Universitat Autònoma de Barcelona

Qualitative theory of differential equations in the plane and in thr space, with emphasis on the center-focus problem and on the Lotka-Volterra systems

Valentín Ramírez Sadovski

Qualitative theory of differential equations in the plane and in the space, with emphasis on the center-focus problem and on the Lotka-Volterra systems

Valentín Ramírez Sadovski
Thesis director: Jaume Llibre i Salò

Contents

Preface

In the theory of ordinary differential equations we can find two fundamental problems. The direct problem which consists in a broad sense in to find the solutions of a given ordinary differential equation, and the inverse problem. An inverse problem of ordinary differential equations is to find the more general differential system satisfying a set of given properties. For instance what are the differential systems in $\mathbb{R}^N$ having a given set of invariant hypersurfaces, or of first integrals?

Probably the first inverse problem appeared in Celestial Mechanics, it was stated and solved by Newton (1687) in Philosophie Naturalis Principia Mathematica, and it concerns with the determination of the potential field of force that ensures the planetary motion in accordance to the observed properties, namely the Kepler's laws.

The first statement of the inverse problem as the problem of finding the more general differential system of first order satisfying a set of given properties was stated by Erugin [21] and developed in [23, 33, 50].

The aim of the present thesis is to state and study the following three inverse problem.

(I) *The inverse approach to the center-focus problem for planar differential systems.*

Let

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}, \tag{1}$$

be the real planar analytic or polynomial vector field associated to the real planar differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{2}$$

where the dot denotes derivative with respect to an independent variable t.

In what follows we assume that origin $O := (0, 0)$ is a singular point, i.e. $P(0, 0) = Q(0, 0) = 0$. The singular point O is a *center* if there exists an open neighborhood U of O where all the orbits contained in $U \setminus \{O\}$ are periodic.

The study of the centers of analytical or polynomial differential systems (2) has a long history. The first works are due to Poincaré [47] and Dulac [19] . Later on were developed by Bendixson [8], Frommer [22], Liapunov [30] and many others.

In the present thesis we shall study the differential system of the form

$$\dot{x} = -y + X, \quad \dot{y} = x + Y, \tag{3}$$

where $X = X(x, y)$ and $Y = Y(x, y)$ are real analytic or polynomials functions in an open neighborhood of O whose Taylor expansions at O do not contain constant and linear terms.

System (3) always has a center or a focus at the origin.

One of the classical problems in the qualitative theory of the differential system (3) is to characterize the local phase portrait in a sufficiently small neighborhood near the origin i.e. distinguishing between a center or focus. This problem is called the *center-focus problem* .

In the study of the center-focus problem the following theorems play a very important role (see for instance [30, 47, 44])

Theorem 1. *For the analytic differential system* (3) *there exists a formal power series* $H = \sum_{n=2}^{\infty} H_n := \frac{1}{2}(x^2 + y^2) + \sum_{n=3}^{\infty} H_n(x, y)$, *where* $H_j = H_j(x, y)$ *is a homogenous polynomial of degree* j *such that* $\frac{dH}{dt} = \sum_{j=1}^{\infty} v_{2k}(x^2 + y^2)^k$, *where* v_{2k} *are the Poincaré-Liapunov constants.*

Assume that the formal power series H converges. If the constants $v_j = 0$ for $j \in \mathbb{N}$ then there exists a local first integral $H := \frac{1}{2}(x^2+y^2)+\sum_{j=3}^{\infty} H_j$, and consequently the origin is a center. If there exists a first non-zero Liapunov constant v_{2k}, then the origin is a stable focus if $v_{2k} < 0$ and unstable if $v_{2k} > 0$. If $v_{2k} = 0$ for $k = 1, \ldots, n-1$ and $v_{2n} \neq 0$ then the system (3) has a *focus of order* n at the origin.

We recall the following definition. Let U be an open and dense set in $\mathbb{R}^2$. We say that a non-constant C^r with $r \geq 1$ function $F\colon U \to \mathbb{R}$ is a *first integral* of the analytic or polynomial vector field $\mathcal{X}$ on U, if $F(x(t), y(t))$ is constant for all values of t for which the solution $(x(t), y(t))$ of $\mathcal{X}$ is defined on U. Clearly F is a first integral of $\mathcal{X}$ on U if and only if $\mathcal{X}F = 0$ on U.

Poincaré and Liapunov proved the next two results, see for instance [47, 30, 26, 44].

Theorem 2. *A planar polynomial differential system*

$$\dot{x} = -y + \sum_{j=2}^{m} X_j(x,y), \quad \dot{y} = x + \sum_{j=2}^{m} Y_j(x,y), \tag{4}$$

of degree m has a center at the origin if and only if it has a first integral of the form

$$H = \sum_{j=2}^{\infty} H_j(x,y) = \frac{1}{2}(x^2+y^2) + \sum_{j=3}^{\infty} H_j(x,y), \tag{5}$$

where X_j, Y_j and H_j are homogenous polynomials of degree j.

Theorem 3. *An analytic planar differential system*

$$\dot{x} = -y + \sum_{j=2}^{\infty} X_j(x,y), \quad \dot{y} = x + \sum_{j=2}^{\infty} Y_j(x,y), \tag{6}$$

has a center at the origin if and only if it has a first integral of the form (5).

From Theorems 1, 2 and 3 it is clear that an analytic or polynomial differential system (3) has a center at the origin if and only if the Poincaré-Liapunov constants $v_k = 0$ for $k \geq 1$ (*Poincaré's criterion*). Moreover, the v_k's are polynomials over $\mathbb{Q}$ in the coefficients of the polynomial differential system. A necessary and sufficient condition to have a center is then the annihilation of all these constants. In view of the Hilbert's basis theorem this occurs if and only if for a finite number of k, $k < j$ and j sufficiently large, $v_k = 0$. Unfortunately, trying to solve the center problem computing the Poincaré-Liapunov constants is in general not possible due to the huge computations.

Although we have an algorithm for computing the Poincaré-Liapunov constants for linear type center, we have no algorithm to determine how many of them need to be zero to imply that all of them are zero for cubic or higher degree polynomial differential systems. Bautin [6] showed in 1939 that for a quadratic polynomial differential system, to annihilate all v_k's it suffices to have $v_k = 0$ for $i = 1, 2, 3$. So the problem of the center is solved for quadratic systems. This problem was solved for the cubic differential systems with homogenous nonlinearities (see for instance [43, 51, 52]).

The analytic function (5) is called the *Poincaré-Liapunov local first integral.*

Theorem 2 is due to Poincaré, and Theorem 3 is due to Liapunov.

Now we shall introduce another criterion for solving the center problem due to Reeb.

We will need the following definitions.

A function $V = V(x, y)$ is an *inverse integrating factor* of system (2) in an open subset $U \subset \mathbb{R}^2$ if $V \in C^1(U), V \not\equiv 0$ in U and

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = V\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right).$$

In short notation this equation can be rewritten as follows

$$\mathcal{Y}(V) := \frac{\partial\left(\dfrac{P}{V}\right)}{\partial x} + \frac{\partial\left(\dfrac{Q}{V}\right)}{\partial y} = 0 \Longleftrightarrow P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \mathrm{div}\mathcal{Y}\ V.$$

We note that $\{V = 0\}$ is formed by orbits of system (2). The function $1/V$ defines an integrating factor in $U\backslash\{V = 0\}$ of system (2) which allows to compute a first integral for (2) in $U\backslash\{V = 0\}$.

If exists an integrating factor then differential system (2) is topological equivalent to the Hamiltonian vector fields, i.e.

$$\frac{\mathcal{Y}}{V} = \frac{\partial H}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial y}.$$

Hence we get that

$$\frac{\partial H}{\partial x} = -\frac{Q}{V} \qquad \frac{\partial H}{\partial y} = \frac{P}{V},$$

thus the function H can be computed as follows (see for instance [29])

$$H = \int_{x_0}^{x}\left(-\frac{Q}{V}\right)dx + \int_{y_0}^{y}\left(\frac{P}{V}\right)\bigg|_{x=x_0} dy.$$

The following theorem holds (see for instance [49])

Theorem 4. *[Reeb 's criterion] (see for instance [49]). The analytic differential system* (6) *has a center at the origin if and only if there is a local nonzero analytic inverse integrating factor of the form $V = 1 + h.o.t.$ in a neighborhood of the origin.*

An analytic inverse integrating factor having the Taylor expansion at the origin $V = 1 + h.o.t.$ is called a *Reeb inverse integrating factor*.

Hence we get that to show that a singular point is a center for system (3) we have two basic mechanisms: we either apply Poincaré–Liapunov Theorem and we show that we have a local analytic first integral, or we apply the Reeb inverse integrating factor.

We observe that the difficulties of computations of the inverse integrating factor or Poincaré-Liapunov first integral for a given differential system are comparable to solving the system itself.

From this point of view it is interesting to state the problem about the determination of the structure of a differential system (3) knowing that it has a given first integral or an integral factor.The first chapter of my thesis is dedicated to the study these inverse problems.

(II) *Weak centers.*

The second chapter of the thesis is dedicated to the determination of differential equations (3) under the condition that they have a Poincaré-Liapunov first integral of the form $H = \frac{1}{2}(x^2+y^2)\,(1+h.o.t.)$.

We find a new class of centers which we call weak centers. We say that a center at the origin of an analytic differential system is a *weak center* if in a neighborhood of the origin it has an analytic first integral of the form $H = \frac{1}{2}(x^2+y^2)\left(1+\sum_{j=1}^{\infty}\Upsilon_j\right)$, where Υ_j is a homogenous polynomial of degree j.

We have characterized the expression of an analytic or polynomial differential system having a weak center at the origin. We prove that the following statements are equivalent.

(a) If an analytic (or polynomial) differential systems has a weak center at the origin then it can be written as

$$\dot{x} = -y(1+\Lambda)+x\Omega, \qquad \dot{y} = x(1+\Lambda)+y\Omega, \tag{7}$$

(see Theorem 9).

(b) Let $\dot{z} = iz+R(z,\bar{z})$ be the system (4) in complex coordinates $z = x+iy$ and $\bar{z} = x-iy$. Then if this system has a weak center at the origin then $R(z,\bar{z}) = z\Phi(z,\bar{z})$ (see Proposition 30). This is a very simple criterium to determine the non existence of a weak center.

(c) If an analytic (or polynomial) differential system (3.30) (or (6)) has a weak center, then the first integral (5) satisfies that its homogenous polynomial H_j for $j = 3,\ldots,k \leq \infty$ is $H_j = H_2\Upsilon_{j-2}$, for $j = 3,\ldots,k \leq \infty$, where Υ_i is a homogenous polynomial of degree i.

Moreover we prove that the uniform isochronous centers and the isochronous holomorphic centers are weak centers.

It is well known the following result (see for instance [7]).

Let $\mathcal{X}$ be an analytic vector field associated to differential system (3). Then $\mathcal{X}$ has either a focus or a center at the origin, and under a formal change of coordinates the differential system associated to $\mathcal{X}$ can be written into the Kirchhoff normal form

$$\begin{aligned}\dot{x} &= -y(1+S_2(x^2+y^2))+xS_1(x^2+y^2),\\ \dot{y} &= x(1+S_2(x^2+y^2))+yS_1(x^2+y^2),\end{aligned}$$

where $S_j = S_j(x^2+y^2)$ for $j=1,2$ are formal series in the variable x^2+y^2 . Clearly these differential equations are particular case of (7).

We have extended the weak conditions of a center given by Alwash and Lloyd in [3] for linear centers with homogenous polynomial nonlinearities (see Proposition 24), to a general analytic or polynomial differential system see Theorem 12. Furthermore the centers satisfying the generalized weak conditions of a center, introduced in Theorem 12, are weak centers.

(III) *Construction of the generalized 3-dimensional Lotka-Volterra systems having a Darboux invariant. Final Evolutions*

The Lotka-Volterra systems in $\mathbb{R}^3$ are the differential systems (see for instance [25])

$$\dot{x}_j = x_j\left(d_j + a_j x_1 + b_j x_2 + c_j x_3\right), \quad \text{for} \quad j=1,2,3, \tag{8}$$

The state space of this system is the set

$$\mathbb{R}^3_+ = \left\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_j \geq 0, \quad \text{for} \quad j=1,2,3.\right\}.$$

Models of Lotka-Volterra systems in $\mathbb{R}^3$ occur frequently in the physical and engineering sciences, as well in the biological. Differential system (8) was introduced independently by Lotka-Volterra in 1920s to model the interaction among the species (see [39, 54, 25])

The applications of Lotka-Volterra model in the population biology is well-known. For example for two dimensional Lotka-Volterra model we have the predator-prey model and for the three dimensional Lotka-Volterra model we have the symmetric and non-symmetric May-Leonard model (see for instance [13, 25, 32]) describing the competitions between three species.

Recently it has become important the generalization of 3-dimensional Lotka-Volterra systems of the form

$$\begin{aligned}\dot{x} &= x(a_0+a_1x+a_2y+a_3z) = X_1,\\ \dot{y} &= y(b_0+b_1x+b_2y+b_3z) = X_2,\\ \dot{z} &= c_0+c_1x+c_2y+c_3z+c_4x^2+c_5xy+c_6xz+c_7y^2+c_8yz+c_9z^2\\ &= X_3.\end{aligned} \tag{9}$$

In particular to study the resistant viral and bacterial strains, and the treatment on their proliferation (see for instance [9]). One framework for studying such systems is the multistrain model study by Castillo-Chavez and Feng [10]. The model, which is an approximation of the full system discussed in [10] was proposed in [53]. The model has only a single susceptible compartment and two infectious compartments corresponding to the two infectious agents. The model equations are

$$\begin{aligned} \dot{x} =& \quad x(-b_1 - \gamma_1 + \nu\, y + \beta_1\, z), \\ \dot{y} =& \quad y(-b_1 - \gamma_2 - \nu\, x + \beta_2\, y), \\ \dot{z} =& \quad z(-b_1 - \beta_1 x - \beta_2 y) + b_1 + \gamma_1 x + \gamma_2 y. \end{aligned}$$

The state space of this system is the set

$$\mathbb{R}^3_+ = \left\{(x, y, z) \in \mathbb{R}^3 : \quad \text{and} \quad x \geq 0,\, y \geq 0 \quad z \geq 0\right\}.$$

In [32] the authors characterized all the final evolution of this model under generic assumptions.

The main objective of this section is to study the global dynamics of the generalized Lotka -Volterra model (9) with the Darboux invariant $I = (x + y + z - 1)e^{-at}$ (see definition below) defined in the positive quadrant of $\mathbb{R}^3$

$$\mathbb{M} = \{x \geq 0, \quad y \geq 0, \quad \forall z \in \mathbb{R}\}. \tag{10}$$

We observe that this differential system does not correspond in general to a biological model, in view of (10), as the z variable can have in this case a negative value for the population.

The dynamics of the obtained differential equations on the invariant plane $\Pi : x + y + z = 1$ is described by the two dimensional Lotka Volterra system. Dynamics of the obtained differential system at infinity produces a cubic planar Kolmogorov system. To solve the center focus problem for this system we apply the results given in the solution of the problem I and II. We characterize all the final evolutions of this model.

0.1 Statement of the main results

0.1.1 Main results for the inverse center problem

One of the main objectives of the present thesis is to analyze the center problem from the inverse point of view. We state and solve the following two inverse problems of analytic and polynomial vector fields.

Problem 5. Inverse Poincaré-Liapunov's Problem *Determine the analytic (polynomial) planar differential systems* (3)*, or the associated vector fields*

$$\mathcal{X} = \sum_{j=1}^{k} \left(X_j \frac{\partial}{\partial x} + Y_j \frac{\partial}{\partial y} \right), \qquad for \quad k \leq \infty, \tag{11}$$

where $X_j = X_j(x,y)$, $Y_j = Y_j(x,y)$ *for* $j \geq 2$ *are homogenous polynomials of degree* j *for which the given function* (5) *is a local analytic first integral.*

Problem 6. Inverse Reeb Problem *Determine the analytic (polynomial) planar vector fields* (11) *for which the*

$$V = 1 + \sum_{j=1}^{\infty} V_j,$$

where $V_j = V_j(x,y)$ *for* $j \geq 2$ *are homogenous polynomials of degree* j*, is a Reeb inverse integrating factor, i.e.*

$$\mathcal{X}(x)\frac{\partial V}{\partial x} + \mathcal{X}(y)\frac{\partial V}{\partial y} = V\left(\frac{\partial \mathcal{X}(x)}{\partial x} + \frac{\partial \mathcal{X}(y)}{\partial y}\right). \tag{12}$$

The solutions of the problems 5 and 6 for analytic and polynomial vector fields are given in the following theorems.

We will denote by $\{f,\, g\}$ the Poisson bracket, i.e.

$$\{f,\, g\} := \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

Theorem 7. *Consider the analytic vector field* $\mathcal{X}$ *associated to the differential system* (6) *Then this vector field has a Poincaré-Liapunov local first integral if and only if it has a Reeb inverse integrating factor. Moreover,*

(*i*) *the analytic differential system* (6) *for which* $H = (x^2 + y^2)/2 + h.o.t.$ *is a local first integral can be written as*

$$\dot{x} = \left(1 + \sum_{j=1}^{\infty} g_j\right)\{H, x\}, \qquad \dot{y} = \left(1 + \sum_{j=1}^{\infty} g_j\right)\{H, y\} \tag{13}$$

where $g_j = g_j(x,y)$ *is an arbitrary homogenous polynomial of degree* j *which we choose in such a way that the series* $\sum_{j=1}^{\infty} g_j$ *converge in the neighborhood of the origin.*

(ii) The differential system (4) for which $V = 1 + \sum_{j=1}^{\infty} V_j$ *is a Reeb integrating factor can be written as*

$$\dot{x} = \left(1 + \sum_{j=1}^{\infty} V_j\right) \{F, x\}, \qquad \dot{y} = \left(1 + \sum_{j=1}^{\infty} V_j\right) \{F, y\},$$

where $F = \sum_{j=2}^{\infty} F_j$ *and* $F_2 = (x^2 + y^2)/2$, $F_j = F_j(x, y)$ *for* $j > 2$ *is an arbitrary homogenous polynomial of degree* j *which we choose in such a way that* $\sum_{j=2}^{\infty} F_j$ *converges, i.e.* F *is an arbitrary Poincaré-Liapunov local first integral.*

Theorem 8. *Consider the polynomial vector field* $\mathcal{X}$ *associated to the differential system (4). Then this polynomial vector field has a Poincaré-Liapunov local first integral if and only if it has a Reeb inverse integrating factor. Moreover, the differential system associated to the vector field* $\mathcal{X}$ *for which* $H = (x^2 + y^2)/2 + h.o.t.$ *is a local first integral can be written as*

$$\begin{aligned} \dot{x} = & \left(1 + \sum_{j=1}^{\infty} g_j\right) \{H, x\} \\ = & \{H_{m+1}, x\} + (1 + g_1)\{H_m, x\} + \ldots + (1 + g_1 + \ldots + g_{m-1})\{H_2, x\}, \\ \dot{y} = & \left(1 + \sum_{j=1}^{\infty} g_j\right) \{H, y\} \\ = & \{H_{m+1}, y\} + (1 + g_1)\{H_m, y\} + \ldots + (1 + g_1 + \ldots + g_{m-1})\{H_2, y\}. \end{aligned} \tag{14}$$

where g_j *are homogenous polynomial of degree* j *for* $j \geq 1$ *such that* $\sum_{j=1}^{\infty} g_j$ *is an analytic function in the neighborhood of the origin and the function*

$$\Psi = \left(1 + \sum_{j=1}^{\infty} g_j\right)^{-1} = 1 - \sum_{k=1}^{\infty} \left(g_k - \sum_{n=1}^{k-1} g_n G_{k-n\ m+1}\right) := 1 - \sum_{k=1}^{\infty} G_{k\ m+1},$$

satisfies the following conditions

$$\{H_{m+1}, \Psi\} + \{H_m, (1 + g_1)\Psi\} + \ldots + \{H_2, (1 + g_1 + \ldots + g_{m-1})\Psi\} = 0. \tag{15}$$

The first integral H becomes

$$H = \frac{1}{2}(x^2 + y^2) + \sum_{j=2}^{\infty} H_j = \tau_1 H_{m+1} + \tau_2 H_m + \ldots + \tau_m H_2, \tag{16}$$

with

$$\tau_s = 1 - s \sum_{j=1}^{\infty} \frac{G_{j\,s}}{m+1+j}. \tag{17}$$

for $s = 2, \ldots, m$ are an analytic function in the neighborhood of the origin, where

$$G_{k\,j} := g_{k+m+1-j} - \sum_{j=1}^{k-1} g_n G_{k-n\;j}, \textit{for} \quad k \geq 1, \quad j = 2, \ldots, m+1. \tag{18}$$

Theorem 7 and 8 are proved in section 1.3.

0.1.2 Main results for the inverse weak center problem

Differential system (3) has a weak center at the origin if it has a local analytic first integral of the form $H = \frac{1}{2}(x^2+y^2)\left(1 + \sum_{j=1}^{\infty} \Upsilon_j(x,y)\right) = H_2\Phi(x,y)$, where Υ_j is a homogenous polynomial of degree j.

One of the important characterization of the weak center consists in that the equation (3) can be written as

$$\dot{x} = -y(1+\Lambda) + x\Omega, \qquad \dot{y} = x(1+\Lambda) + y\Omega, \tag{19}$$

where $\Lambda = \Lambda(x,y)$ and $\Omega = \Omega(x,y)$ are convenient analytic (polynomial) functions. These differential systems are called $\Lambda-\Omega$ *differential systems*. This results is given in the following theorem.

Theorem 9. *A center at the origin of an analytic differential system (6) is a weak* *center at the origin if and only if this system can be written as*

$$
\begin{aligned}
\dot{x} = \ & -y\left(1+\sum_{j=2}^{\infty}\left(\frac{j+1}{2}\Upsilon_{j-1}+\frac{j}{2}g_1\Upsilon_{j-2}+\ldots+\frac{3}{2}g_{j-2}\Upsilon_1+g_{j-1}\right)\right) \\
& +\frac{x}{2}\sum_{j=2}^{\infty}\Big(\{\Upsilon_{j-1},H_2\}+g_1\{\Upsilon_{j-2},H_2\}+\ldots+g_{j-2}\{\Upsilon_1,H_2\}\Big) \\
& := -y(1+\Lambda)+x\Omega, \\
\dot{y} = \ & x\left(1+\sum_{j=2}^{\infty}\left(\frac{j+1}{2}\Upsilon_{j-1}+\frac{j}{2}g_1\Upsilon_{j-2}+\ldots+\frac{3}{2}g_{j-2}\Upsilon_1+g_{j-1}\right)\right) \\
& +\frac{y}{2}\sum_{j=2}^{\infty}\Big(\{\Upsilon_{j-1},H_2\}+g_1\{\Upsilon_{j-2},H_2\}+\ldots+g_{j-2}\{\Upsilon_1,H_2\}\Big) \\
& := x(1+\Lambda)+y\Omega,
\end{aligned}
\tag{20}
$$

where $\Upsilon_0 = 1$, $g_0 = 1$, g_j *and* Υ_j *are homogenous polynomials of degree* j *for* $j \geq 1$ *has the first integral* $H = H_2\left(1+\sum_{j=2}^{\infty}\Upsilon_j\right)$. *Moreover assuming that*

$$
\begin{aligned}
\frac{j+1}{2}\Upsilon_{j-1}+\frac{j}{2}g_1\Upsilon_{j-2}+\ldots+\frac{3}{2}g_{j-2}\Upsilon_1+g_{j-1} &= 0, \\
\{\Upsilon_{j-1},H_2\}+g_1\{\Upsilon_{j-2},H_2\}+\ldots+g_{j-2}\{\Upsilon_1,H_2\} &= 0,
\end{aligned}
\tag{21}
$$

for $j \geq m+1$, *we obtain necessary and sufficient conditions under which the polynomial differential system* (20) *of degree* m *and has the first integral*

$$H = H_2(1+\mu_1\Upsilon_1+\ldots+\mu_{m-1}\Upsilon_{m-1}), \tag{22}$$

where $\mu_j = \mu_j(x,y)$ *is a convenient analytic function in the neighborhood of the origin for* $j = 1,\ldots,m-1$.

Corollary 10. *Define*

$$\Lambda_{j-1} = \frac{j+1}{2}\Upsilon_{j-1}+\frac{j}{2}g_1\Upsilon_{j-2}+\ldots+\frac{3}{2}g_{j-2}\Upsilon_1+g_{j-1}, \quad \textit{for} \quad j>1,$$

a homogenous polynomial of degree $j-1$, *then differential system* (20) *and condi-*

tions (21) *can be rewritten as follows*

$$\begin{aligned}\dot{x} = &\ -y(1+\sum_{j=1}^{\infty}\Lambda_j) + x\sum_{j=2}^{\infty}\frac{1}{j+1}\left(\{H_j,g_1\}+\ldots+\{H_3,g_{j-2}\}+\{H_2,g_{j-1}-\Lambda_{j-1}\}\right)\\ = &\ P,\\ \dot{y} = &\ x(1+\sum_{j=1}^{\infty}\Lambda_j) + y\sum_{j=2}^{\infty}\frac{1}{j+1}\left(\{H_j,g_1\}+\ldots+\{H_3,g_{j-2}\}+\{H_2,g_{j-1}-\Lambda_{j-1}\}\right)\\ = &\ Q,\end{aligned}$$

and

$$\begin{aligned}(j+1)H_{j+1}+jg_1H_j+\ldots+3g_{j-2}H_3+2g_{j-1}H_2 = &\ 0,\\ \{H_j,g_1\}+\ldots+\{H_3,g_{j-2}\}+\{H_2,g_{j-1}\} = &\ 0,\end{aligned}$$

for $j > m$ *respectively. Moreover the following relation holds*

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=\sum_{j=1}^{\infty}\left(\{H_j,g_1\}+\ldots+\{H_3,g_{j-2}\}+\{H_2,g_{j-1}\}\right).$$

A center O of system (2) is a *uniform isochronous center* if the equality $x\dot{y}-y\dot{x}=\kappa(x^2+y^2)$ holds for a nonzero constant κ; or equivalently in polar coordinates (r,θ) such that $x=r\cos\theta$, $y=r\sin\theta$, we have that $\dot{\theta}=\kappa$.

Corollary 11. *An analytic differential system* (19) *has a uniform isochronous center at the origin if and only if*

$$\begin{aligned}\dot{x} = &\ -y+x\sum_{j=2}^{\infty}\frac{1}{j+1}\Big(\{H_j,g_1\}+\ldots+\{H_2,g_{j-1}\}\Big)=-y-x\Omega,\\ \dot{y} = &\ x+y\sum_{j=2}^{\infty}\frac{1}{j+1}\Big(\{H_j,g_1\}+\ldots+\{H_2,g_{j-1}\}\Big)=x+y\Omega,\end{aligned}\tag{23}$$

and

$$0 = (j+1)H_{j+1}+jg_1H_j+\ldots+3g_{j-2}H_3+2g_{j-1}H_2, \quad for \quad j\geq 2.$$

Moreover a polynomial differential system of degree m (19) *has a uniform isochronous center at the origin if and only if* (23) *holds and*

$$\{H_j,g_1\}+\ldots+\{H_3,g_{j-2}\}+\{H_2,g_{j-1}\}=0, \quad for \quad j\geq m+1.$$

The inverse approach to study the uniform isochronous center was given in [38].

The proofs of Theorem 9 and Corollaries 10 and 11 are given in section 2.1

Theorem 12. *[Generalized weak condition of a center of an analytic (polynomial) differential systems] We consider an analytic (polynomial) differential system (6). Then the origin is a weak center if there exists $\mu \in \mathbb{R}\backslash\{0\}$ such that the following relations hold*

$$\begin{aligned}(x^2+y^2)\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right) =& \quad \mu\,(xX+yY),\\ \int_0^{2\pi}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)|_{x=\cos t,\,y=\sin t}dt =& \quad 0,\end{aligned} \tag{24}$$

where $\mu \in \mathbb{R}\backslash\{0\}$. Moreover this differential system can be written as

$$\begin{aligned}\dot{x} =& \quad -y\left(1+q(H_2)+(1-1/\lambda)\Upsilon+\frac{1}{2}\left(x\frac{\partial\,\Upsilon}{\partial\,x}+y\frac{\partial\,\Upsilon}{\partial\,y}\right)\right)+\frac{x}{2}\{\Upsilon,H_2\},\\ \dot{y} =& \quad x\left(1+q(H_2)+(1-1/\lambda)\Upsilon+\frac{1}{2}\left(x\frac{\partial\,\Upsilon}{\partial\,x}+y\frac{\partial\,\Upsilon}{\partial\,y}\right)\right)+\frac{y}{2}\{\Upsilon,H_2\},\end{aligned} \tag{25}$$

with $\lambda = 2/\mu$, and $\Upsilon = \Upsilon(x,y)$ and $q = q(H_2)$ are convenient analytic functions.

If differential system (25) is a polynomial differential system of degree m, i.e. $\Upsilon = \Upsilon(x,y)$ is a polynomial of degree $m-1$ and $q(H_2)) = \sum_{j=1}^{[(m-1)/2]} \alpha_j H_2^{j-1}$, here $[(m-1)/2]$ is the integer part of $(m-1)/2$, α_j is a constant for $j = 1,\ldots,[(m-1)/2]$ such that $1+\alpha_1+\dfrac{\lambda-1}{\lambda}\Upsilon(0,0) \neq 0$, then the system (25) is quasi Darboux-integrable with the first integral F which is given in what follows

(i) If $\lambda \neq 1$ and $\prod_{n=2}^{[m/2]}(n-1/\lambda) \neq 0$, then

$$F = \frac{H_2}{\left(\Upsilon+\dfrac{1+\alpha_1}{1-1/\lambda}+\dfrac{\alpha_2 H_2}{2-1/\lambda}+\ldots+\dfrac{\alpha_m H_2^{[(m-1)/2]-1}}{[(m-1)/2]-1/\lambda}\right)^{\lambda/(\lambda-1)}}. \tag{26}$$

The algebraic curves $H_2 = 0$ and

$$g = \Upsilon+\frac{1+\alpha_1}{1-1/\lambda}+\frac{\alpha_2 H_2}{2-1/\lambda}+\ldots+\frac{\alpha_m H_2^{[(m-1)/2]-1}}{[(m-1)/2]-1/\lambda} = 0,$$

are invariant curve with cofactors $\{g,H_2\}$ and $(1-1/\lambda)\{g,H_2\}$, respectively.

(ii) If $\lambda = 1$ and $1+\alpha_1 \neq 0$, then

$$F = H_2 e^{-\left(\Upsilon+\alpha_2 H_2+\alpha_3 H_2^2/2+\ldots+\alpha_m H_2^{[(m-1)/2]-1}/([(m-1)/2]-1)\right)/(1+\alpha_1)}. \tag{27}$$

The algebraic curves $H_2 = 0$ is invariant with cofactor $\{H_2,\Upsilon\}$.

(iii) *If* $1/[m/2] \leq \lambda = 1/k < 1$ *and* $\left(\alpha_k \prod_{n=2, n\neq k}^{[(m-1)/2]} (n-k)\right) \neq 0$, *then*

$$F = \frac{H_2}{\left(\Upsilon + \dfrac{1+\alpha_1}{1-k} + \displaystyle\sum_{j=2,\, j\neq k}^{[(m-1)/2]} \frac{\alpha_j}{j-k} H_2^{j-1} + \alpha_k H_2^{k-1} \log H_2\right)^{1/(k-1)}}. \tag{28}$$

The algebraic curve $H_2 = 0$ *and the non-polynomial curve*

$$f = \Upsilon + \frac{1+\alpha_1}{1-k} + \sum_{j=2,\, j\neq k}^{[(m-1)/2]} \frac{\alpha_j}{j-k} H_2^{j-1} + \alpha_k H_2^{k-1} \log H_2 = 0,$$

are invariant curves with cofactors $\{\Upsilon, H_2\}$ *and* $(1-k)\{\Upsilon, H_2\}$, *respectively. We observe that* $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(iv) *If* $1/[(m-1)/2] \leq \lambda = 1/k < 1$, $\alpha_k = 0$ *and* $\prod_{n=2, n\neq k}^{[(m-1)/2]} (n-k) \neq 0$, *then*

$$F = \frac{H_2}{\left(\Upsilon + \dfrac{1+\alpha_1}{1-k} + \displaystyle\sum_{j=2,\, j\neq k}^{[(m-1)/2]} \frac{\alpha_j}{j-k} H_2^{j-1}\right)^{1/(k-1)}}. \tag{29}$$

The algebraic curves $H_2 = 0$ *and*

$$g = \Upsilon + \frac{1+\alpha_1}{1-k} + \sum_{j=2,\, j\neq k}^{[(m-1)/2]} \frac{\alpha_j}{j-k} H_2^{j-1} = 0$$

are invariant algebraic curves with cofactors $\{g, H_2\}$ *and* $(1-k)\{g, H_2\}$ *respectively.*

The given first integrals has the following Taylor extension at the origin $F = H_2(1 + h.o.t.)$. *Consequently the origin is a weak center. In an analogous way we can study the analytic case.*

Theorem 12 is proved in section 2.2.

For the $\Lambda - \Omega$ differential systems we get the following conjectures.

Conjecture 13. *The polynomial differential system of degree* m

$$\begin{aligned} \dot{x} &= -y(1 + \mu\,(a_2x - a_1y)) + x(a_1x + a_2y + \Omega_{m-1}), \\ \dot{y} &= x(1 + \mu\,(a_2x - a_1y)) + y(a_1x + a_2y + \Omega_{m-1}), \end{aligned} \tag{30}$$

where $(\mu+m-2)(a_1^2+a_2^2)\neq 0$, *and* $\Omega_{m-1}=\Omega_{m-1}(x,y)$ *is a homogenous polynomial of degree* $m-1$, *has a weak center at the origin if and only if system* (30) *after a linear change of variables* $(x,y)\longrightarrow(X,Y)$ *is invariant under the transformations* $(X,Y,t)\longrightarrow(-X,Y,-t)$.

Theorem 14. *Conjecture 13 holds for* $m=2,3,4,5,6$.

The only difficulty for proving Conjectures 13 for the Λ–Ω systems of degree m with $m>6$ is the huge number of computations for obtaining the conditions that characterize the centers.

The proofs of Theorem 14 is given in subsection 2.4.3.

Conjecture 15. *The polynomial differential system of degree* m

$$\begin{aligned}\dot{x} &= -y(1+\mu(a_2x-a_1y))+x(a_1x+a_2y+\sum_{j=2}^{m-1}\Omega_j(x,y)),\\ \dot{y} &= x(1+\mu(a_2x-a_1y))+y(a_1x+a_2y+\sum_{j=2}^{m-1}\Omega_j(x,y)),\end{aligned} \tag{31}$$

under the assumptions $(\mu+(m-2))(a_1^2+a_2^2)\neq 0$ *and* $\sum_{j=2}^{m-2}\Omega_j\neq 0$, *where* $\Omega_j=\Omega_j(x,y)$ *is a homogenous polynomial of degree* j *for* $j=2,\ldots,m-1$, *has a weak center at the origin if and only if system* (31) *after a linear change of variables* $(x,y)\longrightarrow(X,Y)$ *is invariant under the transformations* $(X,Y,t)\longrightarrow(-X,Y,-t)$. *Moreover differential system* (31) *in the variables* X,Y *becomes*

$$\begin{aligned}\dot{X} &= -Y(1+\mu Y)+X^2\Theta(X^2,Y)=-Y(1+\mu Y)+X\{H_2,\Phi\},\\ \dot{Y} &= X(1+\mu Y)+XY\Theta(X^2,Y)=X(1+\mu Y)+Y\{H_2,\Phi\},\end{aligned}$$

where $\Theta(X^2,Y)$ *is a polynomial of degree* $m-2$, *and* Φ *is a polynomial of degree* $m-1$ *such that* $\{H_2,\Phi\}=X\Theta(X^2,Y)$.

Theorem 16. *Conjecture 15 holds for* $m=2,3$ *and for* $m=4$ *with* $\mu=0$.

The proof of Theorem 16 for $\mu=0$ and $m=2$ goes back to Loud [40]. The proof of Theorem 16 for $\mu=0$ and $m=3$ was done by Collins [15]. Finally the proof of Theorem 16 for $\mu=0$ and $m=4$ goes back to [2, 1, 11]. But in the proof of this last result there is some mistakes. The phase portraits of these systems are classified in [4, 27, 28]. The proof that these centers are weak centers has been done in Theorem 9.

The proof of Theorem 16 is given in section 2.6.

0.1.3 Main results on the generalized 3-dimensional Lotka-Volterra systems having a Darboux invarian

The main result in the study of the differential systems

$$\begin{aligned} \dot{x} &= x(a-b-cx-(d+e)y-fz) = P(x,y,z), \\ \dot{y} &= y(a-g+dx-hy-iz) = Q(x,y,z), \\ \dot{z} &= z(fx+iy+a)+x(cx+ey+b)+y(hy+g)-a = R(x,y,z) \end{aligned} \tag{32}$$

is given in the following theorem.

Theorem 17. *The generalized Lotka-Volterra differential systems* (32) *has* 7920 *different phase portraits.*

We have also solved the center-focus problem for the two dimensional Lotka-Volterra systems (see Proposition 66) and for the two dimensional cubic Kolmogorov systems (see Proposition 68).

Theorem 17 is proved in subsection 3.6.3.

We note that the results of the first two chapters have been published in the papers [36, 35, 37].

Chapter 1

An inverse approach to the center problem

1.1 Introduction

From the inverse Problems 5 and 6 it follows that, either given an analytic function H of the form (5) we shall determine the analytic functions X and Y in (3) in such a way that the function H is a first integral of the differential system (3), or given an analytic function $1+\sum_{j=1}^{\infty} V_j$ in a neighborhood of the origin we shall determine the analytic functions X and Y in (3) in such a way that the analytic differential system (3) has the function V as a Reeb integrating factor.

The solutions of the given inverse problem for analytic case is given in Theorem 7 and for the polynomial case is given in Theorem 8.

The inverse Poincaré-Liapunov's problem (see Problem 5) and inverse Reeb problem (see Problem 6) for the analytic ($k=\infty$) and polynomial ($k<\infty$) planar vector fields has been solved in the Theorems 7 and 8 which provides the expressions of the analytic differential systems (6) in function of its first integral (5) or in function of its Reeb inverse integrating factor.

The solution of the problems 5 and 6 provide the expression for the analytic or polynomial functions X and Y, i.e. we determine all the homogeneous parts of X and Y which are given by

$$\begin{aligned} X_j = &\ \{H_{j+1},x\}+g_1\{H_j,x\}+\ldots+g_{j-1}\{H_2,x\}, \\ Y_j = &\ \{H_{j+1},y\}+g_1\{H_j,y\}+\ldots+g_{j-1}\{H_2,y\}, \qquad \text{for} \quad j\geq 2, \end{aligned} \tag{1.1}$$

where $\{f,g\}:=\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial y}-\dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x}$ is the *Poisson bracket* of the functions f and g, and $g_i=g_i(x,y)$ is an arbitrary homogenous polynomial of degree i. For the Inverse

Poincaré-Liapunov's Problem the polynomials g_i are arbitrary functions such that $1+\sum_{i=1}^{\infty} g_i$ converges in the neighborhood of the origin, and for the Inverse Reeb Problem we obtain

$$\begin{aligned} X_j = &\ \{F_{j+1}, x\} + V_1\{F_j, x\} + \ldots + V_{j-1}\{F_2, x\}, \\ Y_j = &\ \{F_{j+1}, y\} + V_1\{F_j, y\} + \ldots + V_{j-1}\{F_2, y\}, \qquad \text{for} \quad j \geq 2, \end{aligned}$$

where the homogeneous polynomial F_i of degree $i > 2$ is arbitrary, $F_2 = (x^2 + y^2)/2$, and $1 + \sum_{i=1}^{\infty} F_i$ converges in the neighborhood of the origin.

From the solutions of the inverse problem 5 and 6 we obtain the next results.

Corollary 18. *An analytic differential system* (6) *has a center at the origin if and only if either it has a first integral H of the form* (5)*, or it has an inverse Reeb integrating factor V. Moreover differential system* (6) *can be written as* $\dot{x} = V\{H, x\}, \quad \dot{y} = V\{H, y\}$.

From (1.1) we obtain the solutions of the Problem 5 and 6 for polynomial differential system of degree m.

Corollary 19. *A polynomial differential system* (4) *has a center at the origin if and only if it has an analytic first integral H given in* (5)*. Moreover this differential system* (6) *can be written as*

$$\begin{aligned} \dot{x} = &\ \{H_{m+1}, x\} + (1+g_1)\{H_m, x\} + \ldots + (1 + g_1 + \ldots + g_{m-1})\{H_2, x\}, \\ \dot{y} = &\ \{H_{m+1}, y\} + (1+g_1)\{H_m, y\} + \ldots + (1 + g_1 + \ldots + g_{m-1})\{H_2, y\}, \end{aligned}$$

and its first integral can be written $H = \tau_1 H_{m+1} + \tau_2 H_m + \ldots + \tau_m H_2$*, where g_j is a convenient polynomial of degree j for $j = 1, \ldots, m-1$ and $\tau_j = \tau_j(x, y)$ for $j = 1, \ldots, m+1$ are convenient analytic functions in a neighborhood of the origin.*

We observe that the inverse Problem 5 was solve for polynomial vector fields with homogenous nonlinearities in [34].

1.2 Preliminary concepts and results

In the proofs of the results that we provide in this section it plays an important role the following results.

Proposition 20. *The next relation holds* $\int_0^{2\pi} \{H_2, \Psi\}|_{x=\cos t,\, y=\sin t}\, dt = 0$ *for arbitrary C^1 function $\Psi = \Psi(x, y)$ defined in the interval* $[0, 2\pi]$.

Proof. Indeed, if we change $x = \cos t$, $y = \sin t$ then it is easy to show that

$$\{H_2, \Psi\}|_{x=\cos t,\, y=\sin t} = x\frac{\partial \Psi}{\partial y} - y\frac{\partial \Psi}{\partial x}\bigg|_{x=\cos t,\, y=\sin t} = \frac{d\Psi(\cos t,\, \sin t)}{dt}.$$

Hence, $\displaystyle\int_0^{2\pi} \{H_2, \Psi\}|_{x=\cos t,\, y=\sin t}\, dt = \Psi(\cos t, \sin t)|_{t=0}^{t=2\pi} = 0.$ □

The following result is due to Liapunov (see Theorem 1, page 276 of [30]).

Theorem 21. *If all the roots $\lambda_1, \ldots, \lambda_n$ of the equation*

$$\begin{vmatrix} p_{11}-\lambda & p_{21} & \ldots & p_{n1} \\ p_{12} & p_{22}-\lambda & \ldots & p_{n2} \\ \ldots & \ldots & \ldots & \ldots \\ p_{1n} & p_{2n} & \ldots & p_{nn}-\lambda \end{vmatrix}$$

are such that the relation $\lambda = m_1\lambda_1 + \ldots + m_n\lambda_n$, is not vanishing for an arbitrary non-negative integers $m_1, \ldots, m_n$ linked by the expression $m = m_1 + \ldots + m_n \neq 0$. Then for an arbitrary given homogenous polynomial $U = U(x_1, \ldots, x_n)$ of degree m there exists a unique homogenous polynomial $V = V(x_1, \ldots, x_n)$ of degree m which is a solution of the equation

$$\sum_{j=1}^{n} (p_{j1}x_1 + \ldots + p_{jn}x_n)\, \frac{\partial V}{\partial x_j} = U.$$

In particular, for $n = 2$ the partial differential equation

$$x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x} := \{H_2, V\} = U, \tag{1.2}$$

has a unique solution V if and only if

$$\lambda_1 m_1 + \lambda_2 m_2 = i(m_1 - m_2) \neq 0 \quad \textit{with} \quad m = m_1 + m_2 \neq 0.$$

As a simple consequence of Theorem 21 we have the next result.

Corollary 22. *Let $U = U(x, y)$ be a homogenous polynomial of degree m. The linear partial differential equation (1.2) has a unique homogenous polynomial solution V of degree m if m is odd; and if V is a homogenous polynomial solution when m is even then any other homogenous polynomial solution is of the form $V + c(x^2 + y^2)^{m/2}$ with $c \in \mathbb{R}$. Moreover, for m even these solutions exist if and only if* $\displaystyle\int_0^{2\pi} U(x, y)|_{x=\cos t, y=\sin t}\, dt = 0.$

In what follows some examples of planar vector fields having a center are studied.

1.2.1 Hamiltonian system

When system (3) is Hamiltonian, i.e. there exists a function $F = F(x, y)$ such that

$$-y + X(x, y) = -\frac{\partial F(x, y)}{\partial y}, \quad x + Y(x, y) = \frac{\partial F(x, y)}{\partial x}.$$

Hence $F = \frac{1}{2}(x^2 + y^2) + h.o.t.$ is a first integral.

1.2.2 Reversible system

Besides Hamiltonian systems there is another class of systems (3) for which the origin is a center, namely the reversible systems satisfying the following definition.

We say that system (3) is *reversible with respect to the straight line* l through the origin if it is invariant with respect to reversion about l and a reversion of time t (see for instance [16]).

The following criterion goes back to Poincaré see for instance [48], p.122.

Theorem 23. *The origin of system* (3) *is a center if the system is reversible.*

In particular this theorem is applied for the case when (3) is invariant under the transformations $(x, y, t) \longrightarrow (-x, y, -t)$ or $(x, y, t) \longrightarrow (x, -y, -t)$.

1.2.3 Weak condition for a center

The weak condition for a center was due to Alwash and Lloyd [3, 34].

Proposition 24. *The origin is a center of a polynomial differential system of the form*

$$\dot{x} = -y + X_m, \quad \dot{y} = x + Y_m, \tag{1.3}$$

where X_m and Y_m are homogenous polynomial of degree m, if there exists $\mu \in \mathbb{R}$ such that

$$(x^2 + y^2)\left(\frac{\partial X_m}{\partial x} + \frac{\partial Y_m}{\partial y}\right) = \mu\,(xX_m + yY_m)\,,$$

and either $m = 2k$ is even; or $m = 2k - 1$ is odd and $\mu \neq 2k$; or $m = 2k - 1$ is odd, $\mu = 2k$ and

$$\int_0^{2\pi} \left(\frac{\partial X_m}{\partial x} + \frac{\partial Y_m}{\partial y}\right)|_{x=\cos t,\, y=\sin t} dt = 0.$$

In [18] the author proved that if $\mu = 2m$ then system (1.3) has the rational first integral

$$\frac{x^2 + y^2 - 2\,(xY_m - yX_m)}{(x^2 + y^2)^m}.$$

1.2.4 Cauchy-Riemann condition for a center

Another particular case of differential systems with a center are the systems satisfing the Cauchy–Riemann conditions (see for instance [16]).

Proposition 25. *(Cauchy-Riemann condition for a center) Let O be a center of* (2)*. Then O is isochronous center if P and Q satisfy the Cauchy-Riemann equations*

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x} \tag{1.4}$$

A center of system (3) for which (1.4) holds is called a *holomorphic center,* which is also an isochronous center, see for more details [41] and [46]. We recall that a center of system (3) located at the origin is an *isochronous center* if all the periodic solutions in a neighborhood of the origin have the same period.

1.3 The Proofs of Theorem 7 and 8

Proof of Theorem 7 . First we prove the "only if part". Assume that the analytic differential system (6) has a Poincaré-Liapunov local first integral. Then we shall see that it can be written as (13), where $1+\sum_{j=1}^{\infty} g_j$ is the Reeb inverse integrating factor.

Consider a general analytic vector field with a singular point at the origin. Then it can be written as $\mathcal{X} = \left(\sum_{j=1}^{\infty} X_j(x,y)\right)\frac{\partial}{\partial x} + \left(\sum_{j=1}^{\infty} Y_j(x,y)\right)\frac{\partial}{\partial y}$, where X_j and Y_j for $j = 0, 1, \ldots$ are homogenous polynomials of degree j. Since the analytic first integral H (5) starts with $H_2 = (x^2 + y^2)/2$, without loss of generality this implies that $X_1(x,y) = -y$ and $Y_1(x,y) = x$. Hence the following infinite number of equations must be satisfied

$$\begin{aligned}
0 = \quad & \frac{dH}{dt} = \left(x + \frac{\partial H_3}{\partial x} + \ldots\right)(-y + X_2 + X_3 + \ldots) \\
& + \left(y + \frac{\partial H_3}{\partial y} + \ldots\right)(x + Y_2 + Y_3 + \ldots) \\
= \quad & xX_2 + yY_2 + \{H_2, H_3\} \\
+ \quad & +xX_3 + yY_3 + \frac{\partial H_3}{\partial x}X_2 + \frac{\partial H_3}{\partial y}Y_2 + \{H_2, H_4\} \\
& +xX_4 + yY_4 + \frac{\partial H_3}{\partial x}X_3 + \frac{\partial H_3}{\partial y}Y_3 + \frac{\partial H_4}{\partial x}X_2 + \frac{\partial H_4}{\partial y}Y_2 + \{H_2, H_5\} + \ldots \\
& \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
& +xX_n + yY_n + \frac{\partial H_3}{\partial x}X_{n-1} + \frac{\partial H_3}{\partial y}Y_{n-1} + \ldots + \frac{\partial H_n}{\partial x}X_2 + \frac{\partial H_n}{\partial y}Y_2 + \{H_2, H_{n+1}\} \\
& \vdots \qquad \vdots \qquad \vdots \qquad \vdots
\end{aligned}$$

Consequently

$$\begin{aligned}
xX_2 + yY_2 + \{H_2, H_3\} =& \quad 0, \\
xX_3 + yY_3 + \frac{\partial H_3}{\partial x}X_2 + \frac{\partial H_3}{\partial y}Y_2 + \{H_2, H_4\} =& \quad 0, \\
xX_4 + yY_4 + \frac{\partial H_3}{\partial x}X_3 + \frac{\partial H_3}{\partial y}Y_3 + \frac{\partial H_4}{\partial x}X_2 + \frac{\partial H_4}{\partial y}Y_2 + \{H_2, H_5\} =& \quad 0, \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad & \quad \vdots \\
xX_n + yY_n + \frac{\partial H_3}{\partial x}X_{n-1} + \ldots + \frac{\partial H_n}{\partial x}X_2 + \frac{\partial H_n}{\partial y}Y_2 + \{H_2, H_{n+1}\} =& \quad 0, \\
xX_{n+1} + yY_{n+1} + \frac{\partial H_3}{\partial x}X_n + \ldots + \frac{\partial H_{n+1}}{\partial x}X_2 + \frac{\partial H_{n+1}}{\partial y}Y_2 + \{H_2, H_{n+2}\} =& \quad 0, \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad & \quad \vdots
\end{aligned} \tag{1.5}$$

First, we introduce the notations

$$\mathcal{X}_j(*) = \{H_{j+1}, *\} + g_1\{H_j, *\} + \ldots + g_{j-1}\{H_2, *\},$$

where $j \geq 2$ and $g_j = g_j(x, y)$ is a homogenous polynomial in the variables x and y of degree j.

The first equation of (1.5) can be rewritten as follows $x\left(X_2 + \frac{\partial H_3}{\partial y}\right) +$

$y\left(Y_2 - \dfrac{\partial H_3}{\partial x}\right) = 0$. Solving it with respect to X_2 and Y_2 we obtain

$$\begin{aligned} X_2 = & -\frac{\partial H_3}{\partial y} - yg_1 = \{H_3, x\} + g_1\{H_2, x\} := \mathcal{X}_2(x), \\ Y_2 = & \frac{\partial H_3}{\partial x} + xg_1 = \{H_3, y\} + g_1\{H_2, y\} := \mathcal{X}_2(y), \end{aligned}$$

where $g_1 = g_1(x, y)$ is an arbitrary homogenous polynomial of degree one. By substituting these polynomials into the second equation of (1.5) we get

$$x\left(X_3 + \frac{\partial H_4}{\partial y} + g_1\frac{\partial H_3}{\partial y}\right) + y\left(Y_3 - \frac{\partial H_4}{\partial x} - g_1\frac{\partial H_3}{\partial x}\right) = 0.$$

By solving this equation with respect to X_3 and Y_3 we have

$$\begin{aligned} X_3 = & -\frac{\partial H_4}{\partial y} - g_1\frac{\partial H_3}{\partial y} - yg_2 = \{H_4, x\} + g_1\{H_3, x\} + g_2\{H_2,x\} := \mathcal{X}_3(x), \\ Y_3 = & \frac{\partial H_4}{\partial x} + g_1\frac{\partial H_3}{\partial x} + xg_2 = \{H_4, y\} + g_1\{H_3, y\} + g_2\{H_2, y\} := \mathcal{X}_3(y), \end{aligned}$$

where $g_2 = g_2(x, y)$ is an arbitrary homogenous polynomial of degree two. By continuing this process we obtain $X_4, Y_4, \ldots, X_n, Y_n$, i.e.

$$\begin{aligned} X_n = & \{H_{n+1}, x\} + g_1\{H_n, x\} + \ldots + g_{n-1}\{H_2, x\} := \mathcal{X}_n(x), \\ Y_n = & \{H_{n+1}, y\} + g_1\{H_n, y\} + \ldots + gn - 1\{H_2, y\} := \mathcal{X}_n(y), \end{aligned} \tag{1.6}$$

where $g_n = g_n(x, y)$ is an arbitrary homogenous polynomial of degree n. Hence, since $\sum_{j=1}^{\infty} g_j$ converges in a neighborhood of the origin, we get that

$$\begin{aligned} \dot{x} = & -y + X_2 + X_3 + \ldots + X_j + \ldots = -y + \mathcal{X}(x) = -y + \sum_{j=2}^{\infty} \mathcal{X}_j(x) \\ = & \left(1 + \sum_{j=1}^{\infty} g_j\right) \{H, x\}, \\ \dot{y} = & x + Y_2 + Y_3 + \ldots + Y_j + \ldots = x + \mathcal{X}(y) = x + \sum_{j=2}^{\infty} \mathcal{X}_j(y) \\ = & \left(1 + \sum_{j=1}^{\infty} g_j\right) \{H, y\}. \end{aligned}$$

Note that the function $1+\sum_{j=1}^{\infty} g_j$ is an analytic integrating factor of the differential system (13) i.e. it is a Reeb inverse integrating factor. Thus the "only if part" is proved.

Now we prove the " if " part. We assume that system (4) has a Reeb inverse integrating factor. From the equation (12), i.e.

$$\begin{aligned}&(X_1+X_2+X_3+\ldots)\left(\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial x}+\frac{\partial V_3}{\partial x}+\ldots\right)\\&+(Y_1+Y_2+Y_3+\ldots)\left(\frac{\partial V_1}{\partial y}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial y}+\ldots\right)\\=\;&(1+V_1+V_2+\ldots)\left(\frac{\partial X_1}{\partial x}+\frac{\partial Y_1}{\partial y}+\frac{\partial X_2}{\partial x}+\frac{\partial Y_2}{\partial y}+\frac{\partial X_3}{\partial x}+\frac{\partial Y_3}{\partial y}+\ldots\right)\end{aligned}$$

if follows that

$$\begin{aligned}0 &= \frac{\partial X_1}{\partial x}+\frac{\partial Y_1}{\partial y},\\ Y_1\frac{\partial V_1}{\partial y}+X_1\frac{\partial V_1}{\partial x} &= \frac{\partial X_2}{\partial x}+\frac{\partial Y_2}{\partial y},\\ Y_1\frac{\partial V_2}{\partial y}+X_1\frac{\partial V_2}{\partial x}+X_2\frac{\partial V_1}{\partial x}+Y_2\frac{\partial V_1}{\partial y} &= \frac{\partial X_3}{\partial x}+\frac{\partial Y_3}{\partial y}+V_1\left(\frac{\partial X_2}{\partial x}+\frac{\partial Y_2}{\partial y}\right),\\ \vdots\qquad\quad \vdots\qquad\quad \vdots\qquad\quad &\vdots\qquad\quad \vdots\qquad\quad \vdots\end{aligned}\tag{1.7}$$

From the first equation of (1.7) we get that $X_1=-\dfrac{\partial F_2}{\partial y}$, $\quad Y_1=\dfrac{\partial F_2}{\partial x}$, where $F_2=F_2(x,y)$ is an arbitrary homogenous polynomial of degree 2. From the second equation of (1.7) we obtain

$$\frac{\partial}{\partial x}\left(X_2+V_1\frac{\partial F_2}{\partial y}\right)+\frac{\partial}{\partial y}\left(Y_2-V_1\frac{\partial F_2}{\partial x}\right)=0,$$

hence

$$X_2=-\frac{\partial F_3}{\partial y}-V_1\frac{\partial F_2}{\partial y},\qquad Y_2=\frac{\partial F_3}{\partial x}+V_1\frac{\partial F_2}{\partial x},$$

where $F_3=F_3(x,y)$ is an arbitrary homogenous polynomial of degree 3. From the third equation of (1.7) we obtain

$$\frac{\partial}{\partial x}\left(X_3+V_1\frac{\partial F_3}{\partial x}+V_2\frac{\partial F_2}{\partial y}\right)+\frac{\partial}{\partial y}\left(Y_3-V_1\frac{\partial F_3}{\partial y}-V_2\frac{\partial F_2}{\partial x}\right)=0,$$

thus

$$X_3=-\frac{\partial F_4}{\partial y}-V_1\frac{\partial F_3}{\partial y}-V_2\frac{\partial F_2}{\partial y},\quad Y_3=\frac{\partial F_4}{\partial x}+V_1\frac{\partial F_3}{\partial x}+V_2\frac{\partial F_2}{\partial x},$$

where $F_4 = F_4 x, y)$ is an arbitrary homogenous polynomial of degree 4. By continuing this process we get

$$\{F_j, V_1\} + \ldots + \{F_2, V_{j-1}\} = \frac{\partial X_j}{\partial x} + \frac{\partial Y_j}{\partial y},$$

by considering that $\{F_k, V_n\} = \frac{\partial}{\partial y}\left(V_n \frac{\partial F_j}{\partial x}\right) - \frac{\partial}{\partial x}\left(V_n \frac{\partial F_j}{\partial y}\right)$ we deduce the relation

$$\frac{\partial}{\partial x}\left(X_j + V_1 \frac{\partial F_j}{\partial y} + \ldots + \ldots + V_{j-1} \frac{\partial F_2}{\partial y}\right) + \frac{\partial}{\partial y}\left(X_j - V_1 \frac{\partial F_j}{\partial x} - - \ldots - V_{j-1} \frac{\partial F_2}{\partial x}\right) = 0.$$

By using the notation $\tilde{\mathcal{X}}_j(*) = \{F_{j+1}, *\} + V_1\{F_j, *\} + \ldots + V_{j-1}\{F_2, *\}$, where $j \geq 2$, $V_j = V_j(x, y)$ and $F_j = F_j(x, y)$ are homogenous polynomial in the variables x and y of degree j,we get that

$$\begin{aligned}
\tilde{\mathcal{X}}_1(x) = \{F_2, x\} = & \quad X_1, \\
\tilde{\mathcal{X}}_1(y) = \{F_2, y\} = & \quad Y_1, \\
\tilde{\mathcal{X}}_2(x) = \{F_3, x\} + V_1\{F_2, x\} = & \quad X_2, \\
\tilde{\mathcal{X}}_2(y) = \{F_3, y\} + V_1\{F_2, y\} = & \quad Y_2, \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots & \qquad \vdots \\
\tilde{\mathcal{X}}_m(x) = \{F_{m+1}, x\} + V_1\{F_m, x\} + \ldots + V_{m-1}\{F_2, x\} = & \quad X_m, \\
\tilde{\mathcal{X}}_m(y) = \{F_{m+1}, y\} + V_1\{F_m, y\} + \ldots + V_{m-1}\{F_2, y\} = & \quad Y_m,
\end{aligned} \tag{1.8}$$

$$\begin{aligned}
\tilde{\mathcal{X}}_{m+k-1}(x) = \{F_{m+k}, x\} + V_1\{F_{m-1+k}, x\} + \ldots + V_{m+k-2}\{F_2, x\} = & \quad X_{m+k-1}, \\
\tilde{\mathcal{X}}_{m+k-1}(y) = \{F_{m+k}, y\} + V_1\{F_{m-1+k}, y\} + \ldots + V_{m+k-2}\{F_2, y\} = & \quad Y_{m+k-1},
\end{aligned} \tag{1.9}$$

where $k > 1$ and $F_k = F_k(x, y)$ is an arbitrary homogenous polynomial of degree j, for $j \geq 3$, such that the series $F = \sum_{j=2}^{\infty} F_j$, converges at neighborhood of the origin. By considering that we are interesting in studying the linear type center then $X_1 = -y$ and $X_1 = x$ then we have that $F_2 = (x^2 + y^2)/2$. Therefore

$$F = (x^2 + y^2)/2 + F_3 + F_4 + \ldots,$$

is a Poincaré-Liapunov local first integral this prove the " if " part of the theorem.

Moreover, by summing we get

$$\begin{aligned}\dot{x} = &\ -y + \sum_{j=2}^{\infty} X_j = \sum_{j=2}^{\infty} (\{F_{j+1}, x\} + V_1\{F_j, x\} + \ldots + V_{j-1}\{F_2, x\}) \\ = &\ \left(1 + \sum_{j=2}^{\infty} V_j\right) \{F, x\} \\ \dot{y} = &\ x + \sum_{j=2}^{\infty} Y_j = \sum_{j=2}^{\infty} (\{F_{j+1}, y\} + V_1\{F_j, y\} + \ldots + V_{j-1}\{F_2, y\}) \\ = &\ \left(1 + \sum_{j=2}^{\infty} V_j\right) \{F, y\},\end{aligned}$$

Thus the proof of the theorem follows. □

Proof of Corollary 18. Follows trivially from the proof of Theorem 7 (see statement (i) and (ii)). □

Remark 26. *(i) From the "only if" part follows that the arbitrariness which we determine the vector fields with the given Poincaré-Liapunov local first integral is related with the Reeb's inverse integrating factor* $V = 1 + \sum_{j=2}^{\infty} g_j$ *and from the " if" part follows that the arbitrariness which we determine the vector fields with the given Reeb's inverse integrating factor is related with the Poincaré-Liapunov local first integral* $F = (x^2 + y^2)/2 + F_3 + F_4 + \ldots$.

(ii) From the equation

$$\left(1 + \sum_{j=2}^{\infty} g_j\right) \{H, y\} = -y + X, \quad \left(1 + \sum_{j=2}^{\infty} g_j\right) \{H, x\} = x + Y, \tag{1.10}$$

where $X = X_2 + X_3 + \ldots$ *and* $Y = Y_2 + Y_3 + \ldots$, X_j *and* Y_j *are given polynomial of degree* j, *for* $j \geq 2$, *we shall determine the function* H.

By introducing the homogenous polynomial

$$T_1 = g_1 \qquad T_k = g_k - \sum_{j=1}^{k-1} g_j T_{k-j}, \quad \textit{for} \quad k \geq 1,$$

$$Z_j = xY_{j-1} - yX_{j-1}, \qquad X_1 = -y, \qquad Y_1 = x \qquad \textit{for} \quad j \geq 1.$$

After some computations we can prove that the following Taylor expansion holds at the neighborhood of the origin

$$\frac{1}{1+\sum_{j=2}^{\infty} g_j} = 1-\sum_{j=1}^{\infty} T_j,$$

$$x\frac{\partial H}{\partial x}+y\frac{\partial H}{\partial y} = 2H_2+3H_3+\ldots+nH_n+\ldots.$$

Hence from (1.10) *we get that*

$$2H_2+3H_3+\ldots+nH_n+\ldots = \frac{1}{1+\sum_{j=2}^{\infty} g_j}(xY-yX)$$

$$= \left(1-\sum_{j=1}^{\infty} T_j\right)(x(Y_1+Y_2+y_3+\ldots)-y(X_1+X_2+X_3+\ldots))$$

$$= \left(1-\sum_{j=1}^{\infty} T_j\right)(Z_2+Z_3+\ldots+Z_n+\ldots).$$

Thus

$$H_2=\frac{Z_2}{2},\quad H_3=\frac{Z_3-Z_2T_1}{3},\quad H_n=\frac{Z_n-\sum_{j=2}^{n-1} Z_jT_{n-j}}{n},$$

Consequently

$$\begin{aligned} H = & \sum_{n=2}^{\infty} H_j = \sum_{n=2}^{\infty}\left(\frac{Z_n-\sum_{j=2}^{n-1} Z_jT_{n-j}}{n}\right) = \sum_{n=2}^{\infty}\frac{Z_n}{n} - Z_2\sum_{n=3}^{\infty}\frac{T_{n-2}}{n} - Z_3\sum_{n=4}^{\infty}\frac{T_{n-2}}{n} - \ldots \\ = & \sum_{n=2}^{\infty}\frac{Z_n}{n} - Z_2\sum_{n=1}^{\infty}\frac{T_n}{n+2} - Z_3\sum_{n=1}^{\infty}\frac{T_n}{n+3} - \ldots \\ = & \sum_{n=2}^{\infty}\frac{Z_n}{n} - \sum_{n=2}^{\infty} Z_n\sum_{k=1}^{\infty}\frac{T_k}{k+n}\ldots \\ = & \sum_{n=2}^{\infty} Z_n\left(\frac{1}{n}-\sum_{k=1}^{\infty}\frac{T_k}{k+n}\right) = \sum_{n=1}^{\infty} Z_{n+1}\left(\frac{1}{n+1}-\sum_{k=1}^{\infty}\frac{T_k}{k+n+1}\right). \end{aligned}$$

Thus we obtain the expression of H thought the components of the given analytic vector fields and the Reeb inverse integrating factor

$$H = \sum_{n=1}^{\infty} (xY_n - yX_n) \left(\frac{1}{n+1} - \sum_{k=1}^{\infty} \frac{T_k}{k+n+1} \right). \tag{1.11}$$

In particular if the vector field $\mathcal{X}$ is polynomial of degree m then (1.11) *becomes*

$$H = \sum_{n=1}^{m} (xY_n - yX_n) \left(\frac{1}{n+1} - \sum_{k=1}^{\infty} \frac{T_k}{k+n+1} \right).$$

Clearly that in view of the inequality

$$\left| \sum_{k=1}^{\infty} \frac{T_k}{k+n+1} \right| \leq \left| \sum_{k=1}^{\infty} T_k \right|,$$

we obtain that the function $\displaystyle\sum_{k=1}^{\infty} \frac{T_k}{k+n+1}$ *is an anlytic function in the neighborhood of the origin.*

Proof of Theorem 8. Now we assume that the vector field $\mathcal{X}$ is polynomial of degree m. First we prove the "only if" part. From (1.6) it follows that if $X_n = Y_n = 0$ for $n > m$, then

$$\begin{aligned}
\dot{x} = & \; -y + \sum_{j=2}^{\infty} \mathcal{X}_j(x) = -y + \sum_{j=2}^{m} \mathcal{X}_j(x) \\
= & \; -y + \{H_{m+1}, x\} + (1+g_1)\{H_m, x\} + \ldots + (1 + g_1 + \ldots + g_{m-1})\{H_2, x\}, \\
\dot{y} = & \; x + \sum_{j=2}^{\infty} \mathcal{X}_j(y) = x + \sum_{j=2}^{m} \mathcal{X}_j(y) \\
= & \; x + \{H_{m+1}, y\} + (1+g_1)\{H_m, y\} + \ldots + (1 + g_1 + \ldots + g_{m-1})\{H_2, y\}.
\end{aligned} \tag{1.12}$$

Clearly, if $X_n = Y_n = 0$ for $n \geq m+1$, then

$$\begin{aligned}
X_{m+1} = &\ \{H_{m+2}, x\} + g_1\{H_{m+1}, x\} + \ldots + g_m\{H_2, x\} = 0, \\
Y_{m+1} = &\ \{H_{m+2}, y\} + g_1\{H_{m+1}, y\} + \ldots + g_m\{H_2, y\} = 0,, \\
X_{m+2} = &\ \{H_{m+3}, x\} + g_1\{H_{m+2}, x\} + \ldots + g_{m+1}\{H_2, x\} = 0, \\
Y_{m+2} = &\ \{H_{m+3}, y\} + g_1\{H_{m+2}, y\} + \ldots + g_{m+1}\{H_2, y\} = 0,, \\
X_{m+3} = &\ \{H_{m+4}, x\} + g_1\{H_{m+3}, x\} + \ldots + g_{m+2}\{H_2, x\} = 0, \\
Y_{m+3} = &\ \{H_{m+4}, y\} + g_1\{H_{m+3}, y\} + \ldots + g_{m+2}\{H_2, y\} = 0, \\
\vdots \quad & \quad \vdots \quad \vdots \quad \vdots \qquad , \\
X_{m+k} = &\ \{H_{m+k+1}, x\} + g_1\{H_{m+k}, x\} + \ldots + g_{m+k-1}\{H_2, x\} = 0, \\
Y_{m+k} = &\ \{H_{m+k+1}, y\} + g_1\{H_{m+k}, y\} + \ldots + g_{m+k-1}\{H_2, y\} = 0 \quad \text{for} \quad k \geq 4.
\end{aligned} \tag{1.13}$$

this infinity systems of first order partial differential equations can be rewritten as follows

$$\begin{aligned}
X_{m+1} = &\ \{H_{m+2}, x\} + g_1\{H_{m+1}, x\} + \ldots + g_m\{H_2, x\} = 0, \\
Y_{m+1} = &\ \{H_{m+2}, y\} + g_1\{H_{m+1}, y\} + \ldots + g_m\{H_2, y\} = 0,, \\
X_{m+2} = &\ \{H_{m+3}, x\} + (g_2 - g_1^2)\{H_{m+1}, x\} + \ldots + (g_{m+1} - g_1 g_m)\{H_2, x\} = 0, \\
Y_{m+2} = &\ \{H_{m+3}, y\} + (g_2 - g_1^2)\{H_{m+1}, y\} + \ldots + (g_{m+1} - g_1 g_m)\{H_2, y\} = 0, \\
X_{m+3} = &\ \{H_{m+4}, x\} + (g_1^3 - 2g_1 g_2 + g_3)\{H_{m+1}, x\} + \ldots \\
&+ (g_{m+2} + g_1^2 g_m - g_2 g_m - g_1 g_{m+1})\{H_2, x\} = 0, \\
Y_{m+3} = &\ \{H_{m+4}, y\} + (g_1^3 - 2g_1 g_2 + g_3)\{H_{m+1}, y\} + \ldots \\
&+ (g_{m+2} + g_1^2 g_m - g_2 g_m - g_1 g_{m+1})\{H_2, y\} = 0, \\
\vdots \qquad & \quad \vdots \qquad \vdots \qquad \vdots \qquad , \\
X_{m+k} = &\ \{H_{m+k+1}, x\} + G_{k\,m+1}\{H_{m+1}, x\} + \ldots + G_{k\,2}\{H_2, x\} = 0, \\
Y_{m+k} = &\ \{H_{m+k+1}, y\} + G_{k\,m+1}\{H_{m+1}, y\} + \ldots + G_{k\,2}\{H_2, y\} = 0, \quad \text{for} \quad k \geq 4.
\end{aligned} \tag{1.14}$$

where

$$G_{k\,j} = -\sum_{n=1}^{k-1} g_n G_{k-n\,j} + g_{m+k+1-j} \qquad \text{for} \quad j = 1, \ldots, m+1, \quad k \geq 1 \tag{1.15}$$

are homogenous polynomial of degree $m+k+1-j$. This system of partial differential equations of first order is compatible if and only if the following relations hold

$$\{H_{m+k}, g_1\} + \{H_{m+k-1}, g_2\} + \ldots + \{H_2, g_{m+k-1}\} = 0, \tag{1.16}$$

for $k \geq 1$, or equivalently

$$\{H_{m+1},\, G_{k\,m+1}\}+\{H_m,\, G_{k\,m}\}+\ldots+\{H_3,\, G_{k\,3}\}+\{H_2,\, G_{k\,2}\}=0 \qquad \text{for} \quad k \geq 1. \tag{1.17}$$

Hence, in view of Corollary 22 we get that (1.16) has solution if and only if

$$\int_0^{2\pi} \Big(\{H_{m+k}, g_1\} + \{H_{m+k-1}, g_2\} + \ldots + \{H_3, g_{m+k-2}\} \Big)\Big|_{x=\cos t,\, y=\sin t} dt = 0.$$

or equivalently

$$\int_0^{2\pi} \Big(\{H_{m+1},\, G_{k\,m+1}\} + \{H_m,\, G_{k\,m}\} + \ldots + \{H_3,\, G_{k\,3}\} \Big)\Big|_{x=\cos t,\, y=\sin t} dt = 0.$$

Clearly that these conditions always hold if $m+k-1$ is odd, and consequently in this case there exist a unique homogenous polynomial of degree $m+k-1$ We shall study partial differential equations (1.14) under the conditions (1.16).

For $k=1$ from (1.14) we get

$$\begin{aligned} \frac{\partial H_{m+2}}{\partial x} &= -g_1 \frac{\partial H_{m+1}}{\partial x} - g_2 \frac{\partial H_m}{\partial x} - \ldots - g_m \frac{\partial H_2}{\partial x}, \\ \frac{\partial H_{m+2}}{\partial y} &= -g_1 \frac{\partial H_{m+1}}{\partial y} - g_2 \frac{\partial H_m}{\partial y} - \ldots - g_m \frac{\partial H_2}{\partial y}, \end{aligned} \tag{1.18}$$

where $g_m = g_m(x,y)$ is unknown homogenous polynomial of degree m. From the condition

$$\frac{\partial^2 H_{m+2}}{\partial y \partial x} = \frac{\partial^2 H_{m+2}}{\partial x \partial y}$$

we obtain that g_m must satisfy the following first order partial differential equations (see first equation of (1.16))

$$\{H_{m+1}, g_1\} + \{H_m, g_2\} + \ldots + \{H_2, g_m\} = 0.$$

Hence the two first partial differential system (1.14) are compatible, consequently from (1.18) and using that H_j are homogenous polynomials of degree j, for $j = m+2, \ldots, 2$ we get that

$$H_{m+2} = -\frac{1}{m+2}\left((m+1)g_1 H_{m+1} + \ldots + 3g_{m-1}H_3 + 2g_m H_2\right).$$

For $k=2$ system (1.14) becomes

$$\begin{aligned} \{H_{m+3}, x\} + g_1\{H_{m+2}, x\} + \ldots + g_m\{H_3, x\} + g_{m+1}\{H_2, x\} &= 0, \\ \{H_{m+3}, y\} + g_1\{H_{m+2}, y\} + \ldots + g_m\{H_3, y\} + g_{m+1}\{H_2, y\} &= 0, \end{aligned} \tag{1.19}$$

which in view of (1.18) system (1.19) can be written as

$$
\begin{aligned}
&\{H_{m+3},x\}+\left(g_1^2-g_2\right)\{H_{m+1},x\}+\left(g_1g_2-g_3\right)\{H_m,x\}\\
&+\ldots+(g_1g_{m-1}-g_m)\{H_3,x\}+(g_1g_m+g_{m+1})\{H_2,x\}= \ 0,\\
&\{H_{m+3},y\}+\left(g_1^2-g_2\right)\{H_{m+1},y\}+\left(g_1g_2-g_3\right)\{H_m,y\}\\
&+\ldots+(g_1g_{m-1}-g_m)\{H_3,y\}+(g_1g_m+g_{m+1})\{H_2,y\}= \ 0,
\end{aligned}
\tag{1.20}
$$

where $g_{m+1}=g_{m+1}(x,y)$ it is an unknown homogenous polynomial of degree $m+1$ which must satisfy the first order partial differential equation

$$
\begin{aligned}
&\{\left(g_1^2-g_2\right),H_{m+1}\}+\{\left(g_1g_2-g_3\right),H_m\}\\
&+\ldots+\{(g_1g_{m-1}-g_m),H_3\}+\{\left(g_1g_m-g_{m+1}\right),H_2\}=0.
\end{aligned}
$$

Hence, in view of Proposition 22 we get that

$$
\begin{aligned}
\int_0^{2\pi}&\Big(\{\left(g_1^2-g_2\right),H_{m+1}\}+\{\left(g_1g_2-g_3\right),H_m\}\\
&+\ldots+\{(g_1g_{m-1}-g_m),H_3\}\Big)\Big|_{x=\cos t,\,y=\sin t}dt=0.
\end{aligned}
$$

On the other hand, from (1.20) and in view of the fact that H_j are homogenous polynomial of degree j for $j=m+3,\ldots,2$ we get that

$$
\begin{aligned}
H_{m+3}= \ &-\frac{1}{m+3}\Big((m+1)\left(g_1^2-g_2\right)H_{m+1}-m\left(g_1g_2-g_3\right)H_m\\
&+\ldots+3(g_1g_{m-1}-g_m)H_3-2\left(g_1g_m-+g_{m+1}\right)H_2\Big).
\end{aligned}
$$

For $k=3$ system (1.14) becomes

$$
\begin{aligned}
\{H_{m+4},x\}+g_1\{H_{m+3},x\}+\ldots+g_m\{H_4,x\}+g_{m+1}\{H_3,x\}+g_{m+2}\{H_2,x\}= \ &0,\\
\{H_{m+4},x\}+g_1\{H_{m+3},y\}+\ldots+g_m\{H_4,y\}+g_{m+1}\{H_3,y\}+g_{m+2}\{H_2,y\}= \ &0,
\end{aligned}
$$

which in view of (1.18) system (1.19) can be written as

$$
\begin{aligned}
&\{H_{m+4},x\}+\left(-g_1^3+2g_1g_2-g_3\right)\{H_{m+1},x\}\\
&+\left(-g_1^2g_2+g_3g_1+g_2^2-g_4\right)\{H_m,x\}+\ldots+\left(g_1g_{m+1}-g_1^2g_{m-1}-g_{m+2}\right)\{H_2,x\}=0,\\
&\{H_{m+4},y\}+\left(-g_1^3+2g_1g_2-g_3\right)\{H_{m+1},y\}\\
&+\left(-g_1^2g_2+g_3g_1+g_2^2-g_4\right)\{H_m,y\}+\ldots+\left(g_1g_{m+1}-g_1^2g_{m-1}-g_{m+2}\right)\{H_2,y\}=0,
\end{aligned}
\tag{1.21}
$$

where $g_{m+2}=g_{m+2}(x,y)$ it is an unknown homogenous polynomial of degree $m+2$ which must satisfy the first order partial differential equation

$$
\begin{aligned}
0= \ &\{-g_1^3+2g_1g_2-g_3,\,H_{m+1}\}+\{-g_1^2g_2+g_3g_1+g_2^2-g_4,\,H_m\}\\
&+\ldots+\{(g_2-g_1^2)g_m+g_1g_{m+1}-g_{m+2},\,H_2\}.
\end{aligned}
$$

$$\int_0^{2\pi} \Big(\{ (-g_1^3 + 2g_1 g_2 - g_3) \,,\, H_{m+1} \} \\ + \ldots + \{ \big((g_2 - g_1^2) g_{m-1} + g_1 g_m - g_{m+1} \big) \,,\, H_3 \} \Big) \Big|_{x=\cos t,\, y=\sin t} dt = 0.$$

Under this condition system (1.21) is compatible. Hence by using the property of homogenous polynomial we get that

$$H_{m+4} = -\frac{1}{m+4} \sum_{j=1}^{m} (m+2-j) \big(\big(g_j (g_2 - g_1^2) + g_1 g_{j+1} - g_{j+2} \big) H_{m+2-j}.$$

By continuing this process we deduce that under the condition (1.16) or equivalently (1.17) where g_{m+k-1} it is an unknown homogenous polynomial of degree $m+k-2$ which we choose as a solution of the first order partial differential equation (1.16) or equivalently (1.17). Thus after some computations we can prove that

$$H_{m+k+1} = -\frac{\displaystyle\sum_{j=2}^{m+1} j G_{k\,j} H_j}{m+k+1}$$

for $k \geq 5$, where G_{kj} is a convenient homogenous polynomial of degree $m+k+1-j$ given in formula (1.15). By inserting these polynomials in the expression for the first integral H we finally obtain

$$\begin{aligned} H = &\sum_{j=2}^{\infty} H_j = H_2 + H_3 + \ldots + H_{m+1} + \sum_{j=m+2}^{\infty} H_j \\ = &\; H_2 \left(1 - 2 \sum_{j=1}^{\infty} \frac{G_{j2}}{m+1+j} \right) + \ldots + H_{m+1} \left(1 - (m+1) \sum_{j=1}^{\infty} \frac{G_{j\,m+1}}{m+1+j} \right) \\ := &\; \tau_{m+1} H_{m+1} + \tau_m H_m + \ldots + \tau_2 H_2, \end{aligned}$$

Now we prove that from the conditions (1.15) it follows (15) with τ_s given in the formula (17).

Hence, if $\displaystyle\sum_{j=3}^{\infty} g_j$ converges in a neighborhood of the origin. We will prove the following relationship

$$\begin{aligned}
\sum_{j=1}^{\infty} G_{j\,m+1} = &\quad 1-\Psi, \\
\sum_{j=1}^{\infty} G_{j\,m} = &\quad 1-(1+g_1)\Psi, \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \\
\sum_{j=1}^{\infty} G_{j\,2} = &\quad 1-(1+g_1+g_2+\ldots+g_{m-1})\Psi,
\end{aligned} \tag{1.22}$$

We proved only the first relationship. The remaining relationship can be analogously demonstrated

Indeed, from the relations

$$\begin{aligned}
G_{1\,m+1} = &\quad g_1, \\
G_{2\,m+1} = &\quad g_2 - g_1 G_{1\,m+1}, \\
G_{3\,m+1} = &\quad g_3 - g_2 G_{1\,m+1} - g_1 G_{1\,m+1}, \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
G_{k\,m+1} = &\quad g_k - \sum_{n=1}^{k-1} g_n\, G_{k-n\;n+1} \quad \text{for} \quad k \geq 4, \\
\vdots \qquad & \qquad \vdots \qquad\qquad \vdots \qquad\qquad \ddots
\end{aligned}$$

By summing we get that

$$\sum_{k=1}^{\infty} G_{k\,m+1} = \sum_{j=1}^{\infty} g_j \left(1 - \sum_{k=1}^{\infty} G_{k\,m+1}\right),$$

thus

$$\sum_{k=1}^{\infty} G_{k\,m+1} = \frac{\sum_{j=1}^{\infty} g_j}{1+\sum_{j=1}^{\infty} g_j} = 1-\Psi,$$

By summing in (1.17) we get that

$$\begin{aligned} 0 = \ & \{H_{m+1},\, 1-\sum_{j=3}^{\infty} G_{k\,m+1}\} + \{H_m,\, 1-\sum_{j=3}^{\infty} G_{k\,m}\} + \ldots \\ & +\{H_3,\, 1-\sum_{j=3}^{\infty} G_{k\,3}\} + \{H_2,\, 1-\sum_{j=3}^{\infty} G_{k\,2}\} = \{H_{m+1},\, \Psi\} \\ & +\{H_m,\, (1+g_1)\Psi\} + \ldots + \{H_2,\, (1+g_1+\ldots+g_{m-1})\Psi\}. \end{aligned}$$

Hence in view of (1.22) we get (15). On the other hand from (17) it follows that the function τ_s for $s = 2, \ldots, m+1$ are analytic in the neighborhood of the origin. Indeed, from the inequalities

$$|\tau_s| = \left|1 - s\sum_{j=1}^{\infty} \frac{G_{j\,s}}{m+1+j}\right| \leq 1 + \left|\sum_{j=1}^{\infty} G_{j\,s}\right|,$$

hence

$$-1 - \left|\sum_{j=1}^{\infty} G_{j\,s}\right| \leq \tau_s \leq 1 + \left|\sum_{j=1}^{\infty} G_{j\,s}\right|.$$

Therefore τ_s is an analytic function in the neigborhood of the origin.

Thus we obtain that the polynomial differential system (1.12) of degree m can be written as (14) where $1 + \sum_{j=2}^{\infty} g_j$ is the Reeb inverse integrating factor and $\tau_2, \ldots, \tau_{m+1}$ are analytic functions in the neighborhood of the origin. In short the proof of the "only if part" and the statement (i) follows. This proves the "only if part" of the theorem.

Now we prove the " if" part. We assume that $V = 1 + \sum_{j=2}^{\infty} V_j$ is the Reeb inverse integrating factor. From (1.8) and (1.9) it follows that If $X_j = Y_j = 0$ for $j \geq m+1$, then

$$\begin{aligned} X_k = \ & \{F_{k+1}, x\} + V_1\{F_k, x\} + \ldots + V_{k-1}\{F_2, x\} = 0, \\ Y_k = \ & \{F_{k+1}, y\} + V_1\{F_k, y\} + \ldots + V_{k-1}\{F_2, y\} = 0, \end{aligned} \tag{1.23}$$

for $k \geq m+1$.

System of partial differential equations of first order (1.23) is compatible if and only if

$$\{V_1, F_k\} + \{V_2, F_{k-1}\} + \ldots + \{V_{k-1}, F_2\} = 0, \tag{1.24}$$

where $k \geq m+1$. The proof of statement (ii) can be obtained analogously to the proof of statement (i), if we take $g_j = V_j$ and $H_{j+1} = F_{j+1}$ for $j = 1, \ldots, m$.

Finally we observe that from (16) it follows that

$$\frac{\partial F}{\partial y} = \Psi\left(\{H_{m+1}, x\} + (1+g_1)\{H_m, x\} + \ldots + (1+g_1+\ldots+g_{m-1})\{H_2, x\}\right),$$
$$\frac{\partial F}{\partial x} = \Psi\left(\{H_{m+1}, y\} + (1+g_1)\{H_m, y\} + \ldots + (1+g_1+\ldots+g_{m-1})\{H_2, y\}\right),$$

From the condition $\dfrac{\partial^2 F}{\partial x \partial y} = \dfrac{\partial^2 F}{\partial y \partial x}$, we get the condition (15). In short the theorem is proved. □

Proof of Corollary 19. It follows easily from the proof of Theorem 8. □

Remark 27. *From the proof of Theorem 8 it follows that* (15) *is equivalent to the infinite number of first order partial differential equations*

$$\{H_{m+1}, G_{k\,m+1}\}+\{H_m, G_{k\,m}\}+\ldots+\{H_3, G_{k\,3}\}+\{H_2, G_{k\,2}\} = 0 \qquad \text{for} \quad k \geq 1, \tag{1.25}$$

where G_{kj} are the homogenous polynomial (18), *with unknowns the homogenous polynomials g_j of degree $j \geq m$. Hence by Proposition 20 we obtain the conditions*

$$\int_0^{2\pi} \Big(\{H_{m+1}, G_{k\,m+1}\} + \{H_m, G_{k\,m}\} + \ldots + \{H_3, G_{k\,3}\}\Big)\Big|_{x=\cos t,\, y=\sin t} dt = 0.$$

The first condition of (1.25) *(for $k = 1$), by Corollary 22 guarantees the existence of the solution g_m of first equation of* (1.25)*, the second condition (for $k = 2$), again by Corollary 22, guarantees the existence of the solution g_{m+1} of the second equation of* (1.25)*, and so on.*

Conditions (1.25) *and* (27) *are equivalent to the following relations .*

$$\{H_{m+j+1}, g_1\} + \{H_{m+j}, g_2\} + \ldots + \{H_3, g_{m+j-1}\} + \{H_2, g_{m+j}\} = 0,$$
$$\int_0^{2\pi} \left(\{H_{m+j+1}, g_1\} + \{H_{m+j}, g_2\} + \ldots + \{H_3, g_{m+j-1}\}\right)|_{x=\cos t, y=\sin t}\, dt = 0,$$

for $j \geq 0$.

Remark 28. *Theorem 8 can be applied to determine the Poincaré-Liapunov first integral and Reeb inverse integrating factor for the case when the polynomial differential system is given. Indeed, given a polynomial vector field $\mathcal{X}$ of degree m with a linear type center at the origin of coordinates, using* (14) *we determine its first integral H and its Reeb inverse integrating factor. Thus, if in* (3) $X = \sum_{j=2}^{m} X_j$ *and* $Y = \sum_{j=2}^{m} Y_j$ *with X_j and Y_j homogenous polynomials of degree j, from* (14) *equating the terms of the same degree we get*

$$\{H_{j+1},x\}+g_1\{H_j,x\}+\ldots+g_{j-1}\{H_2,x\}=X_j$$
$$\{H_{j+1},y\}+g_1\{H_j,y\}+\ldots+g_{j-1}\{H_2,y\}=Y_j,$$

for $j=2,\ldots,m$. *Then*

$$\begin{array}{rcl}
\dfrac{\partial H_3}{\partial y} &=& -X_2-yg_1,\\
\dfrac{\partial H_3}{\partial x} &=& Y_2-xg_1,\\
\dfrac{\partial H_4}{\partial y} &=& -X_3-g_1\dfrac{\partial H_3}{\partial y}-yg_2,\\
\dfrac{\partial H_4}{\partial x} &=& Y_3-g_1\dfrac{\partial H_3}{\partial x}-xg_2\\
\dfrac{\partial H_5}{\partial y} &=& -X_4-g_1\dfrac{\partial H_4}{\partial y}-g_2\dfrac{\partial H_3}{\partial y}-yg_3,\\
\dfrac{\partial H_5}{\partial x} &=& Y_4-g_1\dfrac{\partial H_4}{\partial x}-g_2\dfrac{\partial H_3}{\partial x}-xg_3,\\
\vdots & \vdots & \vdots \\
\dfrac{\partial H_{m+1}}{\partial y} &=& -X_m-g_1\dfrac{\partial H_m}{\partial y}\ldots-g_{m-2}\dfrac{\partial H_3}{\partial y}-g_{m-1}\dfrac{\partial H_2}{\partial y},\\
\dfrac{\partial H_{m+1}}{\partial x} &=& Y_m-g_1\dfrac{\partial H_m}{\partial x}\ldots-g_{m-2}\dfrac{\partial H_3}{\partial x}-g_{m-1}\dfrac{\partial H_2}{\partial x}\\
\dfrac{\partial H_{m+k+1}}{\partial y} &=& -g_1\dfrac{\partial H_{m+k}}{\partial y}\ldots-g_{m+k-2}\dfrac{\partial H_3}{\partial y}-g_{m+k-1}\dfrac{\partial H_2}{\partial y},\\
\dfrac{\partial H_{m+k+1}}{\partial x} &=& -g_1\dfrac{\partial H_{m+k}}{\partial x}\ldots-g_{m+k-2}\dfrac{\partial H_3}{\partial x}-g_{m+k-1}\dfrac{\partial H_2}{\partial x}\\
\vdots & \vdots & \vdots
\end{array}\tag{1.26}$$

for $k>0$. *From the first two equation of* (1.26) *it follows that* g_1 *must satisfy the first order partial differential equation*

$$\{H_2,g_1\}=\frac{\partial X_2}{\partial x}+\frac{\partial Y_2}{\partial y},\tag{1.27}$$

which in view of Corollary 22 has a unique solution. Substituting g_1 *into the first two equations of* (1.26) *and using the Euler's Theorem for homogenous polynomial we obtain*

$$H_3=\frac{1}{3}\left(xY_2-yX_2-2g_1H_2\right).\tag{1.28}$$

We shall determine g_2 as a solution of the first order partial differential equation

$$\{H_2, g_2\} = \frac{\partial X_3}{\partial x} + \frac{\partial Y_3}{\partial y} - \{H_3, g_1\}, \tag{1.29}$$

where g_1 is a solution of (1.27). Then in view of Corollary 22 we get that under the condition

$$\int_0^{2\pi} \left(\frac{\partial X_3}{\partial x} + \frac{\partial Y_3}{\partial y} - \{H_3, g_1\} \right) \Big|_{x=\cos t,\, y=\sin t} dt = 0, \tag{1.30}$$

g_2 exists and has the form $g_2(x,y) = \tilde{g}_2(x,y) + cH_2$ where c is a constant. Hence from the third and fourth equation of (1.26) we get

$$H_4 = \frac{1}{4}\left(xY_3 - yX_3 - 3g_1H_3 - 2g_2H_2\right). \tag{1.31}$$

We shall determine g_3 as a solution of the first order partial differential equation

$$\{H_2, g_3\} = \frac{\partial X_4}{\partial x} + \frac{\partial Y_4}{\partial y} - \{H_4, g_1\} - \{H_3, g_2\}, \tag{1.32}$$

where g_1, g_2 and H_3, H_4 are solutions of the previous differential equations. Then in view of Corollary 22 we get that there exist an unique solution g_3. Hence from the fifth and sixth equation of (1.26) we get

$$H_5 = \frac{1}{5}\left(xY_4 - yX_4 - 4g_1H_4 - 3g_2H_3 - 2g_3H_2\right).$$

By continuing this process we obtain that if g_{m-1} is a solution of the equation

$$\{H_2, g_{m-1}\} = \frac{\partial X_m}{\partial x} + \frac{\partial Y_m}{\partial y} - \{H_m, g_1\} \ldots - \{H_3, g_{m-2}\}, \tag{1.33}$$

which exist if and only if

$$\int_0^{2\pi} \left(\frac{\partial X_m}{\partial x} + \frac{\partial Y_m}{\partial y} - \{H_m, g_1\} \ldots - \{H_3, g_{m-2}\} \right) \Big|_{x=\cos t,\, y=\sin t} dt = 0, \tag{1.34}$$

where the homogenous $g_{m-2}, \ldots, g_1$ are solutions of the previous first order differentia system, then the homogenous polynomial of degree $m+1$ can be calculated as follows

$$H_{m+1} = -\frac{1}{m+1}\left(xY_m - yX_m - mg_1H_m - \ldots - 2g_{m-1}H_2\right) \tag{1.35}$$

Finally under the conditions

$$\begin{aligned} \{H_2, g_{m+k-1}\} &= -\{H_{m+k}, g_1\} - \{H_{m+k-1}, g_2\} \ldots - \{H_3, g_{m+k-2}\}, \\ 0 &= \int_0^{2\pi} \left(-\{H_{m+k}, g_1\} - \{H_{m+k-1}, g_2\} \ldots - \{H_3, g_{m+k-2}\} \right) \Big|_{x=\cos t,\, y=\sin t} dt, \end{aligned} \tag{1.36}$$

we get that

$$\begin{aligned} H_{m+k+1} = & -\frac{1}{m+k+1}\Big(-(m+k)g_1 H_{m+k} \\ & -(m+k-1)g_2 H_{m+k-1} - \dots - 2g_{m+k-1}H_2\Big) \end{aligned} \tag{1.37}$$

for $k > 0$. *Thus we get the expression of* H *and of the integrating factor* $1+\sum_{j=1}^{\infty} g_j$.

Example 29. *In this example by applying Theorem 8 we prove that the class of Kolmogorov cubic differential system*

$$\begin{aligned} \dot{x} = & -y + l_{20}(x^2-y^2) + l_{11}xy + l_{30}(x^3-xy^2) + l_{21}x^2y := P \\ \dot{y} = & \; x + s_{20}(x^2-y^2) + s_{11}xy + l_{21}xy^2 + l_{30}(x^2y-y^3) := Q, \end{aligned} \tag{1.38}$$

which appear in the last chapter in the study the generalized Lotka-Volterra systems, satisfies the necessary conditions of the center. Indeed, from the formula (14) *this differential system can be written as*

$$\begin{aligned} \dot{x} = & \; \{H_4,x\} + (1+g_1)\{H_3,x\} + (1+g_1+g_2)\{H_2,x\} = P, \\ \dot{y} = & \; \{H_4,y\} + (1+g_1)\{H_3,y\} + (1+g_1+g_2)\{H_2,y\} = Q, \end{aligned}$$

where $H_2 = 1/2(x^2+y^2)$, H_j *and* g_j *are convenient polynomials of degree* j. *We integrated this partial differential systems of first order respect to the unknown polynomials* H_4, H_3, g_2, *and* g_1

$$\begin{aligned} & \{H_4,x\} + (1+g_1)\{H_3,x\} + (1+g_1+g_2)\{H_2,x\} \\ = & -y + l_{20}(x^2-y^2) + l_{11}xy + l_{30}(x^3-xy^2) + l_{21}x^2y = P, \\ & \{H_4,y\} + (1+g_1)\{H_3,y\} + (1+g_1+g_2)\{H_2,y\} \\ = & \; x + s_{20}(x^2-y^2) + s_{11}xy + l_{21}xy^2 + l_{30}(x^2y-y^3) = Q \end{aligned} \tag{1.39}$$

Clearly that (1.39) *is equivalent to the systems*

$$\begin{aligned} \{H_3,x\} + g_1\{H_2,x\} = & \; l_{20}(x^2-y^2) + l_{11}xy := X_2 \\ \{H_3,y\} + g_1\{H_2,y\} = & \; s_{20}(x^2-y^2) + s_{11}xy := Y_2. \\ \{H_4,x\} + g_1\{H_3,x\} + g_2\{H_2,x\} = & \; l_{30}(x^3-xy^2) + l_{21}x^2y := X_3, \\ \{H_4,y\} + g_1\{H_3,y\} + g_2\{H_2,y\} = & \; l_{21}xy^2 + l_{30}(x^2y-y^3) := Y_3. \end{aligned} \tag{1.40}$$

From the first two equations we obtain that

$$\begin{aligned} \frac{\partial H_3}{\partial y} = & -g_1y - l_{20}(x^2-y^2) - l_{11}xy, \\ \frac{\partial H_3}{\partial x} = & -g_1x + s_{20}(x^2-y^2) + s_{11}xy. \end{aligned} \tag{1.41}$$

From the compatibility condition $\dfrac{\partial^2 H_3}{\partial x \partial y} = \dfrac{\partial^2 H_3}{\partial y \partial x}$ *we get that the homogenous polynomial* g_1 *of degree one must be a solution of the following partial differential equation of first order*

$$\{H_2, g_1\} = (2l_{20} + s_{11})x + (l_{11} - 2s_{20})y,$$

which in view of Corollary 12 has a unique solution

$$g_1 = (2s_{20} - l_{11})x + (s_{11} + 2l_{20})y.$$

Inserting g_1 *into* (1.41) *we get after the integration that*

$$H_3 = \frac{1}{3}(l_{11} - s_{20})x^3 - \frac{1}{3}(l_{20} + s_{11})y^3 - l_{20}x^2y - s_{20}xy^2.$$

Inserting g_1 *and* H_3 *into the two last equations of system* (1.40) *and by solving respect to the partial derivative of unknown homogenous polynomial* H_4 *of degree four we get that*

$$\begin{aligned} \frac{\partial H_4}{\partial y} &= -g_1\frac{\partial H_3}{\partial y} - g_2 y - X_3, \\ \frac{\partial H_4}{\partial x} &= -g_1\frac{\partial H_3}{\partial x} - g_2 x + Y_3, \end{aligned} \tag{1.42}$$

From the compatibility condition $\dfrac{\partial^2 H_4}{\partial x \partial y} = \dfrac{\partial^2 H_4}{\partial y \partial x}$ *we deduce thct the unknown homogenous polynomial* g_2 *of degree two must be a solution of the following partial differential equation of first order*

$$\{H_2, g_2\} = \{H_3, g_1\} + \frac{\partial X_3}{\partial x} + \frac{\partial Y_3}{\partial y} \tag{1.43}$$

Again from Corollary 12 has a solution of the form $g_2 + cH_2$ *if and only if*

$$V_1 = \int_0^{2\pi} \{H_3, g_1\}|_{x=\cos t,\, y=\sin t}\, dt + \int_0^{2\pi} \left(\frac{\partial X_3}{\partial x} + \frac{\partial Y_3}{\partial y}\right)\bigg|_{x=\cos t,\, y=\sin t} dt = 0,$$

By considering that

$$\begin{aligned} &\int_0^{2\pi} \left(\frac{\partial X_3}{\partial x} + \frac{\partial Y_3}{\partial y}\right)\bigg|_{x=\cos t,\, y=\sin t} dt \\ &= \int_0^{2\pi} \left(4(l_{30}(x^2 - y^2) + 4l_{21}xy)\right|_{x=\cos t,\, y=\sin t} dt = 0, \\ &\int_0^{2\pi} \{H_3, g_1\}|_{x=\cos t,\, y=\sin t}\, dt = 0, \end{aligned}$$

we get that the condition $V_1 = 0$ *holds and consequently equation* (1.43) *has the following solution*

$$\begin{aligned} g_2 = \ & (s_{11}s_{20} - s_{11}l_{11} - l_{11}l_{20} - 4l_{12})xy + (l_{11}l_{20} + 4s_{20}^2 - 2s_{20}l_{11} + 2l_{20}^2 - l_{21})x^2 \\ & +(2s_{11}l_{20} + 2s_{20}^2 - s_{20}l_{11} + 4l_{20} + l_{21})y^2 - 4\alpha(x^2+y^2), \end{aligned}$$

where α *is an arbitrary constant. Inserting* g_2 *into* (1.42) *after the integration we get that*

$$\begin{aligned} H_4 = \ & \alpha(x^2+y^2)^2 + \frac{1}{4}\left(l_{21} - 2l_{20}^2 - l_{20}s_{11} + l_{11}^2 - l_{11}s_{20} - 2s_{20}^2\right)x^4 \\ & + \frac{1}{4}\left(s_{11}^2 + l_{20}s_{11} - 2l_{20}^2 - 2s_{20}^2 + l_{11}s_{20} - l_{21}\right)y^4 \\ & + \left(2s_{20}l_{20} - l_{11}l_{20} + l_{12}\right)x^3y + \left(s_{11}s_{20} + 2s_{20}l_{20} + l_{12}\right)xy^3, \end{aligned}$$

where α *is an arbitrary constant. Hence system* (1.38)*satisfies the necessary conditions of existence of the centers. In the third chapter we prove that the system has a center at the origin.*

Chapter 2

Weak centers

2.0.1 Analytic and polynomial vector fields with local analytic first integral of the form $H = \frac{1}{2}(x^2+y^2)\,(1+h.o.t.)$

We say that a differential system (3) has a *weak center* at the origin if it has a local analytic first integral of the form $H = \frac{1}{2}(x^2+y^2)\left(1+\sum_{j=1}^{\infty}\Upsilon_j(x,y)\right) = H_2\Phi(x,y)$, where Υ_j is a convenient homogenous polynomial of degree j.

The aim of this section is to study the weak centers for analytic and polynomial differential systems.

In the study of the weak centers plays a fundamental role the differential systems (19). In the next proposition we prove that this system is equivalent to differential system

$$\dot{z} = iz + R(z,\bar{z}), \tag{2.1}$$

where $R = R(z,\bar{z})$ is an analytic (or polynomial) function at the origin, $z = x+iy$ and $\bar{z} = x - iy$.

Proposition 30. *Consider a differential system of the form* (2.1)*. Then this system can be rewritten as* (19) *if and only if* $R(z,\bar{z}) = z\Theta(z,\bar{z})$ *where* Θ *is analytic (or polynomial).*

Proof. Assume that $R(z,\bar{z}) = z\Theta(z,\bar{z})$. By considering that $\Theta(z,\bar{z}) = U(x,y) + iV(x,y)$ then $z\Theta(z,\bar{z}) = xU - yV + i(yU + xV)$, thus from (2.1) it follows that

$$\dot{x} = -y + xU - yV = -y(1+V) + xU, \qquad \dot{y} = x + yU + xV = x(1+V) + yU,$$

hence by comparing with (19) we get that $\Lambda = V$ and $\Omega = U$. The reciprocity it is easy to obtain. Indeed, system (19) can be written as $\dot{z} = iz + z(\Omega + i\Lambda)$. □

Corollary 31. *The center of polynomial differential system is a weak center if and only if* $H_j = H_2\Upsilon_{j-2}$ *for* $j = 3, \ldots, m+1$.

Proof. Follows trivially in view of formula (16). □

The singular point of system (3) located at the origin is an *isochronous center* if all the periodic solutions in a neighborhood of it has the same period.

Corollary 32. *The weak center of a polynomial differential system* (19) *is an isochronous center if and only if*

$$\int_0^{2\pi} \frac{d\theta}{1+\Lambda(r\cos\theta, r\sin\theta)} = 2\pi, \tag{2.2}$$

where (r,θ) *are the polar coordinates, and* r *satisfies that*

$$H(r\cos\theta, r\sin\theta) = r^2/2\left(1+\sum_{j=1}^{\infty} r^j\Upsilon_j(\cos\theta,\sin\theta)\right)$$

is a constant on any periodic solution surrounding the isochronous center.

A center O of system (2) is a *uniform isochronous center* if the equality $x\dot{y} - y\dot{x} = \kappa(x^2+y^2)$ holds for a nonzero constant κ; or equivalently in polar coordinates (r,θ) such that $x = r\cos\theta$, $y = r\sin\theta$, we have that $\dot{\theta} = \kappa$.

Theorem 9 has the following additional corollary.

Corollary 33. *Assume that the planar differential system* (4) *has a center at the origin. Then this center is a holomorphic isochronous center if and only if system* (4) *can be written as* (19), *i.e. is a weak center, with the function* Λ *and* Ω *satisfying the Cauchy–Riemann conditions* $\dfrac{\partial\Omega}{\partial x} - \dfrac{\partial\Lambda}{\partial y} = 0$, $\dfrac{\partial\Omega}{\partial y} + \dfrac{\partial\Lambda}{\partial x} = 0$. *Hence* $\Omega + i(1+\Lambda) = f(z)$ *where* $z = x+iy$, *and* $f = f(z)$ *is a holomorphic function on* $\mathbb{C}$. *Moreover, a polynomial differential system* (19) *with a holomorphic center at the origin is Darboux integrable.*

We observe that the Darboux integrability of polynomial differential system with a holomorphic center at the origin is a well known result (see for instance [45]).

Remark 34. *From Corollaries 33 and 11 it follows that all the uniform isochronous centers and all the holomorphic isochronous centers for polynomial differential systems are always weak centers.*

It is important to observe that there is not a relation between isochronous centers and weak centers, i.e. there exist isochronous centers which are not weak centers and weak centers which are not isochronous centers. Then for instance the quadratic isochronous center

$$\dot{x} = -y - \frac{4x^2}{3}, \quad \dot{y} = x(1 - \frac{16y}{3}),$$

is not a weak center because it has the first integral $H = (9-24y+32x^2)^2/(3-16y)$ *for more details see [11]. On the other hand the quadratic system*

$$\dot{x} = -y - x^2 - 3y^2, \quad \dot{y} = x + 2xy,$$

has a weak center at the origin because it has the first integral $H = (1+2y)(x^2+y^2)$ *but it is not isochronous see [11]. In fact in subsection 2.4.4 we provide all the quadratic system with weak centers (see [34]).*

Now we introduce the following definitions and notations.

Let $\mathbb{R}[x,y]$ be the ring of all real polynomials in the variables x and y, and let $\mathcal{X}$ be the polynomial vector field (2) of degree m. Let $g = g(x,y) \in \mathbb{R}[x,y]\backslash\mathbb{R}$. Then $g = 0$ is an *invariant algebraic curve* of $\mathcal{X}$ if $\mathcal{X}g = P\dfrac{\partial g}{\partial x} + Q\dfrac{\partial g}{\partial y} = Kg$, where $K = K(x,y)$ is a polynomial of degree at most $m-1$, which is called the *cofactor* of $g = 0$. A function $g = g(x,y)$ satisfying that $g = 0$ is an *invariant curve* (i.e. formed by orbits of the vector field $\mathcal{X}$) is called *partial integral.* If $g \in \mathbb{R}[x,y]\backslash\mathbb{R}$ then g is called a *polynomial partial integral* or a *Darboux polynomial.* If the polynomial g is irreducible in $\mathbb{R}[x,y]$, then we say that the invariant algebraic curve $g = 0$ is *irreducible*, and that its *degree* is the degree of the polynomial g. A first integral F of the polynomial vector field (1) is called *Darboux* if

$$F = e^{k(x,y)/h(x,y)} g_1^{\lambda_1}(x,y) \dots g_r^{\lambda_r}(x,y),$$

where k, h, $g_1, ..., g_r$ are polynomials and $\lambda_1, \dots, \lambda_r$ are complex constants. For more details on the so–called Darboux theory of integrability see for instance Chapter 8 of [20].

We introduce the following definition. We say that a polynomial vector field $\mathcal{X}$ of degree m is *quasi–Darboux integrable* if there exist r polynomial partial integrals $g_1, \dots, g_r$ and s non-polynomial C^r, $r > 0$ in $D \subseteq \mathbb{R}^2$ partial integrals $f_1, \dots, f_s$ satisfying

$$\mathcal{X}(f_j) = P\frac{\partial f_j}{\partial x} + Q\frac{\partial f_j}{\partial y} = K_j f_j,$$

where $K_j = K_j(x,y)$ is a convenient polynomials of degree $m-1$, for $j = 1, \dots, s$ such that the function

$$F = e^{k(x,y)/h(x,y)} g_1^{\lambda_1}(x,y) \dots g_r^{\lambda_r}(x,y) f_1^{\kappa_1}(x,y) \dots f_s^{\kappa_s}(x,y),$$

is a first integral, where $k = k(x,y)$, $h = h(x,y)$ are polynomials, and $\lambda_1, \dots, \lambda_r$, $\kappa_1, \dots, \kappa_s$, are complex constants. We observe that a generalization of the Darboux theory was developed in the paper [24], which evidently contains the above definition with another name, but for our aim we shall use the name of quasi–Darboux integrable.

We have the following conjecture.

Conjecture 35. *A polynomial differential system* (19) *having a weak center at the origin is quasi-Darboux integrable.*

This conjecture is supported by several facts which we give below.

Proposition 36. *A polynomial differential system* (19) *with a weak center at the origin is quasi–Darboux integrable in a neighborhood of the origin with the first integral* $H = \frac{1}{2}(x^2+y^2)\left(1+\sum_{j=1}^{m+1}\tau_j(x,y)\Upsilon_j(x,y)\right) := H_2 f(x,y)$, *where* $H_2 = 0$ *is an invariant algebraic curve and* $f = 0$ *is an analytic (non polynomial) invariant curve with cofactor* 2Ω *and* -2Ω *respectively.*

Proof. Since at the origin of system (19) there is a weak center, we have an analytic first integral $H = H_2 f$ in a neighborhood of the origin. So clearly $H_2 = 0$ and $f = 0$ are invariant curves of system (19). It is easy to check that $\dfrac{dH_2}{dt} = 2H_2\Omega$. From the first integral $H = H_2 f$ we get that

$$\frac{dH_2}{dt} f + H_2\frac{df}{dt} = 2H_2\Omega\, f + H_2\frac{df}{dt} = 0.$$

Thus $\dfrac{df}{dt} = -2\Omega\, f$, and the proposition is proved. □

2.0.2 Center problem for analytic or polynomial vector fields with a generalized weak condition of a center

First we prove the following two propositions.

Proposition 37. *Assume that a differential system* (3) *satisfies the relation*

$$(x^2+y^2)\left(\frac{\partial(-y+X)}{\partial x}+\frac{\partial(x+Y)}{\partial y}\right) = \mu\left(x(-y+X)+y(x+Y)\right), \qquad (2.3)$$

with $\mu \in \mathbb{R}\backslash\{0\}$. *Then the system can be written as in* (19) *with*

$$\Omega(x,y) = \frac{2}{\mu}\left(\frac{\partial(-y+X)}{\partial x}+\frac{\partial(x+Y)}{\partial y}\right),$$

and $\Lambda = \Lambda(x,y)$ *an arbitrary analytic function in a neighborhood of the origin. Moreover system* (3) *has the inverse integrating factor* $(x^2+y^2)^{\mu/2}$, *and it can be written as*

$$\dot{x} = (x^2+y^2)^{\mu/2}\{F,x\} \qquad \dot{y} = (x^2+y^2)^{\mu/2}\{F,y\}, \qquad (2.4)$$

with

$$\begin{aligned} F = & \int_\gamma\left(\frac{-Xdy+Ydx}{(x^2+y^2)^{\mu/2}}+\frac{d(x^2+y^2)}{2(x^2+y^2)^{\mu/2}}\right) \\ = & \begin{cases} \dfrac{1}{2-\mu}(x^2+y^2)^{(\mu-2)/2}+\displaystyle\int_\gamma\frac{-Xdy+Ydx}{(x^2+y^2)^{\mu/2}} & \text{if } \mu\neq 2, \\ \log\sqrt{x^2+y^2}+\displaystyle\int_\gamma\frac{-Xdy+Ydx}{(x^2+y^2)}, & \text{if } \mu = 2. \end{cases} \end{aligned} \qquad (2.5)$$

Note that if in (2.3) we have that $\mu = 0$, then system (3) is a Hamiltonian system.

Proposition 38. *Consider the polynomial differential system* (2) *of degree* m *which satisfy the relations*

$$\int_0^{2\pi} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) |_{x=\cos t,\, y=\sin t} dt = 0. \tag{2.6}$$

Then there exist polynomials $\tilde{H} = \sum_{j=3}^{m+1} H_j$ *and* $G = \sum_{j=1}^{m-1} G_j$ *of degree* $m+1$ *and* $m-1$ *respectively such that system* (2) *can be written as*

$$\dot{x} = \{\tilde{H}, x\} + (1+G)\{H_2, x\}, \quad \dot{y} = \{\tilde{H}, y\} + (1+G)\{H_2, y\}. \tag{2.7}$$

Note that we have extended the definition of "weak condition for a center" given in subsection 2.3 for a quasi-homogenous polynomial differential system to a general analytic differential system. Proposition 24 it has been generalized in the Theorem 12 which is proved in section 2.2.

2.1 The Proofs of Theorem 9 and Corollaries 10, 11

Proof of Theorem 9. Necessity We suppose that system (3) has a weak center at the origin . Consequently there exists an analytic local first integral $H = H_2(1 + \sum_{j=1}^{\infty} \Upsilon_j) := H_2\Phi$. Then from Theorem 7 it follows the necessary and sufficient conditions on the existence of a linear type center for an analytic differential system differential system. Thus (13) becomes

$$
\begin{aligned}
\dot{x} = {} & V\{H,x\} = -V\left(y\Phi + H_2\frac{\partial\Phi}{\partial y}\right) = -Vy\left(\Phi + \frac{y}{2}\frac{\partial\Phi}{\partial y} + \frac{x}{2}\frac{\partial\Phi}{\partial x}\right) + V\frac{x}{2}\{\Phi, H_2\} \\
= {} & \left(1+\sum_{j=1}^{\infty} g_j\right)\left(-y\big(1+\sum_{j=1}^{\infty}\frac{j+2}{2}\Upsilon_j\big) + \frac{x}{2}\sum_{j=1}^{\infty}\frac{j+2}{2}\{\Upsilon_j, H_2\}\right) \\
= {} & -y\Big(1+\sum_{j=2}^{\infty}\Big(\frac{j+1}{2}\Upsilon_{j-1} + \frac{j}{2}g_1\Upsilon_{j-2} + \ldots + \frac{3}{2}g_{j-2}\Upsilon_1 + g_{j-1}\Big)\Big) \\
& + \frac{x}{2}\sum_{j=2}^{\infty}\Big(\{\Upsilon_{j-1}, H_2\} + g_1\{\Upsilon_{j-2}, H_2\} + \ldots + g_{j-2}\{\Upsilon_1, H_2\}\Big) \\
= {} & -y\big(1+\sum_{j=2}^{\infty}\Lambda_j\big) + \frac{x}{2}\sum_{j=2}^{\infty}\Omega_j \\
\dot{y} = {} & V\{H,y\} = V\left(x\Phi + H_2\frac{\partial\Phi}{\partial x}\right) = Vx\left(\Phi + \frac{y}{2}\frac{\partial\Phi}{\partial y} + \frac{x}{2}\frac{\partial\Phi}{\partial x}\right) + V\frac{y}{2}\{\Phi, H_2\} \\
= {} & \left(1+\sum_{j=1}^{\infty} g_j\right)\left(x\big(1+\sum_{j=1}^{\infty}\frac{j+2}{2}\Upsilon_j\big) + \frac{y}{2}\sum_{j=1}^{\infty}\frac{j+2}{2}\{\Upsilon_j, H_2\}\right) \\
= {} & x\Big(1+\sum_{j=2}^{\infty}\Big(\frac{j+1}{2}\Upsilon_{j-1} + \frac{j}{2}g_1\Upsilon_{j-2} + \ldots + \frac{3}{2}g_{j-2}\Upsilon_1 + g_{j-1}\Big)\Big) \\
& + \frac{y}{2}\sum_{j=2}^{\infty}\Big(\{\Upsilon_{j-1}, H_2\} + g_1\{\Upsilon_{j-2}, H_2\} + \ldots + g_{j-2}\{\Upsilon_1, H_2\}\Big) \\
= {} & x\big(1+\sum_{j=2}^{\infty}\Lambda_j\big) + \frac{y}{2}\sum_{j=2}^{\infty}\Omega_j.
\end{aligned}
\tag{2.8}
$$

Sufficiency Now we suppose that if the origin is center of (20) then we prove that the origin is a weak center. Indeed, by considering that in this case

$$
\begin{aligned}
X = {} & X_1 + X_2 + X_3 + \ldots + X_n + \ldots = -y(1+\Lambda) + x\Omega \\
= {} & -y - y\Lambda_1 - y\Lambda_2 - \ldots - y\Lambda_n - \ldots + x\Omega_1 + x\Omega_2 + \ldots + x\Omega_n + \ldots \\
Y = {} & Y_1 + Y_2 + Y_3 + \ldots + Y_n + \ldots = x(1+\Lambda) + y\Omega \\
= {} & x + x\Lambda_1 + x\Lambda_2 + \ldots + x\Lambda_n + \ldots + y\Omega_1 + y\Omega_2 + \ldots + y\Omega_n + \ldots .
\end{aligned}
$$

Hence

$$X_n = -y\Lambda_{n-1} + x\Omega_{n-1} \qquad Y_n = x\Lambda_{n-1} + y\Omega_{n-1}, \quad \text{for} \quad n \geq 1,$$

where $\Lambda_0 = \Omega_0 = 1$. Therefore $xY_n - yX_n = 2H_2\Lambda_{n-1}$. Thus from (1.11) we obtain that

$$\begin{aligned} H = & \sum_{n=1}^{\infty}(xY_n - yX_n)\left(\frac{1}{n+1} - \sum_{k=1}^{\infty}\frac{T_k}{k+n+1}\right) \\ = & H_2\sum_{n=1}^{\infty}\frac{\Lambda_{n-1}}{2}\left(\frac{1}{n+1} - \sum_{k=1}^{\infty}\frac{T_k}{k+n+1}\right) := H_2\,\Phi(x,y), \end{aligned}$$

consequently $H = H_2\Phi$ is a first integral, therefore the origin is a weak center.

The second statement we prove as follows. Under the assumption

$$\begin{aligned} &-y\sum_{j=m+1}^{\infty}\left(\frac{j+1}{2}\Upsilon_{j-1} + \frac{j}{2}g_1\Upsilon_{j-2} + \ldots + \frac{3}{2}g_{j-2}\Upsilon_1 + g_{j-1}\right) \\ &+\frac{x}{2}\sum_{j=m+1}^{\infty}\Big(\{\Upsilon_{j-1}, H_2\} + g_1\{\Upsilon_{j-2}, H_2\} + \ldots + g_{j-2}\{\Upsilon_1, H_2\}\Big) = 0 \\ &x\sum_{j=m+1}^{\infty}\left(\frac{j+1}{2}\Upsilon_{j-1} + \frac{j}{2}g_1\Upsilon_{j-2} + \ldots + \frac{3}{2}g_{j-2}\Upsilon_1 + g_{j-1}\right) \end{aligned}$$

$$+\frac{y}{2}\sum_{j=m+1}^{\infty}\Big(\{\Upsilon_{j-1}, H_2\} + g_1\{\Upsilon_{j-2}, H_2\} + \ldots + g_{j-2}\{\Upsilon_1, H_2\}\Big) = 0$$

which is equivalent to the equations

$$\begin{aligned} \frac{j+1}{2}\Upsilon_{j-1} + \frac{j}{2}g_1\Upsilon_{j-2} + \ldots + \frac{3}{2}g_{j-2}\Upsilon_1 + g_{j-1} = & \; 0 \\ \{\Upsilon_{j-1}, H_2\} + g_1\{\Upsilon_{j-2}, H_2\} + \ldots + g_{j-2}\{\Upsilon_1, H_2\} = & \; 0 \end{aligned}$$

for $j > m+1$, from (2.8) we get the following polynomial differential equations of degree m.

$$
\begin{aligned}
\dot{x} = & -y\left(1+\sum_{j=2}^{m}\left(\frac{j+1}{2}\Upsilon_{j-1}+\frac{j}{2}g_1\Upsilon_{j-2}+\ldots+\frac{3}{2}g_{j-2}\Upsilon_1+g_{j-1}\right)\right)\\
& +\frac{x}{2}\sum_{j=2}^{m}\Big(\{\Upsilon_{j-1},H_2\}+g_1\{\Upsilon_{j-2},H_2\}+\ldots+g_{j-2}\{\Upsilon_1,H_2\}\Big),\\
\dot{y} = & \; x\left(1+\sum_{j=2}^{m}\left(\frac{j+1}{2}\Upsilon_{j-1}+\frac{j}{2}g_1\Upsilon_{j-2}+\ldots+\frac{3}{2}g_{j-2}\Upsilon_1+g_{j-1}\right)\right)\\
& +\frac{y}{2}\sum_{j=2}^{m}\Big(\{\Upsilon_{j-1},H_2\}+g_1\{\Upsilon_{j-2},H_2\}+\ldots+g_{j-2}\{\Upsilon_1,H_2\}\Big)
\end{aligned}
$$

Here Υ_j is a convenient homogenous polynomial of degree j, such that $H_2 = (x^2+y^2)/2$, $H_{j+2} = H_2\Upsilon_j$, for $j = 1\ldots m+1$ and g_j is an arbitrary homogenous polynomial of degree j satisfying (15). Consequently in view of (16) we obtain the local first integral (22). Thus the theorem is proved □

Proof of Corollary 10. From the relations $H_j = H_2\Upsilon_{j-1}$ and

$$
H_2\Lambda_{j-1} = \frac{j+1}{2}H_{j+1}+\frac{j}{2}g_1H_j+\ldots+\frac{3}{2}g_{j-2}H_3+g_{j-1}H_2, \quad \text{for} \quad j>1,
$$

it follows that

$$
H_{j+1} = -\frac{2}{j+1}\left(\frac{j}{2}g_1H_j+\ldots+\frac{3}{2}g_{j-2}H_3+g_{j-1}H_2\right)+\frac{2H_2}{j+1}\Lambda_{j-1}.
$$

Inserting H_{j+1} into the relations $\Phi_j = \{H_{j+1},H_2\}+g_1\{H_j,H_2\}+\ldots+g_{j-2}\{H_3,H_2\}$, we obtain that

$$
\begin{aligned}
\Phi_j = & -\frac{j}{j+1}\{g_1H_j,H_2\}+g_1\{H_j,H_2\}+\ldots\\
= & \; \frac{1}{j+1}\left((j+1)g_1\{H_j,H_2\}-\frac{j}{j+1}\left(H_j\{g_1,H_2\}+g_1\{H_j,H_2\}\right)+\ldots\right)\\
= & \; \frac{1}{j+1}\Big(\left(x\frac{\partial g_1}{\partial x}+y\frac{\partial g_1}{\partial y}\right)\left(y\frac{\partial H_j}{\partial x}-x\frac{\partial H_j}{\partial y}\right)-\left(x\frac{\partial H_j}{\partial x}+y\frac{\partial H_j}{\partial y}\right)\left(y\frac{\partial g_1}{\partial x}-x\frac{\partial g_1}{\partial y}\right)+\ldots\Big)\\
= & \; \frac{1}{j+1}\left(2H_2\{H_j,g_1\}+\ldots\right),
\end{aligned}
$$

here we use the relations $g_1 = x\dfrac{\partial g_1}{\partial x}+y\dfrac{\partial g_1}{\partial y}, \quad jH_j = x\dfrac{\partial H_j}{\partial x}+y\dfrac{\partial H_J}{\partial y}$.

Hence after some computations we get that

$$
\Phi_j = \frac{2H_2}{j+1}\left(\{\Lambda_{j-1},H_2\}-\left(\{H_j,g_1\}+\{H_{j-1},g_2\}+\ldots+\{H_3,g_{j-2}\}+\{H_2,g_{j-1}\}\right)\right).
$$

Then the proof of the corollary follows easily. □

In order to illustrate Theorem 9 we study the following polynomial systems.

Example 39. *The following cubic polynomial differential system has a center at the origin (see [55])*

$$\begin{aligned}\dot{x} = &\ -y+\frac{1}{2}\left(x^2-xy-2y^2-xy^2-y^3\right)=-y(1+y+\frac{y^2}{2})+\frac{x}{2}(x-y-y^2),\\ \dot{y} = &\ x+\frac{1}{2}\left(3xy-y^2+xy^2-y^3\right)=x(1+y+\frac{y^2}{2})+\frac{y}{2}(x-y-y^2),\end{aligned}$$

Consequently this system can be rewritten as (19) *with the functions* Ω *and* Λ *determined as follows* $1+\Lambda=\dfrac{1}{2}(1+(y+1)^2$, $\Omega=x-y-y^2$. *and hence the center is a weak center.*

Proof of Corollary 32. First we observe that differential equations (19) in polar coordinates $x=r\cos\theta$, $y=r\sin\theta$ becomes

$$\dot{r}=r\Omega(r\cos\theta,r\sin\theta),\qquad \dot{\theta}=1+\Lambda(r\cos\theta,r\sin\theta),$$

hence in view of that the center is weak center, then the polar coordinates must be such that $H(r\cos\theta,\,r\sin\theta)=C=\text{constant}$. Hence we get that the weak center is an isochronous center if and only if (2.2) holds, thus the corollary is proved. □

Proof of Corollary 11. It trivially follows from Corollary 10 assuming that $\Lambda_j=0$ for $j=1,\dots,\infty$. Hence, if we assume that $\{H_j,g_1\}+\ldots+\{H_2,g_{j-1}\}=0$ for $j>m+1$ then we obtain the conditions under which the polynomial differential system has a uniform isochronous center at the origin. Thus the proof of the corollary follows. □

Proof of Corollary 33. From the Cauchy–Riemann conditions it is easy to obtain condition

$$\frac{\partial\Lambda}{\partial x}+\frac{\partial\Omega}{\partial y}=0,\qquad \frac{\partial\Lambda}{\partial y}-\frac{\partial\Omega}{\partial x}=0\Longleftrightarrow\frac{\partial\left(\Omega+i\Lambda\right)}{\partial\bar{z}}=0,$$

i.e. the functions Λ and Ω are harmonic functions. Moreover differential system (19) in complex coordinates $z=x+i\,y$ and $\bar{z}=x-i\,y$ becomes

$$\dot{z}=iz+z\Phi(z)=f(z),$$

which in view Proposition 30 can be written as (19), thus the holomorphic isochronous center is a weak center.

We observe that the problem on the existence the first integral for the complex differential system was study in particular in [45]. In [36] We have proven that the first integral for holomorphic center can be written as $H_2(1+h.o.t.)$. □

Proof of Proposition 37. From (2.3) it follows that $x(-y+X-\lambda\, x\Omega)+y(x+Y-y\Omega)=0$, where $\lambda=2/\mu$ and $\Omega=\dfrac{\partial(-y+X)}{\partial\, x}+\dfrac{\partial(x+Y)}{\partial\, y}$. Thus

$$-y+X=-\nu\, y+x\Omega, \qquad x+Y=x\nu+y\Omega,$$

where $\nu=\nu(x,y)$ is an arbitrary function. Denoting $\nu=1+\Lambda$ we get that differential equations (3) coincide with (19). On the other hand in view of the relations

$$(-y+X)\frac{\partial H_2}{\partial\, x}+(x+X)\frac{\partial H_2}{\partial\, y}=\lambda\, H_2\Omega=\lambda\, H_2\left(\frac{\partial(-y+X)}{\partial\, x}+\frac{\partial(x+Y)}{\partial\, y}\right),$$

which is equivalent to

$$\frac{\partial}{\partial x}\left(\frac{-y+X}{(x^2+y^2)^{\mu/2}}\right)+\frac{\partial}{\partial y}\left(\frac{x+Y}{(x^2+y^2)^{\mu/2}}\right)=0. \tag{2.9}$$

i.e. H_2^λ is inverse integrating factor. Thus differential system (3) can be written as (2.4) with F given by the formula (2.5). In short corollary is proved. □

Proof of Proposition 38. Suppose that P and Q can be written as in (2.7) where $\tilde{H}$ and G are polynomials, and we shall see that such polynomials exist when (2.6) holds. Then

$$\frac{\partial\tilde{H}}{\partial y}=-yG-P, \quad \frac{\partial\tilde{H}}{\partial x}=-xG+Q.$$

Hence by considering that $\dfrac{\partial^2\tilde{H}}{\partial x\partial y}=\dfrac{\partial^2\tilde{H}}{\partial y\partial x}$, we get that

$$x\frac{\partial G}{\partial y}-y\frac{\partial G}{\partial x}=\{H_2,G\}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}.$$

By considering that (2.6) holds, then in view of Corollary 22 we deduce that there exists a polynomial $G=\sum_{j=1}^{m-1}G_j$ such that

$$x\frac{\partial G_j}{\partial y}-y\frac{\partial G_j}{\partial x}=\{H_2,G_j\}=\frac{\partial P_j}{\partial x}+\frac{\partial Q_j}{\partial y}. \tag{2.10}$$

We can determine the function $\tilde{H}$ as follows

$$\tilde{H}=\int_{x_0}^{x}(-x\sum_{j=1}^{m-1}G_j+Q)dx-\int_{y_0}^{y}(y\sum_{j=1}^{m-1}G_j+P)\Big|_{x=x_0}dy,$$

where $G_j=G_j(x,y)$ is the solution of equation (2.10). In short the proposition is proved. □

We observe that from (2.7) it follows that

$$xP + yQ = \{\tilde{H}, H_2\}, \qquad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \{H_2, G\},$$

thus in view of Proposition 20 we obtain that

$$\int_0^{2\pi} (x\,P(x,y) + y\,Q(x,y))\;|_{x=\cos t,\,y=\sin t} dt = 0,$$
$$\int_0^{2\pi} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)|_{x=\cos t,\,y=\sin t} dt = \quad 0.$$

We consider an analytic differential system (6) under the assumptions (24). The previous result can be extended for the analytic vector field. Thus we have the following proposition.

Proposition 40. *Let* (2) *be an analytic differential system which satisfies the relation* $\int_0^{2\pi} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)|_{x=\cos t,\,y=\sin t} dt = 0$. *Then there exist analytic functions* $\tilde{H} = \sum_{j=2}^{\infty} H_j$ *and* $G = \sum_{j=1}^{\infty} G_j$ *such that*

$$\dot{x} = P = -\frac{\partial \tilde{H}}{\partial y} - yG = \{\tilde{H}, x\} + G\{H_2, x\}, \quad \dot{y} = Q = \frac{\partial H}{\partial x} + xG = \{\tilde{H}, y\} + G\{H_2, y\}.$$

Proof. It is analogous to the proof of Proposition 38. □

2.2 Proof of Theorem 12

Proof of Theorem 12. We shall study only the case when the differential system is a polynomial differential systems of degree m.

It is possible to show that condition (2.3) is equivalent to (2.9). Hence from the first of condition of (24) and in view of Proposition 37 we get that a polynomial differential system (3) can be written as (2.4) with F given in the formula (2.5).

On the other hand in view of Proposition 38 and the second of conditions (24) we get that there exist polynomials $\tilde{H} = \tilde{H}(x,y)$ and $G = G(x,y)$ of degree $m+1$ and $m-1$ respectively, such that the following relations hold

$$\begin{aligned} \dot{x} &= -y + X = \{\tilde{H}, x\} + G\{H_2, x\}, \\ \dot{y} &= x + Y = \{\tilde{H}, y\} + G\{H_2, y\}, \end{aligned} \tag{2.11}$$

Hence $\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = \{H_2, G\}$, $\quad xX + yY = \{\tilde{H}, H_2\}$, consequently condition (2.3) becomes $2H_2\{H_2, G\} = \mu\{\tilde{H}, H_2\}$. Thus

$$\lambda H_2\{H_2, G\} = \{\tilde{H}, H_2\} \Longleftrightarrow \{H_2, \tilde{H} + \lambda H_2 G\} = 0,$$

where $\lambda = 2/\mu$. Therefore

$$\tilde{H} = -\lambda H_2 G + \lambda p(H_2) := H_2 \Upsilon \tag{2.12}$$

here $p(H_2)$ is a polynomial of degree $[(m-1)/2]$, where $[(m-1)/2]$ is the integer part of $m-1)/2$ such that

$$p(H_2) = a_1 H_2 + a_2 H_2^2 + \ldots + \alpha_m H_2^{[m-1)/2]} := H_2 q(H_2) \qquad \text{and} \qquad G = -1/\lambda\,\Upsilon + q(H_2).$$

Thus by putting (2.12) into differential system (2.11) we get

$$\begin{aligned}
\dot{x} = &\ -y - \frac{\partial \tilde{H}}{\partial y} - yG = -H_2 \frac{\partial \Upsilon}{\partial y} - y(1 + \Upsilon + G)\\
= &\ -H_2 \frac{\partial \Upsilon}{\partial y} - y(1 + \frac{\lambda-1}{\lambda}\Upsilon + q(H_2)),\\
= &\ -y\left(1 + q(H_2) + (1-1/\lambda)\Upsilon + 1/2\left(x\frac{\partial \Upsilon}{\partial x} + y\frac{\partial \Upsilon}{\partial y}\right)\right) + \frac{x}{2}\{\Upsilon, H_2\},\\
\dot{y} = &\ x + \frac{\partial \tilde{H}}{\partial x} + xG = H_2 \frac{\partial \Upsilon}{\partial x} + x(1 + \Upsilon + G)\\
= &\ H_2 \frac{\partial \Upsilon}{\partial x} + x(1 + \frac{\lambda-1}{\lambda}\Upsilon + +q(H_2))\\
= &\ x\left(1 + q(H_2) + (1-1/\lambda)\Upsilon + 1/2\left(x\frac{\partial \Upsilon}{\partial x} + y\frac{\partial \Upsilon}{\partial y}\right)\right) + \frac{y}{2}\{\Upsilon, H_2\}.
\end{aligned}$$

Consequently

$$\begin{aligned}
\dot{H}_2 &= H_2\{\Upsilon, H_2\},\\
\dot{\Upsilon} &= -\left(1 + \frac{\lambda-1}{\lambda}\Upsilon + q(H_2)\right)\{\Upsilon, H_2\},
\end{aligned}$$

hence

$$\frac{d\Upsilon}{dH_2} = \frac{1-\lambda}{\lambda H_2}\Upsilon - \frac{1+q(H_2)}{H_2}.$$

After the integration this first order linear ordinary differential equations we have the following solution

$$\Upsilon = H_2^{1/\lambda - 1}\left(C - \int \frac{1+q(H_2)}{H_2^{1/\lambda}} dH_2\right),$$

where C is an arbitrary constant. Consequently we have the following particular cases.

(i) If $\lambda \neq 1$ and $\displaystyle\prod_{n=2}^{[(m-1)/2]} (n - 1/\lambda) \neq 0$, then

$$\Upsilon = H_2^{1/\lambda-1}C - \frac{1+\alpha_1}{1-1/\lambda} - \frac{\alpha_2 H_2}{2-1/\lambda} - \ldots - \frac{\alpha_m H_2^{[(m-1)/2]-1}}{[(m-1)/2]-1/\lambda}.$$

(ii) If $\lambda = 1$, then

$$\Upsilon = C - (1+\alpha_1)\log H_2 - \alpha_2 H_2 - \frac{\alpha_3 H_2^2}{2} - \ldots - \frac{\alpha_m H_2^{[(m-1)/2]-1}}{[(m-1)/2]-1}.$$

(iii) If $1 < \lambda = 1/k \leq 1/[(m-1)/2]$ and $\prod_{n=2, n\neq k}^{[(m-1)/2]} (n-k) \neq 0$, then

$$\begin{aligned}\Upsilon = & -H_2^{k-1}\Big(-C + \frac{1+\alpha_1}{1-k}H_2^{1-k} - \frac{\alpha_2}{2-k}H_2^{2-k} - \ldots \\ & -\alpha_k \log H_2 - \alpha_{k+1}H_2 - \frac{\alpha_{k+2}}{2}H_2^2 - \ldots - \frac{\alpha_m}{[(m-1)/2]-k}H_2^{[(m-1)/2]-k}\Big).\end{aligned}$$

(iv) If $1/[(m-1)/2] \leq \lambda = 1/k < 1$, $\alpha_k = 0$ and $\prod_{n=2, n\neq k}^{[(m-1)/2]} (n-k) \neq 0$, then

$$\Upsilon = H_2^{k-1}C - \frac{1+\alpha_0}{1-k} - \sum_{j=2,\, j\neq k}^{[(m-1)/2]} \frac{\alpha_j}{j-k}H_2^{j-1}.$$

Excluding the constant C in the obtained solutions we deduce the first integrals F given in formula (26), (27), (28), (29). □

We observe that if in the equation (25) the relation $1+\alpha_1+\frac{\lambda-1}{\lambda}\Upsilon(0,0) = 0$, holds then the origin is not linear type. Indeed, under this condition the given equation becomes

$$\begin{aligned}\dot{x} = & -y\left(\sum_{j=1}^{m-1}((1-1/\lambda)+\frac{j}{2})\Upsilon_j + \sum_{j=2}^{[(m-1)/2]} \alpha_j H_2^{j-1}\right) + \frac{x}{2}\{\Upsilon, H_2\}, \\ \dot{y} = & \; x\left(\sum_{j=1}^{m-1}((1-1/\lambda)+\frac{j}{2})\Upsilon_j + \sum_{j=2}^{[(m-1)/2]} \alpha_j H_2^{j-1}\right) + \frac{y}{2}\{\Upsilon, H_2\}.\end{aligned}$$

where $\Upsilon = \sum_{j=1}^{m-1} \Upsilon_j$, and Υ_j is a homogenous polynomial of degree j for $j = 1, \ldots, m-1$.

2.3 $\Lambda - \Omega$ differential system. Center problem part I

. The main objective In this section we study the center problem for Λ-Ω systems of degree m

$$\begin{aligned} \dot{x} = & \; -y(1+b_1x+b_2y)+x(a_1x+a_2y+\Omega_{m-1}), \\ \dot{y} = & \; x(1+b_1x+b_2y)+y(a_1x+a_2y+\Omega_{m-1}), \end{aligned} \tag{2.13}$$

where $\Omega_{m-1} = \Omega_{m-1}(x,y)$ is a homogenous polynomial of degree $m-1$, a_1, a_2 and b_1, b_2 are constants We also state some conjectures related with the existence the center for (2.13).

We provide the following results.

Proposition 41. *The polynomial differential system of degree m*

$$\begin{aligned} \dot{x} = & \; -y(1+(m-2)(a_1y-a_2x))+x(a_1x+a_2y+\Omega_{m-1}), \\ \dot{y} = & \; x(1+(m-2)a_1y-a_2x))+y(a_1x+a_2y+\Omega_{m-1}), \end{aligned} \tag{2.14}$$

where $\Omega_{m-1} = \Omega_{m-1}(x,y)$ is a homogenous polynomial of degree $m-1$, has a weak center at the origin if and only if

$$\int_0^{2\pi} \Omega_{m-1}(\cos t, \sin t)dt = 0. \tag{2.15}$$

Moreover system (2.14) *has the first integral*

$$H = \frac{H_2}{\left(1+\dfrac{m-1}{m+1}G(x,y)+(m-1)\Gamma(H_2)+\Phi(H_2)\right)^{2/(m-1)}} = H_2(1+h.o.t.), \tag{2.16}$$

where G is a polynomial of degree $m-1$ and such that $\{H_2, G\} = -(m+1)(a_1x+a_2y+\Omega_{m-1})$, $\Gamma = \Gamma(H_2)$ is a convenient polynomial on H_2 and $\Phi(H_2)$ is a convenient polynomial if m is even and a convenient function if m is odd (see the proof for the expressions of Γ and Φ).

We observe that if we take $a_1 = a_2 = 0$ in Proposition 41 we obtain following corollary.

Corollary 42. *The polynomial differential system of degree m*

$$\dot{x} = -y + x\Omega_{m-1} \quad \dot{y} = x + y\Omega_{m-1}, \tag{2.17}$$

has a weak uniform center at the origin if and only if (2.15) *holds. Moreover system* (2.17) *has the Poincaré-Liapunov first integral* (2.16) *where G is a homogenous polynomial of degree $m-1$ and such that $\{H_2, G\} = \Omega_{m-1}$.*

Conti in [16] proved the first part of Corollary42, but the second part providing an explicit expression of the first integral of system (2.17) is new.

Remark 43. *The weak centers obtained in Proposition 41 and Corollary 42 in general are not invariant with respect to the transformation* $(x, y, t) \longrightarrow (-x, y, -t)$. *Indeed the polynomial differential system of degree* $2k$ *given by formula* (2.14) *and* (2.17), *when* Ω_{2k-1} *is an arbitrary polynomial of degree* $2k-1$, *in general is not reversible and has the weak center at the origin.*

We recall that a polynomial differential system has a uniform center at the origin if written in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ we obtain that $\dot{\theta}$ is constant.

2.4 The Proofs of Proposition 41, Corollary 42, Theorems 14

2.4.1 Preliminary results

Below we need the following results.

Let

$$x = \kappa_1 X - \kappa_2 Y, \qquad y = \kappa_2 X + \kappa_1 Y, \tag{2.18}$$

be a non-degenerated linear transformation, i.e. $\kappa_1^2 + \kappa_2^2 \neq 0$. Then differential system (19) becomes

$$\begin{aligned} \dot{X} = & -Y\left(1 + \tilde{\Lambda}(X, Y)\right) + X\tilde{\Omega}(X, Y), \\ \dot{Y} = & \ X\left(1 + \tilde{\Lambda}(X, Y)\right) + Y\tilde{\Omega}(X, Y), \end{aligned} \tag{2.19}$$

where $\tilde{\Lambda}(X, Y) = \Lambda(\kappa_1 X - \kappa_2 Y, \kappa_2 X + \kappa_1 Y)$ and $\tilde{\Omega}(X, Y) = \Omega(\kappa_1 X - \kappa_2 Y, \kappa_2 X + \kappa_1 Y)$. Here we say that system (3) is *reversible with respect to a straight line* l through the origin if it is invariant with respect to reversion about l and a reversion of time t (see for instance [16]). In particular Poincaré's Theorem is applied for the case when (3) is invariant under the transformations $(x, y, t) \longrightarrow (-x, y, -t)$, or $(x, y, t) \longrightarrow (x, -y, -t)$.

In the proof of the results which we give later on we need the Poincaré's Theorem (see Theorem 23).

Since a rotation with respect to the origin of coordinates is a particular transformation of type (2.18), when a center of system (19) is invariant with respect to a straight line it is not restrictive to assume that such straight line is the x axis. So the center of system (19) will be invariant by the transformation $(x, y, t) \longrightarrow (-x, y, -t)$ or $(x, y, t) \longrightarrow (x, -y, -t)$. We shall study only the first case, i.e. we shall suppose that the Λ-Ω system is invariant with respect to the transformation $(x, y, t) \longrightarrow (-x, y, -t)$.

The following proposition is easy to prove (see for instance [43]).

Proposition 44. *Differential system* (2.19) *is invariant under the transformation* $(X,Y,t) \longrightarrow (-X,Y,-t)$ *if and only if it can be written as*

$$\begin{aligned} \dot{X} = &\ -Y\left(1+\Theta_1(X^2,Y)\right)+X^2\Theta_2(X^2,Y), \\ \dot{Y} = &\ X\left(1+\Theta_1(X^2,Y)\right)+XY\Theta_2(X^2,Y), \end{aligned} \tag{2.20}$$

Remark 45. *Using the notations of* (2.19) *and* (2.20) *after some computations we can prove that the following relations*

$$\begin{aligned} \tilde{\Lambda}(X,Y) = &\ \sum_{j=1}^{m-1}\tilde{\Lambda}_j = \sum_{j=1}^{m-1}\sum_{k+n=j}\lambda_{kn}X^kY^n = \Theta_1(X^2,Y), \\ \tilde{\Omega}(X,Y) = &\ \sum_{j=1}^{m-1}\tilde{\Omega}_j = \sum_{j=1}^{m-1}\sum_{k+n=j}\omega_{kn}X^kY^n = X\Theta_2(X^2,Y), \end{aligned}$$

hold if and only if

$$\begin{aligned} \lambda_{2l-1,j} = &\ 0 \quad \text{for} \quad l=1,2,\ldots,[m/2] \quad \text{and} \quad j=0,\ldots,m-2l, \\ \omega_{2l,j} = &\ 0 \quad \text{for} \quad l=0,2,\ldots,[(m-1)/2] \quad \text{and} \quad j=0,\ldots,m-1-2l, \end{aligned}$$

where [ ] *denotes the integer part function. Consequently,* $[m^2/4]$ *coefficients of* $\tilde{\Lambda}$ *must be zero, and* $[(m^2+2m-3)/4]$ *coefficients of* $\tilde{\Omega}$ *must be zero.*

Corollary 46. *Polynomial differential system* (2.20) *can be written as*

$$\begin{aligned} \dot{X} = &\ -Y\left(1+\Theta_1(X^2,Y)\right)+X\{H_2,\,\Phi\} = P(X,Y), \\ \dot{Y} = &\ X\left(1+\Theta_1(X^2,Y)\right)+Y\{H_2,\,\Phi\} = Q(X,Y), \end{aligned} \tag{2.21}$$

where $\Phi=\Phi(x,y)$ *is a polynomial of degree at most* $m-1$ *such that* $\{H_2,\,\Phi\} = X\Theta_2(X^2,Y)$. *In particular all uniform isochronous centers which after a linear change of variables* $(x,y) \longrightarrow (X,Y)$ *are invariant under the transformations* $(X,Y,t) \longrightarrow (-X,Y,-t)$ *can be written as*

$$\dot{X} = -Y + X\{H_2,\,\Phi\}, \qquad \dot{Y} = X + Y\{H_2,\,\Phi\},$$

Proof. By considering that

$$\int_0^{2\pi} X\Theta_2(X^2,Y)\big|_{x=\cos t,\,y=\sin t}\,dt = \int_0^{2\pi}\Theta_2(1-\sin^2 t,\sin t)\cos t\,dt = 0.$$

By Corollary 22 we get that there exists a polynomial Φ such that $X\Theta_2(X^2,Y) = \{H_2,\,\Phi\}$. Thus we have the formula (2.21). Since for uniform isochronous centers $\Theta_1(X^2,Y)=0$, the corollary follows. □

Proposition 47. *Any weak center invariant at the origin of system* (2.21) *satisfies that the integral of the divergence on the unit circle is zero.*

Proof. The weak center of of system (2.21) satisfies

$$\begin{aligned}\frac{\partial P}{\partial X}+\frac{\partial Q}{\partial Y} &= 2\{H_2,\,\Phi\}+X\frac{\partial\{H_2,\,\Phi\}}{\partial X}+Y\frac{\partial\{H_2,\,\Phi\}}{\partial Y}+\{H_2,\Theta_1\}\\ &= \left\{H_2, 2\Phi+X\frac{\partial\Phi}{\partial X}+Y\frac{\partial\Phi}{\partial Y}+\Theta_1\right\}.\end{aligned}$$

From Proposition 20 satisfy that

$$\int_0^{2\pi}\left(\frac{\partial P}{\partial X}+\frac{\partial Q}{\partial Y}\right)\Big|_{X=\cos t, Y=\sin t}dt=0.$$

□

2.4.2 Proofs of Proposition 41 and Corollary 42

Proof of Proposition 41. Sufficiency: If $\int_0^{2\pi}\Omega_{m-1}(\cos t,\sin t)dt = 0$ then (2.6) holds with

$$\begin{aligned}P &:= -y(1+(m-2)(a_1y-a_2x))+x(a_1x+a_2y+\Omega_{m-1}),\\ Q &:= x(1+(m-2)(a_1y-a_2x))+y(a_1x+a_2y+\Omega_{m-1}),\end{aligned}$$

because Ω_{m-1} is a homogenous polynomial of degree $m-1$ and from the Euler Theorem for homogenous polynomial we have

$$\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}=(m+1)\left(a_1x+a_2y+\Omega_{m-1}\right).$$

Therefore

$$\int_0^{2\pi}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\Big|_{\substack{x=\cos t\\ y=\sin t}}dt=0.$$

Then in view of Proposition 38 we get that there exist polynomials F and G of degree $m+1$ and $m-1$ respectively, such that system (2.14) can be written as

$$\begin{aligned}\dot{x} &= P=\{F,x\}+(1+G)\{H_2,x\},\\ \dot{x} &= Q=\{F,y\}+(1+G)\{H_2,y\}.\end{aligned}\tag{2.22}$$

After some computations we obtain

$$\begin{aligned}\{F,H_2\} &= 2H_2(a_1x+a_2y+\Omega_{m-1}),\\ \{G,H_2\} &= -(m+1)(a_1x+a_2y+\Omega_{m-1}).\end{aligned}$$

Therefore $\{F+\frac{2H_2}{m+1}G,\,H_2\}=0$. Consequently $F=-\frac{2H_2}{m+1}G+p(H_2)$, where

$$p(H_2)=b_0+b_1H_2+\ldots+b_{[m+1)/2]}H_2^{[(m+1)/2]}$$

is an arbitrary polynomial of degree $\leq 2[(m+1)/2]$, where $[\ ]$ denotes the integer part function. Inserting F into the differential system (2.22) and after some computations we get that

$$\begin{aligned} \dot{x} = & \quad P = -\frac{2H_2}{m+1}\{G,x\} + \left(1 + p'(H_2) + \frac{m-1}{m+1}G\right)\{H_2,x\}, \\ \dot{y} = & \quad Q = -\frac{2H_2}{m+1}\{G,y\} + \left(1 + p'(H_2) + \frac{m-1}{m+1}G\right)\{H_2,y\}, \end{aligned}$$

where $p'(H_2) = \dfrac{dp}{dH_2}$. In order to have that the linear terms of the previous system be $-y$ and x respectively, we need that $b_1 = 0$, because $G(0,0) = 0$. Therefore

$$\begin{aligned} \dot{H}_2 = & \quad x\dot{x} + y\dot{y} = -\frac{2H_2}{m+1}\left(x\{G,x\} + y\{G,y\}\right) = \frac{2H_2}{m+1}\{H_2,G\}, \\ \dot{G} = & \quad \frac{\partial G}{\partial x}\dot{x} + \frac{\partial G}{\partial y}\dot{y} = \left(1 + p'(H_2) + \frac{m-1}{m+1}G\right)\left(\frac{\partial G}{\partial x}\{H_2,x\} + \frac{\partial G}{\partial y}\{H_2,y\right) \\ = & \quad \{H_2,G\}\left(1 + \frac{m-1}{m+1}G + p'(H_2)\right). \end{aligned}$$

Consequently we have the following linear ordinary differential of first order

$$\frac{dG}{dH_2} - \frac{(m-1)G}{2H_2} = \frac{(m+1)(1+p'(H_2))}{2H_2}.$$

Thus after the integration we have the existence of a first integral H given by

$$\begin{aligned} H = & \quad \frac{H_2}{\left(1 + \frac{m-1}{m+1}G(x,y) + (m-1)\Gamma(H_2) + [(m+1)/2]\Phi(H_2)\right)^{2/(m-1)}} \\ = & \quad H_2(1 + O(x,y)), \end{aligned} \tag{2.23}$$

hence the origin is a weak center, where

$$\Gamma(H_2) = \frac{2b_2}{m-3}H_2 + \frac{3b_3}{m-5}H_2^2 + \ldots + \frac{([(m+1)/2]-1)b_{[(m+1)/2]-1}}{[(m1+)/2]-1-(m+1)/2}H_2^{[(m+1)/2]-2},$$

and

$$\Phi(H_2) = \begin{cases} \dfrac{b_{[(m+1)/2]}}{[(m1+)/2]-(m+1)/2}H_2^{[(m+1)/2]-1} & \text{if } m \text{ is even,} \\ b_{[(m+1)/2]}H_2^{(m-1)/2}\log H_2 & \text{if } m \text{ is odd.} \end{cases}$$

We observe that if m is odd then this first integral (2.23) is non analytic at the origin.

Necessity: Now we suppose that the origin of system (2.14) is a center and we must prove that (2.15) holds. Indeed, if the origin is a center then from Theorem 8 it follows that differential system (2.14) can be written as (see (14)), where the homogenous polynomials H_k and g_l for $k = 2, \ldots, m+1$ and $l = 1, \ldots, m-1$ satisfy

$$
\begin{aligned}
\{H_{m+1}, x\} + g_1\{H_m, x\} \ldots + g_{m-1}\{H_2, x\} :=&\ X_m = x\Omega_{m-1},\\
\{H_{m+1}, y\} + g_1\{H_m, y\} \ldots + g_{m-1}\{H_2, y\} =&\ Y_m = y\Omega_{m-1},\\
\{H_m, x\} + g_1\{H_{m-1}, x\} \ldots + g_{m-2}\{H_2, x\} =&\ 0,\\
\{H_m, y\} + g_1\{H_{m-1}, y\} \ldots + g_{m-2}\{H_2, y\} =&\ 0,\\
\vdots \quad \vdots \quad \vdots \quad &\ \ \vdots \quad ,\\
\{H_5, x\} + g_1\{H_4, x\} + g_2\{H_3, x\} + g_3\{H_2, x\} =&\ 0,\\
\{H_5, y\} + g_1\{H_4, y\} + g_2\{H_3, y\} + g_3\{H_2, y\} =&\ 0,\\
\{H_4, x\} + g_1\{H_3, x\} + g_2\{H_2, x\} =&\ 0,\\
\{H_4, y\} + g_1\{H_3, y\} + g_2\{H_2, y\} =&\ 0,\\
\{H_3, x\} + g_1\{H_2, x\} :=&\ X_2\\
=&\ -y(m-2)(a_1y - a_2x) + x(a_1x + a_2y),\\
\{H_3, y\} + g_1\{H_2, y\} :=&\ Y_2\\
=&\ x(m-2)(a_1y - a_2x) + y(a_1x + a_2y).
\end{aligned}
\tag{2.24}
$$

From the last two equations we obtain (see (1.27) and (1.28))

$$g_1 = (m+1)(a_1y - a_2x), \quad H_3 = -\frac{2}{m+1}H_2g_1.$$

After from the third and fourth equations starting from the end of system (2.24) we have (see (1.29) and (1.31))

$$g_2 = -\frac{1}{m+1}g_1^2 + cH_2, \quad H_4 = \frac{H_2}{4(m+1)}\left(8g_1^2 - 2c(m+1)H_2\right).$$

Later from the fifth and sixth equations starting from the end of system (2.24) we get

$$
\begin{aligned}
g_3 =&\ \frac{2(m+3)}{3(m+1)^2}g_1^3 - \frac{(m-1)c}{m+1}g_1H_2,\\
H_5 =&\ \frac{g_1H_2}{15(m+1)^2}\left((28m+54)g_1^2 - \frac{3\,c\,H_2}{2}(m+1)(2m+3)\right),
\end{aligned}
$$

where c is an arbitrary constant. Continuing with these computations we can deduce that

$$H_k = H_2\left(P_{k-2}(g_1) + P_{k-4}(g_1)H_2 + P_{k-6}(g_1)H_2^2 + \ldots + f_{k-2}(g_1, H_2),\right)$$

where

$$f_{k-2}(g_1, H_2) = \begin{cases} aH_2^j, & \text{if } k-2 = 2j, \\ P_1(g_1)H_2^j, & \text{if } k-2 = 2j+1 \end{cases}$$

where $P_k(g_1)$ is a polynomial of degree k for $k = 1, \dots, m+1$.

Inserting the previous expressions into the homogenous polynomial $\dfrac{\partial X_m}{\partial x} + \dfrac{\partial Y_m}{\partial y}$ and by considering that

$$\begin{aligned} \{p(g_1)H_2,\, q(g_1)H_2\} = &\ \{H_2,\, \Psi(g_1)H_2H_2\}, \\ \frac{\partial x\Omega_{m-1}}{\partial x} + \frac{\partial y\Omega_{m-1}}{\partial y} = &\ x\frac{\partial\Omega_{m-1}}{\partial x} + y\frac{\partial\Omega_{m-1}}{\partial y} + 2\Omega_{m-1} = (m+1)\Omega_{m-1}, \end{aligned}$$

where $\Psi(g_1)$ is a convenient function on g_1. we get

$$\{H_m, g_1\} + \{H_{m-1}, g_2\} \ldots + \{H_2, g_{m-1}\} = (m+1)\Omega_{m-1}.$$

After some computations it follows that

$$\{H_m, g_1\} + \{H_{m-1}, g_2\} \ldots + \{H_2, g_{m-1}\} = \{H_2,\, \Phi(g_1, H_2)\} = (m+1)\Omega_{m-1}$$

which we obtain from the compatibility conditions of the first two equations of (2.24). Consequently in view of Proposition 20 we get that $\displaystyle\int_0^{2\pi} \Omega_{m-1}(\cos t, \sin t)dt = 0$. In short the proposition is proved. □

Proof of Corollary 42. It follows trivially from Proposition 41 under the conditions $a_1 = a_2 = 0$. □

Remark 48. *Polynomial differential system* (2.14) *under the condition* (2.15) *can be written as*

$$\begin{aligned} \dot{x} = &\ -y(1 + (m-2)(a_1y - a_2x)) + x(a_1x + a_2y + \Omega_{m-1}) \\ = &\ -y(1 + (m-2)(a_1y - a_2x)) + x\{H_2,\, \Psi\}, \\ \dot{y} = &\ x(1 + (m-2)a_1y - a_2x)) + y(a_1x + a_2y + \Omega_{m-1}) \\ = &\ x(1 + (m-2)(a_1y - a_2x)) + y\{H_2,\, \Psi\}, \end{aligned}$$

where $\{H_2,\, \Psi\} = a_1x + a_2y + \Omega_{m-1}$.

2.4.3 Proofs of Theorem 14

The proof of Theorem 14 follows from the next propositions.

Proposition 49. *A quadratic polynomial differential system*

$$\begin{aligned}\dot{x} = & \quad -y(1 + n_1 x + n_2 y) + x(a_1 x + a_2 y), \\ \dot{y} = & \quad x(1 + n_1 x + n_2 y) + y(a_1 x + a_2 y),\end{aligned} \tag{2.25}$$

has a weak center at the origin if and only if

$$a_1 n_1 + a_2 n_2 = 0. \tag{2.26}$$

Moreover the quadratic differential system (2.25) *satisfying condition* (2.26)*, after a linear change of variables* $(x, y) \longrightarrow (X, Y)$ *it is invariant under the transformations* $(X, Y, t) \longrightarrow (-X, Y, -t)$.

Proof. Sufficiency:. We suppose that (2.26) holds and we shall show that then the origin of system (2.25) is a center. Indeed doing the change of variables given in (2.18) with $\kappa_1^2 + \kappa_2^2 \neq 0$ to system (2.25) we obtain the differential system

$$\begin{aligned}\dot{X} = & \quad -Y\Big(1 + (\kappa_1 n_1 + \kappa_2 n_2)X + (\kappa_1 n_2 - \kappa_2 n_1)Y\Big) \\ & \quad +X\Big((\kappa_1 a_1 + \kappa_2 a_2)X + (\kappa_1 a_2 - \kappa_2 a_1)Y\Big), \\ \dot{Y} = & \quad X\Big(1 + (\kappa_1 n_1 + \kappa_2 n_2)X + (\kappa_1 n_2 - \kappa_2 n_1)Y\Big) \\ & \quad +Y\Big((\kappa_1 a_1 + \kappa_2 a_2)X + (\kappa_1 a_2 - \kappa_2 a_1)Y\Big).\end{aligned}$$

This system can be written as (2.20) if and only if

$$\kappa_2 a_1 - \kappa_1 a_2 = 0, \qquad \kappa_1 n_1 + \kappa_2 n_2 = 0. \tag{2.27}$$

Clearly system (2.27) has a nonzero solution κ_1 and κ_2 if and only if $a_1 n_1 + a_2 n_2 = 0$. Under condition (2.27) the quadratic system is

$$\begin{aligned}\dot{X} = & \quad -Y\Big(1 + (\kappa_1 n_2 - \kappa_2 n_1)Y\Big) + (\kappa_1 a_1 + \kappa_2 a_2)X^2, \\ \dot{Y} = & \quad X\Big(1 + (\kappa_1 n_2 - \kappa_2 n_1)Y\Big) + (\kappa_1 a_1 + \kappa_2 a_2)XY.\end{aligned}$$

This system is invariant under the transformation $(X, Y, t) \longrightarrow (-X, Y, -t)$, so in view of Theorem 23 the origin is a center, which in view of Theorem 9 is a weak center.

Necessity: We now suppose that system (2.25) has a center at the origin. From Theorem 8 we get that quadratic differential system (2.25) can be written as

$$\begin{aligned}\dot{x} = & \quad \{H_3, x\} + (1 + g_1)\{H_2, x\} \\ = & \quad -y(1 + n_1 x + n_2 y) + x(a_1 x + a_2 y) := -y + X_2, \\ \dot{y} = & \quad \{H_3, y\} + (1 + g_1)\{H_2, y\} \\ = & \quad x(1 + n_1 x + n_2 y) + y(a_1 x + a_2 y) := x + Y_2.\end{aligned}$$

Thus

$$\begin{aligned} \{H_3,x\}+g_1\{H_2,x\} = &\ -y(n_1x+n_2y)+x(a_1x+a_2y) := X_2,\\ \{H_3,y\}+g_1\{H_2,y\} = &\ x(n_1x+n_2y)+y(a_1x+a_2y) := Y_2. \end{aligned}$$

By determining the homogenous polynomial g_1 as the unique solution of the equation (see (1.27))

$$\{H_2,g_1\} = \frac{\partial X_2}{\partial x}+\frac{\partial Y_2}{\partial y} = (n_2+3a_1)x+(3a_2-n_1)y,$$

we get that

$$g_1 = (n_1-3a_2)x+(n_2+3a_1)y. \tag{2.28}$$

In view of homogeneity of H_2 and H_3 we obtain that (see (1.28))

$$H_3 = \frac{1}{3}\left(xY_2-yX_2-2g_1H_2\right) = 2H_2(a_2x-a_1y).$$

On the other hand from conditions (1.26) since $X_3 = Y_3 = 0$ we get that

$$\frac{\partial H_4}{\partial y} = -g_1\frac{\partial H_3}{\partial y}-yg_2, \quad \frac{\partial H_4}{\partial x} = -g_1\frac{\partial H_3}{\partial x}-xg_2.$$

We shall determine g_2 as a solution of the first order partial differential equation (see (1.29))

$$\{H_2,g_2\}+\{H_3,g_1\} = 0,$$

where g_1 is given in (2.28). Then in view of Corollary 22 with $V = g_2$ and $U = -\{H_3,g_1\}$, we get that if $\int_0^{2\pi}\{H_3,g_1\}|_{x=\cos t,\,y=\sin t}\,dt = 4\pi(a_1n_1+a_2n_2) = 0$, then g_2 exists and has the form $g_2(x,y) = \tilde{g}_2(x,y)+cH_2$ where c is a constant. Thus we prove the necessity of the condition (2.26). In short the proposition is proved. □

Proposition 50. *A cubic polynomial differential system*

$$\begin{aligned} \dot{x} = &\ -y(1+\mu(a_2x-a_1y))\\ &+x(a_1x+a_2y+a_3x^2+a_4y^2+a_5xy),\\ \dot{y} = &\ x(1+\mu(a_2x-a_1y))\\ &+y(a_1x+a_2y+a_3x^2+a_4y^2+a_5xy), \end{aligned} \tag{2.29}$$

has a weak center at the origin if and only if

$$a_3+a_4 = 0, \qquad a_1a_2a_5+(a_2^2-a_1^2)a_4 = 0, \tag{2.30}$$

Moreover system (2.29) *under condition* (2.30) *and* $(\mu+1)(a_1^2+a_2^2) \neq 0$, *after a linear change of variables* $(x,y) \longrightarrow (X,Y)$ *is invariant under the transformations* $(X,Y,t) \longrightarrow (-X,Y,-t)$.

We observe that Proposition 50 when $\mu = 0$ provides Theorem 2.2 of the Collins [15].

Proof. Sufficiency: First of all we observe that cubic differential system (2.29) under the change of variables (2.18) becomes

$$\begin{aligned}
\dot{X} = & -Y\Big(1+\mu(\kappa_1 a_2-\kappa_2 a_1)X-\mu(\kappa_2 a_2+\kappa_1 a_1)Y\Big)\\
& +\frac{X}{(\kappa_1^2+\kappa_2^2)^2}\Big((\kappa_1^2+\kappa_2^2)\left((\kappa_1 a_1+\kappa_2 a_2)X+(\kappa_1 a_2-\kappa_2 a_1)Y\right)\\
& +(\kappa_1^2 a_3+\kappa_2^2 a_4+\kappa_1\kappa_2 a_5)X^2\\
& +(\kappa_2^2 a_3+\kappa_1^2 a_4-\kappa_1\kappa_2 a_5)Y^2+((\kappa_1^2-\kappa_2^2)a_5+2\kappa_1\kappa_2(a_4-a_3))XY\Big),\\
\dot{Y} = & \; X\Big(1+\mu(\kappa_1 a_1-\kappa_2 a_2)X-\mu(\kappa_1 a_2-\kappa_2 a_1)Y\Big)\\
& +\frac{Y}{(\kappa_1^2+\kappa_2^2)^2}\Big((\kappa_1^2+\kappa_2^2)\left((\kappa_1 a_1+\kappa_2 a_2)X+(\kappa_1 a_2-\kappa_2 a_1)Y\right)\\
& +(\kappa_1^2 a_3+\kappa_2^2 a_4+\kappa_1\kappa_2 a_5)X^2\\
& +(\kappa_2^2 a_3+\kappa_1^2 a_4-\kappa_1\kappa_2 a_5)Y^2+((\kappa_1^2-\kappa_2^2)a_5+2\kappa_1\kappa_2(a_4-a_3))XY\Big),
\end{aligned} \tag{2.31}$$

is invariant under the transformation $(X,Y,t) \longrightarrow (-X,Y,-t)$ if and only if using Proposition 49 we have

$$\begin{aligned}
\kappa_2 a_1-\kappa_1 a_2 = & \; 0,\\
\kappa_1^2 a_3+\kappa_2^2 a_4+\kappa_1\kappa_2 a_5 = & \; 0,\\
+\kappa_2^2 a_3+\kappa_1^2 a_4-\kappa_1\kappa_2 a_5 = & \; 0.
\end{aligned} \tag{2.32}$$

From these conditions it follows that κ_1 and κ_2 such that $\kappa_1^2+\kappa_2^2 \neq 0$ if and only if

$$a_1\kappa_2 - a_2\kappa_1 = 0 \quad \text{and} \quad a_3+a_4=0, \qquad k_1k_2a_5+(k_2^2-k_1^2)a_4=0.$$

We suppose that (2.30) holds and show that then the origin of system (2.29) is a center. Clearly if $a_1^2+a_2^2 \neq 0$, then after the change $x = a_1X - a_2Y, \quad y = a_2X+a_1Y$, we get that system (2.31) coincide with system (2.20) when $\kappa_1 = a_1$ and $\kappa_2 = a_2$, and consequently (2.32) becomes (2.30). Since system (2.31) is invariant under the change $(X,Y,t) \longrightarrow (-X,Y,-t)$ i.e. is reversible by Poincaré Theorem its origin is a center, and by Theorem 9 it is a weak center. Thus the sufficiency of proposition is proved.

Necessity We suppose that the origin of (2.29) is a center. We must show that (2.30) holds. Indeed from Theorem 8 it follows that differential system (2.29) can be written as

$$\begin{aligned}
&\{H_4,\, x\} + (1+g_1)\{H_3, x\} + (1+g_1+g_2)\{H_2, x\}\\
=\;& -y(1+\mu(a_2x-a_1y)) + x(a_1x+a_2y+a_3x^2+a_4y^2+a_5xy),\\
&\{H_4,\, y\} + (1+g_1)\{H_3, y\} + (1+g_1+g_2)\{H_2, y\}\\
=\;& x(1+\mu(a_2x-a_1y)) + y(a_1x+a_2y+a_3x^2+a_4y^2+a_5xy).
\end{aligned}$$

Consequently taking the homogenous parts of these two previous equalities we obtain

$$\begin{aligned}
\{H_4,\, x\} + g_1\{H_3, x\} + g_2\{H_2, x\} =\;& -\frac{\partial H_4}{\partial y} - g_1\frac{\partial H_3}{\partial y} - g_2\frac{\partial\, H_2}{\partial y}\\
=\;& x(a_3x^2+a_4y^2+a_5xy) := x\Omega_2 = X_3,\\
\{H_4,\, y\} + g_1\{H_3, y\} + g_2\{H_2, y\} =\;& \frac{\partial H_4}{\partial x} + g_1\frac{\partial H_3}{\partial x} + g_2\frac{\partial\, H_2}{\partial x}\\
=\;& y(a_3x^2+a_4y^2+a_5xy) := y\Omega_2 = Y_3,\\
\{H_3, x\} + g_1\{H_2, x\} =\;& -\frac{\partial H_3}{\partial y} - g_1\frac{\partial H_2}{\partial y}\\
=\;& -\mu\, y(a_2x-a_1y) + x(a_1x+a_2y),\\
\{H_3, y\} + g_1\{H_2, y\} =\;& \frac{\partial H_3}{\partial x} + g_1\frac{\partial H_2}{\partial x}\\
=\;& \mu\, x(a_2x-a_1y) + y(a_1x+a_2y).
\end{aligned} \tag{2.33}$$

From the last two equations we have that (see (1.27) and (1.28)))

$$g_1 = (\mu-3)(a_2x-a_1y), \quad \text{and} \quad H_3 = 2(a_2x-a_1y)H_2 := \Upsilon_1 H_2.$$

Inserting g_1 and H_3 into the two first equations and taking into account the Euler Theorem for homogenous polynomials we get that $\dfrac{\partial\, X_3}{\partial x}+\dfrac{\partial\, Y_3}{\partial x} = 4\Omega_2$ consequently (see (1.29) and (1.30))

$$\{H_2, g_2\} + \{H_3, g_1\} - 4\Omega_2 = 0,$$

and this equation has solution if and only if (see Corollary 22)

$$\int_0^{2\pi} (\{H_3, g_1\} - 4\Omega_2)|_{x=\cos t,\, y=\sin\, t}\, dt = a_3 + a_4 = 0,$$

Hence the first two equations of system (2.33) have a solution if and only if $a_3+a_4 =$

0. The solutions of (2.33) are

$$\begin{aligned} H_4 = &\Big(9a_2^2 - 3a_2^2\mu - c)x^2 + (4a_1a_2\mu + 4a_4 - 12a_1a_2)yx \\ &+(6a_1^2 + (2a_1^2 + a_2^2)\mu + 3a_2^2 - 2a_2a_5 - c)y^2\Big)H_2/2 := \Upsilon_2 H_2, \\ g_2 = &\ \alpha_1 x^2 + (-6a_1a_2 + 2a_1\mu - 4a_4)yx \\ &+((a_2^2 - a_1^2)(\mu - 3) + 2a_5 + \alpha_1)y^2, \end{aligned}$$

where c is an arbitrary constant. By inserting $a_3 + a_4 = 0$ into (2.29) we get

$$\begin{aligned} \dot{x} = &\ -y(1 - \mu(a_1y - a_2x)) + x(a_1x + a_2y - a_4x^2 + a_4y^2 + a_5xy), \\ \dot{y} = &\ x(1 - \mu(a_2x - a_1y)) + y(a_1x + a_2y - a_4x^2 + a_4y^2 + a_5xy), \end{aligned} \tag{2.34}$$

By calculating g_3 as the unique polynomial homogenous of degree four solution of the equation $\{H_4, g_1\} + \{H_3, g_2\} + \{H_2, g_3\} = 0$, (see (1.32)) and determining H_5 from (1.32) with $X_4 = Y_4 = 0$ we get $H_5 = -4g_1H_4/5 - 3g_2H_3/5 - 2g_3H_2/5$. From equation

$$\{H_5, g_1\} + \{H_4, g_2\} + \{H_3, g_3\} + \{H_2, g_4\} = 0,$$

(see (1.34) with $m = 5$), and in view of Proposition 20 we get that this equation has a solution for the homogenous polynomial g_4 if and only if

$$\begin{aligned} 0 = &\ \int_0^{2\pi} (\{H_5, g_1\} + \{H_4, g_2\} + \{H_3, g_3\})|_{x=\cos t,\, y=\sin t} \\ = &\ 3\pi(\mu + 1)^2(a_4(a_2^2 - a_1^2) + a_1a_2a_5) = 0. \end{aligned}$$

Thus $a_4(a_2^2 - a_1^2) + a_1a_2a_5 = 0$.

The case when $\mu = -1 = -\left.(m-2)\right|_{m=3}$ has been studied in Proposition 41.

If $a_1 = a_2 = 0$ then system (2.34) becomes

$$\begin{aligned} \dot{x} = &\ -y + x(-a_4x^2 + a_4y^2 + a_5xy) := -y + x\Omega_2, \\ \dot{y} = &\ x + y(-a_4x^2 + a_4y^2 + a_5xy) = x + y\Omega_2, \end{aligned}$$

which is a polynomial differential system with homogenous nonlinearities. Consequently we have that

$$\int_0^{2\pi} \Omega_2|_{x=\cos t, y=\sin t}\, dt = \int_0^{2\pi} \left(-a_4x^2 + a_4y^2 + a_5xy\right)\Big|_{x=\cos t, y=\sin t} dt = 0.$$

Hence and in view of Corollary 42 we obtain that the origin is a weak center. In short the proposition is proved. □

Proposition 51. *A quartic polynomial differential system*

$$
\begin{aligned}
\dot{x} = &\ -y(1+\mu(a_2x-a_1y)) \\
&+x(a_1x+a_2y+a_6x^3+a_7y^3+a_8x^2y+a_9xy^2), \\
\dot{y} = &\ x(1+\mu(a_2x-a_1y)) \\
&+y(a_1x+a_2y+a_6x^3+a_7y^3+a_8x^2y+a_9xy^2),
\end{aligned}
\tag{2.35}
$$

where $(\mu+2)(a_1^2+a_2^2 \neq 0$*, has a weak center at the origin if and only if*

(i) $a_1a_2 \neq 0$ *and*

$$
\begin{aligned}
a_6+\frac{1}{2a_2^3}\left(a_7(a_1^3-3a_2^2a_1)+a_9(a_2^3-a_1^2a_2)\right) &= 0, \\
a_8+\frac{1}{2a_2^2a_1}\left(a_7(3a_1^3-3a_1a_2^2)+a_9(a_2^3-3a_1^2a_2)\right) &= 0.
\end{aligned}
\tag{2.36}
$$

(ii) $a_7=a_8=0$ *when* $a_2=0$ *and* $a_1 \neq 0$*,*

(iii) $a_6=a_9=0$ *when* $a_1=0$ *and* $a_2 \neq 0$*.*

Moreover the system (2.35) under the conditions (i) , (ii) and after a linear change of variables $(x,y) \longrightarrow (X,Y)$ *it is invariant under the transformations* $(X,Y,t) \longrightarrow (-X,Y,-t)$*, and in the case (iii) it is invariant under the transformations* $(X,Y,t) \longrightarrow (X,-Y,-t)$*,*

Proof. Sufficiency: Doing the change of variables (2.18) system (2.35) can be written as differential system (2.19) with $m=4$ and

$$
\begin{aligned}
\tilde{\Lambda} = &\ \mu(a_2\kappa_1-a_1\kappa_2)X-\mu(a_1\kappa_1+a_2\kappa_2)Y, \\
\tilde{\Omega} = &\ \left(\kappa_1^3a_6+\kappa^3a_7+a_8\kappa_1^2\kappa_2+a_9\kappa_1\kappa_2^2\right)X^3+\left(-a_6\kappa_2^3+a_7\kappa_1^3+\kappa_1\kappa_2(\kappa_2a_8-\kappa_1a_9)\right)Y^3 \\
&+\left(3\kappa_1\kappa_2(\kappa_2a_7-\kappa_1a_6)+a_8(\kappa_1^3-2\kappa_1\kappa_2^2)+a_9(2\kappa_1\kappa_2^2-\kappa_2^3)\right)YX^2 \\
&+\left(3\kappa_1\kappa_2(\kappa_1a_7+\kappa_1a_6)+a_8(\kappa_2^3-2\kappa_1^2\kappa_2)+a_9(-2\kappa_1\kappa_2^2+\kappa_1^3)\right)Y^2X \\
&+(a_1\kappa_1+a_2\kappa_2)X+(a_2\kappa_1-a_1\kappa_2)Y
\end{aligned}
$$

which is invariant under the transformation $(X,Y,t) \longrightarrow (-X,Y,-t)$ if and only if

$$
\begin{aligned}
\kappa_2a_1-\kappa_1a_2 &= 0, \\
-a_6\kappa_2^3+a_7\kappa_1^3+\kappa_1\kappa_2(\kappa_2a_8-\kappa_1a_9) &= 0, \\
3\kappa_1\kappa_2(\kappa_2a_7-\kappa_1a_6)+a_8(\kappa_1^3-2\kappa_1\kappa_2^2)+a_9(2\kappa_1\kappa_2^2-\kappa_2^3) &= 0.
\end{aligned}
$$

Hence if

(i) $\kappa_1\kappa_2 \neq 0$ then

$$\begin{aligned} \kappa_2 a_1 - \kappa_1 a_2 = & \ 0, \\ a_6 + \frac{1}{2\kappa_2^3}\left(a_7(\kappa_1^3 - 3\kappa_2^2\kappa_1) + a_9(\kappa_2^3 - \kappa_1^2\kappa_2)\right) = & \ 0, \\ a_8 + \frac{1}{2\kappa_2^2\kappa_1}\left(a_7(3\kappa_1^3 - 3\kappa_1\kappa_2^2) + a_9(\kappa_2^3 - 3\kappa_1^2\kappa_2)\right) = & \ 0- \end{aligned}$$

(ii) $\kappa_2 = 0$ $a_1 \neq 0$ then $a_2 = a_7 = a_8 = 0$.

(iii) $\kappa_1 = 0$ $a_2 \neq 0$, then $a_1 = a_6 = a_9 = 0$.

We suppose that statement (i), (ii) or (iii) hold and we will show that then the origin is a center of system (2.35). Indeed, if $a_1a_2 \neq 0$, then after the change $x = a_1X - a_2Y, \quad y = a_2X + a_1Y$, we get that system (2.35) coincide with system (2.20) for $m = 4$ and with $\kappa_1 = a_1$ and $\kappa_2 = a_2$ and consequently by considering that this system is invariant under the change $(X, Y, t) \longrightarrow (-X, Y, -t)$ or $(X, Y, t) \longrightarrow (X, -Y, -t)$, i.e. it is reversible. Hence in view Theorem 23 the origin is a center.

If $a_2 = a_7 = a_8 = 0$ and $a_1 \neq 0$, then differential equations (2.35) become

$$\begin{aligned} \dot{x} = & \ -y(1 - \mu\, a_1 y)) + x^2(a_1 + a_6x^2 + a_9y^2), \\ \dot{y} = & \ x(1 - \mu\, a_1 y)) + yx(a_1 + a_6x^2 + +a_9y^2). \end{aligned}$$

This system is invariant under the change $(x, y, t) \longrightarrow (-x, y, -t)$, i.e. it is reversible. Hence in view Theorem 23 the origin is a center.

If $a_1 = a_6 = a_9 = 0$ and $a_2 \neq 0$, then differential system (2.35) becomes

$$\begin{aligned} \dot{x} = & \ -y(1 + \mu\, a_2 x)) + xy(a_2 + a_7y^2 + a_8x^2), \\ \dot{y} = & \ x(1 + \mu\, a_2 x)) + y^2(a_2 + a_7y^2 + a_8x^2). \end{aligned}$$

this system is invariant under the change $(x, y, t) \longrightarrow (x, -y, -t)$ i.e. it is reversible. Hence in view Theorem 23 the origin is a center.

Thus in view of the Poincaré Theorem (see Theorem 23) we get that in cases (i), (ii) and (iii) the origin is a center. Furthermore by Theorem 9 this center is a weak center.

Necessity: We prove the necessity of the statement (a). We suppose that the origin is a weak center. Indeed, from Theorem 8 it follows that differential system

(2.35) can be written as

$$
\begin{aligned}
& \{H_5,\, x\} + (1+g_1)\{H_4, x\} + (1+g_1+g_2)\{H_3, x\} + (1+g_1+g_2+g_3)\{H_2, x\} \\
= \;& -y(1+\mu(a_2x-a_1y)) + x(a_1x+a_2y+a_6x^3+a_7y^3+a_8x^2y+a_9xy^2) \\
= \;& -y + X_2 + X_4, \\
& \{H_5,\, y\} + (1+g_1)\{H_4, y\} + (1+g_1+g_2)\{H_3, y\} + (1+g_1+g_2+g_3)\{H_2, y\}, \\
= \;& x(1+\mu(a_2x-a_1y)) + y(a_1x+a_2y+a_6x^3+a_7y^3+a_8x^2y+a_9xy^2) \\
= \;& x + Y_2 + Y_4.
\end{aligned}
\tag{2.37}
$$

In view of Corollary 22 and assisted by an algebraic computer system we can obtain the solutions of (2.37), i.e. the homogenous polynomials H_5, H_3, g_1, g_3 of degree odd are unique and the homogenous polynomials H_4, g_2 of degree even are obtained modulo an arbitrary polynomial of the form $c(x^2+y^2)^k$ where $k = 1, 2$. Indeed taking the homogenous part of these equations we obtain

$$
\begin{aligned}
\{H_3, x\} + g_1\{H_2, x\} = &\; -y(\mu(a_2x-a_1y)) + x(a_1x+a_2y), \\
\{H_3, y\} + g_1\{H_2, y\} = &\; x(\mu(a_2x-a_1y)) + y(a_1x+a_2y).
\end{aligned}
$$

The solutions of these equations are

$$
g_1 = (\mu-3)(a_2x-a_1y), \qquad H_3 = 2H_2(a_2x-a_1y).
$$

$$
\begin{aligned}
\{H_4, x\} + g_1\{H_3, x\} + g_2\{H_2, x\} = &\; 0, \\
\{H_4, x\} + g_1\{H_3, x\} + g_2\{H_2, x\} = &\; 0,
\end{aligned}
$$

the compatibility condition of these two last equations becomes of $\{H_3, g_1\} + \{H_2, g_2\} = 0$, and since $\{H_3, g_1\} = a_1a_2(x^2-y^2) + (a_2^2-a_1^2)xy$. By considering that

$\displaystyle\int_0^{2\pi} \{H_3,\, g_1\}|_{x=\cos t, y=\sin t} = 0$, we get that the compatibility conditions $\{H_2, g_2\} + \{H_3, g_1\} = 0$ has solutions. Therefore after some computations we get that $g_2 = (3-\mu)(a_2x-a_1y)^2 + c_2H_2$, where c_2 is a constant. Then from system (2.69) we obtain the solution

$$
H_4 = \frac{H_2}{2}((\mu-3)\left((a_1^2-a_2^2)(x^2-y^2) + 4a_1a_2xy\right) + c_1H_2^2,
$$

where c_1 is a constant. Inserting these previous solutions g_1, H_3, g_2 and H_4 into the partial differential equations

$$
\begin{aligned}
& \{H_5,\, x\} + g_1\{H_4, x\} + g_2\{H_3, x\} + g_3\{H_2, x\} \\
= \;& x(a_6x^3+a_7y^3+a_8x^2y+a_9xy^2) = x\Omega_3 := X_4, \\
& \{H_5,\, y\} + g_1\{H_4, y\} + g_2\{H_3, y\} + g_3\{H_2, y\} \\
= \;& y(a_6x^3+a_7y^3+a_8x^2y+a_9xy^2) = y\Omega_3 := Y_4
\end{aligned}
$$

we get that these differential equations have a unique solution. Indeed, in this case (1.32) becomes

$$\{H_4, g_1\} + \{H_3, g_2\} + \{H_2, g_3\} = 5\Omega_3, \tag{2.38}$$

because $\dfrac{\partial X_4}{\partial x} + \dfrac{\partial Y_4}{\partial y} = 5\Omega_3$, here we have taking into account that Ω_3 is a homogenous polynomial of degree 3. Consequently there exist a unique solution

$$\begin{aligned} g_3 = \;& \left(a_1(a_2^2 + 2a_1^2/3)\mu^2 - a_1(10a_1^2/3 + 4a_2^2)\mu + a_1(4a_1^2 + 3a_3^2) + 5(a_9 + 2a_6)/3\right) y^3 \\ & \left(-a_2(a_1^2 + 2a_2^2/3)\mu^2 + a_2(10a_2^2/3 + 4a_1^2)\mu - (3a_1^2 + 4a_2^2)a_2 - 5/3(a_8 + 2a_7)\right) x^3 \\ & + \left(a_1^3\mu^2 - 2(2a_1^2 + a_2^2)a_2\mu + 3a_1(a_1^2 + 2a_2^2) + 7a_6\right) x^2y \\ & + \left(-a_2^3\mu^2 + 2(2a_2^2 + a_1^2)a_2\mu - 3a_2(a_1^2 + 2a_2^2) - 5a_7\right) xy^2 \\ & + c(1-\mu)(a_1x + a_2y)H_2 \end{aligned}$$

of (2.38) and substituting g_3 into (2.70) we get

$$\begin{aligned} H_5 = \;& -2/3H_2\big(\left(-a_2^3\mu^2 + a_2(2a_2^2 - 3a_1^2)\mu + 3a_2(3a_1^2 + a_2^2) - a_8 - 2a_7\right) x^3 \\ & - \left(a_1^3\mu^2 + a_1(2a_1^2 - 3a_2^2)\mu + 3a_1(3a_2^2 + a_1^2) + a_9 + 2a_6\right) y^3 \\ & \left(3a_1a_2^2\mu^2 + 3a_1(a_1^2 - 4a_2^2)\mu + 9a_1(a_2^2 - a_1^2) + 3a_6\right) yx^2 \\ & - \left(3a_2a_1^2\mu^2 + 3a_2(a_1^2 - 4a_2^2)\mu + 9a_2(a_2^2 - a_1^2) - 3a_7\right) y^2x \\ & cH_2^2(3a_1y - 2a_2x), \end{aligned}$$

where c is a constant.

On the other hand from (27) for $m = 4$ and $j = 0$ we get that

$$\begin{aligned} & \int_0^{2\pi} \left(\{H_5,\, g_1\} + \{H_4,\, g_2\} + \{H_3,\, g_3\}\right)|_{x=\cos t, y=\sin t}\, dt \\ & = -\frac{3\pi(\mu-3)}{3}\left(a_1(3a_7 + a_8) - a_2(3a_6 + a_9)\right) = 0. \end{aligned} \tag{2.39}$$

Hence we get that

(i) If $a_1a_2 \neq 0$ then by introducing the notations

$$\begin{aligned} \lambda_2 = \;& a_6 + \frac{1}{2a_2^3}\left(a_7(a_1^3 - 3a_2^2a_1) + a_9(a_2^3 - a_1^2a_2)\right), \\ \lambda_3 = \;& a_8 + \frac{1}{2a_2^2a_1}\left(a_7(3a_1^3 - 3a_1a_2^2) + a_9(a_2^3 - 3a_1^2a_2)\right), \end{aligned}$$

we get that (2.39) becomes

$$-\frac{3\pi(\mu-3)}{3}\left(3a_2\lambda_2 - a_1\lambda 3\right) = 0.$$

(ii) If $a_1 = 0$ and $a_2 \neq 0$ then $3a_7 + a_8 = 0$.

(iii) If $a_2 = 0$ and $a_1 \neq 0$ then $3a_6 + a_9 = 0$.

Again from (1.33) with $m = 5$ we have the differential equation

$$\{H_5, g_1\} + \{H_4, g_2\} + \{H_3, g_3\} + \{H_2, g_4\} = 0,$$

From this equation we get the homogenous polynomial of degree four g_4 which in view of Corollary 22 can be obtained with an arbitrary term of the form $c(x^2+y^2)^2$, i.e. g_4 is equal to

$$\Big((-9a_1^4a_2 + 11(3a_7 + a_8)a_1^2 - a_2^2(3(11a_7 + a_8) + 54(a_7 + a_8)a_2^2\Big)x^4 + \frac{1}{18a_2}\Big($$
$$(a^4 - a_2^4)(3a_2\mu^3 + 6a_2\mu^2) + a_2^4 - a_2a_9)\mu - (18(3a_7 + a_8)a_1^2 - 36a_2a_1a_9 + a_8)a_1^2$$
$$+(3a_1^2a_2((a_1^2 - 3a_2^2)\mu^3 + 2(a_2^2 - a_1^2)\mu^2 + a_1^2(27a_2^3 + 15a_7 + 5a_8 - 9a_1^2a_2)$$
$$-2a_2(a_1a_9 + 9a_2a_7)\mu + 54a_2(a_2a_7 - a_1a_9) - a_2^2(3(11a_7 + a_8)\Big)x^2y^2$$
$$+\Big(\frac{2}{3}a_2a_1^3(\mu - 2)\mu^2 + (\frac{a_1}{3}(7a_7 + 10a_8) + \frac{5}{2}a_2a_9 - 2a_2a_1^3)\mu - 12a_1a_7\Big)xy^3$$
$$+\Big(\frac{2}{3}a_2a_1^3(\mu - 2)\mu^2 + (a_1(\frac{7a_7}{3} + \frac{4a_8}{3} - 2a_2^3) + \frac{a_2a_9}{3})\mu$$
$$-12a_1a_7 - 4a_1a_8 + 4a_2a_9\Big)x^3y + c(x^2 + y^2)^2,$$

if $a_2 \neq 0$. In analogous form we can obtain the expression of g_4 when $a_1 \neq 0$. Now we determine the homogenous polynomial H_6 from (1.37) with $m = 4$ and $k = 1$ we obtain

$$H_6 = -\frac{5}{6}g_1H_5 - \frac{4}{6}g_2H_4 - \frac{3}{6}g_3H_3 - \frac{2}{6}g_4H_2.$$

Since the integral of the homogenous polynomial of degree 5

$$\int_0^{2\pi} (\{H_6, g_1\} + \{H_5, g_2\} + \{H_4, g_3\} + \{H_3, g_4\})|_{x=\cos t, y=\sin t}\, dt \equiv 0,$$

then we obtain that there is a unique solution for the homogenous polynomial g_5 of degree 5 of the equation

$$\{H_6, g_1\} + \{H_5, g_2\} + \{H_4, g_3\} + \{H_3, g_4\} + \{H_2, g_5\} = 0.$$

see (1.36) with $m = 4$ and $k = 2$.

Calculating the homogenous polynomial H_7 of degree 7 from (1.37) with $m = 4$ and $k = 2$ we get

$$H_7 = -\frac{6}{7}g_1H_6 - \frac{5}{7}g_2H_5 - \frac{4}{7}g_3H_4 - \frac{3}{7}g_4H_3 - \frac{2}{7}g_5H_2,$$

and inserting these polynomials into the integrand of the following homogenous polynomial of degree 6 we have

$$\int_0^{2\pi} (\{H_7, g_1\} + \{H_6, g_2\} + \{H_5, g_3\} + \{H_4, g_4\} + \{H_3, g_5\})|_{x=\cos t, y=\sin t}\, dt = I(a_1, a_2, \mu), \tag{2.40}$$

where I is such that

(iii) if $a_1 = 0$ and $a_2 \neq 0$ then

$$I(a_1, 0, \mu) = -\frac{\pi\, a_1^3(\mu-3)}{63}\left((231\mu^2 + 1230\mu + 1776)a_7 + 7(5\mu+12)(\mu+2)a_9\right) = 0.$$

(ii) if $a_2 = 0$ and $a_1 \neq 0$ then

$$I(0, a_2, \mu) = \frac{5\pi\, a_2^3(\mu-3)}{189}\left(21(5\mu+12)(\mu+2)a_6 + (49\mu^2 + 2\mu - 432)a_9\right) = 0,$$

(i) If $a_1 a_2 \neq 0$ then

$$\begin{aligned} I(a_1, a_2, \mu) = \ & \frac{\pi\,(m-3)}{189}\Big(\left(21(5\mu+12)(\mu+2)(3a_1^2 + 5a_2^2)\right)\lambda_2 + a_1\Big((105a_1^2 + 427a_2^2)\mu^2 \\ & +(426a_1^2 + 2306a_2^2)\mu + 504a_1^2 + 3384a_2^2\Big)\lambda_3\Big) = 0. \end{aligned}$$

By solving equations (2.39) and (2.40) and by considering that the three cases can be represented as a linear system $A\xi = 0$, where $\xi = (a_6, a_9)^T$ in the case (iii), $\xi = (a_7, a_8)^T$ in the case (ii) and $\xi = (\lambda_2, \lambda_3)^T$ in the case (i). By considering that the determinant of the matrix A is $|A| = (\mu-3)(\mu+2)(21\mu^2 + 128\mu + 212)$. Then under the condition $(\mu-3)(\mu+2) \neq 0$ we obtain the necessity of statement (a) of Proposition 51.

Now we study the case when $\mu = 3$. After some computations we get that

$$\begin{aligned} &\int_0^{2\pi} (\{H_5, g_1\} + \{H_4, g_2\} + \{H_3, g_3\})|_{x=\cos t, y=\sin t}\, dt \\ &= -\frac{15\pi}{2}\left(a_1(3a_7 + a_8) - a_2(3a_6 + a_9)\right) = 0. \end{aligned}$$

First we shall study the case when $a_1 = 0$, and $a_2 \neq 0$. From (2.39) we get that $3a_6 + a_9 = 0$. From (2.40) we obtain that the integral is identically zero because $\mu = 3$. On the other hand from the relation

$$\begin{aligned} &\int_0^{2\pi} \Big(\{H_9, g_1\} + \{H_8, g_2\} + \{H_7, g_3\} + \{H_6, g_4\} + \{H_5, g_5\} \\ &+\{H_4, g_6\} + \{H_3, g_7\} + \Big)\Big|_{x=\cos t, y=\sin t}\, dt = I(a_1, a_2) = 0, \end{aligned}$$

where I is a convenient function in a_1 and a_2. If (iii) $a_1 = 0$ and $a_2 \neq 0$ then

$$I(0, a_2) = (105a_2^3a_7 - 105a_2^3a_8 - 864a_2^2c)a_6 + (995a_2^3a_7 + 285a_2^2a_8 - 288a_2^2c)a_9 = 0.$$

By considering that $a_9 = -3a_6$ we obtain that $(3a_7 + a_8)a_6 = 0$. Thus if $3a_7 + a_8 \neq 0$ then $\mu - 3 = a_1 = a_6 = a_9 = 0$. We observe that if $a_1 = 0$ then $\lambda_2 = a_6$ and $\lambda_3 = a_9$. In analogous form we can study the case when $a_2 = 0$ and $a_1 \neq 0$.

If $3a_7 + a_8 = 3a_6 + a_9 = 0$ then we obtain differential system

$$\begin{aligned} \dot{x} = & \ -y(1 + \mu(a_2x - a_1y)) + x\left(a_1x + a_2y + a_6x(x^2 - 3y^2) + a_7y(y^2 - 3x^2)\right), \\ \dot{y} = & \ x(1 + \mu(a_2x - a_1y)) + y\left(a_1x + a_2y + a_6x(x^2 - 3y^2) + a_7y(y^2 - 3x^2)\right), \end{aligned}$$

From (2.36) we have that this system has a weak center at the origin if and only if

$$a_2(3a_1^2 - a_2^2)a_6 + a_1(a_1^2 - 3a_2^2)a_7 = 0.$$

Hence if $a_1 = 0$ then $a_6 = a_9 = 0$, and if $a_2 = 0$ then $a_7 = a_8 = 0$. In short we prove that the origin is a weak center of (2.35) in the case when $\mu = 3$. Thus we obtain the necessity of the condition in statement (a). The necessity and sufficiency of statement (c), i.e. when $\mu + 2 = 0$ follows from Proposition 41. Finally we study the necessity and sufficiency of statement (b). Thus when $a_1 = a_2 = 0$ we obtain that differential system (2.35) becomes

$$\begin{aligned} \dot{x} = & \ -y + x(a_6x^3 + a_7y^3 + a_8xy + a_9xy^2) := -y + x\Omega_3, \\ \dot{y} = & \ x + y(a_6x^3 + a_7y^3 + a_8xy + a_9xy^2) := x + y\Omega_3, \end{aligned}$$

which is a polynomial differential system of degree four with homogenous nonlinearities. Consequently $\int_0^{2\pi} \Omega_3|_{x=\cos t, y=\sin t}\, dt = 0$, then in view of Corollary 42 we get that the origin is a weak center. In short the proposition is proved. □

Proposition 52. *A polynomial differential system of degree five*

$$\begin{aligned} \dot{x} = & \ -y(1 + \mu(a_2x - a_1y)) \\ & +x(a_1x + a_2y + a_{10}x^4 + a_{11}x^3y + a_{12}x^2y^2 + a_{13}xy^3 + a_{14}y^4), \\ \dot{y} = & \ x(1 + \mu(a_2x - a_1y)) \\ & +y(a_1x + a_2y + +a_{10}x^4 + a_{11}x^3y + a_{12}x^2y^2 + a_{13}xy^3 + a_{14}y^4), \end{aligned} \tag{2.41}$$

has a weak center at the origin if and only if

(a)

$$\begin{aligned} a_{12} + 3(a_{10} + a_{14}) = & \ 0, \\ 2a_1^3a_2^3a_{13} - \left(a_1^6 + 7(a_1^2a_2^4 - a_2^4a_1^2)\right)a_{10} - \left(a_1^5a_2 - 4a_1^3a_2^3 + a_1a_2^5\right)a_{11} = & \ 0, \\ 2a_1^2a_2^2a_{14} - \left(a_1^4 - 4a_1^2a_2^2 + a_2^4\right)a_{10} - (a_1^3a_2 - a_1a_2^3)a_{11} = & \ 0. \end{aligned} \tag{2.42}$$

Moreover

(i) *If* $a_1 = 0$ *and* $a_2 \neq 0$ *then* $a_{12} = a_{10} = a_{14} = 0$.

(ii) *If* $a_2 = 0$ *and* $a_1 \neq 0$ *then* $a_{12} = a_{10} = a_{14} = 0$.

(iii) *If* $a_1 a_2 \neq 0$ *then by introducing the notations*

$$\lambda_1 = \ a_{13} - \frac{1}{2a_1^3a_2^3}\Big(\left(a_1^6 + 7(a_1^2a_2^4 - a_1^4a_2^2 - a_2^6\right) a_{10} - \left(a_1^5a_2 - 4a_1^3a_2^3 + a_1a_2^5\right) a_{11}\Big),$$

$$\lambda_2 = \ a_{14} - \frac{1}{2a_1^3a_2^3}\Big(\left(a_1^4 - 4a_1^2a_2^2 + a_2^4\right) a_{10} - (a_1^3a_2 - a_1a_2^3)a_{11}\Big),$$

we get that $\lambda_1 = \lambda_2 = 0$.

(b) $a_1 = a_1 = 0$, *and* $a_{12} + 3(a_{10} + a_{14}) = 0$.

(c) $\mu + 3 = 0$.

Moreover system (2.41) *under the conditions* (2.42) *and* $(\mu + 3)(a_1^2 + a_2^2) \neq 0$ *and after a linear change of variables* $(x, y) \longrightarrow (X, Y)$ *it is invariant under the transformations* $(X, Y, t) \longrightarrow (-X, Y, -t)$, *or* $(X, Y, t) \longrightarrow (X, -Y, -t)$.

Proof. Sufficiency: First we write system (2.41) after the linear change (2.18). Later we observe that the polynomial differential system obtained is invariant under the transformation $(X, Y, t) \longrightarrow (-X, Y, -t)$ if and only if

$$\begin{aligned} \kappa_1 a_2 - \kappa_2 a_1 =& \ \ 0,\\ 2\kappa_1^2\kappa_2^2 a_{12} + (3\kappa_1^4 - 6\kappa_2^2\kappa_1^2 + 3\kappa_2^4)a_{10} + (3\kappa_1^3\kappa_2 - 3\kappa_2^3\kappa_1)a_{11} =& \ \ 0,\\ 2\kappa_1^3\kappa_2^3 a_{13} - (\kappa_1^6 - 7\kappa_1^4\kappa_2^2 + 7\kappa_1^2\kappa_2^4 - \kappa_2^6)a_{10} - (\kappa_1^5\kappa_2 - 4\kappa_2^3\kappa_1^3 + \kappa_1\kappa_2^5)a_{11} =& \ \ 0,\\ 2\kappa_1^2\kappa_2^2 a_{14} - (\kappa_1^4 - 4\kappa_1^2\kappa_2^2 + \kappa_2^4)a_{10} + (\kappa_1\kappa_2^3 - \kappa_1^3\kappa_2)a_{11} =& \ \ 0, \end{aligned}$$

From the last three conditions it follows that $a_{12} + 3(a_{10} + a_{14}) = 0$.

We suppose that (2.42) holds and then the origin is a center of system (2.41), and by Theorem 9 is a weak center. We suppose that $a_1^2 + a_2^2 \neq 0$. Then after the change $x = a_1X - a_2Y, \quad y = a_2X + a_1Y$, we get that system (2.41) coincides with system (2.20) with $\kappa_1 = a_1$ and $\kappa_2 = a_2$, and consequently system (2.41) is invariant under the change $(X, Y, t) \longrightarrow (-X, Y, -t)$. Thus in view of the Poincaré Theorem we get that the origin is a center. Again by Theorem 9 this center is weak.

We observe that under the assumptions of statement (iii) differential system (2.41) becomes

$$\dot{x} = y(1+\mu\, a_2x)+xy(a_2+a_{11}x^3+a_{13}xy^2), \quad \dot{y} = x(1+\mu\, a_2x)+y^2(a_2+a_{11}x^3+a_{13}xy^2).$$

This system is invariant under the transformation $(x, y, t) \longrightarrow (x, -y, -t)$. Under hypothesis of statement (ii) differential system (2.41) becomes

$$\dot{x} = -y(1-\mu\, a_1y)+x^2(a_1+a_{11}x^2y+a_{13}y^3), \quad \dot{y} = x(1-\mu\, a_1y)+yx(a_2+a_{11}x^2y+a_{13}y^2).$$

This system is invariant under the transformation $(x, y, t) \longrightarrow (-x, y, -t)$. Under the assumptions of statement (i) the conditions $\lambda_1 = \lambda_2 = 0$ follows immediately.

Necessity Now we suppose that the origin is a center of (2.41) and we prove that (2.42) holds. Indeed, from Theorem 8 it follows that differential system (2.41) can be written as

$$\begin{aligned}
&\{H_6, x\} + (1+g_1)\{H_5, x\} + (1+g_1+g_2)\{H_4, x\} + (1+g_1+g_2+g_3)\{H_3, x\} \\
&+(1+g_1+g_2+g_3+g_4)\{H_2, x\} \\
=\;& -y(1+\mu(a_2x - a_1y)) + x(a_1x + a_2y + a_{10}x^4 + a_{11}x^3y + a_{12}x^2y^2 + a_{13}xy^3 + a_{14}y^4), \\
&\{H_6, y\} + (1-g_1)\{H_5, y\} + (1+g_1+g_2)\{H_4, y\} + (1+g_1+g_2+g_3)\{H_3, y\} \\
&+(1+g_1+g_2+g_3+g_4)\{H_2, y\} \\
=\;& x(1+\mu(a_2x - a_1y)) + y(a_1x + a_2y + +a_{10}x^4 + a_{11}x^3y + a_{12}x^2y^2 + a_{13}xy^3 + a_{14}y^4),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\{H_3, x\} + g_1\{H_2, x\} &= -y\mu(a_2x - a_1y) + x(a_1x + a_2y), \\
\{H_3, y\} + g_1\{H_2, y\} &= -y\mu(a_2x - a_1y) + y(a_1x + a_2y).
\end{aligned}$$

These equations have the unique solutions g_1 and H_3. Inserting these homogenous polynomials into the equations

$$\begin{aligned}
\{H_4, x\} + g_1\{H_3, x\} + g_2\{H_2, x\} &= 0, \\
\{H_4, y\} + g_1\{H_3, y\} + g_2\{H_2, y\} &= 0,
\end{aligned}$$

we get H_4 and g_2 with arbitrary terms of the form $c_k(x^2+y^2)^k$ with c_k a constant for $k = 1, 2$. Inserting g_1, H_3, g_2 and H_4 into the equations

$$\begin{aligned}
\{H_5, x\} + g_1\{H_4, x\} + g_2\{H_3, x\} + g_3\{H_2, x\} &= 0, \\
\{H_5, y\} + g_1\{H_4, y\} + g_2\{H_3, y\} + g_3\{H_2, y\} &= 0,
\end{aligned}$$

we have a unique solutions g_3 and H_5. Inserting g_j and H_{j+2} for $j = 1, 2, 3$ into the equations

$$\begin{aligned}
&\{H_6, x\} + g_1\{H_5, x\} + g_2\{H_4, x\} + g_3\{H_3, x\} + g_4\{H_2, x\} \\
=\;& x(a_{10}x^4 + a_{11}x^3y + a_{12}x^2y^2 + a_{13}xy^3 + a_{14}y^4) = x\Omega_4, \\
&\{H_6, y\} + g_1\{H_5, y\} + g_2\{H_4, y\} + g_3\{H_3, y\} + g_4\{H_2, y\} \\
=\;& y(a_{10}x^4 + a_{11}x^3y + a_{12}x^2y^2 + a_{13}xy^3 + a_{14}y^4) = y\Omega_4,
\end{aligned}$$

we get that this partial differential system has solution if and only if and only if $a_{12} + 3(a_{10} + a_{14}) = 0$. Indeed from the computability condition

$$\{H_5, g_1\} + \{H_4, g_2\} + \{H_3, g_3\} + \{H_2, g_4\} = 6\Omega_4,$$

we obtain that if

$$\int_0^{2\pi} (\{H_5, g_1\} + \{H_4, g_2\} + \{H_3, g_3\} - 6\Omega_4)|_{x=\cos t, y=\sin t}\, dt$$
$$= a_{12} + 3(a_{10} + a_{14}) = 0.$$

Then there exist solutions g_4 and H_6 which we obtain with arbitrary terms of the form $d_k(x^2+y^2)^k$ with d_k a constant for $k = 1, 2$.

Inserting the homogenous polynomials g_j and H_{j+2} of degree j and $j+2$ respectively, for $j = 1, 2, 3, 4$ into (1.36) for $m = 5$ and $k = 2$ we get

$$\int_0^{2\pi} (\{H_7, g_1\} + \{H_6, g_2\} + \{H_5, g_3\} + \{H_4, g_4\} + \{H_3, g_5\})|_{x=\cos t, y=\sin t}\, dt$$
$$= I_1(a_1, a_2) = 0.$$

where $I_1(a_1, a_2)$ is such that

$$\begin{aligned} I_1(a_1, 0) = &\ 2\pi\, a_1^2\, (\mu+3)(\mu+2)\, (a_{10} - a_{14}) = 0, \\ I_1(0, a_2) = &\ 2\pi\, a_2^2\, (\mu+3)(\mu+2)\, (a_{10} - a_{14}) = 0, \\ I_1(a_1, a_2)|_{a_1a_2 \neq 0} = &\ 2\pi\, (\mu+3)(\mu+2) \left((a_1^2 - a_2^2)\lambda_1 + a_1a_2\lambda_2\right) = 0. \end{aligned}$$

and

$$\int_0^{2\pi} \Big(\{H_9, g_1\} + \{H_8, g_2\} + \{H_7, g_3\} + \{H_6, g_4\} + \{H_5, g_5\}$$
$$+\{H_4, g_6\} + \{H_3, g_7\}\Big)\Big|_{x=\cos t, y=\sin t} dt = I_2(a_1, a_2) = 0,$$

where $= I_2(a_1, a_2)$ is a constant such that

$$\begin{aligned} I_2(a_1, 0)|_{a_{10}=a_{14}} = &\ -\frac{\pi\, a_1^4}{864} \left(2120\mu^3 + 7161\mu^2 + 5202\mu + 7857\right) a_{14} = 0, \\ I_2(0, a_2)|_{a_{10}=a_{14}} = &\ -\frac{\pi\, a_2^4}{864} \left(2120\mu^3 + 7161\mu^2 + 5202\mu + 7857\right) a_{14} = 0 = 0, \\ I_2(a_1, a_2)|_{a_1a_2\neq 0} = &\ \nu_1\lambda_1 + \nu_2\lambda_2 = 0. \end{aligned}$$

where ν_1 and ν_2 are convenient constants such that the system

$$2\pi\, (\mu+3)(\mu+2) \left((a_1^2 - a_2^2)\lambda_1 + a_1a_2\lambda_2\right) := \sigma_1\lambda_1 + \sigma_2\lambda_2 = 0, \quad \nu_1\lambda_1 + \nu_2\lambda_2 = 0,$$

is such that $\Delta := \sigma_1\nu_2 - \sigma_2\nu_1 = (\mu+3)(\mu+2)p(\mu, a_1, a_2)$ with $p(\mu, a_1, a_2)$ a convenient polynomial of degree 3 in the variable μ and $p(0, a_1, a_2) \neq 0$.

Consequently if μ is such that $\Delta \neq 0$ we obtain that $\lambda_1 = \lambda_2 = 0$ The case when $(\mu+2)p(\mu, a_1, a_2) = 0$ could be studied in a similar way to the proof of Proposition 51. The proof of statement (c), i.e. when $\mu = -3$ is studied in Proposition 41. Now we prove statement (b). Thus $a_1 = a_2 = 0$ and the differential system (2.41) under the condition $a_{12} + 3(a_{10} + a_{14}) = 0$ becomes

$$\begin{aligned}\dot{x} =& \ -y + x(a_{10}x^4 + a_{11}x^3y - 3(a_{10} + a_{14})x^2y^2 + a_{13}xy^3 + a_{14}y^4),\\ \dot{y} =& \ x + y(a_{10}x^4 + a_{11}x^3y - 3(a_{10} + a_{14})x^2y^2 + a_{13}xy^3 + a_{14}y^4)\end{aligned}$$

Hence we get that $\Omega_4 = a_{10}x^4 + a_{11}x^3y - 3(a_{10} + a_{14})x^2y^2 + a_{13}xy^3 + a_{14}y^4$ and satisfies the condition $\int_0^{2\pi} \Omega_4|_{x=\cos t, y=\sin t}\, dt = 0$, because $a_{12} + 3(a_{10} + a_{14}) = 0$. Then in view of Corollary 42 we get that the origin is a weak center. Thus the necessity of the statement (b) is proved, and consequently the proposition is proved. □

Proposition 53. *The polynomial differential system of degree six*

$$\begin{aligned}\dot{x} =& \ -y(1 + \mu(a_2x - a_1y))\\ &+x(a_1x + a_2y + a_{15}x^5 + a_{16}y^5 + a_{17}x^4y + a_{18}x^3y^2 + a_{19}x^2y^3 + a_{20}xy^4),\\ \dot{y} =& \ x(1 + \mu(a_2x - a_1y))\\ &+y\left(a_1x + a_2y + a_{15}x^5 + a_{16}y^5 + a_{17}x^4y + a_{18}x^3y^2 + a_{19}x^2y^3 + a_{20}xy^4\right),\end{aligned} \tag{2.43}$$

has a weak center at the origin if and only if

(a) the following conditions hold:

$$\begin{aligned}&8a_1^2a_2^5a_{15} + \left(2a_1^5a_2^2 - 4a_1^3a_2^4 + 2a_1a_2^6\right)a_{19} - \left(3a_1^7 - 15a_1^5a_2^2 + 25a_1^3a_2^4 - 5a_1a_2^6\right)a_{16}\\ &+\left(a_2^7 + 11a_1^4a_2^3 - 9a_1^2a_2^5 - 3a_1^6a_2\right)a_{20} = 0,\\ &8a_1^3a_2^4a_{17} - (15a_1^7 - 55a_1^5a_2^2 + 45a_1^3a_2^4 - 5a_1a_2^6)a_{16} - (10a_1^5a_2^2 - 12a_1^3a_2^4 + 2a_1a_2^6)a_{19}\\ &-(-15a_1^6a_2 + 35a_1^4a_2^3 - 13a_1^2a_2^5 + a_2^7)a_{20} = 0,\\ &2a_1^2a_2^3a_{18} - (5a_1^5 - 10a_1^3a_2^2 + 5a_1a_2^4)a_{16} - (4a_1^3a_2^2 - 2a_1a_2^4)a_{19} - (6a_1^2a_2^3 - 5a_1^4a_2 - a_2^5)a_{20} = 0.\end{aligned}$$

Moreover

(i) if $a_1 = 0$ and $a_2 \neq 0$ then $a_{15} = a_{17} = 0$,

(ii) if $a_2 = 0$ and $a_1 \neq 0$ then $a_{16} = a_{18} = 0$,

(iii) if $a_1a_2 \neq 0$ *then*

$$\begin{aligned}
\lambda_1 = \; & a_{15} + \frac{1}{8a_1^2a_2^5}\Big(\big(2a_1^5a_2^2 - 4a_1^3a_2^4 + 2a_1a_2^6\big)\, a_{19} - \big(3a_1^7 - 15a_1^5a_2^2 + 25a_1^3a_2^4 - 5a_1a_2^6\big)\, a_{16} \\
& + \big(a_2^7 + 11a_1^4a_2^3 - 9a_1^2a_2^5 - 3a_1^6a_2\big)\, a_{20}\Big) = 0, \\
\lambda_2 = \; & a_{17} - \frac{1}{8a_1^3a_2^4}\Big((15a_1^7 - 55a_1^5a_2^2 + 45a_1^3a_2^4 - 5a_1a_2^6)a_{16} + (10a_1^5a_2^2 - 12a_1^3a_2^4 + 2a_1a_2^6)a_{19} \\
& +(-15a_1^6a_2 + 35a_1^4a_2^3 - 13a_1^2a_2^5 + a_2^7)a_{20}\Big) = 0, \\
\lambda_3 = \; & a_{18} + \frac{1}{2a_1^2a_2^3}\Big(- (5a_1^5 - 10a_1^3a_2^2 + 5a_1a_2^4)a_{16} \\
& -(4a_1^3a_2^2 - 2a_1a_2^4)a_{19} - (6a_1^2a_2^3 - 5a_1^4a_2 - a_2^5)a_{20}\Big) = 0.
\end{aligned} \tag{2.44}$$

(b) $a_1 = a_2 = 0$.

(c) $\mu + 4 = 0$.

Moreover, assume that $(\mu+4)(a_1^2+a_2^2) \neq 0$ *then system* (2.43) *under the conditions (i), (ii),(iii) and after a linear change of variables* $(x,y) \longrightarrow (X,Y)$ *it is invariant under the transformations* $(X,Y,t) \longrightarrow (-X,Y,-t)$, *or* $(X,Y,t) \longrightarrow (X,-Y,-t)$.

Proof. Sufficiency: First we observe that the polynomial differential system (2.43) under the conditions (i) becomes

$$\begin{aligned}
\dot{x} = \; & -y(1 - \mu\, a_2x) + xy(a_2 + a_{16}y^4 + a_{17}x^4 + a_{19}x^2y^2), \\
\dot{y} = \; & x(1 - \mu\, a_2x) + y^2(a_2 + a_{16}y^4 + a_{17}x^4 + a_{19}x^2y^2),
\end{aligned}$$

which is invariant under the transformation $(x,y,t) \longrightarrow (x,-y,-t)$, and the polynomial differential system (2.43) under the conditions (ii) becomes

$$\begin{aligned}
\dot{x} = \; & -y(1 + \mu\, a_1y) + x^2(a_1x + a_{15}x^4 + a_{18}x^2y^2 + a_{20}y^4), \\
\dot{y} = \; & x(1 + \mu\, a_1y) + xy(a_1x + a_{15}x^4 + a_{18}x^2y^2 + a_{20}y^4),
\end{aligned}$$

which is invariant under the transformation $(x,y,t) \longrightarrow (-x,y,-t)$

Under the linear change of variables (2.18) the differential system (2.43) is invariant under the transformation $(X,Y,t) \longrightarrow (-X,Y,-t)$ if and only if

$$\begin{aligned}
& \big(2\kappa_1^5\kappa_2^2 - 4\kappa_1^3\kappa_2^4 + 2\kappa_1\kappa_2^6\big)\, a_{19} - \big(3\kappa_1^7 - 15^5\kappa_1\kappa_2^2 + 25\kappa_1^3\kappa_2^4 - 5\kappa_1\kappa_2^6\big)\, a_{16} \\
& + \big(\kappa_2^7 + 11\kappa_1^4\kappa_2^3 - 9\kappa_1^2\kappa_2^5 - 3\kappa_1^6\kappa_2\big)\, a_{20} + 8\kappa_1^2\kappa_2^5a_{15} = 0, \\
& (15\kappa_1^7 - 55\kappa_1^5\kappa_2^2 + 45\kappa_1^3\kappa_2^4 - 5\kappa_1\kappa_2^6)a_{16} + (10\kappa_1^5\kappa_2^2 - 12\kappa_1^3\kappa_2^4 + 2\kappa_1\kappa_2^6)a_{19} \\
& +(-15a_1^6a_2 + 35a_1^4a_2^3 - 13a_1^2a_2^5 + a_2^7)a_{20} - 8a_1^3a_2^4a_{17} = 0, \\
& 2\kappa_1^2\kappa_2^3a_{18} - (5\kappa_1^5 - 10\kappa_1^3\kappa_2^2 + 5\kappa_1\kappa_2^4)a_{16} \\
& -(4\kappa_1^3\kappa_2^2 - 2\kappa_1\kappa_2^4)a_{19} - (6\kappa_1^2\kappa_2^3 - 5\kappa_1^4\kappa_2 - \kappa_2^5)a_{20} = 0.
\end{aligned}$$

We shall study only the case (iii), i.e. when $a_1a_2 \neq 0$. We suppose that (2.44) holds and show that then the origin is a center of system (2.43). For the case when $a_1^2+a_2^2 \neq 0$, after the change $x = a_1X - a_2Y, \qquad y = a_2X + a_1Y$, we get that this system coincides with system (2.21) and with $\kappa_1 = a_1$ and $\kappa_2 = a_2$. Consequently system (2.21) is invariant under the change $(X, Y, t) \longrightarrow (-X, Y, -t)$. Thus in view of the Poincaré Theorem we get that the origin is a center, and by Theorem 9 this center is a weak center.

Necessity, Now we suppose that the origin is a center of (2.43) and we shall prove that (2.44) holds. Indeed, from Theorem 8 it follows that differential system (2.43) can be written as

$$
\begin{aligned}
&\{H_7,\, x\} + (1+g_1)\{H_6, x\} + (1+g_1+g_2)\{H_5, x\} + (1+g_1+g_2+g_3)\{H_4, x\}\\
&+(1+g_1+g_2+g_3+g_4)\{H_3, x\} + (1+g_1+g_2+g_3+g_4+g_5)\{H_2, x\}\\
=\ &-y(1+\mu(a_2x-a_1y)) + x\Big(a_1x + a_2y + a_{15}x^5 + a_{16}y^5 + a_{17}x^4y + a_{18}x^3y^2\\
&+a_{19}x^2y^3 + a_{20}xy^4\Big),\\
&\{H_7,\, y\} + (1-g_1)\{H_6, y\} + (1+g_1+g_2)\{H_5, y\} + (1+g_1+g_2+g_3)\{H_4, y\}\\
&+(1+g_1+g_2+g_3+g_4)\{H_3, y\} + (1+g_1+g_2+g_3+g_4+g_5)\{H_2, y\}\\
=\ &x(1+\mu(a_2x-a_1y)) + y\Big(a_1x + a_2y + a_{15}x^5 + a_{16}y^5 + a_{17}x^4y + a_{18}x^3y^2\\
&+a_{19}x^2y^3 + a_{20}xy^4\Big).
\end{aligned}
$$

This partial differential system has solution for arbitrary a_1, a_2, a_{15}, a_{16}, a_{17}, a_{18}, a_{19}, a_{20}.. After some computations we get that

$$
\begin{aligned}
&\int_0^{2\pi} \Big(\{H_7,\, g_1\} + \{H_6, g_2\} + \{H_5,\, g_3\} + \{H_4, g_4\} + \{H_3,\, g_5\}\Big)\Big|_{x=\cos t,\, y=\sin t} dt\\
&\quad = -\pi(\mu+4)\,(a_1\lambda_2(a_1,a_2) - 5a_2\lambda_1(a_1,a_2) - a_2\lambda_3(a_1,a_2)))\\
&\quad := \nu_{11}\lambda_1 + \nu_{12}\lambda_2 + \nu_{13}\lambda_3 = 0,
\end{aligned} \tag{2.45}
$$

where $\lambda_j = \lambda_j(a_1, a_2, \mu)$ for $j = 1, 2, 3$ are constant defined in (2.44). By continuing the integration of (see equation (1.36) with $m = 6$ and $k = 3$) we get that

$$
\begin{aligned}
&\int_0^{2\pi} \Big(\{H_9,\, g_1\} + \{H_8, g_2\} + \{H_7,\, g_3\} + \{H_6, g_4\} + \{H_5,\, g_5\} + \{H_4, g_6\}\\
&+ \{H_3,\, g_7\}\Big)\Big|_{x=\cos t,\, y=\sin\, t} dt := \nu_{21}\lambda_1 + \nu_{22}\lambda_2 + \nu_{23}\lambda_3 = 0,
\end{aligned} \tag{2.46}
$$

where

$$\begin{aligned}\nu_{21} :=\ & \frac{25a_2\pi}{6912}\Big((-1062a_1^2-558a_2^2)\mu^3+(-8961a_1^2-7501a_2^2)\mu^2\\&+1344c-8460a_1^2-8796a_2^2\Big),\\ \nu_{22} :=\ & -\frac{5a_1\pi}{2304}\Big((2070a_2^2-354a_1^2)\mu^3+(13897a_2^2-2987a_1^2)\mu^2\\&+(448c-4308a_2^2-2820a_1^2)+(16677a_1^2-83367a_2^2-1344c\Big),\\ \nu_{23} :=\ & \frac{5a_2}{6912}\Big(4014a_1^2-2250a_2^2)\mu^3+(27933a_1^2-19799a_2^2)\mu^2\\&+(-11556a_1^2-7764a_2^2+1344c)+99889a_2^2-155763a_1^2-4032c\Big),\end{aligned}$$

where c is an arbitrary constant. Again from equation (1.36) with $m=6$ and $k=5$ we get

$$\int_0^{2\pi}\Big(\{H_{11},\,g_1\}+\{H_{10},g_2\}+\{H_9,\,g_3\}+\{H_8,g_4\}+\{H_7,\,g_5\}+\{H_6,g_6\}\\ +\{H_5,\,g_7\}+\{H_4,g_8\}\{H_3,\,g_9\}\Big)\Big|_{x=\cos t,\,y=\sin t}dt := \nu_{31}\lambda_1+\nu_{32}\lambda_2+\nu_{33}\lambda_3=0, \tag{2.47}$$

where ν_{3j} for $j=1,2,3$, are convenient constants. From (2.45), (2.46) and (2.47) we obtain the linear system with respect to $\lambda_1,\ \lambda_2,\ \lambda_3 : A\lambda=0$, where $A=A(\mu)$ is the coefficient matrix with determinant $|A|$ equal to

$$\begin{vmatrix}\nu_{11}&\nu_{12}&\nu_{13}\\ \nu_{21}&\nu_{22}&\nu_{23}\\ \nu_{31}&\nu_{32}&\nu_{33}\end{vmatrix} = -\frac{\pi^3(a_2^2a_1)^3(\mu+4)}{456192}\Big(12276\mu^5+145467\mu^4+502471\mu^3\\ +480577\mu^2-3775995\mu-2701980\Big)\left(846\mu^3+6149\mu^2-576\mu-34299\right)\neq 0.$$

Consequently if $|A|\neq 0$ then the unique solution of these linear system is the trivial solution $\lambda_1=\lambda_2=\lambda_3=0$. The case when $|A|=0$ could be studied in a similar way of Proposition 51. Thus the necessity of statement (a) is proved. To prove statement (c), i.e. when $\mu=-4$ we apply Proposition 41. Finally we prove statement (b), i.e when $a_1=a_2=0$. Under these conditions system (2.43) becomes

$$\begin{aligned}\dot{x} =\ & -y+x(a_{15}x^5+a_{16}y^5+a_{17}x^4y+a_{18}x^3y^2+a_{19}x^2y^3+a_{20}x^1y^4) := -y+x\Omega_5,\\ \dot{y} =\ & x+y(a_{15}x^5+a_{16}y^5+a_{17}x^4y+a_{18}x^3y^2+a_{19}x^2y^3+a_{20}x^1y^4 := x+y\Omega_5,\end{aligned} \tag{2.48}$$

which is a polynomial differential system of degree six with homogenous nonlinearities. Since $\displaystyle\int_0^{2\pi}\Omega_5|_{x=\cos t,y=\sin t}\,dt=0$, in view of Proposition 42 we obtain that the origin is a weak center of (2.48). Thus the necessity of the proposition is proved and consequently the proposition is proved. □

2.4.4 Classification of quadratic and cubic planar differential system with a weak center

For nondegenerate quadratic center (see for instance [6]) and cubic center (see for instance [43, 52]) the center problem has been solved in terms of algebraic equalities satisfied by the coefficients.

Proposition 54. *For the quadratic polynomial differential system*

$$\begin{aligned} \dot{x} = & -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} = & \; x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2, \end{aligned} \tag{2.49}$$

or equivalently

$$\dot{x} = \{H_2 + H_3, x\} + g_1\{H_2, x\}, \quad \dot{y} = \{H_2 + H_3, y\} + g_1\{H_2, y\},$$

with

$$\begin{aligned} H_3 = & \; \frac{1}{3}(\lambda_2 + \lambda_5)\, x^3 + \lambda_3 x^2 y - \frac{1}{3}(\lambda_4 + \lambda_6) y^3 - \lambda_2 x y^2, \\ g_1 = & \; \lambda_4 y - \lambda_5 x. \end{aligned}$$

the origin is a center if and only if one of the following four conditions holds

$$\begin{aligned} (i) \quad & \lambda_4 = \lambda_5 = 0, \\ (ii) \quad & \lambda_2 = \lambda_5 = 0, \\ (iii) \quad & \lambda_3 - \lambda_6 = 0, \\ (iv) \quad & \lambda_5 = 0, \quad \lambda_4 + 5(\lambda_3 - \lambda_6) = 0, \quad \lambda_3\lambda_6 - 2\lambda_6^2 - \lambda_2^2 = 0. \end{aligned} \tag{2.50}$$

Theorem 55. *A quadratic polynomial differential system (2.49) has a weak center at the origin if and only if a linear change of coordinates x, y and a scaling of time t it can be written as one of the following systems*

$$\begin{aligned} \dot{x} = & -y\,(1 + 3\lambda_3 y) - \lambda_3 x^2, \\ \dot{y} = & \; x\,(1 + 3\lambda_3 y) - \lambda_3 xy; \end{aligned} \tag{2.51}$$

$$\begin{aligned} \dot{x} = & -y\,((1 - \lambda_6 y) - \lambda_3 x^2, \\ \dot{y} = & \; x\,((1 - \lambda_6 y) - \lambda_3 xy; \end{aligned} \tag{2.52}$$

$$\begin{aligned} \dot{x} = & -y\,(1 + \lambda_2 x - \lambda_6 y) - x(\lambda_2 x + \lambda_6 y), \\ \dot{y} = & \; x\,(1 + \lambda_2 x - \lambda_6 y) - y(\lambda_2 x + \lambda_6 y); \end{aligned} \tag{2.53}$$

$$\begin{aligned} \dot{x} = &\ \ -y - \lambda_3 x^2, \\ \dot{y} = &\ \ x - \lambda_3 xy; \end{aligned} \tag{2.54}$$

$$\begin{aligned} \dot{x} = &\ \ -y\,(1-\lambda_6 y) - 2\lambda_6 xy, \\ \dot{y} = &\ \ x\,(1-\lambda_6 y) - 2\lambda_6 y^2. \end{aligned} \tag{2.55}$$

Proof. Indeed, differential system (2.49) can be rewritten as (19) if and only if

$$\lambda_4 + \lambda_6 + 3\lambda_3 = 0, \qquad \lambda_5 + 4\lambda_2 = 0. \tag{2.56}$$

Consequently system (2.49) becomes

$$\begin{aligned} \dot{x} = &\ \ -y(1+\lambda_2 x - \lambda_6 y) - x(\lambda_3 x + \lambda_2 y), \\ \dot{y} = &\ \ x(1+\lambda_2 x - \lambda_6 y) - y(\lambda_3 x + \lambda_2 y), \end{aligned}$$

In view of (2.50) and taking into account the condition (2.56) we get $\lambda_4 = \lambda_2 = \lambda_5 = 0$ and $\lambda_6 = -3\lambda_3$, then we obtain the differential system (2.51); $\lambda_5 = \lambda_2 = 0$, and $\lambda_4 + \lambda_6 + 3\lambda_3 = 0$, then we obtain the differential system (2.52); $\lambda_3 = \lambda_6$, $\lambda_5 = -4\lambda_2$, and $\lambda_4 = -4\lambda_3$, then we obtain the differential system (2.53); $\lambda_5 = \lambda_2 = 0$, $\lambda_4 + 5(\lambda_3 - \lambda_6) = 0$ and $\lambda_6(\lambda_3 - 2\lambda_6) = 0$, then we have either $\lambda_5 = \lambda_2 = \lambda_6 = 0$, $\lambda_4 + 5\lambda_3 = 0$, or $\lambda_5 = \lambda_2 = 0, \lambda_3 = 2\lambda_6$ and $\lambda_4 + 5\lambda_6 = 0$. Therefore we get the differential system (2.54) or (2.55). In short, the theorem is proved. □

Remark 56. *In the paper [12] the classification of isochronous quadratic centers is given. From this classification there are only two isochronous centers which are weak centers, which we obtain from the equation* (2.52) *for* $\lambda_3 = \lambda_6 = -1$, *and* $\lambda_3 = -1$, $\lambda_6 = 0$. *These systems are*

$$\dot{x} = -y + x^2 - y^2, \quad \dot{y} = x + 2xy,$$

and

$$\dot{x} = -y + x^2, \quad \dot{y} = x + xy,$$

respectively.

Proposition 57. *For the cubic polynomial differential system*

$$\begin{aligned} \dot{x} = &\ \ -y + Ax^3 + Bx^2y + Cxy^2 + Dy^3, \\ \dot{y} = &\ \ x + Kx^3 + Lx^2y + M\,x\,y^2 + Ny^3, \end{aligned} \tag{2.57}$$

the origin is a center if and only if one of the following sets of conditions hold

$$\begin{aligned}
(i) \qquad 3A+L+C+3N &= 0,\\
(3A+L)(B+D+K+M)-2(A-N)(B+M) &= 0,\\
2(A+N)\left((3A+L)^2-(B+M)^2\right)&\\
+(3A+L)(B+M)(B+K-D-M) &= 0,\\
(ii) \qquad 3A+L+C+3N &= 0,\\
2A+C-L-2N &= 0,\\
B+3D-3K-M &= 0,\\
B+5D+5K+M &= 0,\\
(A+3N)(3A+N)-16DK &= 0.
\end{aligned} \tag{2.58}$$

Theorem 58. *A cubic polynomial differential system (2.57) has a weak center at the origin if and only if after a linear change of coordinates x, y and a scaling of time t it can be written as one of the following systems If $N \neq 0$,*

$$\begin{aligned}
\dot{x} = &\ -y\left(1+Kx^2+(N+L)xy+\left(\frac{K-B}{2}-\frac{LB+LK}{2N}\right)y^2\right)\\
&+x\left(N(y^2-x^2)+(K+B)xy\right),\\
\dot{y} = &\ x\left(1+Kx^2+(N+L)xy+\left(\frac{K-B}{2}-\frac{LB+LK}{2N}\right)y^2\right)\\
&+y\left(N(y^2-x^2)+(K+B)xy\right);
\end{aligned} \tag{2.59}$$

if $N=0$, then we have either the system

$$\begin{aligned}
\dot{x} = &\ -y\left(1+Kx^2-Dy^2\right)+(K+B)x^2y,\\
\dot{y} = &\ x\left(1+Kx^2-Dy^2\right)+(K+B)xy^2;
\end{aligned} \tag{2.60}$$

or the system

$$\begin{aligned}
\dot{x} = &\ -y\left(1+Lxy-Bx^2-Dy^2\right),\\
\dot{y} = &\ x\left(1+Lxy-Bx^2-Dy^2\right).
\end{aligned} \tag{2.61}$$

Proof. Indeed, differential system (2.57) can be rewritten as (19) if and only if

$$L+C=A+N, \qquad M+D=B+K. \tag{2.62}$$

Consequently (2.57) becomes

$$\begin{aligned}
\dot{x} = &\ -y\left(1+(N-C)xy+Kx^2-Dy^2\right)+x\left((L-N+C)x^2+(B+K)xy+Ny^2\right),\\
\dot{x} = &\ x\left(1+(N-C)xy+Kx^2-Dy^2\right)+y\left((L-N+C)x^2+(B+K)xy+Ny^2\right),
\end{aligned}$$

By solving (2.62) together with (2.58) (i) we deduce that this system have no solution. Thus center of cubic system (2.57) under the conditions (2.58) (i) is not a weak center.

By solving (2.62) together with (2.58) (ii) we deduce that this system have three solution.

(i) If $N \neq 0$ then $A + N = 0$ and $L + C = 0,\ 2ND = N(B - K) + L(B + K)$. This solution provides the differential system (2.59).

If $N = 0$ then there exist two solutions.

(ii) If $N = 0$ then $A = 0$ and $L = 0$. This solution provides the differential system (2.60).

- (iii)] If $N = 0$ then $A = 0$ and $K + B = 0$. This solution provides the differential system (2.61).

  Thus the centers of cubic systems (2.57) under the conditions (2.58) (ii) generate three weak centers.

In short, the theorem is proved. □

Remark 59. *In the paper [12] the classification of the all isochronous cubic center is given. From this classification there are only two isochronous center which are weak center, which we obtain from* (2.59) *for* $N = -1,\ B = K = 0,$ *and* $N = -L = -1,\ K = B = 0,$ *These cubic systems are*

$$\begin{aligned} \dot{x} =& \ -y + x^3 - 3xy^2 = -y(1 + 2xy) + x(x^2 - y^2), \\ \dot{y} =& \ x + 3x^2y - y^3 = x(1 + 2xy) + y(x^2 - y^2), \end{aligned}$$

and

$$\begin{aligned} \dot{x} =& \ -y + x^3 - xy^2 = -y + x(x^2 - y^2), \\ \dot{y} =& \ x + x + x^2y - y^3 = x + y(x^2 - y^2), \end{aligned}$$

respectively.

2.5 $\Lambda - \Omega$ differential system. Center problem part II

In this section by developing the results given in the previous section we study the Conjecture 15.

2.6 Proof of Theorem 16

The proof of Theorem 16 for $m = 2$ and $m = 3$ follows from the proof of Theorem 7 of [35] .

Proposition 60. *The fourth polynomial differential system*

$$
\begin{aligned}
\dot{x} = & \; -y + x\Big(a_1x + a_2y + a_3x^2 + a_4y^2 \\
& + a_5xy + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2\Big) := P, \\
\dot{y} = & \; x + y\Big(a_1x + a_2y + a_3x^2 + a_4y^2 \\
& + a_5xy + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2\Big) := Q,
\end{aligned}
\tag{2.63}
$$

where $a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 \neq 0$ has a weak center at the origin if and only if after a linear change of variables $(x, y) \longrightarrow (X, Y)$ it is invariant under the transformations $(X, Y, t) \longrightarrow (-X, Y, -t)$ or $(X, Y, t) \longrightarrow (X, -Y, -t)$. Moreover,

(i) if $a_1^2 + a_2^2 \neq 0$, then system (2.63) has a weak center at the origin if and only if

$$
\begin{aligned}
a_3 + a_4 = 0, \qquad a_5a_1a_2 + (a_2^2 - a_1^2)a_4 = & \; 0, \\
a_1^3a_7 - a_1^2a_2a_9 + a_1a_2^2a_8 - a_2^3a_6 = & \; 0, \\
3a_1a_2^2a_7 - 3a_1^2a_2a_6 + (a_1^3 - 2a_1a_2^2)a_8 + (2a_1^2a_2 - a_2^3)a_9 = & \; 0.
\end{aligned}
\tag{2.64}
$$

Consequently

(a)

$$
\begin{aligned}
a_3 + a_4 = 0, \qquad a_5 + \frac{(a_2^2 - a_1^2)}{a_1a_2}a_4 = & \; 0, \\
a_6 + \frac{1}{2a_2^3}\left(a_7(a_1^3 - 3a_2^2a_1) + a_9(a_2^3 - a_1^2a_2)\right) = & \; 0, \\
a_8 + \frac{1}{2a_2^2a_1}\left(a_7(3a_1^3 - 3a_1a_2^2) + a_9(a_2^3 - 3a_1^2a_2)\right) = & \; 0.
\end{aligned}
$$

when $a_1a_2 \neq 0$,

(b) $a_2 = a_3 = a_4 = a_7 = a_8 = 0$, when $a_1 \neq 0$,

(c) $a_1 = a_3 = a_4 = a_6 = a_9 = 0$, when $a_2 \neq 0$.

(ii) If $a_1 = a_2 = 0$ and $a_4a_5 \neq 0$ then system (2.63) has a weck center at the origin if and only if

$$
\begin{aligned}
a_3 + a_4 = & \; 0, \\
\lambda\, a_5 + (1 - \lambda^2)a_4 = & \; 0, \\
\lambda^3a_7 - \lambda^2a_9 + \lambda\, a_8 - a_6 = & \; 0, \\
3\lambda^2a_7 + 3\lambda a_6 + \left(\lambda^3 - 2\lambda^2\right)a_8 + \left(2\lambda^2 - 1)\right)a_9 = & \; 0,
\end{aligned}
$$

where $\lambda = \dfrac{a_5 + \sqrt{4a_4^2 + a_5^2}}{2a_4}$. *Moreover the weak center in this case after a linear change of variables* $(x, y) \longrightarrow (X, Y)$ *it is invariant under the transformations* $(X, Y, t) \longrightarrow (-X, Y, -t)$.

(iii) if $a_1 = a_2 = a_3 = a_4 = a_5 = 0$*, then the origin is a weak center.*

Proof. Sufficiency: First of all we observe that the polynomial differential system (2.63) after the linear change of variables (2.18) would be invariant under the transformation $(X, Y, t) \longrightarrow (-X, Y, -t)$ if and only if

$$\begin{aligned} \kappa_2 a_1 - \kappa_1 a_2 &= 0, \\ \kappa_1^2 a_3 + \kappa_2^2 a_4 + \kappa_1\kappa_2 a_5 &= 0, \\ \kappa_2^2 a_3 + \kappa_1^2 a_4 - \kappa_1\kappa_2 a_5 &= 0, \\ \kappa_1^3 a_7 - \kappa_1^2\kappa_2 a_9 + \kappa_1\kappa_2^2 a_8 - \kappa_2^3 a_6 &= 0, \\ 3\kappa_1\kappa_2^2 a_7 - 3\kappa_1^2\kappa_2 a_6 + (\kappa_1^3 - 2\kappa_1\kappa_2^2)a_8 + (2\kappa_1^2\kappa_2 - \kappa_1\,\kappa_2^3)a_9 &= 0. \end{aligned} \tag{2.65}$$

We suppose that (2.65) holds, and consequently the origin of the new system is a center. When $a_1^2 + a_2^2 \neq 0$, after the change $x = a_1X - a_2Y, \quad y = a_2X + a_1Y$, we get that the system has the form of system (2.20) with $m = 4$, here $\kappa_1 = a_1$ and $\kappa_2 = a_2$ and consequently this system is invariant under the change $(X, Y, t) \longrightarrow (-X, Y, -t)$ i.e. it is reversible. Thus in view of the Poincaré Theorem we get that the origin is a center. Hence system (2.63) under conditions (2.64) has a weak center at the origin. Thus the sufficiency under assumption (i) is proved.

When $\kappa_1\kappa_2 \neq 0$ then by solving (2.65) with respect to κ_1 and κ_2, and if we denote by $\kappa_1 = a_1$ and $\kappa_2 = a_2$ we obtain (2.36). For the case when $\kappa_2 = 0$ and $k_1 \neq 0$, then from (2.65) it follows that

$$a_2 = a_3 = a_4 = a_7 = a_8 = 0. \tag{2.66}$$

If (2.66) holds then system (2.63) becomes

$$\begin{aligned} \dot{x} &= -y + x^2\Big(a_1 + a_5y + a_6x^2 + a_9y^2\Big), \\ \dot{y} &= x + yx\Big(a_1 + a_5y + a_6x^2 + a_9y^2\Big), \end{aligned}$$

which is invariant under the change $(x, y, t) \longrightarrow (-x, y, -t)$. If $\kappa_1 = 0$ and $k_2 \neq 0$ then from (2.65) it follows that

$$a_1 = a_3 = a_4 = a_6 = a_9 = 0. \tag{2.67}$$

If (2.67) holds then (2.63) becomes

$$\begin{aligned} \dot{x} &= -y + xy\Big(a_2 + a_5x + a_7y^2 + a_8x^2\Big), \\ \dot{y} &= x + y^2\Big(a_2 + a_5x + a_7y^2 + a_8x^2\Big), \end{aligned}$$

which is invariant under the change $(x, y, t) \longrightarrow (x, -y, -t)$.

When $a_1 = a_2 = 0$ and $a_4 a_5 \neq 0$, then by taking

$$\kappa_1 = \cos\theta := \frac{\lambda}{\sqrt{1+\lambda^2}} \quad \text{and} \quad \kappa_2 = \sin\theta := \frac{1}{\sqrt{1+\lambda^2}},$$

where λ is a solution of the equation $\lambda^2 - \dfrac{a_5}{a_4}\lambda - 1 = 0$. After the rotation $x = \cos\theta\, X - \sin\theta\, Y$, $\quad y = \sin\theta\, X + \cos\theta\, Y$, then in view of (2.65) we get that (2.63) becomes

$$\begin{aligned} \dot{X} = & \; -Y + \frac{1+\lambda^2}{2\lambda} X^2 \left(-2a_4 Y + \frac{(a_9 - 3\lambda\, a_7)}{\sqrt{1+\lambda^2}} Y^2 + \frac{\lambda^3 a_7 - \lambda^2 a_9 - 2\lambda\, a_7}{\sqrt{1+\lambda^2}} X^2\right), \\ \dot{Y} = & \; X + \frac{1+\lambda^2}{2\lambda} XY \left(-2a_4 Y + \frac{(a_9 - 3\lambda\, a_7)}{\sqrt{1+\lambda^2}} Y^2 + \frac{\lambda^3 a_7 - \lambda^2 a_9 - 2\lambda\, a_7}{\sqrt{1+\lambda^2}} X^2\right). \end{aligned}$$

Thus this system is invariant under the change $(X, Y, t) \longrightarrow (-X, Y, -t)$, i.e. it is reversible. thus in view of the Poincaré Theorem we get that the origin is a center. Therefore the sufficiency is proved and (ii) holds.

If $a_1 = a_2 = a_3 = a_4 = a_5 = 0$, then system (2.63) becomes

$$\begin{aligned} \dot{x} &= -y + x\Big(a_6 x^3 + a_9 x y^2 + a_7 y^3 + a_8 x^2 y\Big) = -y + x\Omega_3, \\ \dot{y} &= x + y\Big(a_6 x^3 + a_9 x y^2 + a_7 y^3 + a_8 x^2 y\Big) = x + y\Omega_3, \end{aligned}$$

By considering that $\displaystyle\int_0^{2\pi} \Omega_3(\cos t, \sin t)dt = 0$, then in view of Corollary 4 of [35] we get that the origin is a weak center which in general is not reversible. Thus the sufficiency of the proposition follows.

Necessity in case (i) We shall study only the case (a). The case (b) and (c) can be studied in analogous form. Therefore we assume that $a_1 a_2 \neq 0$. Now we suppose that the origin is a center of (2.63) and we prove that (2.36) holds. Indeed, from Theorem 8 it follows that differential system (2.63) can be written as

$$\begin{aligned} P = & \; \{H_5,\, x\} + (1+g_1)\{H_4, x\} + (1+g_1+g_2)\{H_3, x\} + (1+g_1+g_2+g_3)\{H_2, x\} \\ = & \; -y + x\Big(a_1 x + a_2 y + a_4 y^2 + a_3 x^2 + a_5 xy + a_6 x^3 + a_7 y^3 + a_8 x^2 y + a_9 x y^2\Big), \\ Q = & \; \{H_5,\, y\} + (1+g_1)\{H_4, y\} + (1+g_1+g_2)\{H_3, y\} + (1+g_1+g_2+g_3)\{H_2, y\}, \\ = & \; x + y\Big(a_1 x - a_2 y + a_4 y^2 + a_3 x^2 + a_5 xy + a_6 x^3 + a_7 y^3 + a_8 x^2 y + a_9 x y^2\Big) \end{aligned} \tag{2.68}$$

In view of Corollary 22 and assisted by an algebraic computer we can obtain the solutions of (2.68), i.e. the homogenous polynomials H_5, H_3, g_1, g_3 of degree odd are unique and the homogenous polynomials H_4, g_2 of degree even are obtained modulo an arbitrary polynomial of the form $c(x^2+y^2)^k$ where $k=1,2$. Indeed taking the homogenous part of these equations of degree two we obtain

$$\begin{aligned}\{H_3,x\}+g_1\{H_2,x\} &= x(a_1x+a_2y),\\ \{H_3,y\}+g_1\{H_2,y\} &= y(a_1x+a_2y).\end{aligned}$$

The solutions of these equations are

$$g_1=3(a_1y-a_2x), \qquad H_3=2H_2(a_2x-a_1y).$$

The homogenous part of (2.68) of degree 3 is

$$\begin{aligned}\{H_4,x\}+g_1\{H_3,x\}+g_2\{H_2,x\} &= x(a_4y^2+a_3x^2+a_5xy)=x\Omega_2,\\ \{H_4,y\}+g_1\{H_3,y\}+g_2\{H_2,y\} &= y(a_4y^2+a_3x^2+a_5xy)=y\Omega_2.\end{aligned} \tag{2.69}$$

The compatibility condition of these two last equations becomes of $\{H_3,g_1\}+\{H_2,g_2\}=4\Omega_2$, and by considering that $\{H_3,g_1\}=\{H_2,-3(a_2x-a_1y)^2\}$ since

$$\{H_2,g_2-3(a_2x-a_1y)^2\}=4\Omega_2.$$

Hence, in view of proposition 20 we get that

$$\int_0^{2\pi}\Omega_2(\cos t,\sin t)dt=2\pi(a_3+a_4)=0.$$

So $a_3+a_4=0$. Therefore $g_2=3(a_2x-a_1y)^2-a_4xy-2a_5x^2+c_1H_2$, where c_1 is a constant. Then from system (2.69) by considering that H_4 is a homogenous polynomial of degree four we obtain the solution

$$\begin{aligned}H_4 &= -\frac{1}{4}\left(3g_1H_3+2g_2H_2\right)+c_1H_2^2\\ &= H_2\left(3\left((a_2^2-a^2)x^2-a_1a_2xy\right)+a_5x^2+2a_4xy\right)+c_1H_2^2\end{aligned}$$

Inserting these previous solutions g_1, H_3, g_2 and H_4 into the partial differential equations

$$\begin{aligned}&\{H_5,\,x\}+g_1\{H_4,x\}+g_2\{H_3,x\}+g_3\{H_2,x\}\\ =\;& x(a_6x^3+a_7y^3+a_8x^2y+a_9xy^2)=x\Omega_3:=X_4,\\ &\{H_5,\,y\}+g_1\{H_4,y\}+g_2\{H_3,y\}+g_3\{H_2,y\}\\ =\;& y(a_6x^3+a_7y^3+a_8x^2y+a_9xy^2)=y\Omega_3:=Y_4,\end{aligned} \tag{2.70}$$

we get that these differential equations have a unique solution. Indeed, in this case the compatibility condition is

$$\{H_4,g_1\}+\{H_3,g_2\}+\{H_2,g_3\}=5\Omega_3, \tag{2.71}$$

because $\dfrac{\partial X_4}{\partial x} + \dfrac{\partial Y_4}{\partial y} = 5\Omega_3$, and Ω_3 is a homogenous polynomial of degree 3. Consequently there exists a unique solution g_3 of (2.71) such that

$$
\begin{aligned}
g_3 := \;& \Big(-6a_2a_1^2 - a_2^3 + \frac{11}{3}a_2a_5 - \frac{5}{3}a_1a_4 - \frac{10}{3}a_7 - \frac{5}{3}a_8\Big)x^3 \\
&+\Big((2a_1^3 - a_1a_2^2)\mu^2 + (8a_1^3 - 2a_1a_2^2 - 2a_1a_5 - a_2a_4 - 4a_1c_1)\mu \\
&+6a_1^3 + 3a_1a_2^2 - 2a_1a_5 + 9a_2a_4 + 5a_6 - 4a_1c_1\Big)x^2y \\
&+\Big(-a_2a_1^2\mu^2 + (a_1a_4 + 4a_2c_1 + a_1a_4)\mu - 9a_2a_1^2 + 4c_1a_2 - 9a_1a_4 - 5a_7\Big)xy^2 \\
&\Big(\frac{5}{3}a_1^3\mu^2 + \frac{1}{3}(22a_1^3 - 5a_1a_5 - 5a_2a_4 - 4a_1c_1)\mu \\
&+\frac{1}{3}(21a_1^3 + 5a_1a_5 + 5a_2a_4 + 5a_9 + 10a_6 - 12a_1c_1)\Big)y^3,
\end{aligned}
$$

Thus the homogenous polynomial H_5 can be compute as follows

$$
H_5 = -\frac{1}{5}\left(4g_1H_4 + 3g_2H_3 + 2g_3H_2\right),
$$

using (2.70).

Hence partial differential system (2.70) has a solution if and only if $a_3 + a_4 = 0$. On the other hand from (1.25) for $m = 4$ and assuming that $a_1a_2 \neq 0$ and denoting by

$$
\begin{aligned}
\lambda_1 :=\;& a_5 - \frac{(a_1^2 - a_2^2)a_4}{a_1a_2}, \\
\lambda_2 :=\;& a_6 - \frac{1}{2a_2^3}\left(a_7(a_1^3 - 3a_2^2a_1) + a_9(a_2^3 - a_1^2a_2)\right), \\
\lambda_3 :=\;& a_8 - \frac{1}{2a_2^2a_1}\left(a_7(3a_1^3 - 3a_1a_2^2) + a_9(a_2^3 - 3a_1^2a_2)\right).
\end{aligned}
$$

From Remak 34 with $m = 4$ we get that

$$
\begin{aligned}
I_1 :=\;& \int_0^{2\pi} \left(\{H_5,\, g_1\} + \{H_4,\, g_2\} + \{H_3,\, g_3\}\right)\big|_{x=\cos t, y=\sin t}\, dt \\
=\;& 3/2\pi\left(2a_1a_2\lambda_1 + 2a_2\lambda_2 - 2a_1\lambda_3\right) = 0.
\end{aligned}
$$

Under this condition the first differential equation of (1.25) with $m = 4$ becomes

$$
\{H_5,\, g_1\} + \{H_4,\, g_2\} + \{H_3,\, g_3\} + \{H_2, g_4\} = 0.
$$

It has a solution g_4 which in view of Corollary 22 can be obtained as follows

$$
g_4 = G_4(x, y) + 8c_1x(2a_4y + 2a_5x)H_2 + 4c_2H_2^2,
$$

where $G_4 = G_4(x, y)$ is a convenient homogenous polynomial of degree four, c_2 is a constant. Using formula (1.35) with $k = 1$ $X_5 = Y_5 = 0$ we obtain the homogenous polynomial H_6 as follows

$$H_6 = -\frac{5}{6}g_1 H_5 - \frac{4}{6}g_2 H_4 - \frac{3}{6}g_3 H_3 - \frac{2}{6}g_4 H_2.$$

By considering that the integral of the homogenous polynomial of degree 5

$$\int_0^{2\pi} \left(\{H_6,\, g_1\} + \{H_5,\, g_2\} + \{H_4,\, g_3\} + \{H_3,\, g_4\}\right)\big|_{x=\cos t, y=\sin t}\, dt \equiv 0,$$

then we obtain that there is a unique solution for the homogenous polynomial g_5 of degree 5 of the equation

$$\{H_6,\, g_1\} + \{H_5,\, g_2\} + \{H_4,\, g_3\} + \{H_3,\, g_4\} + \{H_2, g_5\} = 0,$$

which c0mes from the first equation of (27) with $m = 4$ and $j = 1$.

Using formula (1.35) with $k = 2$ $X_6 = Y_6 = 0$ we obtain the homogenous polynomial H_7 as follows

$$H_7 = -\frac{6}{7}g_1 H_5 - \frac{5}{7}g_2 H_5 - \frac{4}{7}g_3 H_4 - \frac{3}{7}g_4 H_3 - \frac{2}{7}g_5 H_2$$

and inserting it into the next integral of the homogenous polynomials of degree 6 we get that

$$\begin{aligned} I_2 := &\int_0^{2\pi} \left(\{H_7,\, g_1\} + \{H_6,\, g_2\} + \{H_4,\, g_3\} + \{H_3,\, g_4\}\right)\big|_{x=\cos t, y=\sin t}\, dt \\ = &\ \pi\left(\nu_2\lambda_1\lambda_2 + \nu_4\lambda_1 + \nu_5\lambda_2 + \nu_6\lambda_3\right). \end{aligned}$$

where

$$\begin{aligned} \nu_4 = &\ -\frac{2\Big(4(a_1a_2)^3 + 16a_1a_2^5 + 2a_2^4a_4 + (5a_1a_2^2 - a_1^3)a_7 + (a_1^2a_2 - a_2^3)a_9\Big)}{a_2^2}, \\ \nu_2 = &\ -4a_2, \quad \nu_5 = \frac{-24a_1^3 - 88a_1a_2^3 - 8a_2^2a_4}{a_1}, \quad \nu_6 = -8a_2(a_1^2 + 3a_2^2) \end{aligned}$$

By solving $I_1 = 0$ and $I_2 = 0$ and assuming that $a_1(4a_2^2 + \lambda_1) + 2a_2a_4 \neq 0$. we get that

$$\begin{aligned} \lambda_2 = &\ \frac{a_1\lambda_1\left(-4a_1a_2^5 - 2a_2^4a_4 + (a_1^3 - 5a_1a_2^2)a_7 + (a_2^3 - a_1^2a_2)a_9\right)}{2a_2^3(a_1(4a_2^2 + \lambda_1) + 2a_2a_4)}, \\ \lambda_3 = &\ \frac{\lambda_1(-4a_1a_2^5 + 2a_1a_2^3\lambda_1 - 2_2^4a_4 + (3a_1^3 - 15a_1a_2^2)a_7 + (3a_2^3 - 3a_1^2a_2)a_9)}{2a_2^3(a_1(4a_2^2 + \lambda_1) + 2a_2a_4)}. \end{aligned} \tag{2.72}$$

By continuing this process we get that the following relations must hold

$$\begin{aligned} I_3 := & \int_0^{2\pi} \left(\{H_9, g_1\} + \{H_8, g_2\} + \{H_7, g_3\} + \ldots + \{H_3, g_7\}\right)\big|_{x=\cos t, y=\sin t}\, dt \\ = & \; p(\lambda_1, \lambda_2, \lambda_3) = 0, \end{aligned}$$

where p is a convenient polynomial of degree five in the variables $\lambda_1, \lambda_2, \lambda_3$. Inserting into I_3 the values of λ_2 and λ_3 from (2.72) we get that the following relations must hold

$$\tilde{p} = p(\lambda_1, \lambda_2, \lambda_3)| = \lambda_1 \left(e_4\lambda_1^4 + e_3\lambda_1^3 + e_2\lambda_2^2 + e_1\lambda_1 + e_0\right) = 0,$$

where

$$\begin{aligned} e_4 = & \; 6550\pi\, a_2^4 a_1^4, \\ e_3 = & \; 41280\pi\, a_2^4 a_1^4 c_1 + r_0^{(3)}, \\ e_2 = & \; \left(99840\pi\, a_2^4 a_1^4 \pi\right) c_1^2 + r_1^{(2)}, \\ e_1 = & \; \left(10a_2a_1(79872a_1^3a_2^5 + 3993a_1^2a_2^4a_4)\right) c_1^2 + r_1^{(1)}, \\ e_0 = & \; \pi\,(20a_1a_2 + 10a_4)\left(79872a_1^3a_2^7 + 39936a_1^2a_2^6a_4\right) c_1^2 + r_1^{(0)}, \end{aligned} \tag{2.73}$$

where $r_j^{(k)}$ is a convenient polynomial of degree j in the variable c_1 for $k = 0, 1, 2, 3,$. Now we show that the polynomial $\tilde{p}$ has only one real root. Indeed from the results given in [42] we get that a quartic polynomial with real coefficients $e_4x^4 + e_3x^3 + e_2x^2 + e_1x + e_0$ with $e_4 \neq 0$ has four complex roots if

$$\begin{aligned} D_2 = & \; 3e_3^2 - 8e_2e_4 \leq 0, \\ D_4 = & \; 256e_4^3e_0^3 - 27e_4^2e_1^4 - 192e_4^2e_1e_0^2e_3 - 27e_3^4e_0^2 - 6e_4e_3^2e_0e_3^2 + e_2^2e_1^2e_3^2 \\ & -4e_4e_2^3e_1^2 + 18e_2e_3^3e_1e_0 + 144e_4e_2e_0^2e_3^2 - 80e_4e_2^2e_0e_3e_1 + 18e_4e_2e_1^3e_3 \\ & -4e_2^3e_0e_3^2 - 4e_3^3e_1^3 + 16e_4e_2^4e_0 - 128e_4^2e_2^2e_0^2 + 144e_4^2e_2e_0e_1^2 > 0. \end{aligned}$$

After some computations we can prove that for the e_j's given in (2.73) for $j = 0, 1, 2, 3, 4$ we get that

$$\begin{aligned} D_2 = & \; \Big(-119500800\pi^2a_1^8a_2^8\Big)c_1^2 + q_1^{(2)}, \\ D_4 = & \; \Big(358428672568945904939289600000000\pi^6a_1^{21}a_2^{27}(2a_1a_2 + a_4)^3\Big)c_1^9 + q_8^{(4)}, \end{aligned}$$

where $q_j^{(k)}$ is a convenient polynomial of degree j in the variable c_1, for $k = 2, 4$. Taking the arbitrary constant c_1 big enough and such that $a_1a_2(2a_1a_2 + a_4)c_1 > 0$ we obtain that the polynomial $\tilde{p}$ has the unique real root $\lambda_1 = 0$, and consequently $\lambda_2 = \lambda_3 = 0$.

Finally we study the case when $2a_1a_2 + a_4$. By repeating the process of the previous case we finally obtain that from the equations $I_j = 0$ for $j = 1, 2, 3$ we get that

$$\begin{aligned} \lambda_3 = & \frac{3a_2}{a_1}\lambda_2, \\ 0 = & \lambda_2\left(174a_2^3\lambda_2 + a_2(87a_1^2 - 29a_2^2)a_9 + a_2(261a_2^2 - 87a_1^2)a_7 \right. \\ & \left. + a_2^3a_1(605a_2^2 - 995a_1^2) + 704a_1a_2^3c_1\right). \end{aligned}$$

By choosing the arbitrary constant properly, we can obtain that the unique solution of $I_j = 0$ for $j = 1, 2, 3$ is $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus the origin is a weak center in this particular case. Thus the necessity of the proposition is proved.

We observe that Proposition 60 provides the necessary and sufficient conditions for the existence of quartic uniform isochronous centers. We observe that this problem was study in [11, 2, 1], but in these papers there are some mistakes. For more details see the appendix.

Proposition 60 can be generalized as follows and the proof is similar

Proposition 61. *The fourth polynomial differential system*

$$\begin{aligned} \dot{x} = & -y(1 + \mu(a_2x - a_1y)) + x\Big(a_1x + a_3x^2 + a_2y + a_4y^2 \\ & + a_5xy + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2\Big), \\ \dot{y} = & x(1 + \mu(a_2x - a_1y)) + y\Big(a_1x + a_2y + a_3x^2 + a_4y^2 \\ & + a_5xy + a_6x^3 + a_7y^3 + a_8x^2y + a_9xy^2\Big), \end{aligned} \tag{2.74}$$

where $(\mu + m - 2)(a_1^2 + a_2^2) + a_3^2 + a_4^2 + a_5^2 \neq 0$ has a weak center at the origin if and only if after a linear change of variables $(x, y) \longrightarrow (X, Y)$ it is invariant under the transformations $(X, Y, t) \longrightarrow (-X, Y, -t)$ or $(X, Y, t) \longrightarrow (X, -Y, -t)$. Moreover,

(i) if $a_1^2 + a_2^2 \neq 0$, then system (2.74) has a weak center at the origin if and only if

$$\begin{aligned} a_3 + a_4 = 0, \qquad a_5a_1a_2 + (a_2^2 - a_1^2)a_4 = & \ 0, \\ a_1^3a_7 - a_1^2a_2a_9 + a_1a_2^2a_8 - a_2^3a_6 = & \ 0, \\ 3a_1a_2^2a_7 - 3a_1^2a_2a_6 + (a_1^3 - 2a_1a_2^2)a_8 + (2a_1^2a_2 - a_2^3)a_9 = & \ 0. \end{aligned}$$

Consequently

(a)

$$
\begin{aligned}
a_3 + a_4 = 0, \qquad a_5 + \frac{(a_2^2 - a_1^2)}{a_1 a_2} a_4 &= 0,\\
a_6 + \frac{1}{2a_2^3}\left(a_7(a_1^3 - 3a_2^2 a_1) + a_9(a_2^3 - a_1^2 a_2)\right) &= 0,\\
a_8 + \frac{1}{2a_2^2 a_1}\left(a_7(3a_1^3 - 3a_1 a_2^2) + a_9(a_2^3 - 3a_1^2 a_2)\right) &= 0.
\end{aligned}
$$

when $a_1 a_2 \neq 0$,

(b) $a_2 = a_3 = a_4 = a_7 = a_8 = 0$, *when* $a_1 \neq 0$,

(c) $a_1 = a_3 = a_4 = a_6 = a_9 = 0$, *when* $a_2 \neq 0$.

(ii) *If* $a_1 = a_2 = 0$ *and* $a_4 a_5 \neq 0$ *then system* (2.74) *has a weak center at the origin if and only if*

$$
\begin{aligned}
a_3 + a_4 &= 0,\\
\lambda\, a_5 + (1 - \lambda^2) a_4 &= 0,\\
\lambda^3 a_7 - \lambda^2 a_9 + \lambda\, a_8 - a_6 &= 0,\\
3\lambda^2 a_7 + 3\lambda a_6 + \left(\lambda^3 - 2\lambda^2\right) a_8 + \left(2\lambda^2 - 1)\right) a_9 &= 0,
\end{aligned}
$$

where $\lambda = \dfrac{a_5 + \sqrt{4a_4^2 + a_5^2}}{2a_4}$. *Moreover the weak center in this case after a linear change of variables* $(x, y) \longrightarrow (X, Y)$ *it is invariant under the transformations* $(X, Y, t) \longrightarrow (-X, Y, -t)$.

(iii) *if* $a_1 = a_2 = a_3 = a_4 = a_5 = 0$, *then the origin is a weak center.*

(iv) $\mu + 2 = a_3 = a_4 = a_5 = 0$, *then the origin is a weak center.*

Remark 62. *A weak center in general is not invariant with respect to a straight line. Indeed, the cubic* Λ-Ω *system with a weak center at the origin [55]*

$$
\begin{aligned}
\dot{x} &= -y\left(1 + y + \frac{y^2}{2}\right) + \frac{x}{2}(x - y - y^2),\\
\dot{y} &= x\left(1 + y + \frac{y^2}{2}\right) + \frac{y}{2}(x - y - y^2),
\end{aligned}
$$

is not invariant with respect to the straight line.

□

2.7 Appendix

The classification of the isochronous centers of Proposition 60 for system (2.63) has been previously studied in the papers [11] and [1]. But in both papers there are some mistakes.

More precisely, in [11] they write system (2.63) in in polar coordinates as follows

$$\dot{r} = P_2(\varphi)r^2 + P_3(\varphi)r^3 + P_4(\varphi)r^4, \quad \dot{\varphi} = 1, \tag{2.75}$$

where

$$\begin{aligned} P_2(\varphi) = &\ R_1 \cos\varphi + r_1 \sin\varphi, \\ P_3(\varphi) = &\ R_2 \cos 2\varphi + r_2 \sin 2\varphi, \\ P_4(\varphi) = &\ R_3 \cos 3\varphi + r_3 \sin 3\varphi + R_4 \cos\varphi + r_4 \sin\varphi. \end{aligned}$$

We note that in [11] there forgot to write the term $r_1 \sin\varphi$. The relations between the parameters of (2.63) and the parameters of system (2.75) are

$$\begin{aligned} R_1 = &\ a_1, \quad r_1 = a_2, \quad R_2 = (a_3 - a_4)/2, \quad r_2 = a_5/2, \\ R_0 = &\ (a_3 + a_4)/2, \quad R_3 = (a_6 - a_9)/4, \quad r_3 = (a_8 - a_7)/4, \\ R_4 = &\ (3a_6 + a_9)/4, \quad r_4 = (3a_7 + a_8)/4. \end{aligned}$$

In [1] they write system (2.63) in complex notation as follows

$$\dot{z} = iz + z\Big(Az + \bar{A}\bar{z} + Bz^2 + 2(b_1 + b_3)z\bar{z} + \bar{B}\bar{z}^2 + Cz^3 + Dz^2\bar{z} + \bar{D}\bar{z}z^2 + \bar{C}\bar{z}^3\Big), \tag{2.76}$$

being $z = x + iy$, $\bar{z} = x - iy$, $A = (a_1 - ia_2)/2$, $B = (b_1 + b_3 - ib_2)/4$, $C = (d_1 - id_2)/8$ and $D = (d_3 - id_4)/8$ where $a_1, a_2, b_1, b_2, b_3, d_1, d_2, d_3, d_4$ are real constants. The relations between the parameters of system (2.63) and the parameters of system (2.76) are

$$\begin{aligned} a_1 = &\ a_1, \quad a_2 = a_2, \\ a_3 = &\ 5(b_1 + b_3)/2, \quad a_4 = 3(b_1 + b_3)/2, \quad a_5 = b_2, \\ a_6 = &\ (d_3 + d_1)/4, \quad a_7 = (d_4 - d_2)/4, \quad a_8 = (d_4 + 3d_2)/4, \quad a_9 = (d_3 - 3d_1)/4. \end{aligned}$$

The following sets of conditions are equivalent

- $r_1 = r_4 = R_0 = R_4 = 0$ and $r_3 \neq 0$ for system (2.75),
- $a_2 = b_1 + b_3 = d_3 = d_4 = 0$ and $b_2 \neq 0$ for system (2.76),
- $a_2 = 3a_7 + a_8 = 3a_6 + a_9 = a_3 + a_4 = 0$ and $a_5 \neq 0$ for system (2.63).

In [11] and [1] they claim that system (2.63) under the previous conditions has a center, but this is uncorrect because such a system has a week focus due to the fact their Liapunov constants are not all zero. Thus its first non-zero Liapunov constant is $\pi a_1^2 a_3/2$. For more details on Liapunov constants see for instance chapter 5 of [20].

Chapter 3

Final evolutions of the generalized 3-dimensional Lotka-Volterra

3.1 Introduction

The Lotka-Volterra systems in $\mathbb{R}^3$ are the differential systems of the form

$$\dot{x}_j = x_j \left(d_j + a_j x_1 + b_j x_2 + c_j x_3 \right), \quad \text{for} \quad j = 1, 2, 3, \tag{3.1}$$

(see for instance [25])

The state space is the set

$$\mathbb{R}^3_+ = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_j \geq 0, \quad \text{for} \quad j = 1, 2, 3. \right\}.$$

Models of Lotka-Volterra systems in $\mathbb{R}^3$ occur frequently in the physical and engineering sciences, as well as in biology. Differential systems (3.1) were introduced independently by Lotka and Volterra in 1920s to model the interaction among the species (see [39, 54, 25])

The applications of the Lotka-Volterra models in the population biology is well-known. For example for the two dimensional Lotka-Volterra model we have the predator-prey model and for the three dimensional Lotka-Volterra model we have the symmetric and non-symmetric May-Leonard models (see for instance [13, 25, 31]) describing the competitions between three species.

Recently it has become important the generalization of the 3-dimensional

Lotka-Volterra systems of differential systems of the form

$$\begin{aligned} \dot{x} = &\; x(a_0 + a_1 x + a_2 y + a_3 z) = X_1, \\ \dot{y} = &\; y(b_0 + b_1 x + b_2 y + b_3 z) = X_2, \\ \dot{z} = &\; c_0 + c_1 x + c_2 y + c_3 z + c_4 x^2 + c_5 xy + c_6 xz + c_7 y^2 + c_8 yz + c_9 z^2 = X_3 \end{aligned} \tag{3.2}$$

In particular to study the resistant viral and bacterial strains, and the treatment on their proliferation (see for instance [9]. One framework for studying such systems is the multistrain model study by Castillo-Chavez and Feng [10]. The model, which is an approximation of the full system discussed in [10] was proposed in [53]. The model has only a single susceptible compartment and two infectious compartments corresponding to the two infectious agents. The model equations are

$$\begin{aligned} \dot{x} = &\; x(-b_1 - \gamma_1 + \nu\, y + \beta_1\, z), \\ \dot{y} = &\; y(-b_1 - \gamma_2 - \nu\, x + \beta_2\, y), \\ \dot{z} = &\; z(-b_1 - \beta_1 x - \beta_2 y) + b_1 + \gamma_1 J_1 + \gamma_2 J_2. \end{aligned} \tag{3.3}$$

The state space is the set

$$\mathbb{R}^3_+ = \left\{(x, y, z) \in \mathbb{R}^3 : z \geq 0 \quad \text{and} \quad y \geq 0,\, x \geq 0\right\}.$$

In [32] the authors characterized all the final evolution of this model under generic assumptions.

The main objective of this section is to study the global dynamics of the generalized Lotka -Volterra model (3.2) with a Darboux invariant $I = (x + y + z - 1)e^{at}$ (see definition below) defined in the positive quadrant of $\mathbb{R}^3$ defined by

$$\mathbb{M} = \{x \geq 0, \quad y \geq 0, \quad \forall z \in \mathbb{R}\}. \tag{3.4}$$

We observe that this differential system does not correspond in general to a biological model, in view of (3.4), become z variable can take in this case negative values.

3.2 Basic definition and preliminary results

In this section we give some preliminary results and definitions necessary in our study of the system (9).

We say that the function $I = I(x, y, z, t)$ is an *invariant* of the differential system $\dot{x} = P(x, y, z)$, $\dot{y} = Q(x, y, z)$, $\dot{z} = R(x, y, z)$, if it is constant on the solutions of this system, i.e.

$$\dot{I} = P\frac{\partial I}{\partial x} + Q\frac{\partial I}{\partial y} + R\frac{\partial I}{\partial z} + \frac{\partial I}{\partial t} = 0.$$

We say that the invariant I is a *Darboux invariant* if $I(x, y, z, t) = I_1(x, y, z)e^{at}$ where a is a non–zero real constant.

Proposition 63. *The most general differential systems* (3.2) *having a Darboux invariant of the form* $I = (x + y + z - 1)e^{-at}$ *are the differential systems*

$$\begin{aligned} \dot{x} = &\ x(a_0 + a_1x + a_2y + a_3z) = X_1, \\ \dot{y} = &\ y(b_0 + b_1x + b_2y + b_3z) = X_2, \\ \dot{z} = &\ c_0 + c_1x - c_2y + c_3z + c_4x^2 + c_5xy + c_6xz + c_7y^2 + c_8yz + c_9z^2 \\ = &\ X_3. \end{aligned} \tag{3.5}$$

Proof. Indeed, a three dimensional generalized Lotka-Volterra system (3.5) has the Darboux invariant $I = (x + y + z - 1)e^{-at}$ if and only if

$$X_1\frac{\partial I}{\partial x} + X_2\frac{\partial I}{\partial y} + X_3\frac{\partial I}{\partial z} + \frac{\partial I}{\partial t} = 0,$$

or equivalently $e^{-at}\left(X_1 + X_2 + X_3 + a(x + y + z - 1)\right) = 0$. Hence we get

$$\begin{aligned} &a(x + y + z - 1) + x(a_0 + a_1x + a_2y + a_3z) + y(b_0 + b_1x + b_2y + b_3z) \\ &+c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5xy + c_6xz + c_7y^2 + c_8yz + c_9z^2 = 0. \end{aligned}$$

Thus the coefficients $a_j,\ b_j$ for $j = 1, 2, 3$ and c_k for $k = 0, 1, \ldots, 9$ are such that

$$\begin{aligned} c_0 = -c_3 = &\ a, \quad c_9 = 0, \quad b_0 + c_2 + a = 0, \quad b_3 + c_8 = 0, \\ b_2 + c_7 = &\ 0, \quad a_0 + c_1 + a = 0, \quad a_3 + c_6 = 0, \\ a_2 + b_1 + c_5 = &\ 0, \quad a_1 + c_4 = 0, \quad b_1 + c_7 = 0, \end{aligned}$$

and taking into account the previous conditions we obtain the systems (3.5) of the statement of the proposition. In short the proposition is proved. □

Corollary 64. *The differential system* (32) *contain as a particular case the differential systems* (3.3).

Proof. Indeed, taking $c = e = h = 0,\ a = -b_1,\ d = -\nu,\ f = -\beta_1,\ i = -\beta_2,\ b = \gamma_1,\ g = \gamma_2,$. we get that (32) becomes (). □

Let $\mathbb{R}[x, y, z]$ be the ring of all real polynomials in the variables x, y and z, and let

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} + R\frac{\partial}{\partial z}$$

be a polynomial vector field in $\mathbb{R}^3$ of degree m. Let $g = g(x, y, z) \in \mathbb{R}[x, y, z]$. Then $g = 0$ is an *invariant algebraic surface* of $\mathcal{X}$ if

$$\mathcal{X}g = P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} + R\frac{\partial g}{\partial z} = Kg$$

where $K = K(x, y, z)$ is a polynomial of degree at most $m - 1$, which is called the *cofactor* of g. A polynomial $g = g(x, y, z)$ satisfying that $g = 0$ is an *invariant algebraic surface* (i.e. formed by orbits of the vector field $\mathcal{X}$) and g is called a *polynomial partial integral* or a *Darboux polynomial*.

It is easy to prove that for the differential system (32) the following relation holds

$$\frac{d}{dt}(x + y + z - 1) = a(x + y + z - 1).$$

Hence we have that the plane $x + y + z - 1 = 0$ is an *invariant plane* Π of (32) with cofactor $K = a$.

Let U be an open subset of $\mathbb{R}^3$. We say that a non-locally constant function $H : U \longrightarrow \mathbb{R}$ is a *first integral* of the differential system

$$\dot{x}_1 = X_1(x_1, x_2), \qquad \dot{x}_2 = X_2(x_1, x_2), \tag{3.6}$$

if $H = H(x_1(t), x_2(t))$ is constant for all values of t for which the solution $(x_1(t), x_2(t))$ is defined and contained in U. Clearly a C^1 function H is a first integral of system (3.6) if and only if

$$\dot{H} = \frac{\partial H}{\partial x_1}X_1 + \frac{\partial H}{\partial x_2}X_2 \equiv 0 \quad \text{in } U.$$

We say that the first integral H is the *Poincaré–Liapunov first integral* if the development in the neighborhood of the equilibrium point translated to the origin of coordinates is of the form

$$H = \frac{1}{2}(x^2 + y^2) + h.o.t..$$

3.2.1 $\alpha-$ and ω-limits of the differential system (32)

In this subsection we prove the strong dependence on the parameter a of the behavior of trajectories of differential system (32).

First we give the following definitions.

In this subsection we study the influence of the parameter a on the trajectories of differential system (32).

We recall the following definitions. Given a differential system a point p in its domain of definition. We say the point q belongs to the the *ω–limit of p* (respectively *α–limit of p*) if there is a sequence $\{t_n\}$ with $t_n \longrightarrow \infty$ (respectively $t_n \longrightarrow -\infty$) such that $\lim\limits_{t_n \to \infty} \Phi_p(t_n) = q$ (respectively $\lim\limits_{t_n \to -\infty} \Phi_p(t_n) = q$), where $\phi_p(t)$ is the solution of the system such that $\Phi_p(0) = p$.

Proposition 65. *Let $I = (x + y + z - 1)e^{-at}$ be a Darboux invariant of differential system (32) and let $\phi_p(t) = (x(t), y(t), z(t))$ be a solution of (32) non contained in the invariant plane $x + y + z.1 = 0$.*

(a) *If $a < 0$, then every $\omega-$ limit is on the closure of the invariant plane $x+y+z = 1$ and every $\alpha-$ limit is at the infinity.*

(b) *If $a > 0$, then every $\alpha-$ limit is on the invariant plane $x+y+z = 1$ inside the Poincaré ball, and every $\omega-$ limit is at the infinity.*

Proof. Statement (a) follows from the relation

$$\lim_{t\to\infty} (x(t) + y(t) + z(t) - 1)\, e^{-at} = C = const \neq 0.$$

In a similar way follows statement (b). □

By solving the equations

$$\begin{aligned} x(a - b - cx - (d + e)y - fz) =&\ 0, \\ y(a - g + dx - hy - iz) =&\ 0, \\ z(fx + iy + a) + x(cx + ey + b) + y(hy + g) - a =&\ 0, \end{aligned}$$

we obtain the finite equilibrium points of the system (32). They are

$$\begin{aligned} \tilde{P}_1 =&\ (0,0,1), \qquad \tilde{P}_2 = \left(0, -\frac{a-g-i}{i-h}, 1 + \frac{a-g-i}{i-h}\right), \\ \tilde{P}_3 =&\ \left(-\frac{a-b-f}{-c+f}, 0, 1 + \frac{a-b-f}{-c+f}\right), \quad \tilde{P}_4 = \left(\frac{I_1}{I_3}, -\frac{I_2}{I_3}, 1 - \frac{I_1 - I_2}{I_3}\right). \end{aligned}$$

where

$$\begin{aligned} I_1 =&\ (f - e - d)(a - g - i) - (a - b - f)(i - h), \\ I_2 =&\ (-c + f)(a - g - i) - (a - b - f)(d + i), \\ I_3 =&\ (-c + f)(-h + i) - (f - e - d)(d + i). \end{aligned}$$

Note that the four finite equilibria points are in the invariant plane $x+y+z-1 = 0$:

We do the change

$$\begin{aligned} a_1 =&\ -c + f, \qquad a_2 = f - e - d, \qquad a_3 = a - b - f, \\ b_1 =&\ d + i, \qquad b_2 = i - h, \qquad b_3 = a - g - i, \end{aligned} \tag{3.7}$$

In these notations the parameters I_1, I_2, I_3, I_4 and the equilibrium points become

$$I_1 = a_2b_3 - a_3b_2, \qquad I_2 = a_1b_3 - a_3b_1, \qquad I_3 = a_1b_2 - a_2b_1, \tag{3.8}$$

and

$$\begin{aligned} \tilde{P}_1 =&\ (0,0,1), \quad \tilde{P}_2 = \left(0, -\frac{b_3}{b_2}, 1 + \frac{b_3}{b_2}\right), \\ \tilde{P}_3 =&\ \left(-\frac{a_3}{a_1}, 0, 1 + \frac{a_3}{a_1}\right), \quad \tilde{P}_4 = \left(\frac{I_1}{I_3}, -\frac{I_2}{I_3}, 1 - \frac{I_1 - I_2}{I_3}\right), \end{aligned}$$

respectively.

Now we determine the local phase portraits of system (32) around these equilibrium points.

As we observe that all these points are on the invariant plane $x+y+z=1$. The points $\tilde{P}_1$, and $\tilde{P}_2$ are also in the the plane $x=0$, and the points $\tilde{P}_1$ and $\tilde{P}_3$ are also in the plane $y=0$.

System (32) restricted to the invariant plane $y=0$ becomes

$$\begin{aligned}\dot{x} = &\quad x(a-b-(a-b-a_3-a_1)x-(a-b-a_3)z)=P,\\ \dot{z} = &\quad z((a-b-a_3)x+a)+x((a-b-a_3-a_1)x+b)-a=Q.\end{aligned}$$

Clearly the equilibrium points on the plane $y=0$ are $P_1=(0,1)$ and $P_2=(-a_3/a_1,\,(a_3+a_1)/a_1)$. The eigenvalues at P_1 are a_3 and a.

To study differential system at the point P_2 we do the translation $x=X-a_3/a_1$, $z=Z+1+a_3/a_1$. Thus we get the system

$$\begin{aligned}\dot{X} = &\quad \frac{(a-b-f)(a-b-a_3-a_1)}{a_1}X+\frac{(a-b-f)(a-b-a_3)}{a_1}Z\\ &\quad +(a_3-a+b)XZ+(b-a+a_3+a_1)X^2,\\ \dot{Z} = &\quad \frac{(a+a_3)a_1-(a-b-a_3)a_3}{a_1}X+\frac{a_1a+(b+a_3-a)a_3}{a_1}Z\\ &\quad +(a-b-a_3-a_1)X^2+(a-b-a_3)XZ.\end{aligned}$$

The eigenvalues at P_3 are $-a_3$ and a.

System (32) restricted to the invariant plane $x=0$ is

$$\begin{aligned}\dot{y} = &\quad y(-(i-b_2)y-iz+i+b_3),\\ \dot{z} = &\quad z(iy+a)+y((i-b_2)y-i+a-b_3)-a.\end{aligned}$$

Clearly the equilibrium points on the plane $x=0$ are $P_1=(0,1)$ and $P_3=(-b_3/b_2,\,(b_3+b_2)/b_2)$.

The eigenvalues at P_1 are b_3 and a.

The eigenvalues at P_2 are $-b_3$ and a.

Now we analyze the point $\tilde{P}_4$ which is in the intersection of the plane $x+y+z=1$ and the set $\mathbb{M}$.

Denoting by M the matrix of the linear part at $\bar{P}_4$, after some computations we get

$$\begin{aligned}\det(M-\lambda I) = &\quad (a-\lambda)\Big(\lambda^2+\frac{I_1a_1-I_2b_2}{I_3}\lambda+\frac{I_1I_2}{I_3}\Big)\\ = &\quad (a-\lambda)(\lambda-\lambda_1^{(4)})(\lambda-\lambda_2^{(4)}).\end{aligned}$$

Hence the eigenvalues of the matrix of the linear part at $\bar{P}_4$, are a, $\lambda_1^{(4)}$ and $\lambda_2^{(4)}$.

We observe that the matrix of the linear part of all deduced differential systems have a as eigenvalue.

From the previous computations we get that the eigenvalue a of the point P_1 corresponds to the z–axis direction. The eigenvalues a_3 and b_3 correspond, respectively, to the eigenvectors following the direction of the intersection of the invariant planes $x = 0$ and $y = 0$ with the plane $x+y+z-1 = 0$. Hence, the points P_2 and P_3, which are adjacent to P_1 along these directions, have the eigenvalues $-a_3$ and $-b_3$, and the eigenvalue a on a direction not contained on the plane $x+y+z = 1$. For the point P_4 we show in section 6 that λ_1^4 and λ_2^4 are eigenvalues of the eigenvector which are on the invariant plane $x+y+z = 1$, so consequently a is the eigenvalue of an eigenvector which is not on this plane. From Proposition 65 we have that the trajectories of the generalized Lotka–Volterra system (32) go from the infinity to the closure of the invariant plane $x+y+z = 1$ in the Poincaré ball if if $a < 0$, and from invariant plane and from the the closure of the invariant plane $x + y + z = 1$ in the Poincaré ball to the infinity if $a > 0$.

From these results it follows that to study the global dynamics of system (32)it is sufficient to study the dynamics on the invariant planes $x = 0$ and $y = 0$ and $x + y + z - 1 = 0$ of system (32) and at the infinity.

To study the behavior of the differential system (32) at infinity of the invariant plane we need the Poincaré compactification in $\mathbb{R}^2$. To study the behavior of system (32) at infinity we need the Poincaré compactification in $\mathbb{R}^3$. The aim of the two next subsection is to give the Poincaré compactification in $\mathbb{R}^2$ and $\mathbb{R}^3$ (see chapter 5 of [20])

3.2.2 The Poincaré compactification in $\mathbb{R}^2$.

Let $\mathcal{X} = P\dfrac{\partial}{\partial x} + Q\dfrac{\partial}{\partial y}$ be a planar quadratic vector field associated to the polynomial differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y).$$

The Poincaré compactified vector field $p(\mathcal{X})$ of $\mathcal{X}$ is an analytic vector field defined on the sphere $\mathbb{S}^2$ as follows (see for instance [20]).

Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3\} \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ and $T_y\mathbb{S}^2$ be the tangent plane to $\mathbb{S}^2$ at the point y. The plane of definition of $\mathcal{X}$ is identified with the plane $T_{(0,0,1)}\mathbb{S}^2$ and let $f : T_{(0,0,1)}\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ be the central projection, i.e. a point $p \in T_{(0,0,1)}\mathbb{S}^2$ is mapped by the central projection into the two points which are in the intersection of the straight line thought p and the origin of coordinates with the sphere $\mathbb{S}^2$. Using the map f we get two copies of $\mathcal{X}$ on $\mathbb{S}^2$ one in the northern hemisphere and the other in the southern one. Let $\mathcal{X}'$ be the vector field $D(f) \circ \mathcal{X}$ on $\mathbb{S}^2 \setminus \mathbb{S}^1$. Here $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$ is the equator of $\mathbb{S}^2$ and corresponds to the infinity of the plane of definition of $\mathcal{X}$.

The vector field $p(\mathcal{X})$ a called the *Poincaré compactification* of $\mathcal{X}$, is the unique analytic extension of $y_3^2\mathcal{X}'$ to $\mathbb{S}^2$. On each hemisphere of $\mathbb{S}^2 \setminus \mathbb{S}^1$ there is a copy of $\mathcal{X}$. The dynamic of $p(\mathcal{X})$ near $\mathbb{S}^1$ provides the dynamics of $\mathcal{X}$ near the infinity. The infinity $\mathbb{S}^1$ is invariant by the Poincaré compactification $p(\mathcal{X})$. The projection $(y_1, y_2, y_3) \longrightarrow (y_1, y_2)$ of the closed northern hemisphere on $y_3 = 0$ is the *Poincaré disc.*

In order to have the expression of $p(\mathcal{X})$ on $\mathbb{S}^2$ we take six local charts (U_j, F_j) and (V_j, F_j) where

$$U_j = \{y \in \mathbb{S}^2 : y_j > 0\}, \quad V_j = \{y \in \mathbb{S}^2 : y_j < 0\},$$

and the diffeomorphisms $F_j : U_j \longrightarrow \mathbb{R}^2$ and $G_j : V_j \longrightarrow \mathbb{R}^2$ for $j = 1, 2, 3$ are the inverses of the central projections from the tangent planes at the points

$$(1,0,0) \quad (-1,0,0), \quad (0,1,0), \quad (0,-1,0), \quad (0,0,1) \quad \text{and} \quad (0,0,-1),$$

respectively. If we denote the points of U_j and V_j by $z = (u, v)$ then their coordinates have different meaning depending on the local chart that we are working. Therefore, after easy computations the differential systems associated to the vector field $p(\mathcal{X})$ are

$$v^2 \left(Q(\frac{1}{v}, \frac{u}{v}) - uP(\frac{1}{v}, \frac{u}{v}), -vP(\frac{1}{v}, \frac{u}{v}) \right) \quad \text{in} \quad U_1,$$

$$v^2 \left(P(\frac{u}{v}, \frac{1}{v}) - uQ(\frac{u}{v}, \frac{1}{v}), -vQ(\frac{u}{v}, \frac{1}{v}) \right) \quad \text{in} \quad U_2.$$

For the V_j's local chart the expressions are the same as the ones of V_j's changed of sign.

The equilibrium points of $p(\mathcal{X})$ which are in $\mathbb{S}^1$ are called *infinite equilibrium points* of $p(\mathcal{X})$ or of $\mathcal{X}$.

3.2.3 The Poincaré compactification in $\mathbb{R}^3$

In $\mathbb{R}^3$ we consider a quadratic polynomial differential system

$$\dot{x} = P^1, \quad \dot{y} = P^2, \quad \dot{z} = P^3,$$

or equivalently its associated polynomial vector field $\mathcal{X} = (P^1, P^2, P^3)$.

Let

$$\mathbb{S}^3 = \{y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : ||y|| = 1\}$$

be the unit sphere in $\mathbb{R}^4$, and $S_+ = \{y \in \mathbb{S}^3 : y_4 > 0\}$ and $S_- = \{y \in \mathbb{S}^3 : y_4 < 0\}$ be the northern and southern hemispheres, respectively. The tangent space to $\mathbb{S}^3$ at the point y is denoted by $Ty\mathbb{S}^3$. Then, the tangent hyperplane $T_{(0,0,0,1)}\mathbb{S}^3 = (x_1, x_2, x_3, 1) \in \mathbb{R}^4 : (x_1, x_2, x_3) \in \mathbb{R}^3$ is identified with $\mathbb{R}^3$. We consider the central projections

$f_+ : \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \longrightarrow \mathbb{S}_+$ and $f_- : \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \longrightarrow \mathbb{S}_-$, defined by

$$f_+ = -\frac{1}{\Delta x}(x_1, x_2, x_3, 1), \quad f_- = \frac{1}{\Delta x}(x_1, x_2, x_3, 1)$$

where $\Delta x = \sqrt{1 + \sum_{j=1}^{3} x_j^2}$. Through these central projections, $\mathbb{R}^3$ can be identified with the northern and the southern hemispheres, respectively. The equator of $\mathbb{S}^3$ is $\mathbb{S}^2 = y \in \mathbb{S}^3 : y_4 = 0$. Clearly, $\mathbb{S}^2$ can be identified with the infinity of $\mathbb{R}^3$. The maps f_+ and f_- define two copies of $\mathcal{X}$, one $Df_+ \circ X$ in the northern hemisphere and the other $Df_- \circ X$ in the southern one. Denote by $\bar{\mathcal{X}}$ the vector field on $\mathbb{S}^3 \setminus \mathbb{S}^2 = \mathbb{S}_+ \cup \mathbb{S}_-$. which restricted to $\mathbb{S}_+$ coincides with $Df_+ \circ \mathcal{X}$ and restricted to $\mathbb{S}_-$ coincides with $Df_- \circ \mathcal{X}$. In what follows we shall work with the orthogonal projection of the closed northern hemisphere to $y_4 = 0$. Note that this projection is a closed ball B of radius one, whose interior is diffeomorphic to $\mathbb{R}^3$ and whose boundary $\mathbb{S}^2$ corresponds to the infinity of $\mathbb{R}^3$. We shall extend analytically the polynomial vector field $\mathcal{X}$ to the boundary, in such a way that the flow on the boundary is invariant. This new vector field on B will be called the *Poincaré compactification* of $\mathcal{X}$, and B will be called the *Poincaré ball.* Poincaré introduced this compactification for polynomial vector fields in $\mathbb{R}^2$, and its extension to $\mathbb{R}^m$ can be found in [14].

The expression for $\bar{\mathcal{X}}(y)$ on $\mathbb{S}_+ \cup \mathbb{S}_+$ is

$$\bar{\mathcal{X}}(y) = y_4 \begin{pmatrix} 1 - y_1^2 & -y_1y_2 & -y_1y_3 \\ -y_1y_2 & 1 - y_2^2 & -y_2y_3 \\ -y_1y_3 & -y_2y_3 & 1 - y_3^2 \\ -y_1y_4 & -y_2y_4 & -y_3y_4 \end{pmatrix} \begin{pmatrix} P^1 \\ P^2 \\ P^3 \end{pmatrix},$$

where $P^n = P^n(y_1/y_4, y_2/y_4, y_3/y_4)$. Written in this way $\bar{\mathcal{X}}(y)$ is a vector field in $\mathbb{R}^4$ tangent to the sphere $\mathbb{S}^3$.

Now we can extend analytically the vector field $\bar{\mathcal{X}}(y)$ to the whole sphere $\mathbb{S}^3$ by $p(\mathcal{X})(y) = y_4^{n-1}\bar{\mathcal{X}}(y)$; this extended vector field $p(\mathcal{X})$ is called the *Poincaré compactification* of $\mathcal{X}$.

As $\mathbb{S}^3$ is a differentiable manifold, to compute the expression for $p(\mathcal{X})$ we can consider the eight local charts $(U_i, F_i), (V_i, G_i)$ where $U_i = \{y \in \mathbb{S}^3 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^3 : y_i < 0\}$ for $i = 1, 2, 3, 4$; the diffeomorphisms $F_i : U_i \longrightarrow \mathbb{R}^3$ and $G_i : V_i \longrightarrow \mathbb{R}^3$ for $i = 1, 2, 3, 4$, are the inverses of the central projections from the origin to the tangent planes at the points $(\pm1, 0, 0, 0), (0, \pm1, 0, 0), (0, 0, \pm1, 0)$ and $(0, 0, 0, \pm1)$, respectively. We now do the computations on U_1

Suppose that the origin $(0, 0, 0, 0)$, the point $(y_1, y_2, y_3, y_4) \in \mathbb{S}^3$ and the point $(1, z_1, z_2, z_3)$ in the tangent plane to $\mathbb{S}^3$ at $(1, 0, 0, 0)$ are collinear, then we have

$$\frac{1}{y_1} = \frac{z_1}{y_2} = \frac{z_2}{y_3} = \frac{z_3}{y_4},$$

and consequently $F_1(y) = \left(\frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{y_4}{y_1}\right) = (z_1, z_2.z_3)$, defines the coordinates on

U_1. As

$$DF_1(y) = \begin{pmatrix} -\frac{y_2}{y_1^2} & \frac{1}{y_1} & 0 & 0 \\ -\frac{y_3}{y_1^2} & 0 & \frac{1}{y_1} & 0 \\ -\frac{y_4}{y_1^2} & 0 & 0 & \frac{1}{y_1} \end{pmatrix},$$

and $y_4^{n-1} = \left(\frac{z_3}{\Delta z}\right)^{n-1}$, the analytical field $p(\mathcal{X})$ becomes

$$\frac{z_3^n}{\Delta\, z^{n-1}} \left(-z_1 P^1 + P^2, -z_2 P^1 + P^3, -z_3 P^1\right),$$

where $P^i = P^i((1/z_3, z_1/z_3, z_2/z_3))$. In a similar way we can deduce the expressions of $p(\mathcal{X})$ in U_2 and U_3 . These are

$$\frac{z_3^n}{\Delta\, z^{n-1}} \left(-z_1 P^2 + P^1, -z_2 P^2 + P^2, -z_3 P^2\right)$$

where $P^i = P^i(z_1/z_3, 1/z_3, z_2/z_3)$ in U_2, and

$$\frac{z_3^n}{\Delta\, z^{n-1}} \left(-z_1 P^3 + P^1, -z_2 P^3 + P^2, -z_3 P^3\right)$$

where $P^i = P^i(z_1/z_3, z_2/z_3, 1/z_3)$ in U_3.

The expression for $p(\mathcal{X})$ in U_4 is $z_3^{n+1}(P_1, P_2, P_3)$ where the component $P_i = P^i(z_1, z_2, z_3)$. The expression for $p(\mathcal{X})$ in the local chart V_i is the same as in U_i multiplied by $(-1)^{n-1}$. When we shall work with the expression of the compactified vector field $p(\mathcal{X})$ in the local charts we shall omit the factor $1/(\Delta\, z)^{n-1}$. We can do that through a rescaling of the time. We remark that all the points on the sphere at infinity in the coordinates of any local chart have $z_3 = 0$.

3.3 Singular points of the system and their local behaviour

3.3.1 Singular points on the invariant planes

Since the plane $x + y + z - 1 = 0$ is invariant for system (32) we shall study the dynamics of this system on this plane.

Clearly on the plane we have that $z = 1 - x - y$. Inserting into (32) we get the following two dimensional Lotka-Volterra system

$$\begin{aligned} \dot{x} = &\; x(a - b - f + (f - c)x + (f - e - d)y), \\ \dot{y} = &\; y(a - g - i + (d + i)x + (i - h)y). \end{aligned} \tag{3.9}$$

We do the change (3.7). Under these changes system (3.9) becomes the two–dimensional Lotka- Volterra system

$$\begin{aligned} \dot{x} = & \; x(a_3 + a_1 x + a_2 y) := P, \\ \dot{y} = & \; y(b_3 + b_1 x + b_2 y) := Q, \end{aligned} \tag{3.10}$$

We remark that these new parameters are independent because the matrix

$$W = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0-1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{pmatrix}.$$

has rank 6 The equilibrium points of this system are

$$\begin{aligned} \tilde{P}_1 = & \; (0,0), \quad \tilde{P}_2 = (0, -\frac{b_3}{b_2}), \\ \tilde{P}_3 = & \; (-\frac{a_3}{a_1}, 0), \quad \tilde{P}_4 = (\frac{I_1}{I_3}, \frac{-I_2}{I_3}). \end{aligned}$$

To study the phase portrait of system (3.10) we do the Poincaré compactification as explained in subsection 3.2.2. Hence we obtain that system (3.10) in the local chart U_1 becomes

$$\begin{aligned} \dot{u} = & \; u(b_1 - a_1 + (b_3 - a_3)v + (b_2 - a_2)u), \\ \dot{v} = & \; -v(a_1 + a_2 u + a_3 v), \end{aligned} \tag{3.11}$$

and in the chart U_2 can be written as

$$\begin{aligned} \dot{u} = & \; u\,(a_2 - b_2 + (a_1 - b_1)u + (a_3 - b_3)v)\,, \\ \dot{v} = & \; -v\,(b_2 + b_1 v + b_3 v)\,. \end{aligned} \tag{3.12}$$

We are interested in studying the infinite singular points of system (3.11) with $v = 0$ and the origin of the system (3.12) if it is a singular point. The chart U_1 has two singular points the origin denoted by P_5 and $P_6 = \left(-\frac{I_4}{I_5}, 0\right)$ if $I_5 \neq 0$, and the origin P_7 of the chart U_2, where

$$I_4 = a_1 - b_1,\; I_5 = a_2 - b_2.$$

3.3.2 Singular points at the infinity

To study the dynamic at the infinity of the region $\mathbb{M}$ we need to do the three–dimensional– Poincaré compactification described in subsection 3.2.3.

The differential system (32) on the chart U_3 is obtained doing the change

$$\tau := \left\{x = \frac{u}{w}, \quad y = \frac{v}{w}, \quad z = \frac{1}{w}\right\},$$

and denoting by

$$\xi(u,v,w) = -cu^2 - hv^2 + aw^2 - buw - evu - gvw,$$

we obtain the particular *Kolmogorov systems*

$$\begin{aligned}
\dot{u} =&\ w^2\,(-u\ R|_\tau + P|_\tau)\\
=&\ u(\xi(u,v,w) - (c+f)u - (d+e+i)v - bw - f),\\
\dot{v} =&\ w^2\,(-v\ R|_\tau + Q|_\tau)\\
=&\ v(\xi(u,v,w) + (d-f)u - (h+i)v - gw - i),\\
\dot{w} =&\ -w^3\ R|_\tau\\
=&\ w(\xi(u,v,w) - fu - iv - aw),
\end{aligned} \tag{3.13}$$

where the quadratic polynomials P, Q and R are given in (32), after the change (1.3).

After the change (3.7) and by considering that $f = a - b - a_3$ we get that (3.13) becomes

$$\begin{aligned}
\dot{u} =&\ u(\tilde{\xi}(u,v,w) + (a_1 - 2f)u + (a_2 - i - f)v + (-a + a_3 + f)w - f),\\
\dot{v} =&\ v(\tilde{\xi}(u,v,w) + (b_1 - i - f)u + (b_2 - 2i)v + (-a + i + b_3)w - i),\\
\dot{w} =&\ w(\tilde{\xi}(u,v,w) - f\,u - iv - aw),
\end{aligned} \tag{3.14}$$

where $\tilde{\xi}(u,v,w)$ is

$$(a_1 - f)u^2 + (a_2 + b_1 - i - f)uv + (a_3 - a + f)wu + (b_2 - i)v^2 + (i - a + b_3)vw + aw^2.$$

The invariant plane $x + y + z - 1 = 0$ in the chart U_3 becomes $u + v - w + 1 = 0$.

The infinite singular points of (3.14) are

$$\begin{aligned}
P_6 &= \left(\frac{I_5}{I_4 - I_5}, \frac{-I_4}{I_4 - I_5}, 0\right), \quad P_7 = (-1,0,0), \quad P_8 = (0,0,0),\\
P_9 &= \left(0, \frac{-i}{h}, 0\right), \quad P_{10} = \left(\frac{-f}{c}, 0, 0\right),\\
P_{11} &= \Big(\frac{ia_2 - b_2 f}{(ia_1 - b_1 f) - (ia_2 - b_2 f) - I_3},\\
&\quad -\frac{ia_1 - b_1 f}{(ia_1 - b_1 f) - (ia_2 - b_2 f) - I_3}, 0\Big).
\end{aligned} \tag{3.15}$$

We observe that the points P_5, P_6 and P_7 are the same points that in the previous section. We want to study the differential equation (3.14) in the plane $w = 0$, at infinity i.e. we study the planar Kolmogorov differential system

$$\begin{aligned} \dot{u} = & \ u(\lambda(u,v) + (a_1 - 2f)u + (a_2 - i - f)v - f) = X_1, \\ \dot{v} = & \ v(\lambda(u,v) + (b_1 - i - f)u + (b_2 - 2i)v - i) = X_2, \end{aligned} \tag{3.16}$$

where $\lambda(u,v) = (a_1 - f)u^2 + (a_2 + b_1 - i - f)uv + (b_2 - i)v^2$,

Its singular points are the projections on the plane $w = 0$ of the points in (3.15) at infinity.

$$\begin{aligned} P_5 & = (0,-1), \quad P_6 = \left(\frac{I_5}{I_4 - I_5}, \frac{-I_4}{I_4 - I_5}\right), \quad P_7 = (-1,0), \\ P_8 & = (0,0), \quad P_9 = \left(0, \frac{-i}{h}\right), \quad P_{10} = \left(\frac{-f}{c}, 0\right), \\ P_{11} & = \left(\frac{ia_2 - b_2 f}{(ia_1 - b_1 f) - (ia_2 - b_2 f) - I_3}, -\frac{ia_1 - b_1 f}{(ia_1 - b_1 f) - (ia_2 - b_2 f) - I_3}\right). \end{aligned}$$

3.3.3 The local dynamic at the singular points.

To study the local dynamic at a singular point of a two-dimensional Lotka-Volterra system we need the determinant Δ and the trace σ of the matrix of the linear part of the system at this point. They can be expressed in terms of the eigenvectors (vaps) of those matrixes: if λ_1,λ_2 are the vaps of the matrix we have $\Delta = \lambda_1\lambda_2$ and $\sigma = \lambda_1 + \lambda_2$. We denote by Δ_j the determinant and by σ_j the trace of the corresponding system's linear part matrix at the point P_j for $j = 1, \dots, 11$.

For system (3.10) and for the points P_1, P_2, P_3 and P_4, after some computations we obtain:

$$\begin{aligned} \Delta_1 = & \ a_3 b_3, & \sigma_1 = a_3 + b_3, \\ \Delta_2 = & \ \frac{b_3 I_1}{b_2}, & \sigma_2 = -b_3 - \frac{I_1}{b_2}, \\ \Delta_3 = & \ -\frac{a_3 I_2}{a_1}, & \sigma_3 = -a_3 + \frac{I_2}{a_1}, \\ \Delta_4 = & \ \frac{-I_1 I_2}{I_3}, & \sigma_4 == \frac{a_1 I_1 - b_2 I_2}{I_3}, \end{aligned}$$

where I_1, I_2 and I_3 are the parameters introduced in the formula (3.8). Analogously for system (3.11) and for the points P_5, P_6 and P_7 we get:

$$\Delta_5 = \ I_4 a_1, \quad \sigma_5 = -a_1 - I_4, \quad \Delta_6 = \frac{I_4 I_3}{I_5}, \quad \sigma_6 = I_4 + \frac{I_3}{I_5},$$

and for the system (3.11) and for the point P_7 we get:

$$\Delta_7 = \quad -I_5 b_2, \qquad \sigma_7 = I_5 - b_2.$$

Finally, for system (3.16) and for the points $P_5, P_6, P_7, P_8, P_9, P_{10}$ and P_{11}, we obtain

$$\begin{aligned}
\Delta_5 = &\quad I_4 a_1, \quad \sigma_5 = a_1 + I_4, \quad \Delta_6 == \frac{I_5 I_4 I_3}{(I_4 - I_5)^2}, \quad \sigma_6 = \frac{I_4 a_2 - I_5 b_1}{I_4 - I_5}, \\
\Delta_7 = &\quad -I_5 b_2, \qquad \sigma_7 = b2 - I_5, \Delta_8 = fi, \quad \sigma_8 = -f - i, \\
\Delta_9 = &\quad \frac{(a_2 i - b_2 f) b_2 i}{(i - b2)^2}, \qquad \sigma_9 = \frac{a_2 i - b_2 f + b_2 i}{b_2 - i}, \\
\Delta_{10} = &\quad \frac{-(a_1 i - b_1 f) a_1 f}{(f - a_1)^2}, \qquad \sigma_{10} = \frac{a_1 i - a_1 f - b_1 f}{f - a_1}, \\
\Delta_{11} = &\quad \frac{((I_5 + b_2) I_4 - I_5 a_1)(-i I_5 + b_2 (f - i))(f I_4 - a_1 (f - i))}{((f - I_5 - b_2) I_4 + (-i + a_1) I_5 - (a_1 - b_2)(f - i))^2}, \\
\sigma_{11} = &\quad \frac{-I_4 b_2 f + I_5 a_1 i}{(f - I_5 - b_2) I_4 + (-i + a_1) I_5 - (a_1 - b_2)(f - i)},
\end{aligned}$$

respectively. We observe that points the P_1, P_2, P_3, P_5, P_6, P_7, P_8, and P_{10} are on the boundary of the region $\mathbb{M}$, so they can only be saddles or nodes, and their local dynamics are completely determined by the corresponding Δ_j and σ_j.

The study of the phase portrait of a differential system in the neighborhood of the equilibrium points P_4 and P_{11} is more complicated, because they can be saddles, nodes, focuss or centers. Thus it is necessary additionally to compute the quantities $\sigma_4^2 - 4\Delta_4$ and $\sigma_{11}^2 - 4\Delta_{11}$. But the main problem is to determine if these points are centers or focuss. Thus we need to solve the center–focus problem. This problem is solved in the following section.

3.4 Center-Focus problem for the points P_4 and P_{11}.

3.4.1 Center-Focus problem for the point P_4

First we shall study the behavior of the differential system in the neighborhood of the equilibrium point P_4. Thus we do the translation

$$x = X + \frac{a_2 b_3 - a_3 b_2}{a_1 b_2 - a_2 b_1}, \qquad y = Y + \frac{a_3 b_1 - a_1 b_3}{a_1 b_2 - a_2 b_1}.$$

Hence differential system (3.10) becomes

$$\begin{aligned}\dot{X} = & \frac{a_1(a_2b_3 - a_3b_2)}{a_1b_2 - a_2b_1}X + \frac{(a_2b_3 - a_3b_2)a_2}{a_1b_2 - a_2b_1}Y + a_1X^2 + a_2YX,\\ \dot{Y} = & \frac{(a_3b_1 - a_1b_3)b_1}{a_1b_2 - a_2b_1}X + \frac{(a_3b_1 - a_1b_3)b_2}{a_1b_2 - a_2b_1}Y + b_1YX + b_2Y^2.\end{aligned} \tag{3.17}$$

The matrix of the linear part is

$$A := \begin{pmatrix} \dfrac{(a_2b_3 - a_3b_2)a_1}{a_1b_2 - a_2b_1} & \dfrac{(a_2b_3 - a_3b_2)a_2}{a_1b_2 - a_2b_1} \\ \\ \dfrac{(a_3b_1 - a_1b_3)b_1}{a_1b_2 - a_2b_1} & \dfrac{(a_3b_1 - a_1b_3)b_2}{a_1b_2 - a_2b_1} \end{pmatrix}.$$

Using the previous notations we get

$$\sigma_4 = \text{trace} A = \frac{a_1I_1 - b_2I_2}{I_3}, \qquad \Delta_4 = \det A = -\frac{I_1I_2}{I_3},$$

To determine the nature of P_4 we additionally need to compute the quantity (see for instance [6, 20])

$$\sigma_4^2 - 4\Delta_4 = \frac{(a_1I_1 - a_2I_2)^2 - 4I_1I_2I_3}{I_3^2}.$$

Consequently if $\sigma_4 \neq 0$ then P_4 is a saddle if $\Delta_4 < 0$, a node if $\Delta_4 > 0$ and $\sigma_4^2 - 4\Delta_4 > 0$, and if $\Delta_4 > 0$ and $\sigma_4^2 - 4\Delta_4 < 0$ then P_4 is a focus.

The main problem in studying the phase portrait of differential equations (3.10) in the neighborhood of P_4 is the case when $\sigma_4 = 0$. This problem is solved in the following proposition.

Proposition 66. *[Solution of the center-focus problem for two dimensional Lotka-Volterra systems] Quadratic differential system* (3.10) *has a center at the point* $P_4 = \left(\dfrac{I_1}{I_3}, -\dfrac{I_2}{I_3}\right)$, *where* I_1, I_2 *and* I_3 *are the parameters given in* (3.8), *if and only if*

$$\sigma_4 = \frac{a_1I_1 - b_2I_2}{I_3} = 0, \qquad \Delta_4\big|_{\sigma_4=0} = -\left.\frac{I_1I_2}{I_3}\right|_{\sigma_4=0} > 0.$$

Moreover differential system (3.10) *under the assumption:*

(a) $b_2(a_1 - b_1) \neq 0$ *has the Poincaré–Liapunov first integral*

$$H_1 = x^{b_2(b_1-a_1)}y^{a_1(a_2-b_2)}\left(b_2(a_2 - b_2)y + (a_1b_2 - b_1a_2)x + b_3(a_2 - b_2)\right)^{a_1b_2-a_2b_1} \tag{3.18}$$

if and only if

$$\sigma_4 = 0 \iff a_3 = \frac{a_1 b_3 (a_2 - b_2)}{b_2(a_1 - b_1)},$$
$$\Delta_4|_{a_3 = \frac{a_1 b_3 (a_2 - b_2)}{b_2(a_1 - b_1)}} > 0 \iff a_1 b_2 (a_1 b_2 - a_2 b_1) < 0.$$

(b) $b_2(a_1 - b_1) = 0$ *has the Poincaré–Liapunov first integral*

(i)

$$H_2 = x^{-b_3} y^{a_3} e^{-b_1 x + a_2 y}.$$

if and only if

$$b_2 = a_1 = 0 \Longrightarrow \sigma_4 = 0, \qquad \Delta_4|_{b_2 = a_1 = 0} > 0 \iff a_3 b_3 < 0,$$

(ii)

$$H_3 = x^{b_2} y^{-a_2} e^{\frac{a_1 x + a_3}{y}}$$

if and only if

$$a_1 - b_1 = b_3 = 0 \Longrightarrow \sigma_4 = 0, \qquad \Delta_4|_{a_1 - b_1 = b_3 = 0} > 0 \iff \frac{b_2}{(a_2 - b_2} > 0.$$

Proof. The center problem for quadratic systems can be solved using the following result due to Bautin (see [5]): The quadratic polynomial differential system

$$\begin{aligned} \dot{U} &= -V - \kappa_3 U^2 + (2\kappa_2 + \kappa_5) UV + \kappa_6 V^2, \\ \dot{V} &= U + (2\kappa_3 + \kappa_4) UV + \kappa_2 (U^2 - V^2) \end{aligned} \tag{3.19}$$

has a center at the origin if and only if on of the following four conditions holds

(i) $\kappa_4 = \kappa_5 = 0$,

(i) $\kappa_2 = \kappa_5 = 0$,

(i) $\kappa_3 - \kappa_6 = 0$,

(i) $\kappa_5 = 0$, $\kappa_4 + 5(\kappa_3 - \kappa_6) = 0$, $\kappa_3\kappa_6 - 2\kappa_6^2 - \kappa_2^2 = 0$.

Now we solve the center problem for the planar Lotka-Volterra systems applying Bautin's result.

Assuming that $b_2(a_1 - b_1) \neq 0$ then by solving the equation $\sigma_4 = 0$ with respect to a_3 then we obtain

$$a_3 = \frac{b_3 a_1 (a_2 - b_2)}{b_2(a_1 - b_1)}. \tag{3.20}$$

Inserting a_3 into (3.17) we get the differential system with with coefficient matrix $\tilde{A} = A|_{(3.20)}$ of the linear part given by

$$\tilde{A} = \begin{pmatrix} \dfrac{b_3 a_1}{a1 - b1} & \dfrac{a_2 b_3}{a_1 - b_1} \\ -\dfrac{a_1 b_1 b_3}{b_2(a_1 - b_1)} & -\dfrac{b_3 a_1}{a_1 - b_1} \end{pmatrix}.$$

On the other hand solving the equation (we suppose that $\Delta_4|_{\sigma_4=0} = \zeta^2 > 0$)

$$\det \tilde{A} = \Delta_4|_{\sigma_4=0} = -\frac{a_1(a_1 b_2 - a_2 b_1) b_3^2}{b_2(a_1 - b_1)^2} = \zeta^2$$

with respect (for example) b_2 we get

$$b_2 = \frac{a_2 a_1 b_1 b_3^2}{\zeta^2 (a_1 - b_1)^2 + a_1^2 b_3^2} := C. \tag{3.21}$$

Now we insert (3.21) into the matrix $\tilde{A}$ we obtain

$$B = \tilde{A}\Big|_{b_2=C} = \begin{pmatrix} \dfrac{b_3 a_1}{a_1 - b_1} & \dfrac{a_2 - b_3}{a_1 - b_1} \\ -\dfrac{a_1 b_1(\zeta^2(a_1 - b_1)^2 + a_1^2 b_3^2)}{a_2 b_3 (a_1 - b_1)} & -\dfrac{b_3 a_1}{a_1 - b_1} \end{pmatrix},$$

where $a_2 b_3 (a_1 - b_1) \neq 0$. Clearly trace$(B) = 0$ and $\det(B) = \zeta^2 > 0$. Consequently its eigenvalues are $\lambda_1 = i|\zeta|$ and $\lambda_2 = -i|\zeta|$.

Hence in view of (3.20) and (3.21) we have that quadratic differential system (3.17) becomes

$$\begin{aligned} \dot{X} = &\ \frac{b_3 a_1}{a_1 - b_1} X + \frac{a_2 - b_3}{a_1 - b_1} Y + \ldots, \\ \dot{Y} = &\ -\frac{a_1 b_1(\zeta^2(a_1 - b_1)^2 + a_1^2 b_3^2)}{a_2 b_3 (a_1 - b_1)} X - \frac{b_3 a_1}{a_1 - b_1} Y + \ldots \end{aligned} \tag{3.22}$$

After the linear change

$$X = b_3 a_2 U, \qquad Y = \zeta(b_1 - a_1)V - a_1 b_3 U,$$

where the matrix

$$S = \begin{pmatrix} b_3 a_2 & 0 \\ -a_1 b_3 & \zeta(b_1 - a_1) \end{pmatrix}$$

is non-degenerated i.e. $\det S = b_3 b_2(b_1 - a_1) \neq 0$, we get that differential system (3.22) becomes

$$\begin{aligned} \dot{U} = & -\zeta V + (b_1 - a_1)a_2 UV, \\ \dot{V} = & \ \zeta U + \frac{b_3 a_2(\zeta^2(a_1^2 - b_1^2) + a_1^2 b_3^2)(a_1 - b_1)}{\zeta^2(a_1 - b_1)^2 + a_1^2 b_3^2} UV \\ & + \frac{\zeta a_1 b_1 b_3 (a_1 - b_1)}{\zeta^2(a_1 - b_1)^2 + a_1^2 b_3^2}(U^2 - V^2). \end{aligned} \tag{3.23}$$

By compare with (3.19) and we obtain that

$$\begin{aligned} \kappa_3 = & \ \kappa_6 = 0, \\ 2\kappa_2 + \kappa_5 = & \ (b_1 - a_1)a_2, \\ 2\kappa_3 + \kappa_4 = & \ \frac{b_3 a_2(\zeta^2(a_1^2 - b_1^2) + a_1^2 b_3^2)(a_1 - b_1)}{\zeta^2(a_1 - b_1)^2 + a_1^2 b_3^2}, \\ \kappa_2 = & \ \frac{\zeta a_1 b_1 b_3 (a_1 - b_1)}{\zeta^2(a_1 - b_1)^2 + a_1^2 b_3^2}. \end{aligned}$$

Consequently, in view of Bautin's result we get that the origin is a center of (3.23) under the condition $b_2 b_3(a_1 - b_1) \neq 0$.

Now we study the case when $b_2 b_3 (a_1 - b_1) = 0$.

(i) First we study the subcase $b_3 = 0$ and $b_2(a_1 - b_1) \neq 0$. From $\sigma_4 = 0$ it follows that $a_3 = 0$. Hence the points $P_4 = P_3 = P_2 = P_1 = (0,0)$, Thus P_4 is on the boundary and consequently can not be a center in this case.

(ii) If $b_2 b_3 \neq 0$ and $a_1 - b_1 = 0$ then the $\text{trace}A|_{a_1=b_1} = -b_3$. So we assume that $b_3 = 0$. On the other hand under these conditions differential system (3.10) becomes

$$\dot{x} = x(a_3 + a_1 x + a_2 y), \qquad \dot{y} = y(a_1 x + b_2 y). \tag{3.24}$$

The equilibrium points of (3.24) are $P_1 = P_2 = (0,0)$, $P_3 = \left(-\dfrac{a_3}{a_1}, 0\right)$ and $P_4 = \left(\dfrac{a_3 b_2}{(a_2 - b_2)a_1}, -\dfrac{a_3}{a_2 - b_2}\right)$. If we consider the case when the point P_3 is in the the first quadrant then $-\dfrac{a_3}{a_1} > 0$ and $\dfrac{b_2}{a_2 - b_2} < 0$. Hence in view of relations $\Delta|_{a_1=b_1,\, b_3=0} = a_3^2 \dfrac{b_2}{a_2 - b_2} < 0$, we get that P_4 is a saddle.

Now we study the case when $-\dfrac{a_3}{a_1} < 0$, i.e. the point P_3 is out of the first quadrant, then P_4 is on the first quadrant if $\dfrac{b_2}{a_2 - b_2} > 0$ and consequently

$\Delta|_{a_1=b_1,\, b_3=0} = a_3^2 \dfrac{b_2}{a_2-b_2} > 0$. We prove that in this case the point P_4 is a center. Indeed after the translation

$$x = X + \frac{a_3 b_2}{(a_2-b_2)a_1}, \qquad y = Y - \frac{a_3}{a_2-b_2},$$

we obtain that system (3.24) becomes

$$\begin{aligned} \dot{X} = &\ \frac{a_3 b_2}{a_2-b_2} X + \frac{a_3 b_2 a_2}{a_1(a_2-b_2)} Y + a_1^2 X^2 + a_2 XY, \\ \dot{Y} = &\ -\frac{a_3 a_1}{a_2-b_2} X - \frac{a_3 b_2}{a_2-b_2} Y + b_2^2 X^2 + a_1 XY, \end{aligned} \tag{3.25}$$

Taking $b_2 = a_2\zeta^2/(a_3^2+\zeta^2)$ we get that the matrix of the linear part has trace zero and determinant equal to ζ^2. Differential system (3.25) becomes

$$\begin{aligned} \dot{X} = &\ \frac{\zeta^2}{a_3} X + \frac{\zeta^2}{a_1 a_3} Y + a_1 X^2 + a_2 XY, \\ \dot{Y} = &\ -\frac{a_1(a_3^2+\zeta^2)}{a_2 a_3} X - \frac{\zeta^2}{a_3} Y + \frac{a_2\zeta^2}{a_3^2+\zeta^2} Y^2 + a_1 XY, \end{aligned} \tag{3.26}$$

After the linear transformation

$$X = -\frac{a_2\zeta^2}{b a_1(a_3^2+\zeta^2)} U + \frac{a_2 a_3 \zeta}{b a_1(a_3^2+\zeta^2)} V, \qquad Y = \frac{U}{b},$$

we obtain that differential system (3.26) becomes

$$\begin{aligned} \dot{V} = &\ -\zeta U - \frac{a_3 a_2 a_3}{b(\zeta^2+a_3^2)} UV, \\ \dot{U} = &\ \zeta V + \frac{a_2 a_3 \zeta}{b(\zeta^2+a_3^2)} \left(V^2 - U^2\right) + \frac{1 a_2 a_3^2}{b(\zeta^2+a_3^2)} UV. \end{aligned} \tag{3.27}$$

Hence in view of Bautin's result with $\kappa_3 = \kappa_6 = 0$ we obtain that the origin is a center of (3.27).

(iv) When $b_2 = 0$ we get that trace$A|_{b_2=0} = -\dfrac{b_3 a_1}{b_1}$. Hence $a_1 b_3 = 0$. If $b_3 = 0$ then $P_4 = \left(0, -\dfrac{a_3}{a_2}\right)$ belong to the boundary and consequently it is not a center. If $a_1 = 0$ then $\det A|_{b_2=a_1=0} = -b_3 a_3$. If $b_2 = a_1 = 0$. Hence if $-a_3 b_3 < 0$ then P_4 is a saddle. We shall study the case when $-a_3 b_3 > 0$. Thus differential system (3.10) under the conditions $b_2 = a_1 = 0$ becomes

$$\dot{x} = x(a_3 + a_2 y), \qquad \dot{y} = y(b_3 + b_1 x). \tag{3.28}$$

We have the equilibrium points $P_1 = (0,0), P_2 = P_7$ and $P_3 = P_5$, but the points P_2 and P_3 go to the infinity and $P_4 = \left(-\frac{b_3}{b_1}, -\frac{a_3}{a_2}\right)$ belong to the first quadrant if $\frac{b_3}{b_1} < 0$ and $\frac{a_3}{a_2} < 0$, consequently by considering that $a_3 b_3 < 0$ we obtain that point $P_6 = (b_1/a_2, 0)$ is out of the first quadrant because $b_1/a_2 < 0$.

After the translation $x = -\frac{b_3}{b_1} + X$ and $y = -\frac{a_3}{a_2} + Y$ system (3.28) becomes

$$\dot{X} = -a_3 r Y + b_1 XY, \qquad \dot{Y} = -\frac{b_3}{r} Y + a_2 XY, \qquad r = \frac{b_1}{a_2}. \tag{3.29}$$

By considering that $-a_3 b_3 > 0$ then after the linear transformation

$$X = \sqrt{\frac{-a_3}{b_3}} V, \qquad Y = \frac{U}{r}, \qquad t = \frac{\tau}{\sqrt{-a_3 b_3}},$$

we obtain that differential system (3.29) is transformed to the system

$$U' = V + \frac{a_2^2}{b_1 |a_3|} UV, \qquad V' = -U + \frac{b_1^2}{a_2 |a_3|} UV.$$

Hence from Bautin's result we get that the origin of the previous system is a center.

□

Proof of Proposition 66 applying Poincaré's result. Another method to solve the center problem is the Poincaré's result (see for instance [47, 34, 36]): A planar polynomial differential system

$$\dot{x} = -y + \sum_{j=2}^{m} X_j(x,y), \quad \dot{y} = x + \sum_{j=2}^{m} Y_j(x,y), \tag{3.30}$$

of degree m has a center at the origin if and only if it has a first integral of the form

$$H = \sum_{j=2}^{\infty} H_j(x,y) = \frac{1}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j(x,y),$$

where X_j, Y_j and H_j are homogenous polynomials of degree j.

Now we prove the proposition using the Poincaré's result.

First we apply the following result (see [20]) : Suppose that a polynomial vector field $\mathcal{X}$ defined in $\mathbb{R}^2$ admits M invariant algebraic surfaces $g_j = 0$ with cofactor K_j for $j = 1, \dots, M$, i.e. the following equations hold

$$\mathcal{X} g_j = X_1 \frac{\partial g_j}{\partial x} + X_2 \frac{\partial g_j}{\partial y} = K_j \, g_j$$

for $j = 1, \dots, M$ If there exist $\mu_j \in \mathbb{R}$ not all zero such that

$$\sum_{j=1}^{M} \mu_j K_j + \mathrm{div}\mathcal{X} = 0, \tag{3.31}$$

then the function

$$\prod_{j=1}^{M} |g_j|^{\mu_j} \tag{3.32}$$

is an integrating factor.

By considering that a two- dimensional Lotka-Volterra system has two invariant straight lines $x = 0$ and $y = 0$ then it has an integrating factor $J = x^{\mu_1} y^{\mu_2}$ if and only if

$$\mu_1 a_3 + \mu_2 b_3 + a_3 + b_3 + (\mu_1 a_1 + \mu_2 b_1 + 2a_1 + b_1)x + (\mu_1 a_2 + \mu_2 b_2 + 2b_2 + a_2)y = 0,$$

consequently

$$\begin{aligned} \mu_1 a_1 + \mu_2 b_1 + 2a_1 + b_1 = &\ 0, \\ \mu_1 a_2 + \mu_2 b_2 + 2b_2 + a_2 = &\ 0, \\ \mu_1 a_3 + \mu_2 b_3 + a_3 + b_3 = &\ 0. \end{aligned} \tag{3.33}$$

Thus this system has a solution if and only if the matrix

$$M = \begin{pmatrix} a_1 & b_1 & -2a_1 - b_1 \\ a_2 & b_2 & -a_2 - 2b_2 \\ a_3 & b_3 & -a_3 - b_3 \end{pmatrix}$$

is such that rang$M < 3$. Consequently

$$\det M = (a_1 b_2 - a_2 b_1)\sigma_4 = a_1(a_2 b_3 - a_3 b_2) - b_2(a_1 b_3 - a_3 b_1) = 0, \tag{3.34}$$

Hence, if $a_1 b_2 - a_2 b_1 \neq 0$ and (3.34) holds, then from (3.33) we obtain that

$$\mu_1 = -1 + \frac{b_2(b_1 - a_1)}{a_1 b_2 - a_2 b_1}, \qquad \mu_2 = -1 + \frac{a_1(a_2 - b_2)}{a_1 b_2 - a_2 b_1}.$$

Clearly if $a_1 b_2 - a_2 b_1 = 0$ then the point P_4 goes at infinity. Thus we shall suppose that always $a_1 b_2 - a_2 b_1 \neq 0$.

Hence the integrating factor becomes

$$J = x^{-1+\frac{b_2(b_1-a_1)}{a_1 b_2 - a_2 b_1}} y^{-1+\frac{a_1(a_2-b_2)}{a_1 b_2 - a_2 b_1}}, \tag{3.35}$$

,

Consequently we have the existence the Integrating factor J, thus it follows that there exists a first integral H_1 such that (see [25])

$$\frac{\partial H_1}{\partial y} = -Jx(a_3 + a_1 x + a_2 y), \quad \frac{\partial H_1}{\partial x} = Jy(b_3 + b_1 x + b_2 y).$$

After the integration we get the Darboux first integral (3.18).

From this first integral we get that the system (3.10) has the additional straight line

$$g_3 = b_2(a_2 - b_2)y + (a_1b_2 - b_1a_2)x + b_3(a_2 - b_2) = 0,$$

with cofactor $a_1x + b_2y$.

After some computations and in view of (3.20) we get that H has the following Taylor expansion at the point $\tilde{P}_4 = P_4|_{a_3=\frac{a_1b_3(a_2-b_2)}{b_2(a_1-b_1)}} = (\frac{b_3}{a_1-b_1}, -\frac{a_1b_3}{b_2(a_1-b_1)})$

$$H_1 = -\frac{b_3}{b_2(a_1-b_1)}\left(a_1b_1x^2 + 2a_1b_2xy + a_2b_2y^2\right) + h.o.t. =< p, \Omega p^T > +h.o.t..$$

where $p = (x, y)$ and

$$\Omega = \begin{pmatrix} -\dfrac{b_1a_1b_3}{b_2(a_1-b_1)} & -\dfrac{a_1b_3}{a_1-b_1} \\ -\dfrac{a_1b_3}{a_1-b_1} & -\dfrac{b_3a_2}{a_1-b_1} \end{pmatrix}.$$

By considering that H_1 is a Poincaré-Liapunov first integral if and only if

$$\det\Omega = \Delta_4|_{a_3=\frac{a_1b_3(a_2-b_2)}{b_2(a_1-b_1)}} = -\frac{a_1(a_1b_2 - a_2b_1)b_3^2}{b_2(a_1-b_1)^2} > 0.$$

then we obtain the proof of the proposition under the condition $b_2(a_1 - b_1) \neq 0$.

From this result we see the advantage of the Poincaré's result over the Bautin's result. The Poincaré's result solves the center problem and additionally gives a information on the existent of the additional straight line $a_1b_3x + a_3b_2y + a_3b_3 = 0$.

Now we study the case when $b_2(a_1 - b_1) = 0$.

(i) If $b_2 = 0$ from $\sigma_4 = a_1a_2b_3 = 0$ we get that if $b_2 = b_3 = 0$ the point $P_4|_{b_2=b_3=0} = (0 - \frac{a_3}{a_1})$ belong to the boundary, consequently it is not a center

If $b_2 = a_1 = 0$, then by considering that in this case $\tilde{\Delta} = \Delta|_{b_2=a_1=0} = -a_2b_3$, then if $-a_2b_3 < 0$ the system has at point $\tilde{P}_4 == P_4|_{a_1=b_2=0}$ a saddle. We study the case when $-a_2b_3 > 0$. Differential system (3.10) under the conditions $b_2 = a_1 = 0$ becomes (3.28). This differential system has the Integrating factor $J = 1/xy$ which is obtained from (3.35) under the conditions $a_1 = b_2 = 0$. Consequently the Darboux first integral is $H_2 = |x|^{-b_3}|y|^{a_3}e^{-b_1x+a_2y}$, which has the following Taylor expansion at the point $\tilde{P}_4 = \left(-\frac{b_3}{b_1}, -\frac{a_3}{a_2}\right)$

$$H_2 = H_2(\tilde{P}_4)\left(\frac{b_1^2}{b_3}x^2 - \frac{a_2^2}{a_3}y^2\right) + h.o.t. =< p, \Omega p^T > +h.o.t,$$

This integral is a Poincaré- Liapunov first integral if and only if $\det\Omega = -\dfrac{b_1^2 a_2^2}{a_3 b_3} > 0$. Consequently the point $\tilde{P}_4$ is a center if $-a_3 b_3 > 0$.

(ii) If $a_1 - b_1 = 0$ then we had proved above that if $\sigma_4|_{a_1=b_1} = b_3 = 0$, then the point $\tilde{P}_4 = P_4|_{a_1=b_1,\, b_3=0}$ is a saddle if $-a_3/a_1 > 0$. Now we study the subcase when $-a_3/a_1 < 0$. Then we have that the Integrating factor becomes $J = (|x|y^2)^{-1}$. After some computations we can prove that the first integral is H_3 which in the neighborhood of the point $P_4 = \left(\dfrac{a_3 b_2}{(a_2-b_2)a_1}, -\dfrac{a_3}{a_2-b_2}\right)$. has the following Taylor expansion

$$H_3 = . =< p, \Omega p^T > +h.o.t.,$$

where $\det\Omega = \left(\dfrac{(a_2-b_2)a_1}{a_3^2}\right)^2 \dfrac{a_2-b_2}{b_2}$. Consequently H_3 is a Poincaré–Liapunov first integral if and only if $\dfrac{a_2-b_2}{b_2} > 0$.

Thus the proposition is proved. □

3.4.2 The center–focus problem for the point P_{11}

. In this subsection we shall study the behavior of differential system (3.16) in the neighborhood of the equilibrium point

$$P_{11} = \left(\frac{b_2 f - i a_2}{a_1(b_2 - i) + b_1(f - a_2) + i a_2 - b_2 f}, \frac{i a_1 - b_1 f}{a_1(b_2 - i) + b_1(f - a_2) + i a_2 - b_2 f}\right).$$

Proposition 67. *Differential system* (3.16) *after the translation*

$$\begin{aligned} u = &\quad X + \frac{b_2 f - i a_2}{a_1(b_2 - i) + b_1(f - a_2) + i a_2 - b_2 f}, \\ v = &\quad Y + \frac{i a_1 - b_1 f}{a_1(b_2 - i) + b_1(f - a_2) + i a_2 - b_2 f}, \end{aligned}$$

can be written as follows

$$\begin{aligned} \dot{X} = &\quad a_{10}X + a_{01}Y + a_{20}X^2 + a_{11}XY + a_{02}Y^2 + a_{30}X^3 \\ &+ a_{21}X^2Y + a_{12}XY^2 = P_1, \\ \dot{Y} = &\quad b_{10}X + b_{01}Y + b_{20}X^2 + b_{11}XY + b_{02}Y^2 + b_{21}X^2Y \\ &+ b_{12}XY^2 + b_{03}Y^3 = P_2, \end{aligned} \tag{3.36}$$

where a_{ij} and b_{ij} are the following constants

$$\begin{aligned}
a_{10} = &\ -(a_2 i - b_2 f)\Big(a_1^2 b_2 - a_1^2 i - a_1 a_2 b_1 + a_1 b_1 f + a_1 b_1 i - a_1 b_2 f + a_1 f i - a_1 i^2 \\
&+ a_2 b_1 f - b_1^2 f - b_1 f^2 + b_1 f i\Big), \\
a_{01} = &\ -(a_2 i - b_2 f)\Big(a_1 a_2 b_2 - a_1 a_2 i - a_1 b_2 f + a_1 b_2 i + a_1 f i - a_1 i^2 - a_2^2 b_1 \\
&+ 2 a_2 b_1 f - b_1 b_2 f - b_1 f^2 + b_1 f i\Big), \\
b_{10} = &\ -(a_1 i - b_1 f)\Big(a_1 a_2 i - a_1 b_1 b_2 - a_1 b_2 f + a_1 b_2 i + a_2 b_1^2 - 2 a_2 b_1 i \\
&- a_2 f i + a_2 i^2 + b_1 b_2 f + b_2 f^2 - b_2 f i\Big), \\
b_{01} = &\ (a_1 i - b_1 f)\Big(a_1 b_2^2 - a_1 b_2 i - a_2^2 i - a_2 b_1 b_2 + a_2 b_1 i + a_2 b_2 f \\
&+ a_2 b_2 i + a_2 f i - a_2 i^2 - b_2^2 f - b_2 f^2 + b_2 f i\Big).
\end{aligned}$$

The matrix of the linear part $M = \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix}$ *is such that*

$$\begin{aligned}
\det M := \Delta_{11} == &\ \frac{(b_2 f - i a_2)(i a_1 - b_1 f)(a_2 b_1 - a_1 b_2)}{(a_1(b_2 - i) + b_1(f - a_2) + i a_2 - b_2 f)^2}, \\
traceM = \sigma_{11} = &\ \frac{a_1 b_2 (i + f) - i a_1 a_2 - f b_1 b_2}{(a_1(b_2 - i) + b_1(f - a_2) + i a_2 - b_2 f}.
\end{aligned}$$

Proof. It is easy to obtain after some computations. □

Clearly if $\sigma_{11}^2 - 4\Delta_{11} > 0$ and $\Delta_{11} > 0$, then the singular point P_{11} is a node, if $\sigma_{11}^2 - 4\Delta_{11} < 0$ and $\Delta_{11} > 0$ then the singular point P_{11} is a focus. We shall study the case when $\sigma_{11} = 0$ and $\Delta_{11} > 0$, i.e. we shall study the center-focus problem for the equation (3.36) (see for instance [5, 36])

Proposition 68. *[Solution of the center-focus problem for a class the Kolmogorov differential system]*

The cubic Kolmogorov differential system (3.16) *has a center at the equilibrium point*

$$P_{11} = \Big(\frac{b_2 f - i a_2}{a_1(b_2 - i) + b_1(f - a_2) + i a_2 - b_2 f}, -\frac{i a_1 - b_1 f}{(i a_1 - b_1 f) - (i a_2 - b_2 f) - I_3}\Big),$$

if and only if

$$\begin{aligned}
\sigma_{11} := &\ \frac{i a_1 (b_2 - a_2) - f b_2 (b_1 - a_1)}{(a_1(b_2 - i) + b_1(f - a_2) + i a_2 - b_2 f} = 0, \\
\Delta_{11}\big|_{\sigma_{11}=0} > 0 = &\ \left.\frac{(b_2 f - i a_2)(i a_1 - b_1 f)(a_2 b_1 - a_1 b_2)}{(a_1(b_2 - i) + b_1(f - a_2) + i a_2 - b_2 f)^2}\right|_{\sigma_{11}=0} > 0.
\end{aligned} \tag{3.37}$$

Moreover differential system (3.16) *under the assumption*

(a) $a_1(b_2 - a_2) \neq 0$ *has the Poincaré–Liapunov first integral*

$$H = u^{b_2(b_1-a_1)}v^{a_1(b_1-b_2)}(1+u+v)^{(a_1-b_1)(a_2-b_2)}$$
$$(b_2(a_1+i-b_1)-a_2 i)u + (b_2(a_1+i-b_2)-a_2 i)v + i(a_2-b_2))^{a_1b_2-a_2b_1} \tag{3.38}$$

if and only if

$$\begin{aligned} \sigma_{11} = 0 \quad &\Longleftrightarrow i = \frac{fb_2(b_1-a_1)}{a_1(b_2-a_2)}. \\ \Delta_{11}|_{i=\frac{fb_2(b_1-a_1)}{a_1(b_2-a_2)}} > 0 \quad &\Longleftrightarrow b_2a_1(a_1b_2-a_2b_1) < 0. \end{aligned} \tag{3.39}$$

(b) $a_1(a_2 - b_2) = 0$ *has the Poincaré–Liapunov first integral*

(i)

$$H = e^{\frac{(a_2+b_1)u+a_2}{1+u+v}}\left(\frac{v}{1+u+v}\right)^f\left(\frac{u}{1+u+v}\right)^{-i} \tag{3.40}$$

if and only if

$$b_2 = a_1 = 0 \Longrightarrow \sigma_{11} = 0, \qquad \Delta_{11}|_{b_2=a_1=0} > 0 \Longleftrightarrow if > 0,$$

(ii)

$$H = e^{\frac{\nu(i-b_2)+i}{u}}\left(\frac{u}{1+u+v}\right)^{b_1}\left(\frac{v}{1+u+v}\right)^{-a_1} \tag{3.41}$$

if and only if

$$b_2 - a_2 = f = 0 \Longrightarrow \sigma_{11} = 0, \qquad \Delta_{11}|_{b_2-a_2=f=0} > 0 \Longleftrightarrow a_1(a_1-b_1) > 0.$$

Proof. By considering that two- dimensional Kolmogorov differential system (3.16) has three invariant straight lines with cofactors K_j for $j = 1, 2, 3$ namely

$$\begin{aligned} g_1 = \quad & u \qquad K_1 = \lambda(u,v) + (a_1 - 2f)u + (a_2 - i - f)v - f, \\ g_2 = \quad & v \qquad K_2 = \lambda(u,v) + (b_1 - i - f)u + (b_2 - 2i)v - i, \\ g_3 = \quad & u + v + 1 \qquad K_3 = \lambda(u,v) - fu - iv, \end{aligned}$$

then this system has a integrating factor (see (3.32)) $J = u^{\mu_1}v^{\mu_2}(1+u+v)^{\mu_3}$ if and only if (see formula (3.31))

$$\mu_1K_1 + \mu_2K_2 + \mu_3K_3 + \text{div}\mathcal{X} = 0,$$

where $\mathcal{X} = (X_1, X_2)$ is the vector field associated to system (3.16). Under the change $\mu_j = -1 + \nu_j$, for $j = 1, 2, 3$, and after some computations we can prove that this equation is equivalent to the linear system with respect to ν_1, ν_2 and ν_3

$$\begin{pmatrix} 1 & 1 & 1 & -4 \\ a_1 - 2f & b_1 - f - i & -f & -(2a_1 - 5f + b_1 - i) \\ a_2 - f - i & b_2 - 2i & -i & -(a_2 - f - 5i + 2b_2) \\ -f & -i & 0 & f + i \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This lineal system has a unique solution if and only $\sigma_{11} = 0$. Under this condition we get that

$$\mu_1 = -1 - \frac{b_2(-b_1 + a_1)}{a_1 b_2 - a_2 b_1}, \ \mu_2 = -1 + \frac{(a_2 - b_2)a_1}{a_1 b_2 - a_2 b_1}, \ \mu_3 = -1 + \frac{(b_1 - a_1)(a_2 - b_2)}{c_1 b_2 - a_2 b_1}.$$

Thus the integrating factor is given by formula (3.45).

First we prove statement (a). Using the previous integrating factor under the condition $a_1(-b_2 + a_2) \neq 0$, we deduce that vector field

$$\tilde{\mathcal{X}} = \mathcal{X}|_{i = \dfrac{(a_1 - b_1)b_2 f}{a_1(-b_2 + a_2)}} = (\tilde{X}_1, \tilde{X}_2),$$

has the Darboux first integral (3.38). Now we prove that H is a Poincaré–Liapunov first integral. We denote by Ω the matrix

$$\Omega = \begin{pmatrix} \dfrac{\partial^2 H}{\partial u \partial u} & \dfrac{\partial^2 H}{\partial u \partial v} \\ \dfrac{\partial^2 H}{\partial u \partial v} & \dfrac{\partial^2 H}{\partial v \partial v} \end{pmatrix}$$

The Darboux first integral (3.38) has the following Taylor expansion in the neighborhood of the point P_{11} $H =< p, \Omega\, p^T > +h.o.t.$, where $p = (u, v)$. After some computations we have that

$$\det \Omega = -\left. J^2 \right|_{p_{11}} \frac{f^2 b_2 a_1 (a_1 b_2 - a_2 b_1)}{(a_1(a_2 - b_2) + f(b_2 - a_1))^2} = \left. J^2 \right|_{p_{11}} \left. \Delta_{11} \right|_{i = \frac{(a_1 - b_1) b_2 f}{a_1(-b_2 + a_2)}} > 0,$$

Thus the Darboux first integral is a Poincaré-Liapunov first integral if and only if $\left.\Delta_{11}\right|_{i = \frac{(a_1 - b_1) b_2 f}{a_1(-b_2 + a_2)}} > 0$, this inequality holds if and only if $b_2 a_1 (a_1 b_2 - a_2 b_1) < 0$. Thus in view of the Poincaré result system (3.16) has a center at the the point P_{11} under the condition (3.39).

Now we prove statement (b). First we study case (i). Under the condition $a_1 = b_2 = 0$ differential system (3.16) becomes

$$\begin{aligned} \dot{u} = & \ u\left(-fu^2 + (a_2 + b_1 - i - f)uv - iv^2 + (b_1 - i - f)u - 2iv - i\right), \\ \dot{v} = & \ v\left(-fu^2 + (a_2 + b_1 - i - f)uv - iv^2 + (b_1 - i - f)u - 2iv - i\right). \end{aligned} \tag{3.42}$$

From (3.24) with $a_1 = b_2 = 0$ we get that the integrating factor (3.24) in this case becomes $\tilde{J} = J|_{a_1=b_2=0} = \dfrac{1}{|uv|(1+u+v)^2}$. Consequently we get that differential system (3.42) is integrable. After some computations we obtain the Darboux first integral (3.40). This function has the following Taylor expansion in the neighborhood of the point $P_{11}|_{b_2=a_1=0}$ $H =< p, \Omega\, p^T > +h.o.t.$, where

$$\det \Omega = if\left(\frac{(b_1a_2 - 1a_2 - b_2f}{a_2^2b_1^2if}\right)^2.$$

Thus the Darboux first integral is a Poincaré-Liapunov first integral if and only if $if > 0$ and consequently system (3.42) has a center at the point $P_{11}|_{b_2=a_1=0}$.

Finally we study case (ii) i.e., when $a_2 - b_2 = f = 0$. Differential system (3.16) under this condition becomes

$$\begin{aligned} \dot{u} = &\ u\left(a_1u^2 + (b_2 + b_1 - i)uv + (b_2 - i)v^2 + a_1u + (b_2 - i)v\right), \\ \dot{v} = &\ v\left(a_1u^2 + (b_2 + b_1 - i)uv + (b_2 - i)v^2 + (b_1 - i)u + (b_2 - 2i)v - i\right). \end{aligned} \tag{3.43}$$

From (3.24) with $a_2 - b_2 = f = 0$ we get that the integrating factor is $\tilde{J} = J|_{a_2-b_2=f=0} = \dfrac{1}{u^2|v(1+u+v)|}$. Consequently we get that differential system (3.43) is integrable. After some computations we get that the first integral of (3.43) is (3.41). This function has the following Taylor expansion in the neighborhood of the point $p_{11}|_{a_2-b_2=f=0}$, $H =< p, \Omega\, p^T > +h.o.t.$, where

$$\det \Omega = A^2a_1(a_1 - b_1),$$

where A is a convenient non-zero constant. Thus the Darboux first integral is a Poincaré-Liapunov first integral if and only if $a_1(a_1 - b_1) > 0$, and consequently system (3.42) has a center at the point $P_{11}|_{a_2-b_2=f=0}$. In short the proposition is proved. □

3.4.3 Non-existence of limit cycles for the differential systems (3.10) and (3.16)

Differential systems (3.10) and (3.16) have not limit cycles as follows from the next result.

Corollary 69. *(i) Two dimensional Lotka-Volterra vector field $\mathcal{X} = (P, Q)$ associated to system* (3.10) *has the Dulac function*

$$J = x^{-1+\frac{b_2(b_1-a_1)}{a_1b_2-a_2b_1}}\,y^{-1+\frac{a_1(a_2-b_2)}{a_1b_2-a_2b_1}},$$

such that

$$div(J\mathcal{X}) = \sigma_4 J, \tag{3.44}$$

Consequently differential system (3.10) *has not periodic solution if* $\sigma_4 \neq 0$. *Moreover if* $\sigma_4 = 0$ *then* J *is the integrating factor.*

(i) *Kolmogorov vector field* $\mathcal{X} = (X_1, X_2)$ *associated to system* (3.16) *has the Dulac function*

$$J = u^{-1-\frac{b_2(-b_1+a_1)}{a_1b_2-a_2b_1}} v^{-1+\frac{(a_2-b_2)a_1}{a_1b_2-a_2b_1}} (1+u+v)^{-1+\frac{(b_1-a_1)(a_2-b_2)}{a_1b_2-a_2b_1}}. \tag{3.45}$$

such that

$$div(J\mathcal{X}) = -(1+u+v)\sigma_{11}J, \tag{3.46}$$

Consequently differential system (3.16) *have not periodic solution if* $\sigma_{11} \neq 0$. *Moreover if* $\sigma_{11} = 0$ *then* J *is the integrating factor.*

Proof. To prove this corollary we apply the Bendixon-Dulac result (see for instance [6, 20]): Let $\mathcal{X}$ be the vector field ¡ssociated to the differential system

$$\dot{x} = X(x, y,) \qquad \dot{y} = Y(x, y,). \tag{3.47}$$

If there exists a C^1 function $J(x, y)$ (called the Dulac function) such that the expression div$(J\mathcal{X})$ has the same sign except possibly in a set of measure 0 in a simply connected region U of the plane, then the system (3.47) has no periodic solutions lying entirely within the region U.

Using this result and in view of (3.44) we get that equation (3.10) has no periodic solutions if $\sigma_4 \neq 0$. On the other hand from (3.46) we obtain that differential (3.16) has no periodic solutions if $\sigma_{11} \neq 0$. The case when $\sigma_4 = 0$ and $\sigma_{11} = 0$ was studied in the previous subsections. □

We recall that statement (i) of Corollary 69 was already proved by Bautin, see [17].

3.4.4 Study of a particular case of system (13)

In this section we study a particular case of system (5).

(i) System (32) can be interpreted as perturbation of the completely integrable three dimensional Lotka -Volterra system

$$\begin{aligned} \dot{x} =& \; x(-dy - fz), \\ \dot{y} =& \; y(dx - iz), \\ \dot{z} =& \; z(fx + iy). \end{aligned} \tag{3.48}$$

We can show that system (3.48) has two independent first integrals

$$H_1 = x + y + z, \qquad H_2 = |x|^i |y|^{-f} |z|^d.$$

To study this differential system we determine z as follows $z = \kappa - x - y$, where $H_1 = \kappa$. Then

$$\begin{aligned} \dot{x} = & \quad x(-dy - f(\kappa - x - y)), \\ \dot{y} = & \quad y(dx - i(\kappa - x - y)). \end{aligned} \tag{3.49}$$

Equilibrium points of (3.49) are $M_1 = (0, 0)$, $M_2 = (0, \kappa)$, $M_3 = (\kappa, 0)$, $M_4 = \left(\frac{i\kappa}{\mu}, -\frac{f\kappa}{\mu}\right)$, where $\mu = d - f + i$.

By introducing the notations

$$a_1 = f, \quad a_2 = f - d, \quad a_3 = -f\kappa, \quad b_1 = i + d, \quad b_2 = i, \quad b_3 = -ik,$$

we obtain that (3.49) becomes

$$\dot{x} = x\left(\frac{a_1 b_3}{b_2} + a_1 x + a_2 y\right), \quad \dot{y} = y(b_3 + (a_1 + b_2 - a_2)x + b_2 y).$$

Clearly that the phase portraits of (3.49) can obtain from the phase portraits of (3.10) under the conditions $a_3 = \dfrac{a_1 b_3}{b_2}$ and $I_4 = I_5$.

(ii) Now we shall study the following system

$$\begin{aligned} \dot{x} = & \quad x(a - ax - (a + b + c)y - (a + d + e)z), \\ \dot{y} = & \quad y\,(c + bx - cy - (c + d + f)z)\,, \\ \dot{z} = & \quad z\,(d + ex + fy - dz)\,, \end{aligned} \tag{3.50}$$

which is the most general three dimensional Lotka-Volterra system having the invariant plane $x + y + z - 1 = 0$. The cofactor of this invariant plane is $-(ax + cy + dz)$. We study differential system (3.50) on the invariant plane. After the change $z = 1 - x - y$ we get the two dimensional Lotka -Volterra system

$$\begin{aligned} \dot{x} = & \quad x(d + e - (e + d)x + (b + c - d - e)y), \\ \dot{y} = & \quad y\,(-d - f + (b + c + d + f)x + (d + f)y)\,. \end{aligned} \tag{3.51}$$

By comparing with differential system (3.10) we get that

$$\begin{aligned} a_3 = & \quad d + e, \quad a_2 = b + c - d - e, \quad a_1 = -e - d, \\ b_3 = & \quad -d - f, \quad b_1 = b + c + d + f, \quad b_2 = d + f. \end{aligned}$$

Consequently system (3.51) becomes

$$\begin{aligned} \dot{x} = & \quad x(-a_3 + a_3 x - a_2 y), \\ \dot{y} = & \quad y(-b_2 + (a_1 - a_2 + b_2)x + b_2 y) \end{aligned} \tag{3.52}$$

hence the following relations hold

$$b_2 + b_3 = 0, \quad a_1 + a_3 = 0, \quad b_1 = b_2 + a_1 - a_2. \tag{3.53}$$

The equilibrium points are $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (0,1)$ and $P_4 = \left(\frac{b_2}{a_2+b_2}, -\frac{a_3}{a_2+b_2}\right)$. The phase portraits of (3.52) can be obtained from phase portraits of system (3.10) under the condition $I_4 = I_5$. The point P_4 is a saddle if $\Delta_4 = \frac{a_3b_2(a_2+a_3)}{a_2+b_2} < 0$, or a center if $\Delta_4 = \frac{a_3b_2(a_2+a_3)}{a_2+b_2} > 0$, because in view of the conditions (3.53) always $\sigma_4 = 0$.

(iii) As we observe in the introduction differential system (32) is a generalization of system (), this differential system on the invariant plane $x+y+z-1=0$ becomes

$$\begin{aligned} \dot{x}_1 = & \ x\left(\beta_1 - \gamma_1 - b - \beta_1 x + (\nu - \beta_1)y\right), \\ \dot{y}_2 = & \ y\left(\beta_2 - b - \gamma_2 - (\nu + \beta_2)x - \beta_2 y\right). \end{aligned}$$

This differential system was study in [32] and has the Darboux invariant.

After the change

$$\begin{aligned} \gamma_1 + b = & \ -a_3 + \beta_1, \ \beta_1 = -a_1, \ \beta_2 = -b_2, \ \nu = -a_1 + a_2, \ \nu - \beta_1 = a_2, \\ \gamma_2 + b = & \ \beta_2 - b_3, \ \nu + \beta_2 = -b_1, \ \beta_2 = -b_2, \end{aligned}$$

and by considering that $\nu + \beta_2 = -b_1 = a_1 - a_2 + b_2$, i.e. $b_1 = a_1 - a_2 + b_2$, then the differential system can be written as

$$\begin{aligned} \dot{x} = & \ x\left(a_3 + a_1 x + a_2 y\right), \\ \dot{y} = & \ y\left(b_3 + (a_1 - a_2 + b_2)x + b_2 y\right). \end{aligned}$$

Clearly this differential equation can be obtained from (3.10) with $b_1 = a_1 - a_2 + b_2$, with is the same $I_4 = I_5$.

3.5 Differential systems (3.36) having a center

For the planar polynomial differential system $\dot{x} = P(x,y)$ and $\dot{y} = Q(x,y)$ we have that the equilibrium point $M_0 = (x_0, y_0)$ is a center of this system if and only it admits a Poincaré-Liapunov first integral H which has the following Taylor expansion at the neighborhood of M_0

$$H(x,y) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + h.o.t. :=< p, \Omega\, p^T > +h.o.t.,$$

where $p = (x,y)$ and Ω is a matrix such that $\det \Omega = a_{11}a_{22} - a_{12}^2 > 0$. Clearly that in his case the given differential system after the translation $x = X + x_0$,

$y = Y + y_0$ and under a convenient linear change can be written this system in the form (3.30).

In this section we shall study the polynomial differential system (3.36) having a center. We prove that polynomial differential system (3.36) with a center can be represented in the form (3.30) for a convenient polynomial of degree three X and Y. In this section we determine this representation under the conditions (3.37).

Proposition 70. *Polynomial differential system* (3.36) *under a conditions* (3.37) *can be written as*

$$\begin{aligned} \dot{U} &= -V + l_{20}(U^2 - V^2) + l_{11}UV + UR_2(U,V) = P, \\ \dot{V} &= U + s_{20}(U^2 - V^2) + s_{11}UV + VR_2(U,V) = Q, \end{aligned} \tag{3.54}$$

where $R_2(U,V) = l_{30}(U^2 - V^2) + l_{21}UV$, s_{20}, s_{11} *and* l_{11}, l_{30}, l_{21} *are convenient rational functions on the parameters* a_1, a_2, b_2, f. *We do not give these expressions because they are very long.*

Proof. We prove this proposition only for the case when By choosing $b_1 = \dfrac{a_1^2 b_2^2 + \varsigma^2}{a_1 a_2 b_2}$, where ς is a non-zero parameter, we obtain that

$$\tilde{\Delta}_{11}\Big|_{b_1 = \frac{a_1^2 b_2^2 + \varsigma^2}{a_1 a_2 b_2}} = \left(\frac{f\varsigma}{a_1(a_2 - b_2) - (a_1 - b_2)f} \right)^2 \geq 0, \qquad a_2 b_2 \neq 0.$$

Introducing the notations $\tilde{X}_j = P_j|_{b_1 = \frac{a_1^2 b_2^2 + \varsigma^2}{a_1 a_2 b_2},\, i = \frac{f b_2 (a_1 - b_1)}{a_1(a_2 - b_2)}}$ we obtain that differential system (3.36) becomes

$$\dot{X} = \tilde{X}_1, \qquad \dot{Y} = \tilde{X}_2, \tag{3.55}$$

with the matrix of the linear part

$$\tilde{M} = M|_{b_1 = \frac{a_1^2 b_2^2 + \varsigma^2}{a_1 a_2 b_2},\, i = \frac{f b_2 (a_1 - b_1)}{a_1(a_2 - b_2)}} = \begin{pmatrix} \tilde{M}_{11} & \tilde{M}_{21} \\ \tilde{M}_{12} & \tilde{M}_{22} \end{pmatrix}$$

where

$$\begin{aligned} \tilde{M}_{11} &= \frac{f}{\rho}\left(b_2 a_1^2 (b_2 - a_2)^2 (a_2 - f)(a_1 - f) + f\varsigma^2 (a_1 a_2 - b_2(a_1 - f))\right), \\ \tilde{M}_{12} &= \frac{f b_2}{\rho}\left(b_2^2 a_1^2 (a_1 - f)^2 (2a_2 - b_2) + (a_2 a_1 (a_2 - f)^2 + f\varsigma^2\right), \\ \tilde{M}_{21} &= \frac{f}{\rho}\left(b_2^2 a_1 - f)^2 ((a_1(a_2 - f))^2 + \varsigma^2) + a_1 a_2 \varsigma^2 (a_1(a_2 - 2b_2) + 2b_2 f)\right), \\ \tilde{M}_{22} &= -\frac{f}{\rho}\left(b_2 a_1^2 (b_2 - a_2)^2 (a_2 - f)(a_1 - f) + f\varsigma^2 (a_1 a_2 - b_2(a_1 - f))\right) = -\tilde{M}_{11}, \\ \rho &= a_1 a_2 (a_2 - b_2)(a_1(a_2 - b_2) - (a_1 - b_2)f)^2. \end{aligned}$$

Hence we get that

$$\det \tilde{M} = \left(\frac{f\varsigma}{a_1(a_2-b_2)-(a_1-b_2)f}\right)^2, \quad \text{trace}\tilde{M} = 0.$$

Now we determine the linear transformation

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \Longleftrightarrow \vec{U} = S\vec{X},$$

such that $S\tilde{M}S^{-1} = E$, where $E = \begin{pmatrix} 0 & -\mu \\ \mu & 0 \end{pmatrix}$. For simplicity we shall study the case when $\mu = 1$.

After some computations we can show that one of these transformation is $S = \begin{pmatrix} 1 & 0 \\ s_3 & s_4 \end{pmatrix}$ where

$$\begin{aligned}
s_3 = {} & \frac{1}{\varrho}\Big(b_2\left((a_1-b_2)^2 + a_1^2(a_2-b_2)^2\right) f^3 \\
& + a_1(a_2-b_2)\Big(a_1b_2(a_2-b_2)(a_2+a_1) + (b_2-a_1)(a_1-3b_2)\Big) f^2 \\
& a_1^2(a_2-b_2)^2\Big(b_2(a_1a_2+1) + 2(b_2-a_1)\Big) f + (a_1(a_2-b_2))^3\Big), \\
s_4 = {} & \frac{1}{\varrho}\Big(\left((a_1-b_2)^2 + a_1^2(a_2-b_2)^2\right) f^2 \\
& + 2a_1(b_2-a_2)\left(a_1a_2(a_2-b_2) + a_1 - b_2\right) f + ar_1(a_2^2+1)(a_2-b_2)^2\Big), \\
\varrho = {} & a_1a_2(a_2-b_2)\Big(a_1(a_2-b_2) - (a_1-b_2)f\Big)^2, \\
f = {} & \frac{a_1(a_2-b_2)}{a_1-b_2+\varsigma}.
\end{aligned}$$

Under the linear transformation

$$X = U, \qquad Y = \frac{1}{s_4}\left(V - s_3U\right),$$

differential system (3.55) becomes (3.54).

The determination of the differential system (3.54) in the neighborhood of the point $P_{11}|_{b_2=a_1=0}$ under the condition $if > 0$, and $P_{11}|_{b_2=a_1=0}$ under the condition $a_1(a_1-b_1) < 0$ can be obtained analogously to the case that we had studied previously. The proof is obtained after some computations and in view of Proposition 67. $\square$

3.6 Global phase portraits. Proof of Theorem 17

In this section we will describe all the possible phase portraits of systems (3.10) whose singular points on the boundary of the region $\mathbb{M}$ are hyperbolic. We recall that an equilibrium point P of a differential system is *hyperbolic* if the real part of the eigenvalues of the linear part of the system at p are non-zero. As we have seen in Proposition 4 to study the global phase portraits we need to study the phase portraits on the invariant plane and on the infinity.

The Δ_j and σ_j for $j = 1, .., 11$ defined in subsection 3.3 can be expressed in terms of the eigenvalues $\lambda_1^{(j)}, \lambda_2^{(j)}$ of the linear part matrix of the corresponding two dimensional system at to the point P_j . The eigenvalues of the linear part matrix of system (3.10) at the points P_j for $j = 1, .., 4$ are

$$\begin{aligned}
\lambda_1^{(1)} &= a_3, \qquad \lambda_2^{(1)} = b_3, \qquad \lambda_1^{(2)} = -b_3,\\
\lambda_2^{(2)} &= -\frac{I_1}{b_2}, \qquad \lambda_1^{(3)} = -a_3, \qquad \lambda_2^{(3)} = \frac{I_2}{a_1},\\
\lambda_1^{(4)} &= \frac{I_1 I_2 + \sqrt{4(-I_1 a_1 + I_2 b_2) I_3 + I_1^2 I_2^2}}{2 I_3},\\
\lambda_2^{(4)} &= \frac{I_1 I_2 - \sqrt{4(-I_1 a_1 + I_2 b_2) I_3 + I_1^2 I_2^2}}{2 I_3}.
\end{aligned}$$

The eigenvalues of the linear part matrix of system (3.11) at the points P_5 and P_6 are

$$\lambda_1^{(5)} = -a1, \qquad \lambda_2^{(5)} = -I_4, \qquad \lambda_1^{(6)} = I_4, \qquad \lambda_2^{(6)} = \frac{I_3}{I_5}$$

The eigenvalues of the linear part matrix of system (3.12) at the points P_7 are

$$\lambda_1^{(7)} = -b_2, \qquad \lambda_2^{(7)} = I_5$$

The eigenvalues of the linear part matrix of system (3.16) at the points P_j for $j = 5, \dots, 11$ are

$$\begin{aligned}
\lambda_1^{(5)} &= a_1, \qquad \lambda_2^{(5)} = I_4, \qquad \lambda_1^{(6)} = \frac{I_3}{I_4 - I_5}, \qquad \lambda_2^{(6)} = -\frac{I_4 I_5}{I_4 - I_5},\\
\lambda_1^{(7)} &= b_2, \qquad \lambda_2^{(7)} = -I_5, \qquad \lambda_1^{(8)} = -f, \qquad \lambda_2^{(8)} = -i,\\
\lambda_1^{(9)} &= \frac{I_9}{h}, \qquad \lambda_2^{(9)} = \frac{b_2 i}{h}, \qquad \lambda_1^{(10)} = -\frac{a_1 f}{c}, \qquad \lambda_2^{(10)} = \frac{I_{10}}{c},\\
\lambda_1^{(11)} &= \frac{-\sigma_{11} + \sqrt{\sigma_{11}^2 - 4\Delta_{11}}}{2}, \qquad \lambda_2^{(11)} = \frac{-\sigma_{11} - \sqrt{\sigma_{11}^2 - 4\Delta_{11}}}{2},
\end{aligned}$$

where $I_9 = a_2 i - b_2 f$ and $I_{10} = a_1 i - b_1 f$.

As we have seen in (3.10) a_1, a_2, a_3, b_1, b_2 and b_3 are independent parameters, and $a_2 = I_5 + b_2$, $b_1 = a_1 - I_4$, so we can express all the eigenvalues of the points in the invariant plane in terms of the independent parameters I_4, I_5, b_2, a_1, a_3 and b_3. Equivalently the eigenvalues of the infinite singularities can be expressed in terms of the independent parameters I_4, I_5, b_2, a_1, f and i. Taking all those parameters and adding the parameter a as in Proposition 65 we have nine parameters that are linearly independent. Then the matrix of those parameters expressed in the original parameters $a, b, c, d, e, f, g, h, i$ of system (32) is

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

These nine parameters are independent because $rank(V) = 9$.

This means that we can obtain the global phase portraits by studying the phase portraits in the invariant plane and at the infinity using the Darboux invariant. In short for determining a global phase portraits it is sufficient to study the phase portraits on the invariant plane at infinity, and since there two invariant surface in a global phase portrait have in common the parameters I_4, I_5, a_1 and b_2, these parameters must be same in order to study the global phase portrait.

We will do our study based on the sign and not in the numerical value of the eigenvalues. For this we need to study the sign of all the parameters previously mentioned in that section plus the auxiliary parameters $I_1, I_2, I_3, I_9, I_10, c, h$, as we will see below. In this case analogous to what we have just stated, the compatibility between the dynamics of the invariant plane and at infinity depend on the parameters corresponding to the points P_4, P_5 and P_6, so it depends on the parameters I_4, I_5, a_1 and b_2 if the point P_6 is out of the region $\mathbb{M}$, and also on the parameter I_3 if the point P_6 is in the region. The signs of these parameters must be the same for the invariant plane dynamics and for the infinity dynamics.

3.6.1 Phase portraits on the invariant plane

We will study all the possible phase portraits on the invariant plane based on the compatibility between the local dynamics of the singular points. The local dynamics of each point can be found studying the sign of the eigenvalues defined in the previous section: if $\lambda_i^{(j)} > 0$ then along the direction of the corresponding eigenvector the point has an unstable behavior, and if $\lambda_i^{(j)} < 0$ a stable one.

As we have just stated, the eigenvalues of the points in the invariant plane can be expressed in terms of the independent parameters I_4, I_5, b_2, a_1, a_3 and b_3. But for studying the sign and not the values of the eigenvalues of the points P_2, P_3

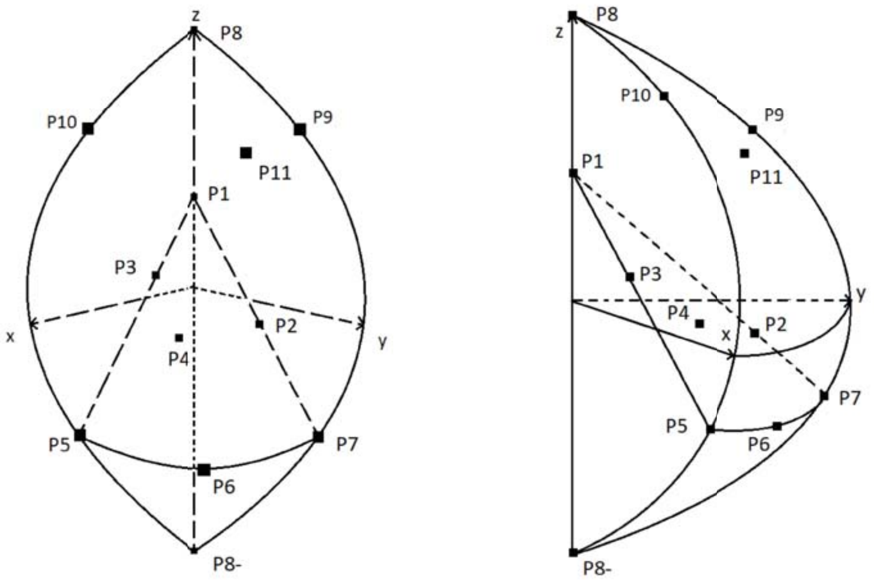

Figure 3.1: Singular points scheme

and P_6 we need to add three dependent parameters, I_1, I_2 and I_3. As there are dependencies between the local dynamics of the points, we will not study the point P_4,because its eigenvalues signs are hard to obtain from the mentioned parameters, and we will find its local dynamics by compatibility with the rest.

From the points P_j for $j = 1, .., 7$, only the points P_1, P_5 and P_7 are always on the boundary of the region $\mathbb{M}$. The pointsP_2, P_3, and P_6 can be in the region $\mathbb{M}$ or outside of it. We will divide our study taking into account the in to the cases by number of points on the boundary of the region $\mathbb{M}$.

Three points on the boundary of $\Pi \cap \mathbb{M}$

Clearly that the points P_2, P_3 and P_6 are outside of the region $\mathbb{M}$ if $a_3/a_1 > 0, b_3/b_2 > 0$ and $I_4/I_5 > 0$. In this case it is sufficient to study the compatibility of the parameters I_4, I_5, a_1, a_3, b_2, and b_3, thus we must distinguish the following cases

$$I_4 > 0, I_5 > 0, a_3 > 0, a_1 > 0, b_3 > 0, b_2 > 0 \ (Fig.3.2),$$
$$I_4 > 0, I_5 > 0, a_3 < 0, a_1 < 0, b_3 < 0, b_2 < 0 \ (Fig.3.3),$$
$$I_4 > 0, I_5 > 0, a_3 > 0, a_1 > 0, b_3 < 0, b_2 < 0 \ (Fig.3.4),$$
$$I_4 > 0, I_5 > 0, a_3 < 0, a_1 < 0, b_3 > 0, b_2 > 0 \ (Fig.3.5).$$

The last case is the only one that has the point P_4 inside of the region $\mathbb{M}$. In this case $\Delta_4 > 0$. This point is a center if $\sigma_4 = 0$, a stable focus if $\sigma_4 < 0$, and an unstable focus if $\sigma_4 > 0$.

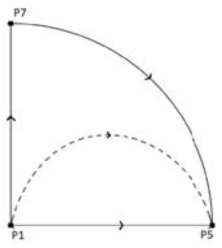

Figure 3.2

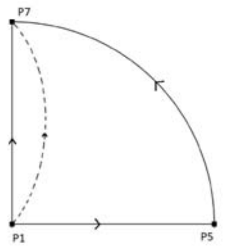

Figure 3.3

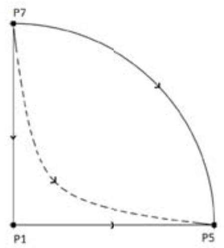

Figure 3.4

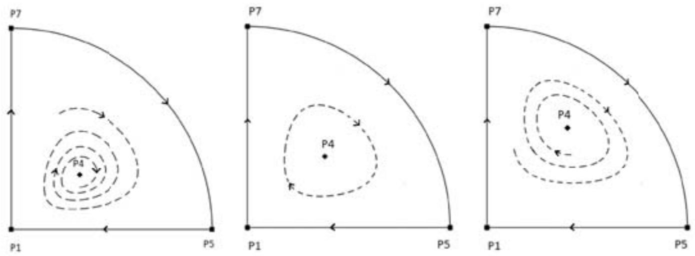

Figure 3.5

Four points on on the boundary of $\Pi \cap \mathbb{M}$

We consider first the case with P_1, P_2, P_5 and P_7 on the boundary of the region $\mathbb{M}$ and the points P_3 and P_6 outside of this region. It must be $a_3/a_1 < 0, b_3/b_2 > 0$ and $I_4/I_5 > 0$. In this case it suffices to study the compatibility of the parameters $I_1, I_4, I_5, a_1, a_3, b_2$, and b_3, thus we must consider the following cases:

$$I_1 < 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 > 0, b_3 > 0, b_2 < 0 \ (Fig.3.6),$$
$$I_1 < 0, I_4 < 0, I_5 < 0, a_3 < 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.7),$$
$$I_1 < 0, I_4 > 0, I_5 > 0, a_3 < 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.8),$$
$$I_1 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 > 0, b_3 > 0, b_2 < 0 \ (Fig.3.9),$$
$$I_1 > 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 > 0, b_3 > 0, b_2 < 0 \ (Fig.3.10),$$
$$I_1 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 > 0, b_3 > 0, b_2 < 0 \ (Fig.3.11),$$
$$I_1 > 0, I_4 > 0, I_5 > 0, a_3 < 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.12),$$

The first and the last two cases have the point P_4 inside of the region $\mathbb{M}$. In the first case $\Delta_4 < 0$, so P_4 is a saddle. In the last two cases $\sigma_4 < 0$ and $\Delta_4 > 0$, so we have a stable node if $\sigma_4^2 - 4\Delta_4 > 0$, and a stable focus if $\sigma_4^2 - 4\Delta_4 < 0$.

Second we consider the case with P_1, P_3, P_5 and P_7 on the boundary of the region $\mathbb{M}$ and the points P_2 and P_6 outside of the region. It must be $a_3/a_1 > 0, b_3/b_2 < 0$ and $I_4/I_5 > 0$. In this case it suffices to study the compatibility of the parameters $I_2, I_4, I_5, a_1, a_3, b_2$, and b_3, and we get the cases:

$$
\begin{aligned}
I_2 &< 0, I_4 > 0, I_5 > 0, a_3 < 0, a_1 > 0, b_3 < 0, b_2 < 0 \ (Fig.3.13),\\
I_2 &< 0, I_4 < 0, I_5 < 0, a_3 < 0, a_1 > 0, b_3 < 0, b_2 < 0 \ (Fig.3.14),\\
I_2 &< 0, I_4 > 0, I_5 > 0, a_3 < 0, a_1 > 0, b_3 > 0, b_2 > 0 \ (Fig.3.15),\\
I_2 &> 0, I_4 > 0, I_5 > 0, a_3 < 0, a_1 > 0, b_3 < 0, b_2 < 0 \ (Fig.3.16),\\
I_2 &> 0, I_4 < 0, I_5 < 0, a_3 < 0, a_1 > 0, b_3 < 0, b_2 < 0 \ (Fig.3.17),\\
I_2 &> 0, I_4 > 0, I_5 > 0, a_3 < 0, a_1 > 0, b_3 > 0, b_2 > 0 \ (Fig.3.18),\\
I_2 &> 0, I_4 < 0, I_5 < 0, a_3 < 0, a_1 > 0, b_3 > 0, b_2 > 0 \ (Fig.3.19),
\end{aligned}
$$

The second, third and e fourth cases have the point P_4 inside of the region $\mathbb{M}$. In the fourth case $\Delta_4 < 0$, consequently P_4 is a saddle. In the second and third cases $\sigma_4 > 0$ and $\Delta_4 > 0$, so we have an unstable node if $\sigma_4^2 - 4\Delta_4 > 0$, and an unstable focus if $\sigma_4^2 - 4\Delta_4 < 0$.

Finally we consider the case with P_1, P_5, P_6 and P_7 on the boundary of the region $\mathbb{M}$ and the point P_2 and P_3 to be outside of this region, thus $a_3/a_1 > 0, b_3/b_2 > 0$ and $I_4/I_5 < 0$. In this case is enough to study the compatibility of the parameters $I_3, I_4, I_5, a_1, a_3, b_2$, and b_3, and we obtain the cases:

$$
\begin{aligned}
I_3 &< 0, I_4 < 0, I_5 > 0, a_3 > 0, a_1 > 0, b_3 > 0, b_2 > 0 \ (Fig.3.20),\\
I_3 &< 0, I_4 > 0, I_5 < 0, a_3 > 0, a_1 > 0, b_3 > 0, b_2 > 0 \ (Fig.3.21),\\
I_3 &> 0, I_4 > 0, I_5 < 0, a_3 > 0, a_1 > 0, b_3 > 0, b_2 > 0 \ (Fig.3.22),\\
I_3 &> 0, I_4 < 0, I_5 > 0, a_3 > 0, a_1 > 0, b_3 < 0, b_2 < 0 \ (Fig.3.23),\\
I_3 &< 0, I_4 < 0, I_5 > 0, a_3 > 0, a_1 > 0, b_3 < 0, b_2 < 0 \ (Fig.3.24),\\
I_3 &> 0, I_4 > 0, I_5 < 0, a_3 > 0, a_1 > 0, b_3 < 0, b_2 < 0 \ (Fig.3.25),\\
I_3 &< 0, I_4 > 0, I_5 < 0, a_3 > 0, a_1 > 0, b_3 < 0, b_2 < 0 \ (Fig.3.26).
\end{aligned}
$$

The second, the fourth and the sixth cases have the point P_4 inside of the region $\mathbb{M}$. In the second case $\Delta_4 < 0$ so P_4 is a saddle. In the fourth case $\sigma_4 < 0$ and $\Delta_4 > 0$ so we have a stable node if $\sigma_4^2 - 4\Delta_4 > 0$ and a stable focus if $\sigma_4^2 - 4\Delta_4 < 0$. In the sixth case $\sigma_4 > 0$ and $\Delta_4 > 0$, so we have an unstable node if $\sigma_4^2 - 4\Delta_4 > 0$, and an unstable focus if $\sigma_4^2 - 4\Delta_4 < 0$.

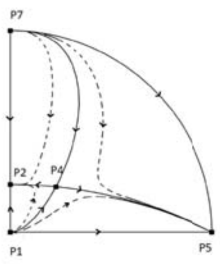

Figure 3.6

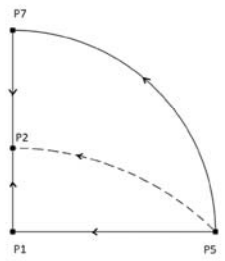

Figure 3.7

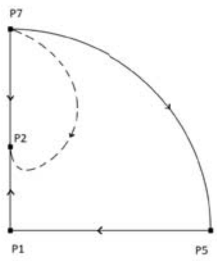

Figure 3.8

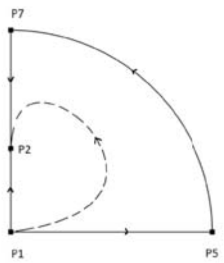

Figure 3.9

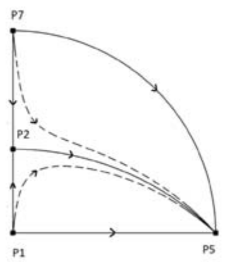

Figure 3.10

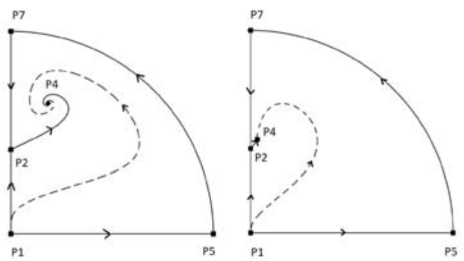

Figure 3.11

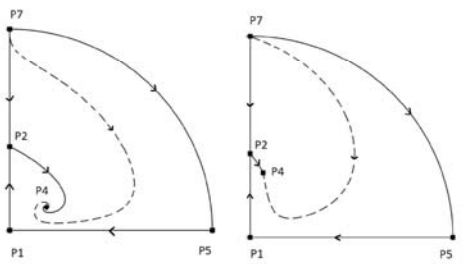

Figure 3.12

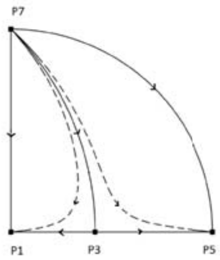

Figure 3.13

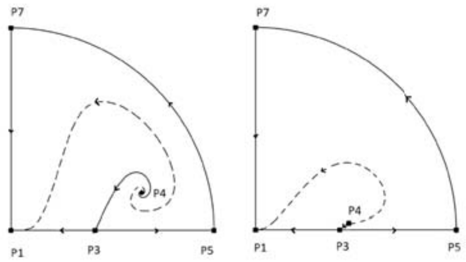

Figure 3.14

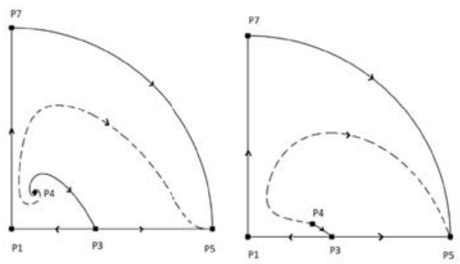

Figure 3.15

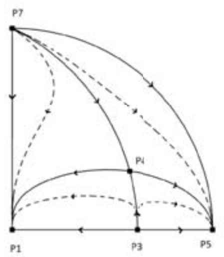

Figure 3.16

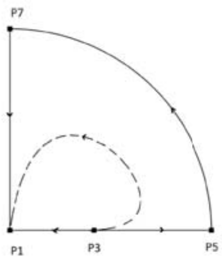

Figure 3.17

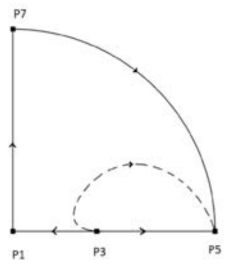

Figure 3.18

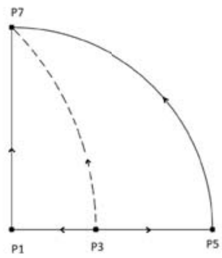

Figure 3.19

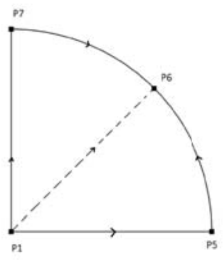

Figure 3.20

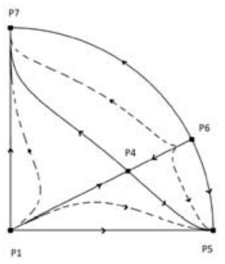

Figure 3.21

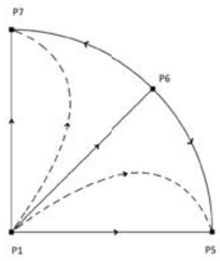

Figure 3.22

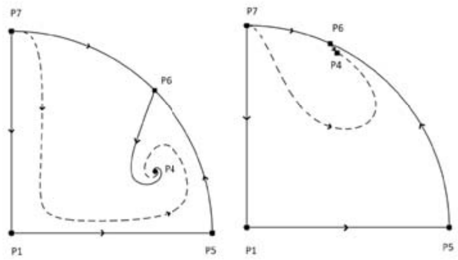

Figure 3.23

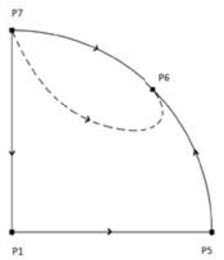

Figure 3.24

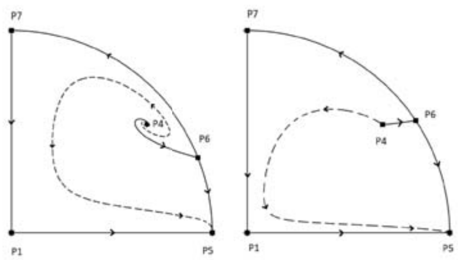

Figure 3.25

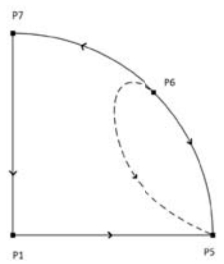

Figure 3.26

Five points on on the boundary of $\Pi \cap \mathbb{M}$

First we consider the case with P_1, P_2, P_3, P_5 and P_7 on the boundary of the region $\mathbb{M}$ and the point P_6 outside of this region, i.e. $a_3/a_1 < 0$, $b_3/b_2 < 0$ and $I_4/I_5 > 0$. In this case is enough to study the compatibility of the parameters $I_1, I_2, I_4, I_5, a_1, a_3, b_2$, and b_3, thus we obtain:

$$I_1 > 0, I_2 > 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.27),$$
$$I_1 < 0, I_2 > 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.28),$$
$$I_1 > 0, I_2 < 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.29),$$
$$I_1 > 0, I_2 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.30),$$
$$I_1 < 0, I_2 < 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.31),$$
$$I_1 < 0, I_2 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.32),$$
$$I_1 > 0, I_2 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.33),$$

$$I_1 < 0, I_2 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.34),$$
$$I_1 < 0, I_2 > 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.35),$$
$$I_1 > 0, I_2 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.36),$$
$$I_1 < 0, I_2 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.37),$$
$$I_1 > 0, I_2 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.38),$$
$$I_1 < 0, I_2 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.39).$$

The second, third, sixth, seventh, eleventh and twelfth cases have the point P_4 inside of the region $\mathbb{M}$. In the second, sixth and eleventh cases $\Delta_4 < 0$, so P_4 is a saddle. In the third and seventh cases $\sigma_4 < 0$ and $\Delta_4 > 0$, so we have a stable node if $\sigma_4^2 - 4\Delta_4 > 0$, and a stable focus if $\sigma_4^2 - 4\Delta_4 < 0$

. In the twelfth case we have $\Delta_4 > 0$. This point is a center if $\sigma_4 = 0$, a stable focus if $\sigma_4 < 0$ and $\sigma_4^2 - 4\Delta_4 < 0$, is a stable node if $\sigma_4 < 0$ and $\sigma_4^2 - 4\Delta_4 > 0$, is an unstable focus if $\sigma_4 > 0$ and $\sigma_4^2 - 4\Delta_4 < 0$, and an unstable node if $\sigma_4 > 0$ and $\sigma_4^2 - 4\Delta_4 > 0$.

Second we consider the case with P_1, P_2, P_5, P_6 and P_7 on the boundary of the region $\mathbb{M}$ and the point P_3 outside of the region. It must be $a_3/a_1 < 0, b_3/b_2 > 0$ and $I_4/I_5 < 0$. In this case it suffices to study the compatibility of the parameters $I_2, I_3, I_4, I_5, a_1, a_3, b_2$, and b_3, and we have:

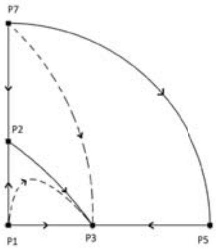

Figure 3.27

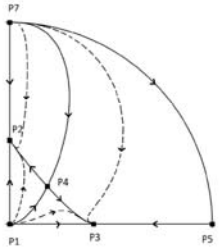

Figure 3.28

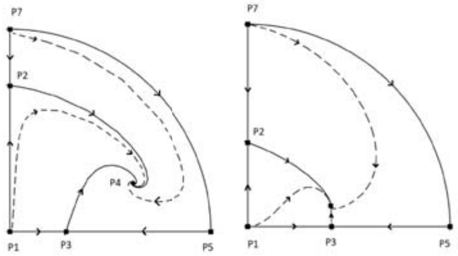

Figure 3.29

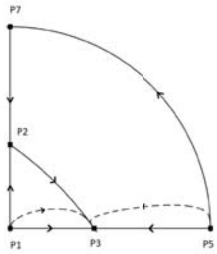

Figure 3.30

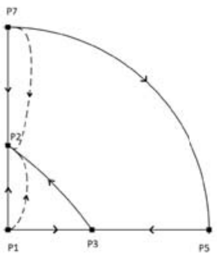

Figure 3.31

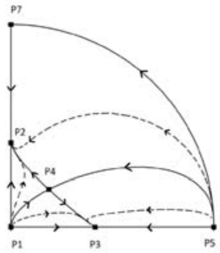

Figure 3.32

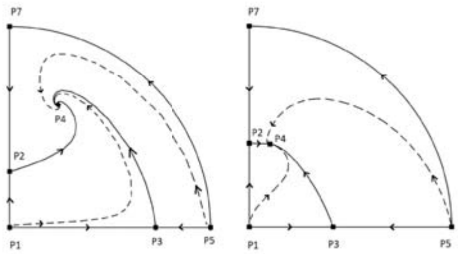

Figure 3.33

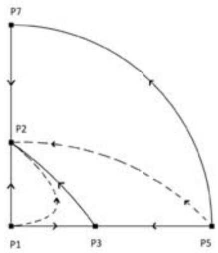

Figure 3.34

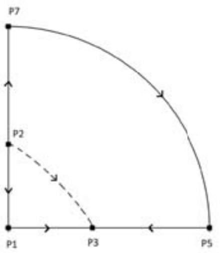

Figure 3.35

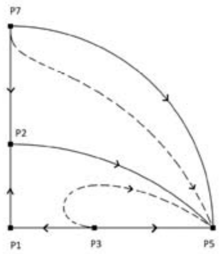

Figure 3.36

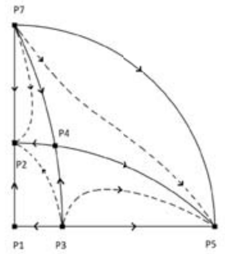

Figure 3.37

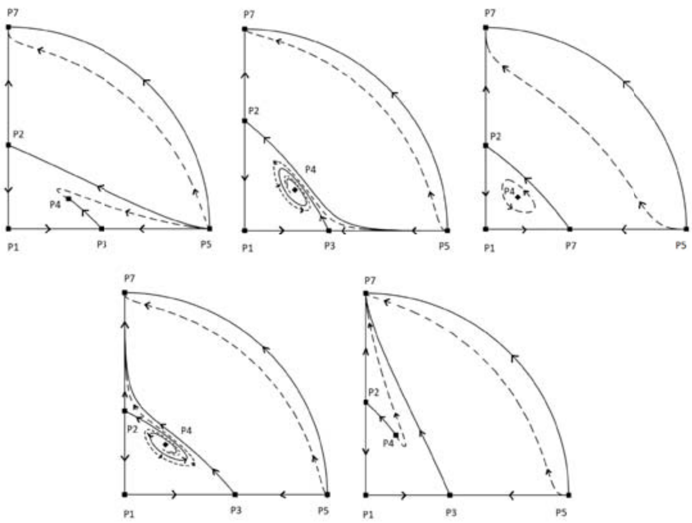

Figure 3.38

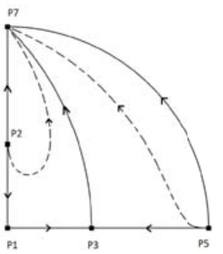

Figure 3.39

$$
\begin{aligned}
I_1 &> 0, I_3 > 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.40),\\
I_1 &< 0, I_3 > 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.41)\\
I_1 &> 0, I_3 < 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.42),\\
I_1 &> 0, I_3 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.43),\\
I_1 &< 0, I_3 < 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.44),\\
I_1 &< 0, I_3 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.45),\\
I_1 &> 0, I_3 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.46),\\
I_1 &< 0, I_3 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.47),\\
I_1 &< 0, I_3 > 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.48)\\
I_1 &> 0, I_3 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.49),\\
I_1 &< 0, I_3 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.50),\\
I_1 &> 0, I_3 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.51),\\
I_1 &< 0, I_3 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.52).
\end{aligned}
$$

The first and the last two cases have the point P_4 inside of the region $\mathbb{M}$. In the first case $\Delta_4 < 0$, so P_4 is a saddle. In the last two cases $\sigma_4 < 0$ and $\Delta_4 > 0$, so we have a stable node if $\sigma_4^2 - 4\Delta_4 > 0$, and a stable focus if $\sigma_4^2 - 4\Delta_4 < 0$.

Finally we consider the case with P_1, P_3, P_5, P_6 and P_7 on the boundary of the region $\mathbb{M}$ and the point P_6 to be outside of the region, i.e. $a_3/a_1 > 0,\ b_3/b_2 < 0$ and $I_4/I_5 < 0$. In this case it suffices to study the compatibility of the parameters $I_3, I_4, I_5, a_1, a_3, b_2,$ and b_3, and we get:

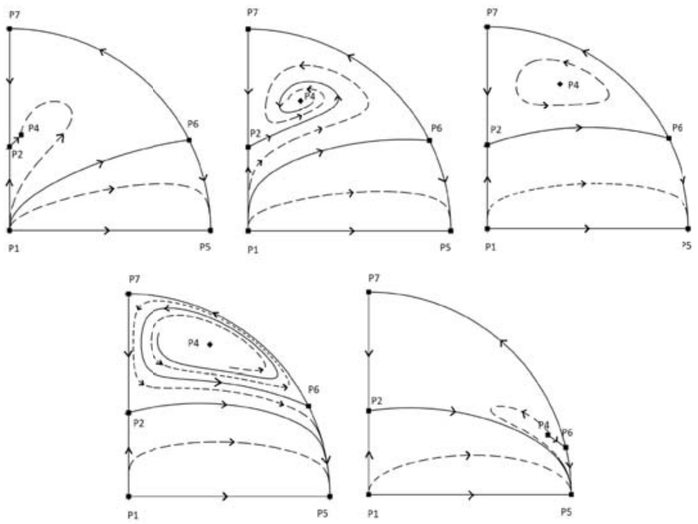

Figure 3.40

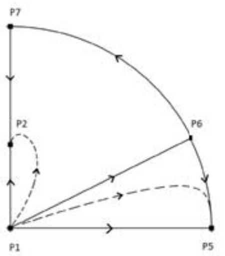

Figure 3.41

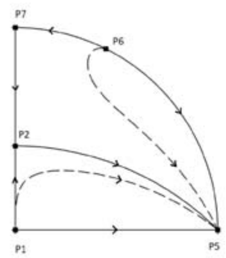

Figure 3.42

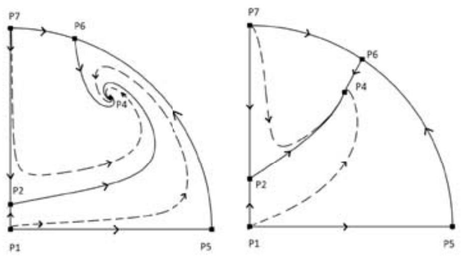

Figure 3.43

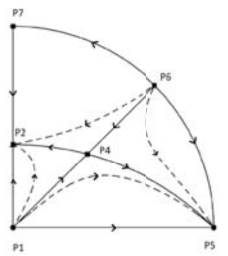

Figure 3.44

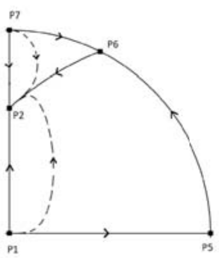

Figure 3.45

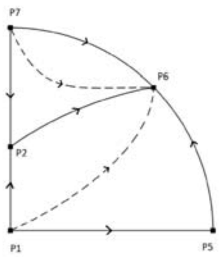

Figure 3.46

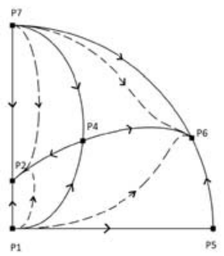

Figure 3.47

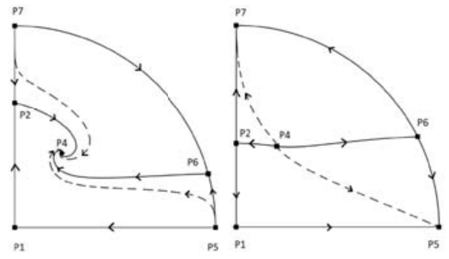

Figure 3.48

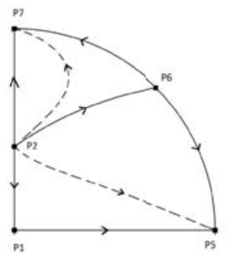

Figure 3.49

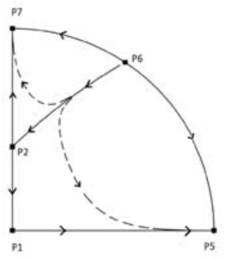

Figure 3.50

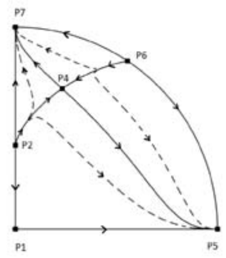

Figure 3.51

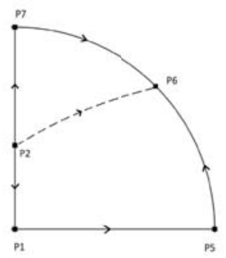

Figure 3.52

$I_2 > 0, I_3 > 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0$ $(Fig.3.53)$,

$I_2 < 0, I_3 > 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0$ $(Fig.3.54)$,

$I_2 > 0, I_3 < 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0$ $(Fig.3.55)$,

$I_2 > 0, I_3 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0$ $(Fig.3.56)$,

$I_2 < 0, I_3 < 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0$ $(Fig.3.57)$,

$I_2 < 0, I_3 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0$ $(Fig.3.58)$,

$I_2 > 0, I_3 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0$ $(Fig.3.59)$,

$I_2 < 0, I_3 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0$ $(Fig.3.60)$,

$I_2 < 0, I_3 > 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0$ $(Fig.3.61)$

$I_2 > 0, I_3 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0$ $(Fig.3.62)$,

$I_2 < 0, I_3 > 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0$ $(Fig.3.63)$,

$I_2 > 0, I_3 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0$ $(Fig.3.64)$,

$I_2 < 0, I_3 < 0, I_4 < 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0$ $(Fig.3.65)$.

The second, fourth and sixth cases have the point P_4 inside of the region $\mathbb{M}$. In the second case $\Delta_4 < 0$ so P_4 is a saddle. In the fourth case $\sigma_4 < 0$ and $\Delta_4 > 0$, so we have a stable node if $\sigma_4^2 - 4\Delta_4 > 0$, and a stable focus if $\sigma_4^2 - 4\Delta_4 < 0$. In the sixth case $\sigma_4 > 0$ and $\Delta_4 > 0$, so we have an unstable node if $\sigma_4^2 - 4\Delta_4 > 0$, and an unstable focus if $\sigma_4^2 - 4\Delta_4 < 0$.

Six points on on the boundary of $\Pi \cap \mathbb{M}$

We consider all the singular points of the system P_j, $j = 1, \ldots, 7$, are on the boundary of the region $\mathbb{M}$, i.e $a_3/a_1 < 0, b_3/b_2 < 0$ and $I_4/I_5 < 0$. In this case we want to study the compatibility of all the parameters $I_1, I_2, I_3, I_4, I_5, a_1, a_3, b_2,$ and b_3, and we obtain

$$I_1 > 0 \quad, I_2 > 0, I_3 < 0, I_4 > 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.66),$$
$$I_1 < 0 \quad, I_2 > 0, I_3 < 0, I_4 > 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.67),$$
$$I_1 > 0 \quad, I_2 > 0, I_3 > 0, I_4 < 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.68),$$
$$I_1 < 0 \quad, I_2 < 0, I_3 < 0, I_4 > 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.69),$$
$$I_1 > 0 \quad, I_2 < 0, I_3 < 0, I_4 < 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.70),$$
$$I_1 > 0 \quad, I_2 < 0, I_3 > 0, I_4 < 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b_2 < 0 \ (Fig.3.71),$$
$$I_1 < 0 \quad, I_2 < 0, I_3 > 0, I_4 < 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 > 0, b2 < 0 \ (Fig.3.72),$$
$$I_1 > 0 \quad, I_2 > 0, I_3 < 0, I_4 > 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.73),$$
$$I_1 < 0 \quad, I_2 > 0, I_3 > 0, I_4 > 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.74)$$
$$I_1 < 0 \quad, I_2 > 0, I_3 < 0, I_4 > 0, I_5 < 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.75),$$
$$I_1 < 0 \quad, I_2 > 0, I_3 > 0, I_4 < 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.76),$$
$$I_1 < 0 \quad, I_2 > 0, I_3 < 0, I_4 < 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.77),$$
$$I_1 < 0 \quad, I_2 < 0, I_3 < 0, I_4 < 0, I_5 > 0, a_3 > 0, a_1 < 0, b_3 < 0, b_2 > 0 \ (Fig.3.78).$$

The second,sixth, tenth and the twelfth cases we have the point P_4 inside of the region $\mathbb{M}$. In the first, tenth and twelfth case $\Delta_4 < 0$ so P_4 is a saddle. In the sixth case $\sigma_4 < 0$ and $\Delta_4 > 0$, so we have a stable node if $\sigma_4^2 - 4\Delta_4 > 0$, and a stable focus if $\sigma_4^2 - 4\Delta_4 < 0$.

3.6.2 Phase portraits at infinity

The eigenvalues of the points at infinity can be expressed in terms of the independent parameters I_4, I_5, b_2, a_1, f and i. But for considering the sign and not the numerical values of the eigenvalues of the points P_9, P_{10} and P_6 we only need to add five dependent parameters, I_9, I_{10}, c, h and I_3. All the points P_j for

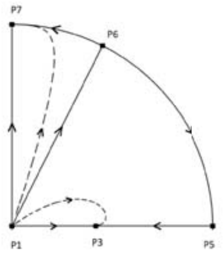

Figure 3.53

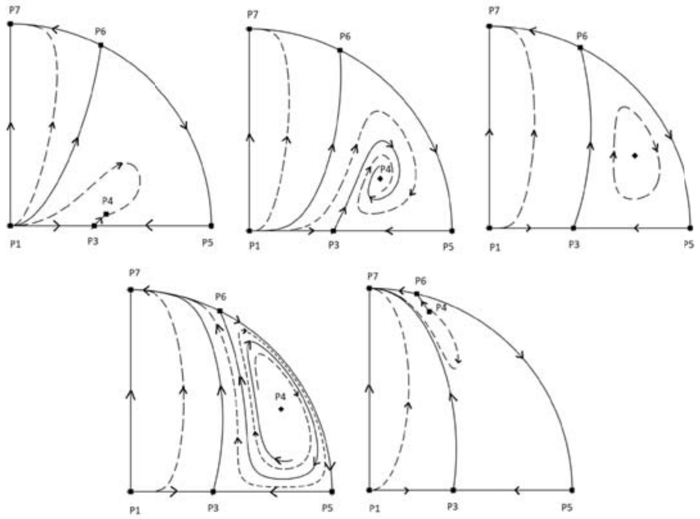

Figure 3.54

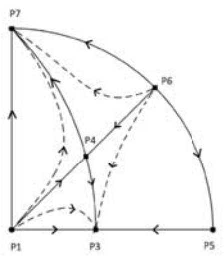

Figure 3.55

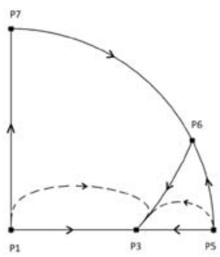

Figure 3.56

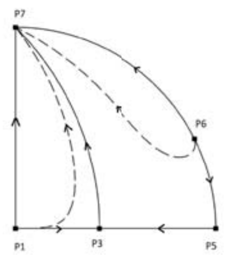

Figure 3.57

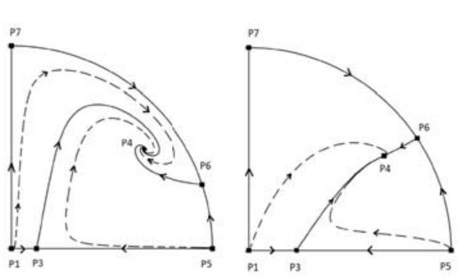

Figure 3.58

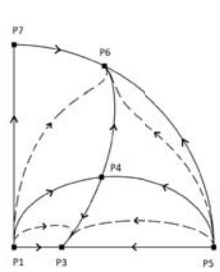

Figure 3.59

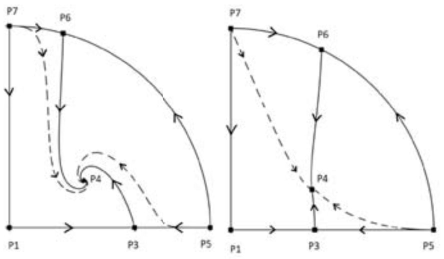

Figure 3.63

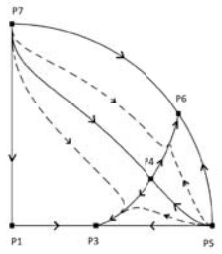

Figure 3.64

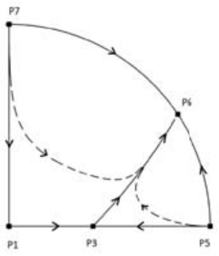

Figure 3.65

$j = 5, \ldots, 11$ with the exception of P_6 and P_{11} are always in the intersection of the infinity with the region $\mathbb{M}$. The points P_6 and P_{11} can be in this intersection or outside of it. For the point P_6 to be in the region, it must be $I_4 I_5 < 0$, so if $I_3 > 0$ then P_6 is a saddle and if $I_3 < 0$ it is a node, stable if $I_4 > 0$ and unstable if $I_4 < 0$. As there are dependencies between the local dynamics of the points, we will not study the point P_{11}, because its eigenvalues signs are hard to obtain from the mentioned parameters, and will find its local dynamic by compatibility with all the rest. Regarding the local dynamic of the point P_6, we will study it by considering the sign of the parameter I_3 only in those cases when we have P_6 and P_{11} both inside of the region $\mathbb{M}$. In the case when only the point P_6 is inside the region, its dynamic is completely determined by the compatibility with the rest of the points.

We will study the different cases based on the signs of the eigenvalues of the points P_5, P_7, P_8, P_9 and P_{10} expressed in the previously mentioned parameters $I_9, I_{10}, f, i, a_1, I_4, b_2, I_5, c, h.$. We will numerate the cases in the following form: the case j corresponds to the system where

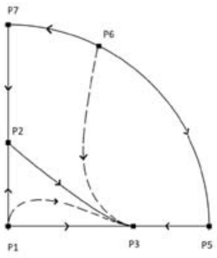

Figure 3.66

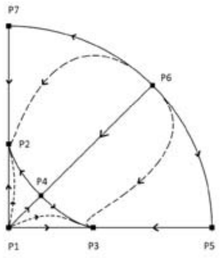

Figure 3.67

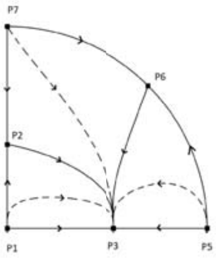

Figure 3.68

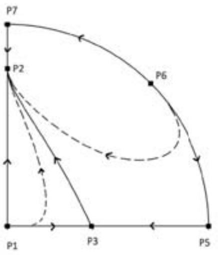

Figure 3.69

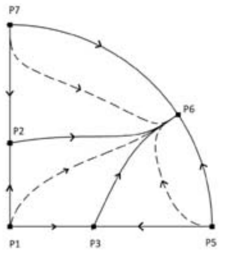

Figure 3.70

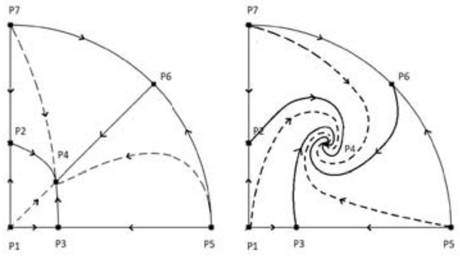

Figure 3.71

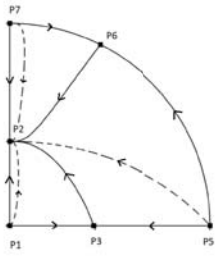

Figure 3.72

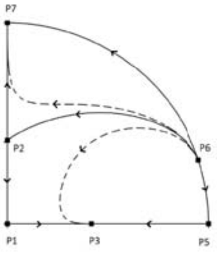

Figure 3.73

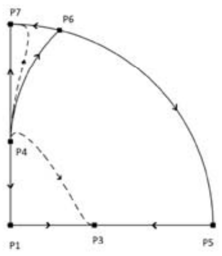

Figure 3.74

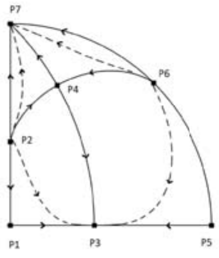

Figure 3.75

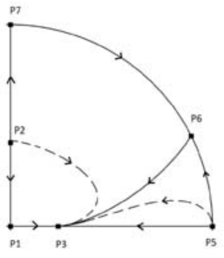

Figure 3.76

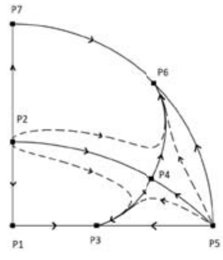

Figure 3.77

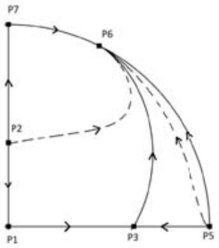

Figure 3.78

$$\begin{aligned}(-1)^{[\frac{j-1}{512}]}I_9 &> 0, \quad (-1)^{[\frac{j-1}{256}]}I_{10} > 0, \quad (-1)^{[\frac{j-1}{128}]}f > 0,\\ (-1)^{([\frac{j-1}{64}]+1)}i &> 0, \quad (-1)^{[\frac{j-1}{32}]}a_1 > 0, \quad (-1)^{[\frac{j-1}{16}]}I_4 > 0,\\ (-1)^{[\frac{j-1}{8}]}b_2 &> 0, \quad (-1)^{[\frac{j-1}{4}]}I_5 > 0, \quad (-1)^{[\frac{j-1}{2}]}c > 0, \quad (-1)^{j-1}h > 0,\end{aligned} \tag{3.56}$$

where $j \in \mathbb{N}$ and $[]$ is the *integer part function*.

We will apply the symmetries defined in section 6.1 in order to reduce the number of compatible cases that we need study.

We define the two following equivalence relations between cases in order to reduce the number of configuration.

Given a phase portrait if the points P_9 and P_{10} are not under the invariant plane then they can be in the northern or southern hemisphere, without modifying the phase portrait. If we look at the coordinates of these points and their eigenvalues we have

$$\begin{aligned}P_9 &= \left(0, \frac{-i}{h}\right)), \qquad P_{10} = \left(\frac{-f}{c}, 0, \right)),\\ \lambda_1^{(9)} &= \frac{I_9}{h}, \quad \lambda_2^{(9)} = \frac{b_2\, i}{h}, \quad \lambda_1^{(10)} = -\frac{a_1 f}{c}, \quad \lambda_2^{(10)} = \frac{I_{10}}{c},\end{aligned}$$

respectively. So in order that the point P_9 change of hemisphere without changing its local phase portrait , the parameter h must changes its sign by the opposite, and in order that the point P_{10} changes of hemisphere without changing its local phase portrait the parameter c has to change its sign by the opposite. We repeat our study in cases. Then in the four first cases the class of equivalence of the case j is defined as all the cases k such that [j/4]=[k/4] where [] is the integer part function. We observe that if the case is compatible and the points P_9 and P_{10} are both over the invariant plane, we have four equivalent cases that are compatible. If one of the points P_9 or P_{10} is under the invariant plane, then we only have two equivalent cases inside the class, the two corresponding to the other point being in the north or south hemisphere, and they are both compatible.

All the compatible cases and they relationships are shown in the table of Fig.79. Only the cases in the first column that have the same case in the column *P9,P10 Equivalence* are drawn, the rest of the phase portrait can be obtained by homotopy from these.

In the cases $j = 2, 9, 34, 54, 65, 73, 93, 105, 777, 792, 797, 833, 841, 861, 873$ the point P_{11} is inside the region $\mathbb{M}$ and P_6 outside of it.

In the cases $j = 54, 65, 73, 105, 777, 833, 841$ we have $\Delta_{11} < 0$ and P_{11} is a saddle.

In the cases $j = 2, 93, 792$ we have $\Delta_{11} > 0$, $\sigma_{11} > 0$ and P_{11} is an unstable focus if $\sigma_{11}^2 - 4\Delta_{11} < 0$ or unstable node if $\sigma_{11}^2 - 4\Delta_{11} > 0$.

In the cases $j = 861, 873$ we have $\Delta_{11} > 0, \sigma_{11} < 0$ and P_{11} is a a stable focus if $\sigma_{11}^2 - 4\Delta_{11} < 0$ or stable node if $\sigma_{11}^2 - 4\Delta_{11} > 0$.

In the cases $j = 9, 34, 797$ we have $\Delta_{11} > 0$. In this cases P_{11} is a center if $\sigma_{11} = 0$, is an unstable focus if $\sigma_{11} > 0$ and $\sigma_{11}^2 - 4\Delta_{11} < 0$, an unstable node if $\sigma_{11} > 0$ and $\sigma_{11}^2 - 4\Delta_{11} > 0$, a stable focus if $\sigma_{11} < 0$ and $\sigma_{11}^2 - 4\Delta_{11} < 0$, and a stable node if $\sigma_{11} < 0$ and $\sigma_{11}^2 - 4\Delta_{11} > 0$.

The cases $j = 6, 38, 69, 77, 89, 774, 781, 845, 849, 857, 889$ have the points P_6 and P_{11} inside of the region $\mathbb{M}$.

In the case $j = 849$ we have $I_3 < 0$ and P_6 is a stable node and P_{11} is a saddle.

In the cases $j = 6, 38, 69, 77, 774, 781$ if $I_3 < 0$ then P_6 is an unstable node and P_{11} is a saddle. If $I_3 > 0$ then P_6 is a saddle and P_{11} is an unstable focus if $\sigma_{11}^2 - 4\Delta_{11} < 0$, or an unstable node if $\sigma_{11}^2 - 4\Delta_{11} > 0$.

In the case $j = 89$ if $I_3 < 0$ then P_6 is an unstable node and P_{11} is a saddle. If $I_3 > 0$ then P_6 is a saddle and P_{11} is a stable focus if $\sigma_{11}^2 - 4\Delta_{11} < 0$, or stable node if $\sigma_{11}^2 - 4\Delta_{11} > 0$.

In the case $j = 857, 889$ if $I_3 < 0$ then P_6 is a stable node and P_{11} is a saddle. If $I_3 > 0$ then P_6 is a stable saddle and P_{11} is a stable focus if $\sigma_{11}^2 - 4\Delta_{11} < 0$, or a stable node if $\sigma_{11}^2 - 4\Delta_{11} > 0$.

In the cases $j = 845$ if $I_3 < 0$ then P_6 is an unstable node and P_{11} is a saddle. If $I_3 > 0$ then P_6 is a saddle, and P_{11} is a center if $\sigma_{11} = 0$, if $\sigma_{11} < 0$ it is a stable focus if $\sigma_{11}^2 - 4\Delta_{11} < 0$, or a stable node if $\sigma_{11}^2 - 4\Delta_{11} > 0$, and if $\sigma_{11} > 0$ it is an unstable focus if $\sigma_{11}^2 - 4\Delta_{11} < 0$, or an unstable node if $\sigma_{11}^2 - 4\Delta_{11} > 0$.

3.6.3 Proof of Theorem 17

Proof. The proof of Theorem 17 is obtained as a consequence of all the results given in this section. Let's consider the process of construction of the global phase portraits described in the beginning of this section: we have 486 possible cases at the invariant plane and 154 cases at infinity. If we classify them by the sign of the common parameters a_1, I_4, b_2, I_5, I_3, we will obtain the number of cases in each group with the same parameters signs (the first two columns of Fig.3.132), that can be combined pair-wise to generate a global phase portrait. By multiplying them, we have the number of compatible combinations in each group, and if we sum them all we have the total number of compatible combinations of both surfaces, which is 3244 cases.

We recall that the cases are classified by the singular points on the boundary of the region $\mathbb{M}$, some of the invariant plane and infinity cases should be accounted more than one time, as we have a bifurcation of the point P_4 for the invariant plane and P_{11} for the infinity. As we have seen, there are two types of bifurcation , node-focus bifurcation, and stable focus/node-center-unstable focus/node bifurcation. The first two are topologically equivalent, but for the second type some additional cases have to be accounted.

For each invariant plane case with this triple bifurcation, the combination with each simple infinity case gives three global cases, and as we have accounted only one, we have two more to add to the global compute , giving a total of 316 cases to add. In the same way, for each combination with triple bifurcation case at infinity we must add eight more cases, so we have to add 144 cases (see Fig.3.133)

For each triple bifurcation case at infinity, in combination with a simple phase portrait at the invariant plane, we have two more cases to add. The eight more cases to add for each triple bifurcation invariant plane case are the same that we have in the previous paragraph. We have to add 256 more cases (see Fig.3.134).

Now that we have accounted all the bifurcations we have a total of 3960 cases.

In short, we have two options in each case for the sign of the parameter a, to be $a < 0$ or $a > 0$, so we have a total of 7920 possible global phase portraits. □

3.7 Examples of global phase portraits

As we have seen in section 3.6, to construct a global phase portrait we must take one case of the 154 cases at the infinity and one of the 486 cases at the

CASE	SIM X/Y	SIM T/-T	SIM COMBINED X/Y & T/-T	P9 / P10 EQUIVALENCE CLASS	I3 SIGN	FIGURE
2	215	255	42	2	-	3.81
4	216	253	41	2	-	
6	199	251	58	6	>0, < 0	3.82
8	200	249	57	6	>0, < 0	
9	245	248	12	9	-	3.83
10	247	247	10	9	-	
11	246	246	11	9	-	
34	223	223	34	34	-	3.84
38	207	219	50	38	>0, < 0	3.85
54	203	203	54	54	-	3.86
65	85	192	172	65	-	3.87
66	87	191	170	65	-	
67	86	190	171	65	-	
68	88	189	169	65	-	
69	69	188	188	69	>0, < 0	3.88
70	71	187	186	69	>0, < 0	
72	72	185	185	69	>0, < 0	
73	117	184	140	73	< 0	3.89
75	118	182	139	73	< 0	
77	101	180	156	77	>0, < 0	3.90
79	102	178	155	77	>0, < 0	
89	113	168	144	89	>0, < 0	3.91
91	114	166	143	89	>0, < 0	
93	97	164	160	93	-	3.92
95	98	162	159	93	-	
105	125	152	132	105	-	3.93
109	109	148	148	109	< 0	3.94
258	727	511	554	258	-	3.95
260	728	509	553	258	-	
262	711	507	570	262	>0	3.96
264	712	505	569	262	>0	
265	757	504	524	265	-	3.97
266	759	503	522	265	-	
267	758	502	523	265	-	
268	760	501	521	265	-	
274	723	495	558	274	< 0	3.98
276	724	493	557	274	< 0	
278	707	491	574	278	-	3.99
280	708	489	573	278	-	
281	753	488	528	281	< 0	3.100
282	755	487	526	281	< 0	
283	754	486	527	281	< 0	
284	756	485	525	281	< 0	
306	731	463	550	306	< 0	3.101
310	715	459	566	310	-	3.102
313	761	456	520	313	< 0	3.103
314	763	455	518	313	< 0	
321	597	448	684	321	-	3.104
322	599	447	682	321	-	
323	598	446	683	321	-	
324	600	445	681	321	-	
325	581	444	700	325	>0	3.105
326	583	443	698	325	>0	
327	582	442	699	325	>0	
328	584	441	697	325	>0	

Figure 3.79

CASE	SIM X/Y	SIM T/-T	SIM COMBINED X/Y & T/-T	P9 / P10 EQUIVALENCE CLASS	I3 SIGN	FIGURE
333	613	436	668	333	>0	3.106
335	614	434	667	333	>0	
337	593	432	688	337	< 0	3.107
338	595	431	686	337	< 0	
339	594	430	687	337	< 0	
340	596	429	685	337	< 0	
341	577	428	704	341	-	3.108
342	579	427	702	341	-	
343	578	426	703	341	-	
344	580	425	701	341	-	
345	625	424	656	345	>0	3.109
347	626	422	655	345	>0	
349	609	420	672	349	-	3.110
351	610	418	671	349	-	
357	589	412	692	357	< 0	3.111
358	591	411	690	357	< 0	
361	637	408	644	361	-	3.112
365	621	404	660	365	< 0	3.113
369	601	400	680	369	< 0	3.114
370	603	399	678	369	< 0	
373	585	396	696	373	-	3.115
374	587	395	694	373	-	
377	633	392	648	377	>0	3.116
381	617	388	664	381	-	3.117
774	967	1019	826	774	>0, < 0	3.118
776	968	1017	825	774	>0, < 0	
777	1013	1016	780	777	-	3.119
778	1015	1015	778	777	-	
781	997	1012	796	781	>0, < 0	3.120
782	999	1011	794	781	>0, < 0	
783	998	1010	795	781	>0, < 0	
784	1000	1009	793	781	>0, < 0	
792	964	1001	829	792	-	3.121
797	993	996	800	797	-	3.122
798	995	995	798	797	-	
799	994	994	799	797	-	
822	971	971	822	822	-	3.123
833	853	960	940	833	-	3.124
834	855	959	938	833	-	
835	854	958	939	833	-	
836	856	957	937	833	-	
841	885	952	908	841	-	3.125
843	886	950	907	841	-	
845	869	948	924	845	>0, < 0	3.126
847	870	946	923	845	>0, < 0	
849	849	944	944	849	< 0	3.127
850	851	943	942	849	< 0	
852	852	941	941	849	< 0	
857	881	936	912	857	>0, < 0	3.128
859	882	934	911	857	>0, < 0	
861	865	932	928	861	-	3.129
863	866	930	927	861	-	
873	893	920	900	873	-	3.130
889	889	904	904	889	>0, < 0	3.131

Figure 3.80

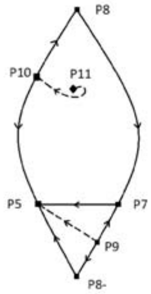

Figure 3.81

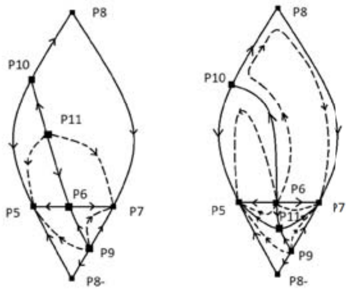

Figure 3.82

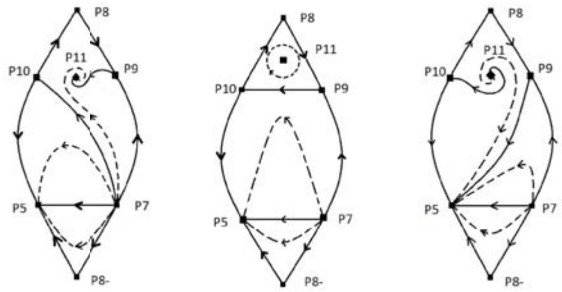

Figure 3.83

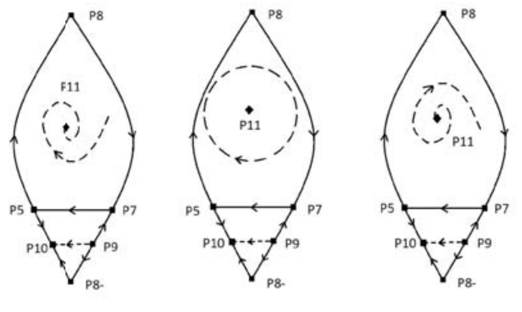

Figure 3.84

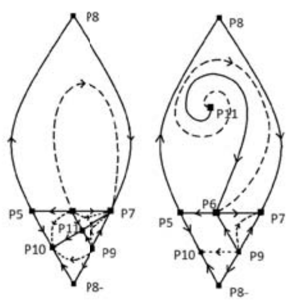

Figure 3.85

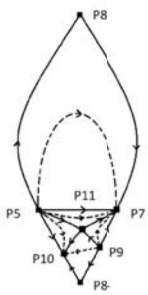

Figure 3.86

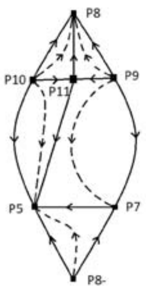

Figure 3.87

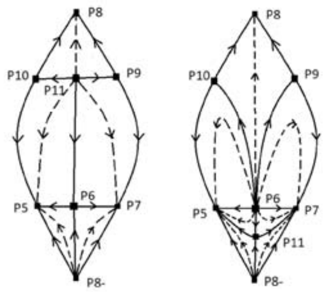

Figure 3.88

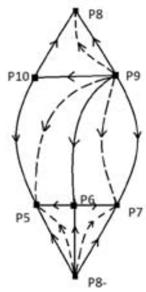

Figure 3.89

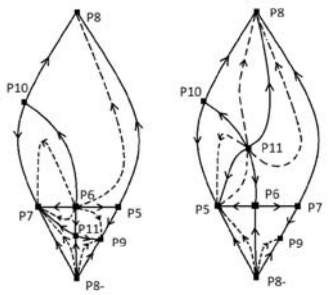

Figure 3.90

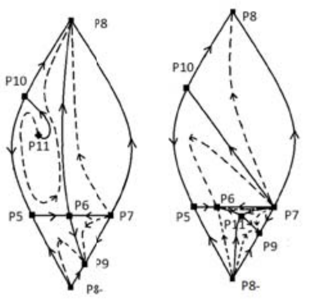

Figure 3.91

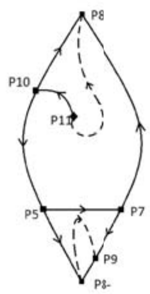

Figure 3.92

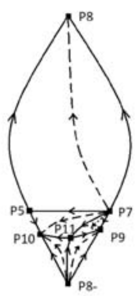

Figure 3.93

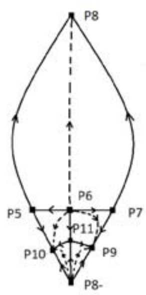

Figure 3.94

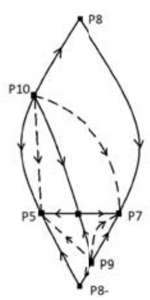

Figure 3.95

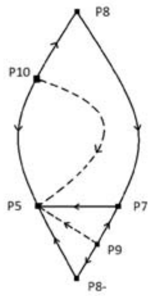

Figure 3.96

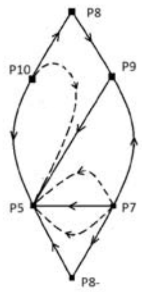

Figure 3.97

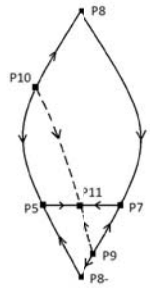

Figure 3.98

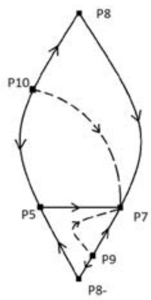

Figure 3.99

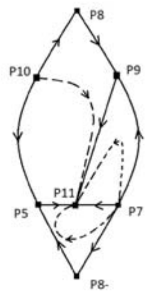

Figure 3.100

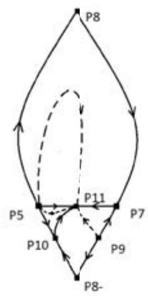

Figure 3.101

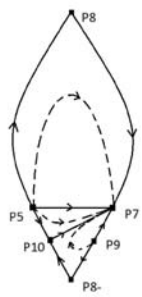

Figure 3.102

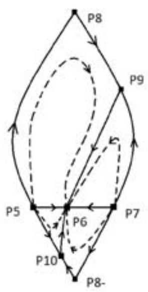

Figure 3.103

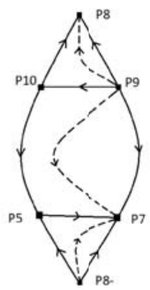

Figure 3.104

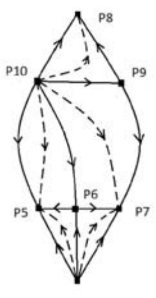

Figure 3.105

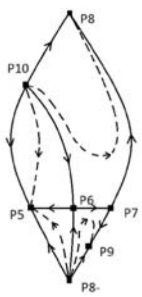

Figure 3.106

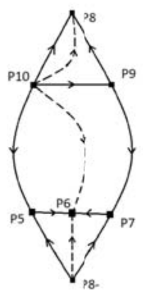

Figure 3.107

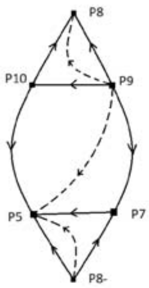

Figure 3.108

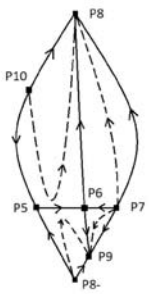

Figure 3.109

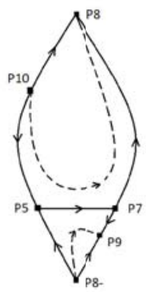

Figure 3.110

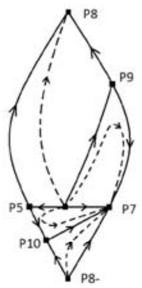

Figure 3.111

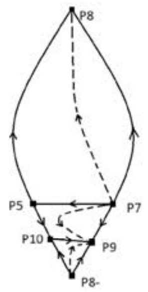

Figure 3.112

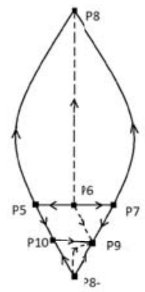

Figure 3.113

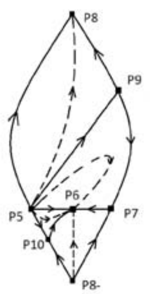

Figure 3.114

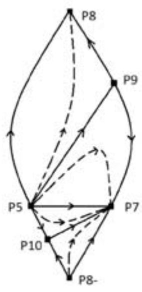

Figure 3.115

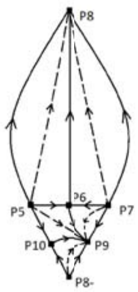

Figure 3.116

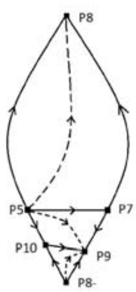

Figure 3.117

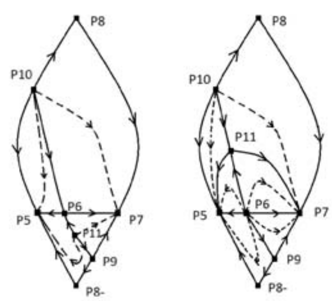

Figure 3.118

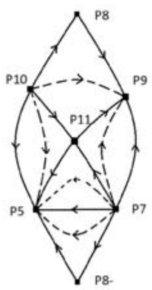

Figure 3.119

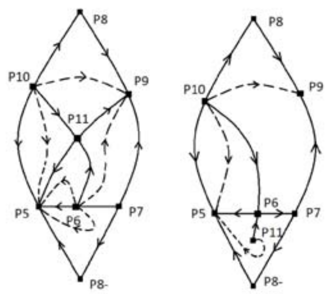

Figure 3.120

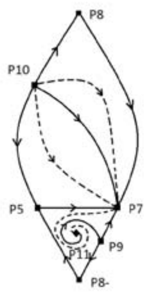

Figure 3.121

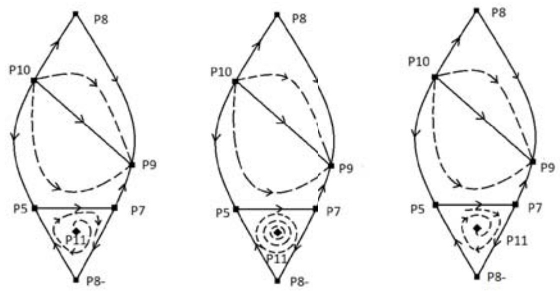

Figure 3.122

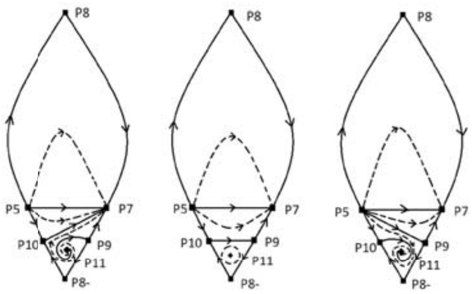

Figure 3.123

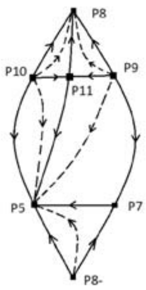

Figure 3.124

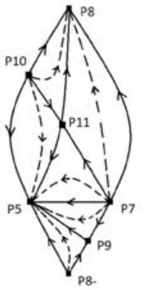

Figure 3.125

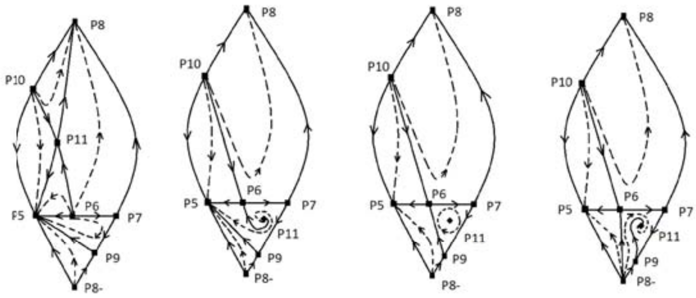

Figure 3.126

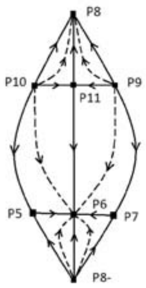

Figure 3.127

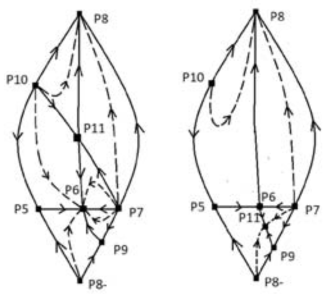

Figure 3.128

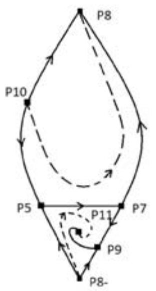

Figure 3.129

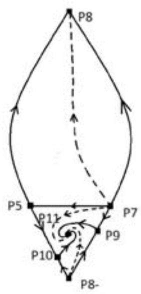

Figure 3.130

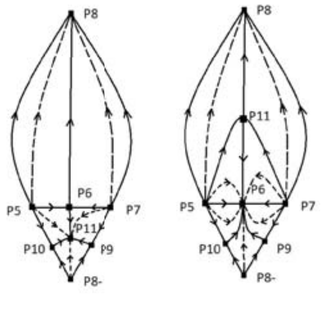

Figure 3.131

Cases	Number of infinity cases	Number of invariant plane cases	Number of global cases
a1<0, I4<0, b2<0, I5<0, I3 -	27	8	**216**
a1<0, I4<0, b2<0, I5>0, I3<0	29	6	**174**
a1<0, I4<0, b2<0, I5>0, I3>0	15	8	**120**
a1<0, I4<0, b2>0, I5<0, I3 -	27	9	**243**
a1<0, I4<0, b2>0, I5<0, I3 <0	2	0	**0**
a1<0, I4<0, b2>0, I5<0, I3>0	2	0	**0**
a1<0, I4<0, b2>0, I5>0, I3<0	22	7	**154**
a1<0, I4<0, b2>0, I5>0, I3>0	17	6	**102**
a1<0, I4>0, b2<0, I5<0, I3<0	3	4	**12**
a1<0, I4>0, b2<0, I5<0, I3>0	16	2	**32**
a1<0, I4>0, b2<0, I5>0, I3 -	27	8	**216**
a1<0, I4>0, b2>0, I5<0, I3<0	17	7	**119**
a1<0, I4>0, b2>0, I5<0, I3>0	22	6	**132**
a1<0, I4>0, b2>0, I5>0, I3 -	17	6	**102**
a1>0, I4<0, b2<0, I5<0, I3 -	17	6	**102**
a1>0, I4<0, b2<0, I5>0, I3<0	22	6	**132**
a1>0, I4<0, b2<0, I5>0, I3>0	17	7	**119**
a1>0, I4<0, b2>0, I5<0, I3 -	27	8	**216**
a1>0, I4<0, b2>0, I5>0, I3<0	16	2	**32**
a1>0, I4<0, b2>0, I5>0, I3>0	3	4	**12**
a1>0, I4>0, b2<0, I5<0, I3<0	17	6	**102**
a1>0, I4>0, b2<0, I5<0, I3>0	22	7	**154**
a1>0, I4>0, b2<0, I5>0, I3 -	27	9	**243**
a1>0, I4>0, b2<0, I5>0, I3<0	2	0	**0**
a1>0, I4>0, b2<0, I5>0, I3>0	2	0	**0**
a1>0, I4>0, b2>0, I5<0, I3<0	15	8	**120**
a1>0, I4>0, b2>0, I5<0, I3>0	29	6	**174**
a1>0, I4>0, b2>0, I5>0, I3 -	27	8	**216**
	486	**154**	**3244**

Figure 3.132

Invariant case	Compatibility class	Infinity simple cases	Infinity triple bifurcation cases
4	a1<0, I4>0, b2>0, I5>0, I3 -	13	4
37	a1<0, I4<0, b2>0, I5<0, I3 -	23	4
39	a1>0, I4>0, b2<0, I5<0, I3>0	22	0
53	a1<0, I4>0, b2>0, I5<0, I3>0	22	0
81	a1>0, I4<0, b2<0, I5<0, I3 -	13	4
114	a1>0, I4>0, b2<0, I5>0, I3 -	23	4
116	a1<0, I4<0, b2>0, I5>0, I3<0	21	1
130	a1>0, I4<0, b2<0, I5>0, I3<0	21	1
	Total	**158**	**18**
	Total to add	**316**	**144**

Figure 3.133

Infinity case	Compatibility class	Simple invariant cases	Triple invariant cases
9	a1>0, I4>0, b2<0, I5>0, I3 -	8	1
12	a1>0, I4>0, b2<0, I5>0, I3 -	8	1
34	a1<0, I4>0, b2>0, I5>0, I3 -	5	1
34	a1<0, I4>0, b2>0, I5>0, I3 -	5	1
223	a1>0, I4<0, b2<0, I5<0, I3 -	5	1
223	a1>0, I4<0, b2<0, I5<0, I3 -	5	1
245	a1<0, I4<0, b2>0, I5<0, I3 -	8	1
248	a1<0, I4<0, b2>0, I5<0, I3 -	8	1
797	a1>0, I4<0, b2<0, I5<0, I3 -	5	1
800	a1>0, I4<0, b2<0, I5<0, I3 -	5	1
822	a1<0, I4<0, b2>0, I5<0, I3 -	8	1
822	a1<0, I4<0, b2>0, I5<0, I3 -	8	1
845	a1>0, I4>0, b2<0, I5<0, I3<0	6	0
869	a1<0, I4>0, b2>0, I5<0, I3<0	7	0
924	a1>0, I4<0, b2<0, I5>0, I3<0	5	1
948	a1<0, I4<0, b2>0, I5>0, I3<0	6	1
971	a1>0, I4>0, b2<0, I5>0, I3 -	8	1
971	a1>0, I4>0, b2<0, I5>0, I3 -	8	1
993	a1<0, I4>0, b2>0, I5>0, I3 -	5	1
996	a1<0, I4>0, b2>0, I5>0, I3 -	5	1
	Total	**128**	**18**
	Total to add	**256**	**144**

Figure 3.134

invariant plane, both having the same signs of the parameters a_1, I_4, b_2, I_5, I_3, and then define the sign of the parameter a. We take the invariant plane case $I_1 < 0, I_4 > 0, I_5 > 0, a_3 > 0, a_1 > 0, b_3 > 0, b_2 < 0, (Fig.3.6)$ and the infinity case j=9 in which, applying (3.56), we have $I_9 > 0, I_{10} > 0, f > 0, i < 0, a_1 < 0, I_4 > 0, b_2 < 0, I_5 > 0, c > 0, h > 0$. The infinity case has a stable focus/node-center-unstable focus/node bifurcation, we will choose the $I_3 = 0$ case (Fig.3.82 with a center). Both cases are shown together in Fig.3.134. We choose $a > 0$, so all the trajectories not belonging to the infinity or the invariant plane go from the invariant plane to the infinity.

Now we can construct the global phase portrait. We have three additional invariant planes: the planes xz, yz, and the invariant plane ψ defined by the points P_1, P_9, P_{10}. We also have an invariant surface ϱ that contains the points $P_4, P_5, P_7, P_{8-}, P_9$ and P_{10}. This invariants and their phase portraits are shown in Fig. 3.135.

The global phase portrait is shown in Fig. 3.136. The plane ψ splits the region $\mathbb{M}$ in two sectors, each of them with a different ω-limit. The trajectories of the upper pyramidal sector with vertices P_1, P_8, P_9 and P_{10} have α-limit P_1 and spin around the trajectory connecting P_1 and P_{11} towards the infinity, having as a ω-limit the orbits around the center P_{11}. The lower sector has as ω-limit P_5 and is splitted in two sub-sectors by the surface ϱ. The trajectories of the sub-sector with vertices P_1, P_2, P_5 and P_{8-} have α-limit P_1 and the trajectories of the sub-sector with vertices P_2, P_5, P_7 and P_{8-} have α-limit P_7.

If we take Fig.3.39 and Fig.3.124, unstable focus subcase (Fig.3.136), and $a < 0$ and we proceed as in the previous case, we have the global phase portrait shown in Fig.3.138. In Fig.3.137 we have the rest of the invariant surfaces in the following order: the planes xz, yz, and the invariant surface triangular sector containing the points P_2, P_{10} and P_5.

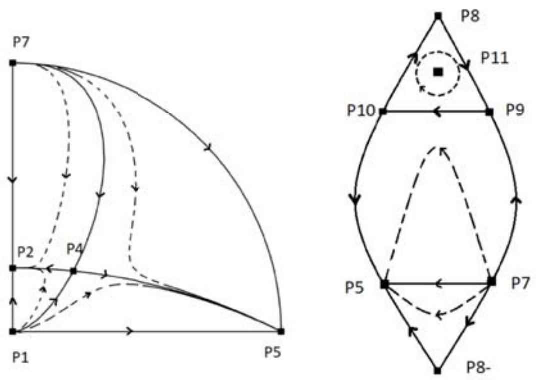

Figure 3.135

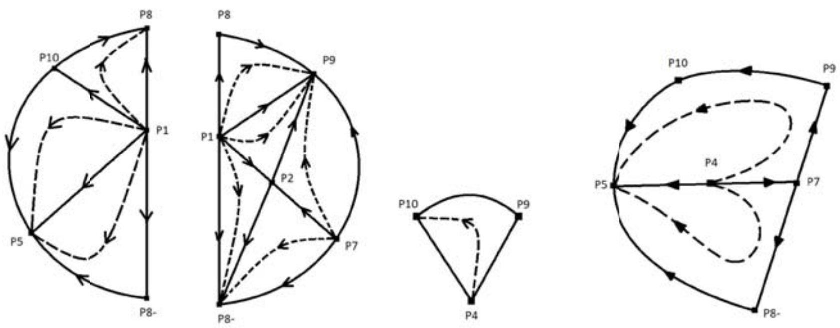

Figure 3.136

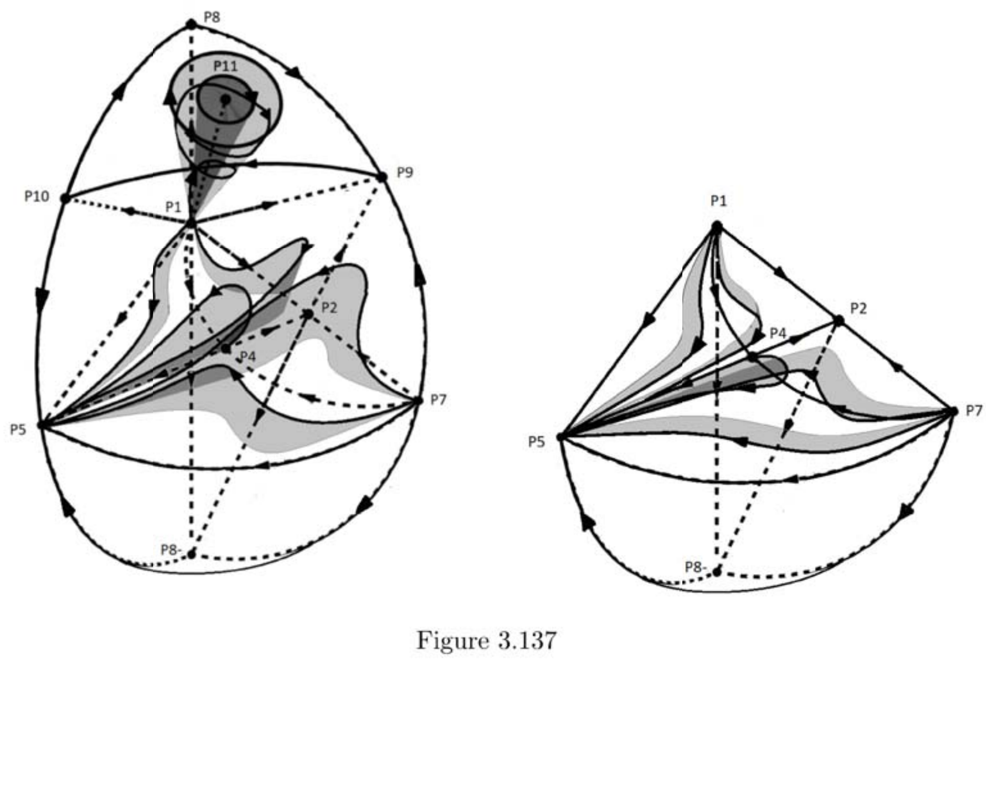

Figure 3.137

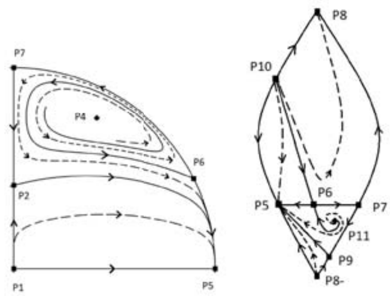

Figure 3.138

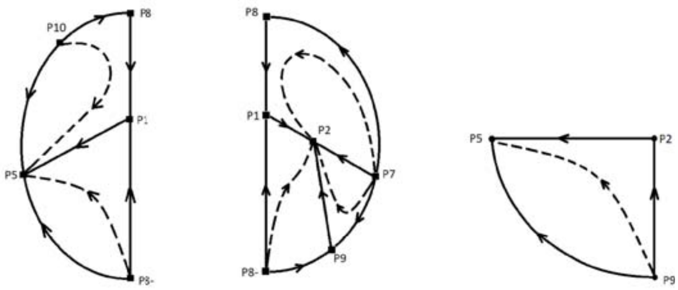

Figure 3.139

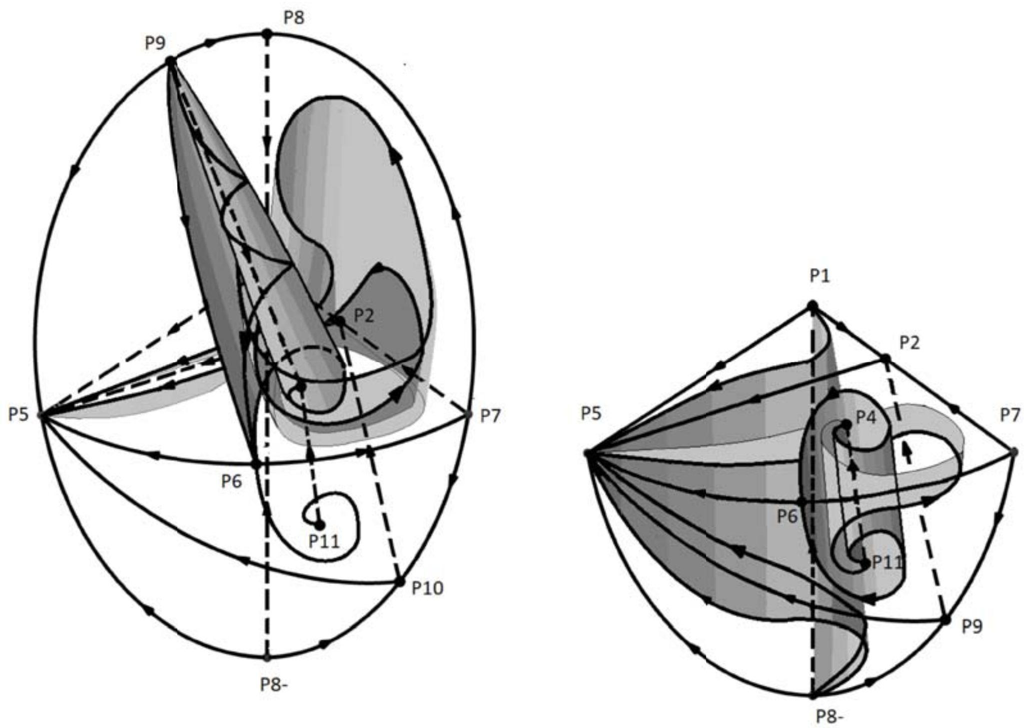

Figure 3.140

Bibliography

[1] A. Algaba and M. Reyes, Computing center conditions for vector fields with constant angular speed, *J. Comp. Appl. Math.* **154** (2003), 143–159.

[2] A. Algaba, M. Reyes, T. Ortega and A. Bravo Campos cuárticos con velocidad angular constante, in Actas: XVI CEDYA Congreso de Ecuaciones Diferenciales y Aplicaciones, VI Congreso de matemática Aplicada , Vol. 2. Las Palmas de Gran Canaria 1999, 1341–1348.

[3] M.A. Alwash and N.G. Lloyd, Non-autonomous equations related to polynomial two dimensional systems, *Proc. Roy. Soc. Edinburgh* **105** A (1987), 129–152.

[4] J. Artés, J. Itikawa and J. Llibre, Uniform isochronous cubic and quartic centers: Revisited, *J. Comp. Appl. Math.,* **313** (2017), 448-453.

[5] N.N. Bautin, On the number of limit cycles appearing with variation of the coefficients from an equilibrium state of the type of a focus or a center, *Mat. Sb. (N.S.),* **30(72)**:1 (1952), 181–196

[6] N.N. Bautin and V.A. Leontovich, Metody i priomy kachestvennogo issledovania dinamicheskij system na ploskosti, *Ed. Nauka,* Moscow 1979.

[7] G. Belitskii, Smooth equivalence of germs of C^∞ of vector fields with one zero or a pair of pure imaginary eigenvalues, *Funct. Anal. Appl.*, **20**, No. 4 (1986), 253–259.

[8] I. Bendixson, Sur les courbes définies par des equations differentielles, *Acta Math.*, **24** (1901), 1–88.

[9] S.M. Blower, P.M. Small and P.C. Hopewell, Control strategies for tuberculosis epidemics: new models for old problems, *Science* **273** (1996), 497–500.

[10] C. Castillo-Chavez and Z. Feng, To treat of not to treat: The case of tuberculosis, *J. Math. Biol.* **35** (1997), 629–656.

[11] I. J. Chavarriga, I.A. García and J. Giné, On the integrability of differential equations defined by the sum of homogeneous vector fields with de-

generate infinity, *International Journal of Bifurcation and Chaos,* 3 (2001), 711–722.

[12] J. Chavarriga and M. Sabatini, A survey on isochronous centers, *Qualitative Ttheory of Dynamical Systems*, **1** (1999), 1–70.

[13] Chia-Weichi, Lih-Ing Wu and Sze-Bi Hsu On the asymmetric May-Leonard model of three competing species, *SIAM J. Appl. Math.* **58**(1) (1998), 211–226.

[14] A. Cima and J. Llibre Bounded polynomial systems, *Trans. Amer. Math. Soc.*, **318** (1990), 420–448.

[15] C.B. Collins, Conditions for a centre in a simple class of cubic systems, *Differential and Integral Equations* **10** (1997), 333–356.

[16] R. Conti, Centers of planar polynomial systems. a review, *Le Matematiche*, Vol. LIII, Fasc. II, (1998), 207–240.

[17] W.A. Coppel, A survey of quadratic systems, *J. Differential Equations* **2** (1966), 293–304.

[18] J. Devlin, Coexisting isochronous centers and non-isochronous centers, *Bull. London Math.* **98** (1996), 495–500.

[19] H. Dulac, Détermination et integration d'une classe d'equations differentielle ayant pour point singulier un centre, *Bull. Sci. Math.*, **32** (1908), 230–252.

[20] F. Dumortier, J. Llibre and J.C. Artés, Qualitative theory of planar differential systems, Universitext, Springer, 2006.

[21] N.P. Erugin, Construction of the whole set of differential equations having a given integral curve, *Akad. Nauk SSSR. Prikl. Mat. Meh.* **16** (1952), 659–670 (in Russian).

[22] M. Frommer, Die Integralkurven einer gewöhnlichen Diferential-gleichung erster Ordnung in der Umgebung rationaler Unbestimmtheitsstellen, *Math. Ann.*, **99** (1928), 222–272.

[23] A.S. Galiullin, Inverse Problems of dynamics, *Mir Publishers*, 1984.

[24] J. Giné, M. Grau and J. Llibre, On the extensions of the Darboux theory of integrability, *Nonlinearity*, **26** (2013), 2221–2229.

[25] J. Hofbauer and K. Sigmund, Evolutionary games and population dynamics, Cambridge University Press, Cambridge, 1988.

[26] Y. Ilyashenko and S. Yakovenko, Lectures on analytic differential equations, *Graduate Studies in Mathematics,* **86**. American Mathematical Society, Providence, RI, 2008.

[27] J. ITIKAWA AND J. LLIBRE, Phase portraits of uniform isochronous quartic centers, *J. Comp. Appl. Math.* **287** (2015), 98–114.

[28] J. ITIKAWA AND J. LLIBRE, Global phase portraits of isochronous centers with quartic homogenous polynomial nonlinearities, *Discrete Contin. Dyn. Syst. Ser. B* **21** (2016), 121–131.

[29] A. KISELIOV, M. KRASNOV, G. MAKARENKO Problemas de ecuaciones diferenciales ordinarias, *Ed. Mir,* Moscow 1979.

[30] M.A. LIAPOUNOFF, Problème général de la stabilité du mouvement, Annals of Mathematics Studies, **17**, Princeton University Press, 1947.

[31] J. LLIBRE, Integrability of polynomial differential systems, Handbook of Differential Equations, Ordinary Differential Equations, *Eds. A. Cañada, P. Drabek and A. Fonda* Elsevier (2004), pp. 437–533.

[32] J. LLIBRE, R. DE OLIVEIRA AND C. VALLS, Final evolutions for simplified multistrain/two-stream model for tuberculosis and dengue fever, *Chaos, Solitons and Fractal,* **118** (2019), 181–186.

[33] J. LLIBRE AND R. RAMÍREZ, Inverse problems in ordinary differential equations and applications, Progress in Math. **313**, Birkhäuser, 2016.

[34] J. LLIBRE, R. RAMÍREZ AND V. RAMÍREZ, An inverse approach to the center-focus problem for polynomial differential system with homogenous nonlinearities, *J. Differential Equations* **263**, (2017), 3327–3369.

[35] J. LLIBRE, R. RAMÍREZ AND V. RAMÍREZ, Center problem for generalized Λ–Ω differential systems, *Electronic Journal of Differential Equations*, Vol. 2018 , No. **184** (2018), pp. 1–23.

[36] J. LLIBRE, R. RAMÍREZ AND V. RAMÍREZ, An inverse approach to the center problem, *Rend. Circ. Mat. Palermo,* **68** (2019), 29– 64.

[37] J. LLIBRE, R. RAMÍREZ AND V. RAMÍREZ, The center problem for the Lambda-Omega differential systems, *J. Differential Equations,* online 1–38.

[38] J. LLIBRE, R. RAMÍREZ, V. RAMÍREZ AND N. SADOVSKAIA, Centers and uniform isochronous centers of planar polynomial differential systems, *J. Dyn.Diff.Equat* **30** (2018), 1295–1310.

[39] A.J. LOTKA, Analytical note on certain rhythmic relations in organic systems, *Proc. Natl. Acad. Sci.* USA, **6** (1920), 410–415.

[40] W.C. LOUD, Behaviour of the period of solutions of certain of certain autonomous systems near centers, Contributions to Differential equations **3** (1964), 21–36.

[41] N. LUKASHEVICH, Isochronism of the center of certain systems of differential equations (Russian), *Diff. Uravnenyia,* **1** (1965), 295–302.

[42] LU YANG, Recent advances on determining the number of real roots of parametric polynomials, *J. Symbolic Computation* **28** (1999), 225–242.

[43] I.G. MALKIN, Criteria for center of a differential equation, (Russian), *Volg. Matem. Sbornik*, **2** (1964), 87–91.

[44] I.G. MALKIN, Stability theory of movements, *Ed. Nauka,* Moscow, (1966) (in Russian).

[45] P. MARDEŜIĆ, C. ROUSSEAU AND B. TONI, Linearization of isochronous center, *J. Diff. Eqs.* **121** (1995), 67–108.

[46] I.I. PLESHKAN, A new method of investigating the isochronism of a system of two differential equations, (Russian), *Diff. Uravnenyia* **5** (1969), 1083–1090.

[47] H. POINCARÉ, Sur l'intégration des équations différentielles du premier ordre et du premier degré I and II, *Rendiconti del Circolo Matematico di Palermo* **5** (1891), 161–191; **11** (1897), 193–239.

[48] V.V. NEMYTSKII AND V.V STEPANOV, Qualitative Theory of Differential Equations, *Princeton Univ. Press,* 1960.

[49] G. REEB, Sur certaines propiétés topologiques des variétés feuilletées , W.T. Wu, G. Reeb (Eds.), Sur les espaces fibrés et les variétés feuilletées, Tome XI, in: Actualités Sci. Indust., vol. **1183**, Hermann et Cie, (1952), Paris.

[50] N. SADOVSKAIA, Inverse problem in theory of ordinary differential equations. Thesis Ph. Dr., Univ. Politécnica de Cataluña, (2002), (in Spanish).

[51] N.A. SAHARNIKOV, Solution of the center focus problem in one case, *Prikl. Mat. Meh.*, **14** (1950), 651–658.

[52] K.S. SIBIRSKII, Method of invariants in the qualitative theory of differential equations, (Russian), *Acad. Sci. Moldavian SSR,* Kishinev, (1968).

[53] P. VAN DEN DRIESSCHE AND J. WATMOUGH, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmition, *Mathematical Biosciencies*, **180** (2002), 29–48.

[54] V. VOLTERRA, Lecons sur la Théorie Mathematique de la Lutte pour la vie, Gauthier-Villars, Paris, 1931.

[55] Z. ZHOU, A new method for research on the center-focus problem of differential systems, *Abstract and Applied Analysis* **2014**, (2014), 1–5.