1 \leqslant i \leqslant m$. So the uniform hypergraph $\mathcal{U}_{2,\Omega}$ is a domination hypergraph and the graphs of the form $G = \overline{K_{\Omega_1}} \vee \cdots \vee \overline{K_{\Omega_m}}$ are domination realizations of $\mathcal{U}_{2,\Omega}$. Conversely, let us prove that if G is a graph such that $\mathcal{D}(G) = \mathcal{U}_{2,\Omega}$ then G is obtained from the complete graph K_Ω by deleting the edges of a perfect matching. So, assume that $\mathcal{D}(G) = \mathcal{U}_{2,\Omega}$. Then from Corollary 2.6 it follows that $\mathcal{N}[G] = \mathrm{Tr}(\mathcal{U}_{2,\Omega})$.

Since $\mathrm{Tr}(\mathscr{U}_{2,\Omega}) = \mathscr{U}_{2m-1,\Omega}$, hence $\mathscr{N}[G] = \mathscr{U}_{2m-1,\Omega}$; therefore all the vertices of G have degree $2m-2$, and consequently, the graph G is obtained from the complete graph K_Ω by deleting the edges of a perfect matching, as we wanted to prove.

From the above we conclude that if $n = 2m$ is even, then $\mathscr{U}_{2,\Omega} = \mathscr{D}(G)$ if and only if G is a graph obtained from K_Ω by deleting the edges of a perfect matching. It is well known that the number of perfect matchings in a complete graph K_{2m} is given by the double factorial $(2m-1)!!$, that is, $(2m)!/(2^m m!)$. Hence, if $n = 2m$ is even, then there are $(2m)!/(2^m m!)$ graphs G such that $\mathscr{D}(G) = \mathscr{U}_{2,\Omega}$. This completes the proof of the third statement.

To finish the proof of the proposition we must show that if $3 \leqslant r \leqslant n-1$, or if $r = 2$ and n is odd, then $\mathscr{U}_{r,\Omega}$ is not a domination hypergraph. Otherwise, assume that there exists a graph G with vertex set $V(G) = \Omega$ and such that $\mathscr{D}(G) = \mathscr{U}_{r,\Omega}$. Since $\mathrm{Tr}(\mathscr{U}_{r,\Omega}) = \mathscr{U}_{n-r+1,\Omega}$, from Corollary 2.6 we get that $\mathscr{N}[G] = \mathscr{U}_{n-r+1,\Omega}$. On one hand, the size of $\mathscr{N}[G]$ is at most n because $V(G) = \Omega$. On the other hand, $\mathscr{U}_{n-r+1,\Omega}$ has size $\binom{n}{n-r+1}$. Therefore $\binom{n}{n-r+1} \leqslant n$, and thus $r = 2$. At this point we have that G is a graph of order n with $\mathscr{N}[G] = \mathscr{U}_{n-r+1,\Omega} = \mathscr{U}_{n-1,\Omega}$. So, G is a $(n-2)$-regular graph of order n, which is not possible if n is odd. This completes the proof of the proposition. □

CHAPTER 3

COMPLETION AND DECOMPOSITION

This chapter presents the main theoretical results of the thesis. Section 3.1 is devoted to the study of completions and decompositions of hypergraphs in a general framework. In Section 3.2 we focus on the domination context. In the other sections, we present specific results for upper domination completions with respect to $\leqslant^+$. Besides the many examples presented here in order to illustrate the theoretical results, on next chapter we use them to perform several computations for some families of hypergraphs.

In the previous chapter we have seen that, in general, a hypergraph is not a domination hypergraph. Therefore, a natural question that arises at this point is to determine how a hypergraph can be transformed into a domination hypergraph. In this chapter we look for *domination completions* of a hypergraph $\mathscr{H}$, that is, we search for graphs G whose associated domination hypergraphs $\mathscr{D}(G)$ are *close* to the hypergraph $\mathscr{H}$. Concretely, by taking into account the partial orders $\leqslant^+$ and $\leqslant^-$ introduced in Chapter 1, we prove that these domination completions of $\mathscr{H}$ exist. Moreover, we will prove that the domination completions of the hypergraph $\mathscr{H}$ define a poset whose minimal elements, the *minimal domination completions*, provide a *decomposition* of the hypergraph $\mathscr{H}$ in the sense that, all the elements of $\mathscr{H}$ can be obtained from the vertex dominating set of suitable graphs. From this we conclude that the hypergraph $\mathscr{H}$ is uniquely determined by its minimal domination completions. In addition, we demonstrate that the minimum number of domination completions appearing in a decomposition of $\mathscr{H}$ allows us to compute a tight lower bound for the distance between the hypergraph $\mathscr{H}$ and the family of the domination hypergraphs.

Throughtout this chapter, we stress the difference of considering hypergraphs with *ground* set Ω or *on* Ω.

3.1 Completions and decompositions of hypergraphs

3.1.1 The four collections of completions of a hypergraph

Let Ω be a finite set. Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a non-empty collection of hypergraphs on Ω and let $\mathscr{H} \in \mathrm{Hyp}(\Omega)$ be a hypergraph on Ω. We wish to identify hypergraphs in Σ that are in some sense close to the hypergraph $\mathscr{H}$. For this, we consider the subset of those hypergraphs from Σ that lie above or below $\mathscr{H}$ with respect to either $\leqslant^+$ or $\leqslant^-$; that is, we associate with $\mathscr{H}$ four different subsets of Σ, one for each choice of order ($\leqslant^+$ or $\leqslant^-$) and side (above or below). We thus define:

$$\Sigma_u^+(\mathscr{H}) = \{\mathscr{H}' \in \Sigma : \mathscr{H} \leqslant^+ \mathscr{H}'\},$$
$$\Sigma_\ell^+(\mathscr{H}) = \{\mathscr{H}' \in \Sigma : \mathscr{H}' \leqslant^+ \mathscr{H}\},$$
$$\Sigma_u^-(\mathscr{H}) = \{\mathscr{H}' \in \Sigma : \mathscr{H} \leqslant^- \mathscr{H}'\},$$
$$\Sigma_\ell^-(\mathscr{H}) = \{\mathscr{H}' \in \Sigma : \mathscr{H}' \leqslant^- \mathscr{H}\},$$

and for $*_1 \in \{+,-\}$ and for $*_2 \in \{u,\ell\}$ we say that a hypergraph $\mathscr{H}'$ in $\Sigma_{*_2}^{*_1}(\mathscr{H})$ is a $\Sigma_{*_2}^{*_1}$-*completion* of $\mathscr{H}$. The completions in $\Sigma_u^{*_1}(\mathscr{H})$ (respectively, in $\Sigma_\ell^{*_1}(\mathscr{H})$) will be called *upper* (respectively, *lower*) completions of $\mathscr{H}$.

Remark 3.1 The computation of all these families for the hypergraphs $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$ yields the results on Table 3.1.

	$\mathscr{H} = \emptyset$	$\mathscr{H} = \{\emptyset\}$	$\mathscr{H} = \{\Omega\}$
$\Sigma_u^+(\mathscr{H})$	Σ	$\Sigma \cap \{\{\emptyset\}\}$	$\Sigma \setminus \{\emptyset\}$
$\Sigma_u^-(\mathscr{H})$	Σ	$\Sigma \setminus \{\emptyset\}$	$\Sigma \cap \{\{\Omega\}\}$
$\Sigma_\ell^+(\mathscr{H})$	$\Sigma \cap \{\emptyset\}$	Σ	$\Sigma \cap \{\emptyset, \{\emptyset\}\}$
$\Sigma_\ell^-(\mathscr{H})$	$\Sigma \cap \{\emptyset\}$	$\Sigma \cap \{\emptyset, \{\emptyset\}\}$	Σ

Table 3.1 The families $\Sigma_{*_2}^{*_1}(\mathscr{H})$ for $\mathscr{H} = \emptyset, \{\emptyset\}, \{\Omega\}$ and $*_1 \in \{+,-\}$ and $*_2 \in \{u,\ell\}$.

The goal of this subsection is to prove that, under certain mild assumptions, these four families of hypergraphs $\Sigma_{*_2}^{*_1}(\mathscr{H})$ are non-empty.

Proposition 3.2 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω. Then the following statements hold.*

1) *If $\mathscr{U}_{1,X} \in \Sigma$ for all non-empty subsets X of Ω, then $\Sigma_u^+(\mathscr{H}) \neq \emptyset$ for all $\mathscr{H} \in \mathrm{Hyp}(\Omega)$ such that $\mathscr{H} \neq \{\emptyset\}$.*

2) *If* $\{X\} \in \Sigma$ *for all non-empty subsets* X *of* Ω*, then* $\Sigma_\ell^+(\mathcal{H}) \neq \emptyset$ *for all* $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ *such that* $\mathcal{H} \neq \{\emptyset\}$*.*

3) *If* $(\mathcal{U}_{1,X})^c \in \Sigma$ *for all non-empty subsets* X *of* Ω*, then* $\Sigma_u^-(\mathcal{H}) \neq \emptyset$ *for all* $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ *such that* $\mathcal{H} \neq \{\Omega\}$*.*

4) *If* $\{X\} \in \Sigma$ *for all proper subsets* X *of* Ω*, then* $\Sigma_\ell^-(\mathcal{H}) \neq \emptyset$ *for all* $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ *such that* $\mathcal{H} \neq \{\Omega\}$*.*

Proof. Set $\Omega = \{x_1, \ldots, x_n\}$.

1) By Lemma 1.15, the fact that $\mathcal{H} \neq \{\emptyset\}$ ensures that $\mathcal{H} \leqslant^+ \mathcal{U}_{1,\Omega} = \{\{x_1\}, \ldots, \{x_n\}\}$. This, together with the assumption that $\mathcal{U}_{1,\Omega} \in \Sigma$, gives $\Sigma_u^+(\mathcal{H}) \neq \emptyset$.

2) Since $\mathcal{H} \neq \{\emptyset\}$, it is clear that for every $A \in \mathcal{H}$ we have $\{A\} \leqslant^+ \mathcal{H}$. By assumption $\{A\} \in \Sigma$ if A is non-empty, hence we conclude that $\{A\} \in \Sigma_\ell^+(\mathcal{H})$ and thus $\Sigma_\ell^+(\mathcal{H})$ is non-empty.

3) If $\mathcal{H} \neq \{\Omega\}$, then $A \subsetneq \Omega$ for all $A \in \mathcal{H}$. Hence, we have that $\mathcal{H} \leqslant^- \mathcal{U}_{n-1,\Omega}$. Thus $\Sigma_u^-(\mathcal{H}) \neq \emptyset$, because $\mathcal{U}_{n-1,\Omega} = (\mathcal{U}_{1,\Omega})^c \in \Sigma$.

   We also give a different proof of this statement based on statement 1). Let $\mathcal{E} = \{Y^c : Y \in \Sigma\} \subseteq \mathrm{Hyp}(\Omega)$. From our assumption we have that $\mathcal{U}_{1,X} \in \mathcal{E}$ for all non-empty subsets X of Ω (because $\mathcal{U}_{1,X} = ((\mathcal{U}_{1,X})^c)^c$ and $(\mathcal{U}_{1,X})^c \in \Sigma$). Moreover, since $\mathcal{H} \neq \{\Omega\}$, the hypergraph $\mathcal{H}^c$ is different from $\{\emptyset\}$. Therefore, by applying the first statement of this proposition to the hypergraph $\mathcal{H}^c$ it follows that $\mathcal{E}_u^+(\mathcal{H}^c)$ is non-empty. By Lemma 1.35 we know that $\Sigma_u^-(\mathcal{H}) = \mathcal{E}_u^+(\mathcal{H}^c)$. So we get that $\Sigma_u^-(\mathcal{H}) \neq \emptyset$.

4) If $\mathcal{H} \neq \{\Omega\}$, then $A \subsetneq \Omega$ for all $A \in \mathcal{H}$. Therefore, we have that $\{A\} \leqslant^- \mathcal{H}$ and $\{A\} \in \Sigma$ for all $A \in \mathcal{H}$. So $\Sigma_\ell^-(\mathcal{H}) \neq \emptyset$.

   We can also prove this statement by applying the statement 2). The proof is analogous to one given in the previous item. Namely, we consider the family of hypergraphs $\mathcal{E} = \{Y^c : Y \in \Sigma\} \subseteq \mathrm{Hyp}(\Omega)$ and we apply the second statement of this proposition to the hypergraph $\mathcal{H}^c$. Since by Lemma 1.35 $\Sigma_\ell^-(\mathcal{H}) = \mathcal{E}_\ell^+(\mathcal{H}^c)$, we get that $\Sigma_\ell^-(\mathcal{H}) \neq \emptyset$. □

The relationships among the four statements of the proposition above are given in the following two remarks and are summarized in Table 3.2.

Remark 3.3 Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ and let $\mathrm{Tr}(\Sigma) \subseteq \mathrm{Hyp}(\Omega)$ be the image of Σ by the transversal map: $\mathrm{Tr}(\Sigma) = \{\mathrm{Tr}(\mathcal{H}') : \mathcal{H}' \in \Sigma\}$. Observe that from Lemma 1.35 it follows that:

$$\Sigma_\ell^+(\mathcal{H}) = \mathrm{Tr}(\mathrm{Tr}(\Sigma)_u^+(\mathrm{Tr}(\mathcal{H})))$$

and that:

$$\Sigma_u^+(\mathcal{H}) = \mathrm{Tr}(\mathrm{Tr}(\Sigma)_\ell^+(\mathrm{Tr}(\mathcal{H}))).$$

In addition, it is clear that for a non-empty subset $X \subseteq \Omega$ we have that $\mathrm{Tr}(\{X\}) = \mathscr{U}_{1,X}$ and that $\mathrm{Tr}(\mathscr{U}_{1,X}) = \{X\}$. Therefore, the statements 1) and 2) of Proposition 3.2 are one the transversal of the other.

Remark 3.4 From the proof of Proposition 3.2 we get that the results of the statements 3) and 4) can be obtained from the results 1) and 2) by considering complementary families. Moreover, we have seen in Remark 3.3 that the statements 1) and 2) are one the transversal of the other. Thereby we conclude that statement 3) can be obtained from statement 4) by considering the complementary of the transversals of the complementary families of hypergraphs. We stress that actually the tranversal operation on the left is Tr^+ while the operation on the right is Tr^- (see Proposition 1.34). We summarize in Table 3.2 these relationships.

$\Sigma_u^+(\mathscr{H})$ Proposition 3.2, 1)	$\overset{(\cdot)^c}{\longleftrightarrow}$	$\Sigma_u^-(\mathscr{H})$ Proposition 3.2, 3)
$\mathrm{Tr}(\cdot) \updownarrow$		$\updownarrow (\mathrm{Tr}((\cdot)^c))^c$
$\Sigma_\ell^+(\mathscr{H})$ Proposition 3.2, 2)	$\overset{(\cdot)^c}{\longleftrightarrow}$	$\Sigma_\ell^-(\mathscr{H})$ Proposition 3.2, 4)

Table 3.2 The relationships between the statements of Proposition 3.2. We stress that actually the tranversal operation on the left is Tr^+ and the operation on the right is Tr^-.

3.1.2 The four collections of optimal completions of a hypergraph

Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a family of hypergraphs and let $\mathscr{H}$ be a hypergraph on Ω. For any choice of order $*_1 \in \{+,-\}$ and side $*_2 \in \{u,\ell\}$, we have introduced in Subsection 3.1.1 the families of completions $\Sigma_{*_2}^{*_1}(\mathscr{H})$.

Assume that $\Sigma_{*_2}^{*_1}(\mathscr{H}) \neq \emptyset$.

We denote by $\Phi_u^{*_1}(\mathscr{H})$ the collection of minimal completions of $\mathscr{H}$ with respect to the order $\leqslant^{*_1}$, where $*_1 \in \{+,-\}$; that is, the minimal elements of the poset $(\Sigma_u^{*_1}(\mathscr{H}), \leqslant^{*_1})$. Analogously, we denote by $\Phi_\ell^{*_1}(\mathscr{H})$ the collection of maximal completions of $\mathscr{H}$ with respect to the order $\leqslant^{*_1}$, where $*_1 \in \{+,-\}$; that is, the maximal elements of the poset $(\Sigma_\ell^{*_1}(\mathscr{H}), \leqslant^{*_1})$.

Observe that, from the definitions, the hypergraphs in $\Phi_{*_2}^{*_1}(\mathscr{H})$ are the hypergraphs in Σ *closest* to $\mathscr{H}$. Therefore we call the hypergraphs of the sets $\Phi_{*_2}^{*_1}(\mathscr{H})$ *the optimal completions of* $\mathscr{H}$. In Table 3.3 we summarize these definitions.

	Order $\leqslant^+$	Order $\leqslant^-$
Upper (u)	$\Phi_u^+(\mathscr{H}) = \min(\Sigma_u^+(\mathscr{H}), \leqslant^+)$	$\Phi_u^-(\mathscr{H}) = \min(\Sigma_u^-(\mathscr{H}), \leqslant^-)$
Lower (ℓ)	$\Phi_\ell^+(\mathscr{H}) = \max(\Sigma_\ell^+(\mathscr{H}), \leqslant^+)$	$\Phi_\ell^-(\mathscr{H}) = \max(\Sigma_\ell^-(\mathscr{H}), \leqslant^+)$

Table 3.3 The optimal completions $\Phi_{*_2}^{*_1}(\mathscr{H})$ of a hypergraph $\mathscr{H}$ depending on the poset considered, $*_1 \in \{+,-\}$ and $*_2 \in \{u, \ell\}$.

3.1.3 The four decompositions of a hypergraph

Let $*_1 \in \{+,-\}$ and $*_2 \in \{u, \ell\}$. We say that a family of hypergraphs $\{\mathscr{H}_1, \dots, \mathscr{H}_r\} \subseteq \Sigma$ is a $\Sigma_{*_2}^{*_1}$-*decomposition* of $\mathscr{H}$ if the following equality holds:

$$\mathscr{H} = \mathscr{H}_1 \square_{*_2}^{*_1} \cdots \square_{*_2}^{*_1} \mathscr{H}_r,$$

where $\square_{*_2}^{*_1} \in \{\overset{+}{\sqcap}, \overset{+}{\sqcup}, \overset{-}{\sqcap}, \overset{-}{\sqcup}\}$, depending on the poset considered (see Table 3.4).

Remark 3.5 If $\{\mathscr{H}_1, \dots, \mathscr{H}_r\} \subseteq \Sigma$ is a $\Sigma_{*_2}^{*_1}$-decomposition of $\mathscr{H}$, then we have that $\mathscr{H}_i \in \Sigma_{*_2}^{*_1}(\mathscr{H})$ for all $i = 1, \dots, r$. Hence, if the hypergraph $\mathscr{H}$ has a $\Sigma_{*_2}^{*_1}$-decomposition, then $\Sigma_{*_2}^{*_1}(\mathscr{H}) \neq \emptyset$.

	Order $\leqslant^+$	Order $\leqslant^-$
Upper (u)	$\square_{*_2}^{*_1} = \overset{+}{\sqcap}$	$\square_{*_2}^{*_1} = \overset{-}{\sqcap}$
Lower (ℓ)	$\square_{*_2}^{*_1} = \overset{+}{\sqcup}$	$\square_{*_2}^{*_1} = \overset{-}{\sqcup}$

Table 3.4 Operations $\square_{*_2}^{*_1}$ that occurs in a $\Sigma_{*_2}^{*_1}$-decomposition of a hypergraph $\mathscr{H}$ depending on the poset considered, where $*_1 \in \{+,-\}$ and $*_2 \in \{u, \ell\}$.

Proposition 3.6 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω. Let $\mathscr{H}$ be a hypergraph on Ω such that $\Sigma_{*_2}^{*_1}(\mathscr{H}) \neq \emptyset$, where $*_1 \in \{+,-\}$ and $*_2 \in \{u, \ell\}$. Then the following statements are equivalents.*

1) *There exists a $\Sigma_{*_2}^{*_1}$-decomposition of $\mathscr{H}$.*

2) *The set of optimal completions $\Phi_{*_2}^{*_1}(\mathscr{H})$ is a $\Sigma_{*_2}^{*_1}$-decomposition of $\mathscr{H}$.*

Proof. We present here the proof for the case $*_1 = +$ and $*_2 = u$. The other cases are formally proved in the same way.

It is enough to prove that the statement 1) implies the statement 2). Let us write $\Phi_u^+(\mathcal{H}) = \{\mathcal{H}_1, \dots, \mathcal{H}_r\}$ and let us consider a Σ_u^+-decomposition of the hypergraph $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_1' \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_s'.$$

Since $\mathcal{H}_i \in \Sigma_u^+(\mathcal{H})$, for all $i \in \{1, \dots, s\}$ there exists a minimal completion $\mathcal{H}_{j_i} \in \Phi_u^+(\mathcal{H})$ such that $\mathcal{H}_{j_i} \leqslant^+ \mathcal{H}_i'$. On one hand we have that $\mathcal{H} \leqslant^+ \mathcal{H}_j$ for all $j \in \{1, \dots, r\}$, so we get that $\mathcal{H} \leqslant^+ \mathcal{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_r$. On the other hand we also have that $\mathcal{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_r \leqslant^+ \mathcal{H}_{j_1} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_{j_s}$ and $\mathcal{H}_{j_1} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_{j_s} \leqslant^+ \mathcal{H}_1' \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_s' = \mathcal{H}$. Hence:

$$\mathcal{H} \leqslant^+ \mathcal{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_r \leqslant^+ \mathcal{H},$$

and from these inequalities we get the equality. □

In view of Proposition 3.6, we can reduce ourselves to the study of the optimal decompositions of a hypergraph; that is, to the case where the hypergraphs appearing in the decomposition are the *closest* to $\mathcal{H}$ (minimal or maximal, depending on $*_2$ being u or ℓ, respectively).

An interesting result arising from the existence of decomposition is the following corollary that characterizes when the poset $(\Sigma_{*_2}^{*_1}(\mathcal{H}), \leqslant^{*_1})$ has a unique minimal element if $*_2 = u$ or a unique maximal element if $*_2 = \ell$.

Corollary 3.7 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω. Let $\mathcal{H}$ be a hypergraph on Ω. Assume that $\mathcal{H}$ has a $\Sigma_{*_2}^{*_1}$-decomposition, where $*_1 \in \{+, -\}$ and $*_2 \in \{u, \ell\}$. Then the following statements are equivalent.*

1) $\Phi_{*_2}^{*_1}(\mathcal{H})$ has only an element.

2) $\mathcal{H} \in \Sigma_{*_2}^{*_1}(\mathcal{H})$.

3) $\Phi_{*_2}^{*_1}(\mathcal{H}) = \{\mathcal{H}\}$.

Proof. It is clear that the second statement implies the first one, and that the third property implies the second one. Thus we only have to prove that the first statement implies the third one. Assume that $\Phi_{*_2}^{*_1}(\mathcal{H}) = \{\mathcal{H}'\}$. By hypothesis, there exists a $\Sigma_{*_2}^{*_1}$-decomposition of $\mathcal{H}$. Therefore, by Proposition 3.6, there also exists a $\Sigma_{*_2}^{*_1}$-decomposition of $\mathcal{H}$ consisting of elements of $\Phi_{*_2}^{*_1}(\mathcal{H})$. Hence $\mathcal{H} = \mathcal{H}'$. □

Moreover, from the existence of decompositions for the hypergraph $\mathcal{H}$ we conclude that the hypergraph $\mathcal{H}$ is univocally determined by the family $\Phi_{*_2}^{*_1}(\mathcal{H})$ of its $\Sigma_{*_2}^{*_1}$-completions; that is, $\mathcal{H}$ is determined by its optimal completions. Namely, we get the following corollary.

Corollary 3.8 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω. Let $\mathcal{H}$ and $\mathcal{H}'$ be hypergraphs on Ω. Assume that both $\mathcal{H}$ and $\mathcal{H}'$ have a $\Sigma_{*_2}^{*_1}$-decomposition, where $*_1 \in \{+, -\}$ and $*_2 \in \{u, \ell\}$. Then the following properties are equivalent.*

1) $\mathcal{H}' = \mathcal{H}$.

2) $\Phi^{*_1}_{*_2}(\mathcal{H}) = \Phi^{*_1}_{*_2}(\mathcal{H}')$.

Proof. That statement 2) implies statement 1) is a consequence of Proposition 3.6. □

3.1.4 Avoidance properties and decompositions

Let $\mathcal{H}$ be a hypergraph on Ω and let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on Ω. Let $*_1 \in \{+,-\}$ and $*_2 \in \{u,\ell\}$. We are going to see that if $\Sigma^{*_1}_{*_2}(\mathcal{H}) \neq \emptyset$, then we have a decomposition for $\mathcal{H}$ if and only if the hypergraph $\mathcal{H}$ fulfills a property which we call the $\Sigma^{*_1}_{*_2}$*-avoidance property* for the hypergraph $\mathcal{H}$.

Avoidance property for $u, \leqslant^+$. We say that the hypergraph $\mathcal{H}$ has the Σ^+_u*-avoidance property* if for all $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H} \leqslant^+ \mathcal{H}'$ and $\mathcal{H} \neq \mathcal{H}'$, there exists $\mathcal{H}_0 \in \Sigma$ such that $\mathcal{H} \leqslant^+ \mathcal{H}_0$ and $\mathcal{H}' \nleqslant^+ \mathcal{H}_0$.

Avoidance property for $\ell, \leqslant^+$. We say that the hypergraph $\mathcal{H}$ has the Σ^+_ℓ*-avoidance property* if for all $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H}' \leqslant^+ \mathcal{H}$ and $\mathcal{H} \neq \mathcal{H}'$, there exists $\mathcal{H}_0 \in \Sigma$ such that $\mathcal{H}_0 \leqslant^+ \mathcal{H}$ and $\mathcal{H}_0 \nleqslant^+ \mathcal{H}'$.

Avoidance property for $u, \leqslant^-$. We say that the hypergraph $\mathcal{H}$ has the Σ^-_u*-avoidance property* if for all $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H} \leqslant^- \mathcal{H}'$ and $\mathcal{H} \neq \mathcal{H}'$, there exists $\mathcal{H}_0 \in \Sigma$ such that $\mathcal{H} \leqslant^- \mathcal{H}_0$ and $\mathcal{H}' \nleqslant^- \mathcal{H}_0$.

Avoidance property for $\ell, \leqslant^-$. We say that the hypergraph $\mathcal{H}$ has the Σ^-_ℓ*-avoidance property* if for all $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H}' \leqslant^- \mathcal{H}$ and $\mathcal{H} \neq \mathcal{H}'$, there exists $\mathcal{H}_0 \in \Sigma$ such that $\mathcal{H}_0 \leqslant^- \mathcal{H}$ and $\mathcal{H}_0 \nleqslant^- \mathcal{H}'$.

Remark 3.9 If $\mathcal{H} \in \Sigma$, then the hypergraph $\mathcal{H}$ satisfies the $\Sigma^{*_1}_{*_2}$-avoidance property for $*_1 \in \{+,-\}$ and $*_2 \in \{u,\ell\}$, because in this case we can take $\mathcal{H}_0 = \mathcal{H}$.

Remark 3.10 It is straightforward to check when one of the hypergraphs $\mathcal{H} = \emptyset$, $\{\emptyset\}$, or $\{\Omega\}$ satisfies the $\Sigma^{*_1}_{*_2}$-avoidance property for $*_1 \in \{+,-\}$ and $*_2 \in \{u,\ell\}$. In Table 3.11 we indicate whether these hypergraphs satisfy this property or give the equivalent condition that the family of hypergraphs $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ must fulfil in order that the hypergraph $\mathcal{H}$ satisfies the corresponding avoidance property.

The following theorems relate the existence of $\Sigma^{*_1}_{*_2}$-decompositions with the satisfiability of the $\Sigma^{*_1}_{*_2}$-avoidance property. We start with the case $*_1 = +$ and $*_2 = u$.

Theorem 3.11 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω. Let $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ such that $\Sigma^+_u(\mathcal{H}) \neq \emptyset$. Then the following statements are equivalent.*

1) *$\mathcal{H}$ has a Σ^+_u-decomposition.*

	$\emptyset$	$\{\emptyset\}$	$\{\Omega\}$
Σ_u^+-a.p.	$\forall \mathcal{H}' \neq \emptyset \; \exists \mathcal{H}_0 \in \Sigma$: $\mathcal{H}' \not\leqslant^+ \mathcal{H}_0$	always	$\forall \mathcal{H}' \neq \emptyset, \{\Omega\} \; \exists \mathcal{H}_0 \in \Sigma$: $\mathcal{H}' \not\leqslant^+ \mathcal{H}_0$
Σ_u^--a.p.	$\forall \mathcal{H}' \neq \emptyset \; \exists \mathcal{H}_0 \in \Sigma$: $\mathcal{H}' \not\leqslant^- \mathcal{H}_0$	$\forall \mathcal{H}' \neq \emptyset, \{\emptyset\} \; \exists \mathcal{H}_0 \in \Sigma$: $\mathcal{H}' \not\leqslant^- \mathcal{H}_0$	always
Σ_ℓ^+-a.p.	always	$\forall \mathcal{H}' \neq \{\emptyset\} \; \exists \mathcal{H}_0 \in \Sigma$: $\mathcal{H}_0 \not\leqslant^+ \mathcal{H}'$	always
Σ_ℓ^--a.p.	always	always	$\forall \mathcal{H}' \neq \{\Omega\} \; \exists \mathcal{H}_0 \in \Sigma$: $\mathcal{H}_0 \not\leqslant^- \mathcal{H}'$

Table 3.5 The $\Sigma_{*_2}^{*_1}$-avoidance properties for the hypergraphs $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$ and for $*_1 \in \{+,-\}$ and $*_2 \in \{u,\ell\}$.

2) $\mathcal{H}$ has the Σ_u^+-avoidance property.

Proof. Assume first that $\mathcal{H}$ has a Σ_u^+-decomposition. Then, by Proposition 3.6, we can write $\mathcal{H} = \mathcal{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_r$, where $\Phi_u^+(\mathcal{H}) = \{\mathcal{H}_1, \ldots, \mathcal{H}_r\}$. Let $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H} \leqslant^+ \mathcal{H}'$ and $\mathcal{H} \neq \mathcal{H}'$. We must demonstrate that there exists $\mathcal{H}_0 \in \Sigma$ such that $\mathcal{H} \leqslant^+ \mathcal{H}_0$ and $\mathcal{H}' \not\leqslant^+ \mathcal{H}_0$. Even more, we claim that there exists $\mathcal{H}_0 \in \Phi_u^+(\mathcal{H})$ such that $\mathcal{H}' \not\leqslant^+ \mathcal{H}_0$. Indeed, it is clear that $\mathcal{H} \leqslant^+ \mathcal{H}_i$ for all $i = 1, \ldots, r$. Moreover, if $\mathcal{H}' \leqslant^+ \mathcal{H}_i$ for all $i = 1, \ldots, r$, then by Remark 1.19 and by hypothesis we have that $\mathcal{H}' \leqslant^+ \mathcal{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_r = \mathcal{H}$. Then we would get $\mathcal{H} = \mathcal{H}'$, a contradiction. Therefore, the hypergraph $\mathcal{H}$ has the Σ_u^+-avoidance property.

Assume now that $\mathcal{H}$ has the Σ_u^+-avoidance property. Set $\Phi_u^+(\mathcal{H}) = \{\mathcal{H}_1, \ldots, \mathcal{H}_r\}$. We know that $\mathcal{H} \leqslant^+ \mathcal{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_r$. Let us denote $\mathcal{H}' = \mathcal{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_r$. Asume that $\mathcal{H} \neq \mathcal{H}'$. In such a case, by hypothesis, there exists $\mathcal{H}_0 \in \Sigma$ such that $\mathcal{H} \leqslant^+ \mathcal{H}_0$ and $\mathcal{H}' \not\leqslant^+ \mathcal{H}_0$. Since $\mathcal{H}_0 \in \Sigma_u^+(\mathcal{H})$, there is an element $\mathcal{H}_i \in \Phi_u^+(\mathcal{H})$ such that $\mathcal{H}_i \leqslant^+ \mathcal{H}_0$. Thus we get $\mathcal{H}' = \mathcal{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_r \leqslant^+ \mathcal{H}_i \leqslant^+ \mathcal{H}_0$, which is a contradiction. Therefore, $\mathcal{H} = \mathcal{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_r$. □

Analogously, we can state the following theorems concerning the other avoidance properties. The proofs are formally identical to the previous one.

Theorem 3.12 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω. Let $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ such that $\Sigma_\ell^+(\mathcal{H}) \neq \emptyset$. Then the following statements are equivalent.*

1) $\mathcal{H}$ has a Σ_ℓ^+-decomposition.

2) *$\mathscr{H}$ has the Σ_ℓ^+-avoidance property.*

Proof. Assume first that $\mathscr{H}$ has a Σ_ℓ^+-decomposition. Then, by Proposition 3.6, we can write $\mathscr{H} = \mathscr{H}_1 \overset{+}{\sqcup} \cdots \overset{+}{\sqcup} \mathscr{H}_r$, where $\Phi_\ell^+(\mathscr{H}) = \{\mathscr{H}_1 \ldots, \mathscr{H}_r\}$. Let $\mathscr{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathscr{H}' \leqslant^+ \mathscr{H}$ and $\mathscr{H}' \neq \mathscr{H}$. We have to show that there exists $\mathscr{H}_0 \in \Sigma$ such that $\mathscr{H}_0 \leqslant^+ \mathscr{H}$ and $\mathscr{H}_0 \nleqslant^+ \mathscr{H}'$. Even more, we claim that there exists $\mathscr{H}_0 \in \Phi_\ell^+(\mathscr{H})$ such that $\mathscr{H}_0 \nleqslant^+ \mathscr{H}'$. Indeed, it is clear that $\mathscr{H}_i \leqslant^+ \mathscr{H}$ for all $i = 1, \ldots, r$. Moreover if $\mathscr{H}_i \leqslant^+ \mathscr{H}'$ for all $\mathscr{H}_i \in \Phi_\ell^+(\mathscr{H})$, then by Remark 1.19 and by hypothesis we have $\mathscr{H} = \mathscr{H}_1 \overset{+}{\sqcup} \cdots \overset{+}{\sqcup} \mathscr{H}_r \leqslant^+ \mathscr{H}'$ and hence $\mathscr{H} = \mathscr{H}'$, which is a contradiction. Therefore, the hypergraph $\mathscr{H}$ has the Σ_ℓ^+-avoidance property.

Assume now that $\mathscr{H}$ has the Σ_ℓ^+-avoidance property. Set $\Phi_\ell^+(\mathscr{H}) = \{\mathscr{H}_1, \ldots, \mathscr{H}_r\}$. We know that $\mathscr{H}_1 \overset{+}{\sqcup} \cdots \overset{+}{\sqcup} \mathscr{H}_r \leqslant^+ \mathscr{H}$. Let us denote $\mathscr{H}' = \mathscr{H}_1 \overset{+}{\sqcup} \cdots \overset{+}{\sqcup} \mathscr{H}_r$. Assume that $\mathscr{H}' \neq \mathscr{H}$. In such a case, by hypothesis, there exists $\mathscr{H}_0 \in \Sigma$ such that $\mathscr{H}_0 \leqslant^+ \mathscr{H}$ and $\mathscr{H}_0 \nleqslant^+ \mathscr{H}'$. Since $\mathscr{H}_0 \in \Sigma_\ell^+(\mathscr{H})$, there exists an element $\mathscr{H}_i \in \Phi_\ell^+(\mathscr{H})$ such that $\mathscr{H}_0 \leqslant^+ \mathscr{H}_i$. Thus we get $\mathscr{H}_0 \leqslant^+ \mathscr{H}_i \leqslant^+ \mathscr{H}_1 \overset{+}{\sqcup} \cdots \overset{+}{\sqcup} \mathscr{H}_r = \mathscr{H}'$, which is a contradiction. Therefore, $\mathscr{H} = \mathscr{H}'$; that is, $\mathscr{H} = \mathscr{H}_1 \overset{+}{\sqcup} \cdots \overset{+}{\sqcup} \mathscr{H}_r$. □

Theorem 3.13 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω. Let $\mathscr{H} \in \mathrm{Hyp}(\Omega)$ such that $\Sigma_u^-(\mathscr{H}) \neq \emptyset$. Then the following statements are equivalent:*

1) *$\mathscr{H}$ has a Σ_u^--decomposition.*

2) *$\mathscr{H}$ has the Σ_u^--avoidance property.*

Proof. Assume that $\mathscr{H}$ has a Σ_u^--decomposition. Then, by Proposition 3.6, we can write $\mathscr{H} = \mathscr{H}_1 \overset{-}{\sqcap} \cdots \overset{-}{\sqcap} \mathscr{H}_r$, where $\Phi_u^-(\mathscr{H}) = \{\mathscr{H}_1, \ldots, \mathscr{H}_r\}$. Let $\mathscr{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathscr{H} \leqslant^- \mathscr{H}'$ and $\mathscr{H} \neq \mathscr{H}'$. We must prove that there exists $\mathscr{H}_0 \in \Sigma$ such that $\mathscr{H} \leqslant^- \mathscr{H}_0$ and $\mathscr{H}' \nleqslant^- \mathscr{H}_0$. In fact, we are going to prove that there exists $\mathscr{H}_0 \in \Phi_u^-(\mathscr{H})$ such that $\mathscr{H}' \nleqslant^- \mathscr{H}_0$. Indeed, it is clear that $\mathscr{H} \leqslant^- \mathscr{H}_i$ for all $i = 1, \ldots, r$. Moreover if $\mathscr{H}' \leqslant^- \mathscr{H}_i$ for all $\mathscr{H}_i \in \Phi_u^-(\mathscr{H})$, then by Remark 1.19 and by hypothesis we have that $\mathscr{H}' \leqslant^- \mathscr{H}_1 \overset{-}{\sqcap} \cdots \overset{-}{\sqcap} \mathscr{H}_r = \mathscr{H}$ and then we would get that $\mathscr{H} = \mathscr{H}'$ which is a contradiction. Hence, $\mathscr{H}$ has the Σ_u^--avoidance property.

Assume now that $\mathscr{H}$ has the Σ_u^--avoidance property. Let $\mathscr{H}' = \mathscr{H}_1 \overset{-}{\sqcap} \cdots \overset{-}{\sqcap} \mathscr{H}_r$, where $\Phi_u^-(\mathscr{H}) = \{\mathscr{H}_1, \ldots, \mathscr{H}_r\}$. We know that $\mathscr{H} \leqslant^+ \mathscr{H}_1 \overset{-}{\sqcap} \cdots \overset{-}{\sqcap} \mathscr{H}_r$. We want to show that $\mathscr{H} = \mathscr{H}'$. Assume on the contrary that $\mathscr{H} \neq \mathscr{H}'$. Observe that $\mathscr{H} \leqslant^- \mathscr{H}'$, by construction. Therefore, by hypothesis, there exists $\mathscr{H}_0 \in \Sigma$ such that $\mathscr{H} \leqslant^- \mathscr{H}_0$ and $\mathscr{H}' \nleqslant^- \mathscr{H}_0$. Then since $\mathscr{H}_0 \in \Sigma_u^-(\mathscr{H})$, there exists an element $\mathscr{H}_i \in \Phi_u^-(\mathscr{H})$ such that $\mathscr{H}_i \leqslant^- \mathscr{H}_0$. Thus, we have that $\mathscr{H}' = \mathscr{H}_1 \overset{-}{\sqcap} \cdots \overset{-}{\sqcap} \mathscr{H}_r \leqslant^- \mathscr{H}_i \leqslant^- \mathscr{H}_0$ which is a contradiction. □

Theorem 3.14 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω. Let $\mathscr{H} \in \mathrm{Hyp}(\Omega)$ such that $\Sigma_\ell^-(\mathscr{H}) \neq \emptyset$. Then the following statements are equivalent:*

1) $\mathcal{H}$ has a Σ_ℓ^--decomposition.

2) $\mathcal{H}$ has the Σ_ℓ^--avoidance property.

Proof. Assume that $\mathcal{H}$ has a Σ_ℓ^--decomposition. Then, by Proposition 3.6, we can write $\mathcal{H} = \mathcal{H}_1 \bar{\sqcup} \cdots \bar{\sqcup} \mathcal{H}_r$, where $\Phi_\ell^-(\mathcal{H}) = \{\mathcal{H}_1, \ldots, \mathcal{H}_r\}$. Let $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H}' \leqslant^- \mathcal{H}$ and $\mathcal{H} \neq \mathcal{H}'$. We must show that there exists $\mathcal{H}_0 \in \Sigma$ such that $\mathcal{H}_0 \leqslant^- \mathcal{H}$ and $\mathcal{H}_0 \nleqslant^- \mathcal{H}'$. In fact, we are going to see that there exists an element $\mathcal{H}_0 \in \Phi_\ell^-(\mathcal{H})$ such that $\mathcal{H}_0 \nleqslant^- \mathcal{H}'$. Indeed, if $\mathcal{H}_i \leqslant^- \mathcal{H}'$ for all $\mathcal{H}_i \in \Phi_\ell^-(\mathcal{H})$, then we have that $\mathcal{H} = \mathcal{H}_1 \bar{\sqcup} \cdots \bar{\sqcup} \mathcal{H}_r \leqslant^- \mathcal{H}'$, and we would get $\mathcal{H} = \mathcal{H}'$, which is a contradiction.

Assume now that $\mathcal{H}$ has the Σ_ℓ^--avoidance property. Let $\mathcal{H}' = \mathcal{H}_1 \bar{\sqcup} \cdots \bar{\sqcup} \mathcal{H}_r$, where $\Phi_u^-(\mathcal{H}) = \{\mathcal{H}_1, \ldots, \mathcal{H}_r\}$. We want to show that $\mathcal{H} = \mathcal{H}'$. Assume on the contrary that $\mathcal{H} \neq \mathcal{H}'$. Observe that $\mathcal{H}_i \leqslant^- \mathcal{H}$ for all $i = 1, \ldots, r$ and so $\mathcal{H}' \leqslant^- \mathcal{H}$. Therefore, there exists $\mathcal{H}_0 \in \Sigma$ such that $\mathcal{H}_0 \leqslant^- \mathcal{H}$ and $\mathcal{H}_0 \nleqslant^- \mathcal{H}'$. Then, since $\mathcal{H}_0 \in \Sigma_\ell^-(\mathcal{H})$, there exists an element $\mathcal{H}_i \in \Phi_\ell^-(\mathcal{H})$ such that $\mathcal{H}_0 \leqslant^- \mathcal{H}_i$. So we have that $\mathcal{H}_0 \leqslant^- \mathcal{H}_i \leqslant^- \mathcal{H}_1 \bar{\sqcap} \cdots \bar{\sqcap} \mathcal{H}_r = \mathcal{H}'$, which is a contradiction. □

3.1.5 Completion and decomposition of hypergraphs under mild conditions

In this subsection we are going to prove that under the assumptions of Proposition 3.2 and some mild hypothesis on Σ, every suitable hypergraph satisfies the avoidance property for the corresponding poset and operation.

Proposition 3.15 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω. Then the following statements hold.*

1) *If $\mathcal{U}_{1,X} \in \Sigma$ for all non-empty subsets X of Ω, then every hypergraph $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H} \neq \emptyset, \{\emptyset\}$ satisfies the Σ_u^+-avoidance property.*

2) *If $\{X\} \in \Sigma$ for all non-empty subsets X of Ω, then every hypergraph $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H} \neq \emptyset, \{\emptyset\}$ satisfies the Σ_ℓ^+-avoidance property.*

3) *If $(\mathcal{U}_{1,X})^c \in \Sigma$ for all non-empty subsets X of Ω, then every hypergraph $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H} \neq \emptyset, \{\Omega\}$ satisfies the Σ_u^--avoidance property.*

4) *If $\{X\} \in \Sigma$ for all proper subsets X of Ω, then every hypergraph $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H} \neq \emptyset, \{\Omega\}$ satisfies the Σ_ℓ^--avoidance property.*

Proof.

1) Let $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H} \leqslant^+ \mathcal{H}'$ and $\mathcal{H} \neq \mathcal{H}'$. We must demonstrate that there exists $\mathcal{H}_0 \in \Sigma$ such that $\mathcal{H} \leqslant^+ \mathcal{H}_0$ and $\mathcal{H}' \nleqslant^+ \mathcal{H}_0$. We observe that if $\mathcal{H} = \{\emptyset\}$, then there is no hypergraph $\mathcal{H}'$ satisfying these properties.

If $\mathscr{H} = \mathscr{U}_{1,X}$ for some non-empty subset $X \subseteq \Omega$, then we can take $\mathscr{H}_0 = \mathscr{H}$. So we may assume that $\mathscr{H} \neq \mathscr{U}_{1,X}$ for all non-empty subsets $X \subseteq \Omega$.

We claim that there exists $B \in \mathscr{H}'$ such that $B \neq A$ for all $A \in \mathscr{H}$. Indeed, if for all $B \in \mathscr{H}'$ there exists $A \in \mathscr{H}$ such that $B = A$, then we have $\mathscr{H}' \subseteq \mathscr{H}$, and consequently $\mathscr{H}' \leqslant^+ \mathscr{H}$. Thus, as $\leqslant^+$ is a partial order, we deduce that $\mathscr{H}' = \mathscr{H}$, which is a contradiction.

Let us denote by $B \in \mathscr{H}'$ such an element. We claim that $\Omega \setminus B \neq \emptyset$. Indeed, otherwise $B = \Omega$, hence $\mathscr{H}' = \{\Omega\}$, so $\mathscr{H}' \leqslant^+ \mathscr{H}$, because $\mathscr{H} \neq \emptyset$. Thus $\mathscr{H}' = \mathscr{H}$, a contradiction.

We consider the hypergraph $\mathscr{H}_0 = \mathscr{U}_{1,\Omega\setminus B}$.

We have that $\mathscr{H}_0 \in \Sigma$, because $\Omega \setminus B \neq \emptyset$.

Moreover, we have that $\mathscr{H}' \nleqslant^+ \mathscr{H}_0$, because $B \in \mathscr{H}'$ but there is no element of $\mathscr{H}_0$ contained in B.

Finally let us prove that $\mathscr{H} \leqslant^+ \mathscr{H}_0$. Let $A \in \mathscr{H}$. As $A \neq B$, there is an element $a \in A$ such that $a \notin B$. Then $\{a\} \subseteq A$ and $\{a\} \in \mathscr{H}_0$. So there exists $A' \in \mathscr{H}_0$ such that $A' \subseteq A$, as we wanted to prove.

2) Let $\mathscr{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathscr{H}' \leqslant^+ \mathscr{H}$ and $\mathscr{H}' \neq \mathscr{H}$. We must demonstrate that there exists $\mathscr{H}_0 \in \Sigma$ such that $\mathscr{H}_0 \leqslant^+ \mathscr{H}$ and $\mathscr{H}_0 \nleqslant^+ \mathscr{H}'$. We observe that if $\mathscr{H} = \emptyset$, then there is no hypergraph $\mathscr{H}'$ satisfying these properties.

   If $\mathscr{H} = \{X\}$, for some non-empty subset $X \subseteq \Omega$, then we can take $\mathscr{H}_0 = \mathscr{H} \in \Sigma_\ell^+(\mathscr{H})$. So assume that $\mathscr{H} \neq \{X\}$ for all non-empty subsets $X \subseteq \Omega$.

   As $\mathscr{H} \nleqslant^+ \mathscr{H}'$, there exists $A_0 \in \mathscr{H}$ such that $A' \nsubseteq A_0$, for all $A' \in \mathscr{H}'$. Let $\mathscr{H}_0 = \{A_0\}$. As $\mathscr{H} \neq \{\emptyset\}$, the subset A_0 is non-empty and hence $\mathscr{H}_0 \in \Sigma$. Moreover, we have that $\mathscr{H}_0 \leqslant^+ \mathscr{H}$ and $\mathscr{H}_0 \nleqslant^+ \mathscr{H}'$, by construction.

3) Let $\Upsilon = \{\mathscr{G}^c : \mathscr{G} \in \Sigma\} \subseteq \mathrm{Hyp}(\Omega)$. By hypothesis, we have that $\mathscr{U}_{1,X} \in \Upsilon$, for all $X \subseteq \Omega$, $X \neq \emptyset$. Moreover, if $\mathscr{H} \neq \{\Omega\}$, then $\mathscr{H}^c \neq \{\emptyset\}$; while $\mathscr{H}^c = \emptyset$, whenever $\mathscr{H} = \emptyset$. Hence, by statement 1), for every hypergraph $\mathscr{H}$ on Ω such that $\mathscr{H} \neq \emptyset, \{\Omega\}$, we get that the hypergraph $\mathscr{H}^c$ satisfies the Υ_u^+-avoidance property. Now let $\mathscr{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathscr{H} \leqslant^- \mathscr{H}'$ and $\mathscr{H} \neq \mathscr{H}'$. Then, by Lemma 1.2 and Lemma 1.35, $\mathscr{H}^c \leqslant^+ (\mathscr{H}')^c$ and $\mathscr{H}^c \neq (\mathscr{H}')^c$. As $\mathscr{H}^c$ has the Υ_u^+-avoidance property, there exists $\mathscr{H}_1 \in \Upsilon$ such that $\mathscr{H}^c \leqslant^+ \mathscr{H}_1$ and $(\mathscr{H}')^c \nleqslant^+ \mathscr{H}_1$. Again by Lemma 1.2 and Lemma 1.35, we have that $\mathscr{H} \leqslant^- \mathscr{H}_1^c$ and $\mathscr{H}' \nleqslant^- \mathscr{H}_1^c$. Moreover, $\mathscr{H}_1^c \in \Sigma$, because $\mathscr{H}_1 \in \Upsilon$. Thus we can take $\mathscr{H}_0 = \mathscr{H}_1^c$.

4) As in the previous proof, we define $\Upsilon = \{\mathscr{G}^c : \mathscr{G} \in \Sigma\}$. Then $\{X\} \in \Upsilon$, for every $X \subseteq \Omega$, $X \neq \emptyset$. Now it is easy to see that $\mathscr{H}$ has the Σ_ℓ^--avoidance property if and only if $\mathscr{H}^c$ has the Υ_ℓ^+-avoidance property. Moreover, it is clear that $\mathscr{H} \neq \emptyset, \{\Omega\}$ if and only if $\mathscr{H}^c \neq \emptyset, \{\emptyset\}$. Therefore, statement 4) is a consequence of statement 2). □

As corollaries of Proposition 3.2, Corollaries 3.7 and 3.8 and Theorems 3.11, 3.12, 3.13 and 3.14 we get the following four results, which are given in [34, 35], but are proved differently.

Corollary 3.16 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω such that $\mathscr{U}_{1,X} \in \Sigma$ for all non-empty subsets X of Ω. If $\mathscr{H}$ is a hypergraph on Ω different from $\emptyset$ and $\{\emptyset\}$, then the set $\Sigma_u^+(\mathscr{H})$ of the Σ_u^+-completions of $\mathscr{H}$ is non-empty, and:*

$$\mathscr{H} = \mathscr{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_r\,,$$

where $\mathscr{H}_1, \ldots, \mathscr{H}_r$ are the minimal elements of the poset $(\Sigma_u^+(\mathscr{H}), \leqslant^+)$. In particular, the following statements hold.

1) *A hypergraph $\mathscr{H}$ on Ω different from $\emptyset$ and $\{\emptyset\}$ belongs to the family Σ if and only if the poset $(\Sigma_u^+(\mathscr{H}), \leqslant^+)$ has a unique minimal element (namely, the hypergraph $\mathscr{H}$).*

2) *If $\mathscr{H}, \mathscr{H}'$ are hypergraphs on Ω different from $\emptyset$ and $\{\emptyset\}$, then $\mathscr{H} = \mathscr{H}'$ if and only if the posets $(\Sigma_u^+(\mathscr{H}), \leqslant^+)$ and $(\Sigma_u^+(\mathscr{H}'), \leqslant^+)$ have the same minimal elements.*

Corollary 3.17 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω such that $\{X\} \in \Sigma$ for all non-empty subsets X of Ω. If $\mathscr{H}$ is a hypergraph on Ω different from $\emptyset$ and $\{\emptyset\}$, then the set $\Sigma_\ell^+(\mathscr{H})$ of the Σ_ℓ^+-completions of $\mathscr{H}$ is non-empty, and:*

$$\mathscr{H} = \mathscr{H}_1 \overset{+}{\sqcup} \cdots \overset{+}{\sqcup} \mathscr{H}_r\,,$$

where $\mathscr{H}_1, \ldots, \mathscr{H}_r$ are the maximal elements of the poset $\big(\Sigma_\ell^+(\mathscr{H}), \leqslant^+\big)$. In particular, the following statements hold.

1) *A hypergraph $\mathscr{H}$ on Ω different from $\emptyset$ and $\{\emptyset\}$ belongs to the family Σ if and only if the poset $\big(\Sigma_\ell^+(\mathscr{H}), \leqslant^+\big)$ has a unique maximal element (namely, the hypergraph $\mathscr{H}$).*

2) *If $\mathscr{H}, \mathscr{H}'$ are hypergraphs on Ω different from $\emptyset$ and $\{\emptyset\}$, then $\mathscr{H} = \mathscr{H}'$ if and only if the posets $\big(\Sigma_\ell^+(\mathscr{H}), \leqslant^+\big)$ and $\big(\Sigma_\ell^+(\mathscr{H}'), \leqslant^+\big)$ have the same maximal elements.*

Corollary 3.18 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω such that $(\mathscr{U}_{1,X})^c \in \Sigma$ for all non-empty subsets X of Ω. If $\mathscr{H}$ is a hypergraph on Ω different from $\emptyset$ and $\{\Omega\}$, then the set $\Sigma_u^-(\mathscr{H})$ of the Σ_u^--completions of $\mathscr{H}$ is non-empty, and:*

$$\mathscr{H} = \mathscr{H}_1 \overset{-}{\sqcap} \cdots \overset{-}{\sqcap} \mathscr{H}_r\,,$$

where $\mathscr{H}_1, \ldots, \mathscr{H}_r$ are the minimal elements of the poset $(\Sigma_u^-(\mathscr{H}), \leqslant^-)$. In particular, the following statements hold.

1) *A hypergraph $\mathscr{H}$ on Ω different from $\emptyset$ and $\{\Omega\}$ belongs to the family Σ if and only if the poset $(\Sigma_u^-(\mathscr{H}), \leqslant^-)$ has a unique minimal element (namely, the hypergraph $\mathscr{H}$).*

2) *If $\mathscr{H}, \mathscr{H}'$ are hypergraphs on Ω different from $\emptyset$ and $\{\Omega\}$, then $\mathscr{H} = \mathscr{H}'$ if and only if the posets $(\Sigma_u^-(\mathscr{H}), \leqslant^-)$ and $(\Sigma_u^-(\mathscr{H}'), \leqslant^-)$ have the same minimal elements.*

Corollary 3.19 *Let $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ be a collection of hypergraphs on a finite set Ω such that $\{X\} \in \Sigma$ for all proper subsets X of Ω. If $\mathscr{H}$ is a hypergraph on Ω different from $\emptyset$ and $\{\Omega\}$, then the set $\Sigma_\ell^-(\mathscr{H})$ of the Σ_ℓ^--completions of $\mathscr{H}$ is non-empty, and:*

$$\mathscr{H} = \mathscr{H}_1 \overset{-}{\sqcup} \cdots \overset{-}{\sqcup} \mathscr{H}_r ,$$

where $\mathscr{H}_1, \ldots, \mathscr{H}_r$ are the maximal elements of the poset $\left(\Sigma_\ell^-(\mathscr{H}), \leqslant^-\right)$. In particular, the following statements hold.

1) *A hypergraph $\mathscr{H}$ on Ω different from $\emptyset$ and $\{\Omega\}$ belongs to the family Σ if and only if the poset $\left(\Sigma_\ell^-(\mathscr{H}), \leqslant^-\right)$ has a unique maximal element (namely, the hypergraph $\mathscr{H}$).*
2) *If $\mathscr{H}, \mathscr{H}'$ are hypergraphs on Ω different from $\emptyset$ and $\{\Omega\}$, then $\mathscr{H} = \mathscr{H}'$ if and only if the posets $\left(\Sigma_\ell^-(\mathscr{H}), \leqslant^-\right)$ and $\left(\Sigma_\ell^-(\mathscr{H}'), \leqslant^-\right)$ have the same maximal elements.*

Remark 3.20 We summarize in Tables 3.6 and 3.7 the relationships among the Corollaries 3.16, 3.17, 3.18, and 3.19. We observe that these results can be obtained from each other by applying suitable operations, as indicated in Table 3.7. Namely, results on the same horizontal line are related by applying complements and results on the same vertical line are related by applying the suitable transversal operation. Indeed, Corollaries 3.16 and 3.17 (left column) are related by the transversal Tr^+ and Corollaries 3.18 and 3.19 (right column) are related by the transversal Tr^- (see Lemma 1.34). See also Remark 3.3 and Remark 3.4.

	Order $\leqslant^+$	Order $\leqslant^-$
Upper (u)	Corollary 3.16	Corollary 3.18
Lower (ℓ)	Corollary 3.17	Corollary 3.19

Table 3.6 The corollaries on the existence of $\Sigma_{*_2}^{*_1}$-completions and decompositions under mild conditions, where $*_1 \in \{+,-\}$ and $*_2 \in \{u,\ell\}$.

Remark 3.21 We observe that in order to apply each one of the previous four corollaries, we need that the family Σ of hypergraphs contains a suitable collection Σ_0. For instance, in Corollary 3.16 we have that $\Sigma_0 = \{\mathscr{U}_{1,X} : X \subseteq \Omega, X \neq \emptyset\}$. Hence, we can always apply these corollaries by taking Σ the corresponding family Σ_0. If we consider this situation for Corollary 3.16, we recover the Proposition 1.45. Concretely, we get the following corollary.

Corollary 3.22 *Let Ω be a finite set. Let $\mathscr{H}$ be a hypergraph on Ω different from $\emptyset$ and $\{\Omega\}$. Set $\mathrm{Tr}(\mathscr{H}) = \{A_1, \ldots, A_m\}$. Then:*

$$\mathscr{H} = \mathscr{U}_{1,A_1} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{U}_{1,A_m} .$$

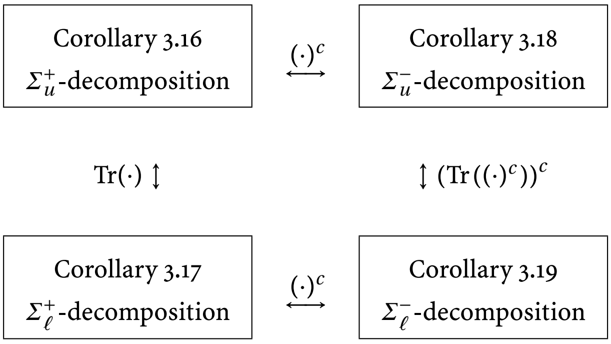


Table 3.7 The relationships between the different $\Sigma_{*_2}^{*_1}$-decompositions, where $*_1 \in \{+,-\}$ and $*_2 \in \{u,\ell\}$. We observe that the transversal operation on the left is Tr^+ whereas the operation appearing on the right is actually Tr^- (see Lemma 1.34).

Proof. Let $\Sigma = \{\mathscr{U}_{1,X} : X \subseteq \Omega, X \neq \emptyset\}$. Observe that if $X \subseteq \Omega$ and $X \neq \emptyset$, then $\mathscr{H} \leqslant^+ \mathscr{U}_{1,X}$ if and only if X is a transversal of $\mathscr{H}$. Hence, the minimal elements of the poset $(\Sigma_u^+(\mathscr{H}), \leqslant^+)$ are exactly those hypergraphs in Σ determined by the elements of $\mathrm{Tr}(\mathscr{H})$; that is, the hypergraphs of the form $\mathscr{U}_{1,X}$ with $X \in \mathrm{Tr}(\mathscr{H})$. □

3.2 Completion of hypergraphs by domination hypergraphs

We will consider the constructions made in Section 3.1 in the specific case that $\Sigma = \mathrm{DomHyp}(\Omega)$ and $\Sigma = \mathrm{DomHyp}_0(\Omega)$.

In general, given a hypergraph $\mathscr{H}$ on a finite set Ω there does not exist a graph G such that $\mathscr{H} \subseteq \mathscr{D}(G)$. Let us show an example.

Example 3.23 Let $n \geqslant 3$. On the finite set $\Omega = \{1, \dots, n\}$, we consider the hypergraph $\mathscr{H} = \mathscr{U}_{n-1,\Omega}$. We claim that $\mathscr{H} \not\subseteq \mathscr{D}(G)$ if G is a graph with vertex set Ω. Indeed, if there exists a graph G with $\mathscr{H} \subseteq \mathscr{D}(G)$, then $\mathscr{H} = \mathscr{D}(G)$. Hence, the hypergraph $\mathscr{U}_{n-1,\Omega}$ is a domination hypergraph. By applying Proposition 2.52, we get that $n = 2$, which is a contradiction.

However, given a hypergraph $\mathscr{H}$ on Ω, we can always find a graph G such that $\mathscr{H} \subseteq D(G) = \mathscr{D}(G)^+$; that is, a graph G such that every element of $\mathscr{H}$ is a dominating set of vertices of G, though not necessarily minimal. For example, if $\mathscr{H} \neq \emptyset, \{\emptyset\}$, then $\mathscr{H} \subseteq \mathscr{P}(\Omega) \setminus \{\emptyset\} = \mathscr{D}(K_\Omega)^+$.

At this point a natural question is to look for graphs G such that $\mathscr{H} \subseteq \mathscr{D}(G)^+$; that is, graphs G such that $\mathscr{H} \leqslant^+ \mathscr{D}(G)$. This fact justifies that we take the partial order $\leqslant^+$ to compare hypergraphs and approximate hypergraphs by domination hypergraphs. For the sake of completeness, we are going to consider the two binary relations $\leqslant^+$ and $\leqslant^-$.

3.2.1 The eight collection of domination completions of hypergraphs

Let $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ be a hypergraph *on* Ω. We define the sets $\mathrm{DomHyp}^{*_1}_{*_2}(\Omega, \mathcal{H})$ as the sets $\Sigma^{*_1}_{*_2}(\mathcal{H})$ where $\Sigma = \mathrm{DomHyp}(\Omega)$ is the family of domination hypergraphs on Ω, and $*_1 \in \{+,-\}$ and $*_2 \in \{u, \ell\}$; that is:

$$\begin{aligned}
\mathrm{DomHyp}^{+}_{u}(\Omega, \mathcal{H}) &= \{\mathcal{H}' \in \mathrm{DomHyp}(\Omega) : \mathcal{H} \leqslant^{+} \mathcal{H}'\},\\
\mathrm{DomHyp}^{+}_{\ell}(\Omega, \mathcal{H}) &= \{\mathcal{H}' \in \mathrm{DomHyp}(\Omega) : \mathcal{H}' \leqslant^{+} \mathcal{H}\},\\
\mathrm{DomHyp}^{-}_{u}(\Omega, \mathcal{H}) &= \{\mathcal{H}' \in \mathrm{DomHyp}(\Omega) : \mathcal{H} \leqslant^{-} \mathcal{H}'\},\\
\mathrm{DomHyp}^{-}_{\ell}(\Omega, \mathcal{H}) &= \{\mathcal{H}' \in \mathrm{DomHyp}(\Omega) : \mathcal{H}' \leqslant^{-} \mathcal{H}\}.
\end{aligned}$$

The elements of these sets are called *upper* or *lower domination completions* of $\mathcal{H}$ with respect to the order $\leqslant^{+}$ or with respect to the order $\leqslant^{-}$, respectively.

Analogously, we can define the sets $\mathrm{DomHyp}^{*_1}_{0,*_2}(\Omega, \mathcal{H})$, with $*_1 \in \{+,-\}$ and $*_2 \in \{u, \ell\}$, by considering domination hypergraphs with ground set Ω; that is by considering $\Sigma = \mathrm{DomHyp}_0(\Omega)$; so:

$$\begin{aligned}
\mathrm{DomHyp}^{+}_{0,u}(\Omega, \mathcal{H}) &= \{\mathcal{H}' \in \mathrm{DomHyp}_0(\Omega) : \mathcal{H} \leqslant^{+} \mathcal{H}'\},\\
\mathrm{DomHyp}^{+}_{0,\ell}(\Omega, \mathcal{H}) &= \{\mathcal{H}' \in \mathrm{DomHyp}_0(\Omega) : \mathcal{H}' \leqslant^{+} \mathcal{H}\},\\
\mathrm{DomHyp}^{-}_{0,u}(\Omega, \mathcal{H}) &= \{\mathcal{H}' \in \mathrm{DomHyp}_0(\Omega) : \mathcal{H} \leqslant^{-} \mathcal{H}'\},\\
\mathrm{DomHyp}^{-}_{0,\ell}(\Omega, \mathcal{H}) &= \{\mathcal{H}' \in \mathrm{DomHyp}_0(\Omega) : \mathcal{H}' \leqslant^{-} \mathcal{H}\}.
\end{aligned}$$

The elements of these sets are called *upper* or *lower domination completions with ground set* Ω of $\mathcal{H}$ with respect to the order $\leqslant^{+}$ or with respect to the order $\leqslant^{-}$, respectively.

Remark 3.24 From a theoretical point of view all these sets of domination completions of a given hypergraph $\mathcal{H}$ make sense. The choice of one or other depend on the application we have in mind. For instance, suppose that we have to design a network such that its collection of inclusion-minimal dominating sets of nodes must be close to a given collection $\mathcal{H}$. If the set of nodes of the network is fixed, then we are looking for domination hypergraphs in the set $\mathrm{DomHyp}^{*_1}_{0,*_2}(\Omega, \mathcal{H})$, for a suitable choice of $*_1 \in \{+,-\}$ and $*_2 \in \{u, \ell\}$, and where Ω is the set of nodes; but if, for example, it is allowed to add some more nodes, then we seek hypergraphs in the set of domination completions $\mathrm{DomHyp}^{*_1}_{*_2}(\Omega, \mathcal{H})$ where Ω is a superset of the set of nodes.

Remark 3.25 Let $\mathcal{H}$ be a hypergraph on a finite set Ω. Let us explore the interpretation of an upper or lower domination completion of $\mathcal{H}$ and the hypergraph $\mathcal{H}$. Recall from Lemma 1.15 the following characterization of the partial orders $\leqslant^{+}$ and $\leqslant^{-}$:

$$\begin{aligned}
\mathcal{H}_1 \leqslant^{+} \mathcal{H}_2 &\iff \forall A \in \mathcal{H}_1\, \exists B \in \mathcal{H}_2 : B \subseteq A,\\
\mathcal{H}_1 \leqslant^{-} \mathcal{H}_2 &\iff \forall A \in \mathcal{H}_1\, \exists B \in \mathcal{H}_2 : A \subseteq B.
\end{aligned}$$

Therefore, we can interpret the domination hypergraphs in $\mathrm{DomHyp}^{*_1}_{*_2}(\Omega, \mathcal{H})$ and in $\mathrm{DomHyp}^{*_1}_{0,*_2}(\Omega, \mathcal{H})$ as follows.

- $(\leqslant^+, u)$: $\mathscr{H} \leqslant^+ \mathscr{D}(G)$ if and only if for all $A \in \mathscr{H}$ there exists $B \in \mathscr{D}(G)$ such that $B \subseteq A$; that is, that $\mathscr{D}(G)$ is an upper domination completion of the hypergraph $\mathscr{H}$ with respect to the order $\leqslant^+$ means that every element of $\mathscr{H}$ is a dominating set of the graph G (though not necessarily minimal).

- $(\leqslant^+, \ell)$: $\mathscr{D}(G) \leqslant^+ \mathscr{H}$ if and only if for all $A \in \mathscr{D}(G)$ there exists $B \in \mathscr{H}$ such that $B \subseteq A$; that is, that $\mathscr{D}(G)$ is a lower domination completion of the hypergraph $\mathscr{H}$ with respect to the order $\leqslant^+$ means that every minimal dominating set of the graph G contains some element of the hypergraph $\mathscr{H}$ (though some elements of $\mathscr{H}$ may not be subsets of any minimal dominating set of the graph G).

- $(\leqslant^-, u)$: $\mathscr{H} \leqslant^- \mathscr{D}(G)$ if and only if for all $A \in \mathscr{H}$ there exists $B \in \mathscr{D}(G)$ such that $A \subseteq B$; that is, that $\mathscr{D}(G)$ is an upper domination completion of the hypergraph $\mathscr{H}$ with respect to the order $\leqslant^-$ means that every element of $\mathscr{H}$ can be completed to a minimal dominating set of the graph G.

- $(\leqslant^-, \ell)$: $\mathscr{D}(G) \leqslant^- \mathscr{H}$ if and only if for all $A \in \mathscr{D}(G)$ there exists $B \in \mathscr{H}$ such that $A \subseteq B$; that is, that $\mathscr{D}(G)$ is a lower domination completion of the hypergraph $\mathscr{H}$ with respect to the order $\leqslant^-$ means that every minimal dominating set of the graph G can be completed to some element of $\mathscr{H}$; that is, *some* elements of $\mathscr{H}$ are dominating sets of the graph G (though other elements of $\mathscr{H}$ may not bear any relationship with the dominating sets of G).

Remark 3.26 It is proved in [29] that, up to adding hyperedges, any hypergraph $\mathscr{H}$ on Ω can be encoded as a closed neighborhood hypergraph with ground set a suitable finite superset Ω^* of Ω. So, by considering the transversal, we get a domination hypergraph with ground set $\Omega^* \supseteq \Omega$ that, in some sense, is close to the hypergraph $\mathscr{H}$. This is not our case because, in the domination completions of $\mathscr{H}$ that we are considering, we can restrict the size of their ground set. Namely, here we consider domination hypergraphs close to $\mathscr{H}$ with ground set Ω or with ground set a subset Ω' of Ω.

We observe that we have that:

$$\begin{array}{ccccc}
\mathrm{DomHyp}^{*_1}_{*_2}(\Omega, \mathscr{H}) & \subseteq & \mathrm{DomHyp}(\Omega) & \subseteq & \mathrm{Hyp}(\Omega) \\
\cup\!\!| & & \cup\!\!| & & \cup\!\!| \\
\mathrm{DomHyp}^{*_1}_{0,*_2}(\Omega, \mathscr{H}) & \subseteq & \mathrm{DomHyp}_0(\Omega) & \subseteq & \mathrm{Hyp}_0(\Omega)
\end{array}$$

and that:

$$\mathrm{DomHyp}^{*_1}_{0,*_2}(\Omega, \mathscr{H}) = \mathrm{DomHyp}^{*_1}_{*_2}(\Omega, \mathscr{H}) \cap \mathrm{Hyp}_0(\Omega).$$

The sets of domination completions $\mathrm{DomHyp}^{*_1}_{*_2}(\Omega, \mathscr{H})$ and $\mathrm{DomHyp}^{*_1}_{0,*_2}(\Omega, \mathscr{H})$ may be different. In general, $\mathrm{DomHyp}^{*_1}_{0,*_2}(\Omega, \mathscr{H}) \subseteq \mathrm{DomHyp}^{*_1}_{*_2}(\Omega, \mathscr{H})$. The following example shows, in the case $*_1 = +$ and $*_2 = u$, that this inclusion can be strict.

Example 3.27 On the finite set $\Omega = \{1,2,3,4\}$, let us consider the hypergraph:

$$\mathcal{H} = \{\{1,2\},\{2,3\},\{3,4\}\},$$

which is not a domination hypergraph, because $\mathcal{H} \neq \mathcal{D}(G)$ for all graphs G with four vertices (see Table 2.3). Let us consider the hypergraph:

$$\mathcal{H}' = \{\{2\},\{3,4\}\}.$$

On one hand, $\mathcal{H}'$ is a domination hypergraph because $\mathcal{H}' = \mathcal{D}(G')$, where G' a graph of type $G_{3,2}$ (see Table 2.2). On the other hand, by Lemma 1.15 we have that $\mathcal{H} \leqslant^+ \mathcal{H}'$. Thus we conclude that $\mathcal{H}' \in \mathrm{DomHyp}^+_u(\Omega,\mathcal{H})$. However, $\mathcal{H}' \notin \mathrm{DomHyp}^+_{0,u}(\Omega,\mathcal{H})$, because $\mathcal{H}' \notin \mathrm{Hyp}_0(\Omega)$.

3.2.2 Existence of domination completions

Remark 3.28 Let Ω be a finite set. The hypergraphs $\emptyset$ and $\{\emptyset\}$ are never domination hypergraphs on Ω nor with ground set Ω. The hypergraph $\{\Omega\}$ is a domination hypergraph on or with ground set Ω if $\Omega \neq \emptyset$; indeed, in this case we have that $\{\Omega\} = \mathcal{D}(\overline{K_\Omega})$. We summarize these calculations in Table 3.8.

$\Sigma = \mathrm{DomHyp}_0(\Omega)$	$\emptyset \notin \Sigma$	$\{\emptyset\} \notin \Sigma$	$\{\Omega\} \in \Sigma$, if $\Omega \neq \emptyset$ (*)
$\Sigma = \mathrm{DomHyp}(\Omega)$	$\emptyset \notin \Sigma$	$\{\emptyset\} \notin \Sigma$	$\{\Omega\} \in \Sigma$, if $\Omega \neq \emptyset$ (*)

Table 3.8 The hypergraphs $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$ and whether they are domination hypergraphs. (*) If $\Omega \neq \emptyset$, then $\{\Omega\} = \mathcal{D}(\overline{K_\Omega})$.

Remark 3.29 Let Ω be a finite set. In Table 3.9 we have computed the upper and lower domination completions of the hypergraphs $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$.

The existence of domination completions with respect to the partial orders $\leqslant^+$ and $\leqslant^-$ is stated in the following proposition.

Proposition 3.30 *Let Ω be a finite set and let $\mathcal{H} \in \mathrm{Hyp}(\Omega)$. Then the following conditions hold.*

1) *The sets $\mathrm{DomHyp}^+_u(\Omega,\mathcal{H})$ and $\mathrm{DomHyp}^+_{0,u}(\Omega,\mathcal{H})$ of upper domination completions are non-empty if and only if $\mathcal{H} \neq \{\emptyset\}$.*

2) *The sets $\mathrm{DomHyp}^-_u(\Omega,\mathcal{H})$ and $\mathrm{DomHyp}^-_{0,u}(\Omega,\mathcal{H})$ of upper domination completions are non-empty.*

	$\mathcal{H} = \emptyset$	$\mathcal{H} = \{\emptyset\}$	$\mathcal{H} = \{\Omega\}$
$\mathrm{DomHyp}^{+}_{0,u}(\Omega,\mathcal{H})$	$\mathrm{DomHyp}_0(\Omega)$	$\emptyset$	$\mathrm{DomHyp}_0(\Omega)$
$\mathrm{DomHyp}^{+}_{u}(\Omega,\mathcal{H})$	$\mathrm{DomHyp}(\Omega)$	$\emptyset$	$\mathrm{DomHyp}(\Omega)$
$\mathrm{DomHyp}^{-}_{0,u}(\Omega,\mathcal{H})$	$\mathrm{DomHyp}_0(\Omega)$	$\mathrm{DomHyp}_0(\Omega)$	$\{\{\Omega\}\}$
$\mathrm{DomHyp}^{-}_{u}(\Omega,\mathcal{H})$	$\mathrm{DomHyp}(\Omega)$	$\mathrm{DomHyp}(\Omega)$	$\{\{\Omega\}\}$
$\mathrm{DomHyp}^{+}_{0,\ell}(\Omega,\mathcal{H})$	$\emptyset$	$\mathrm{DomHyp}_0(\Omega)$	$\{\{\Omega\}\}$
$\mathrm{DomHyp}^{+}_{\ell}(\Omega,\mathcal{H})$	$\emptyset$	$\mathrm{DomHyp}(\Omega)$	$\{\{\Omega\}\}$
$\mathrm{DomHyp}^{-}_{0,\ell}(\Omega,\mathcal{H})$	$\emptyset$	$\emptyset$	$\mathrm{DomHyp}_0(\Omega)$
$\mathrm{DomHyp}^{-}_{\ell}(\Omega,\mathcal{H})$	$\emptyset$	$\emptyset$	$\mathrm{DomHyp}(\Omega)$

Table 3.9 The upper and lower domination completions of the hypergraphs $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$.

3) *The sets* $\mathrm{DomHyp}^{+}_{\ell}(\Omega,\mathcal{H})$ *and* $\mathrm{DomHyp}^{+}_{0,\ell}(\Omega,\mathcal{H})$ *of lower domination completions are non-empty if and only if* $\mathcal{H} \neq \emptyset$.

4) *The set* $\mathrm{DomHyp}^{-}_{\ell}(\Omega,\mathcal{H})$ *of lower domination completions is non-empty if and only if* $\mathcal{H} \neq \emptyset, \{\emptyset\}$.

5) *The set* $\mathrm{DomHyp}^{-}_{0,\ell}(\Omega,\mathcal{H})$ *of lower domination completions is non-empty if and only if* $\mathcal{H} \neq \emptyset, \{\emptyset\}$ *and* $\mathcal{H}$ *has ground set* Ω.

Proof. Since $\mathrm{DomHyp}^{*_1}_{0,*_2}(\Omega,\mathcal{H}) \subseteq \mathrm{DomHyp}^{*_1}_{*_2}(\Omega,\mathcal{H})$, it is enough to proof these statements for $\mathrm{DomHyp}^{*_1}_{0,*_2}(\Omega,\mathcal{H})$.

1) $(u,+)$ and $(u,-)$: we must prove that there exists a graph G with vertex set $V(G) = \Omega$ and such that $\mathcal{H} \leqslant^{+} \mathcal{D}(G)$, respectively $\mathcal{H} \leqslant^{-} \mathcal{D}(G)$. The first case is clear by considering $G = K_\Omega$, the complete graph with vertex set Ω (whenever $\mathcal{H} \neq \{\emptyset\}$); and the second case is also clear by considering $G = \overline{K_\Omega}$, the null graph with vertex set Ω. Indeed, since $\mathcal{D}(K_\Omega) = \{\{x\} : x \in \Omega\} = \mathcal{U}_{1,\Omega}$ and $\mathcal{D}(\overline{K_\Omega}) = \{\Omega\}$, the inequalities $\mathcal{H} \leqslant^{+} \mathcal{D}(K_\Omega)$ and $\mathcal{H} \leqslant^{-} \mathcal{D}(\overline{K_\Omega})$ follow by applying Lemma 1.15.

2) $(\ell,+)$: we must prove that there exists a graph G with vertex set $V(G) = \Omega$ and such that $\mathcal{D}(G) \leqslant^{+} \mathcal{H}$. This result follows by considering $G = \overline{K_\Omega}$, the null graph with vertex set Ω. Indeed, since $\mathcal{D}(\overline{K_\Omega}) = \{\Omega\}$, the inequality $\mathcal{D}(K_\Omega) \leqslant^{-} \mathcal{H}$ follows by applying Lemma 1.15, because $\mathcal{H} \neq \emptyset$. To conclude the proof, we observe that if $\mathcal{H} = \emptyset$, then by Remark 3.1, we get that both sets $\mathrm{DomHyp}^{+}_{\ell}(\Omega,\mathcal{H})$ and $\mathrm{DomHyp}^{+}_{0,\ell}(\Omega,\mathcal{H})$ are empty (see Remark 3.29 and Table 3.9).

3) $(\ell,-)$ with general ground set: if $\mathscr{H} \neq \emptyset, \{\emptyset\}$, then $\mathscr{D}(G) \leqslant^- \mathscr{H}$ where G is the complete graph with vertex set $\mathrm{Gr}(\mathscr{H})$. We observe that, if $\mathscr{H} = \emptyset$, then by Remark 3.1 we have that $\mathrm{DomHyp}_\ell^-(\Omega,\mathscr{H}) = \emptyset$ (see Remark 3.29 and Table 3.9).

4) $(\ell,-)$ with ground set Ω: if $\mathscr{H} \neq \emptyset, \{\emptyset\}$ and has ground set Ω, then $\mathscr{D}(K_\Omega) \leqslant^- \mathscr{H}$. Conversely, if there exists a graph G with $V(G) = \Omega$ such that $\mathscr{D}(G) \leqslant^- \mathscr{H}$, then by Lemma 1.16, we have that $\Omega = \mathrm{Gr}(\mathscr{H})$. □

3.2.3 The eight collections of optimal domination completions of a hypergraph

Let Ω be a finite set and let $\mathscr{H} \in \mathrm{Hyp}(\Omega)$ be a hypergraph on Ω. Let $*_1 \in \{+,-\}$. We say that a domination hypergraph $\mathscr{H}'$ on Ω is a $(*_1,u)$-*minimal domination completion* of $\mathscr{H}$ if $\mathscr{H}'$ is a minimal element of the poset $\left(\mathrm{DomHyp}_u^{*_1}(\Omega,\mathscr{H}), \leqslant^{*_1}\right)$. Let us denote by:

$$\mathrm{MinDomHyp}_u^{*_1}(\Omega,\mathscr{H})$$

the set whose elements are the $(*_1,u)$-minimal domination completions of $\mathscr{H}$, where $*_1 \in \{+,-\}$.

Analogously, we say that a domination hypergraph $\mathscr{H}'$ with ground set Ω is a $(*_1,u)$-*minimal* 0-*domination completion* of the hypergraph $\mathscr{H}$ if $\mathscr{H}'$ is a minimal element of the poset $\left(\mathrm{DomHyp}_{0,u}^{*_1}(\Omega,\mathscr{H}), \leqslant^{*_1}\right)$. Let us denote by:

$$\mathrm{MinDomHyp}_{0,u}^{*_1}(\Omega,\mathscr{H})$$

the set whose elements are the $(*_1,u)$-minimal 0-domination completions of $\mathscr{H}$, where $*_1 \in \{+,-\}$.

If we consider the lower side, we can proceed in a similar way. Namely, for $*_1 \in \{+,-\}$, we say that a domination hypergraph $\mathscr{H}'$ on Ω is a $(*_1,\ell)$-*maximal domination completion* of $\mathscr{H}$ if the hypergraph $\mathscr{H}'$ is a maximal element of the poset $\left(\mathrm{DomHyp}_\ell^{*_1}(\Omega,\mathscr{H}), \leqslant^{*_1}\right)$. Let us denote by:

$$\mathrm{MaxDomHyp}_\ell^{*_1}(\Omega,\mathscr{H})$$

the set whose elements are the $(*_1,\ell)$-maximal domination completions of $\mathscr{H}$, where $*_1 \in \{+,-\}$.

Analogously, we say that a domination hypergraph $\mathscr{H}'$ with ground set Ω is a $(*_1,\ell)$-*minimal* 0-*domination completion* of $\mathscr{H}$ if the hypergraph $\mathscr{H}'$ is a maximal element of the poset $\left(\mathrm{DomHyp}_{0,\ell}^{*_1}(\Omega,\mathscr{H}), \leqslant^{*_1}\right)$. Let us denote by:

$$\mathrm{MaxDomHyp}_{0,\ell}^{*_1}(\Omega,\mathscr{H})$$

the set whose elements are the $(*_1,\ell)$-maximal 0-domination completions of $\mathscr{H}$, where $*_1 \in \{+,-\}$.

Therefore, if we consider the upper completions, we have the following four sets of *optimal domination completions* of the hypergraph $\mathscr{H}$:

$$\begin{aligned}
\mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H}) &= \min\big(\mathrm{DomHyp}^+_u(\Omega,\mathscr{H}),\leqslant^+\big)\\
\mathrm{MinDomHyp}^-_u(\Omega,\mathscr{H}) &= \min\big(\mathrm{DomHyp}^-_u(\Omega,\mathscr{H}),\leqslant^-\big)\\
\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H}) &= \min\big(\mathrm{DomHyp}^+_{0,u}(\Omega,\mathscr{H}),\leqslant^+\big)\\
\mathrm{MinDomHyp}^-_{0,u}(\Omega,\mathscr{H}) &= \min\big(\mathrm{DomHyp}^-_{0,u}(\Omega,\mathscr{H}),\leqslant^-\big).
\end{aligned}$$

Analogously, if we consider the lower completions, we have the following four sets of *optimal domination completions* of the hypergraph $\mathscr{H}$:

$$\begin{aligned}
\mathrm{MaxDomHyp}^+_\ell(\Omega,\mathscr{H}) &= \max\big(\mathrm{DomHyp}^+_\ell(\Omega,\mathscr{H}),\leqslant^+\big)\\
\mathrm{MaxDomHyp}^-_\ell(\Omega,\mathscr{H}) &= \max\big(\mathrm{DomHyp}^-_\ell(\Omega,\mathscr{H}),\leqslant^-\big)\\
\mathrm{MaxDomHyp}^+_{0,\ell}(\Omega,\mathscr{H}) &= \max\big(\mathrm{DomHyp}^+_{0,\ell}(\Omega,\mathscr{H}),\leqslant^+\big)\\
\mathrm{MaxDomHyp}^-_{0,\ell}(\Omega,\mathscr{H}) &= \max\big(\mathrm{DomHyp}^-_{0,\ell}(\Omega,\mathscr{H}),\leqslant^-\big).
\end{aligned}$$

Remark 3.31 Let Ω be a finite set. In Table 3.10 we have computed the upper and lower optimal domination completions of the hypergraphs $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$.

	$\mathscr{H}=\emptyset$	$\mathscr{H}=\{\emptyset\}$	$\mathscr{H}=\{\Omega\}$
$\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H})$	$\{\{\Omega\}\}$	—	$\{\{\Omega\}\}$
$\mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H})$	$\{\{\Omega\}\}$	—	$\{\{\Omega\}\}$
$\mathrm{MaxDomHyp}^-_{0,u}(\Omega,\mathscr{H})$	$\{\{\Omega\}\}$	$\{\{\Omega\}\}$	$\{\{\Omega\}\}$
$\mathrm{MaxDomHyp}^-_u(\Omega,\mathscr{H})$	$\{\{\Omega\}\}$	$\{\{\Omega\}\}$	$\{\{\Omega\}\}$
$\mathrm{MinDomHyp}^+_{0,\ell}(\Omega,\mathscr{H})$	—	$\{\{\Omega\}\}$	$\{\{\Omega\}\}$
$\mathrm{MinDomHyp}^+_\ell(\Omega,\mathscr{H})$	—	$\{\{\Omega\}\}$	$\{\{\Omega\}\}$
$\mathrm{MaxDomHyp}^-_{0,\ell}(\Omega,\mathscr{H})$	—	—	$\{\{\Omega\}\}$
$\mathrm{MaxDomHyp}^-_\ell(\Omega,\mathscr{H})$	—	—	$\{\{\Omega\}\}$

Table 3.10 The upper and lower optimal domination completions of the hypergraphs $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$.

3.2.4 The two collections of minimal domination completions

Even though there are eight collections of optimal domination completions of a hypergraph, in this work we will focus on only two of them, namely the two upper domination completions with respect to the order $\leqslant^+$. So, from now on, when we refer to the *minimal* domination completions of a hypergraph $\mathscr{H}$, we mean the $(+,u)$-minimal domination completions of $\mathscr{H}$ and the $(+,u)$-minimal 0-domination completions of $\mathscr{H}$.

Recall that $\mathrm{DomHyp}_u^+(\Omega,\mathscr{H}) \cap \mathrm{Hyp}_0(\Omega) = \mathrm{DomHyp}_{0,u}^+(\Omega,\mathscr{H})$. The aim of this subsection is to study the difference between the sets of minimal domination completions $\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H})$ and $\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H})$. Namely, we are going to prove that the equality does not hold between the sets of the minimal elements. From the definitions it is clear that:

$$\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}) \cap \mathrm{Hyp}_0(\Omega) \subseteq \mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}),$$

but we are going to see that, in general, this inclusion is not an equality; that is, we are going to prove that if $\mathscr{H}$ is a hypergraph on a finite set Ω, then the minimal domination completions of $\mathscr{H}$ with ground set Ω are minimal 0-domination completions of $\mathscr{H}$, but in general the converse is not true. Actually, from the hypergraphs in Example 3.33 and the calculations performed there, it follows that:

a) There exist hypergraphs $\mathscr{H}$ where the inclusion is strict; that is, hypergraphs $\mathscr{H}$ such that:

$$\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}) \cap \mathrm{Hyp}_0(\Omega) \subsetneq \mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}).$$

Moreover, there exist hypergraphs $\mathscr{H} \in \mathrm{Hyp}(\Omega) \setminus \mathrm{Hyp}_0(\Omega)$ such that:

$$\emptyset \neq \mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}) \cap \mathrm{Hyp}_0(\Omega) \subsetneq \mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}); \tag{3.1}$$

there exists hypergraphs $\mathscr{H} \in \mathrm{Hyp}_0(\Omega)$ with:

$$\emptyset \neq \mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}) \cap \mathrm{Hyp}_0(\Omega) \subsetneq \mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}); \tag{3.2}$$

and, finally, there exist hypergraphs $\mathscr{H}$ such that:

$$\emptyset = \mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}) \cap \mathrm{Hyp}_0(\Omega) \subsetneq \mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}). \tag{3.3}$$

b) There exist hypergraphs $\mathscr{H}$ where the inclusion is an equality; that is, hypergraphs $\mathscr{H}$ with:

$$\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}) \cap \mathrm{Hyp}_0(\Omega) = \mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H});$$

or equivalently, hypergraphs $\mathscr{H}$ with:

$$\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}) \subseteq \mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}).$$

Moreover, there are hypergraphs $\mathscr{H}$ with:

$$\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}) \subsetneq \mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}); \tag{3.4}$$

and there exists hypergraphs $\mathscr{H}$ with:

$$\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}) = \mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}). \tag{3.5}$$

Remark 3.32 The study of the minimal domination completions is completed in Subsection 3.3.5 (hypergraphs with a unique minimal domination completion) and in Subsection 3.3.6 (hypergraphs with the same minimal domination completions). In order to do this study we need the results on decomposition, that we prove in next section.

Example 3.33 On the finite set $\Omega = \{1,2,3,4\}$, let us consider the hypergraphs $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5 \in \mathrm{Hyp}(\Omega)$ defined by:

$$\mathcal{H}_1 = \{\{1,2\},\{1,3\},\{2,3\}\},$$
$$\mathcal{H}_2 = \{\{1,2\},\{2,3\},\{3,4\}\},$$
$$\mathcal{H}_3 = \{\{1,2\},\{1,3,4\},\{2,3,4\}\},$$
$$\mathcal{H}_4 = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}, \text{ and}$$
$$\mathcal{H}_5 = \{\{1\},\{2\},\{3\}\}$$

(observe that $\mathcal{H}_1, \mathcal{H}_5 \notin \mathrm{Hyp}_0(\Omega)$, while $\mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \in \mathrm{Hyp}_0(\Omega)$).

To compute the sets $\mathrm{MinDomHyp}_u^+(\Omega,\mathcal{H})$ and $\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathcal{H})$ of these four hypergraphs we proceed in the following way. First, and by using the explicit computation of the domination hypergraphs of the seventeen non-isomorphic graphs with four or less vertices (see the tables in Section 2.2.4), we compute the set $\mathrm{DomHyp}_u^+(\Omega,\mathcal{H})$ of the domination completions of $\mathcal{H}$; whereas, by taking into account the domination hypergraphs of the eleven non-isomorphic graphs with four vertices, we compute the set $\mathrm{DomHyp}_{0,u}^+(\Omega,\mathcal{H})$ of the domination completions of $\mathcal{H}$ with ground set Ω. At this point, from the descriptions of $(\mathrm{DomHyp}_u^+(\Omega,\mathcal{H}),\leqslant^+)$ and $(\mathrm{DomHyp}_{0,u}^+(\Omega,\mathcal{H}),\leqslant^+)$, we can calculate the set:

$$\mathrm{MinDomHyp}_u^+(\Omega,\mathcal{H}) = \min(\mathrm{DomHyp}_u^+(\Omega,\mathcal{H}),\leqslant^+)$$

of the minimal domination completions of the hypergraph $\mathcal{H}$, as well as the set:

$$\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathcal{H}) = \min(\mathrm{DomHyp}_{0,u}^+(\Omega,\mathcal{H}),\leqslant^+)$$

of the minimal 0-domination completions of $\mathcal{H}$. We apply this procedure to the hypergraphs $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_4$, $\mathcal{H}_5$ on Ω. The minimal domination completions of these hypergraphs are listed below (we omit all the calculations that have been checked with the SAGE program described in Appendix A).

(3.33.1) Minimal domination completions of $\mathcal{H}_1$:

i) $\mathrm{MinDomHyp}_u^+(\Omega,\mathcal{H}_1) = \{\mathcal{H}_{1,i} : 1 \leqslant i \leqslant 3\} \cup \{\mathcal{U}_{2,\Omega}\}$, where:

$$\mathcal{H}_{1,1} = \{\{1\},\{2,3\}\},$$
$$\mathcal{H}_{1,2} = \{\{2\},\{1,3\}\},$$
$$\mathcal{H}_{1,3} = \{\{3\},\{1,2\}\}.$$

ii) $\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{H}_1) = \{\mathscr{H}^0_{1,i} : 1 \leqslant i \leqslant 6\} \cup \{\mathscr{U}_{2,\Omega}\}$, where:

$$\mathscr{H}^0_{1,1} = \{\{1\}, \{2,3\}, \{2,4\}\},$$
$$\mathscr{H}^0_{1,2} = \{\{1\}, \{2,3\}, \{3,4\}\},$$
$$\mathscr{H}^0_{1,3} = \{\{2\}, \{1,3\}, \{1,4\}\},$$
$$\mathscr{H}^0_{1,4} = \{\{2\}, \{1,3\}, \{3,4\}\},$$
$$\mathscr{H}^0_{1,5} = \{\{3\}, \{1,2\}, \{1,4\}\},$$
$$\mathscr{H}^0_{1,6} = \{\{3\}, \{1,2\}, \{2,4\}\}.$$

This is an example of the inclusion (3.1) with $\mathscr{H}_1 \notin \mathrm{Hyp}_0(\Omega)$; that is, $\mathscr{H}_1$ is a hypergraph on Ω that satisfies:

$$\emptyset \neq \mathrm{MinDomHyp}^+_u(\Omega, \mathscr{H}_1) \cap \mathrm{Hyp}_0(\Omega) \subsetneq \mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{H}_1).$$

(3.33.2) Minimal domination completions of $\mathscr{H}_2$:

i) $\mathrm{MinDomHyp}^+_u(\Omega, \mathscr{H}_2) = \{\mathscr{H}_{2,i} : 1 \leqslant i \leqslant 3\}$, where:

$$\mathscr{H}_{2,1} = \{\{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}\},$$
$$\mathscr{H}_{2,2} = \{\{2\}, \{3,4\}\},$$
$$\mathscr{H}_{2,3} = \{\{3\}, \{1,2\}\}.$$

ii) $\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{H}_2) = \{\mathscr{H}^0_{2,i} : 1 \leqslant i \leqslant 3\}$, where:

$$\mathscr{H}^0_{2,1} = \mathscr{H}_{2,1},$$
$$\mathscr{H}^0_{2,2} = \{\{2\}, \{1,3\}, \{3,4\}\},$$
$$\mathscr{H}^0_{2,3} = \{\{3\}, \{1,2\}, \{2,4\}\}.$$

This is an example of the inclusion (3.2) with $\mathscr{H}_2 \in \mathrm{Hyp}_0(\Omega)$; that is, $\mathscr{H}_2$ is a hypergraph with ground set Ω that satisfies:

$$\emptyset \neq \mathrm{MinDomHyp}^+_u(\Omega, \mathscr{H}_2) \cap \mathrm{Hyp}_0(\Omega) \subsetneq \mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{H}_2).$$

(3.33.3) Minimal domination completions of $\mathscr{H}_3$:

i) $\mathrm{MinDomHyp}^+_u(\Omega, \mathscr{H}_3) = \{\mathscr{H}_{3,i} : 1 \leqslant i \leqslant 6\}$, where:

$$\mathscr{H}_{3,1} = \{\{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}\},$$
$$\mathscr{H}_{3,2} = \{\{1,2\}, \{1,4\}, \{2,3\}, \{3,4\}\},$$
$$\mathscr{H}_{3,3} = \{\{1\}, \{2,3,4\}\},$$
$$\mathscr{H}_{3,4} = \{\{2\}, \{1,3,4\}\},$$
$$\mathscr{H}_{3,5} = \{\{3\}, \{1,2\}\},$$
$$\mathscr{H}_{3,6} = \{\{4\}, \{1,2\}\}.$$

ii) $\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{H}_3) = \{\mathscr{H}^0_{3,i} : 1 \leqslant i \leqslant 4\}$, where:

$$\mathscr{H}^0_{3,i} = \mathscr{H}_{3,i}, \quad 1 \leqslant i \leqslant 4.$$

This is an example of the inclusion (3.4) with $\mathscr{H}_3 \in \mathrm{Hyp}_0(\Omega)$; that is, $\mathscr{H}_3$ is a hypergraph with ground set Ω that satisfies:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{H}_3) \subsetneq \mathrm{MinDomHyp}^+_u(\Omega, \mathscr{H}_3).$$

(3.33.4) Minimal domination completions of $\mathscr{H}_4$:

i) $\mathrm{MinDomHyp}^+_u(\Omega, \mathscr{H}_4) = \{\mathscr{H}_{4,i} : 1 \leqslant i \leqslant 7\}$, where:

$$\mathscr{H}_{4,1} = \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\},$$
$$\mathscr{H}_{4,2} = \{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\},$$
$$\mathscr{H}_{4,3} = \{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\},$$
$$\mathscr{H}_{4,4} = \{\{1\},\{2,3,4\}\},$$
$$\mathscr{H}_{4,5} = \{\{2\},\{1,3,4\}\},$$
$$\mathscr{H}_{4,6} = \{\{3\},\{1,2,4\}\},$$
$$\mathscr{H}_{4,7} = \{\{4\},\{1,2,3\}\}.$$

ii) $\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{H}_4) = \{\mathscr{H}^0_{4,i} : 1 \leqslant i \leqslant 7\}$, where:

$$\mathscr{H}^0_{4,i} = \mathscr{H}_{4,i}, \quad 1 \leqslant i \leqslant 7.$$

This is an example of equality (3.5); that is, $\mathscr{H}_4$ is a hypergraph with ground set Ω such that:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{H}) = \mathrm{MinDomHyp}^+_{0u}(\Omega, \mathscr{H}).$$

(3.33.5) Minimal domination completions of $\mathscr{H}_5$:

i) $\mathrm{MinDomHyp}^+_u(\Omega, \mathscr{H}_5) = \{\mathscr{H}_5\}$ because $\mathscr{H}_5$ is a domination hypergraph *on* Ω; namely, $\mathscr{H}_5 = \mathscr{D}(K_{\{1,2,3\}})$.

ii) $\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{H}_5) = \{\mathscr{D}(K_\Omega)\}$.

This is an example of a hypergraph that is a domination hypergraph *on* Ω but not a domination hypergraph *with ground set* Ω; that is, $\mathscr{H}_5$ is a hypergraph on Ω that satisfies the equality (3.3):

$$\mathrm{MinDomHyp}^+_u(\Omega, \mathscr{H}_5) \cap \mathrm{Hyp}_0(\Omega) = \emptyset.$$

Remark 3.34 The description of the minimal domination completions of the hypergraphs $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{H}_3$, $\mathscr{H}_4$ and $\mathscr{H}_5$ will be used later (see Example 3.63 and Examples 3.66 and 3.67).

3.3 Decomposition of hypergraphs by domination hypergraphs

In this section we present different results on decomposition in the framework of domination. First in Subsection 3.3.1 we give those results coming from the general properties of decomposition of hypergraphs under mild conditions (see Subsection 3.1.5). After that, in Subsection 3.3.4 we provide the decomposition theorems arising from the avoidance properties that we will study in Subsection 3.3.2 and 3.3.3. We end this section applying the decomposition to study the hypergraphs having one minimal minimal completion (Subsection 3.3.5), and the hypergraphs having the same minimal domination completions (Subsection 3.3.6).

3.3.1 Domination decompositions from results of Subsection 3.1.5

In Subsection 3.1.5 we have presented four general results (Corollaries 3.16, 3.17, 3.18, and 3.19) concerning the decomposition of a hypergraph $\mathscr{H}$ under some mild conditions on the hypergraph $\mathscr{H}$ and on the collection $\Sigma \subseteq \mathrm{Hyp}(\Omega)$ where we want to decompose the hypergraph $\mathscr{H}$. We are going to apply these results in the case where the collection Σ is related to domination. The following two remarks analyze those mild conditions on Σ.

Remark 3.35 Let Ω be a finite set and let $\Sigma = \mathrm{DomHyp}(\Omega)$. Then the following statements hold.

1) $\mathscr{U}_{1,X} \in \Sigma$ for all non-empty subsets X of Ω (because $\mathscr{U}_{1,X} = \mathscr{D}(K_X)$).

2) $\{X\} \in \Sigma$ for all non-empty subsets X of Ω (because $\{X\} = \mathscr{D}(\overline{K_X})$).

3) In general, $(\mathscr{U}_{1,X})^c \notin \Sigma$ for all non-empty subsets X of Ω. (For example, $(\mathscr{U}_{1,\Omega})^c = \mathscr{U}_{|\Omega|-1,\Omega}$ and by Proposition 2.52, this hypergraph is not a domination hypergraph whenever $n \neq |\Omega|$.)

4) In general, $\{X\} \notin \Sigma$ for all proper subsets X of Ω (because $\{\emptyset\} \notin \Sigma$ and $\emptyset$ is a proper subset of Ω).

Remark 3.36 If Ω is a finite set and $\Sigma = \mathrm{DomHyp}_0(\Omega)$. Then the following statements hold.

1) In general, $\mathscr{U}_{1,X} \notin \Sigma$ for all non-empty proper subsets X of Ω (because $\mathrm{Gr}(\mathscr{U}_{1,X}) = X \neq \Omega$).

2) In general, $\{X\} \notin \Sigma$ for all non-empty subsets X of Ω.

3) In general, $(\mathscr{U}_{1,X})^c \notin \Sigma$ for all non-empty subsets X of Ω.

4) In general, $\{X\} \notin \Sigma$ for all proper subsets X of Ω.

From these remarks it follows that we can only apply the results of Subsection 3.1.5 for $\Sigma = \mathrm{DomHyp}(\Omega)$ and, in this case, in only two of the four possible situations, namely for $(\leqslant^+, u)$ and $(\leqslant^+, \ell)$. In this way, we get the following corollaries.

Corollary 3.37 *If $\mathscr{H}$ is a hypergraph on Ω different from $\emptyset$ and $\{\emptyset\}$, then the set of upper domination completions $\mathrm{DomHyp}_u^+(\Omega,\mathscr{H})$ of $\mathscr{H}$ is non-empty and:*

$$\mathscr{H} = \mathscr{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_r\,,$$

where $\mathscr{H}_1,\ldots,\mathscr{H}_r$ are the minimal elements of the poset $\big(\mathrm{DomHyp}_u^+(\Omega,\mathscr{H}),\leqslant^+\big)$. In particular, the following statements hold.

1) *A hypergraph $\mathscr{H}$ on Ω different from $\emptyset$ and $\{\emptyset\}$ is a domination hypergraph on Ω if and only if the poset $(\mathrm{DomHyp}_u^+(\Omega,\mathscr{H}),\leqslant^+)$ has a unique minimal element (namely, the hypergraph $\mathscr{H}$).*
2) *If $\mathscr{H}$ and $\mathscr{H}'$ are hypergraphs on Ω different from $\emptyset$ and $\{\emptyset\}$, then $\mathscr{H} = \mathscr{H}'$ if and only if the posets $\big(\mathrm{DomHyp}_u^+(\Omega,\mathscr{H}),\leqslant^+\big)$ and $\big(\mathrm{DomHyp}_u^+(\Omega,\mathscr{H}'),\leqslant^+\big)$ have the same minimal elements.*

Proof. It is a direct consequence of Corollary 3.16 and Remark 3.35. □

Corollary 3.38 *If $\mathscr{H}$ is a hypergraph on Ω different $\emptyset$ and $\{\emptyset\}$, then the set of lower dominant completions $\mathrm{DomHyp}_\ell^+(\Omega,\mathscr{H})$ of $\mathscr{H}$ is non-empty and:*

$$\mathscr{H} = \mathscr{H}_1 \overset{+}{\sqcup} \cdots \overset{+}{\sqcup} \mathscr{H}_r\,,$$

where $\mathscr{H}_1,\ldots,\mathscr{H}_r$ are the maximal elements of the poset $(\mathrm{DomHyp}_\ell^+(\Omega,\mathscr{H}),\leqslant^+)$. In particular, the following statements hold.

1) *A hypergraph $\mathscr{H}$ on Ω different from $\emptyset$ and $\{\emptyset\}$ is a domination hypergraph on Ω if and only if the poset $(\mathrm{DomHyp}_\ell^+(\Omega,\mathscr{H}),\leqslant^+)$ has a unique maximal (namely, the hypergraph $\mathscr{H}$).*
2) *If $\mathscr{H}$ and $\mathscr{H}'$ are hypergraphs on Ω different from $\emptyset$ and $\{\emptyset\}$, then $\mathscr{H} = \mathscr{H}'$ if and only if the posets $(\mathrm{DomHyp}_\ell^+(\Omega,\mathscr{H}),\leqslant^+)$ and $(\mathrm{DomHyp}_\ell^+(\Omega,\mathscr{H}'),\leqslant^+)$ have the same maximal elements.*

Proof. It is a direct consequence of Corollary 3.17 and Remark 3.35. □

Remark 3.39 By Remark 3.35, the Corollaries 3.18 and 3.19 are not applicable to the collection $\Sigma = \mathrm{DomHyp}(\Omega)$. Furthermore, by Remark 3.36, the Corollaries 3.16, 3.17, 3.18, and 3.19 are not applicable to the collection $\Sigma = \mathrm{DomHyp}_0(\Omega)$. In order to get analogous decomposition results, we need to use the avoidance properties in the domination context. These properties are studied in the next two subsections.

	$\emptyset$	$\{\emptyset\}$	$\{\Omega\}$
$\mathrm{DomHyp}^+_{0,u}(\Omega)$	no $[\{\Omega\}]$	always	always
$\mathrm{DomHyp}^+_{u}(\Omega)$	no $[\{\Omega\}]$	always	always
$\mathrm{DomHyp}^-_{0,u}(\Omega)$	no $[\{\emptyset\}]$	no $[\{\Omega\}]$	always
$\mathrm{DomHyp}^-_{u}(\Omega)$	no $[\{\emptyset\}, \mathscr{U}_{1,\Omega}]$	no $[\{\Omega\}]$	always
$\mathrm{DomHyp}^+_{0,\ell}(\Omega)$	always	no $[\mathscr{U}_{1,\Omega}]$	always
$\mathrm{DomHyp}^+_{\ell}(\Omega)$	always	no $[\mathscr{U}_{1,\Omega}]$	always
$\mathrm{DomHyp}^-_{0,\ell}(\Omega)$	always	always	always
$\mathrm{DomHyp}^-_{\ell}(\Omega)$	always	always	always

Table 3.11 The $\mathrm{DomHyp}^{*_1}_{*_2}(\Omega)$-avoidance properties and the $\mathrm{DomHyp}^{*_1}_{0,*_2}(\Omega)$-avoidance properties for the hypergraphs $\mathscr{H} = \emptyset$, $\{\emptyset\}$ and $\{\Omega\}$, where $*_1 \in \{+,-\}$ and $*_2 \in \{u, \ell\}$. Enclosed in brackets there are the hypergraphs that prevent the heading hypergraph $\mathscr{H}$ from satisfying the corresponding avoidance property.

3.3.2 The avoidance property $\leqslant^+, u$ for domination hypergraphs

In this subsection we investigate the avoidance properties of a hypergraph on Ω with respect to the order $\leqslant^+$, side u and the collections Σ related to domination. Concretely, Proposition 3.42 states that every hypergraph different from $\emptyset$ and $\{\emptyset\}$ satisfies the $\mathrm{DomHyp}^+_u(\Omega)$-avoidance property and, in addition, satisfies the $\mathrm{DomHyp}^+_{0,u}(\Omega)$-avoidance property except for the hypergraphs $\mathscr{H}$ of the form $\mathscr{U}_{1,\Omega'}$ with a non-empty $\Omega' \subseteq \Omega$. These hypergraphs are studied in Proposition 3.43 and Remark 3.44.

Befor stating our results, let us study in Remark 3.40 the domination avoidance property for the hypergraphs $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$.

Remark 3.40 In Table 3.11 we establish the domination-avoidance properties for the hypergraphs $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$ for both partial orders $\leqslant^+$ and $\leqslant^-$, both sides u and ℓ, and with ground set or not. We also indicate the hypergraphs that prevent $\emptyset$ or $\{\emptyset\}$ from satisfying the corresponding avoidance property.

The key point to obtain the two domination avoidance properties for $\leqslant^+, u$ is the following proposition.

Proposition 3.41 *Let Ω be a finite set. Let $\mathscr{H}$ be a hypergraph on Ω such that $\mathscr{H} \neq \emptyset, \{\emptyset\}$ and such that $\mathscr{H} \neq \mathscr{U}_{1,\Omega'}$ for all non-empty subsets $\Omega' \subsetneq \Omega$. Then for all hypergraph $\mathscr{H}'$ on Ω such that $\mathscr{H} \leqslant^+ \mathscr{H}'$ and $\mathscr{H} \neq \mathscr{H}'$, there exists a domination hypergraph*

$\mathcal{H}_0$ with ground set Ω such that $\mathcal{H} \leqslant^+ \mathcal{H}_0$ and $\mathcal{H}' \nleqslant^+ \mathcal{H}_0$. Moreover, the domination hypergraph $\mathcal{H}_0$ can be chosen to be of the form $\mathcal{H}_0 = \mathcal{D}_{ind}(G_0)$ for some graph G_0 with vertex set Ω.

Proof. Let $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H} \leqslant^+ \mathcal{H}'$ and $\mathcal{H} \neq \mathcal{H}'$. (Observe that if $\mathcal{H} = \{\emptyset\}$, then this hypergraph $\mathcal{H}'$ does not exist.)

We claim that there exists $B \in \mathcal{H}'$ such that $B \neq A$ for all $A \in \mathcal{H}$. Indeed, if for all $B \in \mathcal{H}'$ there exists $A \in \mathcal{H}$ such that $B = A$, then we have $\mathcal{H}' \subseteq \mathcal{H}$, and consequently $\mathcal{H}' \leqslant^+ \mathcal{H}$. Thus, as $\leqslant^+$ is a partial order, we deduce that $\mathcal{H}' = \mathcal{H}$, which is a contradiction.

Let $B \in \mathcal{H}'$ such an element. It is clear that $B \neq \emptyset$ (because if $B = \emptyset$, then $\mathcal{H}' = \{\emptyset\}$ and, since $\mathcal{H} \leqslant^+ \mathcal{H}'$, we get that $\mathcal{H} = \emptyset$ or $\mathcal{H} = \{\emptyset\}$ which is not possible by hypothesis). Moreover, we have that $\Omega \setminus B \neq \emptyset$. Indeed, otherwise $B = \Omega$, hence $\mathcal{H}' = \{\Omega\}$, so $\mathcal{H}' \leqslant^+ \mathcal{H}$ and thus $\mathcal{H}' = \mathcal{H}$, a contradiction.

We claim that there exists $z \in \Omega \setminus B$ such that $\{z\} \notin \mathcal{H}$. Let us proof our claim. If, on the contrary, $\{z\} \in \mathcal{H}$ for all $z \in \Omega \setminus B$, then $\mathcal{U}_{1,\Omega\setminus B} \subseteq \mathcal{H}$. By hypothesis we have $\mathcal{U}_{1,\Omega\setminus B} \neq \mathcal{H}$ (because $\Omega \setminus B \subsetneq \Omega$ since $B \neq \emptyset$). So we get that $\mathcal{U}_{1,\Omega\setminus B} \subsetneq \mathcal{H}$ and hence there exists $A' \in \mathcal{H}$ such that $A' \neq \{z\}$ for all $z \in \Omega \setminus B$. Observe that $A' \subseteq B$, because otherwise there would exist an element $\omega_0 \in A' \cap (\Omega \setminus B)$, hence $\{\omega_0\} \subseteq A' \in \mathcal{H}$ and $\{\omega_0\} \in \mathcal{U}_{1,\Omega\setminus B} \subseteq \mathcal{H}$ and so $A' = \{\omega_0\}$, a contradiction because $A' \neq \{z\}$ for all $z \in \Omega \setminus B$. Thus we have that $A' \subseteq B$ and so, by construction of B, we conclude that $A' \subsetneq B$. Now since $\mathcal{H} \leqslant^+ \mathcal{H}'$, there exists $B' \in \mathcal{H}'$ such that $B' \subseteq A' \subsetneq B$. Hence $B' \subsetneq B$ and $B, B' \in \mathcal{H}$, which is a contradiction.

Let $z \in \Omega \setminus B$ such that $\{z\} \notin \mathcal{H}$. We define the hypergraph $\mathcal{H}_0$ as follows:

$$\mathcal{H}_0 = \{\{x,z\} : x \in B\} \cup \{\{x\} : x \in \Omega \setminus B, x \neq z\}.$$

It is clear that $\mathrm{Gr}(\mathcal{H}_0) = \Omega$, so $\mathcal{H}_0 \in \mathrm{Hyp}_0(\Omega)$. Moreover, $\mathcal{H}_0 = \mathcal{D}(G)$, where G is the graph obtained from K_Ω by deleting the edges $\{x,z\}$, with $x \in B$. We observe further that $\mathcal{D}(G) = \mathcal{D}_{ind}(G)$. We also have that $\mathcal{H}' \nleqslant^+ \mathcal{H}_0$, because $B \in \mathcal{H}'$, but there is no element of $\mathcal{H}_0$ contained in B. Let us finally prove that $\mathcal{H} \leqslant^+ \mathcal{H}_0$; that is, we must demonstrate that if $A \in \mathcal{H}$, then there exists $A_0 \in \mathcal{H}_0$ such that $A_0 \subseteq A$ (Lemma 1.15). Let $A \in \mathcal{H}$. If $A \nsubseteq B \cup \{z\}$, then there is an element $x \in \Omega \setminus B$, $x \neq z$, such that $x \in A$. Hence there exists $\{x\} \in \mathcal{H}_0$ such that $\{x\} \subseteq A$, as we wanted to prove. So now let us assume now that $A \subseteq B \cup \{z\}$. If $z \notin A$, then we have $A \subseteq B$. Since $\mathcal{H} \leqslant^+ \mathcal{H}'$, there is an element $B' \in \mathcal{H}'$ such that $B' \subseteq A \subseteq B$. In such a case, we get $B' = A = B$, which is a contradiction. Hence $z \in A$. But we also know that $\{z\} \notin \mathcal{H}$, so there exists $x \in B$ such that $x \in A$. Therefore we have proved that $\{x,z\} \subseteq A$. This completes the proof because $\{x,z\} \in \mathcal{H}_0$. □

Proposition 3.42 (Domination avoidance property for $\leqslant^+$, u) *Let Ω be a finite set and let $\mathcal{H}$ be a hypergraph on Ω different from $\emptyset$ and $\{\emptyset\}$. Then the following statements hold.*

1) *The hypergraph* $\mathcal{H}$ *satisfies the* $\mathrm{DomHyp}_u^+(\Omega)$*-avoidance.*

2) *If the hypergraph* $\mathcal{H} \neq \mathcal{U}_{1,\Omega'}$ *for all non-empty subsets* $\Omega' \subsetneq \Omega$*, then* $\mathcal{H}$ *satisfies the* $\mathrm{DomHyp}_{0,u}^+(\Omega)$*-avoidance property.*

Proof. We observe that if $\mathcal{H} = \mathcal{U}_{1,\Omega'}$ for some non-empty subset $\Omega' \subseteq \Omega$, then $\mathcal{H} = \mathcal{D}(K_{\Omega'}) \in \mathrm{DomHyp}(\Omega)$. Observe that $\mathcal{H} = \mathcal{D}(K_{\Omega'}) \in \mathrm{DomHyp}_0(\Omega)$ if and only if $\Omega' = \Omega$. Hence $\mathcal{H} = \mathcal{U}_{1,\Omega'}$ verifies the domination avoidance property if $\emptyset \neq \Omega' \subsetneq \Omega$ and verifies the 0-domination avoidance property if $\Omega' = \Omega$.

So we may assume that $\mathcal{H} \neq \mathcal{U}_{1,\Omega'}$ for all non-empty subsets $\Omega' \subseteq \Omega$. Then we can apply Proposition 3.41 to $\mathcal{H}$ and we get that $\mathcal{H}$ satisfies the $\mathrm{DomHyp}_{0,u}^+(\Omega)$-avoidance property and, in particular, also the $\mathrm{DomHyp}_u^+(\Omega)$-avoidance property. □

In order to complete the second statement of the previous proposition, next we deal with the hypergraphs $\mathcal{H}$ of the form $\mathcal{U}_{1,\Omega'}$ for some non-empty subset $\Omega' \subsetneq \Omega$. The study of the hypergraphs of this form will be completed in Subsection 3.3.5.

Proposition 3.43 *Let* Ω *be a finite set and let* Ω' *be a non-empty subset of* Ω*. Then the following statements hold.*

1) $\mathcal{U}_{1,\Omega'} \in \mathrm{DomHyp}(\Omega)$ *and* $\mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\} \in \mathrm{DomHyp}_0(\Omega)$.

2) $\mathcal{U}_{1,\Omega'} \leqslant^+ \mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\}$.

3) *If* $\mathcal{H}$ *is a hypergraph with ground set* Ω*, then:*

$$\mathcal{U}_{1,\Omega'} \leqslant^+ \mathcal{H} \iff \mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\} \leqslant^+ \mathcal{H}.$$

Proof. It is clear that $\mathcal{U}_{1,\Omega'}$ is the domination hypergraph of the complete graph $K_{\Omega'}$ with vertex set $V(K_{\Omega'}) = \Omega'$. Moreover, the hypergraph $\mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\}$ is the domination hypergraph $\mathcal{D}(G)$ of the graph G with vertex set $V(G) = \Omega$ obtained from the complete graph K_Ω by removing the edges of the form $\{w_1, w_2\}$ with $w_1, w_2 \notin \Omega'$. Furthermore it is easy to check that $\mathcal{U}_{1,\Omega'} \leqslant^+ \mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\}$. Hence, as $\leqslant^+$ is a partial order, the proof of the proposition will be completed by showing that if $\mathcal{H}$ is a hypergraph with ground set Ω such that $\mathcal{U}_{1,\Omega'} \leqslant^+ \mathcal{H}$, then $\mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\} \leqslant^+ \mathcal{H}$.

Assume that $\mathcal{H} \in \mathrm{Hyp}_0(\Omega)$ and that $\mathcal{U}_{1,\Omega'} \leqslant^+ \mathcal{H}$. We must demonstrate that $\mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\} \leqslant^+ \mathcal{H}$. Since $\mathcal{U}_{1,\Omega'} \leqslant^+ \mathcal{H}$, by applying Lemma 1.15 we get that for all $\omega \in \Omega'$ there exists $B \in \mathcal{H}$ such that $B \subseteq \{w\}$; hence $B = \{w\}$ or $B = \emptyset$. If $B = \emptyset$, then $\mathcal{H} = \{\emptyset\}$ whose ground set is not Ω (because $\Omega \neq \emptyset$). Thus $B = \{\omega\}$. Therefore, we have $\{w\} \in \mathcal{H}$ for all $\omega \in \Omega'$; that is, $\mathcal{U}_{1,\Omega'} \subseteq \mathcal{H}$. If $\Omega' = \Omega$, then $\mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\} = \mathcal{U}_{1,\Omega} = \mathcal{U}_{1,\Omega'}$ and so we are done. Now, assume that $\Omega' \subsetneq \Omega$. Then there exists $A \in \mathcal{H}$ such that $A \cap \Omega' = \emptyset$; that is, $A \subseteq \Omega \setminus \Omega'$. Therefore, for all $C \in \mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\}$ there exists $A_0 \in \mathcal{H}$ such that $A_0 \subseteq C$, as we wanted to prove. □

Remark 3.44 We observe that Proposition 3.42 can not be applied to hypergraphs of the form $\mathcal{H} = \mathcal{U}_{1,\Omega'}$ for some non-empty subset $\Omega' \subsetneq \Omega$. Actually, from Proposition 3.43 we know that the hypergraphs of the form $\mathcal{H} = \mathcal{U}_{1,\Omega'}$, with $\emptyset \neq \Omega' \subsetneq \Omega$, do not satisfy the $\mathrm{DomHyp}^+_{0,u}(\Omega)$-avoidance property. Indeed, we have that:

1) $\mathcal{H} \leqslant^+ \mathcal{H}'$, where $\mathcal{H}' = \mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\}$;
2) $\mathcal{H} \neq \mathcal{H}'$ (because $\Omega' \neq \Omega$);
3) there exists $\mathcal{H}_0 \in \mathrm{DomHyp}_0(\Omega)$ such that $\mathcal{H} \leqslant^+ \mathcal{H}_0$ (for example, $\mathcal{H}_0 = \mathcal{D}(K_\Omega)$ or $\mathcal{H}_0 = \mathcal{H}$); and
4) there does not exist $\mathcal{H}_0 \in \mathrm{Hyp}_0(\Omega)$ such that $\mathcal{H} \leqslant^+ \mathcal{H}_0$ and $\mathcal{H} \not\leqslant^+ \mathcal{H}_0$ (because for all $\mathcal{H}_0 \in \mathrm{Hyp}_0(\Omega)$ with $\mathcal{H} \leqslant^+ \mathcal{H}_0$ we have $\mathcal{H}' \leqslant^+ \mathcal{H}_0$).

Therefoer, from 1), 2), 3) and 4) it follows that the hypergraph $\mathcal{H} = \mathcal{U}_{1,\Omega'}$ does not satisfy the $\mathrm{DomHyp}^+_{0,u}(\Omega)$-avoidance property.

3.3.3 Other avoidance properties for domination hypergraphs

The following propositions deal with the avoidance property of a hypergraph $\mathcal{H} \in \mathrm{Hyp}(\Omega)$ with respect to $\mathrm{DomHyp}^{*_1}_{*_2}(\Omega)$ and $\mathrm{DomHyp}^{*_1}_{0,*_2}(\Omega)$, for the cases $(*_1, *_2) = (+, \ell)$ and $(*_1, *_2) = (-, \ell)$. Namely, Proposition 3.45 deals with the case $(+, \ell)$ and Proposition 3.47 with the case $(-, \ell)$, respectively. The corresponding property for the case $(-, u)$ remains an open problem.

Proposition 3.45 (Domination avoidance property for $\leqslant^+$, ℓ) *Let Ω be a finite set. If $\mathcal{H}$ is a hypergraph on Ω different from $\emptyset$ and $\{\emptyset\}$, then the hypergraph $\mathcal{H}$ satisfies the* $\mathrm{DomHyp}^+_\ell(\Omega)$*-avoidance property.*

Proof. Let $\mathcal{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathcal{H}' \leqslant^+ \mathcal{H}$ and $\mathcal{H}' \neq \mathcal{H}$ (we observe that if $\mathcal{H} = \emptyset$, then there is no hypergraph $\mathcal{H}'$ satisfying these conditions). We have to show that there exists $\mathcal{H}_0 \in \mathrm{DomHyp}(\Omega)$ such that $\mathcal{H}_0 \leqslant^+ \mathcal{H}$ and $\mathcal{H}_0 \not\leqslant^+ \mathcal{H}'$. Since $\mathcal{H}' \leqslant^+ \mathcal{H}$ and $\mathcal{H}' \neq \mathcal{H}$, we have that $\mathcal{H} \not\leqslant^+ \mathcal{H}'$, and so there exists $A_0 \in \mathcal{H}$ such that $A' \not\subseteq A_0$ for all $A' \in \mathcal{H}'$. If $\mathcal{H} = \{A_0\}$, as $\mathcal{H} \neq \{\emptyset\}$, then $A_0 \neq \emptyset$ and we can take $\mathcal{H}_0 = \mathcal{H} = \mathcal{D}(\overline{K_{A_0}})$. If $\mathcal{H} \neq \{A_0\}$, then there exists $A_1 \in \mathcal{H}$ with $A_1 \neq A_0$. Let $a_0 \in A_0 \setminus A_1$ and consider $\mathcal{H}_0 = \{A_0, \Omega \setminus \{a_0\}\}$. This hypergraph has the following four properties: (1) it has ground set $\mathrm{Gr}(\mathcal{H}_0) = A_0 \cup \Omega \setminus \{a_0\} = \Omega$; (2) it satisfies the inequality $\mathcal{H}_0 \leqslant^+ \mathcal{H}$, because $A_1 \subseteq \Omega \setminus \{a_0\}$; (3) it satisfies $\mathcal{H}_0 \not\leqslant^+ \mathcal{H}'$, by construction; (4) and, finally, $\mathcal{H} = \mathcal{D}(G)$, where G is the graph with vertex set $V(G) = \Omega$, isolated vertices $V_0(G) = A_0 \setminus \{a_0\}$ and a connected component that is a star with center a_0 and leaves $\Omega \setminus A_0$. So $\mathcal{H}_0$ is the hypergraph we seek. □

Remark 3.46 We observe that the hypergraph $\mathcal{H}_0$ appearing in the proof of Proposition 3.45 can be realized with a graph G such that $\mathcal{D}(G) = \mathcal{D}_{ind}(G)$. Moreover, the hypergraph $\mathscr{H}_0$ has ground set Ω if and only if $\mathscr{H} \neq \{X\}$ for all non-empty subsets X of Ω.

Proposition 3.47 (Domination avoidance property for $\leqslant^-$, ℓ) *Let Ω be a finite set. If $\mathscr{H}$ is a hypergraph on Ω different from $\emptyset$, $\{\emptyset\}$ and $\{\Omega\}$, then the hypergraph $\mathscr{H}$ satisfies the* $\mathrm{DomHyp}^-_\ell(\Omega)$*-avoidance property.*

Proof. Let $\mathscr{H}' \in \mathrm{Hyp}(\Omega)$ such that $\mathscr{H}' \leqslant^- \mathscr{H}$ and $\mathscr{H}' \neq \mathscr{H}$. (We observe that if $\mathscr{H} = \emptyset$ or $\mathscr{H} = \{\emptyset\}$, then there is no hypergraph $\mathscr{H}'$ satisfying these conditions.) We have to show that there exists $\mathscr{H}_0 \in \mathrm{DomHyp}(\Omega)$ such that $\mathscr{H}_0 \leqslant^- \mathscr{H}$ and $\mathscr{H}_0 \not\leqslant^- \mathscr{H}'$. Since $\mathscr{H}' \leqslant^- \mathscr{H}$ and $\mathscr{H}' \neq \mathscr{H}$, we have that $\mathscr{H} \not\leqslant^- \mathscr{H}'$. Hence there exists $A_0 \in \mathscr{H}$ such that $A_0 \not\subseteq A'$ for all $A' \in \mathscr{H}'$. Observe that $A_0 \neq \emptyset$ and $A_0 \neq \{\Omega\}$ because $\mathscr{H} \neq \{\Omega\}$ and $\mathscr{H} \neq \{\emptyset\}$. If $\mathscr{H} = \{A_0\}$, then we can take $\mathscr{H}_0 = \mathscr{H} = \mathscr{D}(\overline{K_{A_0}})$ because $A_0 \neq \emptyset$. Else if $\mathscr{H} \neq \{A_0\}$, then we can take $\mathscr{H}_0 = \{A_0\} \cup \mathscr{U}_{1,\mathrm{Gr}(\mathscr{H})\setminus A_0} = \mathscr{D}(\overline{K_{A_0}} \vee K_{\mathrm{Gr}(\mathscr{H})\setminus A_0})$ (observe that $\mathrm{Gr}(\mathscr{H}) \setminus A_0 \neq \emptyset$). Indeed, in this case, we have that $\mathscr{H}_0 \leqslant^- \mathscr{H}$ because for all $\omega \in \mathrm{Gr}(\mathscr{H}) \setminus A_0$ there exists $A_\omega \in \mathscr{H}$ such that $\omega \in A_\omega$; and moreover we have that $\mathscr{H}_0 \not\leqslant^- \mathscr{H}'$ because of A_0. □

Remark 3.48 We observe that the hypergraph $\mathscr{H}_0$ appearing in the proof of Proposition 3.47 can be realized with a graph G such that $\mathscr{D}(G) = \mathscr{D}_{ind}(G)$. Moreover, if the hypergraph $\mathscr{H}$ has ground set Ω, then the ground set of $\mathscr{H}_0$ is also Ω. We recall that the set $\mathrm{DomHyp}^-_{0,\ell}(\Omega, \mathscr{H})$ of lower domination completions is non-empty if and only if $\mathscr{H}$ has ground set Ω (see Proposition 3.30).

Remark 3.49 That all hypergraphs fulfill the domination avoidance property in the case $\leqslant^-, u$ remains an open problem.

3.3.4 Domination decompositions of a hypergraph

In this subsection we focus on the domination decomposition of a hypergraph. Concretely, we are going to prove that the set of minimal upper domination completions of a hypergraph provide a decomposition of the hypergraph; whereas the minimal upper domination completions with ground set determine a decomposition if and only if the hypergraph is not of the form $\mathscr{U}_{1,\Omega'}$ for all non-empty proper subsets Ω' of Ω.

Theorem 3.50 *Let Ω be a finite set and let $\mathscr{H}$ be a hypergraph on Ω. Let us consider the sets of minimal dominations completions:*

$$\mathrm{MinDomHyp}^+_u(\Omega, \mathscr{H}) = \{\mathscr{H}_1, \dots, \mathscr{H}_r\}$$

and:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{H}) = \{\mathscr{H}^0_1, \dots, \mathscr{H}^0_s\}.$$

Then:

$$\mathscr{H} = \mathscr{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_r \leqslant^+ \mathscr{H}^0_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}^0_s$$

and the last inequality turns into an equality if and only if $\mathcal{H} \neq \mathcal{U}_{1,\Omega'}$ for all non-empty subsets $\Omega' \subsetneq \Omega$.

Proof. By applying Proposition 3.6, Theorem 3.11 and Proposition 3.42 it follows the first equality, as well as that the inequality is an equality if $\mathcal{H} \neq \mathcal{U}_{1,\Omega'}$ for all non-empty subsets $\Omega' \subsetneq \Omega$. Moreover, as $\mathcal{H} \leqslant^+ \mathcal{H}_i^0$, by applying Remark 1.19, we get that $\mathcal{H} \leqslant^+ \mathcal{H}_1^0 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_s^0$.

Therefore, the proof of the theorem will be completed by showing that if $\mathcal{H} = \mathcal{U}_{1,\Omega'}$ for some non-empty subset $\Omega' \subsetneq \Omega$, then $\mathcal{H} \neq \mathcal{H}_1^0 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_s^0$.

So, assume that there exists a proper non-empty subset $\Omega' \subsetneq \Omega$ such that $\mathcal{H} = \mathcal{U}_{1,\Omega'}$. In such a case, let us consider the hypergraph $\mathcal{H}^*$ with ground set Ω defined by $\mathcal{H}^* = \mathcal{H} \cup \{\Omega \setminus \Omega'\}$. From Proposition 3.43 it follows that $\mathrm{MinDomHyp}_{0,u}^+(\Omega, \mathcal{H}) = \{\mathcal{H}^*\}$. Therefore, in this case $\mathcal{H}_1^0 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_s^0 = \mathcal{H}^*$ and thus $\mathcal{H} \neq \mathcal{H}_1^0 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_s^0$. □

3.3.5 Domination decompositions and hypergraphs with one minimal domination completion

In Subsection 3.1.3 we have related the existence of decompositions to the fact the poset of completions has a unique element (Corollary 3.7). Here we are going to study this relationship in the case where the completions and decomposition are associated to the domination framework. The main result is Theorem 3.52 and the proof is based on the next lemma.

Lemma 3.51 *Let Ω be a finite set and let $\mathcal{H}$ be a hypergraph on Ω. The following statements hold.*

1) *If $\mathcal{H} = \mathcal{U}_{1,\Omega'}$ for some non-empty subset $\Omega' \subseteq \Omega$, then each one of the posets of domination completions:*

$$\left(\mathrm{DomHyp}_u^+(\Omega, \mathcal{H}), \leqslant^+\right) \quad \textit{and} \quad \left(\mathrm{DomHyp}_{0,u}^+(\Omega, \mathcal{H}), \leqslant^+\right)$$

have a unique minimal element; namely:

$$\mathrm{MinDomHyp}_u^+(\Omega, \mathcal{H}) = \{\mathcal{H}\},$$

whereas:

$$\mathrm{MinDomHyp}_{0,u}^+(\Omega, \mathcal{H}) = \left\{\mathcal{H} \cup \{\Omega \setminus \Omega'\}\right\}.$$

2) *If $\mathcal{H} = \mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\}$ for some non-empty subset $\Omega' \subseteq \Omega$, then each one of the posets of domination completions:*

$$(\mathrm{DomHyp}_u^+(\Omega, \mathcal{H}), \leqslant^+) \quad \textit{and} \quad (\mathrm{DomHyp}_{0,u}^+(\Omega, \mathcal{H}), \leqslant^+)$$

possess the same unique minimal element; namely:

$$\mathrm{MinDomHyp}^+_u(\Omega,\mathcal{H}) = \mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathcal{H}) = \{\mathcal{H}\}.$$

3) *If the poset* $(\mathrm{DomHyp}^+_u(\Omega,\mathcal{H}),\leqslant^+)$ *of domination completions of the hypergraph* $\mathcal{H}$ *has a unique minimal element* $\mathcal{H}_1$*, then* $\mathcal{H} = \mathcal{H}_1$.

4) *If the poset* $(\mathrm{DomHyp}^+_{0,u}(\Omega,\mathcal{H}),\leqslant^+)$ *of domination completions of the hypergraph* $\mathcal{H}$ *has a unique minimal element* $\mathcal{H}^0_1$*, then either* $\mathcal{H} = \mathcal{H}^0_1$ *or there exists a non-empty proper subset* $\Omega' \subsetneq \Omega$ *such that* $\mathcal{H} = \mathcal{U}_{1,\Omega'}$ *and* $\mathcal{H}^0_1 = \mathcal{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\}$.

Proof. The first and the second statement are a direct consequence of Proposition 3.43. Now let us prove the third statement. Namely we are going to show that a contradiction is achieved if $\mathcal{H} \neq \mathcal{H}_1$. So, assume that $\mathcal{H} \neq \mathcal{H}_1$. On one hand, since $\mathcal{H}_1 \in \mathrm{DomHyp}^+_u(\Omega,\mathcal{H})$, the inequality $\mathcal{H} \leqslant^+ \mathcal{H}_1$ holds. On the other hand $\mathcal{H} \neq \mathcal{U}_{1,\Omega'}$ for any non-empty subset $\Omega' \subsetneq \Omega$ because otherwise, if $\mathcal{H} = \mathcal{U}_{1,\Omega'}$, then from the first statement we get that $\mathrm{MinDomHyp}^+_u(\Omega,\mathcal{H}) = \{\mathcal{H}\}$ and so $\mathcal{H} = \mathcal{H}_1$, a contradiction. Thus, since $\mathcal{H}$ and $\mathcal{H}_1$ are two hypergraphs on the finite set Ω, by applying the $\mathrm{DomHyp}^+_u(\Omega)$-avoidance property (Proposition 3.42) we get that there exists a domination hypergraph $\mathcal{H}_0$ on Ω such that $\mathcal{H} \leqslant^+ \mathcal{H}_0$ and $\mathcal{H}_1 \not\leqslant^+ \mathcal{H}_0$; that is, there exists $\mathcal{H}_0 \in \mathrm{DomHyp}^+_u(\Omega,\mathcal{H})$ such that $\mathcal{H}_1 \not\leqslant^+ \mathcal{H}_0$. This is a contradiction because $\mathrm{MinDomHyp}^+_u(\Omega,\mathcal{H}) = \{\mathcal{H}_1\}$, and so $\mathcal{H}_1 \leqslant^+ \mathcal{H}_0$.

To conclude, let us demonstrate the last statement. Since $\mathcal{H}^0_1$ is the unique minimal element of the poset $\left(\mathrm{DomHyp}^+_{0,u}(\Omega,\mathcal{H}),\leqslant^+\right)$, we get that $\mathcal{H} \leqslant^+ \mathcal{H}^0_1$ and that $\mathcal{H}^0_1 \leqslant^+ \mathcal{H}_0$ for all domination completion $\mathcal{H}_0$ of $\mathcal{H}$ with ground set Ω. Therefore, we conclude that either $\mathcal{H} = \mathcal{H}^0_1$ or $\mathcal{H} = \mathcal{U}_{1,\Omega'}$ for some subset $\Omega' \subsetneq \Omega$, because otherwise a contradiction can be obtained by applying the $\mathrm{DomHyp}^+_{0,u}(\Omega)$-avoidance property (Proposition 3.42). Hence, from the first statement of this lemma it follows that $\mathcal{H} = \mathcal{H}^0_1$ or $\mathcal{H} = \mathcal{U}_{1,\Omega'}$ and $\mathcal{H}^0_1 = \mathcal{H} \cup \{\Omega \setminus \Omega'\}$. This completes the proof of the lemma. □

Theorem 3.52 *Let* Ω *be a finite set and let* $\mathcal{H}$ *be a hypergraph on* Ω*. Let us consider the following three conditions:*

C1) *The hypergraph* $\mathcal{H}$ *is a domination hypergraph on* Ω.

C2) *The non-empty poset* $\left(\mathrm{DomHyp}^+_u(\Omega,\mathcal{H}),\leqslant^+\right)$ *has a unique minimal element.*

C3) *The non-empty poset* $\left(\mathrm{DomHyp}^+_{0,u}(\Omega,\mathcal{H}),\leqslant^+\right)$ *has a unique minimal element.*

Then (C1) $\Leftrightarrow$ *(C2)* $\Leftarrow$ *(C3). Moreover, if* $\mathcal{H}$ *satisfies (C1), then* $\mathcal{H}$ *satisfies (C3) if and only if* $\mathcal{H}$ *has ground set* Ω *or* $\mathcal{H} = \mathcal{U}_{1,\Omega'}$ *for some non-empty subset* $\Omega' \subsetneq \Omega$.

Proof. It is clear that if $\mathcal{H} \in \mathrm{DomHyp}(\Omega)$, then $\mathcal{H}$ is the only minimal element of the poset $\left(\mathrm{DomHyp}^+_u(\Omega,\mathcal{H}),\leqslant^+\right)$; that is:

$$\mathrm{MinDomHyp}^+_u(\Omega,\mathcal{H}) = \{\mathcal{H}\}.$$

Therefore condition (C1) implies condition (C2). Moreover, from the third statement of Lemma 3.51 we get that condition (C2) implies condition (C1). So, the equivalence (C1) $\Leftrightarrow$ (C2) holds.

Next let us show that if $\mathscr{H} \in \mathrm{Hyp}(\Omega)$, then condition (C3) implies condition (C2). So, let us assume that the poset $\left(\mathrm{DomHyp}^{+}_{0,u}(\Omega,\mathscr{H}), \leqslant^{+}\right)$ has a unique minimal element $\mathscr{H}^{0}_{1}$. In such a case, from Lemma 3.51 we get that either $\mathscr{H} = \mathscr{H}^{0}_{1}$ or there exists a proper subset $\Omega' \subsetneq \Omega$ such that $\mathscr{H} = \mathscr{U}_{1,\Omega'}$. In any case, $\mathscr{H}$ is a domination hypergraph and so condition (C2) holds.

To finish the proof of the theorem it is enough to show that if $\mathscr{H}$ is a domination hypergraph, then $\mathscr{H}$ satisfies (C3) if and only if $\mathscr{H}$ is a hypergraph with ground set Ω or $\mathscr{H} = \mathscr{U}_{1,\Omega'}$ for some subset $\Omega' \subsetneq \Omega$.

On one hand, it is clear that if $\mathscr{H} \in \mathrm{DomHyp}_0(\Omega)$, then $\mathscr{H} \in \mathrm{DomHyp}^{+}_{0,u}(\mathscr{H},\Omega)$, and so the poset $\left(\mathrm{DomHyp}^{+}_{0,u}(\Omega,\mathscr{H}), \leqslant^{+}\right)$ of domination completions of $\mathscr{H}$ has a unique minimal element, namely:

$$\mathrm{MinDomHyp}^{+}_{0,u}(\Omega,\mathscr{H}) = \{\mathscr{H}\}.$$

On the other hand, from the first statement in Lemma 3.51 it follows that if $\mathscr{H} = \mathscr{U}_{1,\Omega'}$ for some subset $\Omega' \subsetneq \Omega$, then the hypergraph $\mathscr{H} \cup \{\Omega \setminus \Omega'\}$ is the unique minimal element of the poset $\left(\mathrm{DomHyp}^{+}_{0,u}(\Omega,\mathscr{H}), \leqslant^{+}\right)$.

So the proof of the theorem will be completed by showing that if $\mathscr{H}$ is a domination hypergraph on Ω such that $\left(\mathrm{DomHyp}^{+}_{0,u}(\Omega,\mathscr{H}), \leqslant^{+}\right)$ has a unique minimal element $\mathscr{H}_0$, then either $\mathscr{H}$ is a hypergraph with ground set Ω or $\mathscr{H} = \mathscr{U}_{1,\Omega'}$ for some subset $\Omega' \subsetneq \Omega$. This implication follows from the last statement of Lemma 3.51. This completes the proof of the theorem. □

Remark 3.53 In spite of the equivalences in Theorem 3.52, it is interesting to bear in mind that there exist domination hypergraphs $\mathscr{H}$ on the finite set Ω such that the poset $(\mathrm{DomHyp}^{+}_{0,u}(\Omega,\mathscr{H}), \leqslant^{+})$ has a unique minimal element $\mathscr{H}_0$, but $\mathscr{H} \neq \mathscr{H}_0$ (see Example 3.54). Moreover, there exist domination hypergraphs $\mathscr{H}$ on the finite set Ω such that the poset $\left(\mathrm{DomHyp}^{+}_{0,u}(\Omega,\mathscr{H}), \leqslant^{+}\right)$ has two or more minimal elements as it is shown in Example 3.55.

Example 3.54 Let $\Omega = \{1,2,3,4\}$ and let us consider the hypergraph on Ω defined by $\mathscr{H} = \{\{1\},\{2\},\{3\}\}$. Observe that $\mathscr{H} = \mathscr{U}_{1,\Omega'}$, where $\Omega' = \{1,2,3\} \subsetneq \Omega$. Therefore, $\mathscr{H}$ is a domination hypergraph (Proposition 3.43). In addition, by taking into account the Table 2.3, it is not hard to check that the poset $\left(\mathrm{DomHyp}^{+}_{0,u}(\Omega,\mathscr{H}), \leqslant^{+}\right)$ has a unique minimal element. Namely, $\mathrm{MinDomHyp}^{+}_{0,u}(\Omega,\mathscr{H}) = \{\mathscr{H}_0\}$, where $\mathscr{H}_0 = \mathscr{H} \cup \{\Omega \setminus \{1,2,3\}\} = \mathscr{H} \cup \{\{4\}\} = \{\mathscr{D}(K_\Omega)\}$ (observe that this fact is statement 1) of Lemma 3.51). Therefore, $\mathscr{H}$ is a domination hypergraph but $\mathscr{H} \neq \mathscr{H}_0$.

Example 3.55 We now consider the hypergraph $\mathscr{H} = \{\{1\},\{2,3\}\}$ on the finite set $\Omega = \{1,2,3,4\}$. The hypergraph $\mathscr{H}$ is a domination hypergraph on Ω; indeed we have that

$\mathscr{H}' = \mathscr{D}(G)$ with G a graph of type $G_{3,3}$ (see Table 2.2). So we have that $\mathscr{H} \in \mathrm{DomHyp}(\Omega)$ but $\mathscr{H} \notin \mathrm{DomHyp}_0(\Omega)$. Therefore, from Lemma 3.51 and Theorem 3.52, we deduce that the poset $(\mathrm{DomHyp}^+_u(\Omega,\mathscr{H}), \leqslant^+)$ has a unique minimal element; namely:

$$\mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H}) = \{\mathscr{H}\};$$

whereas the poset $\left(\mathrm{DomHyp}^+_{0,u}(\Omega,\mathscr{H}), \leqslant^+\right)$ has two or more minimal elements. Let us show that, in this case, the poset $\left(\mathrm{DomHyp}^+_{0,u}(\Omega,\mathscr{H}), \leqslant^+\right)$ has exactly two minimal elements. From the Table 2.3, it is not hard to check that there are six domination hypergraphs $\mathscr{H}'$ with ground set Ω and such that $\mathscr{H} \leqslant^+ \mathscr{H}'$. Namely, two hypergraphs $\mathscr{H}'$ of the form $\mathscr{H}' = \mathscr{D}(G')$ with G' a graph of type $G_{4,9}$; three hypergraphs $\mathscr{H}' = \mathscr{D}(G')$ with G' of type $G_{4,10}$; and one with G' of type $G_{4,11}$. So $\mathrm{DomHyp}^+_{0,u}(\Omega,\mathscr{H}) = \left\{\mathscr{H}'_1,\dots,\mathscr{H}'_6\right\}$. Now, by a straightforward computation we can show that the minimal elements of the poset $\left(\mathrm{DomHyp}^+_{0,u}(\Omega,\mathscr{H}), \leqslant^+\right)$ are the two hypergraphs $\mathscr{H}'_1, \mathscr{H}'_2$ of the type $\mathscr{H}' = \mathscr{D}(G')$ with G' a graph of type $G_{4,9}$; that is, the hypergraphs $\mathscr{H}'_1 = \{\{1\},\{2,3\},\{2,4\}\}$ and $\mathscr{H}'_2 = \{\{1\},\{2,3\},\{3,4\}\}$. Therefore:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H}) = \left\{\mathscr{H}'_1,\mathscr{H}'_2\right\}.$$

Remark 3.56 Let Ω be a finite set and let $\mathscr{H} \in \mathrm{Hyp}(\Omega)$. Let us consider the sets of minimal dominations completions:

$$\mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H}) = \{\mathscr{H}_1,\dots,\mathscr{H}_r\}$$

and:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H}) = \{\mathscr{H}^0_1,\dots,\mathscr{H}^0_s\}.$$

From Theorem 3.50 we get that if $r=1$, then $\mathscr{H} = \mathscr{H}_1$; while, if $s=1$, then $\mathscr{H} \leqslant \mathscr{H}^0_1$, and $\mathscr{H} = \mathscr{H}^0_1$ if and only if $\mathscr{H} \neq \mathscr{U}_{1,\Omega'}$ for all non-empty subsets $\Omega' \subsetneq \Omega$. Notice that this result is consistent with the statements (3) and (4) in Lemma 3.51 where we analyze the hypergraphs whose posets of domination completions have a unique minimal element; that is, hypergraphs with $r=1$, and hypergraphs with $s=1$. Moreover, recall that from Theorem 3.52 we get that, if $s=1$, then $r=1$. Hence, if $s=1$, then $\mathscr{H} = \mathscr{H}_1$, and so we conclude that $\mathscr{H} = \mathscr{H}_1 \leqslant \mathscr{H}^0_1$. Finally, we observe that from Example 3.55 we have that $r=1$ does not imply that $s=1$.

3.3.6 Domination decompositions and hypergraphs with the same minimal domination completions

In Subsection 3.1.3 we have seen in the general that two hypergraphs are equal if and only if they have the same optimal completions (Corollary 3.8). Here we are going to explore this result in the domination framework. This is done in Theorem 3.57. Observe that from this theorem it follows that any hypergraph is univocally determined by its minimal domination completions.

Theorem 3.57 *Let Ω be a finite set and let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two hypergraphs on Ω. The following statements hold.*

1) $\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}_1) = \mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}_2)$ *if and only if* $\mathscr{H}_1 = \mathscr{H}_2$.
2) $\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_1) = \mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_2)$ *if and only if the hypergraphs* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *fulfill one of the following four cases:*
   (2.1) *either* $\mathscr{H}_1 = \mathscr{H}_2$,
   (2.2) *or there exists a non-empty and proper subset* $\emptyset \subsetneq \Omega' \subsetneq \Omega$ *such that* $\mathscr{H}_1 = \mathscr{U}_{1,\Omega'}$ *and* $\mathscr{H}_2 = \mathscr{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\}$,
   (2.3) *or there exists a non-empty and proper subset* $\emptyset \subsetneq \Omega' \subsetneq \Omega$ *such that* $\mathscr{H}_1 = \mathscr{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\}$ *and* $\mathscr{H}_2 = \mathscr{U}_{1,\Omega'}$,
   (2.4) *or there exist two different elements* $w_{0,1}, w_{0,2} \in \Omega$ *such that* $\mathscr{H}_1 = \mathscr{U}_{1,\Omega\setminus\{w_{0,1}\}}$ *and* $\mathscr{H}_2 = \mathscr{U}_{1,\Omega\setminus\{w_{0,2}\}}$.
3) *If the hypergraphs* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *have the same ground set, then* $\mathscr{H}_1 = \mathscr{H}_2$ *if and only if* $\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_1) = \mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_2)$.

Proof. The first statement of the theorem is a direct consequence of the first equality in Theorem 3.50; whereas the equivalences in the third statement follow from statement (2). Therefore, we must prove the equivalence in the second statement. From Lemma 3.51 it follows that if $\mathscr{H}_1$ and $\mathscr{H}_2$ satisfy (2.2) or (2.3) or (2.4), then:

$$\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_1) = \mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_2).$$

Thus, to conclude we must prove that if:

$$\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_1) = \mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_2),$$

then $\mathscr{H}_1$ and $\mathscr{H}_2$ satisfy either (2.1) or (2.2) or (2.3) or (2.4).

So, let us assume that $\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_1) = \mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_2)$. Observe that if $\mathscr{H}_1$ and $\mathscr{H}_2$ are not of the form $\mathscr{U}_{1,\Omega'}$ with $\Omega' \subsetneq \Omega$, then by applying the second equality in Theorem 3.50, we conclude that $\mathscr{H}_1 = \mathscr{H}_2$. Therefore, without loss of generality, from now on we may assume that there exists a subset $\Omega' \subsetneq \Omega$ such that $\mathscr{H}_1 = \mathscr{U}_{1,\Omega'}$. At this point, the proof will be completed by showing that if $\mathscr{H}_2 \neq \mathscr{H}_1$ and $\mathscr{H}_2 \neq \mathscr{H}_1 \cup \{\Omega \setminus \Omega'\}$, then $|\Omega'| = |\Omega| - 1$ and there exists a subset $\Omega'' \subsetneq \Omega$ of size $|\Omega''| = |\Omega| - 1$ such that $\mathscr{H}_2 = \mathscr{U}_{1,\Omega''}$. Let us prove it.

Since $\mathscr{H}_1 = \mathscr{U}_{1,\Omega'}$, from Lemma 3.51 (1) we get that:

$$\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_1) = \{\mathscr{H}_1 \cup \{\Omega \setminus \Omega'\}\}.$$

Therefore:

$$\mathrm{MinDomHyp}_{0,u}^+(\Omega,\mathscr{H}_2) = \{\mathscr{H}_1 \cup \{\Omega \setminus \Omega'\}\}$$

and thus, since $\mathscr{H}_2 \neq \mathscr{H}_1 \cup \{\Omega \setminus \Omega'\}$, by applying Lemma 3.51 (4) we conclude that there exists $\Omega'' \subsetneq \Omega$ such that $\mathscr{H}_2 = \mathscr{U}_{1,\Omega''}$ and $\mathscr{H}_1 \cup \{\Omega \setminus \Omega'\} = \mathscr{U}_{1,\Omega''} \cup \{\Omega \setminus \Omega''\}$; that is, we have the equality $\mathscr{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\} = \mathscr{U}_{1,\Omega''} \cup \{\Omega \setminus \Omega''\}$. If Ω' has size $|\Omega'| \leqslant |\Omega| - 2$, then from the above equality we conclude that $\Omega' = \Omega''$, and thus $\mathscr{H}_1 = \mathscr{H}_2$, a contradiction because we are assuming $\mathscr{H}_1 \neq \mathscr{H}_2$. Therefore $|\Omega'| > |\Omega| - 2$. Thus, since $\Omega' \subsetneq \Omega$, we conclude that $|\Omega'| = |\Omega| - 1$. So $|\Omega''| = |\Omega| - 1$ because $\mathscr{U}_{1,\Omega'} \cup \{\Omega \setminus \Omega'\} = \mathscr{U}_{1,\Omega''} \cup \{\Omega \setminus \Omega''\}$. This completes the proof of the theorem. □

Remark 3.58 The equivalence in the third statement of the above theorem is not true if the hypergraphs $\mathscr{H}_1$ and $\mathscr{H}_2$ have different ground set. Indeed, let Ω be a finite set and let us consider the following hypergraphs $\mathscr{H}_1 = \mathscr{U}_{1,\Omega\setminus\{w_1\}}$ and $\mathscr{H}_2 = \mathscr{U}_{1,\Omega\setminus\{w_2\}}$ on Ω, where $\omega_1, \omega_2 \in \Omega$. From the statement (2.4) of Theorem 3.57 we get that the equality:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H}_1) = \mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H}_2)$$

holds but $\mathscr{H}_1 \neq \mathscr{H}_2$ if $w_1 \neq w_2$. However, observe that $\mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H}_i) = \{\mathscr{H}_i\}$ (see Lemma 3.51) and so, actually if $w_1 \neq w_2$, then:

$$\mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H}_1) \neq \mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H}_2).$$

3.4 Domination decomposition index of a hypergraph

The decomposition index of a hypergraph can be defined in a general framework where we consider the $\Sigma^{*_1}_{*_2}$-decomposition problem for a family of hypergraphs Σ. However, in this section we are going to study only the decomposition with regard to the domination context with respect to the order $\leqslant^+$ and the side u. Namely, first in Subsection 3.4.1 we define these decomposition indices, and afterwards, in Subsection 3.4.2 we see that this index provides a lower bound for a hypergraph to be a domination hypergraph.

3.4.1 The two domination decomposition indices of a hypergraph

Let $\mathscr{H}$ be a hypergraph on a finite set Ω. In Subsection 3.1.3 we introduced the general definition of decomposition. Here we adapt them to our situation. We say that a family $\{\mathscr{H}_1,\dots,\mathscr{H}_t\} \subseteq \mathrm{DomHyp}(\Omega)$ of $t \geqslant 1$ distinct domination hypergraphs on Ω is a *t-decomposition* of the hypergraph $\mathscr{H}$ if:

$$\mathscr{H} = \mathscr{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_t.$$

A t-0-decomposition is a t-decomposition $\{\mathscr{H}_1,\dots,\mathscr{H}_t\}$ where $\mathscr{H}_1,\dots,\mathscr{H}_t$ are domination hypergraphs with ground set Ω. Let us denote:

$$\mathfrak{D}^+_u(\Omega,\mathscr{H}) = \min\{t : \text{there exists a } t\text{-decomposition of } \mathscr{H}\}$$
$$\mathfrak{D}^+_{0,u}(\Omega,\mathscr{H}) = \min\{t : \text{there exists a } t\text{-0-decomposition of } \mathscr{H}\}.$$

We call $\mathfrak{D}^+_u(\Omega,\mathscr{H})$ the *upper decomposition index* of the hypergraph $\mathscr{H}$ with respect to the order $\leqslant^+$, and we call $\mathfrak{D}^+_{0,u}(\Omega,\mathscr{H})$ the *upper decomposition index of $\mathscr{H}$ with* *ground set* Ω also with respect to the order $\leqslant^+$. Since we are going to consider only the domination decomposition indices of a hypergraph, we will omit the reference to "domination" and we will say "decomposition index" only.

From Theorem 3.50 and Theorem 3.52 we get that, if $\mathscr{H}$ is not a domination hypergraph, then the hypergraphs belonging to $\mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H})$ and to $\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H})$ provide a decomposition of $\mathscr{H}$. (In the ground-set case, this is true if and only if the hypergraph $\mathscr{H}$ is not of the form $\mathscr{U}_{1,\Omega'}$ for a non-empty proper subset Ω' of Ω.)

The next proposition states that, in fact, to compute $\mathfrak{D}^+_u(\Omega,\mathscr{H})$ and $\mathfrak{D}^+_{0,u}(\Omega,\mathscr{H})$ it is enough to consider only those decompositions that consists of minimal domination completions of $\mathscr{H}$.

Proposition 3.59 *Let $\mathscr{H}$ be a hypergraph on a finite set Ω. The following statements hold.*

1) *If $\mathfrak{D}^+_u(\Omega,\mathscr{H}) = \delta$, then there exists a δ-decomposition $\{\mathscr{H}_1,\dots,\mathscr{H}_\delta\}$ of $\mathscr{H}$ where $\mathscr{H}_1,\dots,\mathscr{H}_\delta \in \mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H})$.*

2) *If $\mathfrak{D}^+_{0,u}(\Omega,\mathscr{H}) = \delta$, then there exists a δ-0-decomposition $\{\mathscr{H}_1,\dots,\mathscr{H}_\delta\}$ of $\mathscr{H}$ where $\mathscr{H}_1,\dots,\mathscr{H}_\delta \in \mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H})$.*

Proof. This result can be proved by using the general results (see Proposition 3.6). However, we are going to present a specific proof in order to avoid any difficulty concerning the ground set.

The proof of statement 1) is analogous to the proof of statement 2). Namely, we only must do the following replacements: $\mathfrak{D}^+_{0,u}(\Omega,\mathscr{H})$ by $\mathfrak{D}^+_u(\Omega,\mathscr{H})$; $\mathrm{DomHyp}^+_{0,u}(\Omega,\mathscr{H})$ by $\mathrm{DomHyp}^+_u(\Omega,\mathscr{H})$; the set $\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H})$ by $\mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H})$; and replace "0-decomposition" by "decomposition". Hence, next we only are going to demonstrate the second statement.

To prove statement 2) it is enough to see that any 0-decomposition of $\mathscr{H}$ can be transformed into a 0-decomposition consisting of minimal 0-domination completions of $\mathscr{H}$; that is, we must prove that if $\{\mathscr{H}'_1,\dots,\mathscr{H}'_t\}$ is a t-0-decomposition of $\mathscr{H}$, then there exist distinct $s \leqslant t$ minimal 0-domination completions $\mathscr{H}_{i_1},\dots,\mathscr{H}_{i_s} \in \mathrm{MinDomHyp}^+_{0,u}(\mathscr{H})$ that constitute an s-0-decomposition of $\mathscr{H}$.

So, assume that $\mathscr{H}'_1,\dots,\mathscr{H}'_t \in \mathrm{DomHyp}_0(\Omega)$ are such that $\{\mathscr{H}'_1,\dots,\mathscr{H}'_t\}$ is a t-0-decomposition of $\mathscr{H}$; that is, $\mathscr{H} = \mathscr{H}'_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}'_t$.

We have that $\mathscr{H} = \mathscr{H}'_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}'_t \leqslant^+ \mathscr{H}'_k$ for all k and, as $\mathscr{H}'_k \in \mathrm{DomHyp}_0(\Omega)$, we have that $\mathscr{H}'_k \in \mathrm{DomHyp}^+_{0,u}(\Omega,\mathscr{H})$.

Set $\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H}) = \{\mathscr{H}_1,\dots,\mathscr{H}_r\}$.

Then for all $k \in \{1,\dots,t\}$, there exists $\alpha_k \in \{1,\dots,r\}$ such that $\mathscr{H}_{\alpha_k} \leqslant^+ \mathscr{H}'_k$. Let us write $\{\mathscr{H}_{i_1},\dots,\mathscr{H}_{i_s}\} = \{\mathscr{H}_{\alpha_1},\dots,\mathscr{H}_{\alpha_t}\}$, where $\mathscr{H}_{i_1},\dots,\mathscr{H}_{i_s}$ are different (observe that $s \leqslant \min\{r,t\} \leqslant t$). On one hand, we have that:

$$\mathscr{H}_{i_1} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_{i_s} = \mathscr{H}_{\alpha_1} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_{\alpha_t} \leqslant^+ \mathscr{H}'_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}'_t;$$

and hence we get that $\mathscr{H}_{i_1} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_{i_s} \leqslant^+ \mathscr{H}$. On the other hand, from Remark 1.19, it is clear that:

$$\mathscr{H} \leqslant^+ \mathscr{H}_{i_1} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_{i_s}.$$

Therefore, we conclude that $\mathscr{H} = \mathscr{H}_{i_1} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_{i_s}$; that is, $\{\mathscr{H}_{i_1},\dots,\mathscr{H}_{i_s}\}$ is an s-0-decomposition of $\mathscr{H}$. □

As a consequence of the previous proposition, we get the following results on the decomposition indices.

Proposition 3.60 *Let Ω be a finite set and let $\mathscr{H}$ be a hypergraph on Ω. Then the following statements hold.*

1) *The decomposition index $\mathfrak{D}^+_u(\Omega,\mathscr{H})$ is well defined.*
2) *The decomposition index $\mathfrak{D}^+_{0,u}(\Omega,\mathscr{H})$ is defined if and only if $\mathscr{H} \neq \mathscr{U}_{1,\Omega'}$ for all non-empty and proper subsets Ω' of Ω.*

Proof. The first statement is a direct consequence of Theorem 3.50. Let us prove the second one. If $\mathscr{H} \neq \mathscr{U}_{1,\Omega'}$ for all non-empty and proper subsets Ω' of Ω, then by Theorem 3.50 we have a decomposition and hence $\mathfrak{D}^+_{0,u}(\Omega,\mathscr{H})$ is defined. Conversely, if this decomposition index is defined, then there exists a decomposition of $\mathscr{H}$. Then by Proposition 3.59 the hypergraph $\mathscr{H}$ has a decomposition $\mathscr{H} = \mathscr{H}_{i_1} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_{i_\delta}$, where $\mathscr{H}_{i_j}$ are minimal domination completions of $\mathscr{H}$ with ground set Ω. Therefore, from Theorem 3.50 and Remark 1.19, we have that $\mathscr{H} \leqslant^+ \mathscr{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_s \leqslant^+ \mathscr{H}_{i_1} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_{i_\delta} = \mathscr{H}$, where $\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H}) = \{\mathscr{H}_1,\dots,\mathscr{H}_s\}$. From these inequalities it follows that $\mathscr{H} = \mathscr{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_s$. Then by Theorem 3.50 we get that $\mathscr{H} \neq \mathscr{U}_{1,\Omega'}$ for all non-empty and proper subsets Ω' of Ω. □

Proposition 3.61 *Let Ω be a finite set and let $\mathscr{H}$ be a hypergraph on Ω. Then the following statements hold.*

1) *$\mathfrak{D}^+_u(\Omega,\mathscr{H}) = 1$ if and only if the hypergraph $\mathscr{H}$ is a domination hypergraph.*
2) *$\mathfrak{D}^+_{0,u}(\Omega,\mathscr{H}) = 1$ if and only if the hypergraph $\mathscr{H}$ is a domination hypergraph with ground set Ω.*

3) *If the hypergraph $\mathscr{H}$ is not a domination hypergraph, then:*

$$2 \leqslant \mathfrak{D}_u^+(\Omega, \mathscr{H}) \leqslant \min\{r, \mathfrak{D}_{0,u}^+(\Omega, \mathscr{H})\} \leqslant \mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}) \leqslant s,$$

where r is the number of minimal domination completions of $\mathscr{H}$ and s is the number of minimal 0-domination completions of $\mathscr{H}$.

Proof. Statements 1) and 2) follow easily from Theorem 3.50, Lemma 3.51 and Theorem 3.52. Now if $\mathscr{H}$ is not a domination hypergraph, then $\mathscr{H} \neq \mathscr{U}_{1,\Omega'}$ for all non-empty and proper subsets Ω' of Ω (see Proposition 3.43). In addition, from Proposition 3.59, we get that $\mathfrak{D}_u^+(\Omega, \mathscr{H}) \leqslant r$ and $\mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}) \leqslant s$. Furthermore, from the definition it follows that $\mathfrak{D}_u^+(\Omega, \mathscr{H}) \leqslant \mathfrak{D}_{0,u}^+(\Omega, \mathscr{H})$. From these facts and Theorem 3.50 the third statement follows. □

As far as we know, there is no more general relation between these parameters. Let us show some examples.

Example 3.62 On the finite set $\Omega = \{1,2,3,4\}$ let us consider the hypergraph of Example 3.55; that is, the hypergraph $\mathscr{H} = \{\{1\},\{2,3\}\}$. We know that this hypergraph is a domination hypergraph and that the minimal elements of the poset $\left(\mathrm{DomHyp}_{0,u}^+(\Omega, \mathscr{H}), \leqslant^+\right)$ are $\mathscr{H}_1' = \{\{1\},\{2,3\},\{2,4\}\}$ and $\mathscr{H}_2' = \{\{1\},\{2,3\},\{3,4\}\}$. Thus:

$$\mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H}) = \{\mathscr{H}\},$$
$$\mathrm{MinDomHyp}_{0,u}^+(\Omega, \mathscr{H}) = \{\mathscr{H}_1', \mathscr{H}_2'\}.$$

Hence we conclude that $\mathfrak{D}_u^+(\Omega, \mathscr{H}) = 1$ and $\mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}) = 2$. So $\mathfrak{D}_u^+(\Omega, \mathscr{H}) = 1$ does not implies that $\mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}) = 1$.

Example 3.63 Now, on the finite set $\Omega = \{1,2,3,4\}$ let us consider the hypergraphs $\mathscr{H}_1, \mathscr{H}_2, \mathscr{H}_3, \mathscr{H}_4, \mathscr{H}_5$ of Example 3.33; that is, the hypergraphs:

$$\mathscr{H}_1 = \{\{1,2\},\{1,3\},\{2,3\}\},$$
$$\mathscr{H}_2 = \{\{1,2\},\{2,3\},\{3,4\}\},$$
$$\mathscr{H}_3 = \{\{1,2\},\{1,3,4\},\{2,3,4\}\},$$
$$\mathscr{H}_4 = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}, \text{ and}$$
$$\mathscr{H}_5 = \{\{1\},\{2\},\{3\}\}.$$

In order to compute the values of $\mathfrak{D}_u^+(\Omega, \mathscr{H}_i)$ and $\mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}_i)$ we will use the explicit description of the set:

$$\mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H}_i) = \{\mathscr{H}_{i,1}, \ldots, \mathscr{H}_{i,r_i}\}$$

of minimal domination completions of the hypergraph $\mathscr{H}_i$, and of the set:

$$\mathrm{MinDomHyp}_{0,u}^+(\Omega, \mathscr{H}_i) = \{\mathscr{H}_{i,1}^0, \ldots, \mathscr{H}_{i,s_i}^0\}$$

of the minimal 0-domination completions of the hypergraph $\mathscr{H}_i$ (these descriptions are detailed in Example 3.33). In Table 3.12 we summarize the values of r_i and s_i as well as the values of the decomposition indices $\mathfrak{D}_u^+(\Omega, \mathscr{H}_i)$ and $\mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}_i)$.

	r_i and s_i	$\mathfrak{D}_u^+(\Omega, \mathscr{H}_i)$ and $\mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}_i)$
$i = 1$	$4 < 7$	$2 < 3$
$i = 2$	$3 = 3$	$2 = 2$
$i = 3$	$6 > 4$	$2 = 2$
$i = 4$	$7 = 7$	$3 = 3$
$i = 5$	$1 = 1$	1 and not defined

Table 3.12 Number of minimal domination completions and decomposition indices for the hypergraphs $\mathscr{H}_i$ of Example 3.63.

(3.63.1) Computation of $\mathfrak{D}_u^+(\Omega, \mathscr{H}_1)$ and $\mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}_1)$.
From (3.33.1) we know that:

$$\mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H}_1) = \{\mathscr{H}_{1,i} : 1 \leqslant i \leqslant 3\} \cup \{\mathscr{U}_{2,\Omega}\},$$
$$\mathrm{MinDomHyp}_{0,u}^+(\Omega, \mathscr{H}_1) = \{\mathscr{H}_{1,i}^0 : 1 \leqslant i \leqslant 6\} \cup \{\mathscr{U}_{2,\Omega}\}.$$

Therefore, $\mathscr{H}_1$ is not a domination hypergraph and:

$$2 \leqslant \mathfrak{D}_u^+(\Omega, \mathscr{H}_1) \leqslant 4, \qquad 2 \leqslant \mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}_1) \leqslant 7.$$

It is clear that $\{\mathscr{H}_{1,1}, \mathscr{H}_{1,2}\}$ is a decomposition of $\mathscr{H}_1$. So we conclude that $\mathfrak{D}_u^+(\Omega, \mathscr{H}_1) = 2$. However, on one hand we have that $\{\mathscr{H}_{1,1}^0, \mathscr{H}_{1,2}^0, \mathscr{H}_{1,4}^0\}$ is a 0-decomposition of $\mathscr{H}_1$, and so $\mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}_1) \leqslant 3$; while, on the other hand, it is not hard to check that neither $\{\mathscr{H}_{1,i}^0, \mathscr{H}_{1,j}^0\}$ or $\{\mathscr{H}_{1,i}^0, \mathscr{U}_{2,\Omega}\}$ is a 0-decomposition of $\mathscr{H}_1$, and thus from Proposition 3.59 it follows that $\mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}_1) \neq 2$. Therefore, we conclude that $\mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}_1) = 3$.

(3.63.2) Computation of $\mathfrak{D}_u^+(\Omega, \mathscr{H}_2)$ and $\mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}_2)$.
From (3.33.2) we know that:

$$\mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H}_2) = \{\mathscr{H}_{2,1}, \mathscr{H}_{2,2}, \mathscr{H}_{2,3}\},$$
$$\mathrm{MinDomHyp}_{0,u}^+(\Omega, \mathscr{H}_2) = \{\mathscr{H}_{2,1}^0, \mathscr{H}_{2,2}^0, \mathscr{H}_{2,3}^0\}.$$

Since the hypergraph $\mathscr{H}_2$ is not a domination hypergraph, and $\{\mathscr{H}_{2,1}, \mathscr{H}_{2,2}\}$ is a decomposition of $\mathscr{H}_2$ and $\{\mathscr{H}_{2,1}^0, \mathscr{H}_{2,2}^0\}$ is a 0-decomposition of $\mathscr{H}_2$, we conclude that:

$$\mathfrak{D}_u^+(\Omega, \mathscr{H}_2) = \mathfrak{D}_{0,u}^+(\Omega, \mathscr{H}_2) = 2.$$

Observe that in this case $\{\mathscr{H}_{2,1}, \mathscr{H}_{2,3}\}$ and $\{\mathscr{H}_{2,2}, \mathscr{H}_{2,3}\}$ are also decompositions of $\mathscr{H}_2$, however $\{\mathscr{H}_{2,1}^0, \mathscr{H}_{2,3}^0\}$ is a 0-decomposition of $\mathscr{H}_2$, but $\{\mathscr{H}_{2,2}^0, \mathscr{H}_{2,3}^0\}$ is not a 0-decomposition of $\mathscr{H}_2$.

(3.63.3) Computation of $\mathfrak{D}_u^+(\Omega,\mathcal{H}_3)$ and $\mathfrak{D}_{0,u}^+(\Omega,\mathcal{H}_3)$.

From (3.33.3):

$$\text{MinDomHyp}_u^+(\Omega,\mathcal{H}_3) = \{\mathcal{H}_{3,1},\mathcal{H}_{3,2},\mathcal{H}_{3,3},\mathcal{H}_{3,4},\mathcal{H}_{3,5},\mathcal{H}_{3,6}\},$$
$$\text{MinDomHyp}_{0,u}^+(\Omega,\mathcal{H}_3) = \{\mathcal{H}_{3,1}^0,\mathcal{H}_{3,2}^0,\mathcal{H}_{3,3}^0,\mathcal{H}_{3,4}^0\}$$

where $\mathcal{H}_{3,i}^0 = \mathcal{H}_{3,i}$ for all i. In this case we have that $\{\mathcal{H}_{3,3},\mathcal{H}_{3,4}\}$ is a decomposition of $\mathcal{H}_3$. Therefore:

$$\mathfrak{D}_u^+(\Omega,\mathcal{H}_3) = \mathfrak{D}_{0,u}^+(\Omega,\mathcal{H}_3) = 2,$$

because $\mathcal{H}_3$ is not a domination hypergraph.

(3.63.4) Computation of $\mathfrak{D}_u^+(\Omega,\mathcal{H}_4)$ and $\mathfrak{D}_{0,u}^+(\Omega,\mathcal{H}_4)$.

From (3.33.4) we have that:

$$\begin{aligned}\text{MinDomHyp}_u^+(\Omega,\mathcal{H}_4) &= \text{MinDomHyp}_{0,u}^+(\Omega,\mathcal{H}_4)\\ &= \{\mathcal{H}_{4,1},\dots,\mathcal{H}_{4,7}\}.\end{aligned}$$

Therefore, $\mathcal{H}_4$ is not a domination hypergraphs and, by applying Proposition 3.61 we get that:

$$2 \leqslant \mathfrak{D}_u^+(\Omega,\mathcal{H}_4) = \mathfrak{D}_{0,u}^+(\Omega,\mathcal{H}_4) \leqslant 7.$$

However, on one hand it is clear that $\{\mathcal{H}_{4,4},\mathcal{H}_{4,5},\mathcal{H}_{4,6}\}$ is a decomposition of $\mathcal{H}_4$, and so $\mathfrak{D}_u^+(\Omega,\mathcal{H}_4) \leqslant 3$; while, on the other hand, it is not hard to check that $\{\mathcal{H}_{4,i},\mathcal{H}_{4,j}\}$ is not a decomposition of $\mathcal{H}_4$, and thus from Proposition 3.59 it follows that $\mathfrak{D}_u^+(\Omega,\mathcal{H}_4) \neq 2$. Therefore, we conclude that:

$$\mathfrak{D}_u^+(\Omega,\mathcal{H}_4) = \mathfrak{D}_{0,u}^+(\Omega,\mathcal{H}_4) = 3.$$

(3.63.5) Computation of $\mathfrak{D}_u^+(\Omega,\mathcal{H}_5)$ and $\mathfrak{D}_{0,u}^+(\Omega,\mathcal{H}_5)$.

From (3.33.5) we have that:

$$\text{MinDomHyp}_u^+(\Omega,\mathcal{H}_5) = \{\mathcal{H}_5\}$$

and that:

$$\text{MinDomHyp}_{0,u}^+(\Omega,\mathcal{H}_5) = \{\mathcal{H}_5 \cup \{\{4\}\}\} = \{\mathcal{D}(K_\Omega)\}.$$

Hence, $\mathcal{H}_5$ is a domination hypergraph on Ω and so $\mathfrak{D}_u^+(\Omega,\mathcal{H}_5) = 1$, whereas the index $\mathfrak{D}_{0,u}^+(\Omega,\mathcal{H}_5)$ is not defined because $\mathcal{H}_5$ does not have a 0-decomposition. (In fact, $\mathcal{H}_5 = \mathcal{U}_{1,\Omega'}$, where $\Omega' = \{1,2,3\} \subseteq \Omega$. So it follows from Proposition 3.60 that $\mathfrak{D}_{0,u}^+(\Omega,\mathcal{H}_5)$ is not defined.)

Remark 3.64 We observe that from the previous example we conclude that there are hypergraphs where the inequalities:

$$\mathfrak{D}_u^+(\Omega,\mathcal{H}) \leqslant \min\{r,\mathfrak{D}_{0,u}^+(\Omega,\mathcal{H})\} \leqslant \mathfrak{D}_{0,u}^+(\Omega,\mathcal{H}) \leqslant s$$

are equalities, and that there are hypergraphs where these general inequalities are not equalities.

3.4.2 Domination decomposition index and distance to dominatoin hypergraphs

To finish this section, we present a result on the decomposition index $\mathfrak{D}_u^+(\Omega,\mathscr{H})$ associated with the hypergraph $\mathscr{H}$. Specifically, in Theorem 3.65 we show that $\mathfrak{D}_u^+(\Omega,\mathscr{H})$ allows us to compute a tight lower bound for the *distance* between the hypergraph $\mathscr{H}$ and the family $\mathrm{DomHyp}(\Omega)$ of all the domination hypergraphs on the finite set Ω.

Before stating our theorem, we recall some general notions concerning partially ordered sets (for general references see [37, 41]).

The *cover relation* of a partially ordered set $(X,\preceq)$ is the transitive reflexive reduction of the partial order $\preceq$; that is, an element y of the poset $(X,\preceq)$ *covers* another element x provided that $x \preceq y$ and that there exists no third element z in the poset for which $x \preceq z \preceq y$.

The *Hasse diagram* is a graphical rendering of a partially ordered set $(X,\preceq)$ displayed via the cover relation of the partially ordered set with an implied upward orientation. A point is drawn for each element of the poset, and line segments are drawn between these points according to the following rule: if $x \prec y$ in the poset, then the point corresponding to x appears lower in the drawing than the point corresponding to y, and the line segment between the points corresponding to x and y is included in the drawing if and only if the element y of the poset $(X,\preceq)$ covers the element x.

The *digraph of a partially ordered set* $(X,\preceq)$ is the digraph associated with the Hasse diagram of the poset.

The *distance* between two elements x_1,x_2 of a partially ordered set $(X,\preceq)$ is the distance $d(x_1,x_2)$ between x_1 and x_2 in the digraph of the poset. The distance between an element $x \in X$ and a subset $Y \subseteq X$ in the poset $(X,\preceq)$ is defined as $d(x,Y) = \min\{d(x,y) : y \in Y\}$.

Theorem 3.65 *Let Ω be a finite set and let $\mathscr{H}$ be a hypergraph on Ω. Then $\mathfrak{D}_u^+(\Omega,\mathscr{H})-1$ is a lower bound for the distance between the hypergraph $\mathscr{H} \in \mathrm{Hyp}(\Omega)$ and the set of domination hypergraphs $\mathrm{DomHyp}(\Omega) \subseteq \mathrm{Hyp}(\Omega)$ in the poset $(\mathrm{Hyp}(\Omega),\leqslant^+)$.*

Proof. Let d be the distance between the hypergraph $\mathscr{H}$ and the family of domination hypergraphs $\mathrm{DomHyp}(\Omega)$ in the poset $(\mathrm{Hyp}(\Omega),\leqslant^+)$. Observe that, $d = 0$ if and only if $\mathscr{H}$ is a domination hypergraph if and only if $\mathfrak{D}_u^+(\Omega,\mathscr{H}) = 1$. Therefore, we must prove the inequality $\mathfrak{D}_u^+(\Omega,\mathscr{H}) - 1 \leqslant d$ whenever $d \geqslant 1$.

So assume that $d \geqslant 1$.

Then there exist a sequence:

$$\mathscr{H} = \mathscr{H}_0 \leqslant^+ \mathscr{H}_1 \leqslant^+ \cdots \leqslant^+ \mathscr{H}_{d-1} \leqslant^+ \mathscr{H}_d$$

of hypergraphs on Ω satisfying the following four conditions:

(1) the hypergraph $\mathcal{H}_d$ is a domination hypergraph on Ω;

(2) the hypergraphs $\mathcal{H}_i$ are not domination hypergraphs if $i \neq d$;

(3) $\mathcal{H}_i \neq \mathcal{H}_{i+1}$ for all $i = 0, \dots, d-1$; and

(4) if $\mathcal{H}'$ is a hypergraph on Ω such that $\mathcal{H}_i \leqslant^+ \mathcal{H}' \leqslant^+ \mathcal{H}_{i+1}$, then either $\mathcal{H}' = \mathcal{H}_i$ or $\mathcal{H}' = \mathcal{H}_{i+1}$.

For $i = 0, \dots, d-1$ we have that $\mathcal{H}_i \leqslant^+ \mathcal{H}_{i+1}$. On one hand, from condition (3) we have that $\mathcal{H}_i \neq \mathcal{H}_{i+1}$. On the other hand, from condition (2) it follows that $\mathcal{H}_i \neq \mathcal{U}_{1,\Omega'}$ for any subset $\Omega' \subsetneq \Omega$ (because $\mathcal{H}_i$ is not a domination hypergraph while $\mathcal{U}_{1,\Omega'} = \mathcal{D}(K_{\Omega'})$). Therefore, by applying Proposition 3.42 it follows that there exists a domination hypergraph $\mathcal{H}_{0,i}$ on Ω such that $\mathcal{H}_i \leqslant^+ \mathcal{H}_{0,i}$ and $\mathcal{H}_{i+1} \not\leqslant^+ \mathcal{H}_{0,i}$. Let us define $\mathcal{H}_{i,i+1} = \mathcal{H}_{0,i} \overset{+}{\sqcap} \mathcal{H}_{i+1}$; that is, $\mathcal{H}_{i,i+1}$ is the hypergraph on Ω whose elements are the inclusion-minimal elements of the family $\{A_{0,i} \cup A_{i+1} : A_{0,i} \in \mathcal{H}_{0,i} \text{ and } A_{i+1} \in \mathcal{H}_{i+1}\}$. We claim that $\mathcal{H}_i = \mathcal{H}_{i,i+1}$. Let us prove our claim.

On one hand, from the definition of $\mathcal{H}_{i,i+1}$ and by applying Remark 1.19 we get that $\mathcal{H}_{i,i+1} \leqslant^+ \mathcal{H}_{i+1}$. On the other hand, since $\mathcal{H}_i \leqslant^+ \mathcal{H}_{0,i}$ and $\mathcal{H}_i \leqslant^+ \mathcal{H}_{i+1}$, again from the definition of $\mathcal{H}_{i,i+1}$ and by applying Remark 1.19, now we get that $\mathcal{H}_i \leqslant^+ \mathcal{H}_{i,i+1}$. Therefore $\mathcal{H}_i \leqslant^+ \mathcal{H}_{i,i+1} \leqslant^+ \mathcal{H}_{i+1}$ and hence, from the condition (4) we conclude that either $\mathcal{H}_i = \mathcal{H}_{i,i+1}$ or $\mathcal{H}_{i,i+1} = \mathcal{H}_{i+1}$. So the proof of our claim will be completed if we show that $\mathcal{H}_{i,i+1} \neq \mathcal{H}_{i+1}$. Let us prove it. From the definition of $\mathcal{H}_{i,i+1}$ and by applying Reamrk 1.19 we get that $\mathcal{H}_{i,i+1} \leqslant^+ \mathcal{H}_{0,i}$. Therefore, if $\mathcal{H}_{i,i+1} = \mathcal{H}_{i+1}$, then a contradiction is achieved because $\mathcal{H}_{i+1} \not\leqslant^+ \mathcal{H}_{0,i}$. This completes the proof of our claim.

From our claim:

$$\mathcal{H}_i = \min\left\{A_{0,i} \cup A_{i+1} : A_{0,i} \in \mathcal{H}_{0,i}, A_{i+1} \in \mathcal{H}_{i+1}\right\}$$

for all $i = 0, \dots, d-1$. Hence it is straightforward to prove the equality:

$$\mathcal{H}_0 = \min\left\{A_{0,0} \cup A_{0,1} \cup \cdots \cup A_{0,d-1} \cup A_d : A_{0,i} \in \mathcal{H}_{0,i}, A_d \in \mathcal{H}_d\right\};$$

that is:

$$\mathcal{H}_0 = \mathcal{H}_{0,0} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_{0,d-1} \overset{+}{\sqcap} \mathcal{H}_d.$$

By construction, for all $i = 0, \dots, d-1$ the hypergraphs $\mathcal{H}_{0,i}$ are domination hypergraphs; whereas, by condition (1) we get that the hypergraph $\mathcal{H}_d$ is a domination hypergraph. Moreover, recall that $\mathcal{H} = \mathcal{H}_0$. Hence, we conclude that $\{\mathcal{H}_{0,0}, \dots, \mathcal{H}_{0,d-1}, \mathcal{H}_d\}$ is a $(d+1)$-decomposition of the hypergraph $\mathcal{H}$, and hence $\mathfrak{D}_u^+(\Omega, \mathcal{H}) \leq d+1$. This completes the proof of the theorem. □

From Theorem 3.65 we get that:

$$\mathfrak{D}_u^+(\Omega, \mathcal{H}) - 1 \leq d\left(\mathcal{H}, \mathrm{DomHyp}(\Omega)\right)$$

for all hypergraphs $\mathscr{H}$ on the finite set Ω. To conclude, let us remark that there are examples of hypergraphs $\mathscr{H}$ for which this inequality is an equality (Example 3.66), and examples where the inequality is strict (Example 3.67).

Example 3.66 Let $\Omega = \{1,2,3,4\}$. Let us consider the hypergraph $\mathscr{H}$ on Ω defined as $\mathscr{H} = \{\{1,2\},\{2,3\},\{3,4\}\}$. The decomposition index of this hypergraph is $\mathfrak{D}_u^+(\Omega,\mathscr{H}) = 2$ (see Example 3.63). We claim that the distance d between the hypergraph $\mathscr{H}$ and the family of domination hypergraphs $\mathrm{DomHyp}(\Omega)$ in the poset $(\mathrm{Hyp}(\Omega),\leqslant^+)$ is equal to one. Indeed, on one hand we have that $\mathscr{H} \leqslant^+ \mathscr{H}'$ where $\mathscr{H}'$ is the domination hypergraph $\mathscr{H}' = \{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$ (moreover, from Example 3.33 we get that $\mathscr{H}'$ is a minimal domination completion of $\mathscr{H}$). On the other hand it is not hard to check that, if $\mathscr{H}''$ is a hypergraph on Ω such that $\mathscr{H} \leqslant^+ \mathscr{H}'' \leqslant^+ \mathscr{H}'$, then either $\mathscr{H} = \mathscr{H}''$ or $\mathscr{H}'' = \mathscr{H}'$. Therefore we conclude that $d = 1 = \mathfrak{D}_u^+(\Omega,\mathscr{H}) - 1$, as claimed.

Example 3.67 Finally, on the finite set $\Omega = \{1,2,3,4\}$ now we consider the hypergraph $\mathscr{H} = \{\{1,2\},\{1,3,4\},\{2,3,4\}\}$. This hypergraph also has decomposition index $\mathfrak{D}_u^+(\Omega,\mathscr{H}) = 2$ (see Example 3.63). However, next we are going to prove that $d(\mathscr{H},\mathrm{DomHyp}(\Omega)) \geqslant 2$. To this end it is enough to consider the set:

$$\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}) = \{\mathscr{H}_1,\dots,\mathscr{H}_r\}$$

and to prove that $d(\mathscr{H},\mathscr{H}_i) \geqslant 2$ for all i. Let us do it. From Example 3.33, we get that $\mathrm{MinDomHyp}_u^+(\mathscr{H}) = \{\mathscr{H}_1,\dots,\mathscr{H}_6\}$ where:

$$\begin{aligned}
\mathscr{H}_1 &= \{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\},\\
\mathscr{H}_2 &= \{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\},\\
\mathscr{H}_3 &= \{\{1\},\{2,3,4\}\},\\
\mathscr{H}_4 &= \{\{2\},\{1,3,4\}\},\\
\mathscr{H}_5 &= \{\{3\},\{1,2\}\},\\
\mathscr{H}_6 &= \{\{4\},\{1,2\}\}.
\end{aligned}$$

Now observe that:

$$\begin{aligned}
&\mathscr{H} \leqslant^+ \{\{1,2\},\{1,3\},\{2,3,4\}\} \leqslant^+ \mathscr{H}_1,\\
&\mathscr{H} \leqslant^+ \{\{1,2\},\{2,3\},\{1,3,4\}\} \leqslant^+ \mathscr{H}_2,\\
&\mathscr{H} \leqslant^+ \{\{1,2\},\{1,3\},\{2,3,4\}\} \leqslant^+ \mathscr{H}_3,\\
&\mathscr{H} \leqslant^+ \{\{1,2\},\{2,3\},\{1,3,4\}\} \leqslant^+ \mathscr{H}_4,\\
&\mathscr{H} \leqslant^+ \{\{1,2\},\{1,3\},\{2,3\}\} \leqslant^+ \mathscr{H}_5,\\
&\mathscr{H} \leqslant^+ \{\{1,2\},\{1,4\},\{2,4\}\} \leqslant^+ \mathscr{H}_6.
\end{aligned}$$

Therefore, for all i there exists a hypergraph $\mathscr{H}_i'$ on Ω such that:

$$\mathscr{H} \leqslant^+ \mathscr{H}_i' \leqslant^+ \mathscr{H}_i \quad \text{and} \quad \mathscr{H} \neq \mathscr{H}_i' \neq \mathscr{H}_i.$$

Hence $d(\mathscr{H},\mathscr{H}_i) \geqslant 2$, and so $d(\mathscr{H},\mathrm{DomHyp}(\Omega)) \geqslant 2 > 1 = \mathfrak{D}_u^+(\Omega,\mathscr{H}) - 1$.

CHAPTER 4

COMPUTATION OF MINIMAL DOMINATION COMPLETIONS AND DECOMPOSITION INDICES

In this chapter first we introduce some techniques to compute the minimal domination completions and the decomposition indices of a hypergraph. After that we calculate the minimal domination completions and the decomposition indices of hypergraphs up to order 4 that are not domination hypergraphs. Finally, we also study the minimal domination completions and decomposition indices of some uniform hypergraphs.

4.1 Reduction techniques

In this section we study some reduction techniques for computing the minimal domination completions and the decomposition index of a hypergraph $\mathscr{H}$. We will focus only on $\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H})$ and on $\mathfrak{D}_u^+(\Omega,\mathscr{H})$.

4.1.1 Reduction to hypergraphs with emtpy intersection set

Let $\mathscr{H}$ be a hypergraph on a finite set Ω. Recall that we denote by $\mathrm{Int}(\mathscr{H})$ the intersection of the elements of $\mathscr{H}$; that is, $\mathrm{Int}(\mathscr{H}) = \bigcap_{A\in\mathscr{H}} A$.

The goal of this subsection is to prove that the computation of $\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H})$ and can be reduced to hypergraphs $\mathscr{H}$ with $\mathrm{Int}(\mathscr{H}) = \emptyset$. Namely we are going to prove that, if we are able to calculate the sets $\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}')$ for hypergraphs with intersection $\mathrm{Int}(\mathscr{H}') = \emptyset$, then it is possible to compute the minimal domination completions $\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H})$ of any hypergraph $\mathscr{H}$ (see Proposition 4.2). The same happens with the decomposition index $\mathfrak{D}_u^+(\Omega,\mathscr{H})$ (see Proposition 4.3).

Recall that, by Proposition 2.9, we know that if there exists a graph G such that $\mathscr{H} = \mathscr{D}(G)$, then $\mathrm{Int}(\mathscr{H}) = V_0(G)$, the set of the isolated vertices of G. So, to achieve our purpose, the idea is to add isolated vertices to certain accurate graphs. In order to do this, we will use the following graph constructions.

Let Ω be a finite set and let $x \in \Omega$. Let G be a graph with vertex set $V(G) \subseteq \Omega$. Then we define the graphs $G_{(x)}$ and $G(x)$ as follows:

- $G_{(x)} = G$, if $x \notin V(G)$; while if $x \in V(G)$ and if W_x is the set of vertices $W_x = \{y \in V(G) : y \neq x \text{ and } N[y] \subseteq N[x]\}$, then:
  - the vertex set of the graph $G_{(x)}$ is $V(G_{(x)}) = V(G) \setminus W_x$; and
  - the edges of the graph $G_{(x)}$ are the edges of the subgraph of G induced by $V(G) \setminus (W_x \cup \{x\})$.

  (See Figure 4.1.)
- $G(x) = K_{\{x\}} \cup G$, where $K_{\{x\}}$ is the trivial graph with vertex set $\{x\}$; that is:
  - if $x \in V(G)$, then $G(x) = G$; whereas
  - if $x \notin V(G)$, then $G(x)$ is the graph with vertex set $V(G(x)) = V(G) \cup \{x\}$ and edge set $E(G(x)) = E(G)$.

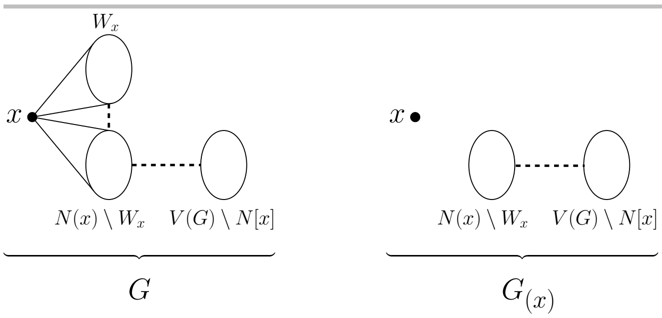


Figure 4.1 The graph $G_{(x)}$.

Lemma 4.1 *Let Ω be a finite set and let $\mathscr{H} \in \mathrm{Hyp}(\Omega)$ be a hypergraph on Ω. Let G be a graph such that $V(G) \subseteq \Omega$ and let us assume that $\mathscr{H} \leqslant^+ \mathscr{D}(G)$. Then $\mathscr{H} \leqslant^+ \mathscr{D}(G_{(x)}) \leqslant^+ \mathscr{D}(G)$ for all $x \in \mathrm{Int}(\mathscr{H})$.*

Proof. Observe that both $\mathscr{D}(G)$ and $\mathscr{D}(G_{(x)})$ are hypergraphs *on* Ω, because $V(G) \subseteq \Omega$ and $V(G_{(x)}) \subseteq \Omega$.

First, let us prove that $\mathscr{D}(G_{(x)}) \leqslant^+ \mathscr{D}(G)$. By the definition of $G_{(x)}$, it is clear that if $D \in \mathscr{D}(G_{(x)})$, then D is a dominating set of G. Hence, there exists $D' \in \mathscr{D}(G)$ such that $D' \subseteq D$. So, $\mathscr{D}(G_{(x)}) \leqslant^+ \mathscr{D}(G)$.

Let us show now the inequality $\mathscr{H} \leqslant^+ \mathscr{D}(G_{(x)})$. Let $A \in \mathscr{H}$. Since $\mathscr{H} \leqslant^+ \mathscr{D}(G)$, there exists $D \in \mathscr{D}(G)$ such that $D \subseteq A$. Observe that the set $C = (D \setminus W_x) \cup \{x\}$ is a dominating set of the graph $G_{(x)}$. Therefore, there exists $D' \in \mathscr{D}(G_{(x)})$ such that $D' \subseteq C$. On one hand, we have that $D \setminus W_x \subseteq D \subseteq A$. On the other hand, $x \in A$ because $x \in \mathrm{Int}(\mathscr{H})$. Hence, it follows that $C \subseteq A$ and consequently $D' \subseteq A$. Therefore, $\mathscr{H} \leqslant^+ \mathscr{D}(G_{(x)})$. □

Proposition 4.2 *Let Ω be a finite set and let $\mathscr{H} \in \mathrm{Hyp}(\Omega)$ be a hypergraph on Ω. Let $x \in \mathrm{Int}(\mathscr{H})$, and let $\mathscr{H}^{\langle x \rangle}$ be the hypergraph on $\Omega \setminus \{x\}$ defined as:*

$$\mathscr{H}^{\langle x \rangle} = \{A \setminus \{x\} : A \in \mathscr{H}\}.$$

Let us assume that $\mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H}^{\langle x \rangle}) = \{\mathscr{H}_1', \dots, \mathscr{H}_r'\}$. Then, the minimal domination completions of $\mathscr{H}$ are:

$$\mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H}) = \{\mathscr{H}_1, \dots, \mathscr{H}_r\}, \quad \mathscr{H}_i = \{A \cup \{x\} : A \in \mathscr{H}_i'\}.$$

Moreover, if $\mathscr{H}_i' = \mathscr{D}(G_i)$, then $\mathscr{H}_i = \mathscr{D}(G_i(x))$.

Proof. Observe that $\mathscr{D}(G_i(x)) \in \mathrm{DomHyp}_0(V(G_i(x))) \subseteq \mathrm{DomHyp}(\Omega)$.

First let us show that $\mathscr{H} \leqslant^+ \mathscr{D}(G_i(x))$. Let $A \in \mathscr{H}$. Then $A \setminus \{x\} \in \mathscr{H}^{\langle x \rangle}$. Since $\mathscr{H}^{\langle x \rangle} \leqslant^+ \mathscr{D}(G_i)$, there exists $D_i \in \mathscr{D}(G_i)$ such that $D_i \subseteq A \setminus \{x\}$. Observe that $D_i \cup \{x\}$ is a dominating set of the graph $G_i(x)$. Therefore, there exists $D_i' \in \mathscr{D}(G_i(x))$ such that $D_i' \subseteq D_i \cup \{x\}$. Hence $D_i' \subseteq A$. So we conclude that $\mathscr{H} \leqslant^+ \mathscr{D}(G_i(x))$.

To complete the proof of the proposition, we must demonstrate that if G is a graph such that $\mathscr{H} \leqslant^+ \mathscr{D}(G)$, then there exists $i_0 \in \{1, \dots, r\}$ such that $\mathscr{D}(G_{i_0}(x)) \leqslant^+ \mathscr{D}(G)$.

So, let G be a graph and assume that $\mathscr{H} \leqslant^+ \mathscr{D}(G)$. From Lemma 4.1 we get that $\mathscr{H} \leqslant^+ \mathscr{D}(G_{(x)}) \leqslant^+ \mathscr{D}(G)$. Hence, for each $A \in \mathscr{H}$, there exists $D \in \mathscr{D}(G_{(x)})$ such that $D \subseteq A$. Observe that $\mathscr{D}(G(x)) = \{D' \cup \{x\} : D' \in \mathscr{D}(G)\}$. Therefore, we have that $x \in D$, and thus $D \setminus \{x\} \subseteq A \setminus \{x\}$. Hence, it follows that $\mathscr{H}^{\langle x \rangle} \leqslant^+ \mathscr{D}(G_{(x)} - \{x\})$ and so, since $\mathscr{D}(G_{(x)} - \{x\})$ is a domination completion of $\mathscr{H}^{\langle x \rangle}$, there exists $i_0 \in \{1, \dots, r\}$ such that $\mathscr{D}(G_{i_0}) \leqslant^+ \mathscr{D}(G_{(x)} - \{x\})$. At this point observe that $\mathscr{D}(G_{i_0}(x)) = \{D' \cup \{x\} : D' \in \mathscr{D}(G_{i_0})\}$. We therefore conclude that $\mathscr{D}(G_{i_0}(x)) \leqslant^+ \mathscr{D}(G)$. This completes the proof of the proposition. □

Proposition 4.3 *Let Ω be a finite set and let $\mathscr{H} \in \mathrm{Hyp}(\Omega)$ be a hypergraph on Ω. Let $x \in \mathrm{Int}(\mathscr{H})$, and let $\mathscr{H}^{\langle x \rangle}$ be the hypergraph on $\Omega \setminus \{x\}$ defined as:*

$$\mathscr{H}^{\langle x \rangle} = \{A \setminus \{x\} : A \in \mathscr{H}\}.$$

Then, $\mathfrak{D}_u^+(\Omega, \mathscr{H}) = \mathfrak{D}_u^+(\Omega, \mathscr{H}^{\langle x \rangle})$.

Proof. Let $\mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H}^{\langle x \rangle}) = \{\mathscr{H}_1', \dots, \mathscr{H}_r'\}$ and let $\mathscr{H}_i = \{A \cup \{x\} : A \in \mathscr{H}_i'\}$. From the definition of decomposition it is easy to prove that a family $\{\mathscr{H}_{i_1}', \dots, \mathscr{H}_{i_s}'\}$ is an s-decomposition of the hypergraph $\mathscr{H}^{\langle x \rangle}$ if and only if the family $\{\mathscr{H}_{i_1}, \dots, \mathscr{H}_{i_s}\}$ is an s-decomposition of the hypergraph $\mathscr{H}$. Hence, the equality $\mathfrak{D}_u^+(\Omega, \mathscr{H}) = \mathfrak{D}_u^+(\Omega, \mathscr{H}^{\langle x \rangle})$ follows from Proposition 3.59 and Proposition 4.2. □

Example 4.4 On the finite set $\Omega = \{1,2,3,4\}$, let us consider the hypergraph:

$$\mathscr{H} = \{\{1,2,4\},\{1,3,4\},\{2,3,4\}\}.$$

Observe that $4 \in \mathrm{Int}(\mathscr{H})$. Therefore, we can apply Propositon 4.2 and Proposition 4.3 in order to compute $\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H})$ and $\mathfrak{D}_u^+(\Omega,\mathscr{H})$. Observe that:

$$\mathscr{H}^{\langle 4\rangle} = \{A \setminus \{4\} : A \in \mathscr{H}\} = \{\{1,2\},\{1,3\},\{2,3\}\}$$

is the hypergraph $\mathscr{H}_1$ of Examples 3.33 and 3.63. Hence, from the computations done in Example 3.33, and by applying Proposition 4.2, we get that:

$$\mathrm{MinDomHyp}_u^+(\Omega,\mathscr{H}) = \left\{\mathscr{H}'_{0,1},\mathscr{H}'_{0,2},\mathscr{H}'_{0,3}\right\},$$

where $\mathscr{H}'_{0,i} = \{\{i,4\},\{j,k,4\}\}$ and $\{i,j,k\} = \{1,2,3\}$; whereas from Example 3.63 and Proposition 4.3 it follows that $\mathfrak{D}_u^+(\Omega,\mathscr{H}) = \mathfrak{D}_u^+(\Omega,\mathscr{H}_0) = 2$.

4.1.2 Reduction to hypergraphs with corank at least two

We start with a simple result that will be used in the lemma and propositions that follow.

Lemma 4.5 *Let Ω be a finite set and let $\mathscr{H}_1$ and $\mathscr{H}_2$ hypergraphs on Ω such that $\mathscr{H}_1 \leqslant^+ \mathscr{H}_2$. Assume that there is an element $\omega \in \Omega$ such that $\{\omega\} \in \mathscr{H}_1$. Then either $\mathscr{H}_2 = \{\emptyset\}$ or $\{\omega\} \in \mathscr{H}_2$. In particular, if $\mathscr{H}_2$ is a domination hypergraph on Ω, then $\{\omega\} \in \mathscr{H}_2$.*

Proof. If $\mathscr{H}_1 \leqslant^+ \mathscr{H}_2$ and $\{\omega\} \in \mathscr{H}_1$, then there exists $A \in \mathscr{H}_2$ such that $A \subseteq \{\omega\}$. Then either $A = \emptyset$, and hence $\mathscr{H}_2 = \{\emptyset\}$, or $A = \{\omega\}$. Observe that $\{\emptyset\}$ is not a domination hypergraph (see Remark 3.28 and Table 3.8). □

Lemma 4.6 *Let Ω be a finite set and let $\mathscr{H}$ be a hypergraph on Ω such that $\{\omega\} \in \mathscr{H}$ for some element $\omega \in \Omega$. Then the following statements hold.*

1) *If G is a graph such that $\mathscr{H} \leqslant^+ \mathscr{D}(G)$, then $G = K_{\{\omega\}} \vee G'$, where $G' = G - \omega$. Moreover we have that $\mathscr{H} \setminus \{\{\omega\}\} \leqslant^+ \mathscr{D}(G')$.*
2) *If G' is a graph such that $\omega \notin V(G')$ and such that $\mathscr{H} \setminus \{\{\omega\}\} \leqslant^+ \mathscr{D}(G')$, then $\mathscr{H} \leqslant^+ \mathscr{D}(G)$, where $G = K_{\{\omega\}} \vee G'$.*

Proof. Assume first that $\mathscr{H} \leqslant^+ \mathscr{D}(G)$ for some graph G. As $\{\omega\} \in \mathscr{H}$, by Lemma 4.5, we have that $\{\omega\} \in \mathscr{D}(G)$. This implies that $\deg(\omega) = |V(G)| - 1$ and therefore the graph G is equal to $K_{\{\omega\}} \vee (G - \omega)$. Moreover, by Proposition 2.49, we have that $\mathscr{D}(G - \omega) = \mathscr{D}(G) \setminus \{\{\omega\}\}$. Hence, $\mathscr{H} \setminus \{\{\omega\}\} \leqslant^+ \mathscr{D}(G - \omega)$.

Assume now that $\mathscr{H} \setminus \{\{\omega\}\} \leqslant^+ \mathscr{D}(G')$ for some graph G' such that $\omega \notin V(G')$. Let $G = K_{\{\omega\}} \vee G'$. We have to demonstrate that $\mathscr{H} \leqslant^+ \mathscr{D}(G)$; that is, if $A \in \mathscr{H}$, then there exists

$B \in \mathscr{D}(G)$ such that $B \subseteq A$. Let $A \in \mathscr{H}$ such that $A \neq \{\omega\}$. Then $A \in \mathscr{H} \setminus \{\{\omega\}\}$ and by hypothesis there exists $B \in \mathscr{D}(G')$ such that $B \subseteq A$. By Proposition 2.49 we have that $\mathscr{D}(G) = \mathscr{D}(G') \cup \{\{\omega\}\}$. So in this case we are done. For the set $\{\omega\} \in \mathscr{H}$, we have that $\{\omega\} \in \mathscr{D}(G)$ also. Hence we have shown that $\mathscr{H} \leqslant^{+} \mathscr{D}(G)$. □

Proposition 4.7 *Let Ω be a finite set and let $\mathscr{H}$ be a hypergraph on Ω. Let $\omega \in \Omega$. Assume that $\{\omega\} \in \mathscr{H}$. Then the following statements hold.*

1) If $\mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H} \setminus \{\{\omega\}\}) = \{\mathscr{H}'_1, \ldots, \mathscr{H}'_r\}$*, then:*

$$\mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H}) = \left\{\mathscr{H}'_1 \cup \{\{\omega\}\}, \ldots, \mathscr{H}'_r \cup \{\{\omega\}\}\right\}.$$

Moreover if $\mathscr{H}'_i = \mathscr{D}(G'_i)$, then $\mathscr{H}_i = \mathscr{D}(K_{\{\omega\}} \vee G'_i)$.

2) If $\mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H}) = \{\mathscr{H}_1, \ldots, \mathscr{H}_r\}$*, then:*

$$\mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H} \setminus \{\{\omega\}\}) = \{\mathscr{H}_1 \setminus \{\{\omega\}\}, \ldots, \mathscr{H}_r \setminus \{\{\omega\}\}\}.$$

Proof. It is a direct consequence of Lemma 4.6 and of the definitions. □

Proposition 4.8 *Let Ω be a finite set and let $\mathscr{H}$ be a hypergraph on Ω. Let $\omega \in \Omega$ such that $\{\omega\} \in \mathscr{H}$. Then $\mathfrak{D}_u^+(\Omega, \mathscr{H}) = \mathfrak{D}_u^+(\Omega, \mathscr{H} \setminus \{\{\omega\}\})$.*

Proof. Set $d = \mathfrak{D}_u^+(\Omega, \mathscr{H})$ and $d_\omega = \mathfrak{D}_u^+(\Omega, \mathscr{H} \setminus \{\{\omega\}\})$. Assume that we have a decomposition of the hypergraph $\mathscr{H}$:

$$\mathscr{H} = \mathscr{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_d$$

with $\mathscr{H}_i \in \mathrm{MinDomHyp}_u^+(\Omega, \mathscr{H})$. By Lemma 4.5 we have that $\{\omega\} \in \mathscr{H}_i$ for all $i = 1, \ldots, d$. Hence, by Proposition 4.7, we get a decomposition of the hypergraph $\mathscr{H} \setminus \{\{\omega\}\}$:

$$\mathscr{H} \setminus \{\{\omega\}\} = \mathscr{H}_1 \setminus \{\{\omega\}\} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_d \setminus \{\{\omega\}\}.$$

Thus, by definition of the decomposition index, we get that $d_\omega \leqslant d$. Assume now that we have a decomposition of the hypergraph $\mathscr{H} \setminus \{\{\omega\}\}$:

$$\mathscr{H} \setminus \{\{\omega\}\} = \mathscr{H}'_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}'_{d_\omega}.$$

Then, by adding the set $\{\omega\}$ to each of the hypergraphs appearing in it, we get a decomposition of $\mathscr{H}$; that is:

$$\mathscr{H} = (\mathscr{H}'_1 \cup \{\{\omega\}\}) \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} (\mathscr{H}'_{d_\omega} \cup \{\{\omega\}\}).$$

Thus we have that $d \leqslant d_\omega$. □

Example 4.9 Let $\Omega = \{1,2,3,4\}$ and let us consider the hypergraph:

$$\mathscr{H} = \{\{1,2\},\{1,3\},\{2,3\},\{4\}\}.$$

The minimal domination completions of the hypergraph $\mathscr{H} \setminus \{\{4\}\}$ are:

$$\{\{1\},\{2,3\}\}, \quad \{\{2\},\{1,3\}\}, \quad \{\{3\},\{1,2\}\}$$

and $\mathfrak{D}^+_u(\Omega, \mathscr{H} \setminus \{\{4\}\}) = 2$ (see Examples 3.33 and 3.63). Hence, by Propositions 4.7 and 4.8, the minimal domination completions of the hypergraph $\mathscr{H}$ are:

$$\{\{1\},\{2,3\},\{4\}\}, \quad \{\{2\},\{1,3\},\{4\}\}, \quad \{\{3\},\{1,2\},\{4\}\}$$

and $\mathfrak{D}^+_u(\Omega, \mathscr{H}) = 2$.

4.2 Hypergraphs of small order

In this section we compute the sets $\mathrm{MinDomHyp}^+_u(\Omega, \mathscr{H})$ and $\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{H})$, and the decomposition indices $\mathfrak{D}^+_u(\Omega, \mathscr{H})$ and $\mathfrak{D}^+_{0,u}(\Omega, \mathscr{H})$, where $\mathscr{H}$ is a hypergraph on a finite set Ω of size $|\Omega| = n \leqslant 4$. Complete results are presented for $n \leqslant 3$, while for $n = 4$ only some partial results are obtained.

4.2.1 Hypergraphs of order 2

In Table 4.1 we can find all the hypergraphs up to isomorphism on the finite set $\Omega = \{1,2\}$, whether they have ground set Ω or not. We observe that all hypergraphs on Ω are domination hypergraphs *on* Ω and have exactly one graph realization (see Tables 2.1 and 2.6). The hypergraph $\{\{1\}\}$ does not have ground set Ω, so $\{\{1\}\} \notin \mathrm{DomHyp}_0(\Omega)$. Thus from Lemma 3.51 we get that:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega, \{\{1\}\}) = \{\{1\},\{2\}\}.$$

We observe that $\mathrm{MinDomHyp}^+_u(\Omega, \{\{1\}\}) = \big\{\{\{1\}\}\big\}$, because $\{\{1\}\} \in \mathrm{DomHyp}(\Omega)$. In this case, we find that $\mathfrak{D}^+_u(\Omega, \{\{1\}\}) = 1$, while $\mathfrak{D}^+_{0,u}(\Omega, \{\{1\}\})$ is not defined (see Proposition 3.61).

4.2.2 Hypergraphs of order 3

In Table 4.2 we can find all the hypergraphs up to isomorphism on the finite set $\Omega = \{1,2,3\}$, with ground set contained in Ω. We observe that all these hypergraphs on Ω are domination hypergraphs *on* Ω, except for $\{\{1,2\},\{1,3\},\{2,3\}\}$, and have exactly one graph realization. This last observation follows from the description of the dominating hypergraphs $\mathscr{D}(G)$ of the non-isomorphic graphs G with $n \leqslant 3$ vertices given in Tables 2.2, 2.1, and 2.2 and by bearing in mind that $\{\{1\}\}$ is the domination hypergraph of the graph G with vertex set $V(G) = \{\{1\}\}$ and edge set $E(G) = \emptyset$. However, there are

$\mathcal{H}$	$\mathrm{DomHyp}_0(\Omega)$	$\mathrm{DomHyp}(\Omega)$	# g. r.
$\{\{1\}\}$	no	yes	1
$\{\{1,2\}\}$	yes	yes	1
$\{\{1\},\{2\}\}$	yes	yes	1

Table 4.1 Non-isomorphic hypergraphs on the set $\Omega = \{1,2\}$, whether they are domination hypergraph and their number of graph realizations (fourth column). See Tables 2.1 and 2.6 for the graph realizations.

three domination hypergraphs on Ω that are not domination hypergraphs with ground set Ω and one hypergraph with ground set Ω that is not a domination hypergraph. For these hypergraphs we calculate their minimal domination completions with ground set or not and the corresponding decomposition index. For the hypergraphs $\mathcal{U}_{1,\{1\}} = \{\{1\}\}$ and $\mathcal{U}_{1,\{1,2\}} = \{\{1\},\{2\}\}$ we can apply Lemma 3.51 and we get that:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathcal{U}_{1,\{1\}}) = \{\{\{1\},\{2,3\}\}\}$$
$$\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathcal{U}_{1,\{1,2\}}) = \{\{\{1\},\{2\},\{3\}\}\}$$

We find these results summarized in Tables 4.2 and 4.3.

$\mathcal{H}$	$\mathrm{DomHyp}_0(\Omega)$	$\mathrm{DomHyp}(\Omega)$	# g. r.
$\{\{1\}\}$	no	yes	1
$\{\{1,2\}\}$	no	yes	1
$\{\{1\},\{2\}\}$	no	yes	1
$\{\{1,2,3\}\}$	yes	yes	1
$\{\{1,2\},\{1,3\},\{2,3\}\}$	no	no	—
$\{\{1,2\},\{1,3\}\}$	yes	yes	1
$\{\{1\},\{2,3\}\}$	yes	yes	1
$\{\{1,2,3\}\}$	yes	yes	1

Table 4.2 Non-isomorphic hypergraphs on the set $\Omega = \{1,2,3\}$, whether they are domination hypergraph and their number of graph realizations (fourth column). See Tables 2.1, 2.2, 2.6, and 2.7 for the graph realizations.

$\mathscr{H}$	M_0	M	$\mathfrak{D}^+_{0,u}$	$\mathfrak{D}^+_u$
$\{\{1\}\}$	$\{\{1\},\{2,3\}\}$	$\{\{1\}\}$	—	1
$\{\{1,2\}\}$	$\{\{1,2\},\{1,3\}\}$, $\{\{1,2\},\{2,3\}\}$	$\{\{1,2\}\}$	2	1
$\{\{1\},\{2\}\}$	$\{\{1\},\{2\},\{3\}\}$	$\{\{1\},\{2\}\}$	—	1
$\{\{1,2\},\{1,3\},\{2,3\}\}$	$\{\{1\},\{2,3\}\}$ $\{\{2\},\{1,3\}\}$ $\{\{3\},\{1,2\}\}$	$\{\{1\},\{2,3\}\}$ $\{\{2\},\{1,3\}\}$ $\{\{3\},\{1,2\}\}$	2	2

Table 4.3 Minimal domination completions of hypergraphs on the set $\Omega = \{1,2,3\}$ that are not domination hypergraphs and their decomposition indices. M_0 and M stand for $\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H})$ and $\mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H})$, respectively.

4.2.3 Hypergraphs of order 4

It is easy to check that there are 28 non-isomorphic hypergraphs on the set $\Omega = \{1,2,3,4\}$ of size four. Namely, we have:

- There are 8 non-isomorphic hypergraphs *on* Ω but not with ground set Ω; that is, in $\mathrm{Hyp}_0(\Omega')$ where $\Omega' \subsetneq \Omega$. See Table 4.4 and Table 4.5.
- There are 20 non-isomorphic hypergraphs with ground set Ω; that is, in $\mathrm{Hyp}_0(\Omega)$. Ten of these hypergraphs are domination hypergraphs with ground set Ω and the other ten hypergraphs are not domination hypergraphs. In Subsection 2.4.4 and Table 2.8 we study the ten hypergraphs that are domination hypergraphs and their graph realizations. For the ten hypergraphs with ground set Ω that are not domination hypergraphs, we list their domination completions in Table 4.6 (see Remark 4.11).

From the above we get that there are 11 non-isomorphic hypergraphs on $\Omega = \{1,2,3,4\}$ that are not domination hypergraphs. As far as we know, there is no general result that allows us to compute the minimal domination completions of all these 11 hypergraphs. In Example 4.10 we compute the minimal domination completions of the hypergraph $\mathscr{H} = \{\{1,3\},\{2,4\}\}$ on the set $\Omega = \{1,2,3,4\}$ by doing an exhaustive description of the poset $(\mathrm{DomHyp}^+_u(\Omega,\mathscr{H}),\leqslant^-)$.

Example 4.10 Let us consider the hypergraph $\mathscr{H} = \{\{1,3\},\{2,4\}\}$ on the set $\Omega = \{1,2,3,4\}$. This hypergraph is not a domination hypergraph (see Example 2.37). Here we are going to show that $\mathscr{H}$ has six minimal domination completions on Ω and that $\mathfrak{D}^+_u(\Omega,\mathscr{H}) = 2$. To this end, first let us compute the set $\mathrm{DomHyp}^+_u(\Omega,\mathscr{H})$ of the upper domination completions of $\mathscr{H}$; that is, the set whose elements are the domination hypergraphs $\mathscr{H}'$ with

$\mathcal{H}$	$\mathrm{DomHyp}_0(\Omega)$	$\mathrm{DomHyp}(\Omega)$	# g. r.
$\{\{1\}\}$	no	yes	1
$\{\{1,2\}\}$	no	yes	1
$\{\{1\},\{2\}\}$	no	yes	1
$\{\{1,2,3\}\}$	no	yes	1
$\{\{1,2\},\{1,3\},\{2,3\}\}$	no	no	—
$\{\{1,2\},\{1,3\}\}$	no	yes	1
$\{\{1\},\{2,3\}\}$	no	yes	1
$\{\{1\},\{2\},\{3\}\}$	no	yes	1

Table 4.4 Hypergraphs $\mathcal{H}$ on the set $\Omega = \{1,2,3,4\}$ such that $\mathrm{Gr}(\mathcal{H}) \subsetneq \Omega$, whether they are domination hypergraph and their number of graph realizations (fourth column). See Tables 2.1, 2.2, 2.3, 2.6, 2.7 and 2.8.

$\mathcal{H} \leqslant^+ \mathcal{H}'$. So we are looking for graphs G' with vertex sets $V(G') \subseteq \Omega$ and such that $\mathcal{H} \leqslant^+ \mathcal{D}(G')$; that is, such that there exist $D_1, D_2 \in \mathcal{D}(G')$ with $D_1 \subseteq \{1,3\}$ and $D_2 \subseteq \{2,4\}$. By taking into account the Tables 2.1, 2.2 and 2.3, it is not hard to check that there are twenty-six hypergraphs $\mathcal{H}'_1, \ldots, \mathcal{H}'_{26}$ satisfying these conditions. Namely:

- four hypergraphs $\mathcal{H}'$ of the form $\mathcal{H}' = \mathcal{D}(G')$, with G' a graph of type $G_{2,2}$;
- four hypergraphs $\mathcal{H}'$ of the form $\mathcal{H}' = \mathcal{D}(G')$, with G' of type $G_{3,3}$;
- four hypergraphs $\mathcal{H}'$ of the form $\mathcal{H}' = \mathcal{D}(G')$, with G' of type $G_{3,4}$;
- two hypergraphs $\mathcal{H}'$ of the form $\mathcal{H}' = \mathcal{D}(G')$, with G' of type $G_{4,4}$;
- one hypergraph $\mathcal{H}'$ of the form $\mathcal{H}' = \mathcal{D}(G')$, with G' of type $G_{4,8}$;
- eight hypergraphs $\mathcal{H}'$ of the form $\mathcal{H}' = \mathcal{D}(G')$, with G' of type $G_{4,9}$;
- two hypergraphs $\mathcal{H}'$ of the form $\mathcal{H}' = \mathcal{D}(G')$, with G' of type $G_{4,10}$; and
- one hypergraph $\mathcal{H}'$ of the form $\mathcal{H}' = \mathcal{D}(G')$, with G' of type $G_{4,11}$.

At this point, the minimal upper domination completions of $\mathcal{H}$ are obtained as the minimal elements of the poset $(\{\mathcal{H}'_1, \ldots, \mathcal{H}'_{26}\}, \leqslant^+)$. In this case, it is a straightforward computation to show that the minimal elements are:

- the four hypergraphs $\mathcal{H}'$ of the type $\mathcal{H}' = \mathcal{D}(G')$, with G' a graph of type $G_{3,3}$, and

$\mathscr{H}$	M_0	M	$\mathfrak{D}^+_{0,u}$	$\mathfrak{D}^+_u$
$\{\{1\}\}$	$\{\{1\},\{2,3,4\}\}$	$\{\{1\}\}$	—	1
$\{\{1,2\}\}$	$\{\{1,2\},\{1,3,4\}\}$, $\{\{1,2\},\{2,3,4\}\}$	$\{\{1,2\}\}$	2	1
$\{\{1\},\{2\}\}$	$\{\{1\},\{2\},\{3,4\}\}$	$\{\{1\},\{2\}\}$	—	1
$\{\{1,2,3\}\}$	$\{\{1,2,3\},\{1,3,4\}\}$, $\{\{1,2,3\},\{2,3,4\}\}$, $\{\{1,2,3\},\{1,2,4\}\}$	$\{\{1,2,3\}\}$	2	1
$\{\{1,2\},\{1,3\}\}$	$\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$, $\{\{1,2\},\{1,3\},\{1,4\}\}$	$\{\{1,2\},\{1,3\}\}$	2	1
$\{\{1,2\},\{1,3\},\{2,3\}\}$	Example 3.33 and Example 3.63	$\{\{1\},\{2,3\}\}$ $\{\{2\},\{1,3\}\}$ $\{\{3\},\{1,2\}\}$	3	2
$\{\{1\},\{2,3\}\}$	$\{\{1\},\{2,3\},\{2,4\}\}$, $\{\{1\},\{2,3\},\{3,4\}\}$	$\{\{1\},\{2,3\}\}$	2	1
$\{\{1\},\{2\},\{3\}\}$	$\{\{1\},\{2\},\{3\},\{4\}\}$	$\{\{1,2,3\}\}\{\{1\},\{2\},\{3\}\}$	—	1

Table 4.5 Minimal domination completions of hypergraphs on the set $\Omega = \{1,2,3,4\}$ with ground set strictly contained in Ω that are not domination hypergraphs and their decomposition indices. M_0 and M stand for $\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H})$ and $\mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H})$, respectively.

- two hypergraphs $\mathscr{H}'$ of the form $\mathscr{H}' = \mathscr{D}(G')$, with G' a graph of type $G_{4,4}$.

That is:

$$\mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H}) = \left\{\mathscr{H}^1_{3,3}, \mathscr{H}^2_{3,3}, \mathscr{H}^3_{3,3}, \mathscr{H}^4_{3,3}, \mathscr{H}^1_{4,4}, \mathscr{H}^2_{4,4}\right\},$$

where:

$$\begin{aligned}
\mathscr{H}^1_{3,3} &= \{\{1\},\{2,4\}\},\\
\mathscr{H}^2_{3,3} &= \{\{2\},\{1,3\}\},\\
\mathscr{H}^3_{3,3} &= \{\{3\},\{2,4\}\},\\
\mathscr{H}^4_{3,3} &= \{\{4\},\{1,3\}\},\\
\mathscr{H}^1_{4,4} &= \{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}, \quad \text{and}\\
\mathscr{H}^2_{4,4} &= \{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}.
\end{aligned}$$

Now, by applying Proposition 3.61 it follows that $2 \leqslant \mathfrak{D}_u^+(\Omega,\mathscr{H}) \leqslant 6$. However, in this case we have that $\mathfrak{D}_u^+(\Omega,\mathscr{H}) = 2$ because $\left\{\mathscr{H}^1_{3,3},\mathscr{H}^3_{3,3}\right\}$ is a 2-decomposition of $\mathscr{H}$. Observe that $\left\{\mathscr{H}^i_{j,k},\mathscr{H}^\ell_{m,n}\right\}$ is a 2-decomposition of $\mathscr{H}$ if and only if:

$$\{(i,j,k),(\ell,m,n)\} = \{(1,3,3),(3,3,3)\},$$

or:

$$\{(i,j,k),(\ell,m,n)\} = \{(1,4,4),(2,4,4)\}.$$

Remark 4.11 (SAGE calculations) Let $\Omega = \{1,2,3,4\}$. Then it is easy to check that there are only 10 non-isomorphic hypergraphs with ground set Ω that are not domination hypergraphs (see Table 2.3 for the list of domination hypergraphs of order 4). Namely, every element of $\mathrm{Hyp}_0(\Omega) \setminus \mathrm{DomHyp}_0(\Omega)$ is isomorphic, as hypergraph, to one hypergraph of the Table 4.6. Under each hypergraph $\mathscr{H}$ of the list, we write the sets $M_0 = \mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{H})$ and $M = \mathrm{MinDomHyp}^+_u(\Omega,\mathscr{H})$ as calculated with the SAGE program of Appendix A.

4.3 Domination completion and decomposition of uniform hypergraphs

Let Ω be a finite set of size $|\Omega| = n$ and let $1 \leqslant r \leqslant n$. Recall that a hypergraph $\mathscr{H}$ on Ω is called r-uniform if $|A| = r$ for all $A \in \mathscr{H}$. Recall also that we denote by $\mathscr{U}_{r,\Omega}$ the r-uniform hypergraph on Ω whose elements are all the subsets of Ω of size r; that is, $\mathscr{U}_{r,\Omega} = \{A \subseteq \Omega : |A| = r\}$.

In this section we will focus on the uniform hypergraphs of the form $\mathscr{U}_{r,\Omega}$. We have seen in Proposition 2.52 that the uniform hypergraphs $\mathscr{U}_{r,\Omega}$ are domination hypergraphs only for $r = 1$, $r = |\Omega| - 1$ and for $r = 2$ and $|\Omega|$ is even. In this section we study the upper minimal domination completions of $\mathscr{U}_{r,\Omega}$ with ground set Ω when this hypergraph is not a domination hypergraph. Specifically, we analyze the minimal domination completions for $\mathscr{U}_{r,\Omega}$ where $r = 2$ and $|\Omega|$ is odd (Subsection 4.3.1), when $r = |\Omega| - 1$ (Subsection 4.3.2), and when r is arbitrary and $|\Omega| \leqslant 5$ (Subsection 4.3.3).

Throughout this section we will only consider hypergraphs with ground set Ω; that is, elements of $\mathrm{Hyp}_0(\Omega)$. Moreover, we will use the partial order $\leqslant^+$ and look for upper domination completions. Therefore, we are going to describe the set of minimal completions $\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{U}_{r,\Omega})$ and to compute the decomposition index $\mathfrak{D}^+_{0,u}(\Omega,\mathscr{U}_{r,\Omega})$.

4.3.1 Minimal domination completions of $\mathscr{U}_{2,\Omega}$

From Proposition 2.52, Lemma 3.51 and Theorem 3.52 we get that:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathscr{U}_{2,\Omega}) = \{\mathscr{U}_{2,\Omega}\} \iff |\Omega| \text{ is even.}$$

Type		Hypergraph $\mathscr{H}$ and its minimal completions M_0 and M
1 ($\mathscr{H}_2$)	$\mathscr{H}$	$\mathscr{U}_{3,\Omega} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$
	$M = M_0$	$\{\{1\},\{2,3,4\}\}$, $\{\{2\},\{1,3,4\}\}$, $\{\{3\},\{1,2,4\}\}$, $\{\{4\},\{1,2,3\}\}$, $\{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$, $\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$, $\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$
2 ($\mathscr{H}_3$)	$\mathscr{H}$	$\{\{1,2,3\},\{1,2,4\},\{1,3,4\}\}$
	$M = M_0$	$\{\{1,4\},\{1,2,3\}\}$, $\{\{1,3\},\{1,2,4\}\}$, $\{\{1,2\},\{1,3,4\}\}$
3 ($\mathscr{H}_5$)	$\mathscr{H}$	$\{\{1,2,3\},\{1,2,4\},\{3,4\}\}$
	M_0	$\{\{3\},\{1,2,4\}\}$, $\{\{4\},\{1,2,3\}\}$, $\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$, $\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$
	M	$\{\{\{1\},\{3,4\}\}$, $\{\{2\},\{3,4\}\}$, $\{\{3\},\{1,2,4\}\}$, $\{\{4\},\{1,2,3\}\}$, $\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$, $\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$
4 ($\mathscr{H}_7$)	$\mathscr{H}$	$\{\{1,2\},\{1,3\},\{2,3,4\}\}$
	M_0	$\{\{1\},\{2,3,4\}\}$, $\{\{2\},\{1,3\},\{1,4\}\}$, $\{\{3\},\{1,2\},\{1,4\}\}$, $\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$
	M	$\{\{1\},\{2,3,4\}\}$, $\{\{2\},\{1,3\}\}$, $\{\{3\},\{1,2\}\}$, $\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$
5 ($\mathscr{H}_6$)	$\mathscr{H}$	$\{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$
	$M = M_0$	$\{\{1\},\{2,3,4\}\}$, $\{\{2\},\{1,3\},\{1,4\}\}$, $\{\{3\},\{1,2\},\{1,4\}\}$, $\{\{4\},\{1,2\},\{1,3\}\}$, $\mathscr{U}_{2,\Omega}$
6 ($\mathscr{H}_{11}$)	$\mathscr{H}$	$\{\{1,2\},\{1,3\},\{1,4\},\{2,4\},\{3,4\}\}$
	$M = M_0$	$\{\{1\},\{2,4\},\{3,4\}\}$, $\{\{4\},\{1,2\},\{1,3\}\}$, $\mathscr{U}_{2,\Omega}$
7 ($\mathscr{H}_{12}$)	$\mathscr{H}$	$\{\{1,2\},\{1,3\},\{1,4\},\{2,3\}\}$
	M_0	$\{\{1\},\{2,3\},\{3,4\}\}$, $\{\{1\},\{2,3\},\{2,4\}\}$, $\{\{2\},\{1,3\},\{1,4\}\}$, $\{\{3\},\{1,2\},\{1,4\}\}$, $\mathscr{U}_{2,\Omega}$
	M	$\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\},\{1,4\}\}$, $\{\{3\},\{1,2\},\{1,4\}\}$, $\mathscr{U}_{2,\Omega}$
8 ($\mathscr{H}_{15}$)	$\mathscr{H}$	$\{\{1,2\},\{1,3\},\{2,4\}\}$
	M_0	$\{\{1\},\{2,3\},\{2,4\}\}$, $\{\{2\},\{1,3\},\{1,4\}\}$, $\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$
	M	$\{\{1\},\{2,4\}\}$, $\{\{2\},\{1,3\}\}$, $\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$
9 ($\mathscr{H}_{16}$)	$\mathscr{H}$	$\{\{1,2\},\{3,4\}\}$
	M_0	$\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$, $\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$
	M	$\{\{1\},\{3,4\}\}$, $\{\{2\},\{3,4\}\}$, $\{\{3\},\{1,2\}\}$, $\{\{4\},\{1,2\}\}$, $\{\{1,2\},\{1,3\},\{2,4\},\{3,4\}\}$, $\{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$
10 ($\mathscr{H}_{17}$)	$\mathscr{H}$	$\{\{1,2\},\{1,3\},\{2,3\},\{4\}\}$
	$M = M_0$	$\{\{1\},\{4\},\{2,3\}\}$, $\{\{2\},\{4\},\{1,3\}\}$, $\{\{3\},\{4\},\{1,2\}\}$

Table 4.6 Minimal domination completion of hypergraphs of order 4 with ground set $\Omega = \{1,2,3,4\}$. Enclosed in parenthesis in the first column there is the corresponding hypergraph of Table 2.8.

In this section we compute the set $\mathrm{MinDomHyp}^{+}_{0,u}(\Omega, \mathcal{U}_{2,\Omega})$ of the minimal upper domination completions of the uniform hypergraph $\mathcal{U}_{2,\Omega}$ with ground set Ω, whenever the finite set Ω has odd size.

Lemma 4.12 *Let Ω' be a non-empty subset of a finite set Ω. Let $\mathcal{H}$ be a hypergraph on Ω and let:*

$$\mathcal{H}[\Omega'] = \{A \in \mathcal{H} : A \subseteq \Omega'\}.$$

Assume that $\mathcal{H}[\Omega'] \neq \emptyset$ and let G' be a graph with vertex set $V(G') \subseteq \Omega'$. Then $\mathcal{H}[\Omega'] \leqslant^{+} \mathcal{D}(G')$ if and only if $\mathcal{H} \leqslant^{+} \mathcal{D}(G' \vee K_{\Omega\setminus\Omega'})$.

Proof. First assuming that $\mathcal{H}[\Omega'] \leqslant^{+} \mathcal{D}(G')$ we are going to prove the inequality $\mathcal{H} \leqslant^{+} \mathcal{D}(G' \vee K_{\{\Omega\setminus\Omega'\}})$. Recall that by Lemma 2.22 we know that:

$$\mathcal{D}(G' \vee K_{\Omega\setminus\Omega'}) = \mathcal{D}(G') \cup \{\{w\} : w \in \Omega \setminus \Omega'\} = \mathcal{D}(G') \cup \mathcal{U}_{1,\Omega\setminus\Omega'}.$$

Therefore, by applying Lemma 1.15, we must demonstrate that if $A \in \mathcal{H}$, then there exists $D \in \mathcal{D}(G') \cup \mathcal{U}_{1,\Omega\setminus\Omega'}$ such that $D \subseteq A$. Let $A \in \mathcal{H}$. If $A \not\subseteq \Omega'$, then there is $\omega \in A \cap (\Omega \setminus \Omega')$ and so we can set $D = \{\omega\}$. Now assume that $A \subseteq \Omega'$. Then $A \in \mathcal{H}[\Omega'] \leqslant^{+} \mathcal{D}(G')$, and hence there exists $D' \in \mathcal{D}(G')$ such that $D' \subseteq A$. Thus, in such a case, we can consider $D = D'$.

Now suppose that $\mathcal{H} \leqslant^{+} \mathcal{D}(G' \vee K_{\Omega\setminus\Omega'})$. We want to prove that $\mathcal{H}[\Omega'] \leqslant^{+} \mathcal{D}(G')$; that is, we must prove that if $A' \in \mathcal{H}[\Omega']$, then there exists $D' \in \mathcal{D}(G')$ such that $D' \subseteq A'$. Let $A \in \mathcal{H}[\Omega']$. Since $A \in \mathcal{H}[\Omega'] \subseteq \mathcal{H}$ and $\mathcal{H} \leqslant^{+} \mathcal{D}(G' \vee K_{\Omega\setminus\Omega'})$, there exists $D \in \mathcal{D}(G' \vee K_{\Omega\setminus\Omega'})$ such that $D \subseteq A$. But $A \subseteq \Omega'$ and, by Lemma 2.22, we get that $\mathcal{D}(G' \vee K_{\Omega\setminus\Omega'}) = \mathcal{D}(G') \cup \mathcal{U}_{1,\Omega\setminus\Omega'}$. Therefore $D \in \mathcal{D}(G')$. Now the proof is completed by setting $D' = D$. □

Theorem 4.13 *Let Ω be a finite set of size $|\Omega| = n$. Assume that n is odd. Then the following statements hold.*

1) *For all $w \in \Omega$, the hypergraph:*

$$\mathcal{H}_{\omega} = \{\{\omega\}\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega\}}$$

is a domination hypergraph with ground set Ω. Moreover, if G is a graph with vertex set Ω, then $\mathcal{D}(G) = \mathcal{H}_{\omega}$ if and only if $G = K_{\{\omega\}} \vee G'$, where G' is a graph with vertex set $\Omega \setminus \{w\}$ and $\mathcal{D}(G') = \mathcal{U}_{2,\Omega\setminus\{\omega\}}$.

2) *The uniform hypergraph $\mathcal{U}_{2,\Omega}$ has n minimal upper domination completions with ground set Ω. Namely, if $\Omega = \{\omega_1, \ldots, \omega_n\}$, then:*

$$\mathrm{MinDomHyp}^{+}_{0,u}(\Omega, \mathcal{U}_{2,\Omega}) = \{\mathcal{H}_{\omega_1}, \ldots, \mathcal{H}_{\omega_n}\}.$$

3) *If $w_{i_1}, w_{i_2} \in \Omega$, $w_{i_1} \neq w_{i_2}$, then $\{\mathcal{H}_{\omega_{i_1}}, \mathcal{H}_{\omega_{i_2}}\}$ is a 2-0-decomposition of $\mathcal{U}_{2,\Omega}$. In particular, $\mathfrak{D}^{+}_{0,u}(\Omega, \mathcal{U}_{2,\Omega}) = 2$.*

Proof. Let $w \in \Omega$. Since $\Omega \setminus \{\omega\}$ has even size, the hypergraph $\mathcal{U}_{2,\Omega\setminus\{\omega\}}$ is a domination hypergraph, and so there exists a graph G_0' with vertex set $V(G_0') = \Omega \setminus \{\omega\}$ such that $\mathcal{U}_{2,\Omega\setminus\{\omega\}} = \mathcal{D}(G_0')$. If $G_0 = K_{\{\omega\}} \vee G_0'$, then G_0 is a graph with vertex set Ω and:

$$\mathcal{D}(G_0) = \mathcal{D}(K_{\{\omega\}} \vee G_0') = \{\{\omega\}\} \cup \mathcal{D}(G_0') = \{\{\omega\}\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega\}} = \mathcal{H}_\omega.$$

So $\mathcal{H}_\omega$ is a domination hypergraph with ground set Ω.

To conclude the proof of the first statement we must demonstrate that if G is a graph with $\mathcal{D}(G) = \{\{\omega\}\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega\}}$, then $G = K_{\{\omega\}} \vee G'$ for some graph G' with vertex set $\Omega \setminus \{w\}$ and $\mathcal{D}(G') = \mathcal{U}_{2,\Omega\setminus\{\omega\}}$. Let $G' = G - \omega$ be the graph obtained by deleting the vertex ω from G. Since $\{\omega\} \in \mathcal{D}(G)$, the vertex ω is universal in G and so $G = K_{\{\omega\}} \vee (G - \omega)$. Moreover, from Lemma 2.22 we have $\mathcal{D}(G) = \{\{\omega\}\} \cup \mathcal{D}(G')$. Thus we conclude that $\mathcal{D}(G') = \mathcal{U}_{2,\Omega\setminus\{\omega\}}$. This completes the proof of the first statement.

Next we are going to prove that $\mathcal{H}_{\omega_1}, \ldots, \mathcal{H}_{\omega_n}$ are the minimal upper domination completions of $\mathcal{U}_{2,\Omega}$ with ground set Ω.

Let $1 \leqslant i \leqslant n$. It is clear that $\mathcal{U}_{2,\Omega} \leqslant^+ \{\{\omega_i\}\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega_i\}}$, that is $\mathcal{U}_{2,\Omega} \leqslant^+ \mathcal{H}_{\omega_i}$.

Moreover, from statement 1), the hypergraph $\mathcal{H}_{\omega_i}$ is a domination hypergraph with ground set Ω. So $\mathcal{H}_{\omega_1}, \ldots, \mathcal{H}_{\omega_n}$ are upper domination completions of $\mathcal{U}_{2,\Omega}$ with ground set Ω; that is:

$$\left\{\mathcal{H}_{\omega_1}, \ldots, \mathcal{H}_{\omega_n}\right\} \subseteq \mathrm{DomHyp}^+_{0,u}(\Omega, \mathcal{H}_{2,\Omega}). \tag{4.1}$$

Now let us prove that:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathcal{U}_{2,\Omega}) \subseteq \left\{\mathcal{H}_{\omega_1}, \ldots, \mathcal{H}_{\omega_n}\right\}. \tag{4.2}$$

In order to do this it is enough to show that if $\mathcal{H}$ is an upper domination completion of $\mathcal{U}_{2,\Omega}$ with ground set Ω, then there exists $\omega \in \Omega$ such that $\mathcal{H}_\omega \leqslant^+ \mathcal{H}$. Let $\mathcal{H}$ be an upper domination completion of $\mathcal{U}_{2,\Omega}$ with ground set Ω. Recall that $\mathcal{U}_{2,\Omega}$ is not a domination hypergraph. So $\mathcal{U}_{2,\Omega} \neq \mathcal{H}$ and hence, since the hypergraph $\mathcal{U}_{2,\Omega}$ consists of all subsets of Ω of size 2, there exists $\omega \in \Omega$ such that $\{\omega\} \in \mathcal{H}$. Therefore we have that $\mathcal{U}_{2,\Omega} \leqslant^+ \mathcal{H}$ and that $\{\omega\} \in \mathcal{H}$, and so $\{\{\omega\}\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega\}} \leqslant^+ \mathcal{H}$; that is, $\mathcal{H}_\omega \leqslant^+ \mathcal{H}$.

From (4.1) and (4.2) we have that:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathcal{U}_{2,\Omega}) = \min\left\{\mathcal{H}_{\omega_1}, \ldots, \mathcal{H}_{\omega_n}\right\}.$$

Observe that if $i \neq j$, then $\mathcal{H}_{\omega_i} \not\leqslant^+ \mathcal{H}_{\omega_j}$. So, we get:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathcal{U}_{2,\Omega}) = \min\left\{\mathcal{H}_{\omega_1}, \ldots, \mathcal{H}_{\omega_n}\right\} = \left\{\mathcal{H}_{\omega_1}, \ldots, \mathcal{H}_{\omega_n}\right\},$$

which completes the proof of the second statement.

Finally, we must prove that if $\omega_1 \neq \omega_2$, then $\left\{\mathcal{H}_{\omega_1}, \mathcal{H}_{\omega_2}\right\}$ is a 2-0-decomposition of $\mathcal{U}_{2,\Omega}$; that is, we must prove that:

$$\mathcal{U}_{2,\Omega} = \mathcal{H}_{\omega_1} \overset{+}{\sqcap} \mathcal{H}_{\omega_2}.$$

Since $\mathcal{H}_{\omega_i} = \{\{\omega_i\}\} \cup \mathcal{U}_{2,\Omega\setminus\{\omega_i\}}$, the union $A_1 \cup A_2$ has at least size two whenever $A_1 \in \mathcal{H}_{\omega_1}$ and $A_2 \in \mathcal{H}_{\omega_2}$ and $w_1 \neq w_2$. Moreover, it is clear that every subset $\{\omega_k, \omega_\ell\}$ with $\omega_k \neq \omega_\ell$ can be obtained as $A_1 \cup A_2$, for some $A_1 \in \mathcal{H}_{\omega_1}$ and $A_2 \in \mathcal{H}_{\omega_2}$. Hence we conclude that:

$$\min\left\{A_1 \cup A_2 : A_1 \in \mathcal{H}_{\omega_1} \text{ and } A_2 \in \mathcal{H}_{\omega_2}\right\} = \mathcal{U}_{2,\Omega};$$

that is, $\mathcal{H}_{\omega_1} \overset{+}{\sqcap} \mathcal{H}_{\omega_2} = \mathcal{U}_{2,\Omega}$. □

To finish this subsection, let us see an example to illustrate this theorem.

Example 4.14 Let $\Omega = \{1,2,3\}$ and let us consider the 2-uniform hypergraph $\mathcal{U}_{2,\Omega}$; that is $\mathcal{U}_{2,\Omega} = \{\{1,2\},\{1,3\},\{2,3\}\}$. By Proposition 2.52, we know that $\mathcal{U}_{2,\Omega}$ is not a domination hypergraph. Thus, by Theorem 3.52, we conclude that $(\mathrm{DomHyp}^+_{0,u}(\Omega,\mathcal{U}_{2,\Omega}), \leqslant^+)$ has at least two minimal elements. The computation of these minimal elements have already been done in Subsection 4.2.2, but here we are going to use Theorem 4.13. By this theorem, we know that:

$$\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathcal{U}_{2,\Omega}) = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\},$$

where $\mathcal{H}_i = \{\{i\}\} \cup \mathcal{U}_{2,\Omega\setminus\{i\}} = \{\{i\},\{j,k\}\}$ and $\{i,j,k\} = \Omega$. Moreover, we have that $\{\mathcal{H}_i, \mathcal{H}_j\}$ is a 2-0-decomposition of $\mathcal{U}_{2,\Omega}$ and $\mathfrak{D}^+_{0,u}(\Omega,\mathcal{U}_{2,\Omega})$.

4.3.2 Minimal domination completions of $\mathcal{U}_{n-1,\Omega}$

From Proposition 2.52 we get that if Ω has size $n \geqslant 3$, then the hypergraph $\mathcal{U}_{n-1,\Omega}$ is not a domination hypergraph. The goal of this section is to provide a complete description of the set $\mathrm{MinDomHyp}^+_{0,u}(\Omega,\mathcal{U}_{n-1,\Omega})$ (Theorem 4.19), and to display their graph realizations (Proposition 4.22). In addition, we present an upper bound for the 0-decomposition index $\mathfrak{D}^+_{0,u}(\Omega,\mathcal{U}_{n-1,\Omega})$ (Proposition 4.24). Up to now, the computation of the exact value of this index remains as an open problem.

In order to prove our results we will use the following four technical lemmas. Three of these lemmas are concerned with graphs that are disjoint union of stars; whereas the last lemma deals with a property of the partial order $\leqslant^+$.

The notions involved in these lemmas are about *stars*. A tree T of order $n \geqslant 2$ is a *star* if it is isomorphic to the complete bipartite graph $K_{1,n-1}$. Observe that a tree T of order $n \geqslant 2$ is a star if and only if T has at most one vertex of degree at least 2, the *center* of the star. If a star T has no vertices of degree at least 2, then T is isomorphic to K_2 and both vertices can be considered as centers of the star. Stars can also be characterized as non-empty connected graphs such that all its edges are incident to a *leaf*, that is, a vertex of degree 1. It is clear that every graph without isolated vertices and such that all its edges have at least one endpoint of degree 1 is a disjoint union of stars. The following result is a direct consequence of this fact.

Lemma 4.15 *If G is a graph without isolated vertices, then G is a disjoint union of stars if and only if $\mathcal{N}[G] = E(G)$.*

Proof. Suppose first that G is a disjoint union of stars. If x is a leaf (that is, a vertex of degree 1), then $N[x] = \{x, y\} \in E(G)$, whereas if x is a vertex of degree $r \geqslant 2$, then $N[x] = \{x, y_1, \ldots, y_r\}$ where $y_1, \ldots, y_r$ are the leaves hanging from x. Therefore, we conclude that $\mathcal{N}[G] = \{N[x] : x \text{ is a leaf}\}$. So, $\mathcal{N}[G] = E(G)$.

Now suppose that $\mathcal{N}[G] = E(G)$. Then, every edge has an endpoint of degree 1, because it is the neighborhood of some vertex. Therefore, G is a disjoint union of stars. □

Lemma 4.16 *Every graph G without isolated vertices contains a spanning subgraph that is a disjoint union of stars.*

Proof. It is sufficient to prove that the statement holds for connected graphs G of order $n \geqslant 2$. We proceed by induction on n. The result is trivial for $n = 2$. Now assume that G is a connected graph of order $n \geqslant 3$. Consider a spanning tree T of G. If T is a star, then the result follows. So we may assume that T is not a star. In such a case T has at least two vertices of degree $\geqslant 2$. Consider an edge of the path joining these two vertices. By removing this edge, we obtain two trees T_1 and T_2 of order at least 2 and without isolated vertices. By inductive hypothesis, both trees contain a spanning subgraph that is a disjoint union of stars. To finish observe that the union of those subgraphs is a spanning subgraph of G that is a disjoint union of stars. □

Lemma 4.17 *If G is the disjoint union of the stars $S_1, \ldots, S_r$, then G has exactly 2^r inclusion-minimal dominating sets. Namely, the dominating sets of G are the sets of vertices of the form:*

$$\{c_j : j \in J\} \cup \left(\bigcup_{i \in \{1,\ldots,r\} \setminus J} L_i \right)$$

where $J \subseteq \{1, \ldots, r\}$, and where c_i and L_i are respectively the center and the set of leaves of the star S_i (whenever S_i is isomorphic to K_2, choose one of the two vertices as the center and the other as the leaf).

Proof. It is clear that a star has exactly two minimal dominating sets (see Remark 2.25). Namely, if the star S is not isomorphic to K_2, then the minimal dominating sets of S are the set of leaves and the set containing only the center; whereas if S is isomorphic to K_2, then the minimal dominating sets of S are the sets containing exactly one vertex. Now, the result follows by applying Lemma 2.22 because if G is the disjoint union of the stars $S_1, \ldots, S_r$, then $\mathscr{D}(G) = \{D_1 \cup \cdots \cup D_r : D_i \in \mathscr{D}(S_i)\}$. □

Lemma 4.18 *If G' be a spanning subgraph of G, then $\mathscr{D}(G') \leqslant^+ \mathscr{D}(G)$.*

Proof. From $V(G) = V(G')$ and $E(G') \subseteq E(G)$, we have that every dominating set of G' is also a dominating set of G. In particular, if $D' \in \mathscr{D}(G')$ then D' contains a minimal dominating set D of G. Therefore, the inequality $\mathscr{D}(G') \leqslant^+ \mathscr{D}(G)$ holds. □

Now, by using these lemmas, we are going to prove the following theorem which provides a complete description of all the minimal upper domination completions of the uniform hypergraph $\mathcal{U}_{n-1,\Omega}$ with ground set Ω.

Theorem 4.19 *If Ω be a finite set of size $n \geqslant 3$, then the upper minimal domination completions of $\mathcal{U}_{n-1,\Omega}$ with ground set Ω are the domination hypergraphs $\mathcal{H}$ of the form $\mathcal{H} = \mathcal{D}(G)$, where G is a disjoint union of stars with vertex set $V(G) = \Omega$; that is:*

$$\mathrm{MinDomHyp}^{+}_{0,u}(\Omega, \mathcal{H}) = \{\mathcal{D}(G) : G \textit{ is a disjoint union of stars with } V(G) = \Omega\}.$$

Proof. Let:

$$\Sigma = \{G : G \text{ is a disjoint union of stars with } V(G) = \Omega\}.$$

First we will prove that $\mathcal{U}_{n-1,\Omega} \leqslant^{+} \mathcal{D}(G)$ for all $G \in \Sigma$; that is, we must show that if $A \in \mathcal{U}_{n-1,\Omega}$, then there exists $D \in \mathcal{D}(G)$ such that $D \subseteq A$. So, let $A \in \mathcal{U}_{n-1,\Omega}$. Then $A = \Omega \setminus \{\omega_0\}$ for some $\omega_0 \in \Omega$. By Lemma 2.14, there exists a minimal dominating set D_0 of G not containing ω_0. Thus, $D_0 \subseteq \Omega \setminus \{\omega_0\}$. So we can set $D = D_0$.

Now, we will prove that if $\mathcal{H}$ is an upper domination completion of $\mathcal{U}_{n-1,\Omega}$ with ground set Ω, then there exists $G \in \Sigma$ such that $\mathcal{D}(G) \leqslant^{+} \mathcal{H}$. So, let $\mathcal{H}$ be an upper domination completion of $\mathcal{U}_{n-1,\Omega}$ with ground set Ω. Then $\mathcal{U}_{n-1,\Omega} \leqslant^{+} \mathcal{H}$ and there is a graph $G_{\mathcal{H}}$ with vertex set Ω such that $\mathcal{H} = \mathcal{D}(G_{\mathcal{H}})$. Notice that if $\mathcal{U}_{n-1,\Omega} \leqslant^{+} \mathcal{D}(G_{\mathcal{H}})$, then $G_{\mathcal{H}}$ has no isolated vertices. Indeed, otherwise the isolated vertex ω_0 should belong to every minimal dominating set of $G_{\mathcal{H}}$ implying that $\Omega \setminus \{\omega_0\} \in \mathcal{U}_{n-1,\Omega}$ does not contain any minimal dominating set of $G_{\mathcal{H}}$, which is a contradiction. Thus, since $G_{\mathcal{H}}$ has no isolated vertices, by Lemma 4.16, there exists a spanning subgraph G of $G_{\mathcal{H}}$ that is a disjoint union of stars. Since G is a spanning subgraph of $G_{\mathcal{H}}$, by Lemma 4.18 it follows that $\mathcal{D}(G) \leqslant^{+} \mathcal{D}(G_{\mathcal{H}})$. Therefore we conclude that $G \in \Sigma$ and $\mathcal{D}(G) \leqslant^{+} \mathcal{H}$.

Finally, it remains to prove that the dominating hypergraphs of distinct disjoint union of stars with vertex set Ω are either equal or non-comparable. In other words, we must demonstrate that if $\mathcal{D}(G) \leqslant^{+} \mathcal{D}(G')$ with $G, G' \in \Sigma$, then $G = G'$. So, let $G, G' \in \Sigma$ with $\mathcal{D}(G) \leqslant^{+} \mathcal{D}(G')$. Then from Lemma 1.35, 3a) and Corollary 2.6 it follows that:

$$\mathcal{N}[G'] = \mathrm{Tr}(\mathcal{D}(G')) \leqslant^{+} \mathrm{Tr}(\mathcal{D}(G)) = \mathcal{N}[G].$$

By applying Lemma 4.15 we get that $\mathcal{N}[G'] = E(G')$ and $\mathcal{N}[G] = E(G)$. Therefore $E(G') \leqslant^{+} E(G)$. Hence $E(G') \subseteq E(G)$ because $E(G')$ and $E(G)$ are 2-uniform hypergraphs. At this point observe that the addition of an edge to a graph that is a disjoint union of stars gives rise to a graph not satisfying this property. Therefore we conclude that $E(G) = E(G')$ and, consequently, $G = G'$. □

Let us illustrate this theorem with some examples.

Example 4.20 Let $\Omega = \{1, 2, 3\}$ and let us consider the uniform hypergraph $\mathcal{U}_{2,\Omega} = \{\{1,2\},\{1,3\},\{2,3\}\}$. The stars with vertex set Ω are the graphs G_1, G_2 and G_3 with edge sets given by $E(G_i) = \{\{i, j\}, \{i, k\}\}$, where $\{i, j, k\} = \Omega$. So by applying Theorem 4.19 we get that:

$$\begin{aligned}\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{U}_{2,\Omega}) &= \{\mathscr{D}(G_1), \mathscr{D}(G_2), \mathscr{D}(G_3)\} \\ &= \{\{\{i\}, \{j, k\}\} : \{i, j, k\} = \{1, 2, 3\}\}.\end{aligned}$$

Example 4.21 Let $\Omega = \{1, 2, 3, 4\}$ and let us consider the uniform hypergraph $\mathscr{U}_{3,\Omega} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. See Example 3.33 where the hypergraph $\mathscr{U}_{3,\Omega}$ is denoted by $\mathscr{H}_4$. There are seven graphs with vertex set Ω that are disjoint unions of stars. Namely, if we set $\{i, j, k, \ell\} = \Omega$, then these graphs are the stars G_i, $i = 1, \ldots, 4$, with edge sets given by $E(G_i) = \{\{i, j\}, \{i, k\}, \{i, \ell\}\}$; and the graphs G_i, $i = 5, 6, 7$, such that $E(G_i) = \{\{i, j\}, \{k, \ell\}\}$. By Theorem 4.19 we get that:

$$\begin{aligned}\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{U}_{3,\Omega}) &= \{\mathscr{D}(G_i) : 1 \le i \le 7\} \\ &= \{\{\{i\}, \{j, k, \ell\}\} : \{i, j, k\} = \Omega\} \cup \\ &\quad \{\{\{i, k\}, \{i, \ell\}, \{j, k\}, \{j, \ell\} : \{i, j, k, \ell\} = \Omega\}.\end{aligned}$$

The following proposition characterizes all graphs that realize an upper minimal domination completion of $\mathscr{U}_{n-1,\Omega}$ with ground set Ω. After its proof we present an example of a minimal domination completion $\mathscr{H}_0$ of the uniform hypergraph $\mathscr{U}_{n-1,\Omega}$ for $n = 8$, as well as the description of all the graph realizations of $\mathscr{H}_0$ (the example is illustrated in Figure 4.2).

Proposition 4.22 *Let G be a graph with vertex set Ω that is a disjoint union of stars, and let G' be a graph with vertex set Ω. Then $\mathscr{D}(G) = \mathscr{D}(G')$ if and only if G' is any graph that can be obtained from G in the following way: choosing a set C formed by exactly one center of every connected component of G and adding to G any set of edges joining vertices of C.*

Proof. Let G' be a graph with vertex set Ω. By applying Corollary 2.6 and Lemma 4.15 we get that $\mathscr{D}(G') = \mathscr{D}(G)$ if and only if $\mathscr{N}[G'] = \mathscr{N}[G]$ if and only if $\mathscr{N}[G'] = E(G)$. It is not hard to prove that $\mathscr{N}[G'] = E(G)$ if and only if the following two conditions are satisfied: $E(G) \subseteq E(G')$, and for every edge $\{x, y\} \in E(G)$ either x or y has degree 1 in G'. Therefore we conclude that $\mathscr{D}(G') = \mathscr{D}(G)$ if and only if G' is obtained from G by adding edges joining vertices of a set containing exactly one center of each star of G. ☐

Example 4.23 Let $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$. By Theorem 4.19, the upper minimal domination completions of $\mathscr{U}_{7,\Omega}$ with ground set Ω are the hypergraphs of the form $\mathscr{D}(G)$ where G is a disjoint union of stars with vertex set Ω. It is straightforward to prove that there are 5041 such graphs G, all of them providing different domination hypergraphs. Therefore:

$$|\,\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{U}_{7,\Omega})| = 5041.$$

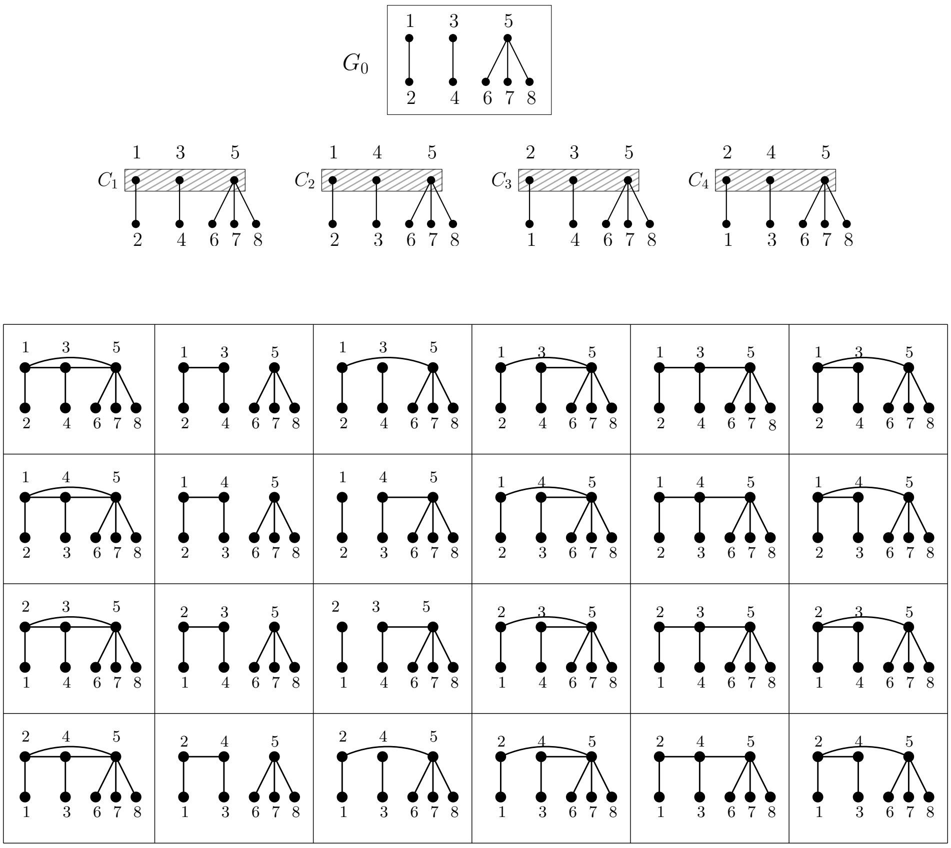


Figure 4.2 The hypergraph $\mathscr{H}_0 = \mathscr{D}(G_0)$ is a minimal domination completion of the uniform hypergraph $\mathscr{U}_{7,\Omega}$ where $\Omega = \{1,2,3,4,5,6,7,8\}$. The graph G_0 together with the 24 graphs obtained by adding edges joining vertices of C_i, $i = 1,\ldots,4$, give raise to all the 25 graph realizations of the domination hypergraph $\mathscr{H}_0$ (see Example 4.23).

One of these graphs G is the graph G_0 obtained as the disjoint union of 3 stars, two of them isomorphic to K_2 and the other one isomorphic to $K_{1,3}$; namely, the graph G_0 with edge set:

$$E(G_0) = \{\{1,2\},\{3,4\},\{5,6\},\{5,7\},\{5,8\}\}.$$

By Lemma 4.17, this graph G_0 has the following $2^3 = 8$ minimal dominating sets:

$$\begin{aligned}\mathscr{D}(G_0) = \{&\{1,3,5\},\{1,3,6,7,8\},\{1,4,5\},\{1,4,6,7,8\},\\ &\{2,3,5\},\{2,3,6,7,8\},\{2,4,5\},\{2,4,6,7,8\}\}.\end{aligned}$$

So, the hypergraph $\mathscr{H}_0 = \mathscr{D}(G_0)$ is an upper minimal domination completion of $\mathscr{U}_{7,\Omega}$ with ground set Ω. In order to obtain all the graph realizations of $\mathscr{H}_0$, we apply Proposition 4.22. In this case we have four possibilities for the set C containing exactly one center of each star. Concretely C is either $C_1 = \{1,3,5\}$, or $C_2 = \{1,4,5\}$, or $C_3 = \{2,3,5\}$, or $C_4 = \{2,4,5\}$. The graphs G' such that its collection of minimal dominating sets is $\mathscr{D}(G') = \mathscr{D}(G_0)$ are obtained by fixing one of the sets C_i and adding edges joining vertices of C_i. It is easy to check that there are exactly 24 different graphs $G' \neq G_0$ obtained in this way (see Figure 4.2).

To conclude this subsection we present an upper bound on the 0-decomposition index $\mathfrak{D}^+_{0,u}(\Omega,\mathscr{U}_{n-1,\Omega})$, where $|\Omega| = n$ (Proposition 4.24). It is worth noting that an exhaustive analysis of all possible cases shows that the equality holds whenever $2 \leqslant n \leqslant 5$. However, it remains an open problem to determine whether the equality holds for $n \geqslant 6$.

Proposition 4.24 *If Ω is a finite set of size $n \geqslant 3$, then:*

$$\mathfrak{D}^+_{0,u}(\Omega,\mathscr{U}_{n-1,\Omega}) \leqslant n-1;$$

that is, there are $n-1$ upper minimal domination completions $\mathscr{H}_1,\dots,\mathscr{H}_{n-1}$ of $\mathscr{U}_{n-1,\Omega}$ with ground set Ω such that:

$$\mathscr{U}_{n-1,\Omega} = \mathscr{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_{n-1}.$$

Proof. Let $\Omega = \{\omega_1,\dots,\omega_n\}$. For $1 \le i \le n-1$, let $\mathscr{H}_i$ be the domination hypergraph $\mathscr{H}_i = \mathscr{D}(S_i)$, where S_i is the star with center ω_i and isomorphic to $K_{1,n-1}$. By Theorem 4.19, the hypergraphs $\mathscr{H}_1,\dots,\mathscr{H}_{n-1}$ are minimal domination completions of $\mathscr{U}_{n-1,\Omega}$ with ground set Ω. Let us show that:

$$\mathscr{U}_{n-1,\Omega} = \mathscr{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_{n-1}.$$

It is clear that $\mathscr{H}_i = \{D_{i,1}, D_{i,2}\}$, where $D_{i,1} = \{\omega_i\}$ and $D_{i,2} = \Omega \setminus \{\omega_i\}$ (see Remark 2.25). On the one hand, the elements of $\mathscr{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_{n-1}$ have size at least $n-1$ and hence the inequality:

$$\mathscr{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathscr{H}_{n-1} \leqslant^+ \mathscr{U}_{n-1,\Omega}$$

holds. On the other hand, if $A \in \mathcal{U}_{n-1,\Omega}$, then $A = \Omega \setminus \{\omega\}$ for some $w \in \Omega$, and thus we get that $A = D_{1,1} \cup \cdots \cup D_{n-1,1}$ if $w = w_n$; whereas $A = D_{i_0,2} \cup \left(\cup_{i \neq i_0} D_{i,1}\right)$ if $w = w_{i_0} \neq w_n$. So, the inequality:

$$\mathcal{U}_{n-1,\Omega} \leqslant^{+} \mathcal{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_{n-1}$$

also holds. Therefore we conclude that:

$$\mathcal{U}_{n-1,\Omega} = \mathcal{H}_1 \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{H}_{n-1},$$

as we wanted to prove. □

Remark 4.25 Let $\Omega = \{1,\ldots,n\}$. In Proposition 4.24 we have used the explicit upper domination completions of the hypergraph $\mathcal{U}_{n-1,\Omega}$ with ground set Ω in order to compute an upper bound of its 0-decomposition index $\mathfrak{D}^{+}_{0,u}(\Omega,\mathcal{U}_{n-1,\Omega})$. Next we are going to do an alternative and direct proof of the inequality $\mathfrak{D}^{+}_{u}(\Omega,\mathcal{H}) \leqslant n-1$. Recall that $\mathfrak{D}^{+}_{u}(\Omega,\mathcal{H}) \leqslant \mathfrak{D}^{+}_{0,u}(\Omega,\mathcal{H})$, by Proposition 3.61. Namely, we are going to show that the decomposition index $\mathfrak{D}^{+}_{u}(\Omega,\mathcal{U}_{n-1,\Omega}) \leq n-1$ by finding an explicit $(n-1)$-decomposition of $\mathcal{U}_{n-1,\Omega}$ into domination hypergraphs *on* Ω.

We have that $\mathcal{U}_{n-1,\Omega} = \mathrm{Tr}(\mathcal{U}_{2,\Omega})$. Hence, by Proposition 1.45 we have:

$$\mathcal{U}_{n-1,\Omega} = \mathrm{Tr}(\mathcal{U}_{2,\Omega}) = \mathcal{U}_{1,\{1,2\}} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{U}_{1,\{n-1,n\}}. \tag{4.3}$$

Now we group together all hypergraphs of the form $\mathcal{U}_{1,\{1,b\}}$, with $b \neq 1$, and we get:

$$\mathcal{U}_{1,\{1,2\}} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{U}_{1,\{1,n-1\}} = \{\{1\},\{2,\ldots,n\}\}\} = \mathcal{D}(G_1),$$

where G_1 is the star with vertex set Ω and center the vertex 1. Similarly, by grouping together now all the hypergraphs of the form $\mathcal{U}_{1,\{2,b\}}$, with $b \neq 1,2$, we get:

$$\mathcal{U}_{1,\{2,3\}} \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{U}_{1,\{2,n-1\}} = \{\{2\},\{3,\ldots,n\}\}\} = \mathcal{D}(G_2),$$

where G_2 is the star with vertex set $\{2,\ldots,n\}$ and center the vertex 2. Proceeding in a similar way with the remaining hypergraphs of the decomposition (4.3), we get:

$$\mathcal{U}_{n-1,\Omega} = \mathcal{D}(G_1) \overset{+}{\sqcap} \cdots \overset{+}{\sqcap} \mathcal{D}(G_{n-1}), \tag{4.4}$$

where G_i is the star with vertex set $\{i,\ldots,n\}$ and center at i. Equality (4.4) provides us with a $(n-1)$-decomposition of $\mathcal{U}_{n-1,\Omega}$ in $\mathrm{DomHyp}^{+}_{u}(\Omega)$. Thus we have that $\mathfrak{D}^{+}_{u}(\Omega,\mathcal{U}_{n-1,\Omega}) \leq n-1$.

4.3.3 Minimal domination completions of $\mathcal{U}_{r,\Omega}$, $|\Omega| \leqslant 5$

The aim of this subsection is to compute the set $\mathrm{MinDomHyp}^{+}_{0,u}(\Omega,\mathcal{U}_{r,\Omega})$, where Ω is a finite set of size $|\Omega| = n \leqslant 5$ and where $1 \leqslant r \leqslant n$. From Proposition 2.52, Lemma 3.51 and Theorem 3.52 we get that:

$$\mathrm{MinDomHyp}^{+}_{0,u}(\Omega,\mathcal{U}_{r,\Omega}) = \{\mathcal{U}_{r,\Omega}\} \Leftrightarrow (r,n) \neq (2,3),(2,5),(3,4),(3,5),(4,5).$$

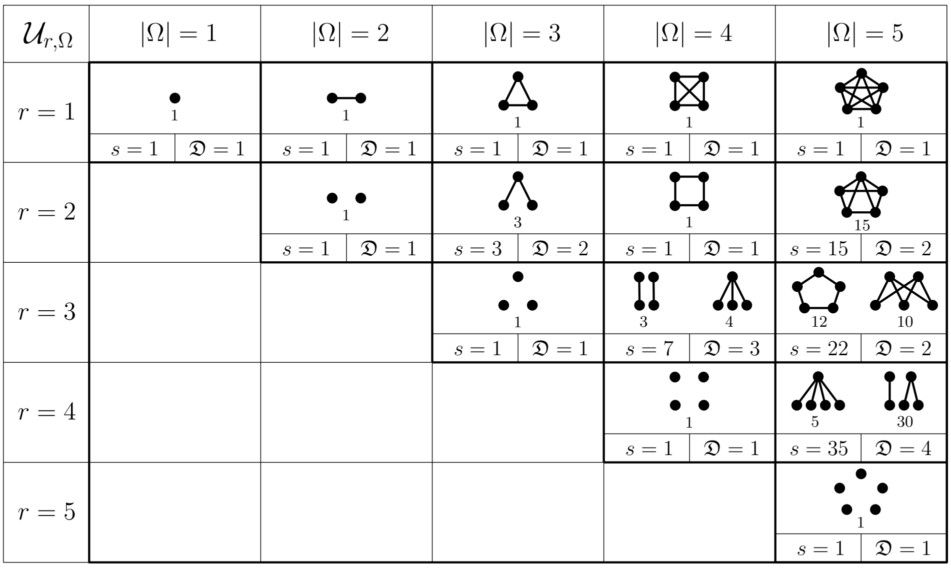


Figure 4.3 For $1 \leqslant n = |\Omega| \leqslant 5$, the upper minimal domination completions of $\mathscr{U}_{r,\Omega}$ with ground set Ω are the domination hypergraphs $\mathscr{D}(G')$, where G' is a graph isomorphic to a graph G in the figure. The number below each graph G denotes the number of different hypergraphs $\mathscr{H} \in \mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{U}_{r,\Omega})$ with $\mathscr{H} = \mathscr{D}(G')$ for some graph with G' isomorphic to G. In addition, in each case, the total number $s = |\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{U}_{r,\Omega})|$ of minimal domination completions of $\mathscr{U}_{r,\Omega}$, and the decomposition index $\mathfrak{D} = \mathfrak{D}^+_{0,u}(\Omega, \mathscr{U}_{r,\Omega})$ are given.

In addition, the results of the the preceding subsections provide a complete description of the set $\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{U}_{r,\Omega})$ for $(r, n) = (2,3), (2,5), (3,4), (4,5)$. Thus, it only remains to calculate the set of minimal domination completions $\mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{U}_{r,\Omega})$ for $(r, n) = (3,5)$.

The goal of this subsection is to prove that, for $n = 5$ the uniform hypergraph $\mathscr{U}_{3,\Omega}$ has 22 upper minimal domination completions with ground set Ω: 12 of the form $\mathscr{D}(G)$, with G isomorphic to a cicle C_5, and 10 of the form $\mathscr{D}(G)$, with G isomorphic to the complete bipartite graph $K_{2,3}$. This result is stated in Theorem 4.26. This and all the other results about the upper minimal domination completions of the uniform hypergraphs $\mathscr{U}_{r,\Omega}$ with ground set Ω, where $1 \leqslant r \leqslant |\Omega| \leqslant 5$, are summarized in Figure 4.3 (in each case, the graphs G in the figure provide the realization of all the upper minimal domination completions $\mathscr{H}$ of $\mathscr{U}_{r,\Omega}$ with ground set Ω; that is, $\mathscr{H} \in \mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{U}_{r,\Omega})$ if and only if $\mathscr{H} = \mathscr{D}(G')$ for some graph G' isomorphic to a graph G in the figure).

Theorem 4.26 *Let Ω be a finite set of size $|\Omega| = 5$. Let $\mathscr{C}_5$ and $\mathscr{K}_{2,3}$ be the families of* *graphs with vertex set Ω, where the graphs of $\mathscr{C}_5$ are exactly those isomorphic to the cycle C_5, whereas the graphs of $\mathscr{K}_{2,3}$ are all those isomorphic to the complete bipartite graph $K_{2,3}$. The following statements hold.*

1) *The upper minimal domination completions of $\mathscr{U}_{3,\Omega}$ with ground set Ω are the domination hypergraphs $\mathscr{D}(G)$, where the graph G is isomorphic to either a cycle C_5 or to a complete bipartite graph $K_{2,3}$; that is:*

$$\mathrm{MinDomHyp}^{+}_{0,u}(\Omega, \mathscr{U}_{3,\Omega}) = \left\{\mathscr{D}(G) : G \in \mathscr{C}_5 \cup \mathscr{K}_{2,3}\right\}.$$

2) *The uniform hypergraph $\mathscr{U}_{3,\Omega}$ has 22 upper minimal domination completions with ground set Ω; that is, $|\mathrm{MinDomHyp}^{+}_{0,u}(\Omega, \mathscr{U}_{3,\Omega})| = 22$.*

3) *The uniform hypergraph $\mathscr{U}_{3,\Omega}$ has 0-decomposition index:*

$$\mathfrak{D}^{+}_{0,u}(\Omega, \mathscr{U}_{3,\Omega}) = 2;$$

that is, there exist $\mathscr{H}, \mathscr{H}' \in \mathrm{MinDomHyp}^{+}_{0,u}(\Omega, \mathscr{U}_{3,\Omega})$ such that:

$$\mathscr{U}_{3,\Omega} = \mathscr{H} \overset{+}{\sqcap} \mathscr{H}'.$$

The rest of this subsection is devoted to prove this theorem.

From now on, throughout this subsection, $\Omega = \{1,2,3,4,5\}$.

First observe that if $G \in \mathscr{C}_5$, then $\mathscr{D}(G)$ contains the five pairs of non-adjacent vertices; while if $G \in \mathscr{K}_{2,3}$, then $\mathscr{D}(G)$ contains both stable sets and the 6 pairs of adjacent vertices (see Figure 4.4). Using these facts, it is easy to check that if $G, G' \in \mathscr{C}_5 \cup \mathscr{K}_{2,3}$, then $\mathscr{D}(G) = \mathscr{D}(G')$ if and only if $G = G'$. Therefore:

$$\begin{aligned}
|\{\mathscr{D}(G) : G \in \mathscr{C}_5 \cup \mathscr{K}_{2,3}\}| &= |\{\mathscr{D}(G) : G \in \mathscr{C}_5\}| + |\{\mathscr{D}(G) : G \in \mathscr{K}_{2,3}\}| \\
&= |\mathscr{C}_5| + |\mathscr{K}_{2,3}| \\
&= 12 + 10 = 22.
\end{aligned}$$

Thus, the statement 2) of the theorem follows from the first one.

Now let us demonstrate the third statement of the theorem. The statement 1) will be proved after doing this.

Recall that the uniform hypergraph $\mathscr{U}_{3,\Omega}$ is not a domination hypergraph (Proposition 2.52). So, $\mathfrak{D}^{+}_{0,u}(\Omega, \mathscr{U}_{3,\Omega}) \geq 2$. The inequality $\mathfrak{D}^{+}_{0,u}(\Omega, \mathscr{U}_{3,\Omega}) \leq 2$ follows from Proposition 4.27. This proposition shows all the ways to obtain $\mathscr{U}_{3,\Omega}$ as $\mathscr{D}(G_1) \overset{+}{\sqcap} \mathscr{D}(G_2)$, when $G_1, G_2 \in \mathscr{C}_5 \cup \mathscr{K}_{2,3}$.

Proposition 4.27 *If $G_1, G_2 \in \mathscr{C}_5 \cup \mathscr{K}_{2,3}$, then the following statements are equivalent.*

$\mathcal{D}(G) = E(K_\Omega) \setminus E(G)$
$= \{\{i_1, i_3\}, \{i_1, i_4\}, \{i_2, i_4\}, \{i_2, i_5\}, \{i_3, i_5\}\}$

$\mathcal{D}(G) = E(G) \cup \{V_1, V_2\}$
$= \{\{i_1, i_2\}, \{i_1, i_4\}, \{i_3, i_2\}, \{i_3, i_4\}, \{i_5, i_2\}, \{i_5, i_4\}, \{i_1, i_3, i_5\}, \{i_2, i_4\}\}$

Figure 4.4 Minimal dominating sets of a graph G with vertex set $V(G) = \{i_1, i_2, i_3, i_4, i_5\}$, isomorphic to C_5 (left) and isomorphic to $K_{2,3}$ (right).

1) $\mathscr{U}_{3,\Omega} = \mathscr{D}(G_1) \overset{+}{\sqcap} \mathscr{D}(G_2)$.

2) $G_1, G_2 \in \mathscr{C}_5$ *and* $E(G_1) \cup E(G_2) = E(K_\Omega)$.

Proof. First consider the case $G_1, G_2 \in \mathscr{C}_5$ with $E(G_1) \cup E(G_2) = E(K_\Omega)$. In such a case, G_2 is the complement of the cycle G_1. So $E(G_1) \cap E(G_2) = \emptyset$, and hence $\mathscr{D}(G_1) = E(G_2)$ and $\mathscr{D}(G_2) = E(G_1)$. Thus, every set $A_1 \cup A_2$, with $A_1 \in \mathscr{D}(G_1)$ and $A_2 \in \mathscr{D}(G_2)$, has size 3 or 4. It is straightforward to check that every element of $\mathscr{U}_{3,\Omega}$ can be obtained as $A_1 \cup A_2$ where $A_1 \in \mathscr{D}(G_1)$ and $A_2 \in \mathscr{D}(G_2)$. Therefore, $\mathscr{U}_{3,\Omega} = \mathscr{D}(G_1) \overset{+}{\sqcap} \mathscr{D}(G_2)$.

The proof of the proposition will be completed by showing that, in any other case, there exists $A_0 \in \mathscr{D}(G_1) \cap \mathscr{D}(G_2)$ with $|A_0| = 2$. Indeed, if there exists $A_0 \in \mathscr{D}(G_1) \cap \mathscr{D}(G_2)$, then $A_0 \in \mathscr{D}(G_1) \overset{+}{\sqcap} \mathscr{D}(G_2)$, and so $\mathscr{D}(G_1) \overset{+}{\sqcap} \mathscr{D}(G_2)$ has at least an element of size $|A_0|$. Hence, if $|A_0| = 2$, then $\mathscr{U}_{3,\Omega} \neq \mathscr{D}(G_1) \overset{+}{\sqcap} \mathscr{D}(G_2)$.

Therefore, we must prove that there exists $A_0 \in \mathscr{D}(G_1) \cap \mathscr{D}(G_2)$ with $|A_0| = 2$ in three cases:

- whenever $G_1, G_2 \in \mathscr{C}_5$ and $E(G_1) \cup E(G_2) \neq E(K_\Omega)$;
- whenever $G_1, G_2 \in \mathscr{K}_{2,3}$; and,
- whenever $G_1 \in \mathscr{C}_5$ and $G_2 \in \mathscr{K}_{2,3}$.

Let us prove this.

- If $G_1, G_2 \in \mathscr{C}_5$ and $E(G_1) \cup E(G_2) \neq E(K_\Omega)$, then there exists $\{x, y\} \in E(K_\Omega) \setminus (E(G_1) \cup E(G_2))$. In this case $\mathscr{D}(G_i) = E(K_\Omega) \setminus E(G_i)$. So, we can set $A_0 = \{x, y\} \in \mathscr{D}(G_1) \cap \mathscr{D}(G_2)$.

- Now let us assume that $G_1, G_2 \in \mathcal{K}_{2,3}$. Then $|E(G_1)| = |E(G_2)| = 6$. So $E(G_1) \cap E(G_2) \neq \emptyset$ and thus there exists $\{x, y\} \in E(G_1) \cap E(G_2)$. Since $E(G_i) \subseteq \mathcal{D}(G_i)$, in this case the subset $A_0 = \{x, y\}$ satisfies the required conditions.

- Finally, suppose that $G_1 \in \mathcal{C}_5$ and $G_2 \in \mathcal{K}_{2,3}$. Then $|E(K_\Omega) \setminus E(G_1)|=5$ and $|E(G_2)| = 6$. So there exists $\{x, y\} \in (E(K_\Omega) \setminus E(G_1)) \cap E(G_2) \subseteq \mathcal{D}(G_1) \cap \mathcal{D}(G_2)$, and thus the proof is completed by setting $A_0 = \{x, y\}$. □

At this point, the proof of Theorem 4.26 will be completed by proving the first statement. This statement follows as a consequence of Propositions 4.28, 4.29 and 4.33. The first two propositions show that the hypergraphs $\mathcal{D}(G)$, where $G \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}$, are upper domination completions of $\mathcal{U}_{3,\Omega}$ with ground set Ω (Proposition 4.28) and that any pair of different such hypergraphs are non comparable (Proposition 4.29); whereas the last proposition (Proposition 4.33) states that the upper minimal domination completions of $\mathcal{U}_{3,\Omega}$ with ground set Ω are hypergraphs of the form $\mathcal{D}(G)$, where $G \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}$. The proof of Proposition 4.33 is involved and requires three technical lemmas concerning the size of the elements of the upper minimal domination completions of $\mathcal{U}_{3,\Omega}$ with ground set Ω and their transversal (Lemmas 4.30, 4.31 and 4.32).

Proposition 4.28 *If $G \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}$, then $\mathcal{U}_{3,\Omega} \leqslant^+ \mathcal{D}(G)$.*

Proof. From Lemma 1.15, we must show that if A is a subset of Ω of size 3, then there exists $D \in \mathcal{D}(G)$ such that $D \subseteq A$. This is clear when $G \in \mathcal{C}_5$, because in such a case every set of three vertices of G contains a pair of two non-adjacent vertices, that are a minimal dominating set of G. Now let assume that $G \in \mathcal{K}_{2,3}$. In this case the result follows by taking into account that the stable set of size 3 is a minimal dominating set of G, and every other set of three vertices contains two adjacent vertices. So any subset of size three contains a minimal dominating set of G. □

Proposition 4.29 *If $G_1, G_2 \in \mathcal{C}_5 \cup \mathcal{K}_{2,3}$ and $\mathcal{D}(G_1) \leqslant^+ \mathcal{D}(G_2)$, then $\mathcal{D}(G_1) = \mathcal{D}(G_2)$.*

Proof. First, suppose that $G_1, G_2 \in \mathcal{C}_5$. Then the hypergraphs $\mathcal{D}(G_1)$ and $\mathcal{D}(G_2)$ contain both exactly five elements of size 2. In such a case it is clear that if $\mathcal{D}(G_1) \leqslant^+ \mathcal{D}(G_2)$, then $\mathcal{D}(G_1) = \mathcal{D}(G_2)$.

Now assume that $G_1, G_2 \in \mathcal{K}_{2,3}$. Then the hypergraphs $\mathcal{D}(G_1)$ and $\mathcal{D}(G_2)$ contain both exactly one element of size 3 and seven elements of size 2. Therefore, from $\mathcal{D}(G_1) \leqslant^+ \mathcal{D}(G_2)$ we deduce that the seven elements of size 2 must be the same. In addition, since there is no inclusion relation between the elements of a hypergraph, the element of size 3 must be also the same. Hence we conclude that $\mathcal{D}(G_1) = \mathcal{D}(G_2)$.

The proof will be completed by showing that the inequality $\mathcal{D}(G_1) \leqslant^+ \mathcal{D}(G_2)$ is not possible neither in the case $G_1 \in \mathcal{K}_{2,3}$ and $G_2 \in \mathcal{C}_5$, nor in the case $G_1 \in \mathcal{C}_5$ and $G_2 \in \mathcal{K}_{2,3}$.

If $G_1 \in \mathscr{K}_{2,3}$ and $G_2 \in \mathscr{C}_5$, then $\mathscr{D}(G_1)$ contains six elements of size 2 while $\mathscr{D}(G_2)$ contains only five elements of size 2. Thus, in such a case, the inequality $\mathscr{D}(G_1) \leqslant^+ \mathscr{D}(G_2)$ is not possible.

Finally, assume $G_1 \in \mathscr{C}_5$ and $G_2 \in \mathscr{K}_{2,3}$. If $\mathscr{D}(G_1) \leqslant^+ \mathscr{D}(G_2)$, then the five elements of size 2 of $\mathscr{D}(G_1)$ must be in $\mathscr{D}(G_2)$; that is, $\mathscr{D}(G_1) \subseteq \mathscr{D}(G_2)$. But the five elements of $\mathscr{D}(G_1)$ correspond to five pairs of non-adjacent vertices of a cycle of order 5, that induce also a cycle of order 5. Nevertheless, there is not possible to induce a cycle of order 5 with five elements of size 2 of $\mathscr{D}(G_2)$. Therefore, $\mathscr{D}(G_1) \leqslant^+ \mathscr{D}(G_2)$ is not possible in that case. This completes the proof of the proposition. □

Lemma 4.30 *Let $\mathscr{H}$ be a hypergraph. If $\mathscr{U}_{3,\Omega} \leqslant^+ \mathscr{H}$, then $|X| \geq 3$ for every $X \in \mathrm{Tr}(\mathscr{H})$.*

Proof. On the contrary, assume that there exists $X \in \mathrm{Tr}(\mathscr{H})$ such that $1 \leq |X| \leq 2$. In such a case, consider a subset $A \subseteq \Omega$ of size $|A| = 3$ satisfying $A \cap X = \emptyset$. Since $|A| = 3$, hence $A \in \mathscr{U}_{3,\Omega}$. Therefore there exists $B \in \mathscr{H}$ contained in A because $\mathscr{U}_{3,\Omega} \leqslant^+ \mathscr{H}$. Hence $B \cap X \subseteq A \cap X$, and so $B \cap X = \emptyset$. This leads us to a contradiction because $B \in \mathscr{H}$ and $X \in \mathrm{Tr}(\mathscr{H})$. □

Lemma 4.31 *Let $\mathscr{H} \in \mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{U}_{3,\Omega})$. If there exists $A \in \mathscr{H}$ such that $|A| = 3$, then $\mathscr{H} = \mathscr{D}(G)$ for some graph $G \in \mathscr{K}_{2,3}$.*

Proof. It is enough to prove that $\mathscr{D}(G) \leqslant^+ \mathscr{H}$ for some $G \in \mathscr{K}_{2,3}$, because by Proposition 4.28, $\mathscr{U}_{3,\Omega} \leqslant^+ \mathscr{D}(G)$, and so, the minimality of $\mathscr{H}$ implies that $\mathscr{D}(G) = \mathscr{H}$.

Without loss of generality we may assume that $A = \{1,2,3\} \in \mathscr{H}$. In such a case, $\{1,4,5\}$, $\{2,4,5\}$ and $\{3,4,5\}$ are in $\mathrm{Tr}(\mathscr{H})$, because all these subsets have non-empty intersection with the elements of $\mathscr{H}$, and there are no elements of cardinality less or equal than 2 in $\mathrm{Tr}(\mathscr{H})$ (Lemma 4.30). Since $\mathscr{H}$ is a domination hypergraph, there exists a graph G_0 such that $\mathscr{H} = \mathscr{D}(G_0)$, and so $\mathrm{Tr}(\mathscr{H}) = \mathscr{N}[G_0]$ (Corollary 2.6). Therefore, $\{1,4,5\}$, $\{2,4,5\}$ and $\{3,4,5\}$ are the closed neighborhoods for some $x,y,z \in \Omega$; that is, $N_{G_0}[x] = \{1,4,5\}$, $N_{G_0}[y] = \{2,4,5\}$ and $N_{G_0}[z] = \{3,4,5\}$. Observe that at least one of the elements $x,y,z \in \Omega$ is different from 4 and 5. So, without loss of generality we may assume that $x \neq 4,5$ and so $x = 1$. Thus $\{1,4,5\} = N_{G_0}[1]$. Hence $\{1,4\},\{1,5\} \in E(G_0)$, and consequently, $N_{G_0}[4] \neq \{2,4,5\},\{3,4,5\}$ and $N_{G_0}[5] \neq \{2,4,5\},\{3,4,5\}$. So we conclude that $N_{G_0}[1] = \{1,4,5\}$, that $N_{G_0}[2] = \{2,4,5\}$, and that $N_{G_0}[3] = \{3,4,5\}$. Hence it follows that $F = \{\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,4\},\{3,5\}\} \subseteq E(G_0)$. At this point let us consider the subgraph G induced by the edges of F. Observe that G is isomorphic to $K_{2,3}$ with stable sets $\{1,2,3\}$ and $\{4,5\}$. So, $G \in \mathscr{K}_{2,3}$. Moreover, the graph G is a spanning subgraph of G_0 and hence, from Lemma 4.18 it follows that $\mathscr{D}(G) \leqslant^+ \mathscr{D}(G_0) = \mathscr{H}$. This completes the proof of the lemma. □

Lemma 4.32 *Let $\mathscr{H} \in \mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{U}_{3,\Omega})$. If $|A| = 2$ for all $A \in \mathscr{H}$, then $\mathscr{H} = \mathscr{D}(G)$ for some graph $G \in \mathscr{C}_5$.*

Proof. Let G_0 be a graph with $\mathscr{H} = \mathscr{D}(G_0)$. Reasoning as in the proof of the previous lemma, now it is enough to prove that $\mathscr{D}(G) \leqslant^+ \mathscr{D}(G_0)$, for some $G \in \mathscr{C}_5$.

To prove this inequality we will use the following four facts.

(i) First, notice that G_0 has no vertex of degree 4, because otherwise there would be an element in $\mathscr{D}(G_0)$ of size 1.

(ii) Second, we claim that if $\{a, b\} \notin \mathscr{H}$, then $\{a, b\} \in E(G_0)$. Let us prove our claim. Assume to the contrary that a and b are non-adjacent in G_0. In such a case, both vertices a and b belong in a minimal dominating set D of $\mathscr{D}(G_0)$ (for instance, we can consider an inclusion-maximal independent set D containing a and b). But $\mathscr{D}(G_0) = \mathscr{H}$ and, by assumption, all the elements of $\mathscr{H}$ have size 2. Therefore we conclude that $\{a, b\} = D \in \mathscr{D}(G_0) = \mathscr{H}$, which is a contradiction. This completes the proof of our claim.

(iii) Next, observe that $\mathscr{N}[G_0] = \mathrm{Tr}(\mathscr{D}(G_0)) = \mathrm{Tr}(\mathscr{H})$ (Corollary 2.6). So it follows that all the elements of $\mathscr{N}[G_0]$ have at least 3 elements, by Lemma 4.30.

(iv) Finally, let us show that $\mathscr{N}[G_0]$ has at least one element X of size 3. Suppose on the contrary that it is not true. If $\mathscr{N}[G_0] = \{\Omega\}$, then G_0 has at least one vertex of degree 4, which is not possible. So, without loss of generality we may assume that $\Omega \notin \mathscr{N}[G_0]$ and that $\{1, 2, 3, 4\} \in \mathscr{N}[G_0]$. In such a case, all subsets of cardinality 4 must be in $\mathscr{N}[G_0]$, because otherwise there exists $j \in \bigcap_{N \in \mathscr{N}[G_0]} N$, so $\deg_{G_0}(j) = 4$, which is a contradiction. Therefore $\mathscr{N}[G_0]$ contains all the subsets of cardinality 4. But this is not possible, because there is no graph of order 5 with all the vertices of degree 3.

At this point, using the previous four facts, we will prove that there exists a graph $G \in \mathscr{C}_5$ such that $\mathscr{D}(G) \leqslant^+ \mathscr{D}(G_0)$.

We distinguish three cases:

- $\mathscr{N}[G_0]$ has exactly one element of size 3;
- $\mathscr{N}[G_0]$ has at least two elements X and Y of size 3 with $|X \cap Y| = 2$; and
- $\mathscr{N}[G_0]$ has at least two elements X and Y of size 3 with $|X \cap Y| = 1$.

First suppose that $\mathscr{N}[G_0]$ has exactly one element of size 3. Hence, by (i) and (ii), it follows that the remaining elements of $\mathscr{N}[G_0]$ have size 4. We may assume that $\{1, 2, 3\} \in \mathscr{N}[G_0]$ and $N_{G_0}[1] = \{1, 2, 3\}$. In such a case:

$$\mathscr{N}[G_0] \subseteq \{\{1, 2, 3\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\}.$$

As $4,5 \notin N_{G_0}[1]$, we have that $1 \notin N_{G_0}[4]$ and $1 \notin N_{G_0}[5]$. Thus, $N_{G_0}[4] = N_{G_0}[5] = \{2,3,4,5\}$. Hence $\{\{1,2\},\{1,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\} \subseteq E(G_0)$. Therefore, G_0 contains a subgraph G that is isomorphic to the cycle C_5 and, by Lemma 4.18, $\mathscr{D}(G) \leqslant^+ \mathscr{D}(G_0)$.

Next suppose that $\mathscr{N}[G_0]$ has at least two elements X and Y of size $|X| = |Y| = 3$ with $|X \cap Y| = 2$. Without loss of generality we may assume that $X = \{1,2,3\}$ and that $Y = \{1,2,5\}$. Since $X, Y \in \mathscr{N}[G_0]$, and since $\{4,5\} \cap X = \emptyset$ and $\{3,4\} \cap Y = \emptyset$, we get that $\{4,5\},\{3,4\} \notin \mathrm{Tr}(\mathscr{N}[G_0]) = \mathscr{D}(G_0)$. Therefore $\{4,5\},\{3,4\} \notin \mathscr{H}$ and thus, from (iii), we conclude that $\{4,5\},\{3,4\} \in E(G_0)$. Hence it follows that $\{\{1,2,3\},\{1,2,5\}\} = \{N_{G_0}[1], N_{G_0}[2]\}$. By symmetry, we may assume that $N_{G_0}[1] = \{1,2,3\}$ and that $N_{G_0}[2] = \{1,2,5\}$. In such a case, $\{\{1,2\},\{1,3\},\{2,5\},\{4,5\},\{3,4\}\} \subseteq E(G_0)$. Hence, G_0 contains a subgraph G that is isomorphic to the cycle C_5 and, by Lemma 4.18, $\mathscr{D}(G) \leqslant^+ \mathscr{D}(G_0)$.

Finally, suppose that $\mathscr{N}[G_0]$ has at least two elements X and Y of size $|X| = |Y| = 3$ with $|X \cap Y| = 1$. Without loss of generality we may assume that $X = \{1,2,3\}$ and that $Y = \{3,4,5\}$. Reasoning as in the preceding case, $\{4,5\}$ and $\{1,2\}$ belong to $E(G_0)$. If $N_{G_0}[3] = \{1,2,3\}$, then $\{3,4,5\}$ must be $N_{G_0}[4]$ or $N_{G_0}[5]$, obtaining respectively that $\{3,4\} \in E(G_0)$ or that $\{3,5\} \in E(G_0)$. So, if $N_{G_0}[3] = \{1,2,3\}$, then we get that $4 \in N_{G_0}[3]$ or that $5 \in N_{G_0}[3]$, a contradiction. Therefore we conclude that $N_{G_0}[3] \neq \{1,2,3\}$ and, by symmetry, we get that $N_{G_0}[3] \neq \{3,4,5\}$. Hence, without loss of generality we may assume that $N_{G_0}[1] = \{1,2,3\}$ and that $N_{G_0}[4] = \{3,4,5\}$. At this point recall that the intersection of the elements of $\mathscr{N}[G_0]$ is empty (because otherwise there would be a vertex u of degree 4, which is not possible by (i)). Set $Z \in \mathscr{N}[G_0]$ such that $3 \notin Z$. If $|Z| = 3$, then either $|X \cap Z| = 2$ or $|Y \cap Z| = 2$, and we proceed as in the preceding case. If $|Z| \neq 3$, then $Z = \{1,2,4,5\}$, and so Z is $N_{G_0}[2]$ or $N_{G_0}[5]$. In any case, $\{2,5\} \in E(G_0)$. Therefore, $\{\{1,2\},\{1,3\},\{2,5\},\{3,4\},\{4,5\}\} \subseteq E(G_0)$. So, G_0 contains a subgraph G that is isomorphic to the cycle C_5 and, by Lemma 4.18, $\mathscr{D}(G) \leqslant^+ \mathscr{D}(G_0)$. □

Proposition 4.33 *If $\mathscr{H} \in \mathrm{DomHyp}^+_{0,u}(\Omega, \mathscr{U}_{3,\Omega})$, then there exists a graph $G \in \mathscr{C}_5 \cup \mathscr{K}_{2,3}$ such that $\mathscr{U}_{3,\Omega} \leqslant^+ \mathscr{D}(G) \leqslant^+ \mathscr{H}$.*

Proof. Let $\mathscr{H}_0 = \mathscr{D}(G_0) \in \mathrm{MinDomHyp}^+_{0,u}(\Omega, \mathscr{U}_{3,\Omega})$ such that $\mathscr{U}_{3,\Omega} \leqslant^+ \mathscr{H}_0 \leqslant^+ \mathscr{H}$. By Lemmas 4.31 and 4.32, it is enough to show that $\mathscr{H}_0$ has either an element of size 3 or all its elements have size 2. Let us prove it.

First observe that for all $A \in \mathscr{H}_0$, we have $|A| \leq 3$. Indeed, suppose on the contrary that there exists $A \in \mathscr{H}_0$ such that $|A| \geq 4$. If $\{a,b,c,d\} \subseteq A$, then $\mathscr{H}_0$ does not contain any subset of $\{a,b,c\} \in \mathscr{U}_{3,\Omega}$, contradicting that $\mathscr{U}_{3,\Omega} \leqslant^+ \mathscr{H}_0$.

From the above, it only remains to prove that, if $\mathscr{H}_0$ has no elements of size 3, then all its elements have size exactly 2. On the contrary, let us assume that there exists $A \in \mathscr{H}_0$ such that $|A| = 1$. We are going to prove that, in such a case, a contradiction is achieved.

Without loss of generality we may assume that $A = \{5\}$. Thus, $\deg_{G_0}(5) = 4$ because $A \in \mathscr{H} = \mathscr{D}(G_0)$. Let $G_1 = G_0 - 5$ be the graph obtained by deleting the vertex 5 from G_0.

It is clear that $G_0 = G_1 \vee K_{\{5\}}$. Since $\mathscr{U}_{3,\Omega} \leqslant^+ \mathscr{H}_0$, hence $\mathscr{U}_{3,\Omega} \leqslant^+ \mathscr{D}(G_1 \vee K_{\{5\}})$, and thus, by applying Lemma 4.12 it follows that:

$$\mathscr{U}_{3,\Omega\setminus\{5\}} = \mathscr{U}_{3,\Omega}[\Omega \setminus \{5\}] \leqslant^+ \mathscr{D}(G_1).$$

Let $\mathscr{D}(G')$ be a minimal domination completion of $\mathscr{U}_{3,\Omega\setminus\{5\}}$ such that:

$$\mathscr{U}_{3,\Omega\setminus\{5\}} \leqslant^+ \mathscr{D}(G') \leqslant^+ \mathscr{D}(G_1).$$

By using Lemma 2.22, it is is easy to check that:

$$\mathscr{U}_{3,\Omega} \leqslant^+ \mathscr{D}(G' \vee K_{\{5\}}) \leqslant^+ \mathscr{D}(G_1 \vee K_{\{5\}}).$$

Therefore, $\mathscr{U}_{3,\Omega} \leqslant^+ \mathscr{D}(G' \vee K_{\{5\}}) \leqslant^+ \mathscr{H}_0$. So, $\mathscr{H}_0 = \mathscr{D}(G' \vee K_{\{5\}})$ because $\mathscr{H}_0$ is an upper minimal domination completion of $\mathscr{U}_{3,\Omega}$ with ground set Ω. By Theorem 4.19 we may assume that G' is isomorphic to $K_{1,3}$ or to $2K_2$. If G' is isomorphic to $K_{1,3}$, then $\mathscr{H}_0 = \mathscr{D}(G' \vee K_{\{5\}}) = \{\{5\}\} \cup \mathscr{D}(G')$ has an element of size 3, which contradicts our assumption. If G' is isomorphic to $2K_2$, then by applying Proposition 4.22 we get that $\mathscr{D}(G') = \mathscr{D}(G'')$ where G'' is a path of order 4 obtained by joining a pair of vertices of different connected components in $2K_2$. Therefore:

$$\mathscr{H}_0 = \mathscr{D}(K_{\{5\}} \vee G') = \{\{5\}\} \cup \mathscr{D}(G') = \{\{5\}\} \cup \mathscr{D}(G'') = \mathscr{D}(K_{\{5\}} \vee G'').$$

But the graph $K_{\{5\}} \vee G''$ contains a spanning subgraph $G''' \in \mathscr{C}_5$. So, from Lemma 4.18 and Proposition 4.28, we have that:

$$\mathscr{U}_{3,\Omega} \leqslant^+ \mathscr{D}(G''') \leqslant^+ \mathscr{D}(K_{\{5\}} \vee G'') = \mathscr{H}_0.$$

Therefore, $\mathscr{H}_0 = \mathscr{D}(G''')$ because $\mathscr{H}_0$ is an upper minimal domination completion of $\mathscr{U}_{3,\Omega}$ with ground set Ω. This leads us to a contradiction because all the dominating sets of the graph G''' have size 2. □

APPENDIX A

SAGE CODE

```
### Domination Hypergraphs, v.2.1. J.L.Ruiz, 11/2016.

# Auxiliary functions
#####################

def toTuples(l):
    return tuple(map(tuple, l))

def toSets(l):
    return Set(map(Set, l))

def isSupersetOfSome(L, D):
    """
    Checks whether D contains some element of L
    """
    return exists(L, lambda E: Set(D).issuperset(Set(E)))[0]

def act(g, s):
    """
    Action of a permutation g on a set, a list or a tuple s.
    Returns a Set. g belongs to SymmetricGroup( s )
    """
    return Set([g(i) for i in s])

# Partial Orders for Hypergraphs
################################

def hgOrderP(H1, H2):
    """
    Partial Order Plus: checks whether for all h1 in H1,
    there exists h2 in H2 such that h2 is a subset of h1.
    """
    flags = dict(zip(map(tuple, H1), [false]*len(H1)))

    for h1 in H1:
        flags[tuple(h1)] = exists(H2, lambda h: set(h).issubset(set(h1)))[0]
```

```
    return all(flags.values())

def hgOrderM(H1, H2):
    """
    Partial Order Minus: checks whether for all h1 in H1,
    there exists h2 in H2 such that h1 is a subset of h2.
    """
    flags = dict(zip(map(tuple, H1), [false]*len(H1)))

    for h1 in H1:
        flags[tuple(h1)] = exists(H2, lambda h: set(h1).issubset(set(h)))[0]

    return all(flags.values())

# Functions related to domination
#################################

def isDominating(D, G):
    """
    Checks whether the set of vertices D of the graph G
    is a dominating set of vertices of G.
    """
    Dc = list( Set(G.vertices()) - Set(D) )
    flags = dict( zip(Dc, [false]*len(Dc)) )

    for v in Dc:
        flags[v] = exists(D, lambda u: G.has_edge(u,v))[0]

    return all(flags.values())

def minimalDominatingSets(G):
    """
    Computes the list of all minimal dominating sets
    of vertices of a graph G.
    """
    domlist = []

    for i in range(1, G.order()+1):
        for D in Subsets(G.vertices(), i):
            if (not isSupersetOfSome(domlist, list(D))) and \
                    isDominating(list(D), G):
                domlist.append( list(D) )

    return domlist

from sage.graphs.inpedendent_sets import IndependentSet

def maximalIndependentSets(G):
    """
    Returns the list of all maximal independent sets of a graph G.
    """
    return IndependentSet(G, maximal=True)
```

```
def DomHyp0(n):
    """
    Computes the set of dominating hypergraphs of graphs of order n
    with vertex set equal to {0,1,2,...,n-1}.
    """
    domset = Set([])
    result = Set([])

    for G in graphs(n):
        dset = toSets( minimalDominatingSets(G) )
        domset = domset.union( Set([dset]) )

    for h in domset:
        for g in SymmetricGroup( range(n) ):
            aux_set = Set( [Set(map(lambda s: act(g,s), h))] )
            result = result.union( aux_set )

    return result

def DomHyp(n):
    """
    Computes the set of dominating hypergraphs of all graphs up to
    order n with vertex set contained in {0,1,...,n-1}.
    """
    domsets = DomHyp0(n)
    X = Set(range(n))

    for k in range(1, n):
        domhyp0 = DomHyp0(k)
        for Y in X.subsets(k):
            dd = dict( zip(range(k), Y) )
            for g in SymmetricGroup( range(k) ):
                for dh in domhyp0:
                    l = toSets([ [dd[g(i)] for i in x] for x in dh])
                    domsets = domsets.union( Set([l]) )

    return domsets

def coDominatingGraphs(n):
    """
    Returns a dictionary with keys all the dominating hypergraphs
    of ALL graphs of order n and values the list of all graphs
    with that dominating hypergraph.
    """
    codomgr = dict()

    for G in graphs(n):
        D = toSets(minimalDominatingSets(G))
        if not codomgr.has_key(D):
            codomgr[D] = [G]
        else:
            codomgr[D] = codomgr[D] + [G]
        for s in SymmetricGroup( G.vertices() ):
            Ds = Set( map( lambda h: act(s, h), D ) )
```

```
            def f(i, j):
                return G.has_edge(s.inverse()(i), s.inverse()(j))
            Gs = Graph([G.vertices(), f])
            if not codomgr.has_key(Ds):
                codomgr[Ds] = [Gs]
            elif not (Gs in codomgr[Ds]):
                codomgr[Ds] = codomgr[Ds] + [Gs]

    return codomgr

# Posets of domination hypergraphs
##################################

def DomPoset(n, H, order="+", side="u", ground="yes"):
    """
    Returns the poset of domination completions of H,
    with respect to 'order', 'side', 'ground' indicated.
    Order: '+' (default) or '-'.
    Side: 'u' (default) or 'l'.
    Ground: 'yes' (default) or 'no'.
    """
    if order == "+":
        hgOrder = hgOrderP
    elif order == "-":
        hgOrder = hgOrderM
    else:
        hgOrder = hgOrderP
        print "Order must be + or -. Using +."

    if ground == "yes":
        domhyp = DomHyp0(n)
    elif ground == "no":
        domhyp = DomHyp(n)
    else:
        domhyp = DomHyp0(n)
        print "Ground must be yes or no. Using yes."

    if side == "u":
        hd = [D for D in domhyp if hgOrder(H,D)]
    elif side == "l":
        hd = [D for D in domhyp if hgOrder(D,H)]
    else:
        hd = [D for D in domhyp if hgOrder(H,D)]
        print "Side must be u or l. Using u."

    return Poset( (tuple(map(toTuples, hd)), hgOrder) )

def MinDomHyp(n, H, order="+", ground="yes"):
    """
    Returns the minimal domination completions of H
    with respect to the 'order' and 'ground' indicated.
    Order: '+' (default) or '-'.
    Ground: 'yes' (default) or 'no'.
    """
```

```
    str_order = order
    str_ground = ground
    ps = DomPoset(n, H, order=str_order, side="u", ground=str_ground)
    return ps.minimal_elements()

def MaxDomHyp(n, H, order="+", ground="yes"):
    """
    Returns the maximal domination hypergraphs of H
    with respect to the 'order' and 'ground' indicated.
    Order: '+' (default) or '-'.
    Ground: 'yes' (default) or 'no'.
    """
    str_order = order
    str_ground = ground
    ps = DomPoset(n, H, order=str_order, side="l", ground=str_ground)
    return ps.maximal_elements()

# Operations
############

def intersectionP(*hypers):
    """
    Computes the intersection 'plus' of the hypergraphs given.
    """
    if len(hypers) == 1:
        return hypers[0]
    elif len(hypers) == 2:
        H1, H2 = hypers
        H1 = toSets(H1)
        H2 = toSets(H2)
        J = Set(h1.union(h2) for h1 in H1 for h2 in H2)

        def f(A, B):
            return Set(A).issubset(Set(B))

        K = Poset( (toTuples(J), f) ).minimal_elements()
        return K
    else:
        return reduce(intersectionP, hypers)

def unionP(*hypers):
    """
    Computes the union 'plus' of the hypergraphs given.
    """
    if len(hypers) == 1:
        return hypers[0]
    elif len(hypers) == 2:
        H1, H2 = hypers
        H1 = toSets(H1)
        H2 = toSets(H2)
        J = H1.union(H2)

        def f(A, B):
            return Set(A).issubset(Set(B))
```

```
        K = Poset( (toTuples(J), f) ).minimal_elements()
        return K
    else:
        return reduce(unionP, hypers)

def intersectionM(*hypers):
    """
    Computes the intersection 'minus' of the hypergraphs given.
    """
    if len(hypers) == 1:
        return hypers[0]
    elif len(hypers) == 2:
        H1, H2 = hypers
        H1 = toSets(H1)
        H2 = toSets(H2)
        J = Set(h1.intersection(h2) for h1 in H1 for h2 in H2)

        def f(A, B):
            return Set(A).issubset(Set(B))

        K = Poset( (toTuples(J), f) ).maximal_elements()
        return K
    else:
        return reduce(intersectionM, hypers)

def unionM(*hypers):
    """
    Computes the union 'minus' of the hypergraphs given.
    """
    if len(hypers) == 1:
        return hypers[0]
    elif len(hypers) == 2:
        H1, H2 = hypers
        H1 = toSets(H1)
        H2 = toSets(H2)
        J = H1.union(H2)

        def f(A, B):
            return Set(A).issubset(Set(B))

        K = Poset( (toTuples(J), f) ).maximal_elements()
        return K
    else:
        return reduce(unionM, hypers)

def transversal(H):
    """
    Computes the usual transversal (or blocker) of the hypergraph H.
    """
    uH = [[x] for x in A for A in H]
    return intersectionP(*uH)
```

APPENDIX B

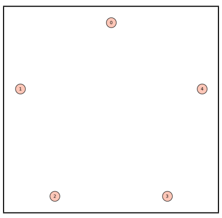
DOMINATION HYPERGRAPHS AND CODOMINATING GRAPHS OF ORDER 5

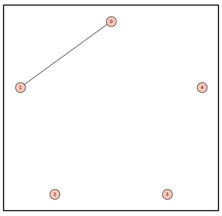
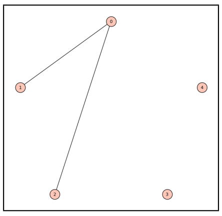
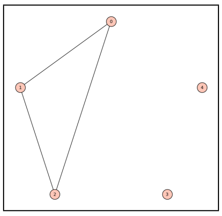
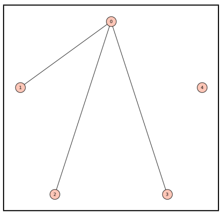
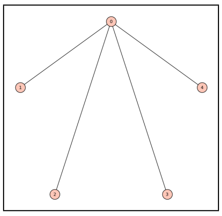
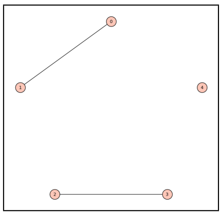
In this appendix we present the results of two calculations. First, in Section B.1 we list all graphs of order 5 up to isomorphism together with their independence-domination and domination hypergraphs. Second, in Section B.2 we list all domination hypergraphs with ground set $\Omega = \{0, 1, 2, 3, 4\}$ with more than one graph realization.

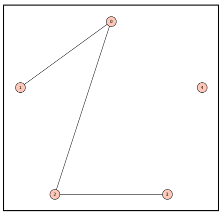
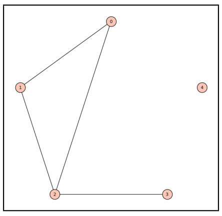
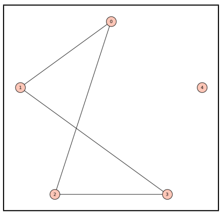
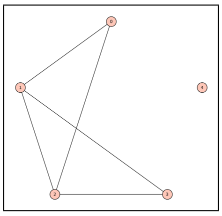
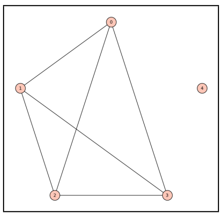
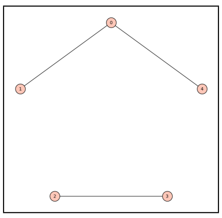
B.1 Domination hypergraphs of graphs of order 5

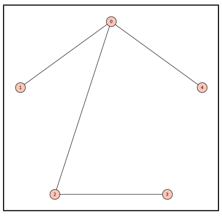
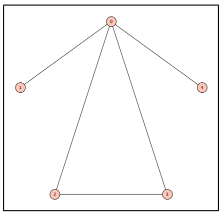
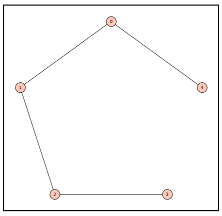
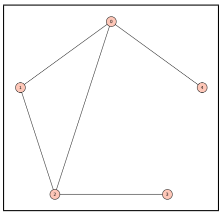
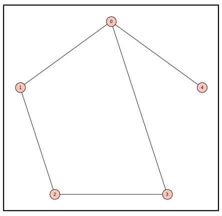
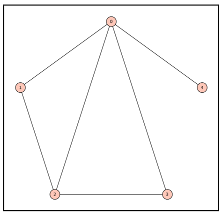
In this section we list the domination hypergraph and the independence-domination hypergraph of all graphs of order 5 up to isomorphism. In the second column of the following table we write first the maximal independent sets (on the first row) of the graph on the left and then the rest of the dominant sets (on the second row), it there are any. So whenever there is only a row in the table, we have $\mathscr{D}(G) = \mathscr{D}_{ind}(G)$ for the corresponding graph G.

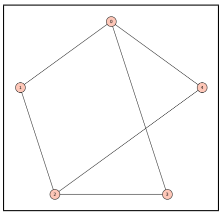
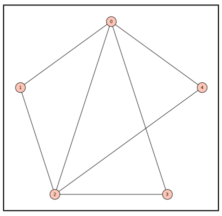
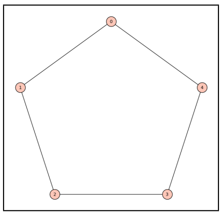
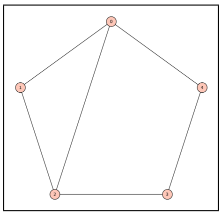
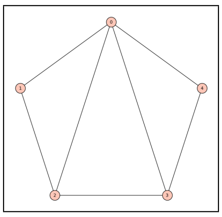
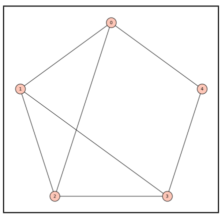
Observe that the vertex labels of the graph G run from 0 to 4 counterclockwise starting from the upper vertex.

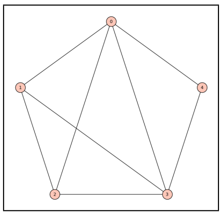
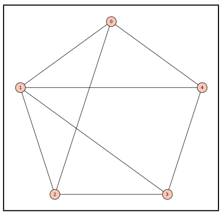
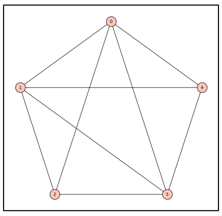
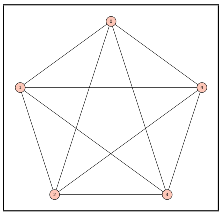
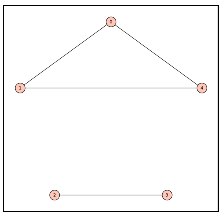
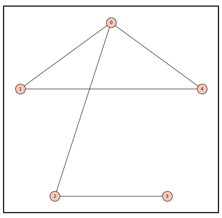
G	$\mathscr{D}_{ind}(G)$ (first line) and $\mathscr{D}(G)$ (first and second line)
	$\{0, 1, 2, 3, 4\}$

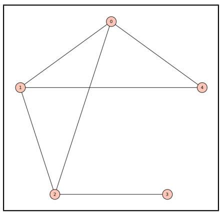
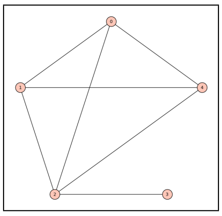
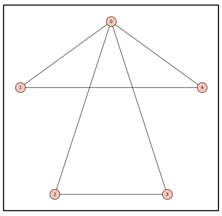
G	$\mathscr{D}_{ind}(G)$ (first line) and $\mathscr{D}(G)$ (first and second line)
	$\{1,2,3,4\}, \{0,2,3,4\}$
	$\{1,2,3,4\}, \{0,3,4\}$
	$\{1,3,4\}, \{0,3,4\}, \{2,3,4\}$
	$\{1,2,3,4\}, \{0,4\}$
	$\{1,2,3,4\}, \{0\}$
	$\{1,3,4\}, \{0,3,4\}, \{0,2,4\}, \{1,2,4\}$

G	$\mathscr{D}_{ind}(G)$ (first line) and $\mathscr{D}(G)$ (first and second line)
	$\{1,3,4\},\{0,3,4\},\{1,2,4\}$ $\{0,2,4\}$
	$\{2,4\},\{1,3,4\},\{0,3,4\}$
	$\{0,3,4\},\{1,2,4\}$ $\{0,1,4\},\{1,3,4\},\{2,3,4\},\{0,2,4\}$
	$\{2,4\},\{0,3,4\},\{1,4\}$
	$\{2,4\},\{3,4\},\{0,4\},\{1,4\}$
	$\{1,3,4\},\{0,2\},\{0,3\},\{1,2,4\}$

G	$\mathscr{D}_{ind}(G)$ (first line) and $\mathscr{D}(G)$ (first and second line)
	$\{1,3,4\},\{0,3\},\{1,2,4\}$ $\{0,2\}$
	$\{1,3,4\},\{0\},\{1,2,4\}$
	$\{2,4\},\{1,3,4\},\{0,2\},\{0,3\}$
	$\{2,4\},\{1,3,4\},\{0,3\}$ $\{0,2\}$
	$\{2,4\},\{1,3,4\},\{0,2\}$ $\{0,3\},\{0,1\}$
	$\{2,4\},\{1,3,4\},\{0\}$

G	$\mathscr{D}_{ind}(G)$ (first line) and $\mathscr{D}(G)$ (first and second line)
	$\{1,3,4\},\{0,2\}$ $\{2,4\},\{2,3\},\{1,2\},\{0,4\},\{0,3\},\{0,1\}$
	$\{1,3,4\},\{2\},\{0\}$
	$\{1,3\},\{2,4\},\{1,4\},\{0,2\},\{0,3\}$
	$\{1,3\},\{2,4\},\{1,4\},\{0,3\}$ $\{0,4\},\{2,3\},\{0,2\}$
	$\{1,3\},\{2,4\},\{1,4\},\{0\}$ $\{2,3\}$
	$\{2,4\},\{1,4\},\{0,3\}$ $\{3,4\},\{2,3\},\{0,2\},\{1,3\},\{0,4\},\{0,1\}$

G	$\mathscr{D}_{ind}(G)$ (first line) and $\mathscr{D}(G)$ (first and second line)
	$\{2,4\},\{1,4\},\{3\},\{0\}$
	$\{2,4\},\{0,3\},\{1\}$ $\{3,4\},\{0,4\},\{2,3\},\{0,2\}$
	$\{2,4\},\{3\},\{0\},\{1\}$
	$\{4\},\{2\},\{3\},\{0\},\{1\}$
	$\{2,4\},\{3,4\},\{0,2\},\{1,3\},\{1,2\},\{0,3\}$
	$\{1,3\},\{1,2\},\{2,4\},\{0,3\},\{3,4\}$ $\{0,2\}$

G	$\mathscr{D}_{ind}(G)$ (first line) and $\mathscr{D}(G)$ (first and second line)
	$\{1,3\},\{2,4\},\{0,3\},\{3,4\}$ $\{1,2\},\{0,2\}$
	$\{1,3\},\{3,4\},\{2\},\{0,3\}$
	$\{1,3\},\{1,2\},\{2,4\},\{0\},\{3,4\}$

B.2 Codominating hypergraphs of order 5

In this section we focus on domination hypergraphs with ground set Ω of size $|\Omega| = 5$ and their different graph realizations. Computations done with SAGE yield the following results:

- there are 414 different domination hypergraphs $\mathscr{H}$ with ground set Ω with exactly one graph realization; that is, such that there is only one graph G with vertex set Ω and $\mathscr{H} = \mathscr{D}(G)$;
- there are 90 different domination hypergraphs with ground set Ω and exactly two graph realizations;
- there are 50 different domination hypergraphs with ground set Ω and exactly three graph realizations;
- there are 30 different domination hypergraphs with ground set Ω and exactly five graph realizations; and

- there are 10 different domination hypergraphs with ground set Ω and exactly thirteen graph realizations.

In the following tables we list only the domination hypergraphs with ground set Ω that have more than one graph realization together with these graphs realizations.

Domination hypergraphs of order 5 with two graph realizations

Hypergraph	Graph realizations (edge sets)
$\{\{1,2\},\{1,3,4\},\{0,2\},\{0,3\}\}$	$\{\{2,4\},\{0,4\},\{2,3\},\{0,1\}\}$
	$\{\{2,4\},\{0,4\},\{2,3\},\{0,2\},\{0,1\}\}$
$\{\{3,4\},\{0,2\},\{0,3\},\{1,2,4\}\}$	$\{\{1,3\},\{0,4\},\{2,3\},\{0,1\}\}$
	$\{\{1,3\},\{0,4\},\{2,3\},\{0,3\},\{0,1\}\}$
$\{\{2,4\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{1,4\},\{0,3\}\}$	$\{\{2,4\},\{3,4\},\{2,3\},\{1,2\},\{0,4\},\{0,1\}\}$
	$\{\{2,4\},\{3,4\},\{2,3\},\{0,2\},\{1,4\},\{0,1\}\}$
$\{\{0,2\},\{0,4\},\{1,4\},\{1,2,3\}\}$	$\{\{2,4\},\{3,4\},\{0,3\},\{0,1\}\}$
	$\{\{2,4\},\{3,4\},\{0,4\},\{0,3\},\{0,1\}\}$
$\{\{3,4\},\{2,3\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{1,4\}\}$	$\{\{2,4\},\{3,4\},\{1,3\},\{1,2\},\{0,3\},\{0,1\}\}$
	$\{\{2,4\},\{2,3\},\{1,3\},\{1,4\},\{0,3\},\{0,1\}\}$
$\{\{2,4\},\{2,3\},\{0,2\},\{1,3\},\{1,4\},\{0,3\},\{0,1\}\}$	$\{\{3,4\},\{2,3\},\{1,2\},\{0,4\},\{0,3\},\{0,1\}\}$
	$\{\{3,4\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{0,3\}\}$
$\{\{2,4\},\{3,4\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{0,1\}\}$	$\{\{2,3\},\{0,2\},\{1,2\},\{0,4\},\{1,4\},\{0,3\}\}$
	$\{\{2,4\},\{2,3\},\{0,2\},\{1,4\},\{0,3\},\{0,1\}\}$
$\{\{2,4\},\{2,3\},\{1,3\},\{0,4\},\{1,4\},\{0,3\},\{0,1\}\}$	$\{\{3,4\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{0,1\}\}$
	$\{\{3,4\},\{0,2\},\{1,2\},\{1,4\},\{0,3\},\{0,1\}\}$
$\{\{1,2\},\{0,2,3\},\{1,4\},\{3,4\}\}$	$\{\{1,3\},\{2,4\},\{0,4\},\{0,1\}\}$
	$\{\{1,3\},\{2,4\},\{0,4\},\{1,4\},\{0,1\}\}$
$\{\{1,2\},\{2,3\},\{0,3\},\{0,1,4\}\}$	$\{\{1,3\},\{2,4\},\{0,2\},\{3,4\}\}$
	$\{\{1,3\},\{2,4\},\{2,3\},\{0,2\},\{3,4\}\}$
$\{\{3,4\},\{0,4\},\{1,2,3\},\{0,1\}\}$	$\{\{2,4\},\{1,4\},\{0,2\},\{0,3\}\}$
	$\{\{2,4\},\{0,4\},\{1,4\},\{0,2\},\{0,3\}\}$
$\{\{2,4\},\{3,4\},\{0,2\},\{1,3\},\{1,4\},\{0,3\},\{0,1\}\}$	$\{\{3,4\},\{2,3\},\{1,3\},\{1,2\},\{0,4\},\{0,1\}\}$
	$\{\{2,3\},\{1,3\},\{1,2\},\{0,4\},\{1,4\},\{0,3\}\}$
$\{\{2,4\},\{1,2\},\{0,3,4\},\{1,3\}\}$	$\{\{0,1\},\{2,3\},\{0,2\},\{1,4\}\}$
	$\{\{0,1\},\{1,2\},\{1,4\},\{0,2\},\{2,3\}\}$
$\{\{2,4\},\{2,3\},\{0,2\},\{1,3\},\{0,4\},\{1,4\},\{0,3\}\}$	$\{\{3,4\},\{2,3\},\{0,2\},\{1,2\},\{0,4\},\{0,1\}\}$
	$\{\{2,4\},\{3,4\},\{0,2\},\{1,2\},\{0,3\},\{0,1\}\}$
$\{\{1,3\},\{1,4\},\{0,3\},\{0,2,4\}\}$	$\{\{1,2\},\{2,3\},\{0,1\},\{3,4\}\}$
	$\{\{1,3\},\{1,2\},\{2,3\},\{0,1\},\{3,4\}\}$

Hypergraph	Graph realizations (edge sets)
{{2,4},{3,4},{0,2},{1,2},{1,4},{0,3},{0,1}}	{{2,4},{2,3},{1,3},{1,2},{0,4},{0,1}}
	{{2,3},{0,2},{1,3},{1,2},{0,4},{1,4}}
{{2,4},{3,4},{2,3},{0,2},{1,3},{0,4},{0,1}}	{{2,4},{2,3},{1,2},{0,4},{1,4},{0,3}}
	{{2,4},{3,4},{0,2},{1,2},{1,4},{0,3}}
{{1,3},{1,2},{0,3},{0,2,4}}	{{3,4},{2,3},{0,1},{1,4}}
	{{1,3},{3,4},{2,3},{0,1},{1,4}}
{{2,4},{3,4},{1,3},{1,2},{0,4},{0,3},{0,1}}	{{3,4},{2,3},{0,2},{1,4},{0,3},{0,1}}
	{{2,3},{0,2},{1,3},{0,4},{1,4},{0,3}}
{{1,3},{0,2,3},{0,4},{1,4}}	{{2,4},{1,2},{0,1},{3,4}}
	{{2,4},{1,2},{1,4},{0,1},{3,4}}
{{1,3},{2,3},{0,2,4},{1,4}}	{{1,2},{0,1},{0,3},{3,4}}
	{{1,3},{1,2},{0,1},{0,3},{3,4}}
{{2,4},{3,4},{0,2},{1,3},{1,2},{1,4},{0,3}}	{{3,4},{2,3},{1,2},{0,4},{1,4},{0,1}}
	{{2,4},{2,3},{1,3},{0,4},{1,4},{0,1}}
{{2,4},{2,3},{1,3},{1,2},{0,4},{1,4},{0,3}}	{{3,4},{2,3},{0,2},{1,2},{1,4},{0,1}}
	{{2,4},{3,4},{0,2},{1,3},{1,2},{0,1}}
{{2,4},{3,4},{2,3},{0,2},{1,4},{0,3},{0,1}}	{{3,4},{2,3},{0,2},{1,3},{1,2},{0,4}}
	{{2,4},{2,3},{1,3},{1,2},{0,4},{0,3}}
{{2,4},{2,3},{0,1,3},{1,4}}	{{1,2},{0,4},{0,2},{3,4}}
	{{2,4},{1,2},{0,4},{0,2},{3,4}}
{{1,2},{0,4},{1,4},{0,2,3}}	{{1,3},{2,4},{0,1},{3,4}}
	{{1,3},{2,4},{1,4},{0,1},{3,4}}
{{2,4},{0,4},{2,3},{0,1,3}}	{{1,2},{1,4},{0,2},{3,4}}
	{{2,4},{1,2},{1,4},{0,2},{3,4}}
{{2,4},{3,4},{0,2},{1,3},{1,2},{0,4},{0,3}}	{{3,4},{2,3},{0,2},{0,4},{1,4},{0,1}}
	{{2,4},{2,3},{0,4},{1,4},{0,3},{0,1}}
{{1,2},{2,3},{0,1,4},{3,4}}	{{1,3},{2,4},{0,2},{0,3}}
	{{1,3},{2,4},{2,3},{0,2},{0,3}}
{{1,2},{0,3,4},{0,2},{1,4}}	{{1,3},{2,4},{2,3},{0,1}}
	{{1,3},{1,2},{2,3},{2,4},{0,1}}
{{1,2},{0,4},{2,3,4},{0,1}}	{{1,3},{1,4},{0,2},{0,3}}
	{{1,3},{1,4},{0,2},{0,3},{0,1}}
{{0,1},{2,3},{0,3},{1,2,4}}	{{1,3},{3,4},{0,4},{0,2}}
	{{1,3},{3,4},{0,4},{0,2},{0,3}}
{{2,4},{1,3,4},{0,2},{0,1}}	{{1,2},{0,4},{2,3},{0,3}}
	{{1,2},{0,4},{2,3},{0,2},{0,3}}
{{1,3},{0,2},{0,3},{1,2,4}}	{{3,4},{0,4},{2,3},{0,1}}

Hypergraph	Graph realizations (edge sets)
	{{3, 4}, {0, 4}, {2, 3}, {0, 3}, {0, 1}}
{{3, 4}, {2, 3}, {0, 2}, {1, 2}, {0, 4}, {1, 4}, {0, 1}}	{{2, 4}, {1, 3}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {0, 2}, {1, 3}, {1, 4}, {0, 3}, {0, 1}}
{{2, 3}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}	{{2, 4}, {3, 4}, {2, 3}, {1, 2}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 1}}
{{1, 4}, {0, 2}, {2, 3, 4}, {0, 1}}	{{1, 3}, {1, 2}, {0, 4}, {0, 3}}
	{{1, 3}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}
{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {0, 1}}	{{2, 4}, {2, 3}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {0, 2}, {1, 3}, {1, 2}, {1, 4}, {0, 3}}
{{2, 4}, {3, 4}, {2, 3}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {1, 2}, {0, 4}}
	{{2, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 4}, {1, 4}}
{{1, 3}, {0, 4}, {2, 3, 4}, {0, 1}}	{{1, 2}, {1, 4}, {0, 2}, {0, 3}}
	{{1, 2}, {1, 4}, {0, 2}, {0, 3}, {0, 1}}
{{2, 4}, {2, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}	{{3, 4}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {1, 4}}
	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {1, 4}, {0, 1}}
{{1, 3, 4}, {2, 3}, {0, 2}, {0, 1}}	{{2, 4}, {1, 2}, {0, 4}, {0, 3}}
	{{2, 4}, {1, 2}, {0, 4}, {0, 2}, {0, 3}}
{{3, 4}, {2, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {0, 4}, {0, 1}}
	{{2, 4}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 3}}
{{3, 4}, {2, 3}, {0, 2}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}	{{2, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 4}, {0, 1}}
	{{2, 4}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {0, 3}}
{{0, 1}, {1, 4}, {2, 3, 4}, {0, 3}}	{{1, 3}, {1, 2}, {0, 4}, {0, 2}}
	{{1, 3}, {1, 2}, {0, 4}, {0, 2}, {0, 1}}
{{2, 4}, {0, 1, 4}, {2, 3}, {0, 3}}	{{1, 3}, {1, 2}, {0, 2}, {3, 4}}
	{{1, 3}, {1, 2}, {2, 3}, {0, 2}, {3, 4}}
{{1, 2}, {0, 3}, {2, 3, 4}, {0, 1}}	{{1, 3}, {0, 4}, {1, 4}, {0, 2}}
	{{1, 3}, {0, 4}, {1, 4}, {0, 2}, {0, 1}}
{{1, 3}, {0, 2}, {2, 3, 4}, {0, 1}}	{{1, 2}, {0, 4}, {1, 4}, {0, 3}}
	{{1, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
{{1, 2}, {0, 3, 4}, {2, 3}, {0, 1}}	{{1, 3}, {2, 4}, {1, 4}, {0, 2}}
	{{1, 3}, {1, 2}, {1, 4}, {0, 2}, {2, 4}}
{{0, 2}, {0, 4}, {1, 2, 3}, {3, 4}}	{{2, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
{{2, 3}, {0, 1, 2}, {3, 4}, {1, 4}}	{{1, 3}, {2, 4}, {0, 4}, {0, 3}}
	{{1, 3}, {2, 4}, {0, 4}, {0, 3}, {3, 4}}
{{1, 3}, {2, 3}, {0, 2, 4}, {0, 1}}	{{1, 2}, {1, 4}, {0, 3}, {3, 4}}
	{{1, 3}, {1, 2}, {1, 4}, {0, 3}, {3, 4}}

Hypergraph	Graph realizations (edge sets)
{{1, 3}, {0, 4}, {0, 3}, {1, 2, 4}}	{{3, 4}, {2, 3}, {0, 2}, {0, 1}}
	{{3, 4}, {2, 3}, {0, 2}, {0, 3}, {0, 1}}
{{3, 4}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}	{{2, 4}, {3, 4}, {2, 3}, {1, 3}, {0, 4}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {1, 4}, {0, 3}, {0, 1}}
{{1, 3}, {3, 4}, {0, 2, 4}, {1, 2}}	{{0, 1}, {2, 3}, {0, 3}, {1, 4}}
	{{1, 3}, {0, 1}, {2, 3}, {0, 3}, {1, 4}}
{{2, 4}, {1, 2}, {0, 4}, {0, 1, 3}}	{{3, 4}, {2, 3}, {0, 2}, {1, 4}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 4}}
{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 4}, {0, 1}}	{{3, 4}, {2, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}
	{{2, 4}, {3, 4}, {1, 3}, {1, 2}, {0, 4}, {0, 3}}
{{3, 4}, {1, 2, 4}, {0, 3}, {0, 1}}	{{1, 3}, {0, 4}, {2, 3}, {0, 2}}
	{{1, 3}, {0, 4}, {2, 3}, {0, 2}, {0, 3}}
{{2, 4}, {3, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 4}, {0, 1}}	{{3, 4}, {2, 3}, {0, 2}, {1, 2}, {1, 4}, {0, 3}}
	{{2, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 4}, {0, 3}}
{{2, 3}, {0, 4}, {0, 1, 2}, {3, 4}}	{{1, 3}, {2, 4}, {1, 4}, {0, 3}}
	{{1, 3}, {2, 4}, {1, 4}, {0, 3}, {3, 4}}
{{2, 4}, {3, 4}, {0, 1, 2}, {1, 3}}	{{0, 4}, {2, 3}, {0, 3}, {1, 4}}
	{{3, 4}, {0, 4}, {2, 3}, {0, 3}, {1, 4}}
{{2, 4}, {1, 2}, {0, 3, 4}, {0, 1}}	{{1, 3}, {2, 3}, {0, 2}, {1, 4}}
	{{1, 3}, {1, 2}, {1, 4}, {0, 2}, {2, 3}}
{{2, 4}, {3, 4}, {0, 2}, {0, 1, 3}}	{{1, 2}, {0, 4}, {1, 4}, {2, 3}}
	{{2, 4}, {1, 2}, {0, 4}, {1, 4}, {2, 3}}
{{2, 4}, {1, 2}, {0, 1, 3}, {3, 4}}	{{0, 4}, {2, 3}, {0, 2}, {1, 4}}
	{{2, 4}, {0, 4}, {2, 3}, {0, 2}, {1, 4}}
{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {1, 2}, {0, 3}, {0, 1}}	{{2, 3}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 1}}
	{{2, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
{{2, 4}, {1, 4}, {0, 2}, {0, 1, 3}}	{{1, 2}, {0, 4}, {2, 3}, {3, 4}}
	{{2, 4}, {1, 2}, {0, 4}, {2, 3}, {3, 4}}
{{1, 3}, {3, 4}, {0, 4}, {0, 1, 2}}	{{2, 4}, {2, 3}, {0, 3}, {1, 4}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 3}, {1, 4}}
{{2, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}	{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 4}, {0, 1}}
	{{3, 4}, {0, 2}, {1, 3}, {1, 2}, {1, 4}, {0, 3}}
{{1, 2}, {1, 3, 4}, {0, 2}, {0, 4}}	{{2, 4}, {2, 3}, {0, 3}, {0, 1}}
	{{2, 4}, {2, 3}, {0, 2}, {0, 3}, {0, 1}}
{{3, 4}, {2, 3}, {0, 2}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}	{{2, 4}, {2, 3}, {1, 3}, {0, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {0, 3}, {0, 1}}
{{1, 3}, {0, 2, 3}, {1, 4}, {2, 4}}	{{1, 2}, {0, 4}, {0, 1}, {3, 4}}

Hypergraph	Graph realizations (edge sets)
	{{1, 2}, {0, 4}, {1, 4}, {0, 1}, {3, 4}}
{{0, 4}, {2, 3}, {0, 3}, {1, 2, 4}}	{{1, 3}, {3, 4}, {0, 2}, {0, 1}}
	{{1, 3}, {3, 4}, {0, 2}, {0, 3}, {0, 1}}
{{2, 4}, {3, 4}, {2, 3}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}	{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 4}, {1, 4}}
	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {1, 4}, {0, 3}}
{{2, 4}, {0, 2, 3}, {1, 4}, {0, 1}}	{{1, 3}, {1, 2}, {0, 4}, {3, 4}}
	{{1, 3}, {1, 2}, {0, 4}, {1, 4}, {3, 4}}
{{1, 3}, {1, 2}, {0, 3, 4}, {0, 2}}	{{2, 4}, {0, 1}, {2, 3}, {1, 4}}
	{{2, 4}, {1, 2}, {1, 4}, {0, 1}, {2, 3}}
{{2, 4}, {1, 3, 4}, {0, 2}, {0, 3}}	{{1, 2}, {0, 4}, {2, 3}, {0, 1}}
	{{1, 2}, {0, 4}, {2, 3}, {0, 2}, {0, 1}}
{{3, 4}, {0, 1, 2}, {0, 3}, {1, 4}}	{{1, 3}, {2, 4}, {0, 4}, {2, 3}}
	{{1, 3}, {2, 4}, {0, 4}, {2, 3}, {3, 4}}
{{1, 3}, {0, 1, 4}, {2, 3}, {0, 2}}	{{2, 4}, {1, 2}, {0, 3}, {3, 4}}
	{{2, 4}, {1, 2}, {2, 3}, {0, 3}, {3, 4}}
{{1, 3, 4}, {2, 3}, {0, 2}, {0, 4}}	{{2, 4}, {1, 2}, {0, 3}, {0, 1}}
	{{2, 4}, {1, 2}, {0, 2}, {0, 3}, {0, 1}}
{{1, 2}, {0, 3, 4}, {1, 4}, {2, 3}}	{{1, 3}, {2, 4}, {0, 2}, {0, 1}}
	{{1, 3}, {1, 2}, {0, 2}, {2, 4}, {0, 1}}
{{0, 4}, {1, 4}, {1, 2, 3}, {0, 3}}	{{2, 4}, {3, 4}, {0, 2}, {0, 1}}
	{{2, 4}, {3, 4}, {0, 4}, {0, 2}, {0, 1}}
{{2, 4}, {3, 4}, {0, 1, 2}, {0, 3}}	{{1, 3}, {0, 4}, {2, 3}, {1, 4}}
	{{1, 3}, {3, 4}, {0, 4}, {2, 3}, {1, 4}}
{{1, 3}, {0, 1, 4}, {2, 3}, {2, 4}}	{{1, 2}, {0, 2}, {0, 3}, {3, 4}}
	{{1, 2}, {2, 3}, {0, 2}, {0, 3}, {3, 4}}
{{2, 4}, {0, 4}, {1, 2, 3}, {0, 1}}	{{3, 4}, {1, 4}, {0, 2}, {0, 3}}
	{{3, 4}, {0, 4}, {1, 4}, {0, 2}, {0, 3}}
{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 1}}	{{2, 4}, {3, 4}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}
{{0, 1, 4}, {2, 3}, {0, 2}, {3, 4}}	{{1, 3}, {1, 2}, {0, 3}, {2, 4}}
	{{1, 3}, {1, 2}, {2, 3}, {0, 3}, {2, 4}}
{{0, 2, 3}, {1, 4}, {0, 1}, {3, 4}}	{{1, 3}, {1, 2}, {0, 4}, {2, 4}}
	{{1, 3}, {1, 2}, {0, 4}, {1, 4}, {2, 4}}
{{2, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 1}}	{{3, 4}, {0, 2}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}
	{{2, 4}, {3, 4}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}
{{2, 4}, {0, 4}, {1, 2, 3}, {0, 3}}	{{3, 4}, {1, 4}, {0, 2}, {0, 1}}
	{{3, 4}, {0, 4}, {1, 4}, {0, 2}, {0, 1}}

Hypergraph	Graph realizations (edge sets)
{{1, 3}, {3, 4}, {0, 2, 4}, {0, 1}}	{{1, 2}, {1, 4}, {0, 3}, {2, 3}}
	{{1, 3}, {1, 2}, {1, 4}, {0, 3}, {2, 3}}

Domination hypergraphs of order 5 with three graph realizations

Hypergraph	Graph realizations (edge sets)
{{1, 3}, {0, 3}, {0, 2, 4}, {1, 2, 4}}	{{3, 4}, {2, 3}, {0, 1}}
	{{3, 4}, {2, 3}, {0, 3}, {0, 1}}
	{{1, 3}, {3, 4}, {2, 3}, {0, 1}}
{{2, 4}, {0, 2}, {3}, {1, 2}, {0, 4},	{{2, 4}, {1, 2}, {0, 4}, {3, 4}, {2, 3}, {1, 3}, {0, 3}, {0, 1}}
{1, 4}, {0, 1}}	{{1, 2}, {0, 4}, {1, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 3}}
	{{2, 4}, {1, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 3}, {0, 1}}
{{0, 4}, {1, 2, 3}, {2, 3, 4}, {0, 1}}	{{1, 4}, {0, 2}, {0, 3}}
	{{0, 4}, {1, 4}, {0, 2}, {0, 3}}
	{{1, 4}, {0, 2}, {0, 3}, {0, 1}}
{{1, 2}, {0, 4}, {1, 4}, {3, 4}, {2, 3},	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 4}, {0, 1}}
{0, 2}, {1, 3}, {0, 3}, {0, 1}}	{{2, 4}, {3, 4}, {2, 3}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}
{{0, 2, 3}, {1, 4}, {2, 3, 4}, {0, 1}}	{{1, 3}, {1, 2}, {0, 4}}
	{{1, 3}, {1, 2}, {0, 4}, {1, 4}}
	{{1, 3}, {1, 2}, {0, 4}, {0, 1}}
{{4}, {2, 3}, {0, 2}, {1, 3}, {1, 2},	{{2, 4}, {1, 2}, {0, 4}, {1, 4}, {3, 4}, {2, 3}, {0, 3}, {0, 1}}
{0, 3}, {0, 1}}	{{2, 4}, {0, 4}, {1, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 1}}
	{{2, 4}, {1, 2}, {0, 4}, {1, 4}, {3, 4}, {0, 2}, {1, 3}, {0, 3}}
{{3, 4}, {1, 3}, {0, 4}, {1, 4}, {2},	{{2, 4}, {1, 2}, {0, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 1}}
{0, 3}, {0, 1}}	{{2, 4}, {1, 2}, {1, 4}, {3, 4}, {2, 3}, {0, 2}, {0, 3}, {0, 1}}
	{{2, 4}, {1, 2}, {0, 4}, {1, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 3}}
{{2, 4}, {1, 2}, {0, 4}, {1, 4}, {3, 4},	{{2, 4}, {3, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 4}, {0, 1}}
{0, 2}, {1, 3}, {0, 3}, {0, 1}}	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 3}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}
{{1, 3, 4}, {0, 1, 3}, {0, 1, 4}, {0, 1, 2},	{{2, 4}, {3, 4}, {0, 2}, {0, 3}}
{1, 2, 3}, {1, 2, 4}}	{{2, 4}, {0, 4}, {2, 3}, {0, 3}}
	{{3, 4}, {0, 4}, {2, 3}, {0, 2}}
{{0, 2, 3}, {0, 4}, {1, 4}, {1, 2, 3}}	{{2, 4}, {3, 4}, {0, 1}}
	{{2, 4}, {3, 4}, {0, 4}, {0, 1}}
	{{2, 4}, {3, 4}, {1, 4}, {0, 1}}

Hypergraph	Graph realizations (edge sets)
{{2, 4}, {0, 4}, {1, 2, 3}, {0, 1, 3}}	{{3, 4}, {1, 4}, {0, 2}}
	{{3, 4}, {0, 4}, {1, 4}, {0, 2}}
	{{2, 4}, {3, 4}, {1, 4}, {0, 2}}
{{1, 3}, {3, 4}, {0, 1, 2}, {0, 2, 4}}	{{2, 3}, {0, 3}, {1, 4}}
	{{3, 4}, {2, 3}, {0, 3}, {1, 4}}
	{{1, 3}, {2, 3}, {0, 3}, {1, 4}}
{{0, 4}, {1, 2, 3}, {0, 3}, {1, 2, 4}}	{{3, 4}, {0, 2}, {0, 1}}
	{{3, 4}, {0, 2}, {0, 3}, {0, 1}}
	{{3, 4}, {0, 4}, {0, 2}, {0, 1}}
{{0, 1, 4}, {2, 3}, {0, 3}, {1, 2, 4}}	{{1, 3}, {3, 4}, {0, 2}}
	{{1, 3}, {3, 4}, {0, 2}, {0, 3}}
	{{1, 3}, {3, 4}, {2, 3}, {0, 2}}
{{2, 4}, {1, 2}, {0, 4}, {1, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 1}}	{{3, 4}, {2, 3}, {0, 2}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}
	{{2, 4}, {3, 4}, {1, 3}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 4}, {0, 3}, {0, 1}}
{{2, 4}, {1, 2}, {0, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 3}, {0, 1}}	{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 1}}
	{{2, 4}, {2, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {1, 2}, {1, 4}, {0, 3}}
{{2, 4}, {1, 2}, {0, 4}, {1, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 3}}	{{3, 4}, {2, 3}, {0, 2}, {1, 2}, {0, 4}, {1, 4}, {0, 1}}
	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {1, 2}, {0, 3}, {0, 1}}
	{{2, 4}, {2, 3}, {1, 3}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
{{1, 3, 4}, {0, 2}, {2, 3, 4}, {0, 1}}	{{1, 2}, {0, 4}, {0, 3}}
	{{1, 2}, {0, 4}, {0, 3}, {0, 1}}
	{{1, 2}, {0, 4}, {0, 2}, {0, 3}}
{{1, 3}, {2, 3, 4}, {0, 2, 4}, {0, 1}}	{{1, 2}, {1, 4}, {0, 3}}
	{{1, 2}, {1, 4}, {0, 3}, {0, 1}}
	{{1, 3}, {1, 2}, {1, 4}, {0, 3}}
{{2, 4}, {3, 4}, {2, 3}, {0}, {1, 3}, {1, 2}, {1, 4}}	{{1, 2}, {0, 4}, {1, 4}, {3, 4}, {2, 3}, {0, 2}, {0, 3}, {0, 1}}
	{{2, 4}, {1, 2}, {0, 4}, {3, 4}, {0, 2}, {1, 3}, {0, 3}, {0, 1}}
	{{2, 4}, {0, 4}, {1, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 3}, {0, 1}}
{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1}, {0, 4}, {0, 3}}	{{1, 2}, {0, 4}, {1, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 1}}
	{{2, 4}, {1, 2}, {1, 4}, {3, 4}, {0, 2}, {1, 3}, {0, 3}, {0, 1}}
	{{2, 4}, {1, 2}, {0, 4}, {1, 4}, {2, 3}, {1, 3}, {0, 3}, {0, 1}}
{{1, 2}, {0, 3, 4}, {2, 3, 4}, {0, 1}}	{{1, 3}, {1, 4}, {0, 2}}
	{{1, 3}, {1, 4}, {0, 2}, {0, 1}}
	{{1, 3}, {1, 2}, {1, 4}, {0, 2}}
{{2, 4}, {1, 2}, {0, 3, 4}, {0, 1, 3}}	{{2, 3}, {0, 2}, {1, 4}}

Hypergraph	Graph realizations (edge sets)
	{{2,4},{2,3},{0,2},{1,4}}
	{{1,2},{1,4},{0,2},{2,3}}
{{3,4},{0,4},{0,1,2},{1,2,3}}	{{2,4},{1,4},{0,3}}
	{{2,4},{0,4},{1,4},{0,3}}
	{{2,4},{3,4},{1,4},{0,3}}
{{1,3},{1,2},{0,3,4},{0,2,4}}	{{0,1},{2,3},{1,4}}
	{{1,2},{1,4},{0,1},{2,3}}
	{{1,3},{0,1},{2,3},{1,4}}
{{2,4},{0,2,3},{1,4},{0,1,3}}	{{1,2},{0,4},{3,4}}
	{{1,2},{0,4},{1,4},{3,4}}
	{{2,4},{1,2},{0,4},{3,4}}
{{0,1,4},{1,3,4},{2,3},{0,2}}	{{2,4},{1,2},{0,3}}
	{{2,4},{1,2},{0,2},{0,3}}
	{{2,4},{1,2},{2,3},{0,3}}
{{0,1,4},{2,3},{0,1,2},{3,4}}	{{1,3},{2,4},{0,3}}
	{{1,3},{2,4},{2,3},{0,3}}
	{{1,3},{2,4},{0,3},{3,4}}
{{1,2},{0,3,4},{2,3},{0,1,4}}	{{1,3},{2,4},{0,2}}
	{{1,3},{1,2},{0,2},{2,4}}
	{{1,3},{2,4},{2,3},{0,2}}
{{1,3,4},{0,3,4},{0,1,4},{2,3,4},	{{1,3},{2,3},{0,2},{0,1}}
{0,2,4},{1,2,4}}	{{1,2},{2,3},{0,3},{0,1}}
	{{1,3},{1,2},{0,2},{0,3}}
{{2,4},{1,2},{0,4},{1,4},{2,3},	{{2,4},{3,4},{2,3},{1,2},{0,4},{0,3},{0,1}}
{0,2},{1,3},{0,3},{0,1}}	{{3,4},{0,2},{1,3},{1,2},{0,4},{1,4},{0,3}}
	{{2,4},{3,4},{2,3},{0,2},{1,3},{1,4},{0,1}}
{{2,4},{1,3,4},{0,2},{0,1,3}}	{{1,2},{0,4},{2,3}}
	{{2,4},{1,2},{0,4},{2,3}}
	{{1,2},{0,4},{2,3},{0,2}}
{{1,2},{1,3,4},{0,3,4},{0,2}}	{{2,4},{2,3},{0,1}}
	{{2,4},{2,3},{0,2},{0,1}}
	{{2,4},{1,2},{2,3},{0,1}}
{{3,4},{0,1,2},{0,3},{1,2,4}}	{{1,3},{0,4},{2,3}}
	{{1,3},{3,4},{0,4},{2,3}}
	{{1,3},{0,4},{2,3},{0,3}}
{{0,2,3},{1,3,4},{0,3,4},{0,1,3},	{{2,4},{1,4},{0,2},{0,1}}
{1,2,3},{2,3,4}}	{{1,2},{0,4},{1,4},{0,2}}

Hypergraph	Graph realizations (edge sets)
	{{2, 4}, {1, 2}, {0, 4}, {0, 1}}
{{1, 3}, {0, 1, 4}, {2, 3}, {0, 2, 4}}	{{1, 2}, {0, 3}, {3, 4}}
	{{1, 3}, {1, 2}, {0, 3}, {3, 4}}
	{{1, 2}, {2, 3}, {0, 3}, {3, 4}}
{{1, 2}, {0, 3, 4}, {1, 4}, {0, 2, 3}}	{{1, 3}, {2, 4}, {0, 1}}
	{{1, 3}, {1, 2}, {2, 4}, {0, 1}}
	{{1, 3}, {2, 4}, {1, 4}, {0, 1}}
{{2, 4}, {3, 4}, {0, 1, 2}, {0, 1, 3}}	{{0, 4}, {2, 3}, {1, 4}}
	{{2, 4}, {0, 4}, {2, 3}, {1, 4}}
	{{3, 4}, {0, 4}, {2, 3}, {1, 4}}
{{2, 4}, {0, 1, 4}, {2, 3}, {0, 1, 3}}	{{1, 2}, {0, 2}, {3, 4}}
	{{1, 2}, {2, 3}, {0, 2}, {3, 4}}
	{{2, 4}, {1, 2}, {0, 2}, {3, 4}}
{{2, 4}, {1, 2}, {0, 4}, {1, 4}, {3, 4},	{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {1, 4}}
{2, 3}, {0, 2}, {0, 3}, {0, 1}}	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}
{{0, 2}, {1, 3, 4}, {1, 2, 3}, {0, 4}}	{{2, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {0, 2}, {0, 3}, {0, 1}}
	{{2, 4}, {0, 4}, {0, 3}, {0, 1}}
{{2, 4}, {0, 4}, {1, 4}, {3, 4}, {2, 3},	{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {0, 1}}
{0, 2}, {1, 3}, {0, 3}, {0, 1}}	{{2, 4}, {3, 4}, {0, 2}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}
{{0, 2, 3}, {1, 4}, {0, 1, 2}, {3, 4}}	{{1, 3}, {2, 4}, {0, 4}}
	{{1, 3}, {2, 4}, {0, 4}, {1, 4}}
	{{1, 3}, {2, 4}, {0, 4}, {3, 4}}
{{2, 4}, {1, 2}, {1, 4}, {3, 4}, {2, 3},	{{3, 4}, {2, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
{0, 2}, {1, 3}, {0, 3}, {0, 1}}	{{2, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 1}}
	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {0, 3}}
{{0, 1}, {2, 3, 4}, {0, 3}, {1, 2, 4}}	{{1, 3}, {0, 4}, {0, 2}}
	{{1, 3}, {0, 4}, {0, 2}, {0, 1}}
	{{1, 3}, {0, 4}, {0, 2}, {0, 3}}
{{0, 2, 3}, {0, 1, 2}, {1, 2, 3}, {2, 3, 4},	{{1, 3}, {0, 4}, {1, 4}, {0, 3}}
{0, 2, 4}, {1, 2, 4}}	{{3, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{1, 3}, {3, 4}, {0, 4}, {0, 1}}
{{0, 2, 3}, {0, 3, 4}, {0, 1, 3}, {0, 1, 4},	{{1, 3}, {1, 2}, {2, 4}, {3, 4}}
{0, 1, 2}, {0, 2, 4}}	{{1, 2}, {1, 4}, {2, 3}, {3, 4}}
	{{1, 3}, {2, 4}, {2, 3}, {1, 4}}

Hypergraph	Graph realizations (edge sets)
$\{\{1,3\},\{0,2,3\},\{1,4\},\{0,2,4\}\}$	$\{\{1,2\},\{0,1\},\{3,4\}\}$
	$\{\{1,3\},\{1,2\},\{0,1\},\{3,4\}\}$
	$\{\{1,2\},\{1,4\},\{0,1\},\{3,4\}\}$
$\{\{1,3,4\},\{0,2\},\{0,3\},\{1,2,4\}\}$	$\{\{0,4\},\{2,3\},\{0,1\}\}$
	$\{\{0,4\},\{2,3\},\{0,2\},\{0,1\}\}$
	$\{\{0,4\},\{2,3\},\{0,3\},\{0,1\}\}$
$\{\{2,4\},\{1,2\},\{0,4\},\{1,4\},\{3,4\},$ $\{2,3\},\{1,3\},\{0,3\},\{0,1\}\}$	$\{\{3,4\},\{2,3\},\{0,2\},\{1,2\},\{1,4\},\{0,3\},\{0,1\}\}$
	$\{\{2,4\},\{3,4\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{0,1\}\}$
	$\{\{2,4\},\{2,3\},\{0,2\},\{1,3\},\{0,4\},\{1,4\},\{0,3\}\}$

Domination hypergraphs of order 5 with five graph realizations

Hypergraph	Graph realizations (edge sets)
$\{\{0,2,3\},\{0,3,4\},\{0,1,2\},\{0,1,4\}\}$	$\{\{1,3\},\{2,4\}\}$
	$\{\{1,3\},\{2,4\},\{1,4\}\}$
	$\{\{1,3\},\{1,2\},\{2,4\}\}$
	$\{\{1,3\},\{2,4\},\{2,3\}\}$
	$\{\{1,3\},\{2,4\},\{3,4\}\}$
$\{\{1,3\},\{4\},\{2,3\},\{0,2\},\{0,1\}\}$	$\{\{2,4\},\{3,4\},\{1,2\},\{0,4\},\{1,4\},\{0,3\},\{0,1\}\}$
	$\{\{2,4\},\{3,4\},\{0,2\},\{1,2\},\{0,4\},\{1,4\},\{0,3\}\}$
	$\{\{2,4\},\{3,4\},\{2,3\},\{1,2\},\{0,4\},\{1,4\},\{0,3\}\}$
	$\{\{2,4\},\{3,4\},\{1,3\},\{1,2\},\{0,4\},\{1,4\},\{0,3\}\}$
	$\{\{2,4\},\{3,4\},\{1,2\},\{0,4\},\{1,4\},\{0,3\}\}$
$\{\{0,1,2\},\{0,1,3\},\{0,2,4\},\{0,3,4\}\}$	$\{\{2,3\},\{1,4\}\}$
	$\{\{2,4\},\{2,3\},\{1,4\}\}$
	$\{\{3,4\},\{2,3\},\{1,4\}\}$
	$\{\{1,2\},\{1,4\},\{2,3\}\}$
	$\{\{1,3\},\{2,3\},\{1,4\}\}$
$\{\{2,4\},\{0,4\},\{2,3\},\{0,3\},\{1\}\}$	$\{\{3,4\},\{2,3\},\{0,2\},\{1,3\},\{1,2\},\{1,4\},\{0,1\}\}$
	$\{\{3,4\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{1,4\},\{0,1\}\}$
	$\{\{2,4\},\{3,4\},\{0,2\},\{1,3\},\{1,2\},\{1,4\},\{0,1\}\}$
	$\{\{3,4\},\{0,2\},\{1,3\},\{1,2\},\{1,4\},\{0,3\},\{0,1\}\}$
	$\{\{3,4\},\{0,2\},\{1,3\},\{1,2\},\{1,4\},\{0,1\}\}$
$\{\{2,4\},\{1,4\},\{0,2\},\{3\},\{0,1\}\}$	$\{\{3,4\},\{2,3\},\{1,3\},\{1,2\},\{0,4\},\{0,3\},\{0,1\}\}$
	$\{\{3,4\},\{2,3\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{0,3\}\}$
	$\{\{3,4\},\{2,3\},\{1,3\},\{1,2\},\{0,4\},\{1,4\},\{0,3\}\}$
	$\{\{2,4\},\{3,4\},\{2,3\},\{1,3\},\{1,2\},\{0,4\},\{0,3\}\}$

Hypergraph	Graph realizations (edge sets)
	{{3, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 4}, {0, 3}}
{{1, 3, 4}, {0, 1, 2}, {0, 1, 3}, {1, 2, 4}}	{{0, 4}, {2, 3}}
	{{2, 4}, {0, 4}, {2, 3}}
	{{3, 4}, {0, 4}, {2, 3}}
	{{0, 4}, {2, 3}, {0, 2}}
	{{0, 4}, {2, 3}, {0, 3}}
{{2, 4}, {1, 2}, {0, 4}, {3}, {0, 1}}	{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 4}, {0, 3}, {0, 1}}
	{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 3}}
	{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 2}, {1, 4}, {0, 3}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 4}, {0, 3}}
	{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 4}, {0, 3}}
{{0, 2, 3}, {0, 1, 2}, {2, 3, 4}, {1, 2, 4}}	{{1, 3}, {0, 4}}
	{{1, 3}, {0, 4}, {1, 4}}
	{{1, 3}, {3, 4}, {0, 4}}
	{{1, 3}, {0, 4}, {0, 1}}
	{{1, 3}, {0, 4}, {0, 3}}
{{2, 4}, {3, 4}, {0, 2}, {0, 3}, {1}}	{{3, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 1}}
	{{2, 3}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 1}}
	{{2, 3}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 1}}
	{{2, 3}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 1}}
{{1, 3}, {2, 4}, {2, 3}, {0}, {1, 4}}	{{3, 4}, {2, 3}, {0, 2}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}
	{{3, 4}, {0, 2}, {1, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {0, 2}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}
	{{3, 4}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}
	{{3, 4}, {0, 2}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}
{{0, 1, 4}, {0, 3, 4}, {2, 3, 4}, {1, 2, 4}}	{{1, 3}, {0, 2}}
	{{1, 3}, {0, 2}, {0, 1}}
	{{1, 3}, {1, 2}, {0, 2}}
	{{1, 3}, {0, 2}, {0, 3}}
	{{1, 3}, {2, 3}, {0, 2}}
{{1, 3}, {0, 4}, {1, 4}, {2}, {0, 3}}	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 2}, {0, 4}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 2}, {1, 4}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 2}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 2}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 2}, {0, 1}}
{{1, 2}, {4}, {2, 3}, {0, 3}, {0, 1}}	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 1}}

Hypergraph	Graph realizations (edge sets)
	$\{\{2,4\},\{3,4\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{1,4\}\}$
	$\{\{2,4\},\{3,4\},\{2,3\},\{0,2\},\{1,3\},\{0,4\},\{1,4\}\}$
	$\{\{2,4\},\{3,4\},\{0,2\},\{1,3\},\{0,4\},\{1,4\},\{0,3\}\}$
	$\{\{2,4\},\{3,4\},\{0,2\},\{1,3\},\{0,4\},\{1,4\}\}$
$\{\{0,1,2\},\{1,2,3\},\{2,3,4\},\{0,2,4\}\}$	$\{\{1,4\},\{0,3\}\}$
	$\{\{0,4\},\{1,4\},\{0,3\}\}$
	$\{\{3,4\},\{1,4\},\{0,3\}\}$
	$\{\{1,4\},\{0,3\},\{0,1\}\}$
	$\{\{1,3\},\{1,4\},\{0,3\}\}$
$\{\{3,4\},\{1,4\},\{2\},\{0,3\},\{0,1\}\}$	$\{\{2,4\},\{3,4\},\{2,3\},\{0,2\},\{1,3\},\{1,2\},\{0,4\}\}$
	$\{\{2,4\},\{2,3\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{0,1\}\}$
	$\{\{2,4\},\{2,3\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{0,3\}\}$
	$\{\{2,4\},\{2,3\},\{0,2\},\{1,3\},\{1,2\},\{0,4\},\{1,4\}\}$
	$\{\{2,4\},\{2,3\},\{0,2\},\{1,3\},\{1,2\},\{0,4\}\}$
$\{\{0,2,3\},\{1,3,4\},\{2,3,4\},\{0,1,3\}\}$	$\{\{1,2\},\{0,4\}\}$
	$\{\{1,2\},\{0,4\},\{1,4\}\}$
	$\{\{2,4\},\{1,2\},\{0,4\}\}$
	$\{\{1,2\},\{0,4\},\{0,1\}\}$
	$\{\{1,2\},\{0,4\},\{0,2\}\}$
$\{\{0,2,3\},\{1,2,3\},\{0,2,4\},\{1,2,4\}\}$	$\{\{3,4\},\{0,1\}\}$
	$\{\{3,4\},\{0,3\},\{0,1\}\}$
	$\{\{1,3\},\{3,4\},\{0,1\}\}$
	$\{\{3,4\},\{0,4\},\{0,1\}\}$
	$\{\{3,4\},\{1,4\},\{0,1\}\}$
$\{\{1,3\},\{3,4\},\{0,4\},\{2\},\{0,1\}\}$	$\{\{2,4\},\{3,4\},\{2,3\},\{0,2\},\{1,2\},\{1,4\},\{0,3\}\}$
	$\{\{2,4\},\{2,3\},\{0,2\},\{1,2\},\{1,4\},\{0,3\},\{0,1\}\}$
	$\{\{2,4\},\{2,3\},\{0,2\},\{1,3\},\{1,2\},\{1,4\},\{0,3\}\}$
	$\{\{2,4\},\{2,3\},\{0,2\},\{1,2\},\{0,4\},\{1,4\},\{0,3\}\}$
	$\{\{2,4\},\{2,3\},\{0,2\},\{1,2\},\{1,4\},\{0,3\}\}$
$\{\{0,3,4\},\{1,2,3\},\{2,3,4\},\{0,1,3\}\}$	$\{\{1,4\},\{0,2\}\}$
	$\{\{1,4\},\{0,2\},\{0,1\}\}$
	$\{\{0,4\},\{1,4\},\{0,2\}\}$
	$\{\{2,4\},\{1,4\},\{0,2\}\}$
	$\{\{1,2\},\{1,4\},\{0,2\}\}$
$\{\{0,1,4\},\{0,2,4\},\{0,1,3\},\{0,2,3\}\}$	$\{\{1,2\},\{3,4\}\}$
	$\{\{1,3\},\{1,2\},\{3,4\}\}$
	$\{\{1,2\},\{2,3\},\{3,4\}\}$

Hypergraph	Graph realizations (edge sets)
	{{1, 2}, {1, 4}, {3, 4}}
	{{2, 4}, {1, 2}, {3, 4}}
{{2, 4}, {1, 3}, {3, 4}, {0}, {1, 2}}	{{3, 4}, {2, 3}, {0, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 3}, {0, 2}, {1, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 3}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {2, 3}, {0, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 3}, {0, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
{{0, 1, 4}, {1, 3, 4}, {0, 1, 2}, {1, 2, 3}}	{{2, 4}, {0, 3}}
	{{2, 4}, {3, 4}, {0, 3}}
	{{2, 4}, {2, 3}, {0, 3}}
	{{2, 4}, {0, 2}, {0, 3}}
	{{2, 4}, {0, 4}, {0, 3}}
{{1, 3}, {1, 2}, {4}, {0, 2}, {0, 3}}	{{2, 4}, {3, 4}, {2, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {0, 4}, {1, 4}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {1, 3}, {0, 4}, {1, 4}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 4}, {1, 4}, {0, 1}}
{{1, 3, 4}, {0, 3, 4}, {0, 2, 4}, {1, 2, 4}}	{{2, 3}, {0, 1}}
	{{2, 3}, {0, 2}, {0, 1}}
	{{1, 2}, {2, 3}, {0, 1}}
	{{2, 3}, {0, 3}, {0, 1}}
	{{1, 3}, {2, 3}, {0, 1}}
{{0, 2, 3}, {1, 3, 4}, {0, 3, 4}, {1, 2, 3}}	{{2, 4}, {0, 1}}
	{{2, 4}, {0, 2}, {0, 1}}
	{{2, 4}, {1, 2}, {0, 1}}
	{{2, 4}, {0, 4}, {0, 1}}
	{{2, 4}, {1, 4}, {0, 1}}
{{1, 2}, {0, 4}, {1, 4}, {0, 2}, {3}}	{{2, 4}, {3, 4}, {2, 3}, {1, 3}, {0, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {1, 3}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {1, 3}, {0, 3}, {0, 1}}
{{3, 4}, {0, 4}, {2, 3}, {0, 2}, {1}}	{{2, 4}, {0, 2}, {1, 3}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {1, 3}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {2, 3}, {1, 3}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {1, 3}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}

Hypergraph	Graph realizations (edge sets)
{{0, 1, 4}, {1, 2, 3}, {0, 1, 3}, {1, 2, 4}}	{{3, 4}, {0, 2}}
	{{3, 4}, {0, 2}, {0, 3}}
	{{3, 4}, {2, 3}, {0, 2}}
	{{3, 4}, {0, 4}, {0, 2}}
	{{2, 4}, {3, 4}, {0, 2}}
{{1, 2}, {2, 3}, {1, 4}, {0}, {3, 4}}	{{2, 4}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {0, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {0, 2}, {1, 3}, {0, 4}, {0, 3}, {0, 1}}
{{0, 1, 4}, {1, 3, 4}, {2, 3, 4}, {0, 2, 4}}	{{1, 2}, {0, 3}}
	{{1, 3}, {1, 2}, {0, 3}}
	{{1, 2}, {0, 3}, {0, 1}}
	{{1, 2}, {2, 3}, {0, 3}}
	{{1, 2}, {0, 2}, {0, 3}}

Domination hypergraphs of order 5 with thirteen graph realizations

Hypergraph	Graph realizations (edge sets)
{{2, 4}, {3, 4}, {0, 2}, {1, 4}, {0, 3}, {0, 1}}	{{1, 3}, {1, 2}, {0, 4}, {2, 3}}
	{{1, 3}, {1, 2}, {0, 4}, {2, 3}, {2, 4}}
	{{1, 3}, {1, 2}, {0, 4}, {1, 4}, {2, 3}}
	{{1, 3}, {1, 2}, {0, 4}, {2, 3}, {3, 4}}
	{{1, 3}, {1, 2}, {0, 4}, {2, 3}, {0, 1}}
	{{1, 3}, {1, 2}, {0, 4}, {2, 3}, {0, 2}}
	{{1, 3}, {1, 2}, {0, 4}, {2, 3}, {0, 3}}
	{{2, 4}, {3, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 4}}
	{{3, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 4}, {1, 4}}
	{{2, 3}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {0, 1}}
	{{2, 4}, {2, 3}, {1, 3}, {1, 2}, {0, 4}, {1, 4}}
	{{2, 3}, {0, 2}, {1, 3}, {1, 2}, {0, 4}, {0, 3}}
	{{2, 3}, {1, 3}, {1, 2}, {0, 4}, {0, 3}, {0, 1}}
{{3, 4}, {2, 3}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}	{{1, 3}, {2, 4}, {0, 4}, {0, 2}}
	{{1, 3}, {2, 4}, {0, 4}, {1, 4}, {0, 2}}
	{{1, 3}, {2, 4}, {0, 4}, {0, 2}, {0, 1}}
	{{1, 3}, {1, 2}, {0, 4}, {0, 2}, {2, 4}}

Hypergraph	Graph realizations (edge sets)
	{{1,3},{2,4},{0,4},{0,2},{0,3}}
	{{1,3},{2,4},{0,4},{0,2},{3,4}}
	{{1,3},{2,4},{0,4},{2,3},{0,2}}
	{{2,4},{0,2},{1,3},{0,4},{1,4},{0,1}}
	{{2,4},{0,2},{1,3},{1,2},{0,4},{0,1}}
	{{2,4},{0,2},{1,3},{1,2},{0,4},{1,4}}
	{{2,4},{2,3},{0,2},{1,3},{0,4},{0,3}}
	{{2,4},{3,4},{2,3},{0,2},{1,3},{0,4}}
	{{2,4},{3,4},{0,2},{1,3},{0,4},{0,3}}
{{2,4},{2,3},{0,2},{1,3},{1,4},{0,1}}	{{1,2},{0,4},{0,3},{3,4}}
	{{1,2},{0,4},{1,4},{0,3},{3,4}}
	{{2,4},{1,2},{0,4},{0,3},{3,4}}
	{{1,3},{1,2},{0,4},{0,3},{3,4}}
	{{1,2},{0,4},{0,1},{0,3},{3,4}}
	{{1,2},{0,4},{0,2},{0,3},{3,4}}
	{{1,2},{0,4},{2,3},{0,3},{3,4}}
	{{3,4},{1,2},{0,4},{1,4},{0,3},{0,1}}
	{{2,4},{3,4},{0,2},{1,2},{0,4},{0,3}}
	{{3,4},{1,3},{1,2},{0,4},{1,4},{0,3}}
	{{3,4},{1,3},{1,2},{0,4},{0,3},{0,1}}
	{{3,4},{2,3},{0,2},{1,2},{0,4},{0,3}}
	{{2,4},{3,4},{2,3},{1,2},{0,4},{0,3}}
{{3,4},{2,3},{0,2},{1,2},{0,4},{1,4}}	{{1,3},{2,4},{0,3},{0,1}}
	{{1,3},{2,4},{0,2},{0,3},{0,1}}
	{{1,3},{1,2},{0,1},{2,4},{0,3}}
	{{1,3},{2,4},{2,3},{0,3},{0,1}}
	{{1,3},{2,4},{1,4},{0,3},{0,1}}
	{{1,3},{2,4},{0,1},{0,3},{3,4}}
	{{1,3},{2,4},{0,4},{0,3},{0,1}}
	{{2,4},{0,2},{1,3},{1,2},{0,3},{0,1}}
	{{2,4},{2,3},{1,3},{1,2},{0,3},{0,1}}
	{{2,4},{2,3},{0,2},{1,3},{0,3},{0,1}}
	{{2,4},{3,4},{1,3},{1,4},{0,3},{0,1}}
	{{2,4},{3,4},{1,3},{0,4},{0,3},{0,1}}
	{{2,4},{1,3},{0,4},{1,4},{0,3},{0,1}}
{{2,4},{2,3},{1,2},{0,4},{0,3},{0,1}}	{{1,3},{3,4},{1,4},{0,2}}
	{{1,3},{3,4},{1,4},{0,2},{0,3}}

Hypergraph	Graph realizations (edge sets)
	{{1, 3}, {3, 4}, {1, 4}, {0, 2}, {0, 1}}
	{{1, 3}, {3, 4}, {0, 4}, {1, 4}, {0, 2}}
	{{1, 3}, {2, 4}, {1, 4}, {0, 2}, {3, 4}}
	{{1, 3}, {3, 4}, {2, 3}, {0, 2}, {1, 4}}
	{{1, 3}, {1, 2}, {1, 4}, {0, 2}, {3, 4}}
	{{3, 4}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 3}}
	{{3, 4}, {0, 2}, {1, 3}, {0, 4}, {1, 4}, {0, 1}}
	{{3, 4}, {0, 2}, {1, 3}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {1, 2}, {1, 4}}
	{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 2}, {1, 4}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {1, 3}, {1, 4}}
{{0, 2}, {1, 3}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}	{{2, 4}, {3, 4}, {2, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 3}, {0, 1}}
	{{2, 4}, {3, 4}, {2, 3}, {0, 2}, {0, 1}}
	{{1, 3}, {2, 4}, {2, 3}, {0, 1}, {3, 4}}
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{{2, 4}, {3, 4}, {0, 2}, {1, 3}, {1, 2}, {0, 3}}	{{0, 1}, {0, 4}, {2, 3}, {1, 4}}
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Hypergraph	Graph realizations (edge sets)
{{2, 4}, {3, 4}, {1, 3}, {1, 2}, {0, 4}, {0, 1}}	{{2, 3}, {0, 2}, {0, 3}, {1, 4}}
	{{2, 4}, {2, 3}, {0, 2}, {0, 3}, {1, 4}}
	{{0, 4}, {2, 3}, {0, 2}, {0, 3}, {1, 4}}
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	{{2, 3}, {0, 2}, {1, 3}, {1, 4}, {0, 3}, {0, 1}}
	{{2, 4}, {2, 3}, {0, 2}, {0, 4}, {1, 4}, {0, 3}}
{{2, 4}, {2, 3}, {1, 3}, {0, 4}, {1, 4}, {0, 3}}	{{1, 2}, {0, 1}, {0, 2}, {3, 4}}
	{{1, 3}, {1, 2}, {0, 1}, {0, 2}, {3, 4}}
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	{{1, 2}, {0, 1}, {0, 2}, {0, 3}, {3, 4}}
	{{0, 1}, {1, 2}, {1, 4}, {0, 2}, {3, 4}}
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	{{3, 4}, {0, 2}, {1, 2}, {0, 4}, {1, 4}, {0, 1}}
{{3, 4}, {2, 3}, {0, 2}, {1, 3}, {0, 4}, {0, 1}}	{{2, 4}, {1, 2}, {1, 4}, {0, 3}}
	{{2, 4}, {1, 2}, {0, 4}, {1, 4}, {0, 3}}
	{{2, 4}, {1, 2}, {1, 4}, {0, 3}, {0, 1}}
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	{{2, 4}, {3, 4}, {2, 3}, {1, 2}, {1, 4}, {0, 3}}

Hypergraph	Graph realizations (edge sets)
	$\{\{2,4\},\{0,2\},\{1,2\},\{0,4\},\{1,4\},\{0,3\}\}$
	$\{\{2,4\},\{3,4\},\{1,3\},\{1,2\},\{1,4\},\{0,3\}\}$

BIBLIOGRAPHY

[1] M. Aigner, E. Triesch: *Reconstructing a Graph from its Neighborhood Lists*. Combinatorics, Probability and Computing 2 (2), 103–113 (1993), doi:10.1017/S0963548300000535.

[2] I. Anderson: Combinatorics of Finite Sets. Dover, Mineola, N.Y., 2002.

[3] C. Berge: Théorie des Graphes et ses Applications. Dunod, Paris, 1958.

[4] C. Berge: Hypergraphs: Combinatorics of Finite Sets. North Holland, Amsterdam, 1989.

[5] G. Birkhoff: Lattice Theory. American Mathematical Society, Providence, Rhode Island, 1973.

[6] E. Boros, V. Gurvich, I. Zverovich: *Neighborhood Hypergraphs of Bipartite Graphs*. J. Graph Theory 58, 69–95 (2008). doi:10.1002/jgt.20293.

[7] N. Bourgeois, F. Della Croce, B. Escoffier, V.Th. Paschos: *Fast algorithms for min independent dominating set*. Discrete Appl. Math. 161, 4–5 (2013) 558–572, doi:10.1016/j.dam.2012.01.003.

[8] A. Brandstadt, V. B. Le, J. P. Spinrad: Graph Classes: A survey, SIAM Monographs on Discrete Mathematics and Applications, Society for Industrial and Applied Mathematiccs, Philadelphia, 1999.

[9] A. Bretto: Hypergraph Theory. An Introduction. Springer, 2013.

[10] G. Chartrand, L. Lesniak: Graphs and Digraphs, fourth ed. Chapman & Hall/CRC, Boca Raton, 2005.

[11] E.J. Cockayne, S.T. Hedetniemi: *Towards a theory of domination*. Networks 7 (1977) 247–261, doi:10.1002/net.3230070305.

[12] E.J. Cockayne, S.T. Hedetniemi, R. Laskar: *Gallai Theorems for Graphs, Hypergraphs and Set Systems*. Discrete mathematics 72 (1988) 35–47, doi:10.1016/0012-365X(88)90192-6.

[13] E.J. Cockayne, C.M. Mynhardt: *Domination-Related Parameters*, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics. Marcel Dekker, New York, 1998, pp. 379–400.

[14] J-F. Couturier, P. Heggernes, P. van't Hof, D. Kratsch: *Minimal dominating sets in graph classes: Combinatorial bounds and enumeration*. Theor. Comput. Sci. 487 (2013) 82–94, doi:10.1016/j.tcs.2013.03.026.

[15] J-F. Couturier, R. Letourneur, M. Liedloff: *On the number of minimal dominating sets on some graph classes*. Theor. Comput. Sci. 562 (2015) 634–642, doi:10.1016/j.tcs.2014.11.006.

[16] B.A. Davey, H.A. Priestley: Introduction to Lattices and Order, second ed. Cambridge University Press, New York, 2002.

[17] P. Duchet: *Hypergraphs*, in: R.L. Graham, M. Grötschel, L. Lovász, Eds., Handbook of Combinatorics, Vol. 1. The MIT Press-North Holland, Amsterdam-Cambridge, 1995, pp. 381–432.

[18] F.V. Fomin, F. Grandoni, A.V. Pyatkin, A.A. Stepanov: *On maximum number of minimal dominating sets in graphs*. Electron. Notes Discrete Math. 22 (2005) 157–162, doi:10.1016/j.endm.2005.06.028.

[19] F.V. Fomin, F. Grandoni, A.V. Pyatkin, A.A. Stepanov: *Combinatorial bounds via measure and conquer: Bounding minimal dominating sets and applications*. ACM Trans. Algorithms 5 (1) (2008) No. 9, doi:10.1145/1435375.1435384.

[20] F. V. Fomin, J. Kratochvíl, D. Lokshtanov, F. Mancini, and J. A. Telle:. *On the complexity of reconstructing H-free graphs from their Star Systems*. Journal of Graph Theory, 68(2): 113–124, 2010. DOI: 10.1002/jgt.20544.

[21] W. Goddard, M.A. Henning: *Independent domination in graphs: A survey and recent results*. Discrete Math. 313 (2013) 839–854, doi:10.1016/j.disc.2012.11031.

[22] G. Grätzer: Lattice Theory. First Concepts and Distributive Lattices. Dover, Mineola, N.Y., 2009.

[23] T.W. Haynes, S.T. Hedetniemi, P.J. Slater: Fundamentals of Domination in Graphs. Marcel Dekker, New York, 1998.

[24] T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.): Domination in Graphs: Advanced Topics. Marcel Dekker, New York, 1998.

[25] S.T. Hedetniemi, R.C. Laskar, Eds.: Topics on Domination, Annals of Discrete Mathematics 48, North-Holland, Amsterdam, 1991.

[26] S.T. Hedetniemi, A.A. McRae, D.A. Parks: *Complexity results*, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics. Marcel Dekker, New York, 1998, pp. 233–270.

[27] M.A. Henning: *A survey of selected recent results on total domination in graphs*. Discrete Math., 309 (1) (2009) 32–63, doi:10.1016/j.disc.2007.12.044.

[28] S. R. Jayaram: *Minimal dominating sets of cardinality two in a graph*. Indian J. pure appl. Math., 28(1):43–46, 1997.

[29] M.M. Kanté, V. Limouzy, A. Mary, L. Nourine: *On the enumeration of minimal dominating sets and related notions*. SIAM J. Discrete Math., 28 (4) (2014) 1916–1929, doi:10.1137/120862612.

[30] D. Kleitman: *On Dedekind's Problem: The Number of Monotone Boolean Functions.*. Proc. Amer. Math. Soc. 21, 677-682, 1969.

[31] D. Kratsch: *Algorithms*, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics. Marcel Dekker, New York, 1998, pp. 191–232.

[32] C.L. Liu: Introdution to Combinatorial Mathematics. McGraw Hill, New York, 1968.

[33] J. Martí-Farré: *From clutters to matroids*. Electron. J. Combin., 21(1)-P1.11, 2014.

[34] J. Martí-Farré, A. de Mier: *Completion and decomposition of a clutter into representable matroids*. Linear Algebra Appl., 472:31–47, 2015, doi:10.1016/j.laa.2015.01.023.

[35] J. Martí-Farré, A. de Mier: *Transformation and decomposition of clutters into matroids*. Advances in Mathematics, 312, 2017, 286–314, doi:10.1016/j.aim.2017.03.022.

[36] J. Martí-Farré, M. Mora, J.L. Ruiz: *Completion and decomposition of hypergraphs into dominating sets of graphs*. Electronic Notes in Discrete Mathematics, 49:213–220, 2015, doi:10.1016/j.endm.2015.06.031.

[37] J. Neggers, H.S. Kim: Basic Posets. World Scientific Publishing Co., Inc., River Edge, NJ (1998).

[38] O. Ore: Theory of Graphs. Amer. Math. Soc. Colloq. Pub. 38, Providence, 1962.

[39] S. Roman: Lattices and Ordered Sets. Springer, New York, 2008.

[40] E. Sampathkumar: *Domination parameters of a graph*, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.), Domination in Graphs: Advanced Topics. Marcel Dekker, New York, 1998, pp. 271–300.

[41] D.A. Simovici, C. Djeraba: Mathematical tools for data mining. Set theory, partial orders, combinatorics. Advanced Information and Knowledge Processing. Springer, London (2014).

[42] V.T. Sós: *Problem with a comment by Lázló Babai*. In: Combinatorics, Vol. II, A. Hajnal and V. T. Sós (Editors), pp.1214. North-Holland, Amsterdam, (1978).

[43] V. I. Voloshin: Introduction to Graph and Hypergraph Theory. Nova Science Publishers, New York, 2009.

[44] D.B. West: Introduction to Graph Theory, second ed. Prentice Hall, Upper Saddle River, 2001.

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