Chapter 2

Energy-based modelling of electromechanical systems

In this chapter the energy–based model of the system is presented in a Hamiltonian formalism. Basic notions on Port-controlled Hamiltonian Systems (PCHS) and Dirac structures are discussed and then applied to the electromechanical systems. The whole system (doubly-fed induction machine and back-to-back converter) is finally presented in as PCHS.

Part of the results of this Chapter can also be found in [4][9][15].

2.1 Port-controlled Hamiltonian Systems

The central paradigm of network modelling of complex systems is to have individual open subsystems with well defined port interfaces, hiding an internal model of variable complexity, and a set of rules describing how the subsystems interact through the port variables.

One implementation of this general idea is what is known as port Hamiltonian systems or port-controlled Hamiltonian systems (PCHS) [56][57] (see also [25] and references therein). Hamiltonian systems are close to the classical Lagrangian methods, both techniques use the state dependent energy or co-energy functions to characterize the dynamics of the different elements. In this approach, energy plays a fundamental role, port variables are conjugated variables such that their product has dimension of power, and the interconnection of subsystems is implemented by means of what is called a Dirac structure, which enforces the preservation of power, and can be seen as a generalization of Tellegen's theorem of circuit theory [58]. PCHS theory allows the coupling of systems from different domains using energy as the linking concept, and provides the mathematical foundation for bond-graph modelling [20][42]. Although originally developed for lumped parameter systems, PCHS theory has been extended to distributed parameter systems as well [88], described by partial differential equations, for which numerical spatial discretization schemes have also been developed [41].

Besides describing systems in a modular, scalable and non domain-specific way, PCHS theory allows a natural implementation of passivity-based control methods [51][87], using energy as the storage function. The clear separation between (a) constitutive relations, given by the energy, or Hamiltonian, function, (b) the structure matrix, describing how energy flows inside the system, and (c) the power ports, some of which may be terminated by dissipative elements, allows the design of controllers with a clear physical interpretation in what is known as Interconnection and Damping Assignment Passivity-Based Control

(IDA-PBC) [66], see Chapter 4.

2.1.1 Port-controlled Hamiltonian Systems in explicit form

Explicit port-controlled Hamiltonian systems have the form

$$\left\{\begin{array}{lcl} \dot{x} & = & (J(x)-R(x))\partial_x H(x)+g(x)u \\ y & = & g^T(x)\partial_x H(x) \end{array}\right. \tag{2.1}$$

where

- $x \in \mathbb{R}^n$ is the vector state.
- $u, y \in \mathbb{R}^m$ are the port variables.
- $H(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is the Hamiltonian function (usually representing the energy function of the system). The ∂_x (or ∂, if no confusion arises) operator defines the gradient of a function of x, and in what follows we will take it as a column vector.
- $J(x) \in \mathbb{R}^{n\times n}$ is the interconnection matrix (which is skew-symmetric, $J(x) = -J(x)^T$). It represents the internal connection between whole elements of the system and define its structure.
- $R(x) \in \mathbb{R}^{n\times n}$ is the dissipation matrix (which, is symmetric and, in physical systems, semi-positive definite $R(x) = R^T \geq$). It represents the losses of the system.
- $g(x) \in \mathbb{R}^{n\times m}$ is an interconnection matrix describing, the port connection of the system outside the world. It yields the flow of energy to/from the system through the port variables, u and y.

A nice feature of port-controlled Hamiltonian systems is their passivity and stability properties.

Proposition 2.1. *Assume a PCHS (2.1) with*

- *A1 - a Hamiltonian function $H(x)$ bounded from below ($H(x) > c$), with a minimum at x^**
- *A2 - an skew-symmetric interconnection matrix $J(x) = -J^T(x)$, and*
- *A3 - a semi-positive definite dissipation matrix $R(x) = R^T(x) \geq 0$,*

*then the system, in a closed-loop (*i.e. *for $u = 0$), is asymptotically stable*[a]*.*

[a] Notice that, if assumption A3 restricts in $R(x) = R^T(x) > 0$, the closed-loop system is globally asymptotically stabe

Proof. Consider the Hamiltonian function $H(x)$, its derivative

$$\dot{H}(x) = (\partial H)^T \dot{x} = (\partial H)^T (J(x) - R(x))\partial H + (\partial H)^T g(x)u$$

with (A2) and $y = g^T(x)\partial H(x)$ the power-balance equation is recovered

$$\dot{H}(x) = -(\partial H)^T R(x) \partial H + y^T u$$

with (A3) and considering $u = 0$

$$\dot{H}(x) \leq 0.$$

This result, with (A1) concludes that the Hamiltonian function is a Lyapunov function. Invoking LaSalle's invariance principle, if the largest invariant set under the dynamics (2.1, with $u = 0$) contained in

$$\left\{ x \in \mathbb{R}^n | (\partial H)^T R \partial H = 0 \right\}$$

equals x^*, then the system is asymptotically stable. □

2.1.2 Dirac structures

As said before, the central mathematical object of the formulation is what is called a Dirac structure, which contains information about the interconnection network. In the simplest case (finite dimensional systems), flows corresponding to the open ports are arranged in a m-dimensional vector space $\mathcal{V}$, $f \in \mathcal{V}$, while the associated efforts are viewed as elements of its dual $\mathcal{V}^*$, $e \in \mathcal{V}^*$. The dual pairing between vectors and forms provides then the product which yields power,

$$\langle e, f \rangle = p.$$

One also needs an *state space* $\mathcal{X}$, with local coordinates $x \in \mathbb{R}^n$ and corresponding tangent and co-tangent spaces $T\mathcal{X}$ and $T^*\mathcal{X}$. These will eventually be associated to the bonds corresponding to the energy storing elements.

Let $\mathcal{B}(x) = T_x\mathcal{X} \times T_x^*\mathcal{X} \times \mathcal{V} \times \mathcal{V}^*$. On $\mathcal{B}(x)$ one can define a symmetric bilinear form

$$\langle f_1^x, e_1^x, f_1, e_1), (f_2^x, e_2^x, f_2, e_2) \rangle_+ = (e_1^x, f_2^x) + (e_2^x, f_2^x) + (e_1, f_2) + (e_2, f_1).$$

A Dirac Structure on $\mathcal{B} = \cup_{x \in \mathcal{X}} \mathcal{B}(x)$ is a smooth subbundle $\mathcal{D} \in \mathcal{B}$ such that, for each x,

$$\mathcal{D}(x) = \mathcal{D}^\perp(x),$$

$$\mathcal{D}^\perp(x) = (f_1^x, e_1^x, f_1, e_1) \mid \langle (f_1^x, e_1^x, f_1, e_1), (f_2^x, e_2^x, f_2, e_2) \rangle_+ = 0, \forall (f_2^x, e_2^x, f_2, e_2) \in \mathcal{D}(x).$$

Dirac structures have the following important properties:

1. $\dim \mathcal{D}(x) = n + m$.
2. If $(f^x, e^x, f, e) \in \mathcal{D}(x)$, then $(e^x, f^x) + (e, f) = 0$.
3. In local coordinates, a Dirac structure can be characterized by $(n + m)$-dimensional square matrices $E(x), F(x)$, satisfying $F(x)E^T(x) + E(x)F^T(x) = 0$ as follows:

$$(f^x, e^x, f, e) \in \mathcal{D}(x) \Longleftrightarrow F(x) \begin{bmatrix} f^x \\ f \end{bmatrix} + E(x) \begin{bmatrix} e^x \\ e \end{bmatrix} = 0. \quad (2.2)$$

Let H be a smooth function on $\mathcal{X}$. The port controlled Hamiltonian system corresponding to $(\mathcal{X}, \mathcal{V}, \mathcal{D}, H)$ is defined by

$$(f^x, e^x, f, e) \in \mathcal{D}(x), \tag{2.3}$$

with

$$\begin{aligned} \dot{x} &= -f^x, & (2.4)\\ e^x &= dH(x), & (2.5) \end{aligned}$$

where the minus sign in (2.4) is introduced for convenience (it is related to an input power convention; see (2.6) below). Substituting (2.4), (2.5) in (2.3) one has, alternatively,

$$(-\dot{x}, dH(x), f, e) \in \mathcal{D}(x).$$

It follows from the self-duality of the Dirac structure that

$$\dot{H} = (e, f), \tag{2.6}$$

which expresses the energy balance, *i.e.* the rate of variation of the system energy equals the power coming into the system. In local coordinates, assuming that $F(x)$ is invertible for all x, one has

$$\begin{bmatrix} \dot{x} \\ -f \end{bmatrix} = F^{-1}(x)E(x) \begin{bmatrix} \partial H(x) \\ e \end{bmatrix}.$$

It is a simple computation to see from $FE^T + EF^T = 0$ that $F^{-1}E$ must be skew-symmetric. Then

$$\begin{bmatrix} \dot{x} \\ -f \end{bmatrix} = \begin{bmatrix} J(x) & g(x) \\ -g^T(x) & J_f(x) \end{bmatrix} \begin{bmatrix} \partial H(x) \\ e \end{bmatrix}.$$

where $J(x), J_f(x)$ are skew-symmetric.

Dissipation may be included by terminating some of the open ports. Replacing $m \to m + m_r$ and setting $e_r = -R(x)f$ with $R^T(x) = R(x) \geq 0$, one gets

$$\dot{H}(x) = (e, f) + (e_r, f_r) = (e, f) - f_r^T R(x) f_r \leq (e, f).$$

Again, in local coordinates the system can be expressed as

$$\left\{ \begin{array}{rcl} \dot{x} &=& (J(x) - R(x))\partial H(x) + g(x)e \\ f &=& g^T(x)\partial H(x) - J_f(x)e. \end{array} \right.$$

and an implicit port-controlled Hamiltonian system is obtained (2.1). In general, one can interconnect several PCHS with open ports using Dirac structures, and the result is again a PCHS, although not necessarily in explicit form. A main feature of the formalism is that the interconnection of Hamiltonian subsystems using a Dirac structure yields again a Hamiltonian system [25].

2.1.3 Interconnection examples

Here we present some elementary examples of interconnection of two PCHS using a Dirac structure. Although the formulation may seem a little overkill for such elementary systems, this has the advantage that the main features and results can be easily checked against physical intuition.

Example 2.2: A point particle subjected to an external force

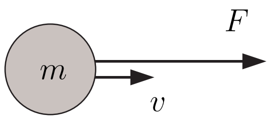

Figure 2.1: A point particle subjected to an external force.

Consider a point particle in one dimension, with mass m and moving at speed v, and an external force F applied to it, Figure 2.1. This dynamical system is described by variables $x = p \in \mathbb{R}$, where $p = mv$, and the energy function

$$H(p) = \frac{1}{2m}p^2.$$

The elements of the Dirac structure description (2.3) are

$$f^p = -\dot{p}, \quad e^p = \partial_p H(p) = \frac{p}{m} = v, \quad e = F, \quad f = v^F,$$

where v^F is the speed of the point where the external force is applied. The physics yields the interconnection laws

$$\dot{p} = F, \quad v = v^F,$$

The first relation is just Newton's second law, while the second one says that the external force is applied to the particle. The two interconnection laws above can be written as a Dirac structure with

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{F(p)} \begin{bmatrix} -\dot{p} \\ v^F \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{E(p)} \begin{bmatrix} v \\ F \end{bmatrix} = 0,$$

which indeed satisfy $FE^T + EF^T = F + F^T = 0$.

Example 2.3: An ideal spring

Consider an ideal spring where $x = q \in \mathbb{R}$ is the equilibrium displacement. The energy function is described by

$$H(q) = \frac{1}{2}kq^2.$$

where k is the spring constant. In this case, the elements of the Dirac structure are

$$f^x = -\dot{q}, \quad e^x = kq, \quad e = F, \quad f = v,$$

where F is the force *on* the spring and v the velocity of the point where it is applied. In this case, physics dictates that

$$\dot{q} = v, \quad F = kq,$$

so that the force on the spring is moving with its displacement and $-F = -kq$ is the force done on whatever is acting on the spring. These two relations can be written again as a Dirac structure:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{F(q)} \begin{bmatrix} -\dot{q} \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}}_{E(q)} \begin{bmatrix} kq \\ F \end{bmatrix} = 0.$$

Example 2.4: A point particle subjected to two external forces

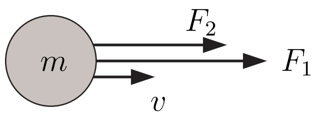

Figure 2.2: A point particle with a mass subjected to two external forces.

Consider now a point particle with mass m subjected to two external forces F_1, F_2, Figure 2.2. As in the first example, the dynamical variable is $x = p \in \mathbb{R}$ and as energy function we have again

$$H(x) = \frac{1}{2m}p^2.$$

The elements of the Dirac structure description are

$$f^p = -\dot{p}, \quad e^p = \frac{p}{m} = v, \quad e_1 = F_1, \quad f_1 = v_1, \quad e_2 = F_2, \quad f_2 = v_2,$$

where e_i, $i = 1, 2$ is the velocity of the point where F_i is applied. Notice that we only add an external force to the first example, and now the interconnection laws are

$$\dot{p} = F_1 + F_2, \quad v_1 = v_2 = v.$$

This is also a Dirac structure,

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{F(p)} \begin{bmatrix} -\dot{p} \\ v_1 \\ v_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{E(p)} \begin{bmatrix} v \\ F_1 \\ F_2 \end{bmatrix} = 0.$$

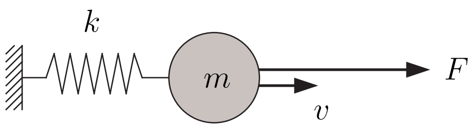

Figure 2.3: A point particle with a mass and an ideal spring subjected to an external force.

Example 2.5: A point particle attached to an ideal spring an subjected to an external force

Consider a point particle with mass m, connected to an ideal spring (with elastic constant k) and subjected besides to an external force F (Figure 2.3). In this example two dynamical variables appears, one from the mass subsystem subjected to two external forces (one of them will be the elastic force of the spring) and another one from the ideal spring. Figure

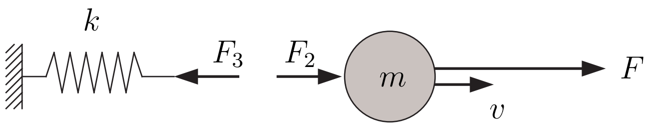

Figure 2.4: The point particle and ideal spring system, decoupled.

2.4 shows these two separated subsystems, which we have already described as a Dirac structure. With the slightly new notation, we have

– point particle:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{F(p)} \begin{bmatrix} -\dot{p} \\ v^F \\ v_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{E(p)} \begin{bmatrix} v \\ F \\ F_2 \end{bmatrix} = 0.$$

– ideal spring:

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}}_{F(q)} \begin{bmatrix} -\dot{q} \\ v_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}}_{E(q)} \begin{bmatrix} kq \\ F_3 \end{bmatrix} = 0.$$

The interconnection dictates that

$$F_3 = -F_2, \quad v_3 = v_2$$

or in a Dirac structure form (this is in fact a *constant* Dirac structure, with no dynamical contents and describing just the topology of the interconnection)

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}}_{F_c} \begin{bmatrix} v_2 \\ v_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}}_{E_c} \begin{bmatrix} F_2 \\ F_3 \end{bmatrix} = 0.$$

Putting everything together, we have

$$-\dot{p}+F+F_2=0, \quad v=v_2=v^F, \quad -\dot{q}+v_3=0, \quad F_3=kq \quad F_3=-F_2, \quad v_3=v_2,$$

or, eliminating the auxiliary interconnection variables v_2, v_3, F_2, F_3,

$$F-kq-\dot{p}=0, \quad v^F=v, \quad \dot{q}=v,$$

which can be written in Dirac structure form as

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{F(p,q)} \begin{bmatrix} -\dot{p} \\ v^F \\ -\dot{q} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{E(p,q)} \begin{bmatrix} v \\ F \\ kq \end{bmatrix} = 0.$$

Notice that the energy function is the sum of the partial energy functions

$$H(p,q)=\frac{1}{2m}p^2+\frac{1}{2}kq^2.$$

Example 2.6: Two point particles subjected to an external force

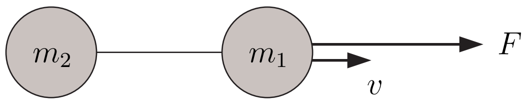

Figure 2.5: Two point particles subjected to an external force.

As a last example consider two point particles with masses m_1 and m_2, glued together and to which an external force F is applied, Figure 2.5. The system can be seen as interconnection of a point particle m_1 subjected to two external forces and a second mass m_2 subjected to an external force, see Figure 2.6.

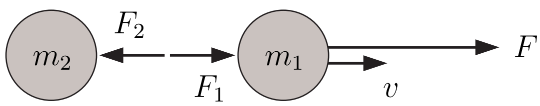

Figure 2.6: The two particle system, decoupled.

Adapting the notation, the subsystems are described by

– point particle m_1:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{F(p_1)} \begin{bmatrix} -\dot{p}_1 \\ v \\ v_1 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{E(p_1)} \begin{bmatrix} v^{p_1} \\ F \\ F_1 \end{bmatrix} = 0.$$

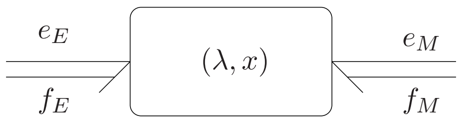

Figure 2.7: A generalized electromechanical system.

– point particle m_2:

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{F(p_2)} \begin{bmatrix} -\dot{p}_2 \\ v_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{E(p_2)} \begin{bmatrix} v^{p_2} \\ F_2 \end{bmatrix} = 0,$$

In this case the interconnection is described by

$$v_1 = v_2, \quad F_1 = -F_2.$$

Putting everything together and getting rid of v_1, v_2, F_1, F_2, one gets

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{F(p_1,p_2)} \begin{bmatrix} -\dot{p}_1 \\ -\dot{p}_2 \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}}_{E(p_1,p_2)} \begin{bmatrix} v^{p_1} \\ v^{p_2} \\ F \end{bmatrix} = 0,$$

which also satisfies (2.2).

2.2 Port-controlled Hamiltonian description of electromechanical systems

An electromechanical system exchanges energy between a mechanical and electrical part by means of geometry variations. In this section a general model, which includes many of the classical electrical machines as well as linear motors and levitating systems [75], is presented.

Consider the system displayed in Figure 2.7[1]. There are n_E generalized electrical ports (e_E, f_E) and n_M generalized mechanical ones (e_M, f_M), and the state variables are denoted by $\lambda \in \mathbb{R}^{n_E}$, $x \in \mathbb{R}^{n_M}$ (we use a magnetic and translation mechanics notation, although the ports can be of any nature).

The equations of motion and the constitutive relations of the ports of this system, namely $\dot{\lambda} = e_E$, $\dot{x} = f_M$, $f_E = \partial_\lambda H_E$, $e_M = \partial_x H_E$, where $H_E = H_E(\lambda, x)$ is the energy

[1] In this Section we display systems using a bond graph representation, which, in a sense, are the graphical counterpart of port-Hamiltonian modelling. This representation is not essential for our purposes, but the reader not familiarized with bond graph theory can, for instance, consult [19][20][48].

function, can be expressed in explicit port-Hamiltonian form as[2]

$$\begin{bmatrix} \dot{\lambda} \\ \dot{x} \end{bmatrix} = \mathbb{O} \begin{bmatrix} \partial_\lambda H_E \\ \partial_x H_E \end{bmatrix} + \mathbb{I} \begin{bmatrix} e_E \\ f_M \end{bmatrix}, \quad (2.7)$$

$$\begin{bmatrix} f_E \\ e_M \end{bmatrix} = \mathbb{I}^T \begin{bmatrix} \partial_\lambda H_E \\ \partial_x H_E \end{bmatrix}. \quad (2.8)$$

This is just the purely electromagnetic part of an electromechanical system. In fact, the electromechanical system always contains some mechanical inertia, independently of whether the port is connected to other systems or not. To model this, consider a generalized mechanical element with n_I ports (e_I, f_I) and state variables $p \in \mathbb{R}^{n_I}$. The dynamical equations of the element, $\dot{p} = e_I$, $f_I = I^{-1}p$, are written in port-Hamiltonian form as

$$\dot{p} = \mathbb{O}\, \partial_p H_I + \mathbb{I}\, e_I, \quad (2.9)$$

$$f_I = \mathbb{I}^T\, \partial_p H_I, \quad (2.10)$$

with

$$H_I(p) = p^T I^{-1} p.$$

This purely mechanic part can be coupled to the electromagnetic part and to the rest of the system (if any), by means of

$$e_I = -B_{IM} e_M + F_I, \quad (2.11)$$

$$f_M = B_{IM}^T f_I, \quad (2.12)$$

$$f_I = v_I, \quad (2.13)$$

where the mechanical ports of the inertia element have been split into one contribution from the electromagnetic part, (e_M, f_M), and the connection to other subsystems, (F_I, v_I), with $F_I, v_I \in \mathbb{R}^{n_I}$. The matrix B_{IM} takes into account the fact that the mechanical ports may be connected to the electromagnetical part in a nontrivial way (or the fact that $n_I \neq n_M$), and the minus sign in (2.11) reflects Newton's third law (e_M is the force *on* the electromagnetic part, so a minus sign must be introduced to get the force on the mechanical element). Notice that the above relations define a Dirac structure in $\mathbb{R}^{n_M+2n_I} \times \mathbb{R}^{n_M+2n_I}$ with coordinates $(e_M, -f_M, -e_I, f_I, F_I, v_I)$, since the $n_M + 2n_I$ equations are clearly independent and can be written as

$$\underbrace{\begin{bmatrix} \mathbb{I} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I} \end{bmatrix}}_{F} \begin{bmatrix} -f_M \\ -e_I \\ v_I \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbb{O} & B_{IM}^T & \mathbb{O} \\ -B_{IM} & \mathbb{O} & \mathbb{I} \\ \mathbb{O} & -\mathbb{I} & \mathbb{O} \end{bmatrix}}_{E} \begin{bmatrix} e_M \\ f_I \\ F_I \end{bmatrix} = 0,$$

with $EF^T + FE^T = E + E^T = 0$. Notice that the two minus signs in $-f_M$ and $-e_I$ correspond to power flowing into the mechanical port of the electromagnetic subsystem and power flowing into the mechanical inertia, respectively, so that

$$F_I^T v_I = e_I^T f_I + e_M^T f_M.$$

The bond graph corresponding to the whole system is displayed in Figure 2.8, where the power flow conventions can be clearly appreciated.

[2] $\mathbb{O}$ denotes the zero matrix of appropriate size, and likewise $\mathbb{I}$ stands for the identity matrix.

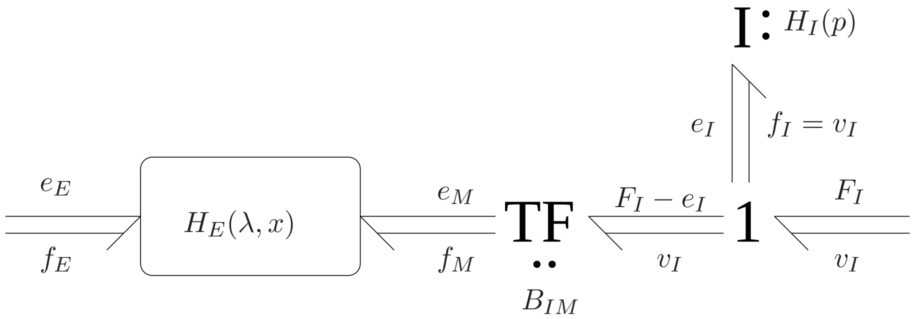

Figure 2.8: Bond graph of a generalized electromechanical system with mechanical inertia included.

From (2.7), (2.8), (2.9), (2.10), (2.11), (2.12) and (2.13), one can express the equations of motion for the state variables in terms of the external inputs (e_E, F_I), and obtain also the corresponding outputs (f_E, v_I). Indeed, eliminating the internal port variables (e_M, f_M) and (e_I, f_I), one gets

$$\begin{aligned} \dot{\lambda} &= e_E, \\ \dot{x} &= B_{IM}^T \partial_p H_I, \\ \dot{p} &= -B_{IM} \partial_x H_E + F_I, \\ f_E &= \partial_\lambda H_E, \\ v_I &= \partial_p H_I. \end{aligned}$$

This can be given a port-Hamiltonian form, with total Hamiltonian

$$H_{EM}(\lambda, x, p) = H_E(\lambda, x) + H_I(p),$$

and

$$\begin{aligned} \begin{bmatrix} \dot{\lambda} \\ \dot{x} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & B_{IM}^T \\ \mathbb{O} & -B_{IM} & \mathbb{O} \end{bmatrix} \begin{bmatrix} \partial_\lambda H_{EM} \\ \partial_x H_{EM} \\ \partial_p H_{EM} \end{bmatrix} + \begin{bmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{bmatrix} \begin{bmatrix} e_E \\ F_I \end{bmatrix}, \\ \begin{bmatrix} f_E \\ v_I \end{bmatrix} &= \begin{bmatrix} \mathbb{I} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{I} \end{bmatrix} \begin{bmatrix} \partial_\lambda H_{EM} \\ \partial_x H_{EM} \\ \partial_p H_{EM} \end{bmatrix}. \end{aligned}$$

In fact, a further generalization can be included, allowing a nontrivial interconnection of the external and electrical ports to the actual power sources. This can be represented by means of constant matrices B_E, B_M of appropriate dimensions, similarly to the transformer matrix in the mechanical side. Denoting by (e_S, f_S), respectively (F_S, v_S), the variables at the source electrical (mechanical) ports, the final port Hamiltonian description is then

$$\begin{aligned} \begin{bmatrix} \dot{\lambda} \\ \dot{x} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & B_{IM}^T \\ \mathbb{O} & -B_{IM} & \mathbb{O} \end{bmatrix} \begin{bmatrix} \partial_\lambda H_{EM} \\ \partial_x H_{EM} \\ \partial_p H_{EM} \end{bmatrix} + \begin{bmatrix} B_E & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \\ \mathbb{O} & B_M \end{bmatrix} \begin{bmatrix} e_S \\ F_S \end{bmatrix}, \\ \begin{bmatrix} f_S \\ v_S \end{bmatrix} &= \begin{bmatrix} B_E^T & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & B_M^T \end{bmatrix} \begin{bmatrix} \partial_\lambda H_{EM} \\ \partial_x H_{EM} \\ \partial_p H_{EM} \end{bmatrix}. \end{aligned} \qquad (2.14)$$

We will now specialize this general description to the case of linear electrical systems, and will also add dissipative effects for both the electrical and the mechanical parts. We will adopt a bottom-top approach and see how the above description arises from the dynamical equations as usually presented in the electrical engineering literature. Under the assumption of linear electrical constitutive relations, and taking a simple inertia form for the mechanical kinetic energy, the total energy is

$$H(\lambda, p, \theta) = \frac{1}{2}\lambda^T L^{-1}(\theta)\lambda + \frac{1}{2}p^T J_m^{-1} p,$$

where $\lambda \in \mathbb{R}^{n_e}$ are the generalized electrical energy variables (again, they may be charges or magnetic fluxes), $p \in \mathbb{R}^{n_m}$ are the generalized mechanical variables (linear or angular, or associated to any other generalized coordinate), and $\theta \in \mathbb{R}^{n_m}$ are the generalized geometric coordinates. $L \in \mathbb{R}^{n_e \times n_e}$ is the inductance (or capacitance) matrix.

One has

$$\begin{aligned} \dot{\lambda} &= -R_e i + B_e v \\ \dot{p} &= -B_r \omega - \tau_e(\lambda, \theta) + B_m \tau_L \\ \dot{\theta} &= J_m^{-1} p \end{aligned} \tag{2.15}$$

where $B_e \in \mathbb{R}^{n_e \times m_e}$ is a matrix indicating how the input voltages $v \in \mathbb{R}^{m_e}$ are connected to the electrical devices, $B_m \in \mathbb{R}^{n_m \times m_m}$ is a matrix indicating how the external applied mechanical torque $\tau_L \in \mathbb{R}^{m_m}$ are connected to the mechanical subsystem, $R_e \in \mathbb{R}^{n_e \times n_e}$ is the electrical damping matrix, $B_r \in \mathbb{R}^{n_m \times n_m}$ is the mechanical damping matrix, $\tau_e \in \mathbb{R}^{n_m}$ is the electrical torque and $i \in \mathbb{R}^{n_e}$, $\omega \in \mathbb{R}^{n_m}$

$$i = L^{-1}(\theta)\lambda = \partial_\lambda H, \quad \omega = J_m^{-1} p = \partial_p H \tag{2.16}$$

are the electrical currents (or voltages) and mechanical velocities $\omega = \dot{\theta}$. The electromechanical energy conversion is given by the constitutive law

$$\tau_e = \partial_\theta H = -\frac{1}{2}\lambda^T L^{-1} \frac{\partial L}{\partial \theta} L^{-1} \lambda, \tag{2.17}$$

where $\frac{\partial L}{\partial \theta}^{-1} = -L^{-1}\frac{\partial L}{\partial \theta}L^{-1}$ has been used. Using (2.16) and (2.17), equations (2.15) can be written in an explicit port-controlled Hamiltonian system (2.1) form as

$$\begin{aligned} \dot{x} \;=\; & \Bigg(\underbrace{\begin{bmatrix} O_{n_e \times n_e} & O_{n_e \times n_m} & O_{n_e \times n_m} \\ O_{n_m \times n_e} & O_{n_m \times n_m} & -I_{n_m \times n_m} \\ O_{n_m \times n_e} & I_{n_m \times n_m} & O_{n_m \times n_m} \end{bmatrix}}_{J} - \underbrace{\begin{bmatrix} R_e & O_{n_e \times n_m} & O_{n_e \times n_m} \\ O_{n_m \times n_e} & B_r & O_{n_m \times n_m} \\ O_{n_m \times n_e} & O_{n_m \times n_m} & O_{n_m \times n_m} \end{bmatrix}}_{R} \Bigg) \partial H \\ & + \underbrace{\begin{bmatrix} B_e & O_{n_e \times m_m} \\ O_{n_m \times m_e} & B_m \\ O_{n_m \times m_e} & 0_{n_m \times m_m} \end{bmatrix}}_{g} \begin{bmatrix} v \\ \tau_L \end{bmatrix} \end{aligned}$$

where $x = [\lambda^T, p^T, \theta^T]^T \in \mathbb{R}^{n_e + n_m}$. This has the form of (2.14), with the addition of a dissipation matrix (notice also that the order of the configuration and mechanical variables has been interchanged).

2.2.1 Examples

In this section two examples of modelling PCHS electromechanical systems are presented, namely a DC motor and a magnetic levitation system (see [75] for additional examples).

Example 2.7: a DC motor

A DC motor is displayed in Figure 2.9. Assuming parasitic resistances in both rotor and field loops (r_a and r_f), and neglecting any mutual inductance effect between both loops, the equations of motion for the variables $\lambda_f = L_f i_f$ (the field flux), $\lambda_a = L_A i_a$ (the armature flux), and $p_m = J_m \omega$ (the mechanical angular momentum of the rotor) are

$$\begin{aligned} \dot{\lambda}_f &= -r_f i_f + v_f \\ \dot{\lambda}_a &= -L_{Af} i_f \omega - r_a i_a + v_a \\ \dot{p}_m &= L_{Af} i_f i_a - B_r \omega - \tau_e \end{aligned} \tag{2.18}$$

where τ_e is the external applied mechanical torque.

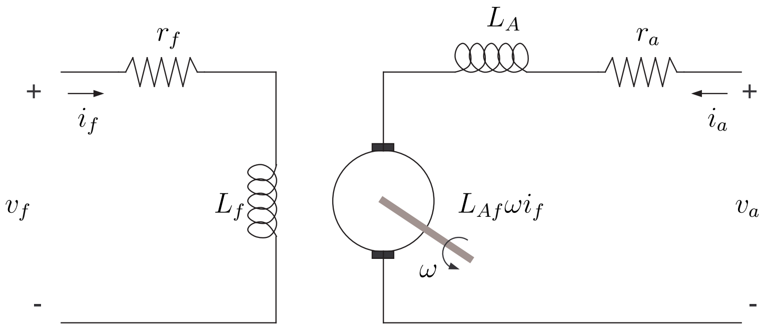

Figure 2.9: Circuit scheme of a DC motor.

For the DC motor, the above formulation has to be modified slightly due to the different role played by the geometric coordinates (the angle in this case), see [15] for further discussion. Direct inspection of equations (2.18) shows that they can be given a PCHS form, with the Hamiltonian variables

$$x = [\lambda_f, \lambda_a, J_m \omega]^T,$$

a Hamiltonian function

$$H(x) = \frac{1}{2}\lambda^T L^{-1} \lambda + \frac{1}{2J_m}\omega^2$$

where $\lambda = [\lambda_f, \lambda_a]^T$ and

$$L = \begin{bmatrix} L_f & 0 \\ 0 & L_A \end{bmatrix}.$$

The interconnection and damping matrices are

$$J = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -L_{Af} i_f \\ 0 & L_{Af} i_f & 0 \end{bmatrix}, \quad R = \begin{bmatrix} r_f & 0 & 0 \\ 0 & r_a & 0 \\ 0 & 0 & B_r \end{bmatrix},$$

and the port matrix is $g = I_3$, with inputs

$$u = [v_f, v_a, \tau_e]^T.$$

Example 2.8: A magnetic levitation system

Figure 2.10 shows a very simplified model of a magnetic levitation system.

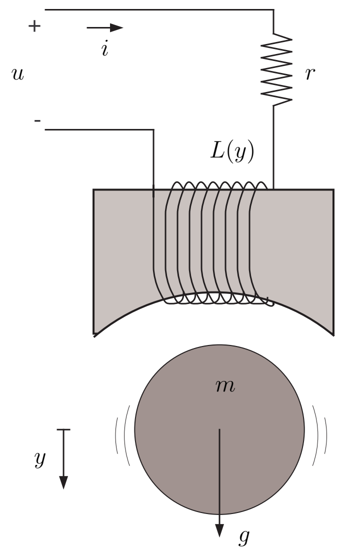

Figure 2.10: A magnetic levitation system.

The flux lines generated by the current at the coil close through the air gap and the iron ball. Since the air gap has a variable reluctance, the system tries to close it, and this counteracts the gravity. The equations of motion are

$$\begin{aligned} \dot{\lambda} &= -ri + u \\ m\dot{v} &= F_m + mg \\ \dot{y} &= v \end{aligned}$$

where $\lambda = L(y)i$ is the linkage flux, r is the resistance of the coil, and F_m the magnetic force, given by

$$F_m = \frac{\partial W_c}{\partial y},$$

where the magnetic co-energy is (assuming a linear magnetic system)

$$W_c = \frac{1}{2}\frac{\partial L}{\partial y}i^2.$$

In general $L(y)$ is a complicated function of the air gap, y. A classical approximation for L for this kind of systems for small y is

$$L(y) = \frac{k}{a+y}$$

with k, a constants.

The system can be expressed as a PCHS taking

$$x = [\lambda, p = mv, y]^T$$

as Hamiltonian variables, and

$$\dot{x} = \left(\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}_{J} - \underbrace{\begin{bmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{R} \right) \partial H + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{g} u$$

with the Hamiltonian function

$$H(x) = \frac{1}{2k}(a + x_3)x_1^2 + \frac{1}{2m}x_2^2 - mgx_3.$$

2.3 Variable structure systems in the PCHS framework

Electronic power converters can be modelled as Variable Structure Systems (VSS). In [34] an extended study on the modelling large class of power converters was presented. Only ideal switches and diodes were considered. Here, ideal means that the switching devices are lossless elements, that is, current conduction occurs only at zero voltage, and viceversa).The main contribution was that a mathematical model of a power converter can be systematically derived using the generalized Hamiltonian formalism as

$$\dot{x} = (J(x, S) - R(x, S))\partial H + g(x, S)u \tag{2.19}$$

where S is a (multi)-index, with values on a finite, discrete set, enumerating the different structure topologies. As is described in the previous section, the state is described by $x \in \mathbb{R}^n$, H is the Hamiltonian function, giving the total energy of the system, J is an antisymmetric matrix, describing how energy flows inside the system, $R = R^T \geq 0$ is a dissipation matrix, and g is an interconnection matrix which yields the flow of energy to/from the system, given by the dual power variables $u \in \mathbb{R}^m$ and $y = g^T(\partial H)^T$.

The main feature is that the for all operating modes the same state variables x, the same Hamiltonian function H and the same dissipation matrix R are considered and the variable topology is captured in the structure matrices (J and g) and the dissipation R.

Power converter are RLC circuits with a variable topology, then the state variables are related to the inductors and capacitors. In the Hamiltonian formalism the variables are fluxes and charges

$$x = [\lambda, q] \in \mathbb{R}^n$$

with a Hamiltonian function

$$H = \frac{1}{2} x^T Q^{-1} x,$$

where Q is the matrix relating between Hamiltonian and Lagrangian variables (fluxes and charges to currents and voltages). The methodology presented in [34] extracts the non-energetic elements (resistors, transformers, diodes and switches), and as a first step considers an LC circuit with ports connected to the disregarded elements. Then the system can be written as

$$\dot{x} = J_x \partial H + G_s u_s + G_r u_r + G_{sw} u_{sw} \tag{2.20}$$

with natural outputs

$$y = \begin{bmatrix} y_s \\ y_r \\ y_{sw} \end{bmatrix} = \begin{bmatrix} G_s^T \\ G_r^T \\ G_{sw}^T \end{bmatrix} \partial H + D \begin{bmatrix} u_s \\ u_r \\ u_{sw} \end{bmatrix} \tag{2.21}$$

where $x \in \mathbb{R}^n$ are the Hamiltonian variables, $J_x \in \mathbb{R}^{n \times n}$ is the interconnection matrix between the energetic storage elements, $y_s, u_s \in \mathbb{R}^s$ are the power variables of the external ports, $y_r, u_r \in \mathbb{R}^r$ are the power variables of the resistances and $y_{sw}, u_{sw} \in \mathbb{R}^{sw}$ are the power variables associates to the switches. G is the associate input matrix for each kind of components (with appropriate dimension) and D is the throughput skew-symmetric matrix which describes the interaction between switches, resistances and sources.

Analyzing each position of the switches the PCHS is obtained. Rather than repeating the general formulation in [34], we present a full example with the full-bridge rectifier that constitutes half of our B2B converter.

Example 2.9: a full-bridge rectifier

The full bridge rectifier, represented in Figure 2.11, is the left hand side of the back-to-back converter discussed in subsection 1.3.1.

It is made by an AC single-phase voltage source $v_i(t) = E \sin(\omega_s t)$, an inductor L, a capacitor C for the DC part and r takes into account all the resistance losses. s_1, s_2, t_1 and t_2 represents ideal switches.

Following the methodology explained above, a PCHS of a full-bridge rectifier can be obtained. First, the system can be written in a form (2.20) as

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \partial H + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v_i + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (-i_{DC}) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} (-e_r) \\ &+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} (-e_{s1}) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} (-e_{s2}) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} (-f_{t1}) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} (-f_{t2}) \end{aligned} \tag{2.22}$$

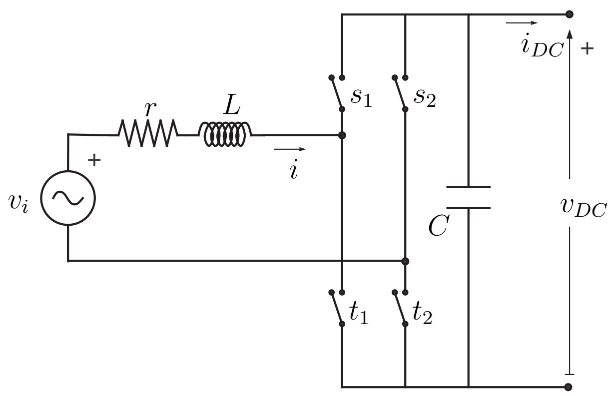

Figure 2.11: Basic scheme of a full-bridge rectifier.

where $x = [\lambda, q]^T$ are the inductor flux ($\lambda = Li$) and the capacitor charge ($q = Cv_{DC}$), and a Hamiltonian function

$$H = \frac{1}{2} x^T Q^{-1} x \tag{2.23}$$

where

$$Q = \begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}$$

and e, f are efforts and fluxes (in a electrical domain, voltages and currents).The corresponding outputs for the source, current port, resistance and switches (2.21) are given by

$$y = \begin{bmatrix} f_{v_i} \\ e_{i_{DC}} \\ f_r \\ f_{s_1} \\ f_{s_2} \\ e_{t_1} \\ e_{t_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \partial H + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_i \\ -i_{DC} \\ -e_r \\ -e_{s_1} \\ -e_{s_2} \\ -f_{t_1} \\ -f_{t_2} \end{bmatrix}. \tag{2.24}$$

From the Ohm's law, the constitutive relation for the resistance is

$$e_r = \frac{\lambda}{L} r. \tag{2.25}$$

Analyzing each one of the two positions of the switch, with (2.24),

– Mode 1, ($S = 1$)

$$\left\{ \begin{array}{l} e_{s_1} = e_{t_2} = 0 \\ f_{s_2} = f_{t_1} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} f_{s_1} = f_{t_2} = \frac{\lambda}{L} \\ e_{s_2} = -\frac{q}{C}, e_{t_1} = -\frac{q}{C} \end{array} \right.$$

– Mode 2, ($S = -1$)

$$\left\{ \begin{array}{l} f_{s_1} = f_{t_2} = 0 \\ e_{s_2} = e_{t_1} = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} e_{s_1} = -\frac{q}{C}, e_{t_2} = -\frac{q}{C} \\ f_{s_2} = f_{t_1} = -\frac{\lambda}{L} \end{array} \right.$$

From the previous analysis, one can write,

$$\begin{cases} e_{s_1} &= \frac{S-1}{2}\frac{q}{C} \\ e_{s_2} &= -\frac{S+1}{2}\frac{q}{C} \\ f_{t_1} &= \frac{S-1}{2}\frac{\lambda}{L} \\ f_{t_2} &= \frac{S+1}{2}\frac{\lambda}{L} \end{cases} \tag{2.26}$$

and replacing (2.25) and (2.26) in (2.22)

$$\begin{bmatrix} \dot{\lambda} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} -r & -S \\ S & 0 \end{bmatrix} \begin{bmatrix} \frac{\lambda}{L} \\ \frac{q}{C} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_i \\ i_{DC} \end{bmatrix} \tag{2.27}$$

which has the form (2.19) with the Hamiltonian variables $x = [\lambda, q]^T$, the Hamiltonian function (2.23), the interconnection and dissipation matrices

$$J = \begin{bmatrix} 0 & -S \\ S & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$g = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

with the input port variables $u = [v_i, i_{DC}]^T$.

Notice that for the averaged approximation models, S takes values in a continuum set.

2.4 Generalized Space State Averaged in a PCHS structure

Our approach to the modelling and control of the B2B converter is based on the combined use of the PCHS and GSSA formalisms. Detailed presentations of GSSA theory and applications can be found in [21][55][79][84].

Averaging techniques for VSS are based on the idea that the change in a state or control variable is small over a given time length, and hence one is not interested on the fine details of the variation. Hence one constructs evolution equations for averaged quantities of the form

$$\langle x\rangle(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\ \mathrm{d}\tau,$$

where $T > 0$ is chosen according to the goals of the problem.

The GSSA expansion tries to improve on this and capture the fine detail of the state evolution by considering a full Fourier series. Thus, one defines

$$\langle x\rangle_k(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)e^{-jk\omega\tau}\ \mathrm{d}\tau, \tag{2.28}$$

with $\omega = 2\pi/T$ and $k \in \mathbb{Z}$. The time functions $\langle x\rangle_k$ are known as index-k averages or k-phasors.

Under standard assumptions about $x(t)$, one gets, for $\tau \in [t-T,t]$ with t fixed,

$$x(\tau) = \sum_{k=-\infty}^{+\infty} \langle x\rangle_k(t)e^{jk\omega\tau}. \tag{2.29}$$

If the $\langle x\rangle_k(t)$ are computed with (2.28) for a given t, then (2.29) just reproduces $x(\tau)$ periodically outside $[t-T,t]$, so it does not yield x outside of $[t-T,t]$ if x is not T-periodic. However, the idea of GSSA is to let t vary in (2.28) so that we really have a kind of "moving" Fourier series:

$$x(\tau) = \sum_{k=-\infty}^{+\infty} \langle x\rangle_k(t)e^{jk\omega\tau}, \quad \forall\tau.$$

If the expected steady state of the system has a finite frequency content, one may select some of the coefficients in this expansion and get a truncated GSSA expansion. The desired steady state can then be obtained from a regulation problem for which appropriate constant values of the selected coefficients are prescribed. A more mathematically advanced discussion is presented in [84].

In order to obtain a dynamical GSSA model we need the following two essential properties:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x\rangle_k(t) = \left\langle \frac{\dot{\mathrm{x}}}{\dot{\mathrm{t}}} \right\rangle_k (t) - jk\omega\langle x\rangle_k(t), \tag{2.30}$$

$$\langle x\rangle y_k = \sum_{l=-\infty}^{+\infty} \langle x\rangle_{k-l}\langle y\rangle_l. \tag{2.31}$$

Notice that $\langle x\rangle_k$ is in general complex and that, if x is real,

$$\langle x\rangle_{-k} = \overline{\langle x\rangle_k}.$$

We will use the notation $\langle x\rangle_k = x_k^R + jx_k^I$, where the averaging notation has been suppressed. In terms of these real and imaginary parts, the convolution property (2.31) becomes (notice that $x_0^I = 0$ for x real, and that the following expressions are, in fact, symmetric in x and y)

$$\begin{aligned}
\langle xy\rangle_k^R &= x_k^R y_0^R \\
&+ \sum_{l=1}^{\infty}\left\{(x_{k-l}^R + x_{k+l}^R)y_l^R - (x_{k-l}^I - x_{k+l}^I)y_l^I\right\} \\
\langle xy\rangle_k^I &= x_k^I y_0^R \\
&+ \sum_{l=1}^{\infty}\left\{(x_{k-l}^I + x_{k+l}^I)y_l^R + (x_{k-l}^R - x_{k+l}^R)y_l^I\right\}
\end{aligned} \tag{2.32}$$

Moreover, the evolution equation (2.30) splits into

$$\begin{aligned}
\dot{x}_k^R &= \left\langle \frac{\dot{\mathrm{x}}}{\dot{\mathrm{t}}} \right\rangle_k^R + k\omega x_k^I, \\
\dot{x}_k^I &= \left\langle \frac{\dot{\mathrm{x}}}{\dot{\mathrm{t}}} \right\rangle_k^I - k\omega x_k^R.
\end{aligned} \tag{2.33}$$

If all the terms in (2.19) have a series expansion in their variables, one can use (2.33) and (2.32) to obtain evolution equations for $x_k^{R,I}$, and then truncate them according to the selected variables. The result is a PCHS description for the truncated GSSA system, to which IDA-PBC regulation techniques can be applied. General formulae for the PCHS description of the full GSSA system, as well as a discussion of the validity of the controller designed for the truncated system, can be found in [16].

Example 2.10: a full-bridge rectifier

The GSSA model of a full-bridge rectifier is studied in [6][39]. Notice that this truncation is basically used to simplify the tracking problem to a regulation one. In this way, it is important a proper selection of the truncated harmonics (based on the control objectives) because the designed controller will only work if the selected harmonics really represent the targeted steady state.

The control objectives are

- the dc value of v_{DC} voltage should be equal to a desired constant V_d, and
- the power factor of the converter should be equal to one. This means that the inductor current should be $i = LI_d \sin(\omega_s t)$, where I_d is an appropriate value to achieve the first objective via energy balance.

It is sensible for the control objectives of the problem to use a truncated GSSA expansion with $\omega = \omega_s$, keeping only the zeroth-order average of the dc-bus voltage, q_0, and the two components of the first harmonic of the inductor current, λ_1^R and λ_1^I. As explained in [39], this selection of coefficients can be further justified if one writes it for $z = \frac{1}{2}q^2$ instead of q, and uses the new control variable $v = -Sq$. In fact, these redefinitions are instrumental in order to fulfill the conditions [16] under which the controller designed for the truncated system can be used for the full system.

With all this, and following the dynamical system obtained in the previous section (equation (2.27)), one gets the PCHS,

$$x = \underbrace{\begin{bmatrix} -r & v \\ -v & 0 \end{bmatrix}}_{J-R} \partial H + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -\sqrt{2z} \end{bmatrix}}_{g} u$$

with the Hamiltonian variables $x = [\lambda, z]^T$, the external inputs $u = [v_i, i_{DC}]^T$ and a Hamiltonian function

$$H(\lambda, z) = \frac{\lambda^2}{2L} + \frac{z}{C}.$$

Next, we apply a GSSA expansion to this system, and set to zero all the coefficients except for $x_1 \triangleq z_0$, $x_2 \triangleq \lambda_1^R$, $x_3 \triangleq \lambda_1^I$, $u_1 \triangleq v_1^R$ and $u_2 \triangleq v_1^I$. Using that i_{DC} is assumed to be locally constant, and that the only nonzero coefficient of v_i is $v_{i1}^I = -\frac{E}{2}$, one gets

$$\begin{aligned} \dot{x}_1 &= -i_{DC}\sqrt{2x_1} - \frac{2}{L}u_1 x_2 - \frac{2}{L}u_2 x_3 \\ \dot{x}_2 &= -\frac{r}{L}x_2 + \omega_s x_3 + \frac{1}{C}u_1 \\ \dot{x}_3 &= -\omega_s x_2 - \frac{r}{L}x_3 - \frac{E}{2} + \frac{1}{C}u_2. \end{aligned} \tag{2.34}$$

This system can be given a PCHS form with

$$J = \begin{bmatrix} 0 & -u_1 & -u_2 \\ u_1 & 0 & \frac{\omega_s L}{2} \\ u_2 & -\frac{\omega_s L}{2} & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{r}{2} & 0 \\ 0 & 0 & \frac{r}{2} \end{bmatrix}$$

$$g = \begin{bmatrix} -\sqrt{2x_1} & 0 \\ 0 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix},$$

the port variables $u = [i_{DC}, E]^T$ and the Hamiltonian function

$$H = \frac{1}{C}x_1 + \frac{1}{L}x_2^2 + \frac{1}{L}x_3^2. \tag{2.35}$$

This model differs from [39] in the $-i_{DC}\sqrt{2x_1}$ term, that now is included in the g matrix. This change is instrumental in achieving the required bidirectional power flow capability, as discussed in Section 4.2.

2.5 Port Hamiltonian model of a DFIM controlled through a B2B converter

In this Section the Port-controlled Hamiltonian model of the flywheel energy storage system is presented. As explained in Section 1, the system is basically composed by the doubly-fed induction machine and the back-to-back converter. The flywheel can be modelled, disregarding the torsion in the shaft, adding an extra inertia to the DFIM, while the local load can be seen as a stator current requirement, see discussion 1.4. These results, including a Bond Graph description, are also presented in [4].

2.5.1 Port-controlled Hamiltonian model of a doubly-fed induction machine

A dq-model of a doubly-fed induction machine is presented in sub-Section 1.2.3. Equations (1.11) and (1.15) describe the dynamics of a DFIM and can be written as a Port-controlled Hamiltonian System, see also [12]. The Hamiltonian variables are (the D subindex refers to the DFIM subsystem) $x_D^T = (\lambda_s^T, \lambda_r^T, J_m\omega) \in \mathbb{R}^5$, or $x_D^T = (\Lambda^T, x_m)$, where $\Lambda^T = (\lambda_s^T, \lambda_r^T) \in \mathbb{R}^4$, λ_s, $\lambda_r \in \mathbb{R}^2$ are the inductor fluxes in dq-coordinates (stator and rotor respectively), $x_m = J_m\omega$ is the mechanical Hamiltonian variable, ω the angular speed of the rotor, and J_m is the total moment of inertia of the rotating parts (including the flywheel). The structure $J_D \in \mathbb{R}^{5\times5}$ and damping $R_D \in \mathbb{R}^{5\times5}$ matrices are

$$J_D = \begin{bmatrix} -\omega_s L_s J_2 & -\omega_s L_{sr} J_2 & O_{2\times1} \\ -\omega_s L_{sr} J_2 & -(\omega_s - \omega_r) L_r J_2 & L_{sr} J_2 i_s \\ O_{1\times2} & L_{sr} i_s^T J_2 & 0 \end{bmatrix}$$

$$R_D = \begin{bmatrix} R_s I_2 & O_{2\times 2} & O_{2\times 1} \\ O_{2\times 2} & R_r I_2 & O_{2\times 1} \\ O_{1\times 2} & O_{1\times 2} & B_r \end{bmatrix},$$

where L are inductances, R are resistances, lower indices s and r refer to stator and rotor respectively, B_r is the mechanical damping, i_s and $i_r \in \mathbb{R}^2$ are the stator and rotor currents and

$$J_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Currents $i^T = [i_s^T, i_r^T] \in \mathbb{R}^4$ and fluxes Λ are related by $\Lambda = \mathcal{L}i$, where the inductance matrix $\mathcal{L}$ is (reminding (1.13))

$$\mathcal{L} = \begin{bmatrix} L_s I_2 & L_{sr} I_2 \\ L_{sr} I_2 & L_r I_2 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

The interconnection matrix is

$$g_D = \begin{bmatrix} I_2 & O_{2\times 2} \\ O_{2\times 2} & I_2 \\ O_{1\times 2} & O_{1\times 2} \end{bmatrix} \in \mathbb{R}^{5\times 4}$$

with the port variables $u^T = [v_s^T, v_r^T] \in \mathbb{R}^4$, where v_s, $v_r \in \mathbb{R}^2$ are the stator and rotor voltages. Finally, the Hamiltonian function is

$$H_D = \frac{1}{2}\Lambda^T \mathcal{L}^{-1} \Lambda + \frac{1}{2J_m} x_m^2.$$

2.5.2 Port-controlled Hamiltonian model of a back-to-back converter

A detailed description of the back-to-back converter is done in Section 1.3. The dynamical equation (1.16) escribes an averaged model of the power converter, where $v_i(t) = E\sin(\omega_s t)$ is a single-phase AC voltage source, L is the inductance (including the effect of any transformer in the source), C is the capacitor of the DC part, r takes into account all the resistance losses (inductor, source and switches), s_k and t_k, $k = 1, 2, 4, 5, 6$. Switch states take values in $\{-1, 1\}$ and t-switches are complementary to s-switches: $t_k = \bar{s_k} = -s_k$. Additionally, $s_2 = \bar{s_1} = -s_1$.

The PCHS averaged model of the full-bridge rectifier is as follows[3]. The Hamiltonian variables are (B subindex refers to the B2B subsystem) $x_B^T = [\lambda, q] \in \mathbb{R}^2$, where λ is the inductor flux and q is the DC charge in the capacitor. The Hamiltonian function is

$$H_B = \frac{1}{2L}\lambda^2 + \frac{1}{2C}q^2,$$

while the structure and damping matrices are

$$J_B = \begin{bmatrix} 0 & -s_1 \\ s_1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2} \qquad R_B = \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

[3]For control design purposes, the full-bridge rectifier modelling is done using the GSSA approach. See Section 4.2 for the GSSA model and for more details on the disregarded harmonics.

The interconnection matrix is

$$g_B = \begin{bmatrix} 1 & O_{1\times 3} \\ 0 & f^T \end{bmatrix} \in \mathbb{R}^{2\times 4}, \quad f = \frac{1}{2}\begin{bmatrix} s_6 - s_4 \\ s_5 - s_6 \\ s_4 - s_5 \end{bmatrix} \in \mathbb{R}^3,$$

with inputs

$$u = \begin{bmatrix} v_i \\ -i_{abc} \end{bmatrix} \in \mathbb{R}^4,$$

where $i_{abc}^T = [i_a, i_b, i_c] \in \mathbb{R}^3$ are the three-phase currents in the inverter part. Notice that the inverter subsystem can be seen as a Dirac structure [25] (see sub-Section 2.1.2) with

$$\begin{aligned} v_{abc} &= f v_{DC} \\ i_{DC} &= f^T i_{abc} \end{aligned}$$

where $v_{abc}^T = [i_a, i_b, i_c] \in \mathbb{R}^3$ are the three-phase voltages and $v_{DC} \in \mathbb{R}$, is the DC voltage, and $i_{DC} \in \mathbb{R}$ is the DC current supplied by the rectifier subsystem.

2.5.3 Port-controlled Hamiltonian model of the whole system

Fig. 2.12 shows the interconnection scheme of the whole system (B2B+DFIM). The *dq*-transformation connects the B2B converter with the DFIM as a Dirac structure.

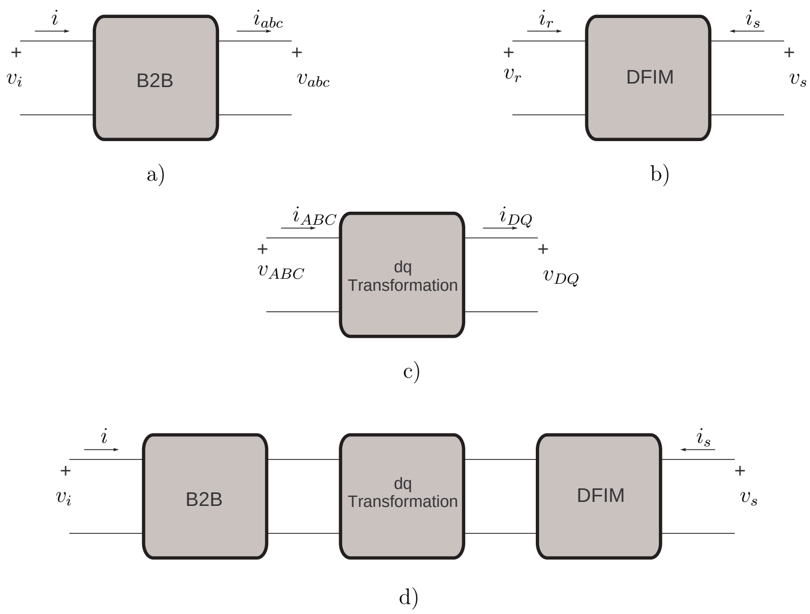

Figure 2.12: Interconnection scheme.

The interconnection relations are

$$v_r = v_{DQ}, \; i_r = i_{DQ}, \; v_{ABC} = v_{abc}, \; i_{ABC} = i_{abc}. \tag{2.36}$$

We use equations (2.36) and introduce a new $\mathcal{K}$ matrix which comes from the dq-transformation of the rotor variables explained in sub-Section 1.2.2,

$$\mathcal{K} = e^{-J_2(\delta-\theta)} T_* \in \mathbb{R}^{3\times 2},$$

with T_* defined so as to remove the homopolar component (which is, in fact equations (1.4), considering an equilibrate system):

$$T_* = \begin{bmatrix} \frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \in \mathbb{R}^{2\times 3}.$$

The variables of the whole PCHS system are $x = [\Lambda^T, J_m\omega_r, \lambda, q]^T \in \mathbb{R}^7$, with energy function

$$H = H_D + H_B = \frac{1}{2}\Lambda^T \mathcal{L}^{-1}\Lambda + \frac{1}{2J_m}x_m^2 + \frac{1}{2L}\lambda^2 + \frac{1}{2C}q^2.$$

The $\mathbb{R}^{7\times 7}$ structure and dissipation matrices are

$$J - R = \begin{bmatrix} \begin{matrix} & J_D - R_D & \end{matrix} & \begin{matrix} O_{2\times 1} & O_{2\times 1} \\ O_{2\times 1} & \mathcal{K}^T f \\ 0 & 0 \end{matrix} \\ \begin{matrix} O_{1\times 2} & O_{1\times 2} & 0 \\ O_{1\times 2} & -f^T\mathcal{K} & 0 \end{matrix} & J_B - R_B \end{bmatrix},$$

and the interconnection matrix and port variables are

$$g = \begin{bmatrix} I_2 & O_{2\times 1} \\ O_2 & O_{2\times 1} \\ O_{1\times 2} & 0 \\ O_{1\times 2} & 1 \\ O_{1\times 2} & 0 \end{bmatrix} \in \mathbb{R}^{7\times 3} \qquad u = [v_s^T, v_i]^T \in \mathbb{R}^3.$$

2.5.4 Simulations of a Hamiltonian model of a DFIM controlled through a B2B converter

In this subsection some simulations of the Port-controlled Hamiltonian System obtained above are presented. The whole system requires some control in order to verify it. More precisely the rectifier part of the back-to-back needs an appropriate signal input, and the rotor voltages of the doubly-fed induction machine has to be synchronously with the mechanical speed. For this reason the following simulations are using the controllers designed is Chapter 4.

The simulation has been performed using the *20-sim*[4] modelling and simulation software. The DFIM parameters used in the simulations are (in SI units): $L_{sr} = 0.041$, $L_s = L_r = 0.042$, $J_m = 0.0005$, $B_r = 0.005$, $R_s = 0.087$, $R_r = 0.0228$, $v_s = \left[\sqrt{\frac{2}{3}}380, 0\right]^T$ and $\omega_s = 2\pi 50$. The B2B parameters are (in SI units): $r = 0.001$, $L = 0.001$, $C = 0.0045$, $E = 68.16$.

For the purposes of testing the model, the desired mechanical speed changes around $\omega = 2\pi 50$ (dotted line in Fig. 4.15) and a desired bus voltage $v_{DC}^d = 150$ has been selected.

Fig. 4.15 shows the desired and simulated mechanical speed. Fig. 2.14 shows the reactive power compensation of the stator side of the DFIM. Fig. 4.30 shows v_{DC}, which remains close to the desired value even in the transient of the machine. Finally, voltage v_i and current i at the single phase source feeding the B2B are depicted in Fig. 2.16, showing that they are nearly in phase.

[4]See www.20sim.com

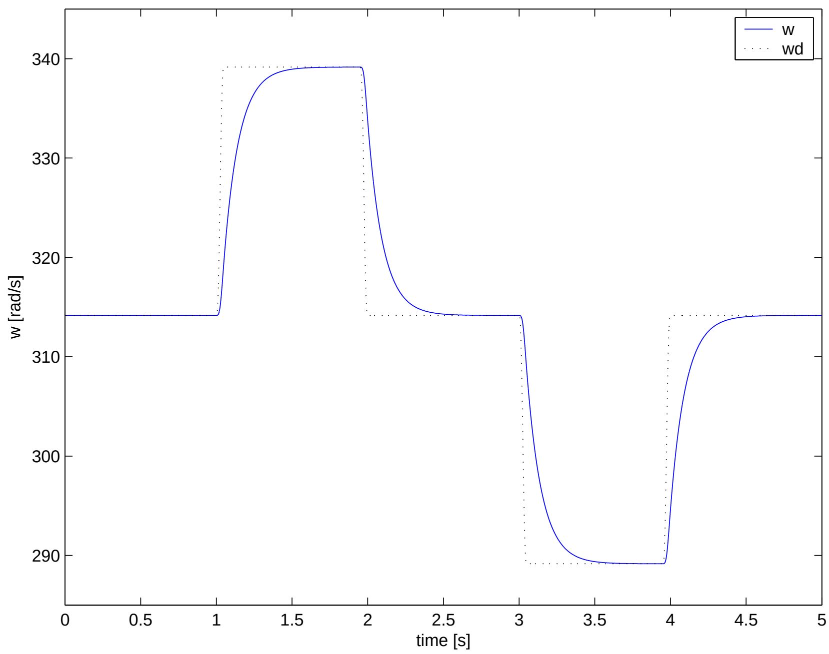

Figure 2.13: Simulation results: Mechanical speed ω.

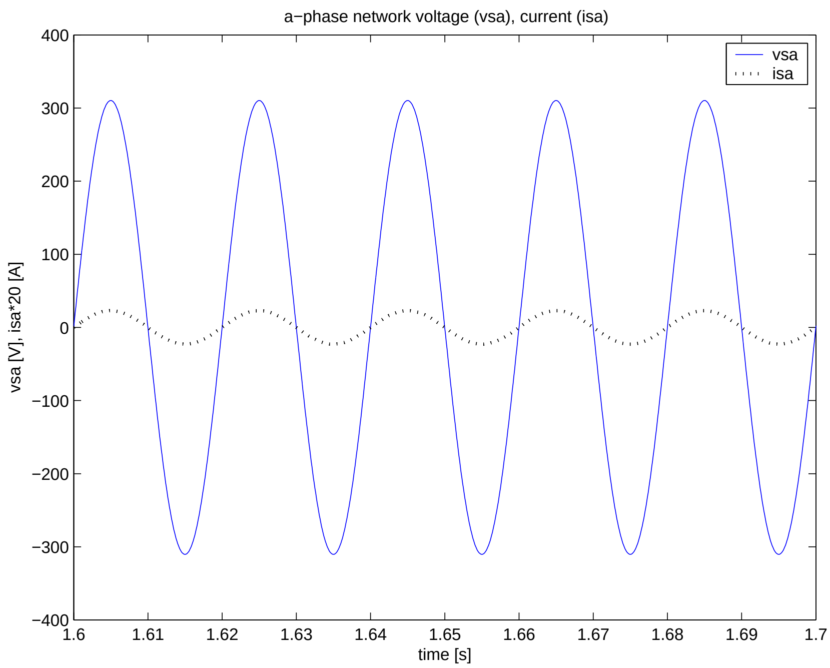

Figure 2.14: Simulation results: Detail of the grid a-phase voltage v_{sa} and current i_{sa}.

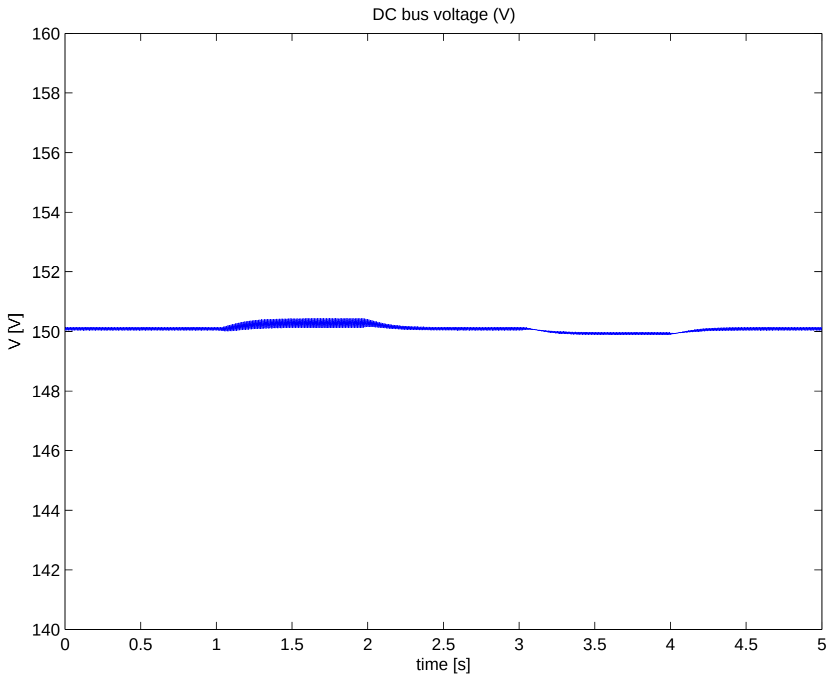

Figure 2.15: Simulation results: DC-bus voltage v_{DC} of the B2B.

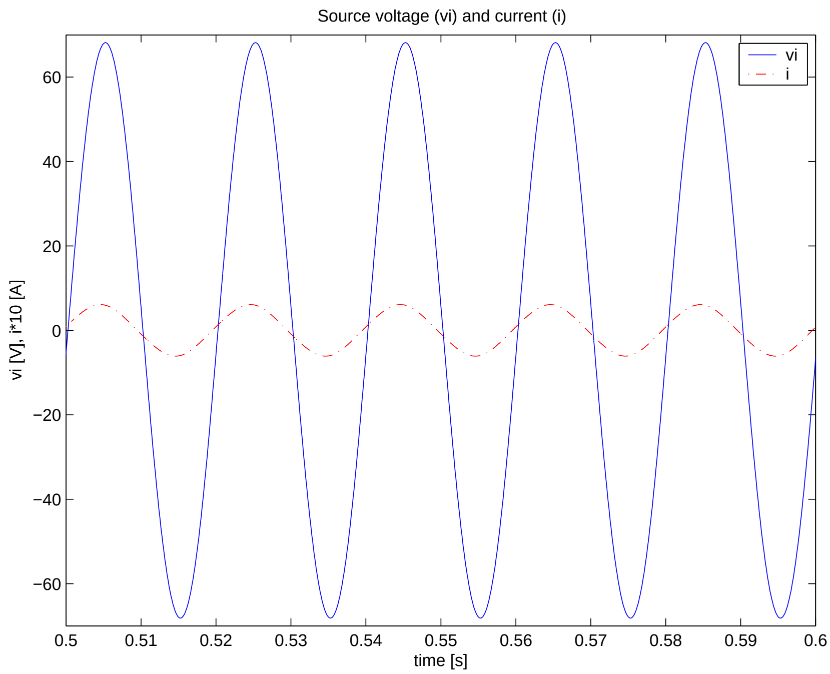

Figure 2.16: Simulation results: Detail of the AC single-phase voltage and current for the B2B.