Universitat Autònoma de Barcelona

ASYMPTOTIC EXPANSIONS, RESURGENCE AND LARGE ORDER BEHAVIOUR OF QUANTUM CHROMODYNAMICS

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A DISSERTATION WRITTEN IN PARTIAL FULFILMENT OF THE DOCTORAL DEGREE IN PHYSICS

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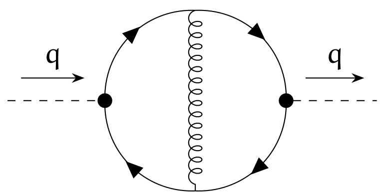

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Asymptotic expansions, resurgence and large order behaviour of quantum chromodynamics

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I, **Matthias Rudolf Jamin**, certify that the dissertation with title Asymptotic expansions, resurgence and large order behaviour of quantum chromodynamics was written by **Ramon Miravitllas Mas** under my supervision in partial fulfilment of the **Doctoral degree in Physics**.

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I am also grateful to Santiago Peris. The second part of this dissertation came to fruition thanks to his constant demand for improvement and his inspiration to keep trying. In retrospect, that was a lot of work!

I think both of them shaped me into the researcher I am today and I want to thank them again for also caring about my future in physics research.

Abstract

For realistic quantum field theories, numerical predictions of physical observables can only be calculated from perturbative expansions in powers of the couplings, the parameters that determine the strength of the field interactions. While the predictive success of quantum field theory is undeniable, these perturbative computations are plagued with divergences.

On one hand, the coefficients of the perturbative expansion are computed from loop integrals that are divergent most of the times. Some of these divergences are associated with unphysical terms that can be subtracted. In other cases, a renormalisation procedure is applied to cancel these divergences, but this entails a choice of theoretical conventions (scale and scheme) which physical observables cannot depend on.

On the other hand, once the loop integrals have been renormalised, the resulting expansion still sums to an infinite answer for all non-vanishing values of the coupling. This is due to the fact that the coefficients of the expansion grow factorially with the order. Still, these expansions can be understood as asymptotic expansions, which encode the limiting behaviour of the observable for small coupling, and whose truncation to an optimal term yields numerical approximations of the observable. This second kind of divergence is in fact not limited to quantum field theories, but it may arise in different contexts of mathematics and physics: for instance, in perturbative approximations to the energy eigenvalues of a quantum mechanic system, or in formal solutions to differential equations.

In Part I of this dissertation, the main object of study is the strong coupling constant and the perturbative expansions of physical observables in quantum chromodynamics. First, we briefly discuss how the loop divergence of the quantum corrected gluon propagator can be absorbed inside the strong coupling constant during the renormalisation. This process, however, comes at the cost of introducing scale and scheme dependences into the coupling, therefore it is not a physical observable of the theory. This motivates a coupling redefinition whose scheme dependence is reduced to a single parameter. We then use this coupling redefinition in phenomenological analysis of physical observables associated to electron-positron scattering, and to Higgs and tau decays into hadrons. We demonstrate that appropriate choices of this scheme parameter can lead to substantial improvements in perturbative predictions of these observables.

In Part II, we discuss the divergence of asymptotic expansions in the context of path integrals. Conventionally, the method of Borel summation assigns a finite answer to the divergent expansion. Still, the Borel sum might not encode the full information of a function, because it misses exponentially small corrections. We then consider a

slight variation of the conventional Borel summation, in which a generalised Borel transform (an inverse Laplace transform) is followed by a directional Laplace transform. These tools allow us to give perhaps better answers to typical problems in Borel summation: missing exponential corrections and ambiguities in the Borel summation. In addition, we define resurgence as a connection between the discontinuity of a function and the coefficients of its asymptotic expansion. From this definition, we can reduce resurgence to the problem of missing exponential corrections in asymptotic expansions and correlate different approaches to resurgence found in the literature.

Contents

Part I

Large order behaviour of QCD

Chapter 1

Scheme variations of the QCD strong coupling

Perturbation theory in the strong coupling α_s is one of the central approaches to predictions in quantum chromodynamics (QCD). Because of confinement, however, α_s is not a physical observable: its definition inherently depends on theoretical conventions such as renormalisation scale and renormalisation scheme. Obviously, physical quantities—which can be measured in the laboratory, either directly or indirectly—should not depend on such choices. Regarding the renormalisation scale, this condition allows to derive the so-called renormalisation group equations (RGE), which have to be satisfied by all physical quantities. For the renormalisation scheme, the situation is more complicated, because order by order the strong coupling can be redefined. For that reason, perturbative computations are performed mainly in convenient schemes, like minimal subtraction (MS) [1] or modified minimal subtraction ($\overline{\text{MS}}$) [2].

The aim of this chapter is to introduce a redefinition of the strong coupling, which we will denote $\widehat{\alpha}_s$, satisfying two properties. First, the scale running of the coupling, described by the β function, is explicitly scheme invariant. Second, the scheme dependence of the coupling can be parameterised by a single parameter C. Hence, in the following, we shall refer to this coupling as the C-scheme coupling. As we will see, variations of C will directly correspond to transformations of the QCD scale invariant parameter Λ.

In Section 1.1, we will first illustrate the renormalisation of the fermion bubble at one loop and how this renormalisation introduces scale and scheme dependences into the α_s coupling. This will motivate the introduction of the C-scheme coupling in Section 1.2. We will then discuss the properties of this coupling redefinition: its scale and scheme evolution, and its perturbative relation to α_s. In Section 1.3, we will briefly

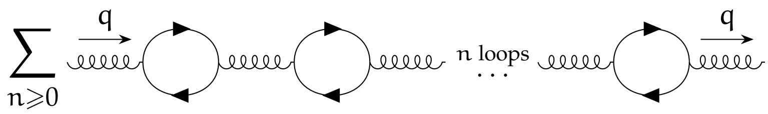


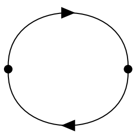
Figure 1.1: Gluon chain.[2]

describe correlators of a quantum field theory in the diagrammatic approach. In particular, we will define the vector correlator of QCD and we will illustrate that this quantity has scale and scheme dependences. Only after taking one derivative with respect to the energy transfer of the correlator, we obtain a physical quantity independent of these artificial parameters. In Section 1.4, we will explain how the C-scheme coupling can be exploited in order to improve perturbative predictions in QCD and we will illustrate this point in the context of the Adler function. Finally, in Section 1.5, we will review the so-called large-β_0 approximation.

1.1 Renormalisation of the gluon chain

In this section, we will briefly introduce scheme and scale dependences of α_s at first order in perturbation theory. For that, let us consider the gluon chain in Figure 1.1, which is the gluon propagator with quantum corrections given by an infinite number of fermion bubbles. First, we consider a single fermion bubble, which is easily obtained from Feynman rules:[1]

$$(\Pi_0)^{ab}_{\mu\nu}(q) =$$



$$= i(g_0\mu^{\epsilon})^2\, N_f T_F \delta^{ab} \int \frac{d^dp}{(2\pi)^d} \frac{\mathrm{Tr}\left[(\not{p}-\not{q})\gamma_\mu \not{p}\gamma_\nu\right]}{(p-q)^2p^2}\,. \tag{1.1}$$

N_f is the number of fermions that contribute to the bubble, a, b are the colour indices of the propagating gluon and $T_F = 1/2$ is a color factor that arises from the trace of the Gell-mann matrices $\mathrm{Tr}\left[\frac{\lambda^a}{2}\frac{\lambda^b}{2}\right] = T_F\delta^{ab}$. The subscripts 0 in Π_0 and the gauge coupling g_0 indicate that these are unrenormalised quantities. The loop integral has already been promoted to $d = 4+2\epsilon$ dimensions in preparation for dimension regularisation. Finally, the renormalisation scale μ has been incorporated so that the energy dimension of the fermion bubble is kept constant with d.

A simple way to compute Eq. 1.1 is to use its known tensor structure

$$(\Pi_0)^{ab}_{\mu\nu}(q) = \delta^{ab}\left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi_0\left(q^2\right), \tag{1.2}$$

[1] In our conventions, we use the $(+,-,-,-)$ metric signature.

[2] Feynman diagrams displayed thorough this work are possible thanks to the LaTeX package TikZ-Feynman [3].

so it remains to compute the scalar factor $\Pi_0(q^2)$ of the fermion bubble. We find

$$\begin{aligned}\Pi_0(q^2) &= -\frac{(\Pi_0)^{\mu}_{\mu}(q^2)}{(d-1)q^2} = -i(g_0\mu^{\epsilon})^2\frac{N_f T_F}{(d-1)q^2}\int\frac{d^d p}{(2\pi)^d}\frac{\mathrm{Tr}\left[(\not p-\not q)\gamma_\mu \not p\gamma^\mu\right]}{(p-q)^2p^2} \\ &= a_0\,2N_f T_F\left(\frac{-q^2}{4\pi\mu^2}\right)^{\epsilon}\frac{\Gamma(2+\epsilon)^2\Gamma(-\epsilon)}{\Gamma(4+2\epsilon)},\end{aligned} \tag{1.3}$$

where we have defined the unrenormalised strong coupling constant[3]

$$a_0 = \frac{(g_0\mu^{\epsilon})^2}{4\pi^2}. \tag{1.4}$$

Now, by linking $n+1$ free gluon propagators with n fermion bubbles, we obtain

$$\frac{i\delta^{ab}}{q^4}\left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\left[-\Pi_0(q^2)\right]^n, \tag{1.5}$$

so, by summing up from $n=0$ to $n=\infty$ fermion bubbles, the gluon chain is given by

$$\frac{-i\delta^{ab}}{q^2}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right)\frac{1}{1+\Pi_0(q^2)} - i\delta^{ab}\xi\frac{q_\mu q_\nu}{q^4}, \tag{1.6}$$

where ξ is the gauge-fixing parameter.

In the background field method [4], the quantity $a_0/(1+\Pi_0)$ becomes renormalised with only the renormalisation constant Z_a of the coupling, given by $a_0 = Z_a a$. The quantity

$$\frac{a_0}{1+\Pi_0} = \frac{Z^a a}{1+Z^a a\,2N_f T_F\left(\frac{-q^2}{4\pi\mu^2}\right)^{\epsilon}\frac{\Gamma(2+\epsilon)^2\Gamma(-\epsilon)}{\Gamma(4+2\epsilon)}}, \tag{1.7}$$

is finite in the limit $\epsilon \to 0$ if we make the following choice for the renormalisation constant:

$$Z_a = \frac{1}{1+\frac{N_f T_F}{3}\left(\frac{1}{\epsilon}-c\right)a}, \tag{1.8}$$

where c is any desired constant. In this case, we have

$$\lim_{\epsilon\to 0}\frac{a_0}{1+\Pi_0} = \frac{a}{1+\Pi}, \tag{1.9}$$

where Π is the renormalised fermion bubble, given by

$$\Pi(q^2) = -a\frac{N_f T_F}{3}\log\left(-\frac{q^2}{\mu^2}e^{C}\right). \tag{1.10}$$

[3]Many times the gauge coupling g is expressed in terms of the parameter $\alpha_s = (g\mu^{\epsilon})^2/(4\pi)$ (with one less factor of π in the denominator). Here we prefer to use the parameter $a = \alpha_s/\pi$, as it simplifies many formulas, but we shall call both a and α_s the strong coupling constant.

The value $C = c - 5/3 + \gamma_E - \log(4\pi)$ parametrises the subtraction scheme of the renormalisation procedure. It is common to consider the $\overline{\text{MS}}$ (modified minimal subtraction scheme), in which $C = -5/3$, but we prefer to not fix this parameter for the upcoming discussions.

Finally, we can compute the fermionic contribution to the first coefficient β_1 of the β function, defined by

$$\beta(a) = -\mu \frac{da(\mu)}{d\mu} = \sum_{n\geqslant 1} \beta_n a^{n+1} \,. \tag{1.11}$$

After some algebra and using the renormalisation constant in Eq. 1.8, we obtain

$$\beta(a) \sim -\lim_{\epsilon\to 0} \frac{2\epsilon a}{Z_a} = \beta_{1f} a^2 \,, \tag{1.12}$$

with

$$\beta_{1f} = -\frac{2N_f T_F}{3} \,. \tag{1.13}$$

At first order in QCD, we still would be missing the contribution of the gluon and ghost loops to the quantum corrected gluon propagator. An analogous computation would yield

$$\beta_{1g} = \frac{11}{6} N_c \tag{1.14}$$

where N_c is the number of colors. Therefore, the first coefficient of the β function becomes

$$\beta_1 = \beta_{1g} + \beta_{1f} = \frac{11N_c - 2N_f}{6} \,, \tag{1.15}$$

which coincides with the result of Eq. A.3. The full set of β coefficients, up to order a^5, is given in Appendix A.1. We emphasise that Eq. 1.15 is independent of the choice of subtraction scheme C. The same is realised for β_2, but not for the coefficients of higher orders.

Now, by solving the differential equation in Eq. 1.11 at first order, we obtain

$$\frac{1}{a(\mu)} = \frac{1}{a(\mu_0)} + \beta_1 \log\left(\frac{\mu}{\mu_0}\right) . \tag{1.16}$$

which is the well-known 1-loop running of the coupling. It is interesting that the combination $a/(1+\Pi)$ is μ independent, because the μ dependence from the renormalised fermion bubble, in Eq. 1.10, cancels with the μ dependence of the running coupling a. We also realise, from Eq. 1.10, that changes in μ can be compensated by changes in the scheme C. Replacing μ by $e^{-C/2}$ and μ_0 by 1 in Eq. 1.16, we obtain the scheme dependence

$$\frac{1}{a(\mu, C)} = \frac{1}{a(\mu, 0)} - \frac{\beta_1}{2} C \,, \tag{1.17}$$

and, by construction, the combination $a/(1+\Pi)$ is also scheme independent. Furthermore, this equation suggests the definition of a scheme invariant coupling

$$\frac{1}{\widehat{a}(\mu)} = \frac{1}{a(\mu, C)} + \frac{\beta_1}{2}C\,, \tag{1.18}$$

where $\widehat{a}(\mu) = a(\mu, 0)$ is just the coupling at the scheme $C = 0$.

1.2 The C-scheme coupling

In full QCD, the construction of a scheme invariant coupling as in Eq. 1.18 does not appear to be possible, at least in a universal sense, independent of any observable. Nonetheless, our aim in this section will be to provide the definition of a QCD coupling, which we also term $\widehat{a}$, whose scheme can be parametrised by a single parameter C. The running of $\widehat{a}$ will be described by a simple β function, only depending on the scheme invariant coefficients β_1 and β_2.

For that, we consider the scale invariant Λ parameter of QCD

$$\Lambda = \mu e^{-1/(\beta_1 a(\mu))} a(\mu)^{-\beta_2/\beta_1^2} \exp\left\{\int_0^{a(\mu)} \frac{da}{\widetilde{\beta}(a)}\right\}, \tag{1.19}$$

where $a(\mu)$ is the QCD coupling at the scale μ and

$$\frac{1}{\widetilde{\beta}(a)} = \frac{1}{\beta(a)} - \frac{1}{\beta_1 a^2} + \frac{\beta_2}{\beta_1^2 a} \tag{1.20}$$

is a combination free of singularities at $a = 0$.

Although Λ by definition is scale independent, it does depend on the scheme. In particular, if the coupling is a in one scheme and a' in another, so that both couplings are related by $a' = a + c_1 a^2 + \mathcal{O}(a^3)$, then the scheme transformation of Λ is given by [5]

$$\Lambda' = \Lambda e^{c_1/\beta_1}\,, \tag{1.21}$$

where Λ (Λ') is the Λ parameter in the a (a') coupling. The Λ parameter only changes with c_1 and is insensitive to the rest of the expansion coefficients.

The scheme dependence of Eq. 1.21 suggests to define a new coupling $\widehat{a}$ by the implicit relation

$$f(\widehat{a}) = \beta_1 \log\left(\frac{\mu}{\Lambda}\right) + \frac{\beta_1}{2}C\,, \tag{1.22}$$

where f is some function to be specified later. The right hand side of the equation has a very simple scheme transformation originating from Eq. 1.21.[4] Thus, the new coupling $\widehat{a}$ has the same property regardless of the choice of the function f.

Combining Eq. 1.19 and Eq. 1.22, we find

$$f(\widehat{a}) - \frac{\beta_1}{2}C = \frac{1}{a} + \frac{\beta_2}{\beta_1}\log(a) - \beta_1 \int_0^a \frac{da}{\widetilde{\beta}(a)}, \tag{1.23}$$

so we choose

$$f(\widehat{a}) = \frac{1}{\widehat{a}} + \frac{\beta_2}{\beta_1}\log(\widehat{a}) \tag{1.24}$$

in order to match both sides of Eq. 1.23. The coupling $\widehat{a}$ is then implicitly defined through the relation

$$\frac{1}{\widehat{a}} + \frac{\beta_2}{\beta_1}\log(\widehat{a}) = \beta_1 \log\left(\frac{\mu}{\Lambda}\right) + \frac{\beta_1}{2}C. \tag{1.25}$$

The choice of f in Eq. 1.24 is not arbitrary, but it is necessary so that the perturbative relation between a and $\widehat{a}$ remains a simple power expansion $\widehat{a} = a + \sum_{n\geqslant 1} c_n a^{n+1}$. It is in this sense that $\widehat{a}$ is a legitimate coupling redefinition. We call $\widehat{a}$ the C-scheme coupling, which was first introduced in [6].

It is interesting to note that $\widehat{a}$ becomes the scheme invariant coupling of Eq. 1.18 when we set $\beta_n = 0$ for $n \geqslant 2$. This is also a reason that motivated the definition of $\widehat{a}$ in full QCD.

The C-scheme coupling has simple properties regarding scale and C transformations. Differentiating Eq. 1.25 with respect to either μ or C, we find

$$-\mu\frac{d\widehat{a}}{d\mu} = \widehat{\beta}(\widehat{a}) = \frac{\beta_1\widehat{a}^2}{1-\beta_2\widehat{a}/\beta_1} = -2\frac{d\widehat{a}}{dC}. \tag{1.26}$$

So changes in the scheme C are completely equivalent to changes in the scale μ. A shift in the scale from μ_1 to μ_2 can be compensated by a shift in the scheme from C_1 to C_2, so that $\mu_1/\mu_2 = e^{(C_1-C_2)/2}$. In addition, because β_1 and β_2 are both scheme independent parameters, then the β function of $\widehat{a}$ is explicitly scheme independent as well. Finally, we comment that the $\widehat{a}$ coupling has a similar running to that of the 't Hooft coupling [7], for which $\beta(a) = \beta_1 a^2 + \beta_2 a^3$.

In Figure 1.2, we display the coupling $\widehat{a}$ as a function of C. As our initial $\overline{\text{MS}}$ input we employ $\alpha_s(M_\tau) = 0.316(10)$ which results from the current PDG average $\alpha_s(M_Z) = 0.1181(13)$ [8] after scale evolution. The yellow band corresponds to the variation within the α_s uncertainties. Below roughly $C = -2$, the relation between $\widehat{a}$ and the $\overline{\text{MS}}$ coupling ceases to be perturbative and breaks down.

[4]For convenience, we reparametrised this scheme transformation in terms of C instead of c_1.

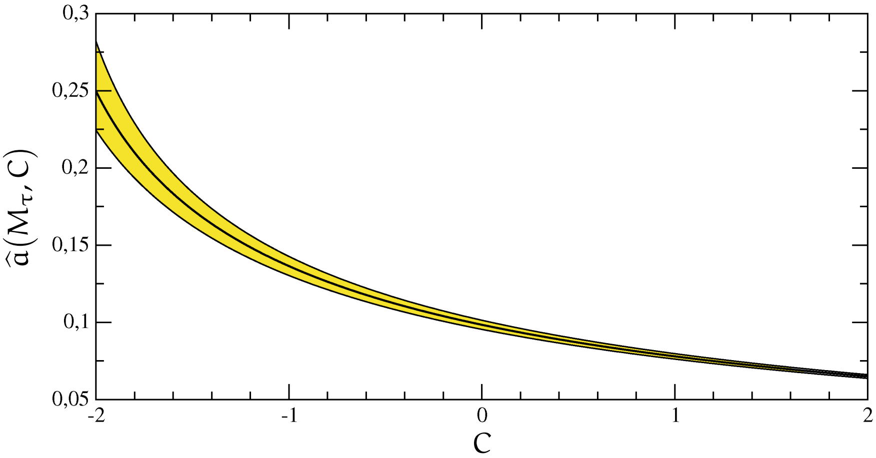


Figure 1.2: The coupling $\widehat{a}$ according to Eq. 1.25 as a function of C, and for the $\overline{\text{MS}}$ input value $\alpha_s(M_\tau) = 0.316(10)$. The yellow band corresponds to the α_s uncertainty.

We can transform from the a coupling to $\widehat{a}$ by numerically solving Eq. 1.22. This is how we obtained the plot of Figure 1.2. Another alternative, which we will employ for the phenomenological analysis of Section 1.4, is to compute the perturbative relation between a and $\widehat{a}$ in two steps. First, we transform into the coupling $\bar{a} = \widehat{a}(C = 0)$ via the relation

$$a = \bar{a} + \left(\frac{\beta_3}{\beta_1} - \frac{\beta_2^2}{\beta_1^2}\right)\bar{a}^3 + \left(\frac{\beta_4}{2\beta_1} - \frac{\beta_2^3}{2\beta_1^3}\right)\bar{a}^4 + \left(\frac{\beta_5}{3\beta_1} - \frac{\beta_2\beta_4}{6\beta_1^2} + \frac{5\beta_3^2}{3\beta_1^2} - \frac{3\beta_2^2\beta_3}{\beta_1^3} + \frac{7\beta_2^4}{6\beta_1^4}\right)\bar{a}^5 + \ldots \quad (1.27)$$

Then, in a second step, the C evolution of Eq. 1.26 can be employed to transform from $\bar{a}$ to the general C-scheme coupling:

$$\bar{a} = \widehat{a} + \frac{\beta_1}{2}C\widehat{a}^2 + \left(\frac{\beta_2}{2}C + \frac{\beta_1^2}{4}C^2\right)\widehat{a}^3 + \left(\frac{\beta_2^2}{2\beta_1}C + \frac{5\beta_1\beta_2}{8}C^2 + \frac{\beta_1^3}{8}C^3\right)\widehat{a}^4 + \left(\frac{\beta_2^3}{2\beta_1^2}C + \frac{9\beta_2^2}{8}C^2 + \frac{13\beta_1^2\beta_2}{24}C^3 + \frac{\beta_1^4}{16}C^4\right)\widehat{a}^5 + \ldots \quad (1.28)$$

To finish this section, we point out to the possibility of defining a fully scheme invariant coupling in full QCD. Since the QCD coupling is not directly measurable, such a definition would have to be based on a particular physical observable, for example the QCD Adler function[5]. In the past, such definitions have been discussed in the literature (see, for instance, [9, 10]). However, then the definition of the coupling is non-universal and its Λ parameter and β function depend on the perturbative expansion coefficients of the physical quantity. For this reason, in this work we prefer to stick to the universal coupling $\widehat{a}$ according to the definition of Eq. 1.25.

[5]We will introduce this physical quantity in Section 1.3.

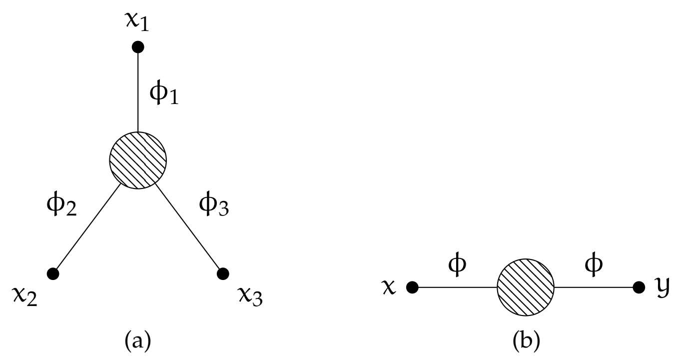


Figure 1.3: n-point correlation functions. The blob denotes the sum of all possible diagrams that are linked to the external vertices.

1.3 The vector correlator and the Adler function

Physical quantities in a quantum field theory are typically expressed in terms of n-point correlation functions, defined by the path integral

$$\langle\Omega|\,T\{\phi_1(x_1)\phi_2(y_2)\cdots\}\,|\Omega\rangle = \frac{1}{Z}\int [d\phi]\; e^{iS[\phi]}\phi_1(x_1)\phi_2(x_2)\cdots \tag{1.29}$$

where ϕ_n are fields, S is the action of the theory and Z is the partition function. Correlation functions have a simple interpretation in terms of Feynman diagrams: they corresponds to the set of connected diagrams with external vertices x_1, x_2, ..., each linked to the diagram with a field line ϕ_1, ϕ_2, ..., respectively (see Figure 1.3a). In particular, the two-point correlator $\langle\Omega|\,T\{\phi(x)\phi(y)\}\,|\Omega\rangle$ is the propagator of a ϕ field with quantum corrections (see Figure 1.3b).

Most of the times it is convenient to rewrite correlators in momentum space by applying Fourier transformations on the space-time variables. In particular, for two-point correlators, we have

$$\Pi(q) = i\int d^4x\, e^{iqx}\,\langle\Omega|\,T\{\phi_1(x)\phi_2(0)\}\,|\Omega\rangle\ . \tag{1.30}$$

In the following, let us consider the vector correlator of QCD, defined by

$$\Pi_V^{\mu\nu}(q) = i\int d^4x\, e^{iqx}\,\langle\Omega|\,T\left\{J^\mu(x)\,J^\nu(0)^\dagger\right\}|\Omega\rangle\ , \tag{1.31}$$

where $J_\mu(x) = {:}\overline{\psi}_a(x)\gamma_\mu\psi_b(x){:}$ is the vector current and ψ_a is a quark field with flavour a. This correlator is specially important in QCD, as it emerges in calculations of the total cross section of e^+e^- scattering into hadrons, and it also governs theoretical predictions of the inclusive decay rate of τ leptons into hadronic final states [11]. Diagrammatically, the vector correlator is given by Figure 1.4.

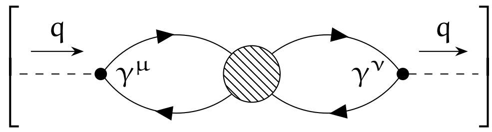
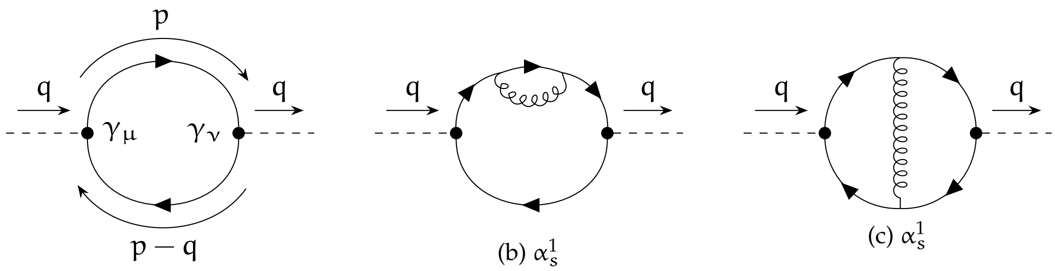
$\Pi_V^{\mu\nu}(q) =$ 

Figure 1.4: Vector correlator in QCD.



(a) α_s^0 (b) α_s^1 (c) α_s^1

Figure 1.5: Leading order contributions in QCD to the vector correlator.

Here we will only consider the massless limit. Since the vector current is conserved in this limit (that is, $\partial^\mu J_\mu(x) = 0$) then $q_\mu \Pi_V^{\mu\nu}(q^2) = 0$, and we can derive the following tensor structure:

$$\Pi_V^{\mu\nu}(q) = \left(q^\mu q^\nu - g^{\mu\nu} q^2\right)\Pi_V(q^2)\,, \tag{1.32}$$

which is valid at all orders in perturbation theory.

At zeroth order in QCD (see Figure 1.5a) and in the massless limit, we can just repeat the same computation in Eq. 1.3. After expanding around $\epsilon = 0$, we obtain

$$\Pi_V(q^2) = \frac{N_c}{12\pi^2}\left[\frac{1}{\epsilon} - \log\left(-\frac{q^2}{\mu^2}\right) - \gamma_E + \log(4\pi) + \frac{5}{3} + \mathcal{O}(\epsilon)\right] + \mathcal{O}(a). \tag{1.33}$$

(We use the same notation introduced in Section 1.1 for the dimensional regularisation). This result clearly depends on the renormalisation scale μ and the subtraction scheme, but we can construct a physical quantity by taking one derivative with respect to $s = q^2$:

$$D(s) = -s\frac{d\Pi_V(s)}{ds}\,, \tag{1.34}$$

which we call the Adler function [12].

The perturbative expansion for the vector correlator in the massless limit is given by

$$\Pi_V^{PT}(s) = -\frac{N_c}{12\pi^2}\sum_{n\geqslant 0} a(\mu)^n \sum_{k=0}^{n+1} c_{nk}\log^k\left(-\frac{s}{\mu^2}\right), \tag{1.35}$$

where the coupling $a(\mu)$ is evaluated at the scale μ. Thus, after derivation with respect to s, the perturbative expansion of the Adler function becomes

$$D_{\mathrm{PT}}(s) = \frac{N_c}{12\pi^2} \sum_{n \geqslant 0} a(\mu)^n \sum_{k=1}^{n+1} k\, c_{nk} \log^{k-1}\left(-\frac{s}{\mu^2}\right). \tag{1.36}$$

As a physical quantity, the Adler function is μ invariant and thus we are free to choose any renormalisation scale. In particular, it is convenient to set $\mu^2 = -s = Q^2 > 0$, which resums the logarithms in Eq. 1.36, yielding the simple expression

$$D_{\mathrm{PT}}(s) = \frac{N_c}{12\pi^2} \sum_{n \geqslant 0} c_{n1} a(Q)^n . \tag{1.37}$$

From our computation at zeroth order in Eq. 1.33, it is straightforward to obtain $c_{n1} = 1$. The front factor $-N_c/(12\pi^2)$ in Eq. 1.35 has been chosen so as to normalise this coefficient to 1.

At first order in QCD, the contribution to Π_V is given by the diagrams in Figure 1.5b and Figure 1.5c. In this case, after a long computation, we obtain

$$\Pi_V(s) = -\frac{N_c}{12\pi^2}\left[1 + a + \mathcal{O}(a^2)\right] \log\left(\frac{-s}{\mu^2}\right) + \text{constants}. \tag{1.38}$$

So, we read the coefficient $c_{11} = 1$. The coefficients c_{n1} are known analytically up to order α_s^4 [13]. For $N_c = N_f = 3$, they are found to be:

$$c_{01} = c_{11} = 1\,, \tag{1.39}$$

$$c_{21} = \frac{299}{24} - 9\zeta_3\,, \tag{1.40}$$

$$c_{31} = \frac{58057}{288} - \frac{779}{4}\zeta_3 + \frac{75}{2}\zeta_5\,, \tag{1.41}$$

$$c_{41} = \frac{78631453}{20736} - \frac{1704247}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7\,, \tag{1.42}$$

where $\zeta_n = \zeta(n)$ is the Riemann ζ function.

1.4 Phenomenological applications of the C-scheme coupling

We now proceed to apply the $\widehat{a}$ coupling introduced in Section 1.2 to concrete cases. Having at our disposal a parameter to investigate scheme variations, we show that appropriate choices of C can lead to substantial improvements in the perturbative predictions of physical quantities.

As our first application, we investigate the perturbative series of the Adler function introduced in Eq. 1.34. To this end, it is convenient to define the reduced Adler function

$\widehat{D}$ as

$$\widehat{D}(s) = \frac{12\pi^2}{N_c} D(s) - 1\,. \tag{1.43}$$

A perturbative expansion of this quantity is given by (with the scale choice $\mu = Q^2$)

$$\widehat{D}(s) = a(Q) + 1.640 a(Q)^2 + 6.371 a(Q)^3 + 49.08 a(Q)^4 + \mathcal{O}\big(a(Q)^5\big)\,, \tag{1.44}$$

where we have used the coefficients of Eq. 1.39–1.42.

Using the perturbative expressions of $\widehat{a}$ in Eq. 1.27 and Eq. 1.28, we rewrite the expansion in Eq. 1.44 in terms of the C-scheme coupling, resulting in

$$\begin{aligned}\widehat{D}(s) = \widehat{a}(Q) &+ (1.640 + 2.25C)\widehat{a}(Q)^2 + (7.682 + 11.38C + 5.063C^2)\widehat{a}(Q)^3 \\ &+ (61.06 + 72.08C + 47.40C^2 + 11.39C^3)\widehat{a}(Q)^4 + \mathcal{O}\big(\widehat{a}(Q)^5\big)\,. \end{aligned}\tag{1.45}$$

Because the coupling α_s does not depend on C, then the physical quantity $\widehat{D}$ is also independent of the scheme parameter. However, since we will approximate Eq. 1.45 with its truncated perturbative series, some residual C dependence will arise. Our aim will be to improve these perturbative predictions by looking for optimal values of C. As we will see, this optimal C is defined so that the contribution from $\mathcal{O}\big(a^5\big)$ becomes minimal and, thus, in the spirit of asymptotic expansions, the optimal truncation of the perturbative series, which is given by the minimal term, is attained at this order.[6]

A graphical representation of Eq. 1.45 is provided in Figure 1.6, where the truncated perturbative expansion of $\widehat{D}$ is plotted as a function of C, at the energy $s = -M_\tau^2$. The yellow band corresponds to an error estimate from the fifth-order contribution. The required coefficient has been taken to be $c_{51} = 283$, as estimated in Ref. [14]. The yellow band then arises by either removing or doubling the $\mathcal{O}(\widehat{a}^5)$ term.

Generally, it is observed that around $C \approx -1$, a region of stability with respect to the C-variation emerges. For comparison, the blue line corresponds to using $c_{51} = 566$ and still doubling the $\mathcal{O}\big(\widehat{a}^5\big)$ correction. Then, no region of stability is found which seems to indicate that such large values of c_{51} are disfavoured. In the red dot, where $C = -0.783$, the $\mathcal{O}\big(\widehat{a}^5\big)$ vanishes, and the $\mathcal{O}\big(\widehat{a}^4\big)$ correction, which is the last included non-vanishing term, has been employed as a conservative uncertainty, in the spirit of asymptotic expansions. Numerically, we find

$$\widehat{D}_{\rm PT}\big(-M_\tau^2, C = -0.783\big) = 0.1343 \pm 0.0070 \pm 0.0067\,, \tag{1.46}$$

where the second error originates from the uncertainty in $\alpha_s(M_\tau)$. The result of

[6]Strictly speaking, this is only true if the asymptotic series has purely alternating components (or purely non-alternating), but not when the two types are mixed, as this opens the possibility of a fortuitous cancellation between the two components.

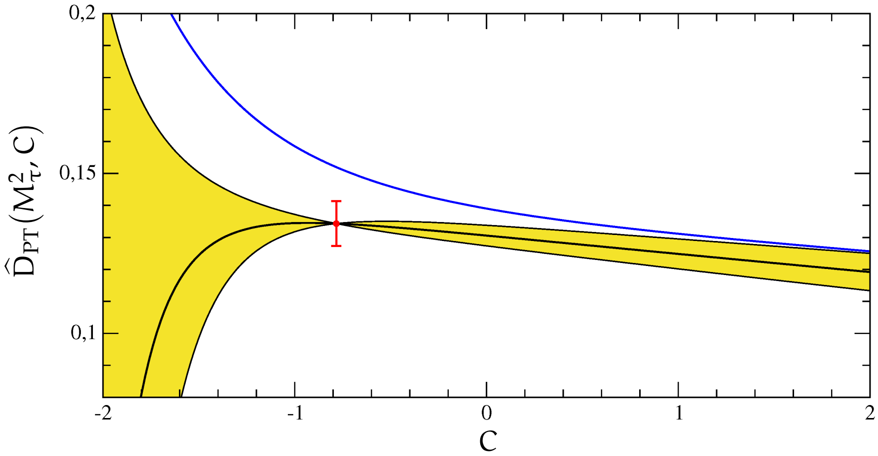


Figure 1.6: $\widehat{D}$ of Eq. 1.45 as a function of C. The yellow band arises from either removing or doubling the fifth-order term. In the red dot, the $\mathcal{O}(\widehat{a}^5)$ vanishes, and $\mathcal{O}(\widehat{a}^4)$ is taken as the uncertainty.

Eq. 1.46 may be compared to the direct $\overline{\mathrm{MS}}$ prediction of Eq. 1.44, which reads

$$\widehat{D}_{PT}(-M_\tau^2) = 0.1316 \pm 0.0029 \pm 0.0060\,. \tag{1.47}$$

Here, the first error is obtained by removing or doubling c_{51}, and the second error again corresponds to the α_s uncertainty.

We may still perform a final comparison of Eq. 1.46 and Eq. 1.47 with the Borel model of the Adler function that was put forward in Ref. [14], and which is based on general knowledge of the renormalon structure for the Borel transform of $\widehat{D}$. Within this model, one obtains

$$\widehat{D}_{PT}(-M_\tau^2) = 0.1354 \pm 0.0127 \pm 0.0058\,. \tag{1.48}$$

In this case, the first uncertainty results from estimates of the perturbative ambiguity that arises from the renormalon singularities. It is seen that this uncertainty is much bigger than the one of Eq. 1.47 and still larger than the one of Eq. 1.46. Therefore, we conclude that the higher-order uncertainty of Eq. 1.47 appears to be underestimated, while Eq. 1.46 seems to provide a more realistic account of the resummed series. Interestingly enough, also its central value is closer to the Borel model result.

Now, we turn to the perturbative expansion for the total τ hadronic width. The central observable is the ratio

$$R_\tau = \frac{\Gamma(\tau^- \to \text{hadrons} + \nu_\tau)}{\Gamma(\tau^- \to e^- + \bar{\nu}_e + \nu_\tau)}\,. \tag{1.49}$$

It can be parameterised as

$$R_\tau = 3\,S_{EW}\big(|V_{ud}|^2 + |V_{us}|^2\big)\big(1 + \delta^{(0)} + \ldots\big)\,, \tag{1.50}$$

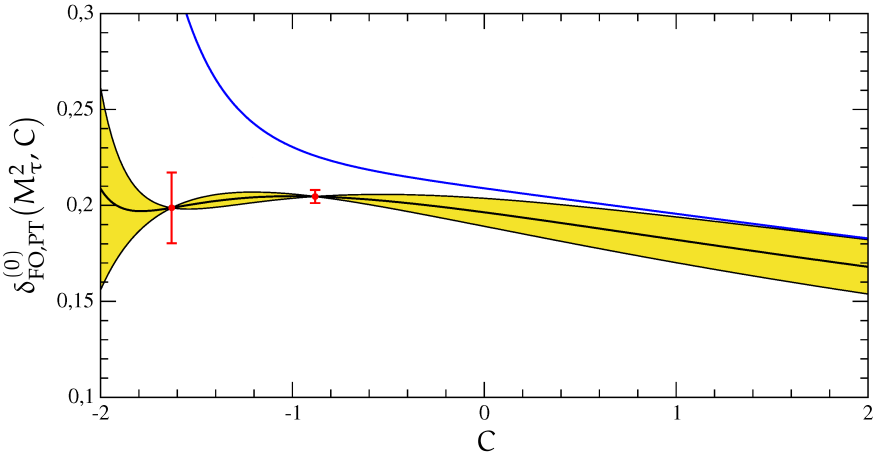


Figure 1.7: $\delta^{(0)}_{\mathrm{FO}}$ of Eq. 1.52 as a function of C. The yellow band arises from either removing or doubling the fifth-order term. In the red dots, the $\mathcal{O}(\widehat{a}^5)$ vanishes, and $\mathcal{O}(\widehat{a}^4)$ is taken as the uncertainty.

where S_{EW} is an electroweak correction, and V_{ud}, V_{us} are Cabibbo-Kobayashi-Maskawa matrix elements. Perturbative QCD of the vector correlator is encoded in $\delta^{(0)}$ (see Refs. [11, 14] for details) and the ellipsis indicate further small subleading corrections.

A complication arises for $\delta^{(0)}$ because, on one hand, it is calculated from a contour integral in the complex energy plane. On the other hand, we seek to resum the scale logarithms $\log(-s/\mu^2)$, and the perturbative prediction depends on whether these logs are resummed before or after performing the contour integration. The first choice is called contour-improved perturbation theory (CIPT) [15] and the second, fixed-order perturbation theory (FOPT).

In FOPT, the perturbative series of $\delta^{(0)}$ in terms of the $\overline{\mathrm{MS}}$ coupling a is given by [13, 14]

$$\delta^{(0)}_{\mathrm{FO}} = a(M_\tau) + 5.202a(M_\tau)^2 + 26.37a(M_\tau)^3 + 127.1a(M_\tau)^4 + \mathcal{O}\big(a(M_\tau)^5\big)\,. \qquad (1.51)$$

Then, in the C-scheme coupling $\widehat{a}$, the expansion for $\delta^{(0)}$ reads

$$\begin{aligned}\delta^{(0)}_{\mathrm{FO}} = \widehat{a}(M_\tau) &+ (5.202 + 2.25C)\widehat{a}(M_\tau)^2 + (27.68 + 27.41C + 5.063C^2)\widehat{a}(M_\tau)^3 \\ &+ (148.4 + 235.5C + 101.5C^2 + 11.39C^3)\widehat{a}(M_\tau)^4 + \mathcal{O}\big(\widehat{a}(M_\tau)^5\big)\,. \qquad (1.52)\end{aligned}$$

In Figure 1.7, we display the truncated expansion of Eq. 1.52 as a function of C. Assuming $c_{51} = 283$, the yellow band again corresponds to removing or doubling the $\mathcal{O}(\widehat{a}^5)$ term. Like for $\widehat{D}$, a nice plateau is found at $C \approx -1$. Taking $c_{51} = 566$ and then doubling the $\mathcal{O}(\widehat{a}^5)$ term results in the blue curve, which does not show stability. Hence, this scenario again is disfavoured. In the red dots, which lie at $C = -0.882$ and $C = -1.629$, the $\mathcal{O}(\widehat{a}^5)$ correction vanishes, and the $\mathcal{O}(\widehat{a}^4)$ term is taken as the uncertainty. The point to the right has a substantially smaller error, and yields

$$\delta^{(0)}_{\mathrm{FO,PT}}\big(C=-0.882\big) = 0.2047 \pm 0.0034 \pm 0.0133\,. \tag{1.53}$$

Once more, the second error covers the uncertainty of $\alpha_s(M_\tau)$. In this case, the direct $\overline{\mathrm{MS}}$ prediction of Eq. 1.51 is

$$\delta^{(0)}_{\mathrm{FO,PT}} = 0.1991 \pm 0.0061 \pm 0.0119\,. \tag{1.54}$$

This value is somewhat lower, but within 1σ of the higher-order uncertainty. Comparing to the Borel model (BM) result of [14], which is given by

$$\delta^{(0)}_{\mathrm{BM}} = 0.2047 \pm 0.0029 \pm 0.0130\,, \tag{1.55}$$

it is found that Eq. 1.53 and Eq. 1.55 are surprisingly similar. In both cases, the parametric α_s uncertainty is substantially larger than the truncation error (especially given the recent increase in the α_s uncertainty provided by the PDG [8]), which underlines the good potential of α_s extractions from hadronic τ decays.

In CIPT, we use Eq. 1.26 to compute contour integrals over the running coupling, and hence the result cannot be given in analytical form. We display the truncated expansion of $\delta^{(0)}_{\mathrm{CI}}$ as a function of C in Figure 1.8. The general behaviour is very similar to FOPT, with the exception that now also for $c_{51} = 566$ a zero of the $\mathcal{O}\big(\hat{a}^5\big)$ term is found. This time, both zeros have similar uncertainties, and employing the point with smaller error (in blue) yields

$$\delta^{(0)}_{\mathrm{CI,PT}}\big(C=-1.246\big) = 0.1840 \pm 0.0062 \pm 0.0084\,. \tag{1.56}$$

As has been discussed many times in the past (see e.g. [14]) the CIPT prediction lies substantially below the FOPT results, and even more so for the C-scheme and Borel model results. On the other hand, the parametric α_s uncertainty in CIPT turns out to be smaller.

In conclusion, we have applied the coupling $\hat{a}$ to investigations of the perturbative series of the reduced Adler function $\widehat{D}$. Our central result is given in Eq. 1.46. Its higher-order uncertainty turned out larger than the corresponding $\overline{\mathrm{MS}}$ prediction of Eq. 1.47, but we consider Eq. 1.46 to be more realistic and conservative.

We also studied the perturbative expansion of the τ hadronic width employing the coupling $\hat{a}$. In this case, our central prediction in FOPT is given in Eq. 1.53. Surprisingly, the result of Eq. 1.53 is very close to the prediction of Eq. 1.55 of the central Borel model developed in Ref. [14], hence providing some support for this approach.

The disparity between FOPT and CIPT predictions for $\delta^{(0)}$ is not resolved by the C-

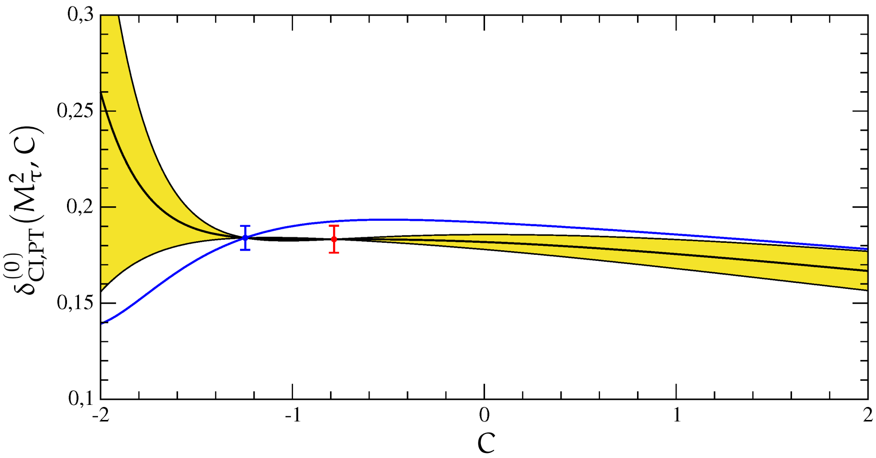


Figure 1.8: $\delta^{(0)}_{\mathrm{CI}}$ as a function of C. The yellow band arises from either removing or doubling the fifth-order term. In the red and blue dots, the $\mathcal{O}(\hat{a}^5)$ vanishes, and $\mathcal{O}(\hat{a}^4)$ is taken as the uncertainty.

scheme. As is seen from Eq. 1.56, the CIPT result turns out substantially lower (as is the case for the $\overline{\mathrm{MS}}$ prediction).

Compared to other celebrated methods used for the optimisation of perturbative predictions, the procedure we present here differs in more than one way. The main difference is that we seek to optimise the perturbative prediction by exploiting its scheme dependence, while the idea behind methods such as BLM [16] or PMC [17, 18] is to obtain a scheme independent result through a well defined algorithm for setting the renormalisation scale, regardless of the intermediate scheme used for the perturbative calculation (which most often is $\overline{\mathrm{MS}}$). Furthermore, some of these methods, such as the «effective charge» [10], involve a process dependent definition of the coupling. Instead, in the procedure described here, we have defined a process independent class of schemes, parameterised by a single continuous parameter C. We then explored variations of this parameter in order to optimise the perturbative series in the spirit of asymptotic expansions. This, however, entails that optimal values of the parameter C depend on the process considered.

1.5 Review of the large-β_0 approximation

To finish this chapter, we review the large-β_0 approximation[7] in the context of the vector correlator. While this approximation does not yield a satisfactory estimate of physical quantities in full QCD, it still provides insights on the large order behaviour of QCD and, in this sense, it inspires strategies to improve on phenomenological analysis (see e.g. [14]). With the same purpose in mind, we will also consider the large-β_0

[7]For historical reasons, we shall speak about the «large-β_0» approximation, although in the notation employed in this work, the leading coefficient of the β function is termed β_1.

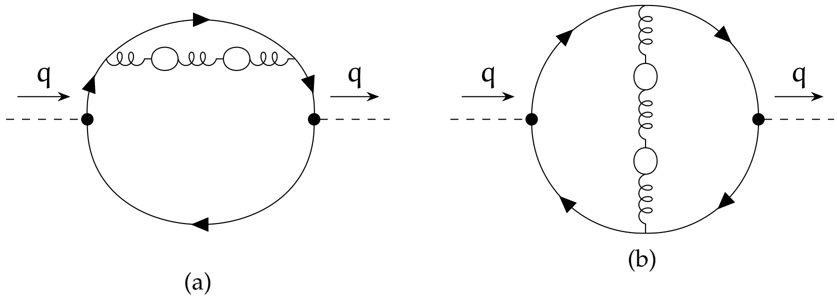


Figure 1.9: Vector correlator in the large-N_f limit.

approximation when discussing the scalar correlator in Chapter 2.

To determine the large-β_0 approximation of a given correlator (below exemplified for the vector correlator), we have to perform two steps:

1. In the first step, we replace the free gluon propagator appearing in the diagrams of Figure 1.5b and Figure 1.5c by the gluon chain of Figure 1.1. The resulting approximation is called the large-N_f limit [19, 20], because only fermion loops are considered in the QCD corrections of the correlator. This process yield the diagrams of Figure 1.9.

2. The diagrams in Figure 1.9 will only depend on the coefficient β_{1f} defined in Eq. 1.13. So, in the second step, we perform the so-called naive non-abelianisation [21], which consists on replacing β_{1f} by the full β_1 coefficient of Eq. 1.15.

The resulting correlator is an approximation of the true result in the sense that, while incorporating corrections to all orders in QCD, it misses many important corrections of the full theory. In the following, using the large-β_0 approximation, we will illustrate how the factorial renormalon divergence characterising renormalisable field theories arises from different momentum regions of the Feynman diagrams.

For the discussion, we consider the modified Adler function of Eq. 1.43 in the large-β_0 approximation. After integrating over the momentum of the big fermion loops appearing in Figure 1.9, and the angles of the gluon chain, we obtain

$$\widehat{D}_{\beta_0}(s) = \frac{1}{4\pi} \int_0^\infty d\tau \, \omega(\tau) \, \frac{\alpha_s}{1+\Pi(s\tau)} \,, \tag{1.57}$$

where $\tau = k^2/s$ (k^2 being the squared momentum of the gluon chain). For the fermion bubble Π, defined in Eq. 1.10, we have already replaced the front factor β_{1f} by β_1 in the process of naive non-abelianisation. The shape function ω is given by [22, Eq. 80]

$$\omega(\tau) = \begin{cases} 8C_F\left\{ \left(\frac{7}{4} - \log\tau\right)\tau + (1+\tau)\big(L_2(-\tau) + \log\tau\log(1+\tau)\big)\right\}, & \tau < 1\,, \\ 8C_F\left\{1 + \log\tau + \left(\frac{3}{4} + \frac{\log\tau}{2}\right)\frac{1}{\tau} \right. & \\ \qquad \left. + (1+\tau)\big(L_2(-\tau^{-1}) - \log\tau\log\big(1+\tau^{-1}\big)\big)\right\}, & \tau > 1\,, \end{cases} \tag{1.58}$$

where L_2 is the dilogarithm and $C_F = (N_c^2 - 1)/(2N_c) = 4/3$ is the Casimir operator. Because we have already seen that the combination $a/(1+\Pi)$ is scale and scheme invariant, then the Adler function $\widehat{D}_{\beta_0}(s)$ in the large-β_0 approximation is also explicitly scale and scheme invariant.

Instead of trying to determine the exact result in Eq. 1.57, we first expand $\widehat{D}_{\beta_0}(s)$ in the coupling α_s:

$$\widehat{D}^{PT}_{\beta_0}(s) = \sum_{n\geqslant 0} c_n^{\beta_0}\alpha_s^{n+1}\,, \tag{1.59}$$

where

$$c_n^{\beta_0} = \frac{1}{4\pi}\int_0^\infty d\tau\,\omega(\tau)\left[\frac{\beta_1}{2\pi}\log\left(\frac{-s\tau}{\mu^2}e^C\right)\right]^n. \tag{1.60}$$

We note that both the coefficients $c_n^{\beta_0}$ and the renormalised coupling α_s depend on scale and scheme choices, even if the Adler function does not.

For large n, the main contribution to the integral in Eq. 1.60 comes from the regions of small τ and large τ, while the middle region $\tau \approx -\mu^2/\big(se^C\big)$ is suppressed because the logarithm becomes 0 around there. Thus, we will only compute the large n behaviour in Eq. 1.60 by approximating ω by its low and large τ asymptotic behaviours:

$$\omega(\tau) \sim 6C_F\tau\,, \quad \tau \to 0\,, \tag{1.61}$$

$$\omega(\tau) \sim \frac{C_F}{\tau^2}\left(\frac{4}{3}\log\tau + \frac{10}{9}\right), \quad \tau \to +\infty\,. \tag{1.62}$$

We split the integral in two energy regions. In the low energy region, corresponding to $\tau \in \big(0, -\mu^2/\big(se^C\big)\big)$, we use Eq. 1.61 and, in the high energy region, corresponding to $\tau \in \big(-\mu^2/\big(se^C\big), \infty\big)$, we use Eq. 1.62. The large n behaviour is then given by

$$c_n^{\beta_0} \sim \frac{C_F}{4\pi}\left\{3\left(\frac{-s}{\mu^2}e^C\right)^{-2}\left(\frac{1}{2}\frac{\beta_1}{2\pi}\right)^n n! \right. \\ \left. + \frac{4}{3}\left(\frac{-s}{\mu^2}e^C\right)\left(-\frac{\beta_1}{2\pi}\right)^n n!\left[n + \frac{11}{6}\right]\right\}, \quad n\to\infty\,. \tag{1.63}$$

The first term comes from the integration in the small τ region (corresponding to small energy going through the gluon chain), thus we denote it as an infrared (IR) factorial divergence. The second term comes from the large τ region, so we denote it as an ultraviolet (UV) factorial divergence. The additional factor n in the second term can be traced back to the logarithm appearing in Eq. 1.62.

From this result we can already compute the two closest singularities in the Borel plane:[8]

$$\mathcal{B}\big[\widehat{D}_{\beta_0}(s)\big](u) = \sum_{n\geqslant 0} \frac{c_n^{\beta_0}}{n!}\left(\frac{u}{\beta_1/(2\pi)}\right)^n \asymp \frac{C_F}{\pi}\frac{3}{2}\left(\frac{-s}{\mu^2}\,e^C\right)^{-2}\frac{1}{2-u} + \frac{C_F}{3\pi}\left(\frac{-s}{\mu^2}\,e^C\right)\left[\frac{1}{(1+u)^2}+\frac{5}{6}\frac{1}{1+u}\right]. \tag{1.64}$$

The singularity at $u=2$ arises from the IR factorial divergence in Eq. 1.63, so we will call it an IR renormalon singularity. The singularity at $u=-1$ comes from the UV factorial divergence, so we call it an UV renormalon singularity.

It is also possible to compute the exact Borel transform in the large-β_0 approximation. For that, we note that the Borel transform of the gluon chain is simply [23]

$$\mathcal{B}\left[\frac{\alpha_s}{1+\Pi(s)}\right](u) = \sum_{n\geqslant 0}\frac{\left[-u\log\left(-s/\mu^2\,e^C\right)\right]^n}{n!} = \left(\frac{-s}{\mu^2}\,e^C\right)^{-u}, \tag{1.65}$$

so the Borel transform of Eq. 1.57 can be written as

$$\mathcal{B}\big[\widehat{D}_{\beta_0}(s)\big](u) = \frac{1}{4\pi}\left(\frac{-s}{\mu^2}\,e^C\right)^{-u}\int_0^\infty d\tau\,\omega(\tau)\,\tau^{-u} = \frac{1}{4\pi}\left(\frac{-s}{\mu^2}\,e^C\right)^{-u}\mathscr{M}[\omega](1-u)\,, \tag{1.66}$$

where $\mathscr{M}$ denotes the Mellin transform.

From Eq. 1.61 and Eq. 1.62, it is easy to check that the Mellin transform $\mathscr{M}[\omega](s)$ converges absolutely in the strip $\mathrm{Re}(s)\in(-1,2)$. This is called the fundamental strip of the Mellin transform, which is the largest strip $\mathrm{Re}(s)\in(a,b)$ where the Mellin transform converges absolutely.

The mapping theorem [24] states that the asymptotic expansion of $\omega(\tau)$ when $\tau\to 0$ determines the position, order and coefficient of the poles of its Mellin transform $\mathscr{M}[\omega](s)$ found to the left of the fundamental strip. Analogously, the asymptotic expansion of $\omega(\tau)$ when $\tau\to+\infty$ determines the singularities found to the right of the fundamental strip. The proof of the theorem results from integrating term by term the asymptotic expansion of ω. In this context, IR renormalons can be defined as the poles

[8] Here we define the Borel transform in terms of the variable $u=\beta_1/(2\pi)\,\zeta$, where ζ would be the standard Borel variable, so that singularities appear at integer positions.

Pole in $\mathscr{M}[\omega](s)$ to the left of the fundamental strip	Behaviour of $\omega(\tau)$ when $\tau \to 0$
$\dfrac{1}{(s-s_0)^{k+1}}$	$+\dfrac{(-1)^k}{k!}\tau^{-s_0}\log^k(\tau)$
Pole in $\mathscr{M}[\omega](s)$ to the right of the fundamental strip	Behaviour of $\omega(\tau)$ when $\tau \to +\infty$
$\dfrac{1}{(s-s_0)^{k+1}}$	$-\dfrac{(-1)^k}{k!}\tau^{-s_0}\log^k(\tau)$

Table 1.1: Mapping theorem for the Mellin transform.

to the left of the fundamental strip, while UV renormalons are defined as the poles to the right.

The explicit formula that relates the singularities of $\mathscr{M}[\omega](s)$ with the asymptotic behaviour of ω is given by the mapping of Table 1.1. To each term of the asymptotic expansion of $\omega(\tau)$ corresponds a pole in $\mathscr{M}[\omega](s)$.

The expression in Eq. 1.58 valid for $\tau < 1$ has one component proportional to $\log(\tau)$, which will yield poles of order 2 according to Table 1.1, and another component proportional to $\log^0(\tau)$, which will yield poles of order 1. Taylor expanding both components around $\tau = 0$ and using the mapping theorem, we obtain

$$\mathscr{M}[\omega](s) \asymp \frac{6C_F}{1+s} - 32C_F \sum_{n\geqslant 2} \frac{(-1)^n}{4n(n-1)}\frac{1}{(s+n)^2} + \frac{(-1)^n(2n-1)}{4n^2(n-1)^2}\frac{1}{s+n}\,, \qquad \mathrm{Re}(s) < -1\,. \tag{1.67}$$

This describes the complete singular information of the poles to the left of the fundamental strip, $\mathrm{Re}(s) < -1$. Analogously, the Taylor expansion of the two components in the second line of Eq. 1.58 around $1/\tau = 0$ yields the complete singular information of the poles to the right of the fundamental strip, $\mathrm{Re}(s) > 2$. By combining both results, the full singular information can be compactly written as

$$\mathscr{M}[\omega](s) = \frac{32C_F}{1+s}\sum_{n\geqslant 2}\frac{(-1)^n n}{[s^2-n^2]^2}\,, \tag{1.68}$$

and there are no missing entire components into this computation (so we may write an equality), because both asymptotic expansions of ω converge to the exact ω. Finally, going back to Eq. 1.66, we have

$$\mathcal{B}\big[\widehat{D}_{\beta_0}(s)\big](u) = \frac{8C_F}{\pi}\left(\frac{-s}{\mu^2}\,e^C\right)^{-u}\frac{1}{2-u}\sum_{n\geqslant 2}\frac{(-1)^n n}{\big[(1-u)^2-n^2\big]^2}\,. \tag{1.69}$$

This result was first obtained in [19] and [20]. It is interesting to mention that in going from ω to the Borel transform of Eq. 1.69, all information about the UV/IR classification of poles is lost. This means that, in principle, one should not be able to recover $\widehat{D}_{\beta_0}$ only from its Borel transform. This observation becomes obvious when considering that to recover ω (and hence $\widehat{D}_{\beta_0}$) from the Mellin transform $\mathscr{M}[\omega]$, we need to compute the inverse Mellin transform, which requires to know the fundamental strip.

To recover the Adler function from its Borel transform, we consider the following Laplace transform:[9]

$$\widehat{D}_{\beta_0}(s) = \frac{2}{\beta_1} \int_0^\infty du\, e^{-2u/(\beta_1 a)} \mathcal{B}\big[\widehat{D}_{\beta_0}(s)\big](u)\,. \tag{1.70}$$

It is very easy to check that, even if the Borel transform depends explicitly on scale and scheme choices, the coupling a in the exponent cancels against these dependences. This is an straightforward verification using the results of Eq. 1.16, Eq. 1.17 and Eq. 1.69.

[9]We note that this Laplace transform is ill-defined, because of the presence of IR renormalon poles along the positive real axis. The principal value prescription shall always be adopted.

Chapter 2

The scalar correlator: large-β_0 and beyond

The scalar correlation function in QCD plays an important role, as it governs the decay of the Higgs into quark-antiquark pairs, and it has been employed in determinations of quark masses from QCD sum rules as well as hadronic decays of the τ lepton. Presently, the perturbative expansion for the scalar correlator is known analytically up to order α_s^4 in the strong coupling [25, 26, 27], and estimates of the next, fifth order have been attempted in the literature. While the decay of the Higgs boson into quark-antiquark pairs is connected to the imaginary part of the scalar correlator $\Psi(s)$ [28], two other physical correlators, $\Psi''(s)$ and $D^L(s)$, have been utilised in QCD sum rule analyses, the former in quark mass extractions [29, 30] and the latter in hadronic τ decays [31, 32, 33]. In this chapter, following the analysis of [34], we shall investigate the perturbative series of all three.

In order to achieve reliable error estimates of missing higher orders in QCD predictions, a better understanding of the perturbative behaviour of the scalar correlator at high orders is desirable. Work along those lines has been performed in Ref. [35], where the scalar correlation function has been calculated in the large-N_f limit, or relatedly the large-β_0 approximation, to all orders in the strong coupling. However, as will be discussed in more detail below, the large-β_0 approximation does not provide a satisfactory representation of the scalar correlator in full QCD. Still, as will be demonstrated, it can serve as a guideline to shed light on the general structure of the scalar correlation function.

Furthermore, while large QCD corrections are found in the case of the correlator $D^L(s)$, the corrections are substantially smaller for $\mathrm{Im}\Psi(s)$ and $\Psi''(s)$. In the large-β_0 approximation this observation can be traced back to the presence of a spurious renormalon pole in the Borel transform at $u = 1$ for $D^L(s)$, whereas $\Psi''(s)$ and $\text{Im}\Psi(s)$ are free from this contribution. We discuss the origin of the additional renormalon pole and its implications, but at any rate conclude that, in view of this fact, the correlator $D^L(s)$ should be avoided in phenomenological analyses.

Additionally, the structure of the Borel transform in the large-β_0 approximation suggests that perturbative predictions can be improved by expressing coupling expansions in terms of the $\widehat{a}$ coupling introduced in Section 1.2. In particular, we find that higher-order corrections tend to become smaller for these expansions in $\widehat{a}$. One reason for this behaviour appears to be that part of the perturbative corrections are resummed into a global prefactor α_s^δ which is present for the scalar correlator.

In Section 2.1, we will collect theoretical expressions for the scalar correlation function $\Psi(s)$ and the corresponding physical correlators $\text{Im}\Psi(s)$, $\Psi''(s)$ and $D^L(s)$, and we will summarise the present knowledge on their perturbative expansions. Furthermore, we will introduce the renormalisation group invariant quark mass $\widehat{m}_\psi$, and we will rewrite the correlators in terms of this mass definition. In Section 2.2, we will review the results of Ref. [35] on the scalar correlation function in the large-β_0 approximation and, in this context, we will discuss the correlators $\Psi''(s)$ and $D^L(s)$. Finally, in Section 2.3, we will investigate two phenomenological applications, $\Psi''(s)$ at the τ mass scale and $\text{Im}\Psi(s)$ for Higgs decay.

2.1 The scalar two-point correlator

The following section shall be concerned with the two-point scalar correlation function, defined by

$$\Psi(q^2) = i \int dx\, e^{iqx} \langle\Omega| T\{j(x)\, j(0)^\dagger\} |\Omega\rangle \,. \tag{2.1}$$

The scalar current $j(x)$ is chosen to arise either from the divergence of the normal-ordered vector current,

$$j(x) = \partial^\mu{:}\bar{u}(x)\gamma_\mu s(x){:} = i(m_u - m_s)\,{:}\bar{u}(x)s(x){:}\,, \tag{2.2}$$

or the interaction of the Higgs boson with quarks,

$$j(x) = m_\psi\,{:}\overline{\psi}(x)\psi(x){:}\,. \tag{2.3}$$

These choices have the advantage of an additional factor of the quark masses, which makes the currents $j(x)$ renormalisation group invariant (RGI). Furthermore, the current in Eq. 2.2 is taken to be flavour non-diagonal, with a particular flavour content that plays a role in hadronic τ decays to strange final states.[1]

The purely perturbative expansion of Ψ is known up to order α_s^4 [25] and takes the general form

$$\Psi_{PT}(s) = -\frac{N_c}{8\pi^2} m(\mu)^2 s \sum_{n\geqslant 0} a(\mu)^n \sum_{k=0}^{n+1} d_{nk} \log^k\left(\frac{-s}{\mu^2}\right) . \tag{2.4}$$

To simplify the notation, we have introduced the generic mass factor $m(\mu)$ which either stands for the combination $(m_u(\mu) - m_s(\mu))$ or $m_\psi(\mu)$.[2] The running quark masses and the QCD coupling are renormalised at the scale μ. As a matter of principle, different scales could be introduced for the renormalisation of coupling and quark masses, but for simplicity, we refrain from this choice.

At each perturbative order n, the only independent coefficients d_{nk} are the d_{n1}. The coefficients d_{n0} depend on the renormalisation prescription and do not contribute in physical quantities, while all remaining coefficients d_{nk} with $k > 1$ can be obtained by means of the RGE. The normalisation in Eq. 2.4 is chosen such that $d_{01} = 1$. Setting the number of colours $N_c = 3$, and employing the $\overline{\text{MS}}$-scheme [2], after tremendous efforts the coefficients d_{n1} up to $\mathcal{O}(\alpha_s^4)$ were found to be [27, 26, 25]:

$$d_{01} = 1\,, \quad d_{11} = \frac{17}{3}\,, \tag{2.5}$$

$$d_{21} = \frac{10801}{144} - \frac{39}{2}\zeta_3 + \left(-\frac{65}{24} + \frac{2}{3}\zeta_3\right) N_f\,, \tag{2.6}$$

$$\begin{aligned} d_{31} = \frac{6163613}{5184} - \frac{109735}{216}\zeta_3 + \frac{815}{12}\zeta_5 + \left(-\frac{46147}{486} + \frac{262}{9}\zeta_3 - \frac{5}{6}\zeta_4 - \frac{25}{9}\zeta_5\right) N_f \\ + \left(\frac{15511}{11664} - \frac{1}{3}\zeta_3\right) N_f^2\,, \end{aligned} \tag{2.7}$$

$$\begin{aligned} d_{41} = \frac{10811054729}{497664} - \frac{3887351}{324}\zeta_3 + \frac{458425}{432}\zeta_3^2 + \frac{265}{18}\zeta_4 + \frac{373975}{432}\zeta_5 - \frac{1375}{32}\zeta_6 \\ - \frac{178045}{768}\zeta_7 + \left(-\frac{1045811915}{373248} + \frac{5747185}{5184}\zeta_3 - \frac{955}{16}\zeta_3^2 - \frac{9131}{576}\zeta_4\right. \\ \left. + \frac{41215}{432}\zeta_5 + \frac{2875}{288}\zeta_6 + \frac{665}{72}\zeta_7\right) N_f + \left(\frac{220313525}{2239488} - \frac{11875}{432}\zeta_3 + \frac{5}{6}\zeta_3^2\right. \\ \left. + \frac{25}{96}\zeta_4 - \frac{5015}{432}\zeta_5\right) N_f^2 + \left(-\frac{520771}{559872} + \frac{65}{432}\zeta_3 + \frac{1}{144}\zeta_4 + \frac{5}{18}\zeta_5\right) N_f^3\,. \end{aligned} \tag{2.8}$$

For future reference, at $N_f = 3$, numerically, the respective coefficients take the values

[1] The $(\bar{u}d)$ flavour content that also arises in hadronic τ decays is obtained by simply replacing the strange quark with a down quark.

[2] In the case of a flavour non-diagonal current, the so-called singlet-diagram contributions are absent, and the perturbative expansion equally applies to the pseudoscalar correlator, up to a replacement of the mass factor $(m_u - m_s)$ by $(m_u + m_s)$.

$$d_{11} = 5.6667\,, \quad d_{21} = 45.846\,, \quad d_{31} = 465.85\,, \quad d_{41} = 5588.7\,. \tag{2.9}$$

The correlator Ψ itself is not related to a measurable quantity. Since it grows linearly with s as s tends to infinity, it satisfies a dispersion relation with two subtraction constants,

$$\Psi(s) = \Psi(0) + s\Psi'(0) + s^2 \int\limits_0^\infty \frac{\rho(s')}{(s')^2(s'-s-i0)}\, ds'\,, \tag{2.10}$$

where $\rho(s) = \mathrm{Im}\Psi(s+i0)/\pi$ is the scalar spectral function. Hence, a possibility to construct a physical quantity other than the spectral function itself, which will be discussed further down below, is to employ the second derivative of $\Psi(s)$ with respect to s. Since the two derivatives remove the two unphysical subtractions in Eq. 2.10, $\Psi''(s)$ is then only related to the spectral function. The corresponding dispersion relation reads

$$\Psi''(s) = 2 \int\limits_0^\infty \frac{\rho(s')}{(s'-s-i0)^3}\, ds'\,, \tag{2.11}$$

and the general perturbative expansion is

$$\Psi''_{\mathrm{PT}}(s) = -\frac{N_c}{8\pi^2}\frac{m(\mu)^2}{s}\sum_{n\geqslant 0} a(\mu)^n \sum_{k=1}^{n+1} k\, d_{nk}\left[\log^{k-1}\left(\frac{-s}{\mu^2}\right) + (k-1)\log^{k-2}\left(\frac{-s}{\mu^2}\right)\right]. \tag{2.12}$$

Being a physical quantity, $\Psi''(s)$ satisfies a homogeneous RGE, and therefore the logarithms can be resummed with the particular scale choice $\mu^2 = -s = Q^2$, leading to the compact expression

$$\Psi''_{\mathrm{PT}}(s) = \frac{N_c}{8\pi^2}\frac{m(Q)^2}{Q^2}\left\{1 + \sum_{n\geqslant 1}(d_{n1} + 2d_{n2})a(Q)^n\right\}. \tag{2.13}$$

In this way, both the running quark mass as well as the running QCD coupling are to be evaluated at the renormalisation scale Q. The dependent coefficients d_{n2} can be calculated from the RGE and they are collected in Appendix A.2. Numerically, at $N_f = 3$, the perturbative coefficients $d''_{n1} = d_{n1} + 2d_{n2}$ of Eq. 2.13 take the values

$$d''_{11} = 3.6667\,, \quad d''_{21} = 14.179\,, \quad d''_{31} = 77.368\,, \quad d''_{41} = 511.83\,. \tag{2.14}$$

It is observed that the coefficients of Eq. 2.14 for the physical correlator are substantially smaller than the d_{n1} of Eq. 2.9.

For the ensuing discussion it will be advantageous to remove the running effects of the quark mass from the remaining perturbative series. This can be achieved by rewriting the running quark masses $m_\psi(\mu)$ in terms of RGI quark masses $\widehat{m}_\psi$, which are defined through the relation

$$m_\psi(\mu) \equiv \widehat{m}_\psi \left[\alpha_s(\mu)\right]^{\gamma_m^{(1)}/\beta_1} \exp\left\{ \int\limits_0^{a(\mu)} da \left[\frac{\gamma_m(a)}{\beta(a)} - \frac{\gamma_m^{(1)}}{\beta_1 a} \right] \right\}. \tag{2.15}$$

(γ_m is the quark-mass anomalous dimension and $\gamma_m^{(n)}$ its perturbative coefficients.) Accordingly, we define a modified perturbative expansion with new coefficients r_n,

$$\Psi''_{\rm PT}(s) = \frac{N_c}{8\pi^2} \frac{\widehat{m}^2}{Q^2} \left[\alpha_s(Q)\right]^{2\gamma_m^{(1)}/\beta_1} \left\{ 1 + \sum_{n\geqslant 1} r_n \, a(Q)^n \right\}, \tag{2.16}$$

which now contain contributions from the exponential factor in Eq. 2.15. At $N_f = 3$ the coefficients r_n take the numerical values

$$r_1 = 5.4568\,, \quad r_2 = 24.287\,, \quad r_3 = 122.10\,, \quad r_4 = 748.09\,. \tag{2.17}$$

The order α_s^4 coefficient r_4 depends on quark-mass anomalous dimensions as well as β function coefficients up to five loops, which for the convenience of the reader in our conventions have been collected in Appendix A.1.

As a second observable, we discuss the imaginary part of the scalar correlator $\mathrm{Im}\Psi(s)$. After resumming the logarithms with the scale choice $\mu^2 = s = M^2$, its general perturbative expansion reads

$$\begin{aligned} \mathrm{Im}\Psi_{\rm PT}(s+i0) &= \frac{N_c}{8\pi} m(M)^2 s \sum_{n\geqslant 0} a(M)^n \sum_{l=0}^{[n/2]} (i\pi)^{2l} d_{n,2l+1} \\ &= \frac{N_c}{8\pi} m(M)^2 s \Big[1 + 5.6667 a(M) + 31.864 a(M)^2 \\ &\qquad\qquad + 89.156 a(M)^3 - 536.84 a(M)^4 + \ldots \Big]. \end{aligned} \tag{2.18}$$

In the first line, $[x]$ denotes the integer value of x, and in the second line, the numerics have again been provided for $N_f = 3$. We remark that in the $\overline{\rm MS}$ scheme the fourth order coefficient turns out to be negative. However, this does not necessarily imply an onset of the dominance of UV renormalons, since the $(i\pi)^{2l}$ terms give a large contribution and contribute to the sign change.

Also for the imaginary part, we introduce a modified perturbative series which results from rewriting the mass factor in terms of the invariant quark mass. This yields

$$\mathrm{Im}\Psi_{\rm PT}(s+i0) = \frac{N_c}{8\pi} \widehat{m}^2 s \left[\alpha_s(M)\right]^{2\gamma_m^{(1)}/\beta_1} \left\{ 1 + \sum_{n\geqslant 0} \bar{r}_n \, a(M)^n \right\}. \tag{2.19}$$

At $N_f = 3$, this time the coefficients $\bar{r}_n$ assume the values

$$\bar{r}_1 = 7.4568\,, \quad \bar{r}_2 = 45.552\,, \quad \bar{r}_3 = 172.64\,, \quad \bar{r}_4 = -204.09\,. \tag{2.20}$$

Besides $\Psi''(s)$ and $\mathrm{Im}\Psi(s)$, in addition, below another physical quantity shall be investigated, which is related to the vector correlator function of Section 1.3. To this end, consider the general decomposition of the vector correlator (Eq. 1.31) into transversal (T) and longitudinal (L) correlators:

$$\begin{aligned} \Pi_V^{\mu\nu}(q) &= \left(q_\mu q_\nu - g_{\mu\nu} q^2\right)\Pi^T(q^2) + q_\mu q_\nu \Pi^L(q^2) \\ &= \left(q_\mu q_\nu - g_{\mu\nu} q^2\right)\left[\Pi^T(q^2) + \Pi^L(q^2)\right] + g_{\mu\nu} q^2 \Pi^L(q^2)\,. \end{aligned} \tag{2.21}$$

The longitudinal component arises only in the massive quark case and it is proportional to the mass difference between the quarks appearing in the vector current. The correlators of the decomposition in the second line, $\Pi^T + \Pi^L$ and Π^L, are free of kinematical singularities and thus should be employed in phenomenological analyses. The longitudinal correlator $\Pi^L(s)$ is related to the scalar correlation function via

$$\Pi^L(s) = \frac{1}{s^2}\left[\Psi(s) - \Psi(0)\right]\,. \tag{2.22}$$

Eq. 2.22 suggests to define a third physical quantity $D^L(s)$ by [31, 32, 33]

$$D^L(s) = -s\frac{d}{ds}\left[s\,\Pi^L(s)\right] = \frac{1}{s}\left[\Psi(s) - \Psi(0)\right] - \Psi'(s)\,. \tag{2.23}$$

Employing Eq. 2.22 and Eq. 2.23, together with the expansion of Eq. 2.13, the general form of the perturbative expansion for D^L reads

$$D^L_{PT}(s) = -\frac{N_c}{8\pi^2} m(\mu)^2 \sum_{n\geqslant 0} a(\mu)^n \sum_{k=1}^{n+1} k\, d_{nk} \log^{k-1}\left(\frac{-s}{\mu^2}\right)\,. \tag{2.24}$$

Comparing Eq. 2.24 to the perturbative expansion of the Adler function in Eq. 1.36, one observes that up to the global prefactor (which however depends on the scale dependent quark mass), they are completely equivalent. Being a physical quantity, also $D^L(s)$ satisfies a homogeneous RGE, and thus again the logarithms in Eq. 2.24 can be resummed with the scale choice $\mu^2 = -s = Q^2$, leading to the simple expression

$$D^L_{PT}(s) = -\frac{N_c}{8\pi^2} m(Q)^2 \sum_{n\geqslant 0} d_{n1}\, a(Q)^n\,. \tag{2.25}$$

From Eq. 2.25 it is again apparent that the only physically relevant coefficients are the d_{n1}. All the rest is encoded in running coupling and quark masses. However, as only the d_{n1} enter, the perturbative behaviour of $D^L(s)$ is substantially worse than that of the correlator $\Psi''(s)$. We shall shed further light on this observation in Section 2.2, where we will discuss the scalar correlator in the large-β_0 approximation.

In analogy to Eq. 2.16 and Eq. 2.19, we can define a new expansion by rewriting the running quark mass in terms of the RGI mass. The corresponding general perturbative expansion for D^L becomes

$$D^L_{PT}(s) = -\frac{N_c}{8\pi^2}\widehat{m}^2\big[\alpha_s(Q)\big]^{2\gamma_m^{(1)}/\beta_1}\left\{1 + \sum_{n\geqslant 0}\widetilde{r}_n\, a(Q)^n\right\}. \tag{2.26}$$

Numerically, at $N_f = 3$, the $\widetilde{r}_n$ are found to be

$$\widetilde{r}_1 = 7.4568\,, \quad \widetilde{r}_2 = 59.534\,, \quad \widetilde{r}_3 = 574.36\,, \quad \widetilde{r}_4 = 6645.3\,. \tag{2.27}$$

2.2 Large-β_0 approximation for the scalar correlator

The large-β_0 approximation for the scalar correlation function was worked out in an impressive tour de force by Broadhurst et al. in Ref. [35]. The approach is to first calculate the large-N_f expansion by summing fermion loop chains in the gluon propagator, and then perform the naive non-abelianisation [21] through the replacement β_{1f} (defined in Eq. 1.13) with the β_1 coefficient.

A correlator Π_S is defined in [35], which is related to the scalar correlator defined in Eq. 2.1 with $\Pi_S = (4\pi)^2\Psi$. In [35, Eq. 32], the scalar correlator in the large-N_f limit was found to be

$$\Psi(s) = -\frac{N_c}{8\pi^2}m(\mu)^2 s\left[L - 2 + \frac{C_F b}{2T_F N_f}H(L,b) + \mathcal{O}\left(\frac{1}{N_f^2}\right) + \mathcal{O}\left(\frac{1}{s}\right)\right], \tag{2.28}$$

where $L = \log(-s/\mu^2)$. The function $H(L,b)$, with $b = T_F N_f a(\mu)/3$, is at the heart of the work [35] and will be discussed in detail below.[3]

Comparing Eq. 2.4 and Eq. 2.28, it immediately follows that

$$\sum_{n\geqslant 1} a(\mu)^n \sum_{k=0}^{n+1} d_{nk} L^k = \frac{C_F b}{2T_F N_f}H(L,b)\,. \tag{2.29}$$

Next, employing the expansion

$$H(L,b) = \sum_{n\geqslant 0} H_{n+1}(L)\, b^{n-1}, \tag{2.30}$$

[3]Some care has to be taken when implementing expressions from Ref. [35], since our convention for the logarithm is $L = \log(-s/\mu^2)$, while in [35], $\log(-\mu^2/s)$ was employed instead.

along the lines of Ref. [35], we obtain

$$\sum_{k=0}^{n+1} d_{nk} L^k = C_F \frac{N_f^{n-1}}{6^n} H_{n+1}(L)\,, \tag{2.31}$$

and in particular

$$d_{n1} = C_F \frac{N_f^{n-1}}{6^n} H_{n+1}^{(1)}\,, \tag{2.32}$$

for the independent coefficients d_{n1}, where $H_{n+1}^{(1)}$ denotes the coefficient of the term of $H_{n+1}(L)$ linear in the logarithm. It remains to arrive at an expression for $H_{n+1}^{(1)}$.

An explicit expression for the $H_{n+1}^{(1)}$ can be pieced together from several formulae presented in Ref. [35], the central of which, for $n \geqslant 1$, reads:

$$n(n+1)H_{n+1}(L) = (n+1)\left[h_{n+2} + 4(L-2)g_{n+1}\right] + 4g_{n+2} + 9(-1)^n \mathcal{D}_{n+1}(L)\,. \tag{2.33}$$

The coefficients h_{n+2} do not concern us here, because they are independent of L. Therefore, we only have to determine expressions for g_n and $\mathcal{D}_n(L)$.

On one hand, the quantities g_n are related to the expansion coefficients of the quark-mass anomalous dimension $\gamma_m(a)$ in the large-N_f limit. In this limit, one finds [36, 35]

$$\gamma_m(a) = -\frac{\mu}{m(\mu)}\frac{dm(\mu)}{d\mu} = \frac{2C_F b}{T_F N_f} g(b) + \mathcal{O}\left(\frac{1}{N_f^2}\right), \tag{2.34}$$

where the function $g(b)$ is given by

$$g(b) = \frac{(3-2b)^2}{(4-2b)} \frac{\Gamma(2-2b)}{\Gamma(2-b)^2} \frac{\sin(\pi b)}{\pi b}\,. \tag{2.35}$$

Then the expansion of $g(b)$, together with an efficient way to generate it, which was also presented in [35], reads:

$$g(b) = \sum_{n\geqslant 0} g_n b^{n-1} = \left[4 - \sum_{n\geqslant 2}\left(\frac{3}{2^n} + \frac{n}{2}\right) b^{n-2}\right] \exp\left(\sum_{l\geqslant 3} \frac{2^l - 3 - (-1)^l}{l} \zeta_l b^l\right). \tag{2.36}$$

For the convenience of the reader, we list the first six coefficients g_n:

$$\begin{aligned} g_1 &= \frac{9}{4}\,, & g_2 &= -\frac{15}{8}\,, \\ g_3 &= -\frac{35}{16}\,, & g_4 &= -\frac{83}{32} + \frac{9}{2}\zeta_3\,, \\ g_5 &= -\frac{195}{64} - \frac{15}{4}\zeta_3 + \frac{27}{4}\zeta_4\,, & g_6 &= -\frac{451}{128} - \frac{35}{8}\zeta_3 - \frac{45}{8}\zeta_4 + \frac{27}{2}\zeta_5\,. \end{aligned} \tag{2.37}$$

Comparing the general expansion of $g(b)$ with the expansion of $\gamma_m(a)$, the relation for the coefficients of the quark-mass anomalous dimension is given by

$$\gamma_m^{(n)} = 4C_F \frac{N_f^{n-1}}{6^n} g_n \,. \tag{2.38}$$

Employing the coefficients g_n of Eq. 2.37, it can easily be verified that the terms with the highest power of N_f in $\gamma_m^{(n)}$ (see Eq. A.8–A.12) are indeed reproduced.

On the other hand, the functions $\mathcal{D}_n(L)$ in the last summand of Eq. 2.33, and the corresponding coefficients $\mathcal{D}_n^{(1)}$ of the linear component in L, can be derived from the following relation:[4]

$$\sum_{n\geqslant 0} \frac{\mathcal{D}_n(L)}{n!} u^n = \big[1 + uG_D(u)\big] e^{-(L-5/3)u} \,. \tag{2.39}$$

The term $-5/3$ in the exponent is particular for the $\overline{\text{MS}}$ scheme, which is employed unless otherwise stated. (Below, we shall, however, generalise our expressions to an arbitrary scheme for the coupling.) The function $G_D(u)$ was found to be [35]

$$\begin{aligned} G_D(u) &= \frac{2}{1-u} - \frac{1}{2-u} + \frac{2}{3}\sum_{p\geqslant 3} \frac{(-1)^p}{(p-u)^2} - \frac{2}{3}\sum_{p\geqslant 1} \frac{(-1)^p}{(p+u)^2} \\ &= \frac{2}{1-u} - \frac{1}{2-u} + \frac{1}{6}\Big[\zeta(2, 2-\tfrac{u}{2}) - \zeta(2, \tfrac{3}{2}-\tfrac{u}{2}) - \zeta(2, 1+\tfrac{u}{2}) + \zeta(2, \tfrac{1}{2}+\tfrac{u}{2})\Big] \\ &= \sum_{k\geqslant 1} \frac{k+3}{3}(2-2^{-k})u^{k-1} - \frac{8}{3}\sum_{l\geqslant 1} \zeta_{2l+1} l(1-4^{-l})u^{2l-1} \,. \end{aligned} \tag{2.40}$$

The first line of Eq. 2.40 explicitly displays the renormalon structure, separated in IR renormalon poles at positive integer u, and UV renormalon poles at negative integer u, while the second lines gives an expression in terms of the Hurwitz ζ function. Finally, the third line provides the Taylor expansion of $G_D(u)$ around $u = 0$, which corresponds to the perturbative expansion.

Inserting the extracted coefficients g_n and $\mathcal{D}_n^{(1)}$ into $H_{n+1}^{(1)}$ derived from Eq. 2.33, it is a simple matter to verify that Eq. 2.32 reproduces the contributions with the highest power of N_f in the coefficients d_{n1} of Eq. 2.5–2.8 for $n \geqslant 1$. To facilitate the comparison, the first few coefficients $\mathcal{D}_n^{(1)}$ and $H_{n+1}^{(1)}$ have been collected in Appendix A.3.

In the following, we will derive an expression for $\Psi''(s)$ of Eq. 2.13 in the large-β_0 approximation. The required second derivative of the function $H(L, b)$ with respect to L can be extracted from expressions provided in Ref. [35], along the lines of the computation above which led to the coefficients d_{n1}. To convert the large-N_f expansion into the large-β_0 approximation, all occurrences of N_f have to be replaced by $-3\beta_1$. Finally, rewriting sums over the $\mathcal{D}_n$ coefficients (and derivatives) in terms of the Borel

[4]The relation to the corresponding coefficients $\widetilde{\Delta}_n$ of [35] is given by $n(n-1)\widetilde{\Delta}_n = -2\mathcal{D}_n^{(1)}$.

transform of the coupling, those sums can be expressed in closed form containing the function $G_D(u)$. This yields

$$\Psi''_{\beta_0}(s) = -\frac{N_c}{8\pi^2}\frac{m(\mu)^2}{s}\Bigg\{1 - \frac{2}{\beta_1}\sum_{n\geqslant 1}\frac{\gamma_m^{(n+1)}}{n}a(\mu)^n + \frac{3C_F}{\beta_1}\int_0^\infty du\, e^{-2u/(\beta_1 a(\mu))}\Big[(1-u)\big[1+uG_D(u)\big]e^{-(L-5/3)u} - 1\Big]\frac{1}{u} + \dots\Bigg\}, \quad (2.41)$$

where the ellipses stand for terms with additional suppression in β_1 or s. Because the integrand contains IR renormalon poles along the path of integration, a prescription has to be specified in order to define the integral. In the present study, the principal value prescription shall always be adopted.

As Ψ'' satisfies a homogeneous RGE, the logarithm can be resummed through the scale choice $\mu^2 = -s = Q^2$. Furthermore, the running of the quark mass is reflected in the terms containing the coefficients of the quark-mass anomalous dimension $\gamma_m^{(n)}$. The leading order coefficient $\gamma_m^{(1)}$ is cancelled by the last term -1 inside the square brackets. Hence, the mass running (except for the leading order) can be resummed by expressing the quark mass in terms of the RGI quark mass $\widehat{m}$ according to Eq. 2.15.

In addition, we rewrite Eq. 2.41 in terms of the scheme dependent large-β_0 coupling defined in Eq. 1.17. This leads to our final formula for Ψ'' in the large-β_0 approximation:

$$\Psi''_{\beta_0}(s) = \frac{N_c}{8\pi^2}\frac{\widehat{m}^2}{Q^2}\big[\alpha_s(Q,C_m)\big]^{2\gamma_m^{(1)}/\beta_1}\Bigg\{1 - 2\frac{\gamma_m^{(1)}}{\beta_1}\log\Big[1 + C_m\frac{\beta_1}{2}a(Q,C_m)\Big] + \frac{2\pi}{\beta_1}\int_0^\infty du\, e^{-2u/(\beta_1 a(Q,C_a))}\,\mathcal{B}\big[\Psi''\big](u) + \dots\Bigg\}, \quad (2.42)$$

where we have introduced two separate constants C_m and C_a, referring to the scheme dependencies of quark mass and coupling, respectively. The Borel transform is given by

$$\begin{aligned}\mathcal{B}\big[\Psi''\big](u) &= \frac{3C_F}{2\pi}e^{-C_a u}\Big[(1-u)G_D(u) - 1\Big] \\ &= \frac{3C_F}{2\pi}e^{-C_a u}\Bigg\{\frac{1}{(2-u)} - \frac{2}{3}\sum_{p\geqslant 3}(-1)^p\left[\frac{(p-1)}{(p-u)^2} - \frac{1}{(p-u)}\right] \\ &\qquad - \frac{2}{3}\sum_{p\geqslant 1}(-1)^p\left[\frac{(p+1)}{(p+u)^2} - \frac{1}{(p+u)}\right]\Bigg\}.\end{aligned} \quad (2.43)$$

The second line again provides the separation of the Borel transform $\mathcal{B}\big[\Psi''\big](u)$ in IR and UV renormalon poles. This expression is analogous to that of the Adler function that we found in Eq. 1.69, but with the scale already set to $\mu^2 = -s$. Except for the linear IR pole at $u = 2$, being related to the gluon condensate, we have quadratic and linear IR poles at all integers $u \geqslant 3$. Furthermore, quadratic and linear UV renormalon poles are found for all integer $u \leqslant -1$. Hence, like for the Adler function, at large orders the perturbative coefficients will be dominated by the quadratic UV renormalon pole at $u = -1$, which lies closest to to the origin.

As is also observed from Eq. 2.42, the perturbative series contains a term without renormalon singularities, which is related to the scheme dependence of the global prefactor $[\alpha_s]^{2\gamma_m^{(1)}/\beta_1}$. This «no-pole» contribution is absent in the scheme with $C_m = 0$ and we note that this scheme can be attained by rewriting the global prefactor in terms of the invariant coupling $\widehat{\alpha}_s(Q)$, defined in Eq. 1.18.

Let us proceed to an investigation of the perturbative expansion for three different choices of the renormalisation scheme. We begin with the $\overline{\text{MS}}$ scheme for both mass and coupling, in which $C_m = C_a = -5/3$. In this scheme, the coefficients r_n, introduced in Eq. 2.16, are found to be

$$r_1^{\beta_0}(\overline{\text{MS}}, \overline{\text{MS}}) = \frac{16}{3} = 5.3333\,, \tag{2.44}$$

$$r_2^{\beta_0}(\overline{\text{MS}}, \overline{\text{MS}}) = \left(\frac{143}{36} - 2\zeta_3\right)\beta_1 = 7.0565\,, \tag{2.45}$$

$$r_3^{\beta_0}(\overline{\text{MS}}, \overline{\text{MS}}) = \left(\frac{1465}{324} - \frac{4}{3}\zeta_3\right)\beta_1^2 = 59.107\,, \tag{2.46}$$

$$r_4^{\beta_0}(\overline{\text{MS}}, \overline{\text{MS}}) = \left(\frac{17597}{2592} + \frac{5}{6}\zeta_3 - \frac{15}{2}\zeta_5\right)\beta_1^3 = 1.2504\,. \tag{2.47}$$

The first entry in the argument of $r_n^{\beta_0}$ refers to the scheme for the mass and the second for the coupling. The numerical values have been given for $N_f = 3$. Comparing to Eq. 2.17, except for the first coefficient r_1, the higher-order coefficients are not at all well represented by the large-β_0 approximation, with a complete failure observed at the fourth order. To obtain a better understanding of this behaviour, the contribution of the lowest-lying renormalon poles to the perturbative large-β_0 coefficients shall be investigated.

In Table 2.1, we present the contributions in percent to the first 12 perturbative coefficients r_n in the large-β_0 approximation and the $\overline{\text{MS}}$ scheme coming from the two lowest-lying UV renormalon poles at $u = -1, -2$ and from the three lowest-lying IR renormalon poles at $u = 2, 3, 4$, as well as the contribution from the no-pole term. It is observed that starting with about the fifth order, the dominance of the lowest-lying UV pole at $u = -1$ sets in. The no-pole term, which does not contain a renormalon singularity, dominates the first two orders. Furthermore, for the fourth order, huge cancellations between the different contributions take place. At this order, only when

	$r_1^{\beta_0}$	$r_2^{\beta_0}$	$r_3^{\beta_0}$	$r_4^{\beta_0}$	$r_5^{\beta_0}$	$r_6^{\beta_0}$
UV_{-1}	25.0	−56.7	31.7	−15025.0	65.2	133.9
UV_{-2}	−6.2	3.5	1.1	618.5	0.4	−0.8
IR_2	18.8	69.1	42.3	10973.5	28.4	−28.4
IR_3	−2.8	−6.3	−1.3	349.9	2.5	−3.8
IR_4	1.6	3.1	0.3	−259.0	−1.3	1.7
No-Pole	62.5	88.6	26.4	3514.5	4.6	−2.2
SUM	98.8	101.3	100.7	172.4	99.8	100.4

	$r_7^{\beta_0}$	$r_8^{\beta_0}$	$r_9^{\beta_0}$	$r_{10}^{\beta_0}$	$r_{11}^{\beta_0}$	$r_{12}^{\beta_0}$
UV_{-1}	89.6	105.7	97.7	101.1	99.5	100.2
UV_{-2}	0.0	−0.1	0.0	0.0	0.0	0.0
IR_2	9.0	−4.9	2.1	−1.0	0.4	−0.2
IR_3	1.5	−0.8	0.3	−0.1	0.1	0.0
IR_4	−0.6	0.3	−0.1	0.0	0.0	0.0
No-Pole	0.3	−0.1	0.0	0.0	0.0	0.0
SUM	99.9	100.1	100.0	100.0	100.0	100.0

Table 2.1: Contribution (in percent) of the lowest-lying ultraviolet (UV) and infrared (IR) renormalon poles as well as the no-pole term to the first 12 perturbative coefficients $r_n^{\beta_0}$ in the $\overline{\mathrm{MS}}$ scheme for both quark mass and renormalon terms.

adding the no-pole term and UV and IR renormalon contributions up to $p = 15$ in Eq. 2.43, a 1% precision on the coefficient $r_4^{\beta_0}$ is reached.

Now, we move to the discussion of renormalisation schemes for which the mass renormalisation is taken at $C_m = 0$, and thus the no-pole, logarithmic term of Eq. 2.42 vanishes. Since the renormalisation scheme in the mass and in the renormalon contribution can be chosen independently, we still have the freedom to employ a different scheme in the latter case. Using the $\overline{\mathrm{MS}}$ scheme in the Borel integral, $C_a = -5/3$, the first four perturbative coefficients are found to be:

$$r_1^{\beta_0}(C = 0, \overline{\mathrm{MS}}) = 2\,, \tag{2.48}$$

$$r_2^{\beta_0}(C = 0, \overline{\mathrm{MS}}) = \left(\frac{31}{12} - 2\zeta_3\right)\beta_1 = 0.8065\,, \tag{2.49}$$

$$r_3^{\beta_0}(C = 0, \overline{\mathrm{MS}}) = \left(\frac{15}{4} - \frac{4}{3}\zeta_3\right)\beta_1^2 = 43.482\,, \tag{2.50}$$

$$r_4^{\beta_0}(C = 0, \overline{\mathrm{MS}}) = \left(\frac{5449}{864} + \frac{5}{6}\zeta_3 - \frac{15}{2}\zeta_5\right)\beta_1^3 = -42.695\,. \tag{2.51}$$

We observe that the first two orders are substantially smaller than in Eq. 2.44–2.47, due to the fact that the no-pole term has effectively been resummed into the global prefactor. The third order is of a similar size and the fourth order turns out to be

	$r_1^{\beta_0}$	$r_2^{\beta_0}$	$r_3^{\beta_0}$	$r_4^{\beta_0}$	$r_5^{\beta_0}$	$r_6^{\beta_0}$
UV_{-1}	66.7	−496.0	43.1	440.0	68.4	131.0
UV_{-2}	−16.7	31.0	1.5	−18.1	0.5	−0.8
IR_2	50.0	604.5	57.6	−321.4	29.7	−27.8
IR_3	−7.4	−55.1	−1.7	−10.2	2.6	−3.7
IR_4	4.2	27.1	0.5	7.6	−1.4	1.7
SUM	96.8	111.5	100.9	97.9	99.8	100.4

	$r_7^{\beta_0}$	$r_8^{\beta_0}$	$r_9^{\beta_0}$	$r_{10}^{\beta_0}$	$r_{11}^{\beta_0}$	$r_{12}^{\beta_0}$
UV_{-1}	89.9	105.6	97.7	101.1	99.5	100.2
UV_{-2}	0.0	−0.1	0.0	0.0	0.0	0.0
IR_2	9.1	−4.9	2.1	−1.0	0.4	−0.2
IR_3	1.5	−0.8	0.3	−0.1	0.1	0.0
IR_4	−0.6	0.3	−0.1	0.0	0.0	0.0
SUM	99.9	100.1	100.0	100.0	100.0	100.0

Table 2.2: Contribution (in percent) of the lowest-lying ultraviolet (UV) and infrared (IR) renormalon poles to the first 12 perturbative coefficients $r_n^{\beta_0}$ in the mixed scheme with $C_m = 0$ for the quark mass and $\overline{\text{MS}}$ in the renormalon terms.

negative, which indicates that the leading UV renormalon singularity is already dominating. This is confirmed by the separated contributions of the lowest-lying IR and UV renormalons, again provided in Table 2.2. This time large cancellations between the lowest-lying UV and IR renormalons take place for the second and fourth order. This cancellation could be the reason for an anomalously small second order coefficient. Like in the $\overline{\text{MS}}$ scheme, dominance of the leading UV renormalon at $u = -1$ sets in at about the fifth order.

To conclude our discussion of the perturbative expansion of Ψ'' in the large-β_0 approximation, we investigate the scheme with $C_m = C_a = 0$ in both no-pole and renormalon contributions. The corresponding first few perturbative coefficients read

$$r_1^{\beta_0}(C=0, C=0) = 2\,, \tag{2.52}$$

$$r_2^{\beta_0}(C=0, C=0) = \left(\frac{11}{12} - 2\zeta_3\right)\beta_1 = -6.6935\,, \tag{2.53}$$

$$r_3^{\beta_0}(C=0, C=0) = \left(\frac{5}{6} + 2\zeta_3\right)\beta_1^2 = 65.558\,, \tag{2.54}$$

$$r_4^{\beta_0}(C=0, C=0) = \left(\frac{37}{32} - \frac{15}{2}\zeta_5\right)\beta_1^3 = -603.31\,. \tag{2.55}$$

In this case, the leading UV renormalon dominates already from the lowest order, which is reflected in the sign-alternating behaviour of the perturbative coefficients.

	$r_1^{\beta_0}$	$r_2^{\beta_0}$	$r_3^{\beta_0}$	$r_4^{\beta_0}$	$r_5^{\beta_0}$	$r_6^{\beta_0}$
UV_{-1}	66.7	134.5	103.0	105.7	101.5	101.3
UV_{-2}	−16.7	−22.4	−9.0	−4.7	−2.3	−1.2
IR_2	50.0	−16.8	3.9	−1.4	0.5	−0.2
IR_3	−7.4	−1.7	0.8	−0.3	0.1	0.0
IR_4	4.2	1.4	−0.4	0.1	0.0	0.0
SUM	96.8	95.0	98.2	99.4	99.8	99.9

Table 2.3: Contribution (in percent) of the lowest-lying ultraviolet (UV) and infrared (IR) renormalon poles to the first 6 perturbative coefficients $r_n^{\beta_0}$ in the scheme with $C_m = C_a = 0$ for both quark mass and renormalon terms.

We also observe the strong growth of the coefficients that signals the asymptotic behaviour of the series.[5]

In Table 2.3, once again we present the contributions in percent to the first 6 perturbative coefficients. As indicated above, in this scheme we find that already the second coefficient $r_2^{\beta_0}$ is largely dominated by the leading UV renormalon at $u = -1$, and for still higher orders the series is fully dominated by this contribution. This behaviour is expected from the exponential factor $\exp(-C_a u)$ in Eq. 2.43, which entails that in the scheme with $C_a = 0$ the residues of the IR renormalon poles are no longer enhanced with respect to the UV renormalons as is the case in the $\overline{\text{MS}}$ scheme.

In an analogous fashion to the derivation of Eq. 2.42, we can derive an expression for the correlation function D^L of Eq. 2.25 in the large-β_0 approximation. It reads

$$D^L_{\beta_0}(s) = -\frac{N_c}{8\pi^2}\widehat{m}^2\left[\alpha_s(Q,C_m)\right]^{2\gamma_m^{(1)}/\beta_1}\left\{1 - 2\frac{\gamma_m^{(1)}}{\beta_1}\log\left[1 + C_m\frac{\beta_1}{2}a(Q,C_m)\right]\right.$$
$$\left. + \frac{2\pi}{\beta_1}\int_0^\infty du\, e^{-2u/(\beta_1 a(Q,C_a))}\frac{3C_F}{2\pi}e^{-C_a u}G_D(u) + \dots\right\}. \quad (2.56)$$

The perturbative expansion of this correlator shall only be discussed in the mixed scheme with $C_m = 0$ for the quark mass and $C_a = -5/3$ for the coupling. Then, the

[5]As an amusing aside, we remark that in this scheme, at each order $n > 1$, only the highest possible ζ function coefficients $\zeta(2\,[n/2+1]-1)$ arise, where $[x]$ denotes the integer value of x. For instance, compare Eq. 2.47 to Eq. 2.55. The ζ_3 term has vanished in the latter equation. We will further investigate the absence of ζ values in Chapter 3.

	$\widetilde{r}_1^{\beta_0}$	$\widetilde{r}_2^{\beta_0}$	$\widetilde{r}_3^{\beta_0}$	$\widetilde{r}_4^{\beta_0}$	$\widetilde{r}_5^{\beta_0}$	$\widetilde{r}_6^{\beta_0}$
UV_{-1}	33.3	−5.8	9.5	−8.6	8.2	−10.6
UV_{-2}	−8.3	−2.9	−1.1	−0.3	−0.1	0.0
UV_1	100.0	138.7	109.9	123.1	99.1	115.3
IR_2	−25.0	−28.2	−16.7	−12.8	−6.4	−4.2
IR_3	−3.7	−4.5	−2.8	−2.3	−1.1	−0.7
SUM	96.3	97.3	98.8	99.2	99.7	99.8

	$\widetilde{r}_7^{\beta_0}$	$\widetilde{r}_8^{\beta_0}$	$\widetilde{r}_9^{\beta_0}$	$\widetilde{r}_{10}^{\beta_0}$	$\widetilde{r}_{11}^{\beta_0}$	$\widetilde{r}_{12}^{\beta_0}$
UV_{-1}	9.6	−13.2	11.3	−16.2	13.1	−19.4
UV_{-2}	0.0	0.0	0.0	0.0	0.0	0.0
UV_1	92.5	114.5	89.2	116.5	87.0	119.5
IR_2	−1.8	−1.2	−0.5	−0.3	−0.1	−0.1
IR_3	−0.3	−0.2	−0.1	0.0	0.0	0.0
SUM	99.9	100.0	100.0	100.0	100.0	100.0

Table 2.4: Contribution (in percent) of the lowest-lying ultraviolet (UV) and infrared (IR) renormalon poles to the first 12 perturbative coefficients $r_n^{\beta_0}$ in the mixed scheme with $C_m = 0$ for the quark mass and $\overline{\text{MS}}$ in the renormalon terms.

coefficients $\widetilde{r}_n$ of Eq. 2.26 in the large-β_0 approximation are given by

$$\widetilde{r}_1^{\beta_0}(C=0,\overline{\text{MS}}) = 4\,, \tag{2.57}$$

$$\widetilde{r}_2^{\beta_0}(C=0,\overline{\text{MS}}) = \left(\frac{25}{4} - 2\zeta_3\right)\beta_1 = 17.3065\,, \tag{2.58}$$

$$\widetilde{r}_3^{\beta_0}(C=0,\overline{\text{MS}}) = \left(\frac{205}{18} - \frac{10}{3}\zeta_3\right)\beta_1^2 = 149.486\,, \tag{2.59}$$

$$\widetilde{r}_4^{\beta_0}(C=0,\overline{\text{MS}}) = \left(\frac{21209}{864} - \frac{25}{6}\zeta_3 - \frac{15}{2}\zeta_5\right)\beta_1^3 = 1071.81\,, \tag{2.60}$$

where, like in the previous cases, the numerical values have been given at $N_f = 3$. It is again observed that the coefficients $\widetilde{r}_n^{\beta_0}$ are substantially worse behaved than the coefficients $r_n^{\beta_0}$.

In Table 2.4, we present the contributions in percent of the three lowest-lying UV renormalon poles at $u = 1, -1, -2$ and the two lowest-lying IR renormalon poles at $u = 2, 3$ in the mixed scheme. The surprising finding, which can also be inferred directly from Eq. 2.40 and Eq. 2.56, is that the function D^L suffers from an additional spurious renormalon pole at $u = 1$. This observation was, of course, already made in Ref. [35]. Because the linear $u = 1$ pole has the larger residue as compared to the UV renormalon pole at $u = -1$, it dominates the perturbative coefficients for a large number of orders, before the quadratic UV pole at $u = -1$ takes over.[6]

The origin of the renormalon pole at $u = 1$ can be understood from Eq. 2.23. In the construction of D^L, the term $\Psi(0)/s$ is subtracted. The subtraction constant $\Psi(0)$ consists of a contribution from the quark condensate and an UV divergent perturbative term proportional to m^4 (see Appendix A.4 for details). The subtraction of this divergent term leads to an ambiguity which results in the emergence of the additional renormalon at $u = 1$, and since it is of UV origin, in Table 2.4 we have labelled the pole accordingly. Generally, because of this spurious renormalon pole, it appears advisable to avoid the correlator D^L in phenomenological analyses.

A detailed discussion of the third physical observable related to the scalar correlator, $\mathrm{Im}\Psi(s + i0)$, in the large-β_0 approximation, has been presented in Ref. [35], and therefore, we shall not repeat it here. We only remark that, like Ψ'', also the spectral function does not suffer from a renormalon pole at $u = 1$. In the case of Ψ'', this pole contribution, which is present in the independent perturbative coefficients d_{n1}, is cancelled by the term $2d_{n2}$ (see Eq. 2.13), which individually also receives contributions from a pole at $u = 1$. In the case of $\mathrm{Im}\Psi(s)$, the $u = 1$ pole contribution in d_{n1} ($l = 0$) is cancelled by the terms $(i\pi)^{2l} d_{n,2l+1}$ ($l \geqslant 1$) in the sum over l of Eq. 2.18.

In conclusion, from the investigation of the scalar correlator in the large-β_0 approximation, it appears advantageous to express at least the global prefactor $[\alpha_s]^{2\gamma_m^{(1)}/\beta_1}$ in terms of the scheme invariant coupling $\widehat{\alpha}_s$ (which is equivalent to choosing $C_m = 0$), such that the quark mass factor is fully scheme independent. In full QCD, this situation can be attained by writing the global prefactor in terms of the C-scheme coupling that we introduced in Section 1.2.

2.3 Phenomenological analysis

Let us now investigate the phenomenological implications of introducing the C-scheme coupling of Section 1.2. We begin by doing this on the basis of the second derivative of the scalar correlator Ψ'', where, as a first step, the coupling in the prefactor, originating from the running of the quark mass, is reexpressed in terms of $\widehat{\alpha}_s$. Defining the quantity $\widehat{\Psi}''(s)$:

$$\Psi''(s) = -\frac{N_c}{8\pi^2}\frac{\widehat{m}^2}{s}\widehat{\Psi}''(s)\,, \tag{2.61}$$

[6] In the scheme with $C_m = C_a = 0$, in which the spurious pole at $u = 1$ is less enhanced, still for many orders large cancellations between the lowest-lying poles at $u = -1$ and $u = 1$ take place.

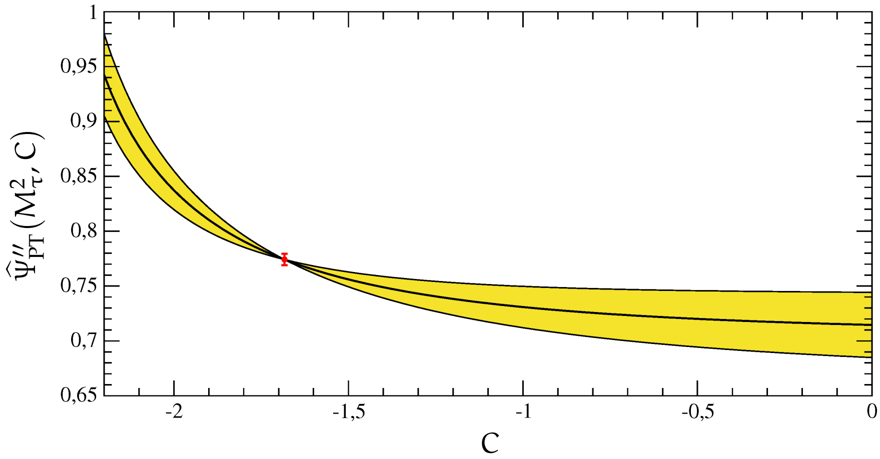


Figure 2.1: $\widehat{\Psi}''$ according to Eq. 2.62 as a function of C for $\alpha_s(M_\tau) = 0.316$. The yellow band corresponds to either removing or doubling the $\mathcal{O}(a^4)$ correction to estimate the respective uncertainty. In the red point, where $\mathcal{O}(a^4)$ vanishes, the third order is taken as the error.

and employing the transformation of the QCD coupling provided in Eq. 1.27 and Eq. 1.28, we find

$$\begin{aligned}\widehat{\Psi}''(s) = \left[\widehat{\alpha}_s(Q)\right]^{8/9}\Big\{1 &+ (5.4568 + 2C)a(Q)\\ &+ (25.452 + 14.469C - 0.25C^2)a(Q)^2\\ &+ (135.29 + 74.006C - 6.2531C^2 + 0.20833C^3)a(Q)^3\\ &+ (824.05 + 367.82C - 56.089C^2 + 9.2479C^3 - 0.24740C^4)a(Q)^4 + \mathcal{O}\big(a(Q)^5\big)\Big\}. \qquad (2.62)\end{aligned}$$

Thus far the coupling $a(Q)$ within the curly brackets is left in the $\overline{\text{MS}}$ scheme. We will proceed with investigating this case numerically and then, in a second step, also rewrite these contributions in terms of $\widehat{a}(Q)$.

To this end, Figure 2.1 displays a numerical account of the behaviour of the truncated expansion of $\widehat{\Psi}''$ as a function of the scheme parameter C. As we are interested in applications to hadronic τ decays, for definiteness, we have chosen $s = -M_\tau^2$. Therefore, we evaluate the perturbative expansion to $\alpha_s(M_\tau) = 0.316(10)$ in the $\overline{\text{MS}}$ scheme, which is obtained from the current PDG average $\alpha_s(M_Z) = 0.1181(13)$ [8] after scale evolution. The coupling $\widehat{\alpha}_s(Q)$ required in the prefactor has been determined by directly solving Eq. 1.25 numerically, not via the expansion of Eq. 1.27 and Eq. 1.28. In order to estimate the uncertainty in the perturbative prediction, the fourth order term is either removed or doubled. The steepest curve in Figure 2.1 then corresponds to setting the $\mathcal{O}(a^4)$ contribution to zero and the flattest one to doubling it. The yellow band hence corresponds to the region of expected values for $\widehat{\Psi}''_{PT}$, depending on the parameter C.

We observe that at $C = -1.683$ the $\mathcal{O}(a^4)$ correction vanishes.[7] The red data point then indicates an estimate where the uncertainty is taken to be the size of the third order term. At this value of C, the third-order correction has already turned negative and, beyond it, also the $\mathcal{O}(a^4)$ contribution changes sign. This is an indication that in the respective region of C the contributions from IR and UV renormalons are more balanced. To obtain a more complete picture, also the uncertainty of α_s should be folded in. Numerically, our result at $C = -1.683$ then reads

$$\widehat{\Psi}''_{\rm PT}\big(-M_\tau^2, C = -1.683\big) = 0.774 \pm 0.005^{+0.058}_{-0.052} = 0.774^{+0.058}_{-0.052}\,, \tag{2.63}$$

where the first error corresponds to the $\mathcal{O}(a^3)$ correction also displayed in Figure 2.1, while the second error results from the current uncertainty in α_s. The total error on the right hand side has been obtained by adding the individual uncertainties in quadrature.

The value in Eq. 2.63 can be compared to the result at $C = 0$,

$$\widehat{\Psi}''_{\rm PT}\big(-M_\tau^2, C = 0\big) = 0.715 \pm 0.030^{+0.040}_{-0.038} = 0.715^{+0.050}_{-0.048}\,. \tag{2.64}$$

The two predictions in Eq. 2.63 and Eq. 2.64 are compatible and have similar uncertainties. At present, the error on α_s is dominant. While in the prediction of Eq. 2.63, the estimated uncertainty from missing higher orders is substantially reduced, its sensitivity to α_s and its uncertainty is increased. This is due to the fact that at $C = -1.683$, symmetrising the error, one finds $\widehat{\alpha}_s = 0.610 \pm 0.045$. This increased sensitivity on α_s may also be seen as a virtue if one aims at an extraction of α_s along the lines of [14, 37, 38, 39]. In this respect, further understanding of the behaviour of the perturbative series, for example, through models for the Borel transform in the spirit of Ref. [14], could be helpful. As a last remark, we point out that at the scale of M_τ, for $C < -2$, the scheme transformation ceases to be perturbative and breaks down. Therefore, such values should be discarded for phenomenology.

We proceed with our second step of also expressing the coupling $a(Q)$ within the curly brackets of Eq. 2.62 in terms of $\widehat{a}(Q)$. As a matter of principle, we could introduce two different scheme constants C_m and C_a, related to mass and coupling renormalisation, respectively, since the global prefactor originates from the quark mass, and the remaining expansion concerns the QCD coupling. To keep the discussion more transparent, however, we prefer to only use a single common constant $C = C_m = C_a$. Then the

[7]This value is surprisingly close to $C = -5/3$ in large-β_0, which enters the construction of the invariant coupling in Eq. 1.18 in the $\overline{\rm MS}$ scheme, though, presumably, this is merely a coincidence.

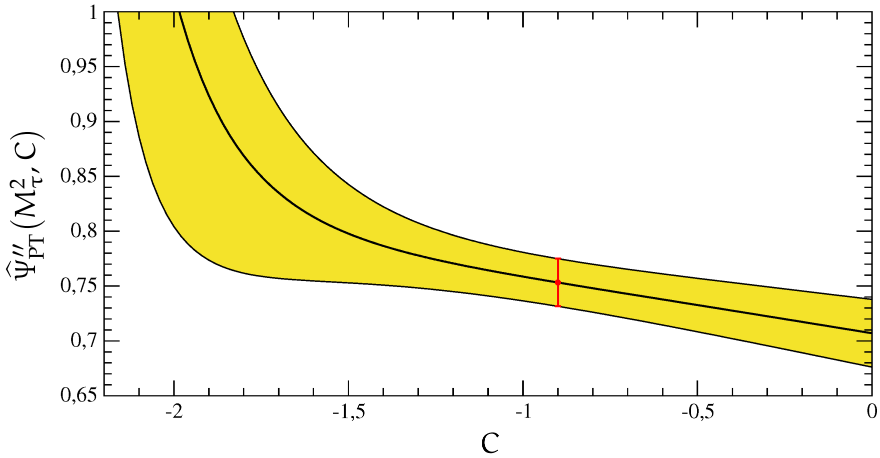


Figure 2.2: $\widehat{\Psi}''$ according to Eq. 2.65 as a function of C for $\alpha_s(M_\tau) = 0.316$. The yellow band corresponds to either removing or doubling the $\mathcal{O}(\widehat{a}^4)$ correction to estimate the respective uncertainty. At the red point, the uncertainty resulting from the $\mathcal{O}(\widehat{a}^4)$ contribution is minimal.

expansion in $\widehat{a}(Q)$ takes the form

$$\begin{aligned}\widehat{\Psi}''(s) = \big[\widehat{\alpha}_s(Q)\big]^{8/9}\Big\{&1 + (5.4568 + 2C)\widehat{a}(Q)\\ &+ (25.452 + 26.747C + 4.25C^2)\widehat{a}(Q)^2\\ &+ (142.44 + 212.99C + 94.483C^2 + 9.2083C^3)\widehat{a}(Q)^3\\ &+ (932.71 + 1625.0C + 1099.8C^2 + 291.95C^3 + 20.143C^4)\widehat{a}(Q)^4 + \mathcal{O}\big(\widehat{a}(Q)^5\big)\Big\}\,. \end{aligned} \tag{2.65}$$

The corresponding graphical representation of this result is displayed in Figure 2.2. In this case, the order $\widehat{a}^4$ correction does not vanish for any sensible value of C. The smallest uncertainty is assumed around $C \approx -0.9$, where we obtain

$$\widehat{\Psi}''_{PT}\big(-M_\tau^2, C = -0.9\big) = 0.753 \pm 0.022^{+0.050}_{-0.046} = 0.753^{+0.055}_{-0.051}\,. \tag{2.66}$$

The first error in Eq. 2.66 is the red band in Figure 2.2 and the second error again corresponds to the uncertainty induced from the error on α_s. In view of the large α_s error, the result in Eq. 2.66 is again fully compatible with Eq. 2.63 and Eq. 2.64.

Let us now turn to the decay of the Higgs boson into quark-antiquark pairs. The corresponding decay width is given by

$$\Gamma\big(H \to q\bar{q}\big) = \frac{\sqrt{2}\,G_F}{M_H}\mathrm{Im}\Psi\big(M_H^2 + i0\big) \equiv \frac{N_c G_F M_H}{4\pi\sqrt{2}}\,\widehat{m}_q^2\,\widehat{R}\big(M_H^2\big)\,, \tag{2.67}$$

where G_F is the Fermi constant. The function $\widehat{R}$ is defined so that the first coefficient of its perturbative expansion is normalised to 1.

We proceed in analogy to the case of Ψ'' by first expressing only the global prefactor

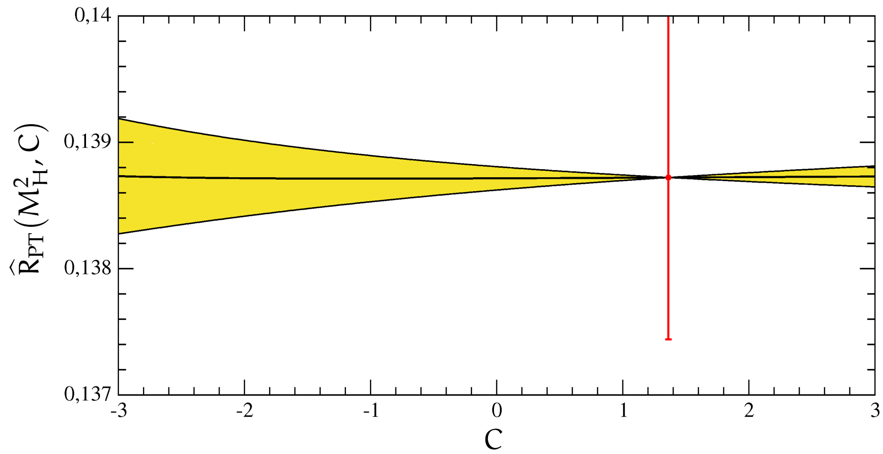


Figure 2.3: $\widehat{R}$ according to Eq. 2.68 as a function of C for $\alpha_s(M_H) = 0.1127$. The yellow band corresponds to either removing or doubling the $\mathcal{O}(a^4)$ correction to estimate the respective uncertainty. In the red point, where $\mathcal{O}(a^4)$ vanishes, the third order is taken as the error.

in terms of the coupling $\widehat{\alpha}_s$. This step results in

$$\begin{aligned}\widehat{R}(s) = \left[\widehat{\alpha}_s(Q)\right]^{24/23}\Big\{&1 + (8.0176 + 2C)a(Q) \\ &+ (46.732 + 18.557C + 0.08333C^2)a(Q)^2 \\ &+ (142.12 + 117.09C - 1.5384C^2 - 0.05093C^3)a(Q)^3 \\ &- (544.67 - 426.17C + 22.522C^2 - 2.2856C^3 - 0.04774C^4)a(Q)^4 + \mathcal{O}\big(a(Q)^5\big)\Big\}\,. \end{aligned} \tag{2.68}$$

Because here we are investigating the Higgs decay, we set the number of flavours $N_f = 5$ and the energy $s = M_H^2$.

For $\alpha_s(M_H) = 0.1127$, in Figure 2.3 we display a graphical representation of the truncated expansion of $\widehat{R}$ as a function of C. Because the coupling is much smaller at the Higgs scale than at the τ scale, the perturbative expansion converges faster in the present case, and thus the typical $\mathcal{O}(a^3)$ term is substantially larger than $\mathcal{O}(a^4)$ at $C = 1.362$, where this latter contribution vanishes. This is obvious from the large error bars in red. The corresponding numerical result reads:

$$\widehat{R}_{PT}\big(M_H^2, C = 1.362\big) = 0.1387 \pm 0.0013 \pm 0.0020 = 0.1387 \pm 0.0024\,, \tag{2.69}$$

where the second error again results from the uncertainty of $\alpha_s(M_H)$. Still, even though the large $\mathcal{O}(a^3)$ uncertainty has been assumed, the current error arising from the α_s input is still bigger.

Also for the Higgs decay, we express the remaining a expansion of Eq. 2.68 in powers

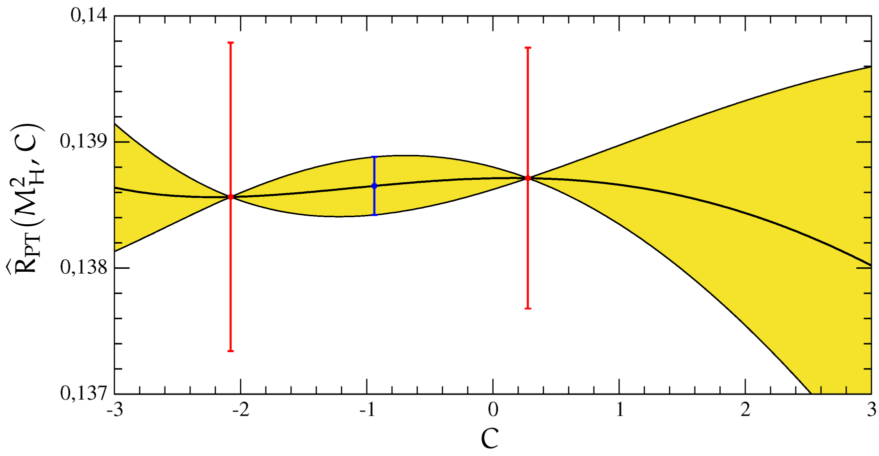


Figure 2.4: $\widehat{R}$ according to Eq. 2.70 as a function of C for $\alpha_s(M_H) = 0.1127$. The yellow band corresponds to either removing or doubling the $\mathcal{O}(\widehat{a}^4)$ correction to estimate the respective uncertainty. In the red points, where $\mathcal{O}(\widehat{a}^4)$ vanishes, the third order is taken as the error.

of $\widehat{a}$. This yields

$$\begin{aligned}\widehat{R}(s) = \left[\widehat{\alpha}_s(Q)\right]^{24/23}\Big\{1 &+ (8.0176 + 2C)\widehat{a}_Q \\ &+ (46.732 + 33.924C + 3.9167C^2)\widehat{a}_Q^2 \\ &+ (141.19 + 315.38C + 103.88C^2 + 7.6157C^3)\widehat{a}_Q^3 \\ &- (524.03 - 1491.9C - 1353.1C^2 - 277.97C^3 - 14.756C^4)\widehat{a}_Q^4 + \mathcal{O}\left(\widehat{a}^5\right)\Big\}, \end{aligned} \tag{2.70}$$

and the corresponding behaviour as a function of C is presented in Figure 2.4. This time we find two values of C where the $\mathcal{O}(\widehat{a}^4)$ correction vanishes, and they are again displayed as the red data points. In both cases, the corresponding uncertainty inferred from the size of the third order is much larger than a typical fourth order term. The corresponding numerical results are given by

$$\widehat{R}_{PT}\left(M_H^2, C = -2.079\right) = 0.1386 \pm 0.0012 \pm 0.0020 = 0.1386 \pm 0.0023\,, \tag{2.71}$$

and

$$\widehat{R}_{PT}\left(M_H^2, C = -0.277\right) = 0.1387 \pm 0.0010 \pm 0.0020 = 0.1387 \pm 0.0022\,, \tag{2.72}$$

where the second error once more is due to the α_s uncertainty and the final errors result from a quadratic average. In a situation like this, in our opinion a conservative estimate of higher-order corrections can be obtained by assuming the maximal $\mathcal{O}(\widehat{a}^4)$ correction between those two points and taking that as the perturbative uncertainty. This approach is shown as the blue point, and the numerical value reads

$$\widehat{R}_{PT}\left(M_H^2, C = -0.94\right) = 0.1387 \pm 0.0002 \pm 0.0020 = 0.1387 \pm 0.0020\,. \tag{2.73}$$

It is clear that now the higher-order uncertainty is completely negligible with respect to the present error in α_s.

To summarise, rewriting the perturbative expansion in terms of the coupling $\widehat{\alpha}_s$ of Eq. 1.25 introduces interesting approaches to improve the convergence of the series for the known low-order corrections, before the asymptotic behaviour sets in. We demonstrated this explicitly for the correlator Ψ'' at the scale M_τ and for the decay of the Higgs boson into quarks, which is related to ImΨ at the scale M_H. In both examples, however, the parametric uncertainty induced by the error on α_s dominates. This is in part due to the recent increase in the α_s uncertainty of the PDG average [8] by more than a factor of two, in view of an earlier analysis of α_s determinations from lattice QCD by the FLAG collaboration [40]. Hence, we expect our findings to increase in importance when the uncertainty on α_s again shrinks in the future. Still, in view of the potential to strengthen the sensitivity on α_s, our approach could also open promising options for improved non-lattice α_s determinations.

2.4 Conclusions

The scalar correlation function is one of the basic QCD two-point correlators with important phenomenological applications for the decay of the Higgs boson to quark-antiquark pairs [28], determinations of light quark masses from QCD sum rules [29, 30] and contributions to hadronic decays of the τ lepton [31, 32, 33]. Presently, the perturbative expansion of the scalar correlator is known up to order α_s^4 in the strong coupling [25].

Three physical functions related to the scalar correlator play a role for phenomenological studies: ImΨ in Higgs decay, Ψ'' in quark-mass extractions and D^L in finite-energy sum rule analyses of hadronic τ decays. From the known perturbative coefficients it is observed that the renormalisation-group resummed D^L only depends on the independent coefficients d_{n1} (see Eq. 2.25), and those corrections turn out much larger than the ones for Ψ'' and ImΨ, for which combinations of the d_{nk} with $k \geqslant 1$ appear. The coefficients d_{nk} with $k \geqslant 2$ are calculable from the renormalisation group equation and only depend on the lower-order d_{n1}, the β function coefficients, and the coefficients of the quark-mass anomalous dimension.

In order to understand this pattern of higher-order corrections better, we reviewed the results for the scalar correlator in the large-β_0 approximation [35], and derived compact expressions for the correlators Ψ'' and D^L in terms of Borel transforms, which directly give access to the renormalon structure of the respective correlators. While this structure in the case of Ψ'' is analogous to the one of the Adler function (double and single IR renormalon poles for $u \geqslant 2$, with only a single pole at $u = 2$, as well as double and single UV poles for $u \leqslant -1$), for the correlator D^L we found an additional single pole at $u = 1$. The origin of this spurious pole, which is suspected to be of UV origin, can be traced back to the divergent subtraction $\Psi(0)/s$ that is performed in the construction of D^L. While the pole at $u = 1$ is present in the coefficients d_{n1}, for Ψ'' and $\mathrm{Im}\Psi$ it is cancelled by corresponding contributions to the dependent coefficients d_{nk} with $k > 1$.

Another feature of the scalar correlator that becomes apparent from the large-β_0 approximation is the appearance of a regular contribution that is related to the renormalisation of the global mass factor m^2. By rewriting this prefactor in terms of the renormalisation group invariant quark mass $\widehat{m}_\psi$, we are left with the logarithmic term in Eq. 2.42, which depends on the leading order coefficients β_1 and $\gamma_m^{(1)}$, as well as the renormalisation scheme of the coupling in the prefactor. Expressing this prefactor in terms of the coupling $\widehat{\alpha}_s$ of Eq. 1.18 (corresponding to $C_m = 0$), which is a scheme invariant coupling in large-β_0, the regular logarithmic contribution is resummed. Improvements in the behaviour of the perturbative series were also discussed in Section 2.2, and it was concluded that this is in part due to shifting the contribution of UV renormalon poles, in particular the lowest-lying one at $u = -1$, to lower orders. Generally, however, it has to be acknowledged that for the scalar correlator the large-β_0 approximation does not provide a satisfactory representation of the full QCD case.

Phenomenological applications of reexpressing in terms of $\widehat{\alpha}_s$ the perturbative series of Ψ'' at the M_τ scale, and $\mathrm{Im}\Psi$ at the M_H scale, were investigated in Section 2.3. To this end, we considered two cases: a first, in which only the α_s prefactor, originating from the quark mass, is rewritten in $\widehat{\alpha}_s$, and the remaining series is kept in the $\overline{\mathrm{MS}}$ scheme; and a second case, in which the whole series is expressed in terms of the coupling $\widehat{\alpha}_s$. Generally, it can be concluded that appropriate choices of C allow for an improvement of the behaviour of the perturbative series for the first few known orders. This is, however, achieved at the expense of an increase in the value of the coupling, either only in the prefactor, or also in the remaining expansion terms, which leads to an increased sensitivity to α_s and to its uncertainty.

In an era in which just recently the error on the PDG average of the strong coupling [8] has increased by more than a factor of two, in view of an earlier analysis of α_s determinations from lattice QCD by the FLAG collaboration [40], we find that in all considered cases the uncertainty of our perturbative predictions is dominated by the error on α_s. Therefore, in the investigated examples, currently, improvements in the perturbative series appear to be a secondary issue. Still, when our knowledge on the value of α_s at some point returns to a precision comparable to previous estimates, the uncertainty due to higher-order corrections becomes of a similar size, and optimising the series by appropriate scheme choices through variation of the parameter C should allow for refined perturbative predictions.

Nevertheless, the increased sensitivity on α_s for certain ranges of C can also be taken as a virtue if one aims at determinations of α_s, for example from hadronic τ decay spectra along the lines of Refs. [14, 37, 38, 39], as this could result in reduced equivalent uncertainties in the $\overline{\text{MS}}$ coupling. In this respect, also analysing models for the Borel transform in the coupling $\widehat{\alpha}_s$, along the lines of Ref. [14], could provide additional helpful insights.

Since a substantial part of the improvements results from rewriting global prefactors of α_s, investigating other observables which include such factors and suffer from large perturbative corrections could be rather promising. These factors may either be explicitly present, like for example in gluonium correlation functions which carry a global factor α_s^2, or may emerge from quark-mass factors, similarly to the scalar correlator or the total semi-leptonic B-meson decay rate, which is proportional to m_b^5. It is to be expected that also in these applications the perturbative expansion can be improved by adequate scheme choices in the coupling $\widehat{\alpha}_s$.

Chapter 3

Absence of even ζ function values in Euclidean physical quantities

In the past, it has been noted several times that even-integer values of the Riemann ζ function are absent in the perturbative expansion of some QCD Euclidean physical quantities[1]. One prominent example is the Adler function up to order α_s^4 [13], and explanations for this behaviour were provided in the literature [41, 42, 43]. However, the regularity for example fails in the scalar quark correlation function [25] and the scalar gluonium correlator [44], both also being analytically available up to order α_s^4 in the $\overline{\text{MS}}$-scheme [2].

In this chapter, we shall demonstrate that both, the Euclidean physical scalar correlation function, as well as the scalar gluonium correlator, up to order α_s^4 are free of even-integer ζ function values when they are appropriately expressed in terms of the C-scheme coupling of Section 1.2 [45]. This result has been later demostrated for several other physical quantities in [46, 47]. We will also give additional arguments to explain why even-integer ζ function values have not yet appeared in the Adler function, but we conjecture that a $\zeta(4)$ term should appear at order α_s^5 in the $\overline{\text{MS}}$-scheme. We will further conjecture that this term should again cancel when the Adler function is expressed in $\widehat{\alpha}_s$ and that the same might hold for other Euclidean physical quantities.

[1]That is, physical quantities that are evaluated for $q^2 = s < 0$. In contrast, we have Minkowskian physical quantities, like the spectral functions, which are evaluated for physical energy $s > 0$.

3.1 Cancellation of even ζ's in the scalar correlator

As we have seen in Chapter 2, in the case of the scalar correlator, a Euclidean physical quantity is given by the second derivative of the correlation function $\Psi(s)$. For concreteness, we set $N_f = 3$ in Eq. 2.5–2.8. Then the coefficients $d''_{n1} = d_{n1} + 2d_{n2}$ of the perturbative expansion of Eq. 2.13 take the values [25]

$$d''_{11} = \frac{11}{3}\,, \tag{3.1}$$

$$d''_{21} = \frac{5071}{144} - \frac{35}{2}\zeta_3\,, \tag{3.2}$$

$$d''_{31} = \frac{1995097}{5184} - \frac{65869}{216}\zeta_3 - \frac{5}{2}\zeta_4 + \frac{715}{12}\zeta_5\,, \tag{3.3}$$

$$\begin{aligned} d''_{41} = \frac{2361295759}{497664} - \frac{25214831}{5184}\zeta_3 + \frac{192155}{216}\zeta_3^2 - \frac{14575}{576}\zeta_4 \\ + \frac{59875}{108}\zeta_5 - \frac{625}{48}\zeta_6 - \frac{52255}{256}\zeta_7\,. \end{aligned} \tag{3.4}$$

We observe that d''_{31} contains a ζ_4 term and d''_{41} has both ζ_4 and ζ_6.

For the ensuing discussion it will be essential to remove the running effects of the quark mass from the remaining perturbative series. This can be achieved by rewriting the running mass $m(Q)$ in terms of the invariant quark mass of Eq. 2.15. After this step, the coefficients r_n defined in the perturbative expansion of Eq. 2.16 become

$$r_1 = \frac{442}{81}\,, \tag{3.5}$$

$$r_2 = \frac{2449021}{52488} - \frac{335}{18}\zeta_3\,, \tag{3.6}$$

$$r_3 = \frac{24657869923}{51018336} - \frac{678901}{1944}\zeta_3 + \frac{18305}{324}\zeta_5\,, \tag{3.7}$$

$$\begin{aligned} r_4 = \frac{378986482023877}{66119763456} - \frac{21306070549}{3779136}\zeta_3 + \frac{601705}{648}\zeta_3^2 + \frac{445}{96}\zeta_4 \\ + \frac{3836150}{6561}\zeta_5 - \frac{3285415}{20736}\zeta_7\,. \end{aligned} \tag{3.8}$$

The r_4 coefficient depends on β function coefficients as well as quark-mass anomalous dimensions up to five loops [48]. Let us remark that the ζ_4 term that is present in d''_{31} and the ζ_6 term in d''_{41} are cancelled by these additional contributions, while ζ_4 still remains in r_4. The respective cancellations have also been observed in Ref. [41] for a related quantity.

As the last step, we reexpress the QCD coupling in terms of the C-scheme coupling using the perturbative relation of Eq. 1.27. For concreteness, we fix $C = 0$. The perturbative expansion of Ψ'' then becomes

$$\Psi''_{\mathrm{PT}}(s) = \frac{N_c}{8\pi^2}\frac{\widehat{m}^2}{Q^2}\left[\widehat{\alpha}_s(Q)\right]^{2\gamma_m^{(1)}/\beta_1}\left\{1+\sum_{n\geqslant 0}\widehat{r}_n\widehat{a}(Q)^n\right\}, \tag{3.9}$$

with the coefficients given by

$$\widehat{r}_1 = \frac{442}{81}, \tag{3.10}$$

$$\widehat{r}_2 = \frac{2510167}{52488} - \frac{335}{18}\zeta_3, \tag{3.11}$$

$$\widehat{r}_3 = \frac{12763567259}{25509168} - \frac{673561}{1944}\zeta_3 + \frac{18305}{324}\zeta_5, \tag{3.12}$$

$$\widehat{r}_4 = \frac{49275071521973}{8264970432} - \frac{10679302931}{1889568}\zeta_3 + \frac{601705}{648}\zeta_3^2 + \frac{117947335}{209952}\zeta_5 - \frac{3285415}{20736}\zeta_7. \tag{3.13}$$

We note that the coefficient β_5 has a ζ_4 term [49, 50]. This contribution then emerges through the global prefactor $[\alpha_s]^{2\gamma_m^{(1)}/\beta_1}$ into the perturbative expansion and cancels with the ζ_4 remaining in Eq. 3.8, such that only odd-integer ζ values persist. Even though we have just provided results for $N_f = N_c = 3$, we can also verify that the cancellation of even ζ values does in fact take place for an arbitrary number of flavours and number of colours. Furthermore, since the transformation in Eq. 1.28 only contains the β function coefficients β_1 and β_2, which are rational, the absence of even ζ values also remains true for a general C in the C-scheme coupling.

We remark that the cancellation of even ζ values only holds for Euclidean quantities, which are evaluated at $s < 0$. For instance, this cancellation will not take place for $\mathrm{Im}\Psi$, which is a Minkowskian physical quantity. As seen from the perturbative expansion in Eq. 2.18, factors π^2 arise from the logarithms $\log\left(-s/\mu^2\right)$ evaluated at the scale $\mu^2 = s > 0$. These powers of π^2 can then be reexpressed in terms of even ζ values like $\zeta_2 = \pi^2/6$, $\zeta_4 = \pi^4/90$, $\zeta_6 = \pi^6/945$ and so on, invalidating our result for Minkowskian physical quantities.

The origin of the ζ values can be traced back to the computation of the scalar correlator from Feynman diagrams. In dimensional regularisation with $d = 4 + 2\epsilon$ dimensions for the loop integrals, only the following three combinations of ζ values might appear:

$$\zeta_3 + \frac{3\epsilon}{2}\zeta_4 - \frac{5\epsilon^3}{2}\zeta_6, \quad \zeta_5 + \frac{5\epsilon}{2}\zeta_6, \quad \zeta_7. \tag{3.14}$$

This implies that if a loop integral is finite for $\epsilon \to 0$, then only odd ζ values will appear in this limit. However, when one considers a physical quantity, the renormalisation process may still introduce even ζ values arising from the renormalisation constants.

So, while loop integrals will never provide even ζ values, they can still appear in physical quantities. This observation was formally proved in Theorem 4 of [42] and, in this example, we have shown that these even ζ values coming from the renormalisation procedure can be eliminated by reexpressing the perturbative expansion of a physical quantity in the C-scheme coupling.

3.2 Cancellation of even ζ's in the gluonium correlator

A basic two-point correlation function that is relevant for the study of the scalar gluonium can be defined as

$$\Pi_{G^2}(q^2) = i \int dx\, e^{iqx} \langle \Omega | T\{J_G(x)\, J_G(0)\} | \Omega \rangle \,, \tag{3.15}$$

where the gluonic current is represented by $J_G(x) = G^a_{\mu\nu}(x)\, G_a^{\mu\nu}(x)$ and $G^a_{\mu\nu}(x)$ is the QCD field-strength tensor. In this case, a perturbative expansion is given by

$$\Pi^{PT}_{G^2}(s) = -\frac{N_c^2 - 1}{4\pi^2} s^2 \sum_{n \geqslant 0} a(\mu)^n \sum_{k=0}^{n+1} g_{nk} \log^k \left(\frac{-s}{\mu^2} \right) . \tag{3.16}$$

In order to be able to define a physical quantity, one should work with a renormalisation group invariant current. In the chiral limit, where the operator $J_G(x)$ does not mix with $m\,\overline{\psi}\psi$ or m^4, such a current can be chosen to be [51]

$$\widehat{J}_G(x) = \frac{\beta(a)}{\beta_1 a} J_G(x) = \frac{\beta(a)}{\beta_1 a} G^a_{\mu\nu}(x)\, G_a^{\mu\nu}(x) \,. \tag{3.17}$$

In analogy to $\Pi_{G^2}(q^2)$, we can then define the two-point correlator for the current $\widehat{J}_G(x)$, which expressed in terms of $\Pi_{G^2}(q^2)$ takes the form:

$$\widehat{\Pi}_{G^2}(s) = \left(\frac{\beta(a)}{\beta_1 a} \right)^2 \Pi_{G^2}(s) \,. \tag{3.18}$$

Because $\Pi_{G^2}(s)$ behaves like s^2 for large s, as can be seen from Eq. 3.16, a Euclidean physical quantity in analogy to the Adler function can be obtained by taking three derivatives of $\widehat{\Pi}_{G^2}(Q^2)$, leading to the definition

$$D_{G^2}(s) = -s \frac{d^3 \widehat{\Pi}_{G^2}(s)}{ds^3} \,. \tag{3.19}$$

The corresponding perturbative expansion then takes the following general form [52]:

$$D^{PT}_{G^2}(s) = \frac{N_c^2-1}{2\pi^2}\left(\frac{\beta(a)}{\beta_1 a}\right)^2 \sum_{n\geqslant 0} a(\mu)^n \sum_{k=1}^{n+1} k\, g_{nk}\left[2\log^{k-1}\left(\frac{-s}{\mu^2}\right)\right.$$
$$\left. + 3(k-1)\log^{k-2}\left(\frac{-s}{\mu^2}\right) + (k-1)(k-2)\log^{k-3}\left(\frac{-s}{\mu^2}\right)\right]. \quad (3.20)$$

Fixing the scale $\mu^2 = -s = Q^2$, we obtain the compact expression

$$D^{PT}_{G^2}(s) = \frac{N_c^2-1}{2\pi^2} a(Q)^2 \sum_{n\geqslant 0} g_n a(Q)^n\,, \quad (3.21)$$

where the coefficients g_n are combinations of the g_{nk} and β function coefficients coming from the front factor $(\beta(a))^2/(\beta_1 a)^2$. Up to order α_s^4, the coefficients g_n can be extracted from the results provided in Ref. [44]. Again at $N_f = N_c = 3$, they are:

$$g_0 = 1\,, \quad g_1 = \frac{104}{9}\,, \quad (3.22)$$

$$g_2 = \frac{87605}{648} - \frac{465}{8}\zeta_3\,, \quad (3.23)$$

$$g_3 = \frac{52031155}{31104} - \frac{216701}{144}\zeta_3 + \frac{10205}{24}\zeta_5\,, \quad (3.24)$$

$$g_4 = \frac{33122537939}{1492992} - \frac{1833382667}{62208}\zeta_3 + \frac{264275}{64}\zeta_3^2 + \frac{1335}{128}\zeta_4 + \frac{1478075}{128}\zeta_5 - \frac{2016175}{576}\zeta_7\,. \quad (3.25)$$

The coefficient g_4 in the $\overline{\text{MS}}$ scheme contains a ζ_4 term.

As in the case of the scalar correlator, we rewrite the perturbative series in Eq. 3.21 in terms of the C-scheme coupling $\hat{a}$:

$$D_{G^2}(s) = \frac{N_c^2-1}{2\pi^2}\hat{a}(Q)^2 \sum_{n\geqslant 0} \hat{g}_n \hat{a}(Q)^n\,. \quad (3.26)$$

For this expansion, the coefficients $\hat{g}_n$ are found to be:

$$\hat{g}_0 = 1\,, \quad \hat{g}_1 = \frac{104}{9}\,, \quad (3.27)$$

$$\hat{g}_2 = \frac{178607}{1296} - \frac{465}{8}\zeta_3\,, \quad (3.28)$$

$$\hat{g}_3 = \frac{20134253}{11664} - \frac{23979}{16}\zeta_3 + \frac{10205}{24}\zeta_5\,, \quad (3.29)$$

$$\hat{g}_4 = \frac{116204856235}{5038848} - \frac{690830641}{23328}\zeta_3 + \frac{264275}{64}\zeta_3^2 + \frac{59594845}{5184}\zeta_5 - \frac{2016175}{576}\zeta_7\,. \quad (3.30)$$

As expected, once again the ζ_4 term in g_4 has been cancelled by the corresponding contribution in β_5 emerging from the global prefactor α_s^2. Again in this we can also verify that the respective cancellation is independent of the number of flavours N_f and colours N_c.

3.3 Absence of even ζ's in the Adler function, an explanation

From the examples of Section 3.1 and Section 3.2, we conjecture that the perturbative coefficients of Euclidean physical quantities are absent of even ζ values to all orders[2] when they are written in the C-scheme coupling.

In the case of the Adler function, we note that the perturbative coefficients quoted in Eq. 1.39–1.42 are already absent of any even ζ function values. Starting from the above conjecture, we can now provide an explanation for the special characteristic of this physical quantity.

As in the previous examples, we transform the Adler function into the C-scheme coupling $\widehat{a}$ at $C = 0$. The corresponding expansion assumes the form

$$D_{PT}(s) = \frac{N_c}{12\pi^2} \sum_{n \geqslant 0} \widehat{c}_{n1} \widehat{a}(Q)^n \,. \tag{3.31}$$

Employing Eq. 1.27, only the coefficients $\widehat{c}_{31}$ and $\widehat{c}_{41}$ turn out different from the $\overline{\text{MS}}$ coefficients. They read:

$$\widehat{c}_{31} = \frac{262955}{1296} - \frac{779}{4}\zeta_3 + \frac{75}{2}\zeta_5 \,, \tag{3.32}$$

$$\widehat{c}_{41} = \frac{357259199}{93312} - \frac{1713103}{432}\zeta_3 + \frac{4185}{8}\zeta_3^2 + \frac{34165}{96}\zeta_5 - \frac{1995}{16}\zeta_7 \,. \tag{3.33}$$

Like c_{31} and c_{41}, the coefficients $\widehat{c}_{31}$ and $\widehat{c}_{41}$ still only include odd ζ values up to ζ_7, because the transformation in Eq. 1.27 only includes β coefficients up to β_4, which have ζ_3 as their sole irrational component. We compare to the previous examples, in which the prefactors $[\alpha_s]^{2\gamma_m^{(1)}/\beta_1}$ for the scalar correlator and α_s^2 for the gluonium correlator provided a contribution from β_5 already at the fourth order when reexpressing the perturbative expansions in the C-scheme coupling. Therefore, the coefficients c_{n1} of the perturbative expansion of the Adler function do not contain any even ζ values up to order α_s^4 due to the fact that the vector current has no anomalous dimension and, hence, no prefactor in α_s arises.

Still, at order α_s^5, we conjecture that the coefficient c_{51} will also contain a ζ_4 term which

[2] Even if presently we can only make this verification up to fourth order.

will cancel against the ζ_4 in β_5 once the Adler function is reexpressed in the C-scheme coupling. Under this assumption, we can predict the component of c_{51} proportional to ζ_4. At $N_c = 3$, but for arbitrary number of quark flavours N_f, we find

$$c_{51}^{\zeta_4} = \left(\frac{2673}{512} - \frac{1627}{4608}N_f + \frac{809}{6912}N_f^2\right)\zeta_4 \,. \tag{3.34}$$

3.4 Conclusions

In this work, we have demonstrated that the Euclidean physical quantities corresponding to the scalar quark and scalar gluonium correlators do not contain even-integer ζ function values in their perturbative coefficients up to the presently analytically available order α_s^4 when the perturbative expansion is written in terms of the C-scheme coupling $\widehat{\alpha}_s$. We have shown this explicitly for the coupling $\widehat{\alpha}_s$ at $C = 0$, but the statement remains true for an arbitrary C, since the relation in Eq. 1.28 only contains β_1 and β_2, which are rational numbers.

In the case of the Adler function, the perturbative coefficients in the $\overline{\text{MS}}$ scheme up to order α_s^4 do not contain even-integer ζ function values. This is related to the fact that the vector current has no anomalous dimension and, hence, no prefactor depending on α_s arises. We conjecture that a ζ_4 term will appear in the order α_s^5 coefficient c_{51}. Assuming that this term is again cancelled in the C-scheme by a corresponding term in the β_5 coefficient, in Eq. 3.34 we predict the component $c_{51}^{\zeta_4}$ proportional to ζ_4 in the perturbative expansion of the Adler function. This has been later confirmed in [47].

To our knowledge, at this moment, the cancellation of even-integer ζ function values for perturbative expansions of Euclidean physical correlators in the C-scheme coupling $\widehat{\alpha}_s$ can only be checked for the scalar quark and scalar gluonium correlation functions, as only these functions are available up to the required order α_s^4. Nonetheless, we conjecture that the same cancellation should also take place for any Euclidean quantities. It is exciting that this claim has been later reinforced in [46, 47].

Appendix **A**

Appendix of Part I

A.1 Renormalisation group functions

In our notation, the QCD β function and mass anomalous dimension are defined as:

$$-\mu\frac{da}{d\mu} = \beta(a) = \beta_1 a^2 + \beta_2 a^3 + \beta_3 a^4 + \beta_4 a^5 + \ldots\,, \tag{A.1}$$

$$-\frac{\mu}{m}\frac{dm}{d\mu} = \gamma_m(a) = \gamma_m^{(1)} a + \gamma_m^{(2)} a^2 + \gamma_m^{(3)} a^3 + \gamma_m^{(4)} a^4 + \ldots \tag{A.2}$$

It is assumed that we work in the minimal subtraction scheme $\overline{\mathrm{MS}}$. To make the presentation self-contained, below we provide the known coefficients of the β function and mass anomalous dimension in this convention. Numerically, for $N_c = 3$, the first five coefficients of the β function are given by [53, 54, 55, 49]

$$\beta_1 = \frac{11}{2} - \frac{1}{3}N_f\,, \tag{A.3}$$

$$\beta_2 = \frac{51}{4} - \frac{19}{12}N_f\,, \tag{A.4}$$

$$\beta_3 = \frac{2857}{64} - \frac{5033}{576}N_f + \frac{325}{1728}N_f^2\,, \tag{A.5}$$

$$\begin{aligned}\beta_4 = \frac{149753}{768} + \frac{891}{32}\zeta_3 - \left(\frac{1078361}{20736} + \frac{1627}{864}\zeta_3\right) N_f \\ + \left(\frac{50065}{20736} + \frac{809}{1296}\zeta_3\right) N_f^2 + \frac{1093}{93312}N_f^3\,,\end{aligned} \tag{A.6}$$

$$
\begin{aligned}
\beta_5 = {} & \frac{8157455}{8192} + \frac{621885}{1024}\zeta_3 - \frac{88209}{1024}\zeta_4 - \frac{144045}{256}\zeta_5 \\
& - \left(\frac{336460813}{995328} + \frac{1202791}{10368}\zeta_3 - \frac{33935}{3072}\zeta_4 - \frac{1358995}{13824}\zeta_5\right) N_f \\
& + \left(\frac{25960913}{995328} + \frac{698531}{41472}\zeta_3 - \frac{5263}{2304}\zeta_4 - \frac{5965}{648}\zeta_5\right) N_f^2 \\
& - \left(\frac{630559}{2985984} + \frac{24361}{62208}\zeta_3 - \frac{809}{6912}\zeta_4 - \frac{115}{1152}\zeta_5\right) N_f^3 \\
& + \left(\frac{1205}{1492992} - \frac{19}{5184}\zeta_3\right) N_f^4 .
\end{aligned} \tag{A.7}
$$

And the first five coefficients of γ_m are given by [56, 48]

$$\gamma_m^{(1)} = 2\,, \tag{A.8}$$

$$\gamma_m^{(2)} = \frac{101}{12} - \frac{5}{18} N_f\,, \tag{A.9}$$

$$\gamma_m^{(3)} = \frac{1249}{32} - \left(\frac{277}{108} + \frac{5}{3}\zeta_3\right) N_f - \frac{35}{648} N_f^2\,, \tag{A.10}$$

$$
\begin{aligned}
\gamma_m^{(4)} = {} & \frac{4603055}{20736} + \frac{1060}{27}\zeta_3 - \frac{275}{4}\zeta_5 - \left(\frac{91723}{3456} + \frac{2137}{72}\zeta_3 - \frac{55}{8}\zeta_4 - \frac{575}{36}\zeta_5\right) N_f \\
& + \left(\frac{2621}{15552} + \frac{25}{36}\zeta_3 - \frac{5}{12}\zeta_4\right) N_f^2 - \left(\frac{83}{7776} - \frac{1}{54}\zeta_3\right) N_f^3\,,
\end{aligned} \tag{A.11}
$$

$$
\begin{aligned}
\gamma_m^{(5)} = {} & \frac{99512327}{82944} + \frac{23201233}{62208}\zeta_3 + \frac{3025}{16}\zeta_3^2 - \frac{349063}{2304}\zeta_4 - \frac{28969645}{15552}\zeta_5 \\
& + \frac{15125}{32}\zeta_6 + \frac{25795}{32}\zeta_7 - \left(\frac{150736283}{746496} + \frac{391813}{1296}\zeta_3 + \frac{2365}{144}\zeta_3^2 \right. \\
& \left. - \frac{1019371}{6912}\zeta_4 - \frac{12469045}{31104}\zeta_5 + \frac{39875}{288}\zeta_6 + \frac{56875}{432}\zeta_7\right) N_f + \left(\frac{660371}{186624} \right. \\
& \left. + \frac{251353}{15552}\zeta_3 + \frac{725}{216}\zeta_3^2 - \frac{41575}{3456}\zeta_4 - \frac{33005}{5184}\zeta_5 + \frac{2875}{432}\zeta_6\right) N_f^2 \\
& + \left(\frac{91865}{746496} + \frac{803}{2592}\zeta_3 + \frac{7}{72}\zeta_4 - \frac{10}{27}\zeta_5\right) N_f^3 \\
& - \left(\frac{65}{31104} + \frac{5}{1944}\zeta_3 - \frac{1}{216}\zeta_4\right) N_f^4 .
\end{aligned} \tag{A.12}
$$

A.2 Renormalisation dependent coefficients of the scalar correlator

The dependent perturbative coefficients d_{nk} (with $k > 1$) appearing in the perturbative expansion of the scalar correlator (see Eq. 2.4) can be expressed in terms of the independent coefficients d_{n1}, and the coefficients of the β-function and the mass anomalous dimension. In particular, the coefficients d_{n2}, which are required in the second derivative of the scalar correlator (see Eq. 2.13), take the form

$$d_{n2} = -\frac{\gamma_m^{(n)}}{2} d_{01} - \frac{1}{4} \sum_{k=1}^{n-1} \left(2\gamma_m^{(n-k)} + k\beta_{n-k}\right) d_{k1} \,. \tag{A.13}$$

Explicitly, at $N_c = 3$ and up to the fourth order, they read:

$$d_{12} = -1 \,, \tag{A.14}$$

$$d_{22} = -\frac{53}{3} + \frac{11}{18} N_f \,, \tag{A.15}$$

$$d_{32} = -\frac{49349}{144} + \frac{585}{8}\zeta_3 + \left(\frac{11651}{432} - \frac{59}{12}\zeta_3\right) N_f - \left(\frac{275}{648} - \frac{1}{9}\zeta_3\right) N_f^2 \,, \tag{A.16}$$

$$\begin{aligned} d_{42} = &-\frac{49573615}{6912} + \frac{535759}{192}\zeta_3 - \frac{30115}{96}\zeta_5 + \left(\frac{56935973}{62208} - \frac{243511}{864}\zeta_3 + \frac{5}{6}\zeta_4 \right. \\ &\left. + \frac{1115}{48}\zeta_5\right) N_f - \left(\frac{6209245}{186624} - \frac{250}{27}\zeta_3 + \frac{25}{36}\zeta_5\right) N_f^2 \\ &+ \left(\frac{985}{2916} - \frac{5}{54}\zeta_3\right) N_f^3 \,. \end{aligned} \tag{A.17}$$

A.3 The coefficients $\mathcal{D}_n^{(1)}$ and $H_n^{(1)}$

Here, we provide the coefficients $\mathcal{D}_n^{(1)}$ and $H_n^{(1)}$ required to predict the perturbative coefficients d_{n1} in the large-β_0 approximation of the scalar correlator up to fifth order. They read:

$$\begin{aligned} \mathcal{D}_1^{(1)} &= -1 \,, & \mathcal{D}_2^{(1)} &= -\frac{22}{3} \,, \\ \mathcal{D}_3^{(1)} &= -\frac{275}{6} + 12\zeta_3 \,, & \mathcal{D}_4^{(1)} &= -\frac{7880}{27} + 80\zeta_3 \,, \\ \mathcal{D}_5^{(1)} &= -\frac{324385}{162} + \frac{1000}{3}\zeta_3 + 600\zeta_5 \,, & \mathcal{D}_6^{(1)} &= -\frac{1224355}{81} + \frac{10000}{9}\zeta_3 + 6000\zeta_5 \,, \end{aligned} \tag{A.18}$$

and

$$\begin{aligned} H_2^{(1)} &= \frac{51}{2} \,, & H_3^{(1)} &= -\frac{585}{8} + 18\zeta_3 \,, \\ H_4^{(1)} &= \frac{15511}{72} - 54\zeta_3 \,, & H_5^{(1)} &= -\frac{520771}{576} + \frac{585}{4}\zeta_3 + \frac{27}{4}\zeta_4 + 270\zeta_5 \,, \\ & & H_6^{(1)} &= \frac{19577503}{4320} - \frac{2021}{6}\zeta_3 - \frac{9}{2}\zeta_4 - \frac{8946}{5}\zeta_5 \,. \end{aligned} \tag{A.19}$$

A.4 The subtraction constant $\Psi(0)$

In order to understand the structure of the subtraction constant $\Psi(0)$, examining the lowest perturbative order is sufficient. For definiteness, we consider the case of the current in Eq. 2.2 that plays a role in hadronic τ decays. $\Psi(0)$ receives contributions from the normal-ordered quark condensate and a perturbative term proportional to m^4. At lowest order it reads:

$$\begin{aligned}\Psi(0) = -(m_u - m_s)\big[\,\langle\Omega|{:}\bar{u}u{:}|\Omega\rangle - \langle\Omega|{:}\bar{s}s{:}|\Omega\rangle\,\big] \\ + 4iN_c(m_u - m_s)\big[m_u I_{m_u} - m_s I_{m_s}\big]\,,\end{aligned} \tag{A.20}$$

where I_m is the UV divergent massive scalar vacuum-bubble integral

$$\begin{aligned} I_m &= \mu^{2\varepsilon}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2 - m^2 + i0)} \\ &= \frac{i}{(4\pi)^2} m^2 \left\{-\frac{1}{\epsilon} - \gamma + \log(4\pi) - \log\left(\frac{m^2}{\mu^2}\right) + 1 + \mathcal{O}(\epsilon)\right\}. \end{aligned} \tag{A.21}$$

The explicit expression for I_m has been provided in dimensional regularisation with $d = 4 + 2\epsilon$, but the particular regularisation scheme is inessential to our argument.

Precisely the same massive scalar vacuum-bubble contribution as in the second term of Eq. A.20 also arises when rewriting the normal-ordered condensates in terms of non-normal-ordered minimally subtracted quark condensates [57, 58]. Therefore, $\Psi(0)$ can also be expressed as

$$\Psi(0) = -(m_u - m_s)\big[\,\langle\Omega|\,\bar{u}u\,|\Omega\rangle - \langle\Omega|\,\bar{s}s\,|\Omega\rangle\,\big]\,, \tag{A.22}$$

which absorbs the mass logarithms in the definition of the quark condensate. Due to a Ward identity, the condensate contribution in $\Psi(0)$ does not receive higher-order corrections, and at least at next-to-leading order, it has been checked that the perturbative term matches the vacuum-bubble structure that arises when rewriting $\langle\Omega|{:}\bar{\psi}\psi{:}|\Omega\rangle$ in terms of $\langle\Omega|\,\bar{\psi}\psi\,|\Omega\rangle$ [59]. It is expected that this behaviour, and hence also the form of Eq. A.22, should remain the same to all orders. As an aside, it may be remarked that for the pseudoscalar channel the combination in Eq. A.22 with flavour sums of quark masses as well as condensates is precisely what appears in the Gell-Mann-Oakes-Renner relation [60, 61].

As we have seen, the subtraction constant $\Psi(0)$ suffers from a UV divergence originating from the perturbative quark-mass correction in Eq. A.20. Even though this contribution can be absorbed in the definition of the quark condensate by rewriting normal-ordered in terms of non-normal-ordered condensates, because of the subtraction of $\Psi(0)/s$, the UV divergence reflects itself in the spurious renormalon at $u = 1$ in the correlation function $D^{L}_{\beta_0}$ of Eq. 2.56.

Part II

Borel summation and resurgence

Chapter 4

Borel summation of divergent series

The asymptotic expansion of a function f encodes the limiting behaviour of $f(z)$, in terms of «simpler» functions, when the variable z approaches a certain point of interest. A typical example of asymptotic expansions are Taylor series. However, these are very special, because they converge in a disc of non-vanishing radius. In general, asymptotic expansions can diverge. That is, their sum gives an infinite answer for all values of the variable except, perhaps, at one point.

In this work, we consider a general field theory with action $S[\phi, g]$, where ϕ is a field and g is a coupling that parametrises the strength of the field interaction. The expectation value of a functional $\mathcal{O}[\phi]$—the objects of interest of the theory—is given by the Euclidean path integral

$$\langle \mathcal{O} \rangle(g) = \frac{1}{Z(g)} \int [d\phi]\, e^{-S[\phi, g]}\, \mathcal{O}[\phi]\,, \tag{4.1}$$

where $Z(g)$ is the partition function.

We assume that the free theory ($g = 0$) reduces to a multivariate normal distribution, so we have a method to compute any desired expectation value of a polynomial in the fields for $g = 0$ (either with Isserlis' or Wick's theorem). Using perturbation theory, we may then compute expectation values in the interacting theory ($g \neq 0$) as an expression in powers of g:

$$\langle \mathcal{O} \rangle(g) \sim \sum_{n \geqslant 0} a_n g^n\,, \quad g \to 0^+, \tag{4.2}$$

which is to be understood as an asymptotic expansion for $\langle \mathcal{O} \rangle(g)$. In fact, the series diverges for all values of $g \neq 0$. The coefficients a_n, given by combinations of expectation values in the free theory, are factorially divergent with n [23, 62, 63].

In this context, it is interesting to ask if we can recover $\langle \mathcal{O} \rangle(g)$ only from its asymptotic expansion. With the method of Borel summation, we can assign a function to any factorially divergent series, so that the divergent series is an asymptotic expansion of the resulting function. Thus, we could be tempted to identify the Borel sum of the asymptotic expansion in Eq. 4.2 with $\langle \mathcal{O} \rangle(g)$ itself. This, however, might not be true in general, because there is not an injective correspondence between asymptotic expansions and functions.[1] For example, the functions

$$\frac{1}{1-x} \quad \text{and} \quad \frac{1}{1-x} + e^{-1/x} \tag{4.3}$$

both have the same asymptotic expansion, $\sum_{n\geqslant 0} x^n$, in the limit $x \to 0^+$, because the exponential term is hidden beyond all orders of this expansion. Therefore, Borel summation of a divergent series only gives one possible answer amongst many.

In this chapter, we will first cover some basics of complex analysis, namely, analytic continuations and Riemann surfaces (in Section 4.1 and Section 4.2, respectively). This is a prerequisite topic for the discussion of the main body of the chapter, from Section 4.3 to Section 4.8, where we will discuss different examples of factorially divergent series, we will sum them with the method of Borel summation and we will determine the properties of the Borel sum as a complex analytic function in the Riemann surface. In Section 4.6, we will understand these properties in the framework of alien calculus and resurgence. In Section 4.7 and Section 4.8, we will move from strictly academic examples to physically motivated examples, employing the ideas presented in the chapter to 0-dimensional path integrals. This will cover the basics of Borel summation and resurgence, which we will need to fully understand the results of Chapter 5.

4.1 Analytic continuation of a complex function

Analytic continuations will be a central tool to this work, so we start by fixing its definition.

Definition 1 (Analytic continuation)**.** Let f_1 and f_2 be holomorphic functions on domains Ω_1 and Ω_2, respectively. Suppose that $\Omega_1 \cap \Omega_2 \neq \varnothing$ and that $f_1 = f_2$ on $\Omega_1 \cap \Omega_2$. Then f_2 is called an analytic continuation of f_1 to Ω_2, and vice versa.

Analytic continuations are used many times in the context of power series. For example, the power series $\sum_{n\geqslant 0} z^n$ is convergent for $|z| < 1$ and, thus, it defines an holomorphic function in this disc, which we denote D. Also, the series is divergent at

[1] This is even true for convergent asymptotic expansions. The problem arises not because an expansion is divergent, but because it is asymptotic.

all points $|z| > 1$. Furthermore, $1/(1-z)$ is an holomorphic function in $\mathbb{C} \setminus \{1\}$ and

$$\sum_{n\geqslant 0} z^n = \frac{1}{1-z} \quad \text{in } D \cap \mathbb{C} \setminus \{1\} = D\,. \tag{4.4}$$

Thus, we say that $1/(1-z)$ provides the analytic continuation of the power series to $\mathbb{C} \setminus \{1\}$ and, in light of the unicity of the analytic continuation, we would normally use both the power series and $1/(1-z)$ interchangeably, as if they were the same function.

Another useful definition is that of analytic continuation along a curve.

Definition 2 (Analytic continuation along a curve)**.** We consider a curve $t \mapsto \gamma(t)$ with $t \in [0,1]$. We define the collection of pairs (f_t, D_t) where D_t is a disc centred at $\gamma(t)$ and f_t is an holomorphic function in D_t. Further suppose that, for each fixed t, values t' close to t satisfy $\gamma(t') \in D_t$ and the functions f_t and $f_{t'}$ coincide in $D_t \cap D_{t'}$. Then, we say that the analytic continuation of f_0 along the curve γ is f_1.

The principal branch of the logarithm, which we denote by Log, is holomorphic in $\mathbb{C} \setminus \mathbb{R}^-$ and takes real values for $z > 0$. As an example, we will compute the analytic continuation of $f_0(z) \equiv \mathrm{Log}(z)$ along the curve $\gamma(t) = e^{2\pi i t}$, $t \in [0,1]$ (a circle of radius 1 that winds once around the origin in a counter-clockwise direction). We will choose the family of D_t as discs of radius 1 centred at $\gamma(t)$. Now we have to determine the family of functions f_t satisfying the conditions of Definition 2.

First, we choose

$$f_t(z) = \mathrm{Log}(z)\,, \quad t \in [0, 1/4]\,. \tag{4.5}$$

The discs D_t never cross the cut $\mathbb{R}^-$ in this interval of t, thus all of the f_t are holomorphic in D_t and they trivially take the same values in the intersection of the discs.

Next, we pick

$$f_t(z) = \mathrm{Log}(z) + \begin{cases} 0 & \text{if } \mathrm{Im}(z) \geqslant 0 \\ 2\pi i & \text{if } \mathrm{Im}(z) < 0\,, \end{cases} \quad t \in (1/4, 3/4]\,. \tag{4.6}$$

It is easy to check that the piecewise part cancels exactly the discontinuity of Log along $\mathbb{R}^-$ and the resulting function is discontinuous along $\mathbb{R}^+$ instead. Thus, Eq. 4.6 is holomorphic in $\mathbb{C} \setminus \mathbb{R}^+$. In particular, the f_t are holomorphic in their corresponding discs D_t. It is obvious that the f_t are equal in the intersection of their discs D_t when $t \in (1/4, 3/4]$. In addition, because the piecewise part does not modify $\mathrm{Log}(z)$ in the region $\mathrm{Im}(z) \geqslant 0$, then for any $t \in (1/4, 3/4]$ and $t' \in [0, 1/4]$, $f_{t'} = f_t$ in the intersection of the corresponding discs.

Finally, we choose

$$f_t(z) = \mathrm{Log}(z) + \begin{cases} 0 \\ 2\pi i \end{cases} + \begin{cases} 2\pi i & \text{if } \mathrm{Im}(z) \geqslant 0 \\ 0 & \text{if } \mathrm{Im}(z) < 0\,, \end{cases} \quad t \in (3/4, 1]\,. \tag{4.7}$$

The second piecewise term ensures that the f_t in Eq. 4.7 are holomorphic in $\mathbb{C} \setminus \mathbb{R}^-$. Moreover, the first piecewise term is the one we have already added in the previous step and it ensures that the f_t in the intervals $t \in (1/4, 3/4]$ and $t \in (3/4, 1]$ coincide in the intersection of the corresponding discs.

Setting $t = 1$ in Eq. 4.7, we conclude that the analytic continuation of $f_0(z) = \mathrm{Log}(z)$ along $\gamma(t) = e^{2\pi i t}$, $t \in [0, 1]$, is

$$\mathrm{cont}_\gamma\big(\mathrm{Log}(z)\big) \equiv f_1(z) = \mathrm{Log}(z) + 2\pi i\,. \tag{4.8}$$

It is easy to be convinced that this computation is insensitive to the details of the curve γ. As long as the curve winds one time around 0 in a counter-clockwise direction, the result remains the same. Of course, the result also does not depend on the exact shape of the discs D_t (actually, they can be an arbitrary domain, as long as they contain the point $\gamma(t)$ and f_t is analytic inside the corresponding domain).

Eq. 4.8 can be generalised to curves γ_k winding $k \in \mathbb{Z}$ times around 0, with negative k understood as $-k$ winds around 0 in a clockwise direction. The generalisation is given by

$$\mathrm{cont}_{\gamma_k}\big(\mathrm{Log}(z)\big) = \mathrm{Log}(z) + 2\pi i k\,. \tag{4.9}$$

From Eq. 4.9 we can compute the analytic continuation along γ_k of many different functions with a single branch point at $z = 0$. For example, the function

$$\sqrt{z} = e^{\frac{1}{2}\mathrm{Log}(z)} \tag{4.10}$$

has the analytic continuations

$$\mathrm{cont}_{\gamma_k}\left(\sqrt{z}\right) = e^{\frac{1}{2}\left(\mathrm{Log}(z) + 2\pi i k\right)} = (-1)^k \sqrt{z}\,. \tag{4.11}$$

4.2 The Riemann surface of the logarithm

The Riemann surface is an extension of the complex plane $\mathbb{C}$ defined, heuristically, by

$$\widetilde{\mathbb{C}} \equiv \left\{ r\underline{e}^{i\theta} \mid r > 0,\, \theta \in \mathbb{R} \right\}, \tag{4.12}$$

where $\underline{e}$ denotes an special exponential function which we define so that it is bijective. That is, each $\theta \in \mathbb{R}$ corresponds to a different element in $\widetilde{\mathbb{C}}$. This is opposite to the standard exponential function, in which values of θ that differ by 2π would yield the same element. The Riemann surface, topologically, can be understood as a infinitely numerable set of planes $\mathbb{C} \setminus \{0\}$ which are continuously and successively connected one to the other in a spiral shape with 0 being the axis of this spiral.

It is a direct consequence of the exponential $\underline{e}$ being bijective that its inverse, which we call $\mathscr{L}\text{og}$, exists and is also bijective. This function is defined by

$$\mathscr{L}\text{og}\left(r\underline{e}^{i\theta}\right) \equiv \log(r) + i\theta \in \mathbb{C}\,, \tag{4.13}$$

where log denotes the standard logarithm of real variable.

We now want to restrict the domain of $\mathscr{L}\text{og}$ from $\widetilde{\mathbb{C}}$ to $\mathbb{C} \setminus \{0\}$, which is equivalent to replacing $\underline{e}$ by e in Eq. 4.13. However, with this replacement, values of θ differing by a multiple of 2π correspond to the same complex number $re^{i\theta}$, but their image is different. This means that the logarithm has become a multivalued function. To solve this problem, we have to choose a branch of $\mathscr{L}\text{og}$. One way to do this is to restrict the range of θ to a semi-open interval I of length 2π. In this way, no two different angles $\theta_1, \theta_2 \in I$ are separated by a multiple of 2π and the logarithm becomes well defined when making this restriction on the angles.

For example, when we fix $I = (-\pi, \pi]$, we are choosing the principal branch of $\mathscr{L}\text{og}$. Another choice is $I = [0, 2\pi)$, which yields a branch of $\mathscr{L}\text{og}$ with a discontinuity along the positive real axis. A more unconventional choice would be $I = (-\pi, 0) \cup [2\pi, 3\pi)$; even if I is not an interval, no two angles inside I are separated by a multiple of 2π. In this last case, the corresponding branch would be discontinuous along all $\mathbb{R}$ (with two cuts originating from $z = 0$).

It is interesting to note the following identity resulting from the definition of Eq. 4.13:

$$\mathscr{L}\text{og}\left(z\underline{e}^{2\pi i k}\right) = \mathscr{L}\text{og}(z) + 2\pi i k\,, \quad z \in \widetilde{\mathbb{C}},\ k \in \mathbb{Z}\,. \tag{4.14}$$

It says that, for each fixed $z \in \widetilde{\mathbb{C}}$, when going up k sheets directly above z in the Riemann surface, the logarithm changes by $2\pi i k$. Comparing with the result of Eq. 4.9, we see that this notion is related to the notion of analytic continuation along a curve that winds k times around 0. Namely, given γ_k and a function $\mathscr{F}$ holomorphic in $\widetilde{\mathbb{C}}$, the relationship reads:

$$\mathscr{F}\left(z\underline{e}^{2\pi i k}\right) = \text{cont}_{\gamma_k}\left(\mathscr{F}(z)\right)\,, \quad z \in \mathbb{C} \setminus \{0\}\,. \tag{4.15}$$

In simple words, the values a function takes in different sheets of the Riemann surface can be computed with an analytic continuation around the origin.

It will be useful to consider the notion of discontinuity of a function from the point of view of the Riemann surface.

Definition 3 (Discontinuity of a multivalued function)**.** The discontinuity of a multivalued function $\mathscr{F}$ defined in the Riemann surface of the logarithm is given by

$$\operatorname{Disc}\mathscr{F}(z) = \mathscr{F}(z) - \mathscr{F}\big(z\underline{e}^{-2\pi i}\big)\,, \quad z \in \widetilde{\mathbb{C}}\,. \tag{4.16}$$

In contrast to the standard definition $\operatorname{Disc} f(z) = f(-z + i0) - f(-z - i0)$ valid for $z > 0$, our definition is an extension to any z in the Riemann surface. This make sense, as the branch cut of a multivalued function is arbitrary (the only restriction is that it has to connect the branch point at $z = 0$ with ∞), thus we may compute the discontinuity of the branch cut at any desired point. Because $f(z)$ and $f\big(z\underline{e}^{-2\pi i}\big)$ are both analytic functions, then the difference $\operatorname{Disc} f(z)$ is itself an analytic function.

4.3 The Borel sum and the Euler series

In this section, we will discuss the most basic properties of Borel summation. All of our results will be motivated in the context of a simple example, the Euler series, but we will make the statements in all generality.

We consider the Euler series, which is the divergent series given by

$$\widetilde{\varphi}(z) = \sum_{n \geqslant 0} \frac{(-1)^n n!}{z^{n+1}}\,, \tag{4.17}$$

and whose Borel transform is

$$\widehat{\varphi}(\zeta) \equiv \mathcal{B}[\widetilde{\varphi}](\zeta) = \sum_{n \geqslant 0} (-1)^n \zeta^n = \frac{1}{1+\zeta}\,. \tag{4.18}$$

We then define the Borel sum of $\widetilde{\varphi}(z)$ as

$$\mathscr{L}[\widetilde{\varphi}](z) \equiv \mathcal{L}\mathcal{B}[\widetilde{\varphi}](z) = \mathcal{L}[\widehat{\varphi}](z) = \int_0^\infty \frac{e^{-z\zeta}}{1+\zeta}\,d\zeta\,, \tag{4.19}$$

where $\mathcal{L}$ is the Laplace transform. This is the conventional method to assign a finite value to a factorially divergent series.

Our first observation is that, if $|\widehat{\varphi}(\zeta)| \leqslant Ke^{A|\zeta|}$ (we say $\widehat{\varphi}$ is exponentially bounded), then

$$\int_0^\infty |d\zeta|\, \left|e^{-z\zeta}\right|\, |\widehat{\varphi}(\zeta)| \leqslant K \int_0^\infty d\zeta\, e^{-(\mathrm{Re}(z)-A)\zeta} \tag{4.20}$$

and the last integral converges if and only if $\mathrm{Re}(z) - A > 0$. Absolute convergence ensures that the Laplace transform defines an analytic function in $\mathrm{Re}(z) > A$.[2]

Before continuing, we state the following theorem, which presents an interpretation for the Borel sum of a divergent series. See [64, Theo. 7.2] for the proof.

Theorem 1. *If* $\widetilde{\varphi}(z) = \sum a_n/z^{n+1}$ *and* $|\widehat{\varphi}(\zeta)| \leqslant K e^{A|\zeta|}$ *in a strip* $S_\sigma = \bigcup_{\zeta_0 \in \mathbb{R}^+} D\left(\zeta_0, \frac{1}{\sigma}\right)$, *then*

$$\left|\mathscr{L}^0[\widetilde{\varphi}](z) - a_0 - a_1/z - \cdots - a_{N-1}/z^N\right| \leqslant L(N+1)!\big(\sigma/|z|\big)^{N+1}, \qquad \mathrm{Re}(z) > A\,, \tag{4.21}$$

(where $L, \sigma > 0$*) and we say* $\mathscr{L}^0[\widetilde{\varphi}]$ *admits* $\widetilde{\varphi}$ *as an uniform asymptotic expansion in* $\mathrm{Re}(z) > A$.

In particular, Eq. 4.21 implies that the divergent series $\widetilde{\varphi}(z)$ is an asymptotic expansion to its own Borel sum (however, Eq. 4.21 is in fact an stronger statement). In this sense, it is legitimate to assign finite values to an object that is by definition divergent and might be considered mathematically ill-defined. In fact, asymptotic expansions of many common functions are divergent, but still they are well-defined mathematical objects that correctly encode the asymptotic behaviour of the function in the limit $z \to \infty$.

Furthermore, Eq. 4.21 gives a numerical interpretation for the divergent series. The truncation of the series in Eq. 4.17 gives a numerical approximation of the Borel sum up to an error $L(N+1)!\big(\sigma/|z|\big)^{N+1}$. For fixed large z, the truncation to the first terms gradually yields better approximations, because $|z|^{N+1}$ increases faster than $(N+1)!$. However, there is a turning point where the factorial starts to grow faster than the powers of $|z|$ and then the approximation becomes worse for each additional term incorporated into the truncated series. This is what we observe in Figure 4.1.

In this work, we will be interested in a more general definition of Borel sum:

Definition 4 (Borel sum along the direction of θ)**.** Given a formal series $\widetilde{\varphi}$, its Borel sum along the direction of θ is

$$\mathscr{L}^\theta[\widetilde{\varphi}](z) \equiv \mathcal{L}^\theta \mathcal{B}[\widetilde{\varphi}](z) = \int_0^{\infty e^{i\theta}} e^{-z\zeta}\, \widehat{\varphi}(\zeta)\, d\zeta\,, \tag{4.22}$$

where the operator $\mathcal{L}^\theta$ is the Laplace transform along the line $(0, \infty e^{i\theta})$.

[2] This conclusion results from a combination of Morera's and Fubini's theorems.

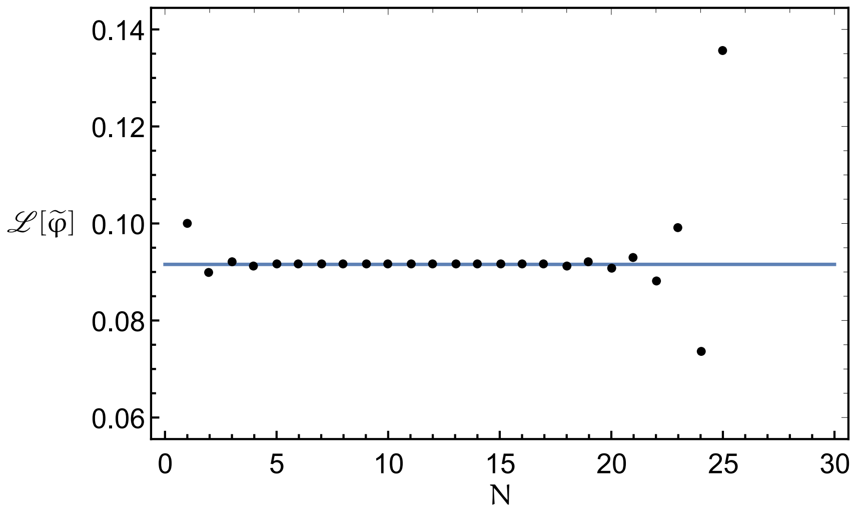


Figure 4.1: Asymptotic approximations to the Borel sum of Eq. 4.19, with $z = 10$, given by the truncated series of Eq. 4.17 up to the term z^N. The blue line corresponds to the exact Borel sum.

The following proposition relates different directions of summation:

Proposition 1. *Let θ_1, θ_2 be angles with $0 \leqslant \theta_2 - \theta_1 < \pi$ and consider the closed path C_R defined by the union of the curves $(0, Re^{i\theta_1})$, $(Re^{i\theta_2}, 0)$ and the arc C_R^{12} of radius R that sweeps from θ_1 to θ_2 (see Figure 4.2a). Suppose that $\widehat{\varphi}$ is analytic in the interior and the boundary of the curve C_R, for all R, and that it satisfies the exponential bound $|\widehat{\varphi}(\zeta)| \leqslant Ke^{A|\zeta|}$ there. Then*

$$\mathcal{L}^{\theta_1}[\widehat{\varphi}](z) = \mathcal{L}^{\theta_2}[\widehat{\varphi}](z)\,, \quad \mathrm{Re}\left(ze^{i\theta_1}\right) > A \cap \mathrm{Re}\left(ze^{i\theta_2}\right) > A\,.^3 \tag{4.23}$$

This is the result in [64, Lem. 9.4].

Proof. If we check that

$$\lim_{R\to\infty} \int_{C_R^{12}} d\zeta\, e^{-z\zeta} \widehat{\varphi}(\zeta) = 0\,, \quad \mathrm{Re}\left(ze^{i\theta_1}\right) > A \cap \mathrm{Re}\left(ze^{i\theta_2}\right) > A\,, \tag{4.24}$$

a direct application of Cauchy's integral theorem yields

$$\begin{aligned} \mathcal{L}^{\theta_1}[\widehat{\varphi}](z) - \mathcal{L}^{\theta_2}[\widehat{\varphi}](z) &= \lim_{R\to\infty} \left(\int_0^{Re^{i\theta_1}} + \int_{Re^{i\theta_2}}^0 + \int_{C_R^{12}} \right) d\zeta\, e^{-z\zeta} \widehat{\varphi}(\zeta) \\ &= \lim_{R\to\infty} \oint_{C_R} d\zeta\, e^{-z\zeta} \widehat{\varphi}(\zeta) = 0\,, \end{aligned} \tag{4.25}$$

for any z in the same region of Eq. 4.24.

It is possible to prove Eq. 4.24 explicitly for all z in the corresponding region, but we will give the proof only for z such that $|z| > A$ and $\arg(z) = -(\theta_2 + \theta_1)/2$. Then we

[3] See Figure 5.1a for a graphical representation of the region $\mathrm{Re}\left(ze^{i\theta}\right) > A$, which is a half-plane.

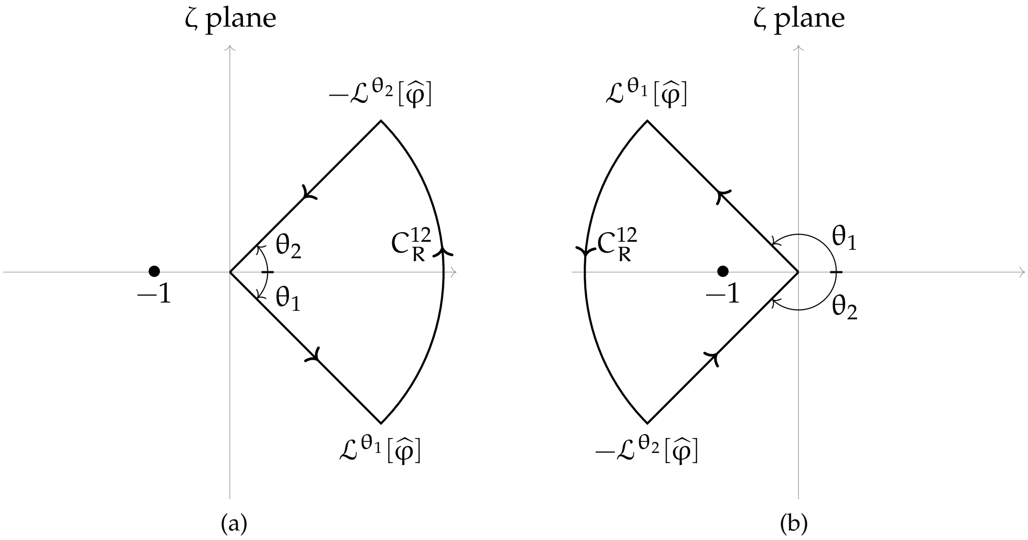


Figure 4.2: Relationships between the Borels sums in Eq. 4.19.

will use analytic continuation to extend the result to the whole region.

We have

$$\begin{aligned}\left|\int_{C_R^{12}} d\zeta\, e^{-z\zeta}\widehat{\varphi}(\zeta)\right| &\leqslant K\int_{C_R^{12}} |d\zeta|\, e^{-\operatorname{Re}(z\zeta)}|\widehat{\varphi}(\zeta)| \\ &\leqslant KR\int_{\theta_1}^{\theta_2} d\theta \exp\left[-R\left(|z|\cos\left(\theta-\frac{\theta_2+\theta_1}{2}\right)-A\right)\right] \xrightarrow{R\to\infty} 0\,.\end{aligned} \tag{4.26}$$

The limit in the last line holds if

$$|z|\cos\left(\theta-\frac{\theta_2+\theta_1}{2}\right)-A>0 \tag{4.27}$$

along the whole interval of integration over θ (see Jordan's lemma for details). Given that $|z|>A$, it is enough to prove that $\theta-(\theta_2+\theta_1)/2\in(-\pi/2,\pi/2)$, for all $\theta\in[\theta_1,\theta_2]$, but this is a direct consequence of the hypothesis that $0\leqslant\theta_2-\theta_1<\pi$.

This completes the proof for the particular case $|z|>A$ and $\arg(z)=-(\theta_2+\theta_1)/2$. To extend the result to $\operatorname{Re}\left(ze^{i\theta_1}\right)>A\cap\operatorname{Re}\left(ze^{i\theta_2}\right)>A$, we observe that $\mathcal{L}^{\theta_1}[\widehat{\varphi}]$ and $\mathcal{L}^{\theta_2}[\widehat{\varphi}]$ are both holomorphic in the half-planes where the Laplace transform converges, and they both take the same values in the line $\left\{z\in\mathbb{C}\mid |z|>A\,,\arg z=-(\theta_2+\theta_1)/2\right\}\subset\operatorname{Re}\left(ze^{i\theta_1}\right)>A\cap\operatorname{Re}\left(ze^{i\theta_2}\right)>A$. Therefore, the two Laplace transforms provide analytic continuations to different regions that must coincide in the intersection $\operatorname{Re}\left(ze^{i\theta_1}\right)>A\cap\operatorname{Re}\left(ze^{i\theta_2}\right)>A$. □

Proposition 1 has very important implications. The Borel sum in Eq. 4.19 is a function initially defined in some half-plane $\mathrm{Re}(z) > A$, but the directional Borel sum of Definition 4 provides analytic continuations to different regions of the complex plane (see Definition 1).

In the case of the Euler series, the directional Borel sums converge in the half-planes $\mathrm{Re}\left(ze^{i\theta}\right) > 0$ (because the exponential bound of $1/(1+\zeta)$ is satisfied with $A = 0$). In this situation, thanks to Proposition 1, the Borel sum of the Euler series does not depend on the particular $\theta \in (-\pi/2, +\pi/2)$ chosen. Thus, concatenating the corresponding half-planes of convergence, the Borel sum becomes a function defined in $\mathbb{C} \setminus \mathbb{R}^-$.

We could be tempted to also admit $\theta \notin (-\pi/2, +\pi/2)$, but then we could find two directions θ_1, θ_2 such that their corresponding contour C_R in Proposition 1 encircles the singularity at $\zeta = -1$ of $\widehat{\varphi}(\zeta) = 1/(1+\zeta)$ (see Figure 4.2b) and the proposition would no longer apply. Let us consider this more general situation:

Proposition 2. *Under the hypothesis of Proposition 1, we now admit that C_R encircles some isolated singularities of $\widehat{\varphi}$. If $\{\zeta_n\}_{n\in\mathbb{N}}$ are the encircled singularities (for $R \to \infty$), then*

$$\mathcal{L}^{\theta_1}[\widehat{\varphi}](z) = \mathcal{L}^{\theta_2}[\widehat{\varphi}](z) + 2\pi i \sum_{n\geqslant 1} \mathrm{Res}\left(e^{-z\zeta}\widehat{\varphi}(\zeta), \zeta = \zeta_n\right) ,$$
$$\mathrm{Re}\left(ze^{i\theta_1}\right) > A \cap \mathrm{Re}\left(ze^{i\theta_2}\right) > A \,. \quad (4.28)$$

While the proof essentially remains the same as in Proposition 1, the residues now contribute into the closed integral along C_R of Eq. 4.25.

In the case of the Euler series, $\widehat{\varphi}$ has a single singularity at $\zeta = -1$ and the residue of $e^{-z\zeta}\widehat{\varphi}(\zeta)$ at this point is e^z. Thus,

$$\mathcal{L}^{\theta_1}[\widehat{\varphi}](z) = \mathcal{L}^{\theta_2}[\widehat{\varphi}](z) + 2\pi i e^z \,, \quad (4.29)$$

whenever θ_1 and θ_2 satisfy the hypothesis of Proposition 2. We think the best explanation to this result is achieved when discussing the problem in the Riemann surface. The meaning of Eq. 4.29 is that, given z, we have two values, differing by $2\pi i e^z$, that we can associate to the Borel sum. This is exactly the same situation for a multivalued function. The directions θ_1, θ_2 probe different regions of the Riemann surface having the same projection in the complex plane. See Figure 4.3. By choosing a branch, we assign a value to each $z \in \mathbb{C}$ amongst all possible values that the multivalued function can attain at z. When we previously fixed $\theta \in (-\pi/2, +\pi/2)$, we were actually making a choice that fixes the branch of the Borel sum.

We will now compute the discontinuity of the Borel sum from Definition 3 using the method of analytic continuation. It is very convenient to compute the discontinuity

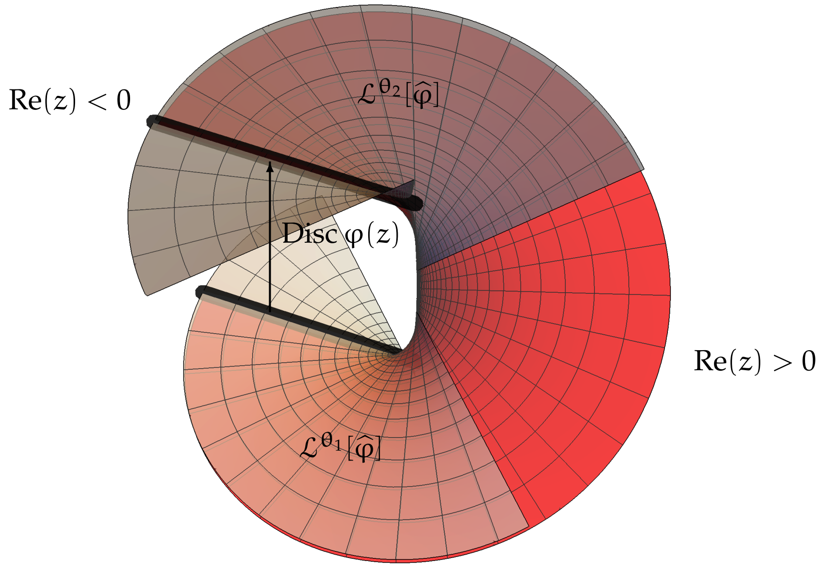


Figure 4.3: Riemann surface of the Borel sum in Eq. 4.19. The regions of analytic continuation given by $\mathcal{L}^{\theta_1}[\widehat{\varphi}]$ and $\mathcal{L}^{\theta_2}[\widehat{\varphi}]$ are depicted in grey.

in this way, because the different directions in the Borel sum of Definition 4 provide analytic continuations of the same function to different regions.

Consider the two directions $\theta_- = -\pi + \epsilon$ and $\theta_+ = +\pi - \epsilon$, with $0 < \epsilon < \pi/2$. Starting from θ_-, we compute successive analytic continuations of the Borel sum by continuously changing θ in the interval $[\theta_-, \theta_+]$. This provides an analytic continuation along the clockwise curve γ_{-1} winding once around the origin. While the Borel sums in the directions θ_- and θ_+ converges in two overlapping half-planes, the two results differ according to Eq. 4.29:

$$\mathrm{cont}_{\gamma_{-1}}\left(\mathcal{L}^{\theta_-}[\widehat{\varphi}](z)\right) = \mathcal{L}^{\theta_+}[\widehat{\varphi}](z) = \mathcal{L}^{\theta_-}[\widehat{\varphi}](z) - 2\pi i e^{z}\,. \tag{4.30}$$

Using Definition 3, Eq. 4.15 and Eq. 4.30, we obtain

$$\mathrm{Disc}\,\varphi(z) = \mathcal{L}^{\theta_-}[\widehat{\varphi}](z) - \mathcal{L}^{\theta_+}[\widehat{\varphi}](z) = 2\pi i e^{z}\,, \tag{4.31}$$

where φ is the Borel sum of $\widetilde{\varphi}$, understood as a function defined in the Riemann surface of the logarithm (see Figure 4.3 for a graphical representation of the discontinuity). This result is in principle valid for $\mathrm{Re}\left(ze^{i\theta_+}\right) > 0 \cap \mathrm{Re}\left(ze^{i\theta_-}\right) > 0$, but we can choose ϵ as small as we want, and in the limit of small ϵ the two half-planes coalesce into $\mathrm{Re}(z) < 0$.

From this computation, we have shown that the singularities in the Borel plane actually encode the discontinuity of the Borel sum φ. We will now check that the properties we have obtained for the Borel sum of the Euler series are correct by an explicit computation in terms of the exponential integral E_1, defined by

$$E_1(z) = \int_z^\infty dt\, \frac{e^{-t}}{t}\,. \tag{4.32}$$

After the change of variable $t = z(1+\zeta)$, $\zeta \in (0, +\infty)$, we obtain

$$E_1(z) = e^{-z} \int_0^\infty d\zeta\, \frac{e^{-z\zeta}}{1+\zeta} = e^{-z} \varphi(z)\,. \tag{4.33}$$

Furthermore, a very convenient expression for the exponential integral is given by

$$E_1(z) = -\gamma - \mathscr{L}\mathrm{og}(z) - \sum_{k\geqslant 1} \frac{(-z)^k}{k!\,k}\,, \tag{4.34}$$

where the series on the right is convergent for all z and thus it defines an entire function. Eq. 4.34 clearly separates its entire component from its multivalued component, the latter appearing in the form of a logarithm. From Eq. 4.33, it is now clear that the Borel sum of the Euler series also inherits this multivalued component, whose discontinuity coincides with our computation in Eq. 4.31.

4.4 The error function

In this example, we introduce the factorially divergent series

$$\widetilde{\Phi}(z) = \sum_{n\geqslant 0} \frac{(-1)^n \Gamma\left(n+\frac{1}{2}\right)}{z^{n+1}}\,, \tag{4.35}$$

which is closely related to the error function (see [65, Ch. 1]). The Borel transform of this series is

$$\widehat{\Phi}(\zeta) \equiv \mathcal{B}[\widetilde{\Phi}](\zeta) = \sum_{n\geqslant 0} \frac{(-1)^n \Gamma\left(n+\frac{1}{2}\right)}{n!} \zeta^n = \frac{\sqrt{\pi}}{\sqrt{1+\zeta}}\,. \tag{4.36}$$

The singularity of this function at $\zeta = -1$ is a branch point, so Proposition 2 is no longer valid. Instead, we have:

Proposition 3. *Under the hypothesis of Proposition 1, we now admit that C_R encircles some isolated singularities or branch points of $\widehat{\varphi}$ (for $R \to \infty$). If $\{\zeta_n\}_{n\in\mathbb{N}}$ are the encircled singu-*

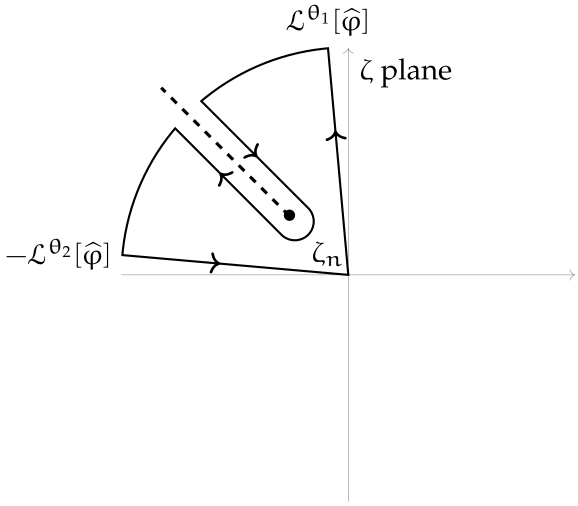


Figure 4.4: Relationships between the family of Borel sums $\mathcal{L}^{\theta}[\widehat{\varphi}]$. Case when $\widehat{\varphi}$ has branch points.

larities, then

$$\begin{aligned}\mathcal{L}^{\theta_1}[\widehat{\varphi}](z) = \mathcal{L}^{\theta_2}[\widehat{\varphi}](z) + \sum_{n\geqslant 1} e^{-z\zeta_n} \int_0^{\infty\zeta_n} d\zeta\, e^{-z\zeta}\, \mathrm{Disc}\, \widehat{\varphi}(\zeta+\zeta_n) \\ + 2\pi i \sum_{n\geqslant 1} \mathrm{Res}\left(e^{-z\zeta}\widehat{\varphi}(\zeta), \zeta=\zeta_n\right), \quad \mathrm{Re}\left(ze^{i\theta_1}\right) > A \cap \mathrm{Re}\left(ze^{i\theta_2}\right) > A\,. \end{aligned} \tag{4.37}$$

For the proof we would consider the path in Figure 4.4. The contour around the branch cut of $\widehat{\varphi}$ yields the integral of the discontinuity appearing in Eq. 4.37.

As we did in the Euler series example, we want to determine the properties of the Borel sum in the Riemann surface and compute its discontinuity. To apply Proposition 3, we first need the discontinuity of $\widehat{\Phi}$, given by

$$\mathrm{Disc}\, \widehat{\Phi}(\zeta-1) = \sqrt{\pi}\left(\frac{1}{\sqrt{\zeta}} - \frac{1}{-\sqrt{\zeta}}\right) = \frac{2\sqrt{\pi}}{\sqrt{\zeta}}\,. \tag{4.38}$$

Then, for $\theta_- = -\pi+\epsilon$, $\theta_+ = +\pi-\epsilon$, we obtain

$$\begin{aligned}\mathrm{cont}_{\gamma_{-1}}\left(\mathcal{L}^{\theta_-}[\widehat{\Phi}](z)\right) = \mathcal{L}^{\theta_+}[\widehat{\Phi}](z) &= \mathcal{L}^{\theta_-}[\widehat{\Phi}](z) + e^z \int_0^{-\infty} e^{-z\zeta}\frac{2\sqrt{\pi}}{\sqrt{\zeta}}\, d\zeta \\ &= \mathcal{L}^{\theta_-}[\widehat{\Phi}](z) + 2\pi i \frac{e^z}{\sqrt{-z}}\,,\end{aligned} \tag{4.39}$$

so the discontinuity of the Borel sum Φ is given by

$$\mathrm{Disc}\, \Phi(z) = \mathcal{L}^{\theta_+}[\widehat{\Phi}](z) - \mathcal{L}^{\theta_-}[\widehat{\Phi}](z) = 2\pi i \frac{e^z}{\sqrt{-z}}\,. \tag{4.40}$$

Let us consider the error function ϕ, which can be written in terms of the Borel sum Φ:

$$\begin{aligned}\phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}u\, e^{-u^2} &= -x\frac{e^{-x^2}}{\pi}\Phi\big(x^2\big) + \begin{cases} +1 & \text{if } \mathrm{Re}(z) > 0 \\ -1 & \text{if } \mathrm{Re}(z) < 0 \end{cases} \\ &= -x\frac{e^{-x^2}}{\pi}\left[\Phi(x^2) - \frac{i\pi e^{x^2}}{{}_1\sqrt{-x^2}}\right],\end{aligned} \tag{4.41}$$

where ${}_1\sqrt{}$ is a branch of the square root which is realised by fixing $\arg(-x^2) \in [0, 2\pi)$. We set $z = x^2$ and consider only the part in square brackets:

$$F(z) = \Phi(z) - \frac{i\pi e^z}{{}_1\sqrt{-z}}. \tag{4.42}$$

On one hand, the error function is entire, while on the other hand, the Borel sum Φ has a multivalued component. The only possibility is that the second term in Eq. 4.42 cancels this multivalued component. Indeed, using Eq. 4.16 and Eq. 4.40, we have, in the Riemann surface,

$$\begin{aligned}F\big(z\underline{e}^{-2\pi i}\big) &= \Phi\big(z\underline{e}^{-2\pi i}\big) - \frac{i\pi e^z}{\sqrt{-z\underline{e}^{-2\pi i}}} = \Phi(z) - \mathrm{Disc}\,\Phi(z) - \frac{i\pi e^z}{-\sqrt{-z}} \\ &= \Phi(z) - \frac{i\pi e^z}{\sqrt{-z}} = F(z),\end{aligned} \tag{4.43}$$

which means that after an analytic continuation of Eq. 4.42 along γ_{-1}, the function F returns to itself, so its discontinuity is 0 as expected.

4.5 The digamma function

In this example, we consider the divergent series

$$\widetilde{\Psi}(z) = -\frac{1}{2z} - \sum_{n\geqslant 1}\frac{B_{2n}}{(2n)z^{2n}}, \tag{4.44}$$

related to the digamma function, where B_{2n} are the Bernoulli numbers. The Borel transform of this series is

$$\widehat{\Psi}(\zeta) = -\frac{1}{2} - \sum_{n\geqslant 1}\frac{B_{2n}}{(2n)!}\zeta^{2n-1} = -\frac{1}{2} - \frac{1}{\zeta}\left(\frac{\zeta}{e^\zeta - 1} - 1 + \frac{\zeta}{2}\right) = \frac{e^\zeta(1-\zeta) - 1}{\zeta(e^\zeta - 1)}. \tag{4.45}$$

Its poles are at the positions $\zeta = 2\pi i k$ with $k \in \mathbb{Z}\setminus\{0\}$ and the residues are given by

$$\mathrm{Res}\left(e^{-z\zeta}\widehat{\Psi}(\zeta),\, \zeta = 2\pi i k\right) = -e^{-2\pi i k z}. \tag{4.46}$$

Consider the Borel sum along the directions $\theta_1 = 0$, $\theta_2 = 3\pi/4$ and $\theta_3 = 5\pi/4$. From Proposition 3, we have the relationships:

$$\mathcal{L}^{\theta_1}[\widehat{\Psi}](z) = \mathcal{L}^{\theta_2}[\widehat{\Psi}](z) + 2\pi i \sum_{k\geqslant 1} e^{-2\pi i k z} = \mathcal{L}^{\theta_2}[\widehat{\Psi}](z) + \pi\cot(\pi z) - i\pi\,, \tag{4.47}$$

$$\mathcal{L}^{\theta_2}[\widehat{\Psi}](z) = \mathcal{L}^{\theta_3}[\widehat{\Psi}](z)\,, \tag{4.48}$$

$$\mathcal{L}^{\theta_3}[\widehat{\Psi}](z) = \mathcal{L}^{\theta_1}[\widehat{\Psi}](z) + 2\pi i \sum_{k\geqslant 1} e^{2\pi i k z} = \mathcal{L}^{\theta_1}[\widehat{\Psi}](z) - \pi\cot(\pi z) - i\pi\,, \tag{4.49}$$

with each equality valid in the corresponding intersection of the half-planes. We can now compute the analytic continuation of the Borel sum $\mathcal{L}^{\theta_1}[\widehat{\Psi}]$ along the curve γ_{-1} in three steps:

(1) While $\mathcal{L}^{\theta_1}[\widehat{\Psi}](z)$ is defined only in $\mathrm{Re}(z) > 0$, the right hand side of Eq. 4.47 provides an analytic continuation to $\mathrm{Re}\left(ze^{i\theta_2}\right) > 0$.

(2) Using Eq. 4.48, the function

$$\mathcal{L}^{\theta_3}[\widehat{\Psi}](z) + \pi\cot(\pi z) - i\pi \tag{4.50}$$

provides an analytic continuation of the preceding analytic continuation from $\mathrm{Re}\left(ze^{i\theta_2}\right) > 0$ to $\mathrm{Re}\left(ze^{i\theta_3}\right) > 0$.

(3) Finally, using Eq. 4.49 in Eq. 4.50, the function

$$\Big(\mathcal{L}^{\theta_1}[\widehat{\Psi}](z) - \pi\cot(\pi z) - i\pi\Big) + \pi\cot(\pi z) - i\pi = \mathcal{L}^{\theta_1}[\widehat{\Psi}](z) - 2\pi i \tag{4.51}$$

provides an analytic continuation of Eq. 4.50 from $\mathrm{Re}\left(ze^{i\theta_3}\right) > 0$ to $\mathrm{Re}(z) > 0$, completing the clockwise circuit around the origin.

Therefore, we have

$$\mathrm{cont}_{\gamma_{-1}}\Big(\mathcal{L}^{\theta_1}[\widehat{\Psi}](z)\Big) = \mathcal{L}^{\theta_1}[\widehat{\Psi}](z) - 2\pi i\,, \tag{4.52}$$

and

$$\mathrm{Disc}\,\Psi(z) = -2\pi i\,, \tag{4.53}$$

where Ψ is the extension of $\mathcal{L}^{\theta_1}[\widehat{\Psi}]$ in the Riemann surface. This result is expected, because the digamma function ψ (which has no branch cut) can be written as

$$\psi(z) = \mathrm{Log}(z) + \Psi(z)\,, \tag{4.54}$$

so the discontinuities from the logarithm and the Borel sum cancel each other.

While the digamma has no branch cut, it still has poles at 0 and at all negative integers.

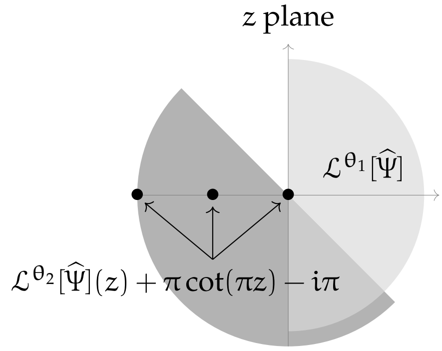


Figure 4.5: An analytic continuation of $\mathcal{L}^{\theta_1}[\widehat{\Psi}]$ to $\mathbb{R}^-$ is given by $\mathcal{L}^{\theta_2}[\widehat{\Psi}](z) + \pi\cot(\pi z) - i\pi$. This analytic continuation has poles in $\mathbb{R}^-$ coming from $\pi\cot(\pi z)$, in correspondence with the poles of the digamma function. The shadowed regions are the half-planes of analyticity of the two Borel sums.

Thus, according to Eq. 4.54, the Borel sum Ψ must incorporate these singularities in some way. Indeed, the analytic continuation of Ψ from $\mathrm{Re}(z) > 0$ to $\mathrm{Re}\left(ze^{i\theta_2}\right) > 0$ (the latter a region containing $\mathbb{R}^-$) is the right hand side of Eq. 4.47 and the term $\pi\cot(\pi z)$ appearing there contains the singularities of the digamma function (see Figure 4.5). It is interesting to verify that the singularities of the Borel transform not only encode the branch cuts of the Borel sum, but also its singularities in the Riemann surface.

We also get the following interesting insight from this example. When the Borel plane has an infinite amount of singularities, each residue introduces a discontinuity to the Borel sum that behaves like $e^{-\zeta_n z}$, where ζ_n is the position of the singularity. However, the sum of all contributions can be of a complete different kind, as we have seen in this case, where the discontinuity in Eq. 4.53 is just a constant (so, it arises as if there were a singularity at $\zeta = 0$).

4.6 Alien calculus and the Airy equation

Alien calculus is a mathematical framework developed by Écalle [66] and it has applications to divergent series arising as formal solutions to a differential equation. While our work will not rely on the results of alien calculus, we still think that it is important to review this aspect of resurgent analysis, because it has seen applications in areas of physics where the functions under study arise as solutions to a given differential equation [67, 68]. For instance, the Painlevé I equation is relevant for two-dimensional quantum gravity [69].

In this section, we will discuss how alien calculus connects to the framework of the present work and we will take the opportunity to define the so-called median resummation of a divergent series in terms of the Stokes automorphism, which is an important operator in alien calculus.

For context, we introduce the Airy equation[4]

$$\varphi''(x) = x\varphi(x)\,. \tag{4.55}$$

It is convenient to perform the change of variable $z \equiv x^{3/2}$. In the z variable, the Airy equation reads

$$\varphi''(z) + \frac{1}{3z}\varphi'(z) - \frac{4}{9}\varphi(z) = 0\,. \tag{4.56}$$

Given that the above differential equation is of order two, we expect two independent solutions. We can check that these solutions can be formally represented by the following divergent series:[5]

$$Z_A(z) = \frac{z^{5/6}}{2\sqrt{\pi}}\, e^{-\frac{2}{3}z} \sum_{n\geqslant 0} \frac{a_n}{z^{n+1}} = \frac{z^{5/6}}{2\sqrt{\pi}}\, e^{-\frac{2}{3}z}\widetilde{A}(z)\,, \tag{4.57}$$

$$Z_B(z) = \frac{z^{5/6}}{2\sqrt{\pi}}\, e^{+\frac{2}{3}z} \sum_{n\geqslant 0} \frac{(-1)^n a_n}{z^{n+1}} = \frac{z^{5/6}}{2\sqrt{\pi}}\, e^{+\frac{2}{3}z}\widetilde{B}(z)\,, \tag{4.58}$$

where $\widetilde{A}$ and $\widetilde{B}$ are the power series of the solutions and

$$a_n = \frac{1}{2\pi}\left(-\frac{3}{4}\right)^n \frac{\Gamma\left(n+\frac{5}{6}\right)\Gamma\left(n+\frac{1}{6}\right)}{n!}\,. \tag{4.59}$$

The Borel transforms of $\widetilde{A}$ and $\widetilde{B}$ are given by

$$\widehat{A}(\zeta) = \sum_{n\geqslant 0} \frac{a_n}{n!}\zeta^n = F\left(\frac{1}{6}, \frac{5}{6}, 1, -\frac{3\zeta}{4}\right), \tag{4.60}$$

$$\widehat{B}(\zeta) = \sum_{n\geqslant 0} \frac{a_n}{n!}(-\zeta)^n = \widehat{A}(-\zeta)\,, \tag{4.61}$$

where F is the ordinary hypergeometric function. Both Borel transforms are holomorphic functions up to a branch cut emerging from the points $\zeta = -4/3$ and $\zeta = +4/3$, respectively.

In Eq. 4.57 and Eq. 4.58, if we take the Borel sums of $\widetilde{A}$ and $\widetilde{B}$, we obtain true solutions to Eq. 4.56. So, from Theorem 1, we conclude that these formal solutions are in fact asymptotic expansions of these true solutions.

Next, we will check that the two true solutions are connected through the singularities in the Borel plane. From the properties of the ordinary hypergeometric function F, we

[4]See [67] for a discussion of the Airy equation along the same lines as will be presented here

[5]These divergent series are formal solutions in the sense that the truncated series at order N satisfies Eq. 4.56 up to terms of order higher than N.

have:

$$\widehat{A}\left(\zeta - \tfrac{4}{3}\right) = -i\widehat{B}(\zeta)\frac{\mathscr{L}\mathrm{og}(\zeta)}{2\pi i} + R(\zeta)\,, \tag{4.62}$$

where R is an entire function. Thus, the discontinuity of this function is given by

$$\mathrm{Disc}\,\widehat{A}\left(\zeta - \tfrac{4}{3}\right) = -i\widehat{B}(\zeta)\,. \tag{4.63}$$

This result can also be derived from the following property of the hypergeometric function:

$$\begin{aligned}\mathrm{Disc}\,F(a,b,c,1+\zeta) = -\frac{2\pi i\,\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}(1-\zeta)^{1-c}\zeta^{c-a-b}&\\ \times F(1-a,1-b,c-a-b+1,-\zeta)\,.&\end{aligned} \tag{4.64}$$

Then, for θ_1, θ_2 close enough to π, Proposition 3 yields

$$\begin{aligned}\mathcal{L}^{\theta_1}[\widehat{A}](z) &= \mathcal{L}^{\theta_2}[\widehat{A}](z) + e^{\frac{4}{3}z}\int_0^{-\infty} d\zeta\, e^{-z\zeta}\,\mathrm{Disc}\,\widehat{A}\left(\zeta - \tfrac{4}{3}\right)\\ &= \mathcal{L}^{\theta_2}[\widehat{A}](z) - ie^{\frac{4}{3}z}\mathcal{L}^{\pi}[\widehat{B}](z)\,.\end{aligned} \tag{4.65}$$

Therefore, the Borel sums of the two solutions of the Airy equation are intimately related. This might seem as a coincidence, but in the following we will prove, using the language of alien calculus, that this is a consequence of the fact that both functions are solutions to the same differential equation.

Let us define the alien derivative Δ_ω (see [64, Def. 28.2]), which is an operator acting on a series $\widetilde{A}$. The definition depends on the singularities of the Borel transform $\widehat{A}$. As a simplification, we will only consider a very restricted case where $\widehat{A}$ has no branch singularities all along the segment $(0,\omega)$ that connects the origin with ω. We feel there is no need to further complicate the discussion and still within this limitation we will be able to address the Airy equation.

Definition 5 (Alien derivative)**.** The alien derivative is an operator that takes a series $\widetilde{A}$ and returns another series $\Delta_\omega\widetilde{A}$ given by:

- If ω is not a singularity of $\widehat{A}$, then $\Delta_\omega\widetilde{A} = 0$.
- If ω is a singularity of $\widehat{A}$ and

$$\widehat{A}\left(\zeta + \omega\right) = \frac{\psi_0}{2\pi i\zeta} + \widehat{\psi}(\zeta)\frac{\mathscr{L}\mathrm{og}(\zeta)}{2\pi i} + R(\zeta)\,, \tag{4.66}$$

  where ψ_0 is a complex constant, and $\widehat{\psi}$ and R are holomorphic functions around the origin, then

$$\Delta_\omega\widetilde{A}(z) = \psi_0 + \sum_{n\geqslant 0}\frac{\widehat{\psi}^{(n)}(0)}{z^{n+1}}\,. \tag{4.67}$$

As a first step, we will compute the alien derivative of $\widetilde{A}$ at the point $\omega = -4/3$. To do so, we we consider Eq. 4.62 and, by definition of the alien derivative, we have

$$\Delta_{-\frac{4}{3}}\widetilde{A}(z) = -i\widetilde{B}(z)\,. \tag{4.68}$$

In the case of $\widetilde{B}$, the Borel transform $\widehat{B}$ in Eq. 4.61 has a single singularity at $\omega = +4/3$ and

$$\widehat{B}\left(\zeta + \tfrac{4}{3}\right) = -i\widehat{A}(\zeta)\frac{\mathscr{L}\mathrm{og}(\zeta)}{2\pi i} + R(\zeta)\,, \tag{4.69}$$

so

$$\Delta_{+\frac{4}{3}}\widetilde{B}(z) = -i\widetilde{A}(z)\,. \tag{4.70}$$

The relations of Eq. 4.68 and Eq. 4.70, which we will call resurgent connections, can be inferred directly from the Airy equation and two important properties of the alien derivative, or rather, the dotted alien derivative:

$$\dot{\Delta}_{\omega} = e^{-\omega z}\Delta_{\omega}\,. \tag{4.71}$$

These properties are:

- The alien derivative satisfies [70]

$$\frac{d}{dz}\Delta_{\omega}\widetilde{\varphi} = \Delta_{\omega}\frac{d\widetilde{\varphi}}{dz} + \omega\Delta_{\omega}\widetilde{\varphi}\,. \tag{4.72}$$

  so the dotted alien derivative $\dot{\Delta}_{\omega}$ commutes with d/dz.

- The alien derivative and also the dotted alien derivative satisfy the Leibniz's rule (or product rule) [70]

$$\Delta_{\omega}(\widetilde{\varphi}\widetilde{\psi}) = (\Delta_{\omega}\widetilde{\varphi})\widetilde{\psi} + \widetilde{\varphi}(\Delta_{\omega}\widetilde{\psi})\,. \tag{4.73}$$

Operating with $\dot{\Delta}_{\omega}$ on Eq. 4.56 and using the two above properties, we obtain a differential equation for $\dot{\Delta}_{\omega}\varphi$, which reads

$$(\dot{\Delta}_{\omega}\,\varphi)''(z) + \frac{1}{3z}(\dot{\Delta}_{\omega}\,\varphi)'(z) - \frac{4}{9}(\dot{\Delta}_{\omega}\,\varphi)(z) = 0\,. \tag{4.74}$$

That is, given a solution

$$\varphi = \sigma_A Z_A + \sigma_B Z_B\,, \tag{4.75}$$

written as a linear combination of the two formal series in Eq. 4.57 and Eq. 4.58, the formal object $\dot{\Delta}_{\omega}\widetilde{\varphi}$ is also a solution to the same equation, so we can ascertain that $\dot{\Delta}_{\omega}\varphi$ is also a linear combination of Z_A and Z_B. Although the coefficients of this latter linear combination will be different from Eq. 4.75, we can argue that the dependence of $\dot{\Delta}_{\omega}\varphi$ on the parameters σ_A, σ_B will be inside the coefficients of its linear combination:

$$\dot{\Delta}_\omega \varphi = S_A^\omega(\sigma_A, \sigma_B) Z_A + S_B^\omega(\sigma_A, \sigma_B) Z_B \,. \tag{4.76}$$

Now, applying the dotted alien derivative on Eq. 4.75 and, then, using Eq. 4.76, we obtain the relation

$$\sigma_A \, \dot{\Delta}_\omega \, Z_A + \sigma_B \, \dot{\Delta}_\omega \, Z_B = S_A^\omega(\sigma_A, \sigma_B) Z_A + S_B^\omega(\sigma_A, \sigma_B) Z_B \,. \tag{4.77}$$

Using the expressions of Z_A and Z_B from Eq. 4.57 and Eq. 4.58, using Eq. 4.71 and setting $\omega = \pm 4/3$, the above equation can be written as

$$\sigma_A e^{+\frac{2}{3}z} \Delta_{-\frac{4}{3}} \widetilde{A} + \sigma_B e^{+2z} \Delta_{-\frac{4}{3}} \widetilde{B} = S_A^{-4/3}(\sigma_A, \sigma_B) e^{-\frac{2}{3}z} \widetilde{A} + S_B^{-4/3}(\sigma_A, \sigma_B) e^{+\frac{2}{3}z} \widetilde{B} \,, \tag{4.78}$$

$$\sigma_A e^{-2z} \Delta_{+\frac{4}{3}} \widetilde{A} + \sigma_B e^{-\frac{2}{3}z} \Delta_{+\frac{4}{3}} \widetilde{B} = S_A^{+4/3}(\sigma_A, \sigma_B) e^{-\frac{2}{3}z} \widetilde{A} + S_B^{+4/3}(\sigma_A, \sigma_B) e^{+\frac{2}{3}z} \widetilde{B} \,. \tag{4.79}$$

Imposing equality between the terms with the same exponential factor, we obtain four equations for the alien derivatives:

$$\begin{aligned} \Delta_{-\frac{4}{3}} \widetilde{A} &= \frac{S_B^{-4/3}(\sigma_A, \sigma_B)}{\sigma_A} \widetilde{B} \,, & \Delta_{-\frac{4}{3}} \widetilde{B} &= 0 \,, \\ \Delta_{+\frac{4}{3}} \widetilde{A} &= 0 \,, & \Delta_{+\frac{4}{3}} \widetilde{B} &= \frac{S_A^{+4/3}(\sigma_A, \sigma_B)}{\sigma_B} \widetilde{A} \,, \end{aligned} \tag{4.80}$$

and also the restrictions

$$S_A^{-4/3}(\sigma_A, \sigma_B) = S_B^{+4/3}(\sigma_A, \sigma_B) = 0 \,. \tag{4.81}$$

Finally, we notice that the left hand sides of Eq. 4.80 are independent of σ_A, σ_B, so the right hand sides must also be so. This fixes the dependences of the remaining coefficients up to a constant factor:

$$S_B^{-4/3}(\sigma_A, \sigma_B) = S_B^{-4/3} \sigma_A \,, \tag{4.82}$$

$$S_A^{+4/3}(\sigma_A, \sigma_B) = S_A^{+4/3} \sigma_B \,, \tag{4.83}$$

and the constants can be determined by comparison with Eq. 4.68 and Eq. 4.70:

$$S_B^{-4/3} = S_A^{+4/3} = -i \,. \tag{4.84}$$

We note that only from the properties of the alien derivative we were able to determine the connection between the two independent solutions of the Airy equation (up to the constants $S_B^{-4/3}$ and $S_A^{+4/3}$). Even the exact shape of the Airy equation was irrelevant, we only needed to know that its solutions can be coded into two parameters, as in Eq. 4.75.

Taking advantage of Eq. 4.75, we can write $Z_A = \partial\varphi/\partial\sigma_A$ and $Z_B = \partial\varphi/\partial\sigma_B$, which are explicit expressions in terms of φ. Thus, Eq. 4.76 can be written as

$$\dot{\Delta}_\omega \varphi = \left(S_A^\omega(\sigma_A, \sigma_B) \frac{\partial}{\partial\sigma_A} + S_B^\omega(\sigma_A, \sigma_B) \frac{\partial}{\partial\sigma_B} \right) \varphi \,, \tag{4.85}$$

from where we can read an explicit expression for the operator $\dot{\Delta}_\omega$. It now makes sense that we call $\dot{\Delta}_\omega$ a «derivative». In the particular cases $\omega = \pm 4/3$, the factors in front of $\partial/\partial\sigma_A$, $\partial/\partial\sigma_B$ can be substituted by their explicit expressions in Eq. 4.81, Eq. 4.82 and Eq. 4.83, yielding

$$\dot{\Delta}_{-\frac{4}{3}} = -i\sigma_A \frac{\partial}{\partial\sigma_B} \,, \quad \dot{\Delta}_{+\frac{4}{3}} = -i\sigma_B \frac{\partial}{\partial\sigma_A} \,. \tag{4.86}$$

Next, we move to an application of the alien derivatives. We define the Stokes automorphism in the direction θ as

$$\mathfrak{S}_\theta \equiv \exp\left(\sum_{\omega\in\Omega(\theta)} \dot{\Delta}_\omega \right) , \tag{4.87}$$

where $\Omega(\theta)$ is a discrete set of points which lie in $(0, \infty e^{i\theta})$. We have the following result [64, Theo. 29.5]:

Proposition 4. *Under the hypothesis of Proposition 1, we now admit that there is a direction* θ *that contains all the singularities between* θ_1 *and* θ_2*. Then*

$$\mathcal{L}^{\theta_1}\mathcal{B}[\widetilde{\varphi}](z) = \mathcal{L}^{\theta_2}\mathcal{B}[\mathfrak{S}_\theta \widetilde{\varphi}](z) \,, \quad \mathrm{Re}\left(z e^{i\theta_1}\right) > A \cap \mathrm{Re}\left(z e^{i\theta_2}\right) > A \,. \tag{4.88}$$

In fact, Proposition 4 is a reformulation of Proposition 3 in the language of alien calculus. Let us check this in the context of the Airy equation. The Borel transform $\widehat{A}$ has a single singularity at $\zeta = -4/3$, which is contained in the direction $\theta = \pi$, so let us consider this direction in the Stokes automorphism:

$$\mathfrak{S}_\pi \widetilde{A} = \exp\left(\dot{\Delta}_{-\frac{4}{3}} \right) \widetilde{A} = \left(1 + \dot{\Delta}_{-\frac{4}{3}} \right) \widetilde{A} = \widetilde{A} - i e^{\frac{4}{3}z} \widetilde{B} \,. \tag{4.89}$$

We note that there is no need to consider further powers of the alien derivative, because $\widehat{B}$ has no singularity at $\zeta = -4/3$, so $(\Delta_{-4/3})^2 \widetilde{A} = -i\Delta_{-4/3}\widetilde{B} = 0$. Proposition 4 then yields

$$\mathcal{L}^{\theta_1}[\widehat{A}](z) = \mathcal{L}^{\theta_2}\mathcal{B}[\mathfrak{S}_\pi \widetilde{A}](z) = \mathcal{L}^{\theta_2}[\widehat{A}](z) - i e^{\frac{4}{3}z} \mathcal{L}^{\theta_2}[\widehat{B}](z) \,. \tag{4.90}$$

We verify that this in fact Eq. 4.65 (which we obtained from Proposition 3), up to direction of the Borel sum of $\widetilde{B}$. However, it is easy to check that this direction is irrelevant, as a direct consequence of Proposition 1.

To recapitulate, we have seen that the Borel sum of the formal solutions to the Airy equation are interconnected, as in Eq. 4.65. Using the framework of alien calculus, we have shown that this is a general feature of divergent series arising as formal solutions to a differential equation. However, we want to remark that in this work we will not exploit these connections of resurgent analysis, because our objects of study will be expectation values of a field theory and their asymptotic expansions in powers of the coupling. These asymptotic expansions do not arise from a differential equation, but rather from quantum fluctuations around the trivial saddle point of the action of the theory.

Therefore, in this work, we will prefer the direct approach of Proposition 3 instead of the framework of alien calculus. Still, we will make use of the so-called median resummation of a divergent series, which is defined in terms of the Stokes automorphism as [71, 72, 73, 74]

$$\mathscr{L}^0_{\mathrm{med}}[\widetilde{\varphi}](z) \equiv \mathcal{L}^{\theta_+}\mathcal{B}\big[\mathfrak{S}_0^{+1/2}\widetilde{\varphi}\big](z) = \mathcal{L}^{\theta_-}\mathcal{B}\big[\mathfrak{S}_0^{-1/2}\widetilde{\varphi}\big](z)\,, \tag{4.91}$$

where $\theta_+ \in (0, +\pi)$ and $\theta_- \in (-\pi, 0)$. This median resummation is a prescription to resum a divergent series $\widetilde{\varphi}$ with singularities in the positive real axis of the Borel plane in such a way that the resulting function is real for $z > 0$. For example, the series $\widetilde{B}$ (whose Borel transform $\widehat{B}$ has a singularity at $\zeta = 3/4$) has the following median resummation:

$$\begin{aligned}\mathscr{L}^0_{\mathrm{med}}[\widetilde{B}](z) &= \mathcal{L}^{\theta_+}[\widehat{B}](z) - \frac{i}{2}e^{-\frac{4}{3}}\mathcal{L}^{\theta_+}[\widehat{A}](z)\\ &= \mathcal{L}^{\theta_-}[\widehat{B}](z) + \frac{i}{2}e^{-\frac{4}{3}}\mathcal{L}^{\theta_-}[\widehat{A}](z)\,.\end{aligned} \tag{4.92}$$

It is easy to check that the above two expressions coincide in $\mathrm{Re}(z) > 0$ and that both yield a real value for $z > 0$.

4.7 0-dimensional quartic interaction theory

We consider a field toy model in which space-time is 0-dimensional and fields are real. That is, fields are functions with domain equal to a single point $\{p\}$ and we can identify each field configuration $\phi(p) = \phi$ as the number $\phi \in \mathbb{R}$ that the field configuration takes at p. We further specify the theory by setting the action

$$S(\phi, m^2, \lambda) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\,. \tag{4.93}$$

In the path integral approach, integration along all field configurations is an integration along $\mathbb{R}$. The partition function of this model is

$$Z(m^2,\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\phi\, e^{-\frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4} = m\sqrt{\frac{3}{2\pi}} \frac{e^{\frac{3m^4}{4\lambda}}}{\sqrt{\lambda}} K_{1/4}\left(\frac{3m^4}{4\lambda}\right), \tag{4.94}$$

where K_n is the modified Bessel function of the second kind.

In this section, we will exemplify the following properties of the path integral of Eq. 4.94:

(A) Quantum fluctuations around the trivial saddle point of the action (a saddle point with zero action) yield an asymptotic expansion for the path integral. The Borel sum of this expansion recovers the path integral.

(B) Quantum fluctuations around non-trivial saddle points (saddle points with non-zero action) contribute to the asymptotic expansion of the discontinuity of the path integral. The Borel sum of this expansion recovers the discontinuity.

(C) As a corollary of both these results and Proposition 3: Quantum fluctuations around the trivial saddle point are related to quantum fluctuations around the non-trivial ones. We call this a resurgent connection, in analogy to the connections of Eq. 4.62 and Eq. 4.69 within the two solutions of the Airy equation.

First, let us compute quantum fluctuations around the trivial saddle point of the action ($\phi = 0$) to obtain the perturbative expansion of Z. The first term in the action of Eq. 4.93 is analogous to the kinetic term and can be represented diagramatically as a propagator line. The second term, proportional to λ, corresponds to interaction vertices with four fields ϕ.

Let us recall how perturbation theory is done in the path integral approach.[6] First, we split e^{-S} in two exponentials: one contains the propagators (free theory) and the other, the interactions. Then we keep the exponential with the propagators as it is, but expand the interaction part in powers of the coupling λ. Finally, we integrate each term of the resulting expansion term by term.

Employing the above procedure, we obtain

$$\begin{aligned} Z_P(m^2,\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\phi\, e^{-S(\phi,m^2,\lambda)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\phi\, e^{-\frac{m^2}{2}\phi^2} \sum_{n\geqslant 0} \frac{1}{n!}\left(\frac{-\lambda}{4!}\right)^n \phi^{4n} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n\geqslant 0} \frac{1}{n!}\left(\frac{-\lambda}{4!}\right)^n \int_{-\infty}^{\infty} d\phi\, e^{-\frac{m^2}{2}\phi^2}\phi^{4n} = \frac{1}{m}\sum_{n\geqslant 0} \frac{\Gamma\left(2n+\frac{1}{2}\right)}{\sqrt{\pi}\, n!}\left(\frac{-\lambda}{3!\, m^4}\right)^n, \end{aligned} \tag{4.95}$$

[6]In Appendix B.1, we review perturbation theory in more detail and in a more general context.

where we used

$$\int_{-\infty}^{\infty} d\phi\, e^{-\frac{m^2}{2}\phi^2}\phi^{2n} = \left(\frac{2}{m^2}\right)^{n+\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right) \tag{4.96}$$

in the last line. Thus, we have the perturbative series

$$Z_P\left(m^2,\lambda\right) = \frac{1}{m\lambda}\sum_{n\geqslant 0}(-1)^n a_n \lambda^{n+1} \tag{4.97}$$

with coefficients

$$a_n = \frac{\Gamma\left(2n+\frac{1}{2}\right)}{\sqrt{\pi}n!}\left(\frac{1}{3!\,m^4}\right)^n = \frac{\Gamma\left(n+\frac{1}{4}\right)\Gamma\left(n+\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)n!}\left(\frac{2}{3m^4}\right)^n. \tag{4.98}$$

Here we used some properties of the Γ function, $2^{1-2z}\sqrt{\pi}\,\Gamma(2z) = \Gamma(z)\Gamma(z+1/2)$ and $\sqrt{2}\pi = \Gamma(1/4)\Gamma(3/4)$, to write the coefficients in a more convenient way.

It is a straightforward verification that Z_P, even if divergent, it correctly encapsulates the asymptotic behaviour of the partition function Z for small positive coupling λ. In addition, the Borel sum of Z_P recovers the path integral Z.

Second, we will compute a non-perturbative expansion for Z. The non-perturbative parts come from quantum fluctuations around non-trivial saddle points of the action. For holomorphic functions extrema are always saddle points, so in this case, the saddle points are given by the equation

$$\frac{dS(\phi,m^2,\lambda)}{d\phi} = m^2\phi + \frac{\lambda}{3!}\phi^3 = 0\,, \tag{4.99}$$

which has the solutions

$$\phi_0 = 0\,, \quad \phi_1 = \pm\sqrt{-\frac{6m^2}{\lambda}}\,. \tag{4.100}$$

On one hand, quantum fluctuations around ϕ_0, for which $S(\phi_0) = 0$, yields perturbation theory, which we have already computed. On the other hand, quantum fluctuations around ϕ_1 (with either of the signs), for which $S(\phi_1) = -3m^4/(2\lambda)$, will yield non-perturbative expansions. The expansion of the action around this point is

$$\begin{aligned} S\left(\phi,m^2,\lambda\right) &= -\frac{3m^4}{2\lambda} - m^2(\phi-\phi_1)^2 \pm \sqrt{\frac{-\lambda m^2}{6}}(\phi-\phi_1)^3 + \frac{\lambda}{24}(\phi-\phi_1)^4 \\ &= \frac{m^2}{4}\phi_1^2 - m^2(\phi-\phi_1)^2 + \frac{m^2}{\phi_1}(\phi-\phi_1)^3 - \frac{m^2}{4\phi_1^2}(\phi-\phi_1)^4\,, \end{aligned} \tag{4.101}$$

where the last line is independent of which sign we choose for ϕ_1. The first term, which is a constant of the fields, will be the origin of an exponential factor in the non-perturbative expansion. The second term is the equivalent of the propagator, but we notice that it has the wrong sign. The propagator will appear inside the partition function as $e^{+m^2(\phi-\phi_1)^2}$, which diverges when ϕ goes to infinity along the real line. Thus, we have to consider a different integration path in the complex plane of ϕ for which the exponential converges to 0 and still goes through ϕ_1. For example, we can choose the path $\phi(y) = \phi_1 + iy/\sqrt{2}$, $y \in \mathbb{R}$ (at the end of the section we will explain why we can change the path). Plugging this expression inside the expansion of Eq. 4.101 we obtain

$$S\big(\phi(y), m^2, \lambda\big) = \frac{m^2}{4}\phi_1^2 + \frac{m^2}{2}y^2 - i\frac{m^2}{2\sqrt{2}\phi_1}y^3 - \frac{m^2}{16\phi_1^2}y^4 \,. \tag{4.102}$$

The non-perturbative expansion of Z is then given by

$$\begin{aligned} Z_{\mathrm{NP}}\big(m^2, \lambda\big) &= 2\frac{i}{\sqrt{2}}\, e^{-\frac{m^2}{4}\phi_1^2} \int\limits_{-\infty}^{\infty} d\phi\, e^{-\frac{m^2}{2}y^2} \sum_{n\geqslant 0} \frac{1}{n!}\left(i\frac{m^2}{2\sqrt{2}\phi_1}y^3 + \frac{m^2}{16\phi_1^2}y^4\right)^n \\ &= i\sqrt{2}\, e^{-\frac{m^2}{4}\phi_1^2} \frac{1}{m\lambda} \sum_{n\geqslant 0} a_n \lambda^{n+1} \,, \end{aligned} \tag{4.103}$$

where the coefficients a_n are already quoted in Eq. 4.98. We note that, even if the action in Eq. 4.102 depends on the sign of ϕ_1, from the symmetry of the integral in Eq. 4.103, we see that Z_{NP} does not depend on this choice (only terms with an even power of y contribute to the integral, and these are always paired with an even power of ϕ_1). From this observation, in Eq. 4.103 we only computed the contribution from one of the saddle points ϕ_1 and multiplied the result by a factor 2 (so Z_{NP} contains the contribution from both saddle points ϕ_1).

Inspired in the discussion of [75, Sec. 2.3], let us argue why Z_{NP} is an asymptotic expansion for the discontinuity of the partition function. Consider a general angle θ and the integral

$$Z_\theta\big(m^2, 1/z\big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty e^{i\theta}}^{\infty e^{i\theta}} d\phi\, e^{-\frac{m^2}{2}\phi^2 - \frac{z^{-1}}{4!}\phi^4} \,, \tag{4.104}$$

(with $z = 1/\lambda$) which converges absolutely in $\mathrm{Re}\left(z^{-1}e^{i4\theta}\right) > 0$, so it defines an holomorphic function in this region. It is easy to check that

$$Z_{\theta_1}\big(m^2, 1/z\big) = Z_{\theta_2}\big(m^2, 1/z\big)\,, \quad \mathrm{Re}\left(z^{-1}e^{i4\theta_1}\right) > 0 \cap \mathrm{Re}\left(z^{-1}e^{i4\theta_2}\right) > 0\,, \tag{4.105}$$

if $|\theta_2 - \theta_1| < \pi/4$, so these integrals provide different analytic continuations of Z. Therefore, we can compute the discontinuity of Z as the difference of two of those integrals with conveniently chosen angles. For example, we consider the angles $\theta_- = -\pi/4$ and $\theta_+ = +\pi/4$, whose respective integrals both converge in $\mathrm{Re}\left(z^{-1}\right) < 0$. Their

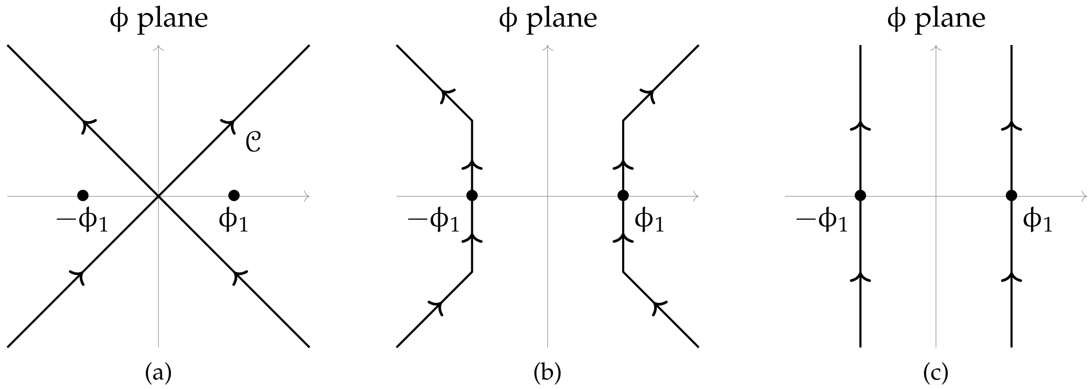


Figure 4.6: Path $\mathcal{C}$ and its successive deformations. The saddle points ϕ_1 and $-\phi_1$ are at the real axis for $\lambda = 1/z > 0$.

difference yields

$$\mathrm{Disc}\, Z\big(m^2, \lambda\big) = Z_{\theta_+}\big(m^2, 1/z\big) - Z_{\theta_-}\big(m^2, 1/z\big) = \frac{1}{\sqrt{2\pi}} \int_{\mathcal{C}} d\phi \, e^{-\frac{m^2}{2}\phi^2 - \frac{z^{-1}}{4!}\phi^4}, \quad (4.106)$$

where $\mathcal{C}$ is the path in Figure 4.6a, which we then conveniently deform into the path of Figure 4.6b. Lastly, we can compute an asymptotic expansion of Disc Z by deforming $\mathcal{C}$ into the path of Figure 4.6c,[7] which is precisely the path we used in the computation of Eq. 4.103.

Using the Borel framework, we will now prove that the expansions Z_P and Z_{NP} are related to one another. Given Eq. 4.97 and Eq. 4.103, it is convenient to define the divergent series $\widetilde{\Phi}_0$ and $\widetilde{\Phi}_1$:

$$Z_P\big(m^2, \lambda\big) = \frac{\widetilde{\Phi}_0(1/\lambda)}{m\lambda}, \qquad \widetilde{\Phi}_0(z) = \sum_{n\geqslant 0} \frac{a_n}{z^{n+1}}, \quad (4.107)$$

$$Z_{NP}\big(m^2, \lambda\big) = i\sqrt{2}\, e^{+\frac{3m^4}{2\lambda}} \frac{\widetilde{\Phi}_1(1/\lambda)}{m\lambda}, \qquad \widetilde{\Phi}_1(z) = \sum_{n\geqslant 0} \frac{(-1)^n a_n}{z^{n+1}}. \quad (4.108)$$

We have the Borel transforms

$$\widehat{\Phi}_0(\zeta) = \sum_{n\geqslant 0} \frac{a_n}{n!}(-\zeta)^n = F\left(\frac{1}{4}, \frac{3}{4}, 1, \frac{-\zeta}{\zeta_0}\right), \quad (4.109)$$

$$\widehat{\Phi}_1(\zeta) = \sum_{n\geqslant 0} \frac{a_n}{n!}\zeta^n = \widehat{\Phi}_0(-\zeta), \quad (4.110)$$

where $\zeta_0 \equiv 3m^4/2$ and F is the ordinary hypergeometric function. We see that $\widehat{\Phi}_0$ has a logarithmic singularity at $\zeta = -\zeta_0$, while $\widehat{\Phi}_1$ has it at $\zeta = \zeta_0$. Around these singular

[7] This last deformation actually changes the region of convergence of the integral in Eq. 4.106 from $\mathrm{Re}\,(z^{-1}) < 0$ to $\mathrm{Re}\,(z^{-1}) > 0$, but the asymptotic expansion is insensitive to this change.

points, we obtain the relations

$$\widehat{\Phi}_0(\zeta-\zeta_0) = -\mathrm{i}\sqrt{2}\,\widehat{\Phi}_1(\zeta)\frac{\mathscr{L}\mathrm{og}(\zeta)}{2\pi\mathrm{i}} + \text{holomorphic}\,, \tag{4.111}$$

$$\widehat{\Phi}_1(\zeta+\zeta_0) = -\mathrm{i}\sqrt{2}\,\widehat{\Phi}_0(\zeta)\frac{\mathscr{L}\mathrm{og}(\zeta)}{2\pi\mathrm{i}} + \text{holomorphic}\,. \tag{4.112}$$

Thus,

$$\mathrm{Disc}\,\widehat{\Phi}_0(\zeta-\zeta_0) = -\mathrm{i}\sqrt{2}\,\widehat{\Phi}_1(\zeta)\,, \tag{4.113}$$

$$\mathrm{Disc}\,\widehat{\Phi}_1(\zeta+\zeta_0) = -\mathrm{i}\sqrt{2}\,\widehat{\Phi}_0(\zeta)\,. \tag{4.114}$$

The above result can also be directly obtained from the discontinuity of the hypergeometric function in Eq. 4.64. These relations are analogous to Eq. 4.62 and Eq. 4.69 which we obtained for the two solutions to the Airy equation. However, in that occasion, these resurgent connections were a consequence of the properties of the alien derivatives, while in the present case there is no differential equation, so we cannot follow the same arguments. Instead, we will prove the resurgent connection of Eq. 4.113 using (A) and (B) and Proposition 3.

Indeed, (A) and (B) imply that

$$\mathrm{Disc}\,\Phi_0(z) = \mathrm{i}\sqrt{2}\,\mathrm{e}^{+\zeta_0 z}\Phi_1(z) = \mathrm{i}\sqrt{2}\,\mathrm{e}^{+\zeta_0 z}\int_0^{-\infty} \mathrm{d}\zeta\,\mathrm{e}^{-z\zeta}\,\widehat{\Phi}_1(\zeta)\,, \tag{4.115}$$

where Φ_0 is the Borel sum of $\widetilde{\Phi}_0$ along $\theta = 0$ (then conveniently analytically continued) and Φ_1 is the Borel sum of $\widetilde{\Phi}_1$ along the direction $\theta = \pi$. Furthermore, by analytic continuation of Φ_0 and Proposition 3, we have

$$\mathrm{Disc}\,\Phi_0(z) = \Phi_0^-(z) - \Phi_0^+(z) = \mathrm{e}^{+\zeta_0 z}\int_0^{-\infty} \mathrm{d}\zeta\,\mathrm{e}^{-z\zeta}\,\mathrm{Disc}\,\widehat{\Phi}_0(\zeta-\zeta_0)\,, \tag{4.116}$$

where the superscripts $+$ and $-$ is a shorthand notation for Borel summation, respectively, along directions above and below the negative real line of the Borel plane. Comparing Eq. 4.115 and Eq. 4.116, we obtain Eq. 4.113, which completes the proof of the corollary in (C).

In passing, we will also determine the branching behaviour of Φ_0. For convenience, we define $\mathcal{L}^{\theta_+}[\widehat{\Phi}_i](z) = \Phi_i^+(z)$ ($i = 0, 1$), and analogously for θ_-, where $\theta_+ \in (0, +\pi)$ and $\theta_- \in (-\pi, 0)$. From Proposition 3, Eq. 4.113 and Eq. 4.114, we have the following relations between the Borel sums of the two expansions $\widetilde{\Phi}_0$ and $\widetilde{\Phi}_1$:

$$\begin{cases} \Phi_0^+(z) - \Phi_0^-(z) = -i\sqrt{2}\, e^{+\zeta_0 z}\Phi_1^\pm(z)\,, \\ \Phi_1^+(z) - \Phi_1^-(z) = 0\,, \end{cases} \qquad \mathrm{Re}(z) < 0\,, \tag{4.117}$$

$$\begin{cases} \Phi_0^-(z) - \Phi_0^+(z) = 0\,, \\ \Phi_1^-(z) - \Phi_1^+(z) = -i\sqrt{2}\, e^{-\zeta_0 z}\Phi_0^\pm(z)\,, \end{cases} \qquad \mathrm{Re}(z) > 0\,. \tag{4.118}$$

Then, computing the analytic continuation of Φ_0^+ to different sheets of the Riemann surface in a similar fashion to the procedure of Section 4.5, we obtain

$$\Phi_0^+\left(z\underline{e}^{2\pi i}\right) = \Phi_0^+(z) + i\sqrt{2}\, e^{+\zeta_0 z}\Phi_1^+(z)\,, \tag{4.119}$$

$$\Phi_0^+\left(z\underline{e}^{i4\pi}\right) = -\Phi_0^+(z)\,, \tag{4.120}$$

$$\Phi_0^+\left(z\underline{e}^{i6\pi}\right) = -\Phi_0^+(z) - i\sqrt{2}\, e^{+\zeta_0 z}\Phi_1^+(z)\,, \tag{4.121}$$

$$\Phi_0^+\left(z\underline{e}^{i8\pi}\right) = \Phi_0^+(z)\,. \tag{4.122}$$

We see that Φ_0^+ returns to itself after 4 turns, so this function has a 4-sheeted Riemann surface.

4.8 0-dimensional sine theory

In all the examples we have considered so far, each function was recovered from the Borel sum of its asymptotic expansion. Nevanlinna's theorem[8] provides sufficient conditions to ensure this result. However, outside of the conditions of the theorem, the Borel sum might not coincide with the function we are trying to recover and this discrepancy originates from the fact that the Borel sum is missing exponential corrections hidden beyond all orders of the asymptotic expansion. In the present section, we will consider such a situation in the context of a 0-dimensional field theory. While quantum fluctuations around the saddle points of the action will still provide the correct asymptotic expansions for the path integral and its discontinuity (as in (A) and (B)), the Borel sums of these expansions will be missing exponential corrections.

We consider the 0-dimensional toy model with fields ϕ defined on $\mathbb{R}$ and with the action

$$S(\phi,\lambda) = \begin{cases} \dfrac{1}{2\lambda}\sin^2\left(\sqrt{\lambda}\phi\right) & \text{if } \phi \in \left[-\dfrac{\pi}{2\sqrt{\lambda}}, +\dfrac{\pi}{2\sqrt{\lambda}}\right] \\ 0 & \text{otherwise.} \end{cases} \tag{4.123}$$

[8] We will review and discuss this theorem in Chapter 5.

In this case, the partition function is given by

$$Z(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{\pi}{2\sqrt{\lambda}}}^{+\frac{\pi}{2\sqrt{\lambda}}} e^{-S(\phi,\lambda)}\, d\phi = \sqrt{\frac{\pi}{2}} \frac{e^{-\frac{1}{4\lambda}}}{\sqrt{\lambda}} I_0\left(\frac{1}{4\lambda}\right), \tag{4.124}$$

where I_n is the modified Bessel function of the first kind, which is an entire function. Either from the second or third expression of Eq. 4.124, we easily see that the partition function has a square root type singularity at $\lambda = 0$. That is,

$$Z\left(\lambda \underline{e}^{2\pi i}\right) = -Z(\lambda)\,, \quad \text{equivalently,} \quad \operatorname{Disc} Z(\lambda) = 2Z(\lambda)\,. \tag{4.125}$$

We will later understand the importance of this observation.

From the solutions of the equation

$$\frac{dS(\phi,\lambda)}{d\phi} = \frac{\sin\left(\sqrt{\lambda}\phi\right)\cos\left(\sqrt{\lambda}\phi\right)}{\sqrt{\lambda}} = 0\,, \tag{4.126}$$

we have the saddle points

$$\phi_0 = 0\,, \quad \phi_1 = \pm\frac{\pi}{2\sqrt{\lambda}}\,. \tag{4.127}$$

On one hand, expanding around the zero action point ϕ_0, we have:

$$S(\phi,\lambda) = \frac{1}{2}\phi^2 - \frac{\lambda}{6}\phi^4 + \frac{\lambda^2}{\phi^6} - \frac{\lambda^3}{630}\phi^8 - \dots\,, \tag{4.128}$$

and the perturbative expansion is given by

$$\begin{aligned} Z_P(\lambda) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\phi^2} \exp\left[\frac{\lambda}{6}\phi^4 - \frac{\lambda^2}{45}\phi^6 + \frac{\lambda^3}{630}\phi^8 - \dots\right] d\phi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\phi^2} \left[1 + \frac{\phi^4}{6}\lambda + \left(-\frac{\phi^6}{45} + \frac{\phi^8}{2\cdot 6^2}\right)\lambda^2 \right. \\ &\qquad \left. + \left(\frac{\phi^8}{630} - \frac{\phi^{10}}{6\cdot 45} + \frac{\phi^{12}}{6\cdot 6^3}\right)\lambda^3 + \dots\right] d\phi\,. \end{aligned} \tag{4.129}$$

Now we use Eq. 4.96 and integrate term by term to obtain

$$\begin{aligned} Z_P(\lambda) &= \left[1 + \frac{3}{6}\lambda + \left(-\frac{5\cdot 3}{45} + \frac{7\cdot 5\cdot 3}{2\cdot 6^2}\right)\lambda^2 \right. \\ &\qquad \left. + \left(\frac{7\cdot 5\cdot 3}{630} - \frac{9\cdot 7\cdot 5\cdot 3}{6\cdot 45} + \frac{11\cdot 9\cdot 7\cdot 5\cdot 3}{6\cdot 6^3}\right)\lambda^3 + \dots\right] \\ &= \left[1 + \frac{1}{2}\lambda + \frac{9}{8}\lambda^2 + \frac{75}{16}\lambda^3 + \dots\right]. \end{aligned} \tag{4.130}$$

The result to all orders is given by [73, 74]

$$Z_P(\lambda) = \frac{1}{\lambda} \sum_{n \geqslant 0} a_n \lambda^{n+1} \,, \tag{4.131}$$

with coefficients

$$a_n = \frac{\Gamma\left(n+\frac{1}{2}\right)^2}{\Gamma\left(\frac{1}{2}\right)^2 n!} 2^n \,. \tag{4.132}$$

On the other hand, the action expanded around $\phi = \phi_1$ (with either of the signs) is given by

$$S(\phi, \lambda) = \frac{1}{2\lambda} - \frac{1}{2}(\phi - \phi_1)^2 + \frac{\lambda}{6}(\phi - \phi_1)^4 - \frac{\lambda^2}{45}(\phi - \phi_1)^6 + \frac{\lambda^3}{630}(\phi - \phi_1)^8 - \dots \tag{4.133}$$

As we noticed in the quartic theory, here the propagator term also has the wrong sign. Thus, we integrate in parallel to the imaginary axis using the parametrisation $\phi(y) = \phi_1 + iy$, $y \in \mathbb{R}$. Then,

$$S(\phi(y), \lambda) = \frac{1}{2\lambda} + \frac{1}{2}y^2 + \frac{\lambda}{6}y^4 + \frac{\lambda^2}{45}y^6 + \frac{\lambda^3}{630}y^8 + \dots \tag{4.134}$$

Now the propagator term has the correct sign. Up to the constant term, the expansion of the action we have obtained is the same as in Eq. 4.128, but replacing λ by $-\lambda$. Plugging Eq. 4.133 inside the partition function, the non-perturbative expansion is given by [73, 74]

$$\begin{aligned} Z_{NP}(\lambda) &= 2ie^{-\frac{1}{2\lambda}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2} \exp\left[-\frac{\lambda}{6}y^4 - \frac{\lambda^2}{45}y^6 - \frac{\lambda^3}{630}y^8 - \dots\right] dy \\ &= 2ie^{-\frac{1}{2\lambda}} \frac{\sqrt{2\pi}}{\lambda} \sum_{n \geqslant 0} (-1)^n a_n \lambda^{n+1} \,, \end{aligned} \tag{4.135}$$

where the coefficients a_n were already quoted in Eq. 4.132.

It is easy to check that the expansions in Eq. 4.131 and Eq. 4.135 are asymptotic expansions for Z and Disc Z, respectively, by comparing to Eq. 4.124 and Eq. 4.125.

As in the case of the quartic theory, it is convenient to define $\widetilde{\Phi}_0$ and $\widetilde{\Phi}_1$:

$$Z_P(\lambda) = \frac{\widetilde{\Phi}_0(1/\lambda)}{\lambda} \,, \qquad \widetilde{\Phi}_0(z) = \sum_{n \geqslant 0} \frac{a_n}{z^{n+1}} \,, \tag{4.136}$$

$$Z_{NP}(\lambda) = 2ie^{-\frac{1}{2\lambda}} \frac{\widetilde{\Phi}_1(1/\lambda)}{\lambda} \,, \qquad \widetilde{\Phi}_1(z) = \sum_{n \geqslant 0} \frac{(-1)^n a_n}{z^{n+1}} \,. \tag{4.137}$$

The Borel transforms of $\widetilde{\Phi}_0$ and $\widetilde{\Phi}_1$ are given by

$$\widehat{\Phi}_0(\zeta) = \sum_{n\geqslant 0} \frac{a_n}{n!}\zeta^n = F\left(\frac{1}{2},\frac{1}{2},1,2\zeta\right), \tag{4.138}$$

$$\widehat{\Phi}_1(\zeta) = \sum_{n\geqslant 0} \frac{a_n}{n!}(-\zeta)^n = \widehat{\Phi}_0(-\zeta). \tag{4.139}$$

We notice that $\widehat{\Phi}_0$ has a logarithmic singularity at $\zeta = 1/2$, while $\widehat{\Phi}_1$, at $\zeta = -1/2$. Considering the expansions around those points, we obtain the relations

$$\widehat{\Phi}_0\left(\zeta+\tfrac{1}{2}\right) = -2i\widehat{\Phi}_1(\zeta)\frac{\mathscr{L}\mathrm{og}(\zeta)}{2\pi i} + \text{holomorphic}, \tag{4.140}$$

$$\widehat{\Phi}_1\left(\zeta-\tfrac{1}{2}\right) = -2i\widehat{\Phi}_0(\zeta)\frac{\mathscr{L}\mathrm{og}(\zeta)}{2\pi i} + \text{holomorphic}. \tag{4.141}$$

So, we have the resurgent connections

$$\mathrm{Disc}\,\widehat{\Phi}_0\big(\zeta+\tfrac{1}{2}\big) = -2i\widehat{\Phi}_1(\zeta), \tag{4.142}$$

$$\mathrm{Disc}\,\widehat{\Phi}_1\big(\zeta-\tfrac{1}{2}\big) = -2i\widehat{\Phi}_0(\zeta). \tag{4.143}$$

In this case, however, we will not be able to prove this connection as we did in the quartic interaction model, because the Borel sums of Z_P and Z_{NP} do not coincide with Z and $\mathrm{Disc}\,Z$ respectively and, therefore, Eq. 4.115 fails. We postpone the proof to Section 5.5 in the next chapter.

In complete analogy to the quartic interaction, and using the same notation, we obtain the following relationships between the Borel sums along different directions:

$$\begin{cases} \Phi_0^+(z) - \Phi_0^-(z) = 0, \\ \Phi_1^+(z) - \Phi_1^-(z) = -2ie^{+\frac{1}{2}z}\Phi_0^\pm(z), \end{cases} \qquad \mathrm{Re}(z) < 0, \tag{4.144}$$

$$\begin{cases} \Phi_0^-(z) - \Phi_0^+(z) = -2ie^{-\frac{1}{2}z}\Phi_1^\pm(z), \\ \Phi_1^-(z) - \Phi_1^+(z) = 0, \end{cases} \qquad \mathrm{Re}(z) > 0. \tag{4.145}$$

From the first line of Eq. 4.145, we observe that the Borel sums $\Phi_0^-(z)$ or $\Phi_0^+(z)$ of the perturbative expansion of Z have an ambiguous imaginary part (depending on the direction of summation) for $z > 0$. This imaginary part can be traced to the presence of a singularity at the positive real axis, $\zeta = 1/2$, in $\widehat{\Phi}_0$. Thus, in contrast to the quartic theory, where the partition function was equal to the Borel sum of its perturbative expansion, in the case of the sine theory, the same situation cannot take place, because the partition function in 4.124 is purely real and non ambiguous for $z > 0$, while the Borel sums $\Phi_0^\pm$ are not.

Instead, we can check that the median resummation (defined in Eq. 4.91) of the perturbative expansion $\widetilde{\Phi}_0$ is real and non-ambiguous for $z > 0$, and it coincides with the partition function in Eq. 4.124 for $\mathrm{Re}(z) > 0$. From Eq. 4.140 and Definition 5, we read explicitly the alien derivative of $\widetilde{\Phi}_0$ at the point 1/2, given by

$$\dot{\Delta}_{\frac{1}{2}}\widetilde{\Phi}_0 = -2ie^{-\frac{1}{2}z}\widetilde{\Phi}_1\,. \tag{4.146}$$

The Stokes automorphism in the direction $\theta = 0$ is $\mathfrak{S}_0 = \exp\left\{\dot{\Delta}_{\frac{1}{2}}\right\}$ and we have

$$\mathfrak{S}_0^{+1/2}\widetilde{\Phi}_0 = \exp\left(+\frac{1}{2}\dot{\Delta}_{\frac{1}{2}}\right)\widetilde{\Phi}_0 = \left(1+\frac{1}{2}\dot{\Delta}_{\frac{1}{2}}\right)\widetilde{\Phi}_0 = \widetilde{\Phi}_0 - ie^{-\frac{1}{2}z}\widetilde{\Phi}_1\,. \tag{4.147}$$

The exponential only has to be expanded up to first order, because $\dot{\Delta}_{1/2}\widetilde{\Phi}_1 = 0$ and, thus, $(\dot{\Delta}_{1/2})^n\widetilde{\Phi}_0 = 0$ for $n \geqslant 2$. The Borel sum of Eq. 4.147 is, by definition, the median resummation of $\widetilde{\Phi}_0$, given by

$$\mathcal{L}^{\theta_+}\mathcal{B}\left[\mathfrak{S}_0^{+1/2}\widetilde{\Phi}_0\right] = \Phi_0^+ - ie^{-\frac{1}{2}z}\Phi_1^+ = \Phi_0^- + ie^{-\frac{1}{2}z}\Phi_1^- = \mathcal{L}^{\theta_-}\mathcal{B}\left[\mathfrak{S}_0^{-1/2}\widetilde{\Phi}_0\right]. \tag{4.148}$$

Both the above expressions are non-ambiguous and they coincide with $Z(1/z)/z$ in $\mathrm{Re}(z) > 0$. The ambiguous Borel sums $\Phi_0^\pm$ were missing exponential corrections that we have correctly recovered with median resummation.

To finish this section, we will make a consideration regarding the branching behaviours of the Borel sums $\Phi_0^\pm$, $\Phi_1^\pm$. Let us recall that Eq. 4.144 and Eq. 4.145 are crucial to determine the analytic continuations of these Borel sums. Repeating a procedure similar to Section 4.5, we can determine the value of the Borel sums in different sheets of the Riemann surface. For example, by going one sheet upwards we obtain,

$$\begin{aligned}\Phi_0^+\left(z\underline{e}^{2\pi i}\right) &= \Phi_0^+(z) + 2ie^{-\frac{1}{2}z}\left(\Phi_1^+(z) + 2ie^{+\frac{1}{2}z}\Phi_0^+(z)\right)\\ &= -3\Phi_0^+(z) + 2ie^{-\frac{1}{2}z}\Phi_1^+(z)\,,\end{aligned} \tag{4.149}$$

$$\Phi_1^+\left(z\underline{e}^{2\pi i}\right) = \Phi_1^+(z) + 2ie^{+\frac{1}{2}z}\Phi_0^+(z)\,. \tag{4.150}$$

Therefore, the median resummation $\Phi \equiv \mathcal{L}^{\theta_+}\mathcal{B}\left[\mathfrak{S}_0^{+1/2}\widetilde{\Phi}_0\right]$ satisfies

$$\begin{aligned}\Phi\left(z\underline{e}^{2\pi i}\right) &= \Phi_0^+\left(z\underline{e}^{2\pi i}\right) - ie^{-\frac{1}{2}z}\Phi_1^+\left(z\underline{e}^{2\pi i}\right)\\ &= -3\Phi_0^+(z) + 2ie^{-\frac{1}{2}z}\Phi_1^+(z) - ie^{-\frac{1}{2}z}\left(\Phi_1^+(z) + 2ie^{+\frac{1}{2}z}\Phi_0^+(z)\right)\\ &= -\Phi_0^+(z) + ie^{-\frac{1}{2}z}\Phi_1^+(z) = -\Phi(z)\,,\end{aligned} \tag{4.151}$$

which is the expected branching behaviour when compared to Eq. 4.125.

In summary, when considered as a function defined on the Riemann surface, the Borel sum of Z_P has a non-zero ambiguous imaginary part for $\lambda > 0$ ($z > 0$). Moreover, the Borel sum has a monodromy incompatible with the function Z itself, as is evident when comparing Eq. 4.149 with Eq. 4.125. Only when we consider the median resummation of the perturbative expansion, which introduces missing exponential corrections to the Borel sum, we solve the two problems simultaneously: the median resummaion takes real values for $\lambda > 0$ and it also has a monodromy compatible with that of Z.

4.9 Conclusions

In this chapter, we have approached the problem of summing factorially divergent series in two ways. In the first approach, in Section 4.3, Section 4.4 and Section 4.5, we presented different divergent series and summed them using the Borel summation method. Then we discussed the analytic properties of these Borel sums. In particular, we saw that the Borel sum is a multivalued function whose discontinuity arises from the singularities in the Borel plane. As a corollary from Theorem 1, we also concluded the divergent series is actually an asymptotic expansion to its own Borel sum.

In the second approach, we presented a particular mathematical problem—a differential equation or a path integral—for which we wanted to obtain a solution. In Section 4.6, we computed two series (Eq. 4.57 and Eq. 4.58) that formally satisfy the Airy equation and the Borel sum of these series were true solutions to the differential equation. The formal solutions, albeit being divergent series, were asymptotic expansions to the true solutions. In Section 4.7, we computed the perurbative expansion of a path integral in a quartic interaction theory and verified that the Borel sum of this expansion recovered the original function. The situation was different for the path integral of the sine theory in Section 4.8. Even though the perturbative series was also an asymptotic expansion to the path integral, its Borel sum did not recover the original function, but instead the median resummation of Eq. 4.148 did. In particular, median resummation provided additional exponential corrections that were missing in the asymptotic expansion.

These situations pose an interesting question: Given an asymptotic expansion as a solution to a mathematical problem, when is its Borel sum a true solution? It is clear from the sine theory that this does not always happen. As we will see in the next chapter, Nevanlinna's theorem gives sufficient conditions under which the Borel sum of an asymptotic expansion recovers the original function and these conditions are related to the absence of exponential corrections to the asymptotic expansion.

Moreover, in this chapter we also discussed the concept of resurgent connections. First we saw that the two formal solutions to the Airy equation are related through the alien derivative, as in Eq. 4.68 and Eq. 4.70. We were able to derive these relations (up to a factor) by using the framework of alien calculus developed by Écalle and supporting the derivation in the existence of a differential equation. Then, the path integral of the quartic interaction theory did not arise as a solution to a differential equation, but we still were able to prove the analogous resurgent connection in Eq. 4.113, which relates quantum fluctuations around the trivial saddle point (perturbative expansion) with quantum fluctuations around the non-trivial saddle points (non-perturbative expansion). In this case, we supported the derivation of the resurgent connection in the observation that the Borel sum of the perturbative expansion coincided with the path integral and the Borel sum of the non-perturbative expansion coincided with the discontinuity of the path integral. Finally, in the case of the sine theory, we again observed the resurgent connection in Eq. 4.142, but in this case the conditions that we needed in the quartic interaction failed. We postponed the derivation of this resurgent connection to Chapter 5, where this connection will be the central topic.

Chapter 5

Resurgence, a problem of missing exponential corrections

Resurgence is concerned with whether the asymptotic expansion of a function, in some way, encodes the full information about this function. In the context of field theories, this is an interesting question, because expectation values most of the times can only be computed within perturbation theory, which yields an asymptotic expansion of the function in powers of the coupling.

In physics, resurgence has been gathering attention over the past years. It has applications in quantum mechanics [76], matrix models [67], supersymmetric gauge theories [77] and topological string theory [68]. For a very exhaustive list of references, see the introduction in [78].

We consider again the expectation value of Eq. 4.1, but now we extend its asymptotic expansion in Eq. 4.2 to the following transseries, which also incorporates potentially missing exponential corrections,

$$\langle \mathcal{O} \rangle (g) \sim \sum_{n \geqslant 0} a_n g^n + i e^{-S_1/g} (-g)^{-\alpha} \sum_{n \geqslant 0} b_n g^n + \dots, \quad g \to 0^+, \tag{5.1}$$

where $S_1 > 0$ and the dots might contain additional exponential corrections, like $e^{-S/g}$, with $S > S_1$. The coefficients a_n are obtained from quantum fluctuations around the trivial saddle point of the action S. That is, a field configuration Φ for which $\delta S[\Phi, g]/\delta\phi = 0$ and $S[\Phi, g] = 0$. The coefficients b_n are computed from non-trivial saddle points, with $S[\Phi, g] = S_1/g \neq 0$.

At first sight, it seems like there is no way that the original asymptotic expansion encodes the full information of the function. Clearly, the coefficients b_n may be completely decoupled from the a_n. Nevertheless, in the context of path integrals, there is indeed a connection between the two sets of coefficients. As described in [79], resurgence is the connection between the large order behaviour of the coefficients a_n and the low order behaviour of the coefficients b_n (and, in fact, this connection also happens between different exponential sectors of the transseries). In this sense, the information in b_n and in the coefficients of other exponential sectors is redundant. In other words, the asymptotic expansion of $\langle \mathcal{O} \rangle (g)$—the first expansion in the transseries of Eq. 5.1—can fully encode the function, at least implicitly.

Closely related, in quantum mechanics, the energy levels of a Hamiltonian can be written as a 1-dimensional Euclidean path integral. For example, the ground energy is given by

$$E(g) = \lim_{T \to +\infty} -\frac{1}{T} \log \operatorname{Tr}\left(e^{-HT}\right), \tag{5.2}$$

where

$$\operatorname{Tr}\left(e^{-HT}\right) = \int_{q(0)=q(T)} [dq]\, e^{-S[\phi, g]} \tag{5.3}$$

and $S[\phi, g]$ is the field version of the Hamiltonian H integrated in the time interval $[0, T]$. As in the case of the field theory, we may compute an asymptotic expansion

$$E(g) \sim \sum_{n \geqslant 0} a_n g^n, \quad g \to 0^+. \tag{5.4}$$

In this scenario, $E(g)$ has a branch cut along $\mathbb{R}^-$ and resurgence is understood as a connection between the leading behaviour, for $g \to 0^-$, of the discontinuity $\operatorname{Disc} E(g)$ and the large order behaviour of the coefficients a_n in Eq. 5.4 [76, 80, 81].[1]

At the same time, the asymptotic behaviour of $\operatorname{Disc} E(g)$ can be computed from the non-trivial saddle points of the action appearing in Eq. 5.3 [76, 81]. In particular, if Φ is a saddle point with $S[\Phi, g] = S_1/g$, then

$$\operatorname{Disc} E(g) \sim 2i\, e^{-S_1/g} (-g)^{-\alpha} \sum_{n \geqslant 0} b_n g^n, \quad g \to 0^-, \tag{5.5}$$

for some b_n and α.

To recapitulate, we have seen that there are two different ways to understand resurgence. In the first case, in a general field theory, we have a connection between the coefficients a_n and b_n in the transseries of Eq. 5.1. In the second case, in quantum mechanics, we have a connection between the coefficients a_n in the asymptotic expansion of Eq. 5.4 and the coefficients b_n in Eq. 5.5, which encode the asymptotic behaviour of the discontinuity. In the present work, we will argue that both interpretations of

[1] Perhaps it is even better to think that the connection is in fact between the coefficients of the asymptotic expansion of $\operatorname{Tr}\left(e^{-HT}\right)$ and the discontinuity of this function. As argued in [76], this connection is then inherited by E through the relation in Eq. 5.2.

resurgence are two sides of the same coin.

As a preparation, in Section 5.1 we will introduce the tools and the notation that we will use thorough this work. These tools are just extensions of the conventional Borel transform and Borel sum. In this section, we will argue why it makes sense to consider these generalisations.

In Section 5.2, we will discuss sufficient conditions that forbid the existence of exponential corrections hidden beyond the asymptotic expansion of a function. This result is known as Nevanlinna's theorem [82] and it will be central to our discussions and conclusions in the forthcoming sections.

In Section 5.4, under the framework of Borel and Laplace transforms, we will formalise the connection previously described between the asymptotic expansion of the ground energy $E(g)$ and its discontinuity (first presented in [80] and valid when $E(g)$ satisfies a dispersion relation in the g plane). We will call this result a resurgent connection, because it is a connection between the trivial saddle point of the action S and the non-trivial saddle points.[2] To introduce the reader in this topic, we will first review the work of [80] in Section 5.3.

Section 5.5 contains the main results of this chapter. There we will exemplify that median resummed series still feature the resurgent connection even if those functions never satisfy a dispersion relation. Then, for median resummed series, we will bring together two apparently unrelated features: the resurgent connection as described in [80] and the connection between the coefficients a_n, b_n in the transseries of Eq. 5.1.

The present chapter is a reproduction of the discussion in [83], but with additional expositions and cross-references to Chapter 4.

5.1 A redifinition of the Borel transform

Because the results presented in this chapter hold for a variety of situations, we will consider a generic complex analytic function f with a power-like asymptotic expansion given by

$$f(z) \sim \sum_{n \geqslant 0} \frac{a_n}{z^{n+1}}, \quad z \to +\infty, \tag{5.6}$$

where the coefficients a_n are factorially divergent, so the series diverges for all z. When relevant, we will frame the discussion again in the context of physics. In particular, f will be an Euclidean path integral (like Eq. 4.1 or Eq. 5.3) and $z = 1/g$.[3] In

[2] In this sense, it coincides with the definition of resurgent connection that we presented in Chapter 4.

[3] On some occasions, for the discussion of quark-hadron duality (see [84] for an introduction), f will be a two-point correlator in quantum chromodynamics and $z = q^2$, where q is the (large) momentum

perturbation theory, we compute an asymptotic expansion of f in powers of small positive g, which corresponds precisely to the expansion in Eq. 5.6.

It is standard to define the Borel transform of the asymptotic series in Eq. 5.6:

$$B(\zeta) = \sum_{n\geqslant 0} \frac{a_n \zeta^n}{n!} \,. \tag{5.7}$$

Because the a_n are factorially divergent, this function converges in a disc around 0. If we can analytically continue the Borel transform to a strip around $\mathbb{R}^+$, then we may verify that the Laplace transform of B satisfies

$$\int_0^\infty d\zeta \, e^{-z\zeta} \, B(\zeta) \sim \sum_{n\geqslant 0} \frac{a_n}{z^{n+1}} \,, \quad z \to +\infty \,. \tag{5.8}$$

by integrating Eq. 5.7 term by term. The formalisation of this result is part (ii) of Nevanlinna's theorem below, and we also briefly presented this result in Theorem 1.

Given that f and the above Laplace transform both have the same asymptotic expansion, one hopes the two functions coincide. However, this might not be true in general, because two functions that differ by an exponentially small term, like e^{-z}, still share the same power-like asymptotic expansion. Part (i) of Nevanlinna's theorem gives sufficient conditions to ensure that these exponential corrections are not present and, thus, to ensure that f coincides with the Laplace transform in Eq. 5.8.

In the present chapter, instead of the conventional Borel transform in Eq. 5.7, we consider the inverse Laplace transform

$$B(\zeta) = \frac{1}{2\pi i} \int_{\mathcal{C}_a} dz \, e^{z\zeta} \, f(z) \,, \tag{5.9}$$

where $\mathcal{C}_a$ is the path $a + iy$, $y \in \mathbb{R}$, with a a constant to the right of all singularities of f.

There are two reasons for this redefinition. First, it applies whether f admits the asymptotic expansion in Eq. 5.6 or not. Second, it clearly reveals that the Borel transform B is related to the singularities of f and, in particular, to its discontinuity.

One can check that this definition coincides with Eq. 5.7 if the $f(z)$ appearing in Eq. 5.9 is replaced by its power-like asymptotic expansion and each term $1/z^{n+1}$ is integrated with the residue theorem. In this sense, the inverse Laplace transform in Eq. 5.9 is an extension on the initial definition of the Borel transform.

We will also use the directional Laplace transform of Definition 4, which has two main

going through the correlator. In this case, Eq. 5.6 has to be understood as the operator product expansion of the correlator and, in all generality, the coefficients a_n can contain logarithms of $z = q^2$.

advantages. First, as we have seen in Chapter 4, it extends the validity of the original asymptotic expansion from $z > 0$ to different regions of the Riemann surface. Second, imaginary ambiguities that may arise in conventional Borel summation can be understood in the following way: the two Borel summations (above and below the positive real axis) have a different imaginary part because they are actually two different analytic continuations of a function with domain in a Riemann surface.

The directional Laplace transform of Definition 4 is also considered in [68, 72, 78, 73, 79, 67, 74] with similar interpretations, but in this work we will emphasise the role of the analytic continuation of the Laplace transform, as in [64].

5.2 Nevanlinna's theorem

It is impossible in general to reconstruct a function solely from the information contained in its power-like asymptotic expansion. For example, two functions that differ by an exponentially suppressed term have the same asymptotic expansion. Nevanlinna's theorem provides sufficient conditions which forbid the presence of exponential corrections hidden beyond the asymptotic expansion and, in those circumstances, the function can be in fact recovered from this expansion by the process of Borel summation. Let us first state the theorem. The proof is presented in detail in Appendix B.2 and sketched in [85, 71].

We review the theorem and its discussion because it contains relevant observations that will prove useful in later sections.

Theorem 2 (Nevanlinna's theorem). *(i) Let* f *be an analytic function in* $\mathrm{Re}(z) > A$ *and satisfy there*

$$f(z) = \sum_{n=0}^{N-1} \frac{a_n}{z^{n+1}} + R_N(z) \tag{5.10}$$

with

$$|R_N(z)| \leqslant L(N+1)!\big(\sigma/|z|\big)^{N+1}, \tag{5.11}$$

where $L > 0$ *and* $\sigma > 0$*. (We say* f *admits* $\sum a_n/z^{n+1}$ *as a uniform 1-Gevrey asymptotic expansion in* $\mathrm{Re}(z) > A$*).*

Under the above hypothesis, the series

$$B(\zeta) = \sum_{n\geqslant 0} \frac{a_n \zeta^n}{n!} \tag{5.12}$$

converges in $|\zeta| < 1/\sigma$ *and has an analytic continuation to the strip* $S_\sigma = \bigcup_{\zeta_0\in\mathbb{R}^+} D(\zeta_0, 1/\sigma)$, *where* $D(\zeta_0, r)$ *is an open disc of centre* ζ_0 *and radius* r.[4] *Furthermore,*

$$|B(\zeta)| \leqslant K e^{A|\zeta|}, \tag{5.13}$$

with $K > 0$*, in every strip* $S_{\sigma'}$ *with* $\sigma' > \sigma$*, and* f *can be recovered from the Laplace transform*

$$f(z) = \int_0^\infty d\zeta\, e^{-z\zeta} B(\zeta)\,, \quad \mathrm{Re}(z) > A\,. \tag{5.14}$$

(ii) If $B(\zeta)$ *is analytic in the strip* $S_{\sigma'}$ *(for all* $\sigma' > \sigma$*) and there satisfies the bound of Eq. 5.13, then the function* f *defined by Eq. 5.14 is analytic in* $\mathrm{Re}(z) > A$ *and admits* $\sum B^{(n)}(0)/z^{n+1}$ *as a uniform 1-Gevrey asymptotic expansion in* $\mathrm{Re}(z) > a$*, for any* $a > A$.

Part (i) of the theorem gives sufficient conditions under which f is uniquely recovered from the coefficients a_n and further presents a particular way to do so: through the Borel summation of Eq. 5.14.

Part (ii) specifies that the sufficient conditions of part (i) are also necessary in the following sense. If Eq. 5.10 and Eq. 5.11 are not satisfied at least in a region of the type $\mathrm{Re}(z) > a$, then f cannot be recovered from its asymptotic expansion using the Borel summation in Eq. 5.14 (although that does not mean f cannot be uniquely recovered through other methods, as we will see in Example 2).

In essence, Nevanlinna's theorem states that, if f satisfies Eq. 5.10 and Eq. 5.11 in $\mathrm{Re}(z) > A$, then the remainder R_N cannot contain any exponential corrections. Let us understand this statement in detail in the following example.

Example 1. *We discuss the exponential function* $f(z) = e^{-z^\alpha}$*, where* $\alpha > 0$*. We will see that* f *fails to satisfy the hypothesis of the theorem.*

First, we compute the asymptotic expansion of f *for* $|\arg(z)| = \theta < \pi/(2\alpha)$*. We consider the limits*

$$\lim_{z\to\infty e^{i\theta}} \frac{f(z)}{1/z^{n+1}} = 0\,, \quad n \in \mathbb{N}\,. \tag{5.15}$$

This means all the coefficients a_n *of the expansion of* f *are 0 and*

$$f(z) \sim 0\,, \quad |\arg(z)| < \pi/(2\alpha)\,. \tag{5.16}$$

These are precisely the directions along which f *is exponentially suppressed. It is clear from this result that* f *cannot be recovered from its asymptotic expansion and, therefore,* f *cannot satisfy the hypothesis of part (i) in the theorem. In the following we will find the precise conditions*

[4] The bound in Eq. 5.11 indicates that the coefficients a_n are, at most, factorially divergent, but there is no restriction on their phase. In particular, they could have fixed sign, which would then be incompatible with the fact that B has no singularities in the positive real axis. In Example 2, we will clarify this apparent contradiction and understand that the condition that f is analytic in $\mathrm{Re}(z) > A$ implicitly forces that the a_n cannot be of fixed sign.

that are not satisfied.

Let us compute the bound in Eq. 5.11 for the remainder R_N*, which in this case coincides with* f*. It is known that for* $x > 0$*,* $e^{-x} \leqslant (N+1)!/x^{N+1}$ *for any* $N \in \mathbb{N}$*. Therefore, choosing a particular direction* θ*, we have the bound*

$$|R_N(z)| = e^{-\operatorname{Re}(z^\alpha)} = e^{-\cos(\alpha\theta)|z|^\alpha} \leqslant (N+1)! \left(\frac{1/\cos(\alpha\theta)}{|z|^\alpha}\right)^{N+1}. \tag{5.17}$$

We now consider three possibilities:

- *If* $\alpha > 1$*, it is enough to consider that* $1/|z|^\alpha < 1/|z|$ *for* $|z| > 1$*. So the bound in Eq. 5.11 is satisfied with* $L = 1$ *and* $\sigma = 1/\cos(\alpha\theta_0)$ *in the region* $|\arg(z)| \leqslant \theta_0 < \pi/(2\alpha)$*. Notice that for any choice of* θ_0*, the region* $|\arg(z)| \leqslant \theta_0$ *never encompasses a half-plane, as demanded by the theorem. So, even if Eq. 5.11 is satisfied, the region where this happens is too small in angle.*

- *If* $\alpha = 1$*, it seems that Eq. 5.11 is satisfied in* $\operatorname{Re}(z) > 0$ *(choosing* θ_0 *as close to* $\pi/2$ *as desired). However, notice that* $\sigma = 1/\cos(\theta_0)$ *increases to infinity as* θ_0 *approaches* $\pi/2$*. In this case, Eq. 5.11 is satisfied in the region* $|\arg(z)| < \pi/2 - \epsilon$ *for any* $\epsilon > 0$*, but the theorem demands* $\epsilon = 0$*.*[5] *We say that* f *admits* 0 *as a 1-Gevrey asymptotic expansion in* $\operatorname{Re}(z) > 0$*, but not uniformly.*

- *If* $\alpha < 1$*, Eq. 5.11 fails for* $|z|$ *high enough no matter which* L *and* σ *are chosen. The bound would be satisfied if* $(N+1)!$ *was replaced by* $(N+1)!^{1/\alpha}$*.*

Now, for a general function f*, if an exponential term* e^{-z^α} *was missing in its asymptotic expansion, then the remainder* R_N*, defined by*

$$R_N(z) = f(z) - \sum_{n=0}^{N-1} \frac{a_n}{z^{n+1}}, \tag{5.18}$$

would contain this exponential term. We would be able to repeat the discussion from the previous example and conclude that some of the hypothesis of part (i) in the theorem fail.

Example 2. *If the hypotheses in part (i) are satisfied, one of the implications is that the Borel transform of* f *must be analytic in some strip* S_σ*. Let us discuss the following function, defined in terms of the exponential integral* E_1*:*

$$f(z) = -e^{-z}E_1(-z) \sim \sum_{n\geqslant 0} \frac{n!}{z^{n+1}}, \quad |z| \to \infty, \tag{5.19}$$

whose Borel transform is $B(\zeta) = 1/(1-\zeta)$*, with an explicit singularity at* $1 \in S_\sigma$*. This implies either* f *is not analytic in any of the regions* $\operatorname{Re}(z) > A$ *or the bound of Eq. 5.11 fails*

[5] When approaching ∞ in parallel to the imaginary line, the exponential term has non-vanishing modulus and, hence, it spoils the validity of the asymptotic expansion in the region with $\epsilon = 0$.

there.

The exponential integral E_1 *is a multivalued function that can be written as*

$$E_1(z) = -\gamma - \log(z) - \sum_{k\geqslant 1} \frac{(-z)^k}{k!\,k}\,. \tag{5.20}$$

Because the series on the right defines an entire function, this expression shows that f *has a logarithmic singularity at* $z = 0$*.*

The branch cut of the logarithm is conventionally placed along $\mathbb{R}^-$*. Thus,* f *has a branch cut along* $\mathbb{R}^+$*. It is obvious in this case that* f *is not analytic in* $\mathrm{Re}(z) > A$ *for any* A*. However, choosing a different branch for the logarithm, the branch cut may be placed along* $\mathbb{R}^-$*. Namely, consider the function*

$$\begin{cases} -e^{-z}E_1(-z) & \textit{if } \mathrm{Im}(z) \leqslant 0 \\ -e^{-z}(E_1(-z) - 2\pi i) & \textit{if } \mathrm{Im}(z) > 0\,. \end{cases} \tag{5.21}$$

This function has the same Borel transform B *as* f *(because the singularities of* f *did not change), but now its branch cut stretches along* $\mathbb{R}^-$*. In moving the cut, we have introduced an exponential term that is not suppressed along* $i\mathbb{R}^+$*. Therefore, by making the function analytic in* $\mathrm{Re}(z) > A$*, the remainder* R_N *no longer satisfies the bound of Eq. 5.11.*

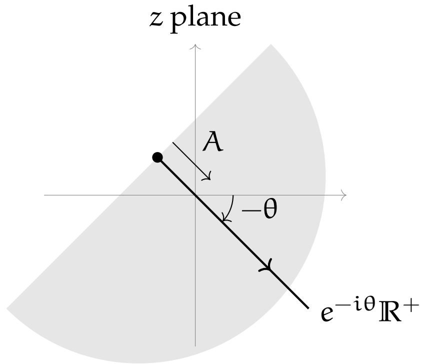
Still, f *can be in fact uniquely recovered from its asymptotic expansion, in the sense that* f *is the only function that has the asymptotic expansion* $\sum n!/z^{n+1}$ *uniformly valid in* $\mathrm{Re}(z) < 0$ *(compared to* $\mathrm{Re}(z) > 0$*). We need a slight modification of Theorem 2. Instead of the region* $\mathrm{Re}(z) > A$ *in part (i) of the theorem, we consider the generalised region* $\mathrm{Re}\left(ze^{i\theta}\right) > A$*, which is the half-plane bisected by the half-line* $e^{-i\theta}\mathbb{R}^+$ *and whose boundary is at a distance* A *from 0 (see Figure 5.1). In addition, the strip where* B *is analytic and satisfies the bound of Eq. 5.13 is replaced by* $S_\sigma(\theta) = \bigcup_{\zeta_0 \in e^{i\theta}\mathbb{R}^+} D(\zeta_0, 1/\sigma)$*. Then,* f *can be recovered from the directional Laplace transform*

$$f(z) = \int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta} B(\zeta)\,, \quad \mathrm{Re}\left(ze^{i\theta}\right) > A\,. \tag{5.22}$$

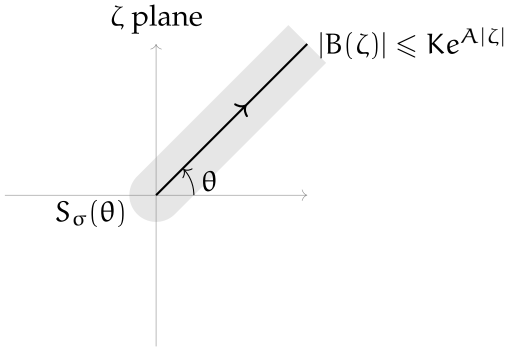
Part (ii) of the theorem may be modified in the same way.

For the example at hand, we might consider $\theta = \pi$*. The function* f *is analytic in* $\mathrm{Re}(z) < 0$ *and admits* $\sum n!/z^{n+1}$ *as a uniform 1-Gevrey asymptotic expansion in that region. Thus* f *can be uniquely recovered from the coefficients* $a_n = n!$ *in* $\mathrm{Re}(z) < 0$ *through Borel summation along the direction* $\theta = \pi$*.*

This example shows that it is not necessary that a function satisfies the hypothesis of part (i) of Nevanlinna's theorem in $\mathrm{Re}(z) > A$, but it is enough if they are satisfied in



(a) $\mathrm{Re}\left(ze^{i\theta}\right) > A$



(b) $S_\sigma(\theta) = \bigcup_{\zeta_0 \in e^{i\theta}\mathbb{R}^+} D(\zeta_0, 1/\sigma)$

Figure 5.1: Regions where the generalisation of Theorem 2 applies.

some half-plane $\mathrm{Re}\left(ze^{i\theta}\right) > A$.

The example also shows that the situations between $f(z)$ and $f(-z)$ are symmetric. For $f(-z)$, the Borel transform is $B(-\zeta)$, which has a pole at $\zeta = -1$, so it does not interfere with the standard Borel summation. For $f(z)$, even if the pole at $\zeta = 1$ interferes with the summation, we may just change the direction of summation. In both cases, the function may be uniquely recovered from its asymptotic expansions, as we would naturally expect.

As a final remark, we notice that $f(z)$ in Eq. 5.19 has an imaginary exponentially small part for $z > 0$, coming from the logarithm in Eq. 5.20. This imaginary part is ambiguous and can also be traced back to the presence of the pole at $\zeta = 1$. The asymptotic expansions of path integrals in powers of the coupling $g = 1/z$ sometimes are non-alternating, as in the sine theory of Section 4.8. Thus the Borel sums of these expansions also have imaginary exponentially small parts for $z > 0$. Nevertheless, we expect that path integrals are real for positive coupling ($z > 0$). Therefore we will always need exponential corrections to cancel those imaginary parts. In particular, this means that these path integrals will never satisfy the conditions of Nevanlinna's theorem or its generalisation.

5.3 The resurgent connection, a first approach by dispersion relations

In the present section, we will review the derivation of the resurgent connection developed in [80], which is based on the existence of a dispersion relation. The purpose of this section is twofold. First, it will serve as an introduction to Section 5.4, where we will give precise conditions that guarantee the resurgent connection. Second, we will argue that the resurgent connection applies to more general situations that those covered by dispersion arguments. In light of the shortcomings of the dispersion arguments, we will drop them at the end of this section.

Given a function f analytic in $\mathbb{C} \setminus \mathbb{R}^-$, we assume that

$$f(z) \sim \sum_{n \geqslant 0} \frac{a_n}{z^{n+1}}, \quad z \to +\infty, \tag{5.23}$$

and

$$\operatorname{Disc} f(z) \sim 2i\, b_0\, e^{-Sz}(-z)^{\alpha-1}, \quad z \to -\infty, \tag{5.24}$$

where $\operatorname{Disc} f(z) = f(z+i0) - f(z-i0)$ with $z < 0$. The resurgent connection is the relation between the large $-z$ behaviour of Disc f and the large order behaviour of the coefficients a_n.

If f is a path integral, this is in fact a connection between large order perturbative physics and low order non-perturbative physics. The coefficients a_n in Eq. 5.23 are computed from quantum fluctuations around the trivial saddle point (saddle points with zero action), while the coefficient b_0 in Eq. 5.24 is computed from fluctuations around non-trivial saddle points (non-zero action).[6] It is in this sense that the connection is «resurgent».

We note that Lipatov's method [63] also describes a connection between the large order behaviour of the a_n and the non-trivial saddle points of the action (although in this case, the connection exists with no mention to the discontinuity of f at all). See [86] for an illustration of this method on different field models.

The power of the resurgent connection is that a single diagram, encoding b_0, is enough to determine the values of the a_n for large n, an information that would require the computation of an infinite number of diagrams otherwise.

To determine the exact resurgent connection, we consider the closed path in Figure 5.2. Using the residue theorem, we have

$$f(z) = -\frac{1}{2\pi i} \int_{\delta}^{R} dw\, \frac{\operatorname{Disc} f(-w)}{w+z} + I_\delta(z) + \mathcal{I}_R(z), \tag{5.25}$$

where $I_\delta(z)$ and $\mathcal{I}_R(z)$ are the integrals of $f(w)/(w-z)/(2\pi i)$ around C_δ and C_R, respectively. The width of $\gamma_{\delta,R}$ is already taken to 0 and so the integral along this path can be written as an integral in $[\delta, R]$ of the discontinuity of f.

First, let us assume that $I_\delta(z)$ does not contribute to the large order behaviour of the a_n. In essence, what happens at small z (around the circle C_δ) should be independent

[6] For example, in [81], where the ground energy of the anharmonic oscillator is discussed, b_0 is computed in this way. We have also carried out this computation for the 0-dimensional quartic interaction of Section 4.7.

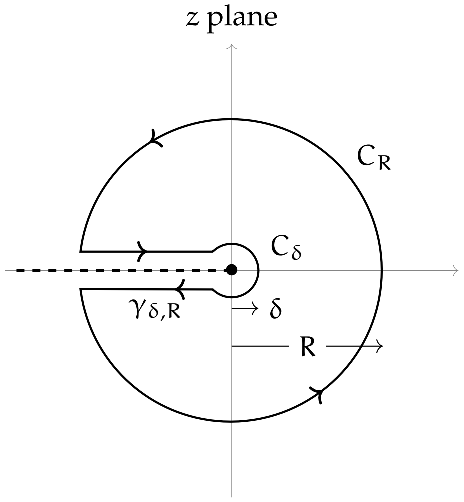


Figure 5.2: Closed path for a dispersion relation. The width of the path $\gamma_{\delta,R}$ around $\mathbb{R}^-$ is already taken to 0.

of the behaviour at large z (encoded in the coefficients a_n). This argument is heuristic at this point, but we will formalise it in Section 5.4. We note that the hypothesis $I_\delta(z) \to 0$ when $\delta \to 0$ was needed in [80] in order to derive the resurgent connection. We are improving on the original discussion by not demanding any condition on $I_\delta(z)$.

Second, we assume that $|\mathcal{I}_R(z)| \to 0$ for $R \to \infty$.[7] This assumption is essential in the discussion of [80]. We will show in Example 3 below that if this hypothesis fails, then the resurgent connection may not take place in general. Still, in Section 5.5, we will be able to identify a less restrictive set of functions for which the connection holds even if $\mathcal{I}_R(z)$ does not vanish for large R (these will be the functions obtained from median resummation).

Finally, we invoke the Taylor expansion of $1/(w+z)$ around $w = 0$ and integrate term by term in Eq. 5.25:

$$\begin{aligned} f(z) - I_\delta(z) &= -\frac{1}{2\pi i}\int_\delta^\infty dw\,\frac{\operatorname{Disc} f(-w)}{w+z} \\ &= -\frac{1}{2\pi i}\int_\delta^\infty dw\,\operatorname{Disc} f(-w)\,\frac{1}{z}\sum_{n\geqslant 0}\left(-\frac{w}{z}\right)^n \\ &\sim \sum_{n\geqslant 0}\frac{1}{z^{n+1}}\left[\frac{(-1)^{n+1}}{2\pi i}\int_\delta^\infty dw\,w^n\operatorname{Disc} f(-w)\right]. \end{aligned} \tag{5.26}$$

Notice that the Taylor expansion of $1/(w+z)$ should only be valid inside the disc of convergence $|w| < |z|$, but the line of integration stretches much beyond this region

[7]Under this assumption, we say f satisfies a dispersion relation (regardless of the contribution from I_δ). Also note that «dispersion relation» is commonly associated with the q^2 plane, where q is some relevant momentum. Here we use this term with no regards to the physical interpretation of z.

for any finite z. This is the typical situation where integrating term by term yields a divergent asymptotic expansion, rather than a convergent series.

Comparing the last line in Eq. 5.26, with the asymptotic expansion $f(z) \sim \sum a_n/z^{n+1}$, we already conclude that

$$a_n \simeq \frac{(-1)^{n+1}}{2\pi i} \int_\delta^\infty dw\, w^n \text{Disc}\, f(-w)\,, \tag{5.27}$$

and the equality is exact up to corrections coming from $I_\delta(z)$.

As we are only interested in the high order behaviour of the a_n, we ignore the contribution form $I_\delta(z)$ and choose δ large enough so that we may replace $\text{Disc}\, f(-z)$ by its behaviour at large $-z$, quoted in Eq. 5.24. After integration we obtain

$$a_n \sim \frac{(-1)^{n+1}}{\pi} \frac{\Gamma(n+\alpha)}{(-S)^{n+\alpha}}\, b_0\,, \quad n \to \infty\,. \tag{5.28}$$

This concludes the derivation of the resurgent connection.

We would like to warn the reader that there is a caveat with this derivation of Eq. 5.28. In Appendix B.3, we discuss an example where all the assumptions in the present section are satisfied, but even then there is no correspondence between Eq. 5.24 and Eq. 5.28. In fact, the reason this derivations is not complete is because we did not keep track of the error between the true a_n and the approximation in Eq. 5.28. We will take care of this issue in Section 5.4.

In the following, we discuss a simple example to understand the importance of the assumption that $\mathcal{I}_R(z)$ vanishes at large R.

Example 3. *Consider the function*

$$g(z) = -e^{-z}\left[E_1(-z) + \log(-z)\right], \tag{5.29}$$

which is the same function as in Example 2, but with an additional exponential term. From Eq. 5.20, it is easy to check that the function of this example is entire. In particular, this means its discontinuity is 0.

An asymptotic expansion for g *is given by*

$$g(z) \sim \sum_{n\geqslant 0} \frac{n!}{z^{n+1}}\,, \quad \text{Re}(z) > 0\,, \tag{5.30}$$

which is the same expansion as in Eq. 5.19, but the region of validity cannot be extended past the imaginary axis. This is because the exponential term $-e^{-z}\log(-z)$ *becomes enhanced in* $\text{Re}(z) < 0$. *In fact, the expansion is not uniformly valid in* $\text{Re}(z) > 0$, *because the modulus of* *the exponential term goes like* $\log|z|$ *along lines parallel to the imaginary axis.*

The function g *does not feature a resurgent connection. If it did, given the 0 discontinuity (smaller than* e^{-Sz} *at large* $-z$ *for all* $S < 0$*), the coefficients* a_n *should be smaller than* $n!/(-S)^n$ *for all* $S < 0$*, which clearly is not the case.*

Of course, this function fails to realise the assumptions that we demanded in the previous derivation. In particular, it fails the hypothesis that $\mathcal{I}_R(z)$ *vanishes at large* R*, due to the presence of the exponential term* $-e^{-z}\log(-z)$*.*

One might think that f will always satisfy a dispersion relation provided we perform enough subtractions. This is correct up to some point. Leading terms in the asymptotic expansion like z^n or $\log^n(z)$ (with $n \geqslant 0$) can be eliminated until the subtracted function vanishes for $|z| \to \infty$, so it satisfies a dispersion relation. But exponential corrections hidden beyond the asymptotic expansion cannot be dealt in the same way, so they will always spoil the dispersion relation.

In spite of this, we notice that the resurgent connection can still take place if the exponential corrections make no contribution to the discontinuity of f. As we will see in Section 5.5, this last observation will be central to the generalisation of the resurgent connection beyond functions that satisfy a dispersion relation. In this sense, our discussion will generalise that of [80].

5.4 The resurgent connection, formal statements

Before presenting the formal statements of the resurgent connection, we will develop some intuition by discussing the particular example below.

Example 4. *The function* $f(z) = e^z E_1(z)$ *has the discontinuity*

$$\operatorname{Disc} f(z) = -2\pi i e^z\,, \quad z < 0\,, \tag{5.31}$$

which can be computed from the logarithm in Eq. 5.20.

The inverse Laplace transform of f *is*

$$B(\zeta) = -\frac{1}{2\pi i}\int_{\gamma_{0,\infty}} dz\, e^{z\zeta}\, f(z) = \frac{1}{2\pi i}\int_0^{-\infty} dz\, e^{z\zeta}\operatorname{Disc} f(z) = \frac{1}{\zeta+1}\,. \tag{5.32}$$

Here we have started from the definition in Eq. 5.9 and deformed the path $\mathcal{C}_a$ *into* $\gamma_{0,\infty}$*. Note that this deformation is possible because* f *goes to 0 for large* $|z|$ *in the region* $\operatorname{Re}(z) \leqslant a$*.*

Using the integral representation of E_1*, one can analytically check that the Laplace transform* *of* B *recovers* f*. Thus, in this case,*

$$f(z) = \int_0^\infty d\zeta\, e^{-z\zeta}\, B(\zeta) \sim \sum_{n\geqslant 0} \frac{(-1)^n n!}{z^{n+1}}\,, \quad \mathrm{Re}(z) > 0\,, \tag{5.33}$$

where the asymptotic expansion is obtained from part (ii) of Nevanlinna's theorem, with $B^{(n)}(0) = (-1)^n n!$.

This is an explicit verification of the resurgent connection, where the discontinuity in Eq. 5.24 fixes the singularities of B *and, in turn, the singularities determine the large order behaviour in Eq. 5.28 of the coefficients* a_n *(in this case, the result is exact). This time, however, we have used the Borel framework as the main tool of the derivation, rather than a dispersion relation.*

In addition, we now check that the converse resurgent connection also holds. That is, the large order behaviour of the coefficients a_n *fixes the singularities in* B *and, in turn, the singularities determine the discontinuity of* f.

We consider a function f *whose asymptotic expansion is* $f(z) \sim \sum (-1)^n n!/z^{n+1}$*, uniformly valid in* $\mathrm{Re}(z) > 0$*. In this case, from part (i) of Nevanlinna's theorem,*

$$B(\zeta) = \sum_{n\geqslant 0} \frac{a_n \zeta^n}{n!} = \frac{1}{\zeta+1}\,; \quad f(z) = \int_0^\infty d\zeta\, e^{-z\zeta}\, B(\zeta)\,, \quad \mathrm{Re}(z) > 0\,. \tag{5.34}$$

As we saw in Section 4.3, the Borel sum has a multivalued component arising from the pole at $\zeta = -1$ *and its discontinuity is given by* $\mathrm{Disc}\, f(z) = -2\pi i e^z$*, as we expected.*

Let us argue why it is somehow anticipated that the discontinuity of a function and the coefficients a_n are related through the singularities in the Borel plane. An heuristic argument by 't Hooft shows that instanton singularities in the Borel plane are determined by the value of the action that each non-trivial saddle point attains [7] (also see [75, Sec. 4.6] for a review).

In quantum mechanics, we know that the discontinuity of the path integral in Eq. 5.3 is computed from the non-trivial saddle points in the action S. Simultaneously, these saddle points also determine the position of the singularities in the Borel plane by the 't Hooft argument. Thus, we conclude that the discontinuity is related to the position of the singularities.

After discussing Example 4, we are now in a good position to formally state the resurgent connection.

Proposition 5 (Resurgent connection)**.** *Let* f *be an analytic function in* $\mathbb{C} \setminus (\mathbb{R}^- + A)$ *and satisfy* $|f(z)| \leqslant |a_0|/|z|$ *in* $\mathbb{C}$ *minus a neighbourhood of* A*. Further assume that*

$$\mathrm{Disc}\, f(z) = 2i\, b_0\, e^{-Sz}(-z)^{\alpha-1} \left[1 + \mathcal{O}\left(\frac{1}{z}\right)\right], \quad z \to -\infty\,, \tag{5.35}$$

with $S < 0$.[8] *Then the Borel transform*

$$B(\zeta) = \frac{1}{2\pi i}\int_{\mathcal{C}_a} dz\, e^{z\zeta}\, f(z)\,, \tag{5.36}$$

with $a > A$, *is analytic in* $\mathrm{Re}(\zeta) > S$ *and is exponentially bounded there by* $|B(\zeta)| \leqslant Ke^{A\,\mathrm{Re}(\zeta)}$. *Furthermore,*

$$f(z) = \int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta}\, B(\zeta) \sim \sum_{n\geqslant 0} \frac{a_n}{z^{n+1}}\,, \quad \mathrm{Re}\left(ze^{i\theta}\right) > a\cos(\theta)\,, \tag{5.37}$$

(the asymptotic expansion being uniformly valid), with $|\theta| < \pi/2$ *and*

$$a_n = B^{(n)}(0) = \frac{(-1)^{n+1}}{\pi}\frac{\Gamma(n+\alpha)}{(-S)^{n+\alpha}}\, b_0\left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right], \quad n \to \infty\,. \tag{5.38}$$

The proof can be found in Appendix B.4. There we repeat the steps in the first part of Example 4, but for a general function. First we define the Borel transform in Eq. 5.36 and check, using Eq. 5.35, that its Taylor coefficients $B^{(n)}(0)$ have the large order behaviour in Eq. 5.38. The tricky part is to prove the equality in Eq. 5.37, but once this is done, the validity of the asymptotic expansion $f(z) \sim \sum a_n/z^{n+1}$ is just a consequence of part (ii) of Nevanlinna's theorem.

A lesson that may be learned from Example 4 and Proposition 5 is that the resurgent connection is always satisfied by functions that can be written as a Laplace transform. We will transfer this result to the discussion of Section 5.5.

Observation 1. As we already pointed out after Eq. 5.28 (also in Appendix B.3), the resurgent connection cannot take place if the error $\mathcal{O}(1/z)$ in Eq. 5.35 contains further exponential corrections $e^{-S_1 z}$ with $\mathrm{Re}(S_1) < \mathrm{Re}(S)$, but with $|S_1| < |S|$.

We have two different orderings in the Borel plane. The closest singularity to 0 determines the leading behaviour of the a_n. This is an ordering in $|\zeta|$. The singularity with the highest real part determines the leading behaviour of $\mathrm{Disc}\, f$. This is an ordering in $\mathrm{Re}(\zeta)$. In fact, this was already observed by [87] in the context of the operator product expansion and quark-hadron duality.

Under the assumptions of the proposition, we impose $S < 0$ in order to deal with Observation 1. The Borel transform B is analytic in $\mathrm{Re}(\zeta) > S$, so the singularities in the Borel plane can only be in the region $\mathrm{Re}(\zeta) \leqslant S$. But, because $S < 0$, any of these singularities will always be farther away from the origin than S. We could admit $S \in \mathbb{C} \setminus \mathbb{R}^+$ in the proposition as long as we have additional assumptions on the Borel plane that forbid the situation of Observation 1. A sketch of the proof for this

[8] See below for a generalisation to $S \in \mathbb{C} \setminus \mathbb{R}^+$.

generalised result can be found in Appendix B.5.

In the following, we also present a kind of «converse» to Proposition 5. In this case, we make no mention to the asymptotic expansion of f, but rather we make assumptions directly over the Borel transform. Of course, if f has the asymptotic expansion $\sum a_n/z^{n+1}$, then the large order behaviour of the a_n determine the singularities of B. It is in this sense that the proposition below is the converse statement.

Proposition 6 is a formalisation of the idea behind [87], where quark-hadron duality was discussed in the framework of Borel transforms. The idea is closely related to alien calculus and, in this context, it was already discussed in [64, p. 100–101]. The main difference between [64] and Proposition 6 is that here we formally take care of error terms.

Proposition 6. *Given* $S \in \mathbb{C} \setminus \mathbb{R}^+$ *and* $\epsilon > 0$, *let* B *be an analytic function in a domain containing* $\mathbb{R}^+$ *and the sector* $|\arg(\zeta - S)| \leqslant \pi/2 + \epsilon$, *from which we subtract the point* S *and the cut arising from the singularity at* S *(see the grey region in Figure 5.3). Further assume that* $|B(\zeta)| \leqslant K e^{A|\zeta|}$ *in the above domain and that*

$$B(\zeta) = -\frac{b_0}{\pi}\frac{\Gamma(\alpha)}{(\zeta - S)^{\alpha}}\Big[1 + \mathcal{O}(\zeta - S)\Big], \quad \zeta \to S\,. \tag{5.39}$$

Then the function

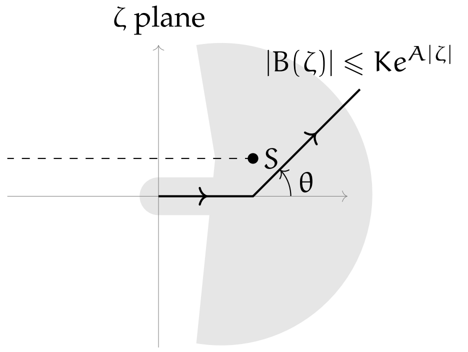
$$f(z) = \int_0^{\infty} d\zeta\, e^{-z\zeta}\, B(\zeta)\,, \quad \mathrm{Re}(z) > A\,, \tag{5.40}$$

admits two analytic continuations (clockwise and anti-clockwise) around a disc of radius A *centred in the origin and their difference yields*

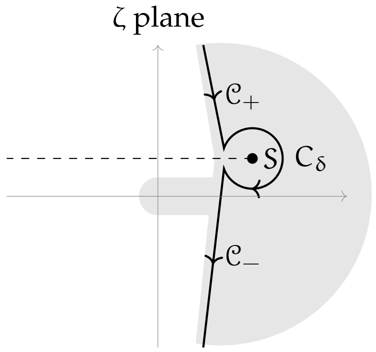
$$\mathrm{Disc}\, f(z) = 2i\, b_0\, e^{-Sz}(-z)^{\alpha-1}\left[1 + \mathcal{O}\left(\frac{1}{z}\right)\right], \quad z \to -\infty\,. \tag{5.41}$$

The proof can be found in Appendix B.6. In summary, if f is the Laplace transform in Eq. 5.40, then we may compute analytic continuations of f by changing the direction of integration. In addition, Disc f may be computed from the difference of the Laplace transform between the two directions $-\pi/2 - \epsilon$ and $+\pi/2 + \epsilon$, as we saw in the second part of Example 4. If necessary, we avoid the singularity at S by considering the paths defined in Figure 5.3a.

Furthermore, the difference between the two Laplace transforms, and therefore Disc f, can be rewritten as an integral along a single path, as that depicted in Figure 5.3b. Let us call $\mathcal{C}$ to this path. For comparison, the analogue if we were to know the full information about the analyticity of B in the whole complex plane would be the contour surrounding the branch cut in Figure 4.4. We note here that there is no need to completely surround the branch cut if we are only looking for an asymptotic approxi-



(a) f_θ



(b) Disc f

Figure 5.3: Contours of integration in the ζ plane needed to define f_θ and Disc f, respectively, for the proof of Proposition 6. The grey region (minus S and its cut) is the domain where B is analytic.

mation to Disc f.

We notice that, given any segment $[a, b]$, we have the bound

$$\left|\int_a^b d\zeta\, e^{-z\zeta}\, B(\zeta)\right| \leqslant \int_a^b |d\zeta|\, e^{-z\,\mathrm{Re}(\zeta)}\, |B(\zeta)| \leqslant m\, |b-a|\, e^{-z \max\left\{\mathrm{Re}(a), \mathrm{Re}(b)\right\}}, \quad (5.42)$$

valid for $z < 0$, where m is the maximum of B inside $[a, b]$. This means that, as long as we can deform the path $\mathcal{C}$ to the left of the Borel plane, we can arbitrarily reduce the values of $\mathrm{Re}(a)$ and $\mathrm{Re}(b)$ and, therefore, the power of the exponential behaviour. The rightmost singularity of B (the point S) prevents further deformation to the left. Thus, we expect that the singularity at S encodes the leading contribution to Disc $f(z)$ for large $-z$.

Proposition 6 is actually a generalisation of the observation in Section 4.3, Section 4.4, Section 4.5 and Section 4.7, where we saw that singularities in the Borel plane determine the discontinuity of the Borel sum. In these examples, we needed to know the properties of B (analyticity, bounds) in the whole complex plane. However, the present proposition is less restrictive regarding these assumptions (B might not even admit an analytic continuation beyond its original domain of analyticity) and, accordingly, we only get an asymptotic approximation to the true discontinuity.

As long as the Laplace transform in Eq. 5.40 is well defined, we may admit that B has a singularity at $\zeta = 0$. This is specially important such as when f is a correlator in quantum chromodynamics, where $z = q^2$ is the momentum of the correlator. In this case, a singularity at $\zeta = 0$ is expected (otherwise, the asymptotic expansion of f would be a simple power-expansion in $1/z$, but we know that the structure of the operator product expansion is much richer, since it contains logarithms). In particular, a singularity at $\zeta = 0$ makes a non-exponential contribution to the discontinuity of the correlator.

5.5 Singularities on the positive real axis of the Borel plane

Until now, culminating in Proposition 5, we have defined the resurgent connection as the relation between the leading behaviours of Disc f and a_n. Even if f satisfies the bound $|f(z)| \leqslant |a_0|/|z|$ (or a dispersion relation), the resurgent connection cannot take place in the situation of Observation 1. However, this situation arises because our knowledge of Disc f is not complete. If we knew the exact discontinuity, we would be able to determine all the singularities in the Borel plane and, therefore, determine the large order behaviour of the a_n to any desired accuracy. In this case, Observation 1 becomes meaningless.

With this in mind, from the theoretical point of view we find it more convenient to define resurgence in the following way: «f satisfies a resurgent connection if Disc f fully encodes all the information about the coefficients a_n». Then, the question whether f satisfies a resurgent connection reduces to the question whether the asymptotic expansion of f is missing exponential corrections or not.[9] In this sense, the discussion is simplified, as we no longer have to deal with Observation 1. For instance, the function in Appendix B.3 would have a resurgent connection, because even if the singularities in its Borel transform, S_1 and S_2, are such that $\mathrm{Re}(S_1) < \mathrm{Re}(S_2)$, but $|S_1| < |S_2|$, the exact coefficients a_n of its asymptotic expansion are in correspondence with its exact discontinuity.

The main lesson from Section 5.4 is then that the resurgent connection (as defined in the previous paragraph) is naturally satisfied by functions expressible as a Laplace transform. A Laplace transform satisfies by default the assumptions in part (i) of Nevanlinna's theorem and, as such, exponential corrections hidden beyond its asymptotic expansion are forbidden. From this observation, it is natural that Laplace transforms always feature a resurgent connection, because the absence of exponential corrections also ensures that these cannot incorporate additional discontinuities that may spoil the connection (as we saw happening in Example 3).

Nevertheless, if the exponential corrections do not incorporate discontinuities (for example, because they are entire functions), then it is clear that the resurgent connection will take place even for functions which are not expressible as a Laplace transform.

In the present section, we will exemplify that median resummed series, which by definition incorporate exponential corrections, still satisfy the resurgent connection. We defined median resummed series in Eq. 4.91 and, in the context of path integrals, they

[9] Hence, the title of this chapter.

arise from the necessity to assign finite and purely real values (when $z > 0$) to divergent series whose Borel transforms have singularities on the positive real axis.

Example 5. *In this example we want to define the median resummation of* $\sum n!/z^{n+1}$ *and discuss its resurgent connection. We have seen that the Borel transform of the above series is* $B(\zeta) = 1/(\zeta - 1)$. *The Laplace transform of* B *along the direction* $\theta = \pi$ *defines a function in* $\mathrm{Re}(z) < 0$ *which can be analytically extended to* $\mathrm{Re}(z) > 0$. *Due to the pole at* $\zeta = 1$, *this function takes values with non-zero imaginary part for* $z > 0$ *and, also, this imaginary part is ambiguous depending on the path of analytic continuation.*

The median resummation of $\sum n!/z^{n+1}$ *is defined by*

$$f(z) = \int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta}\, B(\zeta) \pm i\pi\, e^{-z}\,, \tag{5.43}$$

where the minus sign is chosen when $\theta \in (0, +\pi)$ *and the plus sign, when* $\theta \in (-\pi, 0)$.[10] *By construction,* f *is non-ambiguous and purely real for* $z > 0$.

We denote by f_0 *the Laplace transform in Eq. 5.43 (that is, without the exponential term). On one hand,* f_0 *alone satisfies the assumptions of Proposition 5 (actually* $f_0(-z)$*, but we can change the variable again after determining the connection). On the other hand, the exponential term spoils the condition that* $|f(z)| \leqslant |a_0|/|z|$ *when* z *goes along the imaginary axis.*

By writing f_0 *in terms of the exponential integral* E_1, *we obtain*

$$f(z) = -e^{-z}E_1(-z) \pm i\pi\, e^{-z} = -e^{-z}\left(-\gamma - \log(z) - \sum_{k\geqslant 1} \frac{z^k}{k!\,k}\right). \tag{5.44}$$

In the second equality, we have used Eq. 5.20 and absorbed the ambiguous exponential term inside the logarithm. This process changes the logarithm from $\log(-z)$ *to* $\log(z)$.

Eq. 5.44 explicitly shows that the ambiguous exponential term that arises from median resummation does not alter the discontinuity of the Laplace transform f_0, *whose multivalued component is* $\log(-z)$. *Indeed,* $\log(-z)$ *and* $\log(z)$, *albeit being different functions, have the same discontinuity.*

We gather the following observations from Example 5:

(A) f_0 satisfies the resurgent connection in Proposition 5.

(B) For $z \to +\infty$, f_0 shares the same asymptotic expansion with f.

[10] It might be argued that, for $\theta \in (+\pi/2, +\pi) \cup (-\pi, -\pi/2)$, there should be no exponential correction because there are no singularities in $\mathbb{R}^-$ and, thus, no ambiguity to cancel. Removing the exponential term for those values of θ would break the analytic properties of the resummed series $f(z)$.

(C) $\mathrm{Disc}\, f_0(z) = \mathrm{Disc}\, f(z)$ for $z \in \mathbb{C}$ (in particular $z < 0$) by choosing appropriate branches for $\log(z)$ and $\log(-z)$.

We conclude from the above points that f satisfies the same resurgent connection as the Laplace transform f_0.

To finish this section, we will exemplify that we expect a similar situation for path integrals with singularities on the positive axis of the Borel plane. We consider again the partition function Z of the 0-dimensional sine theory introduced in Section 4.8.

To continue the discussion, we define

$$\Phi(z) = \frac{Z(1/z)}{z}, \tag{5.45}$$

such that $z = 1/\lambda$ and the asymptotic expansion of Φ starts with the power $1/z$.

From the exact result in Eq. 4.124, we can check the validity of the following asymptotic expansions:

$$\Phi(z) \sim \sum_{n\geqslant 0} \frac{a_n}{z^{n+1}}, \quad \mathrm{Re}(z) > 0; \quad \Phi(z) \sim \pm i e^{-\frac{z}{2}} \sum_{n\geqslant 0} \frac{b_n}{z^{n+1}}, \quad \mathrm{Re}(z) < 0, \tag{5.46}$$

where a_n are the coefficients of the perturbative expansion of Z, quoted in Eq. 4.132, and $b_n = (-1)^n a_n$ are the coefficients of the non-perturbative expansion. The $\pm$ encodes the branch cut of Φ. We take the plus sign for z above the cut and the minus sign for z below the cut.

The above result can be compactly written as the transseries

$$\Phi(z) \sim \sum_{n\geqslant 0} \frac{a_n}{z^{n+1}} \pm i e^{-\frac{z}{2}} \sum_{n\geqslant 0} \frac{b_n}{z^{n+1}}, \quad |z| \to \infty. \tag{5.47}$$

The transseries contains both the perturbative expansion Z_P and the non-perturbative corrections Z_{NP}. When $\mathrm{Re}(z) > 0$, the second term is exponentially suppressed and it is hidden for large $|z|$. Similarly, for $\mathrm{Re}(z) < 0$, the second term is exponentially enhanced and the first term is now hidden.

The exponential corrections in Eq. 5.47 would in general spoil the resurgent connection. However, we still verify that the large order behaviour

$$a_n \sim \frac{\Gamma(n)}{\pi} 2^n, \quad n \to \infty, \tag{5.48}$$

is in correspondence with

$$\mathrm{Disc}\,\Phi(z) = 2\Phi(z) \sim 2i\frac{e^{-\frac{z}{2}}}{z}\,, \quad z \to -\infty\,. \tag{5.49}$$

(With parameters $b_0 = 1$, $\alpha = 0$, $S = 1/2$ in Proposition 5).[11] We remark that, a priori, this is not the same connection that is observed in [73, 74]. In these references, they discuss the relation between the coefficients a_n and b_n in the transseries of Eq. 5.47 and here we are concerned with the connection between the a_n and the discontinuity of Φ.

Let us argue why this resurgent connection takes place. From the transseries in Eq. 5.47, we may write Φ as the median resummation of the series $\sum a_n/z^{n+1}$. This yields the result of Eq. 4.148, which written explicitly reads

$$\Phi(z) = \int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta}\, \widehat{\Phi}_0(\zeta) \pm i e^{-\frac{z}{2}} \int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta}\, \widehat{\Phi}_1(\zeta)\,. \tag{5.50}$$

The Borel transforms $\widehat{\Phi}_0$, $\widehat{\Phi}_1$ are defined in Eq. 4.138 and Eq. 4.139, respectively. The minus sign is chosen when $\theta \in (0, +\pi)$, and the plus sign, when $\theta \in (-\pi, 0)$.

Let us call Φ_0 the Laplace transform of $\widehat{\Phi}_0$ as it appears in Eq. 5.50. We recall that $\widehat{\Phi}_0(\zeta)$ has a logarithmic singularity at $\zeta = 1/2$ and that this singularity generates an ambiguous imaginary part in $\Phi_0(z)$ for $z > 0$, which is cancelled against the second term in Eq. 5.50. In this way, $\Phi(z)$ is real for $z > 0$.

In this example we will prove that, instead of (C), we have

(C′) $\mathrm{Disc}\,\Phi_0(z)$ and $\mathrm{Disc}\,\Phi(z)$ (understood as analytic functions in the variable z) differ by an exponentially small term when $z \to -\infty$. Equivalently, the two discontinuities share the same asymptotic expansion in that limit.

Gathering (A), (B), (C′), the conclusion of Example 5 follows again: Φ satisfies the same resurgent connection as the Laplace transform Φ_0. So, let us prove (C′).

The discontinuity of Φ_0 comes from the difference in the two directions above (θ_+) and below (θ_-) the positive real axis of the Borel plane:

$$\begin{aligned} \mathrm{Disc}\,\Phi_0(z) &= \left(\int_0^{\infty e^{i\theta_+}} - \int_0^{\infty e^{i\theta_-}} \right) d\zeta\, e^{-z\zeta}\, \widehat{\Phi}_0(\zeta) \\ &= -e^{-\frac{z}{2}} \int_0^{\infty} d\zeta\, e^{-z\zeta}\, \mathrm{Disc}\,\widehat{\Phi}_0\left(\zeta + \tfrac{1}{2}\right)\,, \quad \mathrm{Re}(z) > 0\,. \end{aligned} \tag{5.51}$$

[11] We only display the connection at leading order, but it is easy to check that the connection also happens between all sub-leading corrections to Eq. 5.48 and Eq. 5.49.

The second term in Eq. 5.50 exactly cancels this discontinuity in $\mathrm{Re}(z) > 0$. The $\pm$ conspire with the singularity in $\widehat{\Phi}_0$ so that when changing the direction θ from below 0 to above 0, there is effectively no singularity in $\widehat{\Phi}_0$. Specifically, we have the cancellation

$$\mathrm{Disc}\,\Phi_0(z) - 2ie^{-\frac{z}{2}} \int_0^\infty d\zeta\, e^{-z\zeta}\, \widehat{\Phi}_1(\zeta) = 0\,, \quad \mathrm{Re}(z) > 0\,. \tag{5.52}$$

The fulfilment of this equation in $\mathrm{Re}(z) > 0$ implies that

$$\mathrm{Disc}\,\widehat{\Phi}_0\left(\zeta + \tfrac{1}{2}\right) = -2i\widehat{\Phi}_1(\zeta)\,, \quad \zeta \geqslant 0\,, \tag{5.53}$$

and the result also has to be satisfied for ζ in the Riemann surface of $\widehat{\Phi}_1$ due to the unicity of the analytic continuation.[12]

Combining Eq. 5.51 and Eq. 5.53, we may write

$$\mathrm{Disc}\,\Phi_0(z) = 2ie^{-\frac{z}{2}} \int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta}\, \widehat{\Phi}_1(\zeta)\,, \quad \mathrm{Re}\left(ze^{i\theta}\right) > 0\,. \tag{5.54}$$

We also introduced a direction θ in the integral such that $\mathrm{Disc}\,\Phi_0$ can be analytically continued from $\mathrm{Re}(z) > 0$ to $\mathrm{Re}(z) < 0$.

Thanks to the ambiguity cancellation, Eq. 5.50 changes smoothly when θ changes from above to below the positive real axis (even if the singularity in $\zeta = 1/2$ of $\widehat{\Phi}_0$ would normally prevent that). Instead, the directions above (θ_+) and below (θ_-) the negative real axis yield two different results for Eq. 5.50 at the same point z. Their difference is defined as the discontinuity of Φ. Namely, we have

$$\mathrm{Disc}\,\Phi(z) = ie^{-\frac{z}{2}} \left(\int_0^{\infty e^{i\theta_-}} + \int_0^{\infty e^{i\theta_+}} \right) d\zeta\, e^{-z\zeta}\, \widehat{\Phi}_1(\zeta)\,, \quad \mathrm{Re}(z) < 0\,. \tag{5.55}$$

Note that the two integrals would normally appear with opposite sings (and then the difference would be exactly 0), but because of the $\pm$, the integrals are summed instead.

Now there are two ways to proceed. We either check that $\mathrm{Disc}\,\Phi_0$ and $\mathrm{Disc}\,\Phi$ differ by an exponentially small term or that they share the same asymptotic expansion. Here we will go with the later. Using part (ii) of Nevanlinna's theorem in the Laplace transform appearing either in Eq. 5.54 (with θ close enough to π) or in Eq. 5.55, we see that in both cases

$$\left.\begin{array}{l}\mathrm{Disc}\,\Phi_0(z)\\ \mathrm{Disc}\,\Phi(z)\end{array}\right\} \sim 2ie^{-\frac{z}{2}} \sum_{n\geqslant 0} \frac{b_n}{z^{n+1}}\,, \quad z \to -\infty\,, \tag{5.56}$$

[12]This in principle would prove the resurgent connection as defined in Section 4.7 and Section 4.8. However, we now want to go a step further and also clarify that this entails the resurgent connection as defined in Section 5.3.

where $b_n = \widehat{\Phi}_1^{(n)}(0)$. This completes the verification.[13]

It might be argued that the cancellation in Eq. 5.52 is a particular feature of the example we have discussed, but in fact, this is a general feature of median resummed series, which ensures that the resulting function is real for $z > 0$.

We also make the following observation from this example. Even if the resurgent connection is originally a connection between the asymptotic expansion of a function and its discontinuity, in this case it is reinterpreted as a connection between the asymptotic expansion of the function and its exponential corrections in the transseries. This is inferred from the realisation that the exponential corrections in Eq. 5.47 are related to the asymptotic expansion of $\mathrm{Disc}\,\Phi$ in Eq. 5.56. Moreover, we can now clarify why quantum fluctuations around the non-trivial saddle points, which we computed in Section 4.8, would yield these exponential corrections. This is so because we expect in general that these saddle points contain information about the discontinuity of the path integral, as we have exemplified in Section 4.7 and Section 4.8.

We finish this section with an important observation. Given an arbitrary series, we haven seen that the function defined as the median resummed series always features a resurgent connection. Nevertheless, given the asymptotic expansion of a function, it could be misleading to think that the median resummation of this expansion always yields the true function. If the median resummed expansion does not coincide with the function, we cannot make any claim regarding its resurgent connection.

In Appendix B.7, we discuss an example within 2-dimensional field theory where the function under study does not coincide with the median resummation of its asymptotic expansion and, in consequence, the function does not satisfy any resurgent connection.[14]

5.6 Conclusions

In the present work, resurgence is defined as a connection between the discontinuity of a function and the coefficients of its asymptotic expansion (Proposition 5). These two elements are related through the singularities in the Borel plane. Schematically:

$$\text{Discontinuity of } f \quad \leftrightharpoons \quad \text{Singularities of } B \quad \leftrightharpoons \quad \begin{array}{c}\text{Coefficients } a_n \text{ in the} \\ \text{asymptotic expansion of } f\end{array}$$

[13] It is interesting to note that $\mathrm{Disc}\,\Phi_0 \neq \mathrm{Disc}\,\Phi$. In fact, the two discontinuities differ by a non-zero exponential correction that arises from the singularity at $\zeta = -1/2$ in B_1.

[14] We might argue that median resummation does not work in the example of Appendix B.7, because it has renormalon singularities, rather than instantons. It could be interesting to study further if this argument is indeed correct.

In the literature (for instance, [79]), resurgence is understood as a connection between different exponential sectors of a transseries. Discussing the 0-dimensional path integral of Section 4.8, we have concluded that these are two sides of the same coin. At this point, this might seem a trivial statement, because in that example the exponential corrections in the transseries encode the asymptotic behaviour of the discontinuity (compare Eq. 5.47 to Eq. 5.56).

To reach the above conclusion, we had to make a detour. We first discussed sufficient conditions that forbid the existence of exponential corrections to the asymptotic expansion of a function, a result known as Nevanlinna's theorem. Under the assumptions of the theorem, a function can be uniquely recovered from its asymptotic expansion with the method of Borel summation. This theorem was interesting in the context of our work, because we later showed that a function free of exponential corrections (thus expressible as a Laplace transform) is the minimal unit featuring a resurgent connection.

We later observed that we can add exponential corrections to these minimal units as long as the corrections do not spoil the original resurgent connection by incorporating additional discontinuities (a canonical example is given when the exponential corrections are entire functions). In particular, we have discussed a special case of exponential corrections which arise from the median resummation of a divergent series with fixed sign coefficients. For all practical purposes, the resurgent connection held in our examples of median resummation as if the exponential corrections were not even present.

Finally, we want to remark again that, in general, a function defined by a path integral does not have to coincide with the median resummation of its asymptotic expansion. For example, in the 2-dimensional path integral of Appendix B.7, it is clear that the median resummation of the asymptotic expansion does not recover the full function. It is beyond the scope of our work to understand when median resummation is enough to recover the true function. A discussion along this line can be found, for instance, in [88].

Appendix **B**

Appendix of Part II

B.1 Isserlis' theorem

In this section, we will use Isserlis' theorem—a result from probability theory and the counterpart of Wick's theorem—to compute correlators. We will also discuss how this result applies to the computation of perturbative expansions in a field theory.

Consider a random vector $\vec{\phi} = (\phi_1, \phi_2, \dots, \phi_N)$ following a normal distribution with mean $\vec{\mu} = 0$ and covariance matrix Σ. Its joint density is given by

$$f(\vec{\phi}) = \frac{\exp\left(-\frac{1}{2}\vec{\phi}\Sigma^{-1}\vec{\phi}\right)}{\sqrt{\det(2\pi\Sigma)}}\,. \tag{B.1}$$

In a field theory, the vector $\vec{\phi}$ is a field and the indices i of its components encompass position, particle type, spin, etc.

An m-point correlator is the expected value of the product of m factors ϕ_i. Denoting the m-point correlators $\langle \phi_1 \phi_2 \cdots \rangle$ as $\langle 12 \cdots \rangle$ to simplify the notation, Isserlis' theorem states that

$$\langle 12 \cdots (2n-1) \rangle = 0\,, \tag{B.2}$$

$$\langle 12 \cdots (2n) \rangle = \sum_{\ell \in \mathcal{P}} \prod_{i=1}^{n} \langle r_{\ell_{1,i}} r_{\ell_{2,i}} \rangle\,, \tag{B.3}$$

where $\mathcal{P}$ is the set of possible pairings of the elements $\{1, 2, \dots, 2n\}$. The pairings can be defined as matrices in $\{1, 2, \dots, 2n\}^{2\times n}$ with all components different from each other and an equivalence where two matrices are the same if one can be made equal to the other by permuting its columns or the two elements within a column. For example,

$$\begin{pmatrix} 1 & 3 & 5 & \dots & 2n-1 \\ 2 & 4 & 6 & \dots & 2n \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 5 & 4 & \dots & 2n-1 \\ 1 & 6 & 3 & \dots & 2n \end{pmatrix} \tag{B.4}$$

are the same matrices within the equivalence.

Eq. B.2 trivializes the computation of m-point correlators whenever m is odd. Eq. B.3 is just the inductive generalization of the case $n = 2$:

$$\langle 1234 \rangle = \langle 12 \rangle \langle 34 \rangle + \langle 13 \rangle \langle 24 \rangle + \langle 14 \rangle \langle 23 \rangle \,. \tag{B.5}$$

For example, in the 6-point correlator $\langle 123456 \rangle$, pairing the 1 to each of the other indices and leaving the rest unpaired, one obtains

$$\langle 123456 \rangle = \langle 12 \rangle \langle 3456 \rangle + \langle 13 \rangle \langle 2456 \rangle + \langle 14 \rangle \langle 2356 \rangle + \langle 15 \rangle \langle 2346 \rangle + \langle 16 \rangle \langle 2345 \rangle \tag{B.6}$$

and now we may apply Isserlis' theorem for $n = 2$ to each term, yielding Eq. B.3.

Isserlis' theorem reduces any arbitrary $2n$-point correlator to a sum of products of the 2-point correlators $\langle ij \rangle$. Each term is the product of n 2-point correlators and there are a total of $(2n)!/(2^n n!) = 3 \cdot 5 \cdot 7 \cdots (2n-1)$ terms in the sum. As examples, the number of terms is equal to 3 for 4-point correlators, 15 for 6-point correlators, 105 for 8-point correlators and 945 for 10-point correlators.

The 2-point correlator $\langle ij \rangle$ can be identified with the line $i \bullet\!\!-\!\!-\!\!\bullet\, j$ in the diagrammatic approach (a line joining the vertex i with j) and its value is just the component (i, j) of the covariance matrix Σ. That is,

$$\langle ij \rangle = \; i \bullet\!\!-\!\!-\!\!\bullet\, j \; = \frac{1}{\sqrt{\det(2\pi\Sigma)}} \int_{\mathbb{R}^N} d\vec{\phi}\, \phi_i \phi_j \exp\left(-\frac{1}{2}\vec{\phi}\Sigma^{-1}\vec{\phi}\right) = \Sigma_{ij} \,. \tag{B.7}$$

This is the analogous of a free propagator in a field theory. Therefore, any m-point correlator can be computed in terms of 2-point correlators, which are just the components of the covariance matrix Σ.

Isserlis' theorem only holds in principle for normal-distributed random vectors with mean $\vec{\mu} = 0$. But, actually, it can also be applied "approximately" to the generalized joint density

$$f_V(\vec{\phi}) = \frac{1}{\mathcal{N}} \frac{\exp\left(-\frac{1}{2}\vec{\phi}\Sigma^{-1}\vec{\phi} + \lambda V(\vec{\phi})\right)}{\sqrt{\det(2\pi\Sigma)}} = f(\vec{\phi}) \frac{e^{\lambda V(\vec{\phi})}}{\mathcal{N}} \,, \tag{B.8}$$

where the factor $\mathcal{N}$ is a proper normalization, so that the density integrates to 1.

The actual computation of the correlator

$$\langle 12\cdots\rangle_V = \int_{\mathbb{R}^N} d\vec{\phi}\, \phi_1\phi_2\cdots f_V(\vec{\phi}) \tag{B.9}$$

is impossible for arbitrary V, so we will restrict to the case where λ is a small parameter with respect which we will perform perturbation theory and V is a polynomial in the components of $\vec{\phi}$ of degree greater than 3. For reasons that we will explain latter, we also consider V "local", which means that we do not allow index mixing like $\phi_1^2\phi_2$.

First, let us compute the normalization factor $\mathcal{N}$ in perturbation theory. Imposing

$$\int_{\mathbb{R}^N} d\vec{\phi}\, f_V(\vec{\phi}) = 1\,, \tag{B.10}$$

we have

$$\begin{aligned}\mathcal{N} &= \int_{\mathbb{R}^N} d\vec{\phi}\, f(\vec{\phi})e^{\lambda V(\vec{\phi})} = \int_{\mathbb{R}^N} d\vec{\phi}\, f(\vec{\phi}) \sum_{n\geqslant 0} \frac{\big(\lambda V(\vec{\phi})\big)^n}{n!} \\ &= \sum_{n\geqslant 0} \frac{\lambda^n}{n!}\langle V(\vec{\phi})^n\rangle = 1 + \lambda\langle V(\vec{\phi})\rangle + \frac{\lambda^2}{2}\langle V(\vec{\phi})^2\rangle + \dots\end{aligned} \tag{B.11}$$

So we have reduced the problem of computing $\mathcal{N}$ in the generalized joint density of Eq. B.9 to the problem of computing the expected values of $V(\vec{\phi})^n$ in the normal density of Eq. B.1. We know how to exactly solve the later thanks to Isserlis' theorem.

Again in perturbation theory, we may rewrite the correlator $\langle 12\cdots\rangle_V$ of the generalized density in terms of correlators in the normal density:

$$\begin{aligned}\langle 12\cdots\rangle_V &= \frac{1}{\mathcal{N}}\int_{\mathbb{R}^N} d\vec{\phi}\, \phi_1\phi_2\cdots f(\vec{\phi})e^{\lambda V(\vec{\phi})} \\ &= \frac{1}{\mathcal{N}}\int_{\mathbb{R}^N} d\vec{\phi}\, \phi_1\phi_2\cdots f(\vec{\phi}) \sum_{n\geqslant 0} \frac{\big(\lambda V(\vec{\phi})\big)^n}{n!} \\ &= \frac{1}{\mathcal{N}}\sum_{n\geqslant 0} \frac{\lambda^n}{n!}\langle 12\cdots V(\vec{\phi})^n\rangle \\ &= \frac{1}{\mathcal{N}}\Big(\langle 12\cdots\rangle + \lambda\langle 12\cdots V(\vec{\phi})\rangle + \frac{\lambda^2}{2}\langle 12\cdots V(\vec{\phi})^2\rangle + \dots\Big)\,.\end{aligned} \tag{B.12}$$

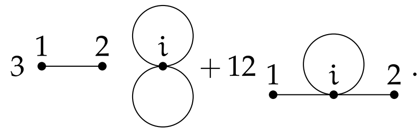
In the diagrammatic approach, the first term in the last equality corresponds to the free theory ($\lambda = 0$), the second term to 1-loop corrections, the third term to 2-loop corrections, and so on. All the above terms can be computed exactly with Isserlis' theorem.

Let us compute an explicit example. Consider the potential $V(\vec{\phi}) = \phi_i^4$, where the sum over i from 1 to N is implicitly understood. From now on, summation is understood whenever free indices appear. Eq. B.12 in this example yields

$$\langle 12\rangle_V = \frac{1}{\mathcal{N}}\left(\langle 12\rangle + \lambda\langle 12i^4\rangle + \frac{\lambda^2}{2}\langle 12i^4j^4\rangle + \dots\right). \tag{B.13}$$

Using Isserlis' theorem, we will expand $\langle 12i^4\rangle$ and $\langle 12i^4j^4\rangle$ in terms of 2-point correlators. For the 1-loop term, we have

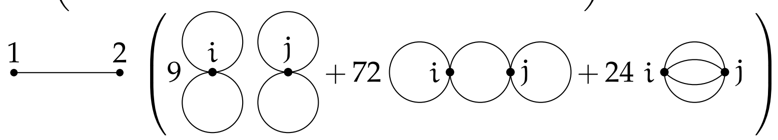
$$\langle 12i^4\rangle = 3\langle 12\rangle\langle ii\rangle^2 + 4\cdot 3\langle 1i\rangle\langle 2i\rangle\langle ii\rangle \tag{B.14}$$



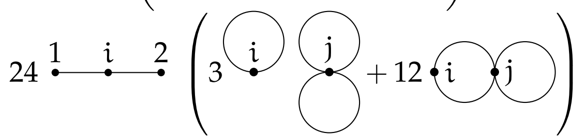
Because $\langle 12i^4\rangle$ is a 6-point correlator, we expect a total of 15 possible pairings of the indices. We have collected the pairings in two groups. In the first group, we have paired the 1 with the 2 (1 way) and the i's with each other (3 ways). Thus, there are a total of 3 pairings that contribute to this group. In the second group, we have paired the 1 with one of the four i's (4 ways) and the 2 with one of the three remaining i's (3 ways). Thus, there are $4\cdot 3 = 12$ pairings contributing to the second group.

For the 2-loop term, we have

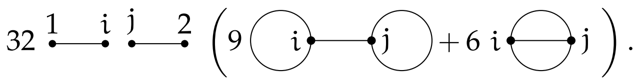
$$\begin{aligned}\langle 12i^4j^4\rangle &= \langle 12\rangle\langle i^4j^4\rangle + 4\cdot 3\cdot 2\langle 1i\rangle\langle 2i\rangle\langle i^2j^4\rangle + 4^2\cdot 2\langle 1i\rangle\langle 2j\rangle\langle i^3j^3\rangle\\ &= \langle 12\rangle\left(3^2\langle ii\rangle^2\langle jj\rangle^2 + (3!)^2\cdot 2\langle ii\rangle\langle jj\rangle\langle ij\rangle^2 + 4!\langle ij\rangle^4\right)\end{aligned}$$



$$+\,24\langle 1i\rangle\langle 2i\rangle\left(3\langle ii\rangle\langle jj\rangle^2 + 4\cdot 3\langle ij\rangle^2\langle jj\rangle\right) \tag{B.15}$$



$$+\,32\langle 1i\rangle\langle 2j\rangle\left(3^2\langle ii\rangle\langle jj\rangle\langle ij\rangle + 3!\langle ij\rangle^3\right)$$



In the first equality, the first term corresponds to pairing the 1 with the 2 (1 way) and leaving the remaining indices untouched. The second term corresponds to pairing the 1 with one of the four i's (4 ways) and then the 2 with one remaining of the three remaining i's (3 ways). We then repeat the same computation pairing the 1 and the 2 with one j each (12 ways). Because there is an implicit sum over i and j, the two preceding groups can be collected together for a total of $4\cdot 3\cdot 2$ terms. Finally, the third term corresponds to the pairing of the 1 with one of the four i's (4 ways) and the 2 with one of the four j's (4 ways). We then repeat the same computation interchanging the roles of i and j, for another (16 ways), so in total the third term appears $4^2 \cdot 2$ times.

In the second equality we have further paired the untouched indices:

(a) $\langle i^4 j^4 \rangle$: We have 3 pairing groups. (i) We pair the i's with each other (3 ways) and the j's with each other (3 ways). (ii) We pair one i with another i (3! ways) and one j with another j (3! ways), while pairing the remaining two i's each with one of the two j's (2 ways). (iii) We pair each i with a j (4! ways).

There is a total of $9 + 72 + 24 = 105$ pairings, as expected from an 8-point correlator.

(b) $\langle i^2 j^4 \rangle$: We have 2 pairing groups. (i) We pair the i's with each other (1 way) and the j's each other (3 ways). (ii) We pair one i with one j (4 ways), the remaining i with another j (3 ways) and the two remaining j's with each other (1 way).

There is a total of $3 + 12 = 15$ pairings, as expected from a 6-point correlator.

(c) $\langle i^3 j^3 \rangle$: We have 2 pairing groups. (i) We pair one i to another i (3 ways), one j to another j (3 ways) and the remaining i and j with each other (1 way). (ii) We pair one i with one j (3 ways), another i with another j (2 ways) and the remaining i and j with each other (1 way).

There is a total of $9 + 6 = 15$ pairings, as expected from a 6-point correlator.

We check that we have performed a total of $105 + 24 \cdot 15 + 32 \cdot 15 = 945$ pairings in $\langle 12 i^4 j^4 \rangle$, which is the number of possible pairings in a 10-point correlator.

Finally, from the result of Eq. B.11, we expand the factor $1/\mathcal{N}$ in powers of λ:

$$\begin{aligned} \frac{1}{\mathcal{N}} &= \frac{1}{1 + \lambda \langle V(\vec{\phi}) \rangle + \frac{1}{2}\lambda^2 \langle V(\vec{\phi})^2 \rangle + \dots} \\ &= 1 - \lambda \langle V(\vec{\phi}) \rangle + \lambda^2 \left[\langle V(\vec{\phi}) \rangle - \frac{\langle V(\vec{\phi})^2 \rangle}{2} \right] + \dots \end{aligned} \tag{B.16}$$

Instead of trying to rewrite this result now in terms of 2-point correlators, it will be more convenient to leave the above expression untouched and instead write Eq. B.14 and Eq. B.15 in terms of $\langle V(\vec{\phi}) \rangle = \langle i^4 \rangle = 3 \langle ii \rangle^2$ and $\langle V(\vec{\phi})^2 \rangle = \langle i^4 j^4 \rangle$:

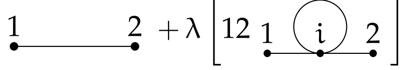
$$\langle 12 i^4 \rangle = \langle 12 \rangle \langle V(\vec{\phi}) \rangle + 12 \langle 1i \rangle \langle 2i \rangle \langle ii \rangle \,, \tag{B.17}$$

$$\begin{aligned} \langle 12 i^4 j^4 \rangle = \langle 12 \rangle \langle V(\vec{\phi})^2 \rangle &+ 24 \langle 1i \rangle \langle 2i \rangle \Big(\langle ii \rangle \langle V(\vec{\phi}) \rangle + 12 \langle ij \rangle^2 \langle jj \rangle \Big) \\ &+ 32 \langle 1i \rangle \langle 2j \rangle \Big(9 \langle ii \rangle \langle jj \rangle \langle ij \rangle + 6 \langle ij \rangle^3 \Big). \end{aligned} \tag{B.18}$$

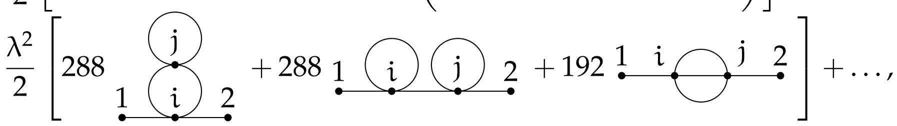
The terms that we have been able to rewrite in terms of V correspond precisely to disconnected diagrams (diagrams with at least one pair of vertices not connected with a line).

Denoting $\mathcal{C}$ as the connected part of $\langle 12\rangle_V$:

$$\begin{aligned}\mathcal{C} = \ & \langle 12\rangle + \lambda\Big[12\langle 1i\rangle\langle 2i\rangle\langle ii\rangle\Big]\end{aligned}$$



$$+\frac{\lambda^2}{2}\Big[288\langle 1i\rangle\langle 2i\rangle\langle ij\rangle^2\langle jj\rangle + \langle 1i\rangle\langle 2j\rangle\Big(288\langle ii\rangle\langle jj\rangle\langle ij\rangle + 192\langle ij\rangle^3\Big)\Big] + \dots \tag{B.19}$$



and using the results of Eq. B.13, Eq. B.16, Eq. B.17 and Eq. B.18, we obtain

$$\begin{aligned}\langle 12\rangle_V = \ & \Big(\langle 12\rangle + \lambda\langle 12i^4\rangle + \frac{\lambda^2}{2}\langle 12i^4j^4\rangle + \dots\Big)\\ & \qquad \times\Big(1 - \lambda\langle V(\vec{\phi})\rangle + \lambda^2\Big[\langle V(\vec{\phi})\rangle - \frac{\langle V(\vec{\phi})^2\rangle}{2}\Big] + \dots\Big)\\ = \ & \mathcal{C} + \lambda\langle 12\rangle\langle V(\vec{\phi})\rangle + \frac{\lambda^2}{2}\langle 12\rangle\langle V(\vec{\phi})^2\rangle + 12\lambda^2\langle 1i\rangle\langle 2i\rangle\langle ii\rangle\langle V(\vec{\phi})\rangle\\ & + \Big(-\lambda\langle V(\vec{\phi})\rangle + \lambda^2\Big[\langle V(\vec{\phi})\rangle - \frac{\langle V(\vec{\phi})^2\rangle}{2}\Big]\Big)\\ & \qquad \times\Big(\langle 12\rangle + \lambda\langle 12\rangle\langle V(\vec{\phi})\rangle + 12\lambda\langle 1i\rangle\langle 2i\rangle\langle ii\rangle\Big) + \dots\end{aligned} \tag{B.20}$$

At order λ^2, all terms containing expectation values of powers of V cancel each other and only the connected part $\mathcal{C}$ remains. This is an explicit verification at this order that the normalization factor $1/\mathcal{N}$ cancels all disconnected diagrams in the computation of $\langle 12\rangle_V$. The same property is satisfied at any order. As a general procedure when computing correlators, one can always forget the normalization factor and only consider the contribution from connected diagrams.

Given the potential $V(\vec{\phi}) = \phi_i^m$, the diagrammatic rules to compute the l-loop term in $\langle 12\cdots(n)\rangle_V$ are as follow:

(a) Draw l vertices, each with m legs, and label them with l indices i_ℓ. These correspond to the inner vertices of the diagram.

(b) Join the legs with each other, up to n free legs. To each free leg we attach an end vertex labelled $1, 2, \dots, n$.

(c) For a diagram constructed as above, we attach the factor $\lambda^l/l!$ to each inner vertex and the the 2-point correlator $\langle ij\rangle$ to each line connecting the vertex i with j. The product of all such factors is the total contribution of the diagram to $\langle 12\cdots(n)\rangle_V$.

B.2 Proof of Nevanlinna's theorem (Theorem 2)

We make the proof for $A > 0$. We can always translate the z variable so this is true.

(i) We define the set of functions

$$b_m(\zeta) = \frac{1}{2\pi i}\int_{\mathcal{C}_a} dz\, e^{z\zeta} z^m f(z)\,, \tag{B.21}$$

for $m \in \mathbb{N}$, $\mathcal{C}_a = \{z \in \mathbb{C} \mid \mathrm{Re}(z) = a\}$ with $a > A$. We notice that b_0 is just the inverse Laplace transform of f and b_m is the m-th derivative of b_0.

Using Eq. 5.10 we obtain the following result

$$\begin{aligned} b_m(\zeta) &= \frac{1}{2\pi i}\int_{\mathcal{C}_a} dz\, e^{z\zeta} z^m \left(\sum_{n=0}^{N-1} \frac{a_n}{z^{n+1}} + R_N(z)\right) \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}_a} dz\, e^{z\zeta} \left(\sum_{n=0}^{N-m-1} \frac{a_{n+m}}{z^{n+1}} + z^m R_N(z)\right). \end{aligned} \tag{B.22}$$

The Laplace transform of natural powers of z is exactly 0, so we have eliminated these terms in the last line.

Now choosing $N = m + 1$, for $\zeta \geqslant 0$ we obtain the bound

$$\begin{aligned} |b_m(\zeta)| &= \left|\frac{1}{2\pi i}\int_{\mathcal{C}_a} dz\, e^{z\zeta} \left(\frac{a_m}{z} + z^m R_{m+1}(z)\right)\right| \\ &= \left|a_m + \frac{1}{2\pi i}\int_{\mathcal{C}_a} dz\, e^{z\zeta} z^m R_{m+1}(z)\right| \\ &\leqslant |a_m| + \frac{e^{a\zeta}}{2\pi}\int_{\mathcal{C}_a} |dz|\,|z|^m \big|R_{m+1}(z)\big| \\ &\leqslant |a_m| + \frac{e^{a\zeta}}{2\pi} L(m+2)!\sigma^{m+2}\int_{\mathcal{C}_a} \frac{|dz|}{|z|^2} = |a_m| + \frac{e^{a\zeta}}{2\pi|a|} L(m+2)!\sigma^{m+2}. \end{aligned} \tag{B.23}$$

We have used $\int_{\mathcal{C}_a} |dz|/|z|^2 = \int_{\mathbb{R}} dy/(y^2 + a^2) = 1/|a|$ in the last line.

Because the bound is valid for any $a > A$, it also is valid in the limit $a \to A$. Lastly, noting that $R_m(z) - R_{m+1}(z) = a_m/z^{m+1}$ we can bound $|a_m| \leqslant L_1(m+2)!\sigma^{m+2}$. Thus, we have proved that

$$|b_m(\zeta)| \leqslant K_1(m+2)!\sigma^{m+2} e^{A\zeta}\,, \quad \zeta \geqslant 0\,. \tag{B.24}$$

This bound will be useful later in the proof.

Next, fixing $m = 0$ in Eq. B.22, let us check that

$$b_0(\zeta) = \sum_{n \geqslant 0} \frac{a_n \zeta^n}{n!} = B(\zeta) \tag{B.25}$$

and that this series converges in $|\zeta| < 1/\sigma$. Indeed, integrating with the residue theorem term by term the series in Eq. B.22 with $\zeta \geqslant 0$, we obtain

$$b_0(\zeta) = \sum_{n=0}^{N-1} \frac{a_n \zeta^n}{n!} + \frac{1}{2\pi i} \int_{\mathcal{C}_a} dz\, e^{z\zeta} R_N(z)\,. \tag{B.26}$$

It is now sufficient to prove that the remainder term in the right goes to 0 as $N \to \infty$ for $0 \leqslant \zeta < 1/\sigma$. Because $\sum a_n \zeta^n/n!$ is a power series, this result is enough to prove that the series converges in all points in the disc $|\zeta| < 1/\sigma$ and defines an analytic function there. We have the following bound for the remainder:

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\mathcal{C}_a} dz\, e^{z\zeta} R_N(z) \right| &\leqslant \frac{e^{a\zeta}}{2\pi} \int_{\mathcal{C}_a} |dz| \left| R_N(z) \right| \leqslant \frac{e^{a\zeta}}{2\pi} L(N+1)! \sigma^{N+1} \int_{\mathcal{C}_a} \frac{|dz|}{|z|^{N+1}} \\ &= \frac{e^{a\zeta}}{2\pi} L(N+1)! \sigma^{N+1} \frac{\sqrt{\pi}\, \Gamma(N/2)}{|a|^N \Gamma\left(\frac{N+1}{2}\right)}\,. \end{aligned} \tag{B.27}$$

The integral in Eq. B.22 does not depend on $a > A$, so we are free to choose a convenient value. If N is large enough, $1/\sigma < N/A$ and we may choose $a = N/\zeta > N\sigma > A$. Using $a = N/\zeta$ and the Stirling formula to deal with the Γ functions at large N in Eq. B.27, we obtain

$$\left| \frac{1}{2\pi i} \int_{\mathcal{C}_a} e^{z\zeta} R_N(z)\, dz \right| \leqslant K_2 (\zeta\sigma)^N N\,, \tag{B.28}$$

which is a bound that clearly goes to 0 for $0 \leqslant \zeta < 1/\sigma$.

Next, let us choose $\zeta_0 \geqslant 0$ and consider the series

$$B_{\zeta_0}(\zeta) = \sum_{m \geqslant 0} \frac{b_m(\zeta_0)}{m!} (\zeta - \zeta_0)^m. \tag{B.29}$$

Using the bound in Eq. B.24, we obtain

$$\begin{aligned} |B_{\zeta_0}(\zeta)| &\leqslant \sum_{m \geqslant 0} \frac{|b_m(\zeta_0)|}{m!} |\zeta - \zeta_0|^m \leqslant K_1 e^{A\zeta_0} \sigma^2 \sum_{m \geqslant 0} (m+1)(m+2) \big(\sigma|\zeta - \zeta_0|\big)^m \\ &= \frac{2K_1 \sigma^2 e^{A\zeta_0}}{\big(1 - \sigma|\zeta - \zeta_0|\big)^3}\,. \end{aligned} \tag{B.30}$$

So, the series in Eq. B.29 converges (absolutely) on the disc $|\zeta - \zeta_0| < 1/\sigma$ and defines an analytic function there. Noting that $b_m = b_0^{(m)} = B^{(m)}$, we come to the conclusion that B_{ζ_0} is just the Taylor expansion of B around $\zeta = \zeta_0$. Concatenating the discs of convergence for each ζ_0 we conclude that B is analytic in the strip S_σ.

In addition, for any $\sigma' > \sigma$,

$$|B_{\zeta_0}(\zeta)| \leqslant \max_{\zeta\in D(\zeta_0,1/\sigma')} \frac{2K_1\sigma^2 e^{A\zeta_0}}{\left(1-\sigma|\zeta-\zeta_0|\right)^3} = \frac{2K_1\sigma^2 e^{A\zeta_0}}{\left(1-\sigma/\sigma'\right)^3}, \quad \zeta \in D(\zeta_0, 1/\sigma'). \tag{B.31}$$

Thus, for each $\zeta = \zeta_0 + iy$, $|y| < 1/\sigma'$, we have proved that

$$|B(\zeta)| \leqslant K e^{A\,\mathrm{Re}(\zeta)} \leqslant K e^{A|\zeta|}, \quad \zeta \in S_{\sigma'}, \tag{B.32}$$

which is the implication in Eq. 5.13 of the theorem.

It remains to prove Eq. 5.14. That is,

$$\int_0^\infty d\zeta\, e^{-z\zeta} \int_{\mathcal{C}_a} dw\, e^{w\zeta} f(w) = 2\pi i\, f(z). \tag{B.33}$$

First we want to interchange the order integration. It is easy to prove this is possible, from Fubini's theorem, by checking that the double integral converges absolutely. Indeed,

$$\int_0^\infty d\zeta \int_{\mathcal{C}_a} |dw| \left|e^{-\zeta(z-w)} f(w)\right| = \left[\int_0^\infty d\zeta\, e^{-\zeta(\mathrm{Re}(z)-a)}\right] \left[\int_{\mathcal{C}_a} |dz| \left|f(z)\right|\right]. \tag{B.34}$$

The first integral in brackets converges because $\mathrm{Re}(z) > a$. The proof that the second integral converges is exactly the same as in Eq. B.23 for $m = 0$. Thus, the double integral converges absolutely.

Interchanging the order of integration, we obtain

$$\begin{aligned}\int_0^\infty d\zeta\, e^{-z\zeta} \int_{\mathcal{C}_a} dw\, e^{w\zeta} f(w) &= \int_{\mathcal{C}_a} dw\, f(w) \int_0^\infty d\zeta\, e^{-\zeta(z-w)} = \int_{\mathcal{C}_a} dw\, \frac{f(w)}{z-w} \\ &= 2\pi i\, f(z).\end{aligned} \tag{B.35}$$

In the last equality we deformed $\mathcal{C}_a$ around $w = z$ and used the residue theorem. This result finishes the proof of the first part of the theorem.

(ii) On one hand, if we assume $B(\zeta)$ is analytic in the strip $S_{\sigma'}$ and there satisfies the bound in Eq. 5.13, then it is easy to prove that the integral in Eq. 5.14 is absolutely convergent for $\mathrm{Re}(z) > A$, so f is an analytic function there.

On the other hand, for any $\zeta_0 \geqslant 0$:

$$\begin{aligned}\left|B^{(m)}(\zeta_0)\right| &= \left|\frac{m!}{2\pi i}\oint_\gamma \frac{B(\zeta)}{(\zeta-\zeta_0)^{m+1}}\, d\zeta\right| \leqslant \frac{m!}{2\pi}\oint_\gamma \frac{|B(\zeta)|}{|\zeta-\zeta_0|^{m+1}}\, |d\zeta| \\ &\leqslant \frac{m!K}{2\pi}\oint_\gamma \frac{e^{A|\zeta|}}{|\zeta-\zeta_0|^{m+1}}\, |d\zeta| \leqslant \frac{m!K}{2\pi} e^{A(\zeta_0+1/\sigma')} \int_0^{2\pi} \frac{d\theta}{(1/\sigma')^m} \\ &= K_1 m! (\sigma')^{m+1} e^{A\zeta_0},\end{aligned} \tag{B.36}$$

where γ is a circle around $\zeta = \zeta_0$ of radius $1/\sigma'$. We have used Cauchy's integral formula in the first equality. From this result we will be able to prove that the function f satisfies Eq. 5.10 and Eq. 5.11 in $\mathrm{Re}(z) > a$ for any $a > A$.

Integrating by parts one time, we obtain

$$f(z) = \left[-B(\zeta)\frac{e^{-z\zeta}}{z}\right]_0^\infty + \frac{1}{z}\int_0^\infty d\zeta\, e^{-z\zeta}B(\zeta) = \frac{B(0)}{z} + \frac{1}{z}\int_0^\infty d\zeta\, e^{-z\zeta}B(\zeta)\,. \tag{B.37}$$

To evaluate the first term, we have used Eq. B.36 to verify that that $|B(\zeta)e^{-z\zeta}| \leqslant K_1\sigma' e^{-\zeta(\mathrm{Re}(z)-A)} \to 0$ when $\zeta \to \infty$ if $\mathrm{Re}(z) > A$. Repeating integration by parts N times, we get

$$f(z) = \sum_{n=0}^{N-1}\frac{B^{(n)}(0)}{z^{n+1}} + R_N(z)\,, \tag{B.38}$$

where

$$R_N(z) = \frac{B^{(N)}(0)}{z^{N+1}} + \frac{1}{z^{N+1}}\int_0^\infty d\zeta\, e^{-z\zeta}B^{(N+1)}(\zeta)\,, \tag{B.39}$$

as obtained after integrating by parts one last time. Now we just have to prove that R_N satisfies the bound in Eq. 5.11 for $\mathrm{Re}(z) > a$. We have

$$\begin{aligned}
|R_N(z)| &\leqslant \frac{\left|B^{(N)}(0)\right|}{|z|^{N+1}} + \frac{1}{|z|^{N+1}}\int_0^\infty d\zeta\, e^{-\zeta\,\mathrm{Re}(z)}\left|B^{(N+1)}(\zeta)\right| \\
&\leqslant K_1(N+1)!\big(\sigma'/|z|\big)^{N+1}\left(\int_0^\infty d\zeta\, e^{-\zeta(\mathrm{Re}(z)-A)}\right) \\
&\leqslant K_1(N+1)!\big(\sigma'/|z|\big)^{N+1}\left(1+\frac{1}{\mathrm{Re}(z)-A}\right) \\
&\leqslant K_1(N+1)!\big(\sigma'/|z|\big)^{N+1}\left(1+\frac{1}{a-A}\right) = L(N+1)!\big(\sigma'/|z|\big)^{N+1}
\end{aligned} \tag{B.40}$$

This verification finishes the last implication of the theorem.

B.3 An illustration of Observation 1

We define

$$f(z) = -e^{-S_1 z}E_1(-S_1 z) - e^{-S_2 z}E_1(-S_2 z)\,. \tag{B.41}$$

Each component of the function has a branch cut conventionally placed along the direction $-\arg(S_1)$ and $-\arg(S_2)$, respectively. Similarly to what we did in Eq. 5.21, we may place the cut along $\mathbb{R}^-$ by adding appropriate exponential terms. By assuming $\mathrm{Re}(S_1), \mathrm{Re}(S_2) < 0$, we ensure that these exponential terms never become enhanced for any $z \in \mathbb{C}$.

We have constructed this function so that its asymptotic expansion is

$$f(z) \sim \sum_{n \geqslant 0} \frac{a_n}{z^{n+1}}, \quad |z| \to \infty, \tag{B.42}$$

with

$$a_n = n! \left(\frac{1}{S_1^{n+1}} + \frac{1}{S_2^{n+1}} \right). \tag{B.43}$$

This function satisfies a dispersion relation, because it goes like $1/z$ for large $|z|$. Its discontinuity, arising from the logarithmic term in Eq. 5.20, is given by

$$\mathrm{Disc}\, f(z) = 2\pi i \left(e^{-S_1 z} + e^{-S_2 z} \right). \tag{B.44}$$

On one hand, the leading behaviour for large n in Eq. B.43 is given by

$$a_n \sim \begin{cases} \dfrac{n!}{S_1^{n+1}} & \text{if } |S_1| < |S_2| \\ \dfrac{n!}{S_2^{n+1}} & \text{if } |S_2| < |S_1|. \end{cases} \tag{B.45}$$

On the other hand, the leading behaviour for large $-z$ in Eq. B.44 is given by

$$\mathrm{Disc}\, f(z) \sim 2\pi i \begin{cases} e^{-S_1 z} & \text{if } \mathrm{Re}(S_1) > \mathrm{Re}(S_2) \\ e^{-S_2 z} & \text{if } \mathrm{Re}(S_2) > \mathrm{Re}(S_1). \end{cases} \tag{B.46}$$

Clearly, there is no correspondence between the two leading behaviours if $|S_1| < |S_2|$ and $\mathrm{Re}(S_1) < \mathrm{Re}(S_2)$, or the other way around.

B.4 Proof of Proposition 5

First of all, we will check that the function

$$B(\zeta) = \frac{1}{2\pi i} \int_{\mathcal{C}_a} dz\, e^{z\zeta} f(z), \tag{B.47}$$

with $a > A$, initially defined for $\zeta > 0$, can be extended to an analytic function in $\mathrm{Re}(\zeta) > S$.

Using $|f(z)| \leqslant |a_0|/|z|$, we can deform $\mathcal{C}_a$ to a contour surrounding the cut of f, similar to Figure 5.2, but with the origin translated to the point A. We obtain

$$B(\zeta) = \frac{1}{2\pi i} \int_{A-\delta}^{-\infty} dz\, e^{z\zeta}\, \mathrm{Disc}\, f(z) + \frac{1}{2\pi i} \int_{C_\delta(A)} dz\, e^{z\zeta} f(z), \tag{B.48}$$

where $C_\delta(A)$ is a circle of radius δ around A. We denote the second term by $E(\zeta)$. It is entire in ζ, because $C_\delta(A)$ is bounded. Choosing δ large enough and using Eq. 5.35, it is easy to check that the first term is absolutely convergent in $\mathrm{Re}(\zeta) > S$, and thus it defines an analytic function there.

In particular, given that $S < 0$, B is analytic at $\zeta = 0$ and we may compute any number of derivatives at this point. We have

$$\begin{aligned} B(\zeta) &= \frac{1}{2\pi i}\int_{-\delta}^{-\infty} dz \sum_{n\geqslant 0} \frac{(z\zeta)^n}{n!}\,\mathrm{Disc}\, f(z) + E(\zeta) \\ &= \sum_{n\geqslant 0} \frac{\zeta^n}{n!}\left[\frac{1}{2\pi i}\int_{-\delta}^{-\infty} dz\, z^n\, \mathrm{Disc}\, f(z)\right] + E(\zeta)\,. \end{aligned} \tag{B.49}$$

Here it is correct to commute the sum with the integral as a consequence of the dominated convergence theorem. From Eq. B.49, we may read the n-th derivative of B at 0:

$$\begin{aligned} B^{(n)}(0) &= \frac{1}{2\pi i}\int_{-\delta}^{-\infty} dz\, z^n\, \mathrm{Disc}\, f(z) + E^{(n)}(0) \\ &= \frac{(-1)^{n+1}}{\pi}\, b_0 \int_{\delta}^{\infty} dz\, e^{Sz} z^{n+\alpha-1}\big[1 + R(z)\big] + E^{(n)}(0) \\ &= \frac{(-1)^{n+1}}{\pi}\, b_0 \left[\frac{\Gamma(n+\alpha)}{(-S)^{n+\alpha}} - \frac{\gamma(n+\alpha, -\delta S)}{(-S)^{n+\alpha}} \right. \\ &\qquad\qquad \left. + \int_{\delta}^{\infty} dz\, e^{Sz}\, z^{n+\alpha-1} R(z)\right] + E^{(n)}(0)\,, \end{aligned} \tag{B.50}$$

where $\gamma(s,x)$ is the lower incomplete gamma function and $|R(z)| \leqslant L/|z|$.

We want to check that $B^{(n)}(0)$ satisfies Eq. 5.38. That is, defining the leading contribution

$$a'_n = \frac{(-1)^{n+1}}{\pi}\frac{\Gamma(n+\alpha)}{(-S)^{n+\alpha}}\, b_0\,, \tag{B.51}$$

we want to prove that

$$\left|\frac{B^{(n)}(0)}{a'_n} - 1\right| \leqslant \frac{M}{n} \tag{B.52}$$

for large enough n. The proof follows from Eq. B.50. We have

$$\begin{aligned} \left|\frac{B^{(n)}(0)}{a'_n} - 1\right| \leqslant \frac{|\gamma(n+\alpha, -\delta S)|}{|\Gamma(n+\alpha)|} &+ \frac{(-S)^{n+\mathrm{Re}(\alpha)}\left|\int_\delta^\infty dz\, e^{Sz}\, z^{n+\alpha-1} R(z)\right|}{|\Gamma(n+\alpha)|} \\ &+ \frac{(-S)^{n+\mathrm{Re}(\alpha)}\left|E^{(n)}(0)\right|}{|\Gamma(n+\alpha)|}\,. \end{aligned} \tag{B.53}$$

We consider the lower bound $|\Gamma(n+\alpha)| \geqslant M_0\Gamma(n+\mathrm{Re}(\alpha))$, valid for small enough M_0 and large enough n.

- Using the asymptotic behaviour $\gamma(s,x) \sim x^s e^{-x}/s$, valid for large s, the first term is bounded by

$$\frac{|\gamma(n+\alpha,-\delta S)|}{|\Gamma(n+\alpha)|} \leqslant \frac{M_1(-\delta S)^n}{\Gamma(n+\mathrm{Re}(\alpha)+1)} = \mathcal{O}\left(\frac{1}{n}\right). \tag{B.54}$$

- Using $|R(z)| \leqslant L/|z|$, the second term is bounded by

$$\frac{(-S)^{n+\mathrm{Re}(\alpha)}\left|\int_\delta^\infty dz\, e^{Sz}\, z^{n+\alpha-1}R(z)\right|}{|\Gamma(n+\alpha)|} \leqslant \frac{M_2(-S)^{n+\mathrm{Re}(\alpha)}\int_0^\infty dz\, e^{Sz}\, z^{n+\mathrm{Re}(\alpha)-2}}{\Gamma(n+\mathrm{Re}(\alpha))} = \mathcal{O}\left(\frac{1}{n}\right). \tag{B.55}$$

- Cauchy inequality yields $\left|E^{(n)}(0)\right| \leqslant m\, n!/r^n$, where m is the maximum of $E(\zeta)$ for ζ along a circle of centre 0 and radius r contained inside the region of analitycity of E. Because E is entire, we are free to choose any r, in particular we may choose $r = -S+1 > 0$. Then the third term is bounded by

$$\frac{(-S)^{n+\mathrm{Re}(\alpha)}\,|E^{(n)}(0)|}{|\Gamma(n+\alpha)|} \leqslant \frac{M_3\Gamma(n+1)}{\Gamma(n+\mathrm{Re}(\alpha))}\left(\frac{-S}{-S+1}\right)^n = \mathcal{O}\left(\frac{1}{n}\right). \tag{B.56}$$

From the above results, Eq. B.52 is realised. Notice that the first and third bounds can be improved to an arbitrary power of $1/n$ (the first is like $1/n!$ and the second, like $1/R^n$, $R > 1$). Actually only the second bound gives the error in Eq. B.52.

It is also in the second bound where the hypothesis $S < 0$ is needed. Otherwise, the remainder term could yield contributions of higher order than a'_n. This is related to Observation 1.

To complete the proof, we still have to check that $|B(\zeta)| \leqslant Ke^{A\,\mathrm{Re}(\zeta)}$ in $\mathrm{Re}(\zeta) > S$ and that, for any $|\theta| < \pi/2$,

$$f(z) = \int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta}\, B(\zeta)\,, \quad \mathrm{Re}\left(ze^{i\theta}\right) > a\cos(\theta)\,. \tag{B.57}$$

Then part (ii) of Nevanlinna's theorem gives the asymptotic expansion $f(z) \sim \sum B^{(n)}(0)/z^{n+1}$, uniformly valid in $\mathrm{Re}\left(ze^{i\theta}\right) > a\cos(\theta)$.

First we will prove the exponential bound on B. The second term in Eq. B.48 is bounded by $|E(\zeta)| \leqslant K_1' e^{A\,\mathrm{Re}(\zeta)} e^{\delta|\zeta|}$ for any $\delta > 0$. Choosing $\delta = 1/|\zeta|$, we find $|E(\zeta)| \leqslant K_1 e^{A\,\mathrm{Re}(\zeta)}$. Furthermore, using Eq. 5.35, we bound the first term in Eq. B.48 with $K_2 e^{A\,\mathrm{Re}(\zeta)}$ (also with the same choice $\delta = 1/|\zeta|$). Thus, B is bounded by

$$|B(\zeta)| \leqslant Ke^{A\,\mathrm{Re}(\zeta)}\,. \tag{B.58}$$

We choose $\theta \in (-\pi/2, +\pi/2)$ and deform the path $\mathcal{C}_a$ into a path $\mathcal{C}_a(\theta)$ parametrized by $w = a + xe^{i(\pi/2-\theta)}$, with $x \in \mathbb{R}$. Assuming that the order of integration can be interchanged, we have, for $\mathrm{Re}\left(ze^{i\theta}\right) > a\cos(\theta)$,

$$\begin{aligned}\int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta}\, B(\zeta) &= \int_0^{\infty e^{i\theta}} d\zeta\, e^{-z\zeta} \left(\frac{1}{2\pi i}\int_{\mathcal{C}_a(\theta)} dw\, e^{w\zeta}\, f(w)\right) \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}_a(\theta)} dw\, f(w) \int_0^{\infty e^{i\theta}} d\zeta\, e^{-(z-w)\zeta} \\ &= \frac{1}{2\pi i}\int_{\mathcal{C}_a(\theta)} dw\, \frac{f(w)}{z-w}\,.\end{aligned} \tag{B.59}$$

In the last line, we deform the path $\mathcal{C}_a(\theta)$ into a circle around $w = z$. This yields $f(z)$ when using the residue theorem. Here we needed that $|f(z)| \leqslant |a_0|/|z|$ for all the deformations of the integration parth.

To prove that the order of integration can be interchanged, it is sufficient to check that the double integral converges absolutely and apply Fubini's theorem. Indeed, using the bound in Eq. B.58, we have

$$\int_0^{\infty e^{i\theta}} |d\zeta|\, e^{-\mathrm{Re}(z\zeta)}\, |B(\zeta)| \leqslant K\int_0^{\infty} d|\zeta|\, e^{-\left(\mathrm{Re}(ze^{i\theta})-A\cos(\theta)\right)|\zeta|}, \tag{B.60}$$

and the last integral converges in $\mathrm{Re}\left(ze^{i\theta}\right) > A\cos(\theta)$.

B.5 Proof of Proposition 5 with $S \in \mathbb{C}\setminus\mathbb{R}^+$

Here we will prove the following generalisation of Proposition 5:

Proposition 7. *Let f be an analytic function in $\mathbb{C}\setminus(\mathbb{R}^- + A)$ and satisfy $|f(z)| \leqslant |a_0|/|z|$ in $\mathrm{Re}(z) > A$ and $|f(z)| \leqslant Ke^{s|z|}$ in $\mathrm{Re}(z) < A$[1] (minus a neighbourhood of A in both cases). Further assume that $\mathrm{Disc}\, f$ satisfies Eq. 5.35 with $S \in \mathbb{C}\setminus\mathbb{R}^+$ and the $\mathcal{O}(1/z)$ terms satisfy Eq. B.55. Then*

$$f(z) = \int_0^{\infty} d\zeta\, e^{-z\zeta}\, B(\zeta) \sim \sum_{n\geqslant 0} \frac{a_n}{z^{n+1}}\,, \quad \mathrm{Re}(z) > a\,, \tag{B.61}$$

(the asymptotic expansion being uniformly valid) where B is defined in Eq. 5.36 and the coefficients a_n satisfy Eq. 5.38.

The proof would go as follows. We consider the Borel transform B in Eq. 5.36. Using

[1] When $\mathrm{Re}(S) > 0$, we expect that f becomes exponentially enhanced somewhere in $\mathrm{Re}(z) < A$. Thus, imposing the bound $|f(z)| \leqslant |a_0|/|z|$ on both half-planes, as we did in Proposition 5, would be too restrictive.

the bound $|f(z)| \leqslant K e^{s|z|}$, valid in $\mathrm{Re}(z) < A$, we can write B as in Eq. B.48 and check that the function is analytic in $\mathrm{Re}(\zeta) > \mathrm{Re}(S)$.

The bounds in Eq. B.54 and Eq. B.56 are still valid, but Eq. B.55 might not due to the fact that S is now complex. This is the reason we are forced to impose this bound in the assumptions of Proposition 7. We find

$$B^{(n)}(0) = \frac{(-1)^{n+1}}{\pi} \frac{\Gamma(n+\alpha)}{(-S)^{n+\alpha}} b_0 \left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right], \tag{B.62}$$

which means that B is also analytic in a disc of radius $|S|$ around the origin.

Finally, to prove Eq. B.61, we repeat the same steps in Appendix B.4 (but only for $\theta = 0$). The hypothesis that $|f(z)| \leqslant |a_0|/|z|$ in $\mathrm{Re}(z) > A$ is used in the last line of Eq. B.59 to deform the path $\mathcal{C}_a(0)$ into a circle around $w = z$.

B.6 Proof of Proposition 6

We define f_θ as the Laplace transform of B along the path in Figure 5.3a, with $\theta \in [-\pi/2 - \epsilon, \pi/2 + \epsilon]$ (going around S if necessary). Using the exponential bound on B, it is easy to check that f_θ is an analytic function in $\mathrm{Re}\left(z e^{i\theta}\right) > A$. Furthermore, the functions f_θ coincide in the intersection of the half-planes of analyticity. Therefore, concatenating the half-planes, we can analytically continue $f = f_0$ around the disc $D(0, A)$ (of radius A and centre 0).

We consider the two directions $\theta_- = -\pi/2 - \epsilon$ and $\theta_+ = +\pi/2 + \epsilon$. These directions define a pair of Laplace transforms whose difference can be written as

$$f_{\theta_-}(z) - f_{\theta_+}(z) = \int_{\mathcal{C}} d\zeta\, e^{-z\zeta}\, B(\zeta), \tag{B.63}$$

where $\mathcal{C} = \mathcal{C}_+ + \mathcal{C}_- + C_\delta$ is the contour in Figure 5.3b. This follows from a convenient deformation (if necessary) of the original paths that define f_θ and the fact that paths in opposite directions cancel each other.

We check that the integrals along $\mathcal{C}_\pm$ are $\mathcal{O}\left(e^{S'z}\right)$ for $z \to -\infty$, where $S' = S - \delta$. Indeed, given the parametrisation $\zeta = S' + x e^{i\theta_\pm}$, with $x \in \mathbb{R}^+$,

$$\begin{aligned} \left|\int_{\mathcal{C}_\pm} d\zeta\, e^{-z\zeta}\, B(\zeta)\right| &\leqslant e^{\mathrm{Re}(S')z} \int_0^\infty dx\, e^{-zx\cos(\theta_\pm)} \left|B\left(S' + x e^{i\theta_\pm}\right)\right| \\ &\leqslant K e^{\mathrm{Re}(S')z} \int_0^\infty dx\, e^{-x(z\cos(\theta_\pm) - A)} = \mathcal{O}\left(e^{S'z}\right). \end{aligned} \tag{B.64}$$

Here we have used the bound $|B(\zeta)| \leqslant K e^{A|\zeta|}$ and verified that $z\cos(\theta_\pm) - A > 0$

for large enough $-z$. The fact that the integral is convergent also proves that $f_{\theta_-}(z) - f_{\theta_+}(z)$ is well defined for large enough $-z$.

$f_{\theta_-}(z) - f_{\theta_+}(z)$ is the difference between the two possible analytical continuations around $D(0, A)$. As we saw on Example 4, this corresponds to

$$\operatorname{Disc} f(z) = f_{\theta_-}(z) - f_{\theta_+}(z)\,, \quad z < 0 \text{ and large enough.} \tag{B.65}$$

This is an exact result at this stage, rather than an approximation.

From Eq. B.64, we see that the leading behaviour in Eq. 5.41 can only come from the integral around C_δ. Choosing δ small enough, we might use Eq. 5.39 to obtain

$$\begin{aligned}\int_{C_\delta} d\zeta\, e^{-z\zeta}\, B(\zeta) &= \int_{C_\delta} d\zeta\, e^{-z\zeta} \left(\frac{-b_0}{\pi} \frac{\Gamma(\alpha)}{(\zeta - S)^\alpha} \left[1 + R(\zeta) \right] \right) \\ &= 2i\, b_0\, e^{-Sz} (-z)^{\alpha - 1} \left[1 + \frac{\sin(\pi\alpha)\Gamma(\alpha)\Gamma(1-\alpha, -\delta z)}{\pi} \right. \\ &\qquad \left. - \frac{1}{(-z)^{\alpha-1}} \frac{\Gamma(\alpha)}{2\pi i} \int_{C_{\delta,0}} d\zeta\, e^{-z\zeta} \frac{R(\zeta + S)}{\zeta^\alpha} \right],\end{aligned} \tag{B.66}$$

where $|R(\zeta + S)| \leqslant K|\zeta|$, $\Gamma(s, x)$ is the upper incomplete gamma function and $C_{\delta,0}$ is a circle of radius δ around 0.

- Using the asymptotic behaviour $\Gamma(s, x) \sim x^{s-1} e^{-x}$ for large x, the first term is bounded by

$$\left| \frac{\sin(\pi\alpha)\Gamma(\alpha)\Gamma(1-\alpha, -\delta z)}{\pi} \right| \leqslant K_1 \frac{e^{\delta z}}{(-\delta z)^\alpha} = \mathcal{O}\left(\frac{1}{z} \right). \tag{B.67}$$

- Using $|R(\zeta + S)| \leqslant K|\zeta|$, the second term is bounded by

$$\left| \frac{1}{(-z)^{\alpha-1}} \frac{\Gamma(\alpha)}{2\pi i} \int_{C_{\delta,0}} d\zeta\, e^{-z\zeta} \frac{R(\zeta + S)}{\zeta^\alpha} \right| = \mathcal{O}\left(\frac{1}{z} \right). \tag{B.68}$$

From the above results, Eq. 5.41 is realised. Notice that the first bound can be improved to an arbitrary power of $1/z$. Actually only the second bound gives the error in Eq. 5.41.

B.7 A path integral with no resurgent connection

In this appendix, we will consider a physically motivated function, the self-energy of the $O(N)$ non-linear sigma model [89]. We will show that this function does not feature a resurgent connection and that this happens because the function is not recovered from the median resummation of its asymptotic expansion. To make this verification, we will start from the asymptotic expansion of the self-energy and we will compute the discontinuity of its median resummation. Then we will check that there are additional exponential corrections to the asymptotic expansion (not arising from median resummation) that also contribute to the discontinuity of the self-energy. Therefore, we will conclude that the discontinuity of the self-energy cannot coincide with the discontinuity of the median resummation.

The self-energy Σ of the $O(N)$ non-linear sigma model, to next-to-leading order in $1/N$, is given by

$$\Sigma(p^2) = \frac{1}{\pi N}\int_{\mathbb{R}^2} d^2k \, \frac{\sqrt{k^2(k^2+4m^2)}}{\log\left[\frac{\sqrt{k^2+4m^2}+\sqrt{k^2}}{\sqrt{k^2+4m^2}-\sqrt{k^2}}\right]} \frac{1}{(p+k)^2+m^2}\,, \tag{B.69}$$

where $m^2 = \mu^2 e^{-1/g(\mu)}$ is the dinamically generated mass of the σ particle and $g(\mu)$ is the coupling of the model at the scale μ. (We follow the same notation as in [90]).

For convenience, we define the variable $z = 1/g(p)$ and the dimensionless function

$$E(z) = \frac{N\Sigma_R\left(m^2 e^z\right)}{m^2 e^z}\,, \tag{B.70}$$

where Σ_R is the renormalised self-energy Σ, obtained after two zero-momentum subtractions. An asymptotic expansion for $E(z)$ is given by (see [90, Eq. 17])

$$E(z) \sim -\log(z) + c + \widetilde{E}(z) = -\log z + c + \sum_{n\geqslant 0} \frac{\sigma_n}{z^{n+1}}\,, \quad z \to +\infty\,, \tag{B.71}$$

where $c = 1.887537\ldots$ and

$$\sigma_n = \begin{cases} -2 & \text{if } n = 0 \\ n!\left\{[1+(-1)^n]\zeta(n+1) - 2\right\} & \text{if } n \geqslant 1\,. \end{cases} \tag{B.72}$$

(ζ is the Riemann ζ-function). We can see the explicit factorial divergence in the coefficients σ_n of the asymptotic expansion. The symbol $\widetilde{E}(z)$ contains only the power-like part of the asymptotic expansion of $E(z)$.

The Borel transform of $\widetilde{E}$ in Eq. B.71 is given by

$$\widehat{E}(t) = \sum_{n\geqslant 0} \frac{\sigma_n t^n}{n!} = \frac{1}{t-1} - \psi(1+t) - \psi(2-t) - 2\gamma\,, \tag{B.73}$$

where ψ is the digamma function and γ is the Euler constant. $\widehat{E}(t)$ has simple poles at $t = k \in \mathbb{Z}\setminus\{0\}$ and is analytic elsewhere. The residue of the poles along the positive real axis are given by

$$r_k = \operatorname{Res}\left(\widehat{E}(t),\, t = k\right) = \begin{cases} +1 & \text{if } k = 1 \\ -1 & \text{if } k \geqslant 2. \end{cases} \tag{B.74}$$

Because we expect that $E(z)$ is real for $z > 0$ ($g > 0$), we consider the median resummed series

$$E_0^{mr}(z) = -\log(z) + c + \int_0^{\infty e^{i\theta}} dt\, e^{-zt}\, \widehat{E}(t) \pm i\pi \sum_{k\geqslant 1} r_k e^{-kz}\,, \tag{B.75}$$

where the minus sign is chosen when $\theta \in (0, +\pi)$ and the plus sign, when $\theta \in (-\pi, 0)$.

Later we will argue that E_0^{mr} does not coincide with E, but still we want to understand the properties of E_0^{mr}, because the singularties in the Borel plane of median resummed series determine the discontinuity of the function (Proposition 6).

Our first observation is that, from $\psi(t) \sim \log(t)$, valid for large $|t|$, we resolve that $\left|\widehat{E}(t)\right| \sim 2\log|t|$. This implies that the Laplace transform in Eq. B.75 defines an analytic function in the half-planes $\operatorname{Re}\left(ze^{i\theta}\right) > 0$ (as a consequence of part (ii) of Nevanlinna's theorem).

By choosing directions $\theta \in (-\pi/2, +\pi/2)$, $\theta \neq 0$, and concatenating the half-planes of analyticity, $E_0^{mr}(z)$ becomes an analytic function in $\mathbb{C} \setminus \mathbb{R}^-$. When changing the direction from below to above the positive real axis of the Borel plane, the two resulting functions coincide in the intersection of the half-planes, thanks to the exponential terms arising from median resummation. Therefore, they provide an analytic continuation of one another (even if the singularities along the positive real axis would normally prevent that).

From the discussion of Section 5.5, we concluded that we might use the result of Proposition 6 for median resummed series. That is, we may obtain the asymptotic behaviour of $\operatorname{Disc} E_0^{mr}(z)$ for large $-z$ from the singularities in the Borel transform.

However, here we face a problem that we did not realise in any of our previous examples. The asymptotic behaviour of the discontinuity is fixed by the singularity with the largest real part in the Borel plane, but in this example there is no such singularity. We have an infinite amount of singularities along the positive axis, each with a larger real part than the previous.

One way to proceed is to compute the contribution to the discontinuity from all singularities along the positive real axis, sum the series in $\operatorname{Re}(z) > 0$, analytically continue the result to $\operatorname{Re}(z) < 0$ and only then extract the leading behaviour for large $-z$.

The contribution to the discontinuity from all positive singularities is given by

$$2\pi i \sum_{k\geqslant 1} r_k e^{-kz} = 2\pi i \frac{e^{-z}\left(1-2e^{-z}\right)}{1-e^{-z}} \,. \tag{B.76}$$

While originally the series on the left only converges in $\mathrm{Re}(z) > 0$, the closed form on the right provides an analytic continuation to $\mathrm{Re}(z) < 0$. The leading behaviour of Eq. B.76 in $\mathrm{Re}(z) < 0$ yields

$$\mathrm{Disc}\, E_0^{\mathrm{mr}}(z) \sim (2\pi i) 2 e^{-z} \,, \quad z \to -\infty \,. \tag{B.77}$$

Effectively, it is as if the singularity with the largest real part were a simple pole at $t = 1$ with residue 2. It is easy to check that the discontinuity from the explicit logarithm in Eq. B.75 and the contributions from negative singularities yield sub-leading corrections to Eq. B.77.

While Eq. B.77 correctly encodes the discontinuity of E_0^{mr}, this result is invalid for the exact function E. We will prove that there are additional exponential corrections to Eq. B.77 which are not captured by median resummation. These exponential corrections will contribute to $\mathrm{Disc}\, E$ on top of Eq. B.77. Therefore, it is impossible that E satisfies a resurgent connection.

The full «Borel representation» of E is given by (see [90, Eq. 14]):

$$E(z) = \int_0^{\infty e^{i\theta}} dt \sum_{n\geqslant 0} (-1)^n e^{-nz} \left(e^{-zt} \left[z\, F_n(t) + G_n(t) \right] - H_n(t) \right) , \tag{B.78}$$

where we have introduced a direction θ in the integral in order to incorporate our framework to the discussion.

The functions F_n, G_n and H_n can be found in [90, App.]. We quote their expressions for $n = 0$:

$$F_0(t) = 1 \,, \tag{B.79}$$

$$G_0(t) = \frac{1}{t} + \frac{1}{t-1} - \psi(1+t) - \psi(2-t) - 2\gamma \,, \tag{B.80}$$

$$H_0(t) = \frac{1}{t} + B_1(t) \,, \tag{B.81}$$

where B_1 is a function analytic in $\mathbb{C} \setminus (\mathbb{R}^- - 2)$.

The integral in Eq. B.78 is well-defined for $\theta = 0$ despite the poles present in G_n and H_n along the positive real axis. The poles completely cancel each other in the sum over n. Namely, the cancellation of the pole at $t = t_0$ occurs between $G_n(t)$ and $H_{n+t_0}(t)$. This is the process of infrared renormalon cancellation, discussed in detail in the series of papers of Ref. [91, 92, 93].

It can be checked that the pole at the position $t = k$ coming from H_k provides the exponential correction $r_k e^{-kz}$ in Eq. B.75. In the same way, the sign changes with the direction θ chosen in Eq. B.78. Thus, this representation already incorporates the exponential corrections arising from median resummation. Still, Eq. B.78 contains additional corrections. Let us make this explicit.

Considering only $n = 0$ in Eq. B.78, we have

$$\int_0^{\infty e^{i\theta}} dt \left(e^{-zt}\left[z F_0(t) + G_0(t)\right] - H_0(t)\right) = 1 + \int_0^{\infty e^{i\theta}} dt\, e^{-zt}\, \widehat{E}(t) - \int_0^{\infty e^{i\theta}} dt \left(H_0(t) - \frac{e^{-zt}}{t}\right). \quad \text{(B.82)}$$

The last integral yields[2]

$$-\int_0^{\infty e^{i\theta}} dt \left(H_0(t) - \frac{e^{-zt}}{t}\right) = -\log(z) + c - 1\,. \quad \text{(B.83)}$$

Thus, we verify that the term $n = 0$ is equal to Eq. B.75 up to the ambiguous exponential terms arising from median resummation. (We are missing them because they come from H_n, with $n \geqslant 1$).

To finish the discussion, it is a simple verification that the poles in G_1 contribute to the discontinuity of $E(z)$. A similar computation to that in Eq. B.76 and Eq. B.77 yields

$$\text{Disc}\, E_1^{\text{mr}}(z) \sim (2\pi i) 2 e^{-z}\,, \quad z \to -\infty\,, \quad \text{(B.84)}$$

where E_1^{mr} is the term $n = 1$ in Eq. B.78 with the necessary exponential corrections to cancel the imaginary ambiguities:

$$E_1^{\text{mr}}(z) = -e^{-z} \int_0^{\infty e^{i\theta}} dt \left(e^{-zt}\left[z F_1(t) + G_1(t)\right] - H_1(t)\right) \pm i\pi\,(\text{exponentials}). \quad \text{(B.85)}$$

This clarifies that, to obtain the real asymptotic behaviour for $\text{Disc}\, E$, the result in Eq. B.77 has to be corrected by adding the contribution in Eq. B.84 and, actually, by adding all contributions from the terms $n \geqslant 1$.

[2]From this computation we come to the conclusion that the pole at $t = 0$ in $G_0(t)$ has nothing to do with renormalisation, as claimed in [90]. Instead, the pole at $t = 0$ encodes the log term appearing in the asymptotic expansion of $E(z)$ (see Eq. B.71). Note that applying Eq. 5.9 with $f(z) = \log(z)$ yields a pole at 0.

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