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# Sylvester matrix rank functions on crossed products and the Atiyah problem

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# Sylvester matrix rank functions on crossed products and the Atiyah problem

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*"A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street."*

David Hilbert (1862-1943)

# *Agraïments*

De tot el llistat (infinit, però per sort numerable!) de persones a les qui haig i vull agraïr una infinitat de coses (aquestes no numerables!), sense cap mena de dubte el primer que ha d'aparèixer en aquest llistat és el meu tutor. Pere, no entenc com m'has pogut aguantar durant tots aquests tres anys! He passat moments molt bons i moments dolents, però sempre m'has sabut guiar pel bon camí i tranquil·litzar-me quan ho veia tot negre! L'únic que puc dir-te és, de veritat, moltes moltes gràcies per tot el suport, dedicació i ajuda que m'has donat en aquests tres anys (i en anys passats també!).

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# *Introduction*

All of the work presented in this thesis appeared while working on the following question.

**Question.** Which is the set of $l^2$-Betti numbers arising from the lamplighter group $\Gamma$?

This question was the initial problem of this thesis. Historically, M. Atiyah [7] introduced in 1976 these particular numbers as homological invariants of a certain kind of homology (nowadays called $l^2$-homology) in order to study Riemannian manifolds $M$ endowed with a free cocompact action of a countable discrete group $G$. His motivation was to generalize the Atiyah-Singer Index Theorem to the noncompact setting.

The $l^2$-Betti numbers can be defined purely in terms of the countable discrete group $G$. Given a subfield $K \subseteq \mathbb{C}$ closed under complex conjugation, one can consider the group algebra $K[G]$ and, more generally, matrix algebras over $K[G]$. Each matrix operator $T \in M_n(K[G])$ can be thought of as a bounded operator $T : l^2(G)^n \to l^2(G)^n$ acting on the left. It turns out that the projection $p_T$ onto $\ker(T)$ belongs to the group von Neumann algebra $\mathcal{N}_n(G)$ of $M_n(K[G])$, which is in fact a *finite* von Neumann algebra, thus endowed with a faithful positive trace $\mathrm{Tr}_{\mathcal{N}_n(G)}$. The $l^2$-Betti numbers are precisely those values that arise when computing the traces of the projections $p_T$, $T \in M_n(K[G])$.

**Definition.** A real positive number $r$ is called an $l^2$-Betti number arising from $G$, with coefficients in $K$, if for some integer $n \geq 1$ there exists a matrix operator $T \in M_n(K[G])$ such that

$$\mathrm{Tr}_{\mathcal{N}_n(G)}(p_T) = r,$$

where $p_T : l^2(G)^n \to l^2(G)^n$ denotes the projection onto $\ker(T)$.

Atiyah computed several $l^2$-Betti numbers in numerous examples, and all of them turned out to be rational, thus giving rise to one of the original questions posted by Atiyah about $l^2$-Betti numbers.

**Question.** Is it possible to obtain irrational values of $l^2$-Betti numbers?

That question motivated a large number of research projects in which stronger statements were formulated (and in some cases proved). One of the strongest versions of Atiyah's original question is the so-called Strong Atiyah Conjecture.

**Strong Atiyah Conjecture.** Is it true that the set of values of $l^2$-Betti numbers is contained in the subgroup of $\mathbb{Q}$ generated by all the elements $\frac{1}{|H|}$, where $H$ ranges over the finite subgroups of $G$?

The lamplighter is precisely the first counterexample to the Strong Atiyah Conjecture, as proven by R. I. Grigorchuk and A. Żuk [42], followed by W. Dicks and T. Schick [25]. More recently, the original Atiyah's question has been solved in the negative, and some authors, including Austin [8], Grabowski [40, 41] and Pichot, Schick and Żuk [86] have found examples of groups having irrational values of $l^2$-Betti numbers. In particular, Grabowski shows in [41] that there are transcendental numbers that appear as $l^2$-Betti numbers of the lamplighter group.

Nevertheless, the Strong Atiyah Conjecture is still open for the class of groups such that there exists an upper bound on the orders of their finite subgroups. In particular it is open for torsion-free groups. For an extensive study of Atiyah's original question and strong versions of it, see [8, 40, 43, 52, 53, 66, 67, 69, 70, 71, 72, 74].

As already said, the initial problem of this thesis was to determine the set of $l^2$-Betti numbers arising from the lamplighter group $\Gamma$, denoted by $\mathcal{C}(\Gamma, K)$. This group is defined to be the semidirect product of $\mathbb{Z}$ copies of the finite group $\mathbb{Z}_2$ by $\mathbb{Z}$, i.e.

$$\Gamma = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}_2 \rtimes_\rho \mathbb{Z}$$

whose automorphism $\rho$ implementing the semidirect product is the well-known Bernoulli shift. A precursor of our work here can be found in the paper [6] by Ara and Goodearl. In this article, the authors attack this problem algebraically, by trying to uncover the structure of what is called the $*$-*regular closure* $\mathcal{R}_{K[\Gamma]}$ of the lamplighter group algebra $K[\Gamma]$ inside $\mathcal{U}(\Gamma)$, the algebra of (unbounded) affiliated operators of the group von Neumann algebra $\mathcal{N}(\Gamma)$ or, more algebraically, the classical ring of quotients of $\mathcal{N}(\Gamma)$ (see Section 1.1.2). The precise connection between the $*$-regular closure $\mathcal{R}_{K[\Gamma]}$ and the set of $l^2$-Betti numbers $\mathcal{C}(\Gamma, K)$ has been provided recently by a result of Jaikin-Zapirain [53], which states that the rank function $\mathrm{rk}_{\mathcal{R}_{K[\Gamma]}}$, obtained from $\mathrm{rk}$[1] by restriction on $\mathcal{R}_{K[\Gamma]}$, is completely determined by its values on matrices over $K[\Gamma]$. Since $\mathcal{R}_{K[\Gamma]}$ is regular, this can be rephrased in the form

$$\phi(K_0(\mathcal{R}_{K[\Gamma]})) = \mathcal{G}(\Gamma, K),$$

where $\phi$ is the state on $K_0(\mathcal{R}_{K[\Gamma]})$ induced by $\mathrm{rk}_{K[\Gamma]}$, and $\mathcal{G}(\Gamma, K)$ the subgroup of $\mathbb{R}$ generated by $\mathcal{C}(\Gamma, K)$. As one of the main results in [6], the authors uncover a portion of the algebraic structure of $\mathcal{R}_{K[\Gamma]}$ by studying the $*$-regular closure $\mathcal{R}_0$ of a certain $*$-subalgebra $\mathcal{A}_0$ of $K[\Gamma]$ in $\mathcal{U}(G)$, and this study leads them to conclude that $\mathcal{G}(\Gamma, K)$ must contain all the rational numbers, namely $\mathbb{Q} \subseteq \mathcal{G}(\Gamma, K)$ [6, Corollary 6.14].

Our initial strategy in this work was to extend the work done by Ara and Goodearl, by considering an increasing sequence of $*$-subalgebras $\mathcal{A}_n$ of $K[\Gamma]$ such that its inductive limit $\mathcal{A}_\infty$ becomes 'big enough' inside $K[\Gamma]$. While working on this problem we realized that there was a natural way of thinking about these 'approximating' $*$-subalgebras $\mathcal{A}_n$ in terms of a concrete dynamical system. The key observation was to realize that the lamplighter group algebra, being a semidirect product of the abelian torsion group $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_2$ by $\mathbb{Z}$, can be thought of as a $\mathbb{Z}$-crossed product $*$-algebra

$$K[\Gamma] \cong K[\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_2] \rtimes_\rho \mathbb{Z} \cong C_K(X) \rtimes_T \mathbb{Z}$$

through the Fourier transform $\mathscr{F} : K[\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_2] \to C_K(X)$, where $X = \prod_{i\in\mathbb{Z}} \mathbb{Z}_2$ is the Pontryagin dual of the group $\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_2$, identified topologically with the Cantor set, $C_K(X)$ denotes the set of locally constant functions $f : X \to K$, and $T : X \to X$ is the homeomorphism of $X$ given by the Bernoulli shift. There is a natural measure $\mu$ on $X$, namely the usual product measure where we take the $\left(\frac{1}{2}, \frac{1}{2}\right)$-measure on each component $\{0, 1\}$, which is ergodic, full and $T$-invariant. This enables us to study the $\mathbb{Z}$-crossed product algebra $C_K(X) \rtimes_T \mathbb{Z}$ by giving '$\mu$-approximations' of the space $X$ (see Section 2.2), which at the level of the algebra correspond to the 'approximating' $*$-subalgebras $\mathcal{A}_n$ given in $K[\Gamma]$ under the previous identification. This construction is motivated by a construction given by Putnam [87, 88].

In the group algebra $K[\Gamma]$ one has a canonical Sylvester matrix rank function $\mathrm{rk}_{K[\Gamma]}$, inherited from the one existing in the $*$-regular ring $\mathcal{U}(\Gamma)$. We can transfer $\mathrm{rk}_{K[\Gamma]}$ to a Sylvester matrix rank function on $C_K(X) \rtimes_T \mathbb{Z}$ by pulling it back through the previous identification, and this gives rise to a $T$-invariant probability measure $\mu$ on $X$. With this observation we have been able to construct, from a fixed $T$-invariant, full and ergodic probability measure and using the construction involving the 'approximating' $*$-subalgebras of $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ developed in Chapter 2, a canonical faithful Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$ on $\mathcal{A}$ not only when $K \subseteq \mathbb{C}$ and $T$ is the Bernoulli shift, but in the more general setting of $K$ being an *arbitrary* field and $T$ an *arbitrary* homeomorphism on a Cantor set $X$. As mentioned, this construction requires the existence of a $T$-invariant probability measure $\mu$ on $X$ which must be ergodic and full. The corresponding Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$ turns out to be unique with respect to the property that the rank of any characteristic function $\chi_U$, being $U \subseteq X$ any clopen set, must equal the measure of $U$ (these results are Theorem 2.3.7 and Proposition 2.3.8). This rank function on $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ gives us a way to define '$l^2$-Betti numbers' arising from the algebra $\mathcal{A}$ which, in the particular case of the lamplighter group algebra, coincide with the $l^2$-Betti numbers arising from $\Gamma$ (see Section 3.2).

In Section 2.3.3 we obtain some results on the structure of the compact convex set $\mathbb{P}(\mathcal{A})$ of Sylvester matrix rank functions on $\mathcal{A}$. We show in Theorem 2.3.15 that, when $X$ is a totally disconnected metrizable compact space (not necessarily infinite) and $T : X \to X$ is any homeomorphism on $X$, then every rank function $\mathrm{rk}$ on the crossed product algebra $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ is induced by some regular ring.

During the course of developing the construction given in Chapter 2, new work by Grabowski on the searching of $l^2$-betti numbers arising from the lamplighter group $\Gamma$ appeared [41]. In this article, the author proved the existence of irrational $l^2$-Betti numbers arising from $\Gamma$, exhibiting a concrete example in [41, Theorem 2]. Very roughly, his idea is to compute $l^2$-Betti numbers by means of decomposing them as an infinite sum of (normalized) dimensions of kernels of finite-dimensional operators (i.e. matrices), and then analyzing the behavior of these finite-dimensional matrices by means of certain graphs in order to determine the global behavior of the dimension of their kernels. We use these ideas in Section 3.2.2, but applied to our construction of Chapter 2. In particular, our main Theorem 3.2.10 gives a whole family of irrational (and even transcendental) $l^2$-Betti numbers arising from $\Gamma$.

[1] Here $\mathrm{rk}$ denotes the canonical rank function on $\mathcal{U}(\Gamma)$ defined through the trace $\mathrm{tr}_{\mathcal{N}(\Gamma)}$, see Section 1.2.2.

Another source of inspiration for the work of this thesis is the work of Elek [29]. To introduce it, we need a historical motivation of the problem.

Murray and von Neumann [80, Theorem XII] proved a uniqueness result for approximately finite von Neumann algebra factors of type $II_1$. This unique factor $\mathcal{R}$ (called the *hyperfinite $II_1$-factor*) plays a central role in the theory of von Neumann algebras. Von Neumann also considered a purely algebraic analogue of this situation, which we briefly explain here. For a field $K$, one considers the sequence

$$M_2(K) \to M_4(K) \to \cdots \to M_{2^n}(K) \to \cdots$$

with respect to the block-diagonal embeddings $x \mapsto \begin{pmatrix} x & \mathbf{0}_{2^n} \\ \mathbf{0}_{2^n} & x \end{pmatrix}$. Its direct limit $\varinjlim_n M_{2^n}(K)$ turns out to be a regular ring which admits a unique rank function rk. The completion of $\varinjlim_n M_{2^n}(K)$ with respect to the induced rank metric, denoted by $\mathcal{M}_K$, is a complete regular ring with a unique rank function, again denoted by rk, which is a *continuous factor*, i.e. a right and left self-injective ring where the set of values of the rank function fills the unit interval $[0,1]$.

It is expected (see e.g. [28, 29, 30]) that the factor $\mathcal{M}_K$ could play a role in algebra which is similar to the role played by the unique hyperfinite factor $\mathcal{R}$ in the theory of operator algebras. In particular, Elek has shown in [29] that, if $\Gamma$ is the lamplighter group, then the continuous factor obtained by taking the rank completion of the $*$-regular closure of $\mathbb{C}[\Gamma]$ in the $*$-algebra $\mathcal{U}(\Gamma)$ is isomorphic to $\mathcal{M}_{\mathbb{C}}$.

This raises the question of what uniqueness properties the von Neumann factor $\mathcal{M}_K$ has. As von Neumann had already shown, $\mathcal{M}_K$ is isomorphic to the factor obtained from any *factor sequence* $(p_i)_i$, that is,

$$\mathcal{M}_K \cong \overline{\varinjlim_n M_{p_n}(K)},$$

where $(p_i)_i$ is a sequence of positive integers converging to infinity and such that $p_i$ divides $p_{i+1}$ for all $i$.

We address this question in Chapter 4, showing in Theorem 4.2.2 that if $\mathcal{B}$ is an ultramatricial $K$-algebra and $\mathrm{rk}_{\mathcal{B}}$ is a nondiscrete extremal pseudo-rank function on $\mathcal{B}$, then the completion of $\mathcal{B}$ with respect to $\mathrm{rk}_{\mathcal{B}}$ is necessarily isomorphic to $\mathcal{M}_K$. We also derive a characterization of the factor $\mathcal{M}_K$ by a local approximation property. Using the latter characterization, we prove in Theorem 4.2.2 that the completion of the algebras $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ considered before with respect to the metric induced by the rank function $\mathrm{rk}_{\mathcal{A}}$ is exactly the algebra $\mathcal{M}_K$. This is connected with a result of Elek [29]

Elek and Jaikin-Zapirain have recently raised the question of whether, for any subfield $K$ of $\mathbb{C}$ closed under complex conjugation, and any countable amenable ICC-group $G$, the rank completion $\overline{\mathcal{R}}_{K[G]}^{\mathrm{rk}}$ of the $*$-regular closure of $K[G]$ in $\mathcal{U}(G)$ is either of the form $M_n(D)$ or of the form $\mathcal{M}_D := D \otimes_K \mathcal{M}_K$, where $D$ is a division ring with center $K$. In view of this question, it is natural to obtain uniqueness results in the more general setting of $D$-rings over a division ring $D$, and also in the setting of rings with involution. We address these questions in the final two sections of Chapter 4.

## *Contents of the thesis*

This thesis consists on five chapters, the first one giving the preliminary background needed in order to follow the other four chapters, which contain the innovative work of the thesis. We start Chapter 1 by providing a brief introduction to von Neumann dimensions (whose values are, in a different context, also called $l^2$-Betti numbers), together with the statements of the Atiyah Conjectures, which concern the question of what kind of values these dimensions can achieve.

We also introduce the notion of von Neumann regular and $*$-regular rings, which play a central role in the next chapters. Many rings of this type carry a natural rank function, which can be used to construct a dimension function over the set of finitely generated (right) projective modules over it. One of the simplest examples one can think about this kind of rings are the matrix algebras over an arbitrary field, together with the usual (normalized) rank of matrices. A not-so-trivial example is given by the algebra of (unbounded) affiliated operators of a finite von Neumann algebra $\mathcal{M}$. In section 1.2.2 we study this example in detail, while we will make use of it later on at this thesis, in Chapters 2 and 3, where it plays a key role.

An important notion introduced in this chapter is the notion of $*$-regular closure of a set inside a $*$-regular ring. There is a whole theory in development concerning the study of the $*$-regular closure, initiated by Jaikin-Zapirain in [53]. As mentioned above, there is a close connection between this construction and the possible range of values that any Sylvester matrix rank function defined on it can achieve, which is crucial in motivating the reason for studying it.

We end the first chapter by discussing a little bit the different (noncommutative) theories about localization in noncommutative rings, ranging from the universal localization (which gives us the Ore localization in some special cases) to other localization theories such as the rational closure and the division closure (see eg. [18, 20]).

In Chapter 2, we consider the crossed product $*$-algebra $\mathcal{A} := C_K(X) \rtimes_T \mathbb{Z}$ induced by a homeomorphism $T : X \to X$ on a totally disconnected, compact, metrizable space $X$. We also assume that $X$ is a measurable space with measure $\mu$, being $\mu$ an ergodic, full $T$-invariant probability measure on $X$.

In Section 2.2 we present a general construction of approximating $\mathcal{A}$ by a suitable subalgebra $\mathcal{A}(E, \mathcal{P})$ by means of approximating the space $X$. We fix a nonempty clopen subset $E$ of $X$, and a finite partition $\mathcal{P}$ of the complement $X \backslash E$. Then $\mathcal{A}(E, \mathcal{P})$ is defined to be the unital $*$-subalgebra of $\mathcal{A}$ generated by the partial isometries $\chi_Z t$, $Z \in \mathcal{P}$. From Proposition 2.2.7 we get that $\mathcal{A}(E, \mathcal{P})$ can be realized as a partial crossed product $*$-algebra, thus interpreting it as an approximation of our crossed product algebra $\mathcal{A}$. We show in Proposition 2.2.14 that $\mathcal{A}(E, \mathcal{P})$ is embeddable into a (possibly infinite) matrix product algebra.

By applying our construction to a decreasing sequence of nonempty clopen sets $\{E_n\}_n$ and taking compatible partitions $\mathcal{P}_n$ of the complements $X \backslash E_n$ (and provided that $X$ be an infinite space), we construct a sequence of approximating $*$-subalgebras $\mathcal{A}_n$ which are embeddable into (possibly infinite) matrix product algebras $\mathfrak{R}_n$, such that its limit $\mathcal{A}_\infty = \varinjlim_n \mathcal{A}_n$ is 'big enough' inside $\mathcal{A}$ (see Proposition 2.3.5), and fits inside $\mathfrak{R}_\infty = \varinjlim_n \mathfrak{R}_n$. Since each matrix product algebra $\mathfrak{R}_n$ carries natural rank functions (satisfying certain compatibility relations concerning the measure $\mu$ on $X$), it is possible to define a rank function $\mathrm{rk}_{\mathfrak{R}_\infty}$ on $\mathfrak{R}_\infty$, thus a Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}_\infty}$ on $\mathcal{A}_\infty$.

This process enables us to embed the whole $*$-algebra $\mathcal{A}$ not into $\mathfrak{R}_\infty$ itself, but into the rank completion $\mathfrak{R}_{\mathrm{rk}}$ of $\mathfrak{R}_\infty$, as we show in Theorem 2.3.7. The natural rank function $\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}$ induces a Sylvester matrix rank function on $\mathcal{A}$, which can be shown to be the unique Sylvester matrix rank function on $\mathcal{A}$ satisfying a certain compatibility property with the measure $\mu$ (Proposition 2.3.8). Moreover, using Theorem 4.2.2, we can idenfity $\mathfrak{R}_{\mathrm{rk}}$ with the well-known von Neumann continuous factor $\mathcal{M}_K$, thus providing an embedding $\mathcal{A} \hookrightarrow \mathcal{M}_K$. It turns out that $\mathcal{M}_K$ is in fact also the rank completion of $\mathcal{A}$. This result is comparable with a well-known result, due to Murray, von Neumann and Connes (Theorem 2.1.3), which states similar results but in a $C^*$-algebraic setting. We give some flavour of it in Section 2.1.

We also study the relations between Sylvester matrix rank functions and probability measures on $X$. In particular, Proposition 2.3.10 reveals that one can reverse the above process, namely that if one starts with an extremal and faithful Sylvester matrix rank function on $\mathcal{A}$, one can construct an ergodic, full and $T$-invariant probability measure on $X$, uniquely determined by exactly the same compatibility property as before.

After that, we devote subsection 2.3.3 to study the structure of the compact convex set $\mathbb{P}(\mathcal{A})$ of Sylvester matrix rank function on $\mathcal{A}$. Our main result in this subsection is Theorem 2.3.15, which states that any Sylvester matrix rank function rk on $\mathcal{A}$ is induced by a regular ring, namely that rk is obtained by pulling back a rank function $\mathrm{rk}_S$ defined on a regular ring $S$ through a homomorphism $\mathcal{A} \to S$. This result does not require $X$ to be infinite.

To conclude, we initiate the study of the $*$-regular closure of $\mathcal{A}$ inside $\mathcal{M}_K$ in order to obtain information about the possible numbers that the rank function $\mathrm{rk}_{\mathcal{A}}$ can achieve. We show that the above approximating sequence of $*$-subalgebras $\mathcal{A}_n$ of $\mathcal{A}$ gives rise to an approximating sequence of $*$-regular rings $\mathcal{R}_n$ of $\mathcal{R}$ in a suitable sense, more specified in Propositions 2.4.5 and 2.4.9. In subsection 2.4.1 we focus basically on uncover a portion of the $*$-regular closures $\mathcal{R}_n$, shedding some light on it in Proposition 2.4.21.

In Chapter 3 we apply our whole machinery from Chapter 2 in order to study some group algebras arising as $\mathbb{Z}$-crossed product algebras, such as the lamplighter group algebra.

We first show how one can relate, in the particular case that $K$ is a subfield of the complex numbers closed under complex conjugation, the group algebra of some special crossed product groups $G = H \rtimes_\rho \mathbb{Z}$ by means of a $\mathbb{Z}$-crossed product algebra through Fourier transform. It is well-known that when $H$ is a countable, discrete and torsion group, then its Pontryagin dual $\widehat{H}$ becomes a totally disconnected, compact metrizable space. We show in Proposition 3.1.1 that, under further hypotheses on the field $K$, we can identify the group algebra of $H$ with the algebra of locally constant functions over $\widehat{H}$ under Fourier transform. In these cases, we obtain an identification

$$K[G] \cong K[H] \rtimes_\rho \mathbb{Z} \cong C_K(\widehat{H}) \rtimes_T \mathbb{Z},$$

where $T : \widehat{H} \to \widehat{H}$ is induced by $\rho : \mathbb{Z} \curvearrowright H$. We observe (Remark 3.1.3) that the resulting explicit formulas for the Fourier transform $\mathscr{F}$ and its inverse $\mathscr{F}^{-1}$ remain valid in *any* field with involution $(K, *)$ of arbitrary characteristic $p$, provided that $K$ satisfy the required hypotheses given in Proposition 3.1.1, so the previous identification $K[G] \cong C_K(\widehat{H}) \rtimes_T \mathbb{Z}$ remain valid if we replace $K \subseteq \mathbb{C}$ by any field with involution $(K, *)$ satisfying some additional hypotheses.

Going back to the case $K \subseteq \mathbb{C}$, the canonical rank function $\mathrm{rk}_{K[G]}$ on $K[G]$ inherited from $\mathcal{U}(G)$ (the algebra of unbounded operators affiliated to the group von Neumann algebra $\mathcal{N}(G)$) gives rise to a $T$-invariant probability measure $\widehat{\mu}$ on $\widehat{H}$, which coincides with the normalized Haar measure on $\widehat{H}$ (Proposition 3.1.4). Once more, this leads us to conclude that for an arbitrary field with involution $(K, *)$ (satisfying the same hypotheses as mentioned above), and by assuming ergodicity of $\widehat{\mu}$, one can construct a canonical Sylvester matrix rank function on $K[G]$ by applying our construction from Chapter 2 to obtain a Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$ on $\mathcal{A} = C_K(\widehat{H}) \rtimes_T \mathbb{Z}$, and pulling it back through the Fourier transform $\mathscr{F} : K[G] \to \mathcal{A}$, so we have indeed reversed the process of finding a suitable rank function, starting with $\mathcal{A}$ and ending with $K[G]$. In this setting, the Atiyah problem for our group algebra $K[G]$(cf. Section 1.1.4) can be translated to a problem on computing ranks inside the $\mathbb{Z}$-crossed product algebra. We also show in Theorem 3.1.6 that, under the identification of $K[G]$ and $\mathcal{A}$ through $\mathscr{F}$, the $*$-regular closure $\mathcal{R}_{K[G]}$ of the group algebra $K[G]$ inside $\mathcal{U}(G)$ (as defined for example in [6, 29, 53]) can be identified with our already defined $*$-regular closure $\mathcal{R}_{\mathcal{A}}$ of the crossed product $\mathcal{A} = C_K(\widehat{H}) \rtimes_T \mathbb{Z}$ inside its rank completion $\mathfrak{R}_{\mathrm{rk}}$, thus giving an alternative way of interpreting it in some cases of interest.

As a particular example, we apply our methods to study the lamplighter group algebra $K[\Gamma]$ in Section 3.2. As mentioned above, this algebra is important because, among other things, it gave the first counterexample to the Strong Atiyah Conjecture, see for example [42], [25]. By using ideas of Grabowski [41] combined with our relation with $l^2$-Betti numbers and values of our canonical Sylvester rank function on $\mathcal{A}$, we have been able to find a whole class of irrational (and even transcendental) $l^2$-Betti numbers arising from $\Gamma$, giving explicit descriptions of the elements, inside matrix algebras over $K[\Gamma]$, that give rise to such values. This is the main result of this section, Theorem 3.2.10.

To end this chapter, we apply our methods to study the particular case of the odometer algebra $C_K(X) \rtimes_T \mathbb{Z}$ with $X = \prod_{i \in \mathbb{N}} \{0, 1\}$, where $T$ is the automorphism $X \to X$ given by addition of $(1, 0, ...)$ with carry over, see Section 3.3. Although it is not possible in this case to realize this crossed product algebra as a group algebra (simply because the crossed product algebra obtained here is simple, see [17]), this example is interesting in its own right because we have been able to *completely* determine the structure of its $*$-regular closure in Theorem 3.3.8, and thus giving a complete description of the set of $l^2$-Betti numbers arising from the algebra $K[\mathbb{Z}(2^\infty)] \rtimes_\rho \mathbb{Z}$ in Theorem 3.3.9. This example has also been studied by Elek in [29], although in there the author does not compute exactly the $*$-regular closure $\mathcal{R}_{\mathcal{A}}$; instead, he computes its rank completion, which he shows that it must be isomorphic to the von-Neumann continuous factor $\mathcal{M}_K$ again. After studying this particular odometer, we realized that the same techniques could also be used to study the general odometer algebra $\mathcal{O}(\overline{n})$, and we do so in Section 3.4, thus giving a complete description of its $*$-regular closure in Theorem 3.4.4 and completely determining the set of $l^2$-Betti numbers arising from the general odometer algebra $\mathcal{O}(\overline{n})$ in Theorem 3.3.9. In particular, it is worth mentioning that from this Theorem it follows that all the (positive part of the) subgroups of $\mathbb{Q}$ containing 1 can appear as the set of $l^2$-Betti numbers arising from some odometer algebra.

Chapter 4 concerns the characterization of the rank completion of some ultramatricial $K$-algebras, being $K$ an arbitrary field. As already mentioned, von Neumann had already shown (and was published later by Halperin [44]) that when one completes any inductive limit of matrix algebras

$$\varinjlim_n M_{p_n}(K)$$

constructed by means of a factor sequence $(p_i)_i$ (so that each $p_i$ is a positive integer dividing $p_{i+1}$), the resulting completion is always a continuous factor, and in fact isomorphic to the von-Neumann continuous factor $\mathcal{M}_K$, independently of the factor sequence chosen. We present in Theorem 4.2.2 a generalization of this result, namely that whenever the rank completion of an ultramatricial $K$-algebra becomes a continuous factor (i.e. when the rank function is nondiscrete and extremal), then this rank completion is necessarily isomorphic to $\mathcal{M}_K$. In fact, we also characterize such $K$-algebras by means of a local property.

We extend, in Section 4.3, the previous result to $D$-rings, being $D$ a division ring. The motivation for doing so is, as explained, a recent question raised by Elek and Jaikin of whether the rank completion $\overline{\mathcal{R}}_{K[G]}^{\mathrm{rk}}$

of the group algebra of an ICC-group $G$ (being in this case $K$ a subfield of the complex numbers closed under complex conjugation) is either of the form $M_n(D)$ or $\mathcal{M}_D := D \otimes_K \mathcal{M}_K$, being $D$ a division ring with center $K$. We have been able to extend condition (2) of Theorem 4.2.2, but we have not found a reasonable analogue of the local condition (3) in this setting. Theorem 4.3.2 provides the main result of that section.

To conclude the chapter, we have considered the corresponding problem for $*$-algebras. Again, the motivation comes from the theory of group algebras: for $K \subseteq \mathbb{C}$ closed under complex conjugation and $G$ a countable discrete group, its group algebra $K[G]$ can be endowed with a natural involution extending the one from $K$, and the completion of the $*$-regular closure of $K[G]$ inside $\mathcal{U}(G)$ is a $*$-regular ring containing $K[G]$ as a $*$-subalgebra. It is then desirable to obtain analogous results as in Theorem 4.2.2 of whether the completion of a standard ultramatricial $*$-algebra $\mathcal{A}$ gives back $\mathcal{M}_K$ again as $*$-algebras. The main results are collected in Theorem 4.4.6, where we have been able to extend condition (2) of Theorem 4.2.2, and also condition (3) although in a somewhat technical way. In the case, however, of $K$ being a $*$-Pythagorean field[2], we can derive a result which is completely analogous to Theorem 4.2.2, and we present it in Corollary 4.4.10.

The results of this chapter have been published in an article at the *Canadian Journal of Mathematics* (2018) [5].

In the last chapter we change our topics from the previous chapters and we concentrate on the study of the structure of KMS states over some particular $C^*$-algebras, namely the ones arising from groupoids and actions of groupoids on graphs.

A KMS state on a $C^*$-algebra $A$ can be thought of as a generalization of a tracial state, but the trace condition is generalized in the presence of dynamics $\alpha : \mathbb{R} \curvearrowright A$. For a state $\phi$ on $A$, we say that $\phi$ satisfies the *KMS condition at inverse temperature* $\beta \in [0, \infty)$ with respect to the dynamics $\alpha$ if

$$\phi(xy) = \phi(y\alpha_{i\beta}x)$$

for every $y \in A$ and *analytic* $x \in A$, meaning that the function $\mathbb{R} \to A, t \mapsto \alpha_t(x)$ can be extended to the whole complex plane $\mathbb{C}$. The theme is that there is always a critical inverse temperature $\beta_c$ below which there are no KMS$_\beta$ states, and above $\beta_c$ the structure of the KMS simplex reflects some of the underlying combinatorial data.

This is in particular the case in our context, where $C^*$-algebras associated to self-similar groupoids are considered [63]. Roughly speaking, a self-similar action of a groupoid on a finite graph $E$ consists of a discrete groupoid $\mathcal{G}$ with unit space identified with the vertices of the graph, and a left action $\mathcal{G} \curvearrowright E^*$ of the groupoid on the path-space $E^*$ of $E$, with the property that for each element $g \in \mathcal{G}$ and each path $\mu \in E^*$ for which $g \cdot \mu$ is defined, there is a unique element $g|_\mu \in \mathcal{G}$ such that

$$g \cdot (\mu\nu) = (g \cdot \mu)(g|_\mu \cdot \nu)$$

for any other path $\nu \in E^*$. This reflects the self-similarity of the action. In [63], the authors show that the self-similar action can be used to transform an arbitrary trace on $C^*(\mathcal{G})$ into a new trace that extends to a KMS state on the Toeplitz algebra $\mathcal{T}(\mathcal{G}, E)$(see subsection 5.1.2), and that this transformation is an isomorphism of the simplex of normalized traces $\mathrm{Tr}(C^*(\mathcal{G}))$ onto the KMS-simplex of $\mathcal{T}(\mathcal{G}, E)$. This chapter is motivated by the observation that this transformation can be considered as a self-mapping of the simplex $\mathrm{Tr}(C^*(\mathcal{G}))$, and so can be iterated. The main result of this chapter is Theorem 5.2.1, which sheds some light on $\mathrm{Tr}(C^*(\mathcal{G}))$. We show that, under certain hypotheses, there exists a 'preferred' trace defined over the groupoid $C^*$-algebra $C^*(\mathcal{G})$, which turns out to be the fixed point of the previous self-mapping $\mathrm{Tr}(C^*(\mathcal{G})) \to \mathrm{Tr}(C^*(\mathcal{G}))$, and that it is precisely the unique trace that extends to a KMS state at the critical inverse temperature $\beta_c$, see Section 5.1.

The main motivation for doing so was to observe that the lamplighter group $\Gamma$ can be thought of as a self-similar group(oid) acting faithfully on a specific graph $E_\Gamma$ given by one vertex and two loops (see Section 5.3), so the hypotheses of our main result applies, giving a 'preferred' trace over $C^*(\Gamma)$ which coincides with the canonical trace tr defined on $C^*(\Gamma)$ by the rule $\mathrm{tr}(u_g) = \delta_{g,e}$, where $e$ is the unit element of $\Gamma$, and $\delta$ is the Kronecker delta. This result is given in Proposition 5.3.1. This connection between the canonical trace over $K[\Gamma] \subseteq C^*(\Gamma)$ and KMS states over a bigger $C^*$-algebra opens a possible analytical approach to attack the problem of computing $l^2$-Beti numbers arising from $\Gamma$, since they are defined to be the value of the trace on the von Neumann group algebra extending the canonical trace tr on projections.

[2] A field with involution $(K, *)$ is called *$*$-Pythagorean* if for any $x, y \in K$, there always exists an element $z \in K$ such that $z^*z = x^*x + y^*y$.

The results of this chapter have been published in an article at the *Journal of Mathematical Analysis and Applications* (2018) [16].

Almost the entire work from this last chapter has been done during a research stay of four months[3] at the School of Mathematics and Applied Statistics from the University of Wollongong, New South Wales (Australia), under the supervision of Professor Aidan Sims. The author would like to thank him and the people from the department in general for their kind hospitality.

[3]The stay had been partitioned into two parts: the first part had a duration of two months and was conducted in 2016, September-October; the second part also had a duration of two months, and was conducted in 2017, September-October.

# Contents

# Chapter 1

# Preliminaries

In this first chapter we will cover the basic notions and theory needed to follow the other chapters of the thesis. We will start by providing a brief introduction about the von Neumann dimension (whose values are, in different contexts, also called $l^2$-Betti numbers), together with the statements of the Atiyah problem, which concern about the question of what kind of values these dimensions can achieve.

We will also introduce the notion of von Neumann regular and $*$-regular rings, which play a central role in the following chapters, especially Chapters 2, 3 and 4. Special rings of this type carry a natural rank function, which can be used to construct a dimension function over the set of finitely generated (right) projective modules over it. One of the simplest examples one can think about these kind of rings are the matrix algebras over an arbitrary field, together with the usual rank of matrices. A not-so-trivial example is given by the algebra of (unbounded) affiliated operators of a prefixed finite von Neumann algebra $\mathcal{M}$. In section 1.2.2 we study this example in detail, while we will make use of it later on in Chapter 3.

We end this chapter by discussing a little bit the different (noncommutative) theories about localization and quasi-invertibility in noncommutative rings, ranging from classical theories to modern ones.

## 1.1 $l^2$ -Betti numbers and the Atiyah problem

When studying Riemannian manifolds $M$ endowed with a free cocompact action $G \curvearrowright M$ of a discrete countable group $G$, Michael Atiyah introduced in 1976 [7] a certain kind of homology on $M$, which is nowadays called *$l^2$-homology*. Atiyah's main motivation was to generalize the Atiyah-Singer Index Theorem to the noncompact setting. Due to its definition, we can apply different tools from functional analysis, for example the theory of von Neumann algebras and Hilbert modules, in order to define a notion of dimension on the resulting $l^2$-homology. This new notion of dimension turned out to be a homological invariant.

Atiyah computed several values of dimensions of this kind, called $l^2$-Betti numbers, in numerous examples, and all of them turned out to be rational numbers. This gave rise to the following natural question.

**Question 1.1.1** (Atiyah)**.** Is it possible to obtain irrational values of $l^2$-Betti numbers?

That question was the beginning of what is now called the *Atiyah Conjecture*, sometimes also called the *Atiyah problem*. Since then, this question has evolved, and different researchers on this topic asked more concrete questions about the possible values of such numbers; the collection of all these questions are referred to as the Atiyah problem. We will give an overview of the problem in Section 1.1.4.

We would like to define $l^2$-Betti numbers from a historical point of view by first defining the classical Betti numbers, followed by an equivalent definition of $l^2$-Betti numbers that the one given by Atiyah. We refer the reader to the original paper of Atiyah [7] and to Lück's book [73] for different definitions of $l^2$-Betti numbers and their equivalence, together with an extensive theory and applications of them to the fields of geometry and $K$-theory.

### 1.1.1 Classical Betti numbers

Let $X$ be a finite CW-complex, and write $X = \bigcup_{k=0}^{n} X^k$ where $X^k$ denotes the $k^{\text{th}}$ skeleton of $X$. So $X^k \backslash X^{k-1}$ consists exactly on the $k$-dimensional cells $e_1^k, ..., e_{n_k}^k$. The *Euler characteristic* $\chi(X)$ of $X$ is defined to be the alternating sum $\sum_{k=0}^{n}(-1)^k n_k$. It generalizes the familiar formula *vertices - edges + faces* for polyhedra.

We put $C_k(X)$ for the free $\mathbb{Z}$-module with basis the $k$-cells $\{e_1^k, ..., e_{n_k}^k\}$, that is $C_k(X) = \bigoplus_{i=1}^{n_k} \mathbb{Z}e_i^k$. After choosing a particular orientation on each $k$-dimensional cell, we get a chain complex

$$\cdots \to C_{k+1}(X) \xrightarrow{\partial_{k+1}} C_k(X) \xrightarrow{\partial_k} C_{k-1}(X) \to \cdots$$

where each connecting map is defined by $\partial_k(e_i^k) = \sum_{\{j : e_j^{k-1} \in \partial e_i^k\}} \epsilon_j e_j^{k-1}$, where $\epsilon_j$ is either 1 or $-1$ depending on the orientation chosen for the cell $e_j^{k-1}$. Therefore we can consider the $k^{\text{th}}$ homology group

$$H_k(X; \mathbb{Z}) = \ker(\partial_k)/\mathrm{Im}(\partial_{k+1}),$$

which has the structure of an abelian group, or $\mathbb{Z}$-module. The $k^{\text{th}}$ Betti number $\beta_k(X; \mathbb{C})$ of $X$ is defined to be the $\mathbb{C}$-dimension of the complexification $H_k(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}$, i.e. $\beta_k(X) = \dim_{\mathbb{C}}(H_k(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C})$. Equivalently, it is the rank of $H_k(X; \mathbb{Z})$ as a $\mathbb{Z}$-module.

**Examples 1.1.2.**

1) For the torus $X = S^1 \times S^1$, one has $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$, $H_1(X; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$, $H_2(X; \mathbb{Z}) \cong \mathbb{Z}$ and $H_k(X; \mathbb{Z}) = 0$ for $k \geq 3$, so

$$\beta_0(X) = 1, \quad \beta_1(X) = 2, \quad \beta_2(X) = 1 \quad \text{and} \quad \beta_k(X) = 0 \text{ for } k \geq 3.$$

More generally, for the $n$-torus $X = T^n = S^1 \times \overset{n}{\cdots} \times S^1$ one makes use of the Künneth's formula

$$H_k(X; \mathbb{Z}) \cong \bigoplus_{i_1 + \cdots + i_l = k} H_{i_1}(S^1; \mathbb{Z}) \otimes \cdots \otimes H_{i_l}(S^1; \mathbb{Z})$$

to obtain the homology of $X$: it is given by $H_k(X; \mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{k}}$ for $0 \leq k \leq n$ and 0 otherwise, so

$$\beta_k(X) = \binom{n}{k} \text{ for } 0 \leq k \leq n, \quad \text{and 0 otherwise.}$$

2) For the $n$-sphere $X = S^n$, one has $H_0(X, \mathbb{Z}) \cong \mathbb{Z}$, $H_n(X; \mathbb{Z}) \cong \mathbb{Z}$ and the others are 0, so

$$\beta_0(X) = 1, \quad \beta_n(X) = 1 \quad \text{and 0 otherwise.}$$

When tensoring $H_k(X; \mathbb{Z})$ with $\mathbb{C}$ we are getting rid of the torsion part of the $k^{\text{th}}$ homology group, so the Betti numbers do not take into account any torsion in the homology. Nevertheless, they are useful topological invariants.

**Theorem 1.1.3** (Properties of Betti numbers)**.** *Let $X, Y$ be two finite connected CW-complexes.*

*(1) Homotopy invariance: if $X, Y$ are homotopy equivalent, then $\beta_k(X) = \beta_k(Y)$.*

*(2) Euler-Poincaré formula: we have $\chi(X) = \sum_{k=0}^{n}(-1)^k \beta_k(X)$.*

*(3) Poincaré duality: if $X = M$ is a closed manifold of dimension $n$, then $\beta_k(M) = \beta_{n-k}(M)$.*

*(4) Künneth's formula: $\beta_k(X \times Y) = \sum_{i+j=k} \beta_i(X)\beta_j(Y)$.*

*(5) $\beta_0(X) = 1$.*

*Proof.* These are well-known results and their proofs can be found in any Algebraic Topology introductory book. □

However, given an invariant for a finite CW-complex $X$, one can extract much more information by passing to the universal cover $\widetilde{X}$ of $X$ and defining an analogous invariant taking into account the action of the fundamental group $\pi = \pi_1(X)$ on $\widetilde{X}$. The $l^2$-Betti numbers arise from this principle applied to the classical Betti numbers.

### 1.1.2 Digression on group von Neumann algebras and Hilbert modules

Let $G$ be a discrete countable group. For any subring $R \subseteq \mathbb{C}$ closed under complex conjugation, one can form the *group $*$-ring* of $G$ with coefficients in $R$, $R[G]$, defined to be the set of finite $R$-combinations of elements of $G$, i.e. consisting of formal finite sums $\sum_\gamma a_\gamma \gamma$ with $a_\gamma \in R$. The sum operation is defined pointwise, the product is induced by the group multiplications (and distributive with respect to the sum), and the $*$-operation is defined towards the rule $(a_\gamma \gamma)^* = \overline{a_\gamma}\gamma^{-1}$.

One can also form the Hilbert space $l^2(G)$ consisting of all square-summable functions $f : G \to \mathbb{C}$ with obvious addition and scalar product, and inner product defined by $\langle f, g\rangle_{l^2(G)} = \sum_\gamma f(\gamma)\overline{g(\gamma)}$. It has a natural basis, naturally identified with $G$, consisting of indicator functions $\xi_\gamma \in l^2(G)$, defined to be 1 over the element $\gamma$ and 0 otherwise.

Observe that $G$ acts faithfully on $l^2(G)$ by right and left multiplication, giving representations of $G$ as bounded operators on $l^2(G)$. We will denote these representations by $\rho : G \to \mathcal{B}(l^2(G))$ and $\lambda : G \to \mathcal{B}(l^2(G))$ respectively. They are commonly called the *right/left regular representations* of $G$ respectively, and their actions are given specifically by

$$(\rho(\gamma)f)(\delta) = f(\delta\gamma) \quad \text{and} \quad (\lambda(\gamma)f)(\delta) = f(\gamma^{-1}\delta), \qquad \text{for } f \in l^2(G), \delta \in G.$$

Note that either $\lambda$ or $\rho$ extend to actions of $R[G]$ on $l^2(G)$ preserving the $*$-operation, namely for an element $T \in R[G]$, the adjoint operator of $\lambda(T)$ (resp. $\rho(T)$) is precisely $\lambda(T^*)$ (resp. $\rho(T^*)$), so we can actually identify $R[G]$ with the image of $\lambda$ (resp. the image of $\rho$) inside $\mathcal{B}(l^2(G))$.

We denote by $\mathcal{N}(G)$ the weak-completion of $\lambda(\mathbb{C}[G])$ inside $\mathcal{B}(l^2(G))$, which is called the *group von Neumann algebra of* $G$. An equivalent algebraic definition can be given, as follows. For a $G$-equivariant bounded operator $T$ we mean a bounded operator on $l^2(G)$ such that $T(\rho(\gamma)f) = \rho(\gamma)T(f)$ for every $f \in l^2(G)$ and $\gamma \in G$ (equivalently, $\rho(\gamma) \circ T = T \circ \rho(\gamma)$ for every $\gamma \in G$). Then $\mathcal{N}(G)$ consists exactly on the set of all $G$-equivariant bounded operators, sometimes denoted also by $\mathcal{B}(l^2(G))^G$.

An important property of the group von Neumann algebra is that it carries a canonical trace $\mathrm{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$, defined through the inner product on $l^2(G)$ by

$$\mathrm{tr}_{\mathcal{N}(G)}(T) = \langle T(\xi_e), \xi_e\rangle_{l^2(G)}.$$

Note that for an element $T = \sum_\gamma a_\gamma \gamma \in R[G]$ its trace is simply $\mathrm{tr}_{\mathcal{N}(G)}(T) = a_e$, the coefficient of the unit element $e \in G$. In the next proposition we show that this trace is:

a) normal, in the sense of [84, Section 3.6]: for each bounded, monotone increasing net of self-adjoint operators $\{T_\alpha\}_\alpha$ in $\mathcal{N}(G)$ with strong limit $T \in \mathcal{N}(G)$, the net $\{\mathrm{tr}_{\mathcal{N}(G)}(T_\alpha)\}_\alpha$ converges to $\mathrm{tr}_{\mathcal{N}(G)}(T)$;

b) positive: $\mathrm{tr}_{\mathcal{N}(G)}(T^*T) \geq 0$ for every $T \in \mathcal{N}(G)$;

c) faithful: if $\mathrm{tr}_{\mathcal{N}(G)}(T^*T) = 0$ for some $T \in \mathcal{N}(G)$, then $T = 0$.

**Proposition 1.1.4.** *The trace $\mathrm{tr}_{\mathcal{N}(G)}$ is normal, faithful and positive. Therefore $\mathcal{N}(G)$ becomes a finite von Neumann algebra. In fact $\mathcal{N}(G)$ is a type $II_1$ factor if and only if all nontrivial conjugacy classes of $G$ are infinite (that is, $G$ is what is called an ICC group).*

*Proof.* Note that, if $\{T_n\}_n$ is a sequence of operators inside $\lambda(\mathbb{C}[G])$ converging weakly to $T \in \mathcal{N}(G)$, then by definition $\mathrm{tr}_{\mathcal{N}(G)}(T_n) = \langle T_n(\xi_e), \xi_e\rangle_{l^2(G)} \xrightarrow{n} \langle T(\xi_e), \xi_e\rangle_{l^2(G)} = \mathrm{tr}_{\mathcal{N}(G)}(T)$. Hence to prove the trace property it is enough to prove it for elements from $\mathbb{C}[G]$, because for general operators $T, S \in \mathcal{N}(G)$ just take two sequences $T_n \xrightarrow{n} T$, $S_m \xrightarrow{m} S$ inside $\mathbb{C}[G]$, and then

$$\mathrm{tr}_{\mathcal{N}(G)}(TS) = \lim_n \lim_m \mathrm{tr}_{\mathcal{N}(G)}(T_n S_m) = \lim_n \lim_m \mathrm{tr}_{\mathcal{N}(G)}(S_m T_n) = \mathrm{tr}_{\mathcal{N}(G)}(ST).$$

By linearity of $\mathrm{tr}_{\mathcal{N}(G)}$, it is enough to prove the equality $\mathrm{tr}_{\mathcal{N}(G)}(\gamma\delta) = \mathrm{tr}_{\mathcal{N}(G)}(\delta\gamma)$ for $\gamma, \delta \in G$; but this is straightforward:

$$\mathrm{tr}_{\mathcal{N}(G)}(\gamma\delta) = \langle \xi_{\gamma\delta}, \xi_e\rangle = \delta_{\gamma,\delta^{-1}} = \langle \xi_{\delta\gamma}, \xi_e\rangle = \mathrm{tr}_{\mathcal{N}(G)}(\delta\gamma).$$

Hence $\mathrm{tr}_{\mathcal{N}(G)}$ is indeed a trace. Normality is proven in exactly the same way as the first computation of the above argument. Let $\{T_\alpha\}_\alpha$ be a bounded, monotone increasing net of self-adjoint operators in $\mathcal{N}(G)$ with strong limit $T \in \mathcal{N}(G)$, so that $T_\alpha(f) \xrightarrow{\alpha} T(f)$ for every $f \in l^2(G)$; then

$$\mathrm{tr}_{\mathcal{N}(G)}(T_\alpha) = \langle T_\alpha(\xi_e), \xi_e\rangle_{l^2(G)} \xrightarrow{\alpha} \langle T(\xi_e), \xi_e\rangle_{l^2(G)} = \mathrm{tr}_{\mathcal{N}(G)}(T),$$

and $\mathrm{tr}_{\mathcal{N}(G)}$ is normal. For positivity, note that for $T \in \mathcal{N}(G)$, $\mathrm{tr}_{\mathcal{N}(G)}(T^*T) = \langle T^*T(\xi_e), \xi_e\rangle = ||T(\xi_e)||^2 \geq 0$. Finally, faithfulness follows from this last computation: $T \in \mathcal{N}(G)$ satisfies $\mathrm{tr}_{\mathcal{N}(G)}(T^*T) = 0$ if and only if $T(\xi_e) = 0$, if and only if $0 = \rho(\gamma^{-1})T(\xi_e) = T(\rho(\gamma^{-1})\xi_e) = T(\xi_\gamma)$ for all $\gamma \in G$, if and only if $T = 0$.

The last statement can be found, for example, in [56, Section 8.6]. □

**Remark 1.1.5.** All the above statements can be easily extended to $k \times k$ matrices: the ring $M_k(R[G])$ acts faithfully on $l^2(G)^k$ in a natural way by letting a matrix act on a column by left multiplication, and then each entry of $M_k(R[G])$ acts on $l^2(G)$ by the right or left regular representations $\rho, \lambda$. We denote these extended actions by $\rho_k, \lambda_k$. We can therefore identify $M_k(R[G])$ with its image $\lambda_k(M_k(R[G])) \subseteq \mathcal{B}(l^2(G)^k)$. We will denote by $\mathcal{N}_k(G)$ the weak-completion of $M_k(\mathbb{C}[G])$ inside $\mathcal{B}(l^2(G)^k)$, which is easily seen to be equal to $M_k(\mathcal{N}(G))$. The previous trace can be extended to a (unnormalized) trace in $\mathcal{N}_k(G)$ by setting, for a matrix $T = (T_{ij}) \in \mathcal{N}_K(G)$,

$$\mathrm{Tr}_{\mathcal{N}_k(G)}(T) = \sum_{i=1}^{k} \mathrm{tr}_{\mathcal{N}(G)}(T_{ii}).$$

A *finitely generated Hilbert (right) G-module* will be any closed subspace $V$ of $l^2(G)^k$ for some $k \geq 1$, invariant by the right action $\rho_k$ of $M_k(\mathbb{C}[G])$, namely for $v \in V$, $\rho_k(T)(v) \in V$ for every $T \in M_k(\mathbb{C}[G])$. It is enough to demand that $\rho_k(\gamma \cdot e_{ij})(v) \in V$ for every $\gamma \in G$ and every matrix unit $e_{ij}$.

For a Hilbert $G$-module $V$, we can decompose our Hilbert space as an orthogonal sum $l^2(G)^k = V \oplus V^\perp$. Let $p_V : l^2(G)^k \to l^2(G)^k$ be the corresponding projection onto $V$.

**Lemma 1.1.6.** *$p_V$ belongs to $\mathcal{N}_k(G)$.*

*Proof.* Since $V$ is invariant under $\rho_k$, so is $V^\perp$. Therefore for $f = f_V + f_{V^\perp} \in l^2(G)^k$ and $T \in M_k(\mathbb{C}[G])$, we have the decomposition $\rho_k(T)f = \rho_k(T)f_V + \rho_k(T)f_{V^\perp} \in V \oplus V^\perp$, hence

$$p_V(\rho_k(T)f) = \rho_k(T)f_V = \rho_k(T)p_V(f).$$

This says that $p_V$ is a $G$-equivariant bounded operator, so it belongs to $\mathcal{N}_k(G)$. □

**Definition 1.1.7.** Let $V \leq l^2(G)^k$ be a Hilbert $G$-module. We define its *von Neumann dimension* as the trace of the projection $p_V$,

$$\dim_{vN}(V) = \mathrm{Tr}_{\mathcal{N}_k(G)}(p_V).$$

**Examples 1.1.8.**

1) For $G$ a finite group, $l^2(G) = \bigoplus_{\gamma\in G} \mathbb{C}\xi_\gamma \cong \mathbb{C}^{|G|}$. If we restrict to the case when $R = \mathbb{C}$, we have an isomorphism of $\mathbb{C}$-vector spaces $\mathbb{C}[G] \cong l^2(G)$ given by $\gamma \mapsto \xi_\gamma$. In this case $\mathcal{B}(l^2(G)) = M_{|G|}(\mathbb{C})$, and $\mathcal{N}(G) = \mathbb{C}[G]$ itself.

   Take $V \leq l^2(G)$ a Hilbert $G$-module, and write $\{v_1, ..., v_n\}$ for an orthonormal basis of $V$ (so $\dim_{\mathbb{C}}(V) = n$). Write $v_i = \sum_{\gamma\in G}\langle v_i, \xi_\gamma\rangle\xi_\gamma$. Since they form an orthonormal basis, we compute

$$1 = \langle v_i, v_i\rangle = \sum_{\gamma\in G} |\langle v_i, \xi_\gamma\rangle|^2.$$

   Here the projection $p_V : l^2(G) \to l^2(G)$ is given by $p_V(f) = \sum_{i=1}^n \langle f, v_i\rangle v_i \in V$. By invariance of $V$, $p_V$ is a $G$-equivariant operator, so $\langle p_V(\xi_\gamma), \xi_\gamma\rangle = \langle \rho(\gamma)p_V(\xi_e), \rho(\gamma)\xi_e\rangle = \langle p_V(\xi_e), \xi_e\rangle$. Hence

$$\dim_{vN}(V) = \mathrm{tr}_{\mathcal{N}(G)}(p_V) = \langle p_V(\xi_e), \xi_e\rangle = \frac{1}{|G|}\sum_{\gamma\in G}\langle p_V(\xi_\gamma), \xi_\gamma\rangle = \frac{1}{|G|}\sum_{i=1}^{n}\sum_{\gamma\in G}|\langle v_i, \xi_\gamma\rangle|^2 = \frac{n}{|G|} = \frac{\dim_{\mathbb{C}}(V)}{|G|}.$$

   In conclusion, for finite $G$ we recover the normalized dimension of $V$ as a $\mathbb{C}$-vector space.

2) Take $G$ to be an abelian group. One can define its Pontryagin dual $\widehat{G}$, which is the set of continuous homomorphisms $\phi : G \to \mathbb{T}$, also called characters. With the compact convergence topology, it is well-known that $\widehat{G}$ becomes a topological abelian group[1]. In fact, since $G$ has the discrete topology, $\widehat{G}$ is compact, so it carries a normalized Haar measure $\mu$.

[1]We refer the reader to [35, Chapter 4] for more information about Pontryagin duality.

Fourier transform gives an isomorphism between Hilbert spaces $\mathcal{F} : l^2(G) \to L^2(\widehat{G}, \mu)$, $\xi_\gamma \mapsto \widehat{\gamma}$, where $\widehat{\gamma}(\phi) = \phi(\gamma)$. Observe that $\mathcal{F}$ is unitary. This induces an isomorphism of operator algebras $\mathcal{B}(l^2(G)) \to \mathcal{B}(L^2(\widehat{G}, \mu))$ given by conjugation by $\mathcal{F}$, $T \mapsto \mathcal{F}T\mathcal{F}^*$. Recall that $L^\infty(\widehat{G}, \mu) \subseteq \mathcal{B}(L^2(\widehat{G}, \mu))$ as an abelian $*$-subalgebra, consisting of multiplication-by-$f$ operators $M_f : L^2(\widehat{G}, \mu) \to L^2(\widehat{G}, \mu)$ for $f \in L^\infty(\widehat{G}, \mu)$. Under this identification, it turns out that $\mathcal{F}\mathcal{N}(G)\mathcal{F}^* = L^\infty(\widehat{G}, \mu)$.

The trace $\operatorname{tr}_{L^\infty(\widehat{G},\mu)} : L^\infty(\widehat{G}, \mu) \cong \mathcal{N}(G) \to \mathbb{C}$ becomes

$$\operatorname{tr}_{L^\infty(\widehat{G},\mu)}(f) = \langle \mathcal{F}^* M_f \mathcal{F}(\xi_e), \xi_e \rangle_{l^2(G)} = \langle M_f \mathcal{F}(\xi_e), \mathcal{F}(\xi_e) \rangle_{L^2(\widehat{G},\mu)} = \int_{\widehat{G}} f(\phi) d\mu(\phi) \quad \text{for } f \in L^\infty(\widehat{G}, \mu).$$

In the particular case $G = \mathbb{Z}$, we have $\widehat{G} = \mathbb{T}$ and the trace becomes

$$\operatorname{tr}_{L^\infty(\mathbb{T},\mu)}(f) = \int_{\mathbb{T}} f(z) d\mu(z).$$

For any Borel subset $\mathcal{U} \subseteq \mathbb{T}$, we can form a Hilbert $\mathbb{Z}$-module $V = V_\mathcal{U} = L^2(\mathcal{U}, \mu) \leq L^2(\mathbb{T}, \mu)$ whose projection is simply $p_V = M_{\chi_\mathcal{U}}$, where $\chi_\mathcal{U}$ denotes the characteristic function of $\mathcal{U}$. Therefore its von Neumann dimension is

$$\dim_{vN}(V) = \operatorname{tr}_{L^\infty(\widehat{G},\mu)}(p_V) = \int_{\mathbb{T}} \chi_\mathcal{U} d\mu(z) = \mu(\mathcal{U}).$$

So in this particular case every real number $t \in [0, 1]$ can occur as the von Neumann dimension of some Hilbert $\mathbb{Z}$-module.

### 1.1.3 Cellular $l^2$-Betti numbers

Let's return to our previous setting. From now on $X$ will be a CW-complex of finite type (meaning that each skeleton $X^k$ is finite dimensional, but $X$ itself may be infinite dimensional). Let $p : \widetilde{X} \to X$ be its universal covering, and put $\pi = \pi_1(X)$ the fundamental group of $X$. We know that $\pi$ acts on $\widetilde{X}$ by deck transformations, so that $X$ is the quotient of $\widetilde{X}$ under this action.

The action $\pi \curvearrowright \widetilde{X}$ induces an action of $\pi$ on the $\mathbb{Z}$-module $C_k(\widetilde{X})$, taking $k$-cells to $k$-cells $\widetilde{e}_i^k \mapsto \widetilde{e}_i^k \cdot \gamma$. With this action, we can turn $C_k(\widetilde{X})$ into a right $\mathbb{Z}[\pi]$-module, where the boundary maps $\partial_k : C_k(\widetilde{X}) \to C_{k-1}(\widetilde{X})$ become $\mathbb{Z}[\pi]$-homomorphisms. Hence we obtain a cellular $\mathbb{Z}[\pi]$-chain complex

$$\cdots \to C_{k+1}(\widetilde{X}) \xrightarrow{\partial_{k+1}} C_k(\widetilde{X}) \xrightarrow{\partial_k} C_{k-1}(\widetilde{X}) \to \cdots$$

Equivalently, one views $C_k(\widetilde{X})$ as the free $\mathbb{Z}[\pi]$-module generated by the (lifts under $p$ of) $k$-cells of $X$, so $C_k(\widetilde{X}) \cong C_k(X) \otimes_\mathbb{Z} \mathbb{Z}[\pi]$.

**Definition 1.1.9.** We define the *cellular $l^2$-chain complex* of $\widetilde{X}$ by

$$C_k^{(2)}(\widetilde{X}) = C_k(\widetilde{X}) \otimes_{\mathbb{Z}[\pi]} l^2(\pi),$$

where we take the $\mathbb{Z}[\pi]$-module structure on $l^2(\pi)$ given by the left action $\lambda : \mathbb{Z}[\pi] \to \mathcal{B}(l^2(\pi))$.

If we pick a cellular basis for $C_k(\widetilde{X})$ one obtain isomorphisms $C_k^{(2)}(\widetilde{X}) \cong l^2(\pi)^k$ as right $\mathbb{Z}[\pi]$-modules. This induces the structure of a Hilbert $\pi$-module on $C_k^{(2)}(\widetilde{X})$, and the previous cellular $\mathbb{Z}[\pi]$-chain complex becomes a *Hilbert chain complex*

$$\cdots \to C_{k+1}^{(2)}(\widetilde{X}) \xrightarrow{\partial_{k+1}^{(2)}} C_k(\widetilde{X})^{(2)} \xrightarrow{\partial_k^{(2)}} C_{k-1}^{(2)}(\widetilde{X}) \to \cdots$$

that is, every map $\partial_k^{(2)} = \partial_k \otimes \mathrm{id}$ is a bounded, $\pi$-equivariant operator between Hilbert $\pi$-modules. One can therefore define the *$l^2$-homology groups* $H_k^{(2)}(\widetilde{X}) = \ker(\partial_k^{(2)})/\overline{\operatorname{Im}(\partial_{k+1}^{(2)})}$. Since we are modding out by the closure of the image of the operator $\partial_{k+1}^{(2)}$, this has the effect that $H_k^{(2)}(\widetilde{X})$ inherits the structure of a Hilbert $\pi$-module, since it is in fact isometrically isomorphic to $\ker(\partial_k^{(2)}) \cap \operatorname{Im}(\partial_{k+1}^{(2)})^\perp$.

**Definition 1.1.10.** We define the $k^{\text{th}}$ $l^2$-Betti number of $X$ to be the von Neumann dimension of the $k^{\text{th}}$ $l^2$-homology group,

$$\beta_k^{(2)}(\widetilde{X}) = \dim_{vN}(H_k^{(2)}(\widetilde{X})).$$

**Theorem 1.1.11** (Properties of $l^2$-Betti numbers). *Let $X, Y$ be two finite connected CW-complexes, and let $\widetilde{X}, \widetilde{Y}$ be the corresponding universal coverings.*

*(1) Homotopy invariance: if $X, Y$ are homotopy equivalent, then $\beta_k^{(2)}(\widetilde{X}) = \beta_k^{(2)}(\widetilde{Y})$.*

*(2) Euler-Poincaré formula: we have $\chi(X) = \sum_{k=0}^n (-1)^k \beta_k^{(2)}(\widetilde{X})$.*

*(3) Poincaré duality: if $X = M$ a closed manifold of dimension $n$, then $\beta_k^{(2)}(\widetilde{M}) = \beta_{n-k}^{(2)}(\widetilde{M})$.*

*(4) Künneth's formula: $\beta_k^{(2)}(\widetilde{X \times Y}) = \sum_{i+j=k} \beta_i^{(2)}(\widetilde{X})\beta_j^{(2)}(\widetilde{Y})$.*

*(5) $\beta_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|}$, where we use the convention that $\frac{1}{|\pi|} = 0$ if $|\pi| = \infty$.*

*(6) Finite coverings: if $X \to Y$ is a finite covering with $d$ sheets, then $\beta_k^{(2)}(\widetilde{X}) = d \cdot \beta_k^{(2)}(\widetilde{Y})$.*

*Proof.* [73, Theorem 1.35]. □

Observe that when $\pi$ is a finite group, in view of Example 1.1.8.1),

$$\beta_k^{(2)}(\widetilde{X}) = \dim_{vN}(H_k^{(2)}(\widetilde{X})) = \frac{1}{|\pi|}\dim_{\mathbb{C}}(H_k^{(2)}(\widetilde{X})) = \frac{1}{|\pi|}\beta_k(\widetilde{X})$$

and we recover the classical Betti numbers for the space $\widetilde{X}$.

Historically, the $l^2$-Betti numbers of the universal cover $\widetilde{M} \to M$ of a closed Riemannian manifold $M$ were first defined by Atiyah in [7] in connection with his $L^2$-Index Theorem, by using the heat kernel defined on $k$-forms on $\widetilde{M}$. We refer the reader to [73, Chapter 1] for the connection between Atiyah's original definition of $l^2$-Betti numbers and our definition using cellular Hilbert chain complexes.

### 1.1.4 $l^2$-Betti numbers for group rings and the Atiyah problem

Let $G$ be again a discrete, countable group. In the particular case of matrix group rings $M_k(K[G])$, being $K$ a subfield of the complex numbers closed under complex conjugation, every matrix operator $A \in M_k(K[G])$ gives rise to an $l^2$-Betti number, in the following way: consider $A$ as an operator $A : l^2(G)^k \to l^2(G)^k$ acting on the left and take $p_A \in \mathcal{N}_k(G)$ to be the projection onto the kernel of $A$, which is a Hilbert $G$-module (so indeed $p_A$ belongs to the von Neumann algebra $\mathcal{N}_k(G)$). Therefore one can consider the von Neumann dimension of $\ker(A)$, which is simply the trace of the projection $p_A$.

**Definition 1.1.12.** A real positive number $r$ is called an *$l^2$-Betti number arising from $G$ with coefficients in $K$* if for some integer $k \geq 1$, there exists a matrix operator $A \in M_k(K[G])$ such that

$$\dim_{vN}(\ker(A)) = \mathrm{Tr}_{\mathcal{N}_k(G)}(p_A) = r.$$

We denote the set of all $l^2$-Betti numbers arising from $G$ with coefficients in $K$ by $\mathcal{C}(G, K)$. It should be noted that this set is always a subsemigroup of $(\mathbb{R}^+, +)$, for if $A_1 \in M_{k_1}(K[G])$ and $A_2 \in M_{k_2}(K[G])$ are two matrix operators such that $\dim_{vN}(\ker(A_1)) = r_1$ and $\dim_{vN}(\ker(A_2)) = r_2$, then $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \in M_{k_1+k_2}(K[G])$ has von Neumann dimension $r_1 + r_2$.

One of the main problems (following this line) is to actually compute the whole set $\mathcal{C}(G, K)$ for fixed $G$, $K$. We are now ready to state the Atiyah Conjecture.

**Conjecture 1.1.13** (The Atiyah Conjecture). *Let $A \in M_k(\mathbb{Q}[G])$. Then $\dim_{vN}(\ker(A)) \in \mathbb{Q}$.*

It is now a well-known fact that this conjecture is false, see for example [8], [41]. However, while some researchers were working on the conjecture before knowing that it was false, they proposed other refined versions of it, and nowadays some of them are still open.

**Conjecture 1.1.14** (The Strong Atiyah Conjecture with coefficients in $K$). *Let $A \in M_k(K[G])$. Then $\dim_{vN}(\ker(A))$ belongs to the subgroup of $\mathbb{Q}$ generated by all the elements $\frac{1}{|H|}$, where $H$ ranges over all the finite subgroups of $G$.*

In this generality, the Strong Atiyah Conjecture does not hold, as proven by R. I. Grigorchuk and A. Żuk in [42], followed by W. Dicks and T. Schick ([25]). Nevertheless, the conjecture is still open for the class of groups such that there exists an upper bound on the orders of their finite subgroups.

**Conjecture 1.1.15** (The Strong Atiyah Conjecture with coefficients in $K$, refined). *Let $A \in M_k(K[G])$. Assume that there exists an upper bound for the orders of finite subgroups of $G$, and let $\text{lcm}(G)$ be the least common multiple of such orders. Then $\dim_{vN}(\ker(A)) \in \frac{1}{\text{lcm}(G)}\mathbb{Z}$.*

This version of the Strong Atiyah Conjecture has been verified in many cases, see for example [71].

We are not going to study these conjectures in full generality. In fact, in Chapter 3 we will exhibit some positive real numbers (most of them irrational and even transcendental) that can appear in $\mathcal{C}(\Gamma, K)$, where $\Gamma$ is the so-called *lamplighter group*, and $K \subseteq \mathbb{C}$ is any subfield of the complex numbers closed under complex conjugation.

Actually, in the same chapter, we will define an analogous set (denoted by $\mathcal{C}(\mathcal{A})$) consisting of positive real numbers that can be achieved by taking *ranks* of matrices over $\mathcal{A}$; we will explain all the details more carefully in the subsequent chapters, but here $\mathcal{A}$ will denote a specific $\mathbb{Z}$-crossed product $*$-algebra, endowed with a 'natural' rank function $\text{rk}_{\mathcal{A}}$, which we will construct in Chapter 2. In fact, due to Proposition 3.1.4, we will deduce that $\mathcal{C}(\mathcal{A})$ and $\mathcal{C}(G, K)$ actually coincide in some cases of interest, hence giving an alternative way of computing $\mathcal{C}(\Gamma, K)$.

## 1.2 Von Neumann regular rings, $*$-regular rings and rank functions

J. Von Neumann introduced the concept of regular rings in his study of rings of operators on Hilbert spaces ([77, 78, 79, 80]), which lead him and F. J. Murray to the discovery of a new mathematical structure which possessed a dimension function. This work led him to the discovery of a new structure with properties resembling those of the lattice $L_n$ that one can form by taking all the linear subspaces of an $n$-dimensional projective space. Previously, K. Menger and G. Birkhoff ([75], [14]) already did this step: they characterized these lattices, forming a class which we will denote by $\mathcal{L}_f$, to be exactly the class of all *complemented, modular, irreducible* lattices satisfying some *chain condition*, so any such a lattice is isomorphic to $L_n$ for some finite $n \geq 1$. These lattices were called *projective geometries*.

In his book [83], Von Neumann dropped this last assumption and added two weaker axioms (that the chain condition already implies), namely the *completeness* property and a certain *continuity* condition. The corresponding lattices satisfying such axioms but *not* satisfying the chain condition were called *continuous geometries*, which forms a class denoted by $\mathcal{L}_\infty$. These new structures resemble the well-known finite dimensional projective geometries $L_n$.

It turns out that one can construct a dimension function (uniquely determined by some normalization conditions) on either projective or continuous geometries, and the surprise was that in this new setting, the set of real numbers achieved from the dimension function constructed for a lattice $L \in \mathcal{L}_\infty$ was seen to fill the whole interval $[0, 1]$, rather than taking only a finite set of values $\{0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\}$, which was already the case for the lattices $L_n \in \mathcal{L}_f$. Such a dimension function can be constructed after defining the notion of *equidimensionality* between elements $x, y \in L \in \mathcal{L}_f \cup \mathcal{L}_\infty$:

$x, y \in L$ are said to be *equidimensional* if they have a common complement $z \in L$.[2]

As its own definition suggests, $x, y$ are equidimensional if and only if their dimensions are the same. We will give the notion of a dimension function later on.

From now on we will follow von Neumann's book [83] on continuous geometries. Just for completeness, we state the definition given by von Neumann of his new axiomatization of lattices. For us, a *continuous geometry* will be a lattice satisfying this definition, so the projective geometries are also included.

[2]See Definition 1.2.1 *d*) for the definition of a complement.

**Definition 1.2.1.** A lattice $(L, \leq)$ is called *complete, complemented, continuous, modular* and *irreducible* (or simply a *continuous geometry*) if the following axioms are satisfied:

a) Modularity: $x \leq z$ implies $(x \vee y) \wedge z = x \vee (y \wedge z)$ for every $y \in L$.

b) Completeness: for every subset $S \subseteq L$, there exists an element $\bigvee S \in L$ which is a *least upper bound* for $S$. Dually, there exists an element $\bigwedge S \in L$ which is a *greatest lower bound* for $S$. We set $0 = \bigwedge L$ and $1 = \bigvee L$.

   We will also denote by $S_1 \vee S_2$ the element $\bigvee(S_1 \cup S_2)$, and dually for $\wedge$.

c) Continuity: for every net $\{x_\alpha\}_\alpha$ of elements of $L$ satisfying either $(i)$ $\alpha < \beta$ implies $x_\alpha \leq x_\beta$ or $(ii)$ $\alpha < \beta$ implies $x_\alpha \geq x_\beta$, then for every $y \in L$

$$\left(\bigvee\{x_\alpha\}_\alpha\right) \wedge y = \bigvee(\{x_\alpha\}_\alpha \wedge y) \quad \text{and} \quad \left(\bigwedge\{x_\alpha\}_\alpha\right) \vee y = \bigwedge(\{x_\alpha\}_\alpha \vee y).$$

d) Complementation: for every $x \in L$ there exists an element $y \in L$ satisfying $x \vee y = 1$ and $x \wedge y = 0$. We refer to such an element $y$ as a complement of $x$.

e) Irreducibility: if $x \in L$ has a unique complement, then either $x = 0$ or 1.

As we have already mentioned, it is possible to define a normalized dimension function over $L$, namely a map $\dim : L \to [0, 1]$ satisfying the following properties:

a) $\dim(0) = 0$, $\dim(1) = 1$.

b) Two elements $x, y \in L$ are equidimensional if and only if $\dim(x) = \dim(y)$.

c) If $x \leq y$, then $\dim(x) \leq \dim(y)$.

d) $\dim(x \wedge y) + \dim(x \vee y) = \dim(x) + \dim(y)$ for every $x, y \in L$.

Theorems 6.9 and 7.4 of [83] guarantees the existence and uniqueness of a dimension function defined over $L$.

It was well-known, before the work of von Neumann, that the lattice $L_n$ of an $n$-dimensional projective space gives rise to a division algebra $D$ sharing a close connection with $L_n$, and in fact he proved that, for $n \geq 4$, $L_n$ can be isomorphically identified with the lattice of all principal right ideals of $M_n(D)$ ([83, Part II, Chapter I]). This observation gave rise to the natural question of whether the same is true for the lattices $L \in \mathcal{L}_\infty$, so one faces the problem of finding a ring $R$ whose lattice of principal right ideals is isomorphic to $L$. It turns out that this can be indeed achieved, provided that the ring $R$ is *regular*. This was the beginning of a whole new theory.

We will review the general theory of regular rings, $*$-regular rings and rank functions defined on them. The major reference of this theory is Goodearl's book [39], apart from von Neumann's book [83].

A unital ring $R$ is called a *regular ring* if for every element $x \in R$ there exists $y \in R$ such that $x = xyx$. Note that, in this case, the element $e = xy$ is an idempotent and generates the same (right) ideal as $x$. In fact, a characterization for regular rings is that every finitely generated (right) ideal of $R$ is generated by a single idempotent (see [39, Theorem 1.1]). Regularity is closed under taking extensions, ideals[3], direct products, matrices, direct limits, among others.

Two idempotents $e, f \in R$ are said to be *equivalent*, denoted by $e \sim f$, if there exists an isomorphism $eR \cong fR$ as right $R$-modules. Equivalently, $e \sim f$ if there exist elements $x \in eRf$, $y \in fRe$ such that $e = xy$ and $f = yx$. To see this equivalence, note first that if $e = xy$, $f = yx$ for some elements $x \in eRf$, $y \in fRe$, then one can define a right $R$-module homomorphism $\varphi : eR \to fR$ given by left multiplication by $y$, so $\varphi(\alpha) = y\alpha$. It is clearly an isomorphism of $R$-modules with inverse given by left multiplication by $x$.

Conversely, if we take an isomorphism of $R$-modules $\varphi : eR \to fR$, then $\varphi(e) = f\widetilde{y}e$ for some $\widetilde{y} \in R$. Analogously $\varphi^{-1}(f) = e\widetilde{x}f$ for some $\widetilde{x} \in R$. Then the elements $x = e\widetilde{x}f$, $y = f\widetilde{y}e$ satisfy the required properties, since

$$e = \varphi^{-1}(\varphi(e)) = \varphi^{-1}(f\widetilde{y}e) = e\widetilde{x}f\widetilde{y}e = xy, \quad f = \varphi^{-1}(\varphi(f)) = \varphi^{-1}(e\widetilde{x}f) = f\widetilde{y}e\widetilde{x}f = yx.$$

[3] Since the definition of regularity on a unital ring does not concert the unit itself, the notion of a regular ideal is analogous: for any element $x$ of the ideal, there exists another element $y$, also in the ideal, such that $x = xyx$.

Regular rings have a rich structure concerning projective modules and idempotents; as one can notice directly from the definition, every element $x \in R$ gives rise to an idempotent, so we have (in principle) a huge source of idempotents in $R$. We state some of the important results. For a finitely generated (right) projective module $P$ over $R$, we denote by $L(P_R)$ the set of all finitely generated submodules of $P$, partially ordered by inclusion, which becomes a complemented, modular lattice with operations

$$A \vee B = A + B, \quad A \wedge B = A \cap B \qquad \text{for } A, B \in L(P_R)$$

([39, Theorem 2.3], [83, Theorem 2.4 of Part II]). In the case $P = R$, since every finitely generated submodule of $R$ is a right ideal of $R$ and $R$ is regular, $L(R_R)$ consists of all principal right ideals generated by a single idempotent, that is $L(R_R) = \{eR \mid e \in R \text{ is idempotent}\}$. If $R$ is simple, it is in particular indecomposable (as a ring), so by [83, Theorem 2.9 of Part II] $L(R_R)$ is *irreducible*. In this case the lattice satisfies axioms $a), b)$ (for finite sets), $d)$ and $e)$.

We now introduce the notion of pseudo rank functions on a regular ring $R$. In fact they can also be defined over any unital ring, but for now we are going to concentrate mainly in the regular case, because of their connection with dimension functions.

**Definition 1.2.2.** A *pseudo-rank function* on a (regular) ring is a real-valued function $\mathrm{rk} : R \to [0, 1]$ satisfying the following properties:

a) $\mathrm{rk}(0) = 0$, $\mathrm{rk}(1) = 1$.

b) $\mathrm{rk}(xy) \leq \mathrm{rk}(x), \mathrm{rk}(y)$ for every $x, y \in R$.

c) If $e, f$ are orthogonal idempotents, then $\mathrm{rk}(e + f) = \mathrm{rk}(e) + \mathrm{rk}(f)$.

If rk satisfies the additional property

d) $\mathrm{rk}(x) = 0$ if and only if $x = 0$,

then rk is called a *rank function* on $R$.

For general properties of pseudo-rank functions over regular rings one can consult [39, Chapter 16]. We summarize some of them into the following proposition.

**Proposition 1.2.3.** *Let $R$ be a regular ring and* rk *a pseudo-rank function on $R$.*

*(i) For elements $x_1, ..., x_n, y_1, ..., y_m \in R$, if $x_1R \oplus \cdots \oplus x_nR$ is isomorphic, as a right $R$-module, to a submodule of $y_1R \oplus \cdots \oplus y_mR$, then*

$$\sum_{i=1}^{n} \mathrm{rk}(x_i) \leq \sum_{j=1}^{m} \mathrm{rk}(y_j).$$

*If moreover it is isomorphic to the full right $R$-module $y_1R \oplus \cdots \oplus y_mR$, then we have equality above.*

*(ii) For any elements $x, y \in R$, $\mathrm{rk}(x + y) \leq \mathrm{rk}(x) + \mathrm{rk}(y)$.*

*(iii) If $u \in R$ is a unit in $R$, then $\mathrm{rk}(x) = \mathrm{rk}(ux)$ for every $x \in R$. As a consequence $\mathrm{rk}(x) = 1$ if $x$ is itself a unit. If moreover* rk *is a rank function, then the converse of this last statement holds, i.e. if $\mathrm{rk}(x) = 1$ for $x \in R$, then $x$ is itself a unit in $R$.*

*Proof.* For $(i)$ and $(ii)$ see [39, Proposition 16.1].

$(iii)$ Take $v \in R$ such that $1 = vu$. Then $\mathrm{rk}(x) = \mathrm{rk}(vux) \leq \mathrm{rk}(ux) \leq \mathrm{rk}(x)$, so we have equality. If $x \in R$ is itself a unit, so that $1 = yx$ for some unit $y \in R$, then $\mathrm{rk}(x) = \mathrm{rk}(yx) = \mathrm{rk}(1) = 1$. Now assume that rk is a rank function, and consider an idempotent $e \in R$ such that $xR = eR$. Then by $(i)$, $1 = \mathrm{rk}(x) = \mathrm{rk}(e)$, so that $\mathrm{rk}(1 - e) = 1 - \mathrm{rk}(e) = 0$. Since rk is a rank function, we must have $e = 1$, so $xR = R$. Analogously we get $Rx = R$, so $x$ is a unit in $R$. □

Every pseudo-rank function rk on a regular ring $R$ defines a pseudo-rank metric $d$ on $R$ by the rule $d(x,y) = \mathrm{rk}(x-y)$. If moreover rk is a rank function, then $d$ is a metric. Note that we can always achieve the situation where $d$ is indeed a metric by factoring through the set of elements having zero rank, i.e. $R \to R/\ker(\mathrm{rk})$. Since the ring operations are continuous with respect to this metric, one can consider the completion $\overline{R}$ of $R$ with respect to $d$. $\overline{R}$ is again a regular ring, and rk can be extended continuously to a rank function $\overline{\mathrm{rk}}$ on $\overline{R}$ such that, with the new metric induced by $\overline{\mathrm{rk}}$, $\overline{R}$ is also complete, and coincides with the natural metric on $\overline{R}$ inherited from the completion process. It turns out that the completion $\overline{R}$ is endowed with an additional ring-theoretic structure, as stated in the next proposition.

**Proposition 1.2.4** (Theorems 19.6 and 19.7 of [39]). *Let $R$ be a regular ring with a pseudo-rank function* rk. *Then the* rk*-completion $\overline{R}$ of $R$ is a regular, right and left self-injective ring. Moreover,* rk *extends uniquely to a continuous rank function $\overline{\mathrm{rk}}$ on $\overline{R}$, and $\overline{R}$ is complete in the $\overline{\mathrm{rk}}$-metric.*

The space of pseudo-rank functions $\mathbb{P}(R)$ on a regular ring $R$ is a Choquet simplex ([39, Theorem 17.5]), and the completion $\overline{R}$ of $R$ with respect to $\mathrm{rk} \in \mathbb{P}(R)$ is a simple ring if and only if rk is an extreme point in $\mathbb{P}(R)$ ([39, Theorem 19.14]).

Regular self-injective rings have a structure theory which somehow resembles the one classifying the factors in the theory of von Neumann algebras. We would like to give a summary of this classification in the ring-theoretic setting of regularity. From now on, $R$ will be a regular, right self-injective ring. An idempotent $e \in R$ is said to be *abelian/directly finite* if the corner $eRe$, which has $e$ as unit, is an *abelian/directly finite* ring[4]. $e$ is called *faithful* if the only central idempotent orthogonal to $e$ is 0.

I) $R$ is of Type $I$ if there exists a faithful abelian idempotent $e \in R$. It is of type $I_f$ if $R$ is directly finite, and of type $I_\infty$ if it is purely infinite[5].

II) $R$ is of Type $II$ if there are no nonzero abelian idempotents, but there exists a faithful directly finite idempotent $e \in R$. It is of type $II_f$ if $R$ is directly finite, and of type $II_\infty$ if it is purely infinite.

III) $R$ is of Type $III$ if there are no nonzero directly finite idempotents.

A characterization for Type $I_f$ is given in [39, Theorem 10.24]: every direct product of matrix rings $\prod_{k=1}^{\infty} M_{n_k}(R_k)$, where each $R_k$ is abelian, is of Type $I_f$, and conversely every Type $I_f$ ring $R$ is isomorphic to a ring of this form. In general, $R$ can be decomposed as a direct product of rings of each type purely.

**Theorem 1.2.5** (Theorem 10.22, together with Theorems 10.13 and 10.21 of [39]). *Any regular, right self-injective ring $R$ can be decomposed as*

$$R = (R_{1f} \times R_{1\infty}) \times (R_{2f} \times R_{2\infty}) \times R_3$$

*where $R_{1f}$ is of type $I_f$ and $R_{1\infty}$ is of type $I_\infty$, $R_{2f}$ is of type $II_f$ and $R_{2\infty}$ is of type $II_\infty$, and $R_3$ is of type $III$.*

In the special case that $R$ admits a rank function, $R$ becomes directly finite, for if $x, y$ are elements of $R$ satisfying $xy = 1$, then $1 = \mathrm{rk}(xy) \leq \mathrm{rk}(x) \leq 1$, so $\mathrm{rk}(x) = 1$ and part *(iii)* of Proposition 1.2.3 tells us that $x$ is a unit in $R$. The relation $xy = 1$ then implies that $y = x^{-1}$ is the inverse of $x$, so $yx = x^{-1}x = 1$. As a consequence, if $R$ is a regular, right self-injective ring admitting a rank function, then $R$ can be decomposed (following the notation above) as

$$R = R_{1f} \times R_{2f}.$$

As a particular case, due to Proposition 1.2.4, any rank completion $(\overline{R}, \overline{\mathrm{rk}})$ of a regular ring $R$ with pseudo-rank function rk is directly finite, so can be decomposed as

$$\overline{R} = \overline{R}_{1f} \times \overline{R}_{2f}.$$

A regular ring $R$ is said to satisfy the *comparability axiom* if for every $x, y \in R$, either $xR$ is isomorphic to a submodule of $yR$ or the other way around. Notationally, either $xR \lesssim yR$ or $yR \lesssim xR$. More generally, $R$ satisfies *general comparability* if for every $x, y \in R$ there exists a central idempotent $e \in R$ such that $exR \lesssim eyR$ and $(1-e)yR \lesssim (1-e)xR$. It is clear that the comparability axiom implies general comparability (take, for instance, either $e = 0$ or 1).

In fact, if a regular ring $R$ satisfies the comparability axiom (or general comparability), we can also compare finitely generated projective $R$-modules.

[4] A unital ring $R$ is called *directly finite* if whenever one has $xy = 1$ for some $x, y \in R$, then $yx = 1$.
[5] A ring $R$ is called *purely infinite* if there are no directly finite central idempotents.

**Proposition 1.2.6** (Propositions 8.2 and 8.8 of [39]). *Let $R$ be a regular ring satisfying* (1) *the comparability axiom or* (2) *general comparability, and let $P, Q$ be two finitely generated projective (right) $R$-modules. Then*

*(i) If $R$ satisfies* (1)*, then $P$ and $Q$ are comparable, in the sense that either $P \lesssim Q$ or $Q \lesssim P$.*

*(ii) If $R$ satisfies* (2)*, then there exists a central idempotent $e \in R$ such that $eP \lesssim eQ$ and $(1-e)Q \lesssim (1-e)P$.*

By [39, Theorem 9.14], every regular, right self-injective ring $R$ satisfies general comparability. If moreover $R$ is simple, then $R$ satisfies the comparability axiom, since in this case the only central idempotents of $R$ are 0 and 1: if $e \in R$ is any central idempotent, then $R = eR \oplus (1-e)R$, so $R$ being simple implies that either $e = 0$ or $1 - e = 0$. In this case, if $R$ carries a rank function rk, we can characterize equivalence of idempotents in terms of the values of their ranks only.

**Proposition 1.2.7.** *Let $R$ be a regular ring satisfying the comparability axiom and admitting a rank function* rk*. Then two idempotents $e, f \in R$ are equivalent if and only if* $\mathrm{rk}(e) = \mathrm{rk}(f)$*. Moreover,* rk *is the unique rank function that $R$ can admit.*

*In particular, this is the case when $R$ is a simple, regular, right and left self-injective ring.*

*Proof.* By part $(i)$ of Proposition 1.2.3, if two idempotents are equivalent then they have the same rank.

Conversely, assume that $\mathrm{rk}(e) = \mathrm{rk}(f)$. Since $R$ satisfies the comparability axiom then either $eR \lesssim fR$ or $fR \lesssim eR$. We can assume without loss of generality that $eR \lesssim fR$. Take an injective right $R$-module homomorphism $\varphi : eR \to fR$. Then $\varphi(eR) = \varphi(e)R$, so $eR \cong \varphi(e)R$ as right $R$-modules. Take $\widetilde{g} \in fR$ an idempotent such that $\varphi(e)R = \widetilde{g}R \leq fR$. Then $g := \widetilde{g}f$ is an idempotent ($g^2 = \widetilde{g}f\widetilde{g}f = \widetilde{g}f = g$) such that $gf = \widetilde{g}f = g$, $\quad fg = f\widetilde{g}f = \widetilde{g}f = g$, so $g \leq f$. Moreover, $g\widetilde{g} = \widetilde{g}f\widetilde{g} = \widetilde{g}$ and $\widetilde{g}g = \widetilde{g}f = g$, so $\widetilde{g}R = gR$. Therefore we obtain the decomposition

$$fR = gR \oplus (f-g)R = \varphi(e)R \oplus (f-g)R.$$

By taking ranks and applying part $(i)$ of Proposition 1.2.3 twice,

$$\mathrm{rk}(e) = \mathrm{rk}(f) = \mathrm{rk}(\varphi(e)) + \mathrm{rk}(f-g) = \mathrm{rk}(e) + \mathrm{rk}(f-g).$$

Hence $\mathrm{rk}(f-g) = 0$. Since rk is a rank function, necessarily $f = g$, and we are done: $eR \cong \varphi(e)R = fR$.

We have already observed that any regular ring $R$ admitting a rank function rk is directly finite, so by [39, Theorem 16.14] rk is the unique rank function that $R$ can admit.

In the particular case that $R$ is a simple, regular, right and left self-injective ring, by [39, Corollary 21.14], there exists a (unique) rank function rk on $R$, so the proposition follows for $R$. □

Let now $R$ be a regular, right and left self-injective, simple ring. By [39, Corollary 13.5], the lattice $L(R_R)$ satisfies all the axioms of a complemented, continuous, modular, irreducible, complete lattice, so one can apply all the theory on continuous geometries to the lattice $L(R_R)$. In particular, every rank function rk on $R$ gives rise to a normalized dimension function defined over $L(R_R)$ by the rule

$$\dim : L(R_R) \to [0,1], \quad \dim(eR) = \mathrm{rk}(e).$$

We check the properties to be a normalized dimension function given after Definition 1.2.1.

a) $\dim(0) = \mathrm{rk}(0) = 0$ and $\dim(R) = \mathrm{rk}(1) = 1$.

b) $eR, fR \in L(R_R)$ are equidimensional if and only if $eR \cong fR$, if and only if $\mathrm{rk}(e) = \mathrm{rk}(f)$ by Proposition 1.2.7, if and only if $\dim(eR) = \dim(fR)$.

c) If $eR \subseteq fR$, then $fe = e$ and so $\dim(eR) = \mathrm{rk}(e) = \mathrm{rk}(fe) \leq \mathrm{rk}(f) = \dim(fR)$.

d) Let $eR, fR \in L(R_R)$. We must show that

$$\dim(eR + fR) + \dim(eR \cap fR) = \dim(eR) + \dim(fR).$$

Take $g, h \in R$ idempotents such that $eR + fR = gR$ and $eR \cap fR = hR$. Let's first assume that $h = 0$. In this case $eR \oplus fR = gR$, and part $(i)$ of Proposition 1.2.3 says

$$\dim(eR) + \dim(fR) = \mathrm{rk}(e) + \mathrm{rk}(f) = \mathrm{rk}(g) = \dim(gR) = \dim(eR + fR).$$

For the general case, take $\overline{h} \in R$ an idempotent such that $\overline{h}R$ is a complement of $hR$ in $fR$, that is $hR \oplus \overline{h}R = fR$. Hence

$$\dim(fR) = \mathrm{rk}(f) = \mathrm{rk}(h) + \mathrm{rk}(\overline{h}) = \dim(hR) + \mathrm{rk}(\overline{h}) = \dim(eR \cap fR) + \mathrm{rk}(\overline{h}).$$

But since $\overline{h}R \subseteq fR$, we compute $\{0\} = \overline{h}R \cap hR = \overline{h}R \cap eR \cap fR = \overline{h}R \cap eR$, and also $gR = eR + fR = eR + hR + \overline{h}R = eR + \overline{h}R = eR \oplus \overline{h}R$. Therefore

$$\dim(eR + fR) = \dim(gR) = \mathrm{rk}(g) = \mathrm{rk}(e) + \mathrm{rk}(\overline{h}) = \dim(eR) + \mathrm{rk}(\overline{h}).$$

Putting everything together,

$$\dim(eR + fR) + \dim(eR \cap fR) = \dim(eR) + \dim(fR).$$

Therefore by [83, Theorems 7.3 and 7.4 of Part I] the range of rk can be

a) either a finite set of values of the form $\left\{0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\right\}$ for some natural number $n \geq 1$, or

b) the whole interval $[0, 1]$.

The pseudo-rank function will be called *discrete* or *continuous* depending on whether its range takes a discrete or a continuous set of values, respectively.

**Examples 1.2.8.**

1) The most common examples of regular rings with a discrete rank function are the finite-dimensional matrix algebras $R_n = M_n(K)$ over an arbitrary field $K$. It admits a unique rank function $\mathrm{rk}_n = \frac{\mathrm{Rk}}{n}$, where Rk is the usual rank defined over matrices. It is clear that the possible set of values for $\mathrm{rk}_n$ is the finite set $\left\{0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\right\}$. In fact, if we denote by $e_{ij}$ to be the standard matrix units in $M_n(K)$, then for $1 \leq m \leq n$ one computes

$$\mathrm{rk}_n(e_{11} + \cdots + e_{mm}) = \frac{\mathrm{Rk}(e_{11} + \cdots + e_{mm})}{n} = \frac{m}{n},$$

so every possible value for $\mathrm{rk}_n$ can be achieved.

2) The following example was due to von Neumann (cf. [82]), and gives an example of a continuous geometry. For a field $K$, take the direct limit $\varinjlim M_{2^n}(K)$ of the sequence

$$M_2(K) \to M_4(K) \to M_8(K) \to \cdots \to M_{2^n}(K) \to \cdots$$

with respect to the block-diagonal embeddings $x \mapsto \begin{pmatrix} x & \mathbf{0}_{2^n} \\ \mathbf{0}_{2^n} & x \end{pmatrix}$. It is a regular ring since each matrix factor $M_{2^n}(K)$ is, and admits a unique rank function rk defined on an element $x = \varinjlim x_n$ to be $\mathrm{rk}(x) = \lim_n \mathrm{rk}_n(x_n)$, where $\mathrm{rk}_n = \frac{\mathrm{Rk}}{2^n}$ is the usual normalized rank on $M_{2^n}(K)$. The completion of $\varinjlim M_{2^n}(K)$ with respect to the induced rank metric, denoted here by $\mathcal{M}_K$, is a complete regular ring with a unique rank function, again denoted by rk, which is a *continuous factor*, i.e. a (right and left) self-injective simple regular ring of type $II_f$, and the set of values of the rank function fills the unit interval $[0, 1]$. To see this, note first that any dyadic rational number $\frac{m}{2^n}$ with $0 \leq m \leq 2^n$ can occur as the rank of some element, for example take $x = \mathrm{Id}_m \oplus 0_{2^n - m} \in M_{2^n}(K)$. Since the dyadic rational numbers are dense in $[0, 1]$, the range of rk cannot be a finite set of the form $\left\{0, \frac{1}{N}, ..., \frac{N-1}{N}, 1\right\}$; therefore in must be the whole interval $[0, 1]$.

Regular rings are also of great interest since every (pseudo-)rank function rk on $R$ can be uniquely extended to a (pseudo-)rank function on matrices over $R$ (see e.g. [39, Corollary 16.10]). This is no longer true if we do not assume $R$ to be regular. The definition that seems to fit in the general setting is the notion of Sylvester matrix rank functions.

**Definition 1.2.9.** Let $R$ be a unital ring. A *Sylvester matrix rank function* rk on $R$ is a function that assigns a nonnegative real number to each matrix over $R$ and satisfies the following conditions:

a) $\mathrm{rk}(M) = 0$ if $M$ is a zero matrix, and $\mathrm{rk}(1) = 1$.

b) $\mathrm{rk}(M_1M_2) \leq \mathrm{rk}(M_1), \mathrm{rk}(M_2)$ for any matrices $M_1$ and $M_2$ which can be multiplied.

c) $\mathrm{rk}\begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$ for matrices $M_1$ and $M_2$.

d) $\mathrm{rk}\begin{pmatrix} M_1 & M_3 \\ 0 & M_2 \end{pmatrix} \geq \mathrm{rk}(M_1) + \mathrm{rk}(M_2)$ for any matrices $M_1$, $M_2$ and $M_3$ of appropriate sizes.

For more theory about Sylvester matrix rank functions we refer the reader to [53] and [92, Part I, Chapter 7]. We summarize some of their properties in the following proposition.

**Proposition 1.2.10.** *Let $R$ be a unital ring and* rk *a Sylvester matrix rank function on $R$.*

*(i) For any matrices $A, B \in M(R)$ of the same size,* $\mathrm{rk}(A + B) \leq \mathrm{rk}(A) + \mathrm{rk}(B)$.

*(ii) If $A \in \mathrm{GL}_n(R)$, then $\mathrm{rk}(A) = n$. Moreover, $\mathrm{rk}(AB) = \mathrm{rk}(B)$ for any matrix $B$ that can be multiplied to the right with $A$.*

*(iii) For any elements $x, y \in R$, $\mathrm{rk}(xy) \geq \mathrm{rk}(x) + \mathrm{rk}(y) - 1$.*

*(iv) If $e \in R$ is a central idempotent, then $\mathrm{rk}(x) = \mathrm{rk}(ex) + \mathrm{rk}((1 - e)x)$ for every $x \in R$.*

*Proof.* *(i)* $\mathrm{rk}(A + B) = \mathrm{rk}\begin{pmatrix} A+B & 0 \\ 0 & 0 \end{pmatrix} = \mathrm{rk}\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) \leq \mathrm{rk}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \mathrm{rk}(A) + \mathrm{rk}(B)$.
*(ii)* $n = \mathrm{rk}(\mathrm{Id}_n) = \mathrm{rk}(AA^{-1}) \leq \mathrm{rk}(A) \leq \mathrm{rk}(\mathrm{Id}_n) = n$, so $\mathrm{rk}(A) = n$. Moreover,

$$\mathrm{rk}(B) = \mathrm{rk}(A^{-1}AB) \leq \mathrm{rk}(AB) \leq \mathrm{rk}(B).$$

*(iii)* $\mathrm{rk}(xy) + 1 = \mathrm{rk}\begin{pmatrix} xy & 0 \\ 0 & 1 \end{pmatrix} = \mathrm{rk}\left(\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} y & 1 \\ 0 & x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}\right)$. Both matrices $\begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$ are invertible in $M_2(R)$, so by *(ii)* we obtain

$$\mathrm{rk}(xy) + 1 = \mathrm{rk}\begin{pmatrix} y & 1 \\ 0 & x \end{pmatrix} \geq \mathrm{rk}(x) + \mathrm{rk}(y).$$

*(iv)* Fix $e \in R$ a central idempotent. Then the matrix $\begin{pmatrix} e & 1-e \\ -(1-e) & e \end{pmatrix}$ is invertible in $M_2(R)$ with inverse $\begin{pmatrix} e & -(1-e) \\ 1-e & e \end{pmatrix}$, so we have, using *(ii)*,

$$\begin{aligned} \mathrm{rk}(x) = \mathrm{rk}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} &= \mathrm{rk}\left(\begin{pmatrix} e & -(1-e) \\ 1-e & e \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e & 1-e \\ -(1-e) & e \end{pmatrix}\right) \\ &= \mathrm{rk}\begin{pmatrix} exe & ex(1-e) \\ (1-e)xe & (1-e)x(1-e) \end{pmatrix} = \mathrm{rk}\begin{pmatrix} ex & 0 \\ 0 & (1-e)x \end{pmatrix} = \mathrm{rk}(ex) + \mathrm{rk}((1-e)x). \end{aligned}$$

□

We denote by $\mathbb{P}(R)$ the compact convex set of Sylvester matrix rank functions on $R$. It is well-known (see for example [39, Proposition 16.20]) that, in case $R$ is a regular ring, this space coincides with the space of pseudo-rank functions on $R$.

As in the case of pseudo-rank functions on a regular ring, a Sylvester matrix rank function rk on a unital ring $R$ gives rise to a pseudo-metric by the rule $d(x, y) = \mathrm{rk}(x - y)$ for $x, y \in R$. We call it *faithful* if its kernel $\ker(\mathrm{rk})$, defined as the set of all element $x \in R$ with zero rank, is exactly $\{0\}$. In this case, $d$ becomes a metric on $R$.

We can always obtain a faithful Sylvester rank function by passing to the quotient $R \to R/\ker(\mathrm{rk})$. The ring operations are continuous with respect to this metric, so one can consider the completion $\overline{R}$ of $R$ with respect to $d$. It is routine to check that rk defines a new Sylvester rank function $\overline{\mathrm{rk}}$ on $\overline{R}$.

### 1.2.1 $*$-regular rings and the $*$-regular closure

We now introduce the notion of a $*$-regular ring, and state some facts about them and their completions with respect to a pseudo-rank function (see for instance [2, 9]). A *$*$-regular ring* is a regular ring endowed with a proper involution, that is, an involution $*$ such that $x^*x = 0$ implies $x = 0$.

The involution is called *positive definite* in case the condition

$$\sum_{i=1}^{n} x_i^* x_i = 0 \quad \Longrightarrow \quad x_i = 0 \text{ for all } 1 \leq i \leq n$$

holds for each positive integer $n$. If $R$ is a $*$-regular ring with positive definite involution, then $M_n(R)$, endowed with the $*$-transpose involution, is also a $*$-regular ring.

For $*$-regular rings, we have a strong property concerning idempotents generating principal right/left ideals of $R$. In fact, if we demand these idempotents to be projections (i.e. elements $e \in R$ such that $e = e^2 = e^*$), then it turns out that there exist unique projections generating a given principal right/left ideal. More generally, we have the following theorem.

**Theorem 1.2.11.** *For a $*$-regular ring $R$, the following hold:*

*(1) For each element $x \in R$, there are unique projections $e, f \in R$ such that $xR = eR$ and $Rx = Rf$; moreover,*

*(2) there exists a unique element $y \in fRe$ such that $xy = e$ and $yx = f$.*

*We will denote by* $\mathrm{LP}(x)$ *the projection $e$, called the left projection of $x$, and by* $\mathrm{RP}(x)$ *the projection $f$, called the right projection of $x$. Moreover, the unique element $y$ of part* (2) *is denoted by $\overline{x}$, and called the relative inverse of $x$.*

*Proof.* Let's prove (1). Since $R$ is regular, there exists an element $w \in R$ such that $(x^*x)w(x^*x) = x^*x$. In this case $x^*x(wx^*x - 1) = 0$, so

$$(wx^*x - 1)^* x^* x(wx^*x - 1) = 0.$$

Since the involution is proper, we must have $x(wx^*x - 1) = 0$, or $x = (xwx^*)x$. Applying $*$ we also get $x^* = x^*(xw^*x^*)$. Consider $e := xwx^*$. Note that

$$e^2 = xwx^*xwx^* = xwx^* = e \quad \text{and} \quad ee^* = xwx^*xw^*x^* = xwx^* = e,$$

so $e^* = ee^* = e$, and $e$ is a projection. It is clear that $xR = eR$, since $x = ex$. By applying the same construction with $x$ replaced by $x^*$, we obtain a projection $f$ such that $x^*R = fR$. If we take $*$ we obtain $Rx = Rf$, as desired.

For the uniqueness part, suppose that there is another projection $e' \in R$ such that $eR = xR = e'R$. In this case we have $e = ee'$ and $e' = e'e$, so $e = e^* = (ee')^* = e'e = e'$. Analogously we get uniqueness for $f$.

For (2), since $xR = eR$ and $Rx = Rf$, $ex = x = xf$, and we can write $e = xz$, $f = wx$ for some $z, w \in R$. Note that $e = e^2 = xze = x(fze)$. Consider the element $y = fze \in fRe$. By construction, it satisfies $xy = e$. For the other equality, we compute

$$f - yx = f(f - yx) = wx(f - yx) = w(xf - xyx) = w(x - ex) = 0.$$

Hence $yx = f$, as required. For uniqueness, suppose that $y' \in fRe$ is another element such that $xy' = e$, $y'x = f$. Then $y' = y'e = y'xy = fy = y$. □

If $e, f$ are projections in a $*$-ring $R$, then we say that $e$ is $*$-equivalent to $f$, written $e \overset{*}{\sim} f$, in case there is $x \in eRf$ such that $e = xx^*$ and $f = x^*x$.

For any subset $S \subseteq R$ of a $*$-regular ring, there exists a smallest $*$-regular subring, denoted by $\mathcal{R}(S, R)$ and termed the $*$-regular closure of $S$ in $R$, of $R$ containing $S$ ([6, Proposition 6.2], see also [72, Proposition 3.1]). In fact, $\mathcal{R}(S, R) = \bigcup_{n \geq 0} \mathcal{R}_n(S, R)$, where $\mathcal{R}_{n+1}(S, R)$ is generated by $\mathcal{R}_n(S, R)$ and the relative inverses in $R$ of the elements of $\mathcal{R}_n(S, R)$, and $\mathcal{R}_0(S, R)$ is the $*$-subring of $R$ generated by the set $S$. It was observed in [53] that $\mathcal{R}_{n+1}(S, R)$ can be described as the subring of $R$ generated by the elements of $\mathcal{R}_n(S, R)$ and the relative inverses of the elements of the form $x^*x$ for $x \in \mathcal{R}_n(S, R)$.

Some properties of the $*$-regular closure are given in the next lemma.

**Lemma 1.2.12.** *Let $S$ be a unital $*$-subring of a $*$-regular ring $R$, and let $\mathcal{R} = \mathcal{R}(S, R)$ be the $*$-regular closure of $S$ in $R$. Write also $\mathcal{R}_n = \mathcal{R}_n(S, R)$. The following holds.*

*i)* $\mathcal{R}(\mathcal{R}, R) = \mathcal{R} = \mathcal{R}(S, \mathcal{R})$.

*ii) Let $J$ be an ideal of $S$, and let $I$ be the ideal of $\mathcal{R}$ generated by $J$. Then $I = \bigcup_{n\geq 0} J_n$, where $J_{n+1}$ is the ideal of $\mathcal{R}_{n+1}$ generated by $J_n$, which coincides with the ideal of $\mathcal{R}_{n+1}$ generated by $J_n$ and the relative inverses in $R$ of the elements of $J_n$, and $J_0 = J$.*

*Proof.* The proofs are routine, but we include them for the convenience of the reader.

For part *i*), note that $S \subseteq \mathcal{R} \subseteq R$. By definition $\mathcal{R}(S, \mathcal{R}) \subseteq \mathcal{R}$, and for the other inclusion note that $\mathcal{R}(S, \mathcal{R})$ is a $*$-regular subring of $R$ containing $S$, so $\mathcal{R} \subseteq \mathcal{R}(S, \mathcal{R})$. The other equality is proved analogously.

For part *ii*), observe that $I = \mathcal{R}J\mathcal{R}$ and each $J_{n+1} = \mathcal{R}_{n+1}J_n\mathcal{R}_{n+1}$. Since each $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$, we have $J_n \subseteq J_{n+1}$.

If $a \in I$, we can write it as a finite combination $a = \sum_{j=1}^m r_j b_j s_j$ with $r_j, s_j \in \mathcal{R}$ and $b_j \in J$. There exists then an index $n_0 \geq 0$ such that $r_j, s_j \in \mathcal{R}_{n_0}$ for all $j = 1, ..., m$. Hence $a = \sum_{j=1}^m r_j b_j s_j \in \mathcal{R}_{n_0} J \mathcal{R}_{n_0} \subseteq \mathcal{R}_{n_0} J_{n_0-1} \mathcal{R}_{n_0} = J_{n_0}$, and we obtain the inclusion $I \subseteq \bigcup_{n\geq 0} J_n$.

Conversely, since each $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$,

$$J_n = \mathcal{R}_n J_{n-1} \mathcal{R}_n = \mathcal{R}_n \mathcal{R}_{n-1} \cdots \mathcal{R}_1 J \mathcal{R}_1 \cdots \mathcal{R}_{n-1} \mathcal{R}_n \subseteq \mathcal{R}_n J \mathcal{R}_n \subseteq \mathcal{R} J \mathcal{R} = I.$$

The result follows. Note that if $r \in J_n$ and $\overline{r}$ is the quasi-inverse of $r$ in $S$, then $\overline{r} \in \mathcal{R}_{n+1}$ and $\overline{r} = \overline{r}r\overline{r}$ belongs to the ideal of $\mathcal{R}_{n+1}$ generated by $J_n$, that is, $\overline{r} \in J_{n+1}$. □

There is a whole theory in development concerning the study of the $*$-regular closure, initiated by Jaikin-Zapirain in [53]. A very useful result connecting the $*$-regular closure and the possible values of any Sylvester matrix rank function defined on it is given in the following proposition, which can be thought of as an analogue of the classical Cramer's rule (see Proposition 1.3.8).

**Proposition 1.2.13** (Corollary 6.2 of [53])**.** *Let $S$ be a unital $*$-subring of a $*$-regular ring $R$, and let $\mathcal{R} = \mathcal{R}(S, R)$ be the $*$-regular closure of $S$ in $R$.*

*Then for any matrices $r_1, ..., r_k \in M_{n\times m}(\mathcal{R})$, there exists a matrix $M \in M_{a\times b}(S)$ and matrices $A_1, ..., A_k \in M_{n\times b}(S)$ such that, for any other square-matrices $T_1, ..., T_k \in M_n(S)$ and any Sylvester matrix rank function* rk *defined on $\mathcal{R}$,*

$$\mathrm{rk}(T_1 r_1 + \cdots + T_k r_k) = \mathrm{rk}\begin{pmatrix} M \\ T_1 A_1 + \cdots + T_k A_k \end{pmatrix} - \mathrm{rk}(M).$$

*In particular, any Sylvester matrix rank function on $\mathcal{R}$ is completely determined by its values on matrices over $S$.*

### 1.2.2 The algebra of (unbounded) affiliated operators of a finite von Neumann algebra

This will be our main example for the rest of the section. Let $\mathcal{H}$ be a Hilbert space. For an (unbounded) operator we will understand a linear map $T : \mathrm{dom}(T) \to \mathcal{H}$, being $\mathrm{dom}(T) \subseteq \mathcal{H}$ a (not necessarily closed) subspace.

We can still define two operations on the set of (unbounded) operators, namely the usual sum and product (composition) of operators, but with domains given by

$$T + S : \mathrm{dom}(T + S) \to \mathcal{H}, \text{ with } \mathrm{dom}(T + S) = \mathrm{dom}(T) \cap \mathrm{dom}(S),$$

$$TS : \mathrm{dom}(TS) \to \mathcal{H}, \text{ with } \mathrm{dom}(TS) = S^{-1}(\mathrm{dom}(T)).$$

When $T : \mathrm{dom}(T) \to \mathrm{ran}(T)$, being $\mathrm{ran}(T) := \mathrm{Im}(T)$, is injective, we can still define an inverse operator for $T$, with domain $\mathrm{ran}(T)$:

$$T^{-1} : \mathrm{ran}(T) \to \mathrm{dom}(T) \subseteq \mathcal{H}.$$

We say that $T : \mathrm{dom}(T) \to \mathcal{H}$ is *closed* if the graph of $T$, defined by $\mathcal{G}(T) = \{(x, Tx) \mid x \in \mathrm{dom}(T)\}$, is a closed subspace of $\mathcal{H} \oplus \mathcal{H}$. Note that, by the Closed Graph Theorem, if $\mathrm{dom}(T) = \mathcal{H}$ and $T$ is a closed operator, then $T$ is bounded. So we are interested in studying those operators for which $\mathrm{dom}(T)$ is a (possibly) proper subspace of $\mathcal{H}$. We call $T$ a *densely defined* operator if $\mathrm{dom}(T)$ is a dense subspace of $\mathcal{H}$.

**Definition 1.2.14.** Let $(\mathcal{M}, \mathrm{tr})$ be a finite von Neumann algebra on $\mathcal{H}$, that is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ closed under the weak (or even strong) operator topology, together with a faithful, normal tracial state tr.

We define $\mathcal{U}$ to be the set of (unbounded) operators $T$ such that

a) $T$ is a closed operator.

b) $T$ is densely defined.

c) $T$ is affiliated to $\mathcal{M}$, meaning that for every bounded operator $S$ commuting with all the operators of $\mathcal{M}$, we have $ST \subset TS$, i.e. the operator $TS$ extends the operator $ST$ in the sense that $\mathrm{dom}(ST) \subseteq \mathrm{dom}(TS)$ and $TS|_{\mathrm{dom}(ST)} = ST$.

By [89, Note 2.11], $\mathcal{U}$ becomes a $*$-regular ring, which is in fact characterized algebraically to be the classical ring of quotients of $\mathcal{M}$ (see [89, Proposition 2.8], also Section 1.3.2). In fact, all the projections $p \in \mathcal{U}$ belong to $\mathcal{M}$ itself[6], so it is possible to define a rank function on $\mathcal{U}$ by means of the trace tr: for an element $u \in \mathcal{U}$, since $\mathcal{U}$ is $*$-regular, by Theorem 1.2.11 there are unique projections $p := \mathrm{LP}(u), q := \mathrm{RP}(u) \in \mathcal{U}$ such that $u\mathcal{U} = p\mathcal{U}$ and $\mathcal{U}u = \mathcal{U}q$. Hence $p, q \in \mathcal{M}$ and they are equivalent, so they have the same trace $\mathrm{tr}(p) = \mathrm{tr}(q)$. We thus define the rank of $u$ to be the value of this trace over the projections $p, q$:

$$\mathrm{rk}_{\mathcal{U}}(u) := \mathrm{tr}(p) = \mathrm{tr}(q).$$

**Lemma 1.2.15.** $\mathrm{rk}_{\mathcal{U}}$ *defines a rank function on* $\mathcal{U}$.

*Proof.* Clearly $\mathrm{rk}(0) = 0$ and $\mathrm{rk}(1) = 1$.

First, note that the rank function satisfies $\mathrm{rk}(u^*) = \mathrm{rk}(u)$ for every $u \in \mathcal{U}$. To see this, note that if $p$ is the unique projection in $\mathcal{M}$ satisfying $u\mathcal{U} = p\mathcal{U}$, then by applying $*$ we get $\mathcal{U}u^* = \mathcal{U}p$, so by uniqueness of the projection $\mathrm{rk}(u^*) = \mathrm{tr}(p) = \mathrm{rk}(u)$.

Now for $u, v \in \mathcal{U}$ write $uv\mathcal{U} = p'\mathcal{U}$ and $u\mathcal{U} = p\mathcal{U}$ for some projections $p, p' \in \mathcal{M}$. Then $p'\mathcal{U} = uv\mathcal{U} \subseteq u\mathcal{U} = p\mathcal{U}$, so $pp' = p'$. By taking $*$ we obtain $p'p = p'$. Hence $p' \leq p$, and $\mathrm{tr}(p') \leq \mathrm{tr}(p)$. We compute

$$\mathrm{rk}(uv) = \mathrm{tr}(p') \leq \mathrm{tr}(p) = \mathrm{rk}(u).$$

By applying the same reasoning with the element $v^*u^*$, we obtain

$$\mathrm{rk}(uv) = \mathrm{rk}(v^*u^*) \leq \mathrm{rk}(v^*) = \mathrm{rk}(v).$$

Now take $e, f \in \mathcal{U}$ orthogonal idempotents, and let $p, q, \overline{p}$ be projections in $\mathcal{U}$ such that $(e+f)\mathcal{U} = \overline{p}\mathcal{U}$, $e\mathcal{U} = p\mathcal{U}$ and $f\mathcal{U} = q\mathcal{U}$. Then $\overline{p}\mathcal{U} = (e+f)\mathcal{U} = e\mathcal{U} \oplus f\mathcal{U} = p\mathcal{U} \oplus q\mathcal{U}$. Define $p \vee q$ to be the least projection which is greater than $p$ and $q$; dually, define $p \wedge q$ to be the greatest projection which is smaller than $p$ and $q$. On one hand, $p\mathcal{U}, q\mathcal{U} \subseteq (p \vee q)\mathcal{U}$, so $\overline{p}\mathcal{U} = p\mathcal{U} \oplus q\mathcal{U} \subseteq (p \vee q)\mathcal{U}$ and we get $\overline{p} \leq p \vee q$. On the other hand, since $p\mathcal{U}, q\mathcal{U} \subseteq \overline{p}\mathcal{U}$, we have that $p, q \leq \overline{p}$, so by definition $p \vee q \leq \overline{p}$ and we obtain equality. Also $(p \wedge q)\mathcal{U} = p\mathcal{U} \cap q\mathcal{U} = \{0\}$, so necessarily $p \wedge q = 0$. Finally,

$$\mathrm{rk}(e+f) = \mathrm{tr}(\overline{p}) = \mathrm{tr}(p \vee q) = \mathrm{tr}(p \vee q) + \mathrm{tr}(p \wedge q) = \mathrm{tr}(p) + \mathrm{tr}(q) = \mathrm{rk}(e) + \mathrm{rk}(f)$$

as required, were we have used the well-known fact that for finite von Neumann algebras, $\mathrm{tr}(p \vee q) + \mathrm{tr}(p \wedge q) = \mathrm{tr}(p) + \mathrm{tr}(q)$ for any projections $p, q \in \mathcal{M}$.

To conclude, if $u \in \mathcal{U}$ is such that $\mathrm{rk}(u) = 0$, then by taking $p$ the unique projection in $\mathcal{U}$ such that $u\mathcal{U} = p\mathcal{U}$ we get $0 = \mathrm{rk}(u) = \mathrm{tr}(p) = \mathrm{tr}(p^*p)$. Since tr is a faithful trace, $p = 0$, and so $u = 0$. □

**Theorem 1.2.16.** *$\mathcal{U}$ becomes a $*$-regular, right and left self-injective ring. Also, $\mathcal{U}$ is complete in the $\mathrm{rk}_{\mathcal{U}}$-metric.*

*If moreover $\mathcal{M}$ is a $II_1$ factor, then $\mathcal{U}$ becomes a continuous factor, i.e. a self-injective, simple regular ring of type $II_f$, and the set of values achieved by its rank function fills the unit interval* $[0, 1]$.

*Proof.* We have already observed that the projections of $\mathcal{U}$ coincide with the projections of $\mathcal{M}$. By [58, Theorem 6.5], the lattice of projections $\mathrm{Proj}(\mathcal{M})$ of $\mathcal{M}$ form a (not necessarily irreducible) continuous geometry; in particular it is continuous. Since the algebra of (unbounded) affiliated operators of $M_2(\mathcal{M})$ can be canonically

[6] This can be proved by using the polar decomposition of elements of $\mathcal{U}$ and the spectral theorem for unbounded operators.

identified with $M_2(\mathcal{U})$, the lattice of projections $\text{Proj}(M_2(\mathcal{U})) = \text{Proj}(M_2(\mathcal{M}))$ is continuous (apply the same Theorem above, [58, Theorem 6.5]), hence by a theorem of Utumi ([97, Corollary 7.5]) the ring $\mathcal{U}$ is right and left self-injective. Therefore $\mathcal{U}$ becomes a $*$-regular, right and left self-injective ring.

Let now $\{e_n\}_{n\geq 1} \subseteq \mathcal{U}$ be central orthogonal idempotents, and note that they are in particular central orthogonal projections[7]. Define the projections $f_n := e_1 + \cdots + e_n$, $e := \bigvee_{n\geq 1} e_n$. The weak continuity of the trace gives

$$\text{rk}_{\mathcal{U}}(f_n) = \text{tr}(f_n) \xrightarrow{n} \text{tr}(e) = \text{rk}_{\mathcal{U}}(e),$$

so by [39, Theorem 21.7 and Proposition 21.8], $\mathcal{U}$ is complete in the $\text{rk}_{\mathcal{U}}$-metric.

For the last part of the theorem, we have commented that $\mathcal{U}$ can be also constructed as the classical ring of quotients of $\mathcal{M}$, and in fact it is true that its center $Z(\mathcal{M})$ is the classical ring of quotients of the center of $\mathcal{M}$. If $\mathcal{M}$ is a $II_1$ factor, then $Z(\mathcal{M}) = \mathbb{C}$, and so $Z(\mathcal{U}) = \mathbb{C}$. By [39, Proposition 19.13 and Theorem 19.14], this implies that $\mathcal{U}$ is a simple ring, and in fact due to the characterization of regular, right self-injective rings (Theorem 1.2.5) and the fact that $\mathcal{U}$ possesses a rank function, it can be decomposed as

$$\mathcal{U} = \mathcal{U}_{1f} \times \mathcal{U}_{2f}$$

where $\mathcal{U}_{1f}$ is of type $I_f$ and $\mathcal{U}_{2f}$ is of type $II_f$. Since $\mathcal{M}$ is assumed to be a $II_1$ factor, it necessarily implies that $\mathcal{U} = \mathcal{U}_{2f}$, so it is of type $II_f$. In particular, the set of values achieved by its rank function fills the unit interval $[0, 1]$. □

## 1.3 Noncommutative localization of rings

In this last section we would like to discuss some techniques for inverting elements in a not necessarily commutative ring $R$.

Classically, for a commutative ring $R$ with unit 1 which is an integral domain we can construct its classical ring of quotients $\mathcal{Q}(R)$, defined to be the set of equivalence classes $ab^{-1}$ of elements $a, b \in R$, with $b \neq 0$, and two classes $a_1b_1^{-1}, a_2b_2^{-1}$ being considered the same if and only if $a_1b_2 = a_2b_1$. As a prototypical example, $\mathcal{Q}(\mathbb{Z}) = \mathbb{Q}$. The fact that $R$ is commutative enables us to define natural operations of sum and product inside $\mathcal{Q}(R)$, which turn it a commutative ring containing $R$ via the embedding $R \hookrightarrow \mathcal{Q}(R)$ given by $a \mapsto a1^{-1}$. Roughly speaking, we are just inverting all the elements of $R$ that are not zero-divisors.

More generally, if one would like to invert a specific set of elements $S \subseteq R$, one must impose some conditions on the set $S$ in order to the inverses be well-defined. That is, one requires that the set $S$ be multiplicative: $1 \in S$, $0 \notin S$, and closed under taking products. These conditions are natural since one want to construct a new ring where the elements of $S$ will become units, and units must be closed under multiplication. One can then construct the localization of $R$ with respect to $S$, denoted by $RS^{-1}$, as defined to be the set of equivalence classes $as^{-1}$ of elements $a \in R$, $s \in S$, and such that two classes $a_1s_1^{-1}, a_2s_2^{-1}$ are considered the same class if and only if there exists an element $t \in S$ satisfying $t(a_1s_2 - a_2s_1) = 0$ (the presence of $t$ is necessary if one wants to ensure transitivity of such equivalence). Again, the commutativity of $R$ gives rise to well-defined operations of sum and product on $RS^{-1}$, turning it into a commutative ring with a natural morphism $\iota : R \to RS^{-1}$ given by $a \mapsto a1^{-1}$, but this time not necessarily injective (its kernel consist of the elements $a \in R$ such that $as = 0$ for some element $s \in S$, so one deduce that it is injective if and only if $S$ does not contain zero-divisors). In particular, any element of $RS^{-1}$ can be written in the form $as^{-1}$ with $a \in R, s \in S$.

The pair $(RS^{-1}, \iota)$ is universal with respect to the property of inverting elements from $S$, that is, if one has another morphism $\varphi : R \to T$ from $R$ to another ring $T$ such that all the elements of $S$ become invertible in $T$ under $\varphi$, one can then uniquely extend the morphism to another one defined over $RS^{-1}$, $\overline{\varphi} : RS^{-1} \to T$, satisfying the usual commutation property $\overline{\varphi} \circ \iota = \varphi$.

When $R$ is a domain, the previous construction $\mathcal{Q}(R)$ is a particular case of this one, taking the set $S$ to be the set of all elements of $R$ that are not zero-divisors (which in such a case is clearly a multiplicative set).

That was a short overview about the topic of localizing elements in the commutative case. It is then natural to try to extend these notions in the noncommutative setting, but things turn out to be much harder to define because of the lack of commutativity. All the theory that we will discuss can be found extensively in [18, 19, 20, 64]. One can also take a look at [68] for noncommutative localization in the special case of group rings.

[7] This is a general fact: if $e \in R$ is a central idempotent in a $*$-regular ring $R$, then the computation $(e - ee^*)(e - ee^*)^* = 0$ together with the fact that the involution is proper implies that $e = ee^* = e^*$, so $e$ is also a projection.

### 1.3.1 Universal localization

Let $R$ be a (not necessarily commutative) ring with unit 1. Given a multiplicative set $S \subseteq R$[8], we aim to construct a new ring containing the inverses of elements of $S$. This can be achieved by using a construction with generators and relations, which yields a universal property for such a ring.

**Theorem 1.3.1** (Proposition 9.2 of [64]). *Following the foregoing notation, there exists a ring $S^{-1}R$ and a $S$-inverting morphism $\iota : R \to S^{-1}R$ (all the elements of $\iota(S)$ are invertible inside $S^{-1}R$) with the usual universal property: given any morphism $\varphi : R \to T$ to another ring $T$ such that $\varphi(S)$ consists of invertible elements in $T$, there exists a unique morphism $\varphi_S : S^{-1}R \to T$ such that $\varphi_S \circ \iota = \varphi$.*

As usual, the pair $(S^{-1}R, \iota)$ is unique up to unique isomorphism. This is called the *universal localization of $R$ with respect to $S$*. The construction is not so hard: one just add extra elements and relations to $R$ in order to achieve invertibility of elements of $S$.

One can easily deduce from this construction that it is very difficult to handle elements of $S^{-1}R$ in practice. For instance, due to the noncommutativity of $R$, the elements of $S^{-1}R$ cannot be written in the simplified form $\iota(a)\iota(s)^{-1}$ anymore; instead, they are sums of products of such elements, like

$$\iota(a_1)\iota(s_1)^{-1}\iota(a_2)\iota(s_2)^{-1} + \iota(s_3)^{-1}\iota(a_3)\iota(s_4)^{-1} - \iota(s_5)^{-1}$$

and even the kernel of $\iota$ does not have an easy description anymore. Nevertheless, the universal localization behaves nicely under taking quotients. Formally, we have the following

**Proposition 1.3.2.** *Let $I$ be a two-sided ideal of $R$, and let $S^{-1}I$ denote the two-sided ideal of $S^{-1}R$ generated by the elements of $\iota(I)$. We have a natural isomorphism $S^{-1}R/S^{-1}I \cong \overline{S}^{-1}(R/I)$, where $\overline{S}$ denotes the image of the set $S$ under the quotient map $\pi : R \to R/I$.*

*Proof.* Since $\iota(I) \subseteq S^{-1}I$, the map $\iota$ induces a well-defined morphism $\iota_I : R/I \to S^{-1}R/S^{-1}I$ given by the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\iota} & S^{-1}R \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_S} \\ R/I & \xrightarrow{\iota_I} & S^{-1}R/S^{-1}I. \end{array}$$

It suffices to check that the pair $(S^{-1}R/S^{-1}I, \iota_I)$ is universal with respect to the set $\overline{S}$. So let $\overline{\varphi} : R/I \to T$ be a morphism such that the elements of $\overline{\varphi}(\overline{S})$ become invertible in the ring $T$. Consider the composition $\varphi = \overline{\varphi} \circ \pi : R \to T$, which satisfies that the elements of $\varphi(S) = \overline{\varphi}(\overline{S})$ become invertible in $T$. By the universal property of the pair $(S^{-1}R, \iota)$, there exists a unique morphism $\varphi_S : S^{-1}R \to T$ such that $\varphi = \varphi_S \circ \iota$. Note that the elements of $S^{-1}I$ are in the kernel of $\varphi_S$, since $\varphi_S(\iota(a)) = \varphi(a) = \overline{\varphi}(\pi(a)) = 0$. Therefore $\varphi_S$ factors through the quotient morphism $\pi_S$, and gives a morphism $\overline{\varphi}_S : S^{-1}R/S^{-1}I \to T$ satisfying $\varphi_S = \overline{\varphi}_S \circ \pi_S$. Putting everything together, we get

$$\overline{\varphi} \circ \pi = \overline{\varphi}_S \circ \pi_S \circ \iota = \overline{\varphi}_S \circ \iota_I \circ \pi$$

so that $\overline{\varphi} = \overline{\varphi}_S \circ \iota_I$. It is easily checked that $\overline{\varphi}_S$ is unique using the fact that $\varphi_S$ is unique.

Hence the pair $(S^{-1}R/S^{-1}I, \iota_I)$ is universal, so isomorphic to $(\overline{S}^{-1}(R/I), \overline{\iota})$ where $\overline{\iota} : R/I \to \overline{S}^{-1}(R/I)$ is the natural morphism given by Theorem 1.3.1 for the ring $R/I$ and the multiplicative set $\overline{S}$. □

### 1.3.2 Classical rings of quotients: Ore localization

Although it is obviously important to have a universal way to formally invert elements of $R$, we have seen that this construction is not so useful when trying to handle concrete elements from $S^{-1}R$. In fact, it can happen that the previous construction leads to the zero ring, even though one starts with a nonzero ring: take for instance $R$ to be $M_2(K)$ for a fixed arbitrary field $K$, and $S = \{1, e_{11}\}$, where $e_{11}$ is the $2 \times 2$ matrix

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

[8] We define *multiplicative* in the same manner as for the commutative case: $1 \in S, 0 \notin S$ and closed under multiplication.

Then in this case $S^{-1}R$ gives back the zero ring. Further, and unlike the commutative case, in general $S^{-1}R$ may not be a domain even if $R$ is.

In this section we shall see that, by imposing some conditions on the set $S$, one ends up with a nice localization ring, called the *(right/left) ring of quotients*, or sometimes the *(right/left) Ore localization ring* when $S$ consists of all the element that are not (right/left) zero-divisors. We would like to maintain the features that one has in classical commutative localization.

**Definition 1.3.3.** Let $R$ be a ring, $S$ a multiplicative subset of $R$. A ring $\overline{R}$ is called a *(right) ring of quotients* if there exists a $S$-inverting morphism $\varphi : R \to \overline{R}$ satisfying:

a) any element of $\overline{R}$ can be written of the form $\varphi(a)\varphi(s)^{-1}$ for $a \in R$, $s \in S$;

b) the kernel of $\varphi$ consists of all the elements $a \in R$ such that $as = 0$ for some $s \in S$.

One can also define the notion of left ring of quotients analogously.

**Theorem 1.3.4** (Theorem 10.6 of [64]). *A (right) ring of quotients for $R$ with respect to $S$ can be constructed if and only if*

*(1) $S$ is a* right Ore set*, meaning that for any elements $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$.*

*(2) $S$ is* right reversible*, meaning that for any $a \in R$, if $sa = 0$ for some $s \in S$, then also $as' = 0$ for some $s' \in S$.*

*Moreover, if we denote by $\mathcal{Q}^r_S(R)$ the ring obtained by this construction, there exists a $S$-inverting morphism $\epsilon : R \to \mathcal{Q}^r_S(R)$ such that the pair $(\mathcal{Q}^r_S(R), \epsilon)$ is universal in the sense of Theorem 1.3.1.*

An analogous result holds by replacing *right* by *left*. As a direct consequence of Theorem 1.3.4, we have that if $S$ is a right Ore set and right reversible (in this case one says that $S$ is a *right denominator set*), then there exists a unique isomorphism $\mathcal{Q}^r_S(R) \cong S^{-1}R$.

*Proof of Theorem 1.3.4.* We will not prove it in full generality, we only aim to give the definition of the operations sum and product of elements of $\mathcal{Q}^r_S(R)$.

First of all, one defines $\mathcal{Q}^r_S(R)$ to be the set of equivalence classes $as^{-1}$ of elements $a \in R$, $s \in \Sigma$, where two classes $a_1s_1^{-1}$, $a_2s_2^{-1}$ are considered the same if and only if there exist elements $b_1, b_2 \in R$ such that $s_1b_1 = s_2b_2 \in S$ and $a_1b_1 = a_2b_2 \in R$.

To define the sum, we use the fact that $S$ is a right Ore set: for two classes $a_1s_1^{-1}, a_2s_2^{-1}$, take elements $b_1 \in R, b_2 \in S$ such that $s = s_1b_1 = s_2b_2 \in S$, so that

$$a_1s_1^{-1} + a_2s_2^{-1} = (a_1b_1)(s_1b_1)^{-1} + (a_2s_2)(s_2b_2)^{-1} = (a_1b_1 + a_2b_2)s^{-1}.$$

For multiplication, the same property for $S$ is needed: given $a_1s_1^{-1}, a_2s_2^{-1}$, take elements $a_3 \in R, s_3 \in S$ such that $s_1a_3 = a_2s_3$, so that

$$a_1s_1^{-1} \cdot a_2s_2^{-1} = (a_1a_3)(s_2s_3)^{-1}.$$

$\epsilon$ is then given by the natural morphism $R \to \mathcal{Q}^r_S(R)$, $a \mapsto a1^{-1}$. □

In the special case that $S$ consists of all the elements that are neither left nor right zero-divisors, $S$ is already right and left reversible, so we only need to demand property (1) of Theorem 1.3.4 in order to ensure existence of a (right) ring of quotients. In this case, we denote it by $\mathcal{Q}^r_{cl}(R)$, also called the *(right) Ore localization ring*, or *(right) classical ring of quotients* of $R$.

Moreover, if $R$ is a domain and $S = R\backslash\{0\}$ is a (right) Ore set, then the construction $\mathcal{Q}^r_S(R)$ leads us to a division ring, and the natural morphism $\epsilon : R \to \mathcal{Q}^r_S(R)$ is an embedding of rings. Therefore we have been able to embed $R$ into a division ring.

### 1.3.3 Σ-rational closure and division closure

In the previous section we have studied a way of embedding $R$ into a division ring when $R$ is a domain and $S = R\backslash\{0\}$ is a (right) Ore set. In the general setting when $R$ need not be a domain nor $R\backslash\{0\}$ a (right) Ore set, the $S$-inverting morphisms are not enough to guarantee the existence, or even good approximations, of such an embedding into a division ring. We shall remedy this by adjoining to $R$ not only the inverses of a set of elements, but of a set of square matrices over $R$ (see [18]).

Let $R$ be a ring, and take $\Sigma \subseteq M(R) = \bigcup_{n=1}^{\infty} M_n(R)$ a set of square matrices over $R$. A morphism $\varphi : R \to T$ to another ring $T$ is said to be $\Sigma$-invertible if every matrix of $\varphi(\Sigma)$ becomes invertible in $T$. The analogous concept for multiplicative sets in the case of matrices is the following.

**Definition 1.3.5.** $\Sigma$ is called *multiplicative* if the following conditions are satisfied:

a) $1 \in \Sigma$, and whenever $A, B$ belongs to $\Sigma$, then $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ belongs to $\Sigma$ too, for any matrix $C$ of appropriate size.

b) If $A$ belongs to $\Sigma$ and one applies permutations of rows and columns, then the resulting matrix still belongs to $\Sigma$.

It is also possible to construct a universal localization of $R$ but for matrices instead of elements $S \subseteq R$.

**Theorem 1.3.6.** *Following the foregoing notation, there exists a ring $\Sigma^{-1}R$ and a $\Sigma$-inverting morphism $\iota : R \to \Sigma^{-1}R$ with the usual universal property: given any $\Sigma$-inverting morphism $\varphi : R \to T$ to another ring $T$, there exists a unique morphism $\varphi_\Sigma : \Sigma^{-1}R \to T$ such that $\varphi = \varphi_\Sigma \circ \iota$.*

This can be found in [18, Chapter 7], and is a generalization of Theorem 1.3.1. The pair $(\Sigma^{-1}R, \iota)$ is unique up to isomorphism, and it is called the *universal $\Sigma$-inverting ring*, or just the *universal localization* of $R$ with respect to $\Sigma$. We also have an analogue of Proposition 1.3.2, as follows.

**Proposition 1.3.7.** *Let $I$ be a two-sided ideal of $R$, and let $\Sigma^{-1}I$ denote the two-sided ideal of $\Sigma^{-1}R$ generated by the elements of $\iota(I)$. We have a natural isomorphism $\Sigma^{-1}R/\Sigma^{-1}I \cong \overline{\Sigma}^{-1}(R/I)$, where $\overline{\Sigma}$ denotes the image of $\Sigma$ under the quotient map $\pi : R \to R/I$.*

Let $\varphi : R \to T$ be a $\Sigma$-invertible morphism. We define the $\Sigma$-*rational closure* of $R$ in $T$, denoted by $\mathcal{R}at_\Sigma(R,T)$ to be the set of all entries of inverses of matrices from $\varphi(\Sigma)$. If $\Sigma$ is multiplicative, by [18, Theorem 7.1.2] $\mathcal{R}at_\Sigma(R,T)$ is a subring of $T$ containing $\varphi(R)$.

When $\Sigma$ is the set of all square matrices over $R$ that become invertible in $T$ under a morphism $\varphi$, we denote the $\Sigma$-rational closure of $R$ in $T$ by $\mathcal{R}at(R,T)$, and simply call it the *rational closure* of $R$ in $T$. It is always a subring of $T$ containing $\varphi(R)$ by [18, Proposition 7.1.1 and Theorem 7.1.2].

A useful result when studying rational closures is Cramer's rule.

**Proposition 1.3.8** (Proposition 7.1.3 of [18])**.** *Let $A$ be an $n \times n$ matrix over $\mathcal{R}at_\Sigma(R,T)$. There exists an integer $m \geq 1$ and invertible matrices $P, Q \in \mathrm{GL}_{n+m}(\mathcal{R}at_\Sigma(R,T))$ such that*

$$B := P(A \oplus Id_m)Q \quad \text{is a } (n+m) \times (n+m) \text{ matrix with coefficients in } \varphi(R).$$

Another way of studying invertibility of elements is via the division closure of $R$. Let $\varphi : R \to T$ be a morphism of rings. We define the *division closure* of $R$ in $T$, denoted by $\mathcal{D}(R,T)$, to be the smallest subring of $T$ containing $\varphi(R)$ and closed under taking inverses of elements, when they exist in $T$ (that is, if $a \in \mathcal{D}(R,T)$ is invertible in $T$ with inverse $a^{-1}$, then $a^{-1} \in \mathcal{D}(R,T)$).

**Lemma 1.3.9.** *If $\varphi : R \to T$ is a morphism of rings, then:*

*i)* $\mathcal{D}(\mathcal{D}(R,T),T) = \mathcal{D}(R,T) = \mathcal{D}(R,\mathcal{D}(R,T))$.

*ii)* $\mathcal{R}at(\mathcal{R}at(R,T),T) = \mathcal{R}at(R,T) = \mathcal{R}at(R,\mathcal{R}at(R,T))$.

*iii)* $\mathcal{D}(R,T) \subseteq \mathcal{R}at(R,T)$.

*Moreover, if $T$ is $*$-regular, then the $*$-regular closure of $R$ in $T$, denoted by $\mathcal{R}(R,T)$, contains the rational closure $\mathcal{R}at(R,T)$*[9].

[9] See Section 1.2.1 for more information about the $*$-regular closure of a $*$-subring of a $*$-regular ring.

*Proof.* The proof of $i)$ is immediate, and $ii)$ can be found in [68, Proposition 3.3]. For $iii)$, take $x \in \mathcal{R}at(R,T)$ which is invertible inside $T$. By Cramer's rule (Proposition 1.3.8), there exists an integer $n \geq 1$ and invertible matrices $P, Q \in \mathrm{GL}_{n+1}(\mathcal{R}at(R,T))$ such that the matrix $P(x \oplus \mathrm{Id}_n)Q$ has entries in $\varphi(R)$. Therefore the matrix $Q^{-1}(x^{-1} \oplus \mathrm{Id}_n)P^{-1}$ belongs, by definition, to $M_{n+1}(\mathcal{R}at(R,T))$, so

$$x^{-1} \oplus \mathrm{Id}_n = Q(Q^{-1}(x^{-1} \oplus \mathrm{Id}_n)P^{-1})P \in M_{n+1}(\mathcal{R}at(R,T)).$$

Hence $x^{-1} \in \mathcal{R}at(R,T)$. This proves that the rational closure is closed under taking inverses of elements, when they exist in $T$. Since it already contains $\varphi(R)$, we deduce that $\mathcal{D}(R,T) \subseteq \mathcal{R}at(R,T)$ by definition of the division closure.

To prove the final part, let $x \in \mathcal{R}at(R,T)$. By definition, $x$ is an entry of some invertible $n \times n$ matrix $A^{-1}$, where $A$ has entries in $\varphi(R)$. Since $\varphi(R) \subseteq \mathcal{R}(R,T)$ and $M_n(\mathcal{R}(R,T))$ is regular, there exists a matrix $B \in M_n(\mathcal{R}(R,T))$ such that $ABA = A$. But inside $M_n(T)$, $A$ is invertible with inverse $A^{-1}$. Therefore $A^{-1} = B \in M_n(\mathcal{R}(R,T))$. Since $x$ was an entry of $A^{-1}$, it follows that $x \in \mathcal{R}(R,T)$. □

# Chapter 2

# Sylvester rank functions on $\mathbb{Z}$-crossed product $*$-algebras and an embedding problem

This chapter, together with the next one, can be thought of as the core of this thesis. In this chapter we concentrate on the study of certain crossed product $*$-algebras by means of a homeomorphism $T : X \to X$ on a totally disconnected, compact, metrizable space $X$. We consider the crossed product $\mathcal{A} := C_K(X) \rtimes_T \mathbb{Z}$ induced by this homeomorphism and the possible Sylvester matrix rank functions that one can construct on $\mathcal{A}$ by means of ergodic $T$-invariant probability measures $\mu$ on $X$.

We present a general construction of approximating $\mathcal{A}$ by a sequence of $*$-subalgebras $\mathcal{A}_n$ which are embeddable into (possibly infinite) matrix product algebras, motivated by a construction given by Putnam [87, 88]. This will enable us to embed the whole $*$-algebra $\mathcal{A}$ into $\mathcal{M}_K$, the well-known von Neumann continuous factor over $K$ (Theorems 2.3.7 and 2.3.9; also see Example 1.2.8 or Chapter 4 Section 4.2 for a detailed description of $\mathcal{M}_K$) and, since $\mathcal{M}_K$ admits a unique Sylvester matrix rank function $\mathrm{rk}_{\mathcal{M}_K}$, it can be restricted to a rank function $\mathrm{rk}_{\mathcal{A}}$ over $\mathcal{A}$. This process gives a way to obtain a unique Sylvester matrix rank function on $\mathcal{A}$ satisfying a certain property (Proposition 2.3.8).

To conclude, we initiate the study of the $*$-regular closure of $\mathcal{A}$ inside $\mathcal{M}_K$ in order to obtain information about the possible numbers that the rank function $\mathrm{rk}_{\mathcal{A}}$ can achieve. In Proposition 2.4.2 we compute the rank completion of this $*$-regular closure, which gives $\mathcal{M}_K$ again.

## 2.1 Motivation coming from the theory of $C^*$-algebras

Here we collect some preliminary information on the relation between traces, measures and states. All this is well-known, see for instance [85]. In this, and in the next chapters, when writing 'measure' we will mean 'Borel regular measure'.

Let $G$ be a countable discrete group acting on a compact metrizable space $X$. In this section, we will denote by $C(X)$ the $C^*$-algebra of complex-valued continuous functions on $X$. By a standard result (see e.g. [85, Theorem 2.8]), the extreme points in the compact convex set of $G$-invariant probability measures on $X$ are precisely the ergodic invariant measures on $X$. By [85, Example 11.31], every $G$-invariant probability measure $\mu$ on $X$ can be extended to a tracial state[1] $\tau$ on the reduced crossed product $C(X) \rtimes_r G$ (using the conditional expectation $E$ onto $C(X)$, see [85, Definition 9.18]). By [85, Theorem 15.22], if the $G$-action is free, then all the tracial states on $C(X) \rtimes_r G$ are obtained this way.

We thus obtain the following well-known fact. Its proof is an easy adaptation of the proof of [85, Theorem 15.21].

[1] A tracial state on a unital $C^*$-algebra $\mathcal{B}$ is a positive linear functional $\tau : \mathcal{B} \to \mathbb{C}$ such that $\tau(1) = 1$ and $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{B}$.

**Theorem 2.1.1.** *Let $\mu$ be a $G$-invariant probability measure on $X$. Assume that $\mu$ is ergodic and almost everywhere free (i.e. for each $g \in G\backslash\{e\}$ the set of fixed points of $g$ has $\mu$-measure 0). If $\tau$ is a tracial state on $C(X) \rtimes_r G$ such that $\tau(f) = \int_X f d\mu$ for all $f \in C(X)$, then necessarily $\tau$ is induced by $\mu$, that is*

$$\tau(a) = \tau(E(a)) = \int_X E(a) d\mu$$

*for all $a \in C(X) \rtimes_r G$, where $E : C(X) \rtimes_r G \to C(X)$ is the canonical conditional expectation onto $C(X)$.*

*Proof.* Let $a \in C(X) \rtimes_r G$ and $\varepsilon > 0$ be given. Since the reduced $C^*$-algebra $C(X) \rtimes_r G$ is the completion of the set of finite formal sums

$$\sum_{g \in G \text{ finite}} b_g u_g, \quad b_g \in C(X)$$

with respect to the reduced norm $\|\cdot\|_r$, we can find an element $b = \sum_{g \in F} b_g u_g$ such that $\|a - b\|_r < \varepsilon/3$, where $F$ is a finite subset of $G$ containing the unit element $e \in G$, and $b_g \in C(X)$. Recall that the product of two element $b_g u_g$ and $b_h u_h$ is defined via the action of $G$ on $C(X)$, that is

$$(b_g u_g)(b_h u_h) = b_g \alpha_g(b_h) u_{gh}$$

where $\alpha_g : C(X) \to C(X)$ is given by $\alpha_g(f)(x) = f(g^{-1}x)$.

Now, since $F$ is a finite set and $\mu$ is almost everywhere free, we can find an open subset $U \subseteq X$ such that $\mu(U) = 1$, and such that $gx \neq x$ for all $g \in F\backslash\{e\}$ and $x \in U$. By regularity of the measure, there exists a compact subset $K$ of $X$ such that $K \subseteq U$ and $\mu(U\backslash K) < \eta^2$, where $\eta$ is such that

$$0 < \eta < \frac{\varepsilon}{3\Big(\sum_{g \in F\backslash\{e\}} \tau(b_g^* b_g)^{1/2}\Big)}.$$

Using the same argument as in [85, Lemma 15.18] and the compactness of $K$, we can find $s_1, ..., s_n \in C(X)$ such that $|s_k(x)| = 1$ for all $k = 1, ..., n$ and all $x \in X$, and such that

$$\frac{1}{n} \sum_{k=1}^n s_k(x) \alpha_g(s_k^*)(x) = 0 \quad \text{for all } x \in K \text{ and all } g \in F\backslash\{e\}.$$

We now consider the map $P : C(X) \rtimes_r G \to C(X) \rtimes_r G$ defined by $P(x) = \frac{1}{n} \sum_{k=1}^n s_k x s_k^*$. Observe that, for $x \in C(X) \rtimes_r G$, $\tau(P(x)) = \frac{1}{n} \sum_{k=1}^n \tau(s_k x s_k^*) = \frac{1}{n} \sum_{k=1}^n \tau(x) = \tau(x)$ since $\tau$ is a tracial state and $s_k^* s_k = 1$ for all $k$, so

$$\begin{aligned} |\tau(a) - \tau(E(a))| &\leq |\tau(a - b)| + |\tau(b) - \tau(E(b))| + |\tau(E(b) - E(a))| \\ &\leq \|a - b\|_r + |\tau(P(b)) - \tau(E(b))| + \|a - b\|_r \\ &< \frac{2\varepsilon}{3} + |\tau(P(b) - E(b))|. \end{aligned}$$

Therefore in order to prove the theorem it only suffices to check that $|\tau(P(b) - E(b))| < \frac{\varepsilon}{3}$, because in that case $|\tau(a) - \tau(E(a))| < \varepsilon$ and so $\tau(a) = \tau(E(a)) = \int_X E(a) d\mu$, since $E(a) \in C(X)$.

We compute $E(b) = b_e u_e = \frac{1}{n} \sum_{k=1}^n s_k b_e u_e s_k^*$, and

$$\begin{aligned} P(b) - E(b) &= \frac{1}{n} \sum_{k=1}^n s_k \Big( \sum_{g \in F} b_g u_g \Big) s_k^* - \frac{1}{n} \sum_{k=1}^n s_k b_e u_e s_k^* \\ &= \frac{1}{n} \sum_{k=1}^n s_k \Big( \sum_{g \in F\backslash\{e\}} b_g u_g \Big) s_k^* = \sum_{g \in F\backslash\{e\}} \Big( \frac{1}{n} \sum_{k=1}^n s_k \alpha_g(s_k^*) \Big) b_g u_g \end{aligned}$$

and thus, using the Cauchy-Schwartz inequality for $\tau$ [2], we obtain

$$\begin{aligned} |\tau(P(b) - E(b))| &\leq \sum_{g \in F\backslash\{e\}} \Big| \tau\Big( \Big( \frac{1}{n} \sum_{k=1}^n s_k \alpha_g(s_k^*) \Big) b_g u_g \Big) \Big| \\ &\leq \sum_{g \in F\backslash\{e\}} \tau\Big( \Big( \frac{1}{n} \sum_{k=1}^n s_k \alpha_g(s_k^*) \Big) \Big( \frac{1}{n} \sum_{k=1}^n s_k \alpha_g(s_k^*) \Big)^* \Big)^{\frac{1}{2}} \tau(b_g^* b_g)^{\frac{1}{2}}. \end{aligned}$$

[2]Recall that this inequality states that $|\tau(xy)|^2 \leq \tau(xx^*)\tau(y^*y)$ for any elements $x, y$ in the $C^*$-algebra.

But now for each $g \in F\backslash\{e\}$, we have

$$\tau\Big(\Big(\frac{1}{n}\sum_{k=1}^{n} s_k\alpha_g(s_k^*)\Big)\Big(\frac{1}{n}\sum_{k=1}^{n} s_k\alpha_g(s_k^*)\Big)^*\Big) = \int_X \Big|\frac{1}{n}\sum_{k=1}^{n} s_k(x)\alpha_g(s_k^*)(x)\Big|^2 d\mu$$
$$= \int_K 0\ d\mu + \int_{X\backslash K} \Big|\frac{1}{n}\sum_{k=1}^{n} s_k(x)\alpha_g(s_k^*)(x)\Big|^2 d\mu \le \mu(X\backslash K) = 1-\mu(K) < \eta^2.$$

It follows that $|\tau(P(b)-E(b))| \le \Big(\sum_{g\in F\backslash\{e\}} \tau(b_g^*b_g)^{1/2}\Big)\eta < \frac{\varepsilon}{3}$, as desired. □

The above ergodic measures induce extremal tracial states on $C(X) \rtimes_r G$, as follows:

**Proposition 2.1.2.** *For each ergodic and almost everywhere free $G$-invariant probability measure $\mu$ on $X$, we define $\varphi_\mu$ to be the induced tracial state on $C(X) \rtimes_r G$ given by Theorem 2.1.1, that is*

$$\varphi_\mu(a) = \int_X E(a)d\mu, \quad a \in C(X) \rtimes_r G.$$

*Then $\varphi_\mu$ is extremal. Moreover, $\varphi_\mu$ is a faithful state if and only if the support of $\mu$ is $X$ (that is, $\mu(U) > 0$ for all nonempty open subset $U$ of $X$).*

*Proof.* Write $\varphi := \varphi_\mu$. Suppose that $\varphi = \alpha\tau_1 + \beta\tau_2$ for some tracial states $\tau_i$ on $C(X) \rtimes_r G$ and for nonnegative numbers $\alpha, \beta$ such that $\alpha + \beta = 1$. Then if we denote by $\tau_i|$ the restriction of $\tau_i$ on $C(X)$, by the Riesz Representation Theorem there are unique $G$-invariant probability measures $\mu_i$ on $X$ such that each $\tau_i|$ is given by

$$\tau_i|(f) = \int_X f d\mu_i, \quad f \in C(X),$$

and it follows that $\mu = \alpha\mu_1 + \beta\mu_2$: indeed, for any continuous function $f \in C(X)$, we have

$$\int_X f d\mu = \varphi(f) = \alpha\tau_1(f) + \beta\tau_2(f) = \alpha\int_X f d\mu_1 + \beta\int_X f d\mu_2 = \int_X f d(\alpha\mu_1 + \beta\mu_2),$$

so by the uniqueness part of the Riesz Representation Theorem, $\mu = \alpha\mu_1 + \beta\mu_2$. Since $\mu$ is extremal ([85, Theorem 2.8]) it follows that either $\alpha = 0$ or $\alpha = 1$, or $\mu_1 = \mu_2$ . Thus either $\varphi = \tau_1$ or $\varphi = \tau_2$, or $\tau_1| = \tau_2|$ are given by integration against $\mu$. But if $\tau_1| = \tau_2|$ are given by integration against $\mu$, then it follows from Theorem 2.1.1 that $\tau_1 = \tau_2$ is given by the formula $\tau_i(a) = \int_X E(a)d\mu = \varphi(a)$. We have shown that $\varphi$ is extremal in $\mathrm{Tr}(C(X) \rtimes_r G)$, the space of tracial states of $C(X) \rtimes_r G$, and so $\tau_1 = \tau_2 = \varphi$.

Let's now prove the second part of the statement. If the support of $\mu$ is not $X$, there exists a nonempty open subset $U$ of $X$ such that $\mu(U) = 0$. Using Urysohn's Lemma, we can find a nonzero and positive continuous function $f \in C(X)$ with $\mathrm{supp}(f) \subseteq U$. But then $\varphi(f) = \int_X f d\mu = \int_{\mathrm{supp}(f)} f d\mu = 0$, so $\varphi$ is not faithful.

Conversely, if the support of $\mu$ is $X$, then the restriction $\tau$ of $\varphi$ to $C(X)$ is a faithful state (given by integration against $\mu$). To see this, assume we have a continuous function $f \in C(X)$ such that $\tau(f^*f) = 0$ and $f \neq 0$. Therefore $f(x) \neq 0$ for some point $x \in X$, and since $f$ is continuous we can find an open set $U \subseteq X$ such that $|f(x)| > 0$ on $x \in U$. Now for any compact $K \subseteq U$, since $f$ is continuous we can find a positive constant $\lambda_K > 0$ such that $|f(x)|^2 \ge \lambda_K$ for every $x \in K$. Then

$$0 = \tau(f^*f) = \int_X |f|^2 d\mu \ge \int_K |f|^2 d\mu \ge \lambda_K\mu(K),$$

so $\mu(K) = 0$. The regularity of the measure implies that $\mu(U) = 0$. This contradicts the fact that the support of $\mu$ is all of $X$, and so we have shown that $\tau$ is faithful. Finally, since the conditional expectation $E\colon C(X) \rtimes_r G \to C(X)$ is faithful ([85, Proposition 9.16]), it follows that $\varphi$ is a faithful state. □

One of our main motivations on writing down this theory comes from the following well-known theorem in the theory of $C^*$-algebras. Recall that the celebrated Murray-von Neumann Theorem ([80]) states that all the hyperfinite $II_1$ factors on separable, infinite-dimensional Hilbert spaces are $*$-isomorphic.

We briefly present the typical model of such hyperfinite $II_1$ factor, denoted by $\mathscr{R}$.

Following Example 1.2.8.2), consider the sequence $M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow M_8(\mathbb{C}) \hookrightarrow \cdots$ with connecting maps

$$\varphi_n : M_{2^n}(\mathbb{C}) \hookrightarrow M_{2^{n+1}}(\mathbb{C}), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

thought as a sequence of (finite-dimensional) $C^*$-algebras together with $*$-homomorphisms. The maps $\varphi_n$ are easily seen to be isometries, so the norm at each $M_{2^n}(\mathbb{C})$ defines a norm on the inductive limit $\varinjlim_n M_{2^n}(\mathbb{C})$ by the rule $\|x\|_{2^\infty} := \lim_n \|x_n\|_{2^n}$. We define the inductive limit $C^*$-algebra $M_{2^\infty}$ to be the enveloping $C^*$-algebra of $\varinjlim_n M_{2^n}(\mathbb{C})$ with respect to the norm $\|\cdot\|_{2^\infty}$, i.e. we complete $\varinjlim_n M_{2^n}(\mathbb{C})$ with respect to the metric induced by the norm $\|\cdot\|_{2^\infty}$, so that $M_{2^\infty} := \overline{\varinjlim_n M_{2^n}(\mathbb{C})}^{\|\cdot\|_{2^\infty}}$.

For each $n \geq 1$, let $\mathrm{tr}_n : M_{2^n}(\mathbb{C}) \to \mathbb{C}$ denote the normalized trace $\mathrm{tr}_n = \frac{1}{2^n}\mathrm{Tr}$. They are tracial states on $M_{2^n}(\mathbb{C})$, and since $\mathrm{tr}_{n+1}(\varphi_n(x)) = \mathrm{tr}_n(x)$ for $x \in M_{2^n}(\mathbb{C})$, we get a unique tracial state on $\varinjlim_n M_{2^n}(\mathbb{C})$ which, by continuity, is extended to a unique tracial state $\tau$ on $M_{2^\infty}$. By using the GNS representation, we can represent $M_{2^\infty}$ as a set of operators acting on some Hilbert space $\mathcal{H}_\tau$; in other words, we can find a Hilbert space $\mathcal{H}_\tau$, a $*$-representation $\pi_\tau : M_{2^\infty} \hookrightarrow \mathcal{B}(\mathcal{H}_\tau)$ and a cyclic vector $\xi_\tau \in \mathcal{H}_\tau$ for $\pi_\tau(M_{2^\infty})$ in such a way that $\tau$ becomes

$$\tau(x) = \langle \pi_\tau(x)(\xi_\tau), \xi_\tau \rangle_{\mathcal{H}_\tau}.$$

We finally let $\mathscr{R}$ be the von Neumann algebra generated by $M_{2^\infty}$ inside $\mathcal{B}(\mathcal{H}_\tau)$, $\mathscr{R} = \pi_\tau(M_{2^\infty})''$. $\tau$ then extends to a normal tracial state $\tau_{\mathscr{R}}$ by the above formula, so that $(\mathscr{R}, \tau_{\mathscr{R}})$ becomes a finite von Neumann algebra of type $II_1$. $\mathscr{R}$ is hyperfinite by construction, and it is fairly easy to see that $\mathscr{R}$ is indeed a factor.

**Theorem 2.1.3.** *Let $G$ be a countable discrete, amenable group acting on a compact metrizable space $X$. Let $\mu$ be an ergodic, almost everywhere free $G$-invariant probability measure on $X$ whose support is $X$, and let $\varphi_\mu$ be the corresponding extremal tracial state on $A = C(X) \rtimes_r G$ given by Theorem 2.1.1.*

*Then we can embed $C(X) \rtimes_r G \hookrightarrow \mathscr{R}$ in such a way that $\varphi_\mu$ extends to the unique tracial state $\tau_{\mathscr{R}}$. In particular, for any clopen subset $U$ of $X$,*

$$\tau_{\mathscr{R}}(\chi_U) = \mu(U).$$

*Proof.* It is well-known that any $II_1$ factor comes equipped with a unique normal tracial state, so the trace $\tau_{\mathscr{R}}$ already constructed is the unique tracial state on $\mathscr{R}$.

We give a general result. Let $A$ be a separable $C^*$-algebra, and let $\varphi$ be any faithful tracial state on $A$. Let $(\mathcal{H}_\varphi, \xi_\varphi, \pi_\varphi)$ be the associated GNS-representation, so $\pi_\varphi : A \hookrightarrow \mathcal{B}(\mathcal{H}_\varphi)$ and

$$\varphi(a) = \langle \pi_\varphi(a)(\xi_\varphi), \xi_\varphi \rangle_{\mathcal{H}_\varphi} \quad \text{for } a \in A.$$

Then $A \cong \pi_\varphi(A) \subseteq \pi_\varphi(A)''$, so $A$ can be embedded in the von Neumann algebra generated by itself inside $\mathcal{B}(\mathcal{H}_\varphi)$. $\varphi$ is an extremal tracial state if and only if $\pi_\varphi(A)''$ is a factor ([76, Theorems 5.1.5 and 5.1.8]). Moreover, if $A$ is nuclear, then $\pi_\varphi(A)''$ is injective ([12]), hence hyperfinite by the equivalence given by Connes in [22]. Since $\varphi$ can be extended to a normal tracial state on $\pi_\varphi(A)''$, it becomes a hyperfinite $II_1$ factor, hence isomorphic to $\mathscr{R}$.

Applying this result to $A = C(X) \rtimes_r G$, since it is a separable nuclear $C^*$-algebra ([12, Theorem 4.2.6]) and $\varphi_\mu$ is extremal and faithful (Proposition 2.1.2), we can embed $C(X) \rtimes_r G$ into $\mathscr{R}$, and moreover

$$\varphi_\mu(a) = \langle \pi_{\varphi_\mu}(a)(\xi_{\varphi_\mu}), \xi_{\varphi_\mu} \rangle_{\mathcal{H}_{\varphi_\mu}} = \tau_{\mathscr{R}}(\pi_{\varphi_\mu}(a)) \quad \text{for } a \in C(X) \rtimes_r G.$$

For the final part, note first that $\chi_U \in C(X)$ since $U$ is clopen. If we identify $C(X) \rtimes_r G \hookrightarrow \mathscr{R}$, then

$$\tau_{\mathscr{R}}(\chi_U) = \varphi_\mu(\chi_U) = \int_X \chi_U \, d\mu = \mu(U). \qquad \square$$

One can see that the factor appearing in the GNS construction above is simply the von Neumann crossed product $L^\infty(X, \mu) \rtimes G$, known as the *group measure construction* in the literature.

We want to obtain analogous results in an algebraic setting: we want to replace traces by Sylvester rank functions, and weak completions by rank completions. For this we will develop an internal construction, based on the work of Putnam et al [87, 88].

## 2.2 A first approximation for $\mathbb{Z}$-crossed product $*$-algebras of the form $\mathcal{A} := C_K(X) \rtimes_T \mathbb{Z}$

We will concentrate on the most basic dynamical system, the one provided by a single homeomorphism $T\colon X \to X$ on a totally disconnected, compact metrizable space $X$. Recall that a probability measure $\mu$ on $X$ is *ergodic* if for every $T$-invariant Borel subset $E$ of $X$ we have that either $\mu(E) = 0$ or $\mu(E) = 1$. $\mu$ is said to be *invariant* in case $\mu(T(E)) = \mu(E)$ for every Borel subset $E$ of $X$.

In what follows $\mu$ will be an ergodic, $T$-invariant probability measure on $X$. We will also often assume that $\mu$ is *full*, that is, its support is the whole space $X$[3].

The following is a simple application of Rokhlin's Lemma. We include a proof for the convenience of the reader.

**Lemma 2.2.1.** *Let $\mu$ be an ergodic $T$-invariant probability measure on $X$, and take $E$ to be a Borel subset of $X$ with positive measure. Consider the first return map $r_E\colon E \to \mathbb{N} \cup \{\infty\}$, defined by*

$$r_E(x) = \min\{l > 0 \mid T^l(x) \in E\}$$

*in case there is $l > 0$ such that $T^l(x) \in E$, and $r_E(x) = \infty$ otherwise. For each $k \in \mathbb{N}$, consider $Y_k^0 = r_E^{-1}(\{k\})$, that is, the set of points of $E$ that return to $E$ for the first time after $k$ iterations by $T$. Also let $Y_\infty = E \backslash \bigsqcup_{k\in\mathbb{N}} Y_k^0$ be the set of points of $E$ that do not return to $E$. For each $1 \le l \le k-1$, we set $Y_k^l = T^l(Y_k^0)$.*

*Then we have that $T(Y_k^l) = Y_k^{l+1}$ for $0 \le l < k-1$, all the sets $Y_k^l$ are mutually disjoint, and the set*

$$Y = Y(E) = \bigsqcup_{k\ge 1} \bigsqcup_{l=0}^{k-1} Y_k^l$$

*satisfies that $\mu(Y) = 1$. In particular, we get*

$$\sum_{k\ge 1} \sum_{l=0}^{k-1} \mu(Y_k^l) = \sum_{k\ge 1} k\mu(Y_k^0) = 1.$$

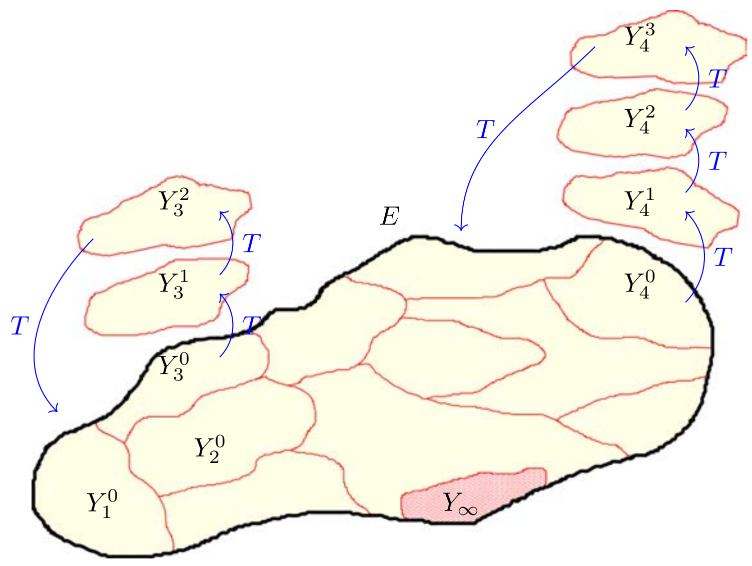



Figure 2.1: A schematic for the sets $Y_k^0$ partitioning the set $E$ (up to a set of measure 0) and the sets $Y_k^l$ partitioning the whole space $X$ (up to a set of measure 0).

*Proof.* Note that each $Y_k^0$ is given by the set $E \cap T^{-1}(X\backslash E) \cap \cdots \cap T^{-k+1}(X\backslash E) \cap T^{-k}(E)$, which are Borel sets. Therefore $Y_\infty = E\backslash \bigsqcup_{k\in\mathbb{N}} Y_k^0$ is also Borel. Moreover, if we choose $E$ to be a clopen set, then all the $Y_k^l$ are also clopen sets, and $Y_\infty$ is a closed set.

[3]It is not true in general that an ergodic $T$-invariant measure is full. For instance, take $T = \mathrm{Id}$ and any one-point mass measure.

We first prove that if $Z$ is a Borel subset of $X$ such that $T(Z) \subseteq Z$, then either $\mu(Z) = 0$ or $\mu(Z) = 1$. Take $Z_0 = \bigcap_{j\geq 0} T^j(Z) \subseteq Z$. Then clearly $T(Z_0) = Z_0$ since $T(Z) \subseteq Z$, so $Z_0$ is a $T$-invariant Borel set. Hence by ergodicity of the measure either $\mu(Z_0) = 0$ or $\mu(Z_0) = 1$. But by invariance and the fact that $T^j(Z) \subseteq Z$ for all $j \geq 0$,

$$\mu(Z\backslash Z_0) = \mu\Big(\bigcup_{j\geq 0} Z\backslash T^j(Z)\Big) \leq \sum_{j\geq 0} \mu(Z\backslash T^j(Z)) = 0,$$

so $\mu(Z) = \mu(Z_0)$ is either 0 or 1, as claimed.

We now show that different sets $Y_k^l, Y_{k'}^{l'}$ are disjoint. Assume that we can find some element $x \in X$ in the intersection $Y_k^l \cap Y_{k'}^{l'}$, that is, such that $x = T^l(y) = T^{l'}(y')$ with $y \in Y_k^0$ and $0 \leq l \leq k-1$, $y' \in Y_{k'}^0$ and $0 \leq l' \leq k'-1$. We can assume without loss of generality that $l \geq l'$. If $l > l'$, $y$ is such that $T^{l-l'}(y) = y' \in E$, so since $y \in Y_k^0$ we must have $k \leq l - l' \leq k - l' - 1$. This is absurd, so necessarily $l = l'$. But in this case $y = y' \in Y_k^0 \cap Y_{k'}^0$. This is only possible for $k = k'$. This says that different sets $Y_k^l, Y_{k'}^{l'}$ are disjoint.

Let's now prove that the $T$-translates of $Y_\infty$ are pairwise disjoint. Assume that we can find an element $x \in T^l(Y_\infty) \cap T^{l'}(Y_\infty)$ for some $0 \leq l, l'$, and assume without loss of generality that $l > l'$. In this case, $y \in E$ is such that $T^{l-l'}(y) = y' \in E$, contradicting the fact that $y \in Y_\infty$. Hence the claim follows.

Finally, let's prove that the $T$-translates of $Y_\infty$ are pairwise disjoint with the sets $Y_k^l$. Take then an element $x \in Y_k^l \cap T^{l'}(Y_\infty)$, so $x = T^l(y) = T^{l'}(z)$ with $y \in Y_k^0$ and $0 \leq l \leq k-1$, $z \in Y_\infty$. If $l < l'$, $T^{l'-l}(z) = y \in E$, which is not possible since $z \in Y_\infty$. If $l = l'$, $z = y \in Y_k^0$, so $T^k(z) = T^k(y) \in E$, which is again not possible since $z \in Y_\infty$. If $l > l'$, $k - (l - l') > 0$, and so $T^{k-(l-l')}(z) = T^k(y) \in E$, a contradiction again. This says that the sets $T^{l'}(Y_\infty)$ and $Y_k^l$ are disjoint.

We now consider

$$Z := \Big(\bigsqcup_{j\geq 0} T^j(Y_\infty)\Big) \sqcup \Big(\bigsqcup_{k\geq 1} \bigsqcup_{l=0}^{k-1} Y_k^l\Big) = \Big(\bigsqcup_{j\geq 0} T^j(Y_\infty)\Big) \sqcup Y.$$

which is Borel. We observe that $T(Z) \subseteq Z$. Indeed, it is clear that $T(Y_k^l) \subseteq Z$ for $0 \leq l < k-1$, and also, $T(Y_k^{k-1}) \subseteq E \subseteq Z$. Therefore by the preceding observation either $\mu(Z) = 0$ or $\mu(Z) = 1$. But it cannot be 0, since it contains $E$, which has positive measure. Hence $\mu(Z) = 1$. Note now that, since $\mu$ is a probability measure,

$$1 \geq \mu\Big(\bigsqcup_{j\geq 0} T^j(Y_\infty)\Big) = \sum_{j\geq 0} \mu(T^j(Y_\infty)) = \sum_{j\geq 0} \mu(Y_\infty)$$

which shows that $\mu(Y_\infty) = 0$. It follows that

$$1 = \mu(Z) = \mu\Big(\bigsqcup_{j\geq 0} T^j(Y_\infty)\Big) + \mu(Y) = \mu(Y) = \sum_{k\geq 1} \sum_{l=0}^{k-1} \mu(Y_k^l). \qquad \square$$

We briefly recall the general construction of the algebraic crossed product of an algebra $A$ by $\mathbb{Z}$. Let $A$ be an algebra and $\alpha : A \to A$ an automorphism of $A$. This automorphism induces a natural action of the infinite cyclic group $\mathbb{Z}$ on $A$ by the rule

$$t^n \cdot a := \alpha^n(a), \quad a \in A.$$

In this manner, we can form the algebraic crossed product algebra $A \rtimes_\alpha \mathbb{Z}$, consisting of formal finite sums $\sum_{n\in\mathbb{Z}} a_n t^n$ where $a_n \in A$ and $t$ is a symbol, with componentwise addition and product given by

$$(at^n)(a't^m) := a(t^n \cdot a')t^{n+m} = a\alpha^n(a')t^{n+m}, \quad a, a' \in A, n, m \in \mathbb{Z}.$$

If moreover $A$ is endowed with an involution $*$ providing the structure of a $*$-algebra and $\alpha$ is a $*$-isomorphism, then $A \rtimes_\alpha \mathbb{Z}$ has also the structure of a $*$-algebra, with involution given by

$$(at^n)^* := t^{-n}a^* = \alpha^{-n}(a^*)t^{-n}, \quad a \in A, n \in \mathbb{Z}.$$

For completeness, we also recall the definition of a *partial* algebraic crossed product by a $\mathbb{Z}$-action. The reader may consult [32] for more information.

A *partial* action of $\mathbb{Z}$ on a $*$-algebra $A$ is a pair $\phi = (\{A_n\}_{n\in\mathbb{Z}}, \{\phi_n\}_{n\in\mathbb{Z}})$ consisting of a collection of self-adjoint two-sided ideals $A_n$ of $A$ and a collection of $*$-isomorphisms $\phi_n : A_{-n} \to A_n$ such that

a) $A_0 = A$ and $\phi_0$ is the identity map.

b) $\phi_n \circ \phi_m \subseteq \phi_{n+m}$, meaning that $\phi_n \circ \phi_m$ is defined on the largest possible domain where the composition makes sense and $\phi_{n+m}$ extends it.

The partial algebraic crossed product of $A$ by $\mathbb{Z}$ with respect to the partial action $\phi$, denoted by $A \rtimes_\phi \mathbb{Z}$, is defined to be the set of all finite formal sums $\sum_{n\in\mathbb{Z}} a_n\delta^n$ with $a_n \in A_n$ and $\delta$ a symbol, with the usual componentwise addition and the product defined by the rule

$$(a_n\delta^n)(b_m\delta^m) := \phi_n(\phi_{-n}(a_n)b_m)\delta^{n+m}.$$

The involution is then defined through the rule

$$(a_n\delta^n)^* := \phi_{-n}(a_n^*)\delta^{-n}.$$

We now return to our previous setting on dynamical systems. From now on $K$ will denote an arbitrary field. From the space $X$ we can construct the algebra of locally constant continuous functions $C_K(X)$, that is, the set of all functions $f : X \to K$ such that for any point $x \in X$, there exists an open neighborhood $U$ of $x$ such that $f$ is constant on $U$. In fact, we can think of $C_K(X)$ as the linear span of characteristic functions $\chi_U$, being $U$ a clopen subset of $X$. To see this, note that we can form an open cover of our space $X = \bigcup_{x\in X} U_x$, where each $U_x$ is an open neighborhood of $x$ such that $f$ is constant on each $U_x$. Since our space $X$ is compact, $X = \bigcup_{i=1}^n U_{x_i}$ for some finite collection $U_{x_1}, ..., U_{x_n}$. Now, $f$ is constant on each $U_{x_i}$, so we can assume (by collecting those $U$'s on which $f$ takes the same value) that the $U_{x_i}$ are disjoint, so $X = \bigsqcup_{i=1}^n U_{x_i}$. In this case each $U_{x_i}$ is a clopen subset of $X$, and it is straightforward to see that

$$f = \sum_{i=1}^n \lambda_i \chi_{U_{x_i}},$$

where $\lambda_i \in K$ is the value of $f$ on $U_{x_i}$.

We can let our homeomorphism $T$ act on $C_K(X)$ by the automorphism given by the rule

$$T(f)(x) := f(T^{-1}(x)), \quad f \in C_K(X), x \in X.$$

Therefore we can apply the algebraic crossed product construction to get the algebra $C_K(X) \rtimes_T \mathbb{Z}$. If moreover $K$ is endowed with an involution $^-$, then $C_K(X)$ has the structure of a $*$-algebra by the rule

$$(f^*)(x) := \overline{f(x)}, \quad f \in C_K(X), x \in X.$$

Hence $C_K(X) \rtimes_T \mathbb{Z}$ becomes a $*$-algebra too, because $T$ is a $*$-automorphism of $C_K(X)$:

$$T(f^*)(x) = f^*(T^{-1}(x)) = \overline{f(T^{-1}(x))} = \overline{T(f)(x)} = T(f)^*(x) \quad \text{for every } f \in C_K(X), x \in X.$$

We start our construction by first approximating our space $X$, and then using this approximation to construct a family of approximating algebras for $C_K(X) \rtimes_T \mathbb{Z}$. First, some definitions.

**Definition 2.2.2.** Let $Y$ be a topological space, endowed with a probability measure $\mu$.

a) By a *partition* of $Y$ we will understand a finite family $\mathcal{P}$ of nonempty, pairwise disjoint clopen subsets of $Y$, such that $Y = \bigsqcup_{Z\in\mathcal{P}} Z$.

   Given two partitions $\mathcal{P}_1, \mathcal{P}_2$ of $Y$, we say that $\mathcal{P}_2$ is finer than $\mathcal{P}_1$ (or $\mathcal{P}_1$ is coarser than $\mathcal{P}_2$), written $\mathcal{P}_1 \precsim \mathcal{P}_2$, if every element $Z \in \mathcal{P}_2$ is contained in a (unique) element $Z' \in \mathcal{P}_1$, that is $Z \subseteq Z'$.

b) By a *quasi-partition* of $Y$ we will understand a finite or countable family $\overline{\mathcal{P}}$ of nonempty, pairwise disjoint clopen subsets of $Y$, such that $Y = \bigsqcup_{\overline{Z}\in\overline{\mathcal{P}}} \overline{Z}$ up to a set of measure 0, that is $\mu\Big(Y \setminus \bigsqcup_{\overline{Z}\in\overline{\mathcal{P}}} \overline{Z}\Big) = 0$.

   Given two quasi-partitions $\overline{\mathcal{P}}_1, \overline{\mathcal{P}}_2$ of $Y$, we say that $\overline{\mathcal{P}}_2$ is finer than $\overline{\mathcal{P}}_1$ (or $\overline{\mathcal{P}}_1$ is coarser than $\overline{\mathcal{P}}_2$), written $\overline{\mathcal{P}}_1 \precsim \overline{\mathcal{P}}_2$, if every element $\overline{Z} \in \overline{\mathcal{P}}_2$ is contained in a (unique) element $\overline{Z}' \in \overline{\mathcal{P}}_1$, that is $\overline{Z} \subseteq \overline{Z}'$.

Note that, in the hypotheses of Lemma 2.2.1, the family $\{Y_k^l\}$ forms a quasi-partition of $X$.

**Definition 2.2.3.** Consider the $*$-algebra $\mathcal{A} := C_K(X) \rtimes_T \mathbb{Z}$. Let $E$ be a nonempty clopen subset of $X$, and let $\mathcal{P}$ be a partition of $X \backslash E$. Define $\mathcal{B} := \mathcal{A}(E, \mathcal{P})$ as the unital $*$-subalgebra of $\mathcal{A}$ generated by the partial isometries $\chi_Z \cdot t$ for $Z \in \mathcal{P}$.

For example, inside $\mathcal{B}$ we can find all the characteristic functions $\chi_Z = (\chi_Z t)(\chi_Z t)^* \in \mathcal{B}$, together with the element $\chi_{X\backslash E} t = \sum_{Z \in \mathcal{P}} \chi_Z t \in \mathcal{B}$ which can be thought of as an approximation for $t \in \mathcal{A}$.

Our first goal is to express $\mathcal{B}$ as a partial algebraic crossed product by a $\mathbb{Z}$-action. Let $\mathcal{B}_0 = C_K(X) \cap \mathcal{B}$ (that is, the set of elements of $\mathcal{B}$ that have degree 0 in $t$), which is a commutative $*$-subalgebra of $\mathcal{B}$. We first give a complete description of $\mathcal{B}_0$ in terms of characteristic functions.

**Lemma 2.2.4.** *The $*$-algebra $\mathcal{B}_0$ is linearly spanned by* 1 *and the projections of the form*

$$\chi_{T^{-r}(Z_{-r}) \cap T^{-r+1}(Z_{-r+1}) \cap \cdots \cap Z_0 \cap T(Z_1) \cap \cdots \cap T^{s-1}(Z_{s-1})}, \tag{2.2.1}$$

*where $Z_{-r}, ..., Z_0, ..., Z_{s-1} \in \mathcal{P}$, and $r, s \geq 0$.*

*Proof.* Recall that for a clopen subset $U$ of $X$, $t\chi_U t^{-1} = T(\chi_U) = \chi_{T(U)}$. We have

$$(\chi_{Z_0} t)(\chi_{Z_1} t) \cdots (\chi_{Z_{s-1}} t)(\chi_{Z_{s-1}} t)^* \cdots (\chi_{Z_0} t)^* = \chi_{Z_0 \cap T(Z_1) \cap \cdots \cap T^{s-1}(Z_{s-1})}$$

and

$$(\chi_{Z_{-1}} t)^* \cdots (\chi_{Z_{-r}} t)^* (\chi_{Z_{-r}} t) \cdots (\chi_{Z_{-1}} t) = \chi_{T^{-r}(Z_{-r}) \cap \cdots \cap T^{-1}(Z_{-1})},$$

which shows that all the projections of the form (2.2.1) belong to $\mathcal{B}_0$.

Let $\mathcal{F}$ be the set of projections of the form (2.2.1) together with 0. Observe that the family $\mathcal{F}$ is closed under products, for if we have two projections

$$\chi_{T^{-r}(Z_{-r}) \cap T^{-r+1}(Z_{-r+1}) \cap \cdots \cap Z_0 \cap T(Z_1) \cap \cdots \cap T^{s-1}(Z_{s-1})}, \quad \chi_{T^{-r'}(Z_{-r'}) \cap T^{-r'+1}(Z'_{-r'+1}) \cap \cdots \cap Z'_0 \cap T(Z'_1) \cap \cdots \cap T^{s'-1}(Z'_{s'-1})},$$

where $Z_{-r}, ..., Z_0, ..., Z_{s-1}, Z'_{-r'}, ..., Z'_0, ..., Z'_{s'-1} \in \mathcal{P}$, and $r, r', s, s' \geq 0$ with for example $r > r'$ and $s > s'$, its product is given by

$$\chi_{T^{-r}(Z_{-r}) \cap \cdots \cap T^{-r'}(Z_{-r'} \cap Z'_{-r'}) \cap \cdots \cap (Z_0 \cap Z'_0) \cap T(Z_1 \cap Z'_1) \cap \cdots \cap T^{s'-1}(Z_{s'-1} \cap Z'_{s'-1}) \cap \cdots \cap T^{s-1}(Z_{s-1})}.$$

Since $\mathcal{P}$ is a partition of $X \backslash E$, it is again of the form (2.2.1) or zero. The other cases for different $r, r', s, s'$ are similar.

Hence, to show the result, it is enough to prove that any product of generators $a_1 \cdots a_n$ of degree 0 in $t$ is of the above form (here each $a_i$ is either of the form $\chi_Z t$ or of the form $(\chi_Z t)^* = t^{-1}\chi_Z$, for $Z \in \mathcal{P}$). An immediate observation is that if $a_1 \cdots a_n$ is of degree 0, then $n$ must be even. We will proceed to show the result by induction on $n$.

Clearly the result is true for $n = 2$, so assume $n > 2$ is even and that each product of at most $n - 2$ generators of degree 0 in $t$ belongs to $\mathcal{F}$. Define $d(i) \in \mathbb{Z}$ by $d(i) = \deg_t(a_1 \cdots a_i)$. Suppose, for instance, that $d(1) = 1$.

If there is $r < n$ such that $d(r) = 0$, then since

$$0 = \deg_t(a_1 \cdots a_n) = d(r) + \deg_t(a_{r+1} \cdots a_n) = \deg_t(a_{r+1} \cdots a_n),$$

we can use induction to conclude that the products $a_1 \cdots a_r$ and $a_{r+1} \cdots a_n$ belong to $\mathcal{F}$, so since the set of projections of the stated form is closed under products, we get the result.

Otherwise we must have $d(r) > 0$ for all $r < n$ and since $d(n) = 0$ we must have that $\deg_t(a_n) = -1$. Then necessarily $a_1 = \chi_{Z_1} t$ and $a_n = t^{-1}\chi_{Z_2}$ for some $Z_1, Z_2 \in \mathcal{P}$, and thus

$$a_1 a_2 \cdots a_{n-1} a_n = \chi_{Z_1}(t a_2 \cdots a_{n-1} t^{-1})\chi_{Z_2}. \tag{2.2.2}$$

By induction, the product $a_2 \cdots a_{n-1}$ belongs to $\mathcal{F}$, and hence $t a_2 \cdots a_{n-1} t^{-1}$ either belongs to $\mathcal{F}$ or it is of the form $\chi_{T(Z'_1) \cap \cdots \cap T^{s-1}(Z'_{s-1})}$, for $Z'_1, \ldots, Z'_s \in \mathcal{P}$. Therefore, the product (2.2.2) is zero if $Z_1 \neq Z_2$ and, if $Z_1 = Z_2$, it belongs to $\mathcal{F}$. In either case, $a_1 \cdots a_n$ belongs to $\mathcal{F}$, as desired. The case where $d(1) = -1$ is similar.

It then follows that $\mathcal{B}$ is the linear span of the given set of projections $\mathcal{F}$. This concludes the proof of the lemma. ∎

We now consider the structure of $\mathcal{B}$ as a partial algebraic crossed product by $\mathbb{Z}$ on $\mathcal{B}_0$. We claim that we can write $\mathcal{B} = \bigoplus_{i\in\mathbb{Z}} \mathcal{B}_i t^i$, where $\mathcal{B}_i = \chi_{X\setminus(E\cup T(E)\cup\cdots\cup T^{i-1}(E))}\mathcal{B}_0$ and $\mathcal{B}_{-i} = \chi_{X\setminus(T^{-1}(E)\cup\cdots\cup T^{-i}(E))}\mathcal{B}_0$ for $i > 0$. Let's show this only for $\mathcal{B}_i$ with $i > 0$, being the other case analogous. Since an element of $\mathcal{B}$ is a linear combination of products of generators $a_1 \cdots a_n$, being each $a_j$ either $\chi_Z t$ or $t^{-1}\chi_Z$ ($Z \in \mathcal{P}$), it is enough to show that if such a product has degree $i$ in $t$, then it can be written as

$$a_1 \cdots a_n = b_i t^i, \quad \text{with } b_i \in \mathcal{B}_0 \text{ satisfying } b_i = \chi_{X\setminus(E\cup\cdots\cup T^{i-1}(E))} b_i.$$

We will show this by induction on $i > 0$. For $i = 1$, take $a_j$ to be a generator such that $\deg_t(a_1 \cdots a_{j-1}) = 0$ and $\deg_t(a_1 \cdots a_{j-1}a_j) = 1$. In this case $a_j = \chi_Z t$ for some $Z \in \mathcal{P}$, and the products $a_1 \cdots a_{j-1}, a_{j+1} \cdots a_n$ both belong to $\mathcal{B}_0$, so

$$a_1 \cdots a_n = (a_1 \cdots a_{j-1})\chi_Z t(a_{j+1} \cdots a_n) = (a_1 \cdots a_{j-1})(ta_{j+1} \cdots a_n t^{-1})\chi_Z \cdot \chi_{X\setminus E} t.$$

We observe that the claim is true in this case. Now assume that the result is true for $i \geq 1$, and take $a_1 \cdots a_n$ of degree $i+1$ in $t$. As before, take $a_j$ to be a generator such that $\deg_t(a_1 \cdots a_{j-1}) = i$ and $\deg_t(a_1 \cdots a_{j-1}a_j) = i+1$. In this case $a_j = \chi_Z t$ for some $Z \in \mathcal{P}$, the product $a_1 \cdots a_{j-1}$ is of the form $b_i t^i$ with $b_i = \chi_{X\setminus(E\cup\cdots\cup T^{i-1}(E))} b_i$ by induction hypothesis, and $a_{j+1} \cdots a_n \in \mathcal{B}_0$. We compute

$$\begin{aligned} a_1 \cdots a_n &= (a_1 \cdots a_{j-1})\chi_Z t(a_{j+1} \cdots a_n) = (b_i t^i)\chi_Z \cdot \chi_{X\setminus E} t(a_{j+1} \cdots a_n) \\ &= \chi_{X\setminus(E\cup\cdots\cup T^{i-1}(E)\cup T^i(E))} \cdot \chi_{T^i(Z)} b_i (t^{i+1} a_{j+1} \cdots a_n t^{-i-1}) t^{i+1}, \end{aligned}$$

which is an element of the desired form. This concludes the induction, and the claim is proved.

Observe that if $b_i t^i \in \mathcal{B}$ then $b_i = \chi_{X\setminus(E\cup\cdots\cup T^{i-1}(E))} b_i$ for $i > 0$ and $b_i = \chi_{X\setminus(T^{-1}(E)\cup\cdots\cup T^{-i}(E))} b_i$ for $i < 0$, and so

$$b_i t^i = b_i (\chi_{X\setminus E} t)^i, \quad b_{-i} t^{-i} = b_{-i} (t^{-1}\chi_{X\setminus E})^i \quad \text{for positive } i.$$

In particular, it is true that $\mathcal{B}_i t^i = \mathcal{B}_0 (\chi_{X\setminus E} t)^i$ and $\mathcal{B}_{-i} t^{-i} = \mathcal{B}_0 (t^{-1}\chi_{X\setminus E})^i$ for $i > 0$.

**Observation 2.2.5.** One needs to be careful with the term $\chi_{X\setminus E} t$ because, although $t$ is invertible with inverse $t^{-1} = t^*$, this is not true for $\chi_{X\setminus E} t$. As a consequence, equalities like

$$(\chi_{X\setminus E} t)^i = (\chi_{X\setminus E} t)^{i+j} (\chi_{X\setminus E} t)^{-j}$$

are no longer true and even meaningful for $i > j > 0$. In the next lemma we summarize the basic arithmetics that one can achieve with these powers.

From now on for $i > 0$, we will write $(\chi_{X\setminus E} t)^{-i}$ for the element $(t^{-1}\chi_{X\setminus E})^i$. We will also understand that $(\chi_{X\setminus E} t)^0$ is 1.

**Lemma 2.2.6.** *Fix $i \geq j \geq 0$. We have the following rules:*

*i)* $(\chi_{X\setminus E} t)^i = (\chi_{X\setminus E} t)^{i-j} (\chi_{X\setminus E} t)^j = (\chi_{X\setminus E} t)^j (\chi_{X\setminus E} t)^{i-j}$.

*ii)* $(\chi_{X\setminus E} t)^{-i} = (\chi_{X\setminus E} t)^{-i+j} (\chi_{X\setminus E} t)^{-j} = (\chi_{X\setminus E} t)^{-j} (\chi_{X\setminus E} t)^{-i+j}$.

*iii)* $(\chi_{X\setminus E} t)^i \neq (\chi_{X\setminus E} t)^{i+j} (\chi_{X\setminus E} t)^{-j} \neq (\chi_{X\setminus E} t)^{-j} (\chi_{X\setminus E} t)^{i+j} \neq (\chi_{X\setminus E} t)^i$, *but: we have the first equality when multiplied (to the left) by the projection* $\chi_{X\setminus(E\cup\cdots\cup T^{i+j-1}(E))}$; *we have the second equality when multiplied (to the left) by the projection* $\chi_{X\setminus(T^{-j}(E)\cup\cdots\cup T^{i+j-1}(E))}$; *we have the third equality when multiplied (to the left) by the projection* $\chi_{X\setminus(T^{-j}(E)\cup\cdots\cup T^{-1}(E))}$.

*iv)* $(\chi_{X\setminus E} t)^{-i} \neq (\chi_{X\setminus E} t)^{-i-j} (\chi_{X\setminus E} t)^j \neq (\chi_{X\setminus E} t)^j (\chi_{X\setminus E} t)^{-i-j} \neq (\chi_{X\setminus E} t)^{-i}$, *but: we have the first equality when multiplied (to the right) by the projection* $\chi_{X\setminus(T^{-j}(E)\cup\cdots\cup T^{-1}(E))}$; *we have the second equality when multiplied (to the right) by the projection* $\chi_{X\setminus(T^{-j}(E)\cup\cdots\cup T^{i+j-1}(E))}$; *we have the third equality when multiplied (to the right) by the projection* $\chi_{X\setminus(E\cup\cdots\cup T^{i+j-1}(E))}$.

The proof of Lemma 2.2.6 is purely computational, so we will not write it down. From now on, we will make use of it without any further reference.

Note that each $\mathcal{B}_i, \mathcal{B}_{-i}$ is an ideal of $\mathcal{B}_0$. Let us define the basic map of the partial action of $\mathbb{Z}$ on $\mathcal{B}_0$ as conjugation by the approximation of $t$, $\chi_{X\setminus E} t$:

$$\varphi_1 \colon \mathcal{B}_{-1} \to \mathcal{B}_1, \quad \varphi_1(b_{-1}) = (\chi_{X\setminus E} t) b_{-1} (\chi_{X\setminus E} t)^*,$$

which is a $*$-isomorphism between $\mathcal{B}_{-1} = \chi_{X\setminus T^{-1}(E)}\mathcal{B}_0$ and $\mathcal{B}_1 = \chi_{X\setminus E}\mathcal{B}_0$ with inverse given by conjugation by $(\chi_{X\setminus E}t)^{-1} = t^{-1}\chi_{X\setminus E}$. Note that since $b_{-1} \in \mathcal{B}_{-1}$, $b_{-1} = \chi_{X\setminus T^{-1}(E)}b_{-1}$. In general for $i \neq 0$ we have a $*$-isomorphism $\varphi_i$ from $\mathcal{B}_{-i}$ onto $\mathcal{B}_i$ which is given by conjugation by $(\chi_{X\setminus E}t)^i$. Using these maps we build a partial action $\varphi$ of $\mathbb{Z}$ on $\mathcal{B}_0$, and we get the following result.

**Proposition 2.2.7.** *There is a canonical $*$-isomorphism $\mathcal{B}_0 \rtimes_\varphi \mathbb{Z} \cong \mathcal{B}$ given by*

$$\Psi\colon \mathcal{B}_0 \rtimes_\varphi \mathbb{Z} \to \mathcal{B}, \quad \Psi\Big(\sum_{i\in\mathbb{Z}} b_i\delta_i\Big) = \sum_{i\in\mathbb{Z}} b_i(\chi_{X\setminus E}t)^i = \sum_{i\in\mathbb{Z}} b_i t^i,$$

*where $b_i \in \mathcal{B}_i$ for $i \in \mathbb{Z}$. Recall that we are using the notation $(\chi_{X\setminus E}t)^i = (t^{-1}\chi_{X\setminus E})^{-i}$ for $i < 0$, and $(\chi_{X\setminus E}t)^0 = 1$.*

*Proof.* Routine. The only nontrivial thing may be to check that the products are preserved. The key observation here is that the product $b_iT^i(b_j)$ belongs to $\mathcal{B}_{i+j}$ for any integer values of $i, j$; this follows by a case-by-case analysis using Lemma 2.2.6. After that, a direct computation shows that the products are indeed preserved under $\Psi$:

$$\begin{aligned}\Psi((b_i\delta_i)(b_j\delta_j)) &= \Psi(\varphi_i(\varphi_{-i}(b_i)b_j)\delta_{i+j}) = \varphi_i(\varphi_{-i}(b_i)b_j)(\chi_{X\setminus E}t)^{i+j}\\ &= (\chi_{X\setminus E}t)^i(\chi_{X\setminus E}t)^{-i}b_i(\chi_{X\setminus E}t)^i b_j(\chi_{X\setminus E}t)^{-i}(\chi_{X\setminus E}t)^{i+j}\\ &= b_iT^i(b_j)(\chi_{X\setminus E}t)^{i+j} = b_iT^i(b_j)t^{i+j} = \Psi(b_i\delta_i)\Psi(b_j\delta_j).\end{aligned}$$ ☐

From Proposition 2.2.7 we can deduce that we are approximating our crossed product $*$-algebra $C_K(X) \rtimes_T \mathbb{Z}$ by a *partial* crossed product $*$-algebra $\mathcal{B}_0 \rtimes_\varphi \mathbb{Z}$, where $\mathcal{B}_0$ is obtained from $C_K(X)$ by means of our nonempty clopen set $E$ and the partition $\mathcal{P}$ of the complement $X\setminus E$.

We summarize in the next lemma the structure of the elements belonging to the ideals $\mathcal{B}_i$, $i \in \mathbb{Z}$, of $\mathcal{B}_0$.

**Lemma 2.2.8.** *A nonzero element $b_i \in \mathcal{B}_i$ ($i \in \mathbb{Z}$) can be written as an orthogonal linear combination of characteristic functions of nonempty sets of the following four different types:*

(*I*) $T^{-N}(Z_{-N}) \cap \cdots \cap T^{M-1}(Z_{M-1})$, *with* $Z_j \in \mathcal{P}$;

(*II*) $T^{-N}(Z_{-N}) \cap \cdots \cap T^{s-2}(Z_{s-2}) \cap T^{s-1}(E)$, *for* $0 \leq s \leq M$ *and* $Z_j \in \mathcal{P}$;

(*III*) $T^{-r}(E) \cap T^{-r+1}(Z_{-r+1}) \cap \cdots \cap T^{M-1}(Z_{M-1})$, *for* $0 \leq r \leq N$ *and* $Z_j \in \mathcal{P}$;

(*IV*) $T^{-r}(E) \cap T^{-r+1}(Z_{-r+1}) \cap \cdots \cap T^{s-2}(Z_{s-2}) \cap T^{s-1}(E)$, *for* $0 \leq r \leq N, 0 \leq s \leq M$ *and* $Z_j \in \mathcal{P}$;

*for some $N, M \geq 0$, where if $i < 0$ then $N \geq -i$ and $r \geq -i$ in (III) and (IV), and if $i > 0$ then $M \geq i$ and $s \geq i$ in (II) and (IV).*

*Proof.* Due to Lemma 2.2.4, we can write a given $b_i \in \mathcal{B}_i$ as a sum

$$b_i = \lambda_0 + \sum_S \lambda_S\chi_S$$

where the sets $S$ are of the form (2.2.1), and $\lambda_0, \lambda_S \in K$. Note that if $i < 0$ then we can take $\lambda_0 = 0$ and all the sets $S$ as in (2.2.1) having $r \geq -i$, and similarly if $i > 0$ we can take $\lambda_0 = 0$ and all the sets $S$ as in (2.2.1) having $s \geq i$.

Take $N$ to be the maximum value of the $r$'s while running through the sets $S$, and $M$ to be the maximum value of the $s$'s. The idea is to expand the element 1 as an orthogonal sum of characteristic functions using the partition $\mathcal{P}$, namely by using the relation

$$1 = \chi_E + \sum_{Z\in\mathcal{P}} \chi_Z.$$

So for a fixed set $S = T^{-r}(Z_{-r}) \cap \cdots \cap T^{s-1}(Z_{s-1})$ with $0 \leq r < N$, we can decompose its characteristic functions as an orthogonal sum as follows:

$$\chi_S = 1 \cdot \chi_S = \chi_{T^{-r-1}(E)\cap S} + \sum_{Z\in\mathcal{P}} \chi_{T^{-r-1}(Z_{-r-1})\cap S}.$$

By further expanding the set $T^{-r-1}(E) \cap S$ to the right, we will end up with a sum of terms of types (III) and (IV); by expanding $T^{-r-1}(Z_{-r-1}) \cap S$ to both sides we will end up with a sum of terms of all types. Of course, we discard the empty sets that appear in this process. Also, if one of the terms appearing in the expansion of $S$ coincides with another term appearing in the expansion of some other set $S'$, we simply collect them by summing the corresponding coefficients. Proceeding in this way, we will end up with an orthogonal sum of the desired form. $\square$

### 2.2.1 A $*$-representation for $\mathcal{B}$

We will assume for the rest of this section that $\mu$ is an ergodic $T$-invariant full probability measure on $X$. We first apply the previous considerations given in Lemma 2.2.1 to the clopen set $E$, and we add into the picture the partition $\mathcal{P}$ of $X \backslash E$. That is, we consider the coarsest quasi-partition $\overline{\mathcal{P}}$ of $X$ such that

a) $\mathcal{P} \cup \{E\} \precsim \overline{\mathcal{P}}$ and $\{Y_k^l\} \precsim \overline{\mathcal{P}}$, where $\{Y_k^l\}$ is the quasi-partition introduced above in Lemma 2.2.1, and

b) if $\overline{Z} \in \overline{\mathcal{P}}$ and $\overline{Z} \subseteq Y_k^0$ for some $k$, then all its translates belong to the quasi-partition too, that is $T^i(\overline{Z}) \in \overline{\mathcal{P}}$ for every $1 \leq i \leq k-1$.

$\overline{\mathcal{P}}$ can be obtained by refining, using $\mathcal{P} \cup \{E\}$, the quasi-partition $\{Y_k^l\}$. It turns out that all the characteristic functions $\chi_{\overline{Z}}$, with $\overline{Z} \in \overline{\mathcal{P}}$, belong to $\mathcal{B}$.

**Lemma 2.2.9.** *The quasi-partition $\overline{\mathcal{P}}$ above consists exactly of all the nonempty subsets of $X$ of the form*

$$W = E \cap T^{-1}(Z_1) \cap T^{-2}(Z_2) \cap \cdots \cap T^{-k+1}(Z_{k-1}) \cap T^{-k}(E) \tag{2.2.3}$$

*for some $k$ and some $Z_1, Z_2, \dots, Z_{k-1} \in \mathcal{P}$ and all its translates by $T^l$, $0 \leq l \leq k-1$; that is, $W$ consists on the set of points $x \in E$ such that $T(x) \in Z_1, T^2(x) \in Z_2, ..., T^{k-1}(x) \in Z_{k-1}$ and $T^k(x) \in E$ again, so they return to $E$ for the first time after $k$ iterations by $T$.*

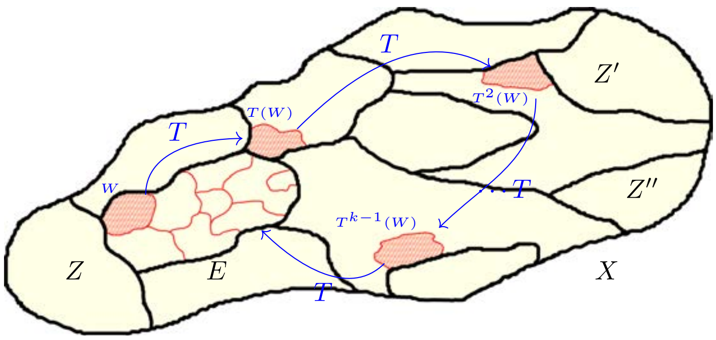



Figure 2.2: A schematic for the quasi-partition $\overline{\mathcal{P}}$, consisting of the sets $W$ and their translates.

*Moreover, each characteristic function $\chi_{\overline{Z}}$ belongs to $\mathcal{B}$ for any $\overline{Z} \in \overline{\mathcal{P}}$.*

*Proof.* Let $\mathbb{V}$ denote the set of all the nonempty sets $W$ of the form (2.2.3), and let $\mathcal{P}'$ be the family of all the translates of all $W \in \mathbb{V}$. For $W = E \cap T^{-1}(Z_1) \cap \cdots \cap T^{-k+1}(Z_{k-1}) \cap T^{-k}(E) \in \mathbb{V}$ we define $|W| = k$, the length of $W$. We will prove that $\mathcal{P}' = \overline{\mathcal{P}}$. We first show:

(1) $\mathcal{P}'$ is a quasi-partition of $X$. Clearly, the sets in $\mathcal{P}'$ are mutually disjoint since $\mathcal{P}$ forms a partition of $X \backslash E$, and the nonempty sets of $\mathbb{V}$ form, for a fixed length $k$, a partition of $Y_k^0 = r_E^{-1}(\{k\})$. Indeed,

$$\begin{aligned}\bigsqcup_{\substack{W \in \mathbb{V} \\ |W|=k}} W &= \bigsqcup_{Z_1, \dots, Z_{k-1} \in \mathcal{P}} E \cap T^{-1}(Z_1) \cap \cdots \cap T^{-k+1}(Z_{k-1}) \cap T^{-k}(E) \\ &= E \cap (X \backslash T^{-1}(E)) \cap \cdots \cap (X \backslash T^{-k+1}(E)) \cap T^{-k}(E) = Y_k^0.\end{aligned}$$

As a consequence, for a fixed $0 \leq l \leq k-1$, the $T^l$-translates of the $W \in \mathbb{V}$ having length $k$ form a partition of $Y_k^l = T^l(Y_k^0)$. Since the family $\{Y_k^l\}$ forms a quasi-partition of $X$, this shows that $\mathcal{P}'$ is a quasi-partition of $X$, because by Lemma 2.2.1,

$$\bigsqcup_{k\geq 1} \bigsqcup_{l=0}^{k-1} \bigsqcup_{\substack{W\in\mathbb{V} \\ |W|=k}} T^l(W) = \bigsqcup_{k\geq 1} \bigsqcup_{l=0}^{k-1} Y_k^l = Y(E).$$

(2) $\mathcal{P}'$ refines $\mathcal{P} \cup \{E\}$ and the family $\{Y_k^l\}$. This is a direct consequence of part (1).

(3) For $\overline{Z} \in \mathcal{P}'$ with $\overline{Z} \subseteq Y_k^0$ for some $k$, then $T^i(\overline{Z}) \in \mathcal{P}'$ for each $1 \leq i \leq k-1$. By construction, all the sets $\overline{Z} \in \mathcal{P}'$ with $\overline{Z} \subseteq Y_k^0$ for some $k$ are the $W \in \mathbb{V}$ having length $k$. It is then clear that all its translates $T^i(W) \in \mathcal{P}'$ for $1 \leq i \leq k-1$.

This shows that $\overline{\mathcal{P}} \precsim \mathcal{P}'$. To show that $\mathcal{P}' \precsim \overline{\mathcal{P}}$, we only have to check that if $Y' \subseteq Y_k^0$ is a nonempty clopen set such that for each $1 \leq i \leq k-1$ the translate $T^i(Y')$ is contained in one of the sets of the partition $\mathcal{P}$, then $Y' \subseteq W$ for some $W \in \mathbb{V}$. But this is clear, since if $T^i(Y') \subseteq Z_i$ for $i = 1, ..., k-1$ where $Z_i \in \mathcal{P}$, and $T^k(Y') \subseteq E$, then $Y' \subseteq E \cap T^{-1}(Z_1) \cap \cdots \cap T^{-k+1}(Z_{k-1}) \cap T^{-k}(E)$. Hence $\mathcal{P}' = \overline{\mathcal{P}}$.

We now check that $\chi_W$ belongs to $\mathcal{B}$. First observe that $\chi_E = 1 - (\chi_{X\backslash E}t)(\chi_{X\backslash E}t)^*$ and $\chi_{T^{-1}(E)} = 1 - (\chi_{X\backslash E}t)^*(\chi_{X\backslash E}t)$ both belong to $\mathcal{B}$. Now by Lemma 2.2.4, we have that $\chi_{T^{-1}(Z_1)\cap\cdots\cap T^{-k+1}(Z_{k-1})} \in \mathcal{B}$ for $Z_1, Z_2, \dots, Z_{k-1} \in \mathcal{P}$. Therefore

$$\chi_W = \chi_E \cdot \chi_{T^{-1}(Z_1)\cap\cdots\cap T^{-k+1}(Z_{k-1})} \cdot (t^{-1}\chi_{X\backslash E})^{k-1}\chi_{T^{-1}(E)}(\chi_{X\backslash E}t)^{k-1} \in \mathcal{B}.$$

Also, for $1 \leq l \leq k-1$, observe that

$$(\chi_{X\backslash E}t)^l \chi_W (t^{-1}\chi_{X\backslash E})^l = \chi_{X\backslash(E\cup T(E)\cup\cdots\cup T^{l-1}(E))} \cdot \chi_{T^l(W)} = \chi_{T^l(W)},$$

and so $\chi_{T^l(W)} \in \mathcal{B}$ too. □

**Proposition 2.2.10.** *For each $W \in \mathbb{V}$, we have $*$-isomorphisms*

$$\chi_W \mathcal{B} \chi_W \cong K, \quad \mathcal{B}\chi_W\mathcal{B} \cong M_{|W|}(K).$$

*Moreover, the element $h_W := \sum_{l=0}^{|W|-1} \chi_{T^l(W)}$ is a unit in the two-sided ideal $\mathcal{B}\chi_W\mathcal{B}$, a central projection in $\mathcal{B}$, and*

$$h_W\mathcal{B} \cong M_{|W|}(K).$$

*In particular, $\chi_W$ is a minimal projection in $\mathcal{B}$* [4].

*Proof.* Fix $W \in \mathbb{V}$. We will prove a more general statement, that is $\chi_{T^l(W)}\mathcal{B}\chi_{T^l(W)} \cong K$ for all $0 \leq l \leq |W|-1$. Write again $\mathcal{B} = \bigoplus_{i\in\mathbb{Z}} \mathcal{B}_i t^i = \bigoplus_{i\in\mathbb{Z}} \mathcal{B}_0(\chi_{X\backslash E}t)^i$, so that $\chi_{T^l(W)}\mathcal{B}\chi_{T^l(W)} = \bigoplus_{i\in\mathbb{Z}} \mathcal{B}_0\chi_{T^l(W)}(\chi_{X\backslash E}t)^i\chi_{T^l(W)}$. For $i > 0$, note that

$$\begin{aligned}
\chi_{T^l(W)}(\chi_{X\backslash E}t)^i\chi_{T^l(W)} &= \chi_{T^l(W)} \cdot \chi_{X\backslash(E\cup\cdots\cup T^{i-1}(E))}t^i\chi_{T^l(W)} \\
&= \chi_{T^l(W)} \cdot \chi_{T^{l+i}(W)} \cdot \chi_{X\backslash(E\cup\cdots\cup T^{i-1}(E))}t^i \\
&= \chi_{T^l(W)} \cdot \chi_{T^{l+i}(W)} \cdot \chi_{X\backslash(E\cup\cdots\cup T^{l+(i-1)}(E))}t^i = 0, \\
\chi_{T^l(W)}(t^{-1}\chi_{X\backslash E})^i\chi_{T^l(W)} &= \Big(\chi_{T^l(W)}(\chi_{X\backslash E}t)^i\chi_{T^l(W)}\Big)^* = 0.
\end{aligned}$$

Therefore $\chi_{T^l(W)}\mathcal{B}\chi_{T^l(W)} = \chi_{T^l(W)}\mathcal{B}_0$. Write now $W = E \cap T^{-1}(Z_1) \cap \cdots \cap T^{-k+1}(Z_{k-1}) \cap T^{-k}(E)$ with $k = |W|$ and $Z_1, ..., Z_{k-1} \in \mathcal{P}$. By Lemma 2.2.4, $\mathcal{B}_0$ is linearly spanned by 1 and the projections of the form

$$p = \chi_{T^{-r}(Z'_{-r})\cap\cdots\cap Z'_0\cap T(Z'_1)\cap\cdots\cap T^{s-1}(Z'_{s-1})} \quad \text{with } Z'_{-r}, ..., Z'_{s-1} \in \mathcal{P} \text{ and } r, s \geq 0.$$

If $s \geq l$, we directly compute $p \cdot \chi_{T^l(W)} = 0$ since $\chi_{T^l(Z'_l)\cap T^l(W)} = 0$. If $0 \leq s < l$,

$$p \cdot \chi_{T^l(W)} = \chi_{T^{-r}(Z'_{-r})\cap\cdots\cap Z'_0\cap\cdots\cap T^s(Z'_s)} \cdot \chi_{T^l(E)\cap T^{l-1}(Z_1)\cap\cdots\cap T^{l-k+1}(Z_{k-1})\cap T^{l-k}(E)}.$$

[4] For an idempotent $e$ in a ring $R$, we say that $e$ is *minimal* if the ideal $eR$ is simple as a right $R$-module.

This is 0 for $r \geq k-l$, and for $r < k-l$, it can be either 0 or $\chi_{T^l(W)}$ again, since $\mathcal{P}$ forms a partition of $X \backslash E$. All these observations together imply that $\chi_{T^l(W)}\mathcal{B}_0 = K\chi_{T^l(W)}$, so $\chi_{T^l(W)}\mathcal{B}\chi_{T^l(W)} = K\chi_{T^l(W)} \cong K$.

Now, by means of previous computations, it is straightforward to see that for general $i,j \in \mathbb{Z}$, we have

$$(\chi_{X\backslash E}t)^i \chi_W (t^{-1}\chi_{X\backslash E})^j = \begin{cases} (\chi_{X\backslash E}t)^i \chi_W (t^{-1}\chi_{X\backslash E})^j & \text{for } 0 \leq i,j \leq |W|-1 \\ 0 & \text{otherwise} \end{cases}. \tag{2.2.4}$$

We then consider

$$e_{ij}(W) := (\chi_{X\backslash E}t)^i \chi_W (t^{-1}\chi_{X\backslash E})^j, \quad 0 \leq i,j \leq |W|-1.$$

Observe that $e_{ll}(W) = (\chi_{X\backslash E}t)^l \chi_W (t^{-1}\chi_{X\backslash E})^l = \chi_{T^l(W)}$ for $0 \leq l \leq |W|-1$. We claim that the set $\{e_{ij}(W)\}_{0 \leq i,j \leq |W|-1}$ is a complete system of matrix units for $\mathcal{B}\chi_W\mathcal{B}$. Indeed, the defining relations for matrix units are satisfied:

$$\begin{aligned} e_{ij}(W)e_{kl}(W) &= (\chi_{X\backslash E}t)^i \chi_W (t^{-1}\chi_{X\backslash E})^j (\chi_{X\backslash E}t)^k \chi_W (t^{-1}\chi_{X\backslash E})^l \\ &= (\chi_{X\backslash E}t)^i t^{-j} \chi_{T^j(W)\cap T^k(W)} \cdot \chi_{X\backslash (E\cup \cdots \cup T^{\max\{j,k\}-1}(E))} t^k (t^{-1}\chi_{X\backslash E})^l \\ &= \delta_{j,k} (\chi_{X\backslash E}t)^i t^{-j} \chi_{T^j(W)} t^j (t^{-1}\chi_{X\backslash E})^l = \delta_{j,k} (\chi_{X\backslash E}t)^i \chi_W (t^{-1}\chi_{X\backslash E})^l = \delta_{j,k} e_{il}(W), \end{aligned}$$

and to prove that $h_W = \sum_{l=0}^{|W|-1} \chi_{T^l(W)} = \sum_{l=0}^{|W|-1} e_{ll}(W)$ is indeed a unit for $\mathcal{B}\chi_W\mathcal{B}$, we first use (2.2.4) to write

$$\begin{aligned} \mathcal{B}\chi_W\mathcal{B} &= \bigoplus_{i,j\in\mathbb{Z}} \mathcal{B}_0 (\chi_{X\backslash E}t)^i \chi_W \mathcal{B}_0 (t^{-1}\chi_{X\backslash E})^j = \bigoplus_{i,j\in\mathbb{Z}} \mathcal{B}_0 (\chi_{X\backslash E}t)^i \chi_W (t^{-1}\chi_{X\backslash E})^j \\ &= \bigoplus_{i,j\geq 0}^{|W|-1} \mathcal{B}_0 e_{ij}(W) = \bigoplus_{i,j\geq 0}^{|W|-1} (\mathcal{B}_0 e_{ii}(W)) e_{ij}(W) = \bigoplus_{i,j\geq 0}^{|W|-1} K e_{ij}(W) \end{aligned}$$

where we have used that $\mathcal{B}_0 e_{ii}(W) = K e_{ii}(W)$. It is now clear that $h_W$ is a unit for $\mathcal{B}\chi_W\mathcal{B}$. We thus get the desired $*$-isomorphism by sending

$$\mathcal{B}\chi_W\mathcal{B} \to M_{|W|}(K), \quad e_{ij}(W) \mapsto e_{ij} \tag{2.2.5}$$

where $\{e_{ij}\}_{0\leq i,j\leq |W|-1}$ is a complete system of matrix units for $M_{|W|}(K)$.

For the second statement, since $\mathcal{B} = \bigoplus_{i\in\mathbb{Z}} \mathcal{B}_0 (\chi_{X\backslash E}t)^i$, it is enough to show that $h_W$ commutes with all the elements $(\chi_{X\backslash E}t)^i$ for $i \in \mathbb{Z}$. By applying the involution, we may assume without loss of generality that $i \geq 1$. By induction, we may further assume that $i = 1$. But for $0 \leq l \leq |W|-1$,

$$\begin{aligned} e_{ll}(W)\cdot\chi_{X\backslash E}t &= \chi_{T^l(W)} \cdot \chi_{X\backslash E}t \\ &= \begin{cases} \chi_{X\backslash E}t \cdot e_{l-1,l-1}(W) & \text{if } 1 \leq l \leq |W|-1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} e_{l,l-1}(W) & \text{if } 1 \leq l \leq |W|-1 \\ 0 & \text{otherwise} \end{cases}, \\ \chi_{X\backslash E}t \cdot e_{ll}(W) &= (\chi_{X\backslash E}t)^{l+1} \chi_W (t^{-1}\chi_{X\backslash E})^l = \begin{cases} e_{l+1,l}(W) & \text{if } 0 \leq l \leq |W|-2 \\ 0 & \text{otherwise} \end{cases}, \end{aligned}$$

so by summing up over $l$ and doing the change $l+1 = l'$, it is clear that $h_W \cdot \chi_{X\backslash E}t = \chi_{X\backslash E}t \cdot h_W$. The result follows.

For the last part, simply observe that the family $\{e_{ij}(W)\}_{0\leq i,j\leq |W|-1}$ is also a complete system of matrix units for the central factor $h_W\mathcal{B}$ of $\mathcal{B}$, so by [64, Theorem 17.5, see also Remark 17.6], there is an isomorphism $h_W\mathcal{B} \cong M_{|W|}(K)$ given by

$$h_W b \mapsto \sum_{i,j=0}^{|W|-1} b_{ij} e_{ij}, \quad \text{with } b_{ij} = e_{1i}(W) \cdot b \cdot e_{j1}(W) \in e_{11}(W)\mathcal{B}e_{11}(W) \cong K$$

which is also a $*$-isomorphism. In fact, one should note that this $*$-isomorphism coincides with the $*$-isomorphism given in (2.2.5), i.e. $h_W\mathcal{B} = \mathcal{B}\chi_W\mathcal{B}$.

It is now straightforward to see that each $\chi_{T^l(W)}$ is a minimal projection in $\mathcal{B}$. □

As a consequence of Proposition 2.2.10 we obtain a $*$-homomorphism from the algebra $\mathcal{B}$ into an infinite matrix product $\mathfrak{R} := \mathfrak{R}(E, \mathcal{P}) = \prod_{W \in \mathbb{V}} M_{|W|}(K)$ given by

$$\pi : \mathcal{B} \to \mathfrak{R}, \quad \pi(b) = (h_W \cdot b)_W.$$

We will show below that this homomorphism is injective, but for that we need a preliminary lemma.

**Lemma 2.2.11.** *Suppose that $b \in \mathcal{B}_0$, $b \neq 0$ can be written as a finite linear combination of the form*

$$b = \sum_{U \in \mathcal{U}} \lambda_U \chi_U,$$

*where the $U \in \mathcal{U}$ are nonempty disjoint clopen subsets of $X$, and $\lambda_U \in K^*$. Then there exists a $W \in \mathbb{V}$ such that $h_W \cdot b \neq 0$.*

*Proof.* Fix one $U \in \mathcal{U}$. Since $\mu$ is a full measure, $\mu(U) > 0$. Also by Lemma 2.2.9, the $T$-translates of elements in $\mathbb{V}$ form a quasi-partition of $X$, so there exists a $W \in \mathbb{V}$ of length $k \geq 1$ such that $U \cap T^l(W) \neq \emptyset$ for some $0 \leq l \leq k-1$. But then

$$h_W \cdot \chi_{U \cap T^l(W)} b = \lambda_U \chi_{U \cap T^l(W)} \neq 0.$$

It follows that $h_W \cdot b \neq 0$. □

Before proving injectivity of $\pi$, we are first interested in computing some images of monomials $b_i t^i$ of $\mathcal{B}$ under $\pi$. So take $b_i t^i$ (resp. $b_{-j} t^{-j}$) with $b_i \in \mathcal{B}_i$ (resp. $b_{-j} \in \mathcal{B}_{-j}$). By Lemma 2.2.4, we can write $b_i$ (resp. $b_j$) as a linear combination of characteristic functions of nonempty sets of the form

$$T^{s-1}(Z'_{s-1}) \cap T^{s-2}(Z'_{s-2}) \cap \cdots \cap Z'_0 \cap T^{-1}(Z'_{-1}) \cap \cdots \cap T^{-r}(Z'_{-r}), \tag{2.2.6}$$

where $r, s \geq 0$. So from now on we will assume that each $b_i$ (resp. $b_j$) is the characteristic function of a set of the form (2.2.6). Note that since $b_i = \chi_{X \setminus (E \cup T(E) \cup \cdots \cup T^{i-1}(E))} b_i$ (resp. $b_{-j} = \chi_{X \setminus (T^{-1}(E) \cup \cdots \cup T^{-j}(E))} b_{-j}$), we can (and will) assume, by expanding these sets if necessary, that $s \geq i$ (resp. $r \geq j$).

**Definition 2.2.12.** Assume that $W$ is in standard form (2.2.3), i.e.

$$W = E \cap T^{-1}(Z_1) \cap \cdots \cap T^{-k+1}(Z_{k-1}) \cap T^{-k}(E)$$

for some $k \geq 1$ and some $Z_1, ..., Z_{k-1} \in \mathcal{P}$. We say that a sequence $(Z'_{s-1}, \ldots, Z'_0, \ldots, Z'_{-r})$ of elements of $\mathcal{P}$ *occurs* in $W$ if there exists $l \geq 0$ such that

$$Z_{l+1} = Z'_{s-1}, \quad Z_{l+2} = Z'_{s-2}, \quad \ldots, \quad Z_{l+s} = Z'_0, \quad Z_{l+s+1} = Z'_{-1}, \quad \ldots, \quad Z_{l+s+r} = Z'_{-r}.$$

That is, if the sequence $(Z'_{s-1}, \ldots, Z'_0, \ldots, Z'_{-r})$ occurs as a subsequence of $(Z_1, Z_2, ..., Z_{k-1})$ displaced $l$ positions to the right. In this case we say that $l$ is an *occurrence* of $(Z'_{s-1}, \ldots, Z'_0, \ldots, Z'_{-r})$ in $W$. Note that a necessary condition for the sequence $(Z'_{s-1}, \ldots, Z'_0, \ldots, Z'_{-r})$ to be an occurrence is that $s + r \leq k - 1$.

Observe that, by definition, if we let $S = T^{s-1}(Z'_{s-1}) \cap \cdots \cap Z'_0 \cap \cdots \cap T^{-r}(Z'_{-r})$ be given by (2.2.6), then $l$ is an occurrence of $(Z'_{s-1}, ..., Z'_0, ..., Z'_{-r})$ in $W$ if and only if $s + r \leq k - 1$ and $T^{l+s}(W) \cap S$ is nonempty, and in this case necessarily $T^{l+s}(W) \cap S = T^{l+s}(W)$.

**Lemma 2.2.13.** *Assume the above notation and that $i, j \geq 0$. We have:*

*i) If $b_i = \chi_S$ with $S$ of the form (2.2.6), then $h_W \cdot b_i t^i$ is nonzero if and only if $(Z'_{s-1}, \ldots, Z'_0, \ldots, Z'_{-r})$ occurs in $W$, and in this case we have*

$$h_W \cdot b_i t^i = \sum_l e_{l+s, l+s-i}(W),$$

*where $l$ ranges over the set of occurrences of $(Z'_{s-1}, \ldots, Z'_0, \ldots, Z'_{-r})$ in $W$.*

*ii) Suppose that $b_{-j} = \chi_S$ with $S$ also given by (2.2.6). Then $h_W \cdot b_{-j} t^{-j}$ is nonzero if and only if $(Z'_{s-1}, \ldots, Z'_0, \ldots, Z'_{-r})$ occurs in $W$, and in this case we have*

$$h_W \cdot b_{-j} t^{-j} = \sum_l e_{l+s, l+s+j}(W),$$

*where $l$ ranges over the set of occurrences of $(Z'_{s-1}, \ldots, Z'_0, \ldots, Z'_{-r})$ in $W$.*

*Observe that in any case the formula is valid, that is*

$$h_W \cdot b_i t^i = \sum_l e_{l+s,l+s-i}(W) \quad \text{whenever } i \in \mathbb{Z},$$

*where $l$ ranges over the set of occurrences of $(Z'_{s-1}, \dots, Z'_0, \dots, Z'_{-r})$ in $W$.*

*Proof.* *i*) This is a simple computation. Write $b_i = \chi_S$ with $S = T^{s-1}(Z'_{s-1}) \cap \cdots \cap Z'_0 \cap \cdots \cap T^{-r}(Z'_{-r})$. We compute $h_W \cdot b_i t^i = \sum_{l=0}^{k-1} \chi_{T^l(W)\cap S} t^i$. By the observation preceding the lemma this sum equals $\sum_l \chi_{T^{l+s}(W)} t^i$ where $l$ ranges over the set of occurrences of $(Z'_{s-1}, \dots, Z'_0, \dots, Z'_{-r})$ in $W$. Since $s \geq i$,

$$\begin{aligned}
\chi_{T^{l+s}(W)} t^i &= (\chi_{X\setminus E} t)^{l+s} \chi_W (t^{-1}\chi_{X\setminus E})^{l+s} t^i = t^{l+s} \chi_{X\setminus(T^{-1}(E)\cup\cdots\cup T^{-l-s}(E))} \cdot \chi_W t^{-l-s} t^i \\
&= t^{l+s} \chi_{X\setminus(T^{-1}(E)\cup\cdots\cup T^{-l-s}(E))} \cdot \chi_W \cdot \chi_{X\setminus(T^{-1}(E)\cup\cdots\cup T^{-l-s+i}(E))} t^{-l-s+i} \\
&= (\chi_{X\setminus E} t)^{l+s} \chi_W (t^{-1}\chi_{X\setminus E})^{l+s-i} = e_{l+s,l+s-i}(W).
\end{aligned}$$

The result follows from this computation. The proof of *ii*) is similar. ☐

There are two relatively special elements inside $\mathcal{B}$ that we are interested in computing their images under $\pi$ for later use. These are the elements $\chi_E = 1 - (\chi_{X\setminus E} t)(\chi_{X\setminus E} t)^*$ and $\chi_{T^{-1}(E)} = 1 - (\chi_{X\setminus E} t)^*(\chi_{X\setminus E} t)$. Their images under $\pi$ are easy to compute: for $W \in \mathbb{V}$, we have $\chi_W \cdot \chi_E = \chi_W$ and $\chi_{T^l(W)} \cdot \chi_E = 0$ for $1 \leq l \leq |W| - 1$, so $h_W \cdot \chi_E = \sum_{l=0}^{|W|-1} \chi_{T^l(W)} \cdot \chi_E = \chi_W = e_{00}(W)$ and

$$\pi(\chi_E) = (e_{00}(W))_W.$$

Also $\chi_{T^l(W)} \cdot \chi_{T^{-1}(E)} = 0$ for $0 \leq l \leq |W| - 2$ and $\chi_{T^{|W|-1}(W)} \cdot \chi_{T^{-1}(E)} = \chi_{T^{|W|-1}(W)}$, so $h_W \cdot \chi_{T^{-1}(E)} = \sum_{l=0}^{|W|-1} \chi_{T^l(W)} \cdot \chi_{T^{-1}(E)} = \chi_{T^{|W|-1}(W)} = e_{|W|-1,|W|-1}(W)$ and

$$\pi(\chi_{T^{-1}(E)}) = (e_{|W|-1,|W|-1}(W))_W.$$

**Proposition 2.2.14.** *With the above hypothesis and notation, we have that the map $\pi \colon \mathcal{B} \to \mathfrak{R}$ is injective. Moreover, the socle of $\mathcal{B}$ is essential and coincides with the ideal generated by $\chi_W$, for $W \in \mathbb{V}$, that is*

$$\mathrm{soc}(\mathcal{B}) = \bigoplus_{W\in\mathbb{V}} \mathcal{B}\chi_W\mathcal{B} = \bigoplus_{W\in\mathbb{V}} h_W\mathcal{B} \cong \bigoplus_{W\in\mathbb{V}} M_{|W|}(K).$$

*Proof.* For injectivity, it is enough to show that the ideal $\bigoplus_{W\in\mathbb{V}} h_W\mathcal{B}$ is essential in $\mathcal{B}$ or, equivalently, that for any nonzero element $b \in \mathcal{B}$, we can always find a $W \in \mathbb{V}$ such that $h_W \cdot b \neq 0$. By writing $b$ as a finite sum

$$b = \sum_{i=-n}^{m} b_i (\chi_{X\setminus E} t)^i = \sum_{i=-n}^{m} b_i t^i$$

with each $b_i \in \mathcal{B}_i$ and $b_{-n} \neq 0$ (where $n \in \mathbb{Z}$), it is enough to show that there exists a $W \in \mathbb{V}$ such that $h_W \cdot b_{-n} \neq 0$. But this follows immediately from Lemmas 2.2.8 and 2.2.11. We obtain that $\pi$ is injective, and also that the ideal $\bigoplus_{W\in\mathbb{V}} h_W\mathcal{B}$ is essential in $\mathcal{B}$.

Now since each $\chi_W$ is a minimal projection by Proposition 2.2.10, it follows that the ideal of $\mathcal{B}$ generated by $\chi_W$ is contained in the socle of $\mathcal{B}$. This says that $\bigoplus_{W\in\mathbb{V}} \mathcal{B}\chi_W\mathcal{B} \subseteq \mathrm{soc}(\mathcal{B})$. In particular, since $h_W\mathcal{B} = \mathcal{B}\chi_W\mathcal{B}$ for any $W \in \mathbb{V}$, this shows that $\mathrm{soc}(\mathcal{B})$ is essential in $\mathcal{B}$, and from the general fact that the socle is contained in any essential ideal ([64, Chapter 3, Exercise 6.12]), we conclude that $\bigoplus_{W\in\mathbb{V}} \mathcal{B}\chi_W\mathcal{B} = \mathrm{soc}(\mathcal{B})$, as required. ☐

There are some terms $b_i t^i$ which are *special*, in the sense that $b_i$ is exactly of the form $\chi_S$, with

$$S = T^{i-1}(Z'_{i-1}) \cap T^{i-2}(Z'_{i-2}) \cap \cdots \cap Z'_0, \tag{2.2.7}$$

that is $s = i$ and $r = 0$ in (2.2.6), and moreover we ask that

$$E \cap T^{-i}(S) \cap T^{-i-1}(E) = E \cap T^{-1}(Z'_{i-1}) \cap \cdots \cap T^{-i}(Z'_0) \cap T^{-i-1}(E) \neq \emptyset.$$

The set of clopen subsets $S$ of $X$ of this form will be denoted by $\mathcal{W}_i$.

Similarly, a term $b_{-j}t^{-j}$ is *special* in case it is of the form $\chi_{S'}$, with

$$S' = T^{-1}(Z'_{-1}) \cap T^{-2}(Z'_{-2}) \cap \cdots \cap T^{-j}(Z'_{-j}), \tag{2.2.8}$$

that is $r = j$ and $s = 0$ in (2.2.6), and moreover we ask that

$$E \cap S' \cap T^{-j-1}(E) = E \cap T^{-1}(Z'_{-1}) \cap \cdots \cap T^{-j}(Z'_{-j}) \cap T^{-j-1}(E) \neq \emptyset.$$

The set of clopen subsets $S'$ of $X$ of this form will be denoted by $\mathcal{W}_{-j}$.

As a term of degree 0 in $t$ we will consider $\chi_{S_0 \cup T^{-1}(S_1)}$, where

$$S_0 = E \cup \Big( \bigcup_{\substack{Z \in \mathcal{P} \\ Z \cap T^{-1}(E) \neq \emptyset}} Z \Big) \quad \text{and} \quad S_1 = E \cup \Big( \bigcup_{\substack{Z \in \mathcal{P} \\ T^{-1}(Z) \cap E \neq \emptyset}} Z \Big).$$

The corresponding clopen set $S_0 \cup T^{-1}(S_1)$ forms a set denoted by $\mathcal{W}_0$.

**Observation 2.2.15.** It is clear that, for $i \geq 1$, the set $\mathcal{W}_i$ is in bijection with the set of all $W \in \mathbb{V}$ having length $i+1$, through the map

$$S \mapsto W(S) := E \cap T^{-i}(S) \cap T^{-i-1}(E).$$

The inverse map will be written as $W \mapsto S(W)$, so that $S(W(S)) = S$ and $W(S(W)) = W$.

Analogously, for $j \geq$, the same set of all $W \in \mathbb{V}$ having length $j+1$ is in bijection with $\mathcal{W}_{-j}$ through the map

$$S' \mapsto W(S') := E \cap S' \cap T^{-j-1}(E).$$

Again, the inverse map will be also denoted by $W \mapsto S'(W)$.

$\mathcal{W}_0$ contains only the clopen $S_0 \cup T^{-1}(S_1)$ which, as a set, it is trivially in bijective correspondence with the set consisting of the only clopen $W \in \mathbb{V}$ with length 1, namely $W = E \cap T^{-1}(E)$, *if* nonempty. We will continue using the previous notation, namely $W \mapsto S(W) := S_0 \cup T^{-1}(S_1)$. Note that, by construction, the element $\chi_{S_0 \cup T^{-1}(S_1)}$ serves as a unit between the special terms, in the sense that

$$\chi_{S_0 \cup T^{-1}(S_1)} \cdot \chi_S t^i = \chi_S t^i = \chi_S t^i \cdot \chi_{S_0 \cup T^{-1}(S_1)} \quad \text{and} \quad \chi_{S_0 \cup T^{-1}(S_1)} \cdot \chi_{S'} t^{-j} = \chi_{S'} t^{-j} = \chi_{S'} t^{-j} \cdot \chi_{S_0 \cup T^{-1}(S_1)},$$

for $\chi_S, \chi_{S'}$ special terms of degrees $i, j \geq 1$, respectively.

The special terms are exactly detected by the representation $\pi$, as follows:

**Lemma 2.2.16.** *With the previous notation,*

*i) Let $b_i t^i = \chi_S t^i$ be a special term, as in* (2.2.7). *Then $h_W \cdot b_i t^i = e_{i,0}(W)$, where $W = W(S)$. Moreover, if $W' \neq W$ is of length $k$ for some $k \geq 0$, then the component of $e_{k-1,0}(W')$ in $h_{W'} \cdot b_i t^i$ is* 0.

*ii) Let $b_{-j}t^{-j} = \chi_{S'} t^{-j}$ be a special term, as in* (2.2.8). *Then $h_W \cdot b_{-j}t^{-j} = e_{0,j}(W)$, where $W = W(S')$. Moreover, if $W' \neq W$ is of length $k$ for some $k \geq 0$, then the component of $e_{0,k-1}(W')$ in $h_{W'} \cdot b_{-j}t^{-j}$ is* 0.

*iii) Let $b_0 = \chi_{S_0 \cup T^{-1}(S_1)}$ be the special term of degree* 0. *Then $h_W \cdot b_0 = e_{0,0}(W)$, where $W = E \cap T^{-1}(E)$ in case being nonempty. Moreover, if $W \neq E \cap T^{-1}(E)$, then the components of $e_{0,0}(W)$ and $e_{|W|-1,|W|-1}(W)$ in $h_W \cdot b_0$ are exactly* 1.

*Proof.* We will only prove *i)*, being the other ones analogous. Take $W = W(S) = E \cap T^{-i}(S) \cap T^{-i-1}(E)$, and note that $T^l(W) \cap S = \emptyset$ for $0 \leq l \leq i-1$. For $l = i$, it gives $T^i(W) \cap S = T^i(W)$. Hence

$$h_W \cdot b_i t^i = \sum_{l=0}^{i} \chi_{T^l(W) \cap S} t^i = \chi_{T^i(W)} t^i = (\chi_{X \setminus E} t)^i \chi_W = e_{i,0}(W).$$

For the second part, it is enough to show that the product $e_{k-1,k-1}(W')\big(h_{W'} \cdot b_i t^i\big) e_{00}(W')$ is zero. This is a straightforward computation:

$$e_{k-1,k-1}(W')\Big(h_{W'} \cdot b_i t^i\Big) e_{00}(W') = \chi_{T^{k-1}(W')} \Big( \sum_{l=0}^{k-1} \chi_{T^l(W') \cap S} t^i \Big) \chi_{W'} = \chi_{T^{k-1}(W') \cap T^i(W') \cap S} t^i.$$

But $W = E \cap T^{-i}(S) \cap T^{-i-1}(E)$, so $T^{k-1}(W') \cap T^i(W') \cap S = (T^{k-1}(W') \cap T^{-1}(E)) \cap (T^i(W') \cap T^i(E)) \cap S = T^{k-1}(W') \cap T^i(W' \cap W)$ which is empty for $W' \neq W$. The result follows. □

## 2.3 Sylvester matrix rank functions on $\mathcal{A}$ and their relation with measures on $X$

In this section we will make use of our previous construction given in Definition 2.2.3 to give a more refined approximation of our algebra $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$. This will enable us to construct, from our measure $\mu$, a Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$ on $\mathcal{A}$, unique with respect to a specific condition implemented by $\mu$ (Theorem 2.3.7 and Proposition 2.3.8). As these results show, the resulting Sylvester matrix rank function is always faithful and extremal in the convex set $\mathbb{P}(\mathcal{A})$[5]. In order for this to work, it is necessary to demand an extra hypothesis to the space $X$: we need the condition that $X$ be *infinite*.

We will also analyze the converse relation, namely how a faithful, extremal Sylvester matrix rank function rk on $\mathcal{A}$ can be used to construct an ergodic, full $T$-invariant probability measure on $X$.

### 2.3.1 Approximation algebras

Throughout this section, $T$ will denote a homeomorphism of an infinite, totally disconnected, compact metrizable space $X$, and $\mu$ will denote a full ergodic $T$-invariant Borel probability measure on $X$. Note that this implies that $\mu$ is atomless, that is, $\mu(\{x\}) = 0$ for all $x \in X$. To see this, assume first that $x \in X$ is not a periodic point for $T$, and consider the Borel set $B = \{T^n(x)\}_{n \geq 0}$. Then

$$1 \geq \mu(B) = \mu\Big(\bigsqcup_{n\geq 0} \{T^n(x)\}\Big) = \sum_{n \geq 0} \mu(\{T^n(x)\}) = \sum_{n\geq 0} \mu(\{x\}),$$

which implies $\mu(\{x\}) = 0$. Now if $x$ is a periodic point for $T$ of period $n > 0$, then $B = \{x, T(x), ..., T^{n-1}(x)\}$ is a $T$-invariant closed set, and so $X \backslash B$ is a $T$-invariant nonempty open set (since $X$ is infinite). By ergodicity, and since $\mu$ is full, necessarily $\mu(X\backslash B) = 1$, and so

$$0 = \mu(B) = \mu(\{x\}) + \cdots + \mu(\{T^{n-1}(x)\}) = n\mu(\{x\}),$$

implying $\mu(\{x\}) = 0$, as claimed.

We now make our construction from Section 2.2.1 to depend on a point $y \in X$. We assume that $\{E_n\}_{n\geq 1}$ is a decreasing sequence of clopen sets of $X$ such that $\bigcap_{n \geq 1} E_n = \{y\}$.

For each $n \geq 1$, let $\mathcal{P}_n$ be a partition of $X \setminus E_n$ such that $\mathcal{P}_{n+1} \cup \{E_{n+1}\}$ is finer than $\mathcal{P}_n \cup \{E_n\}$; so $E_n$ is the disjoint union of $E_{n+1}$ and some of the sets in $\mathcal{P}_{n+1}$.

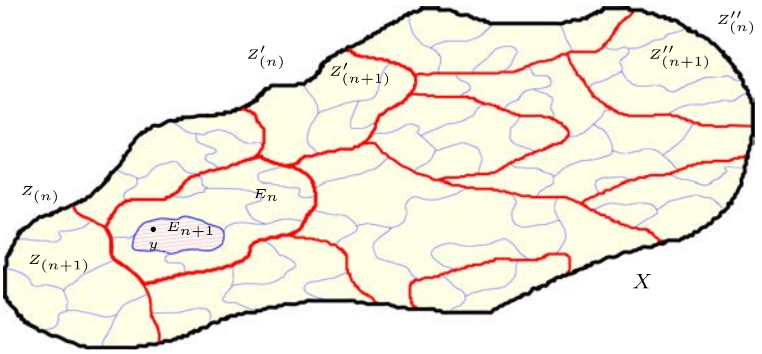



Figure 2.3: A schematic for the decreasing sequence $\{E_n\}_n$ and the partitions $\mathcal{P}_n$ of each complement $X\backslash E_n$.

**Hypothesis 2.3.1.** We require that $\bigcup_{n\geq 1}(\mathcal{P}_n \cup \{E_n\})$ generates the topology of $X$.

Recalling Lemma 2.2.9, each quasi-partition $\overline{\mathcal{P}}_n$ consists of all the $T$-translates of the nonempty subsets of $X$ of the form

$$W = E_n \cap T^{-1}(Z_1) \cap \cdots \cap T^{-k+1}(Z_{k-1}) \cap T^{-k}(E_n)$$

for some $k \geq 1$ and some $Z_1, ..., Z_{k-1} \in \mathcal{P}_n$. We write $\mathbb{V}_n$ for the set of all the $W \in \overline{\mathcal{P}}_n$ of the above form. We thus have $\overline{\mathcal{P}}_n = \{T^l(W) \mid W \in \mathbb{V}_n, 0 \leq l \leq |W| - 1\}$ for all $n$, by Lemma 2.2.9.

[5]We refer the reader to Section 1.2 for the definition of $\mathbb{P}(\mathcal{A})$.

In these conditions, it follows that the quasi-partition $\overline{\mathcal{P}}_{n+1}$ constructed from the partition $\mathcal{P}_{n+1} \cup \{E_{n+1}\}$ is finer than the quasi-partition $\overline{\mathcal{P}}_n$ constructed from the partition $\mathcal{P}_n \cup \{E_n\}$. Indeed, let $W' \in \mathbb{V}_{n+1}$ and write it as

$$W' = E_{n+1} \cap T^{-1}(Z'_1) \cap \cdots \cap T^{-k+1}(Z'_{k-1}) \cap T^{-k}(E_{n+1}) \quad \text{for } Z'_1, ..., Z'_{k-1} \in \mathcal{P}_{n+1}.$$

Since $\mathcal{P}_{n+1} \cup \{E_{n+1}\}$ is finer than $\mathcal{P}_n \cup \{E_n\}$, there exist unique integers $1 \leq k_1 < \cdots < k_r < k$ and unique elements $Z_j \in \mathcal{P}_n$ for $1 \leq j \leq k-1$ such that

$$\begin{array}{ccc} Z'_1 \subseteq Z_1, & Z'_{k_1+1} \subseteq Z_{k_1+1}, & & Z'_{k_r+1} \subseteq Z_{k_r+1}, \\ Z'_2 \subseteq Z_2, & Z'_{k_1+2} \subseteq Z_{k_1+2}, & & Z'_{k_r+2} \subseteq Z_{k_r+2}, \\ \vdots & \vdots & \cdots & \vdots \\ Z'_{k_1-1} \subseteq Z_{k_1-1}, & Z'_{k_2-1} \subseteq Z_{k_2-1}, & & Z'_{k-1} \subseteq Z_{k-1}. \\ \text{and } Z'_{k_1} \subseteq E_n; & \text{and } Z'_{k_2} \subseteq E_n; & & \end{array}$$

Therefore

$$\begin{aligned} W' \subseteq & \Big(E_n \cap T^{-1}(Z_1) \cap \cdots \cap T^{-k_1+1}(Z_{k_1-1}) \cap T^{-k_1}(E_n)\Big) \\ & \cap T^{-k_1}\Big(E_n \cap T^{-1}(Z_{k_1+1}) \cap \cdots \cap T^{-k_2+k_1+1}(Z_{k_2-1}) \cap T^{-k_2+k_1}(E_n)\Big) \\ & \cap \cdots \\ & \cap T^{-k_r}\Big(E_n \cap T^{-1}(Z_{k_r+1}) \cap \cdots \cap T^{-k+k_r+1}(Z_{k-1}) \cap T^{-k+k_r}(E_n)\Big) \\ = & W_0 \cap T^{-k_1}(W_1) \cap \cdots \cap T^{-k_r}(W_r), \end{aligned} \tag{2.3.1}$$

where each $W_i = E_n \cap T^{-1}(Z_{k_i+1}) \cap \cdots \cap T^{-k_{i+1}+k_i+1}(Z_{k_{i+1}-1}) \cap T^{-k_{i+1}+k_i}(E_n)$ belongs to $\mathbb{V}_n$, and that they are not necessarily distinct. From here, it is clear that for $0 \leq l \leq k-1$, $T^l(W')$ is contained in some translate of some $W_i$, and so $\overline{\mathcal{P}}_{n+1}$ is finer than $\overline{\mathcal{P}}_n$.

In this way, we construct a sequence of approximating algebras $\mathcal{A}_n := \mathcal{A}(E_n, \mathcal{P}_n)$ such that $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ given by the embeddings $\iota_n(\chi_Z \cdot t) = \sum_{Z'} \chi_{Z'} \cdot t$ where the sum is over all the $Z' \in \mathcal{P}_{n+1}$ contained in $Z$ (since $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$, for a given $Z \in \mathcal{P}_n$ we can always find elements $Z' \in \mathcal{P}_{n+1}$ whose union is $Z$ up to a $\mu$-null set). By Proposition 2.2.14, we have embeddings $\pi_n : \mathcal{A}_n \to \mathfrak{R}_n$ where $\mathfrak{R}_n = \prod_{W \in \mathbb{V}_n} M_{|W|}(K)$, given by $\pi_n(a) = (h_W \cdot a)_W$.

We build a generalized Bratteli diagram associated to such construction, such that each vertex receives a finite number of edges, and we can order this set of edges in the same way as for the case of an essentially minimal homeomorphism, see for instance [48] for the latter. The only difference is that there are a possibly infinite number of vertices at each level, and that these vertices might emit in principle an infinite number of edges. This can be done as follows.

The vertices at the level $n$ of this generalized Bratteli diagram are the sets $W \in \mathbb{V}_n$, that is, the sets

$$W = E_n \cap T^{-1}(Z_1) \cap \cdots \cap T^{-k+1}(Z_{k-1}) \cap T^{-k}(E_n), \tag{2.3.2}$$

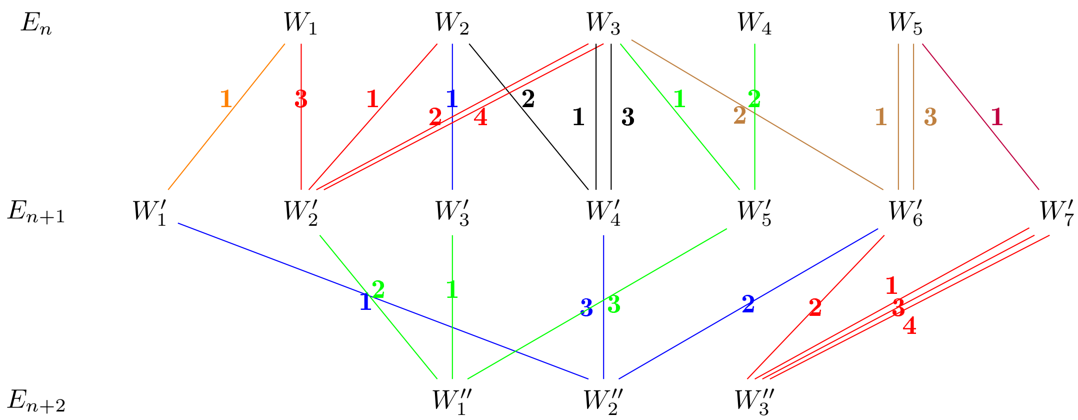
where $Z_1, ..., Z_{k-1} \in \mathcal{P}_n$. There is an arrow from a vertex $W \in \mathbb{V}_n$ to a vertex $W' \in \mathbb{V}_{n+1}$ if $W$ appears as a segment of the sequence corresponding to $W'$; more precisely, if $W$ equals to some $W_i$, being

$$W' \subseteq W_0 \cap T^{-k_1}(W_1) \cap \cdots \cap T^{-k_r}(W_r)$$

as in (2.3.1). Equivalently, if $W' \subseteq T^{-j'}(W)$ or even $W' \cap T^{-j'}(W) \neq \emptyset$ for some $0 \leq j' < |W'| - 1$. If no such $j'$ exists, then there are no arrows from $W$ to $W'$.

The edges ending at $W'$ are linearly ordered according to the integers $j'$. Thus, the set of arrows $W \to W'$ is in bijective correspondence with the set $J(W, W') = \{0 \leq j' < |W'| - 1 \mid W' \subseteq T^{-j'}(W)\}$. Clearly $J(W, W')$ is always a finite set since $W'$ has finite length and the $W$ are disjoint, and each $W'$ receives at least one arrow.

An example of a diagram of this kind is as follows.




Here, the relations between the sets $W, W'$ and $W''$ appearing are

$$\begin{aligned}
&W'_1 \subseteq W_1,\\
&W'_2 \subseteq W_2 \cap T^{-2}(W_3) \cap T^{-5}(W_1) \cap T^{-6}(W_3),\\
&W'_3 \subseteq W_2,\\
&W'_4 \subseteq W_3 \cap T^{-3}(W_2) \cap T^{-5}(W_3),\\
&W'_5 \subseteq W_3 \cap T^{-3}(W_4),\\
&W'_6 \subseteq W_5 \cap T^{-3}(W_3) \cap T^{-6}(W_5),\\
&W'_7 \subseteq W_5;
\end{aligned}
\qquad
\begin{aligned}
&W''_1 \subseteq W'_3 \cap T^{-2}(W'_2) \cap T^{-11}(W'_5),\\
&W''_2 \subseteq W'_1 \cap T^{-1}(W'_6) \cap T^{-10}(W'_4),\\
&W''_3 \subseteq W'_7 \cap T^{-3}(W'_6) \cap T^{-12}(W'_7) \cap T^{-15}(W'_7).
\end{aligned}$$

The corresponding lengths of the cited $W, W'$ and $W''$ are as follows:

$$|W_1| = 1, \quad |W_2| = 2, \quad |W_3| = 3, \quad |W_4| = 4, \quad |W_5| = 3;$$
$$|W'_1| = 1, \quad |W'_2| = 9, \quad |W'_3| = 2, \quad |W'_4| = 8, \quad |W'_5| = 7, \quad |W'_6| = 9, \quad |W'_7| = 3;$$
$$|W''_1| = |W''_2| = |W''_3| = 18.$$

**Proposition 2.3.2.** *Following the above notation, we can embed each $\mathfrak{R}_n$ into $\mathfrak{R}_{n+1}$ via the construction of the generalized Bratteli diagram just mentioned.*

*Proof.* Recall that $\mathfrak{R}_n = \prod_{W \in \mathbb{V}_n} M_{|W|}(K)$, $\mathfrak{R}_{n+1} = \prod_{W' \in \mathbb{V}_{n+1}} M_{|W'|}(K)$. Since each $W'$ receives a finite number of arrows in the diagram, it will be sufficient to define the connecting maps $j_n : \mathfrak{R}_n \to \mathfrak{R}_{n+1}$ on each simple factor $\varphi_W : M_{|W|}(K) \to \mathfrak{R}_{n+1}$, because in this case each $j_n$ will be defined as

$$j_n : \mathfrak{R}_n \to \mathfrak{R}_{n+1}, \quad (a_W)_W \mapsto \sum_W \varphi_W(a_W) \quad \text{(finite sum on each } W'\text{-component).}$$

To get an insight of how we can define each $\varphi_W$, we make an observation: since $\overline{\mathcal{P}}_{n+1}$ is a quasi-partition of $X$, we can decompose our space $X$ via $X = \bigsqcup_{W' \in \mathbb{V}_{n+1}} \bigsqcup_{l=0}^{|W'|-1} T^l(W')$[6], so by intersecting with $W$ one gets

$$W = W \cap X = \bigsqcup_{W' \in \mathbb{V}_{n+1}} \bigsqcup_{l=0}^{|W'|-1} W \cap T^l(W') = \bigsqcup_{W' \in \mathbb{V}_{n+1}} \bigsqcup_{j' \in J(W,W')} T^{j'}(W') \tag{2.3.3}$$

up to a set of measure 0, so that for a fixed $W' \in \mathbb{V}_{n+1}$, $h_{W'} \cdot \chi_W = \sum_{j' \in J(W,W')} \chi_{T^{j'}(W')}$.

We then define $\varphi_W$ to send $e_{00}(W)$ to $\left( \sum_{j' \in J(W,W')} e_{j'j'}(W') \right)_{W'}$, and more generally, $\varphi_W$ is given by the block diagonal $*$-homomorphism

$$\varphi_W(e_{ij}(W)) := \Big( \sum_{j' \in J(W,W')} e_{i+j', j+j'}(W') \Big)_{W'}.$$

[6] This, of course, is true up to a set of measure zero.

Pictorically, for a matrix $M \in M_{|W|}(K)$,

$$\varphi_W(M)_{W'} = \begin{pmatrix} M & & & & \mathbf{0} \\ & 0 & & & \\ & & M & & \\ & & & \ddots & \\ \mathbf{0} & & & & 0 \end{pmatrix} \in M_{|W'|}(K)$$

where we put $M$ at each position $j' \in J(W, W')$. Since every $W \in \mathbb{V}_n$ always emits at least one arrow to some $W' \in \mathbb{V}_{n+1}$ (this is clear since the translates of the $W'$ form a quasi-partition of $X$), the maps $\varphi_W$ are injective, and so $j_n$ is an embedding of $*$-algebras. $\square$

By construction, we obtain commutative diagrams

$$\begin{array}{ccccc} \cdots \mathcal{A}_n & \xrightarrow{\iota_n} & \mathcal{A}_{n+1} & \xrightarrow{\iota_n} & \mathcal{A}_{n+2} \cdots \\ \downarrow \pi_n & & \downarrow \pi_{n+1} & & \downarrow \pi_{n+2} \\ \cdots \mathfrak{R}_n & \xrightarrow{j_n} & \mathfrak{R}_{n+1} & \xrightarrow{j_{n+1}} & \mathfrak{R}_{n+2} \cdots \end{array} \tag{2.3.4}$$

Set now $\mathfrak{R}_\infty = \varinjlim_n (\mathfrak{R}_n, j_n)$ and $\mathcal{A}_\infty = \varinjlim_n (\mathcal{A}_n, \iota_n) = \bigcup_{n \geq 1} \mathcal{A}_n$. Note that each $\mathfrak{R}_n$, being an infinite matrix product, is a regular ring (see Chapter 1, Section 1.2 for more information about regular rings), hence so is its inductive limit $\mathfrak{R}_\infty$. Moreover, by the commutativity of the diagrams (2.3.4) and the fact that each $\pi_n$ is injective, the algebra $\mathcal{A}_\infty$ is obviously a $*$-subalgebra of $\mathfrak{R}_\infty$, through the limit map $\pi_\infty \colon \mathcal{A}_\infty \to \mathfrak{R}_\infty$.

A description of the algebra $\mathcal{A}_\infty$ in terms of the crossed product is given as follows. For an open set $U$ of $X$, we denote by $C_{c,K}(U)$ the ideal of $C_K(X)$ generated by the characteristic functions $\chi_V$, where $V$ is a clopen subset of $X$ such that $V \subseteq U$.

**Lemma 2.3.3.** *Let $\mathcal{A}_y$ be the $*$-subalgebra of $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ generated by $C_K(X)$ and $C_{c,K}(X \backslash \{y\})t$. Then we have $\mathcal{A}_\infty = \mathcal{A}_y$.*

*Proof.* $\mathcal{A}_\infty$ is generated, as a $*$-algebra, by $1 = \chi_X \in \mathcal{A}$ and the partial isometries $\chi_Z t$ for every $Z \in \bigcup_{n \geq 1} \mathcal{P}_n$. It is then clear that $\mathcal{A}_\infty \subseteq \mathcal{A}_y$, because $1 = \chi_X \in C_K(X)$ and $\chi_Z t \in C_{c,K}(X \setminus \{y\})t$ for every $Z \in \bigcup_{n \geq 1} \mathcal{P}_n$ since $y \in E_n$ for all $n \geq 1$.

For the other inclusion, we first check that $C_{c,K}(X \setminus \{y\})t \subseteq \mathcal{A}_\infty$, so let $C$ be a clopen subset of $X$ such that $y \notin C$. Since $C$ is closed and $y \notin C$, there exists an index $n_0 \geq 1$ such that $E_{n_0} \cap C = \emptyset$, and so $E_n \cap C = \emptyset$ for $n \geq n_0$ (because $E_n \subseteq E_{n_0}$ for $n \geq n_0$). Since $C$ is also open and $\bigcup_{n \geq n_0} (\mathcal{P}_n \cup \{E_n\})$ generate the topology of $X$, we can write $C$ as a countable union $C = \bigcup_{i \geq 1} Z_i$ for $Z_i \in \bigcup_{n \geq n_0} \mathcal{P}_n$. But $C$ is also compact, so this countable union is in fact finite, and we can further assume without loss of generality that the $Z_i's$ all belong to the same $\mathcal{P}_N$, and thus are pairwise disjoint. Therefore $C = \bigsqcup_{i=1}^s Z_i$. But now we get that

$$\chi_C t = \sum_{i=1}^s \chi_{Z_i} t \in \mathcal{A}_N \subseteq \mathcal{A}_\infty.$$

This shows that $C_{c,K}(X \backslash \{y\})t \subseteq \mathcal{A}_\infty$. We next show that $C_K(X) \subseteq \mathcal{A}_\infty$. Indeed, if $C$ is a clopen subset of $X$ and $y \notin C$, then the above argument gives that $\chi_C = (\chi_C t)(\chi_C t)^*$ belongs to $\mathcal{A}_\infty$. If $y \in C$ then $X \backslash C$ is a clopen set not containing $y$, so $\chi_{X \backslash C}$ belongs to $\mathcal{A}_\infty$. But then $\chi_C = 1 - \chi_{X \backslash C} \in \mathcal{A}_\infty$. This concludes the proof. $\square$

**Remark 2.3.4.** An analogue of the algebra $\mathcal{A}_y$ appears in the theory of minimal Cantor systems, see e.g. [87], [48], [37]. Let $(X, \varphi)$ be a minimal Cantor system and take $y \in X$. In these papers, the $C^*$-subalgebra $A_y$ of the $C^*$-crossed product $A = C(X) \rtimes_\varphi \mathbb{Z}$ which is generated by $C(X)$ and $C(X \setminus \{y\})u$, where $u$ is the canonical unitary in the crossed product implementing $\varphi$, is considered. It is shown that $A_y$ is an AF-algebra[7].

Although our algebra $\mathcal{A}_y$ is (in general) not ultramatricial, we have shown in Lemma 2.3.3 that $\mathcal{A}_y = \mathcal{A}_\infty$, a direct limit of algebras $\mathcal{A}_n$ which are subalgebras of infinite products of matrix algebras over $K$. This result can be considered as a replacement of being just finite products of full matrix algebras over $K$.

[7]A $C^*$-algebra is called an *AF-algebra* if it is an inductive limit of finite-dimensional $C^*$-algebras.

We may determine how big is the subalgebra $\mathcal{A}_y = \mathcal{A}_\infty$ inside the algebra $\mathcal{A}$ in some cases of interest, namely in the case that our point $y \in X$ is a periodic point for $T$. This is given in the next proposition.

**Proposition 2.3.5.** *Let us assume the above notation. Suppose that $y$ is a periodic point for $T$ with period $l$. Let $I$ be the ideal of $\mathcal{A}$ generated by $C_{c,K}(X\backslash\{y, T(y), \dots T^{l-1}(y)\})$. Then:*

*(i) $I$ is also an ideal of $\mathcal{A}_\infty$, and we have $*$-algebra isomorphisms*

$$\mathcal{A}/I \xrightarrow{\cong} M_l(K[s,s^{-1}]), \qquad \mathcal{A}_\infty/I \xrightarrow{\cong} M_l(K).$$

*(ii) There exists some $M \geq 0$ such that for each $n \geq M$ there is exactly one $W_n \in \mathbb{V}_n$ of length $l$ and containing $y$, and such that the isomorphism $h_{W_n}\mathcal{A}_n \cong M_l(K)$ given during the proof of Proposition 2.2.10 coincides with the restriction of the projection map $q : \mathcal{A}_\infty \to \mathcal{A}_\infty/I$ on $h_{W_n}\mathcal{A}_n$, that is the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{A}_\infty & \xrightarrow{q} & \mathcal{A}_\infty/I \\ \uparrow & & \downarrow \cong \\ h_{W_n}\mathcal{A}_n & \xrightarrow{\cong} & M_l(K) \end{array} \tag{2.3.5}$$

*Moreover, $h_W \in I$ for all $W \in \mathbb{V}_n$, $W \neq W_n$, which means that $h_W$ is the zero matrix under the composition $\mathcal{A}_\infty \to \mathcal{A}_\infty/I \cong M_l(K)$.*

*(iii) $\mathcal{A}_n/(I \cap \mathcal{A}_n) \cong \mathcal{A}_\infty/I \cong M_l(K)$ and $(1 - h_{W_n})\mathcal{A}_n = I \cap \mathcal{A}_n$ for every $n \geq M$.*

*Proof.* (i) It is clear that $I \subseteq \mathcal{A}_\infty$ because the set $X\backslash\{y, T(y), \dots, T^{l-1}(y)\}$ is an invariant open subset of $X$ and so all elements of $I$ are of the form $\sum_{i=-m}^{n} f_i t^i$ where $f_i \in C_{c,K}(X\backslash\{y, T(y), \dots, T^{l-1}(y)\}) \subseteq C_K(X) \subseteq \mathcal{A}_y = \mathcal{A}_\infty$; hence if $U_i$ denotes the support of $f_i$, which is a clopen subset of $X$, then

$$f_i t^i = f_i \cdot \chi_{U_i} t^i = f_i(\chi_{U_i} t)(\chi_{T^{-1}(U_i)} t) \cdots (\chi_{T^{-i+1}(U_i)} t) \in \mathcal{A}_y = \mathcal{A}_\infty.$$

Define a map $\Psi \colon \mathcal{A} \to M_l(K[t,t^{-1}])$ by sending

$$f \in C_K(X),\ f \mapsto \begin{pmatrix} f(y) & & & \mathbf{0} \\ & f(T(y)) & & \\ & & \ddots & \\ \mathbf{0} & & & f(T^{l-1}(y)) \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & & & 0 & t \\ t & 0 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ \mathbf{0} & & & t & 0 \end{pmatrix} = u.$$

It can be verified that $u\Psi(f)u^{-1} = \Psi(T(f))$ by direct computation. It follows from the universal property of the crossed product that there is a unique $*$-homomorphism $\Psi : \mathcal{A} \to M_l(K[t,t^{-1}])$ extending the above assignments. For an element $f_n t^n \in \mathcal{A}$ with $n \geq 0$ we have

$$\Psi(f_n t^n) = \left(\begin{array}{c|c} \mathbf{0}_{\overline{n}\times(l-\overline{n})} & \begin{matrix} f_n(y)t^n & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & f_n(T^{\overline{n}-1}(y))t^n \end{matrix} \\ \hline \begin{matrix} f_n(T^{\overline{n}}(y))t^n & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & f_n(T^{l-1}(y))t^n \end{matrix} & \mathbf{0}_{(l-\overline{n})\times\overline{n}} \end{array}\right)$$

where $\overline{n}$ denotes the unique integer $0 \leq \overline{n} \leq l-1$ such that $n \equiv \overline{n}$ modulo $l$. We can analogously compute it for $n < 0$. In fact, for an arbitrary element $x = \sum_{n\in\mathbb{Z}} f_n t^n \in \mathcal{A}$ one can check that, for $0 \leq i,j \leq l-1$, the $(i,j)$-component of $\Psi(x)$ is given by

$$(\Psi(x))_{i,j} = \sum_{n\in\mathbb{Z}} f_{nl+(i-j)}(T^i(y))t^{nl+(i-j)} = \Big(\sum_{n\in\mathbb{Z}} f_{nl+(i-j)}(T^i(y))t^{nl}\Big)t^{i-j}. \tag{2.3.6}$$

Observe that $I \subseteq \ker(\Psi)$ since every characteristic function $\chi_V$, where $V \subseteq X$ is a clopen subset of $X$ and $y, T(y), ..., T^{l-1}(y) \notin V$, vanishes over this set, and so $\Psi(\chi_V a) = 0$ for any $a \in \mathcal{A}$. Conversely, if $x = \sum_{n\in\mathbb{Z}} f_n t^n \in \ker(\Psi)$, then for any $0 \leq i, j \leq l-1$

$$0 = (\Psi(x))_{i,j} = \sum_{n\in\mathbb{Z}} f_{nl+(i-j)}(T^i(y))t^{nl+(i-j)},$$

which means that each function $f_n$ vanishes on the set $\{y, T(y), ..., T^{l-1}(y)\}$, so $f_n \in I$. Therefore $x \in I$. It is clear that the image of $\Psi$ is given by the subalgebra

$$\mathcal{S}_l := \{X \in M_l(K[t, t^{-1}]) \mid X_{ij} \in K[t^l, t^{-l}]t^{i-j} \text{ for all } 0 \leq i, j \leq l-1\}.$$

That is, each entry is a polynomial in $t, t^{-1}$ of the form $X_{ij} = p_{ij}(t^l, t^{-l})t^{i-j}$, where $p_{ij}(s, s^{-1}) \in K[s, s^{-1}]$. We can construct a $*$-isomorphism between $\mathcal{S}_l$ and the $*$-algebra $M_l(K[s, s^{-1}])$ by defining

$$\overline{\Psi} : \mathcal{S}_l \to M_l(K[s, s^{-1}]), \quad X = (X_{ij}) \mapsto Y = (Y_{ij}) \text{ with } Y_{ij} = p_{ij}(s, s^{-1}).$$

Putting everything together, we obtain a $*$-isomorphism $\mathcal{A}/I \cong \mathcal{S}_l \cong M_l(K[s, s^{-1}])$, as desired.

Now, we restrict the map $\Psi$ to $\mathcal{A}_\infty$. Since $I \subseteq \mathcal{A}_\infty$, the kernel of this restriction is again $I$, so we only need to study its image. Take $x = \sum_{n\in\mathbb{Z}} f_n t^n \in \mathcal{A}_\infty$, so $x \in \mathcal{A}_N$ for some $N \geq 1$. We see from the restrictions of the coefficients (see the paragraph just before Observation 2.2.5, also Lemma 2.3.3) that $f_n = \chi_{X\setminus(E_N\cup\cdots\cup T^{n-1}(E_N))} f_n$ and $f_{-n} = \chi_{X\setminus(T^{-1}(E_N)\cup\cdots\cup T^{-n}(E_N))} f_{-n}$ for $n \geq 1$, so by (2.3.6),

$$(\Psi(x))_{i,j} = f_{i-j}(T^i(y))t^{i-j}.$$

It follows from this that the image of $\mathcal{A}_\infty$ under the composition $\overline{\Psi} \circ \Psi$ is precisely $M_l(K)$.

(ii) Take $M \geq 0$ such that $T(y), ..., T^{l-1}(y) \notin E_M$, so that $T(y), ..., T^{l-1}(y) \notin E_n$ for $n \geq M$, since $E_n \subseteq E_M$ in this case. From now on, fix $n \geq M$. In this case, there are unique sets $Z_1, ..., Z_{l-1} \in \mathcal{P}_n$ such that $T(y) \in Z_1, ..., T^{l-1}(y) \in Z_{l-1}$. Take then

$$W_n := E_n \cap T^{-1}(Z_1) \cap \cdots \cap T^{-l+1}(Z_{l-1}) \cap T^{-l}(E_n),$$

which is nonempty since $y \in W_n$, and $|W_n| = l$. Note that $W_n$ is the unique satisfying these properties, because if there is another $W \in \mathbb{V}_n$ of length $l$ of the form

$$W = E_n \cap T^{-1}(Z_1') \cap \cdots \cap T^{-l+1}(Z_{l-1}') \cap T^{-l}(E_n)$$

and containing $y$, then $T(y) \in Z_1', ..., T^{l-1}(y) \in Z_{l-1}'$, so by uniqueness $Z_i' = Z_i$ for $0 \leq i \leq l-1$, and $W = W_n$.

In order to prove the commutativity of the diagram it is enough to prove that, for $0 \leq i, j \leq l-1$, the elements $e_{ij}(W_n) \in h_{W_n}\mathcal{A}_n \subseteq \mathcal{A}_\infty$ correspond to the matrix units $e_{ij}$ under the composition $\mathcal{A}_\infty \to \mathcal{A}_\infty/I \cong M_l(K)$. By $(i)$, we have the following correspondences under $\Psi$:

$$(\chi_{X\setminus E_n} t)^i \mapsto t^i(e_{i,0} + e_{i+1,1} + \cdots + e_{l-1,l-1-i}),$$

$$(t^{-1}\chi_{X\setminus E_n})^j \mapsto t^{-j}(e_{0,j} + e_{1,j+1} + \cdots + e_{l-1-j,l-1}),$$

$$\chi_{W_n} \mapsto e_{00},$$

so that

$$e_{ij}(W_n) = (\chi_{X\setminus E_n} t)^i \chi_{W_n} (t^{-1}\chi_{X\setminus E_n})^j \overset{\Psi}{\longmapsto} t^{i-j} e_{ij} \overset{\overline{\Psi}}{\longmapsto} e_{ij}$$

as we wanted to show. This proves the commutativity of the diagram (2.3.5). Therefore we obtain a $*$-isomorphism $\mathcal{A}_\infty/I \cong h_{W_n}\mathcal{A}_n$ given by $e_{ij}(W_n) + I \mapsto e_{ij}(W)$. For $W \in \mathbb{V}_n$ with $W \neq W_n$, the idempotents $h_W$ and $h_{W_n}$ are orthogonal, and so $h_W$ is the zero matrix in $h_{W_n}\mathcal{A}_n$ under the previous $*$-isomorphism. That means $h_W \in I$, as required.

(iii) Clearly $1 - h_{W_n} \in \mathcal{A}_n$. Under $\mathcal{A}_\infty/I \cong h_{W_n}\mathcal{A}_n \cong M_l(K)$, the element $(1-h_{W_n}) + I$ corresponds to the zero matrix, so $1 - h_{W_n} \in I$ too.

Define $I_n = (1-h_{W_n})\mathcal{A}_n$. From the previous observation, $I_n \subseteq I \cap \mathcal{A}_n$, and we aim to show the reverse inclusion. Since $h_{W_n}$ is a central idempotent in $\mathcal{A}_n$, we have a decomposition

$$\mathcal{A}_n = (1-h_{W_n})\mathcal{A}_n \oplus h_{W_n}\mathcal{A}_n = I_n \oplus h_{W_n}\mathcal{A}_n$$

so $h_{W_n}\mathcal{A}_n \cong \mathcal{A}_n/I_n$ through $e_{ij}(W_n) \mapsto e_{ij}(W_n) + I_n$. Since $I_n \subseteq I \cap \mathcal{A}_n$, by the modular law be have

$$I \cap \mathcal{A}_n = I \cap (I_n \oplus h_{W_n}\mathcal{A}_n) = I_n \oplus (h_{W_n}\mathcal{A}_n \cap I).$$

Now, we also have a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{A}_n & \hookrightarrow & \mathcal{A}_\infty \\ & & \downarrow & & \downarrow \\ h_{W_n}\mathcal{A}_n & \xrightarrow{\cong} & \mathcal{A}_n/I_n & \longrightarrow & \mathcal{A}_\infty/I, \end{array} \qquad e_{ij}(W_n) \mapsto e_{ij}(W_n) + I_n \mapsto e_{ij}(W_n) + I. \tag{2.3.7}$$

But from $(i)$, the composition $h_{W_n}\mathcal{A}_n \cong \mathcal{A}_n/I_n \to \mathcal{A}_\infty/I$, $e_{ij}(W_n) \mapsto e_{ij}(W_n) + I$ is already a $*$-isomorphism. This has two consequences: first, $\mathcal{A}_n/I_n \cong \mathcal{A}_\infty/I$ canonically, and second, $h_{W_n}\mathcal{A}_n \cap I = \{0\}$, so $I_n = I \cap \mathcal{A}_n$. □

### 2.3.2 A rank function on $\mathcal{A}$

We now study the possible rank functions that the $*$-algebras $\mathcal{A}_\infty$, $\mathcal{A}$ can admit. To start this study, we first concentrate our attention on the approximating algebras $\mathcal{A}_n$ and the embeddings of them into the infinite products of matrices $\pi_n : \mathcal{A}_n \hookrightarrow \mathfrak{R}_n$ already constructed.

We define a rank function $\mathrm{rk}_{\mathfrak{R}_n}$ on $\mathfrak{R}_n = \prod_{W \in \mathbb{V}_n} M_{|W|}(K)$ by taking a concrete convex combination of the normalized rank functions $\mathrm{rk}_{|W|} = \frac{\mathrm{Rk}}{|W|}$ on the matrix algebras $M_{|W|}(K)$[8]. Namely, we take $\alpha_W = |W|\mu(W)$, where $W \in \mathbb{V}_n$, and then we define

$$\mathrm{rk}_{\mathfrak{R}_n}(x) = \sum_{W \in \mathbb{V}_n} \alpha_W \, \mathrm{rk}_{|W|}(x_W) \quad \text{ for } x = (x_W)_W \in \mathfrak{R}_n.$$

Properties $b)$, $c)$ and $d)$ of Definition 1.2.2 are straightforward to check, $\mathrm{rk}_{\mathfrak{R}_n}(0) = 0$ clearly, and for the property $\mathrm{rk}_{\mathfrak{R}_n}(1) = 1$ it is enough to observe that, due to Lemma 2.2.9, we have

$$\sum_{W \in \mathbb{V}_n} \alpha_W = \sum_{k \geq 1} \sum_{\substack{W \in \mathbb{V}_n \\ |W| = k}} k\mu(W) = \mu(X) = 1.$$

So $\mathrm{rk}_{\mathfrak{R}_n}$ is indeed a rank function on $\mathfrak{R}_n$, and a faithful one since $\alpha_W \neq 0$ for all $W \in \mathbb{V}_n$. Moreover, the embeddings $j_n \colon \mathfrak{R}_n \to \mathfrak{R}_{n+1}$ are rank-preserving. To show this, we only have to prove that $\mu(W) = \sum_{W' \in \mathbb{V}_{n+1}} |J(W,W')|\mu(W')$, since then for $x \in \mathfrak{R}_n$,

$$\begin{aligned} \mathrm{rk}_{\mathfrak{R}_{n+1}}(j_n(x)) &= \sum_{W' \in \mathbb{V}_{n+1}} \alpha_{W'} \, \mathrm{rk}_{|W'|}(j_n(x)_{W'}) = \sum_{W' \in \mathbb{V}_{n+1}} \alpha_{W'} \Big( \sum_{W \in \mathbb{V}_n} \mathrm{rk}_{|W'|}(\varphi_W(x_W)_{W'}) \Big) \\ &= \sum_{W' \in \mathbb{V}_{n+1}} \sum_{W \in \mathbb{V}_n} \mu(W')|W'| \, \mathrm{rk}_{|W'|}(\varphi_W(x_W)_{W'}) = \sum_{W \in \mathbb{V}_n} \sum_{W' \in \mathbb{V}_{n+1}} \mu(W') \, \mathrm{Rk}(\varphi_W(x_W)_{W'}) \\ &= \sum_{W \in \mathbb{V}_n} \Big( \sum_{W' \in \mathbb{V}_{n+1}} \mu(W')|J(W,W')| \Big) \mathrm{Rk}(x_W) = \sum_{W \in \mathbb{V}_n} \mu(W)|W| \, \mathrm{rk}_{|W|}(x_W) = \mathrm{rk}_{\mathfrak{R}_n}(x). \end{aligned}$$

But now suppose that $W$ is as in (2.3.2). By virtue of (2.3.3), we can write

$$W = \bigsqcup_{W' \in \mathbb{V}_{n+1}} \bigsqcup_{j' \in J(W,W')} T^{j'}(W') \quad \text{ up to a set of measure 0.}$$

[8] Here Rk denotes the usual rank function of matrices $M \in M_{|W|}(K)$.

From this the above equality follows by invariance of $\mu$:

$$\mu(W) = \sum_{W' \in \mathbb{V}_{n+1}} \sum_{j' \in J(W,W')} \mu(T^{j'}(W')) = \sum_{W' \in \mathbb{V}_{n+1}} |J(W,W')|\mu(W').$$

With this, we can define a faithful rank function over the inductive limit $\mathfrak{R}_\infty = \varinjlim(\mathfrak{R}_n, j_n)$ by setting

$$\mathrm{rk}_{\mathfrak{R}_\infty}(x) = \lim_{n\to\infty} \mathrm{rk}_{\mathfrak{R}_n}(x_n) \quad \text{ if } x = \varinjlim_n x_n,\ x_n \in \mathfrak{R}_n.^9$$

In the next lemma we show that we also have compatibility of our measure $\mu$ and this new rank function defined over $\mathfrak{R}_\infty$.

**Lemma 2.3.6.** *Let $\pi_n \colon \mathcal{A}_n \to \mathfrak{R}_n$ and $\pi_\infty : \mathcal{A}_\infty \to \mathfrak{R}_\infty$ be the canonical inclusions. Then:*

*i) The equality $\mu(Z) = \mathrm{rk}_{\mathfrak{R}_n}(\pi_n(\chi_Z))$ holds for all $Z \in \mathcal{P}_n \cup \{E_n\}$. Moreover,*

*ii) $\mu(U) = \mathrm{rk}_{\mathfrak{R}_\infty}(\pi_\infty(\chi_U))$ for all clopen subset $U$ of $X$.*

*Proof.* Let's prove the first formula. For $Z = E_n$, by the computation done after Lemma 2.2.13, $\pi_n(\chi_{E_n}) = (e_{00}(W))_W$, so

$$\mathrm{rk}_{\mathfrak{R}_n}(\pi_n(\chi_{E_n})) = \sum_{W \in \mathbb{V}_n} \alpha_W \,\mathrm{rk}_{|W|}(e_{00}(W)) = \sum_{W \in \mathbb{V}_n} \mu(W) = \mu\Big(\bigsqcup_{W \in \mathbb{V}_n} W\Big) = \mu(E_n),$$

where we have used that the sets $\{W\}_{W \in \mathbb{V}_n}$ forms a quasi-partition of $E_n$, see Lemma 2.2.9. For $Z \in \mathcal{P}_n$, also by Lemma 2.2.9 the set $\{Z \cap \overline{W}\}_{\overline{W} \in \overline{\mathcal{P}}_n} = \{Z \cap T^l(W)\}_{\substack{W \in \mathbb{V}_n \\ 0 \leq l \leq |W|-1}}$ is a quasi-partition of $Z$. Therefore if $W$ is as in (2.3.2), then $h_W \cdot \chi_Z = \sum_{j : Z_j = Z} e_{jj}(W)$, so

$$\begin{aligned}
\mathrm{rk}_{\mathfrak{R}_n}(\pi_n(\chi_Z)) &= \sum_{W \in \mathbb{V}_n} \alpha_W \,\mathrm{rk}_{|W|}\Big(\sum_{j : Z_j = Z} e_{jj}(W)\Big) = \sum_{W \in \mathbb{V}_n} \mu(W)|\{j \mid Z_j = Z\}| \\
&= \sum_{W \in \mathbb{V}_n} \sum_{j : Z_j = Z} \mu(T^j(W) \cap Z) = \sum_{W \in \mathbb{V}_n} \sum_{l=0}^{|W|-1} \mu(T^l(W) \cap Z) \\
&= \mu\Big(\bigsqcup_{W \in \mathbb{V}_n} \bigsqcup_{l=0}^{|W|-1} T^l(W) \cap Z\Big) = \mu(Z).
\end{aligned}$$

As a consequence, $\mu(Z) = \mathrm{rk}_{\mathfrak{R}_\infty}(\pi_\infty(\chi_Z))$ for all $Z \in \bigcup_{n\geq 1}(\mathcal{P}_n \cup \{E_n\})$. Since $\bigcup_{n\geq 1}(\mathcal{P}_n \cup \{E_n\})$ generates the topology of $X$, every clopen subset $U$ of $X$ can be written as a finite (disjoint) union of elements of the partitions $\mathcal{P}_n \cup \{E_n\}$, so we get that $\mu(U) = \mathrm{rk}_{\mathfrak{R}_\infty}(\pi_\infty(\chi_U))$. □

Using this rank function we will define rank functions over $\mathcal{A}_\infty, \mathcal{A}$. To this aim, we would like to embed our whole algebra $\mathcal{A}$ inside $\mathfrak{R}_\infty$, but this is (in general) not possible. What we will do is to embed $\mathcal{A}$ inside the *rank completion* $\mathfrak{R}_{\mathrm{rk}}$ of $\mathfrak{R}_\infty$ with respect to its rank function $\mathrm{rk}_{\mathfrak{R}_\infty}$.

From now on we will not write down explicitly the maps $\pi_n, \pi_\infty$ and $j_n$, so we will identify

$$\begin{array}{ccccccccccc}
\mathcal{A}_n & \hookrightarrow & \mathcal{A}_{n+1} & \hookrightarrow & \mathcal{A}_{n+2} & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{A}_\infty & \hookrightarrow & \mathcal{A} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \\
\mathfrak{R}_n & \hookrightarrow & \mathfrak{R}_{n+1} & \hookrightarrow & \mathfrak{R}_{n+2} & \hookrightarrow & \cdots & \hookrightarrow & \mathfrak{R}_\infty & \hookrightarrow & \mathfrak{R}_{\mathrm{rk}}
\end{array} \tag{2.3.8}$$

whenever convenient.

[9] In fact, for $x \in \mathfrak{R}_n$ already, $\mathrm{rk}_{\mathfrak{R}_\infty}(j_{n,\infty}(x)) = \mathrm{rk}_{\mathfrak{R}_n}(x)$ where $j_{n,\infty}$ denotes the canonical map $\mathfrak{R}_n \to \mathfrak{R}_\infty$.

**Theorem 2.3.7.** *Let $\mathfrak{R}_{\mathrm{rk}}$ be the rank completion of the regular rank ring $\mathfrak{R}_\infty$ with respect to the rank function $\mathrm{rk}_{\mathfrak{R}_\infty}$. We denote by $\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}} := \overline{\mathrm{rk}_{\mathfrak{R}_\infty}}$ the rank function on $\mathfrak{R}_{\mathrm{rk}}$ extended from $\mathrm{rk}_{\mathfrak{R}_\infty}$*[10]*. We then have an embedding*

$$\mathcal{A} \hookrightarrow \mathfrak{R}_{\mathrm{rk}}$$

*that induces a faithful Sylvester matrix rank function, denoted by $\mathrm{rk}_{\mathcal{A}}$, on $\mathcal{A}$. In turn, the natural inclusion $\mathcal{A}_\infty \subseteq \mathcal{A}$ induces a faithful Sylvester matrix rank function, denoted by $\mathrm{rk}_{\mathcal{A}_\infty}$, on $\mathcal{A}_\infty$.*

*Moreover, we have $\overline{\mathcal{A}_\infty}^{\mathrm{rk}_{\mathcal{A}_\infty}} = \overline{\mathcal{A}}^{\mathrm{rk}_{\mathcal{A}}} = \mathfrak{R}_{\mathrm{rk}}$.*

*Proof.* The function $\mathrm{rk}_{\mathfrak{R}_\infty}$ is a faithful rank function on $\mathfrak{R}_\infty$, and in fact a faithful Sylvester matrix rank function since $\mathfrak{R}_\infty$ is regular (and so it extends uniquely to matrices over $\mathfrak{R}_\infty$ of arbitrary size), so there is an embedding of $\mathfrak{R}_\infty$ into its completion $\mathfrak{R}_{\mathrm{rk}}$, which is a regular self-injective $\mathrm{rk}_{\mathfrak{R}_\infty}$-complete ring (Proposition 1.2.4). This shows that $\mathcal{A}_\infty \hookrightarrow \mathfrak{R}_\infty \subseteq \mathfrak{R}_{\mathrm{rk}}$, and we will simply identify $\mathcal{A}_\infty \subseteq \mathfrak{R}_\infty$ (so we will omit the map $\pi_\infty$ for notational convenience). Now we show that there is a natural embedding of $\mathcal{A}$ into $\mathfrak{R}_{\mathrm{rk}}$.

Observe that $\{\chi_{X \setminus E_n} t\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{R}_{\mathrm{rk}}$, because for $n \geq m$ and using Lemma 2.3.6,

$$\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(\chi_{X\setminus E_n} t - \chi_{X\setminus E_m} t) \leq \mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(\chi_{E_m \setminus E_n}) = \mu(E_m \setminus E_n) \leq \mu(E_m) \xrightarrow[m\to\infty]{} \mu(\{y\}) = 0.$$

Therefore, since $\mathfrak{R}_{\mathrm{rk}}$ is $\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}$-complete, we may consider the element $u := \lim_n \chi_{X\setminus E_n} t \in \mathfrak{R}_{\mathrm{rk}}$. It is an invertible element inside $\mathfrak{R}_{\mathrm{rk}}$ with inverse $\lim_n t^{-1}\chi_{X\setminus E_n}$, since

$$\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(1 - (\chi_{X\setminus E_n} t)(t^{-1}\chi_{X\setminus E_n})) = \mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(\chi_{E_n}) = \mu(E_n) \xrightarrow[n\to\infty]{} 0,$$

$$\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(1 - (t^{-1}\chi_{X\setminus E_n})(\chi_{X\setminus E_n} t)) = \mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(\chi_{T^{-1}(E_n)}) = \mu(T^{-1}(E_n)) = \mu(E_n) \xrightarrow[n\to\infty]{} 0.$$

Moreover, the condition $u\chi_C u^{-1} = \chi_{T(C)} = T(\chi_C)$ is also satisfied for every clopen subset $C$ of $X$, and so we get that $ufu^{-1} = T(f)$ for every $f \in C_K(X)$. Indeed, in rank we have

$$\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(\chi_{T(C)} - (\chi_{X\setminus E_n} t)\chi_C(t^{-1}\chi_{X\setminus E_n})) = \mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(\chi_{T(C)} - \chi_{T(C)}\chi_{X\setminus E_n}) \leq \mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(\chi_{E_n}) = \mu(E_n) \xrightarrow[n\to\infty]{} 0$$

and so $u\chi_C u^{-1} = \chi_{T(C)}$, as required. It follows from the universal property of the crossed product that there is a unique homomorphism

$$\Phi\colon \mathcal{A} = C_K(X) \rtimes_T \mathbb{Z} \to \mathfrak{R}_{\mathrm{rk}}, \quad \sum_{i\in\mathbb{Z}} f_i t^i \mapsto \sum_{i\in\mathbb{Z}} f_i u^i \quad \text{for } f_i \in C_K(X).$$

This map clearly extends the injective homomorphism $\mathcal{A}_\infty \subseteq \mathfrak{R}_\infty \subseteq \mathfrak{R}_{\mathrm{rk}}$. To show that it is injective, it suffices to check that $a = \sum_{i=0}^n f_i u^i$ is never 0 in $\mathfrak{R}_{\mathrm{rk}}$ whenever $f_0 \neq 0$ [11] and all $f_i \in C_K(X)$. But if $f_0 \neq 0$, and $C$ denotes the support of $f_0$, taking $s$ big enough so that $n\mu(E_s) < \mu(C)$, we have

$$\begin{aligned}
\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(\chi_{X\setminus(E_s\cup\cdots\cup T^{n-1}(E_s))}\cdot\chi_C) &= \mu(X \setminus (E_s \cup \cdots \cup T^{n-1}(E_s) \cup C^c)) \\
&= 1 - \mu(E_s \cup \cdots \cup T^{n-1}(E_s) \cup C^c) \\
&\geq 1 - (\mu(E_s) + \cdots + \mu(T^{n-1}(E_s)) + \mu(C^c)) = \mu(C) - n\mu(E_s) > 0
\end{aligned}$$

hence $\chi_{X\setminus(E_s\cup\cdots\cup T^{n-1}(E_s))} f_0 = \chi_{X\setminus(E_s\cup\cdots\cup T^{n-1}(E_s))}\cdot\chi_C f_0 \neq 0$, and moreover

$$\chi_{X\setminus(E_s\cup\cdots\cup T^{n-1}(E_s))}\Big(\sum_{i=0}^n f_i u^i\Big) = \sum_{i=0}^n (\chi_{X\setminus(E_s\cup\cdots\cup T^{n-1}(E_s))} f_i)(\chi_{X\setminus E_s} t)^i \in \mathcal{A}_s \subseteq \mathcal{A}_\infty,$$

and this is nonzero because the map $\mathcal{A}_\infty \subseteq \mathfrak{R}_\infty \subseteq \mathfrak{R}_{\mathrm{rk}}$ is injective.

We thus get the inclusions $\mathcal{A}_\infty \subseteq \mathcal{A} \subseteq \mathfrak{R}_{\mathrm{rk}}$, where we identify $u$ with $t$. Clearly $\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}$ induce faithful Sylvester matrix rank functions, given by restriction, on either $\mathcal{A}_\infty$ and $\mathcal{A}$.

Note that for each $n \geq 1$, $\mathcal{A}_n$ is dense in $\mathfrak{R}_n$ with respect to the $\mathrm{rk}_{\mathfrak{R}_n}$-metric, because by Proposition 2.2.14, $\mathrm{soc}(\mathcal{A}_n) = \bigoplus_{W\in\mathbb{V}_n} M_{|W|}(K)$, which is dense in $\mathfrak{R}_n = \prod_{W\in\mathbb{V}_n} M_{|W|}(K)$. To see this, note that for an element

[10] See Section 1.2, specifically the paragraph preceding Proposition 1.2.4.

[11] We can reduce to this case by multiplying to the right the element $a$ by a suitable power of $u$.

$x \in \mathfrak{R}_n$, we can consider the sequence of elements $\{x_k\}_{k\geq 1}$ defined by $x_k = \Big(\sum_{\substack{W\in\mathbb{V}_n \\ |W|\leq k}} h_W\Big)x \in \mathrm{soc}(\mathcal{A}_n)$. A simple computation, using Lemmas 2.3.6 and 2.2.1, gives

$$\begin{aligned}\mathrm{rk}_{\mathfrak{R}_n}(x - x_k) \leq \mathrm{rk}_{\mathfrak{R}_n}\Big(1 - \sum_{\substack{W\in\mathbb{V}_n \\ |W|\leq k}} h_W\Big) &= \mathrm{rk}_{\mathfrak{R}_n}\Big(\chi_X - \sum_{\substack{W\in\mathbb{V}_n \\ |W|\leq k}} \sum_{l=0}^{|W|-1} \chi_{T^l(W)}\Big) = \\ &= \mu\Big(\bigsqcup_{\substack{W\in\mathbb{V}_n \\ |W|> k}} \bigsqcup_{l=0}^{|W|-1} T^l(W)\Big) = \sum_{\substack{W\in\mathbb{V}_n \\ |W|> k}} |W|\mu(W) \xrightarrow[k\to\infty]{} 0,\end{aligned}$$

so $x_k \xrightarrow{k} x$ in rank. It follows that $\mathcal{A}_\infty$ is dense in $\mathfrak{R}_\infty$, and hence in $\mathfrak{R}_{\mathrm{rk}}$ with respect to the $\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}$-metric, so we also get that $\overline{\mathcal{A}_\infty}^{\mathrm{rk}_{\mathcal{A}_\infty}} = \overline{\mathcal{A}}^{\mathrm{rk}_{\mathcal{A}}} = \mathfrak{R}_{\mathrm{rk}}$. □

It follows that the rank function $\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}$ on $\mathfrak{R}_{\mathrm{rk}}$ restricts to a faithful Sylvester matrix rank function on $\mathcal{A}$ such that $\mathrm{rk}_{\mathcal{A}}(\chi_U) = \mu(U)$ for each clopen subset $U$ of $X$ (2.3.6). We now investigate the uniqueness of this rank function, first over $\mathfrak{R}_\infty$ and then over $\mathcal{A}$ itself, in the next proposition.

**Proposition 2.3.8.** *Following the above notation,*

*(i) the rank function $\mathrm{rk}_{\mathfrak{R}_\infty}$ is a faithful Sylvester matrix rank function on $\mathfrak{R}_\infty$, and it is uniquely determined by the following property: for every clopen subset $U$ of $X$, $\mathrm{rk}_{\mathfrak{R}_\infty}(\pi_\infty(\chi_U)) = \mu(U)$.*

*(ii) the rank function $\mathrm{rk}_{\mathcal{A}}$ from Theorem 2.3.7 is a faithful Sylvester matrix rank function on $\mathcal{A}$, and it is uniquely determined by the same property as in (i), that is, for every clopen subset $U$ of $X$, $\mathrm{rk}_{\mathcal{A}}(\chi_U) = \mu(U)$.*

*Moreover,* $\mathrm{rk}_{\mathfrak{R}_\infty} \in \partial_e\mathbb{P}(\mathfrak{R}_\infty)$ *and* $\mathrm{rk}_{\mathcal{A}} \in \partial_e\mathbb{P}(\mathcal{A})$[12].

*Proof.* We first prove (*i*). As we have already mentioned in the proof of Theorem 2.3.7, $\mathrm{rk}_{\mathfrak{R}_\infty}$ is a faithful Sylvester matrix rank function on $\mathfrak{R}_\infty$ because of regularity of the ring. Hence to prove uniqueness of the Sylvester matrix rank function it suffices to check that if $N$ is another Sylvester matrix rank function on $\mathfrak{R}_\infty$ satisfying the required compatibility of the measure, then the restriction of $N$ to $\mathfrak{R}_\infty$ is $\mathrm{rk}_{\mathfrak{R}_\infty}$.

Since $\mathfrak{R}_\infty = \varinjlim_n \mathfrak{R}_n$, given $n \geq 1$ we consider the restriction $N_n$ of $N$ to $\mathfrak{R}_n$, which is a pseudo-rank function on $\mathfrak{R}_n$, so that $N = \lim_n N_n$. Since $\mathfrak{R}_n = \prod_{W\in\mathbb{V}_n} M_{|W|}(K)$, the restriction of $N_n$ on each simple factor $M_{|W|}(K)$, denoted by $N_n|_W$ is an unnormalized pseudo-rank function on $M_{|W|}(K)$ for each $W \in \mathbb{V}_n$. Since there exists a unique normalized (pseudo-)rank function on $M_{|W|}(K)$ (which we denoted by $\mathrm{rk}_{|W|}$), there must exists a positive constant $\beta_W$ such that $N_n|_W = \beta_W \,\mathrm{rk}_{|W|}$. More generally, consider any finite subset $S \subseteq \{W \in \mathbb{V}_n\}$. Then $\Big(\sum_{W\in S} h_W\Big)\mathfrak{R}_n = \bigoplus_{W\in S} M_{|W|}(K)$. Take the restriction of $N_n$ to $\bigoplus_{W\in S} M_{|W|}(K)$, $N_n|_S$, which turns out to be again an unnormalized pseudo-rank function on $\bigoplus_{W\in S} M_{|W|}(K)$. Hence it can be written as a combination of the unique normalized rank functions on each simple factor $M_{|W|}(K)$, i.e.

$$N_n|_S = \sum_{W\in S} \beta_W \,\mathrm{rk}_{|W|} \quad \text{for some } \beta_W \geq 0 \text{ satisfying } \sum_{W\in S} \beta_S = \sum_{W\in S} N_n(h_W).$$

For a fixed $W' \in S$, we would like to compute the factor $\beta_{W'}$. To this aim, consider the element $x = (x_W)_W \in \bigoplus_{W\in S} M_{|W|}(K)$ given by $x_W = \delta_{W,W'} h_{W'}$. Then

$$\beta_{W'} = N_n|_S(x) = N_n\Big(\Big(\sum_{W\in S} h_W\Big)x\Big) = N_n(h_{W'}) = \alpha_{W'}.$$

We have used the required compatibility property of the measures to compute each $N_n(h_{W'})$, as follows:

$$N_n(h_{W'}) = N_n\Big(\sum_{j=0}^{|W'|-1} e_{jj}(W')\Big) = |W'|N_n(e_{00}(W')) = |W'|N_n(\chi_{W'}) = |W'|\mu(W') = \alpha_{W'}.$$

[12]For a compact convex set $\Delta$, the notation $\partial_e\Delta$ refers to the set of extreme points of $\Delta$.

Finally, for an arbitrary element $x \in \mathfrak{R}_n$, we compute

$$N_n\Big(\Big(\sum_{W\in S} h_W\Big)x\Big) = N_n|_S\Big(\Big(\sum_{W\in S} h_W\Big)x\Big) = \sum_{W\in S} \alpha_W \operatorname{rk}_{|W|}(x_W) = \operatorname{rk}_{\mathfrak{R}_n}\Big(\Big(\sum_{W\in S} h_W\Big)x\Big).$$

This says that $N_n$ and $\operatorname{rk}_{\mathfrak{R}_n}$ coincide on $\bigoplus_{W\in S} M_{|W|}(K)$.

Now fix $k \geq 1$, and consider the finite set $S_k = \{W \in \mathbb{V}_n \mid |W| \leq k\}$. For $x \in \mathfrak{R}_n$, we have the estimate

$$\begin{aligned}
|N_n(x) - \operatorname{rk}_{\mathfrak{R}_n}(x)| &\leq \Big|N_n(x) - N_n\Big(\Big(\sum_{W\in S_k} h_W\Big)x\Big)\Big| + \Big|\operatorname{rk}_{\mathfrak{R}_n}\Big(\Big(\sum_{W\in S_k} h_W\Big)x\Big) - \operatorname{rk}_{\mathfrak{R}_n}(x)\Big| \\
&\leq N_n\Big(1 - \sum_{W\in S_k} h_W\Big) + \operatorname{rk}_{\mathfrak{R}_n}\Big(1 - \sum_{W\in S_k} h_W\Big) \\
&= N_n\Big(\chi_X - \sum_{W\in S_k}\sum_{l=0}^{|W|-1} \chi_{T^l(W)}\Big) + \operatorname{rk}_{\mathfrak{R}_n}\Big(\chi_X - \sum_{W\in S_k}\sum_{l=0}^{|W|-1} \chi_{T^l(W)}\Big) \\
&= 2\mu\Big(\bigsqcup_{\substack{W\in\mathbb{V}_n\\ |W|>k}} \bigsqcup_{l=0}^{|W|-1} T^l(W)\Big) = 2\sum_{\substack{W\in\mathbb{V}_n\\ |W|>k}} |W|\mu(W) \xrightarrow[k\to\infty]{} 0,
\end{aligned}$$

where we have used part $(i)$ of Proposition 1.2.10 for the second inequality, and the limit tends to zero by Lemma 2.2.1. Therefore $N_n = \operatorname{rk}_{\mathfrak{R}_n}$ for all $n \geq 1$, and so $N = \lim_n N_n = \lim_n \operatorname{rk}_{\mathfrak{R}_n} = \operatorname{rk}_{\mathfrak{R}_\infty}$.

$(ii)$ Let $N$ be a Sylvester matrix rank function on $\mathcal{A}$ such that $N(\chi_U) = \mu(U)$ for every clopen subset $U$ of $X$. We first check that the restriction $N_n$ of $N$ on $\mathcal{A}_n$ equals the restriction $\operatorname{rk}_{\mathcal{A}_n}$ of $\operatorname{rk}_{\mathcal{A}}$ on $\mathcal{A}_n$. Since for any finite subset $S \subseteq \{W \in \mathbb{V}_n\}$ we have the identification $\Big(\sum_{W\in S} h_W\Big)\mathcal{A}_n = \bigoplus_{W\in S} h_W\mathcal{A}_n \cong \bigoplus_{W\in S} M_{|W|}(K)$, it follows from the same arguments as in $(i)$ that $N_n(a) = \operatorname{rk}_{\mathcal{A}_n}(a)$ for every $a \in \mathcal{A}_n$, and so the restriction $N_\infty$ of $N$ on $\mathcal{A}_\infty$ coincides with $\operatorname{rk}_{\mathcal{A}_\infty}$.

Now, to show that $N = \operatorname{rk}_{\mathcal{A}}$ on $\mathcal{A}$, it suffices to check that for each algebra generator $a$ of $\mathcal{A}$ and for each $\varepsilon > 0$ there is $b \in \mathcal{A}_\infty$ such that $N(a-b) < \frac{\varepsilon}{2}$ and $\operatorname{rk}_{\mathcal{A}}(a-b) < \frac{\varepsilon}{2}$, since in this case we would have the estimate

$$|N(a) - \operatorname{rk}_{\mathcal{A}}(a)| \leq |N(a) - N(b)| + |\operatorname{rk}_{\mathcal{A}}(a) - \operatorname{rk}_{\mathcal{A}}(b)| \leq N(a-b) + \operatorname{rk}_{\mathcal{A}}(a-b) < \varepsilon$$

for all $\varepsilon > 0$, by using again part $(i)$ of Proposition 1.2.10 for the second inequality, and consequently $N = \operatorname{rk}_{\mathcal{A}}$ on $\mathcal{A}$. This is clear for $a \in C_K(X)$ since $C_K(X) \subseteq \mathcal{A}_\infty$, and it is also clear for $t$, because $\chi_{X\setminus E_n} t \in \mathcal{A}_n$ and

$$N(t - \chi_{X\setminus E_n} t) \leq N(\chi_{E_n}) = \mu(E_n) \xrightarrow[n\to\infty]{} 0, \quad \operatorname{rk}_{\mathcal{A}}(t - \chi_{X\setminus E_n} t) \leq \operatorname{rk}_{\mathcal{A}}(\chi_{E_n}) = \mu(E_n) \xrightarrow[n\to\infty]{} 0.$$

To show that $N$ and $\operatorname{rk}_{\mathcal{A}}$ coincide on matrices over $\mathcal{A}$, consider a concrete matrix algebra $M_l(\mathcal{A})$. The same argument as above works exactly the same in this case, we sketch it for convenience. For a finite subset $S \subseteq \{W \in \mathbb{V}_n\}$, consider the element

$$H_S = \begin{pmatrix} \sum_{W\in S} h_W & & \\ & \overset{l}{\ddots} & \\ & & \sum_{W\in S} h_W \end{pmatrix} \in M_l(\mathcal{A})$$

We have the identification $H_S \cdot M_l(\mathcal{A}_n) = M_l\Big(\big(\sum_{W\in S} h_W\big)\mathcal{A}_n\Big) \cong \bigoplus_{W\in S} M_l(h_W\mathcal{A}_n) \cong \bigoplus_{W\in S} M_l(K) \otimes M_{|W|}(K)$, so the restriction of $N$ on $M_l(\mathcal{A}_n)$ coincides with the restriction of $\operatorname{rk}_{\mathcal{A}}$ on $M_l(\mathcal{A}_n)$, and hence they coincide on $M_l(\mathcal{A}_\infty) = \varinjlim_n M_l(\mathcal{A}_n)$. As before, to show that $N = \operatorname{rk}_{\mathcal{A}}$ on $M_l(\mathcal{A})$, we only need to check that for each algebra generator $A$ of $M_l(\mathcal{A})$ and for each $\varepsilon > 0$ there is $B \in M_l(\mathcal{A}_\infty)$ such that $N(A-B) < \frac{\varepsilon}{2}$ and $\operatorname{rk}_{\mathcal{A}}(A-B) < \frac{\varepsilon}{2}$. This is clear for $a \in M_l(C_K(X))$ since $C_K(X) \subseteq \mathcal{A}_\infty$, and it is also clear for $t \cdot \mathrm{Id}_l$, because $\chi_{X\setminus E_n} t \cdot \mathrm{Id}_l \in M_l(\mathcal{A}_n)$ and

$$N(t\cdot \mathrm{Id}_l - \chi_{X\setminus E_n} t \cdot \mathrm{Id}_l) \leq l \cdot N(\chi_{E_n}) = l \cdot \mu(E_n) \xrightarrow[n\to\infty]{} 0,$$

$$\operatorname{rk}_{\mathcal{A}}(t\cdot \mathrm{Id}_l - \chi_{X\setminus E_n} t \cdot \mathrm{Id}_l) \leq l \cdot \operatorname{rk}_{\mathcal{A}}(\chi_{E_n}) = l \cdot \mu(E_n) \xrightarrow[n\to\infty]{} 0.$$

Let's show that $\mathrm{rk}_{\mathcal{A}}$ is extremal. Suppose we have a convex combination $\mathrm{rk}_{\mathcal{A}} = \alpha N_1 + \beta N_2$, where $N_1$ and $N_2$ are Sylvester matrix rank functions on $\mathcal{A}$. Assume that $\alpha \neq 0, 1$. We first show that each Sylvester matrix rank function $N_i$ induces a $T$-invariant probability measure $\mu_i$ on $X$. For this, we will use an argument similar to the one given in [90, Lemma 5.1]. We define premeasures $\overline{\mu}_i$ over the algebra of clopen sets $\mathbb{K}$ of $X$, by the rule

$$\overline{\mu}_i : \mathbb{K} \to [0,1], \quad \overline{\mu}_i(U) = \overline{N}_i(\chi_U).$$

Indeed, $\overline{\mu}_i(\emptyset) = N_i(0) = 0$, and if $\{U_n\}_{n\geq 1}$ is a sequence of disjoint clopen sets of $X$ such that its union $U$ is also clopen, then $U$ is compact, and therefore it can be written as $U = U_{n_1} \sqcup \cdots \sqcup U_{n_m}$ for some clopen $U_{n_i}$, and

$$\overline{\mu}_i(U) = N_i(\chi_U) = \sum_{i=1}^{m} N_i(\chi_{U_{n_i}}) = \sum_{n\geq 1} \overline{\mu}_i(U_n).$$

So each $\overline{\mu}_i$ is a premeasure, and by [34, Theorem 1.14] they can be uniquely extended to measures $\mu_i$ on the Borel $\sigma$-algebra of $X$, and it is straightforward to show that each $\mu_i$ is a $T$-invariant probability measure on $X$.

Now, we necessarily have the equality $\mu = \alpha\mu_1 + \beta\mu_2$ since

$$\mu(U) = \mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(\chi_U) = \alpha N_1(\chi_U) + \beta N_2(\chi_U) = \alpha\mu_1(U) + \beta\mu_2(U) \quad \text{for every } U \in \mathbb{K}.$$

Since $\mu$ is extremal ([85, Theorem 2.8]) and $\alpha \neq 0, 1$, we obtain that $\mu_1 = \mu_2 = \mu$. This says that $N_i$ are Sylvester matrix rank functions on $\mathcal{A}$ satisfying $N_i(\chi_U) = \mu_i(U) = \mu(U)$ for each $U \in \mathbb{K}$. By the uniqueness property of part *(ii)*, we get that $N_i = \mathrm{rk}_{\mathcal{A}}$. It follows that $\mathrm{rk}_{\mathcal{A}}$ is extremal.

To show that $\mathrm{rk}_{\mathfrak{R}_\infty}$ is extremal, suppose again that we have a convex combination $\mathrm{rk}_{\mathfrak{R}_\infty} = \alpha N_1 + \beta N_2$, where $N_1$ and $N_2$ are pseudo-rank functions on $\mathfrak{R}_\infty$. Assume that $\alpha \neq 0, 1$. Then it is clear that each $N_i$ is continuous with respect to $\mathrm{rk}_{\mathfrak{R}_\infty}$, denoted by $N_i << \mathrm{rk}_{\mathfrak{R}_\infty}$, in the sense of [39, Definition on page 287], and therefore by [39, Proposition 19.12], $N_i$ extend to continuous pseudo-rank functions $\overline{N}_i$ on $\mathfrak{R}_{\mathrm{rk}}$ such that $\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}} = \alpha\overline{N}_1 + \beta\overline{N}_2$. Since we have an identification $\mathcal{A} \subseteq \mathfrak{R}_{\mathrm{rk}}$ given by Theorem 2.3.7, the argument above can be used to show that $\mathrm{rk}_{\mathfrak{R}_\infty} \in \partial_e\mathbb{P}(\mathfrak{R}_\infty)$. $\square$

We can exactly compute the rank completion $\mathfrak{R}_{\mathrm{rk}}$ of $\mathfrak{R}_\infty$ (and of $\mathcal{A}$): it is the well-known von Neumann continuous factor $\mathcal{M}_K$, which is defined as the completion of $\varinjlim_n M_{2^n}(K)$ with respect to its unique rank function (see Example 1.2.8.2) or Chapter 4 for details). Moreover, when the involution $*$ on $K$ is positive definite, we can deduce from Theorem 4.4.6 that there is a $*$-isomorphism between $\mathfrak{R}_{\mathrm{rk}}$ and $\mathcal{M}_K$, where the latter has the involution induced from the $*$-transpose involution on each matrix algebra $M_{2^n}(K)$. The above of course applies when $K$ is a subfield of $\mathbb{C}$ which is invariant under complex conjugation. This generalizes a result of Elek [29].

**Theorem 2.3.9.** *There is an isomorphism of algebras $\mathfrak{R}_{\mathrm{rk}} \cong \mathcal{M}_K$, the von Neumann continuous factor over $K$. Moreover, if $(K,*)$ is a field with positive definite involution, then $\mathfrak{R}_{\mathrm{rk}}$ is a $*$-regular ring in a natural way, and $\mathfrak{R}_{\mathrm{rk}} \cong \mathcal{M}_K$ as $*$-algebras over $K$.*

*Proof.* Since $\mathrm{rk}_{\mathfrak{R}_\infty}$ is extremal (Proposition 2.3.8), it follows from [39, Theorem 19.14] that $\mathfrak{R}_{\mathrm{rk}} = \overline{\mathfrak{R}_\infty}^{\mathrm{rk}_{\mathfrak{R}_\infty}}$ is a simple ring. So $\mathfrak{R}_{\mathrm{rk}}$ is a *continuous factor* in the sense of Definition 4.2.1, that is, a simple, (right and left) self-injective regular ring of type $II_f$. Moreover, there is a countably dimensional dense subalgebra of $\mathfrak{R}_{\mathrm{rk}}$, namely $\mathcal{A}$, and clearly condition (3) in Theorem 4.2.2 is satisfied (because it is satisfied for the dense subalgebra $\mathfrak{R}_\infty$ of $\mathfrak{R}_{\mathrm{rk}}$). It follows that $\mathfrak{R}_{\mathrm{rk}} \cong \mathcal{M}_K$, the von Neumann continuous factor.

Now assume that $(K,*)$ is a field with positive definite involution. Then each $\mathfrak{R}_n = \prod_{W\in\mathbb{V}_n} M_{|W|}(K)$ is a $*$-regular ring, where each factor $M_{|W|}(K)$ of $\mathfrak{R}_n$ has the $*$-transpose involution, and the connecting maps $j_n \colon \mathfrak{R}_n \to \mathfrak{R}_{n+1}$ are given by block-diagonal maps (see Proposition 2.3.2), so in particular are $*$-homomorphisms. Therefore $\mathfrak{R}_\infty$ is a $*$-regular ring, and by [45, Proposition 1], the completion $\mathfrak{R}_{\mathrm{rk}}$ of $\mathfrak{R}_\infty$ is also a $*$-regular ring. One can easily show that $\mathcal{A}$ sits inside $\mathfrak{R}_{\mathrm{rk}}$ as a $*$-subalgebra, i.e. that the homomorphism defined in the proof of Proposition 2.3.7

$$\Phi : \mathcal{A} \hookrightarrow \mathfrak{R}_{\mathrm{rk}}, \quad \sum_{i\in\mathbb{Z}} f_i t^i \mapsto \sum_{i\in\mathbb{Z}} f_i u^i$$

preserves the involution.

Now the local condition (3) in Theorem 4.4.6 is actually somewhat more difficult to check in this case. Given positive integers $n, k$, if we define $S_k = \{W \in \mathbb{V}_n \mid |W| \leq k\}$, we have the estimate

$$\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}\Big(1 - \sum_{W \in S_k} h_W\Big) = \sum_{\substack{W \in \mathbb{V}_n \\ |W|>k}} |W|\mu(W) \xrightarrow[k\to\infty]{} 0$$

as we have showed in the proof of Proposition 2.3.8. Therefore there exists $k_n$ such that $\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}(1 - H_n) < \frac{1}{2^n}$, being $H_n$ the projection $\sum_{W\in S_{k_n}} h_W \in \mathfrak{R}_n$. We approximate $\mathfrak{R}_\infty$ by the unital $*$-subalgebras

$$\mathfrak{R}'_n := H_n\mathfrak{R}_n \oplus (1 - H_n)K.$$

Since $H_n\mathfrak{R}_n = \Big(\sum_{W\in S_{k_n}} h_W\Big)\mathfrak{R}_n \cong \bigoplus_{W\in S_{k_n}} M_{|W|}(K)$, these algebras are $*$-isomorphic to standard matricial $*$-algebras. Although the sequence of projections $(j_{n,\infty}(H_n))$ is not increasing, there are unital $*$-homomorphisms $j'_{n,m} \colon \mathfrak{R}'_n \to \mathfrak{R}'_m$ for $n \leq m$, defined by

$$j'_{n,m}(H_n x + (1 - H_n)\lambda) = H_m \cdot j_{n,m}(H_n x + (1 - H_n)\lambda) + (1 - H_m)\lambda,$$

for $x \in \mathfrak{R}_n$ and $\lambda \in K$. Here $j_{n,m} \colon \mathfrak{R}_n \to \mathfrak{R}_m$ is the natural $*$-homomorphism. Moreover, since each $j_{n,m}$ is given by block-diagonal maps, so are the $j'_{n,m}$. Observe that $(\mathfrak{R}'_n, j'_{n,n+1})$ is not a directed system, but for $z = H_n x + (1 - H_n)\lambda \in \mathfrak{R}'_n$ and all $m \geq n$, we have the estimate

$$\mathrm{rk}_{\mathfrak{R}_m}(j_{n,m}(z) - j'_{n,m}(z)) = \mathrm{rk}_{\mathfrak{R}_m}((1 - H_m)j_{n,m}(z) - (1 - H_m)\lambda) \leq \mathrm{rk}_{\mathfrak{R}_m}(1 - H_m) < \frac{1}{2^m}.$$

Consequently, the proof of the implication (2) $\implies$ (3) in Theorem 4.4.6 can be adapted to the present setting, and we obtain that condition (3) in Theorem 4.4.6 holds. This theorem then gives that $\mathfrak{R}_{\mathrm{rk}} = \overline{\mathfrak{R}_\infty}^{\mathrm{rk}_{\mathfrak{R}_\infty}}$ is $*$-isomorphic to $\mathcal{M}_K$, as desired. ☐

Theorem 2.3.7 and Proposition 2.3.8 state that, given an ergodic, full and $T$-invariant probability measure $\mu$ on $X$, one can construct an extremal faithful Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$ on $\mathcal{A}$, unique with respect to the property that

$$\mathrm{rk}_{\mathcal{A}}(\chi_U) = \mu(U) \quad \text{for every clopen subset } U \text{ of } X.$$

In the next proposition we prove that the converse of this construction can also be made.

**Proposition 2.3.10.** *Let* rk *be an extremal faithful Sylvester matrix rank function on $\mathcal{A}$. Then there exists an ergodic, full and $T$-invariant probability measure $\mu_{\mathrm{rk}}$ on $X$, uniquely determined by the property that*

$$\mu_{\mathrm{rk}}(U) = \mathrm{rk}(\chi_U) \quad \textit{for every clopen subset } U \textit{ of } X.$$

*Proof.* It is clear that rk induces a finitely additive probability measure on the algebra of clopen subsets of $X$ by the rule

$$\overline{\mu}_{\mathrm{rk}}(U) = \mathrm{rk}(\chi_U) \quad \text{for every clopen subset } U \text{ of } X,$$

which, by the same argument as in the proof of Proposition 2.3.8, can be uniquely extended to a Borel probability measure $\mu_{\mathrm{rk}}$ on $X$. It is clearly $T$-invariant since, for $U \subseteq X$ a clopen set,

$$\mu_{\mathrm{rk}}(T(U)) = \mathrm{rk}(\chi_{T(U)}) = \mathrm{rk}(t\chi_U t^{-1}) = \mathrm{rk}(\chi_U) = \mu_{\mathrm{rk}}(U).$$

By [91, Theorem 2.18], $\mu_{\mathrm{rk}}$ is regular. We now show that $\mu_{\mathrm{rk}}$ is an ergodic measure. Suppose that it is not ergodic. Then there is a Borel subset $B$ of $X$ such that $\alpha := \mu_{\mathrm{rk}}(B) \in (0, 1)$. By regularity of the measure, and since the clopen subsets of $X$ form a basis for the topology, there are nonempty clopen subsets $\{U_i\}_{i\geq 1}$ in $X$ such that $\mu_{\mathrm{rk}}(B\triangle U_i) < \frac{1}{2^i}$ for all $i \geq 1$. In particular each $\mu_{\mathrm{rk}}(B\backslash U_i), \mu_{\mathrm{rk}}(U_i\backslash B) < \frac{1}{2^i}$, and

$$\lim_{i\to\infty} \mu_{\mathrm{rk}}(U_i) = \lim_{i\to\infty} \Big(\mu_{\mathrm{rk}}(U_i\backslash B)+\mu_{\mathrm{rk}}(U_i\cap B)\Big) = \lim_{i\to\infty} \mu_{\mathrm{rk}}(U_i\cap B) = \lim_{i\to\infty} \Big(\mu_{\mathrm{rk}}(U_i\cap B)+\mu_{\mathrm{rk}}(B\backslash U_i)\Big) = \mu_{\mathrm{rk}}(B).$$

We then define

$$N_1(M) = \alpha^{-1} \lim_{i\to\infty} \mathrm{rk}(\chi_{U_i} M) \quad \text{and} \quad N_2(M) = (1 - \alpha)^{-1} \lim_{i\to\infty} \mathrm{rk}(\chi_{X\backslash U_i} M)$$

for every matrix $M$ over $\mathcal{A}$. Note that

$$N_1(1) = \alpha^{-1} \lim_{i\to\infty} \mathrm{rk}(\chi_{U_i}) = \alpha^{-1} \lim_{i\to\infty} \mu_{\mathrm{rk}}(U_i) = 1,$$

$$N_2(1) = (1-\alpha)^{-1} \lim_{i\to\infty} \mathrm{rk}(\chi_{X\backslash U_i}) = (1-\alpha)^{-1} \lim_{i\to\infty} \mu_{\mathrm{rk}}(X\backslash U_i) = 1,$$

and since rk is a Sylvester matrix rank function, it is straightforward to check that each $N_i$ satisfies properties *b*), *c*) and *d*) of Definition 1.2.9 for being Sylvester matrix rank functions. To see that they are distinct Sylvester matrix rank functions, take $j \geq 1$ such that $\mu(B\triangle U_j) < \frac{1}{2}\min\{\alpha, 1-\alpha\}$; then

$$N_1(\chi_{U_j}) = \alpha^{-1} \lim_{i\to\infty} \mu(U_i \cap U_j) = \alpha^{-1} \lim_{i\to\infty} \mu(U_i \cap U_j \cap B) = \alpha^{-1}\mu(U_j \cap B) = \alpha^{-1}(\mu(B) - \mu(B\backslash U_j)) > \frac{1}{2},$$

$$N_2(\chi_{U_j}) = (1-\alpha)^{-1} \lim_{i\to\infty} \mu((X\backslash U_i) \cap U_j) = (1-\alpha)^{-1} \lim_{i\to\infty} \mu((X\backslash B) \cap (X\backslash U_i) \cap U_j) \leq (1-\alpha)^{-1}\mu(B\backslash U_j) < \frac{1}{2}.$$

Since $\mathrm{rk} = \alpha N_1 + (1-\alpha)N_2$, this contradicts the fact that rk is extremal in $\mathbb{P}(\mathcal{A})$.

For the fullness of the measure, suppose that $V$ is a nonempty open subset of $X$ such that $\mu_{\mathrm{rk}}(V) = 0$. We can find a nonempty clopen subset $U \subseteq V$, so in particular $\mathrm{rk}(\chi_U) = \mu_{\mathrm{rk}}(U) \leq \mu_{\mathrm{rk}}(V) = 0$; this contradicts the fact that rk is faithful. □

### 2.3.3 The space $\mathbb{P}(\mathcal{A})$

In this section we obtain some results on the structure of the compact convex set $\mathbb{P}(\mathcal{A})$ of Sylvester matrix rank functions on $\mathcal{A}$. Recall that $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ with $T$ a homeomorphism of a totally disconnected, metric, compact space $X$, *not necessarily infinite* throughout this section.

Let $R$ be a unital ring. Following [52], we denote by $\mathbb{P}_{\mathrm{reg}}(R)$ the set of all Sylvester matrix rank functions rk on $R$ that are induced by some regular ring, that is, $\mathrm{rk} \in \mathbb{P}_{\mathrm{reg}}(R)$ if and only if there is a regular rank ring $(S, \mathrm{rk}_S)$ and a ring homomorphism $\varphi\colon R \to S$ such that $\mathrm{rk}(A) = \mathrm{rk}_S(\varphi(A))$ for every matrix $A$ over $R$.

We first investigate the relation between Sylvester matrix rank functions on $C_K(X)$ and Borel probability measures on $X$.

**Lemma 2.3.11.** *There is a natural identification $\mathbb{P}(C_K(X)) = M(X)$, where $M(X)$ denotes the compact convex set of probability measures on $X$. Under this identification, the set $M^{\mathbb{Z}}(X)$ of $T$-invariant probability measures corresponds to the set $\mathbb{P}^{\mathbb{Z}}(C_K(X))$ of $T$-invariant Sylvester matrix rank functions on $C_K(X)$.*

*Proof.* Note that $R = C_K(X)$ is a commutative von Neumann regular ring for each field $K$. Hence the set $\mathbb{P}(R)$ coincides with the set of pseudo-rank functions on $R$ (see Section 1.2 for the definition and properties of pseudo-rank functions on regular rings). Now it is clear that a pseudo-rank function rk on $R$ induces a finitely additive probability measure on the algebra of clopen subsets of $X$ by the rule

$$\overline{\mu}_{\mathrm{rk}}(U) = \mathrm{rk}(\chi_U) \quad \text{for every clopen subset } U \text{ of } X,$$

which by the same argument as in the proof of Proposition 2.3.8, can be uniquely extended to a Borel probability measure $\mu_{\mathrm{rk}}$ on $X$.

Conversely, any Borel probability measure $\mu$ induces a pseudo-rank function $\mathrm{rk}_\mu$ on $R$, as follows. As we have already observed at the beginning of Section 2.2 after Lemma 2.2.1, each element $a \in R$ can be written in the form $a = \sum_{i=1}^n \lambda_i \chi_{U_i}$, where $\lambda_i \in K$ and $\{U_i\}$ forms a partition of $X$, where each $U_i$ is a clopen subset of $X$. We define

$$\mathrm{rk}_\mu(a) = \sum_{i:\lambda_i\neq 0} \mu(U_i).$$

Let's check that it is indeed a pseudo-rank function on $R$. Clearly $\mathrm{rk}_\mu(0) = 0$ and $\mathrm{rk}_\mu(1) = 1$. If $a = \sum_{i=1}^n \lambda_i \chi_{U_i}, b = \sum_{j=1}^m \eta_j \chi_{V_j}$ with $\{U_i\}, \{V_j\}$ partitions of $X$ consisting of clopen sets, and $\lambda_i, \eta_j \in K$, then $ab = \sum_{i,j} \lambda_i \eta_j \chi_{U_i \cap V_j}$, and so

$$\mathrm{rk}_\mu(ab) = \sum_{i:\lambda_i\neq 0} \sum_{j:\eta_j\neq 0} \mu(U_i \cap V_j) \leq \sum_{i:\lambda_i\neq 0} \sum_{j} \mu(U_i \cap V_j) = \sum_{i:\lambda_i\neq 0} \mu(U_i) = \mathrm{rk}_\mu(a).$$

Symmetrically we get $\mathrm{rk}_\mu(ab) \leq \mathrm{rk}_\mu(b)$. To conclude, take $a = \chi_U$ and $b = \chi_V$ two orthogonal idempotents of $R$, so $U, V$ are disjoint clopen subsets of $X$. Then

$$\mathrm{rk}_\mu(a+b) = \mu(U) + \mu(V) = \mathrm{rk}_\mu(a) + \mathrm{rk}_\mu(b)$$

as required.

In this way, we obtain a canonical identification between $\mathbb{P}(R)$ and $M(X)$, since it is easily checked that $\mathrm{rk}_{\mu_{\mathrm{rk}}} = \mathrm{rk}$ and $\mu_{\mathrm{rk}_\mu} = \mu$. Now if $\mu$ is $T$-invariant and $a = \sum_{i=1}^n \lambda_i \chi_{U_i}$ is an element of $C_K(X)$ with $\{U_i\}$ a partition of $X$ consisting of clopen sets, then $T(a) = \sum_{i=1}^n \lambda_i \chi_{T(U_i)}$ and

$$\mathrm{rk}_\mu(T(a)) = \sum_{i:\lambda_i \neq 0} \mu(T(U_i)) = \sum_{i:\lambda_i \neq 0} \mu(U_i) = \mathrm{rk}_\mu(a).$$

Hence $\mathrm{rk}_\mu$ is $T$-invariant. Conversely, if rk is a $T$-invariant Sylvester matrix rank function, then

$$\mu_{\mathrm{rk}}(T(U)) = \mathrm{rk}(\chi_{T(U)}) = \mathrm{rk}(T(\chi_U)) = \mathrm{rk}(\chi_U) = \mu(U) \quad \text{for every clopen set } U.$$

Since the extension to a Borel probability measure is unique, we conclude that $\mu$ is also $T$-invariant. □

**Proposition 2.3.12.** *Continue with the above notation. For each* $\mu \in \partial_e M^{\mathbb{Z}}(X)$ *there exists* $\mathrm{rk} \in \partial_e \mathbb{P}(\mathcal{A}) \cap \mathbb{P}_{\mathrm{reg}}(\mathcal{A})$ *such that* $\mathrm{rk}(\chi_U) = \mu(U)$ *for all clopen subset* $U$ *of* $X$.

*Proof.* Note that, by [85, Theorem 2.8], $\partial_e M^{\mathbb{Z}}(X)$ is the set of ergodic $T$-invariant Borel probability measures on $X$. Now if $\mu \in \partial_e M^{\mathbb{Z}}(X)$, then following the first observations given at the beginning of Section 2.3.1,

a) either there is a periodic point $x \in X$ of $T$, of period $l \geq 1$, such that $\mu(\{x\}) = \frac{1}{l}$, and the support of $\mu$ is the orbit $\mathcal{O}(x)$ of $x$, or

b) $X$ is atomless and the action is essentially free, in the sense that the set of periodic points is a $\mu$-null set (see [61, Remark 2.3]).

In the former case, we follow the idea given in the proof of Proposition 2.3.5. We construct a map $\rho : \mathcal{A} \to M_l(K)$ by the rules

$$f \in C_K(X),\ f \mapsto \begin{pmatrix} f(x) & & & \mathbf{0} \\ & f(T(x)) & & \\ & & \ddots & \\ \mathbf{0} & & & f(T^{l-1}(x)) \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & & & 0 & 1 \\ 1 & 0 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ \mathbf{0} & & & 1 & 0 \end{pmatrix} = u.$$

It can be verified that $u\rho(f)u^{-1} = \rho(T(f))$ by direct computation. It follows from the universal property of the crossed product that there is a unique algebra homomorphism $\rho : \mathcal{A} \to M_l(K)$ extending the above assignments. $M_l(K)$ is a regular rank ring, with unique normalized rank function $\frac{\mathrm{Rk}}{l}$, so it induces a Sylvester matrix rank function on $\mathcal{A}$ by $\mathrm{rk}(a) = \frac{\mathrm{Rk}(\rho(a))}{l}$ for $a \in \mathcal{A}$. If $U$ is a clopen subset of $X$,

$$\mathrm{rk}(\chi_U) = \frac{\mathrm{Rk}(\rho(\chi_U))}{l} = \frac{1}{l}|\{j \mid T^j(x) \in U\}| = \sum_{j=0}^{l-1} \mu(U \cap \{T^j(x)\}) = \mu(U),$$

and since $\mu$ is an ergodic measure, rk is extremal by the same arguments as in the proof of Proposition 2.3.8. It is not difficult to see, using the previous computation, that the restriction of rk on $C_K(X)$ gives the Sylvester matrix rank function $\mathrm{rk}_\mu$ constructed in Lemma 2.3.11. Therefore $\mathrm{rk} \in \partial_e \mathbb{P}(\mathcal{A}) \cap \mathbb{P}_{\mathrm{reg}}(\mathcal{A})$.

In the latter case, we may restrict to the closed subspace $X' := \mathrm{supp}(\mu)$ of $X$, which is an infinite, totally disconnected, compact metric space. Since $\mu$ is $T$-invariant, $T$ restricts to a homeomorphism of $X'$ and the restriction of $\mu$ to $X'$ is a full ergodic $T$-invariant probability measure. It follows from Theorem 2.3.7 and Proposition 2.3.8 that there is $\mathrm{rk} \in \partial_e \mathbb{P}(\mathcal{A}') \cap \mathbb{P}_{\mathrm{reg}}(\mathcal{A}')$, where $\mathcal{A}' := C_K(X') \rtimes_T \mathbb{Z}$, such that rk induces $\mathrm{rk}_\mu$ on $C_K(X)$. Considering the canonical projection

$$P : C_K(X) \rtimes_T \mathbb{Z} \to C_K(X') \rtimes_T \mathbb{Z},$$

we see that $\mathrm{rk}_{\mathcal{A}} = \mathrm{rk} \circ P \in \partial_e \mathbb{P}(\mathcal{A}) \cap \mathbb{P}_{\mathrm{reg}}(\mathcal{A})$, as desired. It is straightforward to check that $\mathrm{rk}_{\mathcal{A}}$ satisfies the desired compatibility property with the measure $\mu$. □

**Remark 2.3.13.** In the case where $\mu \in \partial_e M^{\mathbb{Z}}(X)$ is a measure concentrated in the orbit of a periodic point, we cannot expect neither uniqueness of the extremal Sylvester matrix rank function on $\mathcal{A}$ extending $\mathrm{rk}_\mu$ nor regularity of all the extensions, essentially because of the appearance of isotropy.

Consider, for example, the case of a fixed point $X = \{x\}$, with associated measure $\mu$ satisfying $\mu(\{x\}) = 1$, and $K$ being any field of characteristic different from 2. We obtain an extremal Sylvester matrix rank function $\mathrm{rk}'$ by pulling back the unique Sylvester matrix rank function on $K$ via the homomorphism

$$\mathcal{A} \cong K[t, t^{-1}] \to K[t, t^{-1}]/(t - \alpha) \cong K,$$

the first isomorphism given by $f \mapsto f(x)$, $t \mapsto t$, and $\alpha \in K \backslash \{0, 1\}$. This Sylvester matrix rank function induces the same measure $\mu$ as in Proposition 2.3.12, but the rank functions are clearly different, since $\mathrm{rk}'(t-1) = 1$ and $\mathrm{rk}(t-1) = 0$. This shows the nonuniqueness statement above.

To continue, we need the following result from [54].

**Proposition 2.3.14.** *Let $A = K[t, t^{-1}]$. Then $\mathbb{P}(A) = \mathbb{P}_{\mathrm{reg}}(A)$.*

**Theorem 2.3.15.** *Let $T$ be a homeomorphism on a totally disconnected compact metric space $X$ and let $\mathbb{P}^{\mathbb{Z}}(C_K(X)) = M^{\mathbb{Z}}(X)$ be the space of $T$-invariant measures on $X$, which we identify with the set of $T$-invariant Sylvester matrix rank functions on $C_K(X)$. Set, as before, $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$. Then we have $\mathbb{P}(\mathcal{A}) = \mathbb{P}_{\mathrm{reg}}(\mathcal{A})$.*

*Proof.* By [53, Proposition 5.9], it suffices to show that all extremal Sylvester matrix rank functions on $\mathcal{A}$ are regular. Let $\mathrm{rk} \in \partial_e \mathbb{P}(\mathcal{A})$, and let $\mu_{\mathrm{rk}}$ be the ergodic, full, $T$-invariant probability measure on $X$ given by Proposition 2.3.10.

Assume first that $\mu_{\mathrm{rk}}$ is a measure concentrated in the orbit of a periodic point $x$, of period $l$. In this case, rk induces an extremal Sylvester matrix rank function on $C_K(\mathcal{O}(x)) \rtimes_T \mathbb{Z}$, which is $*$-isomorphic to $M_l(K[t^l, t^{-l}])$ via the map

$$f \in C_K(X), \; f \mapsto \begin{pmatrix} f(x) & & & \mathbf{0} \\ & f(T(x)) & & \\ & & \ddots & \\ \mathbf{0} & & & f(T^{l-1}(x)) \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & & & 0 & t \\ t & 0 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ \mathbf{0} & & & t & 0 \end{pmatrix}$$

and so, by Proposition 2.3.14, rk is a regular Sylvester matrix rank function.

If the support of $\mu_{\mathrm{rk}}$ is infinite then, since $\mu_{\mathrm{rk}}$ is an ergodic $T$-invariant measure, the arguments in Proposition 2.3.12 apply to give that $\mathrm{rk} \in \mathbb{P}_{\mathrm{reg}}(\mathcal{A})$.

Thus in any case we get that $\mathrm{rk} \in \mathbb{P}_{\mathrm{reg}}(\mathcal{A})$, and the proof is complete. □

## 2.4 The $*$-regular closure $\mathcal{R}_{\mathcal{A}}$

We continue with our setting, assuming now that $(K, *)$ is a $*$-field with positive definite involution. Since we have endowed the algebra $\mathcal{A} = C_K(X) \rtimes_T X$ with a unique Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$ such that $\mathrm{rk}_{\mathcal{A}}(\chi_U) = \mu(U)$ for every clopen subset $U$ of $X$ (Proposition 2.3.8), it is natural to ask which is the set of positive real numbers reached by such a rank function. In other words, it would be interesting to determine exactly the set

$$\mathcal{C}(\mathcal{A}) := \mathrm{rk}_{\mathcal{A}} \Big( \bigcup_{i=1}^{\infty} M_i(\mathcal{A}) \Big) \subseteq \mathbb{R}^+.$$

As in Definition 1.1.12, this always has the structure of a semigroup, inherited from $(\mathbb{R}^+, +)$.

We will see in Section 3.1 that, in the particular case of some group algebras $K[G]$, this set is exactly the set of $l^2$-Betti numbers arising from the canonical rank function inherited from $\mathcal{U}(G)$, the classical ring of quotients of the von Neumann algebra $\mathcal{N}(G)$ of the group $G$[13]. In fact, we will prove that the $*$-regular closure $\mathcal{R}_{\mathcal{A}} := \mathcal{R}(\mathcal{A}, \mathfrak{R}_{\mathrm{rk}})$ of $\mathcal{A}$ in the $*$-regular ring $\mathfrak{R}_{\mathrm{rk}}$ corresponds exactly to the $*$-regular closure of the group algebra $K[G]$ inside the $*$-regular ring $\mathcal{U}(G)$.

[13] See Sections 1.1.2 and 1.1.4 for a survey on group von Neumann algebras.

With the aim of computing $\mathcal{C}(\mathcal{A})$, we follow the same strategy as in [6], so we aim to study the $*$-regular closure $\mathcal{R}_{\mathcal{A}}$. The reason for this is given by the following proposition, which is motivated by Proposition 1.2.13.

**Proposition 2.4.1.** *With the foregoing notation, the subgroup of* $(\mathbb{R},+)$ *generated by* $\mathcal{C}(\mathcal{A})$ *coincides with the subgroup of* $(\mathbb{R},+)$ *generated by the set*

$$\mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}(\mathcal{R}_{\mathcal{A}}) = \{\mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}(r) \mid r \in \mathcal{R}_{\mathcal{A}}\},$$

*where* $\mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}$ *is the restriction of* $\mathrm{rk}_{\mathfrak{R}_{\mathrm{rk}}}$ *to* $\mathcal{R}_{\mathcal{A}}$*. Equivalently, it coincides with the image of the state*

$$\phi : K_0(\mathcal{R}_{\mathcal{A}}) \to \mathbb{R}, \quad [p]-[q] \mapsto \mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}(p) - \mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}(q).$$

*Proof.* We write $\mathcal{S}_1$ for the subgroup of $(\mathbb{R},+)$ generated by $\mathcal{C}(\mathcal{A})$, and $\mathcal{S}_2$ for the subgroup of $(\mathbb{R},+)$ generated by $\mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}(\mathcal{R}_{\mathcal{A}})$. By Proposition 1.2.13, we clearly have $\mathcal{S}_2 \subseteq \mathcal{S}_1$.

For the other inclusion note first that, since $\mathcal{R}_{\mathcal{A}}$ is a $*$-regular ring with positive definite involution, each matrix algebra $M_n(\mathcal{R}_{\mathcal{A}})$ is $*$-regular too; hence for each $A \in M_n(\mathcal{R}_{\mathcal{A}})$ there exists a projection $P \in M_n(\mathcal{R}_{\mathcal{A}})$ such that $\mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}(A) = \mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}(P)$ (recall Theorem 1.2.11). We conclude that $\mathcal{C}(\mathcal{A})$ is contained in the set of positive real numbers of the form $\mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}(P)$, where $P$ ranges over projections in matrices over $\mathcal{R}_{\mathcal{A}}$. Now each such projection $P$ is equivalent to a diagonal one ([39, Proposition 2.10]), that is, one of the form

$$\begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_r \end{pmatrix} \quad \text{for some projections } p_1, \dots, p_r \in \mathcal{R}_{\mathcal{A}},$$

so that $\mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}(P) = \mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}(p_1) + \cdots + \mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}(p_r) \in \mathcal{S}_2$, and $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

The last part of the proposition follows easily from the first one, since $\phi(K_0(\mathcal{R}_{\mathcal{A}})) = \mathcal{S}_2$. □

The first thing we notice is that we can completely determine the rank completion of $\mathcal{R}_{\mathcal{A}}$ thanks to Theorem 2.3.7.

**Proposition 2.4.2.** *With the above notation,* $\overline{\mathcal{R}_{\mathcal{A}}}^{\mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}} \cong \mathcal{M}_K$*, where* $\mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}$ *denotes the Sylvester matrix rank function of* $\mathcal{R}_{\mathcal{A}}$ *inherited from the regular ring* $\mathfrak{R}_{\mathrm{rk}}$*.*

*Proof.* Since $\mathcal{A} \subseteq \mathcal{R}_{\mathcal{A}} \subseteq \mathfrak{R}_{\mathrm{rk}}$ and $\overline{\mathcal{A}}^{\mathrm{rk}_{\mathcal{A}}} = \mathfrak{R}_{\mathrm{rk}} \cong \mathcal{M}_K$ due to Theorem 2.3.7, the result follows. □

We will make use of our sequence $\{\mathcal{A}_n\}_{n\geq 1}$ of approximating algebras to approximate $\mathcal{R}_{\mathcal{A}}$ in a suitable way.

In our present setting, the rings $\mathfrak{R}_n$, $\mathfrak{R}_\infty$ and $\mathfrak{R}_{\mathrm{rk}}$ become $*$-regular and all the connecting maps in the commutative diagram below become $*$-homomorphisms.

$$\begin{array}{ccccccccccc}
\mathcal{A}_n & \xhookrightarrow{\iota_n} & \mathcal{A}_{n+1} & \xhookrightarrow{\iota_{n+1}} & \mathcal{A}_{n+2} & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{A}_\infty & \hookrightarrow & \mathcal{A} \\
\downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n+1}} & & \downarrow{\scriptstyle \pi_{n+2}} & & & & \downarrow{\scriptstyle \pi_\infty} & & \downarrow \\
\mathfrak{R}_n & \xhookrightarrow{j_n} & \mathfrak{R}_{n+1} & \xhookrightarrow{j_{n+1}} & \mathfrak{R}_{n+2} & \hookrightarrow & \cdots & \hookrightarrow & \mathfrak{R}_\infty & \hookrightarrow & \mathfrak{R}_{\mathrm{rk}}
\end{array} \tag{2.4.1}$$

We can then consider the $*$-regular closure of $\mathcal{A}_n$ inside $\mathfrak{R}_n$, which we will denote by $\mathcal{R}_n = \mathcal{R}(\mathcal{A}_n, \mathfrak{R}_n)$. Similarly we let $\mathcal{R}_\infty = \mathcal{R}(\mathcal{A}_\infty, \mathfrak{R}_\infty)$.

**Proposition 2.4.3.** *We have inclusions* $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$*, and moreover* $\bigcup_{n\geq 1} \mathcal{R}_n = \mathcal{R}_\infty$*. Therefore the diagram* (2.4.1) *extends to a commutative diagram*

$$\begin{array}{ccccccccccc}
\mathcal{A}_n & \hookrightarrow & \mathcal{A}_{n+1} & \hookrightarrow & \mathcal{A}_{n+2} & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{A}_\infty & \hookrightarrow & \mathcal{A} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\mathcal{R}_n & \hookrightarrow & \mathcal{R}_{n+1} & \hookrightarrow & \mathcal{R}_{n+2} & \hookrightarrow & \cdots & \hookrightarrow & \mathcal{R}_\infty & \hookrightarrow & \mathcal{R}_{\mathcal{A}} \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\mathfrak{R}_n & \hookrightarrow & \mathfrak{R}_{n+1} & \hookrightarrow & \mathfrak{R}_{n+2} & \hookrightarrow & \cdots & \hookrightarrow & \mathfrak{R}_\infty & \hookrightarrow & \mathfrak{R}_{\mathrm{rk}}
\end{array} \tag{2.4.2}$$

*Proof.* Since $\mathcal{A}_n \subseteq \mathcal{A}_{n+1} \cap \mathfrak{R}_n \subseteq \mathcal{R}_{n+1} \cap \mathfrak{R}_n \subseteq \mathfrak{R}_n$, and $\mathcal{R}_{n+1} \cap \mathfrak{R}_n$ is $*$-regular, we have by definition that $\mathcal{R}_n \subseteq \mathcal{R}_{n+1} \cap \mathfrak{R}_n \subseteq \mathcal{R}_{n+1}$, since $\mathcal{R}_n$ is the smallest $*$-regular subring of $\mathfrak{R}_n$ containing $\mathcal{A}_n$. In particular, this shows the commutativity of the left sides of the diagram. The proof for the right sides is similar: $\mathcal{A}_\infty \subseteq \mathcal{A} \cap \mathfrak{R}_\infty \subseteq \mathcal{R}_\mathcal{A} \cap \mathfrak{R}_\infty \subseteq \mathfrak{R}_\infty$, and since $\mathcal{R}_\mathcal{A} \cap \mathfrak{R}_\infty$ is $*$-regular, again we have $\mathcal{R}_\infty \subseteq \mathcal{R}_\mathcal{A} \cap \mathfrak{R}_\infty \subseteq \mathcal{R}_\mathcal{A}$.

For the second part, note that each $\mathcal{A}_n \subseteq \mathcal{R}_n \subseteq \bigcup_{n\geq 1} \mathcal{R}_n \subseteq \mathfrak{R}_\infty$, hence $\mathcal{A}_\infty \subseteq \bigcup_{n\geq 1} \mathcal{R}_n \subseteq \mathfrak{R}_\infty$. It is easy to check, using that $\mathcal{R}_n \subseteq \mathcal{R}_{n+1}$, that $\bigcup_{n\geq 1} \mathcal{R}_n$ is $*$-regular, so by definition $\mathcal{R}_\infty \subseteq \bigcup_{n\geq 1} \mathcal{R}_n$. The other inclusion is trivial because each $\mathcal{R}_n \subseteq \mathcal{R}_\infty$, so the equality follows. □

The following lemma gives some examples of elements that appear inside $\mathcal{R}_\mathcal{A}$.

**Lemma 2.4.4.** *Take $p(x) = \lambda_0 + \lambda_1 x + \cdots + \lambda_k x^k \in K[x]$ a polynomial with $\lambda_0 \neq 0$. Then $p(t) \in \mathcal{A}$ is invertible in $\mathcal{R}_\mathcal{A}$.*

*Moreover, $\mathcal{R}_\mathcal{A}$ contains a copy of the rational function field $K(u)$, so that $K(u) \subseteq \mathcal{R}_\mathcal{A}$.*[14]

*Proof.* Inside $\mathfrak{R}_{\text{rk}}$, we know that $u = \lim_n \pi_n(\chi_{X\setminus E_n} t)$, where the limit is taken with respect to the $\text{rk}_{\mathfrak{R}_{\text{rk}}}$-metric (see Theorem 2.3.7). Hence $p(u) = \lim_n \pi_n(p(\chi_{X\setminus E_n} t))$. Note that, under $\pi_n : \mathcal{A}_n \hookrightarrow \mathfrak{R}_n \subseteq \mathfrak{R}_{\text{rk}}$, $\pi_n(\chi_{X\setminus E_n} t) = (h_W \cdot \chi_{X\setminus E_n} t)_{W \in \mathbb{V}_n}$. We compute

$$h_W \cdot \chi_{X\setminus E_n} t = (e_{00}(W) + \cdots + e_{|W|-1,|W|-1}(W))\chi_{X\setminus E_n} t = e_{10}(W) + \cdots + e_{|W|-1,|W|-2}(W) =: u_W,$$

so

$$\pi_n(p(\chi_{X\setminus E_n} t)) = \lambda_0 \operatorname{Id}_W + \lambda_1 u_W + \cdots + \lambda_k u_W^k.$$

These are all lower triangular matrices inside each simple factor $M_{|W|}(K)$, and since $\lambda_0 \neq 0$, they are invertible. Hence $\pi_n(p(\chi_{X\setminus E_n} t))$ is invertible inside $\mathfrak{R}_n \subseteq \mathfrak{R}_{\text{rk}}$, and so is its limit $\lim_n \pi_n(p(\chi_{X\setminus E_n} t)) = p(u)$.

Since $u$ is clearly invertible in $\mathcal{R}_\mathcal{A}$, it follows that $K(u) \subseteq \mathcal{R}_\mathcal{A}$: for polynomials $p(u) = \lambda_0 + \lambda_1 u + \cdots + \lambda_k u^k$ and $q(u) = u^i(\mu_0 + \mu_1 u + \cdots + \mu_r u^r)$ with $\lambda's, \mu's \in K$ and $\mu_0 \neq 0$, we have

$$p(u)q(u)^{-1} = u^{-i}(\lambda_0 + \lambda_1 u + \cdots + \lambda_k u^k)(\mu_0 + \mu_1 u + \cdots + \mu_r u^r)^{-1} \in \mathcal{R}_\mathcal{A}.$$ □

In what follows we will measure what is the difference between $\mathcal{R}_\infty$ and $\mathcal{R}_\mathcal{A}$ in the case where $y$ is a periodic point (cf. Proposition 2.3.5), and we will uncover the structure of this difference. We start with an easy proposition concerning the structure of $\mathcal{R}_\infty$.

**Proposition 2.4.5.** *Let us assume the above notation and the one from Proposition 2.3.5, so $I$ denotes the ideal of $\mathcal{A}$ generated by $C_{c,K}(X\setminus\{y, T(y), ..., T^{l-1}(y)\})$. Let $\widetilde{I} = \mathcal{R}_\infty I \mathcal{R}_\infty$ be the ideal of $\mathcal{R}_\infty$ generated by $I$. Then:*

*(i) $\widetilde{I} = \bigcup_{n\geq M}(1 - h_{W_n})\mathcal{R}_n$, and there is a $*$-isomorphism*

$$\mathcal{R}_\infty/\widetilde{I} \cong M_l(K).$$

*(ii) If $\mathcal{R}$ denotes the $*$-subalgebra of $\mathcal{R}_\mathcal{A}$ generated by $\widetilde{I}, h_{W_M}\mathcal{A}_M$ and $K[t, t^{-1}]$, then $\widetilde{I}$ is also an ideal of $\mathcal{R}$, $\mathcal{A}$ is contained in $\mathcal{R}$, and there is a $*$-isomorphism*

$$\mathcal{R}/\widetilde{I} \cong M_l(K[t^l, t^{-l}]).$$

Note that, since the ideal $\widetilde{I}$ is already $*$-regular and the quotient $\mathcal{R}/\widetilde{I}$ is not $*$-regular but 'almost', the $*$-subalgebra $\mathcal{R}$ is not the $*$-regular closure $\mathcal{R}_\mathcal{A}$, but 'almost'. We will see later how one should modify $\mathcal{R}$ in order to obtain the whole $\mathcal{R}_\mathcal{A}$. First, we prove Proposition 2.4.5

*Proof.* (i) For $n \geq M$, let $I_n = I \cap \mathcal{A}_n = (1 - h_{W_n})\mathcal{A}_n$ and $\widetilde{I}_n = \mathcal{R}_n I_n \mathcal{R}_n$, the ideal of $\mathcal{R}_n$ generated by $I_n$.

<u>Claim 1</u>: $\widetilde{I} = \bigcup_{n\geq M} \widetilde{I}_n$.

Clearly each $\widetilde{I}_n = \mathcal{R}_n I_n \mathcal{R}_n \subseteq \mathcal{R}_\infty I \mathcal{R}_\infty = \widetilde{I}$, so $\bigcup_{n\geq M} \widetilde{I}_n \subseteq \widetilde{I}$. For the other inclusion, first recall that $I \subseteq \mathcal{A}_\infty$. If $a \in \widetilde{I} = \mathcal{R}_\infty I \mathcal{R}_\infty$, we can write it as a finite combination $a = \sum_{j=1}^m r_j b_j s_j$ with $r_j, s_j \in \mathcal{R}_\infty$ and $b_j \in I \subseteq \mathcal{A}_\infty$. There exists then an index $n_0 \geq M$ such that $r_j, s_j \in \mathcal{R}_{n_0}$ and $b_j \in I \cap \mathcal{A}_{n_0}$ for all $j = 1, ..., m$. Therefore $a = \sum_{j=1}^m r_j b_j s_j \in \mathcal{R}_{n_0} I_{n_0} \mathcal{R}_{n_0} = \widetilde{I}_{n_0}$, and we obtain the inclusion $\widetilde{I} \subseteq \bigcup_{n\geq M} \widetilde{I}_n$.

[14] Recall that under the injection $\mathcal{A} \hookrightarrow \mathfrak{R}_{\text{rk}}$, $t$ is identified with $u$.

Claim 2: $\widetilde{I}_n = (1 - h_{W_n})\mathcal{R}_n$.

Since $I_n = (1 - h_{W_n})\mathcal{A}_n$ and taking into account that $(1 - h_{W_n})$ is central in $\mathcal{R}_n$, we compute

$$\widetilde{I}_n = \mathcal{R}_n I_n \mathcal{R}_n = \mathcal{R}_n(1 - h_{W_n})\mathcal{A}_n\mathcal{R}_n = \mathcal{R}_n(1 - h_{W_n})\mathcal{R}_n = (1 - h_{W_n})\mathcal{R}_n,$$

as required.

Using Claims 1 and 2, $\widetilde{I} = \bigcup_{n\geq M} \widetilde{I}_n = \bigcup_{n \geq M}(1 - h_{W_n})\mathcal{R}_n$.

Claim 3: For $m \geq n \geq M$, $\mathcal{R}_n/\widetilde{I}_n \cong \mathcal{R}_m/\widetilde{I}_m \cong M_l(K)$ canonically, through the maps

$$e_{ij}(W_n) + \widetilde{I}_n \mapsto e_{ij}(W_m) + \widetilde{I}_m \mapsto e_{ij}.$$

To see this, note first that since $\mathcal{A}_n \subseteq \mathcal{R}_n \subseteq \mathfrak{R}_n$ and $h_{W_n}\mathcal{A}_n = h_{W_n}\mathfrak{R}_n \cong M_l(K)$, we have $h_{W_n}\mathcal{R}_n \cong M_l(K)$ through $e_{ij}(W_n) \mapsto e_{ij}$. Now each $h_{W_n}$ is a central idempotent in $\mathcal{R}_n$, so we again have decompositions

$$\mathcal{R}_n = (1 - h_{W_n})\mathcal{R}_n \oplus h_{W_n}\mathcal{R}_n = \widetilde{I}_n \oplus h_{W_n}\mathcal{R}_n.$$

Hence $\mathcal{R}_n/\widetilde{I}_n \cong h_{W_n}\mathcal{R}_n \cong M_l(K) \cong h_{W_m}\mathcal{R}_m \cong \mathcal{R}_m/\widetilde{I}_m$ through the cited maps $e_{ij}(W_n) + \widetilde{I}_n \mapsto e_{ij} \mapsto e_{ij}(W_m)+\widetilde{I}_m$. Note that the $*$-isomorphism $\mathcal{R}_n/\widetilde{I}_n \cong \mathcal{R}_m/\widetilde{I}_m$ is induced by the restriction map $\mathcal{R}_n \hookrightarrow \mathcal{R}_m$, which sends $e_{ij}(W_n)$ to $e_{ij}(W_m)+$ terms inside $\widetilde{I}_m$.

Now fix $n \geq M$, and consider the composition $\mathcal{R}_n \hookrightarrow \mathcal{R}_\infty \to \mathcal{R}_\infty/\widetilde{I}$. Since $\widetilde{I}_n \subseteq \widetilde{I}$, we can factor this $*$-homomorphism through the quotient $\mathcal{R}_n/\widetilde{I}_n$, to get a $*$-homomorphism

$$\mathcal{R}_n/\widetilde{I}_n \to \mathcal{R}_\infty/\widetilde{I}, \quad r + \widetilde{I}_n \mapsto r + \widetilde{I}.$$

From the last claim it follows easily that, for $n, m \geq M$, the diagram

$$\begin{array}{ccc} \mathcal{R}_n/\widetilde{I}_n & \longrightarrow & \mathcal{R}_\infty/\widetilde{I} \\ \wr\Vert & \nearrow & \\ \mathcal{R}_m/\widetilde{I}_m & & \end{array}$$

is commutative. This proves surjectivity of $\mathcal{R}_n/\widetilde{I}_n \to \mathcal{R}_\infty/\widetilde{I}$, as follows. For $r \in \mathcal{R}_\infty = \bigcup_{n\geq 1} \mathcal{R}_n$, $r \in \mathcal{R}_{n_0}$ for some $n_0 \geq M$, and so $r + \widetilde{I} = h_{W_{n_0}}r + (1 - h_{W_{n_0}})r + \widetilde{I} = h_{W_{n_0}}r + \widetilde{I}$. Then $h_{W_{n_0}}r + \widetilde{I}_{n_0} \in \mathcal{R}_{n_0}/\widetilde{I}_{n_0} \cong \mathcal{R}_n/\widetilde{I}_n$, and using the previous diagram we are done.
Injectivity is easy: since $\mathcal{R}_n/\widetilde{I}_n \cong M_l(K)$ is simple, it is enough to show that the element $h_{W_n}$ does not lie inside $\widetilde{I}$. But if $h_{W_n} \in (1 - h_{W_{n_0}})\mathcal{R}_{n_0}$ for some $n_0 \geq M$, then $h_{W_n} = (1 - h_{W_{n_0}})h_{W_n}$, and so $h_{W_n}h_{W_{n_0}} = 0$. This is a contradiction, since (in general) for $n \geq m \geq M$, $W_n \subseteq W_m$, and $h_{W_n}h_{W_m} = h_{W_n}$.
From this we obtain the desired $*$-isomorphism $\mathcal{R}_\infty/\widetilde{I} \cong \mathcal{R}_n/\widetilde{I}_n \cong M_l(K)$.

(ii) We first show that $\widetilde{I}$ is stable under multiplication by elements of $K[t, t^{-1}]$.

Claim 4: $t\widetilde{I} = \widetilde{I}$.

Take $a \in (1 - h_{W_n})\mathcal{R}_n$ for some $n \geq M$. Note that it is enough to show that $t(1 - h_{W_n}) \in \widetilde{I}$, because in this case $ta = t(1 - h_{W_n})a \in \widetilde{I}$, and we would have shown the inclusion $t\widetilde{I} \subseteq \widetilde{I}$. But

$$t(1 - h_{W_n}) = t\chi_{X\backslash(W_n \cup T(W_n)\cup \cdots \cup T^{l-1}(W_n))} = \chi_{X\backslash(T(W_n)\cup T^2(W_n)\cup\cdots\cup T^l(W_n))}t$$

and $\chi_{X\backslash(T(W_n)\cup T^2(W_n)\cup\cdots\cup T^l(W_n))} \in C_{c,K}(X\backslash\{y\})$, so by Lemma 2.3.3 we deduce that $t(1 - h_{W_n}) \in \mathcal{A}_\infty$, hence $t(1 - h_{W_n}) \in \widetilde{I}$. To prove the other inclusion $t^{-1}\widetilde{I} \subseteq \widetilde{I}$, the same trick applies: write

$$\begin{aligned} t^{-1}(1 - h_{W_n}) &= t^{-1}\big(\chi_{X\backslash E_n} - (h_{W_n} - \chi_{W_n})\big) + t^{-1}(\chi_{E_n} - \chi_{W_n}) \\ &= t^{-1}\chi_{X\backslash E_n}\big(\chi_{X\backslash E_n} - (h_{W_n} - \chi_{W_n})\big) + t^{-1}\chi_{E_n\backslash W_n}. \end{aligned}$$

The first term already belongs to $\widetilde{I}$ because $t^{-1}\chi_{X\backslash E_n} \in \mathcal{A}_\infty$, and since $\chi_{E_n\backslash W_n} \in C_{c,K}(X\backslash\{y\})$, Lemma 2.3.3 applies again to deduce that $t^{-1}\chi_{E_n\backslash W_n} \in \mathcal{A}_\infty$ too. Therefore $t^{-1}(1 - h_{W_n}) \in \widetilde{I}$, and $t^{-1}a = t^{-1}(1 - h_{W_n})a \in \widetilde{I}$.
In conclusion $t\widetilde{I} = \widetilde{I}$, as claimed.

As a consequence $p(t)\widetilde{I}, \widetilde{I}p(t) \subseteq \widetilde{I}$ for every Laurent polynomial in $t$, and so $\widetilde{I}$ is an ideal of $\mathcal{R}$.

Claim 5: $\widetilde{I}$ is a proper ideal of $\mathcal{R}$.

This follows from the fact that $\mathrm{rk}_{\mathcal{A}}(1 - h_{W_n}) < 1$ for all $n \geq M$.

Claim 6: $\mathcal{A} \subseteq \mathcal{R}$.

Assume that $h_{W_n}\mathcal{A}_n \subseteq \mathcal{R}$ for some $n \geq M$. Under the quotient map $\mathcal{A}_\infty \to \mathcal{A}_\infty/I \cong M_l(K)$ we know (see the proof of Proposition 2.3.5(ii)) that the matrix units $e_{ij}(W_n)$ correspond to the matrix units $e_{ij}$, hence the differences $e_{ij}(W_{n+1}) - e_{ij}(W_n)$ correspond to 0 inside $\mathcal{A}_\infty/I$, and so they belong to the ideal $I \subseteq \widetilde{I} \subseteq \mathcal{R}$ for all $0 \leq i,j \leq l-1$. Since $e_{ij}(W_n) \in h_{W_n}\mathcal{A}_n \subseteq \mathcal{R}$ already, we deduce that $e_{ij}(W_{n+1}) \in \mathcal{R}$ too, and so the whole algebra $h_{W_{n+1}}\mathcal{A}_{n+1}$ lives inside $\mathcal{R}$. An induction then says that $h_{W_n}\mathcal{A}_n \subseteq \mathcal{R}$ for all $n \geq M$. Hence,

$$\mathcal{A}_n = (1 - h_{W_n})\mathcal{A}_n \oplus h_{W_n}\mathcal{A}_n \subseteq \widetilde{I} + h_{W_n}\mathcal{A}_n \subseteq \mathcal{R}$$

for $n \geq M$, so $\mathcal{A}_\infty \subseteq \mathcal{R}$. In particular (see Lemma 2.3.3) $C_K(X) \subseteq \mathcal{R}$, and since $K[t,t^{-1}] \subseteq \mathcal{R}$ already, we obtain $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z} \subseteq \mathcal{R}$, as claimed.

We consider the quotient $\mathcal{R}/\widetilde{I}$. Since $1 - h_{W_M} \in \widetilde{I}$, we get that the family $\{e_{ij}(W_M) + \widetilde{I}\}_{0 \leq i,j \leq l-1}$ is a complete system of matrix units for $\mathcal{R}/\widetilde{I}$, hence by [64, Theorem 17.5, see also Remark 17.6], there is an isomorphism $\mathcal{R}/\widetilde{I} \cong M_l(T)$, being $T$ the centralizer of the family $\{e_{ij}(W_M) + \widetilde{I}\}_{0 \leq i,j \leq l-1}$ in $\mathcal{R}/\widetilde{I}$. The isomorphism is given by

$$s \mapsto \sum_{i,j=0}^{l-1} s_{ij} e_{ij}(W_M), \quad \text{with } s_{ij} = \sum_{k=0}^{l-1} e_{ki}(W_M) \cdot s \cdot e_{jk}(W_M) \in T,$$

which is also a $*$-isomorphism. We thus only need to prove that $T = K[t^l, t^{-l}]$. The inclusion $K[t^l, t^{-l}] \subseteq T$ is clear, since $e_{ij}(W_M) = (\chi_{X \setminus E_M} t)^i \chi_{W_M} (t^{-1}\chi_{X\setminus E_M})^j = \chi_{T^i(W_M)} t^{i-j}$, and so

$$t^l e_{ij}(W_M) - e_{ij}(W_M)t^l = (\chi_{T^{i+l}(W_M)} - \chi_{T^i(W_M)})t^{i-j+l} \in \widetilde{I}$$

due to the fact that $y$ is a periodic point of period $l$. Therefore $\mathcal{R}/\widetilde{I} \cong M_l(T) \supseteq M_l(K[t^l, t^{-l}])$. In order to prove equality, we only need to check that the element $t \in \mathcal{R}$ belongs to $M_l(K[t^l,t^{-l}])$ under the previous isomorphism. But this is easy:

$$t + \widetilde{I} = t h_{W_M} + \widetilde{I} = \sum_{i=0}^{l-1} t e_{ii}(W_M) + \widetilde{I} = \sum_{i=0}^{l-2} e_{i+1,i}(W_M) + t^l e_{0,l-1}(W_M) + \widetilde{I} \mapsto \begin{pmatrix} 0 & & & 0 & t^l \\ 1 & 0 & & & 0 \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ \mathbf{0} & & & 1 & 0 \end{pmatrix}.$$

So we obtain the desired $*$-isomorphism $\mathcal{R}/\widetilde{I} \cong M_l(K[t^l, t^{-l}])$. □

To continue, we need a couple of technical lemmas, together with a definition.

**Definition 2.4.6.** Let $R$ be a nonunital ring. We say that a family $E \subseteq R$ of idempotents is a *left local unit for $R$* if for every $r_1, ..., r_n \in R$ there exists an idempotent $e \in E$ such that

$$er_i = r_i \quad \text{for all } 1 \leq i \leq n.$$

The concept of *right local unit* is defined analogously.

Note that, in the case that $R$ is a ring endowed with an involution $*$ and $E$ is a left local unit for $R$, then $E^* = \{e^* \mid e \in E\}$ is a right local unit for $R$.

Recall that $\widetilde{I}$ is the ideal of $\mathcal{R}_\infty$ generated by $I$. We write $\mathcal{S}_0$ for the $*$-subalgebra of $\mathcal{R}_{\mathcal{A}}$ generated by $\widetilde{I}, h_{W_M}\mathcal{A}_M$ and $K(t)$[15]. It may be the case that $\widetilde{I}$ is not an ideal of $\mathcal{S}_0$ anymore; nevertheless, we have the following result.

[15] Compare with $\mathcal{R}$ from Proposition 2.4.5.

**Lemma 2.4.7.** *Denote by $\overline{I}_0$ the ideal of $\mathcal{S}_0$ generated by $\widetilde{I}$, and consider*

$$E = \{p(t^l)^{-1}(1 - h_{W_n})p(t^l) \in \overline{I}_0 \mid p(t) \in K[t]\backslash\{0\}, n \geq M\}.$$

*Then $E$ is a left local unit for $\overline{I}_0$.*

*Proof.* Note that every element of $\overline{I}_0$ is a sum of elements of the form

$$p_1(t)q_1(t)^{-1}e_{i_1,j_1}(W_M)\cdots p_s(t)q_s(t)^{-1}e_{i_s,j_s}(W_M)p_{s+1}(t)q_{s+1}(t)^{-1}(1 - h_{W_n})y, \tag{2.4.3}$$

where $p_k, q_k \in K[t]\backslash\{0\}$, $0 \leq i_k, j_k \leq l-1$, $n \geq M$ and $y \in \mathcal{S}_0$. In fact, since $\widetilde{I}$ is stable under multiplication by $K[t, t^{-1}]$ (see the proof of Proposition 2.4.5), the product $p_{s+1}(t)(1 - h_{W_n})$ belongs to $\widetilde{I}$, so we can assume that $p_{s+1}(t) = 1$.

<u>Claim</u>: each element of the form (2.4.3) can be further written as a sum of elements of the form

$$q(t^l)^{-1}(1 - h_{W_n})\widetilde{y} \quad \text{for some } q \in K[t^l]\backslash\{0\}, n \geq M \text{ and } \widetilde{y} \in \mathcal{S}_0.$$

Since the field extension $K(t)/K(t^l)$ has degree $l$, with basis $\{1, t, ..., t^{l-1}\}$, we can write $q_{s+1}(t)^{-1}$ as

$$q_{s+1}(t)^{-1} = \sum_{i=0}^{N} t^i g_i(t^l)^{-1}$$

for some $N \geq 0$ and polynomials $g_i \in K[t^l]\backslash\{0\}$. Therefore we can assume, without loss of generality, that $q_{s+1}$ is a polynomial in $t^l$.

Now recall that, modulo the ideal $\widetilde{I}$, the matrix units $e_{ij}(W_M)$ commute with the element $t^l$, since

$$t^l e_{ij}(W_M) - e_{ij}(W_M)t^l = \big(\chi_{T^{i+l}(W_M)} - \chi_{T^i(W_M)}\big)t^{i-j+l} \in I \subseteq \widetilde{I}.$$

As a consequence $b_s := q_{s+1}(t^l)e_{i_s,j_s}(W_M) - e_{i_s,j_s}(W_M)q_{s+1}(t^l)$ belongs to $\widetilde{I}$, so there exists an integer $n_s \geq M$ such that $b_s = (1 - h_{W_{n_s}})b_s$; therefore

$$e_{i_s,j_s}(W_M)q_{s+1}(t^l)^{-1} - q_{s+1}(t^l)^{-1}e_{i_s,j_s}(W_M) = q_{s+1}(t^l)^{-1}(1 - h_{W_{n_s}})b_s q_{s+1}(t^l)^{-1},$$

so that

$$\begin{aligned} p_1(t)q_1(t)^{-1}&e_{i_1,j_1}(W_M)\cdots p_s(t)q_s(t)^{-1}e_{i_s,j_s}(W_M)q_{s+1}(t^l)^{-1}(1 - h_{W_n})y = \\ &p_1(t)q_1(t)^{-1}e_{i_1,j_1}(W_M)\cdots p_s(t)q_s(t)^{-1}q_{s+1}(t^l)^{-1}e_{i_s,j_s}(W_M)(1 - h_{W_n})y \\ &+ p_1(t)q_1(t)^{-1}e_{i_1,j_1}(W_M)\cdots p_s(t)q_s(t)^{-1}q_{s+1}(t^l)^{-1}(1 - h_{W_{n_s}})b_s q_{s+1}(t^l)^{-1}(1 - h_{W_n})y. \end{aligned}$$

Since $e_{i_s,j_s}(W_M) \in \mathcal{R}_M \subseteq \mathcal{R}_n$ and $1 - h_{W_n}$ is central in $\mathcal{R}_n$, the first term becomes

$$p_1(t)q_1(t)^{-1}e_{i_1,j_1}(W_M)\cdots p_s(t)\widetilde{q}_s(t)^{-1}(1 - h_{W_n})y'$$

with $\widetilde{q}_s = q_s q_{s+1} \in K[t]\backslash\{0\}$ and $y' = e_{i_s,j_s}(W_M)y \in \mathcal{S}_0$, and the second term becomes

$$p_1(t)q_1(t)^{-1}e_{i_1,j_1}(W_M)\cdots p_s(t)\widetilde{q}_s(t)^{-1}(1 - h_{W_{n_s}})y''$$

with now $y'' = b_s q_{s+1}(t^l)^{-1}(1 - h_{W_n})y \in \mathcal{S}_0$. Again, due to the fact that $K[t, t^{-1}]\widetilde{I} \subseteq \widetilde{I}$, we can assume that $p_s = 1$ in each of these terms. Now an induction, repeating exactly the same steps, shows the claim.

Therefore each element $x \in \overline{I}_0$ can be written as a sum of element of the form $q(t^l)^{-1}(1 - h_{W_n})y$ for some $q \in K[t]\backslash\{0\}$, $n \geq M$ and $y \in \mathcal{S}_0$.

Let now $x_1, ..., x_n \in \overline{I}_0$. By the above, we can assume that each $x_i$ is a monomial of the form $q_i(t^l)^{-1}(1 - h_{W_{n_i}})y_i$, with $q_i \in K[t^l]\backslash\{0\}$, $n_i \geq M$ and $y_i \in \mathcal{S}_0$. Consider the polynomial $q := q_1 \cdots q_n \in K[t^l]\backslash\{0\}$. We see that, for each $1 \leq i \leq n$, the result of multiplying $x_i$ by $q$ to the left is always an element of the form $\widetilde{x}_i y_i$, where $\widetilde{x}_i \in \widetilde{I}$. Therefore there exists $N \geq M$ such that $(1 - h_{W_N})q(t^l)x_i = q(t^l)x_i$ for all $1 \leq i \leq n$. The lemma follows by taking the idempotent $e = q(t^l)^{-1}(1 - h_{W_N})q(t^l)$. □

As a consequence of Lemma 2.4.7, the ideal $\overline{I}_0$ must be a proper ideal of $\mathcal{S}_0$. To see this, assume that $1 \in \overline{I}_0$, so that there exists $e = p(t^l)^{-1}(1-h_{W_n})p(t^l)$ satisfying $e = 1$. But this is absurd since then $1 = \mathrm{rk}_{\mathcal{R}_\mathcal{A}}(e) = \mathrm{rk}_\mathcal{A}(1-h_{W_n}) < 1$.

At this moment we could argue as in the proof of Proposition 2.4.5 and compute the quotient $\mathcal{S}_0/\overline{I}_0$. It turns out that this quotient is $*$-isomorphic to $M_l(K(t^l))$, which is a $*$-regular ring. The problem we encounter here is that, in general, the ideal $\overline{I}_0$ might not be $*$-regular. To fix this problem, we consider $\overline{I}_1$ to be the nonunital subalgebra of $\mathcal{R}_\mathcal{A}$ generated by $\overline{I}_0$ and the relative inverses $\overline{x}$ of elements $x \in \overline{I}_0$. Note that it is in fact a $*$-subalgebra, since we always have the equality $\overline{x^*} = \overline{x}^*$, and $\overline{I}_0$ is $*$-closed.

From now on, we let $\mathscr{P}$ to be the set of all the left projections $LP(q(t^l)^{-1}(1-h_{W_n})q(t^l))$, for $n \geq M$ and $q(t) \in K[t]\backslash\{0\}$. So for each $p \in \mathscr{P}$, there is an idempotent $e \in E$ such that $p = LP(e)$; in particular $ep = p$ and $pe = e$. Note that $\mathscr{P} \subseteq \overline{I}_1$.

**Lemma 2.4.8.** *The following statements hold:*

*i) The set $\mathscr{P}$ is a local unit for $\overline{I}_1$.*

*ii) If $\mathcal{S}_1$ denotes the $*$-subalgebra of $\mathcal{R}_\mathcal{A}$ generated by $\overline{I}_1, h_{W_M}\mathcal{A}_M$ and $K(t)$, then $\overline{I}_1$ is a proper ideal of $\mathcal{S}_1$, and there is a $*$-isomorphism*

$$\mathcal{S}_1/\overline{I}_1 \cong M_l(K(t^l)).$$

*Proof.* For i), let $x_1, ..., x_n \in \overline{I}_1$. We can assume that each $x_i$ is a monomial of one of the following forms:

$$(I) \quad r_1\overline{r_2}\cdots \quad \text{with } r_i \in \overline{I}_0;$$
$$(II) \quad \overline{r_1}r_2\cdots \quad \text{with } r_i \in \overline{I}_0.$$

Consider the sets

$$J_1 = \{r \in \overline{I}_0 \mid r \text{ appears as a first term in one of the } x_i\},$$
$$J_2 = \{r^* \in \overline{I}_0 \mid \overline{r} \text{ appears as a first term in one of the } x_i\},$$

so that $J = J_1 \cup J_2$ is a finite subset of $\overline{I}_0$. By Lemma 2.4.7, there exists an idempotent $e \in E$ such that $er = r$ for all $r \in J$. Take $p = LP(e) \in \mathscr{P}$, so $pe = e$ and $ep = p$.

<u>For $J_1$</u>: for an element $r \in J_1$, we compute $pr = per = er = r$.

<u>For $J_2$</u>: for an element $r \in \overline{I}_0$ such that $r^* \in J_2$, we compute $pr^* = per^* = er^* = r^*$, so by taking $*$ we conclude that $rp = r$. Multiplying to the left by the relative inverse $\overline{r}$, we get $(\overline{r}r)p = \overline{r}r$, which is a projection. Hence $\overline{r}r = (\overline{r}r)^* = (\overline{r}rp)^* = p\overline{r}r$, and

$$p\overline{r} = p\overline{r}r\overline{r} = \overline{r}r\overline{r} = \overline{r}.$$

We conclude that $px_i = x_i$ for all $1 \leq i \leq n$. Since $p$ is a projection, part i) follows.

Now ii). By i), it is immediate to check that $\overline{I}_1 \subseteq \mathcal{S}_1$ is proper. To show that it is an ideal, it is enough to show that $p(t)\overline{I}_1, \overline{I}_1 p(t) \subseteq \overline{I}_1$ for all $p(t) \in K(t)$ and that $e_{ij}(W_M)\overline{I}_1, \overline{I}_1 e_{ij}(W_M) \subseteq \overline{I}_1$ for all $0 \leq i,j \leq l-1$. By taking $*$, we only need to show that $p(t)\overline{I}_1, e_{ij}(W_M)\overline{I}_1 \subseteq \overline{I}_1$.

Let $p(t) \in K(t)$ and $a \in \overline{I}_1$. We can assume that $a$ is a monomial of the previous forms (I), (II). If $a = ra'$ for some $r \in \overline{I}_0$ and $a' \in \overline{I}_1$, then

$$p(t)a = \underbrace{p(t)r}_{\in \overline{I}_0} a' \in \overline{I}_1.$$

If $a = \overline{r}a'$ for $r \in \overline{I}_0$ and $a' \in \overline{I}_1$, then consider $p \in \mathscr{P}$ such that $p\overline{r} = \overline{r}$ and $e \in E$ such that $p = LP(e)$, so that $ep = p$ and $pe = e$. Then

$$p(t)a = p(t)\overline{r}a' = p(t)p\overline{r}a' = p(t)ep\overline{r}a' = \underbrace{p(t)e}_{\in \overline{I}_0} a \in \overline{I}_1,$$

as required. Similar computations can be used to show that $e_{ij}(W_M)\overline{I}_1 \subseteq \overline{I}_1$.

Finally, by using exactly the same argument as in the proof of Proposition 2.4.5, we obtain the desired $*$-isomorphism $\mathcal{S}_1/\overline{I}_1 \cong M_l(K(t^l))$. ☐

We are now ready to determine the $*$-regular closure $\mathcal{R}_{\mathcal{A}}$.

**Proposition 2.4.9.** *Following the previous assumptions and caveats, we define $\overline{I}_n$ to be the nonunital subalgebra of $\mathcal{R}_{\mathcal{A}}$ generated by $\overline{I}_{n-1}$ and the relative inverses of elements of $\overline{I}_{n-1}$, starting from the previous $\overline{I}_0$. Let also $\mathcal{S}_n$ be the $*$-subalgebra of $\mathcal{R}_{\mathcal{A}}$ generated by $\overline{I}_n, h_{W_M}\mathcal{A}_M$ and $K(t)$. Then:*

*(i) $\overline{I}_n$ admits the set $\mathscr{P}$ as a local unit;*

*(ii) $\overline{I}_n$ is a proper ideal of $\mathcal{S}_n$, and $\mathcal{S}_n/\overline{I}_n \cong M_l(K(t^l))$ for all $n \geq 0$;*

*(iii) $\overline{I}_\infty = \bigcup_{n\geq 0} \overline{I}_n$ is a proper $*$-regular ideal of $\mathcal{R}_{\mathcal{A}}$, and $\mathcal{R}_{\mathcal{A}}/\overline{I}_\infty \cong M_l(K(t^l))$.*

*Proof.* Note first that each $\overline{I}_n$ is also a $*$-subalgebra of $\mathcal{R}_{\mathcal{A}}$.

(i) and (ii) follows easily by induction, taking into account that the same arguments from the proof of Lemma 2.4.8 apply here to obtain the results.

For (iii), let $\mathcal{S}_\infty$ be the $*$-subalgebra of $\mathcal{R}_{\mathcal{A}}$ generated by $\overline{I}_\infty, h_{W_M}\mathcal{A}_M$ and $K(t)$. Clearly $\overline{I}_\infty \subseteq \mathcal{S}_\infty$ is proper, since $1 \notin \overline{I}_n$ for any $n \geq 0$. To prove that $\overline{I}_\infty$ is an ideal of $\mathcal{S}_\infty$, we only need to show that $K(t)\overline{I}_\infty \subseteq \overline{I}_\infty$ and that $e_{ij}(W_M)\overline{I}_\infty \subseteq \overline{I}_\infty$ for all $0 \leq i, j \leq l-1$. For the first inclusion, take $p(t) \in K(t)$ and $a \in \overline{I}_\infty$. Then $a \in \overline{I}_n$ for some $n \geq 0$, so by (ii) we have $p(t)a \in \overline{I}_n \subseteq \overline{I}_\infty$. The second inclusion is obtained analogously.

To show that $\overline{I}_\infty$ is $*$-regular, take $x \in \overline{I}_\infty$. Then $x \in \overline{I}_n$ for some $n \geq 0$, and its relative inverse $\overline{x}$ belongs to $\overline{I}_{n+1} \subseteq \overline{I}_\infty$. Since $\overline{I}_\infty$ is $*$-closed, we conclude that $\overline{I}_\infty$ is $*$-regular.

Therefore $\overline{I}_\infty$ is a $*$-regular ideal of $\mathcal{S}_\infty$ and, just as before, its quotient $\mathcal{S}_\infty/\overline{I}_\infty$ is $*$-isomorphic to $M_l(K(t^l))$, which is $*$-regular. It follows from [39, Lemma 1.3] that $\mathcal{S}_\infty$ is $*$-regular. Since $\mathcal{A} \subseteq \mathcal{S}_\infty \subseteq \mathcal{R}_{\mathcal{A}}$, we obtain that $\mathcal{S}_\infty = \mathcal{R}_{\mathcal{A}}$. □

In conclusion, in the case that there exists a periodic point $y \in X$ of finite period $l$, we have been able to determine part of the ideal structure of the $*$-regular closure $\mathcal{R}_{\mathcal{A}}$: for each such point $y \in X$ one can apply the above process to construct a maximal ideal $\overline{I}_\infty$ of $\mathcal{R}_{\mathcal{A}}$, thus proving that, in particular, $\mathcal{R}_{\mathcal{A}}$ is not simple. In fact, the construction of the ideal $\overline{I}_\infty$ not only depends on the periodic point $y \in X$, but on the whole orbit $\mathcal{O}(y) = \{y, T(y), ..., T^{l-1}(y)\}$, and in addition any other periodic point $x \in X$ not belonging to the orbit $\mathcal{O}(y)$ will give rise to a different maximal ideal $\overline{I}_{\infty,x}$. It is therefore reasonable to think that, in order to uncover the whole structure of $\mathcal{R}_{\mathcal{A}}$ in the case of the existence of a periodic point, it is crucial to understand the structure of the ideal(s) $\overline{I}_\infty$, and in particular the structure of $\mathcal{R}_\infty = \bigcup_{n\geq 1} \mathcal{R}_n$, which in turn can be studied by studying their pieces $\mathcal{R}_n$. Therefore, in the next section we will concentrate on uncovering a part of the structure of the $*$-regular closure $\mathcal{R}_n$, for a fixed $n$.

### 2.4.1 Localization

We return to the general setting we had in Section 2.2, with the extra hypothesis that $K$ is now a field with a positive definite involution $*$. We fix a clopen subset $E$ of $X$ and a partition $\mathcal{P}$ of $X \setminus E$. Recall from Section 2.2 that we denote by $\mathcal{B}$ the unital $*$-subalgebra of $\mathcal{A}$ generated by the partial isometries $\{\chi_Z t\}_{Z\in\mathcal{P}}$, and we write $\mathcal{B} = \bigoplus_{i\in\mathbb{Z}} \mathcal{B}_i t^i$, with the restrictions already indicated, that is

$$\mathcal{B}_i = \chi_{X\setminus(E\cup T(E)\cup\cdots\cup T^{i-1}(E))}\mathcal{B}_0 \quad \text{and} \quad \mathcal{B}_{-i} = \chi_{X\setminus(T^{-1}(E)\cup\cdots\cup T^{-i}(E))}\mathcal{B}_0 \qquad \text{for } i > 0.$$

We also write $\pi$ for the canonical map $\pi\colon \mathcal{B} \to \mathfrak{R}$ given by $\pi(b) = (h_W \cdot b)_W$, where $\mathfrak{R} = \prod_{W\in\mathbb{V}} M_{|W|}(K)$.

We now pass to power series. We aim to follow the same steps as in [6, Section 7] to study the $*$-regular closure $\mathcal{R}_{\mathcal{B}} = \mathcal{R}(\mathcal{B}, \mathfrak{R})$[16]. However, the situation here is much more involved.

The first step is to consider, from $\mathcal{B}$, a skew power series ring $\mathcal{B}_0[[t;T]]$ by considering infinite sums

$$\sum_{i\geq 0} b_i(\chi_{X\setminus E}t)^i = \sum_{i\geq 0} b_i t^i, \text{ where } b_i \in \mathcal{B}_i \text{ for all } i \geq 0.$$

Similarly we can consider $\mathcal{B}_0[[t^{-1};T^{-1}]]$. Now, given a $W \in \mathbb{V}$, only a finite number of terms in the infinite sum $\sum_{i\geq 0} b_i t^i$ can be nonzero in the factor corresponding to $W$, since by Lemma 2.2.6 and for $i \geq 0$,

$$h_W \cdot (\chi_{X\setminus E}t)^i = e_{i0}(W) + e_{i+1,1}(W) + \cdots + e_{|W|-1,|W|-1-i}(W),$$

[16] Note that, in the notation used in Section 2.4, $\mathcal{R}_{\mathcal{B}} = \mathcal{R}_n$ in case $\mathcal{B}$ is one of the $*$-subalgebras $\mathcal{A}_n$.

and it is exactly zero for $i \geq |W|$. We have a similar situation for $\mathcal{B}_0[[t^{-1};T^{-1}]]$. Therefore we obtain representations

$$\pi_+ : \mathcal{B}_0[[t;T]] \to \mathfrak{R}, \quad b \mapsto (h_W \cdot b)_W \quad \text{and} \quad \pi_- : \mathcal{B}_0[[t^{-1};T^{-1}]] \to \mathfrak{R}, \quad b \mapsto (h_W \cdot b)_W$$

by lower (resp. upper) triangular matrices, extending the canonical one $\pi : \mathcal{B} \to \mathfrak{R}$. We will be mainly interested in the first one $\pi_+$.

We have the following key property.

**Lemma 2.4.10.** *Let $x = \sum_{i\geq 0} b_i t^i \in \mathcal{B}_0[[t;T]]$. Then $x$ is invertible in $\mathcal{B}_0[[t;T]]$ if and only if $b_0$ is invertible in $\mathcal{B}_0$. Analogously for the elements of $\mathcal{B}_0[[t^{-1},T^{-1}]]$.*

*Proof.* Assume first that $x = \sum_{i\geq 0} b_i t^i$ is invertible in $\mathcal{B}_0[[t;T]]$. There exists $y = \sum_{i\geq 0} \widetilde{b}_i t^i$ in $\mathcal{B}_0[[t;T]]$ such that $xy = yx = 1$. In particular, $b_0\widetilde{b}_0 = 1$, and so $b_0$ is invertible in $\mathcal{B}_0$.

Conversely, assume that $b_0$ is invertible in $\mathcal{B}_0$. We can then assume that $b_0 = 1$, so that $x = 1 - y$, where the order of $y$ in $t$ is greater or equal to 1. We then have

$$x^{-1} = (1-y)^{-1} = 1 + y + y^2 + \cdots \in \mathcal{B}_0[[t;T]],$$

hence $x$ is invertible in $\mathcal{B}_0[[t;T]]$. □

As we already mentioned in the previous paragraph, we are going to uncover some structure of the $*$-regular closure of $\pi(\mathcal{B}) \cong \mathcal{B}$ inside $\mathfrak{R}$, $\mathcal{R}_{\mathcal{B}}$. For this purpose, we have the following definitions.

**Definition 2.4.11.** Recalling that $\mathcal{B} = \bigoplus_{i\in\mathbb{Z}} \mathcal{B}_i t^i$,

a) we denote by $\mathcal{B}_+$ the set of elements of $\mathcal{B}$ supported in nonnegative degrees in $t$, that is $\mathcal{B}_+ = \bigoplus_{i\geq 0} \mathcal{B}_i t^i$. Clearly $\mathcal{B}_+ \subseteq \mathcal{B}_0[[t;T]]$. The division closure[17] of $\mathcal{B}_+$ in $\mathcal{B}_0[[t;T]]$ will be denoted by $\mathcal{D}_+$.

b) we set $\mathcal{B}_- = \bigoplus_{i\geq 0} \mathcal{B}_{-i} t^{-i}$. Again, it is clear that $\mathcal{B}_- \subseteq \mathcal{B}_0[[t^{-1},T^{-1}]]$, and we denote by $\mathcal{D}_-$ the division closure of $\mathcal{B}_-$ in $\mathcal{B}_0[[t^{-1};T^{-1}]]$.

Note that $\mathcal{B}_- \cap \mathcal{B}_+ = \mathcal{B}_0$, and we have decompositions $\mathcal{B} = \mathcal{B}_- \oplus (\mathcal{B}_+/\mathcal{B}_0) = (\mathcal{B}_-/\mathcal{B}_0) \oplus \mathcal{B}+$ as vector spaces.

In order to study the division closures $\mathcal{D}_+, \mathcal{D}_-$, we need the following known lemma.

**Lemma 2.4.12** (cf. [60]). *Let $\mathcal{S}$ be a unital $*$-subalgebra of $C_K(X)$ generated by a family of characteristic functions of the form $\{\chi_C\}_C$, where $C$ are clopen subsets of $X$. Then $\mathcal{S}$ is a $*$-regular ring, and every nonzero element of $\mathcal{S}$ can be expressed in the form*

$$\sum_{i=1}^n \lambda_i \chi_{K_i},$$

*where $\lambda_i \in K^\times$ for all $1 \leq i \leq n$, and $\{K_i\}_{i=1}^n$ are mutually disjoint clopen subsets of $X$ such that $\chi_{K_i} \in \mathcal{S}$ for all $1 \leq i \leq n$.*

*In particular, the $*$-subalgebra $\mathcal{B}_0$ of $\mathcal{B}$ is $*$-regular.*

*Proof.* If $a = \sum_{i=1}^n \lambda_i \chi_{K_i}$ is as in the statement, then

$$\Big(\sum_{i=1}^n \lambda_i \chi_{K_i}\Big)\Big(\sum_{i=1}^n \lambda_i^{-1} \chi_{K_i}\Big)\Big(\sum_{i=1}^n \lambda_i \chi_{K_i}\Big) = \Big(\sum_{i=1}^n \chi_{K_i}\Big)\Big(\sum_{i=1}^n \lambda_i \chi_{K_i}\Big) = \sum_{i=1}^n \lambda_i \chi_{K_i},$$

hence $\overline{a} = \sum_{i=1}^n \lambda_i^{-1} \chi_{K_i}$ is the relative inverse of $a$, so that $\mathcal{S}$ is $*$-regular.

It remains to show that each element of $\mathcal{S}$ can be written in the stated form. It is clear that each element of $\mathcal{S}$ is a $K$-linear combination of functions of the form $\chi_{L_i}$, where $L_i$ is a clopen subset of $X$ and $\chi_{L_i} \in \mathcal{S}$, since every product $\chi_{C_1}\chi_{C_2}\cdots\chi_{C_t}$ belongs to $\mathcal{S}$ and equals $\chi_L$, where $L = C_1 \cap \cdots \cap C_t$ is clopen. Therefore, every nonzero element $a$ of $\mathcal{S}$ can be written as $a = \sum_{i=1}^n \lambda_i \chi_{L_i}$, with $\{L_i\}$ clopen subsets of $X$ such that $\chi_{L_i} \in \mathcal{S}$. We now show that this sum can be chosen to be an orthogonal sum. This is done by induction on $n$.

[17] See Section 1.3.3 for the definition of the division closure.

The result is clear for $n = 1$, so assume that $n \geq 1$, that $a = \sum_{i=1}^{n+1} \lambda_i \chi_{L_i}$ with $\{L_i\}$ clopen subsets of $X$ such that $\chi_{L_i} \in \mathcal{S}$, and that $\sum_{i=1}^{n} \lambda_i \chi_{L_i} = \sum_{j=1}^{m} \mu_j \chi_{K_j}$ where now $\{K_j\}$ are mutually disjoint clopen subets of $X$ such that $\chi_{K_j} \in \mathcal{S}$. We compute

$$a = \sum_{j=1}^{m} \mu_j \chi_{K_j} + \lambda_{n+1}\chi_{L_{n+1}} = \sum_{j=1}^{m} (\mu_j + \lambda_{n+1})\chi_{K_j \cap L_{n+1}} + \sum_{j=1}^{m} \mu_j \chi_{K_j \backslash L_{n+1}} + \sum_{j=1}^{m} \lambda_{n+1}\chi_{L_{n+1}\backslash K_j}.$$

The clopen sets $\{K_j \cap L_{n+1}\}_j \cup \{K_j \backslash L_{n+1}\}_j \cup \{L_{n+1} \backslash K_j\}_j$ are clearly disjoint. Moreover,

$$\chi_{K_j \cap L_{n+1}} = \chi_{K_j} \cdot \chi_{L_{n+1}} \in \mathcal{S}, \quad \chi_{K_j \backslash L_{n+1}} = \chi_{K_j} - \chi_{K_j \cap L_{n+1}} \in \mathcal{S}, \text{ and } \chi_{L_{n+1}\backslash K_j} = \chi_{L_{n+1}} - \chi_{L_{n+1}\cap K_j} \in \mathcal{S},$$

so all of their characteristic functions belong to $\mathcal{S}$. This completes the induction step. One should compare this idea with the idea used in Lemma 2.2.8 of expanding the corresponding sets by using suitable partitions of $X$.

Now, the fact that $\mathcal{B}_0$ is a $*$-regular ring follows from the fact that, due to Lemma 2.2.4, it is generated a family of characteristic functions of the above form. □

**Proposition 2.4.13.** *With the preceding notation, we have:*

*(i)* $\mathcal{D}_+$ *coincides with the rational closure*[18] *of* $\mathcal{B}_+$ *in* $\mathcal{B}_0[[t;T]]$*, and similarly for* $\mathcal{D}_-$ *and* $\mathcal{B}_-$*.*

*(ii)* $\pi_+(\mathcal{D}_+)$ *is the division closure of* $\pi_+(\mathcal{B}_+)$ *in* $\mathfrak{R}$*.*

*(iii)* $\pi_+(\mathcal{D}_+) \subseteq \mathcal{R}_\mathcal{B}$*, and similarly,* $\pi_-(\mathcal{D}_-) \subseteq \mathcal{R}_\mathcal{B}$*.*

*(iv)* $\pi_+(\mathcal{D}_+)^* = \pi_-(\mathcal{D}_-)$*.*

*Proof.* (i) This is a standard observation (see e.g. [3, Observation 1.18]), which we reproduce here for the convenience of the reader. We will only deal with the case for $\mathcal{D}_+$, being the other one analogous.

It is always true (part *iii*) of Lemma 1.3.9) that the division closure is contained inside the rational closure, so we only need to prove the reverse inclusion. Let $x$ be an element of the rational closure of $\mathcal{B}_+$ in $\mathcal{B}_0[[t;T]]$, so that $x$ is an entry of some square matrix $A^{-1}$ invertible over $\mathcal{B}_0[[t;T]]$, whose inverse $A$ has entries inside $\mathcal{B}_+$. Writing $A = \sum_{i=0}^{\infty} A_i t^i$, we obtain that $A_0$ is an invertible matrix over $\mathcal{B}_0$. Multiplying by $A_0^{-1}$ we can thus assume that the constant term in the above infinite sum is the identity matrix. With this assumption we obtain, using Lemma 2.4.10, that all the diagonal entries of $A$ are invertible in $\mathcal{B}_+$, and since $\mathcal{D}_+$ is inversion closed, they are also invertible in $\mathcal{D}_+$.

Now, by applying to $A$ a sequence of elementary row transformations, we may further assume that $A$ is a diagonal matrix. Hence $A$ is invertible over $\mathcal{D}_+$, which in particular implies that its inverse $A^{-1}$ has entries inside $\mathcal{D}_+$, and so $x \in \mathcal{D}_+$.

(ii) Recall that $\pi_+$ is an injective homomorphism from $\mathcal{B}_0[[t;T]]$ into $\mathfrak{R}$. We first show that $\pi_+(\mathcal{B}_0[[t;T]])$ is division closed in $\mathfrak{R}$.

For this, let $x = \sum_{i\geq 0} b_i t^i$ be an element in $\mathcal{B}_0[[t;T]]$ such that $\pi_+(x)$ is invertible in $\mathfrak{R}$. Observe that each component of $\pi_+(x)$ is an invertible matrix, with diagonal coming exclusively from elements of $\mathcal{B}_0$. It follows that $\pi_+(b_0) = \pi(b_0)$ must be invertible in $\mathfrak{R}$. But since $\mathcal{B}_0$ is regular (Lemma 2.4.12), there exists $\widetilde{b}_0$ in $\mathcal{B}_0$ such that $b_0\widetilde{b}_0 b_0 = b_0$. Applying $\pi$ and taking into account that $\pi(b_0)$ is invertible in $\mathfrak{R}$, we get that $\pi(b_0)^{-1} = \pi(\widetilde{b}_0)$, and so $b_0$ is in fact invertible in $\mathcal{B}_0$. It follows from Lemma 2.4.10 that $x$ is invertible in $\mathcal{B}_0[[t;T]]$, as required.

Now we use the following general fact: if $R \subseteq S \subseteq T$ are unital embeddings of unital rings, and $S$ is division closed in $T$, then the division closure of $R$ in $T$ equals the division closure of $R$ in $S$, that is $\mathcal{D}(R,T) = \mathcal{D}(R,S)$[19]. Using this and the fact just proved that $\pi_+(\mathcal{B}_0[[t;T]])$ is division closed in $\mathfrak{R}$, we deduce that

$$\mathcal{D}(\pi_+(\mathcal{B}_+), \mathfrak{R}) = \mathcal{D}(\pi_+(\mathcal{B}_+), \pi_+(\mathcal{B}_0[[t;T]])) = \pi_+(\mathcal{D}(\mathcal{B}_+, \mathcal{B}_0[[t;T]])) = \pi_+(\mathcal{D}_+),$$

as desired.

[18] See Section 1.3.3 for the definition of the rational closure.

[19] The proof of this general fact is straightforward and left as an easy exercise.

(iii) By $(ii)$, we have that $\pi_+(\mathcal{D}_+) = \mathcal{D}(\pi_+(\mathcal{B}_+), \mathfrak{R})$ which is contained in the division closure of $\pi(\mathcal{B})$ in $\mathfrak{R}$, $\mathcal{D}(\pi(\mathcal{B}), \mathfrak{R})$. This last one is contained in $\mathcal{R}_{\mathcal{B}}$, by Lemma 1.3.9. Hence $\pi_+(\mathcal{D}_+) \subseteq \mathcal{R}_{\mathcal{B}}$. Similarly $\pi_-(\mathcal{D}_-) \subseteq \mathcal{R}_{\mathcal{B}}$.

(iv) First observe that $\pi_+(\mathcal{B}_+)^* = \pi_-(\mathcal{B}_-)$ and $\pi_+(\mathcal{B}_0[[t;T]])^* = \pi_-(\mathcal{B}_0[[t^{-1}, T^{-1}]])$. The reason is that, for $x = \sum_{i\geq 0} b_i t^i \in \mathcal{B}_0[[t;T]]$ (resp. $\in \mathcal{B}_+$), we have

$$\pi_+(x)^* = \pi_-\Big(\sum_{i\geq 0} t^{-i} b_i^*\Big) = \pi_-\Big(\sum_{i\geq 0} T^{-i}(b_i^*) t^{-i}\Big).$$

The element $b_i^*$ is computed in the $*$-algebra $\mathcal{B}_0$. Also, by the description of $\mathcal{B}$ as a partial crossed product (Proposition 2.2.7), it follows that $T^{-i}(b_i^*) \in \mathcal{B}_{-i}$ and so $\sum_{i\geq 0} T^{-i}(b_i^*)t^{-i} \in \mathcal{B}_0[[t^{-1}; T^{-1}]]$ (resp. $\in \mathcal{B}_-$). Conversely, for $x = \sum_{i\geq 0} b_{-i} t^{-i} \in \mathcal{B}_0[[t^{-1}; T^{-1}]]$ (resp. $\in \mathcal{B}_-$), analogous arguments show that $\pi_-(x) \in \pi_+(\mathcal{B}_0[[t;T]])^*$ (resp. $\in \pi_+(\mathcal{B}_+)^*$), , and so the observation follows.

Now,

$$\pi_+(\mathcal{D}_+)^* = \mathcal{D}(\pi_+(\mathcal{B}_+)^*, \pi_+(\mathcal{B}_0[[t;T]])^*) = \mathcal{D}(\pi_-(\mathcal{B}_-), \pi_-(\mathcal{B}_0[[t^{-1}; T^{-1}]])) = \pi_-(\mathcal{D}_-). \qquad \square$$

We have two subalgebras $\pi_+(\mathcal{D}_+)$ and $\pi_-(\mathcal{D}_-) = \pi_+(\mathcal{D}_+)^*$ of $\mathcal{R}_{\mathcal{B}}$. We will write $\mathfrak{D}$ for the $*$-subalgebra of $\mathcal{R}_{\mathcal{B}}$ generated by $\pi_+(\mathcal{D}_+)$, which coincides with the subalgebra generated by $\pi_+(\mathcal{D}_+)$ and $\pi_-(\mathcal{D}_-)$. Intuitively, we obtain $\mathfrak{D}$ by adjoining inverses of elements of $\mathcal{B}_+$ and $\mathcal{B}_-$.

Consider now a certain subset $\mathcal{S}[[t;T]]$ of $\mathcal{B}_0[[t;T]]$, namely the set of those elements

$$\sum_{i\geq 0} b_i(\chi_{X\setminus E} t)^i = \sum_{i\geq 0} b_i t^i$$

such that each $b_i \in \mathcal{B}_i$ belongs to span$\{\chi_S \mid S \in \mathcal{W}_i$[20]$\}$. The subset $\mathcal{S}[[t;T]]$ is always a linear subspace of $\mathcal{B}_0[[t;T]]$, but it might not be a subalgebra. We will, however, see in the next chapter, Section 3.2, that in the special case of $\mathcal{A}$ being the lamplighter group algebra, $\mathcal{S}[[t;T]]$ is indeed an algebra, and even an integral domain.

However, $\mathcal{S}[[t;T]]$ certainly is a subalgebra of $\mathcal{B}_0[[t;T]]$ when it is endowed with the multiplicative structure given by the Hadamard product $\odot$, which is defined by

$$\Big(\sum_{i\geq 0} b_i t^i\Big) \odot \Big(\sum_{j\geq 0} b_j' t^j\Big) := \sum_{i\geq 0} (b_i b_i') t^i.$$

**Observation 2.4.14.** Each $b_i$ belongs to the linear span of all the $\chi_S$ with $S \in \mathcal{W}_i$, hence they can be written as $b_i = \sum_{S\in\mathcal{W}_i} \lambda_S \chi_S$ for $\lambda_S \in K$. Let $b_i, b_i'$ be two given elements of this form:

$$b_i = \sum_{S\in\mathcal{W}_i} \lambda_S \chi_S, \quad b_i' = \sum_{S'\in\mathcal{W}_i} \mu_{S'} \chi_{S'}.$$

Since the sets $S \in \mathcal{W}_i$ are of the form $T^{i-1}(Z_{i-1}') \cap T^{i-2}(Z_{i-2}') \cap \cdots \cap Z_0'$ for $i > 0$ or $S_0 \cup T^{-1}(S_1)$ in case $i = 0$, we see that for $S, S' \in \mathcal{W}_i$, $S \cap S' = \emptyset$ if they are different, hence $\chi_{S\cap S'} = \delta_{S,S'}\chi_S$. Therefore the Hadamard product of $\mathcal{S}[[t;T]]$ can be written as

$$\Big(\sum_{i\geq 0} b_i t^i\Big) \odot \Big(\sum_{j\geq 0} b_j' t^j\Big) = \sum_{i\geq 0} (b_i b_i') t^i = \sum_{i\geq 0} \Big(\sum_{S\in\mathcal{W}_i} \lambda_S \mu_S \chi_S\Big) t^i.$$

In fact, we can also define an involution $^-$on $\mathcal{S}[[t;T]]$ given by

$$\overline{\sum_{i\geq 0} \Big(\sum_{S\in\mathcal{W}_i} \lambda_S \chi_S\Big) t^i} := \sum_{i\geq 0} \Big(\sum_{S\in\mathcal{W}_i} \overline{\lambda_S} \chi_S\Big) t^i.$$

These operations turn $\mathcal{S}[[t;T]]$ into a commutative $*$-algebra $(\mathcal{S}[[t;T]], \odot, ^-)$. Observe that $\mathcal{S}[[t;T]]$ is a $*$-regular algebra by Lemma 2.4.12. We show that it can be identified with the center of the algebra $\mathfrak{R}$, and also with a certain corner of $\mathfrak{R}$. To do so, we first fix some notation: we will denote the projections $\pi(\chi_C) \in \mathfrak{R}$ by $p_C$ for any clopen $C \subseteq X$.

[20] Recall that these are the sets of the form (2.2.7), together with the set $S_0 \cup T^{-1}(S_1)$.

**Proposition 2.4.15.** *We have an isomorphism of $*$-algebras $\mathcal{S}[[t;T]] \overset{\Psi}{\cong} Z(\mathfrak{R})$, the center of $\mathfrak{R}$. In particular, we have a $*$-isomorphism $\mathcal{S}[[t;T]] \cong p_E \mathfrak{R} p_E$ given by $d \mapsto \Psi(d) p_E$ for $d \in \mathcal{S}[[t;T]]$.*

*Proof.* Recalling Observation 2.2.15, the set $\mathcal{W}_i$ is in bijection with the set of all the $W \in \mathbb{V}$ having length $i+1$, so we can write an element $\sum_{i \geq 0} b_i t^i \in \mathcal{S}[[t;T]]$ as

$$\sum_{i\geq 0} b_i t^i = \sum_{i \geq 0} \Big( \sum_{S \in \mathcal{W}_i} \lambda_S \chi_S \Big) t^i = \sum_{W \in \mathbb{V}} (\lambda_{S(W)} \chi_{S(W)}) t^{|W|-1}.$$

By writing $\mathfrak{R} = \prod_{W \in \mathbb{V}} M_{|W|}(K)$, we have $Z(\mathfrak{R}) = \prod_{W \in \mathbb{V}} K$. We define a map $\Psi : \mathcal{S}[[t;T]] \to Z(\mathfrak{R})$ by

$$\sum_{i \geq 0} \Big( \sum_{S \in \mathcal{W}_i} \lambda_S \chi_S \Big) t^i = \sum_{W \in \mathbb{V}} (\lambda_{S(W)} \chi_{S(W)}) t^{|W|-1} \mapsto (\lambda_{S(W)} \cdot h_W)_W.$$

That is, the $W$-component of the vector $\Psi\Big( \sum_{i \geq 0} \Big( \sum_{S \in \mathcal{W}_i} \lambda_S \chi_S \Big) t^i \Big)$ is given by the diagonal matrix $\lambda_{S(W)} \cdot h_W$. It is straightforward to check that it is indeed an isomorphism of $*$-algebras. Since $Z(\mathfrak{R}) \cong p_E \mathfrak{R} p_E$ trivially, through the map $(\lambda_W \cdot h_W)_W \mapsto (\lambda_W \cdot e_{00}(W))_W$ (see the computations following Lemma 2.2.13), the result follows. □

Our next step is to prove the following formulas, which will be useful later.

**Lemma 2.4.16.** *For $A, B \in \mathcal{S}[[t;T]]$, the following formulas hold inside $\mathfrak{R}$:*

$$p_E \cdot \pi_+(A)^* \cdot p_{T^{-1}(E)} \cdot \pi_+(B) \cdot p_E = \Psi(\overline{A} \odot B) p_E,$$

$$p_{T^{-1}(E)} \cdot \pi_+(A) \cdot p_E \cdot \pi_+(B)^* \cdot p_{T^{-1}(E)} = \Psi(A \odot \overline{B}) p_{T^{-1}(E)}.$$

*Proof.* We only prove the first formula, being the second one analogous. It is enough to check it for a fixed component $W \in \mathbb{V}$, that is, it is enough to prove the equality

$$h_W \cdot p_E \cdot \pi_+(A)^* \cdot p_{T^{-1}(E)} \cdot \pi_+(B) \cdot p_E = \Psi(\overline{A} \odot B) h_W \cdot p_E$$

for a fixed $W \in \mathbb{V}$. Write $A = \sum_{i \geq 0} \Big( \sum_{S \in \mathcal{W}_i} \lambda_S \chi_S \Big) t^i$ and $B = \sum_{j \geq 0} \Big( \sum_{S' \in \mathcal{W}_j} \mu_{S'} \chi_{S'} \Big) t^j$. We first compute, for a fixed $S \in \mathcal{W}_i$ and by using Lemma 2.2.16, the terms

$$e_{|W|-1,|W|-1}(W) \cdot (h_W \cdot \chi_S t^i) \cdot e_{00}(W) = \delta_{W,W(S)} e_{i0}(W) = \delta_{W,W(S)} e_{|W|-1,0}(W).[21]$$

Therefore

$$\begin{aligned} & h_W \cdot p_E \cdot \pi_+(A)^* \cdot p_{T^{-1}(E)} \cdot \pi_+(B) \cdot p_E \\ & = e_{00}(W) \cdot \Big( \sum_{i \geq 0} \sum_{S \in \mathcal{W}_i} \lambda_S h_W \cdot \chi_S t^i \Big)^* \cdot e_{|W|-1,|W|-1}(W) \cdot \Big( \sum_{j \geq 0} \sum_{S' \in \mathcal{W}_j} \mu_{S'} h_W \cdot \chi_{S'} t^j \Big) \cdot e_{00}(W) \\ & = e_{00}(W) \cdot (\lambda_{S(W)} e_{|W|-1,0}(W))^* \cdot e_{|W|-1,|W|-1}(W) \cdot (\mu_{S(W)} e_{|W|-1,0}(W)) \cdot e_{00}(W) \\ & = \overline{\lambda_{S(W)}} \mu_{S(W)} e_{00}(W) = \Psi(\overline{A} \odot B) h_W \cdot p_E, \end{aligned}$$

so the result follows. □

In fact, these formulas can be generalized. In order to do this, we first need to define an idempotent map

$$P : \mathcal{B}_0[[t;T]] \to \mathcal{S}[[t;T]]$$

as follows.

**Lemma 2.4.17.** *With the above notation, there exists an idempotent linear map $P : \mathcal{B}_0[[t;T]] \to \mathcal{S}[[t;T]]$ such that for each $x \in \mathcal{B}_0[[t;T]]$, we have*

$$p_{T^{-1}(E)} \cdot \pi_+(x) \cdot p_E = p_{T^{-1}(E)} \cdot \pi_+(P(x)) \cdot p_E.$$

[21] Note that the condition $\delta_{W,W(S)}$ already encodes the fact that the term is 0 if $|W| \neq i+1$.

*Proof.* For $i \geq 1$, let $V_i$ be the linear subspace of $\mathcal{B}_i$ given by $\text{span}\{\chi_C \mid C \in \mathcal{W}_i\}$, and let $V_i'$ be the linear subspace of $\mathcal{B}_i$ spanned by all the projections $\chi_C$, where

(∗) $C$ is a nonempty clopen subset of $X$ of the form (2.2.6) with either $s > i$ or $r > 0$.

Claim 1: $\mathcal{B}_i = V_i + V_i'$.

As observed after (2.2.6), $\mathcal{B}_i$ is spanned by all the functions $\chi_C$, where $C$ is a clopen subset of $X$ of the form (2.2.6) with $s \geq i$. If $s > i$ or $r > 0$, then $C$ is of the form (∗), so that $\chi_C \in V_i'$. So we can assume that $s = i$ and $r = 0$. Furthermore, if $T^i(E) \cap C \cap T^{-1}(E)$ is nonempty, then $C \in \mathcal{W}_i$ and so $\chi_C \in V_i$. So we can further assume that $T^i(E) \cap C \cap T^{-1}(E) = \emptyset$.
We can then write

$$C = \Big(T^i(E) \cap C\Big) \sqcup \Big( \bigsqcup_{Z \in \mathcal{P}} T^i(Z) \cap C\Big).$$

If $T^i(E) \cap C = \emptyset$, then $\chi_C$ is a sum of terms of the form (∗), so that $\chi_C \in V_i'$. If $T^i(E) \cap C \neq \emptyset$, we can further decompose $C$ as

$$\begin{aligned} C &= \Big(T^i(E) \cap C\Big) \sqcup \Big( \bigsqcup_{Z \in \mathcal{P}} T^i(Z) \cap C\Big) \\ &= \Big(T^i(E) \cap C \cap T^{-1}(E)\Big) \sqcup \Big( \bigsqcup_{Z \in \mathcal{P}} T^i(E) \cap C \cap T^{-1}(Z)\Big) \sqcup \Big( \bigsqcup_{Z \in \mathcal{P}} T^i(Z) \cap C\Big) \\ &= \Big( \bigsqcup_{Z \in \mathcal{P}} T^i(E) \cap C \cap T^{-1}(Z)\Big) \sqcup \Big( \bigsqcup_{Z \in \mathcal{P}} T^i(Z) \cap C\Big) \end{aligned}$$

using that $T^i(E) \cap C \cap T^{-1}(E) = \emptyset$. Note that for each $Z \in \mathcal{P}$, either $C \cap T^{-1}(Z)$ is empty or it is of the form (∗); in the latter case we can write $T^i(E) \cap C \cap T^{-1}(Z)$ as

$$T^i(E) \cap C \cap T^{-1}(Z) = \Big(C \cap T^{-1}(Z)\Big) \Big\backslash \Big( \bigsqcup_{Z' \in \mathcal{P}} T^i(Z') \cap C \cap T^{-1}(Z)\Big),$$

if nonempty. Therefore $\chi_C$ is a linear combination of terms of the form (∗), and thus $\chi_C \in V_i'$.

Claim 2: $V_i \cap V_i' = \{0\}$.

Assume that $b \in V_i \cap V_i'$. Then we can write $b = \sum_{C \in \mathcal{W}_i} \lambda_C \chi_C$, with $\lambda_C \in K$. Since $b \in V_i'$ we have

$$0 = \Big( \sum_{C' \in \mathcal{W}_i} \chi_{T^i(E) \cap C' \cap T^{-1}(E)} \Big) b = \Big( \sum_{C' \in \mathcal{W}_i} \chi_{T^i(E) \cap C' \cap T^{-1}(E)} \Big) \Big( \sum_{C \in \mathcal{W}_i} \lambda_C \chi_C \Big) = \sum_{C \in \mathcal{W}_i} \lambda_C \chi_{T^i(E) \cap C \cap T^{-1}(E)}$$

But now observe that $\{\chi_{T^i(E) \cap C \cap T^{-1}(E)}\}_{C \in \mathcal{W}_i}$ is a family of mutually orthogonal nonzero projections, so we get that $\lambda_C = 0$ for all $C \in \mathcal{W}_i$, and so $b = 0$.

Therefore $\mathcal{B}_i = V_i \oplus V_i'$ for $i \geq 1$. In the base case $i = 0$, we need to distinguish between two different cases, depending on whether the intersection $E \cap T^{-1}(E)$ is empty or not. Recall that the special term of degree 0 is given by $\chi_{S_0 \cup T^{-1}(S_1)}$, where

$$S_0 = E \cup \Big( \bigcup_{\substack{Z \in \mathcal{P} \\ Z \cap T^{-1}(E) \neq \emptyset}} Z \Big) \quad \text{and} \quad S_1 = E \cup \Big( \bigcup_{\substack{Z \in \mathcal{P} \\ T^{-1}(Z) \cap E \neq \emptyset}} Z \Big).$$

$E \cap T^{-1}(E) = \emptyset$: In this case, we take $V_0'$ to be any linear subspace of $\mathcal{B}_0$ complementing $K \cdot \chi_{S_0 \cup T^{-1}(S_1)}$ and containing the family of pairwise orthogonal projections $\chi_{Z \cap T^{-1}(Z')}$, for $Z, Z' \in \mathcal{P}$ satisfying $Z \cap T^{-1}(E) = \emptyset$ and $T^{-1}(Z') \cap E = \emptyset$ (observe that this family is orthogonal to the element $\chi_{S_0 \cup T^{-1}(S_1)}$), so that $\mathcal{B}_0 = K \cdot \chi_{S_0 \cup T^{-1}(S_1)} \oplus V_0'$.

$E \cap T^{-1}(E) \neq \emptyset$: Define here $V_0'$ to be the linear subspace of $\mathcal{B}_0$ spanned by all the projections $\chi_C$, where

$$(*) \quad C \text{ is a nonempty clopen subset of } X \text{ of the form (2.2.6),}$$

and let $V_0 = K \cdot \chi_{E\cap T^{-1}(E)}$. Analogous computations as in the case for $i \geq 1$ show that there is a decomposition $\mathcal{B}_0 = V_0 \oplus V_0'$. In particular, $V_0'$ has codimension 1. If we show that $\chi_{S_0\cup T^{-1}(S_1)} \notin V_0'$ then, since $K \cdot \chi_{S_0\cup T^{-1}(S_1)}$ has dimension 1, we necessarily obtain the decomposition $\mathcal{B}_0 = K \cdot \chi_{S_0\cup T^{-1}(S_1)} \oplus V_0'$. But if $\chi_{S_0\cup T^{-1}(S_1)} \in V_0'$, we could write it as

$$\chi_{S_0\cup T^{-1}(S_1)} = \sum_{r,s\geq 0} \sum_{\text{some } Z_i} \lambda_{r,s,Z_i} \chi_{T^{-r}(Z_{-r})\cap\cdots\cap T^{-1}(Z_{-1})\cap Z_0\cap\cdots\cap T^{s-1}(Z_{s-1})}.$$

Multiplying by $\chi_{E\cap T^{-1}(E)}$ we would get $\chi_{E\cap T^{-1}(E)} = \chi_{E\cap T^{-1}(E)} \cdot \chi_{S_0\cup T^{-1}(S_1)} = 0$, a contradiction. Hence $\chi_{S_0\cup T^{-1}(S_1)} \notin V_0'$, and we have the desired decomposition.

We can now define $P$ as the projection onto the first component in the decomposition

$$\mathcal{B}_0[[t;T]] = \mathcal{S}[[t;T]] \oplus \Big(\prod_{i\geq 0} V_i' t^i\Big).$$

We check the last condition in the statement. Take $x \in \mathcal{B}_0[[t;T]]$, $x = \sum_{i\geq 0} b_i t^i$. We can write it as

$$x = P(x) + \sum_{i\geq 0} \sum_{C \text{ as in } (*)} \lambda_C \chi_C t^i$$

in the case $E \cap T^{-1}(E) \neq \emptyset$, and as

$$x = P(x) + \sum_{\substack{\text{some } C \\ \chi_C \in V_0'}} \lambda_C \chi_C + \sum_{i\geq 1} \sum_{C \text{ as in } (*)} \lambda_C \chi_C t^i$$

if $E \cap T^{-1}(E) = \emptyset$. Note that in the latter case, $p_{T^{-1}(E)} \cdot \chi_C \cdot p_E = 0$ for all $\chi_C \in V_0'$; hence to prove the formula it is enough to check that, for a fixed $W = E \cap T^{-1}(Z_1) \cap \cdots \cap T^{-k+1}(Z_{k-1}) \cap T^{-k}(E) \in \mathbb{V}$, and $C$ of the form $(*)$, then $h_W \cdot p_{T^{-1}(E)} \cdot \chi_C t^i \cdot p_E = 0$. We compute

$$h_W \cdot p_{T^{-1}(E)} \cdot \chi_C t^i \cdot p_E = \chi_{T^{k-1}(W)\cap T^{-1}(E)\cap C\cap T^i(E)} t^i$$

This is zero in case $C$ is of the form $(*)$, since either $C \subseteq T^i(X\backslash E)$ or $C \subseteq T^{-1}(X\backslash E)$. The result follows. □

We can now generalize the formulas in Lemma 2.4.16.

**Lemma 2.4.18.** *For $x, y \in \mathcal{B}_0[[t;T]]$, the following formulas hold:*

$$p_E \cdot \pi_+(x)^* \cdot p_{T^{-1}(E)} \cdot \pi_+(y) \cdot p_E = \Psi(\overline{P(x)} \odot P(y)) p_E,$$

$$p_{T^{-1}(E)} \cdot \pi_+(x) \cdot p_E \cdot \pi_+(y)^* \cdot p_{T^{-1}(E)} = \Psi(P(x) \odot \overline{P(y)}) p_{T^{-1}(E)}.$$

*Proof.* By Lemma 2.4.17 we have $p_{T^{-1}(E)} \cdot \pi_+(x) \cdot p_E = p_{T^{-1}(E)} \cdot \pi_+(P(x)) \cdot p_E$ for all $x \in \mathcal{B}_0[[t;T]]$. Taking the involution on both sides, we get that $p_E \cdot \pi_+(x)^* \cdot p_{T^{-1}(E)} = p_E \cdot \pi_+(P(x))^* \cdot p_{T^{-1}(E)}$ for all $x \in \mathcal{B}_0[[t;T]]$. Now we obtain form Lemma 2.4.16

$$p_E \cdot \pi_+(x)^* \cdot p_{T^{-1}(E)} \cdot \pi_+(y) \cdot p_E = p_E \cdot \pi_+(P(x))^* \cdot p_{T^{-1}(E)} \cdot \pi_+(P(y)) \cdot p_E = \Psi(\overline{P(x)} \odot P(y)) p_E,$$

as desired. □

Recall that $\mathcal{S}[[t;T]]$ is a unital $*$-regular commutative algebra under the product $\odot$. The unit is of course the element $e = \sum_{i\geq 0}(\sum_{C\in\mathcal{W}_i} \chi_C) t^i$. We obtain:

**Proposition 2.4.19.** *Take $s = \sum_{Z\in\mathcal{P}} \chi_Z t = \chi_{X\setminus E} t \in \mathcal{B}_1 t$. Then $u := (1-s)^{-1} = 1 + s + s^2 + \cdots \in \mathcal{B}_0[[t;T]]$ satisfies that $P(u) = e$, where $e$ is the unit element of $(\mathcal{S}[[t;T]], \odot)$. We have the formulas*

$$p_E \cdot \pi_+(u)^* \cdot p_{T^{-1}(E)} \cdot \pi_+(x) \cdot p_E = \Psi(P(x))p_E,$$

$$p_{T^{-1}(E)} \cdot \pi_+(x) \cdot p_E \cdot \pi_+(u)^* \cdot p_{T^{-1}(E)} = \Psi(P(x))p_{T^{-1}(E)}$$

*for all $x \in \mathcal{B}_0[[t;T]]$. In particular, inside $\mathcal{R}_\mathcal{B}$, the ideal generated by $p_E$ coincides with the ideal generated by $p_{T^{-1}(E)}$.*

*Proof.* We first note that

$$s^i = (\chi_{X\setminus E} t)^i = \chi_{X\setminus (E\cup\cdots\cup T^{i-1}(E))} t^i = \sum_{Z_0,Z_1,\dots,Z_{i-1}\in\mathcal{P}} \chi_{Z_0\cap T(Z_1)\cap\cdots\cap T^{i-1}(Z_{i-1})} t^i$$

for $i \geq 1$. It follows from this formula and the computations done in Lemma 2.4.17 that $P(s^i) = \sum_{C\in\mathcal{W}_i} \chi_C t^i$. To compute $P(1)$, we first observe that

$$1 = \chi_{S_0\cup T^{-1}(S_1)} + \chi_{X\setminus S_0 \cap X\setminus T^{-1}(S_1)} = \chi_{S_0\cup T^{-1}(S_1)} + \sum_{\substack{Z\in\mathcal{P} \\ Z\cap T^{-1}(E)=\emptyset}} \sum_{\substack{Z'\in\mathcal{P} \\ T^{-1}(Z')\cap E=\emptyset}} \chi_{Z\cap T^{-1}(Z')}.$$

By definition, the second part of this expression belongs to the complement $V_0'$, so that $P(1) = \chi_{S_0\cup T^{-1}(S_1)}$. Putting everything together, it is clear that $P(u) = e$. Now the desired formulas follow from Lemma 2.4.18. In particular, we have

$$p_E \cdot \pi_+(u)^* \cdot p_{T^{-1}(E)} \cdot \pi_+(u) \cdot p_E = \Psi(P(u))p_E = \Psi(e)p_E = p_E,$$

$$p_{T^{-1}(E)} \cdot \pi_+(u) \cdot p_E \cdot \pi_+(u)^* \cdot p_{T^{-1}(E)} = \Psi(P(u))p_{T^{-1}(E)} = \Psi(e)p_{T^{-1}(E)} = p_{T^{-1}(E)}.$$

One deduces from this that the ideal generated by $p_E$ coincides with the ideal generated by $p_{T^{-1}(E)}$ inside $\mathcal{R}_\mathcal{B}$. ∎

We now define a $*$-algebra $\mathcal{Q}$ which is similar to the one defined in [6, Lemma 6.10].

**Definition 2.4.20.** With the above notation, we define the $*$-algebra $\mathcal{Q}$ as the $*$-regular closure of $P(\mathcal{D}_+)$ in the $*$-regular algebra $(\mathcal{S}[[t;T]], \odot, ^-)$. In other words, $\mathcal{Q}$ is the smallest $*$-regular subalgebra of $\mathcal{S}[[t;T]]$ containing $P(\mathcal{D}_+)$.

This $*$-algebra uncovers a portion of $\mathcal{R}_\mathcal{B}$, as follows:

**Proposition 2.4.21.** *We have an embedding $\Psi(\mathcal{Q})p_E \subseteq p_E \mathcal{R}_\mathcal{B} p_E$. In particular, $\Psi(\mathcal{Q})p_E \subseteq \mathcal{R}_\mathcal{B}$.*

*Proof.* Let $x \in \mathcal{D}_+$. Due to Lemma 2.4.18, we obtain

$$\Psi(P(x))p_E = \Psi(P(u) \odot P(x)) = p_E \cdot \pi_+(u)^* \cdot p_{T^{-1}(E)} \cdot \pi_+(x) \cdot p_E.$$

By Proposition 2.4.13, we have that $\pi_+(\mathcal{D}_+) \subseteq \mathcal{R}_\mathcal{B}$, so all the factors of the right hand side of the above equality belong to $\mathcal{R}_\mathcal{B}$. It follows that $\Psi(P(x))p_E \in p_E\mathcal{R}_\mathcal{B}p_E$, and so $\Psi(P(\mathcal{D}_+)) \subseteq p_E\mathcal{R}_\mathcal{B}p_E$.

By Proposition 2.4.15, the map $d \mapsto \Psi(d)p_E$ defines a $*$-isomorphism from $\mathcal{S}[[t;T]]$ onto $p_E\mathfrak{R}p_E$. It follows that $\Psi(\mathcal{Q})p_E$ is the $*$-regular closure of $\Psi(P(\mathcal{D}_+))p_E$ in $p_E\mathfrak{R}p_E$. Since $\Psi(P(\mathcal{D}_+))p_E \subseteq p_E\mathcal{R}_\mathcal{B}p_E$ and $p_E\mathcal{R}_\mathcal{B}p_E$ is $*$-regular, we obtain that $\Psi(\mathcal{Q})p_E \subseteq p_E\mathcal{R}_\mathcal{B}p_E$. This show the result. ∎

Observe that it is not clear whether $P(\mathcal{D}_+)$ is a subalgebra of $\mathcal{S}[[t;T]]$. Nevertheless, this is not relevant for our result.

**Remark 2.4.22.** At this point we may define a $*$-subalgebra $\mathcal{E}$ of $\mathcal{R}_\mathcal{B}$ as the $*$-subalgebra of $\mathcal{R}_\mathcal{B}$ generated by $\mathfrak{D}$ and $\Psi(\mathcal{Q})p_E$. However, we believe that in general this algebra will be smaller than $\mathcal{R}_\mathcal{B}$ (equivalently, will not be regular). It seems that in order to advance in the understanding of $\mathcal{R}_\mathcal{B}$ we need to study their ideals.

# Chapter 3

# Special cases: the lamplighter group algebra and the odometer algebra

In this chapter we are going to apply our construction from Chapter 2 in order to study some group algebras arising as $\mathbb{Z}$-crossed product algebras, such as the lamplighter group algebra and the odometer algebra. From now on, $(K, *)$ will denote a field with positive definite involution.

First of all, we will show how one can relate the group algebra of some special groups $G$ by means of a $\mathbb{Z}$-crossed product algebra through Fourier transform, sometimes called Pontryagin duality (Proposition 3.1.1). In this setting, the Atiyah problem for our group algebra (see Section 1.1.4) will be translated to a problem on computing ranks inside the $\mathbb{Z}$-crossed product algebra.

After that, we will apply our methods in order to study the lamplighter group algebra $K[\Gamma]$. This algebra is important because, among other things, it gave the first counterexample to the Strong Atiyah Conjecture (Conjecture 1.1.14), see for example [42], [25]. One of our main results in that section is Theorem 3.2.10, which gives a class of irrational (and even transcendental) numbers that appear as von Neumann dimensions of elements of matrix algebras over $K[\Gamma]$.

To end this chapter, we study the case of the 2-odometer algebra $C_K(X) \rtimes_T \mathbb{Z}$ with $X = \prod_{i\in\mathbb{N}}\{0,1\}$, where $T$ is the automorphism $X \to X$ given by addition of $(1, 0, ...)$ with carry over. In this case, it is not possible to realize this crossed product algebra as a group algebra since the crossed product algebra is simple (cf. [17]), but this example is interesting in its own right because the dynamical system defining it is a minimal one, hence we have been able to completely determine the structure of its $*$-regular closure in Theorem 3.3.8, and thus giving a complete description of the set of $l^2$-Betti numbers arising from the algebra $K[\mathbb{Z}(2^\infty)] \rtimes_\rho \mathbb{Z}$ in Theorem 3.3.9. More generally, we also study the case of the $\overline{n}$-odometer algebra, being $\overline{n} = (n_i)_i$ a sequence of natural numbers $n_i \geq 2$, and thus determining its $*$-regular closure in Theorem 3.4.4 and its set of $l^2$-Betti numbers in Theorem 3.4.6. The general $\overline{n}$-odometer algebra has been widely studied, see for instance [24, Chapter VIII.4], [87] for $C^*$-algebraic versions of it.

## 3.1 Relating our construction with some group algebras and the Atiyah problem

Let $G$ be a countable, discrete group, and let $K$ be, throughout this section, a subfield of $\mathbb{C}$ closed under complex conjugation. We first recall some constructions explained in Chapter 1. Consider $K[G]$ the group algebra of $G$. We denote by $\mathcal{N}(G)$ the von Neumann algebra of $G$ and by $\mathcal{U}(G)$ the classical ring of quotients of $\mathcal{N}(G)$ (Sections 1.1.2 and 1.2.2). We have inclusions $\mathcal{A} \subseteq \mathcal{N}(G) \subseteq \mathcal{U}(G)$.

$\mathcal{U}(G)$ is a $*$-regular ring possessing a rank function $\mathrm{rk}_{\mathcal{U}(G)}$ (Lemma 1.2.15), so the algebra $K[G]$ becomes also a rank ring (but not regular) when endowed with the rank function inherited from $\mathcal{U}(G)$, which we will denote by $\mathrm{rk}_{K[G]}$. We briefly recall how one can construct the rank function $\mathrm{rk}_{\mathcal{U}(G)}$, and more generally its extension to $M_n(\mathcal{U}(G))$.

The von Neumann algebra $\mathcal{N}(G)$ carries a natural normal, faithful and positive trace $\mathrm{tr}_{\mathcal{N}(G)}$ (Proposition 1.1.4). Since $\mathcal{U}(G)$ is $*$-regular, for every element $u \in \mathcal{U}(G)$ there exist unique projections $LP(u), RP(u) \in \mathcal{U}(G)$ such that $u\mathcal{U}(G) = LP(u) \cdot \mathcal{U}(G)$ and $\mathcal{U}(G)u = \mathcal{U}(G) \cdot RP(u)$ (Theorem 1.2.11). Both $LP(u), RP(u)$ belong to the von Neumann algebra $\mathcal{N}(G)$, so we define the rank function $\mathrm{rk}_{\mathcal{U}(G)}$ by the rule

$$\mathrm{rk}_{\mathcal{U}(G)}(u) := \mathrm{tr}_{\mathcal{N}(G)}(LP(u)) = \mathrm{tr}_{\mathcal{N}(G)}(RP(u)).$$

All of this can be extended to $k \times k$ matrices (cf. Remark 1.1.5), so we obtain a unique Sylvester matrix rank function $\mathrm{Rk}_{\mathcal{U}(G)}$ over matrix algebras $\mathcal{U}_k(G) = M_k(\mathcal{U}(G))$ extending canonically $\mathrm{rk}_{\mathcal{U}(G)}$, so that

$$\mathrm{Rk}_{\mathcal{U}(G)}(U) := \mathrm{Tr}_{\mathcal{N}_k(G)}(LP(U)) = \mathrm{Tr}_{\mathcal{N}_k(G)}(RP(U))$$

for any matrix $U \in \mathcal{U}_k(G)$, where $LP(U)$ and $RP(U)$ are the left and right projections of $U$ inside $\mathcal{U}_k(G)$, respectively. In particular, we obtain a Sylvester matrix rank function $\mathrm{Rk}_{K[G]}$ over $K[G]$ by restricting the previous Sylvester matrix rank function $\mathrm{Rk}_{\mathcal{U}(G)}$ on each matrix algebra $M_k(K[G]) \subseteq \mathcal{U}_k(G)$. We will see in the next subsection that, when the group algebra can be identified with a $\mathbb{Z}$-crossed product algebra, this Sylvester matrix rank function coincides with the Sylvester matrix rank function given in Theorem 2.3.7 when considering a suitable measure $\mu$ on a suitably constructed totally disconnected, metrizable compact space $X$.

Recall Definition 1.1.12: a real positive number $r$ is called an $l^2$-Betti number arising from $G$ with coefficients in $K$ if for some $k \geq 1$, there exists a matrix operator $A \in M_k(K[G])$ such that

$$\dim_{vN}(\ker(A)) = \mathrm{Tr}_{\mathcal{N}_k(G)}(p_A) = r,$$

where $p_A \in \mathcal{N}_k(G)$ denotes the orthogonal projection onto the kernel of $A$. Equivalently, one can also compute the von Neumann dimension by the formula

$$\dim_{vN}(\ker(A)) = k - \mathrm{Rk}_{K[G]}(A).$$

We denote by $\mathcal{C}(G,K)$ the set of all $l^2$-Betti numbers arising from $G$ with coefficients in $K$, so that

$$\mathcal{C}(G,K) = \mathrm{Rk}_{K[G]}\Big(\bigcup_{i=1}^{\infty} M_i(K[G])\Big) \subseteq \mathbb{R}^+.$$

Since $\mathcal{U}(G)$ is $*$-regular (Theorem 1.2.16), we can consider the $*$-regular closure of the group algebra $K[G]$ inside $\mathcal{U}(G)$, denoted by $\mathcal{R}_{K[G]} = \mathcal{R}(K[G],\mathcal{U}(G))$. It will help us later on when studying the set $\mathcal{C}(G,K)$, see Proposition 2.4.1; the same proof in this context states that the subgroup of $(\mathbb{R},+)$ generated by $\mathcal{C}(G,K)$ coincides with the subgroup of $(\mathbb{R},+)$ generated by the set

$$\mathrm{Rk}_{\mathcal{U}(G)}(\mathcal{R}_{K[G]}) = \{\mathrm{Rk}_{\mathcal{U}(G)}(r) \mid r \in \mathcal{R}_{K[G]}\}.$$

### 3.1.1 Some group algebras arising as $\mathbb{Z}$-crossed product algebras

We can use our given construction to study $K[G]$ in some special cases of interest by using Pontryagin duality, as follows. For a topological, second countable, locally compact abelian group $H$ one can define its Pontryagin dual $\widehat{H}$, which is the set of continuous homomorphisms $\phi : H \to \mathbb{T}$, also called characters. With the compact convergence topology, it is well-known that $\widehat{H}$ becomes a topological, metrizable, locally compact abelian group. If $H$ is a countable discrete group then $\widehat{H}$ is compact, and if moreover $H$ is a torsion group then $\widehat{H}$ is totally disconnected. We refer the reader to [35, Chapter 4] for more information about Pontryagin duality and the proofs of all these statements.

Suppose now that $H$ is a countable discrete, torsion abelian group, and that $\mathbb{Z}$ acts on $H$ by automorphisms via $\rho : \mathbb{Z} \curvearrowright H$. We write $G$ for the semi-direct product group $H \rtimes_\rho \mathbb{Z}$, so $G$ is generated by $t$ (the generator of the $\mathbb{Z}$-part) and by any set $S$ consisting of generators of $H$[1]. We also denote by $\widetilde{\rho} : \mathbb{Z} \curvearrowright K[H]$ the action on the group algebra $K[H]$ which extends $\rho$ by linearity.

This action $\rho$ induces an action $\widehat{\rho}$ of $\mathbb{Z}$ on $\widehat{H}$ by continuous functions, defined by

$$\widehat{\rho}(n)(\phi)(h) := \phi(\rho(-n)(h))$$

for $n \in \mathbb{Z}$, $\phi \in \widehat{H}$ and $h \in H$, where now $\widehat{H}$ becomes a totally disconnected, compact metrizable space. In particular, if we put $T := \widehat{\rho}(1)$, then the action $\widehat{\rho}$ is generated by $T$, in the sense that

$$\widehat{\rho}(n)(\phi) = T^n(\phi) \quad \text{for any } \phi \in \widehat{H} \text{ and } n \in \mathbb{Z}.$$

Note that $T : \widehat{H} \to \widehat{H}$ defines a homeomorphism of the space $\widehat{H}$.

[1] This crossed product construction can be generalized by replacing $\mathbb{Z}$ with any other countable discrete group $\Lambda$, as in [8, Section 2]. However, we will stick into the $\Lambda = \mathbb{Z}$ case for our purposes.

**Proposition 3.1.1.** *Let $\mathcal{O} \subseteq \mathbb{N}$ be the set $\mathcal{O} = \{n \in \mathbb{N} \mid$ there exists an element $g \in H$ of order $n\}$. Assume that, for any $n \in \mathcal{O}$, $K$ contains all the $n^{th}$-roots of unity. We will denote by $\xi_n$ a primitive one.*

*Then we can identify the group algebra $K[G] \cong K[H] \rtimes_{\widetilde{\rho}} \mathbb{Z}$ with $C_K(\widehat{H}) \rtimes_T \mathbb{Z}$ via the Fourier transform*

$$\mathscr{F} : K[H] \to C_K(\widehat{H})$$

*by sending an element $h \in H$ of order $n$ to the element*

$$\chi_{U_{h,0}} + \overline{\xi_n}\chi_{U_{h,1}} + \cdots + \overline{\xi_n}^j \chi_{U_{h,j}} + \cdots + \overline{\xi_n}^{n-1} \chi_{U_{h,n-1}}$$

*where each $U_{h,j} = \{\phi \in \widehat{H} \mid \phi(h) = \xi_n^j\}$, and $\chi_{U_{h,j}}$ denotes the characteristic function of the clopen $U_{h,j}$, and then extending it to a map $K[H] \rtimes_{\widetilde{\rho}} \mathbb{Z} \to C_K(\widehat{H}) \rtimes_T \mathbb{Z}$ by sending $t$ to the generator of $\mathbb{Z}$, also denoted by $t$, on the $\mathbb{Z}$-crossed product.*

See [8], also [40] where the author states an analogous result but in the realm of measured groupoids. Before giving the proof, it is instructive to look at a specific example; in fact, this example was the motivation of most of the constructions and results presented in this thesis. This is an example that Elek also studied in [29].

**Example 3.1.2.** The lamplighter group $\Gamma$ is defined to be the wreath product of the finite group of two elements $\mathbb{Z}_2$ by $\mathbb{Z}$. In other words,

$$\Gamma = \mathbb{Z}_2 \wr \mathbb{Z} = \Big(\bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_2\Big) \rtimes_\sigma \mathbb{Z}$$

where the semidirect product is taken with respect to the Bernoulli shift, that is $\sigma : \mathbb{Z} \curvearrowright \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_2$ is defined by the rule

$$\sigma(n)(a)_i = a_{i+n} \quad \text{for } a = (a_i)_i \in \bigoplus_{i\in\mathbb{Z}} \mathbb{Z}_2.$$

In terms of generators and relations, if we denote by $t$ the generator corresponding to $\mathbb{Z}$, by $a_i$ the generator corresponding to the $i^{\text{th}}$ copy of $\mathbb{Z}_2$, and by 1 the unit element of $\Gamma$, we have the relations $a_i^2 = 1, a_i a_j = a_j a_i$ and $t a_i t^{-1} = a_{i-1}$ for any $i, j \in \mathbb{Z}$. Notationally, we have the following presentation for $\Gamma$:

$$\Gamma = \langle t, \{a_i\}_{i\in\mathbb{Z}} \mid a_i^2,\ a_i a_j a_i a_j,\ t a_i t^{-1} a_{i-1} \text{ for } i, j \in \mathbb{Z}\rangle.$$

Now the Fourier transform gives a $*$-isomorphism

$$\mathscr{F} : K[\Gamma] \to C_K(X) \rtimes_T \mathbb{Z}$$

where here $X = \prod_{i\in\mathbb{Z}}\{0,1\}$ is the Cantor set and $T$ is the shift map, namely $T(x)_i = x_{i+1}$ for $x \in X$. The isomorphism is given by the identifications

$$1 \mapsto \chi_X, \quad t \mapsto t, \quad a_i \mapsto \chi_{U_i} - \chi_{\overline{U_i}} \quad \Big(\text{or equivalently } \frac{1+a_i}{2} \mapsto \chi_{U_i}\Big)$$

where $U_i = \{x \in X \mid x_i = 0\}$ and $\overline{U_i}$ is its complement in $X$. Note that, in particular, the elements $e_i = \frac{1+a_i}{2}$ are idempotents in $K[\Gamma]$, and so are $f_i = 1 - e_i$. They correspond to the characteristic functions of the clopen sets consisting of all the elements in $X$ having a 0 (resp. a 1) at the $i^{\text{th}}$ component.

*Proof of Proposition 3.1.1.* Write $\mu$ for the counting measure of $H$. Then the Fourier transform $\mathscr{F} : K[H] \to C_K(\widehat{H})$ is defined by the formula

$$\mathscr{F}(f)(\phi) = \int_H f(h)\overline{\phi(h)}d\mu(h), \quad f \in K[H], \phi \in \widehat{H}.$$

In fact, $\mathscr{F}$ is defined even for functions $f \in L^1(H,\mu)$, and in such a case its range lies in $C_0(\widehat{H})$ (see [35, Proposition 4.13]). However, in our situation, for a given element $h \in H$ we think of it as a function in $L^1(H,\mu)$ by the usual Kronecker delta function $h(g) = \delta_{h,g}$; we have

$$\mathscr{F}(h)(\phi) = \overline{\phi(h)}$$

and it is straightforward to check that indeed $\mathscr{F}(h) = \chi_{U_{h,0}} + \cdots + \overline{\xi_n}^{n-1}\chi_{U_{h,n-1}}$, so $\mathscr{F} : K[H] \to C_K(\widehat{H})$ is well-defined. It is well-known that the Fourier transform respects the corresponding products and involutions, and by [35, Theorem 4.21] there exists a unique suitably normalized Haar measure $\widehat{\mu}$ on $\widehat{H}$ (in fact, since $H$ is discrete and $\mu$ is the counting measure, $\widehat{\mu}$ corresponds to the normalized Haar measure, so that $\widehat{\mu}(\widehat{H}) = 1$) such that $\mathscr{F}$ defines a unitary isomorphism

$$\mathscr{F} : L^2(H, \mu) \to L^2(\widehat{H}, \widehat{\mu})$$

whose inverse map $\mathscr{F}^{-1}$ is given by integration agains $\widehat{\mu}$:

$$\mathscr{F}^{-1}(\widehat{f})(h) = \int_{\widehat{H}} \widehat{f}(\phi)\phi(h)d\widehat{\mu}(\phi), \quad \widehat{f} \in L^2(\widehat{H}, \widehat{\mu}), h \in H.$$

To conclude the proof, it is enough to check that $\mathscr{F}^{-1} : C_K(\widehat{H}) \to K[H]$ is also well-defined. Take then $\widehat{f} = \chi_U \in C_K(\widehat{H})$ and fix an $h \in H$ with order $n$. We compute

$$\mathscr{F}^{-1}(\chi_U)(h) = \int_U \phi(h)d\widehat{\mu}(\phi) = \sum_{j=0}^{n-1} \int_{U\cap U_{h,j}} \phi(h)d\widehat{\mu}(\phi) = \sum_{j=0}^{n-1} \xi_n^j \cdot \widehat{\mu}(U \cap U_{h,j}).$$

We only need to check that this computation gives 0 for all but finitely many $h$'s. Now, the clopen sets $U_{g,i}$ form a subbasis for the topology of $\widehat{H}$, i.e. the sets

$$U_{(g_1,i_1;\dots;g_l,i_l)} = U_{g_1,i_1} \cap \cdots \cap U_{g_l,i_l}, \qquad g_t \in H, \quad 0 \leq i_t \leq o(g_t) - 1, \quad 1 \leq t \leq l,$$

generate the topology of $\widehat{H}$, where $o(g)$ stands for the order of $g$. For a clopen $U \subseteq \widehat{H}$, it is immediate to see that $U$ can be written as a finite disjoint union of sets of the previous form, so it is enough to restrict our attention to the case when $U = U_{g_1,i_1} \cap \cdots \cap U_{g_l,i_l}$. Much more, since $\mathscr{F}^{-1}$ is a homomorphism,

$$\mathscr{F}^{-1}(\chi_U) = \mathscr{F}^{-1}(\chi_{U_{g_1,i_1}}) \cdots \mathscr{F}^{-1}(\chi_{U_{g_l,i_l}}),$$

so we may further assume that $U = U_{g,i}$ for some $g \in H$ and $0 \leq i \leq o(g) - 1$. We need to describe the values $\widehat{\mu}(U \cap U_{h,j}) = \widehat{\mu}(U_{g,i} \cap U_{h,j})$. Write $m = o(g)$ and recall that $n = o(h)$, so that $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$. We can further assume that $i = 0$. To see this, we first define a character on the finite group $\langle g \rangle$ by the rule

$$\phi_g : \langle g \rangle \to \mathbb{T}, \quad \phi_g(g) = \xi_m$$

Since $\mathbb{T}$ is a *divisible* group, it is injective in the category of abelian groups, so the previous character extends to a character on $H$, which we still denote by $\phi_g$. In this situation, by using left invariance of the Haar measure $\widehat{\mu}$, we get

$$\widehat{\mu}(U_{g,i} \cap U_{h,j}) = \widehat{\mu}(\overline{\phi}_g^i \cdot (U_{g,i} \cap U_{h,j})) = \widehat{\mu}(U_{g,0} \cap U_{h,j+N \text{ (mod } n)}),$$

so the effect of 'moving' the set $U_{g,i} \cap U_{h,j}$ by $\overline{\phi}_g^i$ results in an overall shift in $j$ by a fixed amount $N$, depending only on $\phi_g$ and $i$. Therefore

$$\mathscr{F}^{-1}(\chi_{U_{g,i}})(h) = \sum_{j=0}^{n-1} \xi_n^j \cdot \widehat{\mu}(U_{g,0} \cap U_{h,j+N \text{ (mod } n)}) = \overline{\xi}_n^N \cdot \sum_{j=0}^{n-1} \xi_n^j \cdot \widehat{\mu}(U_{g,0} \cap U_{h,j}) = \overline{\xi}_n^N \mathscr{F}^{-1}(\chi_{U_{g,0}})(h).$$

So from now on we will assume that $U = U_{g,0}$.

Consider now the finite group $\langle g \rangle \cap \langle h \rangle$, which is generated by some power of the element $h$, so we write

$$\langle g \rangle \cap \langle h \rangle = \langle h^k \rangle \quad \text{for some } 1 \leq k \leq n \text{ and } k \text{ dividing } n.$$

Note that in this case, the intersection $U_{g,0} \cap U_{h,j}$ is empty if $j \notin \frac{n}{k}\mathbb{Z}$; hence

$$\mathscr{F}^{-1}(\chi_{U_{g,0}})(h) = \sum_{j=0}^{k-1} \xi_n^{\frac{n}{k}j} \cdot \widehat{\mu}(U_{g,0} \cap U_{h,\frac{n}{k}j}).$$

We can then define a (well-defined) character on the finite group $\langle g, h\rangle$ by the rules

$$\phi_{g,h} : \langle g, h\rangle \to \mathbb{T}, \qquad \phi_{g,h}(h) = \xi_n^{\frac{n}{k}}, \quad \phi_{g,h}(g) = 1.$$

Again, $\phi_{g,h}$ extends to a character on $H$, still denoted by $\phi_{g,h}$. By further using the left invariance of $\widehat{\mu}$,

$$\widehat{\mu}(U_{g,0} \cap U_{h,\frac{n}{k}j}) = \widehat{\mu}(\overline{\phi}_{g,h}^{j} \cdot (U_{g,0} \cap U_{h,\frac{n}{k}j})) = \widehat{\mu}(U_{g,0} \cap U_{h,0}) \quad \text{for every } 0 \leq j \leq k-1,$$

so we end up with

$$\mathscr{F}^{-1}(\chi_{U_{g,0}})(h) = \widehat{\mu}(U_{g,0} \cap U_{h,0}) \cdot \sum_{j=0}^{k-1} \xi_n^{\frac{n}{k}j}.$$

This gives $\widehat{\mu}(U_{g,0} \cap U_{h,0})$ for $k = 1$, and zero otherwise. Since there are only finitely many values of $h$ satisfying $h \in \langle g\rangle$, the result follows.

Note that for $h \in \langle g\rangle$, $U_{g,0} \cap U_{h,0} = U_{g,0}$, and the value of its measure can be explicitly computed to be $\frac{1}{m}$ by using again left invariance of $\widehat{\mu}$. Hence we obtain an explicit formula for $\mathscr{F}^{-1}(\chi_{U_{g,0}})$, namely

$$\mathscr{F}^{-1}(\chi_{U_{g,0}}) = \frac{1}{m}(e + g + \cdots + g^{m-1})$$

and more generally

$$\mathscr{F}^{-1}(\chi_{U_{g,i}}) = \frac{1}{m}(e + \xi_m^i g + \cdots + \xi_m^{(m-1)i} g^{m-1}) \quad \text{for } 0 \leq i \leq m-1.$$

The extension of $\mathscr{F}$ to the respective crossed products is straightforward once we observe that the diagram

$$\begin{array}{ccc} K[H] & \xrightarrow{\mathscr{F}} & C_K(\widehat{H}) \\ {\scriptstyle \widetilde{\rho}(1)}\big\downarrow & & \big\downarrow{\scriptstyle T,\quad T(f)(x)=f(T^{-1}(x))} \\ K[H] & \xrightarrow{\mathscr{F}} & C_K(\widehat{H}) \end{array}$$

commutes. For this, it is enough to show that $T(U_{h,i}) = U_{\rho(h),i}$ for a fixed $h \in H$ of order $n$, and $0 \leq i \leq n-1$. But this is easy:

$$T(U_{h,i}) = \{T(\phi) \in \widehat{H} \mid \phi(h) = \xi_n^i\} = \{\phi \in \widehat{H} \mid \phi(\rho(h)) = \xi_n^i\} = U_{\rho(h),i}.$$

Hence $\mathscr{F}$ extends to a $*$-isomorphism $\mathscr{F} : K[H] \rtimes_{\widetilde{\rho}} \mathbb{Z} \to C_K(\widehat{H}) \rtimes_T \mathbb{Z}$. □

**Remark 3.1.3.** Certainly, in the case $K = \mathbb{C}$ there is no need to involve the whole machinery used during the course of the proof of Proposition 3.1.1. The reason for working on the proof as explicitly as stated is that the resulting formulas for $\mathscr{F}, \mathscr{F}^{-1}$ remain valid in *any* field with involution of arbitrary characteristic $p$, provided that $p$ does not divide any of the natural numbers $n \in \mathcal{O}$, that $K$ contains all the $n^{\text{th}}$-roots of unity for any $n \in \mathcal{O}$, and that one interprets $\overline{\xi}_n$ as $\xi_n^{-1}$ in the corresponding field. Of course, in this general setting one needs to prove by hand that indeed the two maps $\mathscr{F}, \mathscr{F}^{-1}$ defined this way are $*$-isomorphisms, inverse of each other, but it is a matter of computation to check that this is indeed the case.

Recall that we have a rank function $\mathrm{rk}_{K[G]}$ on $K[G]$ given by the restriction of the rank function naturally arising from $\mathcal{U}(G)$, which we denoted by $\mathrm{rk}_{K[G]}$. Our question now is whether we can find a measure $\widehat{\mu}$ on the space $\widehat{H}$ such that, when applying the construction given in Section 2.3.2, we end up with a rank function $\mathrm{rk}_{\mathcal{A}}$ on $\mathcal{A} = C_K(\widehat{H}) \rtimes_T \mathbb{Z}$ that coincides with $\mathrm{rk}_{K[G]}$ under the Fourier transform $\mathscr{F}$. The answer to this question is affirmative, and in fact $\widehat{\mu}$ coincides with the normalized Haar measure on $\widehat{H}$[2], as we show in the next proposition.

**Proposition 3.1.4.** *Let $K \subseteq \mathbb{C}$ be a subfield of $\mathbb{C}$ closed under complex conjugation and containing all the $n^{th}$ roots of unity for every $n \in \mathcal{O}$. Then from $\mathrm{rk}_{K[G]}$ we can construct a full, $T$-invariant probability measure $\widehat{\mu}$ on $\widehat{H}$ which coincides with the normalized Haar measure on $\widehat{H}$.*

*If moreover $\mathrm{rk}_{K[G]}$ is extremal in $\mathbb{P}(K[G])$, then $\widehat{\mu}$ is ergodic, and when applying the construction given in Section 2.3.2 to $\widehat{\mu}$ we end up with a Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$ on $\mathcal{A} = C_K(\widehat{H}) \rtimes_T \mathbb{Z}$ such that $\mathrm{rk}_{K[G]} = \mathrm{rk}_{\mathcal{A}} \circ \mathscr{F}$.*

[2] This is the reason for keeping the notation $\widehat{\mu}$ for such a measure.

*Proof.* We first define a finitely additive probability measure $\overline{\mu}_{K[G]}$ on the algebra $\mathbb{K}$ of clopen subsets of $\widehat{H}$ by the rule

$$\overline{\mu}_{K[G]}(U) = \mathrm{rk}_{K[G]}(\mathscr{F}^{-1}(\chi_U)) \quad \text{for every clopen subset } U \text{ of } \widehat{H}$$

which, by the same argument as in the proof of Proposition 2.3.8, can be uniquely extended to a Borel probability measure $\mu_{K[G]}$ on $\widehat{H}$. Invariance of $\mu_{K[G]}$ follows from the fact that $t$ is an invertible element:

$$\begin{aligned}\mu_{K[G]}(T(U)) &= \mathrm{rk}_{K[G]}(\mathscr{F}^{-1}(\chi_{T(U)}))\\ &= \mathrm{rk}_{K[G]}(t\mathscr{F}^{-1}(\chi_U)t^{-1})\\ &= \mathrm{rk}_{K[G]}(\mathscr{F}^{-1}(\chi_U)) = \mu_{K[G]}(U) \quad \text{for every clopen } U \subseteq X.\end{aligned}$$

The fullness of the measure is easy: let $U \subseteq \widehat{H}$ be a nonempty open subset with null measure, and consider $V \subseteq U$ a nonempty clopen subset of $\widehat{H}$. Then $\mathrm{rk}_{K[G]}(\mathscr{F}^{-1}(\chi_V)) = \mu_{K[G]}(V) \leq \mu_{K[G]}(U) = 0$, which contradicts the fact that $\mathrm{rk}_{K[G]}$ is a rank function on $K[G]$.

If moreover $\mathrm{rk}_{K[G]}$ is extremal, then again an argument similar to the one given in the proof of Proposition 2.3.10 proves that $\mu_{K[G]}$ is ergodic. In this case, $\mu_{K[G]}$ gives rise to a rank function $\mathrm{rk}_{\mathcal{A}}$ on $\mathcal{A} = C_K(\widehat{H}) \rtimes_T \mathbb{Z}$ such that

$$\mathrm{rk}_{\mathcal{A}}(\chi_U) = \mu_{K[G]}(U) = \mathrm{rk}_{K[G]}(\mathscr{F}^{-1}(\chi_U)) \quad \text{for every clopen subset } U \text{ of } \widehat{H}.$$

By the uniqueness part of Proposition 2.3.8, this implies that $\mathrm{rk}_{K[G]} = \mathrm{rk}_{\mathcal{A}} \circ \mathscr{F}$, as required.

Finally, to prove that $\mu_{K[G]}$ coincides with the normalized Haar measure $\widehat{\mu}$ on $\widehat{H}$, just note that for any clopen $U \subseteq \widehat{H}$, $\mathscr{F}^{-1}(\chi_U)$ is a projection in $K[G]$, so its rank coincides with its trace, and we obtain

$$\begin{aligned}\mu_{K[G]}(U) &= \mathrm{rk}_{K[G]}(\mathscr{F}^{-1}(\chi_U)) = \mathrm{tr}_{K[G]}(\mathscr{F}^{-1}(\chi_U))\\ &= \mathscr{F}^{-1}(\chi_U)(e) = \int_{\widehat{H}} \chi_U(\phi)\phi(e)d\widehat{\mu}(\phi) = \int_U d\widehat{\mu}(\phi) = \widehat{\mu}(U).\end{aligned}$$

□

**Remarks 3.1.5.**

1) Proposition 3.1.4 can be thought of as a particular case of the one considered in [8, Section 2]. Let us briefly summarize its content in our context.

   The homeomorphism $T : \widehat{H} \to \widehat{H}$ can be extended to a $*$-automorphism of the commutative $*$-algebra $L^\infty(\widehat{H}, \widehat{\mu})$ consisting of (classes of) bounded measurable functions $f : \widehat{H} \to \mathbb{C}$, also denoted by $T$, via

$$T(f)(x) := f(T^{-1}(x)), \quad f \in L^\infty(\widehat{H}, \widehat{\mu}), x \in \widehat{H}.$$

   Therefore we can construct the algebraic crossed product $L^\infty(\widehat{H}, \widehat{\mu}) \rtimes_T \mathbb{Z}$[3]. We will denote by $t$ the symbol corresponding to the $\mathbb{Z}$-generator, so elements from $L^\infty(\widehat{H}, \widehat{\mu}) \rtimes_T \mathbb{Z}$ will be formal finite sums

$$\sum_{n \in \mathbb{Z}} f_n t^n, \quad f_n \in L^\infty(\widehat{H}, \widehat{\mu}).$$

   Let $L^2(\widehat{H}, \widehat{\mu})$ denote the Hilbert space of (classes of) measurable functions $g : \widehat{H} \to \mathbb{C}$ with finite 2-norm, that is

$$||g||_2^2 := \int_{\widehat{H}} |g(x)|^2 d\widehat{\mu}(x) < \infty,$$

   with the usual scalar product $\langle g, h\rangle_2 := \int_{\widehat{H}} g(x)\overline{h(x)}d\widehat{\mu}(x)$, and consider $\mathcal{H} = l^2(\mathbb{Z}, L^2(\widehat{H}, \widehat{\mu}))$ the Hilbert space of $L^2(\widehat{H}, \widehat{\mu})$-valued functions on $\mathbb{Z}$; so a general element in $\mathcal{H}$ can be written as an infinite sum

$$\sum_{m \in \mathbb{Z}} g_m t^m, \quad g_m \in L^2(\widehat{H}, \widehat{\mu}),$$

   and satisfies the finiteness condition $\sum_{m\in\mathbb{Z}} ||g_m||_2^2 < \infty$. The scalar product in $\mathcal{H}$ is given by

$$\Big\langle \sum_{m \in \mathbb{Z}} g_m t^m, \sum_{m \in \mathbb{Z}} h_m t^m \Big\rangle_{\mathcal{H}} := \sum_{m \in \mathbb{Z}} \int_{\widehat{H}} g_m(x)\overline{h_m(x)}d\widehat{\mu}(x).$$

[3]See Section 2.2 for the construction of the algebraic crossed product.

Observe that $L^\infty(\widehat{H},\widehat{\mu}) \rtimes_T \mathbb{Z}$ acts faithfully on $\mathcal{H}$ by left multiplication, giving a representation $\lambda$ of $L^\infty(\widehat{H},\widehat{\mu}) \rtimes_T \mathbb{Z}$ as bounded operators on $\mathcal{H}$. Specifically, the representation $\lambda : L^\infty(\widehat{H},\widehat{\mu}) \rtimes_T \mathbb{Z} \to \mathcal{B}(\mathcal{H})$ is given by

$$\lambda\Big(\sum_{\substack{n\in\mathbb{Z}\\ \text{finite}}} f_n t^n\Big)\Big(\sum_{m\in\mathbb{Z}} g_m t^m\Big) := \sum_{m\in\mathbb{Z}}\Big(\sum_{\substack{n\in\mathbb{Z}\\ \text{finite}}} f_n T^n(g_m) t^{n+m}\Big) \in \mathcal{B}(\mathcal{H}), \qquad \text{for } f_n \in L^\infty(\widehat{H},\widehat{\mu}), g_m \in L^2(\widehat{H},\widehat{\mu}),$$

where again, for an element $g \in L^2(\widehat{H},\widehat{\mu})$, $T(g) \in L^2(\widehat{H},\widehat{\mu})$ is defined by $T(g)(x) := g(T^{-1}(x))$ for $x \in \widehat{H}$.

We denote by $\mathcal{N}(T)$ to be the weak-completion of $\lambda(L^\infty(\widehat{H},\widehat{\mu}) \rtimes_T \mathbb{Z})$ inside $\mathcal{B}(\mathcal{H})$. We have a canonical trace on $L^\infty(\widehat{H},\widehat{\mu}) \rtimes_T \mathbb{Z}$, defined on an element $f = \sum_{n\in\mathbb{Z}} f_n t^n$ by

$$\mathrm{tr}_{L^\infty(\widehat{H},\widehat{\mu})\rtimes_T\mathbb{Z}}(f) = \langle f \cdot \mathrm{id}_{\widehat{H}}, \mathrm{id}_{\widehat{H}}\rangle_{\mathcal{H}} = \int_{\widehat{H}} f_0(x) d\widehat{\mu}(x),$$

which extends to a normal, positive, faithful trace over $\mathcal{N}(T)$ denoted by $\mathrm{tr}_{\mathcal{N}(T)}$. The identification $K[G] \overset{\mathscr{F}}{\cong} C_K(\widehat{H}) \rtimes_T \mathbb{Z} \subseteq L^\infty(\widehat{H},\widehat{\mu}) \rtimes_T \mathbb{Z}$ extends to a trace-preserving isomorphism of von Neumann algebras $\mathscr{F}_{vN} : \mathcal{N}(G) \overset{\cong}{\to} \mathcal{N}(T)$, which in turn extends to a rank-preserving isomorphism between the respective algebras of affiliated operators, making the diagram

$$\begin{array}{ccc} \mathcal{U}(G) & \xrightarrow{\cong} & \mathcal{U}(T) \\ \cup\!| & & \cup\!| \\ \mathcal{N}(G) & \xrightarrow{\cong} & \mathcal{N}(T) \end{array}$$

commutative, where $\mathcal{U}(G)$ (respectively, $\mathcal{U}(T)$) is the algebra of (unbounded) affiliated operators of $\mathcal{N}(G)$ (respectively, $\mathcal{N}(T)$).

The construction $\mathcal{N}(T)$ is also know in the literature as the *group-measure space von Neumann algebra.*

2) An important observation is that, once we have proven that the measure $\widehat{\mu}$ on $\widehat{H}$ is $T$-invariant, this property does not depend on the base field $K$ anymore. So, by assuming now that $K$ is *any* field with involution of arbitrary characteristic $p$ (with $p$ not dividing any natural number $n \in \mathcal{O}$) and containing all the $n^{\text{th}}$ roots of unity for any $n \in \mathcal{O}$, *and* by assuming *ergodicity* of $\widehat{\mu}$, we can apply our construction from Chapter 2 (specifically, Theorem 2.3.7) to obtain a *canonical* Sylvester matrix rank function on $K[G]$, by simply defining $\mathrm{rk}_{K[G]} = \mathrm{rk}_{\mathcal{A}} \circ \mathscr{F}$.

We can use Proposition 3.1.4 to prove that, in certain cases, the $*$-regular closure of the group algebra $K[G]$ inside $\mathcal{U}(G)$, which we denoted by $\mathcal{R}_{K[G]}$, can be identified with $\mathcal{R}_{\mathcal{A}}$, the $*$-regular closure of $\mathcal{A}$ inside the rank completion $\mathfrak{R}_{\mathrm{rk}}$ of $\mathcal{A}$ with respect to its rank function $\mathrm{rk}_{\mathcal{A}}$.

**Theorem 3.1.6.** *Consider the same notation and hypotheses as in Proposition 3.1.4, and assume that* $\mathrm{rk}_{K[G]}$ *is extremal in* $\mathbb{P}(K[G])$.

*Then we obtain a $*$-isomorphism $\mathcal{R}_{K[G]} \cong \mathcal{R}_{\mathcal{A}}$. In fact, we have commutative diagrams as follows.*[4]

$$\begin{array}{ccccc} \mathcal{A} & \hookrightarrow & \mathcal{R}_{\mathcal{A}} & \hookrightarrow & \mathfrak{R}_{\mathrm{rk}} \\ \wr\!| & & \wr\!| & & \downarrow \\ K[G] & \hookrightarrow & \mathcal{R}_{K[G]} & \hookrightarrow & \mathcal{U}(G) \end{array} \tag{3.1.1}$$

*Moreover, the rank completion of $\mathcal{R}_{K[G]}$ with respect to $\mathrm{rk}_{K[G]}$ is $*$-isomorphic to $\mathcal{M}_K$, the von Neumann continuous factor over $K$.*

*Proof.* Recall from Theorem 2.3.7 that $\mathfrak{R}_{\mathrm{rk}}$ can also be obtained by completing $\mathcal{A}$ with respect to its rank $\mathrm{rk}_{\mathcal{A}}$. Now since $\mathcal{U}(G)$ is complete with respect to the $\mathrm{rk}_{\mathcal{U}(G)}$-metric (see Theorem 1.2.16), Proposition 3.1.4

[4] Recall Proposition 2.4.3.

gives that the completion $\mathfrak{R}_{\mathrm{rk}}$ of $\mathcal{A}$ with respect to $\mathrm{rk}_{\mathcal{A}}$ also fits inside $\mathcal{U}(G)$, making the previous diagram commutative. In turn, since $\mathfrak{R}_{\mathrm{rk}}$ is itself $*$-regular, we see that

$$\mathcal{R}_{K[G]} = \mathcal{R}(K[G], \mathcal{U}(G)) \cong \mathcal{R}(\mathcal{A}, \mathfrak{R}_{\mathrm{rk}}) = \mathcal{R}_{\mathcal{A}},$$

as required. The last part follows from Theorem 2.3.9. □

## 3.2 The lamplighter group algebra

In this section we are going to concentrate on the study of the group algebra of the lamplighter group $\Gamma$. We will work mainly with the presentation of $\Gamma$ given in Example 3.1.2.2), so $\Gamma$ is generated by $\{a_i\}_{i\in\mathbb{Z}}$ and $t$, with relations $a_i^2 = 1, a_i a_j = a_j a_i$ and $t a_i t^{-1} = a_{i-1}$ for every $i, j \in \mathbb{Z}$.

In this case, the hypotheses given in Remark 3.1.3 translates to the fact that the caracteristic of our field $K$ has to be different from 2. Then the Fourier transform gives a $*$-isomorphism

$$\mathscr{F} : K[\Gamma] \to C_K(X) \rtimes_T \mathbb{Z}, \qquad \begin{cases} e_i = \frac{1+a_i}{2} \mapsto \chi_{U_i} \\ t \mapsto t \end{cases}$$

where $X = \prod_{i\in\mathbb{Z}}\{0,1\}$ is the Cantor set, $T$ the shift map defined by $T(x)_i = x_{i+1}$ for $x \in X$, and $U_i$ is the clopen set consisting of all points $x \in X$ having a 0 at the $i^{\text{th}}$ component.

We will follow the same notation as in [40, Section 3]: given $\epsilon_{-k}, ..., \epsilon_l \in \{0,1\}$, the cylinder set $\{x = (x_i) \in X \mid x_{-k} = \epsilon_{-k}, ..., x_l = \epsilon_l\}$ will be denoted by $[\epsilon_{-k} \cdots \underline{\epsilon_0} \cdots \epsilon_l]$. So for example $U_0 = [\underline{0}]$, and the characteristic function $\chi_{[\underline{0}]}$ is identified with the projection $e_0$ under $\mathscr{F}$; also, $\chi_{[0\underline{1}1]}$ is identified with $e_{-1}f_0f_1$.

It is then clear that a basis for the topology of $X$ is given by the collection of clopen sets consisting of all the cylinder sets, that is

$$\{[\epsilon_{-k} \cdots \underline{\epsilon_0} \cdots \epsilon_l]\}_{\substack{\epsilon_i\in\{0,1\} \\ k,l\geq 0}}.$$

We have a natural measure $\mu$ on $X$ given by the usual product measure, where we take the $\left(\frac{1}{2},\frac{1}{2}\right)$-measure on each component $\{0,1\}$. It is well-known (cf. [59, Example 3.1]) that $\mu$ is an ergodic, full and shift-invariant probability measure on $X$. In fact, note that under the Fourier transform, we have

$$\mathscr{F}^{-1}(\chi_{[\epsilon_{-k}\cdots\underline{\epsilon_0}\cdots\epsilon_l]}) = \Big(\frac{1+(-1)^{\epsilon_{-k}}a_{-k}}{2}\Big)\cdots\Big(\frac{1+(-1)^{\epsilon_0}a_0}{2}\Big)\cdots\Big(\frac{1+(-1)^{\epsilon_l}a_l}{2}\Big)$$

and so, since its rank in $K[\Gamma]$ coincides with its trace in $K[\Gamma]$ (it is a projection), we obtain the equality

$$\begin{aligned} \mathrm{rk}_{K[\Gamma]}(\mathscr{F}^{-1}(\chi_{[\epsilon_{-k}\cdots\underline{\epsilon_0}\cdots\epsilon_l]})) &= \mathrm{tr}_{K[\Gamma]}\Big(\Big(\frac{1+(-1)^{\epsilon_{-k}}a_{-k}}{2}\Big)\cdots\Big(\frac{1+(-1)^{\epsilon_0}a_0}{2}\Big)\cdots\Big(\frac{1+(-1)^{\epsilon_l}a_l}{2}\Big)\Big) \\ &= \frac{1}{2^{l+k+1}} = \mu([\epsilon_{-k}\cdots\underline{\epsilon_0}\cdots\epsilon_l]). \end{aligned}$$

It follows from Proposition 3.1.4 that $\mathrm{rk}_{K[\Gamma]} \circ \mathscr{F}^{-1}$ coincides with the Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$, where $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$, obtained from $\mu$ by applying the construction from Section 2.3.2[5]. In particular, the set of $l^2$-Betti numbers arising from $\Gamma$ with coefficients in $K$ can be also computed by means of $\mathrm{rk}_{\mathcal{A}}$,

$$\mathcal{C}(\Gamma, K) = \mathrm{rk}_{\mathcal{A}}\Big(\bigcup_{i=1}^{\infty} M_i(\mathcal{A})\Big) = \mathcal{C}(\mathcal{A}).$$

From now on we will identify $K[\Gamma] \cong C_K(X) \rtimes_T \mathbb{Z}$. We can then apply our constructions from Chapter 2 to study $K[\Gamma]$. We take $E_n = [1\ldots\underline{1}\ldots 1]$ (with $2n+1$ one's) for the sequence of clopen sets, whose intersection gives the point $y = (..., 1, 1, \underline{1}, 1, 1, ...) \in X$ which is a fixed point for the shift map $T$ [6]. We take the partitions $\mathcal{P}_n$ of the complements $X\backslash E_n$ to be the obvious ones, namely

$$\mathcal{P}_n = \{[00\ldots\underline{0}\ldots 00], [00\ldots\underline{0}\ldots 01], ..., [01\ldots\underline{1}\ldots 11]\}.$$

[5] Of course this has to be the case, since $\mu$ as taken is exactly the normalized Haar measure of the group $X = \prod_{i\in\mathbb{Z}}\mathbb{Z}_2$.

[6] It can also be done by taking an even number of one's at each level $n$; we are taking an odd number for comfort.

We will follow the notation of Section 2.3.1: write $\mathcal{A}_n := \mathcal{A}(E_n, \mathcal{P}_n)$ for the unital $*$-subalgebra of $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ generated by the partial isometries $\chi_Z t$ for $Z \in \mathcal{P}_n$. It is easily seen that $\mathcal{A}_n$ coincides with the unital $*$-subalgebra of $\mathcal{A}$ generated by the partial isometries $s_i = e_i t$ for $i = -n, ..., n$, where recall that each $e_i$ is the projection in $K[\Gamma]$ given by $\frac{1+a_i}{2}$ (equivalently, the characteristic function of the clopen set $[0_i]$), and we put $f_i = 1 - e_i$[7]. Indeed,

$$e_i t = \sum_{\substack{[\epsilon_{-n} \cdots \underline{\epsilon_0} \cdots \epsilon_n] \in \mathcal{P}_n \\ \epsilon_i = 0}} \chi_{[\epsilon_{-n} \cdots \underline{\epsilon_0} \cdots \epsilon_n]} t$$

and if $Z = [\epsilon_{-n} \dots \underline{\epsilon_0} \dots \epsilon_n]$ with some $\epsilon_i = 0$, then

$$\chi_Z t = (\widehat{f_{-n} e_{-n}}) \cdots (\widehat{f_{-(i-1)} e_{-(i-1)}})(e_i t)(\widehat{\widehat{f_{i+1} e_{i+1}}}) \cdots (\widehat{\widehat{f_{n+1} e_{n+1}}})$$

where

$$\widehat{f_j e_j} = \epsilon_j f_j + (1 - \epsilon_j) e_j = \begin{cases} e_j & \text{if } \epsilon_j = 0 \\ f_j & \text{if } \epsilon_j = 1 \end{cases} \quad \text{and} \quad \widehat{\widehat{f_j e_j}} = \epsilon_{j-1} f_j + (1 - \epsilon_{j-1}) e_j = \begin{cases} e_j & \text{if } \epsilon_{j-1} = 0 \\ f_j & \text{if } \epsilon_{j-1} = 1 \end{cases}.$$

Note that each $e_j$ belongs to the corresponding $*$-subalgebra, since $e_j = (e_j t)(e_j t)^*$ and $e_{n+1} = (e_n t)^*(e_n t)$. We have, for each $n \geq 1$, inclusions $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ which correspond to the embeddings $\iota_n$ from Section 2.3.1.

The quasi-partition $\overline{\mathcal{P}}_n$ consists here of the translates of the sets $W \in \mathbb{V}_n$ which are either

$$W_0 = E_n \cap T^{-1}(E_n) = [11 \dots \underline{1} \dots 111] \quad \text{of length 1 (there are } 2n+2 \text{ one's)}$$

or

$$\begin{aligned} W_1 &= E_n \cap T^{-1}(Z_n) \cap \cdots \cap T^{-n-1}(Z_0) \cap T^{-2n-1}(Z_{-n}) \cap T^{-2n-2}(E_n) \\ &= [11 \dots \underline{1} \dots 11011 \dots 1 \dots 11] \quad \text{of length } 2n+2 \text{ (there are } 4n+2 \text{ one's, and a zero)} \end{aligned}$$

with each $Z_i = [11 \dots 0_i \dots 11]$ having all one's and a zero in the $i^{\text{th}}$ position, or of the form

$$W(*, *, ..., *, *) = [11 \dots \underline{1} \dots 110 * * \cdots * * 011 \dots 1 \dots 11] \quad \text{of length } (2n+3) + l$$

where here $l \geq 0$ is the number of $*$, and each $*$ can be either a zero or a one, but with at most $2n$ consecutive one's (if there were $2n+1$ consecutive one's, then we would end inside $E_n$ again).

In this particular case it can be checked by hand that indeed $\overline{\mathcal{P}}_n$ forms a quasi-partition of $X$, namely that

$$\sum_{k \geq 1} \sum_{\substack{W \in \mathbb{V}_n \\ |W| = k}} k \mu(W) = 1.$$

First, a definition. From now on we will write $m = 2n + 1$.

**Definition 3.2.1.** For $k \geq 0$ an integer, we define the $k^{\text{th}}$ $m$-acci number, denoted by $\text{Fib}_m(k)$, recursively by setting

$$\text{Fib}_m(0) = 0 \quad , \quad \text{Fib}_m(1) = \text{Fib}_m(2) = 1 \quad , \quad \text{Fib}_m(3) = 2 \quad , \cdots , \quad \text{Fib}_m(m-1) = 2^{m-3}$$

and for $r \geq 0$,

$$\text{Fib}_m(m + r) = \text{Fib}_m(m + r - 1) + \cdots + \text{Fib}_m(r).$$[8]

**Lemma 3.2.2.** *For $k \geq 2$, $\text{Fib}_m(k)$ is exactly the number of possible sequences $(\epsilon_1, ..., \epsilon_l)$ of length $l = k - 2$ that one can construct with zeroes and ones, but having at most $m - 1$ consecutive one's.*

*Proof.* For $2 \leq k \leq m - 1$ the result is clear. For $k \geq m$ a simple combinatorics argument gives the result. □

[7] Compare with [6], where the authors study the algebra $\mathcal{A}_1$.

[8] This sequence is also known in the literature as the $m$-step Fibonacci sequence.

Hence in our particular case,

$$\sum_{k\geq 1}\sum_{\substack{W\in\mathbb{V}_n\\|W|=k}} k\mu(W) = \mu(W_0) + (2n+2)\mu(W_1) + \sum_{l\geq 0}\sum_{\substack{(\epsilon_1,...,\epsilon_l) \text{ with } \epsilon_i\in\{0,1\}\\ \text{and at most } 2n\\ \text{consecutive one's}}} (2n+3+l)\mu(W(\epsilon_1,...,\epsilon_l))$$
$$= \frac{1}{2^{m+1}} + \frac{1}{2^{2m}}\sum_{k\geq 1}\frac{(m+k)\mathrm{Fib}_m(k)}{2^k}.$$

This sum can be computed to be 1 by using the next lemma.

**Lemma 3.2.3.** *We have:*

$$\sum_{k=1}^{\infty}\frac{\mathrm{Fib}_m(k)}{2^k} = 2^{m-1}, \qquad \sum_{k=1}^{\infty}\frac{k\mathrm{Fib}_m(k)}{2^k} = 2^{2m} - (m+1)2^{m-1}.$$

Indeed, by using these summation rules,

$$\sum_{k\geq 1}\sum_{\substack{W\in\mathbb{V}_n\\|W|=k}} k\mu(W) = \frac{1}{2^{m+1}} + \frac{m}{2^{2m}}\sum_{k\geq 1}\frac{\mathrm{Fib}_m(k)}{2^k} + \frac{1}{2^{2m}}\sum_{k\geq 1}\frac{k\mathrm{Fib}_m(k)}{2^k}$$
$$= \frac{1}{2^{m+1}} + \frac{m}{2^{2m}}2^{m-1} + \frac{1}{2^{2m}}(2^{2m} - (m+1)2^{m-1}) = \frac{m+1}{2^{m+1}} + \Big(1 - \frac{m+1}{2^{m+1}}\Big) = 1.$$

*Proof of Lemma 3.2.3.* The proofs are not difficult but the computations can become a little bit tough. Let's compute the first one, so put $K = \sum_{k=1}^{\infty}\frac{\mathrm{Fib}_m(k)}{2^k}$. We will use the recurrence relation for the $\mathrm{Fib}_m(k)$. First note that the first term equals $\frac{\mathrm{Fib}_m(1)}{2} = \frac{1}{2}$ and for $2 \leq k \leq m-1$, $\frac{\mathrm{Fib}_m(k)}{2^k} = \frac{2^{k-2}}{2^k} = \frac{1}{4}$. Also

$$\mathrm{Fib}_m(m) = \mathrm{Fib}_m(m-1) + \cdots + \mathrm{Fib}_m(0) = 2^{m-3} + \cdots + 2 + 1 + 1 + 0 = 2^{m-2},$$

so $\frac{\mathrm{Fib}_m(m)}{2^m} = \frac{1}{4}$ too. Putting everything together, $\sum_{k=1}^{m}\frac{\mathrm{Fib}_m(k)}{2^k} = \frac{m+1}{4}$. In general, for $2 \leq r \leq m$, $S_r := \sum_{k=1}^{r}\frac{\mathrm{Fib}_m(k)}{2^k} = \frac{r+1}{4}$. We can decompose the initial sum as

$$K = \sum_{k=1}^{m}\frac{\mathrm{Fib}_m(k)}{2^k} + \sum_{k=1}^{\infty}\frac{\mathrm{Fib}_m(m+k)}{2^{m+k}} = \frac{m+1}{4} + \sum_{i=1}^{m}\frac{1}{2^i}\Big(\sum_{k=1}^{\infty}\frac{\mathrm{Fib}_m(m+k-i)}{2^{m+k-i}}\Big)$$
$$= \frac{m+1}{4} + \sum_{i=1}^{m-2}\frac{1}{2^i}\Big(\sum_{k=1}^{\infty}\frac{\mathrm{Fib}_m(m+k-i)}{2^{m+k-i}}\Big) + \frac{1}{2^{m-1}}\Big(K - \frac{1}{2}\Big) + \frac{1}{2^m}K$$
$$= \frac{m+1}{4} + \sum_{i=1}^{m-2}\frac{1}{2^i}\Big(\sum_{k=1}^{\infty}\frac{\mathrm{Fib}_m(m+k-i)}{2^{m+k-i}}\Big) + \Big(\frac{1}{2^{m-1}} + \frac{1}{2^m}\Big)K - \frac{1}{2^m}.$$

But for $1 \leq i \leq m-2$, $K = S_{m-i} + \sum_{k=1}^{\infty}\frac{\mathrm{Fib}_m(m+k-i)}{2^{m+k-i}}$, so

$$K = \frac{m+1}{4} + \sum_{i=1}^{m-2}\frac{1}{2^i}\Big(K - S_{m-i}\Big) + \Big(\frac{1}{2^{m-1}} + \frac{1}{2^m}\Big)K - \frac{1}{2^m} = \frac{m+1}{4} + \Big(\sum_{i=1}^{m}\frac{1}{2^i}\Big)K - \Big(\sum_{i=1}^{m-2}\frac{S_{m-i}}{2^i} + \frac{1}{2^m}\Big).$$

We compute $\sum_{i=1}^{m}\frac{1}{2^i} = 1 - \frac{1}{2^m}$, and

$$\sum_{i=1}^{m-2}\frac{S_{m-i}}{2^i} = \frac{1}{4}\sum_{i=1}^{m-2}\frac{m-i+1}{2^i} = \frac{m-1}{4} - \frac{1}{2^m},$$

where we have used the sum $\sum_{j=1}^{n} jx^j = \frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}$ for real $x \neq 1$. Putting everything together

$$K = \frac{m+1}{4} + \Big(1 - \frac{1}{2^m}\Big)K - \frac{m-1}{4},$$

so the result $K = 2^{m-1}$ follows. For the second sum, put $J = \sum_{k=1}^{\infty} \frac{k\mathrm{Fib}_m(k)}{2^k}$. As before, $\frac{\mathrm{Fib}_m(1)}{2} = \frac{1}{2}$ and for $2 \le k \le m$, $\frac{k\mathrm{Fib}_m(k)}{2^k} = \frac{k2^{k-2}}{2^k} = \frac{k}{4}$. Putting everything together, $\sum_{k=1}^{m} \frac{k\mathrm{Fib}_m(k)}{2^k} = \frac{1}{2} + \sum_{k=2}^{m} \frac{k}{4} = \frac{m^2+m+2}{8}$. In general, for $2 \le r \le m$, $N_r := \sum_{k=1}^{r} \frac{k\mathrm{Fib}_m(k)}{2^k} = \frac{1}{2} + \sum_{k=2}^{r} \frac{k}{4} = \frac{r^2+r+2}{8}$. We decompose the initial sum as

$$J = \sum_{k=1}^{m} \frac{k\mathrm{Fib}_m(k)}{2^k} + \sum_{k=1}^{\infty} \frac{(m+k)\mathrm{Fib}_m(m+k)}{2^{m+k}} = \frac{m^2+m+2}{8} + \sum_{k=1}^{\infty} \frac{(m+k)\mathrm{Fib}_m(m+k)}{2^{m+k}}.$$

Let's analize the second sum. We have

$$\begin{aligned} \frac{(m+k)\mathrm{Fib}_m(m+k)}{2^{m+k}} &= \sum_{i=1}^{m} \frac{(m+k)\mathrm{Fib}_m(m+k-i)}{2^{m+k}} \\ &= \sum_{i=1}^{m} \frac{(m+k-i)\mathrm{Fib}_m(m+k-i)}{2^{m+k}} + \sum_{i=1}^{m} \frac{i\mathrm{Fib}_m(m+k-i)}{2^{m+k}} \end{aligned}$$

so

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(m+k)\mathrm{Fib}_m(m+k)}{2^{m+k}} &= \sum_{i=1}^{m} \frac{1}{2^i}\Big(\sum_{k=1}^{\infty} \frac{(m+k-i)\mathrm{Fib}_m(m+k-i)}{2^{m+k-i}}\Big) + \sum_{i=1}^{m} \frac{i}{2^i}\Big(\sum_{k=1}^{\infty} \frac{\mathrm{Fib}_m(m+k-i)}{2^{m+k-i}}\Big) \\ &= \sum_{i=1}^{m-2} \frac{1}{2^i}\Big(\sum_{k=1}^{\infty} \frac{(m+k-i)\mathrm{Fib}_m(m+k-i)}{2^{m+k-i}}\Big) + \frac{1}{2^{m-1}}\Big(\sum_{k=1}^{\infty} \frac{(k+1)\mathrm{Fib}_m(k+1)}{2^{k+1}}\Big) + \frac{1}{2^m}\Big(\sum_{k=1}^{\infty} \frac{k\mathrm{Fib}_m(k)}{2^k}\Big) \\ &\quad + \sum_{i=1}^{m-2} \frac{i}{2^i}\Big(\sum_{k=1}^{\infty} \frac{\mathrm{Fib}_m(m+k-i)}{2^{m+k-i}}\Big) + \frac{m-1}{2^{m-1}}\Big(\sum_{k=1}^{\infty} \frac{\mathrm{Fib}_m(k+1)}{2^{k+1}}\Big) + \frac{m}{2^m}\Big(\sum_{k=1}^{\infty} \frac{\mathrm{Fib}_m(k)}{2^k}\Big). \end{aligned}$$

But for $1 \le i \le m-2$, $J = N_{m-i} + \sum_{k=1}^{\infty} \frac{(m+k-i)\mathrm{Fib}_m(m+k-i)}{2^{m+k-i}}$ and $2^{m-1} = \sum_{k=1}^{\infty} \frac{\mathrm{Fib}_m(k)}{2^k} = S_{m-i} + \sum_{k=1}^{\infty} \frac{\mathrm{Fib}_m(m+k-i)}{2^{m+k-i}}$, so

$$\sum_{k=1}^{\infty} \frac{(m+k)\mathrm{Fib}_m(m+k)}{2^{m+k}} = \Big(\sum_{i=1}^{m} \frac{1}{2^i}\Big)J - \sum_{i=1}^{m-2} \frac{N_{m-i}}{2^i} + \Big(\sum_{i=1}^{m} \frac{i}{2^i}\Big)2^{m-1} - \sum_{i=1}^{m-2} \frac{iS_{m-i}}{2^i} - \frac{m}{2^m}.$$

We use the sums $\sum_{j=1}^{n} jx^j = \frac{x(1-x^n)}{(1-x)^2} - \frac{nx^{n+1}}{1-x}, \sum_{j=1}^{n} j^2x^j = \frac{x(1+x)(1-x^n)}{(1-x)^3} - \frac{2nx^{n+1}}{(1-x)^2} - \frac{n^2x^{n+1}}{1-x}$ (for real $x \neq 1$) to compute

$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}, \qquad \sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}, \qquad \sum_{i=1}^{n} \frac{i^2}{2^i} = 6 - \frac{6+4n+n^2}{2^n};$$

$$\sum_{i=1}^{m-2} \frac{N_{m-i}}{2^i} = \frac{1}{8}\sum_{i=1}^{m-2} \frac{(m-i)^2+(m-i)+2}{2^i} = \frac{m^2-3m+6}{8} - \frac{1}{2^{m-1}},$$

$$\sum_{i=1}^{m-2} \frac{iS_{m-i}}{2^i} = \frac{m+1}{4}\Big(\sum_{i=1}^{m-2} \frac{i}{2^i}\Big) - \frac{1}{4}\Big(\sum_{i=1}^{m-2} \frac{i^2}{2^i}\Big) = \frac{m}{2} - 1 + \frac{1}{2^{m-1}} - \frac{n}{2^m}.$$

Putting everything together,

$$\sum_{k=1}^{\infty} \frac{(m+k)\mathrm{Fib}_m(m+k)}{2^{m+k}} = \Big(1 - \frac{1}{2^m}\Big)J + \Big(2 - \frac{m+2}{2^m}\Big)2^{m-1} - \frac{m^2+m+2}{8} + \frac{1}{2}$$

and finally

$$J = \Big(1 - \frac{1}{2^m}\Big)J + 2^m - \frac{m+1}{2},$$

so the result $J = 2^{2m} - (m+1)2^{m-1}$ follows. □

The representations $\pi_n : \mathcal{A}_n \hookrightarrow \mathfrak{R}_n$, $x \mapsto (h_W \cdot x)_W$ can be explicitly computed:

a) for $W = W_0$, we write
$$q_{0,W} = \chi_{[11\dots\underline{1}\dots111]};$$

b) for $W = W_1$ and $0 \leq i \leq m$, we write
$$q_{i,W} = \chi_{T^i(W)} = t^i q_{0,W} t^{-i} \quad \text{with } q_{0,W} = \chi_{[11\dots\underline{1}\dots11011\dots1\dots11]};$$

c) for $W = W(\epsilon_1, ..., \epsilon_l)$ and $0 \leq i \leq m + l + 1$, we write
$$q_{i,W} = \chi_{T^i(W)} = t^i q_{0,W} t^{-i} \quad \text{with } q_{0,W} = \chi_{[11\dots\underline{1}\dots110\epsilon_1\dots\epsilon_l011\dots1\dots11]}.$$

The elements $q_{i,W}$ belong to the $*$-subalgebra $\mathcal{A}_n$, and in fact
$$h_W = \sum_{i=0}^{|W|-1} q_{i,W} \quad \text{(cf. Proposition 2.2.10)}.$$

Therefore here each $\mathfrak{R}_n = K \times \prod_{k\geq 1} M_{m+k}(K)^{\mathrm{Fib}_m(k)}$. Moreover, by Theorem 3.1.6, we can identify $\mathcal{R}_{K[\Gamma]} \cong \mathcal{R}_{\mathcal{A}}$, and in fact the $*$-regular closure of each $\mathcal{A}_n$ inside $\mathcal{U}(\Gamma)$ coincides with $\mathcal{R}_n$, i.e. $\mathcal{R}_n \cong \mathcal{R}(\mathcal{A}_n, \mathcal{U}(\Gamma))$, and the same for $\mathcal{A}_\infty$, $\mathcal{R}_\infty \cong \mathcal{R}(\mathcal{A}_\infty, \mathcal{U}(\Gamma))$, where we adapt the notation of Chapter 2, Section 2.3. In particular, Theorem 2.3.9 applies in this case to give the following result, already proved by Elek in [29] in this particular case of the lamplighter group $\Gamma$ and $K = \mathbb{C}$.

**Proposition 3.2.4.** *Take $\mathcal{R}_{\mathrm{rk}}$ to be the rank completion of $\mathcal{R}_{K[\Gamma]}$ inside $\mathcal{U}(\Gamma)$ with respect to $\mathrm{rk}_{\mathcal{U}(\Gamma)}$. Then $\mathcal{R}_{\mathrm{rk}} \cong \mathcal{M}_K$ as $*$-algebras over $K$, where $\mathcal{M}_K$ denotes the von Neumann continuous factor over $K$.*

*Proof.* Since $\mathcal{R}_{K[\Gamma]} \cong \mathcal{R}_{\mathcal{A}}$ and the respective ranks coincide due to Theorem 3.1.6, the result follows by the same proof as in Theorem 2.3.9. □

### 3.2.1 The algebra of special terms for the lamplighter group algebra

We now interpret the results in Section 2.4.1 for the example of the lamplighter group algebra. In particular, as promised in Section 2.4.1, we show that the corresponding algebra of special terms $\mathcal{S}_n[[t;T]]$ is an integral domain. Our notation here is a little bit different from that section: we put $\mathcal{A}_{n,0}[[t;T]]$ to denote the set of infinite sums
$$\sum_{i\geq 0} b_i(\chi_{X\setminus E_n} t)^i = \sum_{i\geq 0} b_i t^i, \quad \text{where } b_i \in \mathcal{A}_{n,i} = \chi_{X\setminus(E_n\cup\cdots\cup T^{i-1}(E_n))}\mathcal{A}_{n,0}$$
with $\mathcal{A}_{n,0} = C_K(X) \cap \mathcal{A}_n$. We then have a representation of $\mathcal{A}_{n,0}[[t;T]]$ extending $\pi_n$, which we will also denote by $\pi_n$, so
$$\pi_n : \mathcal{A}_{n,0}[[t;T]] \to \mathfrak{R}_n, \quad a \mapsto (h_W \cdot a)_W.$$

We write $\mathcal{S}_n[[t;T]]$ to denote the subset of $\mathcal{A}_{n,0}[[t;T]]$ consisting of those elements $\sum_{i\geq 0} b_i(\chi_{X\setminus E_n} t)^i$ such that each $b_i$ belongs to $\mathrm{span}\{\chi_S \mid S \in \mathcal{W}_i\}$. These are easy to describe here: the special term of degree 0 is given by
$$S_0 = T^{-1}(S_1) = [\underbrace{1\dots\underline{1}\dots11}_{2n}] \quad \longleftrightarrow \quad \chi_{S_0} = f_{-n+1}\cdots f_0 \cdots f_{n-1} f_n;^{9}$$
the special one of degree $i = 2n+1$ is
$$S = [\underbrace{11\dots1\dots1}_{2n} 0 \underbrace{1\dots\underline{1}\dots11}_{2n}] \quad \longleftrightarrow \quad \chi_S = f_{-3n} f_{-3n+1} \cdots f_{-2n} \cdots f_{-n-1} e_{-n} f_{-n+1} \cdots f_0 \cdots f_{n-1} f_n;$$

[9] Note that in this particular case the sets $S_0$ and $T^{-1}(S_1)$ defining the special element of degree 0 coincide, giving the set $[1\dots\underline{1}\dots11]$.

and with degree $i \geq 2n+2$, we have the elements

$$S = [\underbrace{11\ldots1\ldots1}_{2n}0\underbrace{**\cdots**}_{i-(2n+2)}0\underbrace{1\ldots\underline{1}\ldots11}_{2n}]$$
$$\updownarrow$$
$$\chi_S = f_{-n-i+1}f_{-n-i+2}\cdots f_{-i+1}\cdots f_{-i+n}e_{-i+n+1}*_{-i+n+2}\cdots *_{-n-1}e_{-n}f_{-n+1}\cdots f_0\cdots f_{n-1}f_n$$

with $*_j \in \{e_j, f_j\}$ with no more than $2n+1$ consecutive $f$'s. As remarked in Section 2.4.1, the set $\mathcal{S}_n[[t;T]]$ is always a linear subspace of $\mathcal{A}_{n,0}[[t;T]]$, but in general is not a subalgebra. Nevertheless, the next lemma shows that the lamplighter algebra has some special properties that are reflected in $\mathcal{S}_n[[t;T]]$.

**Lemma 3.2.5.** *In this particular case of the lamplighter group algebra, the space $\mathcal{S}_n[[t;T]]$ becomes a subalgebra of $\mathcal{A}_{n,0}[[t;T]]$, and even an integral domain.*

*Proof.* First one should note that the special term of degree 0 becomes the unit element in $\mathcal{S}_n[[t;T]]$.

We show that if $S \in \mathcal{W}_i$, $S' \in \mathcal{W}_j$ then $S \cap T^i(S') \in \mathcal{W}_{i+j}$ (here $i, j \geq 2n+2$, the other cases can be also checked in a similar way). We have

$$\chi_S = f_{-n-i+1}\cdots f_{-i+1}\cdots f_{-i+n}e_{-i+n+1}a_{-i+n+2}\cdots a_{-n-1}e_{-n}f_{-n+1}\cdots f_0\cdots f_n,$$

$$\chi_{S'} = f_{-n-j+1}\cdots f_{-j+1}\cdots f_{-j+n}e_{-j+n+1}b_{-j+n+2}\cdots b_{-n-1}e_{-n}f_{-n+1}\cdots f_0\cdots f_n,$$

with $a_i, b_i \in \{e_i, f_i\}$ with no more than $2n+1$ consecutive $f$'s, so that

$$\chi_S t^i \cdot \chi_{S'} t^j = \chi_{S\cap T^i(S')}t^{i+j} = f_{-n-j-i+1}\cdots f_{-j-i+1}\cdots f_{-j-i+n}e_{-j-i+n+1}b_{-j-i+n+2}\cdots b_{-n-i-1}e_{-n-i}$$
$$\cdot f_{-n-i+1}\cdots f_{-i}f_{-i+1}\cdots f_{-i+n}e_{-i+n+1}a_{-i+n+2}\cdots a_{-n-1}e_{-n}f_{-n+1}\cdots f_0\cdots f_n.$$

Now it is clear that $S \cap T^i(S') \in \mathcal{W}_{i+j}$. This shows that $\mathcal{S}_n[[t;T]]$ is a subalgebra of $\mathcal{A}_{n,0}[[t;T]]$. To show that $\mathcal{S}_n[[t;T]]$ is a domain, consider two nonzero elements $a, b \in \mathcal{S}_n[[t;T]]$ , and let $\chi_S t^i$ and $\chi_{S'} t^j$ be terms in the support of $a$ and $b$ respectively, of smallest degree. By the computation above $\chi_S t^i \cdot \chi_{S'} t^j = \chi_{S\cap T^i(S')}t^{i+j}$ is a nonzero term of smallest degree in $ab$. This shows that $ab \neq 0$. Note that the special term $\chi_{S_0 \cup T^{-1}(S_1)}$ is the unit of the algebra $\mathcal{S}_n[[t;T]]$. ☐

Define $\mathcal{S}_n[t;T] \subseteq \mathcal{S}_n[[t;T]]$ the set of elements of $\mathcal{S}_n[[t;T]]$ with finite support, i.e. of the form $\sum_{i=0}^r b_i t^i$ with $b_i$ belonging to the linear span of the special elements of degree $i$, and $r \geq 0$.

**Proposition 3.2.6.** *$\mathcal{S}_n[t;T]$ is a free $K$-algebra with infinite generators, and $\mathcal{S}_n[[t;T]]$ is a free power series $K$-algebra with infinite generators.*

*Proof.* A special term

$$f_{-n-i+1}\cdots f_{-i+1}\cdots f_{-i+n}e_{-i+n+1}a_{-i+n+2}\cdots a_{-n-1}e_{-n}f_{-n+1}\cdots f_0\cdots f_n t^i$$

is said to be *pure* if there are no more than $2n-1$ consecutive $f$'s in the $a_{-i+n+2}\cdots a_{-n-1}$ part. Denote by $\mathcal{P}_u$ the set of pure elements. We then have that every special term $\chi_S t^i$ can be written uniquely as a product of pure terms, so we obtain an isomorphism

$$K\langle\langle x_b \mid b \in \mathcal{P}_u\rangle\rangle \cong \mathcal{S}_n[[t;T]], \quad x_b \mapsto b,\ 1 \mapsto \chi_{S_0}$$

which restricts to an isomorphism $K\langle x_b \mid b \in \mathcal{P}_u\rangle \cong \mathcal{S}_n[t;T]$, where $S_0 = [1\ldots\underline{1}\ldots11]$ corresponds to the special term of degree 0. ☐

### Examples with the first two levels of the lamplighter

In this small subsection we will present two examples concerning the structure of the algebras $\mathcal{S}_n[[t;T]]$ or, more generally, of the algebra $\mathcal{S}_E[[t;T]]$ defined as in 2.4.1[10].

[10]Note, however, that the notation used in that section is different from the one we are using now: in the new one we are emphasizing the clopen $E \subseteq X$.

As our first example, we take $E = [1\underline{1}]$ and the obvious partition $\mathcal{P}$ of the complement $X \backslash E$, namely $\mathcal{P} = \{[0\underline{0}], [0\underline{1}], [1\underline{0}]\}$. The quasi-partition $\overline{\mathcal{P}}$ consists of the $T$-translates of the sets

$$W = \begin{cases} \text{either } [1\underline{1}1] \text{ of length } 1, \text{ or} \\ [1\underline{1}0^{k_1}10^{k_2}1\cdots 0^{k_r}11] \text{ of length } k = k_1 + \cdots + k_r + (r+1), \text{ with } r \geq 1 \text{ and each } k_i \geq 1 \end{cases}$$

where $0^{k_i}$ denotes $0 \cdots 0$ $k_i$ times. Indeed,

$$\sum_{k \geq 1} \sum_{\substack{W \in \mathbb{V} \\ |W| = k}} |W| \mu(W) = \frac{1}{2^3} + \sum_{r \geq 1} \sum_{k_1, \dots, k_r \geq 1} \frac{k_1 + \cdots + k_r + (r+1)}{2^{k_1 + \cdots + k_r + (r+3)}} = \frac{1}{8} + \sum_{r \geq 1} \frac{3r+1}{2^{r+3}} = 1,$$

where we have used the formulas $\sum_{k \geq 1} k x^k = \frac{x}{(1-x)^2}$ and $\sum_{k \geq 1} x^k = \frac{x}{1-x}$, valid for all real values $x \in (-1, 1)$.

It is easily seen that the unital $*$-subalgebra $\mathcal{A}_E$ generated by 1 and the elements $\chi_Z t$ with $Z \in \mathcal{P}$ is the same as the $*$-subalgebra generated by 1 and the partial isometries $e_{-1}t$ and $e_0 t$.

The special elements are as follows: the special term of degree 0 is given by

$$S_0 = [\underline{1}] \quad \longleftrightarrow \quad \chi_S = f_0,$$

and with degree $i = k_1 + \cdots + k_r + r \geq 2$, we have the elements

$$S = [10^{k_1}10^{k_2}1\cdots 0^{k_r}\underline{1}] \quad \longleftrightarrow \quad f_{-i}e_{-i+1}\cdots f_{-k_r-1}e_{-k_r}\cdots e_{-1}f_0 t^i$$

with $r \geq 1$, $k_1, ..., k_r \geq 1$. A *pure* term here is one of the form

$$\chi_{[10^k\underline{1}]} t^{k+1} = f_{-k-1}e_{-k}\cdots e_{-1}f_0 t^{k+1}$$

for $k \geq 1$. Therefore, each pure term is parametrized by a positive integer, so here

$$\mathcal{S}_E[[t;T]] \cong K\langle\langle \{x_k\}_{k \geq 2} \rangle\rangle, \quad \chi_{[10^k\underline{1}]} t^{k+1} \mapsto x_{k+1}.$$

For example, the term $\chi_{[10^{k_1}10^{k_2}1\cdots 0^{k_r}\underline{1}]} t^{k_1 + \cdots + k_r + r}$ corresponds to the monomial $x_{k_r+1}\cdots x_{k_1+1}$ (note the noncommutativity of the variables), which has degree $k_1 + \cdots + k_r + r$ with respect to the graded homomorphism $\deg : S_E[[t;T]] \to \mathbb{Z}^*$ such that $\deg(x_k) = k$ for $k \geq 2$.

Let's do the next case $E = [1\underline{1}1]$, which is the first of our levels commented at the beginning of Subsection 3.2.1. The partition $\mathcal{P}$ of the complement $X \backslash E$ is again the obvious one, so the quasi-partition $\overline{\mathcal{P}}$ consists of the $T$-translates of the sets

$$W = \begin{cases} \text{either } [1\underline{1}11] \text{ of length } 1, \text{ or} \\ [1\underline{1}10^{k_1^{(1)}}1\cdots 0^{k_{r_1}^{(1)}}110^{k_1^{(2)}}1\cdots 0^{k_{r_2}^{(2)}}11\cdots\cdots 10^{k_{r_n}^{(n)}}111] \text{ of length} \\ \qquad k = \sum_{i=1}^n \left( \sum_{j=1}^{r_i} k_j^{(i)} + r_i \right) + n + 1, \text{ with } n \geq 1, \text{ each } r_i \geq 1 \text{ and each } k_j^{(i)} \geq 1. \end{cases}$$

Here the unital $*$-subalgebra $\mathcal{A}_E$ generated by 1 and the elements $\chi_Z t$ with $Z \in \mathcal{P}$ is the same as the $*$-subalgebra generated by 1 and the partial isometries $e_{-1}t, e_0 t$ and $e_1 t$.

The special elements are as follows: the special term of degree 0 is given by $S_0 = [\underline{1}1]$, and with degree $i = \left( \sum_{j=1}^{r_i} k_j^{(i)} + r_i \right) + n \geq 3$ we have the elements

$$S = [110^{k_1^{(1)}}1\cdots 0^{k_{r_1}^{(1)}}110^{k_1^{(2)}}1\cdots 0^{k_{r_2}^{(2)}}11\cdots\cdots 10^{k_{r_n}^{(n)}}\underline{1}1]$$

with $n \geq 1$, $r_i \geq 1$ and $k_j^{(i)} \geq 1$. A *pure* term here is one of the form

$$\chi_{[110^{k_1}10^{k_2}1\cdots 0^{k_r}\underline{1}1]} t^{k_1 + \cdots + k_r + r + 1}$$

for $r \geq 1$ and $k_1, ..., k_r \geq 1$. Therefore, each pure term is parametrized by a sequence of positive integers $\overrightarrow{k} = (k_1, ..., k_r) \in \bigcup_{r \geq 1} (\mathbb{Z}^+)^r$, so that

$$\mathcal{S}_E[[t;T]] \cong K\langle\langle \{x_{\overrightarrow{k}}\}_{\overrightarrow{k} \in \bigcup_{r \geq 1} (\mathbb{Z}^+)^r} \rangle\rangle, \quad \chi_{[110^{k_1}10^{k_2}1\cdots 0^{k_r}\underline{1}1]} t^{k_1 + \cdots + k_r + r + 1} \mapsto x_{(k_1, \dots, k_r)}$$

with $\mathrm{Fib}_2(k-2)$ elements of degree $k = k_1 + \cdots + k_r + r + 1$. The nonpure terms in the above step (i.e. taking $E = [1\underline{1}]$) correspond to the pure terms in the present step.

### 3.2.2 Some computations of $l^2$-Betti numbers

We will now focus on computing some $l^2$-Betti numbers of elements from matrix algebras over $K[\Gamma]$. Our goal is to obtain elements that give rise to irrational values. We will use ideas from [41], although applied to our construction.

Recall that, in the case of the lamplighter group algebra, we have $K[\Gamma] \cong \mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$ with $T$ being the Bernoulli shift map, and moreover the Sylvester matrix rank functions $\mathrm{rk}_{K[\Gamma]}$ and $\mathrm{rk}_{\mathcal{A}}$ coincide under this identification[11]. Therefore, given a matrix element $A \in M_l(K[\Gamma])$, one can compute its von Neumann dimension by the formula

$$\dim_{vN}(\ker(A)) = l - \mathrm{rk}_{\mathcal{A}}(A)$$

where $\mathrm{rk}_{\mathcal{A}}(A)$ is the value of the Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$ on the matrix $A$.

We will take advantage of our construction done in the previous chapter: we have a faithful representation $\pi_E : \mathcal{A}_E \to \mathfrak{R}_E$ with $\mathcal{A}_E = \mathcal{A}(E, \mathcal{P})$ and $\mathfrak{R}_E = \prod_{W \in \mathbb{V}_E} M_{|W|}(K)$ for each choice of a clopen set $E \subseteq X$ together with a partition $\mathcal{P}$ of its complement $X \backslash E$. We extend this representation to a faithful representation, also denoted by $\pi_E$, over matrix algebras $\pi_E : M_l(\mathcal{A}_E) \cong M_l(K) \otimes \mathcal{A}_E \to M_l(\mathfrak{R}_E) \cong \prod_{W \in \mathbb{V}_E} M_l(K) \otimes M_{|W|}(K)$ in the canonical way. Hence, $\mathrm{rk}_{\mathcal{A}}$ can be computed over elements in $M_l(\mathcal{A}_E)$ to be

$$\mathrm{rk}_{\mathcal{A}}(A) = \sum_{W \in \mathbb{V}_E} |W| \mu(W) \, \mathrm{Rk}_{|W|}(\pi_E(A)_W) = \sum_{W \in \mathbb{V}_E} \mu(W) \, \mathrm{Rk}(\pi_E(A)_W)$$

where $\mathrm{Rk}_{|W|}$ is the canonical extension of $\mathrm{rk}_{|W|}$ from $M_{|W|}(K)$ to $M_l(K) \otimes M_{|W|}(K)$, and Rk is the usual rank of matrices. We then obtain the following proposition, which will be useful later on when computing von Neumann dimensions of elements $A \in M_l(K[\Gamma])$.

**Proposition 3.2.7.** *With the above notation, for a given element $A \in M_l(\mathcal{A}_E)$, we have the formula*

$$\dim_{vN}(\ker(A)) = \sum_{W \in \mathbb{V}_E} \dim(\ker(\pi_E(A)_W)) \mu(W).$$

*Proof.* It is just a matter of computation: noting that $\mathrm{Rk}(\pi_E(A)) = |W| l - \dim(\ker(\pi_E(A)_W))$, we have

$$\begin{aligned}
\dim_{vN}(\ker(A)) &= l - \mathrm{rk}_{\mathcal{A}}(A) = l - \sum_{W \in \mathbb{V}_E} \mu(W) \, \mathrm{Rk}(\pi_E(A)_W) \\
&= l - \Big( l \sum_{W \in \mathbb{V}_E} |W| \mu(W) - \sum_{W \in \mathbb{V}_E} \dim(\ker(\pi_E(A)_W)) \mu(W) \Big) \\
&= \sum_{W \in \mathbb{V}_E} \dim(\ker(\pi_E(A)_W)) \mu(W).
\end{aligned}$$

□

Fix $\{e_{ij}\}_{0 \le i,j \le l-1}$ to be a full system of matrix units for $M_l(K)$, so that for a fixed $W \in \mathbb{V}_E$, the family $\{e_{ij} \otimes e_{i'j'}(W)\}_{\substack{0 \le i,j \le l-1 \\ 0 \le i',j' \le |W|-1}}$ is a full system of matrix units for $M_l(K) \otimes M_{|W|}(K)$. In particular,

$$\pi_E(e_{ij} \otimes 1)_W = \sum_{i'=0}^{|W|-1} e_{ij} \otimes e_{i'i'}(W) = e_{ij} \otimes h_W.$$

Let now $M_l(K) \otimes M_{|W|}(K)$ act on the $K$-vector space $K^l \otimes K^{|W|}$, with $K$-basis $\{e_i \otimes e_{i'}(W)\}_{\substack{0 \le i \le l-1 \\ 0 \le i' \le |W|-1}}$, by multiplication to the right, that is

$$(e_{ij} \otimes e_{i'j'}(W)) \cdot (e_a \otimes e_b(W)) = \delta_{j,a} \delta_{j',b} e_i \otimes e_{i'}(W).$$

There is a canonical scalar product on $K^l \otimes K^{|W|}$, given by the bilinear form with matrix the identity matrix associated to the previous basis for $K^l \otimes K^{|W|}$, namely

$$\langle e_i \otimes e_{i'}(W), e_j \otimes e_{j'}(W) \rangle = \delta_{i,j} \delta_{i',j'}.$$

[11] Recall that $\mathrm{rk}_{\mathcal{A}}$ is the unique Sylvester matrix rank function associated to the normalized Haar measure on $X = \prod_{i \in \mathbb{Z}} \mathbb{Z}_2$, see Proposition 3.1.4.

**Lemma 3.2.8.** *The entry of* $\pi_E(A)_W$ *corresponding to the* $e_{ij} \otimes e_{i'j'}(W)$ *component is given by*

$$\langle \pi_E(A)_W \cdot (e_j \otimes e_{j'}(W)), e_i \otimes e_{i'}(W) \rangle.$$

*Proof.* Trivial. □

One can then think of the matrix $\pi_E(A)_W$ as the adjacency-labeled matrix of an edge-labeled graph $E_A(W)$, where we have an arrow from $(e_j, e_{j'}(W))$ to $(e_i, e_{i'}(W))$ of label

$$d_{(j,j'),(i,i')} = \langle \pi_E(A)_W \cdot (e_j \otimes e_{j'}(W)), e_i \otimes e_{i'}(W) \rangle.$$

Here *adjacency-labeled* matrix means that the entry of $\pi_E(A)_W$ corresponding to the $e_{ij} \otimes e_{i'j'}(W)$ component is exactly $d_{(j,j'),(i,i')}$, in accordance with Lemma 3.2.8.

There is an example of such a graph in Figure 3.1, corresponding to the matrix

$$\pi_E(A)_W = d_1 e_{22} \otimes e_{21}(W) + d_2 e_{l-2,l-2} \otimes e_{21}(W) + d_3 e_{l-1,l-2} \otimes e_{22}(W) + d_4 e_{22} \otimes e_{|W|-2,|W|-2}(W).$$

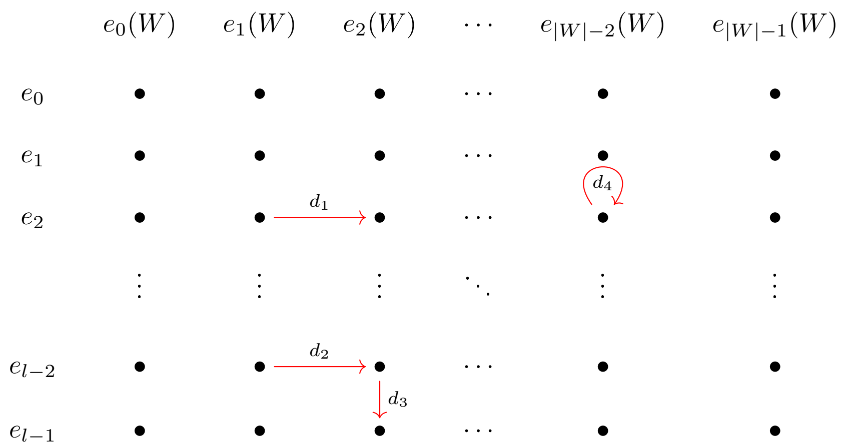



Figure 3.1: An example of a graph $E_A(W)$

If we denote by $\mathcal{G}_A(W)$ the set consisting of the connected components of the graph $E_A(W)$, and by $A_C$ ($C \in \mathcal{G}(W)$) the corresponding adjacency-labeled matrix, then the matrix $\pi_E(A)_W$ is similar to the block-diagonal matrix

$$\begin{pmatrix} A_{C_1} & & & \\ & A_{C_2} & & \\ & & \ddots & \\ & & & A_{C_r} \end{pmatrix}, \quad \text{where } \mathcal{G}_A(W) = \{C_1, ..., C_r\}.$$

As a consequence, we have the formula

$$\dim(\ker(\pi_E(A)_W)) = \sum_{C \in \mathcal{G}_A(W)} \dim(\ker(A_C)), \tag{3.2.1}$$

which we will use throughout in computing dimensions of kernels.

The following lemma is an adaptation of [41, Lemma 20] to our notation.

**Lemma 3.2.9** (Flow Lemma at each vertex $e_i \otimes e_{i'}(W)$)**.** *An element* $\alpha = \sum_{j,j'} \lambda_{(j,j')} e_j \otimes e_{j'}(W) \in K^l \otimes K^{|W|}$ *belongs to the kernel of the matrix* $\pi_E(A)_W$ *if and only if, for every vertex* $e_i \otimes e_{i'}(W)$,

$$\sum_{j,j'} \lambda_{(j,j')} d_{(j,j'),(i,i')} = 0.$$

*Proof.* Trivial; see the proof of [41, Lemma 20]. □

To see how exactly the Flow Lemma 3.2.9 works explicitly, we refer the reader to the appendix given in [41], where some applications of it in concrete examples are described.

We are now ready to give examples of elements $A \in M_l(\mathcal{A})$ with irrational von Neumann dimension. We apply our construction from Chapter 2 with the clopen subset $E = [1\underline{1}]$ and the obvious partition $\mathcal{P}$ of the complement $X \backslash E$ which already appeared in the previous section. Recall that the quasi-partition $\overline{\mathcal{P}}$ consists of the $T$-translates of the sets

$$W = \begin{cases} \text{either } [1\underline{1}1] \text{ of length } 1, \text{ or} \\ [1\underline{1}0^{k_1}10^{k_2}1\cdots 0^{k_r}11] \text{ of length } k = k_1 + \cdots + k_r + (r+1), \text{ with } r \geq 1 \text{ and each } k_i \geq 1 \end{cases}$$

where $0^{k_i}$ denotes $0 \cdots 0$ $k_i$ times.

Our main result of this section is the following:

**Theorem 3.2.10.** *Fix $n \geq 0$. For $0 \leq i \leq n$, take $p_i(x) = a_{0,i} + a_{1,i}x + \cdots + a_{m_i,i}x^{m_i}$ polynomials with positive integer coefficients of degrees at least 1, and $d_1, ..., d_n \geq 2$ natural numbers. Then there exists an element $A$ inside some matrix algebra over $\mathcal{A}_E$ such that*

$$\dim_{vN}(\ker(A)) = q_0 + q_1 \sum_{k \geq 2} \frac{1}{2^{p_0(k) + p_1(k)d_1^k + \cdots + p_n(k)d_n^k}}, \tag{3.2.2}$$

*where $q_0, q_1$ are nonzero rational numbers. We get a bunch of irrational and transcendental $l^2$-Betti numbers.*

*Proof.* Since the proof of the theorem is quite technical and complicated, we will start by giving some examples of elements $A$ whose von Neumann dimension behaves as before, first by just considering a single polynomial $p(k)$ and then by considering $p(k)d^k$. After that, we will present an explicit element having von Neumann dimension as in (3.2.2).

We start by giving a concrete example, $p(x) = 2 + x + x^2$. Consider the element from $M_{10}(\mathcal{A}_E)$ given by

$$A = \begin{pmatrix}
-\chi_{[0\underline{0}]}t^{-1} & 0 & 0 & 0 & 0 & -\chi_{[\underline{0}1]} & \chi_{[0\underline{1}0]}t^{-1} & 0 & 0 & 0 \\
-\chi_{[\underline{0}]} & -\chi_{[0\underline{0}]}t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\chi_{[1\underline{0}]} & -\chi_{[\underline{0}0]}t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\chi_{[\underline{0}1]} & -\chi_{[0\underline{0}]}t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\chi_{[\underline{0}]} & -\chi_{[0\underline{0}]}t^{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\chi_{[1\underline{0}]} & -\chi_{[\underline{0}0]}t & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\chi_{[\underline{0}10]}t^2 & 0 & 0 & 0 & 0 & 0 & -\chi_{[\underline{0}0]}t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\chi_{[0\underline{1}0]}t & \chi_{[\underline{0}]} & -\chi_{[\underline{0}0]}t & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\chi_{[\underline{0}1]} & 0
\end{pmatrix}$$
$$+ (\chi_{[0\underline{0}]} + \chi_{[1\underline{0}0]}) \cdot (\mathrm{Id}_{10} - e_{66} - e_{99})$$

or, in a more concise way, we write

$$\begin{aligned}
A = &- \chi_{[0\underline{0}]}t^{-1} \cdot e_{00} - \chi_{[\underline{0}]} \cdot e_{10} - \chi_{[0\underline{0}]}t^{-1} \cdot e_{11} \\
&- \chi_{[1\underline{0}]} \cdot e_{21} - \chi_{[\underline{0}0]}t \cdot e_{22} - \chi_{[\underline{0}1]} \cdot e_{32} \\
&- \chi_{[0\underline{0}]}t^{-1} \cdot e_{33} - \chi_{[\underline{0}]} \cdot e_{43} - \chi_{[0\underline{0}]}t^{-1} \cdot e_{44} \\
&- \chi_{[1\underline{0}]} \cdot e_{54} - \chi_{[\underline{0}0]}t \cdot e_{55} - \chi_{[\underline{0}1]} \cdot e_{05} \\
&- \chi_{[\underline{0}10]}t^2 \cdot e_{71} - \chi_{[\underline{0}0]}t \cdot e_{77} + \chi_{[\underline{0}]} \cdot e_{87} \\
&- \chi_{[\underline{0}0]}t \cdot e_{88} - \chi_{[\underline{0}1]} \cdot e_{98} \\
&- \chi_{[0\underline{1}0]}t \cdot e_{86} + \chi_{[0\underline{1}0]}t^{-1} \cdot e_{06} \\
&+ (\chi_{[0\underline{0}]} + \chi_{[1\underline{0}0]}) \cdot (\mathrm{Id}_{10} - e_{66} - e_{99}).
\end{aligned}$$

For this element, and if we take $W = [1\underline{1}1]$, we observe that $\pi_E(A)_W$ equals the zero $10 \times 10$ matrix, so its kernel has dimension 10. Figure 3.2 gives the prototypical graph $E_A(W)$ that appears in the case one takes a $W$ of the second form $[1\underline{1}0^{k_1}10^{k_2}1 \cdots 0^{k_r}11]$ with length $k = k_1 + \cdots + k_r + (r+1)$.

Here we have four different types of connected components $C$ for the graph, namely

a) $C_1$, given by the graphs with only one vertex

$$\bullet$$

Note that in this case $\dim(\ker(A_{C_1})) = 1$, and we have $10 + (2k_1 - 1) + (9 + 2k_2 - 1) + \cdots + (9 + 2k_r - 1) + 10 = 11 + 8r + 2(k_1 + \cdots + k_r)$ connected components of this kind.

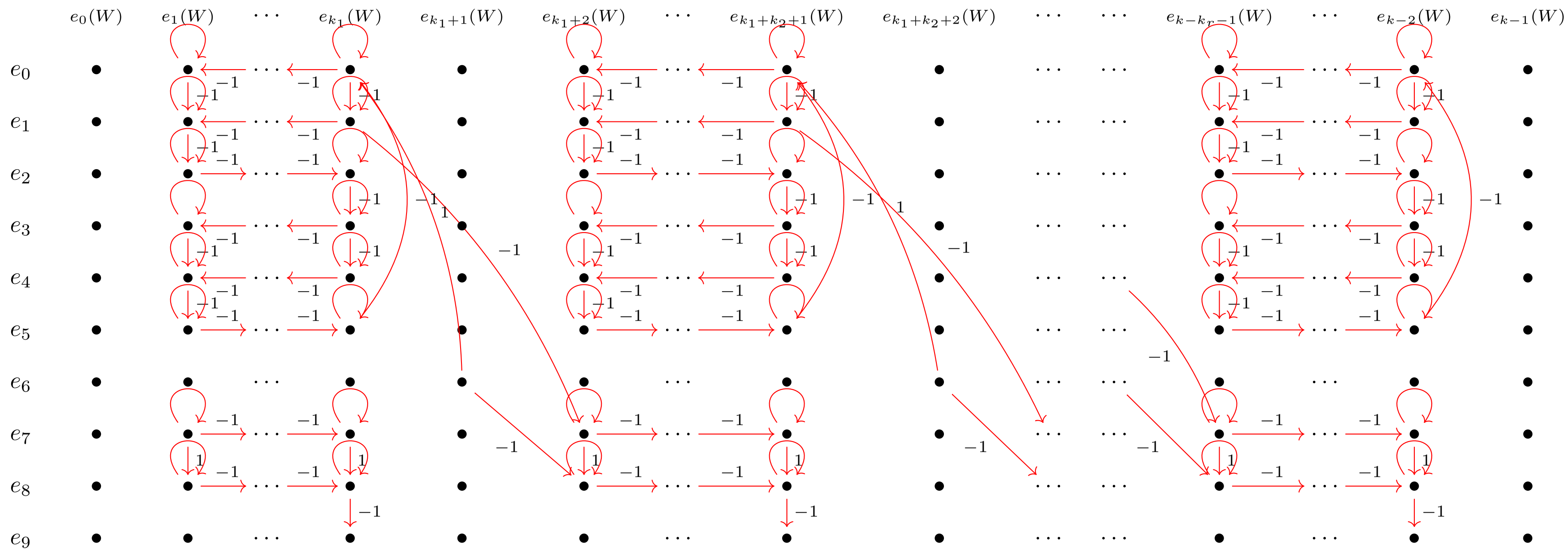
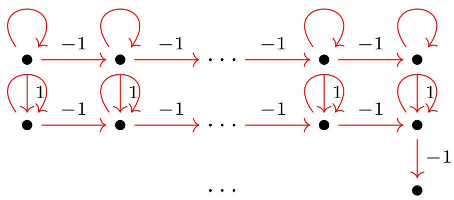



Figure 3.2: The graph $E_A(W)$ for a $W = [1\underline{1}0^{k_1}10^{k_2}1\cdots0^{k_r}11]$ of length $k$. Each loop should be labeled with a 1.[a]

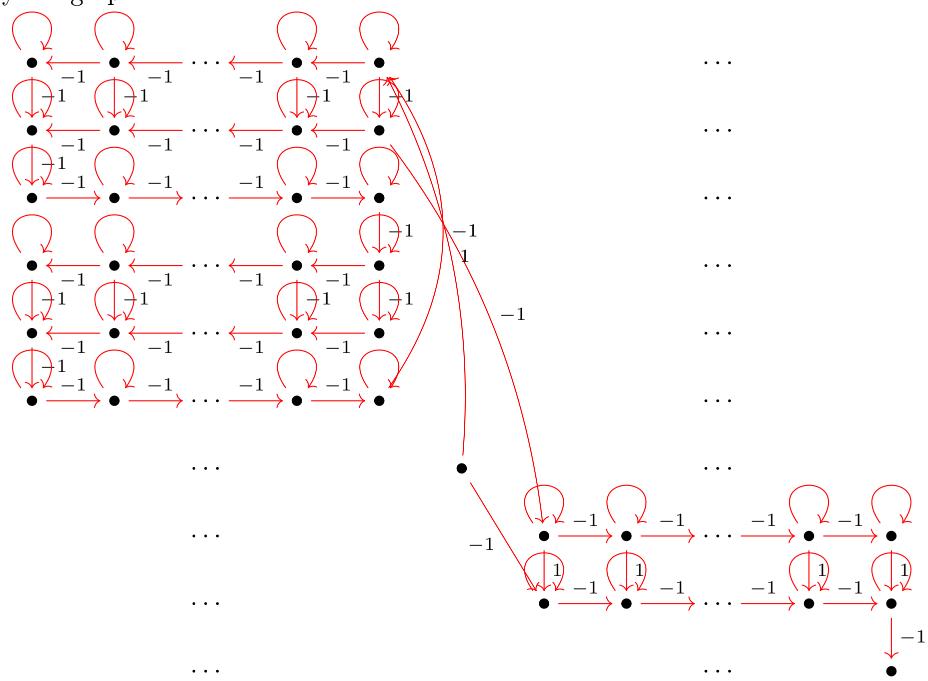
[a]It should be mentioned here that this example is similar to the one given in [41], although some differences are present: apart from the fact that the element is not the same, our construction realizes the 'levels' $e_i$ as a result of allowing matrix elements over the algebra $\mathcal{A}_E$.

b) $C_2$, given by the graph



If we apply Lemma 3.2.9 to this graph, we get that $\dim(\ker(A_{C_2})) = 1$. We only have one connected component of this kind.

c) $C_3$, given by the graphs

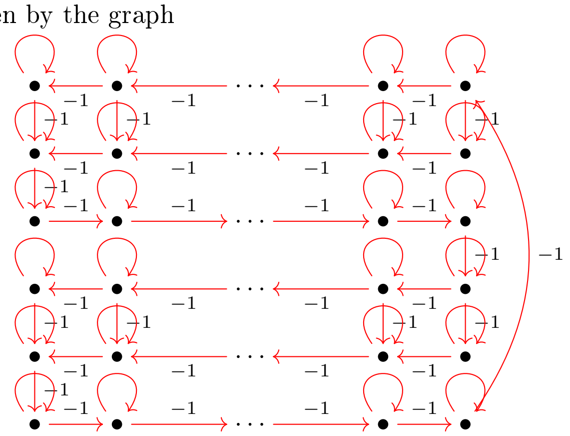


These graphs are the key part of the element $A$, since they will give rise to the irrationality of the value of its von Neumann dimension. Again, by Lemma 3.2.9, it is easy to compute the dimension of the kernel:

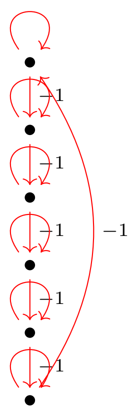
$$\dim(\ker(A_{C_3})) = \left\{ \begin{array}{ll} 2 & \text{if } k_{i+1} = k_i^2 - 1 \\ 1 & \text{otherwise} \end{array} \right\} = 1 + \delta_{k_{i+1}, k_i^2 - 1}.$$

We have $r - 1$ connected components of this kind.

d) $C_4$ finally, given by the graph



for $k_r \geq 2$



for $k_r = 1$

Once more, by virtue of Lemma 3.2.9, we get

$$\dim(\ker(A_{C_4})) = \left\{ \begin{array}{cc} 1 & \text{if } k_r = 1 \\ 0 & \text{otherwise} \end{array} \right\} = \delta_{k_r,1}.$$

Again, we only have one connected component of this kind.

By making use of the formula (3.2.1) and Proposition 3.2.7, we compute

$$\begin{aligned}
\dim_{vN}(\ker(A)) &= \sum_{W\in\mathbb{V}} \dim(\ker(\pi_E(A)_W))\mu(W) = \sum_{W\in\mathbb{V}} \sum_{C\in\mathcal{G}_A(W)} \dim(\ker(A_C))\mu(W) \\
&= \frac{10}{2^3} + \sum_{r\geq 1} \sum_{k_1,\dots,k_r\geq 1} \frac{11+8r+2(k_1+\cdots+k_r)+1+(\delta_{k_2,k_1^2-1}+\cdots+\delta_{k_r,k_{r-1}^2-1})+(r-1)+\delta_{k_r,1}}{2^{k_1+\cdots+k_r+(r+3)}} \\
&= \frac{10}{8} + \frac{1}{8}\sum_{r\geq 1}\frac{11+9r}{2^r} + \frac{1}{8}\sum_{r\geq 1}\frac{4r}{2^r} + \frac{2^3}{8}\sum_{r\geq 1}\frac{1}{2^r}\Big(\sum_{k_1\geq 2}\frac{1}{2^{k_1+k_1^2+2}}+\cdots+\sum_{k_{r-1}\geq 2}\frac{1}{2^{k_{r-1}+k_{r-1}^2+2}}\Big) + \frac{1}{8}\sum_{r\geq 1}\frac{1}{2^{r+1}} \\
&= \frac{10}{8} + \frac{37}{8} + \sum_{k\geq 2}\frac{1}{2^{k^2+k+2}} + \frac{1}{16} = \frac{95}{16} + \sum_{k\geq 2}\frac{1}{2^{k^2+k+2}}
\end{aligned}$$

that is,

$$\dim_{vN}(\ker(A)) = \frac{95}{16} + \sum_{k\geq 2}\frac{1}{2^{k^2+k+2}} \approx 5,941467524...$$

which is an irrational number (its binary expansion is clearly nonperiodic).

Having this example in mind, we now construct an element $A$ with von Neumann dimension of the form

$$q_0 + q_1 \sum_{k\geq 2}\frac{1}{2^{p(k)}},$$

where $q_0, q_1$ are nonzero rational numbers, and $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a degree $n \geq 2$ polynomial with positive integer coefficients. We consider the element from $M_{3n+4}(\mathcal{A}_E)$ given by

$$\begin{aligned}
A = &- \chi_{[0\underline{0}]}t^{-1}\cdot e_{00} - \chi_{[\underline{0}]}\cdot e_{10} - \chi_{[0\underline{0}]}t^{-1}\cdot e_{11} \\
&- \chi_{[1\underline{0}]}\cdot e_{21} - \chi_{[\underline{0}0]}t\cdot e_{22} \\
&- \chi_{[\underline{0}1]}\cdot e_{32} - (a_1-1)\chi_{[\underline{0}1]}\cdot e_{02} \\
&+ \sum_{i=1}^{n-2}\Big( - \chi_{[0\underline{0}]}t^{-1}\cdot e_{3i,3i} - \chi_{[\underline{0}]}\cdot e_{3i+1,3i} \\
&\qquad - \chi_{[0\underline{0}]}t^{-1}\cdot e_{3i+1,3i+1} - \chi_{[1\underline{0}]}\cdot e_{3i+2,3i+1} \\
&\qquad - \chi_{[\underline{0}0]}t\cdot e_{3i+2,3i+2} - \chi_{[\underline{0}1]}\cdot e_{3i+3,3i+2} - a_{i+1}\chi_{[\underline{0}1]}\cdot e_{0,3i+2}\Big) \\
&- \chi_{[0\underline{0}]}t^{-1}\cdot e_{3n-3,3n-3} - \chi_{[\underline{0}]}\cdot e_{3n-2,3n-3} \\
&- \chi_{[0\underline{0}]}t^{-1}\cdot e_{3n-2,3n-2} - \chi_{[1\underline{0}]}\cdot e_{3n-1,3n-2} \\
&- \chi_{[\underline{0}0]}t\cdot e_{3n-1,3n-1} - a_n\chi_{[\underline{0}1]}\cdot e_{0,3n-1} \\
&- \chi_{[\underline{0}10]}t^2\cdot e_{3n+1,1} - \chi_{[\underline{0}0]}t\cdot e_{3n+1,3n+1} + \chi_{[\underline{0}]}\cdot e_{3n+2,3n+1} \\
&- \chi_{[\underline{0}0]}t\cdot e_{3n+2,3n+2} - \chi_{[\underline{0}1]}\cdot e_{3n+3,3n+2} \\
&- \chi_{[0\underline{1}0]}t\cdot e_{3n+2,3n} + \chi_{[0\underline{1}0]}t^{-1}\cdot e_{0,3n} \\
&+ \left(\chi_{[0\underline{0}]} + \chi_{[1\underline{0}0]}\right)\left(\mathrm{Id}_{3n+4} - e_{3n,3n} - e_{3n+3,3n+3}\right).
\end{aligned}$$

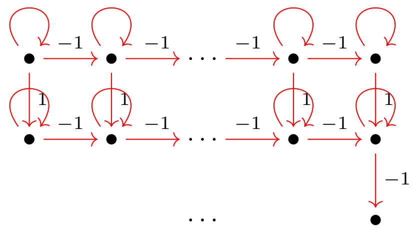
Again, if we take $W = [1\underline{1}1]$, $\pi(A)_W$ gives the zero $(3n+4)\times(3n+4)$ matrix, so its kernel has dimension $3n+4$. For a $W = [1\underline{1}0^{k_1}10^{k_2}1\cdots 0^{k_r}11]$ of length $k = k_1+\cdots+k_r+(r+1)$, its graph $E_A(W)$ has again four different types of connected components $C$, namely

a) $C_1$, given by the graphs with only one vertex

$$\bullet$$

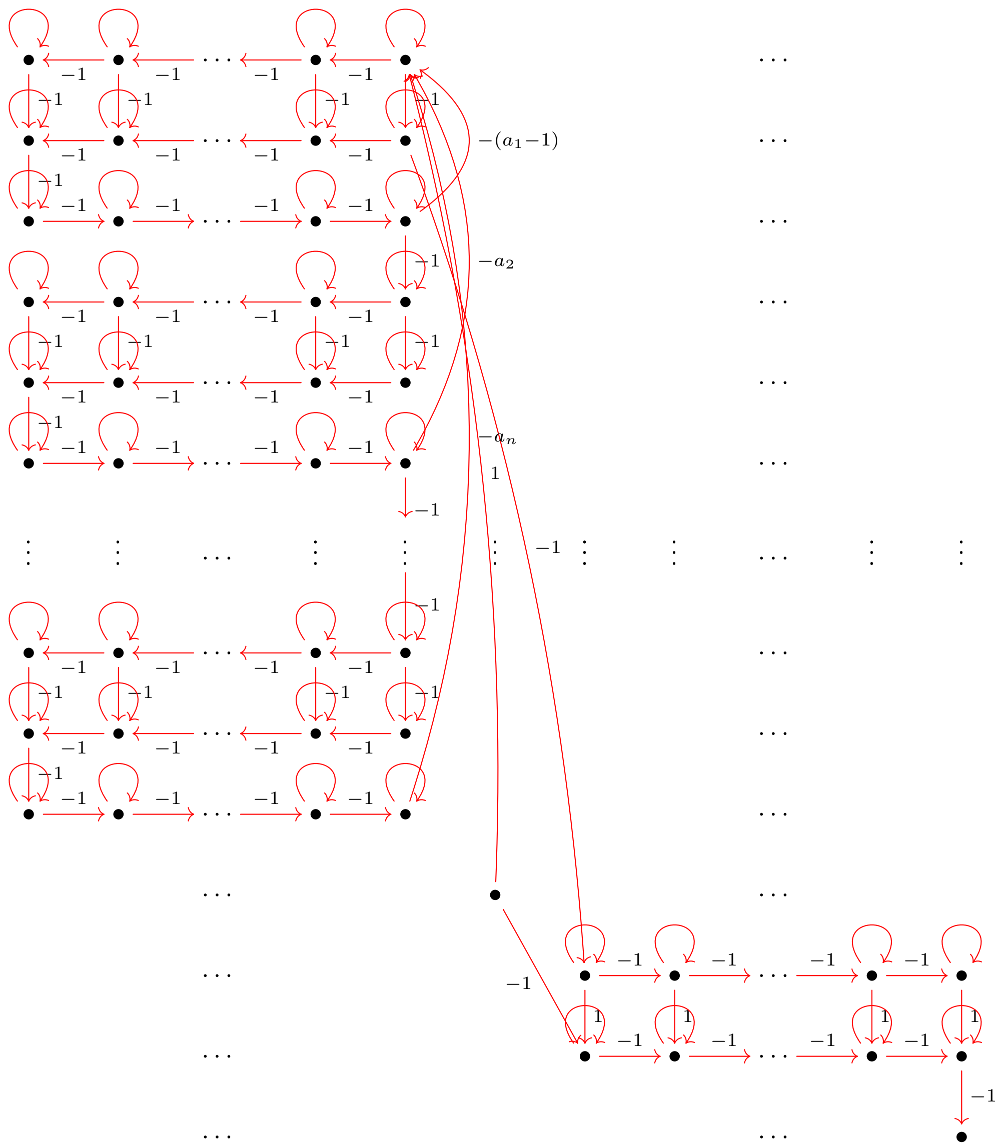
In this case $\dim(\ker(A_{C_1})) = 1$, and we have $3n+4+(2k_1-1)+(3n+3+2k_2-1)+\cdots+(3n+3+2k_r-1)+3n+4 = 3n+5+(3n+2)r+2(k_1+\cdots+k_r)$ connected components of this kind.

b) $C_2$, given by the graph




Applying Lemma 3.2.9 to this graph, we get $\dim(\ker(A_{C_2})) = 1$. We only have one connected component of this kind.

c) $C_3$, given by the graphs

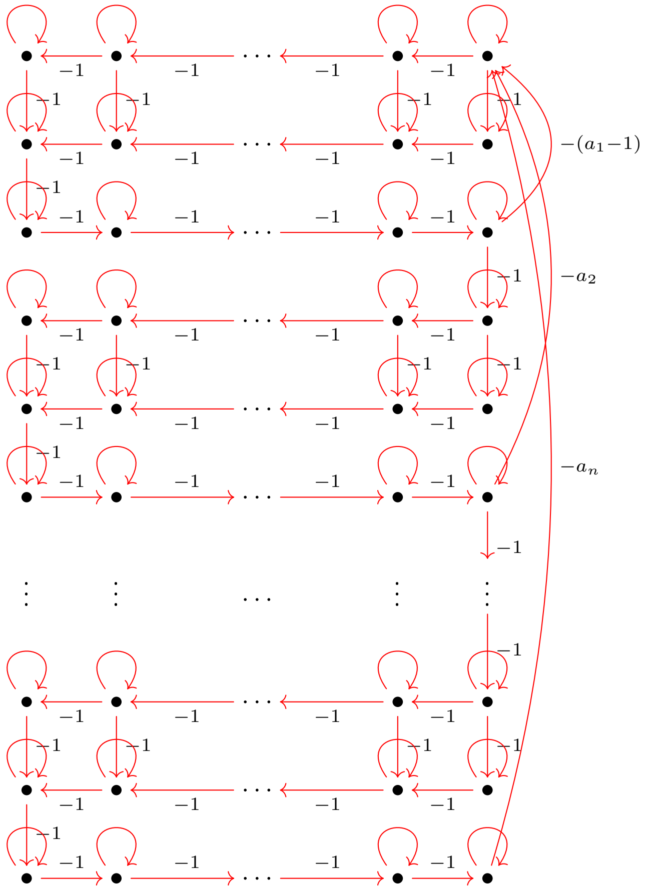



Again, these graphs will be the key point of the element $A$, giving rise to the irrationality of the value $\dim_{vN}(\ker(A))$. Lemma 3.2.9 applied to these graphs tells us that

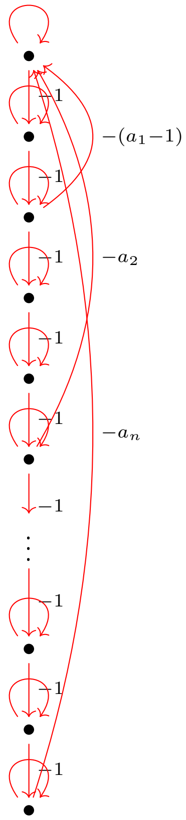
$$\dim(\ker(A_{C_3})) = \left\{ \begin{array}{ll} 2 & \text{if } k_{i+1} = p(k_i) - k_i - a_0 - 1 \\ 1 & \text{otherwise} \end{array} \right\} = 1 + \delta_{k_{i+1}, p(k_i) - k_i - a_0 - 1}.$$

We have $r - 1$ connected components of this kind.

d) Finally $C_4$, given by the graph



for $k_r \geq 2$



for $k_r = 1$

By Lemma 3.2.9 we get $\dim(\ker(A_{C_4})) = 0$ unless our polynomial is of the form $p(x) = a_0 + x + x^2$ (that is $n = 2$ and $a_1 = a_2 = 1$), in such a case we get

$$\dim(\ker(A_{C_4})) = \left\{ \begin{array}{cc} 1 & \text{if } k_r = 1 \\ 0 & \text{otherwise} \end{array} \right\} = \delta_{k_r,1}.$$

We only have one connected component of this type.

We restrict our attention to the case where $p(x)$ is *not* of the form $p(x) = a_0 + x + x^2$, since this particular case was already handled in the preceding example. Due to (3.2.1) and Proposition 3.2.7, we get

$$\begin{aligned}
\dim_{vN}(\ker(A)) &= \sum_{W \in \mathbb{V}} \dim(\ker(\pi_E(A)_W))\mu(W) = \sum_{W \in \mathbb{V}} \sum_{C \in \mathcal{G}_A(W)} \dim(\ker(A_C))\mu(W) \\
&= \frac{3n+4}{2^3} + \sum_{r \geq 1} \sum_{k_1,\ldots,k_r \geq 1} \frac{(3n+5)+(3n+2)r+2(k_1+\cdots+k_r)+1}{2^{k_1+\cdots+k_r+(r+3)}} \\
&\qquad + \sum_{r \geq 1} \sum_{k_1,\ldots,k_r \geq 1} \frac{(r-1)+\delta_{k_2,p(k_1)-k_1-a_0-1}+\cdots+\delta_{k_r,p(k_{r-1})-k_{r-1}-a_0-1}}{2^{k_1+\cdots+k_r+(r+3)}} \\
&= \frac{3n+4}{8} + \frac{1}{8}\sum_{r \geq 1} \frac{(3n+5)+(3n+7)r}{2^r} + \frac{2^{a_0+1}}{8} \sum_{r \geq 1} \frac{1}{2^r} \Big( \sum_{k_1 \geq 2} \frac{1}{2^{p(k_1)}} + \cdots + \sum_{k_{r-1} \geq 2} \frac{1}{2^{p(k_{r-1})}} \Big)
\end{aligned}$$

$$= \frac{3n+4}{8} + \frac{9n+19}{8} + \frac{2^{a_0+1}}{8} \sum_{k \geq 2} \frac{1}{2^{p(k)}} = \frac{12n+23}{8} + \frac{2^{a_0+1}}{8} \sum_{k \geq 2} \frac{1}{2^{p(k)}}$$

that is,

$$\dim_{vN}(\ker(A)) = \frac{12n+23}{8} + \frac{2^{a_0+1}}{8} \sum_{k \geq 2} \frac{1}{2^{p(k)}}$$

which is an irrational number since the degree of the polynomial is at least 2, so the binary expansion of the value is nonperiodic.

Let's do a step further: we now construct an element $A$ with von Neumann dimension of the form

$$q_0 + q_1 \sum_{k \geq 1} \frac{1}{2^{k+p(k)d^k}},$$

where again $q_0, q_1$ are nonzero rational numbers, $p(x)$ is a polynomial satisfying the same conditions as before (but we consider here that its degree is at least 1), and $d \geq 2$ is a natural number. We consider the element from $M_{3n+5}(\mathcal{A}_E)$ given by

$$\begin{aligned}
A = & - d\chi_{[1\underline{0}]} \cdot e_{01} - \chi_{[\underline{0}0]} t \cdot e_{00} \\
& - a_0 \chi_{[\underline{0}1]} \cdot e_{10} - d\chi_{[0\underline{0}]} t^{-1} \cdot e_{11} \\
& - \chi_{[\underline{0}]} \cdot e_{21} - d\chi_{[0\underline{0}]} t^{-1} \cdot e_{22} \\
& - d\chi_{[1\underline{0}]} \cdot e_{32} - \chi_{[\underline{0}0]} t \cdot e_{33} \\
& - \chi_{[\underline{0}1]} \cdot e_{43} - a_1 \chi_{[\underline{0}1]} \cdot e_{13} \\
& + \sum_{i=1}^{n-2} \Big( - \chi_{[0\underline{0}]} t^{-1} \cdot e_{3i+1,3i+1} - \chi_{[\underline{0}]} \cdot e_{3i+2,3i+1} \\
& \quad - \chi_{[0\underline{0}]} t^{-1} \cdot e_{3i+2,3i+2} - \chi_{[1\underline{0}]} \cdot e_{3i+3,3i+2} \\
& \quad - \chi_{[\underline{0}0]} t \cdot e_{3i+3,3i+3} - \chi_{[\underline{0}1]} \cdot e_{3i+4,3i+3} - a_{i+1} \chi_{[\underline{0}1]} \cdot e_{1,3i+3} \Big) \\
& - \chi_{[0\underline{0}]} t^{-1} \cdot e_{3n-2,3n-2} - \chi_{[\underline{0}]} \cdot e_{3n-1,3n-2} \\
& - \chi_{[0\underline{0}]} t^{-1} \cdot e_{3n-1,3n-1} - \chi_{[1\underline{0}]} \cdot e_{3n,3n-1} \\
& - \chi_{[\underline{0}0]} t \cdot e_{3n,3n} - a_n \chi_{[\underline{0}1]} \cdot e_{1,3n} \\
& - \chi_{[\underline{0}10]} t^2 \cdot e_{3n+2,2} - \chi_{[\underline{0}0]} t \cdot e_{3n+2,3n+2} + \chi_{[\underline{0}]} \cdot e_{3n+3,3n+2} \\
& - \chi_{[\underline{0}0]} t \cdot e_{3n+3,3n+3} - \chi_{[\underline{0}1]} \cdot e_{3n+4,3n+3} \\
& - \chi_{[0\underline{1}0]} t \cdot e_{3n+3,3n+1} + \chi_{[0\underline{1}0]} t^{-1} \cdot e_{1,3n+1} \\
& + \left( \chi_{[0\underline{0}]} + \chi_{[1\underline{0}0]} \right) \left( \mathrm{Id}_{3n+5} - e_{3n+1,3n+1} - e_{3n+4,3n+4} \right).
\end{aligned}$$

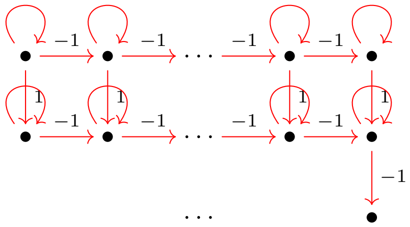
For $W = [1\underline{1}1]$, $\pi(A)_W$ gives the zero $(3n+5) \times (3n+5)$ matrix, so its kernel has dimension $3n+5$. For a $W = [1\underline{1}0^{k_1}10^{k_2}1 \cdots 0^{k_r}11]$ of length $k = k_1 + \cdots + k_r + (r-1)$, its graph $E_A(W)$ has again four different types of connected components $C$, namely

a) $C_1$, given by the graphs with only one vertex

$$\bullet$$

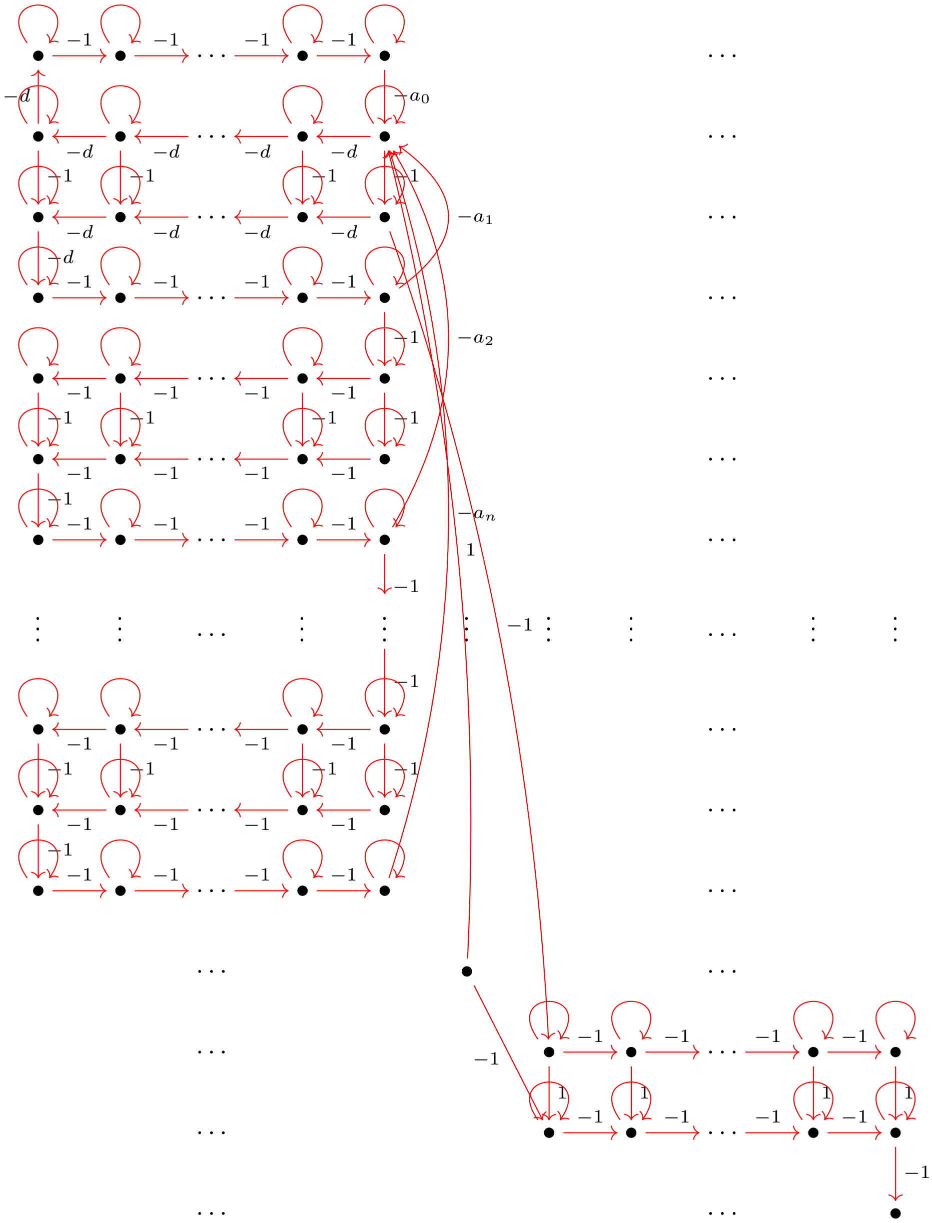
Here $\dim(\ker(A_{C_1})) = 1$, and we have $(3n+5) + (2k_1 - 1) + (3n+3+2k_2) + \cdots + (3n+3+2k_r) + (3n+5) = 3n+6+(3n+3)r + 2(k_1 + \cdots + k_r)$ connected components of this type.

b) $C_2$, given by the graph



We get, using Lemma 3.2.9, $\dim(\ker(A_{C_2})) = 1$. We only have one connected component of this type.

c) $C_3$, given by the graphs

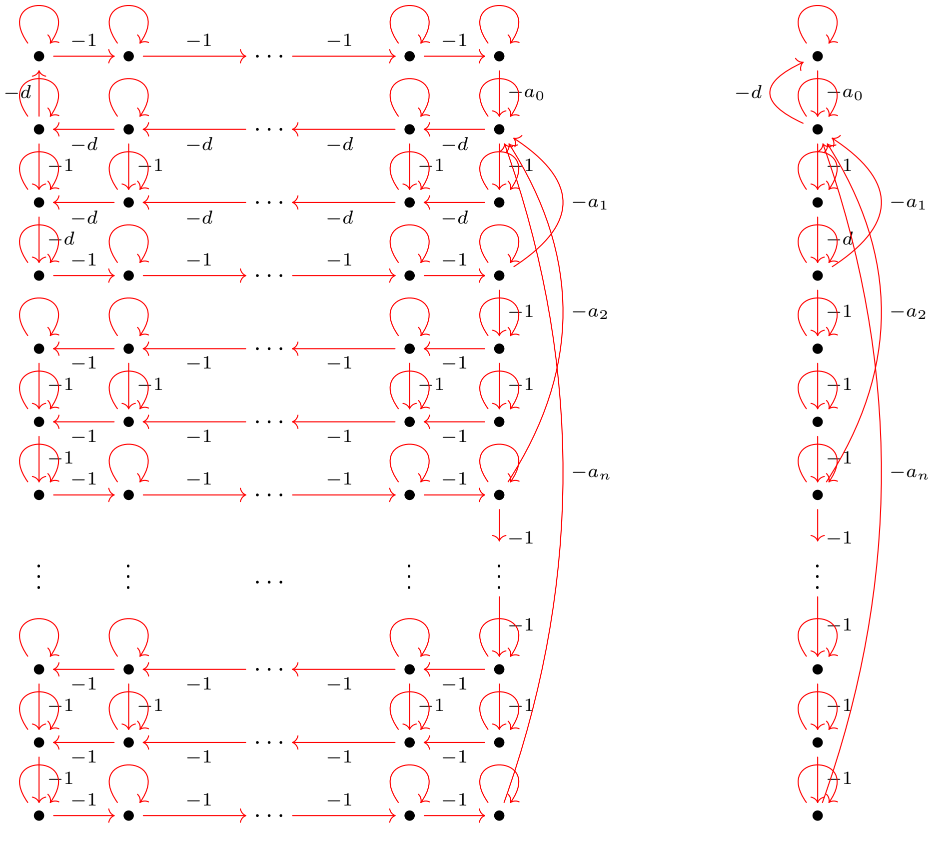


Lemma 3.2.9 applied to these graphs gives

$$\dim(\ker(A_{C_3})) = \left\{ \begin{array}{ll} 2 & \text{if } k_{i+1} = p(k_i)d^{k_i} - 1 \\ 1 & \text{otherwise} \end{array} \right\} = 1 + \delta_{k_{i+1}, p(k_i)d^{k_i}-1}.$$

We have $r - 1$ connected components of this kind.

d) Finally $C_4$, given by the graph




for $k_r \geq 2$      for $k_r = 1$

We have here $\dim(\ker(A_{C_4})) = 0$ again by using Lemma 3.2.9 for every possible value of $k_r$. We only have one connected component of this type.

Due to (3.2.1) and Proposition 3.2.7, we get

$$\begin{aligned}
\dim_{vN}(\ker(A)) &= \sum_{W\in\mathbb{V}} \dim(\ker(\pi_E(A)_W))\mu(W) = \sum_{W\in\mathbb{V}}\sum_{C\in\mathcal{G}_A(W)} \dim(\ker(A_C))\mu(W) \\
&= \frac{3n+5}{2^3} + \sum_{r\geq 1}\sum_{k_1,\dots,k_r\geq 1} \frac{(3n+6)+(3n+3)r+2(k_1+\cdots+k_r)+1}{2^{k_1+\cdots+k_r+(r+3)}} \\
&\qquad + \sum_{r\geq 1}\sum_{k_1,\dots,k_r\geq 1} \frac{(r-1)+\delta_{k_2,p(k_1)d^{k_1}-1}+\cdots+\delta_{k_r,p(k_{r-1})d^{k_{r-1}}-1}}{2^{k_1+\cdots+k_r+(r+3)}} \\
&= \frac{3n+5}{8} + \frac{1}{8}\sum_{r\geq 1}\frac{(3n+6)+(3n+8)r}{2^r} \\
&\qquad + \frac{2}{8}\sum_{r\geq 1}\frac{1}{2^r}\Big(\sum_{k_1\geq 2}\frac{1}{2^{k_1+p(k_1)d^{k_1}}}+\cdots+\sum_{k_{r-1}\geq 2}\frac{1}{2^{k_{r-1}+p(k_{r-1})d^{k_{r-1}}}}\Big) \\
&= \frac{3n+5}{8} + \frac{9n+22}{8} + \frac{2}{8}\sum_{k\geq 1}\frac{1}{2^{k+p(k)d^k}} = \frac{12n+27}{8} + \frac{2}{8}\sum_{k\geq 1}\frac{1}{2^{k+p(k)d^k}}
\end{aligned}$$

that is,

$$\dim_{vN}(\ker(A)) = \frac{12n+27}{8} + \frac{1}{4}\sum_{k\geq 1}\frac{1}{2^{k+p(k)d^k}}$$

which is an irrational number, and even transcendental (see e.g. [95]).

After these examples one can derive the pattern in order to obtain an exponent of the form

$$p_0(k) + p_1(k)d_1^k + \cdots + p_n(k)d_n^k$$

by simply adding more levels, i.e. by considering matrices of higher dimension, and gluing the corresponding graphs in an appropriate way. We write down the corresponding element that gives rise to such a pattern. If we let $N = m_0 + \cdots + m_n$ to be the sum of the degrees of the previous polynomials, then the element $A$ realizing the preceding pattern belongs to $M_{3N+n+5}(\mathcal{A}_E)$, and is given explicitly by

$$\left.\begin{aligned}
A = &- \chi_{[0\underline{0}]}t^{-1} \cdot e_{11} - \chi_{[\underline{0}]} \cdot e_{21} - \chi_{[0\underline{0}]}t^{-1} \cdot e_{22} \\
&- \chi_{[1\underline{0}]} \cdot e_{32} - \chi_{[\underline{0}0]}t \cdot e_{33} - \chi_{[\underline{0}1]} \cdot e_{43} - (a_{1,0} - 1)\chi_{[\underline{0}1]} \cdot e_{03} \\
&+ \sum_{j=1}^{m_0-2} \Big( - \chi_{[0\underline{0}]}t^{-1} \cdot e_{3j+1,3j+1} - \chi_{[\underline{0}]} \cdot e_{3j+2,3j+1} - \chi_{[0\underline{0}]}t^{-1} \cdot e_{3j+2,3j+2} \\
&\qquad - \chi_{[1\underline{0}]} \cdot e_{3j+3,3j+2} - \chi_{[\underline{0}0]}t \cdot e_{3j+3,3j+3} \\
&\qquad - \chi_{[\underline{0}1]} \cdot e_{3j+4,3j+3} - a_{j+1,0}\chi_{[\underline{0}1]} \cdot e_{0,3j+3}\Big) \\
&- \chi_{[0\underline{0}]}t^{-1} \cdot e_{3m_0-2,3m_0-2} - \chi_{[\underline{0}]} \cdot e_{3m_0-1,3m_0-2} - \chi_{[0\underline{0}]}t^{-1} \cdot e_{3m_0-1,3m_0-1} \\
&- \chi_{[1\underline{0}]} \cdot e_{3m_0,3m_0-1} - \chi_{[\underline{0}0]}t \cdot e_{3m_0,3m_0} - a_{m_0,0}\chi_{[\underline{0}1]} \cdot e_{0,3m_0}
\end{aligned}\right\} 1^{\text{st}} \text{ polynomial}$$

$$- \chi_{[\underline{0}1]} \cdot e_{3m_0+2,1}$$

$$\left.\begin{aligned}
&- d_1\chi_{[1\underline{0}]} \cdot e_{3m_0+1,3m_0+2} - \chi_{[\underline{0}0]}t \cdot e_{3m_0+1,3m_0+1} - a_{0,1}\chi_{[\underline{0}1]} \cdot e_{0,3m_0+1} \\
&- d_1\chi_{[0\underline{0}]}t^{-1} \cdot e_{3m_0+2,3m_0+2} - \chi_{[\underline{0}]} \cdot e_{3m_0+3,3m_0+2} \\
&- d_1\chi_{[0\underline{0}]}t^{-1} \cdot e_{3m_0+3,3m_0+3} - d_1\chi_{[1\underline{0}]} \cdot e_{3m_0+4,3m_0+3} \\
&- \chi_{[\underline{0}0]}t \cdot e_{3m_0+4,3m_0+4} - \chi_{[\underline{0}1]} \cdot e_{3m_0+5,3m_0+4} - a_{1,1}\chi_{[\underline{0}1]} \cdot e_{0,3m_0+4} \\
&+ \sum_{j=1}^{m_1-2} \Big( - \chi_{[0\underline{0}]}t^{-1} \cdot e_{3m_0+3j+2,3m_0+3j+2} - \chi_{[\underline{0}]} \cdot e_{3m_0+3j+3,3m_0+3j+2} \\
&\qquad - \chi_{[0\underline{0}]}t^{-1} \cdot e_{3m_0+3j+3,3m_0+3j+3} - \chi_{[1\underline{0}]} \cdot e_{3m_0+3j+4,3m_0+3j+3} \\
&\qquad - \chi_{[\underline{0}0]}t \cdot e_{3m_0+3j+4,3m_0+3j+4} - \chi_{[\underline{0}1]} \cdot e_{3m_0+3j+5,3m_0+3j+4} \\
&\qquad - a_{j+1,1}\chi_{[\underline{0}1]} \cdot e_{0,3m_0+3j+4}\Big) \\
&- \chi_{[0\underline{0}]}t^{-1} \cdot e_{3(m_0+m_1)-1,3(m_0+m_1)-1} - \chi_{[\underline{0}]} \cdot e_{3(m_0+m_1),3(m_0+m_1)-1} \\
&- \chi_{[0\underline{0}]}t^{-1} \cdot e_{3(m_0+m_1),3(m_0+m_1)} - \chi_{[1\underline{0}]} \cdot e_{3(m_0+m_1)+1,3(m_0+m_1)} \\
&- \chi_{[\underline{0}0]}t \cdot e_{3(m_0+m_1)+1,3(m_0+m_1)+1} - a_{m_1,1}\chi_{[\underline{0}1]} \cdot e_{0,3(m_0+m_1)+1}
\end{aligned}\right\} 2^{\text{nd}} \text{ polynomial}$$

$$- \chi_{[\underline{0}1]} \cdot e_{3(m_0+m_1)+3,1}$$

$$\vdots$$

$$- \chi_{[\underline{0}1]} \cdot e_{3(N-m_n)+2,1}$$

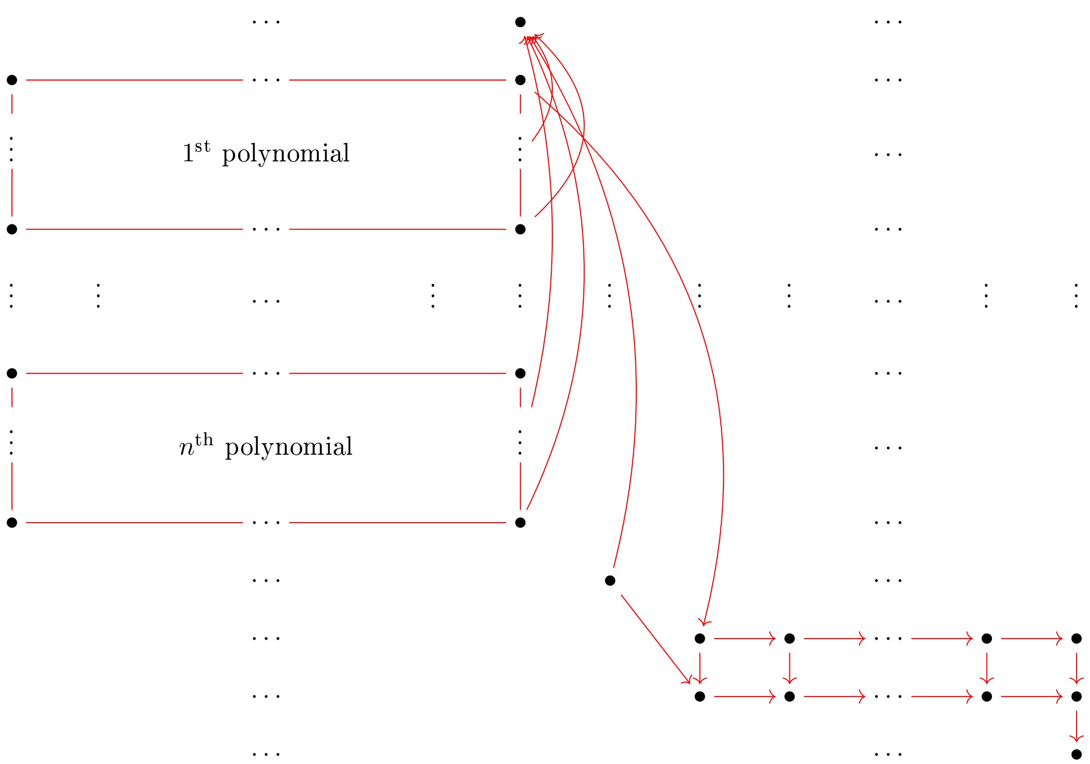
$$\left.\begin{aligned}
&- d_n\chi_{[1\underline{0}]} \cdot e_{3(N-m_n)+n,3(N-m_n)+n+1} - \chi_{[\underline{0}0]}t \cdot e_{3(N-m_n)+n,3(N-m_n)+n} \\
&- a_{0,n}\chi_{[\underline{0}1]} \cdot e_{0,3(N-m_n)+n} - d_n\chi_{[0\underline{0}]}t^{-1} \cdot e_{3(N-m_n)+n+1,3(N-m_n)+n+1} \\
&- \chi_{[\underline{0}]} \cdot e_{3(N-m_n)+n+2,3(N-m_n)+n+1} - d_n\chi_{[0\underline{0}]}t^{-1} \cdot e_{3(N-m_n)+n+2,3(N-m_n)+n+2} \\
&- d_n\chi_{[1\underline{0}]} \cdot e_{3(N-m_n)+n+3,3(N-m_n)+n+2} - \chi_{[\underline{0}0]}t \cdot e_{3(N-m_n)+n+3,3(N-m_n)+n+3}
\end{aligned}\right\} n^{\text{th}} \text{ polynomial}$$

$$
\begin{aligned}
&- \chi_{[\underline{0}1]} \cdot e_{3(N-m_n)+n+4,3(N-m_n)+n+3} - a_{1,n}\chi_{[\underline{0}1]} \cdot e_{0,3(N-m_n)+n+3} \\
&+ \sum_{j=1}^{m_n-2} \Big( - \chi_{[0\underline{0}]} t^{-1} \cdot e_{3(N-m_n)+3j+n+1,3(N-m_n)+3j+n+1} \\
&\qquad - \chi_{[\underline{0}]} \cdot e_{3(N-m_n)+3j+n+2,3(N-m_n)+3j+n+1} \\
&\qquad - \chi_{[0\underline{0}]} t^{-1} \cdot e_{3(N-m_n)+3j+n+2,3(N-m_n)+3j+n+2} \\
&\qquad - \chi_{[1\underline{0}]} \cdot e_{3(N-m_n)+3j+n+3,3(N-m_n)+3j+n+2} \\
&\qquad - \chi_{[\underline{0}0]} t \cdot e_{3(N-m_n)+3j+n+3,3(N-m_n)+3j+n+3} \\
&\qquad - \chi_{[\underline{0}1]} \cdot e_{3(N-m_n)+3j+n+4,3(N-m_n)+3j+n+3} \\
&\qquad - a_{j+1,n}\chi_{[\underline{0}1]} \cdot e_{0,3(N-m_n)+3j+n+3} \Big) \\
&- \chi_{[0\underline{0}]} t^{-1} \cdot e_{3N+n-2,3N+n-2} - \chi_{[\underline{0}]} \cdot e_{3N+n-1,3N+n-2} \\
&- \chi_{[0\underline{0}]} t^{-1} \cdot e_{3N+n-1,3N+n-1} - \chi_{[1\underline{0}]} \cdot e_{3N+n,3N+n-1} \\
&- \chi_{[\underline{0}0]} t \cdot e_{3N+n,3N+n} - a_{m_n,n}\chi_{[\underline{0}1]} \cdot e_{0,3N+n} \\
&\qquad\quad - \chi_{[\underline{0}10]} t^2 \cdot e_{3N+n+2,1} \\
&\qquad\quad + \chi_{[01\underline{0}]} t^{-1} \cdot e_{0,3N+n+1} \\
&\qquad\quad - \chi_{[01\underline{0}]} t \cdot e_{3N+n+3,3N+n+1} \\
&- \chi_{[\underline{0}0]} t \cdot e_{3N+n+2,3N+n+2} + \chi_{[\underline{0}]} \cdot e_{3N+n+3,3N+n+2} \\
&- \chi_{[\underline{0}0]} t \cdot e_{3N+n+3,3N+n+3} - \chi_{[\underline{0}1]} \cdot e_{3N+n+4,3N+n+3} \\
&+ \left(\chi_{[0\underline{0}]} + \chi_{[1\underline{0}0]}\right) \left(\mathrm{Id}_{3N+n+5} - e_{00} - e_{3N+n+1,3N+n+1} - e_{3N+n+4,3N+n+4}\right).
\end{aligned}
$$

$n^{\text{th}}$ polynomial

last graph

The elements in between connect the different polynomials $p_i$, and the contributions (monomials) of the polynomials (that is, the sum $p_0(k) + p_1(k)d_1^k + \cdots + p_n(k)d_n^k$) are accumulated in the $\chi_{[\underline{0}1]} \cdot e_{00}$ component. A simplified schematic of a prototypical graph appearing here is as follows.



Using the same procedure as before, a straightforward (but quite tedious) computation allows us to conclude the proof. We leave the details to the reader. $\square$

To conclude this section, it is interesting to compute some rational values. In [25], the authors computed the von Neumann dimension of the element defined, in our notation from Chapter 2, by $e_0t+t^{-1}e_0 = \chi_{X\setminus E_0}t+t^{-1}\chi_{X\setminus E_0}$, which belongs to $\mathcal{A}_0$. We will compute, in general, the von Neumann dimension of the element $a_n = \chi_{X\setminus E_n}t+t^{-1}\chi_{X\setminus E_n}$, belonging to the $*$-subalgebra $\mathcal{A}_n$. Under $\pi_n$, we obtain the element

$$(0, (t_{m+k}+t^*_{m+k}, \overset{\mathrm{Fib}_m(k)}{\cdots}, t_{m+k}+t^*_{m+k})_{k\geq 1})$$

inside $\mathfrak{R}_n = K \times \prod_{k\geq 1} M_{m+k}(K)^{\mathrm{Fib}_m(k)}$, where recall $m = 2n+1$, and $t_r$ is the $r\times r$ matrix given by

$$\begin{pmatrix} 0 & & & & \mathbf{0} \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & \ddots & 0 & \\ \mathbf{0} & & & 1 & 0 \end{pmatrix}.$$

It is then straightforward to show that

$$\dim(\ker(t_{m+k}+t^*_{m+k})) = \begin{cases} 1 & \text{if } k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

so by Proposition 3.2.7,

$$\begin{aligned} \dim_{vN}(\ker(a_n)) &= \sum_{W\in\mathbb{V}_n} \dim(\ker(\pi_n(a_n)_W))\mu(W) \\ &= \frac{1}{2^{m+1}} + \sum_{k\geq 1} \dim(\ker(t_{m+k}+t^*_{m+k}))\frac{\mathrm{Fib}_m(k)}{2^{2m+k}} = \frac{1}{2^{m+1}} + \sum_{k\geq 1}\frac{\mathrm{Fib}_m(2k)}{2^{2m+2k}}. \end{aligned}$$

This sum can be computed by using the next lemma.

**Lemma 3.2.11.** *We have*

$$\sum_{k\geq 1}\frac{\mathrm{Fib}_m(2k)}{2^{2k}} = 2^{m-1}\frac{2^{m+1}-1}{1+2^{m+2}}, \qquad \sum_{k\geq 0}\frac{\mathrm{Fib}_m(2k+1)}{2^{2k+1}} = 2^{m-1}\frac{2+2^{m+2}-2^{m+1}}{1+2^{m+2}}.$$

*Proof.* Put $K = \sum_{k\geq 1}\frac{Fib_m(k)}{2^k}, K_{\text{even}} = \sum_{k\geq 1}\frac{Fib_m(2k)}{2^{2k}}, K_{\text{odd}} = \sum_{k\geq 0}\frac{Fib_m(2k+1)}{2^{2k+1}}$, so that $K = K_{\text{even}}+K_{\text{odd}}$. We already know from Lemma 3.2.3 that $K = 2^{m-1}$. Once again we will use the recurrence relation for the $\mathrm{Fib}_m(k)$. First note that for $1\leq k\leq n$, $\frac{\mathrm{Fib}_m(2k)}{2^{2k}} = \frac{2^{2k-2}}{2^{2k}} = \frac{1}{4}$, so for $1\leq r\leq n$ we have that $S_r^{\text{even}} := \sum_{k=1}^r \frac{\mathrm{Fib}_m(2k)}{2^{2k}} = \frac{r}{4}$. Similarly, the first term of the second sum is $\frac{\mathrm{Fib}_m(1)}{2} = \frac{1}{2}$, and for $1\leq k\leq n$, $\frac{\mathrm{Fib}_m(2k+1)}{2^{2k+1}} = \frac{2^{2k-1}}{2^{2k+1}} = \frac{1}{4}$ since

$$\mathrm{Fib}_m(m) = \mathrm{Fib}_m(m-1)+\cdots+\mathrm{Fib}_m(0) = 2^{m-3}+\cdots+2+1+1+0 = 2^{m-2},$$

so for $0\leq r\leq n$ we have that $S_r^{\text{odd}} := \sum_{k=0}^r\frac{\mathrm{Fib}_m(2k+1)}{2^{2k+1}} = \frac{1}{2}+\frac{r}{4} = \frac{r+2}{4}$.

We can decompose the initial sum as

$$\begin{aligned} K_{\text{even}} &= \sum_{k=1}^{n}\frac{\mathrm{Fib}_m(2k)}{2^{2k}} + \sum_{k=0}^{\infty}\frac{\mathrm{Fib}_m(m+2k+1)}{2^{m+2k+1}} = \frac{n}{4} + \sum_{i=1}^{m}\frac{1}{2^i}\Big(\sum_{k\geq 0}\frac{\mathrm{Fib}_m(m+2k+1-i)}{2^{m+2k+1-i}}\Big) \\ &= \frac{n}{4} + \sum_{i=1}^{n}\frac{1}{2^{2i}}\Big(\sum_{k\geq 0}\frac{\mathrm{Fib}_m(m+2k+1-2i)}{2^{m+2k+1-2i}}\Big) + \sum_{i=0}^{n}\frac{1}{2^{2i+1}}\Big(\sum_{k\geq 0}\frac{\mathrm{Fib}_m(m+2k+1-2i-1)}{2^{m+2k+1-2i-1}}\Big) \\ &= \frac{n}{4} + \sum_{i=1}^{n-1}\frac{1}{2^{2i}}\Big(\sum_{k\geq 0}\frac{\mathrm{Fib}_m(2(n+k+1-i))}{2^{2(n+k+1-i)}}\Big) + \frac{K_{\text{even}}}{2^{2n}} + \sum_{i=0}^{n-1}\frac{1}{2^{2i+1}}\Big(\sum_{k\geq 0}\frac{\mathrm{Fib}_m(2(n+k-i)+1)}{2^{2(n+k-i)+1}}\Big) + \frac{K_{\text{odd}}}{2^{2n+1}}. \end{aligned}$$

But for $1 \leq i \leq n-1$, $\sum_{k\geq 0} \frac{\mathrm{Fib}_m(2(n+k+1-i))}{2^{2(n+k+1-i)}} = K_{\text{even}} - S^{\text{even}}_{n-i}$, and for $0 \leq i \leq n-1$, $\sum_{k\geq 0} \frac{\mathrm{Fib}_m(2(n+k-i)+1)}{2^{2(n+k-i)+1}} = K_{\text{odd}} - S^{\text{odd}}_{n-1-i}$, so

$$\begin{aligned} K_{\text{even}} &= \frac{n}{4} + \sum_{i=1}^{n-1} \frac{1}{2^{2i}}\Big(K_{\text{even}} - S^{\text{even}}_{n-i}\Big) + \frac{1}{2^{2n}} K_{\text{even}} + \sum_{i=0}^{n-1} \frac{1}{2^{2i+1}}\Big(K_{\text{odd}} - S^{\text{odd}}_{n-1-i}\Big) + \frac{1}{2^{2n+1}} K_{\text{odd}} \\ &= \frac{n}{4} + \Big(\sum_{i=1}^{n} \frac{1}{2^{2i}} - \sum_{i=0}^{n} \frac{1}{2^{2i+1}}\Big) K_{\text{even}} + \Big(\sum_{i=0}^{n} \frac{1}{2^{2i+1}}\Big) K - \Big(\sum_{i=1}^{n-1} \frac{S^{\text{even}}_{n-i}}{2^{2i}} + \sum_{i=0}^{n-1} \frac{S^{\text{odd}}_{n-1-i}}{2^{2i+1}}\Big). \end{aligned}$$

We compute $\sum_{i=1}^{n} \frac{1}{2^{2i}} = \frac{1}{3} - \frac{1}{3\cdot 2^{2n}}$, $\sum_{i=0}^{n} \frac{1}{2^{2i+1}} = \frac{2}{3} - \frac{1}{6\cdot 2^{2n}}$, and

$$\sum_{i=1}^{n-1} \frac{S^{\text{even}}_{n-i}}{2^{2i}} = \frac{1}{4}\sum_{i=1}^{n-1} \frac{n-i}{2^{2i}} = \frac{n}{12} - \frac{1}{9}\Big(1 - \frac{1}{2^{2n}}\Big), \qquad \sum_{i=0}^{n-1} \frac{S^{\text{odd}}_{n-1-i}}{2^{2i+1}} = \frac{1}{8}\sum_{i=0}^{n-1} \frac{n+1-i}{2^{2i}} = \frac{n}{6} + \frac{1}{9}\Big(1 - \frac{1}{2^{2n}}\Big)$$

where we have used the sum $\sum_{j=1}^{n-1} jx^j = \frac{x(1-x^{n-1})}{(1-x)^2} - \frac{(n-1)x^n}{1-x}$ for real $x \neq 1$. Putting everything together,

$$K_{\text{even}} = \frac{n}{4} - \frac{1}{3}\Big(1 + \frac{1}{2^{2n+1}}\Big) K_{\text{even}} + \Big(\frac{2}{3} - \frac{1}{6\cdot 2^{2n}}\Big) K - \frac{n}{4},$$

so the result $K_{\text{even}} = \frac{2^{m+1}-1}{1+2^{m+2}} K$ follows. Since $K = K_{\text{even}} + K_{\text{odd}}$, $K_{\text{odd}}$ also gives the stated value. $\square$

To conclude,

$$\dim_{vN}(\ker(a_n)) = \frac{1}{2^{m+1}} + \frac{1}{2^{2m}} K_{\text{even}} = \frac{3}{1+2^{m+2}} = \frac{3}{1+2^{2n+3}}.$$

Note that, for $n = 0$, $\dim_{vN}(\ker(a_0)) = \frac{1}{3}$, and we recover the result given by Dicks and Schick. Also, as $n \to \infty$, this value tends to zero, as expected since $a_n \to t + t^{-1}$ in rank, which is invertible inside $\mathfrak{R}_{\text{rk}}$.

## 3.3 The odometer algebra

In this section we concentrate in studying the odometer algebra. We first recall how it is defined.

Let $X$ be the compact space $X = \prod_{i\in\mathbb{N}}\{0,1\}$ and let $T$ be the homeomorphism $X \to X$ given by the odometer, namely for $x = (x_i)_i \in X$, $T$ is given by

$$T(x) = x + (1,0,...) = \begin{cases} (1, x_2, x_3, ...) & \text{if } x_1 = 0, \\ (0, ..., 0, 1, x_{n+2}, ...) & \text{if } x_1 = \cdots = x_n = 1 \text{ and } x_{n+1} = 0, \\ (0,0,...) & \text{if } x = (1)_i. \end{cases}$$

Note that the odometer action is just adding $(1,0,...)$ with respect to the binary arithmetics.

Let $(K,*)$ be any field with a positive definite involution $*$. We consider again the $*$-algebra $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$. We will show that $\mathcal{A}$ is isomorphic to the $*$-algebra $\mathcal{P}$ considered in [29, Section 5]. For our convenience, we are going to denote such a $*$-algebra by $\mathcal{PO}_K$. In that paper, only the case $K = \mathbb{C}$ is considered.

We first recall some definitions from [29].

**Definition 3.3.1.** A function $P : \mathbb{Z}\times\mathbb{Z} \to K$ is said to be a *periodic operator* if there exists some $n \geq 1$ such that

a) $P(x,y) = 0$ if $|x-y| > 2^n$, and

b) $P(x,y) = P(x+2^n, y+2^n)$.

We call the value $2^n$ the *period* of $P$. The set of periodic operators $\mathcal{PO}_K$ is a $*$-subalgebra of the $K$-algebra of row-and-column finite matrices with coefficients in $K$, which is endowed with the $*$-transpose involution.

Let $J$ be the periodic operator defined by $J(x+1,x) = 1$ for all $x \in \mathbb{Z}$ and $J(y,x) = 0$ if $y \neq x+1$. We now show that $\mathcal{A}$ is $*$-isomorphic to $\mathcal{PO}_K$.

**Proposition 3.3.2.** *There is a $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{PO}_K$ which sends $C_K(X)$ to the diagonal $*$-subalgebra $\Delta$ of $\mathcal{PO}_K$ and the generator $t$ of $\mathbb{Z}$ to the operator $J$.*

*Proof.* For $n \geq 0$, and $0 \leq l \leq 2^n - 1$, consider the representation of $l$ in binary expression, say $l = \epsilon_1 + 2\epsilon_2 + \cdots + 2^{n-1}\epsilon_n$, and the corresponding basic clopen subset

$$U_{n,l} = [\underline{\epsilon_1}\epsilon_2 \cdots \epsilon_n] = \{x \in X \mid x_i = \epsilon_i \text{ for } 1 \leq i \leq n\},$$

and the set of projections $e_{n,l} = \chi_{U_{n,l}}$ in $C_K(X)$ given by the characteristic functions on these clopen sets $U_{n,l}$. Note that they satisfy

$$\sum_{l=0}^{2^n-1} e_{n,l} = 1, \qquad e_{n,l} = e_{n+1,l} + e_{n+1,l+2^n}$$

for every $n \geq 0$. The algebra $C_K(X)$ is then a commutative ultramatricial algebra with a binary tree as a Bratteli diagram. We define a map

$$\varphi : C_K(X) \to \Delta, \quad e_{n,l} \mapsto d_{n,l}$$

where $d_{n,l}$ is the diagonal periodic operator of period $2^n$ defined by $d_{n,l}(l,l) = 1$ and $d_{n,l}(k,k) = 0$ for $k \neq l$ and $0 \leq k \leq 2^n - 1$. It is easily checked that $\varphi$ provides a $*$-isomorphism from $C_K(X)$ to $\Delta$. Let's show that it can be extended to a $*$-isomorphism $\mathcal{A} \to \mathcal{PO}_K$.

Note that

$$te_{n,l}t^{-1} = t\chi_{U_{n,l}}t^{-1} = \chi_{T(U_{n,l})} = \chi_{U_{n,(l+1) \bmod 2^n}} = e_{n,(l+1) \bmod 2^n}.$$

Moreover, we also have that

$$Jd_{n,l}J^{-1} = d_{n,(l+1) \bmod 2^n}$$

for all $n \geq 0$ and $0 \leq l \leq 2^n - 1$. It follows that we can extend $\varphi$ to a $*$-isomorphism $\varphi : \mathcal{A} \to \mathcal{PO}_K$ by sending $t$ to $J$. $\square$

When $K$ is a subfield of $\mathbb{C}$ closed under complex conjugation and containing all the $2^{nth}$ roots of unity for all $n \geq 0$, the algebra $\mathcal{A}$ is isomorphic to the algebra $\mathcal{P}$ considered in [29, Section 5]. Since Elek works with $K = \mathbb{C}$, we have no need to change our notation here since our algebra $\mathcal{PO}_K$ coincides in this case with his algebra $\mathcal{P}$.

In fact, if $K$ is a field of characteristic different from 2 and containing all the $2^{nth}$ roots of unity for all $n \geq 0$, we can give a description of $C_K(X)$ as the group algebra of a concrete group, namely the Prüfer 2-group $\mathbb{Z}(2^\infty)$. This group can be defined as the subgroup of the complex numbers that are $2^{nth}$ roots of 1 for some $n \geq 0$, that is,

$$\mathbb{Z}(2^\infty) = \{z \in \mathbb{C} \mid z^{2^n} = 1 \text{ for some } n \geq 1\}.$$

In terms of generators and relations, if we denote by 1 the unit element of $\mathbb{Z}(2^\infty)$ and by $g_n$ a primitive $2^{nth}$ root of unity, we have the following presentation for $\mathbb{Z}(2^\infty)$:

$$\langle \{g_n\}_{n\geq 1} \mid g_1^2 = 1, g_{n+1}^2 = g_n \text{ for } n \geq 1\rangle.$$

Take now a collection $\{\xi_{2^n}\}_{n\geq 1}$ of primitive $2^{nth}$ roots of unity in $K$ such that $\xi_{2^{n+1}}^2 = \xi_{2^n}$. The Pontryagin dual of the Prüfer 2-group $\mathbb{Z}(2^\infty)$ can be identified with the group $\prod_{i\in\mathbb{N}} \mathbb{Z}_2 = X$ (whose operation is addition by carry-over) by means of

$$\prod_{i\in\mathbb{N}} \mathbb{Z}_2 \xrightarrow{\cong} \widehat{\mathbb{Z}(2^\infty)}, \quad x = (a_1, a_2, ...) \mapsto \phi_x \quad \text{where} \quad \phi_x(g_n) = \xi_{2^n}^{a_1+2a_2+\cdots+2^{n-1}a_n}.$$

Under this identification, it is clear that the clopen subset $U_{g_n,l} = \{\phi \in \widehat{\mathbb{Z}(2^\infty)} \mid \phi(g_n) = \xi_{2^n}^l\}$ corresponds to the clopen $U_{n,l} = [\underline{\epsilon_1}\epsilon_2 \cdots \epsilon_n] = \{x \in X \mid x_i = \epsilon_i \text{ for } 1 \leq i \leq n\}$, where $l = \epsilon_1 + \cdots + 2^{n-1}\epsilon_n$.

We now consider the group algebra $K[\mathbb{Z}(2^\infty)]$. By Fourier transform[12] the homeomorphism $T : X \to X$ given by carry-over induces a $\mathbb{Z}$-action on $K[\mathbb{Z}(2^\infty)]$ by means of the diagram

$$\begin{array}{ccc} K[\mathbb{Z}(2^\infty)] & \xrightarrow{\mathscr{F}} & C_K(X) \\ {\scriptstyle \rho(1)} \downarrow & & \downarrow {\scriptstyle T, \quad T(f)(x) = f(T^{-1}(x))} \\ K[\mathbb{Z}(2^\infty)] & \xrightarrow{\mathscr{F}} & C_K(X) \end{array}$$

[12]See Proposition 3.1.1.

That is, $\mathbb{Z}$ acts on $K[\mathbb{Z}(2^\infty)]$ by automorphisms by the rule

$$\rho : \mathbb{Z} \curvearrowright K[\mathbb{Z}(2^\infty)], \quad \rho(1)(g_n) = \xi_{2^n} g_n.$$

In the next proposition we show that the group algebra $K[\mathbb{Z}(2^\infty)]$ is $*$-isomorphic to the $*$-subalgebra $\Delta$ of diagonal periodic operators, and the $\mathbb{Z}$-crossed product $K[\mathbb{Z}(2^\infty)] \rtimes_\rho \mathbb{Z}$ is $*$-isomorphic to the whole $*$-algebra $\mathcal{PO}_K$, hence $*$-isomorphic to $C_K(X) \rtimes_T \mathbb{Z}$ by Proposition 3.3.2.

**Proposition 3.3.3.** *Let $K$ be a field of characteristic different from 2 and containing all the $2^{nth}$ roots of unity for all $n \geq 0$. There exists then a $*$-isomorphism $\psi : K[\mathbb{Z}(2^\infty)] \to \Delta$ which extends to a $*$-isomorphism $\psi : K[\mathbb{Z}(2^\infty)] \rtimes_\rho \mathbb{Z} \to \mathcal{PO}_K$ by sending the generator $t$ of $\mathbb{Z}$ to the periodic operator $J$.*

*Proof.* This is a direct consequence of Proposition 3.1.1: we have $*$-isomorphisms

$$\psi : K[\mathbb{Z}(2^\infty)] \xrightarrow{\mathscr{F}} C_K(X) \xrightarrow{\cong} \Delta$$

which we know that extend to $*$-isomorphisms on the respective crossed products

$$\psi : K[\mathbb{Z}(2^\infty)] \rtimes_\rho \mathbb{Z} \xrightarrow{\mathscr{F}} C_K(X) \rtimes_T \mathbb{Z} \xrightarrow{\cong} \mathcal{PO}_K.$$ □

**Remarks 3.3.4.**

1) An alternative proof for Proposition 3.3.3 is given in [29, Lemma 5.2]: the author defines, for each $n \geq 0$ and $0 \leq l \leq 2^n - 1$, the diagonal periodic operator $E_{n,l}$ by

$$E_{n,l}(x,x) = \overline{\xi}_{2^n}^{xl},$$

hence establishing a $*$-isomorphism $\psi : K[\mathbb{Z}(2^\infty)] \to \Delta$ by sending $g_n \mapsto E_{n,1}$. This coincides with our given isomorphism. Also,

$$JE_{n,l}J^{-1} = \xi_{2^n}^l E_{n,l}$$

for all $n \geq 0$ and $0 \leq l \leq 2^n - 1$, so $\psi$ extends to a $*$-isomorphism $\psi : \mathcal{A} \to \mathcal{PO}_K$ by sending $t$ to $J$.

2) It is worth mentioning that in this particular example it is not possible to realize the corresponding algebra $K[\mathbb{Z}(2^\infty)] \rtimes_\rho \mathbb{Z}$ as a group algebra $K[G]$, simply because the crossed product is a simple algebra, as mentioned at the beginning of this chapter. Nevertheless, it is an interesting example because we are able to use the whole machinery from Chapter 2 in order to fully study it, giving for example explicit descriptions concerning its $*$-regular close and the set of $l^2$-Betti numbers arising from it (see the next subsections 3.3.1 and 3.3.2).

The problem we encounter here is that, since the odometer algebra cannot be realized as a group algebra, there is no canonical rank function on it even if $K = \mathbb{C}$. Let's try to analyze the situation. In this discussion, we let $K = \mathbb{C}$.

What we have is a canonical $T$-invariant Sylvester matrix rank function $\mathrm{rk}_{K[\mathbb{Z}(2^\infty)]}$ on the group algebra $K[\mathbb{Z}(2^\infty)]$ (the one inherited from $\mathcal{U}(\mathbb{Z}(2^\infty))$) which, under $\mathscr{F}$, gives a $T$-invariant Sylvester matrix rank function $\mathrm{rk}_{C_K(X)}$ on $C_K(X)$. By Lemma 2.3.11, $\mathrm{rk}_{C_K(X)}$ corresponds to a $T$-invariant probability measure $\mu$ on $X$, and in fact it is easy to compute its value on any clopen set of the form $U_{n,l} = [\underline{\epsilon_1}\epsilon_2 \cdots \epsilon_n]$, where $l = \epsilon_1 + \cdots + 2^{n-1}\epsilon_n$:

$$\begin{aligned}\mu(U_{n,l}) &= \mathrm{rk}_{K[\mathbb{Z}(2^\infty)]}(\mathscr{F}^{-1}(\chi_{U_{n,l}})) = \mathrm{tr}_{K[\mathbb{Z}(2^\infty)]}(\mathscr{F}^{-1}(\chi_{U_{n,l}})) \\ &= \mathrm{tr}_{K[\mathbb{Z}(2^\infty)]}\Big(\frac{1}{2^n}(e + \xi_{2^n}^l g_n + \cdots + \xi_{2^n}^{(2^n-1)l} g_n^{2^n-1})\Big) = \frac{1}{2^n}.\end{aligned}$$

One observes that $\mu$ coincides with the usual product measure, where we take the $\left(\frac{1}{2}, \frac{1}{2}\right)$-measure on each component $\{0,1\}$. It is well-known (cf. [24, Section VIII.4]) that $\mu$ is an ergodic, full and $T$-invariant probability measure on $X$, which in turn coincides with the Haar measure $\widehat{\mu}$ on $X = \widehat{\mathbb{Z}(2^\infty)}$. Hence we can apply our construction from Chapter 2 (specifically, Theorem 2.3.7), to obtain a canonical Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$ on $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z}$, therefore a *canonical* Sylvester matrix rank function on $K[\mathbb{Z}(2^\infty)] \rtimes_\rho \mathbb{Z}$ by pulling back $\mathrm{rk}_{\mathcal{A}}$ under $\mathscr{F}$.

This observation enables us to construct, for *any* field $K$ of characteristic different from 2 and containing all the $2^{nth}$ roots of unity for all $n \geq 0$, a *canonical* Sylvester matrix rank function on $K[\mathbb{Z}(2^\infty)] \rtimes_\rho \mathbb{Z}$ by pulling back $\mathrm{rk}_{\mathcal{A}}$ under $\mathscr{F}$.

### 3.3.1 The $*$-regular closure $\mathcal{R}_{\mathcal{A}}$

In his article [29], Elek computed the rank completion $\overline{\mathcal{R}}_{K[\Gamma]}^{\text{rk}}$ of the $*$-regular closure of the lamplighter group algebra for the case $K = \mathbb{C}$ by first computing the rank completion of the $*$-regular closure of the odometer algebra and then showing that these two must be isomorphic, by a famous theorem of Ornstein and Weiss. We have been able to exactly compute the $*$-regular closure of the odometer algebra $\mathcal{A} \cong K[\mathbb{Z}(2^\infty)] \rtimes_\rho \mathbb{Z}$ in the case $K$ being any field of characteristic different from 2 and containing all the $2^{n\text{th}}$ roots of unity for all $n \geq 0$.

First of all, we first show that our $*$-regular closure $\mathcal{R}_{\mathcal{A}}$ as defined can be identified with his $*$-regular closure defined in [29, Section 2] for crossed product algebras in the case $K = \mathbb{C}$, the field of complex numbers. His definition uses the Murray-von Neumann construction for group-measure spaces, as defined in Remark 3.1.5.2). So here $X = \prod_{i\in\mathbb{N}}\{0,1\}$.

We denote by $\lambda$ the left-regular representation $\lambda : L^\infty(X,\widehat{\mu}) \rtimes_T \mathbb{Z} \to \mathcal{B}(\mathcal{H})$, where $\mathcal{H} = l^2(\mathbb{Z}, L^2(X,\widehat{\mu}))$. Note that $\mathcal{A} = C_K(X) \rtimes_T \mathbb{Z} \subseteq L^\infty(X,\widehat{\mu}) \rtimes_T \mathbb{Z}$. We denote by $\mathcal{N}(\mathcal{A})$ the weak-completion of $\lambda(\mathcal{A})$ inside $\mathcal{B}(\mathcal{H})$, which coincides with the weak-completion of $\lambda(L^\infty(X,\widehat{\mu}) \rtimes_T \mathbb{Z})$, denoted by $\mathcal{N}(T)$. One obtains in this way a diagram

$$\begin{array}{ccccc} \mathcal{A} & \hookrightarrow & \mathcal{R}_{\mathcal{A}} & \hookrightarrow & \mathfrak{R}_{\text{rk}} \\ \cap\!\!| & & & & \\ L^\infty(X,\widehat{\mu}) \rtimes_T \mathbb{Z} & \hookrightarrow & \mathcal{R}_T & \hookrightarrow & \mathcal{U}(T) \end{array}$$

where $\mathcal{U}(T)$ is the algebra of (unbounded) affiliated operators of $\mathcal{N}(T)$, $\text{rk}_{\mathcal{U}(T)}$ is its canonical rank function, and $\mathcal{R}_T := \overline{\mathcal{N}(T)}^{\text{rk}_{\mathcal{U}(T)}}$. The rank function $\text{rk}_{\mathcal{U}(T)}$, when restricted to $\mathcal{A}$, coincides with $\text{rk}_{\mathcal{A}}$ by the uniqueness part of Proposition 2.3.8, since for any clopen subset $U$ of $X$

$$\text{rk}_{\mathcal{U}(T)}(\chi_U) = \text{tr}_{L^\infty(X,\widehat{\mu})\rtimes_T\mathbb{Z}}(\chi_U) = \int_X \chi_U d\widehat{\mu}(x) = \mu(U).$$

So an argument similar to the one given in the proof of Proposition 3.1.6 extends the above diagram to a commutative one

$$\begin{array}{ccccc} \mathcal{A} & \hookrightarrow & \mathcal{R}_{\mathcal{A}} & \hookrightarrow & \mathfrak{R}_{\text{rk}} \\ \cap\!\!| & & \cong & & \downarrow \\ L^\infty(X,\widehat{\mu}) \rtimes_T \mathbb{Z} & \hookrightarrow & \mathcal{R}_T & \hookrightarrow & \mathcal{U}(T). \end{array}$$

We are now ready to compute $\mathcal{R}_{\mathcal{A}}$ for a general field $K$ satisfying the above hypotheses. We will follow the same notation as introduced in Section 3.2 for the case of the lamplighter group algebra. We take $E_n = [\underline{1}1...1]$ (with $n$ one's) for the sequence of clopen sets, whose intersection gives the point $y = (\underline{1}, 1, 1, ...) \in X$. We take the partitions $\mathcal{P}_n$ of the complements $X \backslash E_n$ to be the obvious ones, namely

$$\mathcal{P}_n = \{[\underline{0}0 \cdots 00], [\underline{1}0 \cdots 00], ..., [\underline{1}1 \cdots 10]\}.$$

The unital $*$-subalgebra $\mathcal{A}_n$ is then generated by the partial isometries $\chi_Z t$ for $Z \in \mathcal{P}_n$.

The quasi-partition $\overline{\mathcal{P}}_n$ is really simple in this case: write $Z_{n,0} = [\underline{0}0 \cdots 00]$, and $Z_{n,l} = T^l(Z_{n,0})$ for $1 \leq l \leq 2^n - 2$. Note that these clopen sets form exactly the partition $\mathcal{P}_n$, and that $T(E_n) = Z_{n,0}$, $T(Z_{n,2^n-2}) = E_n$. Therefore there is only one possible $W \in \mathbb{V}_n$, which is of length $2^n$ and given by

$$W = E_n \cap T^{-1}(Z_{n,0}) \cap T^{-2}(Z_{n,1}) \cap \cdots \cap T^{-2^n+1}(Z_{n,2^n-2}) \cap T^{-2^n}(E_n) = E_n.$$

Clearly,

$$\sum_{k\geq 1} \sum_{\substack{W\in\mathbb{V}_n \\ |W|=k}} k\mu(W) = 2^n \mu(E_n) = 1$$

as we already know.

The representations $\pi_n : \mathcal{A}_n \hookrightarrow \mathfrak{R}_n, x \mapsto (h_W \cdot x)_W$ become $*$-isomorphisms, with $\mathfrak{R}_n = M_{2^n}(K)$. Indeed, for $W = E_n$, we have $e_{00}(W) = \chi_{E_n}$, and for $1 \leq i \leq 2^n - 1$, $e_{ii}(W) = \chi_{T^i(E_n)} = \chi_{Z_{n,i-1}}$. Therefore $h_{E_n} = \sum_{i=0}^{2^n-1} e_{ii}(W) = \chi_X = 1$. The embeddings $\iota_n : \mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$ become the block-diagonal embeddings

$$M_{2^n}(K) \to M_{2^{n+1}}(K), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

so that $\mathcal{A}_\infty = \varinjlim_n \mathcal{A}_n \cong \varinjlim_n M_{2^n}(K)$. Note that $\mathcal{A}_\infty$ is already $*$-regular; the problem is that it does not contain $\mathcal{A}$, in fact it is contained in $\mathcal{A}$. Roughly speaking, what we need to adjoin to $\mathcal{A}_\infty$ in order to get the whole algebra $\mathcal{A}$ is the part corresponding to the generator of $\mathbb{Z}$, namely the element $t$. This is what we are going to do next.

**Definition 3.3.5.** For every $n \geq 1$, let $\mathcal{A}_n(t)$ be the unital $*$-subalgebra of $\mathcal{A}$ generated by $\mathcal{A}_n$ and $t$.

We can completely characterize these $*$-subalgebras.

**Proposition 3.3.6.** *There exists a $*$-isomorphism $\mathcal{A}_n(t) \cong M_{2^n}(K[t^{2^n}, t^{-2^n}])$.*

*Proof.* Write $e_{ij}^{(n)} := e_{ij}(E_n) \in \mathcal{A}_n$ for $0 \leq i,j \leq 2^n - 1$. These form a complete system of matrix units inside $\mathcal{A}_n(t)$, so by [64, Theorem 17.5, see also Remark 17.6], there is an isomorphism $\mathcal{A}_n(t) \cong M_{2^n}(T)$, being $T$ the centralizer of the family $\{e_{ij}^{(n)}\}_{0 \leq i,j \leq 2^n - 1}$ in $\mathcal{A}_n(t)$. The isomorphism is given by

$$a \mapsto \sum_{i,j=0}^{2^n-1} a_{ij} e_{ij}^{(n)}, \qquad \text{with } a_{ij} = \sum_{k=0}^{2^n-1} e_{ki}^{(n)} \cdot a \cdot e_{jk}^{(n)} \in T$$

which is also a $*$-isomorphism. We thus only need to prove that $T = K[t^{2^n}, t^{-2^n}]$. The inclusion $K[t^{2^n}, t^{-2^n}] \subseteq T$ is clear, since

$$t^{2^n} e_{ij}^{(n)} t^{-2^n} = t^{2^n} t^i \chi_{E_n} t^{-j} t^{-2^n} = t^i \chi_{T^{2^n}(E_n)} t^{-j} = t^i \chi_{E_n} t^{-j} = e_{ij}^{(n)}.$$

Therefore $\mathcal{A}_n(t) \cong M_{2^n}(T) \supseteq M_{2^n}(K[t^{2^n}, t^{-2^n}])$. In order to prove equality, we only need to check that the element $t \in \mathcal{A}_n(t)$ belongs to $M_{2^n}(K[t^{2^n}, t^{-2^n}])$ under the previous isomorphism. But this is easy:

$$t = \sum_{i=0}^{2^n-1} t e_{ii}^{(n)} = \sum_{i=0}^{2^n-2} e_{i+1,i}^{(n)} + t^{2^n} e_{0,2^n-1}^{(n)} \in M_{2^n}(K[t^{2^n}, t^{-2^n}]).$$

Henceforth we obtain the desired $*$-isomorphism. □

The obvious inclusion map $\mathcal{A}_n(t) \hookrightarrow \mathcal{A}_{n+1}(t)$ translates to an embedding from $M_{2^n}(K[t^{2^n}, t^{-2^n}])$ to $M_{2^{n+1}}(K[t^{2^{n+1}}, t^{-2^{n+1}}])$ given by

$$e_{ij} \mapsto e_{ij} + e_{i+2^n, j+2^n}, \quad t^{2^n} \mathrm{Id}_{2^n} \mapsto \begin{pmatrix} \mathbf{0}_{2^n} & t^{2^{n+1}} \mathrm{Id}_{2^n} \\ \mathrm{Id}_{2^n} & \mathbf{0}_{2^n} \end{pmatrix}.$$

**Corollary 3.3.7.** *$\mathcal{A}$ is $*$-isomorphic to the direct limit $\varinjlim_n M_{2^n}(K[t^{2^n}, t^{-2^n}])$ with respect to the previous embeddings.*

*Proof.* First one should note that the $*$-subalgebra of $\mathcal{A}$ generated by $t$ and $\mathcal{A}_\infty$ is $\mathcal{A}$ itself, since $C_K(X) \subseteq \mathcal{A}_\infty$ by Lemma 2.3.3. Now each $\mathcal{A}_n \subseteq \mathcal{A}_n(t)$, so $\mathcal{A}_\infty = \varinjlim_n \mathcal{A}_n \subseteq \varinjlim_n \mathcal{A}_n(t)$. But $t \in \varinjlim_n \mathcal{A}_n(t)$ too, so $\mathcal{A} = \varinjlim_n \mathcal{A}_n(t) \cong \varinjlim_n M_{2^n}(K[t^{2^n}, t^{-2^n}])$ by Proposition 3.3.6 above. □

We are now ready to compute $\mathcal{R}_\mathcal{A}$.

**Theorem 3.3.8.** *There is a $*$-isomorphism $\mathcal{R}_\mathcal{A} \cong \varinjlim_n M_{2^n}(K(t^{2^n}))$, where we specify the transition maps $M_{2^n}(K(t^{2^n})) \hookrightarrow M_{2^{n+1}}(K(t^{2^{n+1}}))$ during the course of the proof.*

*Proof.* We have embeddings $\mathcal{A}_n(t) \hookrightarrow \mathcal{A} \hookrightarrow \mathcal{R}_\mathcal{A} \hookrightarrow \mathfrak{R}_{\text{rk}}$. By Lemma 2.4.4, the field of fractions of the Laurent polynomials $K[t^{2^n}, t^{-2^n}] \subseteq \mathcal{A}_n(t)$, which is $K(t^{2^n})$, sits inside $\mathcal{R}_\mathcal{A}$. Hence there is, for each $n \geq 1$, a commutative diagram

$$\begin{array}{ccccc} \mathcal{A}_n(t) & \hookrightarrow & M_{2^n}(K(t^{2^n})) & \hookrightarrow & \mathcal{R}_\mathcal{A} \\ \big\downarrow & & & & \parallel \\ \mathcal{A}_{n+1}(t) & \hookrightarrow & M_{2^{n+1}}(K(t^{2^{n+1}})) & \hookrightarrow & \mathcal{R}_\mathcal{A} \end{array}$$

The embedding $\mathcal{A}_n(t) \hookrightarrow \mathcal{A}_{n+1}(t)$ extends uniquely to a $*$-homomorphism

$$M_{2^n}(K(t^{2^n})) \to M_{2^{n+1}}(K(t^{2^{n+1}})).$$

This is straightforward to see, once we prove that for any nonzero element $q(t) \in K[t^{2^n}, t^{-2^n}]$, the corresponding matrix in $M_{2^{n+1}}(K[t^{2^{n+1}}, t^{-2^{n+1}}])$ becomes invertible in $M_{2^{n+1}}(K(t^{2^{n+1}}))$. By multiplying with a suitable power of $t^{\pm 2^n}$, it is sufficient to consider the case when $q(t) = \lambda_0 + \lambda_1 t^{2^n} + \cdots + \lambda_{2r+1}(t^{2^n})^{2r+1}$ with $\lambda_0 \neq 0$. But

$$q(t) \mapsto \lambda_0 \begin{pmatrix} \mathrm{Id}_{2^n} & \mathbf{0}_{2^n} \\ \mathbf{0}_{2^n} & \mathrm{Id}_{2^n} \end{pmatrix} + \lambda_1 \begin{pmatrix} \mathbf{0}_{2^n} & t^{2^{n+1}}\mathrm{Id}_{2^n} \\ \mathrm{Id}_{2^n} & \mathbf{0}_{2^n} \end{pmatrix} + \cdots + \lambda_{2r} \begin{pmatrix} \mathbf{0}_{2^n} & t^{2^{n+1}}\mathrm{Id}_{2^n} \\ \mathrm{Id}_{2^n} & \mathbf{0}_{2^n} \end{pmatrix}^{2r} + \lambda_{2r+1} \begin{pmatrix} \mathbf{0}_{2^n} & t^{2^{n+1}}\mathrm{Id}_{2^n} \\ \mathrm{Id}_{2^n} & \mathbf{0}_{2^n} \end{pmatrix}^{2r+1}$$

and $\begin{pmatrix} \mathbf{0}_{2^n} & t^{2^{n+1}}\mathrm{Id}_{2^n} \\ \mathrm{Id}_{2^n} & \mathbf{0}_{2^n} \end{pmatrix}^2 = t^{2^{n+1}} \begin{pmatrix} \mathrm{Id}_{2^n} & \mathbf{0}_{2^n} \\ \mathbf{0}_{2^n} & \mathrm{Id}_{2^n} \end{pmatrix}$, so if we define

$$\begin{aligned} q_{\text{even}}(t) &:= \lambda_0 + \lambda_2 t^{2^{n+1}} + \lambda_4 (t^{2^{n+1}})^2 + \cdots + \lambda_{2r}(t^{2^{n+1}})^r \in K[t^{2^{n+1}}, t^{-2^{n+1}}], \\ q_{\text{odd}}(t) &:= \lambda_1 + \lambda_3 t^{2^{n+1}} + \lambda_5 (t^{2^{n+1}})^2 + \cdots + \lambda_{2r+1}(t^{2^{n+1}})^r \in K[t^{2^{n+1}}, t^{-2^{n+1}}], \end{aligned}$$

then $q(t)$ is mapped to the matrix

$$\begin{pmatrix} q_{\text{even}}(t)\mathrm{Id}_{2^n} & q_{\text{odd}}(t) t^{2^{n+1}}\mathrm{Id}_{2^n} \\ q_{\text{odd}}(t)\mathrm{Id}_{2^n} & q_{\text{even}}(t)\mathrm{Id}_{2^n} \end{pmatrix}$$

which is invertible in $M_{2^{n+1}}(K(t^{2^{n+1}}))$ with inverse

$$\frac{1}{q_{\text{even}}(t)^2 - t^{2^{n+1}} q_{\text{odd}}(t)^2} \begin{pmatrix} q_{\text{even}}(t)\mathrm{Id}_{2^n} & -q_{\text{odd}}(t) t^{2^{n+1}}\mathrm{Id}_{2^n} \\ -q_{\text{odd}}(t)\mathrm{Id}_{2^n} & q_{\text{even}}(t)\mathrm{Id}_{2^n} \end{pmatrix}.$$

Notice that this matrix is well-defined since the polynomial $q_{\text{even}}(t)^2 - t^{2^{n+1}} q_{\text{odd}}(t)^2$ has nonzero constant term, so is invertible in $K(t^{2^{n+1}})$.

Therefore the previous commutative diagrams extend to commutative diagrams

$$\begin{array}{ccccc} \mathcal{A}_n(t) & \hookrightarrow & M_{2^n}(K(t^{2^n})) & \hookrightarrow & \mathcal{R}_{\mathcal{A}} \\ \downarrow & & \downarrow & & \| \\ \mathcal{A}_{n+1}(t) & \hookrightarrow & M_{2^{n+1}}(K(t^{2^{n+1}})) & \hookrightarrow & \mathcal{R}_{\mathcal{A}} \\ \downarrow & & \downarrow & & \| \\ \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \| \\ \mathcal{A} & \hookrightarrow & \varinjlim_n M_{2^n}(K(t^{2^n})) & \hookrightarrow & \mathcal{R}_{\mathcal{A}} \end{array}$$

But $\varinjlim_n M_{2^n}(K(t^{2^n}))$ is already $*$-regular since each factor $M_{2^n}(K(t^{2^n}))$ is, and contains also $\mathcal{A}$, so $\mathcal{R}_{\mathcal{A}} \cong \varinjlim_n M_{2^n}(K(t^{2^n}))$ as required. □

In particular, since $\varinjlim_n M_{2^n}(K(t^{2^n}))$ has a unique rank function rk, the rank function $\mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}$ on $\mathcal{R}_{\mathcal{A}}$ is also unique, and they agree under the previous $*$-isomorphism.

### 3.3.2 Determining $\mathcal{C}(\mathcal{A})$

In this last subsection we are going to compute explicitly the set $\mathcal{C}(\mathcal{A})$ consisting of all positive real values that the Sylvester matrix rank function $\mathrm{rk}_{\mathcal{A}}$ can achieve. First, let

$$\mathcal{C}(\mathcal{R}_{\mathcal{A}}) := \mathrm{rk}_{\mathcal{R}_{\mathcal{A}}}\Big(\bigcup_{i=1}^{\infty} M_i(\mathcal{R}_{\mathcal{A}})\Big) \subseteq \mathbb{R}^+$$

and note that $\mathcal{C}(\mathcal{A}) \subseteq \mathcal{C}(\mathcal{R}_{\mathcal{A}})$.

**Theorem 3.3.9.** $\mathcal{C}(\mathcal{A}) = \mathcal{C}(\mathcal{R}_\mathcal{A}) = \mathbb{Z}\left[\frac{1}{2}\right]^+$.

*Proof.* The argument is similar to the one given in the proof of Proposition 2.4.1. Since $\mathcal{R}_\mathcal{A}$ is a $*$-regular ring with positive definite involution, each matrix algebra $M_i(\mathcal{R}_\mathcal{A})$ is also a $*$-regular ring. Hence for each $A \in M_i(\mathcal{R}_\mathcal{A})$, there exists a projection $P \in M_i(\mathcal{R}_\mathcal{A})$ such that $\mathrm{rk}_{\mathcal{R}_\mathcal{A}}(A) = \mathrm{rk}_{\mathcal{R}_\mathcal{A}}(P)$ (recall Theorem 1.2.11). We conclude that $\mathcal{C}(\mathcal{R}_\mathcal{A})$ is equal to the set of positive real numbers of the form $\mathrm{rk}_{\mathcal{R}_\mathcal{A}}(P)$, where $P$ ranges over projections in matrices over $\mathcal{R}_\mathcal{A}$.

Now each such projection $P$ is equivalent to a diagonal one ([39, Proposition 2.10]), that is, one of the form

$$\begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_r \end{pmatrix} \quad \text{for some projections } p_1, ..., p_r \in \mathcal{R}_\mathcal{A},$$

so that $\mathrm{rk}_{\mathcal{R}_\mathcal{A}}(P) = \mathrm{rk}_{\mathcal{R}_\mathcal{A}}(p_1) + \cdots + \mathrm{rk}_{\mathcal{R}_\mathcal{A}}(p_r)$. But since $\mathcal{R}_\mathcal{A} \cong \varinjlim_n M_{2^n}(K(t^{2^n}))$, the set of ranks of elements in $\mathcal{R}_\mathcal{A}$ is exactly $\mathbb{Z}\left[\frac{1}{2}\right]^+ \cap [0,1]$. Therefore $\mathrm{rk}_{\mathcal{R}_\mathcal{A}}(P) \in \mathbb{Z}\left[\frac{1}{2}\right]^+$. This proves the inclusions

$$\mathcal{C}(\mathcal{A}) \subseteq \mathcal{C}(\mathcal{R}_\mathcal{A}) \subseteq \mathbb{Z}\left[\frac{1}{2}\right]^+.$$

The inclusion $\mathbb{Z}\left[\frac{1}{2}\right]^+ \subseteq \mathcal{C}(\mathcal{A})$ is straightforward, since

$$\mathrm{rk}_\mathcal{A}(e_{00}^{(n)} + \cdots + e_{mm}^{(n)}) = \frac{m+1}{2^n} \quad \text{for } 0 \leq m \leq 2^n - 1.$$

□

## 3.4 The general odometer algebra

In this section we concentrate in studying the generalized odometer algebra. We first recall how it is defined.

Fix a sequence of natural numbers $\overline{n} = (n_i)_{i \in \mathbb{N}}$ satisfying $n_i \geq 2$ for all $i \in \mathbb{N}$, and consider $X_i$ to be the finite space $\{0, 1, ..., n_i - 1\}$. We can thus form the compact space $X = \prod_{i \in \mathbb{N}} X_i$. Let $T$ be the homeomorphism $X \to X$ given by the odometer, namely for $x = (x_i)_i \in X$, $T$ is defined by

$$T(x) = x + (1, 0, ...) = \begin{cases} (x_1 + 1, x_2, x_3, ...) & \text{if } x_1 \neq n_1 - 1, \\ (0, ..., 0, x_{m+1} + 1, x_{m+2}, ...) & \text{if } x_1 = n_1 - 1, ..., x_m = n_m - 1 \text{ and } x_{m+1} \neq n_m - 1, \\ (0, 0, ...) & \text{if } x = (n_i - 1)_i. \end{cases}$$

Note that the odometer action is just addition of $(1, 0, ...)$ by carry-over.

Let $(K, *)$ be any field with a positive definite involution $*$. We consider again the crossed product $*$-algebra $\mathcal{O}(\overline{n}) := C_K(X) \rtimes_T \mathbb{Z}$. We can define a measure $\mu$ on $X$ by taking the usual product measure, where we consider the measure on each component $X_i$ assigning mass $\frac{1}{n_i}$ on each point in $X_i$. It is well-known (e.g. [24, Section VIII.4]) that $\mu$ is an ergodic, full and $T$-invariant probability measure on $X$, which in turn coincides with the Haar measure $\widehat{\mu}$ on $X$ if we consider $X$ as an abelian group with operation given again by carry-over. Hence we can apply our construction from Chapter 2 (specifically, Theorem 2.3.7), to obtain a canonical Sylvester matrix rank function $\mathrm{rk}_{\mathcal{O}(\overline{n})}$ on $\mathcal{O}(\overline{n})$.

### 3.4.1 The $*$-regular closure $\mathcal{R}_{\mathcal{O}(\overline{n})}$

By using similar techniques from the last two sections, we have been able to exactly compute the $*$-regular closure $\mathcal{R}_{\mathcal{O}(\overline{n})}$ of the generalized odometer algebra $\mathcal{O}(\overline{n})$. We will follow exactly the same steps as in the previous section.

First, define new integers $p_m = n_1 \cdots n_m$ for $m \in \mathbb{N}$. At each level $m \geq 1$, we take $E_m = [\underline{0}0...0]$ (with $m$ zero's) for the sequence of clopen sets, whose intersection gives the point $y = (\underline{0}, 0, 0, ...) \in X$. We take the partitions $\mathcal{P}_m$ of the complements $X \backslash E_n$ to be the obvious ones, namely

$$\mathcal{P}_m = \{[\underline{1}0 \cdots 0], ..., [\underline{(n_1 - 1)}0 \cdots 0], ..., [\underline{(n_1 - 1)}(n_2 - 1) \cdots (n_m - 1)]\}.$$

The unital $*$-subalgebra $\mathcal{O}(\overline{n})_m$ is then generated by the partial isometries $\chi_Z t$ for $Z \in \mathcal{P}_m$.

The quasi-partition $\overline{\mathcal{P}}_m$ is really simple again: write $Z_{m,0} = [\underline{1}0\cdots 0]$, and $Z_{m,l} = T^l(Z_{m,0})$ for $1 \leq l \leq p_m - 2$. Note that these clopen sets form exactly the partition $\mathcal{P}_m$, and that $T(E_m) = Z_{m,0}$, $T(Z_{m,p_m-2}) = E_m$. Therefore there is only one possible $W \in \mathbb{V}_m$, which is of length $p_m$ and given by

$$W = E_m \cap T^{-1}(Z_{m,0}) \cap T^{-2}(Z_{m,1}) \cap \cdots \cap T^{-p_m+1}(Z_{m,p_m-2}) \cap T^{-p_m}(E_m) = E_m.$$

Clearly,

$$\sum_{k\geq 1} \sum_{\substack{W\in\mathbb{V}_m \\ |W|=k}} k\mu(W) = p_m\mu(E_m) = 1$$

as we already know.

The representations $\pi_m : \mathcal{O}(\overline{n})_m \hookrightarrow \mathfrak{R}_m, x \mapsto (h_W \cdot x)_W$ become $*$-isomorphisms, with $\mathfrak{R}_m = M_{p_m}(K)$. Indeed, for $W = E_m$, we have $e_{00}(W) = \chi_{E_m}$, and for $1 \leq i \leq p_m - 1$, $e_{ii}(W) = \chi_{T^i(E_m)} = \chi_{Z_{m,i-1}}$. Therefore $h_{E_m} = \sum_{i=0}^{p_m-1} e_{ii}(W) = \chi_X = 1$. The embeddings $\iota_m : \mathcal{O}(\overline{n})_m \hookrightarrow \mathcal{O}(\overline{n})_{m+1}$ become the block-diagonal embeddings

$$M_{p_m}(K) \to M_{p_{m+1}}(K), \quad x \mapsto \begin{pmatrix} x & & \mathbf{0} \\ & \overset{n_{m+1}}{\ddots} & \\ \mathbf{0} & & x \end{pmatrix}$$

so that $\mathcal{O}(\overline{n})_\infty = \varinjlim_m \mathcal{O}(\overline{n})_m \cong \varinjlim_m M_{p_m}(K)$. Note that $\mathcal{O}(\overline{n})_\infty$ is already $*$-regular; the problem again is that it does not contain $\mathcal{O}(\overline{n})$, in fact it is contained in $\mathcal{O}(\overline{n})$. We need to adjoin to $\mathcal{O}(\overline{n})_\infty$ the element $t$ in order to get the whole algebra $\mathcal{O}(\overline{n})$. This is what we are going to do next.

**Definition 3.4.1.** For every $m \geq 1$, we denote by $\mathcal{O}(\overline{n})_m(t)$ to be the unital $*$-subalgebra of $\mathcal{O}(\overline{n})$ generated by $\mathcal{O}(\overline{n})_m$ and $t$.

We can completely characterize these $*$-subalgebras.

**Proposition 3.4.2.** *There exists a $*$-isomorphism $\mathcal{O}(\overline{n})_m(t) \cong M_{p_m}(K[t^{p_m}, t^{-p_m}])$.*

*Proof.* The proof is exactly the same as in Proposition 3.3.6. The elements $e_{ij}^{(m)} := e_{ij}(E_m) \in \mathcal{O}(\overline{n})_m$, for $0 \leq i,j \leq p_m - 1$, form a complete system of matrix units inside $\mathcal{O}(\overline{n})_m(t)$, so there is an isomorphism $\mathcal{O}(\overline{n})_m(t) \cong M_{p_m}(T)$, being $T$ the centralizer of the family $\{e_{ij}^{(m)}\}_{0\leq i,j\leq p_m-1}$ in $\mathcal{O}(\overline{n})_m(t)$. The isomorphism is given explicitly by

$$a \mapsto \sum_{i,j=0}^{p_m-1} a_{ij}e_{ij}^{(m)}, \quad \text{with } a_{ij} = \sum_{k=0}^{p_m-1} e_{ki}^{(m)} \cdot a \cdot e_{jk}^{(m)} \in T$$

which is also a $*$-isomorphism. We only need to prove that $T = K[t^{p_m}, t^{-p_m}]$. Since

$$t^{p_m} e_{ij}^{(m)} t^{-p_m} = t^{p_m} t^i \chi_{E_m} t^{-j} t^{-p_m} = t^i \chi_{T^{p_m}(E_m)} t^{-j} = t^i \chi_{E_m} t^{-j} = e_{ij}^{(m)}$$

and

$$t = \sum_{i=0}^{p_m-1} t e_{ii}^{(m)} = \sum_{i=0}^{p_m-2} e_{i+1,i}^{(m)} + t^{p_m} e_{0,p_m-1}^{(m)} \in M_{p_m}(K[t^{p_m}, t^{-p_m}]),$$

we deduce that $T = K[t^{p_m}, t^{-p_m}]$, so we obtain the desired $*$-isomorphism. □

The obvious inclusion map $\mathcal{O}(\overline{n})_m(t) \hookrightarrow \mathcal{O}(\overline{n})_{m+1}(t)$ translates to an embedding from $M_{p_m}(K[t^{p_m}, t^{-p_m}])$ to $M_{p_{m+1}}(K[t^{p_{m+1}}, t^{-p_{m+1}}])$ given by

$$e_{ij} \mapsto \sum_{k=0}^{n_{m+1}-1} e_{i+kp_m, j+kp_m}, \quad t^{p_m}\mathrm{Id}_{p_m} \mapsto \begin{pmatrix} \mathbf{0}_{p_m} & & & \mathbf{0}_{p_m} & t^{p_{m+1}}\mathrm{Id}_{p_m} \\ \mathrm{Id}_{p_m} & \mathbf{0}_{p_m} & & & \mathbf{0}_{p_m} \\ & \ddots & \overset{n_{m+1}}{\ddots} & & \\ & & \ddots & \mathbf{0}_{p_m} & \\ \mathbf{0}_{p_m} & & & \mathrm{Id}_{p_m} & \mathbf{0}_{p_m} \end{pmatrix}.$$

**Corollary 3.4.3.** *$\mathcal{O}(\overline{n})$ is $*$-isomorphic to the direct limit $\varinjlim_m M_{p_m}(K[t^{p_m}, t^{-p_m}])$ with respect to the previous embeddings.*

*Proof.* First one should note that the $*$-subalgebra of $\mathcal{O}(\overline{n})$ generated by $t$ and $\mathcal{O}(\overline{n})_\infty = \varinjlim_m \mathcal{O}(\overline{n})_m(t)$ is $\mathcal{O}(\overline{n})$ itself, since $C_K(X) \subseteq \mathcal{O}(\overline{n})_\infty$ by Lemma 2.3.3. Now each $\mathcal{O}(\overline{n})_m \subseteq \mathcal{O}(\overline{n})_m(t)$, so $\mathcal{O}(\overline{n})_\infty = \varinjlim_m \mathcal{O}(\overline{n})_m \subseteq \varinjlim_m \mathcal{O}(\overline{n})_m(t)$. But $t \in \varinjlim_m \mathcal{O}(\overline{n})_m(t)$ too, so $\mathcal{O}(\overline{n}) = \varinjlim_m \mathcal{O}(\overline{n})_m(t) \cong \varinjlim_m M_{p_m}(K[t^{p_m}, t^{-p_m}])$ by Proposition 3.4.2 above. □

We are now ready to compute $\mathcal{R}_{\mathcal{O}(\overline{n})}$.

**Theorem 3.4.4.** *There is a $*$-isomorphism $\mathcal{R}_{\mathcal{O}(\overline{n})} \cong \varinjlim_m M_{p_m}(K(t^{p_m}))$, where we specify the transition maps $M_{p_m}(K(t^{p_m})) \hookrightarrow M_{p_{m+1}}(K(t^{p_{m+1}}))$ during the course of the proof.*

*Proof.* We have embeddings $\mathcal{O}(\overline{n})_m(t) \hookrightarrow \mathcal{O}(\overline{n}) \hookrightarrow \mathcal{R}_{\mathcal{O}(\overline{n})} \hookrightarrow \mathfrak{R}_{\text{rk}}$. By Lemma 2.4.4, the field of fractions of the Laurent polynomials $K[t^{p_m}, t^{-p_m}] \subseteq \mathcal{O}(\overline{n})_m(t)$, which is $K(t^{p_m})$, sits inside $\mathcal{R}_{\mathcal{O}(\overline{n})}$. Hence there is, for each $m \geq 1$, a commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}(\overline{n})_m(t) & \hookrightarrow & M_{p_m}(K(t^{p_m})) & \hookrightarrow & \mathcal{R}_{\mathcal{O}(\overline{n})} \\
\big\downarrow & & & & \| \\
\mathcal{O}(\overline{n})_{m+1}(t) & \hookrightarrow & M_{p_{m+1}}(K(t^{p_{m+1}})) & \hookrightarrow & \mathcal{R}_{\mathcal{O}(\overline{n})}
\end{array}$$

The embedding $\mathcal{O}(\overline{n})_m(t) \hookrightarrow \mathcal{O}(\overline{n})_{m+1}(t)$ extends uniquely to a $*$-homomorphism

$$M_{p_m}(K(t^{p_m})) \to M_{p_{m+1}}(K(t^{p_{m+1}})).$$

This is straightforward to see, once we prove that for any nonzero element $q(t) \in K[t^{p_m}, t^{-p_m}]$, the corresponding matrix in $M_{p_{m+1}}(K[t^{p_{m+1}}, t^{-p_{m+1}}])$ becomes invertible in $M_{p_{m+1}}(K(t^{p_{m+1}}))$. By multiplying with a suitable power of $t^{\pm p_m}$, it is sufficient to consider the case when $q(t) = \lambda_0 + \lambda_1 t^{p_m} + \cdots + \lambda_{kn_{m+1}}(t^{p_m})^{kn_{m+1}}$ with $\lambda_0 \neq 0$. But

$$q(t) \mapsto \begin{pmatrix} q_0(t)\mathrm{Id}_{p_m} & q_{n_{m+1}-1}(t)t^{p_{m+1}}\mathrm{Id}_{p_m} & \cdots & q_1(t)t^{p_{m+1}}\mathrm{Id}_{p_m} \\ q_1(t)\mathrm{Id}_{p_m} & q_0(t)\mathrm{Id}_{p_m} & \cdots & q_2(t)t^{p_{m+1}}\mathrm{Id}_{p_m} \\ \vdots & \vdots & \ddots & \vdots \\ q_{n_{m+1}-1}(t)\mathrm{Id}_{p_m} & q_{n_{m+1}-2}(t)\mathrm{Id}_{p_m} & \cdots & q_0(t)\mathrm{Id}_{p_m} \end{pmatrix}$$

where

$$\begin{aligned} q_0(t) &:= \lambda_0 + \lambda_{n_{m+1}} t^{p_{m+1}} + \cdots + \lambda_{kn_{m+1}}(t^{p_{m+1}})^k \in K[t^{p_{m+1}}, t^{-p_{m+1}}], \\ q_i(t) &:= \lambda_i + \lambda_{n_{m+1}+i} t^{p_{m+1}} + \cdots + \lambda_{(k-1)n_{m+1}+i}(t^{p_{m+1}})^{k-1} \in K[t^{p_{m+1}}, t^{-p_{m+1}}] \quad \text{for } 1 \leq i \leq n_{m+1}-1, \end{aligned}$$

which is an invertible matrix in $M_{p_{m+1}}(K(t^{p_{m+1}}))$ since its determinant is of the form

$$\lambda_0^{p_{m+1}} + t \cdot (\text{polynomial in } t),$$

so invertible in $K(t^{p_{m+1}})$ because $\lambda_0 \neq 0$.

Therefore the previous commutative diagrams extend to commutative diagrams

$$\begin{array}{ccccc}
\mathcal{O}(\overline{n})_m(t) & \hookrightarrow & M_{p_m}(K(t^{p_m})) & \hookrightarrow & \mathcal{R}_{\mathcal{O}(\overline{n})} \\
\big\downarrow & & \big\downarrow & & \| \\
\mathcal{O}(\overline{n})_{m+1}(t) & \hookrightarrow & M_{p_{m+1}}(K(t^{p_{m+1}})) & \hookrightarrow & \mathcal{R}_{\mathcal{O}(\overline{n})} \\
\big\downarrow & & \big\downarrow & & \| \\
\vdots & & \vdots & & \vdots \\
\big\downarrow & & \big\downarrow & & \| \\
\mathcal{A} & \hookrightarrow & \varinjlim_n M_{p_m}(K(t^{p_m})) & \hookrightarrow & \mathcal{R}_{\mathcal{O}(\overline{n})}
\end{array}$$

But $\varinjlim_n M_{p_m}(K(t^{p_m}))$ is already $*$-regular since each factor $M_{p_m}(K(t^{p_m}))$ is, and contains also $\mathcal{A}$, so $\mathcal{R}_{\mathcal{O}(\overline{n})} \cong \varinjlim_m M_{p_m}(K(t^{p_m}))$ as required. $\square$

In particular, since $\varinjlim_m M_{p_m}(K(t^{p_m}))$ has a unique rank function rk, the rank function $\mathrm{rk}_{\mathcal{R}_{\mathcal{O}(\overline{n})}}$ on $\mathcal{R}_{\mathcal{O}(\overline{n})}$ is also unique, and they agree under the previous $*$-isomorphism.

### 3.4.2 Determining $\mathcal{C}(\mathcal{O}(\overline{n}))$

In this last subsection we are going to compute explicitly the set $\mathcal{C}(\mathcal{O}(\overline{n}))$ consisting of all positive real values that the Sylvester matrix rank function $\mathrm{rk}_{\mathcal{O}(\overline{n})}$ can achieve. First, let

$$\mathcal{C}(\mathcal{R}_{\mathcal{O}(\overline{n})}) := \mathrm{rk}_{\mathcal{R}_{\mathcal{O}(\overline{n})}}\Big(\bigcup_{i=1}^{\infty} M_i(\mathcal{R}_{\mathcal{O}(\overline{n})})\Big) \subseteq \mathbb{R}^+$$

and note that $\mathcal{C}(\mathcal{O}(\overline{n})) \subseteq \mathcal{C}(\mathcal{R}_{\mathcal{O}(\overline{n})})$.

First, we need some preliminary definitions.

**Definition 3.4.5.** For each sequence $\overline{n} = (n_1, n_2, ...)$ of positive integers $n_i \geq 2$, one may associate to it the *supernatural number*

$$n = \prod_{i \geq 1} n_i = \prod_{q \in P} q^{\varepsilon_q(n)},$$

where $P = \{q_1, q_2, ...\}$ is the set of prime numbers ordered with respect to its natural order, and each $\varepsilon_q(n) \in \{0\} \cup \mathbb{N} \cup \{\infty\}$.

As in [65, Definition 7.4.2], from any supernatural number $n$ one can construct an additive subgroup of $\mathbb{Q}$, denoted by $\mathbb{Z}(n)$, consisting of those fractions $\frac{a}{b}$ with $a \in \mathbb{Z}$, and $b \in \mathbb{Z}\backslash\{0\}$ being of the form

$$b = \prod_{q \in P} q^{\varepsilon_q(b)},$$

where $\varepsilon_q(b) \leq \varepsilon_q(n)$ for all $q \in P$, and $\varepsilon_q(b) = 0$ for all but finitely many $q$'s.

If $n$ comes from a sequence $\overline{n} = (n_1, n_2, ...)$ as above, $\mathbb{Z}(n)$ is exactly the additive subgroup of $\mathbb{Q}$ consisting of those fractions of the form

$$\frac{a}{n_1 \cdots n_r}, \quad \text{where } a \in \mathbb{Z}, \text{ and } r \in \mathbb{N}.$$

**Theorem 3.4.6.** $\mathcal{C}(\mathcal{O}(\overline{n})) = \mathcal{C}(\mathcal{R}_{\mathcal{O}(\overline{n})}) = \mathbb{Z}(n)^+$.

*Proof.* As in the proof of Theorem 3.3.9, $\mathcal{C}(\mathcal{R}_{\mathcal{O}(\overline{n})})$ is equal to the set of positive real numbers of the form $\mathrm{rk}_{\mathcal{R}_{\mathcal{O}(\overline{n})}}(P)$, where $P$ ranges over projections in matrices over $\mathcal{R}_{\mathcal{O}(\overline{n})}$. Each such projection $P$ is equivalent to a diagonal one, so it is of the form

$$\begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_r \end{pmatrix} \quad \text{for some projections } p_1, ..., p_r \in \mathcal{R}_{\mathcal{O}(\overline{n})},$$

so that $\mathrm{rk}_{\mathcal{R}_{\mathcal{O}(\overline{n})}}(P) = \mathrm{rk}_{\mathcal{R}_{\mathcal{O}(\overline{n})}}(p_1) + \cdots + \mathrm{rk}_{\mathcal{R}_{\mathcal{O}(\overline{n})}}(p_r)$. But since $\mathcal{R}_{\mathcal{O}(\overline{n})} \cong \varinjlim_m M_{p_m}(K(t^{p_m}))$, the set of ranks of elements in $\mathcal{R}_{\mathcal{O}(\overline{n})}$ is exactly $\mathbb{Z}(n)^+ \cap [0,1]$. Therefore $\mathrm{rk}_{\mathcal{R}_{\mathcal{O}(\overline{n})}}(P) \in \mathbb{Z}(n)^+$. This proves the inclusions

$$\mathcal{C}(\mathcal{O}(\overline{n})) \subseteq \mathcal{C}(\mathcal{R}_{\mathcal{O}(\overline{n})}) \subseteq \mathbb{Z}(n)^+.$$

The inclusion $\mathbb{Z}(n)^+ \subseteq \mathcal{C}(\mathcal{O}(\overline{n}))$ is straightforward, since

$$\mathrm{rk}_{\mathcal{O}(\overline{n})}(e_{00}^{(m)} + \cdots + e_{ll}^{(m)}) = \frac{l+1}{p_m} \quad \text{for } 0 \leq l \leq p_m - 1.$$ $\square$

# Chapter 4

# Generalizing a result of von Neumann

This chapter concerns about the characterization of the rank completion of some ultramatricial $K$-algebras, being $K$ an arbitrary field. We show in Theorem 4.2.2 that, whenever the rank completion of an ultramatricial $K$-algebra becomes a separable continuous factor (i.e. a continuous factor $\mathcal{Q}$ containing a dense subalgebra, with respect to its rank metric, of countable $K$-dimension), then its rank completion is isomorphic to a well-known continuous ring, know as the von Neumann continuous factor, and denoted by $\mathcal{M}_K$. We also characterize such $K$-algebras by means of a local property.

We extend, in Sections 4.3 and 4.4, the previous result to $D$-rings (being $D$ a division ring) and ultramatricial $*$-algebras (when $K$ is a field endowed with a positive definite involution). We have not found a reasonable analogue of the local condition in the case of $D$-rings, but we have succeeded, in a some technical way, for the case of fields with involution.

Throughout all this chapter we will *not* make use of the same notations used in the previous chapters, in the sense that $\mathcal{A}$ will not stand for any $\mathbb{Z}$-crossed product algebra anymore, just like $\mathcal{B}$ will not stand for any approximation algebra neither. In terms of notation, this chapter is independent of the other previous ones.

## 4.1 Introduction

Murray and von Neumann showed in [80, Theorem XII] a uniqueness result for approximately finite von Neumann algebra factors of type $II_1$. This unique factor $\mathcal{R}$ is called the *hyperfinite $II_1$-factor* and plays an important role in the theory of von Neumann algebras. It was shown later by Connes [21] that the factor $\mathcal{R}$ is characterized, among $II_1$-factors, by various other properties, such as self-injectivity (in the operator space sense), semidiscreteness or Property P. It is in particular known (e.g. [94, Theorem 3.8.2]) that, for an infinite countable discrete group $\Gamma$ whose nontrivial conjugacy classes are all infinite (termed ICC-groups), the group von Neumann algebra $\mathcal{N}(\Gamma)$ is isomorphic to $\mathcal{R}$ if and only if $\Gamma$ is an amenable group.

Von Neumann also considered a purely algebraic analogue of the above situation, namely the example we have in Chapter 1, Example 1.2.8.2). We briefly recall it. For a field $K$, take the direct limit $\varinjlim_n M_{2^n}(K)$ of the sequence

$$M_2(K) \to M_4(K) \to \cdots \to M_{2^n}(K) \to \cdots$$

with respect to the block-diagonal embeddings $x \mapsto \begin{pmatrix} x & \mathbf{0}_{2^n} \\ \mathbf{0}_{2^n} & x \end{pmatrix}$. It is a (von Neumann) regular ring which admits a unique rank function rk defined on an element $x = \varinjlim_n x_n$ to be $\mathrm{rk}(x) = \lim_n \mathrm{rk}_n(x_n)$, where $\mathrm{rk}_n = \frac{\mathrm{Rk}}{2^n}$ is the usual normalized rank on $M_{2^n}(K)$. The completion of $\varinjlim_n M_{2^n}(K)$ with respect to the induced rank metric, denoted by $\mathcal{M}_K$, is a complete regular ring with a unique rank function, again denoted by rk, which is a *continuous factor*, i.e., a right and left self-injective ring where the set of values of the rank function fills the unit interval $[0, 1]$.

There are recent evidences [28, 29, 30] that the factor $\mathcal{M}_K$ could play a role in algebra which is similar to the role played by the unique hyperfinite factor $\mathcal{R}$ in the theory of operator algebras. In particular, Elek has shown in [29] that, if $\Gamma = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$ is the lamplighter group, then the continuous factor $c(\Gamma)$ obtained by taking the rank completion of the $*$-regular closure of $\mathbb{C}[\Gamma]$ in the $*$-algebra $\mathcal{U}(\Gamma)$ of unbounded operators affiliated to $\mathcal{N}(\Gamma)$, is isomorphic to $\mathcal{M}_{\mathbb{C}}$.

This raises the question of what uniqueness properties the von Neumann factor $\mathcal{M}_K$ has, and whether we can formulate similar characterizations to those in the seminal paper by Connes [21]. As von Neumann had already shown (and was published later by Halperin [44]), $\mathcal{M}_K$ is isomorphic to the factor obtained from any *factor sequence* $(p_i)_i$, that is,

$$\mathcal{M}_K \cong \overline{\varinjlim_n M_{p_n}(K)},$$

where $(p_i)_i$ is a sequence of positive integers converging to infinity and such that $p_i$ divides $p_{i+1}$ for all $i$. Here the direct limit is taken with respect to the block-diagonal embeddings

$$M_{p_i}(K) \to M_{p_{i+1}}(K), \quad x \mapsto \begin{pmatrix} x & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & x \end{pmatrix},$$

and the completion is taken with respect to the unique rank function on the direct limit rk, defined as before.

The purpose of this chapter is to obtain stronger uniqueness properties of the factor $\mathcal{M}_K$. Specifically, we show that if $\mathcal{B}$ is an ultramatricial $K$-algebra and $\mathrm{rk}_{\mathcal{B}}$ is a nondiscrete extremal pseudo-rank function on $\mathcal{B}$, then the completion of $\mathcal{B}$ with respect to $\mathrm{rk}_{\mathcal{B}}$ is necessarily isomorphic to $\mathcal{M}_K$. We also derive a characterization of the factor $\mathcal{M}_K$ by a local approximation property (see Theorem 4.2.2). This was used in Chapters 2 and 3 to generalize Elek's result to arbitrary fields $K$ of characteristic $\neq 2$, using a concrete approximation of the group algebra $K[\Gamma]$ by matricial algebras. It is also worth to mention that, as a consequence of our result and [38, Theorem 2.8], one obtains that the center of an algebra $\mathcal{Q}$ satisfying properties (ii) or (iii) in Theorem 4.2.2 is the base field $K$.

Elek and Jaikin-Zapirain have recently raised the question of whether, for any subfield $K$ of $\mathbb{C}$ closed under complex conjugation, and any countable amenable ICC-group $G$, the rank completion $\overline{\mathcal{R}}^{\mathrm{rk}}_{K[G]}$ of the $*$-regular closure of $K[G]$ in $\mathcal{U}(G)$ is either of the form $M_n(D)$ or of the form $\mathcal{M}_D := D \otimes_K \mathcal{M}_K$, where $D$ is a division ring with center $K$. In view of this question, it is natural to obtain uniqueness results in the slightly more general setting of $D$-rings over a division ring $D$, and also in the setting of rings with involution, since in the above situation, the algebras have a natural involution which is essential even to define the corresponding completions. We address these questions in the final two sections.

## 4.2 Von Neumann's continuous factor

For a field $K$, a *matricial* $K$-algebra is a $K$-algebra which is isomorphic to an algebra of the form

$$M_{n_1}(K) \times M_{n_2}(K) \times \cdots \times M_{n_k}(K)$$

for some positive integers $n_1, n_2, ..., n_k$. An *ultramatricial* $K$-algebra is an algebra which is isomorphic to a direct limit $\varinjlim_n \mathcal{A}_n$ of a sequence of matricial $K$-algebras $\mathcal{A}_n$ and unital algebra homomorphisms $\varphi_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$.

Let $K$ be a field. Write $\mathcal{M} = \mathcal{M}_K$ for the rank completion of the direct limit $\varinjlim_n M_{2^n}(K)$ with respect to its unique rank function. Von Neumann proved a uniqueness property for $\mathcal{M}$. We are going to extend it to ultramatricial algebras. The proof follows the steps in the paper by Halperin [44] (based on von Neumann's proof), but our proof is considerably more involved. Indeed, we will obtain a uniqueness result for the class of continuous factors which have a local matricial structure.

**Definition 4.2.1.** By a *continuous factor* we understand a simple, regular, (right and left) self-injective ring $\mathcal{Q}$ of type $II_f$ (see Section 1.2 for the definition of the types and for a survey of the structure theory of regular self-injective rings).

It follows from Proposition 1.2.7 that $\mathcal{Q}$ admits a unique rank function, denoted here by $\mathrm{rk}_{\mathcal{Q}}$, and that $\mathcal{Q}$ is complete in the $\mathrm{rk}_{\mathcal{Q}}$-metric. Therefore, as we have already explained in Section 1.2, the range of $\mathrm{rk}_{\mathcal{Q}}$ can be either a finite set of values of the form $\left\{0, \frac{1}{n}, ..., \frac{n-1}{n}, 1\right\}$ for some natural number $n \geq 1$, or the whole interval $[0, 1]$. In fact, since $\mathcal{Q}$ is complete of type $II_f$, it follows easily that $\mathrm{rk}_{\mathcal{Q}}(\mathcal{Q}) = [0, 1]$ indeed.

The adjective "continuous" used here refers to the fact that $\mathrm{rk}_{\mathcal{Q}}$ takes a "continuous" set of values, in contrast to the algebra of finite matrices (type $I_n$), where the rank function takes only a finite number of values (see

Example 1.2.8.1)). Note however that any regular self-injective ring $R$ is a right (left) continuous regular ring, in the technical sense that the lattice of principal right (left) ideals $L(R_R)$ is continuous, see [39, Corollary 13.5]. However, the latter property will play no explicit role in the present chapter.

Since $\mathcal{Q}$ is a simple ring, the discussion following Proposition 1.2.6 asserts that $\mathcal{Q}$ satisfies the comparability axiom. We shall use this property thoroughly without explicitly mentioning it.

We will show the following result, among others.

**Theorem 4.2.2.** *Let $\mathcal{Q}$ be a continuous factor, and assume that there exists a dense $K$-subalgebra (with respect to the $\mathrm{rk}_{\mathcal{Q}}$-metric topology) $\mathcal{Q}_0 \subseteq \mathcal{Q}$ of countable $K$-dimension. Then the following are equivalent:*

*(1) $\mathcal{Q} \cong \mathcal{M}$.*

*(2) $\mathcal{Q} \cong \overline{\mathcal{B}}$ for a certain ultramatricial $K$-algebra $\mathcal{B}$, where the completion of $\mathcal{B}$ is taken with respect to the metric induced by an extremal pseudo-rank function on $\mathcal{B}$.*

*(3) For any $\varepsilon > 0$ and $x_1, ..., x_n \in \mathcal{Q}$, there exists a matricial $K$-subalgebra $\mathcal{A}$ of $\mathcal{Q}$ and elements $y_1, ..., y_n \in \mathcal{A}$ such that*

$$\mathrm{rk}_{\mathcal{Q}}(x_i - y_i) < \varepsilon \qquad \text{for } i = 1, \ldots, k.$$

The implication (1) $\implies$ (2) is trivial, since $\mathcal{M}$ is already the completion of the ultramatricial $K$-algebra $\varinjlim_n M_{2^n}(K)$ with respect to its unique rank function, which is extremal.

For (2) $\implies$ (3), if $\mathcal{B} = \varinjlim_m \mathcal{A}_m$ with respect to unital algebra homomorphisms $\varphi_m : \mathcal{A}_m \to \mathcal{A}_{m+1}$, then for every $\varepsilon > 0$ and elements $x_1, ..., x_n \in \mathcal{Q}$ there are elements $y_1, ..., y_n \in \varinjlim_m \mathcal{A}_m$ being close to the $x_i$ up to $\varepsilon$ in rank, that is

$$\mathrm{rk}_{\mathcal{Q}}(x_i - y_i) < \varepsilon \qquad \text{for } i = 1, ..., k.$$

Since $y_1, ..., y_n \in \varinjlim_m \mathcal{A}_m$, there exists an integer $N \geq 1$ such that $\mathcal{A}_N$ contains all of them. But $\mathcal{A}_N$ is already a matricial $K$-subalgebra of $\mathcal{Q}$, so the implication is proved.

The hard implication is (3) $\implies$ (1). For its proof, we will use a method similar to the one used in [44]. However the technical complications are much higher here.

We first prove a lemma, and show that the implication (3) $\implies$ (1) holds assuming that the hypotheses of the lemma are satisfied. After this is done, we will show how to construct (using (3)) the sequences, algebras, and homomorphisms appearing in the lemma.

First, let's fix some notation. Given a factor sequence $(p_i)_i$, the natural block-diagonal embeddings

$$M_{p_i}(K) \to M_{p_{i+1}}(K), \quad x \mapsto \begin{pmatrix} x & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & x \end{pmatrix}$$

will be denoted by $\gamma_{i+1,i}$. If $j > i$, the map $\gamma_{j,i} \colon M_{p_i}(K) \to M_{p_j}(K)$ will denote the composition

$$\gamma_{j,i} = \gamma_{j,j-1} \circ \cdots \circ \gamma_{i+1,i},$$

and the map $\gamma_{\infty,i} \colon M_{p_i}(K) \to \varinjlim_n M_{p_n}(K)$ will stand for the canonical map into the direct limit. By [44], there is an isomorphism $\mathcal{M}_K \cong \overline{\varinjlim_n M_{p_n}(K)}$, where the completion is taken with respect to the unique rank function on the direct limit. We henceforth will identify $\mathcal{M} = \mathcal{M}_K$ with the algebra $\overline{\varinjlim_n M_{p_n}(K)}$.

**Notation 4.2.3.** Finally, for $(X, d)$ a metric space, $A, Y$ subsets of $X$ and $\varepsilon > 0$, we write $A \subseteq_\varepsilon Y$ in case each element of $A$ can be approximated by an element of $Y$ up to $\varepsilon$ with respect to the metric $d$, that is, for each $a \in A$ there exists an element $y \in Y$ such that $d(x, y) < \varepsilon$.

**Lemma 4.2.4.** *Let $\mathcal{Q}$ be a continuous factor with unique rank function $\mathrm{rk}_{\mathcal{Q}}$. Assume that there exists a dense $K$-subalgebra $\mathcal{Q}_0$ of $\mathcal{Q}$ of countable dimension, and let $\{x_n\}_n$ be a $K$-basis of $\mathcal{Q}_0$.*

*Let $\theta \in (0,1)$ be a real number. Assume further that we have constructed two strictly increasing sequences $(q_i)_i$ and $(p_i)_i$ of natural numbers such that $p_i$ divides $p_{i+1}$ and satisfying*

*a) $1 > \frac{p_1}{q_1} > \cdots > \frac{p_i}{q_i} > \frac{p_{i+1}}{q_{i+1}} > \cdots > \theta$, $\lim_{i \to \infty} \frac{p_i}{q_i} = \theta$ and* *b)* $\frac{p_{i+1}}{q_{i+1}} - \theta < \frac{1}{2}\left(\frac{p_i}{q_i} - \theta\right)$ *for* $i \geq 0$.

*We also demand* $p_0 = q_0 = 1$. *Moreover, suppose that there exists a sequence of positive numbers* $\varepsilon_i < \frac{p_i}{q_i} - \theta$, *and matricial* $K$*-subalgebras* $\mathcal{A}_i \subseteq \mathcal{Q}$ *together with algebra homomorphisms* $\rho_i : M_{p_i}(K) \to \mathcal{Q}$ *satisfying the following properties:*

*i)* $\mathrm{rk}_{\mathcal{Q}}(\rho_i(1)) = \frac{p_i}{q_i}$ *for all* $i$.

*ii) For each* $i$ *and each* $x \in \rho_i(1)\mathcal{A}_i\rho_i(1)$, *there exists* $y \in M_{p_{i+1}}(K)$ *such that*

$$\mathrm{rk}_{\mathcal{Q}}(x - \rho_{i+1}(y)) < \frac{p_i}{q_i} - \theta\,.$$

*iii) For each* $z \in M_{p_i}(K)$, *we have*

$$\mathrm{rk}_{\mathcal{Q}}(\rho_i(z) - \rho_{i+1}(\gamma_{i+1,i}(z))) < \frac{p_i}{q_i} - \theta\,.$$

*iv)* $\mathrm{span}\{x_1, ..., x_i\} \subseteq_{\varepsilon_i} \mathcal{A}_i$.

*Then there exists an isomorphism* $\psi : \mathcal{M} \to e\mathcal{Q}e$, *with* $e \in \mathcal{Q}$ *an idempotent such that* $\mathrm{rk}_{\mathcal{Q}}(e) = \theta$.

*Proof.* For $j > i \geq 1$, we have the following diagram

$$\begin{array}{ccccccccc}
M_{p_i}(K) & \xrightarrow{\gamma_{i+1,i}} & M_{p_{i+1}}(K) & \xrightarrow{\gamma_{i+2,i+1}} & \cdots & \xrightarrow{\gamma_{j,j-1}} & M_{p_j}(K) & \xrightarrow{\gamma_{j+1,j}} & \cdots \\
\downarrow{\scriptstyle \rho_i} & & \downarrow{\scriptstyle \rho_{i+1}} & & & & \downarrow{\scriptstyle \rho_j} & & \\
\mathcal{Q} & = & \mathcal{Q} & = & \cdots & = & \mathcal{Q} & = & \cdots
\end{array}$$

which may be, in general, not commutative. We are going to construct a "limit" version of it which indeed commutes. Fix a positive integer $i \geq 1$ and write $\delta_i = \frac{p_i}{q_i} - \theta$. Note that by *b)*, $\delta_i < \frac{1}{2}\delta_{i-1} < 2^{-i}$. For $z \in M_{p_i}(K)$, we consider the sequence in $\mathcal{Q}$ given by $\{\rho_j(\gamma_{j,i}(z))\}_{j \geq i}$. It is a simple computation to show, using *iii)*, that for $h > j \geq i$ we have

$$\begin{aligned}
\mathrm{rk}_{\mathcal{Q}}(\rho_h(\gamma_{h,i}(z)) - \rho_j(\gamma_{j,i}(z))) &= \mathrm{rk}_{\mathcal{Q}}\left(\sum_{k=j}^{h-1} \rho_{k+1}(\gamma_{k+1,i}(z)) - \rho_k(\gamma_{k,i}(z))\right) \\
&\leq \sum_{k=j}^{h-1} \mathrm{rk}_{\mathcal{Q}}(\rho_{k+1}(\gamma_{k+1,i}(z)) - \rho_k(\gamma_{k,i}(z))) < \sum_{k=j}^{h-1} \delta_k < \sum_{k=j}^{h-1} 2^{-k} < 2^{-j+1} \xrightarrow[j \to \infty]{} 0\,.
\end{aligned} \tag{4.2.1}$$

As a consequence, the sequence $\{\rho_j(\gamma_{j,i}(z))\}_{j \geq i}$ is Cauchy in $\mathcal{Q}$, so convergent. We can therefore define algebra homomorphisms $\psi_i : M_{p_i}(K) \to \mathcal{Q}$ by

$$\psi_i(z) = \lim_j \rho_j(\gamma_{j,i}(z)) \in \mathcal{Q}.$$

Now the previous diagram with the $\rho$'s replaced with the $\psi$'s commute, as the following computation shows:

$$\psi_{i+1}(\gamma_{i+1,i}(z)) = \lim_j \rho_j(\gamma_{j,i+1}(\gamma_{i+1,i}(z))) = \lim_j \rho_j(\gamma_{j,i}(z)) = \psi_i(z).$$

So the maps $\{\psi_i\}_i$ give a well-defined algebra homomorphism $\psi : \varinjlim_n M_{p_n}(K) \to \mathcal{Q}$, defined by

$$\psi(\gamma_{\infty,i}(z)) = \psi_i(z) \quad \text{for } z \in M_{p_i}(K).$$

Let's extend it to the rank completion of $\varinjlim_n M_{p_n}(K)$. Denote by $\{e_{kl}\}_{1 \leq k,l \leq p_i}$ the standard matrix units of $M_{p_i}(K)$. By *i)* and the fact that the $e_{kk}$ are mutually orthogonal equivalent idempotents, their images $\rho_i(e_{kk})$ all have the same rank in $\mathcal{Q}$, which equals $\mathrm{rk}_{\mathcal{Q}}(\rho_i(e_{kk})) = \frac{1}{q_i}$as the following computation shows:

$$\frac{p_i}{q_i} = \mathrm{rk}_{\mathcal{Q}}(\rho_i(1)) = \mathrm{rk}_{\mathcal{Q}}(\rho_i(e_{11})) + \cdots + \mathrm{rk}_{\mathcal{Q}}(\rho_i(e_{p_ip_i})) = p_i \cdot \mathrm{rk}_{\mathcal{Q}}(\rho_i(e_{kk}))$$

Now for $z \in M_{p_i}(K)$, if we denote by $\mathrm{rk}_{p_i} = \frac{\mathrm{Rk}_{p_i}}{p_i}$ the unique rank function on $M_{p_i}(K)$, there exist invertible matrices $P, Q \in M_{p_i}(K)$ such that $PzQ = \mathrm{Id}_N \oplus 0_{p_i - N} = e_{11} + \cdots + e_{NN}$, being $N$ the usual matrix rank of $z$. By applying $\rho_i$ and taking into account that the $P, Q$ are invertible,

$$\mathrm{rk}_{\mathcal{Q}}(\rho_i(z)) = \mathrm{rk}_{\mathcal{Q}}(\rho_i(PzQ)) = \mathrm{rk}_{\mathcal{Q}}(\rho_i(e_{11})) + \cdots + \mathrm{rk}_{\mathcal{Q}}(\rho_i(e_{NN})) = \frac{\mathrm{Rk}_{p_i}(z)}{q_i} = \frac{p_i}{q_i}\mathrm{rk}_{p_i}(z) = \frac{p_i}{q_i}\mathrm{rk}_{\mathcal{M}}(\gamma_{\infty,i}(z)).$$

Therefore

$$\mathrm{rk}_{\mathcal{Q}}(\psi(\gamma_{\infty,i}(z))) = \lim_j \mathrm{rk}_{\mathcal{Q}}(\rho_j(\gamma_{j,i}(z))) = \lim_j \frac{p_j}{q_j} \cdot \mathrm{rk}_{\mathcal{M}}(\gamma_{\infty,i}(z)) = \theta \cdot \mathrm{rk}_{\mathcal{M}}(\gamma_{\infty,i}(z)).$$

It follows that $\mathrm{rk}_{\mathcal{Q}}(\psi(x)) = \theta \cdot \mathrm{rk}_{\mathcal{M}}(x)$ for every $x \in \varinjlim_n M_{p_n}(K)$, and thus $\psi$ can be extended to a unital algebra homomorphism $\psi : \mathcal{M} \to e\mathcal{Q}e$, where $e := \psi(1) = \varinjlim_n \rho_n(1)$, which also satisfies the identity $\mathrm{rk}_{\mathcal{Q}}(\psi(z)) = \theta \cdot \mathrm{rk}_{\mathcal{M}}(z)$ for all $z \in \mathcal{M}$. In particular, $\mathrm{rk}_{\mathcal{Q}}(e) = \theta$.

Clearly, the previous identity shows that $\psi$ is injective, for if $x \in \mathcal{M}$ is such that $\psi(x) = 0$,

$$0 = \mathrm{rk}_{\mathcal{Q}}(\psi(x)) = \theta \cdot \mathrm{rk}_{\mathcal{M}}(x),$$

so $x = 0$ since $\mathrm{rk}_{\mathcal{M}}$ is a rank function. To prove surjectivity onto $e\mathcal{Q}e$, let $x \in \mathcal{Q}$ and fix $\eta > 0$. Take $i$ large enough so that

$$\varepsilon_i < \frac{\eta}{10}, \qquad \delta_i < \frac{\eta}{5}, \qquad \mathrm{rk}_{\mathcal{Q}}(e - \rho_i(1)) < \frac{\eta}{5},$$

and such that there exists an element $\widetilde{x} \in \mathrm{span}\{x_1, ..., x_i\}$ satisfying $\mathrm{rk}_{\mathcal{Q}}(x - \widetilde{x}) < \frac{\eta}{10}$ (this is possible due to the fact that $\{x_n\}_n$ is a $K$-basis for the dense subalgebra $\mathcal{Q}_0$). By $iv)$, there exists $y \in \mathcal{A}_i$ so that $\mathrm{rk}_{\mathcal{Q}}(\widetilde{x} - y) < \frac{\eta}{10}$; hence $\mathrm{rk}_{\mathcal{Q}}(x - y) < \frac{\eta}{5}$. We thus have, on one hand,

$$\begin{aligned}\mathrm{rk}_{\mathcal{Q}}(exe - \rho_i(1)y\rho_i(1)) &\le \mathrm{rk}_{\mathcal{Q}}(exe - ex\rho_i(1)) + \mathrm{rk}_{\mathcal{Q}}(ex\rho_i(1) - ey\rho_i(1)) + \mathrm{rk}_{\mathcal{Q}}(ey\rho_i(1) - \rho_i(1)y\rho_i(1)) \\ &\le \mathrm{rk}_{\mathcal{Q}}(e - \rho_i(1)) + \mathrm{rk}_{\mathcal{Q}}(x - y) + \mathrm{rk}_{\mathcal{Q}}(e - \rho_i(1)) < \frac{3\eta}{5}.\end{aligned}$$

On the other hand, since $\rho_i(1)y\rho_i(1) \in \rho_i(1)\mathcal{A}_i\rho_i(1)$, it follows from $ii)$ that there exists $z \in M_{p_{i+1}}(K)$ such that

$$\mathrm{rk}_{\mathcal{Q}}(\rho_i(1)y\rho_i(1) - \rho_{i+1}(z)) < \delta_i\,.$$

Also, for $i + 1 < h$, we get from (4.2.1)

$$\mathrm{rk}_{\mathcal{Q}}(\rho_h(\gamma_{h,i+1}(z)) - \rho_{i+1}(z)) < \sum_{k=i+1}^{h-1} \delta_k < \delta_{i+1} \sum_{k=0}^{h-i-2} 2^{-k} < 2\delta_{i+1} < \delta_i\,,$$

and so letting $h \to \infty$ leads to

$$\mathrm{rk}_{\mathcal{Q}}(\psi(\gamma_{\infty,i+1}(z)) - \rho_{i+1}(z)) \le \delta_i < \frac{\eta}{5}\,.$$

Using the above inequalities, we obtain

$$\begin{aligned}\mathrm{rk}_{\mathcal{Q}}(exe - \psi(\gamma_{\infty,i+1}(z))) \le\, &\mathrm{rk}_{\mathcal{Q}}(exe - \rho_i(1)y\rho_i(1)) + \mathrm{rk}_{\mathcal{Q}}(\rho_i(1)y\rho_i(1) - \rho_{i+1}(z)) \\ &+ \mathrm{rk}_{\mathcal{Q}}(\rho_{i+1}(z) - \psi(\gamma_{\infty,i+1}(z))) \le \frac{3\eta}{5} + \frac{\eta}{5} + \frac{\eta}{5} = \eta.\end{aligned}$$

Now, by choosing a decreasing sequence of positive numbers $\eta_n \xrightarrow{n} 0$, it follows that for each $n$ there exists $w_n \in \mathcal{M}$ satisfying $\mathrm{rk}_{\mathcal{Q}}(exe - \psi(w_n)) < \eta_n$, so that $\lim_n \psi(w_n) = exe$ in $\mathcal{Q}$. But by using the relation between the ranks of $\mathcal{M}$ and $\mathcal{Q}$, we compute

$$\mathrm{rk}_{\mathcal{M}}(w_n - w_m) = \theta^{-1} \cdot \mathrm{rk}_{\mathcal{Q}}(\psi(w_n) - \psi(w_m)) \xrightarrow[n,m \to \infty]{} 0,$$

so we conclude that $\{w_n\}_n$ is a Cauchy sequence in $\mathcal{M}$, hence convergent to some $w \in \mathcal{M}$ satisfying $\psi(w) = \lim_n \psi(w_n) = exe$. This shows that $\psi$ is surjective, proving the lemma. □

We now show how Theorem 4.2.2 follows from Lemma 4.2.4, assuming we are able to show that the hypotheses of that lemma are satisfied. This indeed follows as in [44].

*Proof of the implication* (3) $\Longrightarrow$ (1) *of Theorem 4.2.2.* Take $\theta = 1/2$ and apply Lemma 4.2.4 to obtain an isomorphism $\psi : \mathcal{M} \to e\mathcal{Q}e$, where $\mathrm{rk}_{\mathcal{Q}}(e) = 1/2$. Since $\mathrm{rk}_{\mathcal{Q}}(e) = \frac{1}{2} = \mathrm{rk}_{\mathcal{Q}}(1-e)$, Proposition 1.2.7 says that $e$ and $1-e$ are equivalent idempotents in $\mathcal{Q}$, so that there exist elements $x, y \in \mathcal{Q}$ such that $xy = e$ and $yx = 1-e$. We now obtain an isomorphism of $K$-algebras $\mathcal{Q} \cong M_2(e\mathcal{Q}e)$ by the rule $\alpha \mapsto \begin{pmatrix} e\alpha e & e\alpha y \\ x\alpha e & x\alpha y \end{pmatrix}$, and this gives rise to a chain of isomorphisms

$$\mathcal{M} \cong M_2(\mathcal{M}) \cong M_2(e\mathcal{Q}e) \cong \mathcal{Q},$$

where the first one is given by extending to the respective rank completions the isomorphism $\varinjlim_n M_{2^n}(K) \to M_2(\varinjlim_n M_{2^n}(K))$, $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. This proves the theorem. $\square$

It remains to show that the hypotheses of Lemma 4.2.4 are satisfied. We need a preliminary lemma, which might be of independent interest.

**Lemma 4.2.5.** *Let $p$ be a positive integer. Then there exists a constant $K(p)$, depending only on $p$, such that*

*i) for any field $K$,*

*ii) any $\varepsilon > 0$,*

*iii) any pair $\mathcal{A} \subseteq \mathcal{B}$ where $\mathcal{B}$ is a unital $K$-algebra and $\mathcal{A}$ is a unital regular $K$-subalgebra of $\mathcal{B}$,*

*iv) any pseudo-rank function* rk *on $\mathcal{B}$, and*

*v) any algebra homomorphism $\rho : M_p(K) \to \mathcal{B}$ such that*

$$\{\rho(e_{ij})\}_{1 \le i,j \le p} \subseteq_\varepsilon \mathcal{A}$$

*with respect to the* rk*-metric (where $e_{ij}$ denote the canonical matrix units in $M_p(K)$),*

*there exists an algebra homomorphism $\psi : M_p(K) \to \mathcal{A}$ which is close to $\rho$ in rank, namely*

$$\mathrm{rk}(\rho(e_{ij}) - \psi(e_{ij})) < K(p)\varepsilon \quad \textit{for } 1 \le i, j \le p.$$

*Proof.* We proceed by induction on $p$. Let $p = 1$, and let $K, \varepsilon, \mathcal{A}, \mathcal{B}, \mathrm{rk}$ and $\rho\colon K \to \mathcal{B}$ be as in the statement. Then $\rho(1)$ is an idempotent in $\mathcal{B}$ and, by assumption, there is $x \in \mathcal{A}$ such that $\mathrm{rk}(\rho(1) - x) < \varepsilon$. By [39, Lemma 19.3], there exists an idempotent $g \in \mathcal{A}$ such that $x - g \in \mathcal{A}(x - x^2)$, so $\mathrm{rk}(x-g) \le \mathrm{rk}(x - x^2)$. It follows that

$$\begin{aligned} \mathrm{rk}(\rho(1) - g) &\le \mathrm{rk}(\rho(1) - x) + \mathrm{rk}(x - g) \le \mathrm{rk}(\rho(1) - x) + \mathrm{rk}(x - x^2) \\ &\le \mathrm{rk}(\rho(1) - x) + \mathrm{rk}(x - \rho(1)) + \mathrm{rk}(\rho(1)^2 - x\rho(1)) + \mathrm{rk}(x\rho(1) - x^2) \le 4\,\mathrm{rk}(\rho(1) - x) < 4\varepsilon. \end{aligned}$$

Therefore we can take $K(1) = 4$, and define $\psi : K \to \mathcal{A}$ by $\psi(1) = g$.

Now assume that $p \ge 2$ and that there is a constant $K(p-1)$ satisfying the property corresponding to $p-1$. Let $K, \varepsilon, \mathcal{A}, \mathcal{B}, \mathrm{rk}$ and $\rho : M_p(K) \to \mathcal{B}$ be as in the statement. We identify $M_{p-1}(K)$ with the subalgebra of $M_p(K)$ generated by $e_{ij}$ with $1 \le i, j \le p-1$. By the induction hypothesis, there is a set of $(p-1) \times (p-1)$ matrix units $x_{ij} \in \mathcal{A}$ (so that $x_{ij}x_{kl} = \delta_{jk}x_{il}$ for $1 \le i, j, k, l \le p-1$) satisfying

$$\mathrm{rk}(\rho(e_{ij}) - x_{ij}) < K(p-1)\varepsilon \quad \text{for all } 1 \le i, j \le p-1.$$

Also by hypothesis, there are $z_{1p}, z_{p1} \in \mathcal{A}$ such that $\mathrm{rk}(\rho(e_{1p}) - z_{1p}) < \varepsilon$ and $\mathrm{rk}(\rho(e_{p1}) - z_{p1}) < \varepsilon$. Structurally,

$$\left(\begin{array}{ccc|c} x_{11} & \cdots & x_{1,p-1} & z_{1p} \\ \vdots & \ddots & \vdots & \times \\ x_{p-1,1} & \cdots & x_{p-1,p-1} & \times \\ \hline z_{p1} & \times & \times & \times \end{array}\right) \tag{4.2.2}$$

Our first task is to modify $z_{1p}$ and $z_{p1}$ in order to obtain new elements $z'_{1p}$ and $z'_{p1}$ such that

$$z'_{1p}x_{i1} = 0 = x_{1i}z'_{p1} \qquad \text{for} \quad 1 \leq i \leq p-1, \tag{4.2.3}$$

with suitable bounds on the ranks. To get the desired elements, we proceed by induction on $i$. We will only prove the result for the position $(1,p)$. The element in the position $(p,1)$ is built in a similar way.

We start with $i=1$. We use that $\mathcal{A}$ is regular to obtain an idempotent $g_1 \in \mathcal{A}$ such that

$$z_{1p}x_{11}\mathcal{A} = g_1\mathcal{A}.$$

Note that

$$\begin{aligned}\mathrm{rk}(g_1) &= \mathrm{rk}(z_{1p}x_{11}) = \mathrm{rk}(z_{1p}x_{11} - \rho(e_{1p})\rho(e_{11}))\\ &\leq \mathrm{rk}(z_{1p}x_{11} - \rho(e_{1p})x_{11}) + \mathrm{rk}(\rho(e_{1p})x_{11} - \rho(e_{1p})\rho(e_{11})) < \varepsilon + K(p-1)\varepsilon = (K(p-1)+1)\varepsilon.\end{aligned}$$

Now take $z_{1p}^{(1)} := (1-g_1)z_{1p}$. Since $z_{1p}x_{11} \in g_1\mathcal{A}$ and $g_1$ is an idempotent, $g_1z_{1p}x_{11} = z_{1p}x_{11}$, so we get that $z_{1p}^{(1)}x_{11} = (1-g_1)z_{1p}x_{11} = 0$ and that

$$\mathrm{rk}(z_{1p}^{(1)} - \rho(e_{1p})) = \mathrm{rk}(z_{1p} - g_1z_{1p} - \rho(e_{1p})) \leq \mathrm{rk}(z_{1p} - \rho(e_{1p})) + \mathrm{rk}(g_1) < (K(p-1)+2)\varepsilon.$$

Suppose that, for $1 \leq i-1 \leq p-1$, we have constructed elements $z_{1p}^{(1)}, ..., z_{1p}^{(i-1)}$ in $\mathcal{A}$ such that $z_{1p}^{(l)}x_{j1} = 0$ for all fixed $1 \leq l \leq i-1$ and all $1 \leq j \leq l$, and

$$\mathrm{rk}(z_{1p}^{(i-1)} - \rho(e_{1p})) < \big((2^{i-1}-1)K(p-1) + 2^{i-1}\big)\varepsilon.$$

Let's construct $z_{1p}^{(i)}$: take an idempotent $g_{i-1} \in \mathcal{A}$ such that $z_{1p}^{(i-1)}x_{i1}\mathcal{A} = g_{i-1}\mathcal{A}$. We have the bound in rank of $g_{i-1}$ given by

$$\begin{aligned}\mathrm{rk}(g_{i-1}) &= \mathrm{rk}(z_{1p}^{(i-1)}x_{i1}) = \mathrm{rk}(z_{1p}^{(i-1)}x_{i1} - \rho(e_{1p})\rho(e_{i1}))\\ &\leq \mathrm{rk}(z_{1p}^{(i-1)}x_{i1} - \rho(e_{1p})x_{i1}) + \mathrm{rk}(\rho(e_{1p})x_{i1} - \rho(e_{1p})\rho(e_{i1})) < \big(2^{i-1}K(p-1) + 2^{i-1}\big)\varepsilon.\end{aligned}$$

Define $z_{1p}^{(i)} := (1-g_{i-1})z_{1p}^{(i-1)}$. Since $z_{1p}^{(i-1)}x_{i1} \in g_{i-1}\mathcal{A}$ and $g_{i-1}$ is an idempotent, $g_{i-1}z_{1p}^{(i-1)}x_{i1} = z_{1p}^{(i-1)}x_{i1}$, so we get that $z_{1p}^{(i)}x_{i1} = (1-g_{i-1})z_{1p}^{(i-1)}x_{i1} = 0$ and that

$$\begin{aligned}\mathrm{rk}(z_{1p}^{(i)} - \rho(e_{1p})) &= \mathrm{rk}(z_{1p}^{(i-1)} - g_{i-1}z_{1p}^{(i-1)} - \rho(e_{1p}))\\ &\leq \mathrm{rk}(z_{1p}^{(i-1)} - \rho(e_{1p})) + \mathrm{rk}(g_{i-1}) < \big((2^i-1)K(p-1) + 2^i\big)\varepsilon.\end{aligned}$$

After all these constructions, we simply take $z'_{1p} = z_{1p}^{(p-1)}$, and the element $z'_{p1} := z_{p1}^{(p-1)}$ built in a similar fashion. These elements $z'_{1p}, z'_{p1} \in \mathcal{A}$ satisfy (4.2.3) and are such that

$$\max\{\mathrm{rk}(z'_{1p} - \rho(e_{1p})), \mathrm{rk}(z'_{p1} - \rho(e_{p1}))\} < \big((2^{p-1}-1)K(p-1) + 2^{p-1}\big)\varepsilon. \tag{4.2.4}$$

The next step is to convert $z'_{1p}$ and $z'_{p1}$ into mutually quasi-inverse elements in $\mathcal{A}$, so that the new products $z'_{1p}z'_{p1}$ and $z'_{p1}z'_{1p}$ become idempotents. Indeed, we will also replace in addition our original elements $x_{1i}, x_{i1}$, for $1 \leq i \leq p-1$, by another elements $y_{1i}, y_{i1}$, for $1 \leq i \leq p$, in order to get a coherent family of partial matrix units, i.e. elements satisfying

$$y_{1i}y_{j1} = \delta_{i,j}y_{11} \quad \text{ for } 1 \leq i,j \leq p. \tag{4.2.5}$$

For this we will use [39, Lemma 19.3] and its proof. Consider the element $x'_{11} := x_{11}z'_{1p}z'_{p1}x_{11} \in \mathcal{A}$, and note that

$$\begin{aligned}\mathrm{rk}(x'_{11} - \rho(e_{11})) &= \mathrm{rk}(x_{11}z'_{1p}z'_{p1}x_{11} - \rho(e_{11})\rho(e_{1p})\rho(e_{p1})\rho(e_{11}))\\ &\leq 2\,\mathrm{rk}(x_{11} - \rho(e_{11})) + \mathrm{rk}(z'_{1p} - \rho(e_{1p})) + \mathrm{rk}(z'_{p1} - \rho(e_{p1}))\\ &< 2K(p-1)\varepsilon + 2\big((2^{p-1}-1)K(p-1) + 2^{p-1}\big)\varepsilon = \big(K(p-1)+1\big)2^p\varepsilon,\end{aligned}$$

where we have used the bound given by the induction hypothesis and (4.2.4). Therefore, we get

$$\begin{aligned}\mathrm{rk}(x'_{11}-(x'_{11})^2) \leq& \mathrm{rk}(x'_{11}-\rho(e_{11}))+\mathrm{rk}(\rho(e_{11})^2-x'_{11}\rho(e_{11}))\\ &+\mathrm{rk}(x'_{11}\rho(e_{11})-(x'_{11})^2) < 3(K(p-1)+1)2^p\varepsilon.\end{aligned}$$

Now using [39, Lemma 19.3] and its proof, we can find an idempotent $g \in \mathcal{A}$ such that $g \in x'_{11}\mathcal{A} \cap \mathcal{A}x'_{11}$, $x'_{11} - g \in \mathcal{A}(x'_{11}-(x'_{11})^2)$ and $x'_{11}g = g$. It follows that $gx'_{11}g = g$, and since $x_{11}x'_{11}x_{11} = x'_{11}$, we also obtain $x_{11}g = g = gx_{11}$, so $g \leq x_{11}$. Therefore

$$g = gx'_{11}g = gx_{11}z'_{1p}z'_{p1}x_{11}g = gz'_{1p}z'_{p1}g.$$

Set

$$y_{11} := g \leq x_{11}, \qquad \begin{cases} y_{1i} := gx_{1i} \\ y_{1p} := gz'_{1p}\end{cases} \text{for } 2 \leq i \leq p-1\ , \qquad \begin{cases} y_{i1} := x_{i1}g \\ y_{p1} := z'_{p1}g\end{cases} \text{for } 2 \leq i \leq p-1.$$

Note that, with these definitions, (4.2.5) is satisfied:

$$\begin{cases} y_{1p}y_{j1} = gz'_{1p}x_{j1}g = 0 \\ y_{1p}y_{p1} = gz'_{1p}z'_{p1}g = g = y_{11}\end{cases} \text{and} \quad \begin{cases} y_{1i}y_{j1} = gx_{1i}x_{j1}g = \delta_{ij}gx_{11}g = \delta_{ij}y_{11} \\ y_{1i}y_{p1} = gx_{1i}z'_{p1}g = 0\end{cases} \text{for } 1 \leq i,j \leq p-1.$$

Hence the elements $y_{1i}, y_{j1}$ for $1 \leq i,j \leq p$ form a coherent family of partial matrix units. If we define $y_{ij} := y_{i1}y_{1j}$ for $1 \leq i,j \leq p$, we obtain that $\{y_{ij}\}$ forms a system of $p \times p$ matrix units in $\mathcal{A}$:

$$y_{ij}y_{kl} = y_{i1}y_{1j}y_{k1}y_{1l} = \delta_{j,k}y_{i1}y_{1l} = \delta_{j,k}y_{il}.$$

Therefore we can define an algebra homomorphism $\psi : M_p(K) \to \mathcal{A}$ by the rule $\psi(e_{ij}) = y_{ij}$.

Let's now check that $\psi$ is close to $\rho$ in rank. We have the estimate

$$\begin{aligned}\mathrm{rk}(y_{11}-\rho(e_{11})) &\leq \mathrm{rk}(y_{11}-x'_{11})+\mathrm{rk}(x'_{11}-\rho(e_{11}))\\ &\leq \mathrm{rk}(x'_{11}-(x'_{11})^2)+\mathrm{rk}(x'_{11}-\rho(e_{11}))\\ &< 3(K(p-1)+1)2^p\varepsilon + (K(p-1)+1)2^p\varepsilon = (K(p-1)+1)2^{p+2}\varepsilon.\end{aligned}$$

Using this inequality and (4.2.4), we obtain

$$\begin{aligned}\mathrm{rk}(y_{1p}-\rho(e_{1p})) &= \mathrm{rk}(gz'_{1p}-\rho(e_{11})\rho(e_{1p}))\\ &\leq \mathrm{rk}(gz'_{1p}-\rho(e_{11})z'_{1p})+\mathrm{rk}(\rho(e_{11})z'_{1p}-\rho(e_{11})\rho(e_{1p}))\\ &\leq \mathrm{rk}(g-\rho(e_{11}))+\mathrm{rk}(z'_{1p}-\rho(e_{1p})) < (K(p-1)+1)2^p\varepsilon + \big((2^{p-1}-1)K(p-1)+2^{p-1}\big)\varepsilon\\ &= \big((2^{p+2}+2^{p-1}-1)K(p-1)+2^{p+2}+2^{p-1}\big)\varepsilon.\end{aligned}$$

Similar computations give that $\mathrm{rk}(y_{p1}-\rho(e_{p1})) < \big((2^{p+2}+2^{p-1}-1)K(p-1)+2^{p+2}+2^{p-1}\big)\varepsilon$. For $2 \leq i \leq p-1$, we compute a bound for the other matrix units using the induction hypothesis:

$$\begin{aligned}\mathrm{rk}(y_{1i}-\rho(e_{1i})) &= \mathrm{rk}(gx_{1i}-\rho(e_{11})\rho(e_{1i}))\\ &\leq \mathrm{rk}(gx_{1i}-\rho(e_{11})x_{1i})+\mathrm{rk}(\rho(e_{11})x_{1i}-\rho(e_{11})\rho(e_{1i}))\\ &\leq \mathrm{rk}(g-\rho(e_{11}))+\mathrm{rk}(x_{1i}-\rho(e_{1i})) < (K(p-1)+1)2^{p+2}\varepsilon + K(p-1)\varepsilon\\ &= \big((2^{p+2}+1)K(p-1)+2^{p+2}\big)\varepsilon.\end{aligned}$$

Similarly $\mathrm{rk}(y_{i1}-\rho(e_{i1})) < \big((2^{p+2}+1)K(p-1)+2^{p+2}\big)\varepsilon$. Putting everything together, we get the common upper bound

$$\max_{1\leq i\leq p}\{\mathrm{rk}(y_{1i}-\rho(e_{1i})),\mathrm{rk}(y_{i1}-\rho(e_{i1}))\} \leq \big((2^{p+2}+2^{p-1}-1)K(p-1)+2^{p+2}+2^{p-1}\big)\varepsilon.$$

Finally for any $1 \leq i,j \leq p$,

$$\begin{aligned}\mathrm{rk}(y_{ij}-\rho(e_{ij})) &= \mathrm{rk}(y_{i1}y_{1j}-\rho(e_{i1})\rho(e_{1j}))\\ &\leq \mathrm{rk}(y_{i1}y_{1j}-y_{i1}\rho(e_{1j}))+\mathrm{rk}(y_{i1}\rho(e_{1j})-\rho(e_{i1})\rho(e_{1j}))\\ &\leq \big((2^{p+3}+2^p-2)K(p-1)+2^{p+3}+2^p\big)\varepsilon.\end{aligned}$$

This concludes the proof, if we take $K(p) := (2^{p+3}+2^p-2)K(p-1)+2^{p+3}+2^p$. □

We now show that the hypotheses of Lemma 4.2.4 are indeed satisfied (assuming condition (3) in Theorem 4.2.2). This is obtained from the next Lemma by applying induction (starting with $p_0 = q_0 = 1$ and $\mathcal{A}_0 = K$).

**Lemma 4.2.6.** *Let $\mathcal{Q}$ be a continuous factor with unique rank function $\mathrm{rk}_{\mathcal{Q}}$. Assume that there exists a dense $K$-subalgebra $\mathcal{Q}_0$ of $\mathcal{Q}$ of countable $K$-dimension, and let $\{x_n\}_n$ be a $K$-basis of $\mathcal{Q}_0$. Assume that $\mathcal{Q}$ satisfies condition (3) in Theorem 4.2.2, and let $\theta \in (0,1)$.*

*Let $p$ be a positive integer such that there exist an algebra homomorphism $\rho : M_p(K) \to \mathcal{Q}$, a matricial $K$-subalgebra $\mathcal{A} \subseteq \mathcal{Q}$, a positive integer $m$ and $\varepsilon > 0$ such that*

*i)* $\mathrm{rk}_{\mathcal{Q}}(\rho(1)) = \frac{p}{q} > \theta$ *for some positive integer $q$.*

*ii)* $\{\rho(e_{ij}) \mid i,j = 1,...,p\} \subseteq_\varepsilon \mathcal{A}$, *and* $\mathrm{span}\{x_1,...,x_m\} \subseteq_\varepsilon \mathcal{A}$.

*iii)* $\varepsilon < \frac{1}{48K(p)p^2}\Big(\frac{p}{q} - \theta\Big)$, *where $K(p)$ is the constant introduced in Lemma 4.2.5.*

*Then there exist positive integers $p', g, q'$, with $p' = gp$, a real number $\varepsilon' > 0$, an algebra homomorphism $\rho' : M_{p'}(K) \to \mathcal{Q}$ and a matricial $K$-subalgebra $\mathcal{A}' \subseteq \mathcal{Q}$ such that the following conditions hold:*

*1)* $\mathrm{rk}_{\mathcal{Q}}(\rho'(1)) = \frac{p'}{q'}$.

*2)*
$$0 < \frac{p'}{q'} - \theta < \frac{1}{2}\Big(\frac{p}{q} - \theta\Big).$$

*3) For each $x \in \rho(1)\mathcal{A}\rho(1)$, there exists $y \in M_{p'}(K)$ such that*
$$\mathrm{rk}_{\mathcal{Q}}(x - \rho'(y)) < \frac{p}{q} - \theta.$$

*4) For each $z \in M_p(K)$, we have*
$$\mathrm{rk}_{\mathcal{Q}}(\rho(z) - \rho'(\gamma(z))) < \frac{p}{q} - \theta,$$
*where $\gamma : M_p(K) \to M_{p'}(K)$ is the canonical unital homomorphism sending $z$ to* $\begin{pmatrix} z & & \mathbf{0} \\ & \overset{g}{\ddots} & \\ \mathbf{0} & & z \end{pmatrix}$.

*5)* $\{\rho'(e'_{ij}) \mid i,j = 1,...,p'\} \subseteq_{\varepsilon'} \mathcal{A}'$, *and* $\mathrm{span}\{x_1,...,x_m,x_{m+1}\} \subseteq_{\varepsilon'} \mathcal{A}'$, *where $\{e'_{ij} \mid i,j = 1,...,p'\}$ denote the canonical matrix units in $M_{p'}(K)$.*

*6)* $\varepsilon' < \frac{1}{48K(p')p'^2}\Big(\frac{p'}{q'} - \theta\Big)$.

*Proof.* We denote by $e_{ij}$, for $1 \leq i,j \leq p$, the canonical matrix units in $M_p(K)$. Set $f' := \rho(e_{11})$, which is an idempotent in $\mathcal{Q}$. By *i)* and the fact that the $e_{kk}$ are mutually orthogonal equivalent idempotents,
$$\mathrm{rk}_{\mathcal{Q}}(f') = \mathrm{rk}_{\mathcal{Q}}(\rho(e_{11})) = \frac{1}{p}\big(\mathrm{rk}_{\mathcal{Q}}(\rho(e_{11})) + \cdots + \mathrm{rk}_{\mathcal{Q}}(\rho(e_{pp}))\big) = \frac{1}{q}.$$
Because of *ii)* we can apply Lemma 4.2.5 to obtain an algebra homomorphism $\psi : M_p(K) \to \mathcal{A}$ such that $\mathrm{rk}_{\mathcal{Q}}(\rho(e_{ij}) - \psi(e_{ij})) < K(p)\varepsilon$ for all $1 \leq i,j \leq p$.

Set $f = \psi(e_{11})$ which is an idempotent in $\mathcal{A}$, and observe that
$$\mathrm{rk}_{\mathcal{Q}}(f - f') < K(p)\varepsilon. \tag{4.2.6}$$
Since $\mathcal{A}$ is matricial, $\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_r \cong M_{n_1}(K) \times \cdots \times M_{n_r}(K)$ for some positive integers $n_1,...,n_r$, so we can write
$$f = f_1 + \cdots + f_k$$
for some $k \leq r$, where $f_1,...,f_k$ are nonzero mutually orthogonal idempotents belonging to different simple factors $\mathcal{A}_i$ of $\mathcal{A}$, i.e. each $f_i \in \mathcal{A}_i$ is isomorphic to an idempotent in $M_{n_i}(K)$. We consider, for each $1 \leq i \leq k$,

a set of matrix units $\{f_{jl}^{(i)}\}_{1\le j,l\le r_i}$ inside $f_i\mathcal{A}f_i = f_i\mathcal{A}_i f_i$[1] such that each $f_{jj}^{(i)}$ is a minimal idempotent in $\mathcal{A}_i$, and such that

$$\sum_{j=1}^{r_i} f_{jj}^{(i)} = f_i, \text{ the unit of the corner } f_i\mathcal{A}_i f_i\,.$$

We can think of this decomposition of $f$ using matrices, as follows:

$$f \cong \left( \begin{pmatrix} \begin{pmatrix} f_{11}^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_{r_1r_1}^{(1)} \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}, \; \dots, \; \begin{pmatrix} \begin{pmatrix} f_{11}^{(k)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_{r_kr_k}^{(k)} \end{pmatrix} & 0 \\ 0 & 0 \end{pmatrix}, \; 0, \; \dots, \; 0 \right)$$

with maybe some zero matrices in between the $f_i$'s. Note that, since the $f_{jj}^{(i)}$ are all equivalent inside $f_i\mathcal{A}f_i$,

$$\mathrm{rk}_{\mathcal{Q}}(f) = \sum_{i=1}^k \mathrm{rk}_{\mathcal{Q}}(f_i) = \sum_{i=1}^k \sum_{j=1}^{r_i} \mathrm{rk}_{\mathcal{Q}}(f_{jj}^{(i)}) = \sum_{i=1}^k r_i\, \mathrm{rk}_{\mathcal{Q}}(f_{11}^{(i)}),$$

so that by [39, Lemma 19.1] and (4.2.6) we get the bound

$$\Big|\frac{1}{q} - \sum_{i=1}^k r_i\, \mathrm{rk}_{\mathcal{Q}}(f_{11}^{(i)})\Big| = |\,\mathrm{rk}_{\mathcal{Q}}(f') - \mathrm{rk}_{\mathcal{Q}}(f)| \le \mathrm{rk}_{\mathcal{Q}}(f - f') < K(p)\varepsilon. \tag{4.2.7}$$

We now approximate each real number $\mathrm{rk}_{\mathcal{Q}}(f_{11}^{(i)})$ by a rational number $\frac{p_i}{q_i}$. Concretely, we set

$$\delta := \frac{1}{48p(\sum_{i=1}^k r_i)}\Big(\frac{p}{q} - \theta\Big) > 0\,,$$

and take positive integers $p_i, q_i$ such that $0 < \mathrm{rk}_{\mathcal{Q}}(f_{11}^{(i)}) - \frac{p_i}{q_i} < \delta$. By taking common denominator, we may assume that $q_i = q'$ for all $i = 1, ..., k$. Let $\alpha'$ be such that $\frac{1}{\alpha'} = \sum_{i=1}^k r_i \frac{p_i}{q'}$, and observe that, by using *iii)* and (4.2.7), we have

$$\Big|\frac{p}{q} - \frac{p}{\alpha'}\Big| \le p\Big|\frac{1}{q} - \mathrm{rk}_{\mathcal{Q}}(f)\Big| + p\Big|\mathrm{rk}_{\mathcal{Q}}(f) - \frac{1}{\alpha'}\Big| = p\Big|\frac{1}{q} - \sum_{i=1}^k r_i\, \mathrm{rk}_{\mathcal{Q}}(f_{11}^{(i)})\Big| + \sum_{i=1}^k r_i \Big|\mathrm{rk}_{\mathcal{Q}}(f_{11}^{(i)}) - \frac{p_i}{q'}\Big|$$

$$< K(p)p\epsilon + p\Big(\sum_{i=1}^k r_i\Big)\delta < \frac{1}{48p}(p/q - \theta) + \frac{1}{48}(p/q - \theta) \le \frac{1}{24}(p/q - \theta).$$

So in particular

$$\Big|\frac{p}{q} - \frac{p}{\alpha'}\Big| < \frac{1}{8}\Big(\frac{p}{q} - \theta\Big). \tag{4.2.8}$$

What we have now is an approximation of $\mathrm{rk}_{\mathcal{Q}}(f)$ given by rational numbers, induces by the approximations of each $\mathrm{rk}_{\mathcal{Q}}(f_{11}^{(i)})$ by $\frac{p_i}{q'}$, in such a way that

$$\frac{1}{q} = \mathrm{rk}_{\mathcal{Q}}(f) = r_1\, \mathrm{rk}_{\mathcal{Q}}(f_{11}^{(1)}) + \cdots + r_k\, \mathrm{rk}_{\mathcal{Q}}(f_{11}^{(k)}) \quad \text{and} \quad \frac{1}{\alpha'} = r_1 \frac{p_1}{q'} + \cdots + r_k \frac{p_k}{q'}$$

are close enough. The problem we encounter now is that $\alpha'$ may not be an integer like $q$. We remedy this situation by approximating the whole fraction $\frac{p}{\alpha'}$ by another rational one close enough to $\frac{p}{q}$, while maintaining the same proportions between the fractions $r_i \frac{p_i}{q'}$ in the expansion of $\frac{1}{\alpha'}$. We take

$$\lambda_i = \frac{r_i p_i/q'}{1/\alpha'} = \frac{\alpha' r_i p_i}{q'} \qquad \text{and} \qquad \varepsilon_i = \frac{p_i/q'}{p/\alpha'}\Big(\frac{p}{q} - \theta\Big) = \frac{\alpha' p_i}{pq'}\Big(\frac{p}{q} - \theta\Big).$$

[1] It is possible that $f_i$ coincides with the unit of the simple factor $\mathcal{A}_i$, but in general it may not be the case, so we should consider the corner $f_i\mathcal{A}f_i \cong M_{r_i}(K)$, which is isomorphic to a corner of the full matrix algebra $M_{n_i}(K)$.

and note that $\sum_{i=1}^k \lambda_i = 1$. Moreover, each $\lambda_i$ and $\varepsilon_i$ (and of course $\frac{p_i}{q'}$) does not depend on replacing $p_i$ and $q'$ by $p_iN$ and $q'N$ respectively, for any integer $N \geq 1$, so we can assume that $p_i$ and $q'$ are arbitrarily large. Taking $p_i$ large enough, we claim that we can find nonnegative integers $p_i'$, for $1 \leq i \leq k$, such that

$$\frac{5\varepsilon_i}{8} < \frac{p_i}{q'} - \frac{p_i'}{q'} < \frac{6\varepsilon_i}{8} \tag{4.2.9}$$

for $i = 1, ..., k$. Indeed, we can choose $q'$ big enough so that $\frac{\varepsilon_i q'}{8} > 1$ and thus there is a positive integer $N$ in the open interval $\big(5\frac{\varepsilon_i q'}{8}, 6\frac{\varepsilon_i q'}{8}\big)$. We want to estimate

$$6\frac{\varepsilon_i q'}{8} = 6\frac{p_i}{8}\Big(\frac{\alpha'}{p}\Big)\Big(\frac{p}{q} - \theta\Big). \tag{4.2.10}$$

By using (4.2.8), we obtain a lower bound for $\frac{p}{\alpha'}$, namely

$$\frac{7}{8}\Big(\frac{p}{q} - \theta\Big) = \Big(\frac{p}{q} - \theta\Big) - \frac{1}{8}\Big(\frac{p}{q} - \theta\Big) < \frac{p}{\alpha'} - \theta < \frac{p}{\alpha'},$$

so applying this lower bound to (4.2.10) we get

$$6\frac{\varepsilon_i q'}{8} < 6\frac{p_i}{8}\frac{8}{7} = \frac{6}{7}p_i < p_i\,.$$

This says that $N \leq 6\frac{\varepsilon_i q'}{8} < p_i$, so we can write $N = p_i - p_i'$ for some *nonnegative* integer $p_i'$, which satisfies

$$5\frac{\varepsilon_i q'}{8} < p_i - p_i' < 6\frac{\varepsilon_i q'}{8}.$$

We thus see that (4.2.9) holds. Multiplying these inequalities by $r_i$ and summing over $i$, we obtain

$$\frac{5}{8p}\Big(\frac{p}{q} - \theta\Big) < \frac{1}{\alpha'} - \Big(\sum_{i=1}^k r_i \frac{p_i'}{q'}\Big) < \frac{3}{4p}\Big(\frac{p}{q} - \theta\Big).$$

Hence, setting $g = \sum_{i=1}^k r_i p_i'$ and $p' = pg$, we get

$$\frac{5}{8}\Big(\frac{p}{q} - \theta\Big) < \frac{p}{\alpha'} - \frac{p'}{q'} < \frac{3}{4}\Big(\frac{p}{q} - \theta\Big).$$

Finally, using (4.2.8), we end up with

$$\frac{1}{2}\Big(\frac{p}{q} - \theta\Big) < \frac{p}{q} - \frac{p'}{q'} < \frac{7}{8}\Big(\frac{p}{q} - \theta\Big). \tag{4.2.11}$$

Therefore $p', q'$ are the integers we are looking for. Our next task is to construct a system of $p' \times p'$ matrix units inside $\mathcal{Q}$ in order to define an algebra homomorphism $\rho' : M_{p'}(K) \to \mathcal{Q}$ satisfying the required properties.

Now, since $\mathrm{rk}_{\mathcal{Q}}$ takes all the values of the interval $[0, 1]$, there exists an idempotent $e$ in $\mathcal{Q}$ such that $\mathrm{rk}_{\mathcal{Q}}(e) = \frac{1}{q'}$, and the inequality $\frac{p_i'}{q'} < \mathrm{rk}_{\mathcal{Q}}(f_{11}^{(i)})$ can be thought of as

$$p_i' \cdot \mathrm{rk}_{\mathcal{Q}}(e) < \mathrm{rk}_{\mathcal{Q}}(f_{11}^{(i)}).$$

We can now apply the comparability property to the projective $\mathcal{Q}$-modules $(e\mathcal{Q})^{p_i'}$ and $f_{11}^{(i)}\mathcal{Q}$ to conclude that

$$\text{either} \quad e\mathcal{Q} \oplus \overset{p_i'}{\cdots} \oplus e\mathcal{Q} \lesssim f_{11}^{(i)}\mathcal{Q} \quad \text{or} \quad f_{11}^{(i)}\mathcal{Q} \lesssim e\mathcal{Q} \oplus \overset{p_i'}{\cdots} \oplus e\mathcal{Q}.$$

But due to the previous relation with the ranks, the second option is impossible (recall part $(i)$ of Proposition 1.2.3). Hence we obtain the comparison $e\mathcal{Q} \oplus \overset{p_i'}{\cdots} \oplus e\mathcal{Q} \lesssim f_{11}^{(i)}\mathcal{Q}$, that is $(e\mathcal{Q})^{p_i'}$ is isomorphic, as a right $\mathcal{Q}$-module, to a submodule of $f_{11}^{(i)}\mathcal{Q}$. Write $\phi$ for the map realizing this isomorphism. For each $l = 1, ..., p_i'$, define the injective $\mathcal{Q}$-module homomorphism $\iota_l : e\mathcal{Q} \to (e\mathcal{Q})^{p_i'}$ by sending an element $x \in e\mathcal{Q}$ to the vector in $(e\mathcal{Q})^{p'_i}$ having $x$ at the $l^{\text{th}}$ position and 0 otherwise. Consider the compositions $\phi \circ \iota_l : e\mathcal{Q} \to f_{11}^{(i)}\mathcal{Q}$, which are right $\mathcal{Q}$-module homomorphisms. In fact, if we denote by $x_l^{(i)}$ the image of $e$ under $\phi \circ \iota_l$, then this composition consists exactly of left multiplication by $x_l^{(i)}$, so $e\mathcal{Q} \cong x_l^{(i)}\mathcal{Q}$. Take now $\widetilde{e}_l^{(i)}$ to be an idempotent generating the right ideal $x_l^{(i)}\mathcal{Q}$, so that $x_l^{(i)}\mathcal{Q} = \widetilde{e}_l^{(i)}\mathcal{Q} \leq f_{11}^{(i)}\mathcal{Q}$. This, together with the previous isomorphism, implies that $e\mathcal{Q} \cong \widetilde{e}_l^{(i)}\mathcal{Q}$, so the idempotents $e$, $\widetilde{e}_l^{(i)}$ are all equivalent. In particular, the rank of all these idempotents are the same, and equal to $\mathrm{rk}_{\mathcal{Q}}(e) = \frac{1}{p'}$, and all the $\widetilde{e}_l^{(i)}$ are equivalent to $\widetilde{e}_1^{(1)}$, so there exist elements $\widetilde{x}_{(1,l)}^{(1,i)} \in \widetilde{e}_1^{(1)}\mathcal{Q}\widetilde{e}_l^{(i)}, \widetilde{y}_{(l,1)}^{(i,1)} \in \widetilde{e}_l^{(i)}\mathcal{Q}\widetilde{e}_1^{(1)}$ such that

$$\widetilde{e}_1^{(1)} = \widetilde{x}_{(1,l)}^{(1,i)} \cdot \widetilde{y}_{(l,1)}^{(i,1)}, \quad \widetilde{e}_l^{(i)} = \widetilde{y}_{(l,1)}^{(i,1)} \cdot \widetilde{x}_{(1,l)}^{(1,i)}.$$

Consider $e_l^{(i)} := f_{11}^{(i)}\widetilde{e}_l^{(i)}f_{11}^{(i)} = \widetilde{e}_l^{(i)}f_{11}^{(i)}$. These are new idempotents with the property that $e_l^{(i)} \leq f_{11}^{(i)}$, and their ranks are the same as $e$:

$$\mathrm{rk}_{\mathcal{Q}}(e) = \mathrm{rk}_{\mathcal{Q}}(\widetilde{e}_l^{(i)}) = \mathrm{rk}_{\mathcal{Q}}(e_l^{(i)}\widetilde{e}_l^{(i)}) \leq \mathrm{rk}_{\mathcal{Q}}(e_l^{(i)}) \leq \mathrm{rk}_{\mathcal{Q}}(\widetilde{e}_l^{(i)}) = \mathrm{rk}_{\mathcal{Q}}(e).$$

Moreover, if we let $x_{(1,l)}^{(1,i)} := f_{11}^{(1)}\widetilde{x}_{(1,l)}^{(1,i)}f_{11}^{(i)}$ and $y_{(l,1)}^{(i,1)} := f_{11}^{(i)}\widetilde{y}_{(l,1)}^{(i,1)}f_{11}^{(1)}$, we compute

$$x_{(1,l)}^{(1,i)}y_{(l,1)}^{(i,1)} = f_{11}^{(1)}\widetilde{x}_{(1,l)}^{(1,i)}f_{11}^{(i)}\widetilde{y}_{(l,1)}^{(i,1)}f_{11}^{(1)} = f_{11}^{(1)}\widetilde{e}_1^{(1)}f_{11}^{(1)} = e_1^{(1)},$$

$$y_{(l,1)}^{(i,1)}x_{(1,l)}^{(1,i)} = f_{11}^{(i)}\widetilde{y}_{(l,1)}^{(i,1)}f_{11}^{(1)}\widetilde{x}_{(1,l)}^{(1,i)}f_{11}^{(i)} = f_{11}^{(i)}\widetilde{e}_l^{(i)}f_{11}^{(i)} = e_l^{(i)},$$

so all the $e_l^{(i)}$ are equivalent to $e_1^{(1)}$ through the explicit equivalence already showed. Also, since each $e_l^{(i)} \leq f_{11}^{(i)}$, we have decompositions of $f_{11}^{(i)}\mathcal{Q}$ given by

$$e_l^{(i)}\mathcal{Q} \oplus (f_{11}^{(i)} - e_l^{(i)})\mathcal{Q} = f_{11}^{(i)}\mathcal{Q}$$

This implies that the $e_1^{(i)}, ..., e_{p'_i}^{(i)}$ are mutually orthogonal idempotents in $f_{11}^{(i)}\mathcal{Q}$. Indeed, note first that for $l \neq m$, $e_l^{(i)}\mathcal{Q} \cap e_m^{(i)}\mathcal{Q} = \{0\}$, since if $x$ is an element of the intersection, $x = e_l^{(i)}y = e_m^{(i)}z = \phi(\iota_l(e)y) = \phi(\iota_m(e)z)$ for some $y, z \in \mathcal{Q}$, so $\iota_l(e)y = \iota_m(e)z$. But the $\iota_k(e), \iota_m(e)$ are orthogonal idempotents inside $(e\mathcal{Q})^{p'_i}$, so $\iota_l(e)y = \iota_l(e)\iota_m(e)z = 0$, and $x = 0$. This, together with the previous decompositions of the right $\mathcal{Q}$-module $f_{11}^{(i)}$, implies that $e_m^{(i)}\mathcal{Q}$ is a submodule of the complement $(f_{11}^{(i)} - e_l^{(i)})\mathcal{Q}$, and so the idempotents $e_m^{(i)}, e_l^{(i)}$ are indeed orthogonal.

We summarize what we have so far.

a) We had a decomposition in orthogonal idempotents $f = f_1 + \cdots + f_k$, so we can think the corner $f\mathcal{Q}f$ as

$$f\mathcal{Q}f = \begin{pmatrix} f_1\mathcal{Q}f_1 & f_1\mathcal{Q}f_2 & \cdots & f_1\mathcal{Q}f_k \\ f_2\mathcal{Q}f_1 & f_2\mathcal{Q}f_2 & \cdots & f_2\mathcal{Q}f_k \\ \vdots & \vdots & \ddots & \vdots \\ f_k\mathcal{Q}f_1 & f_k\mathcal{Q}f_2 & \cdots & f_k\mathcal{Q}f_k \end{pmatrix}.$$

b) Each $f_i$ can be further decomposed in orthogonal idempotents $f_i = f_{11}^{(i)} + \cdots + f_{r_ir_i}^{(i)}$, so we can think the corners $f_i\mathcal{Q}f_j$ as

$$f_i\mathcal{Q}f_j = \begin{pmatrix} f_{11}^{(i)}\mathcal{Q}f_{11}^{(j)} & f_{11}^{(i)}\mathcal{Q}f_{22}^{(j)} & \cdots & f_{11}^{(i)}\mathcal{Q}f_{r_jr_j}^{(j)} \\ f_{22}^{(i)}\mathcal{Q}f_{11}^{(j)} & f_{22}^{(i)}\mathcal{Q}f_{22}^{(j)} & \cdots & f_{22}^{(i)}\mathcal{Q}f_{r_jr_j}^{(j)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{r_ir_i}^{(i)}\mathcal{Q}f_{11}^{(j)} & f_{r_ir_i}^{(i)}\mathcal{Q}f_{22}^{(j)} & \cdots & f_{r_ir_i}^{(i)}\mathcal{Q}f_{r_jr_j}^{(j)} \end{pmatrix},$$

that is,

$$f\mathcal{Q}f = \begin{pmatrix} \begin{pmatrix} f_{11}^{(1)}\mathcal{Q}f_{11}^{(1)} & \cdots & f_{11}^{(1)}\mathcal{Q}f_{r_1r_1}^{(1)} \\ \vdots & \ddots & \vdots \\ f_{r_1r_1}^{(1)}\mathcal{Q}f_{11}^{(1)} & \cdots & f_{r_1r_1}^{(1)}\mathcal{Q}f_{r_1r_1}^{(1)} \end{pmatrix} & \cdots & \begin{pmatrix} f_{11}^{(1)}\mathcal{Q}f_{11}^{(k)} & \cdots & f_{11}^{(1)}\mathcal{Q}f_{r_kr_k}^{(k)} \\ \vdots & \ddots & \vdots \\ f_{r_1r_1}^{(1)}\mathcal{Q}f_{11}^{(k)} & \cdots & f_{r_1r_1}^{(1)}\mathcal{Q}f_{r_kr_k}^{(k)} \end{pmatrix} \\ \vdots & \ddots & \vdots \\ \begin{pmatrix} f_{11}^{(k)}\mathcal{Q}f_{11}^{(1)} & \cdots & f_{11}^{(k)}\mathcal{Q}f_{r_1r_1}^{(1)} \\ \vdots & \ddots & \vdots \\ f_{r_kr_k}^{(k)}\mathcal{Q}f_{11}^{(1)} & \cdots & f_{r_kr_k}^{(k)}\mathcal{Q}f_{r_1r_1}^{(1)} \end{pmatrix} & \cdots & \begin{pmatrix} f_{11}^{(k)}\mathcal{Q}f_{11}^{(k)} & \cdots & f_{11}^{(k)}\mathcal{Q}f_{r_kr_k}^{(k)} \\ \vdots & \ddots & \vdots \\ f_{r_kr_k}^{(k)}\mathcal{Q}f_{11}^{(k)} & \cdots & f_{r_kr_k}^{(k)}\mathcal{Q}f_{r_kr_k}^{(k)} \end{pmatrix} \end{pmatrix}.$$

Having all these results at hand, we are now going to construct a set of matrix units inside the corner $f\mathcal{Q}f$ (recall $f = \psi(e_{11})$ was the idempotent approximating $f' = \rho(e_{11})$).

For $1 \leq i \leq k$ and $1 \leq l \leq p'_i$, define $h^{(1,1)}_{(1,1),(1,1)} := e^{(1)}_1$, $h^{(i,i)}_{(1,1),(l,l)} := e^{(i)}_l$, $h^{(1,i)}_{(1,1),(1,l)} := x^{(1,i)}_{(1,l)}$ and $h^{(i,1)}_{(1,1),(l,1)} := y^{(i,1)}_{(l,1)}$, so that

$$h^{(1,1)}_{(1,1),(1,1)} = h^{(1,i)}_{(1,1),(1,l)} \cdot h^{(i,1)}_{(1,1),(l,1)}, \qquad h^{(i,i)}_{(1,1),(l,l)} = h^{(i,1)}_{(1,1),(l,1)} \cdot h^{(1,i)}_{(1,1),(1,l)}.$$

In particular,

$$h^{(1,1)}_{(1,1),(1,1)} \cdot h^{(1,i)}_{(1,1),(1,l)} \cdot h^{(i,i)}_{(1,1),(l,l)} = h^{(1,i)}_{(1,1),(1,l)}, \qquad h^{(i,i)}_{(1,1),(l,l)} \cdot h^{(i,1)}_{(1,1),(l,1)} \cdot h^{(1,1)}_{(1,1),(1,1)} = h^{(i,1)}_{(1,1),(l,1)}.$$

All the idempotents $e^{(i)}_l$ belong to $f^{(i)}_{11}\mathcal{Q}f^{(i)}_{11}$. How can we move them between different corners $f^{(i)}_{jj}$? We know that all the $f^{(i)}_{jj}$ are equivalent to $f^{(i)}_{11}$, the equivalence given by the matrix units $f^{(i)}_{j1}$ and $f^{(i)}_{1j}$, namely $f^{(i)}_{jj} = f^{(i)}_{j1}f^{(i)}_{1j}$ and $f^{(i)}_{11} = f^{(i)}_{1j}f^{(i)}_{j1}$. Therefore we can define, for $1 \leq i \leq k$, $1 \leq j \leq r_i$ and $1 \leq l \leq p'_i$,

$$h^{(i,1)}_{(j,1),(l,1)} := f^{(i)}_{j1} \cdot h^{(i,1)}_{(1,1),(l,1)}, \qquad h^{(1,i)}_{(1,j),(1,l)} := h^{(1,i)}_{(1,1),(1,l)} \cdot f^{(i)}_{1j}.$$

This is a coherent family of $g \times g$ partial matrix units (recall that $g = \sum_{i=1}^k r_i p'_i$), since for $1 \leq i_1, i_2 \leq k$, $1 \leq j_1 \leq r_{i_1}$, $1 \leq j_2 \leq r_{i_2}$, $1 \leq l_1 \leq p'_{i_1}$ and $1 \leq l_2 \leq p'_{i_2}$,

$$\begin{aligned} h^{(1,i_2)}_{(1,j_2),(1,l_2)} \cdot h^{(i_1,1)}_{(j_1,1),(l_1,1)} &= h^{(1,i_2)}_{(1,1),(1,l_2)} \cdot f^{(i_2)}_{1j_2} \cdot f^{(i_1)}_{j_11} \cdot h^{(i_1,1)}_{(1,1),(l_1,1)} \\ &= \delta_{i_1,i_2}\delta_{j_1,j_2} h^{(1,i_1)}_{(1,1),(1,l_2)} \cdot h^{(i_1,1)}_{(1,1),(l_1,1)} = \delta_{i_1,i_2}\delta_{j_1,j_2}\delta_{l_1,l_2} h^{(1,1)}_{(1,1),(1,1)} \end{aligned}$$

Therefore, the elements

$$h^{(i_1,i_2)}_{(j_1,j_2),(l_1,l_2)} := h^{(i_1,1)}_{(j_1,1),(l_1,1)} h^{(1,i_2)}_{(1,j_2),(1,l_2)}$$

form a system of $g \times g$ matrix units inside $f\mathcal{Q}f$, and we can now define an algebra homomorphism $\rho' \colon M_{p'}(K) = M_p(K) \otimes M_g(K) \to \mathcal{Q}$ by the rule

$$\rho'\big(e_{ij} \otimes e^{(i_1,i_2)}_{(j_1,j_2),(l_1,l_2)}\big) = \psi(e_{i1}) h^{(i_1,i_2)}_{(j_1,j_2),(l_1,l_2)} \psi(e_{1j}),$$

where $\{e^{(i_1,i_2)}_{(j_1,j_2),(l_1,l_2)}\}$ is a complete system of matrix units in $M_g(K)$. In order to show that it is well-defined, it is only required to check that the images of the matrix units $e_{ij} \otimes e^{(i_1,i_2)}_{(j_1,j_2),(l_1,l_2)}$ satisfies the corresponding matrix units relations, but this is a matter of computation:

$$\begin{aligned} \rho'\big(e_{ij} \otimes e^{(i_1,i_2)}_{(j_1,j_2),(l_1,l_2)}\big) \cdot \rho'\big(e_{i'j'} \otimes e^{(i'_1,i'_2)}_{(j'_1,j'_2),(l'_1,l'_2)}\big) &= \psi(e_{i1}) h^{(i_1,i_2)}_{(j_1,j_2),(l_1,l_2)} \psi(e_{1j}) \cdot \psi(e_{i'1}) h^{(i'_1,i'_2)}_{(j'_1,j'_2),(l'_1,l'_2)} \psi(e_{1j'}) \\ &= \psi(e_{i1}) h^{(i_1,i_2)}_{(j_1,j_2),(l_1,l_2)} \cdot f \cdot h^{(i'_1,i'_2)}_{(j'_1,j'_2),(l'_1,l'_2)} \psi(e_{1j'}) = \delta_{i_2,i'_1}\delta_{j_2,j'_1}\delta_{l_2,l'_1}\ \psi(e_{i1}) h^{(i_1,i'_2)}_{(j_1,j'_2),(l_1,l'_2)} \psi(e_{1j'}) \\ &= \delta_{i_2,i'_1}\delta_{j_2,j'_1}\delta_{l_2,l'_1}\ \rho'\big(e_{ij'} \otimes e^{(i_1,i'_2)}_{(j_1,j'_2),(l_1,l'_2)}\big) \end{aligned}$$

It remains to verify properties 1)-6).

1) Inside $M_{p'}(K) = M_p(K) \otimes M_g(K)$,

$$1 = \sum_{k=1}^{p} \sum_{i=1}^{k} \sum_{j=1}^{r_i} \sum_{l=1}^{p'_i} e_{kk} \otimes e^{(i,i)}_{(j,j),(l,l)},$$

so

$$\rho'(1) = \sum_{k=1}^{p} \sum_{i=1}^{k} \sum_{j=1}^{r_i} \sum_{l=1}^{p'_i} \psi(e_{k1}) h^{(i,i)}_{(j,j),(l,l)} \psi(e_{1k}).$$

The idempotents $\psi(e_{k1}) h^{(i,i)}_{(j,j),(l,l)} \psi(e_{1k})$ are all pairwise orthogonal, and equivalent to $e_1^{(1)}$, through the equivalences

$$\psi(e_{k1}) h^{(i,i)}_{(j,j),(l,l)} \psi(e_{1k}) = \Big(\psi(e_{k1}) h^{(i,1)}_{(j,1),(l,1)}\Big)\Big(h^{(1,i)}_{(1,j),(1,l)} \psi(e_{1k})\Big),$$
$$e_1^{(1)} = \Big(h^{(1,i)}_{(1,j),(1,l)} \psi(e_{1k})\Big)\Big(\psi(e_{k1}) h^{(i,1)}_{(j,1),(l,1)}\Big).$$

Therefore if we take ranks, we obtain

$$\mathrm{rk}_{\mathcal{Q}}(\rho'(1)) = \sum_{k=1}^{p} \sum_{i=1}^{k} \sum_{j=1}^{r_i} \sum_{l=1}^{p'_i} \mathrm{rk}_{\mathcal{Q}}\left(\psi(e_{k1}) h^{(i,i)}_{(j,j),(l,l)} \psi(e_{1k})\right) = \sum_{k=1}^{p} \sum_{i=1}^{k} \sum_{j=1}^{r_i} \sum_{l=1}^{p'_i} \mathrm{rk}_{\mathcal{Q}}(e_1^{(1)}) = pg \, \mathrm{rk}_{\mathcal{Q}}(e_1^{(1)}) = \frac{p'}{q'},$$

as desired.

2) This follows from (4.2.11).

3) Let $x \in \rho(1)\mathcal{A}\rho(1)$. Then we can write

$$x = \rho(1) x' \rho(1) = \sum_{a,b=1}^{p} \rho(e_{a1}) f' \rho(e_{1a}) x' \rho(e_{b1}) f' \rho(e_{1b})$$

for some $x' \in \mathcal{A}$. Now by *ii)* we can approximate each $\rho(e_{1a}), \rho(e_{b1})$ by an element of $\mathcal{A}$ up to $\varepsilon$ in rank:

$$\mathrm{rk}_{\mathcal{Q}}(\rho(e_{1a}) - x_{1a}), \;\; \mathrm{rk}_{\mathcal{Q}}(\rho(e_{b1}) - x_{b1}) < \varepsilon \quad \text{for some} \quad x_{1a}, x_{b1} \in \mathcal{A}.$$

Thus we can consider the element

$$\widetilde{x} = \sum_{a,b=1}^{p} \psi(e_{a1}) \underbrace{f x_{1a} x' x_{b1} f}_{x_{ab} \in f\mathcal{A}f} \psi(e_{1b}) \in \mathcal{A}$$

so that

$$\begin{aligned} \mathrm{rk}_{\mathcal{Q}}(x - \widetilde{x}) &\leq \sum_{a,b=1}^{p} \mathrm{rk}_{\mathcal{Q}}(\rho(e_{a1}) f' \rho(e_{1a}) x' \rho(e_{b1}) f' \rho(e_{1b}) - \psi(e_{a1}) f x_{1a} x' x_{b1} f \psi(e_{1b})) \\ &\leq \sum_{a,b=1}^{p} \Big( \mathrm{rk}_{\mathcal{Q}}(\rho(e_{a1}) - \psi(e_{a1})) + \mathrm{rk}_{\mathcal{Q}}(\rho(e_{1b}) - \phi(e_{1b})) + 2\,\mathrm{rk}_{\mathcal{Q}}(f - f') \\ &\quad + \mathrm{rk}_{\mathcal{Q}}(\rho(e_{1a}) - x_{1a}) + \mathrm{rk}_{\mathcal{Q}}(\rho(e_{b1}) - x_{b1}) \Big) < p^2(4K(p) + 2)\varepsilon < 5p^2 K(p)\varepsilon, \end{aligned}$$

using $K(p) \geq 4$ for the last inequality. Now if we recall that $f\mathcal{A}f \cong M_{r_1}(K) \times \cdots \times M_{r_k}(K)$, we can write each $x_{ab} \in f\mathcal{A}f$ in the form

$$x_{ab} = \sum_{i=1}^{k} \sum_{j,j'=1}^{r_i} \lambda(a,b)^{(i)}_{jj'} f^{(i)}_{jj'}$$

for some scalars $\lambda(a,b)^{(i)}_{jj'} \in K$. Take the element

$$y = \sum_{a,b=1}^{p} e_{ab} \otimes \Big( \sum_{i=1}^{k} \sum_{j,j'=1}^{r_i} \lambda(a,b)^{(i)}_{jj'} \Big( \sum_{l=1}^{p'_i} e^{(i,i)}_{(j,j'),(l,l)} \Big) \Big) \in M_{p'}(K).$$

Denoting

$$c_{bij'} = \sum_{a=1}^{p}\sum_{j=1}^{r_i} \psi(e_{a1})\lambda(a,b)^{(i)}_{jj'} f^{(i)}_{j1}, \quad d_{bij'} = f^{(i)}_{11}\psi(e_{1b}),$$

we can rewrite

$$\widetilde{x} = \sum_{b=1}^{p}\sum_{i=1}^{k}\sum_{j'=1}^{r_i} c_{bij'} f^{(i)}_{11} d_{bij'}, \quad \rho'(y) = \sum_{b=1}^{p}\sum_{i=1}^{k}\sum_{j'=1}^{r_i} c_{bij'}\Big(\sum_{l=1}^{p'_i} h^{(i,i)}_{(1,1),(l,l)}\Big) d_{bij'},$$

so that

$$\mathrm{rk}_{\mathcal{Q}}(\widetilde{x} - \rho'(y)) \le \sum_{b=1}^{p}\sum_{i=1}^{k}\sum_{j'=1}^{r_i} \mathrm{rk}_{\mathcal{Q}}\left(c_{bij'}\Big(f^{(i)}_{11} - \sum_{l=1}^{p'_i} h^{(i,i)}_{(1,1),(l,l)}\Big) d_{bij'}\right) \le p\sum_{i=1}^{k} r_i\, \mathrm{rk}_{\mathcal{Q}}\left(f^{(i)}_{11} - \sum_{l=1}^{p'_i} h^{(i,i)}_{(1,1),(l,l)}\right).$$

Taking into account that all the idempotents $h^{(i,i)}_{(1,1),(l,l)} = e^{(i)}_l$ are pairwise orthogonal and $\sum_{l=1}^{p'_i} e^{(i)}_l \le f^{(i)}_{11}$, the rank of the difference $f^{(i)}_{11} - \sum_{l=1}^{p'_i} e^{(i)}_l$ is the difference of the ranks, so

$$\mathrm{rk}_{\mathcal{Q}}\left(f^{(i)}_{11} - \sum_{l=1}^{p'_i} e^{(i)}_l\right) = \mathrm{rk}_{\mathcal{Q}}(f^{(i)}_{11}) - \sum_{l=1}^{p'_i}\mathrm{rk}_{\mathcal{Q}}(e^{(i)}_l) = \mathrm{rk}_{\mathcal{Q}}(f^{(i)}_{11}) - \frac{p'_i}{q'}.$$

Hence the previous computation gives

$$\mathrm{rk}_{\mathcal{Q}}(\widetilde{x} - \rho'(y)) \le p\sum_{i=1}^{k} r_i\Big(\mathrm{rk}_{\mathcal{Q}}(f^{(i)}_{11}) - \frac{p'_i}{q'}\Big) = p\sum_{i=1}^{k} r_i\,\mathrm{rk}_{\mathcal{Q}}(f^{(i)}_{11}) - \frac{p'}{q'} = p\Big(\sum_{i=1}^{k} r_i\,\mathrm{rk}_{\mathcal{Q}}(f^{(i)}_{11}) - \frac{1}{q}\Big) + \Big(\frac{p}{q} - \frac{p'}{q'}\Big).$$

Using (4.2.7) and (4.2.11), we get

$$\mathrm{rk}_{\mathcal{Q}}(\widetilde{x} - \rho'(y)) < pK(p)\varepsilon + \frac{7}{8}\Big(\frac{p}{q} - \theta\Big) \le \frac{1}{48p}\Big(\frac{p}{q} - \theta\Big) + \frac{7}{8}\Big(\frac{p}{q} - \theta\Big) \le \frac{43}{48}\Big(\frac{p}{q} - \theta\Big).$$

Putting everything together,

$$\mathrm{rk}_{\mathcal{Q}}(x - \rho'(y)) \le \mathrm{rk}_{\mathcal{Q}}(x - \widetilde{x}) + \mathrm{rk}_{\mathcal{Q}}(\widetilde{x} - \rho'(y)) < 5p^2K(p)\varepsilon + \frac{43}{48}\Big(\frac{p}{q} - \theta\Big) < \frac{5}{48}\Big(\frac{p}{q} - \theta\Big) + \frac{43}{48}\Big(\frac{p}{q} - \theta\Big) = \frac{p}{q} - \theta,$$

as required.

4) Suppose now that $x = \rho(z)$ for some $z = \sum_{a,b=1}^{p} \mu_{ab}e_{ab} \in M_p(K)$, with $\mu_{ab} \in K$. Then we can see that, using the same construction of $y$ given in 3), we obtain that $\lambda(a,b)^{(i)}_{jj'} = \mu_{ab}$ for all $1 \le i \le k$, $1 \le j, j' \le r_i$. Therefore here

$$\begin{aligned} y &= \sum_{a,b=1}^{p} e_{ab} \otimes \Big(\sum_{i=1}^{k}\sum_{j,j'=1}^{r_i} \mu_{ab}\Big(\sum_{l=1}^{p'_i} e^{(i,i)}_{(j,j'),(l,l)}\Big)\Big) \\ &= \Big(\sum_{a,b=1}^{p} \mu_{ab}e_{ab}\Big) \otimes \Big(\sum_{i=1}^{k}\sum_{j,j'=1}^{r_i}\sum_{l=1}^{p'_i} e^{(i,i)}_{(j,j'),(l,l)}\Big) = z \otimes \Big(\sum_{i=1}^{k}\sum_{j,j'=1}^{r_i}\sum_{l=1}^{p'_i} e^{(i,i)}_{(j,j'),(l,l)}\Big) = \gamma(z) \end{aligned}$$

since $\sum_{i=1}^{k}\sum_{j,j'=1}^{r_i}\Big(\sum_{l=1}^{p'_i} e^{(i,i)}_{(j,j'),(l,l)}\Big)$ is the unit of $M_g(K)$.

5) and 6) To conclude the proof, just take $\varepsilon' > 0$ satisfying $\varepsilon' < \frac{1}{48K(p')p'^2}\Big(\frac{p'}{q'} - \theta\Big)$ and, using condition (3) in Theorem 4.2.2, consider a matricial $K$-subalgebra $\mathcal{A}'$ such that $\{\rho'(e'_{ij}) \mid i,j = 1,\dots,p'\} \subseteq_{\varepsilon'} \mathcal{A}'$ and $\mathrm{span}\{x_1,...,x_m,x_{m+1}\} \subseteq_{\varepsilon'} \mathcal{A}'$. □

## 4.3 $D$-rings

Let $D$ be a division ring. A $D$-ring is a unital ring $R$ together with a unital ring homomorphism $\iota\colon D \to R$. If $(R_1, \iota_1)$ and $(R_2, \iota_2)$ are $D$-rings, a $D$-ring homomorphism $\varphi : R_1 \to R_2$ is a ring homomorphism such that $\iota_2 = \varphi \circ \iota_1$. A *matricial $D$-ring* is a $D$-ring $\mathcal{A}$ which is isomorphic, as a $D$-ring, to a finite direct product

$$M_{n_1}(D) \times \cdots \times M_{n_r}(D),$$

where the structure of $D$-ring of the latter is the canonical one, i.e. viewing $d \in D$ inside the matrix product as the element $(d\,\mathrm{Id}_{n_1}, ..., d\,\mathrm{Id}_{n_r})$. A $D$-ring $\mathcal{A}$ is an *ultramatricial $D$-ring* if it is isomorphic, as a $D$-ring, to a direct limit $\varinjlim_n \mathcal{A}_n$ of a sequence $(\mathcal{A}_n, \varphi_n)$ of matricial $D$-rings $\mathcal{A}_n$ and $D$-ring homomorphisms $\varphi_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$.

We now consider a generalization of Theorem 4.2.2 to $D$-rings. We have not found a reasonable analogue of the local condition (3) in this setting, but we are able to extend condition (2). The reason we consider this generalization is the question raised by Elek and Jaikin-Zapirain of whether the completion of the $*$-regular closure of the group algebra of a countable amenable ICC-group is isomorphic to either $M_n(D)$, $n \geq 1$, or to $\mathcal{M}_D$, for some division ring $D$. Here $\mathcal{M}_D$ stands for the completion of $\varinjlim_n M_{2^n}(D)$ with respect to its unique rank function rk, which will be denoted inside the completion by $\mathrm{rk}_{\mathcal{M}_D}$.

Throughout this section, $D$ will denote a division ring, and $K$ will stand for the center of $D$. We start with a simple lemma concerning the $D$-ring $D \otimes_K \mathcal{M}_K$. Here $\iota : D \to D \otimes_K \mathcal{M}_K$ is given by $d \mapsto d \otimes 1$. Note that, even though $\varinjlim_n M_{2^n}(D) \cong \varinjlim_n \left(D \otimes_K M_{2^n}(K)\right) \cong D \otimes_K \left(\varinjlim_n M_{2^n}(K)\right)$, it may happen that $\mathcal{M}_D$ is not isomorphic to $D \otimes_K \mathcal{M}_K$, so a priori we cannot infer anything about the possible rank functions on $D \otimes_K \mathcal{M}_K$.

**Lemma 4.3.1.** *There is a unique rank function* $\mathrm{rk}_\otimes$ *on the (possibly nonregular) simple $D$-ring $D \otimes_K \mathcal{M}_K$, and $D \otimes_K \left(\varinjlim_n M_{2^n}(K)\right) \cong \varinjlim_n M_{2^n}(D)$ is dense in $D \otimes_K \mathcal{M}_K$ with respect to the* $\mathrm{rk}_\otimes$*-metric.*

*Proof.* The ring $D \otimes_K \mathcal{M}_K$ is simple by [19, Corollary 7.1.3].

We denote by $\mathrm{rk}_{\mathcal{M}_K}$ the unique rank function on $\mathcal{M}_K$. Let $x = \sum_{i=1}^k d_i \otimes x_i \in D \otimes_K \mathcal{M}_K$. Since $x_i \in \mathcal{M}_K$, we can take elements $\{x_{i,m}\}_m \subseteq \varinjlim_n M_{2^n}(K)$ approximating $x_i$ in rank, that is $\mathrm{rk}_{\mathcal{M}_K}(x_i - x_{i,m}) \overset{m}{\to} 0$. For each $m$, set

$$x_m := \sum_{i=1}^k d_i \otimes x_{i,m} \in D \otimes_K \left(\varinjlim_n M_{2^n}(K)\right) \cong \varinjlim_n M_{2^n}(D).$$

Then, for any rank function $S$ on $D \otimes_K \mathcal{M}_K$, we have

$$\begin{aligned}|S(x) - S(x_m)| &\leq S(x - x_m) \leq \sum_{i=1}^k S(d_i \otimes (x_i - x_{i,m})) \\ &\leq \sum_{i=1}^k S((d_i \otimes 1)(1 \otimes (x_i - x_{i,m}))) \leq \sum_{i=1}^k S(1 \otimes (x_i - x_{i,m})).\end{aligned}$$

The function $S(1 \otimes -)$ defines a rank function on $\mathcal{M}_K$, so by uniqueness we must have $S(1 \otimes -) = \mathrm{rk}_{\mathcal{M}_K}(-)$. Therefore

$$|S(x) - S(x_m)| \leq \sum_{i=1}^k \mathrm{rk}_{\mathcal{M}_K}(x_i - x_{i,m}) \leq \max_{i=1,\ldots,k}\{\mathrm{rk}_{\mathcal{M}_K}(x_i - x_{i,m})\} \overset{m}{\to} 0$$

This ensures that $S(x)$ is completely determined by the values of $S$ over $\varinjlim_n M_{2^n}(D)$. Since the restriction $S|$ of $S$ over this ring gives again a rank function, by uniqueness of the rank function rk on $\varinjlim_n M_{2^n}(D)$ we must have $S| = \mathrm{rk}$. This tells us that

$$S(x) = \lim_m \mathrm{rk}_D(x_m).$$

This shows at once that there is a unique rank function, which will be denoted by $\mathrm{rk}_\otimes$, on $D \otimes_K \mathcal{M}_K$, and that $D \otimes_K \left(\varinjlim_n M_{2^n}(K)\right) \cong \varinjlim_n M_{2^n}(D)$ is dense in $D \otimes_K \mathcal{M}_K$ with respect to the $\mathrm{rk}_\otimes$-metric. ☐

**Theorem 4.3.2.** *Let $\mathcal{A}$ be an ultramatricial $D$-ring, and let $\mathrm{rk}_\mathcal{A}$ be an extremal pseudo-rank function on $\mathcal{A}$ such that the completion $\mathcal{Q}$ of $\mathcal{A}$ with respect to $\mathrm{rk}_\mathcal{A}$ is a continuous factor. Then there is an isomorphism of $D$-rings $\mathcal{Q} \cong \mathcal{M}_D$.*

*Proof.* We can assume that $\mathcal{A} = \varinjlim_n (\mathcal{A}_n, \varphi_n)$, where each $\mathcal{A}_n$ is a matricial $D$-ring

$$\mathcal{A}_n = M_{m_1}(D) \times \cdots \times M_{m_{r_n}}(D)$$

with the canonical structure of $D$-rings, and each map $\varphi_n \colon \mathcal{A}_n \to \mathcal{A}_{n+1}$ is an injective morphism of $D$-rings.

Write $\mathcal{B}_n = C_{\mathcal{A}_n}(D)$ for the centralizer of $D$ in $\mathcal{A}_n$, i.e. all the elements of $\mathcal{A}_n$ that commute with $D$. It can be explicitly computed in this case, giving

$$\mathcal{B}_n = C_{\mathcal{A}_n}(D) = C_{M_{m_1}(D)}(D) \times \cdots C_{M_{m_{r_n}}(D)}(D) = M_{m_1}(K) \times \cdots \times M_{m_{r_n}}(K)$$

since each $C_{M_{m_i}(D)}(D) = M_{m_i}(K)$ because $K$ is the center of $D$. Hence each $\mathcal{B}_n$ is a matricial $K$-algebra, and

$$D \otimes_K \mathcal{B}_n = D \otimes_K (M_{m_1}(K) \times \cdots \times M_{m_{r_n}}(K)) \cong (D \otimes_K M_{m_1}(K)) \times \cdots \times (D \otimes_K M_{m_{r_n}}(K)) \cong \mathcal{A}_n.$$

Moreover, we have $\varphi_n(\mathcal{B}_n) \subseteq \mathcal{B}_{n+1}$ for all $n \geq 1$, so we obtain an inductive limit of matricial $K$-algebras $\mathcal{B} = \varinjlim_n (\mathcal{B}_n, (\varphi_n)|_{\mathcal{B}_n})$, which is such that

$$D \otimes_K \mathcal{B} = D \otimes_K \big(\varinjlim_n \mathcal{B}_n\big) \cong \varinjlim_n D \otimes_K \mathcal{B}_n \cong \varinjlim_n \mathcal{A}_n = \mathcal{A}.$$

From now on we will simply identify $\mathcal{A} = D \otimes_K \mathcal{B}$. Now, since $\mathcal{B} \subseteq \mathcal{A}$, we have an induced map $\mathbb{P}(\mathcal{A}) \to \mathbb{P}(\mathcal{B})$ sending (pseudo-)rank functions of $\mathcal{A}$ to (pseudo-)rank functions of $\mathcal{B}$ given by restriction on $\mathcal{B}$. In fact, using that each factor $M_m(K), M_m(D)$ has a unique rank function, compatible with respect to the isomorphism $M_m(D) \cong D \otimes_K M_m(K)$, it is clear that each rank function on $\mathcal{B}_n$ can be uniquely extended to a rank function on $\mathcal{A}_n$, and therefore each rank function on $\mathcal{B}$ extends uniquely to a rank function on $\mathcal{A}$. This shows that the previous map $\mathbb{P}(\mathcal{A}) \to \mathbb{P}(\mathcal{B})$ is bijective, and it is straightforward to show that it is in fact an affine homeomorphism. Consequently, the restriction $\mathrm{rk}_{\mathcal{B}}$ of $\mathrm{rk}_{\mathcal{A}}$ to $\mathcal{B}$ is also an extremal pseudo-rank function on $\mathcal{B}$.

Moreover, since $\mathrm{rk}_{\mathcal{A}}(\mathcal{A}) = \mathrm{rk}_{\mathcal{A}}(\mathcal{B})$, it follows that $\mathrm{rk}_{\mathcal{A}}(\mathcal{B})$ is a dense subset of the unit interval, which implies that the completion $\mathcal{Q}_{\mathcal{B}}$ of $\mathcal{B}$ in the $\mathrm{rk}_{\mathcal{B}}$-metric is a continuous factor over $K$. We can then apply Theorem 4.2.2 to $\mathcal{Q}_{\mathcal{B}}$, to conclude that there is an algebra isomorphism $\psi' : \mathcal{M}_K \to \mathcal{Q}_{\mathcal{B}}$, which induces naturally an isomorphism of $D$-rings

$$\psi := \mathrm{id}_D \otimes \psi' \colon D \otimes_K \mathcal{M}_K \to D \otimes_K \mathcal{Q}_{\mathcal{B}}.$$

Observe that $D \otimes_K \mathcal{Q}_{\mathcal{B}}$ can be realized as a subset of $\mathcal{Q}$, since for an element $d \otimes x \in D \otimes_K \mathcal{Q}_{\mathcal{B}}$, there exists a sequence $\{x_m\}_m \subseteq \mathcal{B}$ approximating $x$ in rank. Hence, the sequence of elements $d \otimes x_m \in D \otimes_K \mathcal{B} = \mathcal{A}$ is Cauchy in rank:

$$\mathrm{rk}_{\mathcal{A}}(d \otimes x_m - d \otimes x_{m'}) = \mathrm{rk}_{\mathcal{A}}((d \otimes 1)(1 \otimes (x_m - x_{m'}))) \leq \mathrm{rk}_{\mathcal{B}}(x_m - x_{m'}) \overset{m,m'}{\to} 0,$$

so convergent inside $\mathcal{Q}$, where we identify its limit to be the previous element $d \otimes x$, but inside $\mathcal{Q}$. Since

$$\mathcal{A} = D \otimes_K \mathcal{B} \subseteq D \otimes_K \mathcal{Q}_{\mathcal{B}} \subseteq \mathcal{Q},$$

it follows that $\psi(D \otimes_K \mathcal{M}_K) = D \otimes_K \mathcal{Q}_{\mathcal{B}}$ is dense in $\mathcal{Q}$. By Lemma 4.3.1, $\psi(D \otimes_K (\varinjlim_n M_{2^n}(K)))$ is dense in $\psi(D \otimes_K \mathcal{M}_K)$ with respect to the restriction of $\mathrm{rk}_{\mathcal{Q}}$ to it, therefore $\psi(D \otimes_K (\varinjlim_n M_{2^n}(K)))$ is dense in $\mathcal{Q}$. Hence, the restriction of $\psi$ to $D \otimes_K (\varinjlim_n M_{2^n}(K)) \cong \varinjlim_n M_{2^n}(D)$ gives a rank-preserving isomorphism of $D$-rings from $\varinjlim_n M_{2^n}(D)$ onto a dense $D$-subring of $\mathcal{Q}$, and thus it can be uniquely extended to an isomorphism from $\mathcal{M}_D$ onto $\mathcal{Q}$. □

## 4.4 Fields with involution

In this section, we will consider the corresponding problem for $*$-algebras. Again, the motivation comes from the theory of group algebras. If $K$ is a subfield of $\mathbb{C}$ closed under complex conjugation, and $G$ is a countable discrete group, then there is a natural involution on the group algebra $K[G]$, and the completion of the $*$-regular closure of $K[G]$ in $\mathcal{U}(G)$ is a $*$-regular ring containing $K[G]$ as a $*$-subalgebra. It would be thus desirable to find conditions under which this completion is $*$-isomorphic to $\mathcal{M}_K$, where $\mathcal{M}_K$ is endowed with the involution induced from the involution on $\varinjlim_n M_{2^n}(K)$, which is in turn obtained by endowing each algebra $M_{2^n}(K)$ with the $*$-transpose involution.

We will work with $*$-algebras over a field with positive definite involution $(F,*)$. The involution on $M_n(F)$ will always be the $*$-transpose involution.

A $*$-algebra $\mathcal{A}$ is *standard matricial* if

$$\mathcal{A} = M_{n_1}(F) \times \cdots \times M_{n_r}(F)$$

for some positive integers $n_1, ..., n_r$.

Let $\mathcal{A} = M_{n_1}(F) \times \cdots \times M_{n_r}(F)$, $\mathcal{B} = M_{m_1}(F) \times \cdots \times M_{m_s}(F)$ be two standard matricial $*$-algebras. A *standard map* between $\mathcal{A}$ and $\mathcal{B}$ is a block-diagonal $*$-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, i.e. of the form

$$(A_1, ..., A_r) \mapsto \left( \begin{pmatrix} A_1 & & & & & & \\ & \overset{l_{11}}{\ddots} & & & & & \\ & & A_1 & & & & \\ & & & \ddots & & & \\ & & & & A_r & & \\ & & & & & \overset{l_{1r}}{\ddots} & \\ & & & & & & A_r \end{pmatrix}, \; ..., \; \begin{pmatrix} A_1 & & & & & & \\ & \overset{l_{s1}}{\ddots} & & & & & \\ & & A_1 & & & & \\ & & & \ddots & & & \\ & & & & A_r & & \\ & & & & & \overset{l_{sr}}{\ddots} & \\ & & & & & & A_r \end{pmatrix} \right).$$

For more information, see [2, p.232].

A *standard ultramatricial* $*$-algebra is a direct limit of a sequence $\mathcal{A}_1 \xrightarrow{\Phi_1} \mathcal{A}_2 \xrightarrow{\Phi_2} \mathcal{A}_3 \xrightarrow{\Phi_3} \cdots$ of standard matricial $*$-algebras $\mathcal{A}_n$ and standard maps $\Phi_n \colon \mathcal{A}_n \to \mathcal{A}_{n+1}$. An *ultramatricial* $*$-algebra is a $*$-algebra which is $*$-isomorphic to the direct limit of a sequence of standard matricial $*$-algebras $\mathcal{A}_n$ and $*$-algebra homomorphisms $\Phi_n \colon \mathcal{A}_n \to \mathcal{A}_{n+1}$.

Let $\mathcal{A}$ be a $*$-algebra which is $*$-isomorphic to a standard matricial algebra, through a $*$-isomorphism

$$\psi \colon \mathcal{A} \to M_{n_1}(F) \times \cdots \times M_{n_r}(F).$$

We say that a projection (i.e. a self-adjoint idempotent) $p$ in $\mathcal{A}$ is *standard* (with respect to $\psi$) in case that for each $1 \leq i \leq r$, the $i^{\text{th}}$ component $\psi(p)_i$ of $\psi(p)$ is a diagonal projection in $M_{n_i}(F)$.

Recall that two idempotents $e, f$ in a ring $R$ are *equivalent*, written $e \sim f$, if there are elements $x \in eRf$ and $y \in fRe$ such that $e = xy$ and $f = yx$. If moreover $e, f$ are projections of a $*$-ring $R$, then we say that $e$ is $*$-equivalent to $f$, written $e \overset{*}{\sim} f$, in case there is an element $x \in eRf$ such that $e = xx^*$ and $f = x^*x$. Due to Theorem 1.2.11, in a $*$-regular ring, for every element $x \in R$ there exist unique projections $e = \mathrm{LP}(x)$ and $f = \mathrm{RP}(x)$, called the *left* and the *right* projections of $x$, such that $xR = eR$ and $Rx = Rf$. Furthermore, there exists a unique element $y \in fRe$, termed the *relative inverse* of $x$, such that $xy = e$ and $yx = f$. It is denoted simply by $\overline{x}$.

**Definition 4.4.1.** A $*$-regular ring $R$ satisfies the condition $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$ in case the $*$-equivalence $\mathrm{LP}(x) \overset{*}{\sim} \mathrm{RP}(x)$ holds for each $x \in R$.

Observe that $R$ satisfies $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$ if and only if equivalent projections of $R$ are $*$-equivalent ([2, Lemma 1.1]). In general this condition is not satisfied for a $*$-regular ring, but many $*$-regular rings satisfy it. It is worth to mention that for a field $F$ with positive definite involution, $M_n(F)$ satisfies $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$ for all $n \geq 1$ if and only if $F$ is $*$-Pythagorean [47, Theorem 4.9] (see also [46, Theorem 4.5], [2, Theorem 1.12]).

The following result is relevant for our purposes.

**Theorem 4.4.2** (Theorem 3.5 of [2])**.** *Let $(F,*)$ be a field with positive definite involution, let $\mathcal{A}$ be a standard ultramatricial $*$-algebra, and let* rk *be a pseudo-rank function on $\mathcal{A}$. Then the type II part of the rank completion of $\mathcal{A}$ is a $*$-regular ring satisfying* $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$.

As a consequence of this result, the $*$-algebra $\mathcal{M}_F$ always satisfies $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$, independently of whether the field $F$ is $*$-Pythagorean or not. Hence, if we want to find an analogue of Theorem 4.2.2, we need to find a condition that guarantees the fulfillment of the property $\mathrm{LP} \overset{*}{\sim} \mathrm{RP}$ on $\mathcal{Q}$.

We collect, for the convenience of the reader, some properties of a pseudo-rank function on a $*$-regular ring.

**Lemma 4.4.3.** *Let* rk *be a pseudo-rank function on a* $*$*-regular ring* $R$*. The following hold:*

*i) The involution is isometric, that is,* $\mathrm{rk}(r^*) = \mathrm{rk}(r)$ *for each* $r \in R$.

*ii)* $\mathrm{rk}(\overline{r} - \overline{s}) \leq 3\,\mathrm{rk}(r - s)$ *for all* $r, s \in R$.

*iii)* $\mathrm{rk}(\mathrm{LP}(r) - \mathrm{LP}(s)) \leq 4\,\mathrm{rk}(r - s)$*, and* $\mathrm{rk}(\mathrm{RP}(r) - \mathrm{RP}(s)) \leq 4\,\mathrm{rk}(r - s)$ *for all* $r, s \in R$.

*iv) Suppose that* $e_1, e_2, f_1, f_2$ *are projections in* $R$ *such that* $f_1 \overset{*}{\sim} f_2$ *and* $\mathrm{rk}(e_i - f_i) \leq \varepsilon$ *for* $i = 1, 2$*. Then there exist subprojections* $e'_i \leq e_i$ *such that* $e'_1 \overset{*}{\sim} e'_2$ *and* $\mathrm{rk}(e_i - e'_i) \leq 5\varepsilon$ *for* $i = 1, 2$.

*Proof.* (a) See the proof of Proposition 1 in [45], or [52, Proposition 5.11].

(b) In [10, p.310], it is shown that $\mathrm{rk}(\overline{r} - \overline{s}) \leq 19\,\mathrm{rk}(r - s)$, and the authors comment that K. R. Goodearl has reduced 19 to 5. Here we show that indeed it can be reduced to 3.

Let $e = r\overline{r}$, $f = \overline{r}r$, $g = s\overline{s}$ and $h = \overline{s}s$. We claim that the element $r^*r + (1 - f)$ is invertible in $R$. Since $R$ is $*$-regular, it follows that $Rr^*r = Rf$ (see the proof of Theorem 1.2.11). Take $x \in R$ such that $xr^*r = f$, and we can further assume that $fx = x$. Then $r^*rx^* = f$ too. We compute:

$$(r^*r + (1 - f))((1 - f) + fx^*) = r^*r(1 - f) + r^*rfx^* + (1 - f) = f + (1 - f) = 1$$

since $r^*rf = r^*r$. The claim now follows. Analogously, we find that the element $ss^* + (1 - g)$ is invertible in $R$.

Hence, using that $r^*e = r^*$ and $hs^* = s^*$, we get

$$\begin{aligned}\mathrm{rk}(\overline{r} - \overline{s}) &= \mathrm{rk}((r^*r + (1 - f))(\overline{r} - \overline{s})(ss^* + (1 - g))) = \mathrm{rk}(r^*ss^* + r^*(1 - g) - r^*rs^* - (1 - f)s^*) \\ &\leq \mathrm{rk}(r^*ss^* - r^*rs^*) + \mathrm{rk}(r^*(1 - g) - (1 - f)s^*) = \mathrm{rk}(r - s) + \mathrm{rk}(r^* - s^*) + \mathrm{rk}(fs^* - r^*g) \\ &= 2\,\mathrm{rk}(r - s) + \mathrm{rk}(f(s^* - r^*)g) \leq 3\,\mathrm{rk}(r - s),\end{aligned}$$

as required.

(c) Using (b), we get

$$\mathrm{rk}(\mathrm{RP}(r) - \mathrm{RP}(s)) = \mathrm{rk}(\overline{r}r - \overline{s}s) \leq \mathrm{rk}((\overline{r} - \overline{s})r) + \mathrm{rk}(\overline{s}(r - s)) \leq 3\,\mathrm{rk}(r - s) + \mathrm{rk}(r - s) = 4\,\mathrm{rk}(r - s).$$

The proof for LP is similar.

(d) We follow the idea in [2, proof of Lemma 2.6]. Since $f_1 \overset{*}{\sim} f_2$, there exists a partial isometry $w \in f_1Rf_2$ such that $f_1 = ww^*$ and $f_2 = w^*w$. Consider the self-adjoint element $a = e_1 - e_1ww^*e_1$ and set $p_1 := \mathrm{LP}(a)$. Since it is self-adjoint, we have $\mathrm{LP}(a) = \mathrm{RP}(a^*) = \mathrm{RP}(a)$, and $p_1 \leq e_1$ because $e_1a = ae_1 = a$. Then

$$\mathrm{rk}(p_1) = \mathrm{rk}(a) = \mathrm{rk}(e_1 - e_1f_1e_1) \leq \mathrm{rk}(e_1 - f_1) \leq \varepsilon.$$

Set $p'_1 := e_1 - p_1$. Then $\mathrm{rk}(e_1 - p'_1) = \mathrm{rk}(p_1) \leq \varepsilon$ and, since $p'_1ap'_1 = (e_1 - p_1)a(e_1 - p_1) = 0$, if we set $w' := p'_1w$ we realize that

$$w'(w')^* = p'_1ww^*p'_1 = p'_1e_1p'_1 = p'_1.$$

Now observe that $(w')^*w' = w^*p'_1w \leq w^*w = f_2$. Consider the elements

$$b = e_2 - e_2(w')^*w'e_2, \qquad e''_2 = \mathrm{LP}(b) = \mathrm{RP}(b^*) = \mathrm{RP}(b).$$

We have $e''_2 \leq e_2$ because $e_2b = be_2 = b$, and we can also give an estimate of its rank:

$$\begin{aligned}\mathrm{rk}(e''_2) = N(b) &= \mathrm{rk}(e_2 - e_2(w')^*w'e_2) \leq \mathrm{rk}(e_2 - (w')^*w') \leq \mathrm{rk}(e_2 - f_2) + \mathrm{rk}(w^*w - w^*p'_1w) \\ &\leq \varepsilon + \mathrm{rk}(w^*f_1w - w^*p'_1w) \leq \varepsilon + \mathrm{rk}(f_1 - p'_1) \leq \varepsilon + \mathrm{rk}(f_1 - e_1) + \mathrm{rk}(e_1 - p'_1) \leq 3\varepsilon.\end{aligned}$$

Set $e'_2 = e_2 - e''_2$. As before, since $e'_2be'_2 = (e_2 - e''_2)b(e_2 - e''_2) = 0$, if set set $w'' := w'e'_2$ we get

$$(w'')^*w'' = e'_2(w')^*w'e'_2 = e'_2e_2e'_2 = e'_2$$

and $\mathrm{rk}(e_2 - e'_2) = \mathrm{rk}(e''_2) \leq 3\varepsilon$. Write $e'_1 = w''(w'')^*$. Then $e'_i \leq e_i$ for $i = 1, 2$, $e'_1 \overset{*}{\sim} e'_2$, and

$$\begin{aligned}\mathrm{rk}(e_1 - e'_1) &= \mathrm{rk}(e_1 - p'_1) + \mathrm{rk}(p'_1 - e'_1) \leq \varepsilon + \mathrm{rk}(w'(w')^* - w''(w'')^*) = \varepsilon + \mathrm{rk}(w'f_2(w')^* - w'e'_2(w')^*) \\ &\leq \varepsilon + \mathrm{rk}(f_2 - e'_2) \leq \varepsilon + \mathrm{rk}(f_2 - e_2) + \mathrm{rk}(e_2 - e'_2) \leq 5\varepsilon.\end{aligned}$$

□

**Lemma 4.4.4.** *Let $R$ be a $*$-regular ring, and assume that $R$ is complete with respect to a rank function* rk. *Then $R$ satisfies* LP $\overset{*}{\sim}$ RP *if and only if, given equivalent projections $p, q \in R$ and $\varepsilon > 0$, there exist $*$-equivalent subprojections $p' \leq p$ and $q' \leq q$ such that* $\mathrm{rk}(p - p') < \varepsilon$, $\mathrm{rk}(q - q') < \varepsilon$.

*Proof.* The "only if" direction follows trivially from [2, Lemma 1.1].

For the "if" direction, suppose that $p$ and $q$ are equivalent projections of $R$. By hypothesis, there are $*$-equivalent subprojections $p_1 \leq p$ and $q_1 \leq q$ such that

$$\mathrm{rk}(p_1 - p) < 2^{-1}, \quad \mathrm{rk}(q_1 - q) < 2^{-1}.$$

Set $p_1' := p - p_1$ and $q_1' := q - q_1$. We obtain decompositions $pR = p_1'R \oplus p_1R$ and $qR = q_1'R \oplus q_1R$. Since $p \sim q$ and $p_1 \sim q_1$, we have isomorphisms $pR \cong qR$ and $p_1R \cong q_1R$, so by [39, Theorems 19.7 and 4.14] we obtain an isomorphism $p_1'R \cong q_1'R$ which gives rise to an equivalence $p_1' \sim q_1'$. By hypothesis, there are again $*$-equivalent subprojections $p_2 \leq p_1'$ and $q_2 \leq q_1'$ such that

$$\mathrm{rk}(p - (p_1 + p_2)) = \mathrm{rk}(p_1' - p_2) < 2^{-2}, \quad \mathrm{rk}(q - (q_1 + q_2)) = \mathrm{rk}(q_1' - q_2) < 2^{-2}.$$

Note that $p_1, p_2$ are orthogonal projections, such like $q_1, q_2$. By applying the same procedure, we can inductively construct a sequence $\{p_n\}_n$ of pairwise orthogonal subprojections of $p$, and $\{q_n\}_n$ pairwise orthogonal subprojections of $q$ such that $p_n \leq p - \sum_{i=1}^{n-1} p_i$, $q_n \leq q - \sum_{i=1}^{n-1} q_i$, $p_n \overset{*}{\sim} q_n$ and

$$\mathrm{rk}\left(p - \sum_{i=1}^{n} p_i\right) < 2^{-n}, \quad \mathrm{rk}\left(q - \sum_{i=1}^{n} q_i\right) < 2^{-n}$$

for every $n \geq 1$. In particular, $p = \lim_n \sum_{i=1}^n p_i$ and $q = \lim_n \sum_{i=1}^n q_i$. Let $w_n \in p_nRq_n$ be partial isometries realizing the equivalences $p_n \overset{*}{\sim} q_n$, so $p_n = w_nw_n^*$ and $q_n = w_n^*w_n$. Then for $n \geq m$,

$$\mathrm{rk}\left(\sum_{i=1}^{n} w_i - \sum_{j=1}^{m} w_j\right) \leq \sum_{i=m+1}^{n} \mathrm{rk}(w_i) \leq \sum_{i=m+1}^{n} \mathrm{rk}(p_i) \leq \sum_{i=m+1}^{n} \mathrm{rk}\left(p - \sum_{j=1}^{i-1} p_j\right) < \sum_{i=m+1}^{n} 2^{-i+1} < 2^{-m+1}.$$

It follows that the sequence $\{\sum_{i=1}^n w_i\}_n$ converges to an element $w \in R$, and moreover

$$\left(\sum_{i=1}^{n} w_i\right)\left(\sum_{j=1}^{n} w_j\right)^* = \sum_{i,j=1}^{n} w_iq_iq_jw_j^* = \sum_{i=1}^{n} w_iw_i^* = \sum_{i=1}^{n} p_i \xrightarrow{n} p.$$

Hence $ww^* = p$. Similarly we obtain $w^*w = q$. Therefore $R$ satisfies condition LP $\overset{*}{\sim}$ RP (by [2, Lemma 1.1]). ☐

In order to state the local condition in our main result of this section, we need the following somewhat technical definition.

**Definition 4.4.5.** Let $R$ be a unital $*$-regular ring with a pseudo-rank function rk, and let $\mathcal{A}$ be a unital $*$-subalgebra which is $*$-isomorphic to a standard matricial $*$-algebra. We say that a projection $p \in \mathcal{A}$ is *hereditarily quasi-standard* if

a) $p$ is $*$-equivalent in $\mathcal{A}$ to a standard projection of $\mathcal{A}$, and
b) for each subprojection $p' \leq p$, $p' \in \mathcal{A}$, and each $\varepsilon > 0$ there exists a unital $*$-subalgebra $\mathcal{A}'$ of $R$ and a projection $p'' \in \mathcal{A}'$ satisfying the following properties:
   1) $\mathcal{A}'$ is $*$-isomorphic to a standard matricial $*$-algebra,
   2) $p''$ is $*$-equivalent in $\mathcal{A}'$ to a standard projection of $\mathcal{A}'$,
   3) $p'' \leq p'$ and $\mathrm{rk}(p' - p'') < \varepsilon$, and
   4) $\mathcal{A} \subseteq \mathcal{A}'$.

We can now state the following analogue of Theorem 4.2.2. By a continuous $*$-factor over $F$ we mean a $*$-regular ring $\mathcal{Q}$ which is a $*$-algebra over $F$, and which is a continuous factor in the sense of Definition 4.2.1.

**Theorem 4.4.6.** *Let $(F, *)$ be a field with positive definite involution. Let $\mathcal{Q}$ be a continuous $*$-factor over $F$, and assume that there exists a dense $F$-subalgebra (with respect to the $\mathrm{rk}_{\mathcal{Q}}$-metric topology) $\mathcal{Q}_0 \subseteq \mathcal{Q}$ of countable $F$-dimension. The following are equivalent:*

*(1) $\mathcal{Q} \cong \mathcal{M}_F$ as $*$-algebras.*

*(2) $\mathcal{Q}$ is isomorphic, as a $*$-algebra, to $\overline{\mathcal{B}}$ for a certain standard ultramatricial $*$-algebra $\mathcal{B}$, where the completion of $\mathcal{B}$ is taken with respect to the metric induced by an extremal pseudo-rank function on $\mathcal{B}$.*

*(3) For every $\varepsilon > 0$, elements $x_1, ..., x_n \in \mathcal{Q}$, and projections $p_1, p_2 \in \mathcal{Q}$, there exist a $*$-subalgebra $\mathcal{A}$ of $\mathcal{Q}$ which is $*$-isomorphic to a standard matricial $*$-algebra, elements $y_1, ..., y_n \in \mathcal{A}$, and hereditarily quasi-standard projections $q_1, q_2 \in \mathcal{A}$ such that*

$$\mathrm{rk}_{\mathcal{Q}}(p_j - q_j) < \varepsilon \quad \textit{for } j = 1, 2, \qquad \textit{and} \qquad \mathrm{rk}_{\mathcal{Q}}(x_i - y_i) < \varepsilon \quad \textit{for } 1 \leq i \leq n.$$

*Proof.* Clearly (1) $\implies$ (2), since $\mathcal{M}_F$ is already the completion of the standard ultramatricial $*$-algebra $\varinjlim_n M_{2^n}(F)$ with respect to its unique rank function, which is extremal.

Let's prove (2) $\implies$ (3). Write $\mathcal{B} = \varinjlim_n \mathcal{B}_n$ as a direct limit of a sequence of standard matricial $*$-algebras $\mathcal{B}_n$ and standard maps $\Phi_n \colon \mathcal{B}_n \to \mathcal{B}_{n+1}$. Write $\Phi_{ji} \colon \mathcal{B}_i \to \mathcal{B}_j$ for the composition maps $\Phi_{j-1} \circ \cdots \circ \Phi_i$ for $i < j$, and write $\theta_i \colon \mathcal{B}_i \to \overline{\mathcal{B}} \cong \mathcal{Q}$ for the canonical map. We identify $\mathcal{Q}$ with $\overline{\mathcal{B}}$.

We will show that the desired $*$-subalgebra $\mathcal{A}$ satisfying the required conditions is of the form $\theta_j(\mathcal{B}_j)$. Note that these $*$-subalgebras are indeed $*$-isomorphic to $\mathcal{B}_j$, which are standard matricial $*$-algebras.

For a given $\varepsilon > 0$ and elements $x_1, ..., x_n \in \mathcal{Q}$, there are elements $y_1, ..., y_n \in \mathcal{B} = \varinjlim_n \mathcal{B}_n$ being close to the $x_i$ up to $\varepsilon$ in rank, that is

$$\mathrm{rk}_{\mathcal{Q}}(x_i - y_i) < \varepsilon \quad \text{for } i = 1, ..., k$$

Since $y_1, ..., y_n \in \varinjlim_n \mathcal{B}_n$, there exists an integer $N \geq 1$ such that $\theta_N(\mathcal{B}_N)$ contains all of them. Now the projections. Note that, since the algebras of the form $\theta_j(\mathcal{B}_j)$ form an increasing sequence, it is enough to deal with a single projection, since if we can find, for $i = 1, 2$, hereditarily quasi-standard projections $q_i$ satisfying the required properties and belonging to some $\theta_{N_i}(\mathcal{B}_{N_i})$, then $\theta_N(\mathcal{B}_N), \theta_{N_i}(\mathcal{B}_{N_i}) \subseteq \theta_M(\mathcal{B}_M)$ for $M \geq N, N_i$, and the required $*$-subalgebra $\mathcal{A}$ could be chosen to be $\theta_M(\mathcal{B}_M)$.

Let then $p$ be a projection in $\mathcal{Q}$ and let $\varepsilon > 0$. There exists an integer $i \geq 1$ and an element $x \in \mathcal{B}_i$ such that $\mathrm{rk}_{\mathcal{Q}}(p - \theta_i(x)) < \frac{\varepsilon}{8}$. Write $p' := \mathrm{LP}(x) \in \mathcal{B}_i$. We first claim that $\theta_i(p') = \theta_i(\mathrm{LP}(x)) = \mathrm{LP}(\theta_i(x))$. To prove this, by uniqueness of the projection LP, it is enough to prove that $\theta_i(p')$ is a projection satisfying $\theta_i(p')\mathcal{B} = \theta_i(x)\mathcal{B}$. But this is easy: $\theta_i(p')$ is clearly a projection since the $\theta_i$'s are $*$-homomorphisms, and if $\overline{x}$ is the relative inverse of $x$, then

$$\theta_i(x) = \theta_i(p'x) = \theta_i(p')\theta_i(x), \quad \theta_i(p') = \theta_i(x\overline{x}) = \theta_i(x)\theta_i(\overline{x}).$$

This proves the equality of $\mathcal{B}$-modules $\theta_i(p')\mathcal{B} = \theta_i(x)\mathcal{B}$, as required.

Therefore $\theta_i(p') = \mathrm{LP}(\theta_i(x))$[2]. Now, using Lemma 4.4.3, we can estimate

$$\mathrm{rk}_{\mathcal{Q}}(p - \theta_i(p')) = \mathrm{rk}_{\mathcal{Q}}(\mathrm{LP}(p) - \theta_i(\mathrm{LP}(x))) = \mathrm{rk}_{\mathcal{Q}}(\mathrm{LP}(p) - \mathrm{LP}(\theta_i(x))) \leq 4\,\mathrm{rk}_{\mathcal{Q}}(p - \theta_i(x)) < \frac{\varepsilon}{2},$$

so that $\mathrm{rk}_{\mathcal{Q}}(p - \theta_i(p')) < \frac{\varepsilon}{2}$.

Since $\mathcal{B}_i$ is standard matricial, it is of the form $M_{m_1}(F) \times \cdots \times M_{m_r}(F)$ for some positive integers $m_1, ..., m_r$. Each component $p'_j$ of $p' \in \mathcal{B}_i$ is a projection inside $M_{m_j}(F)$, so there exists an invertible matrix $x_j \in M_{m_j}(F)$ such that $x_j p'_j x_j^{-1}$ is diagonal, hence a standard projection $g_j$ inside $M_{m_j}(F)$. We then observe that the elements $x = (x_j p'_j)_j, y = (x_j^{-1} g_j)_j \in \mathcal{B}_i$ define an equivalence between the projections $p'$ and $g = (g_j)_j$, since $xy = (x_j p'_j x_j^{-1} g_j)_j = (g_j)_j = g$ and $yx = (x_j^{-1} g_j x_j p'_j)_j = (p'_j)_j = p'$. By the proof of [2, Theorem 3.5], there are $j > i$ and projections $\widetilde{p}, \widetilde{g} \in \mathcal{B}_j$ such that $\widetilde{p} \leq \Phi_{ji}(p')$, $\widetilde{g} \leq \Phi_{ji}(g)$ with $\widetilde{g}$ a standard projection, $\widetilde{p} \overset{*}{\sim} \widetilde{g}$, and moreover

$$\mathrm{rk}_{\mathcal{Q}}(\theta_i(p') - \theta_j(\widetilde{p})) < \frac{\varepsilon}{2} \quad \text{and} \quad \mathrm{rk}_{\mathcal{Q}}(\theta_i(g) - \theta_j(\widetilde{g})) < \frac{\varepsilon}{2}.$$

[2] Note that this is general, i.e. if $f : A \to B$ is a $*$-homomorphism between two $*$-regular rings, then for any element $x \in A$, one has the equality $f(\mathrm{LP}(x)) = \mathrm{LP}(f(x))$. The same holds for $RP$.

As a consequence, $\widetilde{p}$ is $*$-equivalent to a standard projection in $\mathcal{B}_j$, and

$$\mathrm{rk}_{\mathcal{Q}}(p - \theta_j(\widetilde{p})) \leq \mathrm{rk}_{\mathcal{Q}}(p - \theta_i(p')) + \mathrm{rk}_{\mathcal{Q}}(\theta_i(p') - \theta_j(\widetilde{p})) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Take $\mathcal{A} := \theta_j(\mathcal{B}_j)$ and $q := \theta_j(\widetilde{p})$. To conclude, we need to show that $q$ is indeed a hereditarily quasi-standard projection. Clearly, property *a)* in Definition 4.4.5 is satisfied. To show property *b)*, take a subprojection $q' = \theta_j(\widetilde{p}')$ of $q = \theta_j(\widetilde{p})$, where $\widetilde{p}'$ is a subprojection of $\widetilde{p}$, and $\delta > 0$. We then use the same argument as above but now applied to the projection $\widetilde{p}'$ of $\mathcal{B}_j$ and to $\delta > 0$ to obtain $k \geq j$ and a projection $\theta_k(\widetilde{p}'')$ in the $*$-subalgebra $\theta_k(\mathcal{B}_k)$ such that the pair $(\theta_k(\widetilde{p}''), \theta_k(\mathcal{B}_k))$ satisfies properties 1) - 4) in Definition 4.4.5 (with $\varepsilon$ replaced with $\delta$).

Finally we prove (3) $\implies$ (1). We first show that $\mathcal{Q}$ satisfies LP $\overset{*}{\sim}$ RP. Let $p_1, p_2$ be equivalent projections of $\mathcal{Q}$ and $\varepsilon > 0$. We will apply Lemma 4.4.4.

Since $p_1, p_2$ are equivalent projections, we can choose $x \in p_1 \mathcal{Q} p_2$ and $y \in p_2 \mathcal{Q} p_1$ such that $xy = p_1$ and $yx = p_2$. Observe that, by uniqueness, necessarily $y = \overline{x}$, the relative inverse of $x$ in $\mathcal{Q}$. Using (3), we can find a $*$-subalgebra $\mathcal{A}$ of $\mathcal{Q}$, which is $*$-isomorphic to a standard matricial $*$-algebra, hereditarily quasi-standard projections $q_1, q_2 \in \mathcal{A}$ such that $\mathrm{rk}_{\mathcal{Q}}(p_i - q_i) < \varepsilon$ and $q_i \overset{*}{\sim} e_i$ in $\mathcal{A}$ for some standard projections $e_1, e_2 \in \mathcal{A}$, and an element $x_1 \in \mathcal{A}$ such that $\mathrm{rk}_{\mathcal{Q}}(x - x_1) < \varepsilon$.

Now set $x_1' := q_1 x_1 q_2 \in \mathcal{A}$, and note that we have the estimate

$$\begin{aligned} \mathrm{rk}_{\mathcal{Q}}(x - x_1') &\leq \mathrm{rk}_{\mathcal{Q}}(p_1 x p_2 - p_1 x q_2) + \mathrm{rk}_{\mathcal{Q}}(p_1 x q_2 - q_1 x q_2) + \mathrm{rk}_{\mathcal{Q}}(q_1 x q_2 - q_1 x_1 q_2) \\ &\leq \mathrm{rk}_{\mathcal{Q}}(p_2 - q_2) + \mathrm{rk}_{\mathcal{Q}}(p_1 - q_1) + \mathrm{rk}_{\mathcal{Q}}(x - x_1) \leq \varepsilon + \varepsilon + \varepsilon = 3\epsilon. \end{aligned}$$

It follows from part *iii)* of Lemma 4.4.3 that, with $q_1'' := \mathrm{LP}(x_1') \in \mathcal{A}$ and $q_2'' := \mathrm{RP}(x_1') \in \mathcal{A}$,

$$\mathrm{rk}_{\mathcal{Q}}(p_i - q_i'') \leq 4\, \mathrm{rk}_{\mathcal{Q}}(x - x_1') < 12\varepsilon \quad \text{ for } i = 1, 2.$$

Observe that $q_1'' = LP(x_1') \sim RP(x_1') = q_2''$ (so they have the same rank), that $q_i'' \leq q_i$, and the complement of $q_i''$ in $q_i$ is "small" in rank:

$$\mathrm{rk}_{\mathcal{Q}}(q_i - q_i'') \leq \mathrm{rk}_{\mathcal{Q}}(q_i - p_i) + \mathrm{rk}_{\mathcal{Q}}(p_i - q_i'') < \varepsilon + 12\varepsilon = 13\varepsilon.$$

We still need to modify the projections $q_i''$ in order to get $*$-equivalence of these, rather than just equivalence.

Since $q_i \overset{*}{\sim} e_i$ in $\mathcal{A}$, there exist partial isometries $w_i \in q_i \mathcal{A} e_i$ such that $w_i w_i^* = q_i$ and $w_i^* w_i = e_i$. Then $e_i'' := w_i^* q_i'' w_i$ are equivalent projections in $\mathcal{A}$ such that $e_i'' \leq e_i$, and moreover

$$\mathrm{rk}_{\mathcal{Q}}(e_i - e_i'') = \mathrm{rk}_{\mathcal{Q}}(w_i^* q_i w_i - w_i^* q_i'' w_i) \leq \mathrm{rk}_{\mathcal{Q}}(q_i - q_i'') < 13\varepsilon.$$

A schematic for our actual situation is as follows.

$$\begin{array}{ccccccccc} q_1'' & \leq & q_1 & \overset{*}{\sim} & e_1 & \geq & e_1'' \\ \wr & & & & & & \wr \\ q_2'' & \leq & q_2 & \overset{*}{\sim} & e_2 & \geq & e_2'' \end{array}$$

We now use the fact that $\mathcal{A}$ is ($*$-isomorphic to) a standard matricial $*$-algebra: the restriction of $\mathrm{rk}_{\mathcal{Q}}$ to $\mathcal{A}$ is a convex combination of the normalized rank functions on the different simple factors of $\mathcal{A}$, so the above information enables us to build standard projections $e_i' \leq e_i$ such that $e_1' \overset{*}{\sim} e_2'$, and

$$\mathrm{rk}_{\mathcal{Q}}(e_i - e_i') < 13\varepsilon \quad \text{ for } i = 1, 2.$$

This in turn gives us projections $q_i' \leq q_i$ (through the $*$-equivalences $q_i \overset{*}{\sim} e_i$) such that $q_1' \overset{*}{\sim} q_2'$ and

$$\mathrm{rk}_{\mathcal{Q}}(q_i - q_i') < 13\varepsilon \quad \text{ for } i = 1, 2.$$

The last step is to transfer these to $p_1, p_2$. For this, observe that

$$\mathrm{rk}_{\mathcal{Q}}(p_i - q_i') \leq \mathrm{rk}_{\mathcal{Q}}(p_i - q_i) + \mathrm{rk}_{\mathcal{Q}}(q_i - q_i') < \varepsilon + 13\varepsilon = 14\varepsilon.$$

Since moreover $q_1'$ and $q_2'$ are $*$-equivalent, it follows from part $iv$) of Lemma 4.4.3 that there exist projections $p_i' \le p_i$ in $\mathcal{Q}$ such that $p_1' \overset{*}{\sim} p_2'$ and

$$\mathrm{rk}_{\mathcal{Q}}(p_i - p_i') < 5 \cdot 14\varepsilon = 70\varepsilon.$$

We can now apply Lemma 4.4.4 to conclude that $\mathcal{Q}$ satisfies LP $\overset{*}{\sim}$ RP.

Now (1) is shown by using the same method employed in Section 4.2. The first thing to notice here is that exactly the same proof given in [44], showing that there is an isomorphism $\mathcal{M}_F \cong \varinjlim_n M_{p_n}(F)$ for any factor sequence $(p_i)_i$, works for fields with positive definite involution, provided that one replaces 'idempotents' by 'projections', and 'homomorphisms' by '$*$-homomorphisms'. We henceforth will identify $\mathcal{M}_F$ with the $*$-algebra $\varinjlim_n M_{p_n}(F)$.

We only need to prove a variant of Lemma 4.2.4 with $*$-algebra homomorphisms $\rho_i : M_{p_i}(F) \to \mathcal{Q}$ instead of just algebra homomorphisms. For this, new versions of Lemmas 4.2.5 and 4.2.6 are required, as follows.

**Lemma 4.4.7** ($*$-version of Lemma 4.2.5)**.** *Let $p$ be a positive integer. Then there exists a constant $K^*(p)$, depending only on $p$, such that*

*i) for any field with involution $(F, *)$,*

*ii) any $\varepsilon > 0$,*

*iii) any pair $\mathcal{A} \subseteq \mathcal{B}$ where $\mathcal{B}$ is a unital $*$-algebra over $F$ and $\mathcal{A}$ is a unital $*$-regular subalgebra of $\mathcal{B}$,*

*iv) any pseudo-rank function* rk *on $\mathcal{B}$ such that $\mathrm{rk}(b^*) = \mathrm{rk}(b)$ for all $b \in \mathcal{B}$, and*

*v) any $*$-algebra homomorphism $\rho : M_p(F) \to \mathcal{B}$ such that*

$$\{\rho(e_{ij})\}_{1 \le i,j \le p} \subseteq_\varepsilon \mathcal{A}$$

*with respect to the* rk*-metric (where $e_{ij}$ denote the canonical matrix units in $M_p(K)$),*

*there exists a $*$-algebra homomorphism $\psi : M_p(F) \to \mathcal{A}$ which is close to $\rho$ in rank, namely*

$$\mathrm{rk}(\rho(e_{ij}) - \psi(e_{ij})) < K^*(p)\varepsilon \quad \text{for } 1 \le i,j \le p.$$

*If, in addition, we are given a projection $f \in \mathcal{A}$ such that $\mathrm{rk}(\rho(e_{11}) - f) < \varepsilon$, then the map $\psi$ can be built with the additional property that $\psi(e_{11}) \le f$.*

*Proof of Lemma 4.4.7.* The proof follows the same steps as the proof of Lemma 4.2.5. There is only an additional degree of approximation due to the fact that we need projections instead of idempotents. Proceeding by induction on $p$, just as in the proof of Lemma 4.2.5, we start with $*$-matrix units $\{x_{ij}\}$ for $1 \le i,j \le p-1$, so that $x_{ij}^* = x_{ji}$ for all $i,j$, and we have to define new elements $y_{1i}$, for $i = 1, \ldots, p$, so that the family $y_{ij} = y_{1i}^* y_{1j}$, $1 \le i,j \le p$, is the desired new family of $*$-matrix units. To this end, one only needs to replace the idempotents $g_i$ found in that proof by the projections $\mathrm{LP}(g_i)$. Using Lemma 4.4.3, one can easily control the corresponding ranks.

The last part is proven by the same kind of induction, starting with $\psi(1) = f$ for the case $p = 1$. □

**Lemma 4.4.8** ($*$-version of Lemma 4.2.6)**.** *Assume that $\mathcal{Q}$ satisfies condition* (3) *in Theorem 4.4.6. Let $\theta \in (0,1)$, and let $\{x_n\}_n$ be an $F$-basis of $\mathcal{Q}_0$.*

*Let $p$ be a positive integer such that there exist a $*$-algebra homomorphism $\rho : M_p(F) \to \mathcal{Q}$, a $*$-subalgebra $\mathcal{A} \subseteq \mathcal{Q}$ which is $*$-isomorphic to a standard matricial $*$-algebra, a hereditarily quasi-standard projection $g \in \mathcal{A}$, a positive integer $m$ and $\varepsilon > 0$ such that*

*i) $\mathrm{rk}_{\mathcal{Q}}(\rho(1)) = \frac{p}{q} > \theta$ for some positive integer $q$.*

*ii) $\mathrm{rk}_{\mathcal{Q}}(\rho(e_{11}) - g) < \varepsilon$, where $\{e_{ij} \mid i,j = 1, ..., p\}$ are the canonical matrix units of $M_p(F)$.*

*iii) $\{\rho(e_{ij}) \mid i,j = 1, \ldots, p\} \subseteq_\varepsilon \mathcal{A}$, and $\mathrm{span}\{x_1, ..., x_m\} \subseteq_\varepsilon \mathcal{A}$.*

*iv) $\varepsilon < \frac{1}{48K^*(p)p^2}\left(\frac{p}{q} - \theta\right)$, where $K^*(p)$ is the constant introduced in Lemma 4.4.7.*

*Then there exist positive integers $p', t, q'$, with $p' = tp$, a real number $\varepsilon' > 0$, a $*$-algebra homomorphism $\rho' : M_{p'}(F) \to \mathcal{Q}$, a $*$-subalgebra $\mathcal{A}' \subseteq \mathcal{Q}$, which is $*$-isomorphic to a standard matricial $*$-algebra and a hereditarily quasi-standard projection $g' \in \mathcal{A}'$ such that the following conditions hold:*

*1)* $\mathrm{rk}_{\mathcal{Q}}(\rho'(1)) = \frac{p'}{q'}$.

*2)*

$$0 < \frac{p'}{q'} - \theta < \frac{1}{2}\Big(\frac{p}{q} - \theta\Big).$$

*3) For each $x \in \rho(1)\mathcal{A}\rho(1)$ there exists $y \in M_{p'}(F)$ such that*

$$\mathrm{rk}_{\mathcal{Q}}(x - \rho'(y)) < \frac{p}{q} - \theta.$$

*4) For each $z \in M_p(F)$, we have*

$$\mathrm{rk}_{\mathcal{Q}}(\rho(z) - \rho'(\gamma(z))) < \frac{p}{q} - \theta,$$

*where $\gamma : M_p(F) \to M_{p'}(F)$ is the canonical unital $*$-homomorphism sending $z$ to* $\begin{pmatrix} z & & \mathbf{0} \\ & \overset{t}{\ddots} & \\ \mathbf{0} & & z \end{pmatrix}$.

*5)* $\mathrm{rk}_{\mathcal{Q}}(\rho'(e'_{11}) - g') < \varepsilon'$, *where $\{e'_{ij} \mid i,j = 1,...,p'\}$ are the canonical matrix units of $M_{p'}(F)$.*

*6)* $\{\rho'(e'_{ij}) \mid i,j = 1,...,p'\} \subseteq_{\varepsilon'} \mathcal{A}'$, *and* $\mathrm{span}\{x_1,...,x_m,x_{m+1}\} \subseteq_{\varepsilon'} \mathcal{A}'$.

*7)* $\varepsilon' < \frac{1}{48K^*(p')p'^2}\Big(\frac{p'}{q'} - \theta\Big)$.

*Proof of Lemma 4.4.8.* The proof is very similar to the proof of Lemma 4.2.6, so we will only indicate the points where the proof has to be modified.

We denote by $e_{ij}$, for $1 \le i,j \le p$, the canonical matrix units in $M_p(F)$. Note that $e_{ij}^* = e_{ji}$ for all $i,j$. Set $f' := \rho(e_{11})$, which is a projection in $\mathcal{Q}$. By *i*),

$$\mathrm{rk}_{\mathcal{Q}}(f') = \mathrm{rk}_{\mathcal{Q}}(\rho(e_{11})) = \frac{1}{p}\big(\mathrm{rk}_{\mathcal{Q}}(\rho(e_{11})) + \cdots + \mathrm{rk}_{\mathcal{Q}}(\rho(e_{pp}))\big) = \frac{1}{q}.$$

By hypothesis, there is a hereditarily quasi-standard projection $g$ in the $*$-subalgebra $\mathcal{A}$ such that $\mathrm{rk}_{\mathcal{Q}}(f' - g) < \varepsilon$. Now because of *iii*) we can apply Lemma 4.4.7 to obtain a $*$-algebra homomorphism $\psi : M_p(F) \to \mathcal{A}$ such that $\psi(e_{11}) \le g$ and $\mathrm{rk}_{\mathcal{Q}}(\rho(e_{ij}) - \psi(e_{ij})) < K^*(p)\varepsilon$ for all $1 \le i,j \le p$.

Due to condition *b*) in Definition 4.4.5, there exists another $*$-subalgebra $\mathcal{A}'$ of $\mathcal{Q}$ which is $*$-isomorphic to a standard matricial $*$-algebra and contains $\mathcal{A}$, and a projection $f \in \mathcal{A}'$, which is $*$-equivalent in $\mathcal{A}'$ to a standard projection of $\mathcal{A}'$, such that $f \le \psi(e_{11})$ and $\mathrm{rk}_{\mathcal{Q}}(\psi(e_{11}) - f) < K^*(p)\varepsilon - \mu$, where

$$\mu = \max\{\mathrm{rk}_{\mathcal{Q}}(\rho(e_{ij}) - \psi(e_{ij})) \mid 1 \le i,j \le p\}.$$

Now, by setting $\psi'(e_{ij}) = \psi(e_{i1})f\psi(e_{1j})$, we obtain that $\psi'$ is a $*$-algebra homomorphism $M_p(F) \to \mathcal{A}'$, and that

$$\begin{aligned}\mathrm{rk}_{\mathcal{Q}}(\rho(e_{ij}) - \psi'(e_{ij})) &\le \mathrm{rk}_{\mathcal{Q}}(\rho(e_{ij}) - \psi(e_{ij})) + \mathrm{rk}_{\mathcal{Q}}(\psi(e_{i1})\psi(e_{11})\psi(e_{1j}) - \psi(e_{i1})f\psi(e_{1j})) \\ &\le \mathrm{rk}_{\mathcal{Q}}(\rho(e_{ij}) - \psi(e_{ij})) + \mathrm{rk}_{\mathcal{Q}}(\psi(e_{11}) - f) < \mu + (K^*(p)\varepsilon - \mu) = K^*(p)\varepsilon,\end{aligned}$$

so that, after changing notation, we may assume that $f = \psi(e_{11})$, and that $f$ is $*$-equivalent in $\mathcal{A}$ to a standard projection of $\mathcal{A}$.

Since $\mathcal{A}$ is a standard matricial $*$-algebra, we can write

$$f = f_1 + \cdots + f_k,$$

where $f_1,...,f_k$ are nonzero mutually orthogonal projections belonging to different simple factors of $\mathcal{A}$. Since $f$ is $*$-equivalent in $\mathcal{A}$ to a standard projection, there exists, for each $1 \le i \le k$, a set of $*$-matrix units

$\{f_{jl}^{(i)}\}_{1\le j,l\le r_i}$ inside $f_i\mathcal{A}f_i$ such that each $f_{jj}^{(i)}$ is a minimal projection in the simple factor to which $f_i$ belongs, and such that

$$\sum_{j=1}^{r_i} f_{jj}^{(i)} = f_i, \text{ the unit of the corner } f_i\mathcal{A}f_i.$$

Now the proof follows the same steps as the one of Lemma 4.2.6. The idempotent $e$ built in that proof can be replaced now by a projection since $\mathcal{Q}$ is $*$-regular and, since $\mathcal{Q}$ satisfies LP $\overset{*}{\sim}$ RP, we can construct $p_i'$ mutually orthogonal subprojections of $f_{11}^{(i)}$. Using this and the fact that $(f_{jl}^{(i)})^* = f_{lj}^{(i)}$ for all $i,j,l$, one builds a system of matrix units inside $f\mathcal{Q}f$

$$\Big\{h^{(i_1,i_2)}_{(j_1,j_2),(u_1,u_2)} \mid 1\le 1i_1, i_2\le 1k,\ 1\le 1j_1\le 1r_{i_1},\ 1\le 1j_2\le 1r_{i_2},\ 1\le 1u_1\le 1p'_{i_1},\ 1\le 1u_2\le 1p'_{i_2}\Big\},$$

satisfying all the conditions stated in the proof of Lemma 4.2.6, and in addition

$$(h^{(i_1,i_2)}_{(j_1,j_2),(u_1,u_2)})^* = h^{(i_2,i_1)}_{(j_2,j_1),(u_2,u_1)}$$

for all allowable indices.

We can now define a $*$-algebra homomorphism $\rho' : M_{p'}(F) = M_p(F)\otimes M_t(F)\to\mathcal{Q}$ by the rule

$$\rho'\big(e_{ij}\otimes e^{(i_1,i_2)}_{(j_1,j_2),(u_1,u_2)}\big) = \psi(e_{i1})h^{(i_1,i_2)}_{(j_1,j_2),(u_1,u_2)}\psi(e_{1j}),$$

where $\{e^{(i_1,i_2)}_{(j_1,j_2),(u_1,u_2)}\}$ is a complete system of $*$-matrix units in $M_t(F)$. The verification of properties 1) - 7) is done in the same way, using condition (3), as in the proof of Lemma 4.2.6. □

**Lemma 4.4.9** ($*$-version of Lemma 4.2.4). *Let $\theta\in(0,1)$, and let $\{x_n\}_n$ be an $F$-basis of $\mathcal{Q}_0$. Assume that we have constructed two strictly increasing sequences $(q_i)_i$ and $(p_i)_i$ of natural numbers such that each $p_i$ divides $p_{i+1}$ and satisfying*

*a)* $1 > \frac{p_1}{q_1} > \cdots > \frac{p_i}{q_i} > \frac{p_{i+1}}{q_{i+1}} > \cdots > \theta$, $\lim_{i\to\infty}\frac{p_i}{q_i} = \theta$ *and*

*b)* $\frac{p_{i+1}}{q_{i+1}} - \theta < \frac{1}{2}\left(\frac{p_i}{q_i} - \theta\right)$ *for* $i\ge 0$.

*We also demand $p_0 = q_0 = 1$. Suppose further that there exists a sequence of positive numbers $\varepsilon_i < \frac{p_i}{q_i} - \theta$ and $*$-subalgebras $\mathcal{A}_i\subseteq\mathcal{Q}$ which are $*$-isomorphic to standard matricial $*$-algebras, together with $*$-algebra homomorphisms $\rho_i : M_{p_i}(F)\to\mathcal{Q}$ satisfying the following properties:*

*i)* $\mathrm{rk}_{\mathcal{Q}}(\rho_i(1)) = \frac{p_i}{q_i}$ *for all* $i$.

*ii) For each $i$ and each $x\in\rho_i(1)\mathcal{A}_i\rho_i(1)$, there exists $y\in M_{p_i+1}(F)$ such that*

$$\mathrm{rk}_{\mathcal{Q}}(x-\rho_{i+1}(y)) < \frac{p_i}{q_i} - \theta.$$

*iii) For each $z\in M_{p_i}(F)$, we have*

$$\mathrm{rk}_{\mathcal{Q}}(\rho_i(z) - \rho_{i+1}(\gamma_{i+1,i}(z))) < \frac{p_i}{q_i} - \theta.$$

*iv)* $\mathrm{span}\{x_1,\dots,x_i\}\subseteq_{\varepsilon_i}\mathcal{A}_i$.

*Then there exists a $*$-isomorphism $\psi : \mathcal{M}_F\to p\mathcal{Q}p$, with $p\in\mathcal{Q}$ a projection such that $\mathrm{rk}_{\mathcal{Q}}(p) = \theta$.*

*Proof.* Again, the proof is very similar to the proof of Lemma 4.2.4, so we will only indicate the points where the proof has to be modified.

Using *iii*), it is easy to show that the sequence $\{\rho_j(\gamma_{j,i}(z))\}_{j\ge i}$ is Cauchy in $\mathcal{Q}$, so convergent. The algebra homomorphisms $\psi_i : M_{p_i}(F)\to\mathcal{Q}$ defined by

$$\psi_i(z) = \lim_j \rho_j(\gamma_{j,i}(z))$$

turn out to be $*$-algebra homomorphisms (since the involution is isometric by part $i$) of Lemma 4.4.3) commuting with the $\gamma$'s, so we get a $*$-algebra homomorphism $\psi : \varinjlim_n M_{p_n}(F) \to \mathcal{Q}$ defined by

$$\psi(\gamma_{\infty,i}(z)) = \psi_i(z) \quad \text{for } z \in M_{p_i}(F).$$

Following the same steps as in the proof of Lemma 4.2.4, we obtain that $\mathrm{rk}_{\mathcal{Q}}(\psi(x)) = \theta \cdot \mathrm{rk}_{\mathcal{M}_F}(x)$ for all $x \in \varinjlim_n M_{p_n}(F)$, so $\psi$ can be extended to a unital $*$-algebra homomorphism $\psi : \mathcal{M}_F \to p\mathcal{Q}p$, where $p := \psi(1) = \lim_n \rho_n(1)$, which also satisfies the identity $\mathrm{rk}_{\mathcal{Q}}(\psi(z)) = \theta \cdot \mathrm{rk}_{\mathcal{M}_F}(z)$ for all $z \in \mathcal{M}_F$. In particular, $\mathrm{rk}_{\mathcal{Q}}(p) = \theta$. Injectivity and surjectivity of $\psi$ follows exactly as in the proof of Lemma 4.2.4. □

Now take $\theta = \frac{1}{2}$. Lemma 4.4.8 enables us to build the sequence of $*$-algebra homomorphisms $\rho_i : M_{p_i}(F) \to \mathcal{Q}$ satisfying the properties stated in Lemma 4.4.9, so we obtain a $*$-isomorphism $\psi : \mathcal{M}_F \to p\mathcal{Q}p$, where $\mathrm{rk}_{\mathcal{Q}}(p) = \frac{1}{2}$. In particular, $\mathrm{rk}_{\mathcal{Q}}(p) = \frac{1}{2} = \mathrm{rk}_{\mathcal{Q}}(1-p)$, so $p \sim 1-p$. Since $\mathcal{Q}$ satisfies LP $\overset{*}{\sim}$ RP, $p \overset{*}{\sim} 1-p$, so that there exists a partial isometry $w \in p\mathcal{Q}(1-p)$ such that $p = ww^*$ and $1-p = w^*w$. We then obtain a $*$-isomorphism $\mathcal{Q} \cong M_2(p\mathcal{Q}p)$ by the rule $\alpha \mapsto \begin{pmatrix} p\alpha p & p\alpha w^* \\ w\alpha p & w\alpha w^* \end{pmatrix}$, and this gives rise to a chain of $*$-isomorphisms

$$\mathcal{M}_F \cong M_2(\mathcal{M}_F) \cong M_2(p\mathcal{Q}p) \cong \mathcal{Q},$$

where the first one is given by extending to the respective rank completions the $*$-isomorphism $\varinjlim_n M_{2^n}(F) \to M_2(\varinjlim_n M_{2^n}(F))$, $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. This proves the theorem. □

In case the base field with involution $(F,*)$ is $*$-Pythagorean, we can derive a result which is completely analogous to Theorem 4.2.2, as follows.

**Corollary 4.4.10.** *Let $(F,*)$ be a $*$-Pythagorean field with positive definite involution. Let $\mathcal{Q}$ be a continuous $*$-factor over $F$, and assume that there exists a dense $F$-subalgebra (with respect to the $\mathrm{rk}_{\mathcal{Q}}$-metric topology) $\mathcal{Q}_0 \subseteq \mathcal{Q}$ of countable $F$-dimension. Then the following are equivalent:*

*(1) $\mathcal{Q} \cong \mathcal{M}_F$ as $*$-algebras.*

*(2) $\mathcal{Q}$ is isomorphic, as a $*$-algebra, to $\overline{\mathcal{B}}$ for a certain ultramatricial $*$-algebra $\mathcal{B}$, where the completion of $\mathcal{B}$ is taken with respect to the metric induced by an extremal pseudo-rank function on $\mathcal{B}$.*

*(3) For every $\varepsilon > 0$ and elements $x_1, ..., x_n \in \mathcal{Q}$, there exist a matricial $*$-subalgebra $\mathcal{A}$ of $\mathcal{Q}$, and elements $y_1, ..., y_n \in \mathcal{A}$ such that*

$$\mathrm{rk}_{\mathcal{Q}}(x_i - y_i) < \varepsilon \qquad \textit{for } i = 1, ..., k.$$

*Proof.* This follows from Theorem 4.4.6, by using the fact that $M_n(F)$ satisfies LP $\overset{*}{\sim}$ RP for all $n \geq 1$ ([47, Theorem 4.9]) and [2, Proposition 3.3]. Note that, since $M_n(F)$ satisfies LP $\overset{*}{\sim}$ RP for all $n \geq 1$, every projection of a standard matricial $*$-algebra is hereditarily quasi-standard. □

# Chapter 5

# KMS states on groupoids over graph algebras

In this last chapter we change our topics from the previous chapters and we concentrate on the study of the structure of KMS states over some particular $C^*$-algebras, namely the ones arising from groupoids and actions of groupoids on graphs. We will not review all the theory of $C^*$-algebras needed, the reader may consult [26, 55, 56, 76, 84] for an extensive source of this theory.

The main result of this chapter is Theorem 5.2.1, which leads some light on the structure of the simplex of normalized traces of the groupoid $C^*$-algebra $C^*(\mathcal{G})$.

Almost the entire work from this chapter has been done during a research stay of four months[1] at the School of Mathematics and Applied Statistics from the University of Wollongong, New South Wales (Australia), under the co-supervision of the Professor Aidan Sims. The author would like to thank him and the people from the department in general for their kind hospitality.

## 5.1 Introduction and preliminaries

There has been a lot of recent interest in the structure of KMS states for the natural gauge actions on $C^*$-algebras associated to algebraic and combinatorial objects (see, for example, [1, 15, 23, 33, 49, 50, 51, 57, 96]). The theme is that there is a critical inverse temperature $\beta_c$ below which the system admits no KMS states, and above this critical inverse temperature the structure of the KMS simplex reflects some of the underlying combinatorial data. For example, for $C^*$-algebras of strongly-connected finite directed graphs, the critical inverse temperature is the logarithm of the spectral radius of the graph, there is a unique KMS state at this inverse temperature, and at supercritical inverse temperatures the extreme KMS states are parametrized by the vertices of the graph [31, 49].

A particularly striking instance of this phenomenon appeared recently in the context of $C^*$-algebras associated to self-similar groups [81, 62] and, more generally, self-similar actions of groupoids on graphs in a recent work by Laca, Raeburn, Ramagge and Whittaker [63]. Roughly speaking, a self-similar action of a groupoid on a finite directed graph $E$ consists of a discrete groupoid $\mathcal{G}$ with unit space identified with $E^0$ and a left action of $\mathcal{G}$ on the path-space of $E$ with the property that for each groupoid element $g$ and each path $\mu$ for which $g \cdot \mu$ is defined, there is a unique groupoid element $g|_\mu$ such that

$$g \cdot (\mu\nu) = (g \cdot \mu)(g|_\mu \cdot \nu) \quad \text{for any other path } \nu.$$

In [63], the authors first show that at supercritical inverse temperatures, the KMS states on the Toeplitz algebra $\mathcal{T}(\mathcal{G}, E)$ of the self-similar action are determined by their restrictions to the embedded copy of $C^*(\mathcal{G})$. They then show that the self-similar action can be used to transform an arbitrary trace on $C^*(\mathcal{G})$ into a new trace on the same $C^*$-algebra that extends to a KMS state on the Toeplitz algebra $\mathcal{T}(\mathcal{G}, E)$, and that this transformation is an isomorphism of the trace simplex of $C^*(\mathcal{G})$ onto the KMS-simplex of $\mathcal{T}(\mathcal{G}, E)$. The transformation is quite natural: given a trace $\tau$ on $C^*(\mathcal{G})$ and given $g \in \mathcal{G}$, the value of the transformed trace

[1] The stay had been partitioned into two parts: the first part had a duration of two months and was conducted in 2016, September-October; the second part also had a duration of two months, and was conducted in 2017, September-October.

at the generator $u_g \in C^*(\mathcal{G})$ is a weighted infinite sum of the values of the original trace on restrictions $g|_\mu$ of $g$ such that $g \cdot \mu = \mu$; so the transformed trace at $u_g$ reflects the proportion – as measured by the initial trace – of the path-space of $E$ that is fixed by $g$. Building on this analysis, Laca, Raeburn, Ramagge and Whittaker proved that if $E$ is strongly connected and the self-similar action satisfies an appropriate finite-state condition, then $\mathcal{T}(\mathcal{G}, E)$ admits a unique KMS state at the critical inverse temperature and this is the only state that factors through the quotient $\mathcal{O}(\mathcal{G}, E)$ determined by the Cuntz–Krieger relations for $E$. So the KMS structure picks out a "preferred trace" on the groupoid $C^*$-algebra $C^*(\mathcal{G})$, namely the restriction of the above mentioned KMS state. Some enlightening examples of this are discussed in [63, Section 9].

This chapter is motivated by the observation that, for a fixed inverse temperature $\beta > \beta_c$, the transformation described in the preceding paragraph is a self-mapping $\chi_\beta$ of the simplex of normalized traces of $C^*(\mathcal{G})$, and so can be iterated. This raises a natural question: for which initial traces $\tau$ and at which supercritical inverse temperatures $\beta$ does the sequence $\{\chi_\beta^n(\tau)\}_{n\geq 1}$ converge, and what information about the self-similar action do the limit traces – that is, the fixed points for $\chi_\beta$ – encode? Our main result, Theorem 5.2.1, gives a very satisfactory answer to this question: the hypotheses of [63] (namely that $E$ is strongly connected and the action $\mathcal{G} \curvearrowright E$ satisfies the finite-state condition) seem to be exactly the hypotheses needed to guarantee that $\chi_\beta$ admits a unique fixed point for every supercritical $\beta$, that this fixed point is a universal attractor for $\chi_\beta$, and that it is precisely the preferred trace that extends to a KMS state at the critical inverse temperature.

After this introduction, we would like to devote the rest of the section to give some preliminary definitions and results concerning self-similar actions of groupoids on graphs and KMS states on $C^*$-algebras.

### 5.1.1 A survey on KMS states

Consider a $C^*$-algebra $A$ together with a strongly continuous homomorphism $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$. An element $x \in A$ is called *analytic* if the function $t \mapsto \alpha_t(x)$ extends to an analytic function from $\mathbb{C}$ to $A$. The set $A^a$ of analytic elements is a dense $^*$-subalgebra of $A$ (see for example [84, Chapter 8]).

**Definition 5.1.1.** We say that a state[2] $\phi$ of $A$ satisfies the Kubo–Martin–Schwinger (KMS) condition at inverse temperature $\beta \in [0, \infty)$ with respect to $\alpha$ if it satisfies

$$\phi(xy) = \phi(y\alpha_{i\beta}(x)) \quad \text{for all analytic } x \in A \text{ and all } y \in A.$$

We call such a $\phi$ a *KMS$_\beta$ state* for $(A, \alpha)$. For $\beta = 0$ we also required $\alpha$-invariance of $\phi$, that is $\phi \circ \alpha_t = \phi$ for all $t \in \mathbb{R}$.

Note that this condition generalizes the trace condition in the presence of the dynamics $\alpha$, but part of it is twisted by $\alpha$ along 'imaginary time'. In statistical mechanics, the dynamics $\alpha$ describes the time evolution of a system, and the KMS condition is considered to be a condition characterizing the state of the system at thermal equilibrium.

It is well-known that a state $\phi$ is KMS$_\beta$ if and only if there exists a set $S$ of analytic elements such that $\operatorname{span} S$ is an $\alpha$-invariant dense subspace of $A$, and $\phi$ satisfies the KMS condition at all $x, y \in S$ (cf. [84, Proposition 8.12.3]).

**Proposition 5.1.2.** *If $\phi$ is a KMS$_\beta$ state, then $\phi$ is $\alpha$-invariant.*

*Proof.* See [84, Proposition 8.12.4]. □

**Examples 5.1.3.**

1) Let $A = M_n(\mathbb{C})$ a finite-dimensional $C^*$-algebra, with canonical trace Tr. For each positive matrix $a \in A$, define the state
$$\phi_a : A \to \mathbb{C}, \quad \phi_a(x) = \frac{\mathrm{Tr}(ax)}{\mathrm{Tr}(a)}.$$
Since any $*$-automorphism of $A$ is given by conjugation with some unitary in $A$, any time evolution $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ arises as $\alpha_t(x) = e^{itH}xe^{-itH}$ for some self-adjoint matrix $H \in A$ (also called a *Hamiltonian*). We show that $\phi_a$ is a KMS$_\beta$ state with respect to $\alpha$ if and only if $a = e^{-\beta H}$.

[2]That is, a positive linear functional of $A$ of norm 1.

Clearly if $a = e^{-\beta H}$,

$$\phi_a(y\alpha_{i\beta}(x)) = \frac{\mathrm{Tr}(e^{-\beta H} y e^{-\beta H} x e^{\beta H})}{\mathrm{Tr}(a)} = \frac{\mathrm{Tr}(e^{-\beta H} xy)}{\mathrm{Tr}(a)} = \phi_a(xy),$$

so $\phi_a$ is a KMS$_\beta$ state. Conversely, if $\phi_a(xy) = \phi_a(y\alpha_{i\beta}(x))$ for all positive matrix $x \in A$ and all matrix $y \in A$, we have

$$\mathrm{Tr}((ax - e^{-\beta H} x e^{\beta H} a)y) = \mathrm{Tr}(a)(\phi_a(xy) - \phi_a(y\alpha_{i\beta}(x))) = 0$$

for all matrix $y \in A$. Hence $ax = e^{-\beta H} x e^{\beta H} a$ for all positive matrix $x \in A$. Equivalently, $e^{\beta H} a$ commutes with all positive matrix $x \in A$, and hence commutes with any element in $A$. Therefore $e^{\beta H} a$ must be a multiple of the identity matrix $\mathrm{Id}_n$, i.e. $a = \lambda e^{-\beta H}$ for $\lambda \in \mathbb{C}\backslash\{0\}$. Noticing that $\phi_{\mu a} = \phi_a$ for any $\mu \neq 0$, we can assume that $a = e^{-\beta H}$, as required.

2) Recall the construction of the hyperfinite $II_1$ factor $\mathscr{R}$ preceding Theorem 2.1.3: we have the sequence of $C^*$-algebras $M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C}) \hookrightarrow M_8(\mathbb{C}) \hookrightarrow \cdots$ with connecting maps

$$\varphi_n : M_{2^n}(\mathbb{C}) \hookrightarrow M_{2^{n+1}}(\mathbb{C}), \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

The norm at each $M_{2^n}(\mathbb{C})$ defines a norm $\|\cdot\|_{2^\infty}$ on the inductive limit. Its completion $\overline{\varinjlim_n M_{2^n}(\mathbb{C})}^{\|\cdot\|_{2^\infty}}$ is sometimes called the *Fermion algebra*, and is denoted by $M_{2^\infty}$. We can also view each $M_{2^n}(\mathbb{C}) = \bigotimes_{k=1}^n M_2(\mathbb{C})$, so that $M_{2^\infty} = \overline{\bigotimes_{k\geq 1} M_2(\mathbb{C})}^{\|\cdot\|_{2^\infty}}$.

For a fixed sequence $\{\mu_n\}_{n\geq 1}$ in $\mathbb{R}^+$, we can define a dynamics $\alpha : \mathbb{R} \to \mathrm{Aut}(M_{2^\infty})$, uniquely characterized by the fact that for each $t \in \mathbb{R}$, the dynamics $\alpha_t$ is given on each simple factor $M_{2^n}(\mathbb{C})$ by conjugation with the element $u_t^n = \bigotimes_{k=1}^n \begin{pmatrix} 1 & 0 \\ 0 & \mu_n^{it} \end{pmatrix}$, that is

$$\alpha_t : M_{2^n}(\mathbb{C}) \to M_{2^n}(\mathbb{C}), \quad A \mapsto u_t^n A u_{-t}^n.$$

We show that there exists KMS$_\beta$ states for every value of $\beta$. Recall that we have a unique tracial state $\tau$ on $M_{2^\infty}$, obtained by extending by continuity the tracial state tr on the inductive limit $\varinjlim_n M_{2^n}(\mathbb{C})$, where each $\mathrm{tr}_n : M_{2^n}(\mathbb{C}) \to \mathbb{C}$ is the normalized tracial state $\mathrm{tr}_n = \frac{1}{2^n}\mathrm{Tr}$. $\tau$ is an example of a KMS$_0$ state.

Now fix $\beta > 0$, and consider the sequence of matrices

$$h_n = \bigotimes_{k=1}^n \begin{pmatrix} \frac{2\mu_k^\beta}{1+\mu_k^\beta} & 0 \\ 0 & \frac{2}{1+\mu_k^\beta} \end{pmatrix} \in M_{2^n}(\mathbb{C}).$$

We define a state on $M_{2^n}(\mathbb{C})$ by $\phi_n(A) = \mathrm{tr}_n(h_n A)$. It is straightforward to see that it gives rise, by continuity, to a KMS$_\beta$ state $\phi : M_{2^\infty} \to \mathbb{C}$ with respect to $\alpha$.

### 5.1.2 A survey on self-similar groupoids

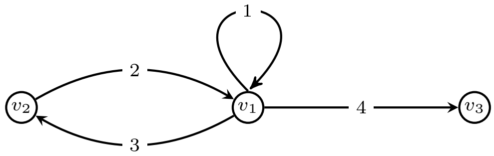
A *groupoid* is a countable small category $\mathcal{G}$ with inverses. Equivalently, a groupoid can be seen as a group where the operation is no longer defined for all the elements of $\mathcal{G}$. In this chapter, we will use $d$ and $c$ for the domain and codomain maps $\mathcal{G} \to \mathcal{G}^{(0)}$ to distinguish them from the range and source maps on directed graphs, where $\mathcal{G}^{(0)}$ denotes the set of objects. We also write $\mathcal{G}^{(2)} = \{(g,h) \mid d(g) = c(h)\}$ for the set of composable elements $g, h \in \mathcal{G}$, so $gh$ is defined if and only if $(g,h) \in \mathcal{G}^{(2)}$. For $u \in \mathcal{G}^{(0)}$, we write

$$\mathcal{G}_u = \{g \in \mathcal{G} \mid d(g) = u\} \quad \text{and} \quad \mathcal{G}^u = \{g \in \mathcal{G} \mid c(g) = u\}.$$

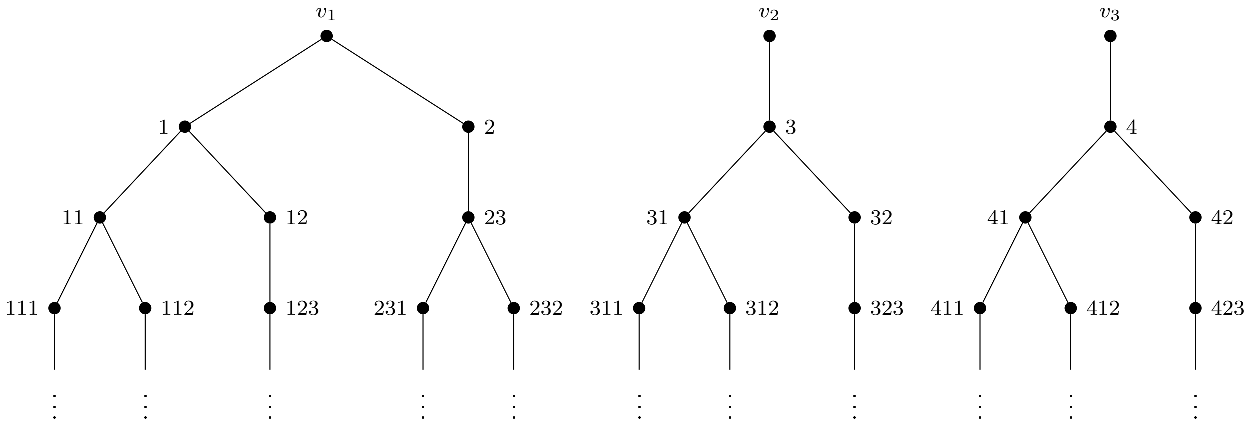
Consider a finite directed graph $E = (E^0, E^1, r, s)$. For $n \geq 2$, write $E^n$ for the paths of length $n$ in $E$, that is $E^n = \{e_1 e_2 \ldots e_n \mid e_i \in E^1, r(e_{i+1}) = s(e_i)\}$. We write $E^* := \bigcup_{n\geq 0} E^n$. We can visualize the set $E^*$ as indexing the vertices of a forest $T = T_E$ given by

$$T^0 = E^* \quad \text{and} \quad T^1 = \{(\mu, \mu e) \in E^* \mid \mu \in E^*, e \in E^1 \text{ and } s(\mu) = r(e)\}.$$

**Example 5.1.4.** For the graph $E$ given by




we have the forest $T_E$ given by




Throughout this chapter, we write $A_E$ for the adjacency matrix of a directed graph $E$, that is the integer $|E^0| \times |E^0|$ matrix with entries $A_E(v, w) = |vE^1w|$ for $v, w \in E^0$.

We are interested in self-similar actions of groupoids on directed graphs $E$ as introduced and studied in [63]. To describe these, first recall that a *partial isomorphism* of the forest $T_E$ corresponding to a directed graph $E$ as above consists of a pair $(v, w) \in E^0 \times E^0$ and a bijection $g : vE^* \to wE^*$ such that

a) $g|_{vE^k} : vE^k \to wE^k$ is bijective for $k \geq 1$.

b) $g(\mu e) \in g(\mu)E^1$ for $\mu \in vE^*$ and $e \in E^1$ with $r(e) = s(\mu)$.

It turns out that the set of partial isomorphisms, denoted by $\mathrm{PIso}(T_E)$, can be endowed with a groupoid structure, as the following propositions shows.

**Proposition 5.1.5** (Proposition 3.2 of [63]). *The set of partial isomorphisms of $T_E$ forms a groupoid, with*

*(i) unit space $E^0$, where the identify morphisms are the partial isomorphisms $id_v : vE^* \to vE^*$ given by the identity map on $vE^*$ for every $v \in E^0$,*

*(ii) the inverse of a partial isomorphism $g : vE^* \to wE^*$ is the standard inverse map $g^{-1} : wE^* \to vE^*$, and*

*(iii) the groupoid multiplication given by composition of maps.*

We define the domain and codomain maps $d, c : \mathrm{PIso}(T_E) \to E^0$ by $d(g) = v$ and $c(g) = w$ for $g$ a partial isomorphism $g : vE^* \to wE^*$.

**Definition 5.1.6.** Let $E$ be a directed graph, and let $\mathcal{G}$ be a groupoid with unit space $E^0$. A *faithful action* of $\mathcal{G}$ on $T_E$ is an injective groupoid homomorphism $\theta : \mathcal{G} \to \mathrm{PIso}(T_E)$ that is the identity map on unit spaces. We write $g \cdot \mu$ rather than $\theta_g(\mu)$ for $g \in \mathcal{G}$ and $\mu \in E^*$ with $d(g) = r(\mu)$. The action $\theta$ is *self-similar* if for each $g \in \mathcal{G}$ and $\mu \in d(g)E^*$ there exists $g|_\mu \in \mathcal{G}$ such that $d(g|_\mu) = s(\mu)$ and

$$g \cdot (\mu\nu) = (g \cdot \mu)(g|_\mu \cdot \nu) \quad \text{for all } \nu \in s(\mu)E^*. \tag{5.1.1}$$

We also say that $\theta$ is *finite-state* if for every element $g \in \mathcal{G}$, the set $\{g|_\mu \mid \mu \in d(g)E^*\}$ is a finite subset of $\mathcal{G}$.

The faithfulness condition ensures that for each $g \in \mathcal{G}$ and $\mu \in d(g)E^*$, there is a *unique* element $g|_\mu \in \mathcal{G}$ satisfying (5.1.1). Throughout this chapter, we will write $\mathcal{G} \curvearrowright E$ to indicate that the groupoid $\mathcal{G}$ acts faithfully on the directed graph $E$.

By Proposition 3.6 of [63], self-similar groupoid actions have the following properties, which we will use without comment henceforth: for $g, h \in \mathcal{G}$, $\mu \in d(g)E^*$, and $\nu \in s(\mu)E^*$,

a) $g|_{\mu\nu} = (g|_\mu)|_\nu$,

b) $\mathrm{id}_{r(\mu)}|_\mu = \mathrm{id}_{s(\mu)}$,

c) if $(g, h) \in \mathcal{G}^{(2)}$, then $\big(g|_{h\cdot\mu}, h|_\mu\big) \in \mathcal{G}^{(2)}$, and $(gh)|_\mu = g|_{h\cdot\mu}h|_\mu$, and

d) $(g^{-1})|_\mu = (g|_{g^{-1}\cdot\mu})^{-1}$.

**Example 5.1.7.** Consider the same graph $E$ as in Example 5.1.4. We define two partial isomorphisms of $T_E$, $\{a, b\}$, by the rules

$$\begin{cases} a \cdot 1 = 3, & a|_1 = a \\ a \cdot 2 = 2, & a|_2 = \mathrm{id}_{v_2} \\ b \cdot 3 = 4, & b|_3 = a \end{cases}$$

that is, $a : v_1E^* \to v_2E^*$ and $b : v_2E^* \to v_3E^*$ are defined recursively by $a \cdot (1\mu) = 3(a \cdot \mu)$ for all $\mu \in v_1E^*$ and $a \cdot (2\mu) = 2\mu$ for all $\mu \in v_2E^*$, and $b \cdot (3\mu) = 4(a \cdot \mu)$ for all $\mu \in v_1E^*$. It is then easily verified that the subgroupoid $\mathcal{G}_E$ of $\mathrm{PIso}(T_E)$ generated by the set of partial isomorphisms $\{a, b, \mathrm{id}_{v_1}, \mathrm{id}_{v_2}, \mathrm{id}_{v_3}\}$ is self-similar and acts faithfully on $E$.

### 5.1.3 The Toeplitz $C^*$-algebra of a self-similar groupoid

The Toeplitz algebra of a self-similar action $\mathcal{G} \curvearrowright E$ is defined in [63] as follows (see also [36]). A *Toeplitz representation* $(v, q, t)$ of $(\mathcal{G}, E)$ in a unital $C^*$-algebra $B$ is a triple of maps $v : \mathcal{G} \to B$, $q : E^0 \to B$, $t : E^1 \to B$ such that

a) $(q, t)$ is a Toeplitz–Cuntz–Krieger family in $B$, that is

(V) $\{q_v\}_{v\in E^0}$ is a family of pairwise orthogonal projections,

(E) $t_e q_{s(e)} = q_{r(e)} t_e = t_e$ for all edges $e \in E^1$,

(TCK1) $t_e^* t_f = \delta_{e,f} q_{s(e)}$ for all $e, f \in E^1$, and

(TCK2) $\{t_e t_e^*\}_{e\in E^1}$ is a family of pairwise orthogonal projections satisfying $t_e t_e^* \le q_{r(e)}$ for every $e \in E^1$,

and it is also required that $\sum_{w\in E^0} q_w = 1_B$;

b) $\{v_g : g \in \mathcal{G}\}$ is a family of partial isometries on $B$ satisfying $v_g v_h = \delta_{d(g),t(h)} v_{gh}$ and $v_{g^{-1}} = v_g^*$ for all $g, h \in \mathcal{G}$, and $v_w = q_w$ for $w \in \mathcal{G}^{(0)} = E^0$;

c) $v_g t_e = \delta_{d(g),r(e)} t_{g\cdot e} v_{g|_e}$ for $g \in \mathcal{G}$ and $e \in E^1$; and

d) $v_g q_w = \delta_{d(g),w} q_{g\cdot w} v_g$ for all $g \in \mathcal{G}$ and $w \in E^0$.

**Definition 5.1.8.** Standard arguments show that there exists a universal $C^*$-algebra $\mathcal{T}(\mathcal{G}, E)$ generated by a Toeplitz representation $\{u, p, s\}$; we call it the *Toeplitz algebra* of the self-similar action $\mathcal{G} \curvearrowright E$.

We have

$$\mathcal{T}(\mathcal{G}, E) = \overline{\mathrm{span}}\{s_\mu u_g s_\nu^* \mid \mu, \nu \in E^*, g \in \mathcal{G}_{s(\nu)}^{s(\mu)}\}$$

by [63, Proposition 4.5]. The argument of the paragraph following the statement of [63, Theorem 6.1] applied with $\pi_\tau$ replaced by a faithful representation of $C^*(\mathcal{G})$ shows that there is an embedding

$$C^*(\mathcal{G}) \hookrightarrow \mathcal{T}(\mathcal{G}, E) \quad \text{via} \quad \delta_g \mapsto u_g.$$

We also find a copy of the Toeplitz algebra $\mathcal{T}C^*(E)$ considered in [49], which is the $C^*$-subalgebra of $\mathcal{T}(\mathcal{G}, E)$ generated by the universal Toeplitz-Cuntz-Krieger family $(p, s)$.

**Definition 5.1.9.** Following [63, Proposition 4.7], the *Cuntz–Pimsner* algebra of $(\mathcal{G}, E)$, denoted $\mathcal{O}(\mathcal{G}, E)$, is defined to be the quotient of $\mathcal{T}(\mathcal{G}, E)$ by the ideal $I$ generated by $\big\{p_v - \sum_{e\in vE^1} s_e s_e^* \mid v \in E^0\big\}$.

Note that $1_{\mathcal{O}(\mathcal{G},E)} = \sum_{\mu\in E^n} s_\mu s_\mu^*$ for any $n \ge 1$.

### 5.1.4 Dynamics on $\mathcal{T}(\mathcal{G}, E)$ and $\mathcal{O}(\mathcal{G}, E)$

The universal property of $\mathcal{T}(\mathcal{G}, E)$ yields a strongly continuous gauge action $\gamma : \mathbb{T} \to \operatorname{Aut}(\mathcal{T}(\mathcal{G}, E))$ such that

$$\gamma_z(u_g) = u_g, \quad \gamma_z(q_v) = q_v, \quad \text{and} \quad \gamma_z(t_e) = zt_e$$

for all $t \in \mathbb{R}, g \in \mathcal{G}, v \in E^0$ and $e \in E^1$. It gives rise to a dynamics $\sigma : \mathbb{R} \to \operatorname{Aut}(\mathcal{T}(\mathcal{G}, E))$ via $\sigma_t = \gamma_{e^{it}}$ for $t \in \mathbb{R}$. Note that, since each $p_v - \sum_{e \in vE^1} s_e s_e^*$ is fixed by $\sigma$, the dynamics $\sigma$ descends to a dynamics, also denoted by $\sigma$, on $\mathcal{O}(\mathcal{G}, E)$.

## 5.2 A fixed-point theorem, and the preferred trace on $C^*(\mathcal{G})$

As we have already explained at the beginning of this chapter, recent results of Laca, Raeburn, Ramagge and Whittaker [63] show that any self-similar action of a groupoid on a graph $\mathcal{G} \curvearrowright E$ determines a 1-parameter family of self-mappings of the trace space of the groupoid $C^*$-algebra $C^*(\mathcal{G})$. More specifically, if we let $\rho(A_E)$ denote the spectral radius of the adjacency matrix $A_E$, then Proposition 5.1 of [63] shows that there are no KMS$_\beta$ states for $(\mathcal{T}(\mathcal{G}, E), \sigma)$ for $\beta < \log \rho(A_E)$, and in [63, Theorem 6.1], given a trace $\tau \in \operatorname{Tr}(C^*(\mathcal{G}))$, the authors show that for $\beta > \log \rho(A_E)$, the series

$$Z(\beta, \tau) := \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k} \tau(u_{s(\mu)})$$

converges to a positive real number, and that there is a KMS$_\beta$ state $\Psi_{\beta,\tau}$ on the Toeplitz algebra $\mathcal{T}(\mathcal{G}, E)$ given by

$$\Psi_{\beta,\tau}(s_\mu u_g s_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|} Z(\beta, \tau)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \Big( \sum_{\substack{\lambda \in s(\mu)E^k \\ g \cdot \lambda = \lambda}} \tau(u_{g|_\lambda}) \Big). \tag{5.2.1}$$

They show that the map $\tau \mapsto \Psi_{\beta,\tau}$ is a homeomorphism from the simplex of tracial states of $C^*(\mathcal{G})$ to the KMS$_\beta$-simplex of $\mathcal{T}(\mathcal{G}, E)$. In particular, it is easy to see that the map

$$\tau \mapsto \Psi_{\beta,\tau}|_{C^*(\mathcal{G})} \tag{5.2.2}$$

determines a self-mapping $\chi_\beta : \operatorname{Tr}(C^*(\mathcal{G})) \to \operatorname{Tr}(C^*(\mathcal{G}))$. We investigate the fixed points for these self-mappings parametrized by $\beta > \log \rho(A_E)$, under the same hypotheses that Laca et al. used to prove that $\mathcal{T}(\mathcal{G}, E)$ admits a unique KMS$_{\log \rho(A_E)}$ state. We prove that for any supercritical value of the parameter $\beta$, the associated self-mapping admits a unique fixed point, which is in fact a universal attractor. This fixed point is precisely the trace that extends to a KMS$_{\log \rho(A_E)}$ state on $\mathcal{T}(\mathcal{G}, E)$.

Our main theorem is the following; its proof occupies the remainder of the section.

**Theorem 5.2.1.** *Let $E$ be a finite strongly connected graph*[3]*, suppose that $\mathcal{G} \curvearrowright E$ is a faithful self-similar action of a groupoid $\mathcal{G}$ on $E$, and suppose that $\beta > \log \rho(A_E)$. If $\mathcal{G} \curvearrowright E$ is finite state, then*

*(1) the map $\chi_\beta : \operatorname{Tr}(C^*(\mathcal{G})) \to \operatorname{Tr}(C^*(\mathcal{G}))$ of (5.2.2) has a unique fixed point $\theta$;*

*(2) for any $\tau \in \operatorname{Tr}(C^*(\mathcal{G}))$ we have $\chi_\beta^n(\tau) \xrightarrow{w^*} \theta$;*

*(3) $\theta$ is the unique trace on $C^*(\mathcal{G})$ that extends to a KMS$_{\log \rho(A_E)}$ state of $\mathcal{T}(\mathcal{G}, E)$.*

We start with a straightforward observation about the map $\chi_\beta$ of (5.2.2).

**Lemma 5.2.2.** *Let $\mathcal{G} \curvearrowright E$ be a faithful self-similar action of a groupoid $\mathcal{G}$ on a finite strongly connected graph $E$, and suppose that $\beta > \log \rho(A_E)$. Then the map $\chi_\beta$ is weak$^*$-continuous. Moreover, if $\tau \in \operatorname{Tr}(C^*(\mathcal{G}))$ is such that the sequence $\{\chi_\beta^n(\tau)\}_{n \geq 1}$ weak$^*$-converges to some $\theta$, then $\theta \in \operatorname{Tr}(C^*(\mathcal{G}))$ and $\chi_\beta(\theta) = \theta$.*

[3]This means that every vertex is reachable from every other vertex. Equivalently, $A_E$ is an irreducible matrix, i.e. for each pair of indices $i, j$ there exists a power of $A_E$ such that the $(i, j)$-component of it is strictly positive, that is $(A_E^k)_{i,j} > 0$ for some integer $k \geq 1$.

*Proof.* The map $\mathrm{Tr}(C^*(\mathcal{G})) \to \mathrm{KMS}(\mathcal{T}(\mathcal{G}, E)), \tau \mapsto \Psi_{\beta,\tau}$ is a homeomorphism and hence continuous, and restriction of states to a subalgebra is clearly continuous, so $\chi_\beta : \mathrm{Tr}(C^*(\mathcal{G})) \to \mathrm{Tr}(C^*(\mathcal{G}))$ is continuous. Hence if $\chi_\beta^n(\tau) \xrightarrow{w^*} \theta$, then $\theta \in \mathrm{Tr}(C^*(\mathcal{G}))$ because the trace simplex of a unital $C^*$-algebra is weak$^*$-compact, and then $\chi_\beta(\theta) = \chi_\beta(\lim_n \chi_\beta^n(\tau)) = \lim_n \chi_\beta^{n+1}(\tau) = \theta$. □

**Proposition 5.2.3.** *Let $\mathcal{G} \curvearrowright E$ be a faithful self-similar action of a groupoid $\mathcal{G}$ on a finite graph $E$, and fix $\beta > \log \rho(A_E)$. Let $\chi_\beta : \mathrm{Tr}(C^*(\mathcal{G})) \to \mathrm{Tr}(C^*(\mathcal{G}))$ be the map (5.2.2). For $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$, define*

$$N(\beta, \tau) := e^\beta(1 - Z(\beta, \tau)^{-1}).$$

*(i) If $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$ is a fixed point for $\chi_\beta$, then for each $g \in \mathcal{G}$, we have*

$$N(\beta, \tau)^n \tau(u_g) = \sum_{\mu \in E^n | g\cdot\mu=\mu} \tau(u_{g|_\mu}) \qquad \textit{for all } n \geq 1. \tag{5.2.3}$$

*(ii) If $E$ is strongly connected with adjacency matrix $A_E$, and $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$ satisfies (5.2.3), then $m := (\tau(u_v))_{v\in E^0}$ is the Perron–Frobenius eigenvector of $A_E$[4], and $N(\beta,\tau) = \rho(A_E)$.*

*Proof.* (i) For each $g \in \mathcal{G}$ we have

$$\begin{aligned}\tau(u_g) = \chi_\beta(\tau)(u_g) &= Z(\beta,\tau)^{-1} \sum_{k=0}^{\infty} e^{-\beta k}\Big(\sum_{\substack{\mu\in E^k\\ g\cdot\mu=\mu}} \tau(u_{g|_\mu})\Big)\\ &= Z(\beta,\tau)^{-1}\Big[\tau(u_g) + e^{-\beta}\sum_{k=0}^{\infty} e^{-\beta k}\Big(\sum_{\substack{\mu\in E^{k+1}\\ g\cdot\mu=\mu}} \tau(u_{g|_\mu})\Big)\Big].\end{aligned}$$

Note that the map $(e,\nu) \mapsto e\nu$ is a bijection between the sets $\{(e,\nu) \in E^1 \times E^k \mid s(e) = r(\nu),\ g\cdot e = e \text{ and } g|_e \cdot \nu = \nu\}$ and $\{\mu \in E^{k+1} \mid g\cdot\mu = \mu\}$, so the definition of $\Psi_{\beta,\tau}$ yields

$$\begin{aligned}\tau(u_g) &= Z(\beta,\tau)^{-1}\tau(u_g) + \sum_{\substack{e\in E^1\\ g\cdot e=e}} \Big(Z(\beta,\tau)^{-1} e^{-\beta}\sum_{k=0}^{\infty} e^{-\beta k}\Big(\sum_{\substack{\nu\in s(e)E^k\\ g|_e\cdot\nu=\nu}} \tau(u_{(g|_e)|_\nu})\Big)\Big)\\ &= Z(\beta,\tau)^{-1}\tau(u_g) + \sum_{\substack{e\in E^1\\ g\cdot e=e}} \Psi_{\beta,\tau}(s_e u_{g|_e} s_e^*).\end{aligned} \tag{5.2.4}$$

We have $\Psi_{\beta,\tau}(s_e u_{g|_e} s_e^*) = \delta_{s(e),c(g)}\delta_{s(e),d(g)} e^{-\beta}\Psi_{\beta,\tau}(u_{g|_e}) = e^{-\beta}\chi_\beta(\tau)(u_{g|_e})$; applying this and rearranging (5.2.4) gives

$$e^\beta\big(1 - Z(\beta,\tau)^{-1}\big)\tau(u_g) = \sum_{\substack{e\in E^1\\ g\cdot e=e}} \chi_\beta(\tau)(u_{g|_e}) = \sum_{\substack{e\in E^1\\ g\cdot e=e}} \tau(u_{g|_e}).$$

Statement (i) now follows from an induction on $n$.

(ii) Using (5.2.3) for $\tau$ with $n = 1$ at the first step, we see that for $v \in E^0$,

$$m_v = N(\beta,\tau)^{-1}\sum_{e\in vE^1} \tau(u_{s(e)}) = N(\beta,\tau)^{-1}\sum_{w\in E^0} A_E(v,w)\tau(u_w) = N(\beta,\tau)^{-1}(A_E m)_v.$$

Hence, since $1 = \tau(1) = \sum_{v\in E^0} m_v$, the vector $m$ is a unimodular nonnegative eigenvector for the irreducible matrix $A_E$ and has eigenvalue $N(\beta,\tau)$. So the Perron–Frobenius theorem [93, Theorem 1.6] shows that $m$ is the Perron–Frobenius eigenvector, and $N(\beta,\tau) = \rho(A_E)$. □

We now turn our attention to the situation where $E$ is strongly connected and $\mathcal{G} \curvearrowright E$ is finite-state, and aim to show that $\chi_\beta$ admits a unique fixed point. The strategy is to show that if $C^*(\mathcal{G})$ admits a trace $\theta$ satisfying (5.2.3), then for any other trace $\tau$ we have $\chi_\beta^n(\tau) \xrightarrow{w^*} \theta$. From this it will follow first that $\chi_\beta^n$ admits at most one fixed point, and second that a trace $\theta$ is a fixed point for $\chi_\beta$ if and only if it satisfies (5.2.3). We start with an easy result from Perron–Frobenius theory.

[4] That is, the unique unimodular eigenvector whose components are strictly positive.

**Lemma 5.2.4.** *Let $A \in M_n(\mathbb{R})$ be an irreducible matrix, and take $\beta > \log \rho(A)$.*

*i) The matrix $I - e^{-\beta}A$ is invertible, and $A_{vN} := (I - e^{-\beta}A)^{-1}$ is primitive; indeed, every entry of $A_{vN}$ is strictly positive.*

*ii) Let $m^A$ be the Perron–Frobenius eigenvector of $A$. Then $m^A$ is also the Perron–Frobenius eigenvector of $A_{vN}$, and $\rho(A_{vN}) = (1 - e^{-\beta}\rho(A))^{-1}$.*

*Proof.* The proof of *i)* is easy. The matrix $I - e^{-\beta}A$ is invertible because $e^{\beta} > \rho(A)$, and so $e^{\beta}$ does not belong to the spectrum of $A$. As in, for example, [27, Section VII.3.1], we have

$$A_{vN} := (I - e^{-\beta}A)^{-1} = \sum_{k=0}^{\infty} e^{-k\beta}A^k.$$

Fix two indices $1 \leq i, j \leq n$. Since $A$ is irreducible, we have $(A^k)_{i,j} > 0$ for some $k = k_{i,j} \geq 0$, and since $A^l_{i,j} \geq 0$ for all $l \geq 1$, we deduce that $(A_{vN})_{i,j} \geq e^{-k\beta}(A^k)_{i,j} > 0$.

For *ii)*, we compute $A_{vN}^{-1}m^A = (I - e^{-\beta}A)m^A = (1 - e^{-\beta}\rho(A))m^A$. Multiplying through by $(1 - e^{-\beta}\rho(A))^{-1}A_{vN}$ shows that $m^A$ is a positive eigenvector of $m^A$ with eigenvalue $(1 - e^{-\beta}\rho(A))^{-1}$, so the result follows from uniqueness of the Perron–Frobenius eigenvector of $A_{vN}$ (cf. [93, Theorem 1.6]). ☐

**Notation 5.2.5.** Henceforth, given a self-similar action $\mathcal{G} \curvearrowright E$ of a groupoid $\mathcal{G}$ on a finite graph $E$ and a trace $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$, we will denote by $x^{\tau} \in [0,1]^{E^0}$ the vector

$$x^{\tau} = \big(\tau(u_v)\big)_{v \in E^0}.$$

**Proposition 5.2.6.** *Let $\mathcal{G} \curvearrowright E$ be a faithful self-similar action of a groupoid $\mathcal{G}$ on a finite strongly connected graph $E$. Fix $\beta > \log \rho(A_E)$, and let $A_{vN} := (I - e^{-\beta}A_E)^{-1}$. Let $\chi_\beta : \mathrm{Tr}(C^*(\mathcal{G})) \to \mathrm{Tr}(C^*(\mathcal{G}))$ be the map (5.2.2). Fix $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$. Then*

$$x^{\chi_\beta^n(\tau)} = \|A_{vN}^n x^{\tau}\|_1^{-1} A_{vN}^n x^{\tau}. \tag{5.2.5}$$

*Proof.* For $v \in E^0$, the definition of $\chi_\beta$ gives

$$\begin{aligned} x_v^{\chi_\beta(\tau)} &= \chi_\beta(\tau)(u_v) = Z(\beta,\tau)^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\Big(\sum_{\mu \in vE^k} \tau(u_{s(\mu)})\Big) \\ &= Z(\beta,\tau)^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{w \in E^0} A_E^k(v,w)\tau(u_w) \\ &= Z(\beta,\tau)^{-1}\sum_{k=0}^{\infty} e^{-\beta k}(A_E^k x^{\tau})_v = Z(\beta,\tau)^{-1}(A_{vN}x^{\tau})_v, \end{aligned}$$

so an induction gives $x^{\chi_\beta^n(\tau)} = Z(\beta, \chi_\beta^{n-1}(\tau))^{-1} \cdots Z(\beta,\tau)^{-1}A_{vN}^n x^{\tau}$. Since both $x^{\chi_\beta^n(\tau)}$ and $x^{\tau}$ have unit 1-norm, we have $Z(\beta, \chi_\beta^{n-1}(\tau))^{-1} \cdots Z(\beta,\tau)^{-1} = \|A_{vN}^n x^{\tau}\|_1^{-1}$, and the result follows. ☐

Our next result shows that for any $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$, the sequence $x^{\chi_\beta^n(\tau)}$ converges exponentially fast to the Perron–Frobenius eigenvector of $A_E$.

**Theorem 5.2.7.** *Let $\mathcal{G} \curvearrowright E$ be a faithful self-similar action of a groupoid $\mathcal{G}$ on a finite strongly connected graph $E$. Fix $\beta > \log \rho(A_E)$, and let $A_{vN} := (I - e^{-\beta}A_E)^{-1}$ and $m = m^E$ be the Perron–Frobenius eigenvector of $A_E$. Let $\chi_\beta : \mathrm{Tr}(C^*(\mathcal{G})) \to \mathrm{Tr}(C^*(\mathcal{G}))$ be the map (5.2.2). Fix $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$. Then $x^{\chi_\beta(\tau)} \xrightarrow{n} m^E$ exponentially fast and $Z(\beta, \chi_\beta^n(\tau)) \xrightarrow{n} \rho(A_{vN})$ exponentially fast.*

*Proof.* Since $E$ is strongly connected, Lemma 5.2.4 shows that $m$ is the (right) Perron–Frobenius eigenvector of $A_{vN} := (I - e^{-\beta}A_E)^{-1}$. Write $\widetilde{m}$ for the left Perron–Frobenius eigenvector of $A_{vN}$ such that $\widetilde{m} \cdot m = 1$. Let $r := \widetilde{m} \cdot x^{\tau}$. Then $r > 0$ because every entry of $\widetilde{m}$ is strictly positive, and $x^{\tau}$ is a nonnegative nonzero vector.

Proposition 5.2.6 implies that

$$x_v^{\chi_\beta^n(\tau)} - m_v = \frac{\rho(A_{vN})^n}{\big\|A_{vN}^n x^{\tau}\big\|_1}\Big[\big(\rho(A_{vN})^{-n}A_{vN}^n x^{\tau} - rm\big)_v + \big(r - \big\|\big(\rho(A_{vN})^{-n}A_{vN}^n x^{\tau}\big)\big\|_1\big)m_v\Big]. \tag{5.2.6}$$

By [93, Theorem 1.2], there exist a real number $0 < \lambda < 1$, a positive constant $C$, and an integer $s \geq 0$ such that for large $n$ we have

$$\rho(A_{vN})^{-n}(A_{vN}^n)_{i,j} - (m \cdot \widetilde{m}^t)_{i,j} \leq Cn^s\lambda^n \quad \text{for all indices } i, j.$$

Moreover, since $Cn^s(\lambda'/\lambda)^n \to 0$ for any $0 < \lambda' < \lambda < 1$, we can assume that $C = 1$ and $s = 0$. So for large $n$, we have

$$\left|\rho(A_{vN})^{-n}(A_{vN}^n x^\tau)_v - rm_v\right| \leq \lambda^n.$$

Since $v \in E^0$ was arbitrary, by summing up over all the vertices we deduce that

$$\left|\rho(A_{vN})^{-n}\|A_{vN}^n x^\tau\|_1 - r\right| \leq |E^0|\lambda^n.$$

Hence $\rho(A_{vN})^{-n}\left\|A_{vN}^n x^\tau\right\|_1 \stackrel{n}{\to} r$ exponentially fast. Making this approximation twice in (5.2.6), we obtain

$$|x_v^{\chi_\beta^n(\tau)} - m_v| \leq \frac{(1 + |E^0|)}{\rho(A_{vN})^{-n}\left\|A_{vN}^n x^\tau\right\|_1}\lambda^n,$$

which converges exponentially fast to 0. Hence $x^{\chi_\beta^n(\tau)} \to m$ exponentially fast.

For the second statement, using Proposition 5.2.6 at the third equality, we calculate

$$\begin{aligned} Z(\beta, \chi_\beta^n(\tau)) &= \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\mu \in E^k} \chi_\beta^n(\tau)(u_{s(\mu)}) \\ &= \left\|A_{vN}x^{\chi_\beta^n(\tau)}\right\|_1 = \frac{\|A_{vN}^{n+1}x^\tau\|_1}{\|A_{vN}^n x^\tau\|_1} = \frac{\rho(A_{vN})^{-(n+1)}\left\|A_{vN}^{n+1}x^\tau\right\|_1}{\rho(A_{vN})^{-n}\left\|A_{vN}^n x^\tau\right\|_1}\rho(A_{vN}). \end{aligned}$$

We saw that $\rho(A_{vN})^{-(n+1)}\left\|A_{vN}^{n+1}x^\tau\right\|_1$ converges to $r > 0$ exponentially fast, so the ratio

$$\frac{\rho(A_{vN})^{-(n+1)}\left\|A_{vN}^{n+1}x^\tau\right\|_1}{\rho(A_{vN})^{-n}\left\|A_{vN}^n x^\tau\right\|_1}$$

converges exponentially fast to 1. □

The following estimate is needed for our key technical result, Theorem 5.2.9.

**Lemma 5.2.8.** *Let $\mathcal{G} \curvearrowright E$ be a faithful finite-state self-similar action of a groupoid $\mathcal{G}$ on a finite strongly connected graph $E$. Fix $\beta > \log \rho(A_E)$, and let $A_{vN} := (I - e^{-\beta}A_E)^{-1}$ and $m = m^E$ be the Perron–Frobenius eigenvector of $A_E$. For $g \in \mathcal{G} \setminus E^0$, $v \in E^0$, and $k \geq 0$, define*

$$\mathcal{G}_g^k(v) := \{\mu \in d(g)E^k v \mid g \cdot \mu = \mu\} \quad \text{and} \quad \mathcal{F}_g^k(v) := \{\mu \in \mathcal{G}_g^k(v) \mid g|_\mu = v\}.$$

*Then for $g \in \mathcal{G}$, we have the estimate*

$$\sum_{k=0}^{\infty} e^{-\beta k} \sum_{v \in E^0} |\mathcal{G}_g^k(v) \setminus \mathcal{F}_g^k(v)| m_v < \rho(A_{vN}) m_{d(g)}.$$

*Proof.* The argument of [63, Lemma 8.7] shows that there exists $k = k(g) > 0$ such that

$$\sum_{v \in E^0} |\mathcal{G}_g^{nk}(v) \setminus \mathcal{F}_g^{nk}(v)| m_v \leq (\rho(A_E)^{k(g)} - 1)^n m_{d(g)}$$

for all $n \geq 0$. For each $k \in \mathbb{N}$ we also have

$$\sum_{v \in E^0} |\mathcal{G}_g^k(v)| m_v \leq \sum_{v \in E^0} |d(g)E^k v| m_v = (A_E^k m)_{d(g)} = \rho(A_E)^k m_{d(g)}.$$

Combining these estimates and using Lemma 5.2.4 *ii*) at the final step, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} e^{-\beta k} \sum_{v\in E^0} |\mathcal{G}_g^k(v)\setminus \mathcal{F}_g^k(v)| m_v &= \sum_{k\neq k(g)} e^{-\beta k} \sum_{v\in E^0} |\mathcal{G}_g^k(v)\setminus \mathcal{F}_g^k(v)| m_v + e^{-\beta k(g)} \sum_{v\in E^0} |\mathcal{G}_g^{k(g)}(v)\setminus \mathcal{F}_g^{k(g)}(v)| m_v \\ &\leq \sum_{k\neq k(g)} e^{-\beta k}\rho(A_E)^k m_{d(g)} + e^{-\beta k(g)}(\rho(A_E)^{k(g)} - 1) m_{d(g)} \\ &< \sum_{k=0}^{\infty} e^{-\beta k}\rho(A_E)^k m_{d(g)} = \rho(A_{vN}) m_{d(g)}. \qquad \square\end{aligned}$$

We are now ready to prove a converse to Proposition 5.2.3($i$), under the hypotheses that $E$ is strongly connected and the action of $\mathcal{G}$ on $E$ is finite-state.

**Theorem 5.2.9.** *Let $\mathcal{G} \curvearrowright E$ be a faithful finite-state self-similar action of a groupoid $\mathcal{G}$ on a finite strongly connected graph $E$. Fix $\beta > \log \rho(A_E)$. Let $\chi_\beta : \mathrm{Tr}(C^*(\mathcal{G})) \to \mathrm{Tr}(C^*(\mathcal{G}))$ be the map (5.2.2). Suppose that $\theta \in \mathrm{Tr}(C^*(\mathcal{G}))$ satisfies (5.2.3). Then for any $\tau \in \mathrm{Tr}(C^*(\mathcal{G}))$, we have $\chi_\beta^n(\tau) \overset{w^*}{\to} \theta$. In particular, $\theta$ is a fixed point for $\chi_\beta$.*

*Proof.* We will prove that for each $g \in \mathcal{G}$ there are constants $0 < \lambda < 1$ and $K, D > 0$ such that

$$|\chi_\beta^n(\tau)(u_g) - \theta(u_g)| < (nK + D)\lambda^{n-1} m_{d(g)}$$

for all $n \geq 0$. Since $(nK + D)K\lambda^{n-1} \overset{n}{\to} 0$ exponentially fast in $n$, the first statement will then follow from an $\frac{\varepsilon}{3}$-argument, since $\mathrm{span}\{u_g\}_{g\in\mathcal{G}}$ is a dense subset for $C^*(\mathcal{G})$.

To simplify notation, define $\tau_0 := \tau$ and $\tau_n := \chi_\beta^n(\tau)$ for $n \geq 1$. For $g \in \mathcal{G}$ and $n \geq 0$, let

$$\Delta_n(g) := \tau_n(u_g) - \theta(u_g).$$

Fix $g \in \mathcal{G}$; note that, if $c(g) \neq d(g)$, then $\tau_n(u_g) = \theta(u_g) = 0$ by [63, Proposition 7.2], so we may assume that $c(g) = d(g)$. Since the action is finite-state, the set $\{g|_\mu \mid \mu \in d(g)E^*\}$ is finite. By Lemma 5.2.8, there is a constant $\alpha < 1$ such that

$$\sum_{k=0}^{\infty} e^{-\beta k} \sum_{v\in E^0} |\mathcal{G}_{g|_\mu}^k(v)\setminus \mathcal{F}_{g|_\mu}^k(v)| m_v \leq \alpha\rho(A_{vN}) m_{d(g|_\mu)} \tag{5.2.7}$$

for all $\mu \in E^*$. Also, since $\theta$ satisfies (5.2.3), we have

$$\theta(u_g) = N(\beta,\theta)^{-k} \sum_{\substack{\mu\in E^k \\ g\cdot\mu=\mu}} \theta(u_{g|_\mu}) \quad \text{for all } k \geq 0.$$

Consequently,

$$\sum_{k=0}^{\infty} e^{-\beta k} \sum_{\substack{\mu\in E^k \\ g\cdot\mu=\mu}} \theta(u_{g|_\mu}) = \sum_{k=0}^{\infty} e^{-\beta k} N(\beta,\theta)^k \theta(u_g) = \big(1 - e^{-\beta}N(\beta,\theta)\big)^{-1}\theta(u_g).$$

Since $N(\beta,\theta) = e^\beta(1 - Z(\beta,\theta)^{-1})$ by definition, we can rearrange to obtain

$$\theta(u_g) = Z(\beta,\theta)^{-1} \sum_{k=0}^{\infty} e^{-\beta k} \sum_{\substack{\mu\in E^k \\ g\cdot\mu=\mu}} \theta(u_{g|_\mu}).$$

Using this, and applying the definition of $\chi_\beta$ at the third equality, we calculate

$$\begin{aligned}\Delta_{n+1}(g) = \tau_{n+1}(u_g) - \theta(u_g) &= \chi_\beta(\tau_n)(u_g) - Z(\beta,\theta)^{-1}\sum_{k=0}^{\infty} e^{-\beta k} \sum_{\substack{\mu\in E^k \\ g\cdot\mu=\mu}} \theta(u_{g|_\mu}) \\ &= Z(\beta,\tau_n)^{-1}\sum_{k=0}^{\infty} e^{-\beta k} \sum_{\substack{\mu\in E^k \\ g\cdot\mu=\mu}} \tau_n(u_{g|_\mu}) - Z(\beta,\theta)^{-1}\sum_{k=0}^{\infty} e^{-\beta k} \sum_{\substack{\mu\in E^k \\ g\cdot\mu=\mu}} \theta(u_{g|_\mu}).\end{aligned}$$

Since the sums are absolutely convergent, we can rewrite each $\theta(u_{g|_\mu})$ as $\tau_n(u_{g|_\mu}) - \Delta_n(g|_\mu)$ and rearrange to obtain

$$\Delta_{n+1}(g) = \big(Z(\beta,\tau_n)^{-1} - Z(\beta,\theta)^{-1}\big)\sum_{k=0}^{\infty} e^{-\beta k}\Big(\sum_{\substack{\mu\in E^k\\ g\cdot\mu=\mu}} \tau_n(u_{g|_\mu})\Big) + Z(\beta,\theta)^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{\substack{\mu\in E^k\\ g\cdot\mu=\mu}} \Delta_n(g|_\mu). \tag{5.2.8}$$

Now, $\theta$ satisfies (5.2.3), so Proposition 5.2.3($ii$) combined with the definition of $N(\beta,\theta)$ imply that $Z(\beta,\theta) = \big(1-e^{-\beta}N(\beta,\theta)\big)^{-1} = \big(1-e^{-\beta}\rho(A)\big)^{-1}$, and then Lemma 5.2.4 $ii$) gives $Z(\beta,\theta) = \rho(A_{vN})$. Also, by definition of $\chi_\beta$, we have

$$\sum_{k=0}^{\infty} e^{-\beta k}\sum_{\substack{\mu\in E^k\\ g\cdot\mu=\mu}} \tau_n(u_{g|_\mu}) = Z(\beta,\tau_n)\tau_{n+1}(u_g).$$

Making these substitutions in (5.2.8), we obtain

$$\Delta_{n+1}(g) = \big(Z(\beta,\tau_n)^{-1} - \rho(A_{vN})^{-1}\big)Z(\beta,\tau_n)\tau_{n+1}(u_g) + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{\substack{\mu\in E^k\\ g\cdot\mu=\mu}} \Delta_n(g|_\mu).$$

With $\mathcal{G}_g^k(v)$ and $\mathcal{F}_g^k(v)$ defined as in Lemma 5.2.8, the preceding expression for $\Delta_{n+1}(g)$ becomes

$$\begin{aligned}\Delta_{n+1}(g) &= \big(Z(\beta,\tau_n)^{-1} - \rho(A_{vN})^{-1}\big)Z(\beta,\tau_n)\tau_{n+1}(u_g)\\ &\quad + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\Big(\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)} \Delta_n(g|_\mu) + \sum_{\mu\in\mathcal{F}_g^k(v)} \Delta_n(g|_\mu)\Big).\end{aligned} \tag{5.2.9}$$

The Cauchy–Schwarz inequality implies that for any $h\in\mathcal{G}$,

$$|\tau_{n+1}(u_h)|^2 = |\tau_{n+1}(u_h^* u_{c(h)})|^2 \le \tau_{n+1}(u_h^* u_h)\tau(u_{c(h)}^* u_{c(h)}) = \tau_{n+1}(u_{d(h)})\tau_{n+1}(u_{c(h)}).$$

Since our fixed $g$ satisfies $d(g) = c(g)$, taking square roots in the preceding estimate gives $|\tau_{n+1}(u_g)| \le \tau_{n+1}(u_{d(g)})$. Applying this combined with the triangle inequality to the right-hand side of (5.2.9), we obtain

$$\begin{aligned}|\Delta_{n+1}(g)| &\le \big|Z(\beta,\tau_n)^{-1} - \rho(A_{vN})^{-1}\big|Z(\beta,\tau_n)\tau_{n+1}(u_{d(g)})\\ &\quad + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\Big(\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)} |\Delta_n(g|_\mu)| + \sum_{\mu\in\mathcal{F}_g^k(v)} |\Delta_n(g|_\mu)|\Big),\end{aligned}$$

which, using that $g|_\mu = v$ for $\mu\in\mathcal{F}_g^k(v)$, becomes

$$\begin{aligned}|\Delta_{n+1}(g)| &\le \big|Z(\beta,\tau_n)^{-1} - \rho(A_{vN})^{-1}\big|Z(\beta,\tau_n)\tau_{n+1}(u_{d(g)})\\ &\quad + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)} |\Delta_n(g|_\mu)|\\ &\quad + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\mu\in\mathcal{F}_g^k(v)} |\Delta_n(v)|.\end{aligned}$$

Since $\big(Z(\beta,\tau_n)^{-1} - \rho(A_{vN})^{-1}\big)Z(\beta,\tau_n) = \rho(A_{vN})^{-1}\big(\rho(A_{vN}) - Z(\beta,\tau_n)\big)$, we obtain

$$\begin{aligned}|\Delta_{n+1}(g)| &\le \rho(A_{vN})^{-1}\big|\rho(A_{vN}) - Z(\beta,\tau_n)\big|\tau_{n+1}(u_{d(g)})\\ &\quad + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)} |\Delta_n(g|_\mu)|\\ &\quad + \rho(A_{vN})^{-1}\sum_{\mu\in d(g)E^*} e^{-\beta|\mu|}|\Delta_n(s(\mu))|.\end{aligned}$$

By Theorem 5.2.7 there are positive constants $\lambda_0$, $K_1$ and $K_2$ with $\lambda_0 < 1$ such that $|\rho(A_{vN}) - Z(\beta, \tau_n)| < K_1\lambda_0^n$ for all $n \geq 0$ and $|\Delta_n(v)| = |\tau_n(u_v) - m_v| < K_2\lambda_0^n$ for all $v \in E^0$ and $n \geq 0$. Thus we obtain

$$\begin{aligned}|\Delta_{n+1}(g)| &\leq K_1\lambda_0^n \rho(A_{vN})^{-1}\tau_{n+1}(u_{d(g)}) \\ &\quad + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)} \big|\Delta_n(g|_\mu)\big| \\ &\quad + K_2\lambda_0^n\rho(A_{vN})^{-1}\sum_{\mu\in d(g)E^*} e^{-\beta|\mu|}.\end{aligned}$$

Theorem 3.1(a) of [49] shows that $\sum_{\mu\in d(g)E^*} e^{-\beta|\mu|}$ converges, and since the entries of the Perron–Frobenius eigenvector $m$ are strictly positive, $l := \max_{v\in E^0}\{m_v^{-1}\}$ is finite. So $K := \frac{2l}{\rho(A_{vN})}\max\{K_1, K_2\sum_{\mu\in E^*} e^{-\beta|\mu|}\}$ satisfies

$$|\Delta_{n+1}(g)| \leq K\lambda_0^n m_{d(g)} + \rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\mu\in\mathcal{G}_g^k(v)\setminus\mathcal{F}_g^k(v)}\big|\Delta_n(g|_\mu)\big|. \tag{5.2.10}$$

Since both $\lambda_0$ and the constant $\alpha$ of (5.2.7) are less than 1, the quantity $\lambda := \max\{\lambda_0, \alpha\}$ is less than 1. Let $D := l\max_{\mu\in d(g)E^*}\{|\tau(u_{g|_\mu})| + |\theta(u_{g|_\mu})|\}$, which is finite because $\mathcal{G} \curvearrowright E$ is finite state. Let $g|_{E^*} := \{g|_\mu \mid \mu \in E^*\} \subseteq \mathcal{G}$. We will prove by induction that $|\Delta_n(h)| \leq (nK + D)\lambda^{n-1}m_{d(h)}$ for all $n \geq 0$ and for all $h \in g|_{E^*}$.

The base case $n = 0$ is trivial because each $|\Delta_0(h)| = |\tau(u_h) - \theta(u_h)| \leq |\tau(u_h)| + |\theta(u_h)| \leq Dl^{-1} \leq D\lambda^{-1}m_{d(h)}$. Now suppose as an inductive hypothesis that $|\Delta_n(h)| \leq (nK + D)\lambda^{n-1}m_{d(h)}$ for all $h \in g|_{E^*}$, and fix $h \in g|_{E^*}$. Applying the inductive hypothesis on the right-hand side of (5.2.10), and then using that $h|_{E^*} \subseteq g|_{E^*}$ and invoking (5.2.7) gives

$$\begin{aligned}|\Delta_{n+1}(h)| &\leq K\lambda_0^n m_{d(h)} + (nK + D)\lambda^{n-1}\rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}\sum_{\nu\in\mathcal{G}_h^k(v)\setminus\mathcal{F}_h^k(v)} m_{d(h|_\nu)} \\ &= K\lambda_0^n m_{d(h)} + (nK + D)\lambda^{n-1}\rho(A_{vN})^{-1}\sum_{k=0}^{\infty} e^{-\beta k}\sum_{v\in E^0}|\mathcal{G}_h^k(v)\setminus\mathcal{F}_h^k(v)|m_v \\ &\leq K\lambda_0^n m_{d(h)} + (nK + D)\lambda^{n-1}\alpha m_{d(h)},\end{aligned}$$

and since $\lambda_0, \alpha \leq \lambda$ we deduce that

$$|\Delta_{n+1}(h)| \leq ((n+1)K + D)\lambda^n m_{d(h)}.$$

The claim follows by induction, and in particular we have $|\Delta_n(g)| \leq (nK + D)\lambda^{n-1}m_{d(g)}$ for all $n \geq 0$, as claimed. This proves the first statement.

The second statement follows immediately from Lemma 5.2.2. □

*Proof of Theorem 5.2.1.* (1) Let $m = m^E$ be the Perron–Frobenius eigenvector of $A_E$. For $v \in \mathcal{G}^{(0)} = E^0$, let $c_v := m_v$. Fix $g \in \mathcal{G} \setminus E^0$. By [63, Proposition 8.2], the sequence

$$\Big(\rho(A_E)^{-n}\sum_{v\in E^0}\big|\{\mu \in E^n \mid g\cdot\mu = \mu, g|_\mu = v\}\big|m_v\Big)_{n\geq 1}$$

converges to some $c_g \in [0, m_{d(g)}]$. By [63, Theorem 8.3], there is a $\mathrm{KMS}_{\log\rho(A_E)}$ state $\psi$ of $\mathcal{T}(\mathcal{G}, E)$ that factors through $(\mathcal{O}(\mathcal{G}, E))$. This $\psi$ satisfies

$$\psi(s_\mu u_g s_\nu^*) = \begin{cases}\rho(A_E)^{-|\mu|}c_g & \text{if } \mu = \nu \text{ and } d(g) = c(g) = s(\mu) \\ 0 & \text{otherwise}\end{cases}.$$

In particular, $\theta := \psi|_{C^*(\mathcal{G})}$ belongs to $\mathrm{Tr}(C^*(\mathcal{G}))$. We claim that $\theta$ is a fixed point for $\chi_\beta$. By the final statement of Theorem 5.2.9, it suffices to show that $\theta$ satisfies (5.2.3). Proposition 8.1 of [63] shows that $x^\theta = \big(\theta(u_v)\big)_{v\in E^0}$

is equal to $m$. Using this, we see that

$$Z(\beta,\theta)=\sum_{v\in E^0}\sum_{k=0}^{\infty}e^{-\beta k}\sum_{\mu\in vE^k}\theta(u_{s(\mu)})=\sum_{v\in E^0}\sum_{k=0}^{\infty}e^{-\beta k}\sum_{w\in E^0}A_E^k(v,w)m_w$$
$$=\sum_{v\in E^0}\sum_{k=0}^{\infty}e^{-\beta k}(A_E^k m)_v=\sum_{k=0}^{\infty}e^{-\beta k}\rho(A_E)^k\sum_{v\in E^0}m_v=(1-e^{-\beta}\rho(A_E))^{-1}.$$

Hence $N(\beta,\theta)=e^{\beta}(1-Z(\beta,\theta)^{-1})=\rho(A_E)$. Since $1_{\mathcal{O}(\mathcal{G},E)}=\sum_{v\in E^0}p_v=\sum_{e\in E^1}s_es_e^*$, we have

$$\theta(u_g)=\psi(u_g)=\sum_{e\in E^1}\psi(u_gs_es_e^*)=\sum_{e\in E^1}\delta_{d(g),r(e)}\psi(s_{g\cdot e}u_{g|_e}s_e^*)$$
$$=\sum_{e\in E^1}\delta_{d(g),r(e)}\delta_{g\cdot e,e}\delta_{d(g|_e),s(e)}\delta_{t(g|_e),s(e)}\rho(A_E)^{-1}\theta(u_{g|_e})=N(\beta,\theta)^{-1}\sum_{\substack{e\in E^1\\ g\cdot e=e}}\theta(u_{g|_e}).$$

Now an easy induction shows that $\theta$ satisfies relation (5.2.3).

It remains to prove that $\theta$ is the unique fixed point for $\chi_\beta$. For this, suppose that $\theta'$ is a fixed point for $\chi_\beta$, so $\chi_\beta^n(\theta')\overset{w^*}{\to}\theta'$. Since $\theta$ satisfies (5.2.3), Theorem 5.2.9 shows that $\chi_\beta^n(\theta')\overset{w^*}{\to}\theta$, so $\theta'=\theta$.

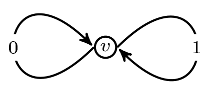
(2) This follows immediately from Theorem 5.2.9 because $\theta$ satisfies (5.2.3).

(3) The trace $\theta$ of part (1) extends to a $\mathrm{KMS}_{\log\rho(A_E)}$ state of $\mathcal{T}(\mathcal{G},E)$ by construction. If $\phi$ is any $\mathrm{KMS}_{\log\rho(A_E)}$ state of $\mathcal{T}(\mathcal{G},E)$, then it restricts to a $\mathrm{KMS}_{\log\rho(A_E)}$ state of the $C^*$-subalgebra $\mathcal{T}C^*(E)$, so it follows from [49, Theorem 4.3(a)] that $\phi$ agrees with $\psi$ on $\mathcal{T}C^*(E)$, and in particular $(\phi(u_v))_{v\in E^0}$ is equal to the Perron–Frobenius eigenvector $m^E$. So [63, Proposition 8.1] shows that $\phi$ factors through $\mathcal{O}(\mathcal{G},E)$. By construction, $\psi$ also factors through $\mathcal{O}(\mathcal{G},E)$. By [63, Theorem 8.3(2)], there is a unique KMS state on $\mathcal{O}(\mathcal{G},E)$, and we deduce that $\phi=\psi$. In particular, $\phi|_{C^*(\mathcal{G})}=\psi|_{C^*(\mathcal{G})}=\theta$. □

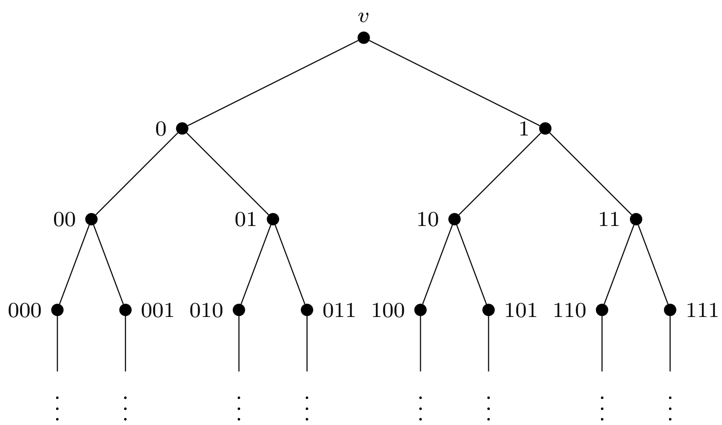
## 5.3 The lamplighter group as a self-similar group(oid)

To conclude this chapter, we would like to connect the study already made in Chapter 3 of the lamplighter group, which is given by the wreath product $\Gamma=\mathbb{Z}_2\wr\mathbb{Z}$, see Example 3.1.2.

Let $E_\Gamma$ be the finite graph given by one vertex $v$ and two loops $0,1$ around $v$, namely




Its forest $T_{E_\Gamma}$ is easy to describe:



Since the graph $E_\Gamma$ contains only one vertex, the set of partial isomorphisms $\mathrm{PIso}(T_{E_\Gamma})$ forms not even a groupoid, but a group under composition of maps. We define two (partial) isomorphisms $a, b : vE^* \to vE^*$ recursively by the rules

$$\begin{cases} a \cdot (0\mu) = 1(b \cdot \mu), \\ a \cdot (1\mu) = 0(a \cdot \mu) \end{cases} \quad \text{and} \quad \begin{cases} b \cdot (0\mu) = 0(b \cdot \mu), \\ b \cdot (1\mu) = 1(a \cdot \mu) \end{cases}$$

for all paths $\mu \in vE^*$. By a theorem of Grigorchuk and Żuk [42, Theorem 2], the subgroup $\mathcal{G}_{E_\Gamma}$ of $\mathrm{PIso}(T_{E_\Gamma})$ generated by $\{a, b\}$ is isomorphic to the lamplighter group $\Gamma$, through the isomorphism

$$\mathcal{G}_{E_\Gamma} \cong \Gamma, \quad b^{-1}a \mapsto a_0, \; b \mapsto t.$$

In particular, $\Gamma$ is a self-similar group(oid) which acts faithfully on $E_\Gamma$. Note that $A_E = (2)$ here, so $\rho(A_E) = 2$.

**Proposition 5.3.1.** *For $\Gamma$ the lamplighter group, the trace $\theta$ from Theorem 5.2.1(3) equals the canonical trace $\mathrm{tr}_\Gamma$ defined on $C^*(\Gamma)$ by the rule $\mathrm{tr}(u_g) = \delta_{g,e}$, where $e$ is the unit element of $\Gamma$, and $\delta$ is the Kronecker delta.*

*Proof.* By a reformulation, in our notation, of the proof of [42, Proposition 9], we deduce that for an element $g \in \Gamma$ different from the identity element, there exists an integer $n_0 \geq 1$ such that

$$|\Gamma_g^{kn_0}| \leq (2^{n_0} - 1)^k \quad \text{for any } k \geq 0,$$

where $\Gamma_g^n = \{\mu \in E^n \mid g \cdot \mu = \mu\}$. Therefore by [63, Proposition 8.2],

$$c_g = \lim_k 2^{-kn_0} |\{\mu \in E^{kn_0} \mid g \cdot \mu = \mu, \; g|_\mu = v\}| \leq \lim_k 2^{-kn_0} |\Gamma_g^{kn_0}| \leq \lim_k (1 - 2^{-n_0})^k = 0.$$

Of course, for $g = e \in \Gamma$, $c_e = \lim_n 2^{-n}|E^n| = 1$. This tells us precisely that $\theta(u_g) = c_g = \delta_{g,e} = \mathrm{tr}_\Gamma(u_g)$ for all $g \in \Gamma$, as required. □

Recall from Section 1.1.2 that the canonical trace $\mathrm{tr}_\Gamma$ over the group algebra $\mathbb{C}[\Gamma]$ can be extended to a trace over $k \times k$ matrices $M_k(\mathbb{C}[\Gamma])$, and this in turn extends to a normal, faithful and positive trace over the von Neumann algebra $\mathcal{N}_k(\Gamma)$ of $M_k(\mathbb{C}[\Gamma])$ inside $\mathcal{B}(l^2(\Gamma))$ (see Proposition 1.1.4).

Since the von Neumann dimension of an element $T \in M_k(\mathbb{C}[\Gamma])$ is defined to be the trace of the projection onto its kernel (Definition 1.1.7), Proposition 5.3.1 opens a possible analytical approach to attack the problem of computing $l^2$-Betti numbers arising from $\Gamma$, by trying to study the role of the unique $\mathrm{KMS}_{\log 2}$ state that its restriction gives back the trace $\mathrm{tr}_\Gamma$.

# Bibliography


[1] Z. Afsar, A. an Huef, I. Raeburn, *KMS states on $C^*$-algebras associated to a family of $*$-commuting local homeomorphisms*, J. Math. Anal. Appl. **464** (2018), 965–1009.

[2] P. Ara, *Matrix rings over *-regular rings and pseudo-rank functions*, Pacific J. Math. **129** (1987), 209–241.

[3] P. Ara, M. Brustenga, *The regular algebra of a quiver*, J. Algebra **309** (2007), 207–235.

[4] P. Ara, J. Claramunt, *Approximating the group algebra of the lamplighter by finite-dimensional algebras*, in preparation.

[5] P. Ara, J. Claramunt, *Uniqueness of the von Neumann continuous factor*, Canad. J. Math. **70** (2018), 961–982.

[6] P. Ara, K. R. Goodearl, *The realization problem for some wild monoids and the Atiyah problem*, Trans. Amer. Math. Soc. **369** (2017), 5665–5710.

[7] M. F. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*, in *Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974)*, Soc. Math. France, Paris, **32–33** (1976), 43–72.

[8] T. Austin, *Rational group ring elements with kernels having irrational dimension*, Proc. London Math. Soc. **107** (2013), 1424–1448.

[9] S. K. Berberian, *Baer $*$-rings*. Die Grundlehren der mathematischen Wissenschaften, Band 195. Springer-Verlag, New York-Berlin, 1972.

[10] B. Blackadar, D. Handelman, *Dimension functions and traces on C*-algebras*, J. Funct. Anal. **45** (1982), 297–340.

[11] O. Bratteli, *Inductive Limits of Finite Dimensional $C^*$-Algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.

[12] N. P. Brown, N. Ozawa, *$C^*$-Algebras and Finite-Dimensional Approximations*. Graduate studies in mathematics, Volume 88. American Mathematica Society, 2008.

[13] J. Berstel, C. Reutenauer, *Rational Series and Their Languages*. Monographs in Theoretical Computer Science, Volume 12. Springer-Verlag, Berlin, 1988.

[14] G. Birkhoff, *Combinatorial relations in projective geometries*, Ann. of Math. **36** (1935), 743–748.

[15] T. M. Carlsen, N. S. Larsen, *Partial actions and KMS states on relative graph $C^*$-algebras*, J. Funct. Anal. **271** (2016), 2090–2132.

[16] J. Claramunt, A. Sims, *Preferred traces on $C^*$-algebras of self-similar groupoids arising as fixed points*, J. Math. Anal. Appl. **466** (2018), 806–818.

[17] L. O. Clark, R. Exel, E. Pardo, A. Sims, C. Starling, *Simplicity of algebras associated to non-Hausdorff groupoids*, preprint 2018 (arXiv:1806.04362 [math.OA]).

[18] P. M. Cohn, *Free Rings and Their Relations*. Second Edition, LMS Monographs, Volume 19. Academic Press, London, 1985.

[19] P. M. Cohn, *Algebra. Vol. 3.* Second edition, John Wiley & Sons, Ltd., Chichester, 1991.

[20] P. M. Cohn, *Free Ideal Rings and Localization in General Rings.* New Mathematical Monographs. Cambridge Univ. Press, cambridge, 2006.

[21] A. Connes, *Classification of injective factors. Cases $II_1$, $II_\infty$, $III_\lambda$, $\lambda \neq 1$*, Ann. of Math. (2) **104** (1976), 73–115.

[22] A. Connes, *On the equivalence between injectivity and semidiscreteness for operator algebras, in "Algèbres d'opérateurs et leurs application en physique théorique"*, Edition CNRS (1979), 107–112.

[23] J. Christensen, K. Thomsen, *Finite digraphs and KMS states*, J. Math. Anal. Appl. **433** (2016), 1626–1646.

[24] K. R. Davidson, *$C^*$-Algebras by Example*, Fields Institute monographs. American Mathematical Soc., 1996.

[25] W. Dicks, T. Schick, *The Spectral Measure of Certain Elements of the Complex Group Ring of a Wreath Product*, Geometriae Dedicata **93** (2002), 121–137.

[26] J. Dixmier, *$C^*$-algebras*, North-Holland mathematical library, Volume 15. North-Holland, 1982.

[27] N. Dunford, J. T. Schwartz, *Linear Operators, Part I.* Interscience, New York, 1958.

[28] G. Elek, *Connes embeddings and von Neumann regular closures of amenable group algebras*, Trans. Amer. Math. Soc. **365** (2013), 3019–3039.

[29] G. Elek, *Lamplighter groups and von Neumann continuous regular rings*, Proc. Amer. Math. Soc. **144** (2016), 2871–2883.

[30] G. Elek, *Infinite dimensional representations of finite dimensional algebras and amenablility*, Math. Annalen **369** (2017), 397–439.

[31] M. Enomoto, M. Fujii, Y. Watatani, *KMS states for gauge action on $O_A$*, Math. Japon. **29** (1984), 607–619.

[32] R. Exel, *Partial Dynamical Systems, Fell Bundles and Applications.* Mathematical Surveys and Monographs, Volume 224. American Mathematical Society, 2017.

[33] C. Farsi, E. Gillaspy, S. Kang, J. A. Packer, *Separable representations, KMS states, and wavelets for higher-rank graphs*, J. Math. Anal. Appl. **434** (2016), 241–270.

[34] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, Pure and Applied Mathematics (New York). Wiley Interscience, New York, 1984.

[35] G. B. Folland, *A Course in Abstract Harmonic Analysis.* Textbooks in Mathematics, Volume 29. CRC Press, 2016.

[36] N. J. Fowler, I. Raeburn, *The Toeplitz Algebra of a Hilbert Bimodule*, Indiana Univ. Math. J. **48** (1999), 155–181.

[37] T. Giordano, I. F. Putnam, C. F. Skau, *Topological orbit equivalence and $C^*$-crossed products*, J. Reine Angew. Math. **469** (1995), 51–111.

[38] K. R. Goodearl, *Centers of regular self-injective rings*, Pacific J. Math. **76** (1978), 381–395.

[39] K. R. Goodearl, *Von Neumann Regular Rings.* Pitman, London, 1979; Second Edition, Krieger, Malabar, Fl., 1991.

[40] L. Grabowski, *On Turing dynamical systems and the Atiyah problem*, Invent. Math. **198** (2014), 27–69.

[41] L. Grabowski, *Irrational $l^2$-invariants arising from the lamplighter group*, Groups Geom. Dyn. **10** (2016), 795–817.

[42] R. I. Grigorchuk, A. Żuk, *The Lamplighter Group as a Group Generated by a 2-state Automaton, and its Spectrum*, Geometriae Dedicata **87** (2001), 209–244.

[43] R. I. Grigorchuk, P. A. Linnell, T. Schick, A. Żuk, *On a conjecture of Atiyah*, C.R. Acad. Sci. Paris **331** (2000), 663-668.

[44] I. Halperin, *Von Neumann's manuscript on inductive limits of regular rings*, Canad. J. Math. **20** (1968), 477–483.

[45] D. Handelman, *Completions of rank rings*, Canad. Math. Bull. **20** (1977), 199–205.

[46] D. Handelman, *Coordinatization applied to finite Baer *-rings*, Trans. Amer. Math. Soc. **235** (1978), 1–34.

[47] D. Handelman, *Rings with involution as partially ordered abelian groups*, Rocky Mountain J. Math. **11** (1981), 337–381.

[48] R. H. Herman, I. F. Putnam, C. F. Skau, *Ordered Bratteli diagrams, dimension groups and topological dynamics*, Internat. J. Math. **3** (1992), 827–864.

[49] A. an Huef, M. Laca, I. Raeburn, A. Sims, *KMS States on the $C^*$-Algebras of Finite Graphs*, J. Math. Anal. Appl. **405** (2013), 388–399.

[50] A. an Huef, M. Laca, I. Raeburn and A. Sims, *KMS states on the $C^*$-algebra of a higher-rank graph and periodicity in the path space*, J. Funct. Anal. **268** (2015), 1840–1875.

[51] M. Ionescu, A. Kumjian, *Hausdorff Measures and KMS States*, Indiana Univ. Math. J. **62** (2013), 443-463.

[52] A. Jaikin-Zapirain, *$L^2$-Betti numbers and their analogues in positive characteristic*, to appear in Proceedings of St. Andrews 2017.

[53] A. Jaikin-Zapirain, *The base change in the Atiyah and the Lück approximation conjectures*, preprint 2018 (http://verso.mat.uam.es/ andrei.jaikin/preprints/sac.pdf).

[54] A. Jaikin-Zapirain, D. López-Álvarez, *On the space of Sylvester matrix rank functions*, in preparation.

[55] R. V. Kadison, J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras.* Volume *I Elementary Theory*, Academic Press Inc., London, 1983.

[56] R. V. Kadison, J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras.* Volume *II Advanced Theory*, Academic Press Inc., London, 1986.

[57] E. T. A. Kakariadis, *KMS states on Pimsner algebras associated with $C^*$-dynamical systems*, J. Funct. Anal. **269** (2015), 325–354.

[58] I. Kaplansky, *Projections in Banach Algebras*, Ann. of Math. (2) **53** (1951), 235–249.

[59] A. S. Kechris, B.D. Miller, *Topics in orbit equivalence.* Lecture Notes in Mathematics, 1852. Springer-Verlag, Berlin, 2004.

[60] K. Keimel, *Algèbres commutatives engendrées par leurs éléments idempotents*, Canad. J. Math. **22** (1970), 1071–1078.

[61] D. Kerr, H. Li, *Ergodic theory. Independence and dichotomies.* Springer Monographs in Mathematics, Springer, Cham 2016.

[62] M. Laca, I. Raeburn, J. Ramagge, M. F. Whittaker, *Equilibrium States on the Cuntz–Pimsner Algebras of Self-Similar Actions*, J. Funct. Anal. **266** (2014), 6619–6661.

[63] M. Laca, I. Raeburn, J. Ramagge, M. F. Whittaker, *Equilibrium states on operator algebras associated to self-similar actions of groupoids on graphs*, Adv. Math. **331** (2018), 268–325.

[64] T. Y. Lam, *Lectures on Modules and Rings.* Graduate texts in mathematics, Volume 189. Springer-Verlag, New York-Berlin, 1999.

[65] F. Larsen, N. Laustsen, M. Rørdam, *An Introduction to $K$-Theory for $C^*$-Algebras*, London Math. Soc. Student Texts. Cambridge Univ. Press, Cambridge, 2000.

[66] P. A. Linnell, *Zero Divisors and Group von Neumann Algebras*, Pacific J. Math. **149** (1991), 349–363.

[67] P. A. Linnell, *Division Rings and Group von Neumann Algebras*, Forum Math. **5** (1993), 561–576.

[68] P. A. Linnell, *Noncommutative localization in group rings*, London Math. Soc. Lecture Note Ser., 330. Cambridge Univ. Press, Cambridge, 2006.

[69] P. A. Linnell, *Embedding Group Algebras into Finite von Neumann Regular Rings*, Trends Math. *Modules and Comodules* (2008), 295–300.

[70] P. A. Linnell, W. Lück, T. Schick, *The Ore Condition, Affiliated Operators, and the Lamplighter Group*, Proceedings of ICTP Trieste conference on High dimensional manifold topology (2001), 315–321.

[71] P. A. Linnell, T. Schick, *Finite Group Extensions and the Atiyah Conjecture*, J. Amer. Math. Soc. **20** (2007), 1003–1051.

[72] P. A. Linnell, T. Schick, *The Atiyah Conjecture and Artinian Rings*, Pure and Appl. Math. **8** (2012), 313–327.

[73] W. Lück, *$L^2$-Invariants: Theory and Applications to Geometry and $K$-Theory.* A Series of Modern Surveys in Matematics, Volume 44. Springer-Verlag, Berlin, 2002.

[74] W. Lück, P. A. Linnell, *Localization, Whitehead groups and the Atiyah conjecture*, Annals of $K$-Theory **3** (2018), 33-53.

[75] K. Menger, *New foundations of projective and affine geometry*, Ann. of Math. **37** (1936), 456–482

[76] G. J. Murphy, *$C^*$-Algebras and Operator Theory.* Academic Press Inc., Boston, MA, 1990.

[77] F. J. Murray, J. von Neumann, *On rings of operators*, Ann. of Math. **37** (1936), 116–229.

[78] F. J. Murray, J. von Neumann, *On rings of operators. II*, Trans. Amer. Math. Soc. **41** (1937), 208–248.

[79] F. J. Murray, J. von Neumann, *On rings of operators. III*, Ann. of Math. **41** (1940), 94–161.

[80] F. J. Murray, J. von Neumann, *On rings of operators. IV*, Ann. of Math. **44** (1943), 716–808.

[81] V. Nekrashevych, *Self-similar groups.* Mathematical Surveys and Monographs, Volume 12. Illustrated Edition, AMS, 2005.

[82] J. von Neumann, *Examples of Continuous Geometry*, Proc. Natl. Acad. Sci. USA **22** (1936), 101–108.

[83] J. von Neumann, *Continuous geometry.* Princeton Landmarks in Mathematics. Princeton University Press, New Jersey, 1988.

[84] G. K. Pedersen, *$C^*$-Algebras and Their Automorphism Groups.* LMS Monographs, Volume 14. Academic Press, London, 1979.

[85] N. C. Phillips, *An introduction to crossed product $C^*$-algebras and minimal dynamics*, notes 2017 (http://pages.uoregon.edu/ncp/Courses/CRMCrPrdMinDyn/Notes_20170205.pdf).

[86] M. Pichot, T. Schick, A. Żuk, *Closed manifolds with transcendental $L^2$-Betti numbers*, J. London Math. Soc. **92** (2015), 371–392.

[87] I. F. Putnam, *The $C^*$-algebras associated with minimal homeomorphisms of the Cantor set*, Pacific J. Math. **136** (1989), 329–353.

[88] I. F. Putnam, *On the topological stable rank of certain transformation group $C^*$-algebras*, Ergodic Theory Dynam. Systems **10** (1990), 197–207.

[89] H. Reich, *Group von Neumann Algebras and Related Algebras*, Ph.D. Thesis, Göttingen 1998.

[90] M. Rørdam, A. Sierakowski, *Purely infinite $C^*$-algebras arising from crossed products*, Erg. Theor. Dynam. Syst. **32** (2012), 273–293.

[91] W. Rudin, *Real and complex analysis*, New York: McGraw-Hill, 1987.

[92] A. H. Schofield, *Representation of rings over skew fields*. London Math. Soc. Lecture Note Ser., 92. Cambridge Univ. Press, Cambridge, 1985.

[93] E. Seneta, *Non-negative Matrices and Markov Chains*. Second Edition, Springer-Verlag, New York, 1981.

[94] A. M. Sinclair, R. R. Smith, *Finite von Neumann algebras and masas*. London Math. Soc. Lecture Note Ser., 351. Cambridge Univ. Press, Cambridge, 2008.

[95] T. aki Tanaka, *Transcendence of the values of certain series with Hadamard's gaps*, Arch. Math. (Basel) **78** (2002), 202–209.

[96] K. Thomsen, *KMS weights on graph $C^*$-algebras*, Adv. Math. **309** (2017), 334–391.

[97] Y. Utumi, *On Continuous Rings and Self Injective Rings*, Trans. Amer. Math. Soc. **118** (1965), 158–173.