Universitat Autònoma de Barcelona

Singular integrals, rectifiability and elliptic measure

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To my parents and to Alessio

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Introduction

Singular integral operators are an important object of study in Harmonic Analysis, with applications that range from Geometric Measure Theory to Partial Differential Equations.

The aim of this doctoral thesis is to investigate few questions of different nature in this very active field of research. The common interest of the problems we consider is to extract information on the geometry of sets and measures in the euclidean space (the regularity of a curve, the density of a measure, rectifiabily and uniform rectifiability) from analytic properties of associated singular integrals. We mostly deal with the Cauchy integral, the Riesz transform, and the gradient of the single layer potential.

The body of the thesis consists of four chapters, each one based on a different article: [Pul18],[Pul19b],[PPT18] and [Pul19a] respectively. We decided to make each chapter accessible independently at the cost of the repetition of some statements.

In the present introduction we provide some historical background, we motivate the research and we state our main results. This preliminary discussion is complemented by a more accurate and complete one in the first section of each chapter, where the reader can also find further bibliographic references.

The background

Singular integrals. Given $K \in L^p(\mathbb{R})$ and $f \in L^q(\mathbb{R})$, $1 \leq p, q \leq \infty$, the classical Young's inequality for their convolution

$$K * f(x) = \int K(x-y) f(y) dy \tag{0.0.1}$$

reads

$$\|K * f\|_{L^r(\mathbb{R})} \leq \|K\|_{L^p(\mathbb{R})} \|f\|_{L^q(\mathbb{R})},$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Hence, in the presence of a kernel K that belongs to $L^1(\mathbb{R})$, the integral (0.0.1) exists almost-everywhere and $T\colon f \mapsto K * f$ is a bounded operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$.

The situation is more delicate in the absence of integrability for K. Let us consider the prototypical case $K(x) = (2\pi x)^{-1}$, $x \neq 0$, and observe that K is an odd function. Given $\varepsilon > 0$ and a Schwartz function f, the integral

$$H_\varepsilon f(x) := \frac{1}{2\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

exists for all x and, moreover, one can prove that there exists almost everywhere the limit

$$Hf(x) := \lim_{\varepsilon \to 0} H_\varepsilon f(x) := \mathrm{p.\,v}\, \frac{1}{2\pi} \int \frac{f(y)}{x-y} dy.$$

The operator H is called *Hilbert transform*, H_ε is its *truncation at level* ε and 'p. v' stands for *principal value.* This operator was introduced by David Hilbert in 1905 and one of its early applications was found by by Marcel Riesz, who investigated its relation with conjugate harmonic functions. For a more in-depth treatment of the Hilbert transform we refer, for example, to [Gra08, Chapter 4].

Among the properties of the Hilbert transform, we highlight that, using the Fourier transform, one can prove that it defines an isometry on $L^2(\mathbb{R})$, i.e.

$$\|Hf\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}, \tag{0.0.2}$$

and $H^2 f = H(Hf) = -Id$. Moreover, H defines a bounded liner operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$, $1 < p < \infty$ and from $L^1(\mathbb{R})$ to the weak-L^1 space $L^{1,\infty}(\mathbb{R})$.

A systematic study of singular integrals was started by Alberto Calderón and Antoni Zygmund in their milestone article [CZ52]. They considered convolution operators of the form

$$T_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy, \qquad \varepsilon > 0,$$

$\Omega\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ being an homogeneous function of degree 0 with null integral average on the unit sphere $\mathbb{S}^{n-1}$ and having the regularity property

$$|\Omega(x_1) - \Omega(x_2)| \leq \omega(|x_1 - x_2|),$$

for some increasing function $\omega\colon [0,1] \to \mathbb{R}$, $\omega(t) \geq t$, with the Dini-type regularity

$$\int_0^1 \omega(t)\frac{dt}{t} < \infty.$$

They introduced a decomposition technique for L^p functions, now commonly referred to as *Calderón-Zygmund decomposition*, which they used to prove that T_ε defines a bounded operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $1 < p < \infty$ with operator norm not depending on ε. We remark that they also proved the existence of principal values for this class of operators and they motivated their study presenting an application to the *first derivatives of the Newton potential of a single layer* associated with a planar mass distribution.

In this thesis we consider singular integral operators of the type

$$T_\mu f(x) = \int K(x,y) f(y) d\mu(y),$$

μ being a Radon measure on the Euclidean space and where the previous writing has to be understood in a proper sense. These operators will not necessarily be of convolution type, but they have *standard kernels*: K is a (possibly vector-valued) function on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,y) : x = y\}$ such that there exist $0 < \alpha \leq 1$ and $C \geq 0$ for which

$$|K(x,y)| \leq \frac{C}{|x-y|^n}$$

and

$$|K(x_1,y) - K(x_2,y)| + |K(y,x_1) - K(y,x_2)| \leq C\frac{|x_1-x_2|^\alpha}{|x-y|^{n+\alpha}}, \text{ for } |x_1 - x_2| < \frac{1}{2}|x-y|.$$

For some of the operators we consider, the previous relations hold just locally (see Chapter 3, Section 2). We also say that a function K with the size and regularity conditions above is a *Calderón-Zygmund kernel*. These kernels are very general and the existence of principal values is a delicate question in this framework. For this reason, the operator T_μ is said to be *bounded on* $L^p(\mu)$ if there exists a constant $C > 0$ such that the truncates at level $\varepsilon > 0$

$$T_{\mu,\varepsilon} f(x) = \int_{|x-y|>\varepsilon} K(x,y) f(y) d\mu(y) \tag{0.0.3}$$

are bounded on $L^p(\mu)$ with constant C not depending on ε, i.e.

$$\|T_{\mu,\varepsilon} f\|_{L^p(\mu)} \leq C\|f\|_{L^p(\mu)}. \tag{0.0.4}$$

Bounds of the type (0.0.4) first appeared in the work of Riesz on the Hilbert transform [Rie27].

Calderón-Zygmund kernels constitute a wide class and the L^2-boundedness of the associated operators is in general not guaranteed, not even in the case μ is the Lebesgue measure. The study of criteria to determine their boundedness constitutes an interesting field of research; we mention the $T1$ and Tb-theorems, for which we refer to the survey [Hof10].

In the presence of a *doubling measure* μ, i.e. a measure such that

$$\mu\big(B(x,2r)\big) \leq C\mu\big(B(x,r)\big), \text{ for some } C > 0,$$

which satisfies the growth condition

$$\mu\big(B(x,r)\big) \leq Cr^n, \qquad x \in \mathbb{R}^n,\, r > 0 \tag{0.0.5}$$

for some positive constant C, the $L^2(\mu)$-boundedness of T_μ implies that the operator is also bounded on $L^p(\mu)$ for $1 < p < \infty$ and from $L^1(\mu)$ to $L^{1,\infty}(\mu)$. Calderón-Zygmund theory has also been investigated in the context of non-doubling measures, motivated by the applications to analytic capacity and related topics. For more details we refer to the book by Xavier Tolsa [Tol14]. The condition (0.0.5) is necessary for the L^2-boundedness of an ample class of singular integral operators (see [Dav91, Proposition 1.4]). Another object which is interesting for applications and that is used in the present manuscript is the *maximal singular integral*

$$T_{\mu,*} f(x) = \sup_{\varepsilon>0} |T_{\mu,\varepsilon} f(x)|.$$

The study of the boundedness of this operator is closely related to that of the L^2-boundedness of T_μ: for T_μ bounded on $L^2(\mu)$, if (0.0.5) holds and $T_\mu f$ is defined in a proper principal value or weak limit sense, Cotlar's inequality provides the almost everywhere pointwise bound

$$T_{\mu,*} f(x) \leq CM(T_\mu f)(x) + CMf(x), \tag{0.0.6}$$

M denoting the Hardy-Littlewood maximal function

$$Mf(x) = \sup_{r>0} \frac{1}{\mu\big(B(x,r)\big)} \int_{B(x,r)} |f| d\mu.$$

For more details, see [Tol14, Theorem 2.18].

Singular integrals and rectifiability The interest of singular integral operators goes even beyond Harmonic Analysis. As in the case of the Hilbert transform and of the operators in (0.0.2) considered by Calderón and Zygmund, the boundedness of singular integrals is naturally linked to a cancellation property of the positive and negative parts of the kernel close to its singularities. To have this cancellation for an (appropriate) operator T_μ, some degree of flatness for the measure μ is typically necessary.

The pioneering result in the investigation of the geometry of measures via singular integrals was the proof by Calderón [Cal77] of the boundedness of the Cauchy transform, which is a convolution operator with kernel $K(z) = z^{-1}$ for $z \in \mathbb{C}$, on Lipschitz graphs with small Lipschitz constant. Few years later, Ronald Coifmann, Alan McIntosh and Yves Meyer [CMM82] achieved the same result without the smallness assumpion.

Lipschitz graphs constitute a particularly relevant case because they are the building blocks for the measure-theoretic analogue of differentiable manifolds. A set $E \subset \mathbb{R}^n$ is called *d-rectifiable* if there exists a countable collection of possibly rotated Lipschitz d-graphs $\{\Gamma_j\}_j$ such that

$$\mathcal{H}^d\Big(E \cap \bigcup_j \Gamma_j\Big) = 0.$$

Again, a Radon measure μ is *d-rectifiable* if it vanishes outside a d-rectifiable set F and it is absolutelly continuous with respect to $\mathcal{H}^d|_F$.

Many characterizatization of rectifiable sets are available in the literature; see, for example, the book by Pertti Mattila [Mat95].

Due to its qualitative nature, rectifiability is a too weak notion to develop a consistent theory of singular integrals. For example, when E is a curve, the growth condition (0.0.5) for $\mu = \mathcal{H}^1|_E$ is both necessary and sufficient for the boundedness of the associated Cauchy transform (see [Dav84]). On the other side, the prototypical example of a set with Hausdorff dimension 1 and whose respective Cauchy transform is not bounded on L^2 can be found in John Garnett's *quarter Cantor set*. This is an example of a *purely unrectifiable* set: a set E is called purely d-unrectifiable if $\mathcal{H}^d(F \cap E) = 0$ for every d-rectifiable set F.

A quantitative study of rectifiability in connection with singular integrals was initiated by Guy David in [Dav84] and [Dav88] and further developed, among other works, by David together with Stephen Semmes in [DS91] and [DS93].

A measure μ in $\mathbb{R}^d$ is called *n-Ahlfors-David* regular (also abbreviated n-AD-regular or just *AD*-regular when n is clear from the context) if there exists some constant $C > 0$ such that

$$C^{-1} r^n \leq \mu(B(x,r)) \leq C\, r^n \quad \text{ for all } x \in \operatorname{supp}(\mu) \text{ and } 0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu)).$$

A set $E \subset \mathbb{R}^d$ is n-AD-regular if the measure $\mathcal{H}^n|_E$ is n-AD-regular.

The set E is called *uniformly n-rectifiable* if it is n-AD-regular and there exist $\theta, M > 0$ such that for all $x \in E$ and all $r > 0$ there is a Lipschitz mapping g from the ball $B_n(0,r)$ in $\mathbb{R}^n$ to $\mathbb{R}^d$ with $\operatorname{Lip}(g) \leq M$ such that

$$\mathcal{H}^n\Big(E \cap B(x,r) \cap g\big(B_n(0,r)\big)\Big) \geq \theta r^n.$$

A measure μ is called uniformly n-rectifiable if it is n-AD-regular and its support is a uniformly n-rectifiable set. Roughly speaking, a set is uniformly rectifiable if it can be covered in a significative amount by Lipschitz images (with controlled Lipschitz constant) at all scales.

David and Semmes (see [Dav84] and [DS91]) proved that, under the background assumption of AD-regularity, a set E is uniformly rectifiable if and only if all its associated singular integral operators of convolution type with odd kernel $K(\cdot)$ that satisfies the regularity condition

$$|\nabla^j K(x)| \leq C|x|^{-d-j}, \qquad \text{for all } j = 0, 1, 2, \cdots \text{ and } x \neq 0 \tag{0.0.7}$$

are bounded on $L^2(\mathcal{H}^n|_E)$. Hence, a control on a wide enough class of singular integrals detects rectifiability. It is legitimate to ask, at this point, if and up to which extent this class is superabundant. For example, are the n components of the d-Riesz transform

$$\mathcal{R}_\mu f(x) = \int \frac{x-y}{|x-y|^{n+d}} f(y) d\mu(y), \tag{0.0.8}$$

the latter expression to be interpreted formally, enough to control uniform d-rectifiability? Weather the L^2-boundedness of the Riesz transform implies uniform rectifiability is a challenging problem that attracted the interest of the experts since the 90's and it is often referred to as the *David and Semmes problem.*

The solution to this question is known (and affirmative) in the cases $d = 1$, thanks to the work of Pertti Mattila, Mark Melnikov and Joan Verdera [MMV96], and $d = n - 1$, by the work of Fedor Nazarov, Xavier Tolsa and Alexander Volberg [NTV14a]. The intermediate cases are still open. The one-dimensional case was solved via the study of the so-called *Menger curvature*; this argument has the advantage of being elegant and direct. However, it is not applicable to the higher dimensional case and the proof in [NTV14a] required a study of the fine structure of the measure and a variational argument inspired by potential theory, previously used in this context also in [ENV14]. One of the main obstructions to the application of the codimension-one methods to the general case is the absence of an adequate substitute of the maximum principle, which plays a crucial role in the proof. Using their result and a covering argument inspired by the work of Hervé Pajot [Paj02], Nazarov, Tolsa and Volberg also proved what follows.

Theorem ([NTV14b]). *Let $E \subset \mathbb{R}^n$ be a set with $\mathcal{H}^n(E) < \infty$. If $\mathcal{R}_{\mathcal{H}^{n-1}|_E}$ is bounded on $L^2(\mathcal{H}^{n-1}|_E)$, then E is $(n-1)$-rectifiable.*

This theorem is the higher dimensional analogue of a previous result by David and Jean-Christophe Léger for $n = 2$ (see [Lég99]).

On the plane, the boundedness of operators associated to other classes of kernels is known to imply uniform rectifiability; for more details on this topic, we refer to [CMT] and the references therein.

Harmonic measure. Considering a domain $\Omega \subset \mathbb{R}^n$ which is regular for the Dirichlet problem, every function $f \in C_0(\partial\Omega)$ has an extension u_f to Ω which is harmonic. Moreover, fixing $p \in \Omega$, which is called *pole*, an application of the Riesz representation theorem ensures the existence of a Radon measure ω^p on the boundary of the domain such that

$$u_f(p) = \int f(y)\, d\omega^p(y).$$

By the maximum principle, the choice $f \equiv 1$ shows that ω^p is a probability measure. Harmonic measure with pole at p can also be interpreted as the probability that a particle, initially positioned at p and moving according to a Brownian motion, first exits Ω through the prescribed part of the boundary. This probabilistic point of view was proposed by Shizuo Kakutani in the 40's.

One of the earliest geometric results on harmonic measure dates back to the work of F. and M. Riesz [RR20], who proved that harmonic measure is mutually absolutelly continuous with respect to the arc-length if $\Omega \subset \mathbb{R}^2$ is simply connected and $\partial\Omega$ has finite length. Christopher Bishop and Peter Jones showed in [BJ90] that this results can be localized to a portion of the boundary and, in addition, that the topology of the domain is crucial for this result to hold: mutual absolute continuity may fail if the domain is not simply connected. For other counterexamples see also [Wu86] and [Zie74].

A vast literature is available also on the investigation of the converse direction. The recent article by Jonas Azzam, Steve Hofmann, José María Martell, Svitlana Mayboroda, Mihalis Mourgoglou, Xavier Tolsa and Alexander Volberg [Azz+16b] proves the conjecture of Bishop (see [Bis92]) that if the harmonic measure is mutually absolutelly continuous with respect to the Hausdorff measure on $E \subset \partial\Omega$, then there exists a rectifiable set $F \subset E$ such that $\omega^p(E \setminus F) = 0$. This work is based on the solution of the David-Semmes problem by Nazarov, Tolsa and Volberg in the codimension 1 case. The link of the Riesz transform with the harmonic measure is evident in the relation

$$c_n \nabla_x G_\Omega(x,p) = \nabla\Theta(x-p) + R_{\omega^p}1(x), \tag{0.0.9}$$

where c_n is a dimensional constant, G_Ω is the Green function of the domain and Θ stands for the fundamental solution of the laplacian.

Related results can be found in [HM01] and [HMU14]. Among the books on harmonic measure, we highlight John Garnett and Donald Marshall's monograph [GM08].

Bishop also proposed the following other problem on harmonic measure and rectifiability: if two domains are such that the respective harmonic measures (with poles in each domain) are mutually absolutely continuous in some subset of a common portion of their boundaries, can that region be covered by a rectifiable set, possibly leaving out subsets which are negligible for the harmonic measure? This is a *free boundary problem*. Bishop's question was answered positively in its full generality by Azzam, Mourgoglou, Tolsa and Volberg in [Azz+16d] where the authors, inspired by the techniques in [KPT09] studied the blowups at the points of mutual absolute continuity of the harmonic measures. Their work is based on the previous results in [AMT17b], where the problem is solved under a non-degeneracy assumption on the boundary of the domain called *capacity-density-condition*.

A study of triple points for harmonic measure can be found in Boris Tsirelson's article [Tsi97] (see also [TV18a]).

The organization of the manuscript

The first two chapters explore what geometric informations are nested into general properties of Calderón-Zygmund operators which, in the case we consider, will mostly be the Cauchy transform.

Chapter 1 focuses on Cotlar's inequality (0.0.6) for the Cauchy integral. More precisely, we chose to work in the framework of a remarkable variant that appeared in the works [MOV11] and [Mat+10].

In particular, the second article proves that if we have a kernel of the form $K(x) = P(x)/|x|^{n+d}$, P being an odd homogeneous polynomial of degree d, and we consider the odd higher order Riesz transform

$$Tf(x) = \int K(x-y)f(y)dy,$$

the summand Mf on the right hand side of (0.0.6) is not necessary if we dominate the maximal Riesz transform by the second order iteration of the Hardy-Littlewood maximal function. Namely,

$$T_*f(x) \le C\,M^2(Tf)(x), \quad x \in \mathbb{R}^n. \tag{0.0.10}$$

The natural question at this point is, in the spirit of Calderón's work, weather this inequality generalizes to rectifiable curves. This would make it possible to try to use it in relation with uniform rectifiability. Daniel Girela-Sarrión provided a negative answer to this question also in the case of the Cauchy transform on a Lipschitz graph (see [Gir13]): it suffices to be in the presence of a corner singuarity on the graph Γ for the pointwise bound

$$\mathcal{C}_*f(x) \le CM^2(\mathcal{C}f)(x) \tag{0.0.11}$$

to fail.

However, on the positive side, Girela-Sarrión proved that (0.0.10) holds if Γ is a closed C^1 curve with the additional regularity hypothesis that the modulus of continuity $\omega(z,\delta)$ of the unit tangent vector to the curve at z satisfies the logarithmic condition

$$\omega(z,\delta) \le C\frac{1}{|\log\delta|} \tag{0.0.12}$$

for $z \in \Gamma$ and δ small enough and $C > 0$. In particular, (0.0.11) is verified if Γ is a $C^{1,\alpha}$ curve, $0 < \alpha < 1$. The question we want to investigate, at this point, is how to characterize Jordan curves which present this improved Cotlar's inequality. In particular, it was not clear if the inequality (0.0.10) for the Cauchy transform implies the existence of tangents at every point of the curve.

To answer this question we first have to drop the C^1 assumption for Γ and to work in the slightly more general framework of *chord-arc curves*, i.e. of rectifiable Jordan curves such that for some constant $C > 0$

$$\ell(z_1, z_2) \le C|z_1 - z_2|, \qquad z_1, z_2 \in \Gamma,$$

where $\ell(z_1,z_2)$ indicates the shortest arc of Γ joining the two points. This condition corresponds to impose that the curve has bounded turning. We also find natural for our problem to ask that Γ does not present angles at small scales, since Girela-Sarrión showed that corners prevent (0.0.6) to hold. For this reason we assume that the curves under study are *asymptotically conformal*: for every $\delta > 0$ there exists $\varepsilon > 0$ such that whenever $z_1, z_2 \in \Gamma$ are such that $|z_1 - z_2| < \varepsilon$, then

$$|z_1 - z| + |z_2 - z| \le (1+\delta)|z_1 - z_2| \tag{0.0.13}$$

for z belonging to the shortest arc of Γ connecting z_1 and z_2. This condition serves to our scope also because asymptotic conformality does not imply that the curve is C^1. The main result of the chapter is the following.

Theorem. *Let Γ be a closed asymptotically conformal chord-arc curve, let $\gamma\colon \mathbb{R} \to \mathbb{C}$ be a periodic bilipschitz parametrization of Γ with period T and let $\mathcal{C}$ be the Cauchy*

Integral on Γ*, i.e.*

$$\mathcal{C}f(z) = \lim_{\varepsilon\to 0} \frac{1}{\pi i} \int_{\Gamma\setminus\gamma(\{\tau:\, |\tau-t|<\varepsilon\})} \frac{1}{w-z} f(w)\, dw, \qquad z\in\Gamma.$$

Then the estimate

$$\mathcal{C}_* f(z) \leq C\, M^2(\mathcal{C}f)(z), \quad z\in\Gamma, \quad f\in L^2(\Gamma,\mu),$$

holds if and only if there exists $C>0$ *such that*

$$\big|\gamma(x+\varepsilon)+\gamma(x-\varepsilon)-2\gamma(x)\big| \leq C\,\frac{\varepsilon}{|\log\varepsilon|}, \tag{0.0.14}$$

for each ε *satisfying* $0<\varepsilon<T$ *and for each* $x\in\mathbb{R}$.

Observe that the previous statement includes the condition (0.0.14) on γ, that can be interpreted as a second order finite difference analogue of (0.0.12). Furthermore, it may be thought as a control of the curvature of Γ. The exposition would not be complete without an example that shows the applicability of this result. We were able to encounter an example of a curve for which (0.0.14) holds but which is *not* differentiable at a point and which, in this way, provides a better understanding of (0.0.6). In particular, this curve shows that logarithmic spiralings of the curve are critical and that a faster winding does not agree with the improved Cotlar's inequality. The correct identification of the principal branch of the complex logarithm to use in our setting requires a detailed discussion.

Many results have been produced about geometric conditions for the boundedness of singular integrals. In *Chapter 2* we analyze which are the properties of a meausure μ on $\mathbb{C}$ that determine when the Cauchy transform, formally given by the expression

$$\mathcal{C}_\mu f(z) = \int \frac{f(w)}{z-w}\, d\mu(w), \qquad z\in\mathbb{C},$$

defines a linear operator which is *compact*. First, we have to clarify what we mean by compactness in our singular integral context. A reasonable request is that a Cauchy transform which is compact on $L^2(\mu)$ should also be bounded on the same space. Under this assumption it is known that the principal values of this operator exist (see [Tol98]), so we understand that the Cauchy transform is compact on $L^2(\mu)$ if it is bounded and the principal value operator is compact on $L^2(\mu)$. We start the investigation from two toy models: the 1-dimensional Hausdorff measure on an interval on the line and the Lebesgue measure on a planar disk. It is possible to give a direct proof that the Cauchy transform of the disk is compact on L^2, contrarily to what happens in the case of an interval. Those proofs also suggest that the (upper) density

$$\Theta^*_\mu(z) := \limsup_{\ell(Q)\to 0} \frac{\mu(Q)}{\ell(Q)}, \tag{0.0.15}$$

Q ranging on the cubes centered at z, plays a central role in the determination of the compactness of $\mathcal{C}_\mu$. The next statement involves also the notion of Menger curvature $c(\mu)$

$$c(\mu)^2 := \iiint \frac{1}{R(x,y,z)^2} d\mu(x)d\mu(y)d\mu(z)$$

and $R(x,y,z)$ indicates the radius of the circumference passing through the three points. We were able to characterize the compactness of the Cauchy transform as follows.

Theorem. *Let μ be a compactly supported positive Radon measure on $\mathbb{C}$ without atoms. The following conditions are equivalent:*

(a) $\mathcal{C}_\mu$ is compact from $L^2(\mu)$ to $L^2(\mu)$.

(b) the two following properties hold:

(1) $\Theta^*_\mu(z) = 0$ uniformly, which means that the limit in (0.0.15) is 0 uniformly in $z \in \mathbb{C}$.

(2) $c^2(\mu|_Q)/\mu(Q) \to 0$ as $\ell(Q) \to 0$, where $\mu|_Q$ stands for the restriction of μ to the cube Q.

(c) the truncated operators $\mathcal{C}_{\varepsilon,\mu}$ converge as $\varepsilon \to 0$ in the operator norm of the space of bounded linear operators from $L^2(\mu)$ to $L^2(\mu)$.

We remark that few other researches (of different nature) on the compactness of singular integrals have been recently conducted by Paco Villarroya (see e.g. [Vil15]).

The *two remaining chapters* deal with the investigation of elliptic equivalents of the recent important results for the Riesz transform and rectifiability of harmonic measure. Codimension 1 plays a central role, so we find it convenient to denote by $\mathbb{R}^{n+1}$ the ambient space.

Both the David-Semmes problem and Bishop's questions on harmonic measure have a natural elliptic formulation. Let $A(\cdot)$ be an $(n+1)\times(n+1)$ uniformly elliptic matrix with essentially bounded real coefficients and let $\mathcal{E}_A(x,y)$ be the fundamental solution of the partial differential equation

$$L_A u = \operatorname{div}\big(A(\cdot)\nabla u\big) = 0$$

in $\mathbb{R}^{n+1}$. The fundamental solution can be constructed under this hypothesis (see [HK07], which extends the classical techniques of [GW82]). Let μ be a compactly supported n-AD-regular measure in $\mathbb{R}^{n+1}$ and consider the associated operator formally defined as

$$T_\mu f(x) = \int \nabla_x \mathcal{E}_A(x,y)\, f(y)\, d\mu(y).$$

The operator T_μ is commonly referred to as a *gradient of the single layer potential* and in the case $A \equiv Id$ coincides with the Riesz transform (modulo a multiplicative dimensional constant). Imposing the further hypothesis of Hölder continuity on the coefficients of the matrix A, $\nabla_x \mathcal{E}(x,y)$ presents (locally) the properties of a Calderón-Zygmund kernel (see [CMT19, Section 2]). Under the Hölder continuity assumption for A, José Conde-Alonso, Mihalis Mourgoglou and Xavier Tolsa first proved that the gradient of the single layer potential is bounded on uniformly rectifiable sets, then they proved that T_μ is not bounded on $L^2(\mu)$ if the measure is *totally lower irregular*. This is an elliptic extension of the main result in [ENV14], which inspired some techniques used in [NTV14a]. Even though several other essential difficulties appear, the link of this scenario with the Riesz transform one is evident: a crucial component of the proof in [CMT19] is to compare the gradient of the fundamental solution at (x,y) to the analogous object for the matrix with constant coefficients $A(x)$, which behaves as the Riesz transform under an affine transformation (depending on x).

In *Chapter 3*, which is the result of a joint work with Laura Prat and Xavier Tolsa, we show that if A is Hölder continuous and uniformly elliptic, μ is an n-AD regular measure on $\mathbb{R}^{n+1}$ with compact support and T_μ is bounded in $L^2(\mu)$, then

μ is uniformly n-rectifiable. This extends the solution of the codimension 1 David-Semmes problem for the Riesz transform to the gradient of the single layer potential. The compactness assumption for $\operatorname{supp}\mu$ is a direct consequence of the lack of scale invariance for the Hölder space and it cannot be dropped in the statement without asking for more properties for the matrix A.

The general scheme of the proof resembles that of [NTV14a] which in turns consists of proving that the *BAUP condition* holds. This criterion asserts that an AD-regular measure is uniformly rectifiability if and only if the (adapted) *cubes* of a dyadic-type lattice $\mathcal{D}_\mu$ built on the support of μ, which is often called *David and Semmes lattice*, that are not *bilaterally approximable by a union of planes* satisfy the Carleson measure condition

$$\sum_{P\subset Q, P \text{ non-BAUP}} \mu(P) \leq C\mu(Q)$$

for every $Q \in \mathcal{D}_\mu$.

However, new difficulties appear because of the different context: in general, the gradient of the fundamental solution does not present the properties of the Riesz transform of being symmetric and homogeneous. This makes an approximation argument for the measure significantly more delicate: we introduced a reflection argument for the matrix A at the scale of a properly chosen flat portion of the measure.

Combining our result with that of Conde-Alonso, Mourgoglou and Tolsa and arguing as in [NTV14b] we can prove that given $E \subset \mathbb{R}^{n+1}$ with finite Hausdorff measure $\mathcal{H}^n$, if $T_{\mathcal{H}^n|_E}$ is bounded in $L^2(\mathcal{H}^n|_E)$, then E is n-rectifiable.

Following the methods of [Azz+16b], this has a direct application to *elliptic measure*, which is the elliptic analogue of harmonic measure. The existence of the elliptic measure for Wiener regular domains (i.e. regular for the Dirichlet problem) follows from the Riesz representation theorem and the work of Littman, Stampacchia and Weinberger [LSW63] on generalized solutions of the Dirichlet problem for equations in divergence form with bounded measurable coefficients.

Definition (Elliptic measure)**.** Let $\Omega \subset \mathbb{R}^{n+1}$ be a Wiener regular domain and let $L_A = \operatorname{div}(A\nabla\cdot)$ be a uniformly elliptic operator with bounded measurable coefficients. Given $p \in \Omega$, the elliptic measure associated with L_A and with pole at p is the probability measure $\omega^p_{L_A}$ on $\partial\Omega$ such that for every $f \in C(\partial\Omega)$

$$u_f(p) = \int f(y)d\omega^p_{L_A}(y),$$

u_f denoting the generalized solution of the Dirichlet problem with data f in the sense of [LSW63].

The elliptic measure is linked to the gradient of the single layer potential by a formula of the same type of (0.0.9), namely

$$\nabla_x G_\Omega(x,p) = c_n \nabla_x \mathcal{E}(x,p) + T_{\omega^p} 1(x),$$

where c_n is a dimensional constant and G_Ω denotes the Green's funtion associated with L_A and Ω.

We show that, under the Hölder continuity assumption for A, if the elliptic measure is absolutelly continuous with respect to surface measure, then it is rectifiable.

This is consistent with the rectifiability result for harmonic measure in [Azz+16b]. In this case the elliptic setting does not bring crucial difficulties to the proof, which is a variation (once the rectifiability result in terms of T_μ is known) of that in the case of the harmonic measure.

Chapter 4 may be regarded as a continuation of the program started in the previous chapter. The main result is a local quantitative uniform rectifiability criterion for measures which are not AD-regular that is formulated in terms of the gradient of the single layer potential operator. Broadly speaking the theorem asserts that given a measure μ, there is a scale with the following feature: if we know that the measure is very flat at that scale and the $L^2(\mu)$-mean oscillation of $T_\mu 1$ is small, then a big portion of μ can be covered (at that level) by a uniformly n-rectifiable set. Both the flatness, quantified via Jones' β-numbers, and the oscillation of the gradient of the layer potential depend on proper densities of μ together with the $L^2(\mu)$-operator norm of T_μ at the level of the scale. Due to its technicality, we prefer to postpone the precise statement of our theorem to the first section of Chapter 4.

This result generalizes the previous study [GT18] by Girela-Sarrión and Tolsa, who worked with the Riesz transform. It can be interpreted as an higher-dimensional analogue of the theorem of David and Léger (see [Lég99]), who formulated a recfiability criterion which involves the Menger curvature of μ. Here, the role of the curvature is assumed by the oscillation of the potential.

This investigation is motivated also by the relevant applications of [GT18] to harmonic measure. Indeed, their result was crucial to fully solve in [AMT17b] and [Azz+16d] the two-phase problem proposed by Bishop. At the end of the manuscript we outline the proof of the following analogue for the elliptic measure associated with an operator in divergence form defined by a Hölder continuous matrix.

Theorem. *Let $n \geq 2$ and let A be a Hölder continuous uniformly elliptic matrix. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ be two Wiener-regular domains and, for $p_i \in \Omega_i$, $i \in \{1,2\}$, let $\omega_i^{p_i}$ be the respective elliptic measures in Ω_i associated with L_A and with pole p_i. Suppose that E is a Borel set such that $\omega_1^{p_1}|_E \ll \omega_2^{p_2}|_E \ll \omega_1^{p_1}|_E$. Then there exists an n-rectifiable set $F \subset E$ with $\omega_1^{p_1}(E \setminus F) = 0$ such that $\omega_1^{p_1}|_F$ and $\omega_2^{p_2}|_F$ are mutually absolutely continuous with respect to $\mathcal{H}^n|_F$.*

The proof is an adaptation of that of [Azz+16d], which in turns uses a blow-up method at the points of mutual absolute continuity of the two elliptic measures. A detailed analysis of the blowup for elliptic measures which serves to our scopes was conducted in [AM17].

For the proof of the quantitative rectifiability theorem we follow the same strategy of that of [GT18]. However, as is Chapter 3, the nature of the gradient of the single layer potential brings some difficulties which make the application of the variational argument more delicate. This requires a change of variable arguments, together with an elliptic homogenization technique to estimate the fundamental solution at large scales.

Some notation and abbreviations

$A \lesssim B$	there exists $C > 0$ (possibly depending on fixed parameters) such that $A \leq CB$
$A \gtrsim B$	$B \lesssim A$
$A \approx B$	$A \lesssim B$ and $B \lesssim A$
$B(x,r)$	open ball of center x and radius r
$r(B)$	radius of the ball B
$A(x,r,R)$	open annulus centered at x with inner radius r and outer radius R
$\mathbb{S}^{n-1}$	unit sphere in $\mathbb{R}^n$
$\operatorname{dist}(x,E)$	distance of the point x from the set E
$\operatorname{dist}_H(\cdot,\cdot)$	Hausdorff distance between two sets
$\ell(Q)$	side length of the cube Q
$\mathcal{L}^n$	Lebesgue measure on $\mathbb{R}^n$
$\int f(x)dx$	$\int f(x)d\mathcal{L}^n(x)$
$L^p(\mu)$	Lebesque spaces associated with the measure μ on $\mathbb{R}^n$
$L^p(\mathbb{R}^n)$, $\|\cdot\|_p$	Lebesque spaces associated with $\mathcal{L}^n$ and its norm
BMO	space of functions with bounded mean oscillation
C^α	space of α-Hölder continuous functions
Lip	space of Lipschitz functions
H	Hilbert transform
$\mathcal{C}_\mu$	Cauchy transform associated with the measure μ
$\mathcal{C}_{\mu,\varepsilon}$	ε truncation of the Cauchy transform
$\mathcal{R}^n_\mu$, $\mathcal{R}_\mu$	(n-)Riesz transform associated with the measure μ
$\mathcal{R}_{\mu,\varepsilon}$	ε truncation of the Riesz transform
T_μ, $T_{\mu,\varepsilon}$	gradient of the single layer potential and its truncation
$\mathcal{C}_{\mu,*}$, $\mathcal{R}_{\mu,*}$, $T_{\mu,*}$	maximal singular integrals
$\mathcal{H}^d$	d-dimesional Hausdorff measure
$\mu\lfloor_E$	restriction of the measure μ on $\mathbb{R}^n$ to the set $E \subseteq R^n$
$\phi_\sharp\mu$	image measure of μ via ϕ
$m_{\mu,E}(f)$	integral average $\fint_E f\,d\mu$
M, M_μ	Hardy-Littlewood maximal function
$\Theta^{n,*}_\mu(x)$, $\Theta^*_\mu(x)$	(n-)upper density of μ at x
$\Theta^n_{\mu,*}(x)$, $\Theta_{\mu,*}(x)$	(n-)lower density of μ at x
$\Theta^n_\mu(x)$, $\Theta_\mu(x)$	(n-)density of μ at x
$c_\mu(\cdot)$, $c(\mu)$	Menger curvature of μ
n-AD-regular	n-Ahlfors-David-regular
BAUP	Bilateral Approximation by a Union of Planes
β_μ, $\beta_{\mu,1}$	β-numbers of Jones
α_μ	α-number of Tolsa
$\mathcal{D}_\mu$	David-Semmes (Chapter 3) or David-Mattila (Chapter 4) lattice associated with the measure μ
x_Q, $\ell(Q)$	center and side length of $Q \in \mathcal{D}_\mu$

L_A	operator in divergence form associated with the matrix A
$\Theta(x, y; A_0)$	fundamental solution to L_{A_0}, A_0 being an elliptic matrix
$= \Theta(x - y; A_0)$	with constant coefficients
$\mathcal{E}_A(\cdot, \cdot)$, $\mathcal{E}(\cdot, \cdot)$	fundamental solution to L_A, A being a uniformly elliptic matrix
	with possibly variable coefficients
$\nabla_1 \mathcal{E}(\cdot, \cdot)$	gradient of $\mathcal{E}$ with respect to the first variable
$\nabla_2 \mathcal{E}(\cdot, \cdot)$	gradient of $\mathcal{E}$ with respect to the second variable
$\omega^p, \omega^p_{L_A}$	harmonic/elliptic measure with pole at p

Chapter 1

Estimates for the maximal Cauchy Integral on chord-arc curves

1.1 Introduction

Consider a homogeneous smooth Calderón-Zygmund operator in $\mathbb{R}^n$

$$Tf(x) = \text{p. v.} \int f(x-y)\, K(y)\, dy \equiv \lim_{\varepsilon\to 0} T_\varepsilon f(x), \quad x \in \mathbb{R}^n,$$

where T_ε is the truncation at level ε defined by

$$T_\varepsilon f(x) = \int_{|y|>\varepsilon} f(x-y)K(y)\, dy, \quad x \in \mathbb{R}^n,$$

and f is in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Here the kernel K is of class C^∞ off the origin, homogeneous of order $-n$ and with zero integral on the unit sphere

$$\{x \in \mathbb{R}^n : |x| = 1\}.$$

Let T_* be the maximal singular integral

$$T_* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|, \quad x \in \mathbb{R}^n.$$

A classical fact relating T_* and the standard Hardy-Littlewood maximal operator M is Cotlar's inequality, which reads

$$T_* f(x) \le C \left(M(Tf)(x) + Mf(x) \right), \quad x \in \mathbb{R}^n. \tag{1.1.1}$$

Combining this with the L^p estimates $\|Tf\|_p \le C \|f\|_p$ and $\|Mf\|_p \le C \|f\|_p$, $1 < p < \infty$ one gets $\|T_* f\|_p \le C \|f\|_p$, $1 < p < \infty$.

It was discovered in [MOV11] that if T is an even higher order Riesz transform, that is, if $K(x) = P(x)/|x|^{n+d}$, with P an even homogeneous polynomial of degree d, then one can get rid of the second term in the right hand side of (1.1.1), namely,

$$T_* f(x) \le C\, M(Tf)(x), \quad x \in \mathbb{R}^n. \tag{1.1.2}$$

Hence $\|T_* f\|_p \le C \|Tf\|_p$, $1 < p < \infty$, in this case. However, if T is an odd higher order Riesz transform, then (1.1.2) may fail and the right substitute turns out to be (see [Mat+10])

$$T_* f(x) \le C\, M^2(Tf)(x), \quad x \in \mathbb{R}^n, \tag{1.1.3}$$

where M^2 stands for the iteration of M.

Inequalities of the type (1.1.2) and (1.1.3) were first considered in relation to the David-Semmes problem (see [MOV11],[Mat+10] and [Ver11]) and later on were studied in the context of the Cauchy singular integral on Lipschitz graphs and C^1 curves by Girela-Sarrión in [Gir13]. Let Γ be either a Lipschitz graph or a closed chord-arc curve in the plane, let T be the Cauchy Singular Integral and M the Hardy-Littlewood maximal operator, both with respect to the arc-length measure, and let T_* be the maximal Cauchy Integral. Precise definitions will be given below. Girela-Sarrión showed in [Gir13] that the presence at a point z of the curve of a non-zero angle prevents (1.1.3), with x replaced by z, to hold. This agrees with the intuition that (1.1.3) should help in finding tangent lines, but suggests that it is a condition definetely stronger than the mere existence of tangents. It was also shown in [Gir13] that if Γ is a closed C^1 curve with the property that the modulus of continuity $\omega(z,\delta)$ of the unit tangent vector satisfies

$$\omega(z,\delta) \leq C\,\frac{1}{\log(\frac{1}{\delta})}, \quad z \in \Gamma, \quad \delta < 1/2, \tag{1.1.4}$$

then (1.1.3) holds with $x \in \mathbb{R}^n$ replaced by $z \in \Gamma$. Observe that condition (1.1.4) quantifies the absence of corners in a curve for which (1.1.3) holds. In this chapter we study the validity of inequality (1.1.3) in the context of chord-arc curves. A chord-arc curve is a rectifiable Jordan curve Γ in the plane with the property that there exists a positive constant C such that, given any two points $z_1, z_2 \in \Gamma$ one has

$$\ell(z_1, z_2) \leq C\,|z_1 - z_2|,$$

where $\ell(z_1,z_2)$ is the length of the shortest arc in Γ joining z_1 and z_2. Equivalently Γ is a bilipschitz image of the unit circle (see [Pom92], Theorem 7.9). Then Γ can be parametrized by a periodic function $\gamma\colon \mathbb{R} \to \Gamma$ of period T satisfying the bilipschitz condition

$$\frac{1}{L}\,|x-y| \leq |\gamma(x) - \gamma(y)| \leq L\,|x-y|, \quad x, y \in \mathbb{R}, \quad |x-y| \leq \frac{T}{2}, \tag{1.1.5}$$

for some positive constant L. We say, by slightly abusing language, that γ is a bilipschitz parametrization of Γ. One can take, for instance, the T-periodic extension of the arc-length parametrization of Γ with T being the length of Γ.

One can easily define the maximal Hardy-Littlewood operator and the Cauchy Integral on a chord-arc curve. Given $z \in \Gamma$ let $t \in \mathbb{R}$ be such that $z = \gamma(t)$. Set

$$\Gamma_{z,r} \coloneqq \gamma(\{\tau\colon |\tau - t| < r\}).$$

One should look at $\Gamma_{z,r}$ as "balls" of radius r centered at z adapted to the parametrization γ. Indeed, owing to the bilipschitz condition (1.1.5), each $\Gamma_{z,r}$ contains and is contained in a disc in Γ of radius comparable to r, for $r < T$. It will be more convenient to work with $\Gamma_{z,r}$ than with the euclidean discs $D(z,r) \cap \Gamma$, where $D(z,r)$ stands for the planar disc of center z and radius r.

Denote by μ the arc-length measure on Γ. For $f \in L^1(\Gamma,\mu)$ and $z \in \Gamma$, we define the Hardy-Littlewood maximal function on the curve Γ as

$$Mf(z) \coloneqq \sup_{r>0} \frac{1}{\mu(\Gamma_{z,r})} \int_{\Gamma_{z,r}} |f|\,d\mu.$$

The Cauchy Integral is defined as

$$\mathcal{C}f(z) = \text{p. v.}\,\frac{1}{\pi\, i}\int_{\Gamma}\frac{1}{w-z}\,f(w)\,dw \equiv \lim_{\varepsilon\to 0}\mathcal{C}_{\varepsilon}f(z), \quad z\in\Gamma,$$

where

$$\mathcal{C}_{\varepsilon}f(z) = \frac{1}{\pi i}\int_{\Gamma\setminus\Gamma_{z,\varepsilon}}\frac{f(w)}{w-z}dw$$

is the truncated Cauchy Integral at level ε. The maximal Cauchy Integral is

$$\mathcal{C}_{*}f(z) := \sup_{\varepsilon>0}\big|\mathcal{C}_{\varepsilon}f(z)\big|.$$

We remark that this definition of Cauchy Integral is slightly different from that of Cauchy transform used in the next chapter of the manuscript. In particular, in Chapter 2 we truncate with respect to standard euclidean balls. We hope that, being the other context more measure-theoretic, this will not cause confusion in the reader.

Our aim is to investigate under what conditions on Γ one has the inequality

$$\mathcal{C}_{*}f(z) \le C\,M^{2}(\mathcal{C}f)(z), \quad z\in\Gamma, \quad f\in L^{2}(\Gamma,\mu),$$

where C is a positive constant. Since we know that angles prevent the above inequality to hold, we need to require on Γ a condition that excludes them. One such a condition is asymptotic conformality. Given two points $z_1, z_2 \in \Gamma$ let $A(z_1, z_2)$ be the arc in Γ joining the two points and having smallest diameter (there is only one if the two points are sufficiently close). The Jordan curve Γ is said to be asymptotically conformal if, given a positive number δ there exists a positive ε, so that for any two points $z_1, z_2 \in \Gamma$ satisfying $|z_1 - z_2| < \varepsilon$ one has

$$|z_1 - z| + |z_2 - z| \le (1+\delta)|z_1 - z_2|, \quad z\in A(z_1, z_2).$$

Our main result reads as follows.

Theorem. *Let $\mathcal{C}$ be the Cauchy Integral on an asymptotically conformal chord-arc curve Γ and let γ be a bilipschitz parametrization of Γ. Then the estimate*

$$\mathcal{C}_{*}f(z) \le C\,M^{2}(\mathcal{C}f)(z), \quad z\in\Gamma, \quad f\in L^{2}(\Gamma,\mu), \tag{1.1.6}$$

holds if and only if there exists $C > 0$ such that

$$\big|\gamma(x+\varepsilon)+\gamma(x-\varepsilon)-2\gamma(x)\big| \le C\,\frac{\varepsilon}{|\log\varepsilon|}, \tag{1.1.7}$$

for each ε satisfying $0 < \varepsilon < T$ and for each $x \in \mathbb{R}$.

One should recall that condition (1.1.7) implies that γ is differentiable almost everywhere in the ordinary sense and the derivative is a function of vanishing mean oscillation (see [WZ59]). Therefore, for chord arc curves satisfying the background assumption of asymptotical conformality, inequality (1.1.6) is equivalent to the precise form of differentiability described in terms of second order differences in (1.1.7). Also notice that if γ is the arc-length parametrization of a C^1 curve, (1.1.4) implies (1.1.7), so that the Theorem generalizes Girela-Sarrión's result.

In Section 1.2 we prove a couple of Lemmas which allow to express condition (1.1.6) in an equivalent form in terms of a function related to the geometry of Γ. Section 1.3 is devoted to take care of a technical question, namely, that it is enough to estimate truncations at small enough levels. In Section 1.4 we prove the Theorem by means of three lemmas, one on them making the connection between the function carrying the geometrical information and the second difference condition (1.1.7). In Section 1.5 we present an example of a spiraling domain that enjoys the equivalent conditions in the Theorem but whose boundary is not of class C^1. This example justifies the efforts made in order to extend the condition (1.1.4) to a less regular case since new geometric behaviors can be detected.

Our terminology and notation are standard. We let C denote a constant independent of the relevant variables under consideration and which may vary at each occurrence. The notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \leq CB$. We write $A \gtrsim B$ if $B \lesssim A$. The disc centered at z of radius r is denoted by $D(z, r)$.

1.2 Two preliminary Lemmas

The beginning of the proof follows the ideas of [Gir13], so that we will be rather concise. Given a function $f \in L^1(\Gamma, \mu)$ we denote by $m_{\Gamma_{z,\varepsilon}}(f) = ⨍_{\Gamma_{z,\varepsilon}} f(w)\, d\mu(w)$ the mean of f on $\Gamma_{z,\varepsilon}$ with respect to the arc length measure μ. We let $K_{z,\varepsilon}$ denote the Cauchy kernel truncated at the point z at level ε, that is,

$$K_{z,\varepsilon}(w) = \frac{1}{\pi\, i} \frac{1}{w - z}\, \chi_{\Gamma \setminus \Gamma_{z,\varepsilon}}(w), \quad w \in \Gamma.$$

Set $g_{z,\varepsilon} = \mathcal{C}(K_{z,\varepsilon})$ and let $N > 1$ be a big number to be chosen later. Following [Gir13, p.673] we obtain the identity

$$-\mathcal{C}_\varepsilon f(z) = I_\varepsilon + II_\varepsilon + III_\varepsilon,$$

where

$$\begin{aligned} I_\varepsilon &:= \int_{\Gamma_{z,N\varepsilon}} \mathcal{C}f(w) \left(g_{z,\varepsilon}(w) - m_{\Gamma_{z,N\varepsilon}}(g_{z,\varepsilon})\right) dw, \\ II_\varepsilon &:= m_{\Gamma_{z,N\varepsilon}}(g_{z,\varepsilon}) \int_{\Gamma_{z,N\varepsilon}} \mathcal{C}f(w) dw, \\ III_\varepsilon &:= \int_{\Gamma \setminus \Gamma_{z,N\varepsilon}} \mathcal{C}f(w) g_{z,\varepsilon}(w) dw. \end{aligned} \tag{1.2.1}$$

Following closely the argument in [Gir13] one can prove that

$$\begin{aligned} |I_\varepsilon| &\leq C\, M^2(\mathcal{C}f)(z), \\ |II_\varepsilon| &\leq C\, M(\mathcal{C}f)(z), \end{aligned}$$

where the constant C does not depend on the choice of N. Since clearly $M(g) \leq M^2(g)$ for any g, we are left with the task of estimating III_ε. The next lemma provides an expression for III_ε in terms of a function encoding the smoothness of Γ. To state the lemma first we need to clarify the definition of a branch of the logarithm of $w - z$, as a function of w with $z \in \Gamma$ fixed, in an appropriate region.

Given $z \in \Gamma$ let Δ_z be a curve connecting z and ∞ in the unbounded component of $\mathbb{C} \setminus \Gamma$. Such curves exist and indeed we will construct a special one in Section 4 (under the additional assumption of asymptotic conformality). Hence $\mathbb{C} \setminus \Delta_z$ is a simply connected domain containing $\Gamma \setminus \{z\}$ and so there exists in $\mathbb{C} \setminus \Delta_z$ a branch of

$\log(w-z)$. In particular, if $z=\gamma(x)$ for some $x\in\mathbb{R}$, the expressions $\log(\gamma(x+\varepsilon)-\gamma(x))$ and $\log(\gamma(x-\varepsilon)-\gamma(x))$ make sense for $0<\varepsilon<T$.

Lemma 1.2.1. *Let Γ be a chord-arc curve and γ a bilipschitz parametrization of Γ. Let $z\in\Gamma$ and let x be a real number such that $\gamma(x)=z$. Then for almost every $w\in\Gamma\backslash\Gamma_{z,N\varepsilon}$ we have*

$$\mathcal{C}(K_{z,\varepsilon})(w)=\frac{1}{\pi^2(z-w)}\big[F(x,\varepsilon)+G_{z,\varepsilon}(w)\big],$$

where

$$F(x,\varepsilon)=\log(\gamma(x+\varepsilon)-\gamma(x))-\log(\gamma(x-\varepsilon)-\gamma(x))+\pi i$$

and

$$|G_{z,\varepsilon}(w)|\leq\frac{C\,\varepsilon}{|z-w|}. \tag{1.2.2}$$

Proof. Take $w\in\Gamma\backslash\Gamma_{z,N\varepsilon}$. Then

$$\begin{aligned}\mathcal{C}(K_{z,\varepsilon})(w)&=-\frac{1}{\pi^2}\lim_{\delta\to0}\int_{\Gamma\backslash(\Gamma_{w,\delta}\cup\Gamma_{z,\varepsilon})}\frac{1}{(\zeta-z)(\zeta-w)}\,d\zeta\\&=-\frac{1}{\pi^2}\frac{1}{w-z}\lim_{\delta\to0}\int_{\Gamma\backslash(\Gamma_{w,\delta}\cup\Gamma_{z,\varepsilon})}\left(\frac{1}{\zeta-w}-\frac{1}{\zeta-z}\right)d\zeta.\end{aligned}$$

Let $y\in\mathbb{R}$ with $\gamma(y)=w$. Then the latest integral in the above formula is

$$\begin{aligned}&\log\left(\gamma(y-\delta)-\gamma(y)\right)-\log\left(\gamma(x+\varepsilon)-\gamma(y)\right)+\log\left(\gamma(x-\varepsilon)-\gamma(y)\right)\\&-\log\left(\gamma(y+\delta)-\gamma(y)\right)-\Big(\log\left(\gamma(y-\delta)-\gamma(x)\right)\\&-\log\left(\gamma(x+\varepsilon)-\gamma(x)\right)+\log\left(\gamma(x-\varepsilon)-\gamma(x)\right)-\log\left(\gamma(y+\delta)-\gamma(x)\right)\Big).\end{aligned}$$

Assume that γ is differentiable at the point y and the derivative $\gamma'(y)$ does not vanish. Then we have that

$$\lim_{\delta\to0}\Big(\log\left(\gamma(y-\delta)-\gamma(y)\right)-\log\left(\gamma(y+\delta)-\gamma(y)\right)\Big)=\pi i,$$

because the curve Δ_w lies in the unbounded component of $\mathbb{C}\setminus\Gamma$, and then to the right hand side of Γ, oriented according to the parametrization γ. Taking the limit as δ goes to 0 we obtain

$$\begin{aligned}\mathcal{C}(K_{z,\varepsilon})(w)=-\frac{1}{\pi^2}\frac{1}{w-z}\Big(&\big(\log(\gamma(x+\varepsilon)-\gamma(x))-\log(\gamma(x-\varepsilon)-\gamma(x))+\pi i\big)\\&-\big(\log(\gamma(x+\varepsilon)-\gamma(y))-\log(\gamma(x-\varepsilon)-\gamma(y))\big)\Big).\end{aligned}$$

Define

$$G_{z,\varepsilon}(w)=\log\left(\gamma(x-\varepsilon)-\gamma(y)\right)-\log\left(\gamma(x+\varepsilon)-\gamma(y)\right).$$

It remains to show the decay inequality (1.2.2). According to the choice of Δ_w we have a well defined branch of $\log(\gamma(x+t)-w)$, $-\varepsilon<t<\varepsilon$. Thus

$$G_{z,\varepsilon}(w)=-\int_{-\varepsilon}^{\varepsilon}\frac{d}{dt}\log(\gamma(x+t)-w)\,dt=-\int_{-\varepsilon}^{\varepsilon}\frac{\gamma'(x+t)}{\gamma(x+t)-w}\,dt. \tag{1.2.3}$$

Since $w = \gamma(y) \in \Gamma \setminus \Gamma_{z,N\varepsilon}$, we have $y \notin (x - N\varepsilon, x + N\varepsilon)$ and so

$$|w - z| = |\gamma(y) - \gamma(x)| \geq \frac{|y - x|}{L} \geq \frac{N\varepsilon}{L},$$

which gives , taking $N \geq 2L^2$,

$$\begin{aligned} |w - \gamma(x + t)| &\geq |w - z| - |\gamma(x) - \gamma(x + t)| \\ &\geq \frac{|w - z|}{2} + \frac{N\varepsilon}{2L} - L\varepsilon \\ &\geq \frac{|w - z|}{2}. \end{aligned}$$

Hence, by (1.2.3),

$$|G_{z,\varepsilon}(w)| \leq \int_{-\varepsilon}^{\varepsilon} \frac{|\gamma'(x + t)|}{|\gamma(x + t) - w|}\, dt \leq \frac{4L\varepsilon}{|w - z|}.$$ □

Lemma 1.2.2. *Let Γ be a chord-arc curve and γ a bilipschitz parametrization of Γ. Then the inequality*

$$\mathcal{C}_*(f)(z) \leq C\, M^2(\mathcal{C}f)(z), \quad z \in \Gamma, \quad f \in L^2(\Gamma, \mu), \tag{1.2.4}$$

is equivalent to

$$|F(x, \varepsilon)||\log(\varepsilon)| \leq C \quad 0 < \varepsilon < T, \quad x \in \mathbb{R}. \tag{1.2.5}$$

Proof. Assume that (1.2.5) holds. Then by Lemma 1.2.1

$$\begin{aligned} III_\varepsilon &= \int_{\Gamma \setminus \Gamma_{z,N\varepsilon}} \mathcal{C}f(w)\, \mathcal{C}(K_{z,\varepsilon})(w)\, dw \\ &= \frac{F(x, \varepsilon)}{\pi^2} \int_{\Gamma \setminus \Gamma_{z,N\varepsilon}} \frac{\mathcal{C}f(w)}{z - w}\, dw + \frac{1}{\pi^2} \int_{\Gamma \setminus \Gamma_{z,N\varepsilon}} \mathcal{C}f(w)\, \frac{G_{z,\varepsilon}(w)}{z - w}\, dw \\ &= F(x, \varepsilon)\, IV_\varepsilon + V_\varepsilon, \end{aligned}$$

where the last identity is a definition of the terms IV_ε and V_ε. One can break the domain of integration in the integrals in IV_ε and V_ε into a union of dyadic annuli

$$A_j = \gamma\{y \in \mathbb{R} \colon N\varepsilon\, 2^j < |y - x| \leq N\varepsilon\, 2^{j+1}\}, \quad j = 0, 1, \ldots$$

then perform standard estimates and apply (1.2.2) to get, thanks to the quadratic decay of the integrand,

$$|V_\varepsilon| \leq C\, M(\mathcal{C}f)(z). \tag{1.2.6}$$

For IV_ε one only has a first order decay, which gives

$$|IV_\varepsilon| \leq C \left| \log\left(\frac{NL}{\varepsilon}\right) \right| M(\mathcal{C}f)(z),$$

thus completing the proof of the sufficient condition.

Assume now (1.2.4). Recalling that $III_\varepsilon = F(x, \varepsilon)\, IV_\varepsilon + V_\varepsilon$ and (1.2.6), we obtain

$$\left|F(x, \varepsilon)\, IV_\varepsilon\right| \leq C\, M^2(\mathcal{C}f)(z), \quad z \in \Gamma, \quad f \in L^2(\Gamma, \mu). \tag{1.2.7}$$

The Cauchy Singular Integral operator T is an isomorphism of $L^2(\Gamma, \mu)$ onto itself. This is proved in Lemma 1 of [Gir13, p. 661] for Lipschitz graphs, and the same proof works in our context. Thus (1.2.7) can be rewritten as

$$\left|F(x,\varepsilon)\int_{\Gamma\setminus\Gamma_{z,N\varepsilon}}\frac{g(w)}{z-w}\,dw\right|\leq C\,M^2(g)(z),\quad z\in\Gamma,\quad g\in L^2(\Gamma,\mu). \tag{1.2.8}$$

To simplify the notation take $x=0=\gamma(x)$. Assume first that $0<\varepsilon<1$. Apply (1.2.8) with g the characteristic function of $\gamma((\varepsilon^n,\varepsilon))$, where n is a large integer to be chosen. Then

$$|F(0,\varepsilon)|\left|\int_{\varepsilon^n}^{\varepsilon}\frac{\gamma'(t)}{\gamma(t)}\,dt\right|\leq C$$

and

$$\begin{aligned}\left|\int_{\varepsilon^n}^{\varepsilon}\frac{\gamma'(t)}{\gamma(t)}\,dt\right| &= |\log(\gamma(\varepsilon))-\log(\gamma(\varepsilon^n))|\\ &\geq |\log(|\gamma(\varepsilon)|)-\log(|\gamma(\varepsilon^n)|)|\\ &\geq \log\left(\frac{1}{L^2\,\varepsilon^{n-1}}\right)\\ &\geq -2\log(L)+(n-2)\log\left(\frac{1}{\varepsilon}\right)+\log\left(\frac{1}{\varepsilon}\right)\\ &\geq |\log(\varepsilon)|\end{aligned}$$

provided $n=n(\varepsilon)$ is large enough so that $-2\log(L)+(n-2)\log(1/\varepsilon)\geq 0$. Therefore (1.2.5) follows in this case.

If $1\leq\varepsilon<T$ then we take as g the characteristic function of $\gamma((\varepsilon^{-n},\varepsilon))$. In this case we get

$$\left|\int_{\varepsilon^{-n}}^{\varepsilon}\frac{\gamma'(t)}{\gamma(t)}\,dt\right|\geq -2\log(L)+n\log(\varepsilon)+\log(\varepsilon)\geq|\log(\varepsilon)|$$

provided n is chosen so that $-2\log(L)+n\log(\varepsilon)\geq 0$. □

1.3 Reduction to estimating truncations at small levels

In this section we reduce the proof of (1.1.6) to estimating the truncations $\mathcal{C}_\varepsilon f$ for small ε. In the previous section we showed that the estimate of $\mathcal{C}_\varepsilon f$ can be reduced to that of the term III_ε in (1.2.1).

Lemma 1.3.1. *If ε_0 is a given positive number, then there exists a large positive number $N=N(L)$ so that*

$$\left|\int_{\Gamma\setminus\Gamma_{z,N\varepsilon}}\mathcal{C}f(w)\,g_{z,\varepsilon}(w)dw\right|\leq C\,M(\mathcal{C}f)(z),\quad z\in\Gamma,\quad \varepsilon_0<\varepsilon,$$

for a positive constant $C=C(\varepsilon_0,L)$.

The small number ε_0 will be chosen in the next section.

Proof. Recall that

$$\begin{aligned}
g_{z,\varepsilon}(w) &= \mathcal{C}(K_{z,\varepsilon})(w) \\
&= -\frac{1}{\pi^2}\,\text{p. v.}\int_{\Gamma\setminus\Gamma_{z,\varepsilon}} \frac{1}{(\zeta-w)(\zeta-z)}\,d\zeta \\
&= -\frac{1}{\pi^2}\frac{1}{w-z}\,\text{p. v.}\int_{\Gamma\setminus\Gamma_{z,\varepsilon}} \left(\frac{1}{\zeta-w}-\frac{1}{\zeta-z}\right)d\zeta \\
&= -\frac{1}{\pi^2}\frac{1}{w-z}\,\text{p. v.}\int_{\Gamma\setminus\Gamma_{z,\varepsilon}} \frac{1}{\zeta-w}\,d\zeta + \frac{1}{\pi^2}\frac{1}{w-z}\,\text{p. v.}\int_{\Gamma\setminus\Gamma_{z,\varepsilon}} \frac{1}{\zeta-z}\,d\zeta \\
&= h(w)+k(w),
\end{aligned}$$

where in the last identity we defined $h(w)$ and $k(w)$.

Applying the bilipschitz character of γ we conclude that

$$|k(w)| \leq \frac{1}{\pi^2}\frac{L^2}{N\,\varepsilon_0^2}\,\text{length}(\Gamma), \quad w\in\Gamma\setminus\Gamma_{z,N\varepsilon}, \quad \varepsilon_0<\varepsilon\,. \tag{1.3.1}$$

The estimate of $h(w)$ is a little trickier. We have

$$h(w) = -\frac{1}{\pi^2}\frac{1}{w-z}\,\text{p. v.}\int_{\Gamma}\frac{1}{\zeta-w}\,d\zeta + \frac{1}{\pi^2}\frac{1}{w-z}\,\text{p. v.}\int_{\Gamma_{z,\varepsilon}}\frac{1}{\zeta-w}\,d\zeta.$$

A simple application of Cauchy's Theorem gives that, if Γ has a tangent at w,

$$\text{p. v.}\int_{\Gamma}\frac{1}{\zeta-w}\,d\zeta = \pi i.$$

As before, the bilipschitz character of γ yields

$$|w-z| \geq \frac{N\,\varepsilon}{L}, \quad w\in\Gamma\setminus\Gamma_{z,N\varepsilon}$$

and

$$|w-\zeta| \geq |w-z|-|z-\zeta| \geq \varepsilon\left(\frac{N}{L}-L\right), \quad w\in\Gamma\setminus\Gamma_{z,N\varepsilon} \quad \zeta\in\Gamma_{z,\varepsilon}.$$

Choose N so that $N/L-L\geq 1$. Then

$$|w-\zeta| \geq \varepsilon, \quad w\in\Gamma\setminus\Gamma_{z,N\varepsilon} \quad \zeta\in\Gamma_{z,\varepsilon}.$$

Gathering all the previous estimates we finally get

$$|h(w)| \leq \frac{1}{\pi}\frac{L}{N\,\varepsilon_0}+\frac{1}{\pi^2}\frac{\text{length}(\Gamma)}{\varepsilon_0}, \quad w\in\Gamma\setminus\Gamma_{z,N\varepsilon}, \quad \varepsilon_0<\varepsilon\,. \tag{1.3.2}$$

Hence (1.3.1) and (1.3.2) yield

$$|g_{z,\varepsilon}(w)| \leq C, \quad w\in\Gamma\setminus\Gamma_{z,N\varepsilon}, \quad \varepsilon_0<\varepsilon,$$

where $C=C(\varepsilon_0,N,L,\text{length}(\Gamma))$ is a constant depending on ε_0,N,L and $\text{length}(\Gamma)$.

Therefore

$$\left|\int_{\Gamma\setminus\Gamma_{z,N\varepsilon}} \mathcal{C}f(w)\,g_{z,\varepsilon}(w)dw\right| \leq C\int_{\Gamma}|\mathcal{C}f(w)|\,d\mu(w) \leq C\,\text{length}(\Gamma)M(\mathcal{C}f)(z),$$

which completes the proof of the lemma. □

1.4 The proof of the Theorem

For $z \neq 0$ let $\mathrm{Arg}(z)$ denote the principal argument of z, so that $0 \leq \mathrm{Arg}(z) < 2\pi$.

Lemma 1.4.1. *Given $\alpha > 0$ there exists a positive number $\varepsilon_0 = \varepsilon_0(L)$ with the following property. Assume that $0 < \varepsilon_1 \leq \varepsilon_0$, $\varepsilon_1/2 < \varepsilon \leq \varepsilon_1$ and that for a fixed $x \in \mathbb{R}$ we have $\gamma(x) = 0$. If $\gamma(x-\tau)$, $\tau > 0$, satisfies*

$$\frac{\varepsilon_1}{2L} < |\gamma(x-\tau)| < L\,\varepsilon_1,$$

then, for some θ such that $\gamma(x-\tau) = |\gamma(x-\tau)|e^{i\theta}$, we have

$$\big|\theta - \big(\mathrm{Arg}(\gamma(x+\varepsilon)) + \pi\big)\big| < \alpha.$$

Proof. Consider the triangle with vertices $0, \gamma(x-\tau)$ and $\gamma(x+\varepsilon)$ and side lengths $A = |\gamma(x-\tau)|$, $B = |\gamma(x+\varepsilon)|$ and $C = |\gamma(x+\varepsilon) - \gamma(x-\tau)|$. By the cosine Theorem

$$C^2 = A^2 + B^2 - 2AB\cos(\phi),$$

where ϕ is the angle opposite to the side C. In other terms

$$1 + \cos(\phi) = \frac{(A+B-C)(A+B+C)}{2AB}.$$

By asymptotic conformality, given $\delta > 0$ there exists $\eta_0 > 0$ such that $C = |\gamma(x+\varepsilon) - \gamma(x-\tau)| < \eta_0$ implies $A + B \leq (1+\delta)C$. The bilipschitz property of γ (1.1.5) yields $\varepsilon_1/2L^2 \leq \tau \leq L^2\,\varepsilon_1$. Hence

$$1 + \cos(\phi) \leq \delta L^4 \frac{(\varepsilon_1 + \tau)^2}{\varepsilon_1\,\tau} \leq 2\delta L^6(1+L^2)^2.$$

Taking $\theta = \mathrm{Arg}(\gamma(x+\varepsilon)) + \phi$ we see that $|\theta - (\mathrm{Arg}(\gamma(x+\varepsilon)) + \pi)| < \alpha$ provided δ is small enough. Since

$$|\gamma(x+\varepsilon) - \gamma(x-\tau)| \leq L(\varepsilon + \tau) \leq \varepsilon_0\,L(1+L^2),$$

one has to choose ε_0 so that $\varepsilon_0\,L(1+L^2) \leq \eta_0$, which shows the correct dependence of ε_0 and completes the proof of the Lemma. □

Given a point $z \in \Gamma$ we want now to construct a special Jordan arc Δ_z connecting z to ∞ in the complement of Γ. Assume, without loss of generality, that $z = 0$. Take $x \in \mathbb{R}$ with $\gamma(x) = 0$. Let ε_0 be the number given in the preceding Lemma and define, for $j = 0, 1, 2, \ldots$, a polar rectangle by

$$R_j = \Big\{ w = |w|e^{i\theta} \colon \frac{\varepsilon_0}{2^{j+1}L} < |w| < \frac{\varepsilon_0\,L}{2^j} \quad \text{and} \quad \Big|\theta - \mathrm{Arg}\Big(\gamma\Big(x + \frac{\varepsilon_0}{2^j}\Big)\Big) + \pi\Big| < \alpha \Big\}.$$

Applying Lemma 1.4.1 with $\varepsilon = \varepsilon_1 = \varepsilon_0/2^j$ we conclude that

$$\{\gamma(x-\tau) \colon 0 < \tau\} \cap \Big\{ w \colon \frac{\varepsilon_0}{2^{j+1}L} < |w| < \frac{\varepsilon_0\,L}{2^j} \Big\} \subset R_j.$$

We need to introduce another polar rectangle

$$S_j = R_j \cap \Big\{ w \colon \frac{\varepsilon_0\,L}{2^{j+1}} < |w| \Big\}, \quad j = 0, 1, 2, \ldots$$

We define inductively $\Delta_z = \Delta_0$ on S_j by just requiring that the Jordan arc $\Delta_0 \cap \overline{S_j}$ lies in the unbounded component of the complement of Γ, $\overline{S_j}$ being the closure of S_j. We then connect $\Delta_0 \cap \overline{S_0}$ with ∞ by a Jordan arc in the complement of Γ, with the only precaution of not reentering the disc $D(0, \varepsilon_0)$ once Δ_0 has left it.

It is worth pointing out that the axis of two consecutive polar rectangles R_j and R_{j+1} make an angle less than α. This follows by the defining property of ε_0 (see the proof of Lemma 1.4.1).

Lemma 1.4.2.

$$\log(\gamma(x-\varepsilon)) - \pi i = \log(-\gamma(x-\varepsilon)), \quad x \in \mathbb{R}, \quad 0 < \varepsilon \leq \varepsilon_0 \,.$$

Proof. We know that

$$\log(\gamma(x-\varepsilon)) - \pi i = \log(-\gamma(x-\varepsilon)) + 2\pi m i \tag{1.4.1}$$

for some integer m. Our goal is to compute the difference

$$\log(\gamma(x-\varepsilon)) - \log(-\gamma(x-\varepsilon))$$

by the integral

$$\int_\varsigma \frac{1}{z}\, dz,$$

where ς is an appropriately chosen Jordan arc connecting $-\gamma(x-\varepsilon)$ to $\gamma(x-\varepsilon)$ in the complement of Δ_0.

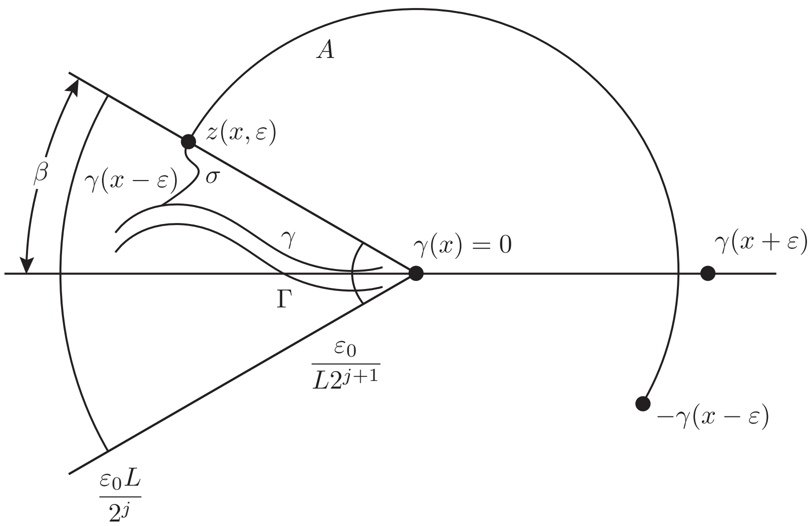

FIGURE 1.1: The curve ς

Assume that $\varepsilon_0 / 2^{j+1} < \varepsilon \leq \varepsilon_0 / 2^j$, for some non-negative integer j. Define N as the smallest integer satisfying

$$\frac{L\,\varepsilon_0}{2^{j+N}} \leq \frac{\varepsilon_0}{L 2^{j+1}}.$$

This is equivalent to $L^2 \leq 2^{N-1}$ and so N depends only on L. Hence $R_k \subset D(0, \varepsilon_0 / L\, 2^{j+1})$, $k \geq j+N$, and, in particular, R_k, $k \geq j+N$, does not intersect the circumference $\partial D(0, |\gamma(x-\varepsilon)|)$.

The angle between the axis of the polar rectangle R_{j+l} and that of R_j is not greater than $l\alpha \leq N\alpha$, $l = 1, 2, ..., N-1$. Set $\beta = N\alpha$, so that β can be as small as desired by taking $\alpha = \alpha(L)$ appropriately. We conclude that

$$R_{j+l} \subset \{w\colon w = |w|e^{i\theta} \text{ with } \big|\theta - \operatorname{Arg}\big(\gamma(x+\varepsilon)+\pi\big)\big| < \beta\}, \quad l = 1,2,\dots,N-1.$$

We are now ready to define the Jordan arc ς. Let $z(x,\varepsilon)$ be the point at the intersection of the circumference $\partial D(0,|\gamma(x-\varepsilon)|)$ and the ray

$$\{w\colon w = |w|e^{i\theta} \text{ with } \theta = \operatorname{Arg}\big(\gamma(x+\varepsilon)+\pi\big) - \beta\}.$$

Let A stand for the arc in $\partial D(0,|\gamma(x-\varepsilon)|)$ having $-\gamma(x-\varepsilon)$ as initial point and $z(x,\varepsilon)$ as end point (counterclockwise oriented).

There exists a rectifiable Jordan arc σ joining the points $z(x,\varepsilon)$ and $\gamma(x-\varepsilon)$ in the bounded component of the complement of Γ with the property that

$$\operatorname{length}(\sigma) \le C\,|z(x,\varepsilon) - \gamma(x-\varepsilon)|.$$

This can be seen readily as follows. Set $\tilde\gamma(e^{ix}) = \gamma(x)$, $x \in \mathbb{R}$. Then $\tilde\gamma$ is a bilipschitz homeomorphism between $\mathbb{T}$ and Γ and thus can be extended to a global bilipschitz homeomorphism of the plane onto itself (see [Tuk80],[Tuk81]). The existence of the arc σ is then easily proved by transferring the question via the extended bilipschitz homeomorphism.

Define $\varsigma = A \cup \sigma$, oriented as already specified. Note that ς lies in the complement of Δ_0, by the previous discussion, in particular, the definition of N and β. Therefore

$$\log(\gamma(x-\varepsilon)) - \log(-\gamma(x-\varepsilon)) = \int_\varsigma \frac{1}{z}\,dz.$$

On one hand we have

$$\int_A \frac{1}{z}\,dz = \pi i + O(\beta)$$

and on the other hand

$$\Big|\int_\sigma \frac{1}{z}\,dz\Big| \le \frac{C\,|z(x,\varepsilon)-\gamma(x-\varepsilon)|}{|\gamma(x-\varepsilon)|} \le C\,\beta = O(\beta).$$

If β is small enough so that $O(\beta) < \pi$, then, by (1.4.1), we get that $m = 0$, and the lemma is proved. □

We need a final lemma, which concludes the proof of the Theorem.

Lemma 1.4.3. *Let Γ be an asymptotically conformal chord-arc curve and let γ be a bilipschitz parametrization of Γ (in the sense of (1.1.5)). Then there exists a constant $C > 1$ and a positive number ε_0 such that*

$$C^{-1}\,\frac{|\gamma(x+\varepsilon)+\gamma(x-\varepsilon)-2\gamma(x)|}{\varepsilon} \le |F(x,\varepsilon)| \le C\,\frac{|\gamma(x+\varepsilon)+\gamma(x-\varepsilon)-2\gamma(x)|}{\varepsilon}, \tag{1.4.2}$$

for $x \in \mathbb{R}$ and $0 < \varepsilon < \varepsilon_0$.

Proof. Without loss of generality assume that $\gamma(x) = 0$. Let ε_0 be the small number provided by Lemma 1.4.1. By the construction of the arc Δ_0 described in the proof of Lemma 1.4.1 we have that the segment joining $-\gamma(x-\varepsilon)$ and $\gamma(x+\varepsilon)$ lies in the complement of Δ_0. We have, by Lemma 1.4.2,

$$\begin{aligned} F(x,\varepsilon) &= \log\big(\gamma(x+\varepsilon)\big) - \log\big(\gamma(x-\varepsilon)\big) + \pi i \\ &= \log\big(\gamma(x+\varepsilon)\big) - \log\big(-\gamma(x-\varepsilon)\big) \end{aligned}$$

and so

$$\begin{aligned} F(x,\varepsilon) &= \int_0^1 \frac{d}{dt}\log\big(-\gamma(x-\varepsilon)+t(\gamma(x+\varepsilon)+\gamma(x-\varepsilon))\big)\,dt \\ &= \int_0^1 \frac{\gamma(x+\varepsilon)+\gamma(x-\varepsilon)}{-\gamma(x-\varepsilon)+t(\gamma(x+\varepsilon)+\gamma(x-\varepsilon))}\,dt. \end{aligned}$$

Set, to simplify the notation, $a=-\gamma(x-\varepsilon)$, $b=\gamma(x+\varepsilon)$ and let θ denote the angle between a and b. By Lemma 1.4.1 we know that θ is as small as we wish. In particular we can assume that $\cos\theta \geq 1/2$. Thus, using the cosine Theorem,

$$\begin{aligned} |a+t(b-a)|^2 &= (1-t)^2|a|^2 + t^2|b|^2 + 2(1-t)t|a||b|\cos\theta \\ &\geq \frac{1}{2}\big((1-t)|a|+t|b|\big)^2 \geq \frac{\varepsilon^2}{2L^2}, \end{aligned}$$

and

$$|F(x,\varepsilon)| \leq \frac{\sqrt{2}L}{\varepsilon}|\gamma(x+\varepsilon)+\gamma(x-\varepsilon)|,$$

which is the upper estimate in (1.4.2).

For the lower estimate we set $z_t = -\gamma(x-\varepsilon)+t(\gamma(x+\varepsilon)+\gamma(x-\varepsilon))$. Since $\mathrm{Re}(z_t) \geq |z_t|/2$ and $|z_t| \leq 2L\,\varepsilon$

$$\begin{aligned} \Big|\int_0^1 \frac{1}{z_t}\,dt\Big| \geq \mathrm{Re}\int_0^1 \frac{1}{z_t}\,dt &= \int_0^1 \frac{\mathrm{Re}(z_t)}{|z_t|^2}\,dt \\ &\geq \int_0^1 \frac{1}{2|z_t|}\,dt \geq \frac{1}{4L\,\varepsilon}. \end{aligned}$$

To complete the proof of the Theorem one only needs to combine Lemmas 1.2.2, 1.3.1 and 1.4.3. □

Remark. Let $a=\gamma(x)-\gamma(x-\varepsilon)$, $b=\gamma(x+\varepsilon)-\gamma(x)$ and let $\alpha(x,\varepsilon)$ be the angle spanned by a and b. For a bilipschitz parametrization γ such that

$$c\,|x-y| \leq |\gamma(x)-\gamma(y)| \leq C\,|x-y|, \quad x,y\in\mathbb{R}, \quad |x-y|\leq \frac{T}{2},$$

we have the estimate

$$|\gamma(x+\varepsilon)+\gamma(x-\varepsilon)-2\gamma(x)|^2 \leq 2C^2\varepsilon^2 - 2c^2\varepsilon^2\cos\alpha(x,\varepsilon).$$

So, in the general case, we can guarantee just a linear decay of the second finite difference $|\gamma(x+\varepsilon)+\gamma(x-\varepsilon)-2\gamma(x)|$ and the logarithmic condition (1.1.7) gives informations about the local behavior of the best constants c and C around x and about the decay of $\alpha(x,\varepsilon)$ for ε small. This remark will be useful in the next section.

1.5 An example

In this section we provide an example of curve γ which is not C^1 but for which the improved Cotlar's inequality (1.1.6) holds. The curve will be constructed in a recursive way and will be parametrized by arc-length. Without loss of generality, we will focus on defining a curve which is not closed. Indeed, possibly by connecting the ends of this curve in a smooth way, we can reduce to the same environment of the previous sections. The idea in the construction of the example is that the curve should resemble a suitable spiraling sequence of smoothened corners of decreasing aperture.
Let $0 < \alpha < \pi/2$. Let $F_\alpha\colon [0,1] \to \mathbb{R}$ be the function with support in $[1/4, 3/4]$ which is linear in $[1/4, 1/2]$ and $[1/2, 3/4]$ with slope $\tan\alpha$ in $[1/4, 1/2]$ and $-\tan\alpha$ in $[1/2, 3/4]$. In other words

$$F_\alpha(t) := \max\Big\{0, \Big(\frac{1}{4} - \Big|t - \frac{1}{2}\Big|\Big) \tan\alpha\Big\}.$$

Let $\xi > 0$. For $t \in \mathbb{R}$ we define the function

$$\eta_\xi(t) := \eta\Big(\frac{t}{\xi}\Big)\frac{1}{\xi},$$

where η is a smooth, even and positive function such that $\operatorname{supp}\eta \subset [-1,1]$ and $\int \eta(t)dt = 1$. For $0 < \xi < 1/100$ we define the regularized function

$$\lambda_\alpha := F_\alpha * \eta_\xi.$$

We will call the curve $\Lambda_\alpha := \big(t, \lambda_\alpha(t)\big)_{t\in[0,1]}$ *α-patch.*

An α-patch has the following properties:

- Λ_α is the graph of a function $\lambda_\alpha\colon [0,1] \to \mathbb{R}$ which is symmetric around $1/2$.
- if we denote by $[a,b]$ the segment joining the points $a, b \in \mathbb{R}^2$, then Λ_α contains the segments $I_\alpha := [(0,0), (1/4-\xi, 0)]$, $II_\alpha := [(1/4+\xi, \xi\tan\alpha), (1/2-\xi, (1/4-\xi)\tan\alpha)]$, $III_\alpha := [(1/2+\xi, (1/4-\xi)\tan\alpha), (3/4-\xi, \xi\tan\alpha)]$ and $IV_\alpha := [(3/4+\xi, 0), (1, 0)]$. We denote by C^i_α, $i = 1, 2, 3$ the remaining three non-affine parts of the graph. Precisely, C^1_α joins the segments I_α and II_α, C^2_α the segments II_α and III_α and C^3_α the segments III_α and IV_α.
- the function λ_α is convex on the intervals below C^1_α and C^3_α and concave on the interval below C^2_α.

The idea is that the α-patch is a smoothened corner, as shown in Figure 2.

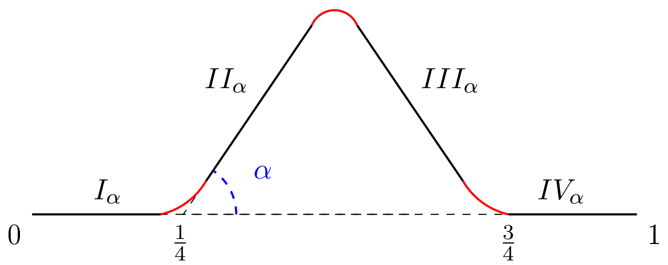

FIGURE 1.2: An α-patch

Remark 1. Let us denote by $\tau(\alpha)$ the difference between the length of the (non-smoothened) graph of F_α and the length of Λ_α. For what follows, we need to estimate its behavior for small values of α. It suffices to observe that

$$\begin{aligned}\tau(\alpha) &:= \text{length}(F_\alpha) - \text{length}(\Lambda_\alpha)\\ &= \int_0^1 \left(\sqrt{1+|f'_\alpha * \eta_\xi|^2(t)}\right) - \left(\sqrt{1+|f'_\alpha|^2(t)}\right) dt \\ &= \int_0^1 \frac{|f'_\alpha * \eta_\xi|^2(t) - |f'_\alpha|^2(t)}{\left(\sqrt{1+|f'_\alpha * \eta_\xi|^2(t)}\right) + \left(\sqrt{1+|f'_\alpha|^2(t)}\right)} dt \le 2\|f'_\alpha\|_\infty = 2\tan\alpha.\end{aligned} \tag{1.5.1}$$

Definition of the curve Γ

Let $\alpha_j := 1/j$ for $j = 1, 2, \dots$ positive integer. For the sake of notational convenience we replace the subscript α_j by j; for instance, we write Λ_j for Λ_{α_j}, I_j for $I_{\alpha_j}, \dots, IV_j$ for IV_{α_j} and C^i_j for $C^i_{\alpha_j}$. Moreover, $\tau_j := \tau(\alpha_j)$. Now we can define Γ according to the following recursive steps:

- $\Gamma_1 := \Lambda_1$.
- We would like to glue on II_1 an appropriate rescaled, translated and rotated copy $\tilde{\Lambda}_2$ of Λ_2. The angle of rotation is α_1. The scaling factor and the translation are chosen so that the origin of $\tilde{\Lambda}_2$ is $(1/4, 0)$ and the end is $\big(1/2, (\tan\alpha)/4\big)$. Denote by $\widetilde{II}_2$ the image of II_2 via the same affinity which maps Λ_2 to $\tilde{\Lambda}_2$; let us use the tilde to denote the images of the other parts of the patch via the same map, too. Delete the segment II_1 from Λ_1 and add $\tilde{\Lambda}_2$. Now the endings of $\tilde{\Lambda}_2$ should be deleted in order to make a connection with Λ_1. The precise expression for the second step curve is

$$\Gamma_2 := \big((\Lambda_1 \setminus II_1) \cup \tilde{\Lambda}_2\big) \setminus \big((\tilde{I}_2 \cup \widetilde{IV}_2) \setminus II_1\big).$$

- given Γ_n, which is a "gluing" of affine copies $\tilde{\Lambda}_j$ of Λ_j for $j \in \{1, \dots, n\}$, where $\widetilde{II}_n$ is the image of II_j under the same affinity which maps Λ_j to $\tilde{\Lambda}_j$, we define

$$\Gamma_{n+1} := ((\tilde{\Lambda}_n \setminus \widetilde{II}_n) \cup \tilde{\Lambda}_{n+1}) \setminus ((\tilde{I}_{n+1} \cup \widetilde{IV}_{n+1}) \setminus \widetilde{II}_n),$$

  where $\tilde{\Lambda}_{n+1}$ is an re-scaled copy of Λ_{n+1} rotated by an angle $\sum_{j=1}^{n+1} \alpha_j$ whose vertices coincide with the images of $(1/4, 0)$ and $\big(1/2, \tan\alpha/4\big)$ via the transformation of the plain that sends Λ_n to $\tilde{\Lambda}_n$.

Then, $\{\Gamma_n\}_n$ converges in the Hausdorff distance (a similar case is presented, for example, in [Fal90]) and we can simply define $\Gamma := \lim_n \Gamma_n$. Let us now state an

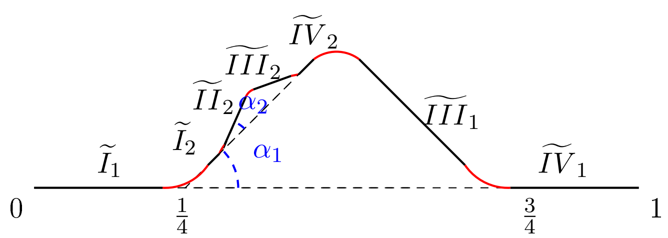

FIGURE 1.3: The second step in the construction of the curve Γ

estimate that we will use in what follows.

Lemma 1.5.1. *Given $0 < \alpha < \pi/2$ and $z_1, z_2 \in \Lambda_\alpha$, we have*

$$\ell(z_1, z_2) \le \frac{|z_1 - z_2|}{\cos\alpha}, \tag{1.5.2}$$

where $\ell(z_1, z_2)$ denotes the length of the arc of Λ_α joining z_1 and z_2.

Proof. Let $t_1 := \lambda_\alpha^{-1}(z_1)$ and $t_2 := \lambda_\alpha^{-1}(z_2)$. We have $|t_1 - t_2| \le |z_1 - z_2|$. Moreover, because of the way we constructed Λ_α, we have that $|\lambda'_\alpha(t)| \le \tan\alpha$ for every $t \in [0,1]$. Collecting all these observations,

$$\begin{aligned}\ell(z_1, z_2) &= \int_{t_1}^{t_2} \sqrt{1 + |\lambda'_\alpha(t)|^2} dt \le \int_{t_1}^{t_2} \sqrt{1 + |\tan\alpha|^2} dt \\ &= |t_2 - t_1|\sqrt{1 + |\tan\alpha|^2} = \frac{|t_2 - t_1|}{\cos\alpha} \le \frac{|z_2 - z_1|}{\cos\alpha}.\end{aligned}$$

□

Remark 2. Notice that the inequality (1.5.2) keeps holding for a scaling of Λ_α, in particular for the $\tilde\Lambda_j$, $j \in \mathbb{N}$.

Let us define $L_1 = 1/2$ and, for $n > 1$,

$$L_n := 2^{-2n+1}\Big(\prod_{j=1}^{n-1} \cos\alpha_j\Big)^{-1},$$

which is half of the diameter of the rescaled patch $\tilde\Lambda_n$ in the construction of the curve Γ. Indeed, some trigonometry gives

$$L_1 = \frac{1}{2}, L_2 = \frac{1}{2}\Big(\frac{1}{2}L_1\frac{1}{\cos\alpha_1}\Big), L_3 = \frac{1}{2}\Big(\frac{1}{2}L_2\frac{1}{\cos\alpha_2}\Big), \cdots, L_n = \frac{1}{2}\Big(\frac{1}{2}L_{n-1}\frac{1}{\cos\alpha_{n-1}}\Big).$$

Observe that the definition of L_n does not depend on α_n because the scaling of $\tilde\Lambda_n$ is determined just by the previous $(n-1)$ angles. We will use L_n as a quantifier of the scale.

Lemma 1.5.2. *For every $\delta > 0$ there exists $k \in \mathbb{N}$ big enough such that for $z_1, z_2 \in \Gamma \cap (\bigcup_{j=k}^{\infty} \tilde\Lambda_j)$ we have*

$$\ell(z_1, z_2) \le (1+\delta)|z_1 - z_2|. \tag{1.5.3}$$

Proof. Let us start with some geometrical observation.
Let $k \in \mathbb{N}$ and $\zeta_1, \zeta_2 \in \Gamma$. Suppose, moreover, that $\zeta_1 \in \tilde I_k$ and $\zeta_2 \in \widetilde{IV}_k$. It is useful to define

$$R_k := \ell(\zeta_1, \zeta_2) - |\zeta_1 - \zeta_2|.$$

Observe that the definition of R_k does not depend on the choice of ζ_1 and ζ_2 in the respective segments. In particular, by the construction of the curve Γ and by the definition of the error term τ_j in (1.5.1), it is not difficult to check that we have

$$R_k = \Big(3\sum_{j=k+1}^{\infty} L_j - L_k\Big) - \sum_{j=k+1}^{\infty} 2L_j\tau_j. \tag{1.5.4}$$

The term between parentheses in the right hand side is the length of the gluing of the 'non-regularized' α-patches in the construction and the second sum is an error term due to the smoothing in the definition of α-patch.
Because of how we chose L_j and τ_j, the quantity R_k represents the error we make in estimating the length of the arch of the curve between $\zeta_1 \in \tilde I_k$ and $\zeta_2 \in \widetilde{IV}_k$ compared to $|\zeta_1 - \zeta_2|$. The presence of factor $2L_j$ in the last sum in the right hand side of (1.5.4) is due to the fact that the diameter of $\tilde\Lambda_j$ is equal to $2L_j$ and, thus, the error term τ_j

has to be rescaled by that value. It turns out that

$$\frac{R_k}{L_k} \to 0 \quad \text{as } k \to \infty, \tag{1.5.5}$$

which justifies the interpretation of R_k as an error term. Indeed, recalling that $\cos\alpha_l \geq \cos\alpha_k$ for $l \geq k$, we have

$$\begin{aligned}\frac{3}{L_k}\sum_{j=k+1}^{\infty} L_j &= \sum_{j=k+1}^{\infty}\frac{3}{4^{j-k}}\Big(\prod_{l=k}^{j-1}\cos\alpha_l\Big)^{-1}\\ &\leq 3\sum_{j=k+1}^{\infty}\Big(\frac{1}{4\cos\alpha_k}\Big)^{j-k} = \frac{3}{4\cos\alpha_k - 1}\end{aligned}$$

and the last term tends to 1 as $k\to\infty$. Moreover, using (1.5.1) and since $L_j \leq 2^{k-j}L_k$ for $j > k$, we have that

$$\frac{1}{L_k}\sum_{j=k+1}^{\infty} 2L_j\tau_j \lesssim \tau_{k+1}\sum_{j=k+1}^{\infty} 2^{k-j} \to 0 \quad \text{as } k\to\infty,$$

so that (1.5.5) follows.

Let us combine this observation with (1.5.2) to prove (1.5.3). Let $z_1, z_2 \in \Gamma$. Observe that each point of Γ belongs to $\widetilde{\Lambda}_j$ for at most two different j. Let k_1 be the maximum index such that $z_1 \in \widetilde{\Lambda}_{k_1}$ and let k_2 be the maximum index such that $z_2 \in \widetilde{\Lambda}_{k_2}$. The rest of the proof works with minor changes if we take the minimum instead of the maximum in the definitions of k_1 and k_2. The use of this indices helps to make the calculations more systematic.
Without loss of generality, suppose $k_1 \leq k_2$. If $k_1 = k_2$, the points belong to the image of the same patch. We have two possible scenarios depending on the relative position of these points. The definition of R_k and the estimate (1.5.2) allow us to write

$$\ell(z_1,z_2) \leq \frac{|z_1 - z_2|}{\cos\alpha_{k_1}}, \tag{1.5.6}$$

if the point are at a distance $|z_1 - z_2| \leq L_{k_1+1}$. For $|z_1 - z_2| \geq L_{k_1+1}/4$, we have to consider the additional error term R_{k_1+1}, which comes from the 'spiraling' part of the curve. In particular

$$\begin{aligned}\ell(z_1,z_2) &\leq \frac{|z_1 - z_2|}{\cos\alpha_{k_1}} + R_{k_1+1}\\ &\leq \frac{|z_1 - z_2|}{\cos\alpha_{k_1}} + \frac{R_{k_1+1}}{4L_{k_1+1}}|z_1 - z_2|,\end{aligned}$$

so that, invoking (1.5.5), the lemma is proven in the case $k_1 = k_2$.

Let us consider the other case, $k_1 < k_2$. If $z_2 \in \widetilde{\Lambda}_{k_1}$, (1.5.6) easily applies because the two points belong to the image of the same patch. So we can suppose $z_2 \notin \widetilde{\Lambda}_{k_1}$. In this case

$$|z_1 - z_2| \geq \frac{L_{k_1+1}}{4}. \tag{1.5.7}$$

Let $z_2' \in \widetilde{II}_{k_1}$ be the orthogonal projection of z_2 on the segment $\widetilde{II}_{k_1}$. The idea now is, by means of projections, to reduce to the case in which the points belong to the image of the same patch. For this purpose it is also useful to use the length of the arcs of the m-th step curve Γ_m that we used to define Γ. By the triangular inequality and denoting by

$$h_{k_1+1} := \min\{h \colon \widetilde{\Lambda}_{k_1+1} \subset [0,h]n_V + V \text{ for some affine line } V \text{ with normal } n_V\} \tag{1.5.8}$$

the width of $\tilde{\Lambda}_{k_1+1}$, we have

$$|z_1 - z_2'| \leq |z_1 - z_2| + h_{k_1+1}. \tag{1.5.9}$$

Let us remark that, by construction of Γ,

$$\frac{h_{k_1+1}}{L_{k_1+1}} \to 0 \text{ as } k \to \infty. \tag{1.5.10}$$

Given $m \in \mathbb{N}$ and $u, v \in \Gamma_m$, it is useful to denote by $l_m(u,v)$ the length of the arc of Γ_m joining u and v. Now we want to prove that

$$\ell(z_1, z_2) \leq \ell_{k_1}(z_1, z_2') + R_{k_1+1}. \tag{1.5.11}$$

Let us just consider the case $z_1 \in \tilde{I}_{k_1}$, since the other cases are analogous. If $z_{k_2} \in \tilde{I}_{k_1+1}$ or $z_{k_2} \in \widetilde{IV}_{k_1+1}$, (1.5.11) holds trivially because $z_2 = z_2'$. Otherwise, let ζ be a point on $\widetilde{IV}_{k_1+1}$ and let us consider the quantities $\ell(z_2, \zeta)$ and $|z_2' - \zeta|$. Observe that the consideration below does not depend on the auxiliary point ζ of $\widetilde{IV}_{k_1+1}$ we choose. Clearly $\ell(z_2, \zeta) \geq |z_2' - \zeta|$ and, because of the definition of R_{k_1+1}, the equality

$$\ell(z_1, z_2) + \ell(z_2, \zeta) = R_{k_1+1} + \ell_{k_1}(z_1, z_2') + |z_2' - \zeta|,$$

holds. So

$$\ell(z_1, z_2) = \ell_{k_1}(z_1, z_2') + R_{k_1+1} + (|z_2' - \zeta| - \ell(z_2, \zeta)) \leq \ell_{k_1}(z_1, z_2') + R_{k_1+1}.$$

The proof of the lemma is now over: indeed using (1.5.2), (1.5.5), (1.5.7), (1.5.9) and (1.5.10) we get

$$\begin{aligned}
\frac{\ell(z_1, z_2)}{|z_1 - z_2|} &\leq \frac{\ell_{k_1}(z_1, z_2')}{|z_1 - z_2|} + \frac{R_{k_1+1}}{|z_1 - z_2|} \\
&\leq \frac{|z_1 - z_2'|}{|z_1 - z_2| \cos \alpha_{k_1}} + \frac{R_{k_1+1}}{|z_1 - z_2|} \\
&\leq \frac{1}{\cos \alpha_{k_1}} + \frac{4h_{k_1+1}}{\cos \alpha_{k_1} L_{k_1+1}} + \frac{4R_{k_1+1}}{L_{k_1+1}} \to 1 \quad \text{as } k_1 \to \infty.
\end{aligned}$$

□

A rectifiable curve Γ is said *asymptotically smooth* if, denoting by $\ell(w_1, w_2)$ the length of the shortest arc of Γ between $w_1, w_2 \in \Gamma$,

$$\frac{\ell(w_1, w_2)}{|w_1 - w_2|} \to 1 \text{ as } |w_1 - w_2| \to 0, \qquad w_1, w_2 \in \Gamma.$$

As shown in [Pom78], an asymptotically smooth curve is also asymptotically conformal.

Proposition 1.5.1. Γ *is asymptotically smooth but not* C^1.

Proof. Let $\tilde{z}_j' \in \Gamma$ be the image of the point z_{α_j}' via the map which sends Λ_j to $\widetilde{\Lambda}_j$. We have that the curve Γ is not C^1 at the point $z_0 := \lim_j z_j$, where z_j is an arbitrary point of $\tilde{\Lambda}_j$. Indeed, by our choice of the angles in the construction, $\sum_j \alpha_j = +\infty$ and the curve spirals close to z_0.

Let us now turn prove that the curve is asymptotically smooth.

Notice that we may write $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \{z_0\}$, where Γ_1 and Γ_2 are smooth curves. Then, for every couple of points $\{z_1, z_2\}$ in one of those two smooth components we can exploit the smoothness to state that for every δ there exists $\bar{\varepsilon}$ such that for $\varepsilon < \bar{\varepsilon}$ and $|z_1 - z_2| = \varepsilon$ we have

$$\ell(z_1, z_2) \leq (1 + \delta)\varepsilon.$$

This, together with the result of Lemma 1.5.2 concludes the proof. □

Let us consider the arc-length parametrization γ of Γ. Being Γ asymptotically smooth, γ is bilipschitz. In particular,

$$\frac{1}{C}|x - y| \leq |\gamma(x) - \gamma(y)| \leq |x - y|$$

for a constant $C > 1$ and $x, y \in [0, L(\Gamma)]$. As in Remark 1.4 we denote by $\alpha(x, \varepsilon)$ the angle between the vectors $\gamma(x) - \gamma(x - \varepsilon)$ and $\gamma(x + \varepsilon) - \gamma(x)$. Because of the geometrical considerations in Remark 1.4, we have that

$$|\gamma(x + \varepsilon) + \gamma(x - \varepsilon) - 2\gamma(x)|^2 \leq \varepsilon^2 \Big(2 - \frac{2}{C^2} \cos \alpha(x, \varepsilon)\Big) \tag{1.5.12}$$

for $\varepsilon > 0$ and $x \in [0, L(\Gamma)]$. Now we want to prove the estimate

$$|\gamma(x + \varepsilon) + \gamma(x - \varepsilon) - 2\gamma(x)| \lesssim \frac{\varepsilon}{|\log \varepsilon|}.$$

Being Γ smooth off the point z_0 and arguing as in [Gir13], the logarithmic condition (1.1.4) and the estimate (1.1.6) are satisfied off that point. Hence it suffices to prove (1.1.6) for $\gamma(x) \in \bigcup_{k \geq k_0} \tilde{\Lambda}_k \cap \Gamma$ and k_0 big enough. To do that, we will study the behavior of the angle $\alpha(x, \varepsilon)$ and of the local value of the bilipschitz constant of γ close to the point z_0.

Being the curve asymptotically smooth, as a corollary of Lemma 1.4.1 we know that $\alpha(x, \varepsilon) \to 0$ for ε small. Then, the second factor in the right hand side of (1.5.12) behaves as

$$2 - \frac{2}{C^2} \cos \alpha(x, \varepsilon) = \Big[2 - \frac{2}{C^2}\Big] + \frac{2}{C^2} \alpha(x, \varepsilon)^2 + O\big(\alpha(x, \varepsilon)^4\big)$$

for $\varepsilon \to 0$.

Let $x_0 := \gamma^{-1}(z_0)$. For $\varepsilon > 0$, we denote by C_ε the smallest constant such that

$$\frac{1}{C_\varepsilon}|x - y| \leq |\gamma(x) - \gamma(y)| \leq |x - y|$$

holds for $x, y \in [x_0 - \varepsilon, x_0 + \varepsilon]$, i.e. the local value of the lower bilipschitz constant close to x_0.

Using this notation, to our purposes it suffices to prove that

$$|\alpha(x, \varepsilon)| \lesssim \big|\log \varepsilon\big|^{-1}$$

and

$$\Big[1 - \frac{1}{C_\varepsilon}\Big] \lesssim \big|\log \varepsilon\big|^{-1} \tag{1.5.13}$$

for ε small and $\gamma(x)$ close enough to z_0.

The following two lemmas respectively prove the estimate for the angle and the estimate for C_ε.

Lemma 1.5.3. *For every ε_0 there exists an integer k_0 such that*

$$|\alpha(x,\varepsilon)| \lesssim |\log\varepsilon|^{-1}$$

for $\varepsilon < \varepsilon_0$, $|x-x_0| < \varepsilon_0$ and $\gamma(x-\varepsilon) \in \bigcup_{k=k_0}^{\infty} \tilde{\Lambda}_k \cap \Gamma$.

Proof. Let $\varepsilon > 0$ and $z = \gamma(x) \in \Gamma$. Moreover, let us define $z_\pm := \gamma(x \pm \varepsilon)$. Let k be the maximum index such that $z \in \tilde{\Lambda}_k$ and let $k_\pm$ be the maximum index such that $z_\pm \in \tilde{\Lambda}_{k_\pm}$. Without loss of generality, we will prove the lemma for $x < x_0$. Let us proceed with some geometrical consideration.

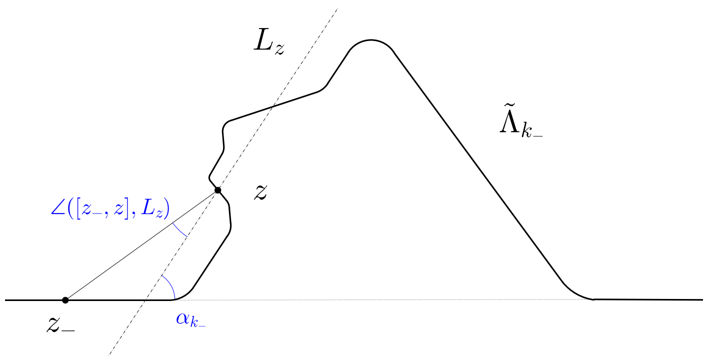

FIGURE 1.4: A schematic representation of the setting of the proof of Lemma 1.5.3.

Let L_z denote the line passing through z and parallel to the segment $\widetilde{II}_{k_-}$. Due to the definition of the angle $\alpha(x,\varepsilon)$, we can fix the line L_z and bound $|\alpha(x,\varepsilon)|$ by the absolute value of the smallest angle $\angle([z_-,z],L_z)$ that L_z forms with the segment $[z_-,z]$ plus the absolute value of the smallest angle $\angle([z,z_+],L_z)$ that L_z forms with the segment $[z,z_+]$.

If z belongs to $\tilde{\Lambda}_{k_-}$, due to the properties of the α_{k_-}-patch, the arc $\gamma([x-\varepsilon,x])$ is entirely contained in a cone of vertex z and aperture $\angle([z_-,z],L_z)$. By elementary geometric considerations, we can write

$$\big|\angle([z_-,z],L_z)\big| \le \alpha_{k_-}. \tag{1.5.14}$$

Again, due to few geometric observations (that are not substantial for the sequel and we decide to omit in order to make the proof more concise) and to the way Γ is defined, it is not difficult to see that

$$\big|\angle([z_+,z],L_z)\big| \le 2\alpha_{k_-}. \tag{1.5.15}$$

We are left to consider the case $z \notin \widetilde{\Lambda}_{k_-}$. As we observed in Lemma 1.5.2, in this case we have $|z_- - z| \ge L_{k_-+1}/4$. Moreover, $\bigcup_{j=k_-+1}^{\infty} \widetilde{\Lambda}_j \cap \Gamma$ is contained in a rectangle whose base lays on $\widetilde{II}_{k_-}$, whose length is smaller than, say, $5L_{k_-+1}/3$ and with height h_{k_-+1} (for its definition we refer to (1.5.8) in Lemma 1.5.2). We recall that

$$\frac{h_j}{L_j} \to 0 \text{ for } j \to \infty.$$

Now observe that $z_+ \in \bigcup_{j=k_-}^{\infty} \widetilde{\Lambda}_j \cap \Gamma$. For every point z in this rectangle, using that $|z - z_+| \gtrsim L_{k_-+1}$, it holds that

$$|\angle([z_-, z], L_z)| \lesssim \alpha_{k_-} \tag{1.5.16}$$

and

$$|\angle([z, z_+], L_z)| \lesssim \alpha_{k_-}. \tag{1.5.17}$$

Joining (1.5.14),(1.5.15),(1.5.16) and (1.5.17), we get

$$|\alpha(z, \varepsilon)| \lesssim \alpha_{k_-}.$$

Then, by the construction of Γ and the definition of L_m, $L_{m+1}/L_m \leq 1/2$ for every m, that by iteration leads to

$$L_m \leq 2^{-m}.$$

Now, if $\gamma(x - \varepsilon) \in \widetilde{\Lambda}_{k_-}$ for k_- big enough, we have that $\varepsilon \lesssim L_{k_-}$ so that

$$k_- \gtrsim |\log \varepsilon|$$

for ε small enough. So, gathering all the considerations and recalling that $\alpha_{k_-} = 1/k_-$, we get the desired result. □

Lemma 1.5.4. *There exists $\varepsilon_1 > 0$ such that the inequality (1.5.13) holds for $\varepsilon < \varepsilon_1$.*

Proof. Let us consider $z_1, z_2 \in \Gamma$. Let k_1 be the maximum index such that $z_1 \in \tilde{\Lambda}_{k_1}$ and k_2 the maximum index such that $z_2 \in \tilde{\Lambda}_{k_2}$. Without loss of generality, $k_1 \leq k_2$ and $\gamma^{-1}(z_1) \leq \gamma^{-1}(\tilde{z})$. The idea is to prove that C_ε^{-1} is greater than a quantity which approximates $\cos \alpha_{k_1}$. It is convenient to split the study into different cases.

If $k_1 = k_2$ and $\gamma^{-1}(z_2) < \bar{x}$ or $k_2 = k_1 + 1$ and $z_2 \in \widetilde{I}_{k_1+1}$, then (1.5.2) gives

$$|z_1 - z_2| \geq \cos \alpha_{k_1} \ell(z_1, z_2).$$

If $k_1 = k_2$ and $\gamma^{-1}(z_2) > \bar{x}$ or $k_2 = k_1 + 1$ and $z_2 \in \widetilde{IV}_{k_1+1}$, then we can write

$$|z_1 - z_2| \geq \cos \alpha_{k_1} \big(\ell(z_1, z_2) - R_{k_1+1}\big) = \Big(\cos \alpha_{k_1} - \cos \alpha_{k_1} \frac{R_{k_1+1}}{l(z_1, z_2)}\Big) \ell(z_1, z_2)$$

and we recall that

$$\frac{R_{k_1+1}}{\ell(z_1, z_2)} \lesssim \frac{R_{k_1+1}}{L_{k_1+1}} \to 0 \qquad \text{for} \qquad k_1 \to \infty.$$

In the remaining cases, we know from the proof of Lemma 1.5.2 that

$$|z_1 - z_2| \geq \Big(\cos \alpha_{k_1} - \cos \alpha_{k_1} \frac{h_{k_1+1}}{\ell(z_1, z_2)} - \cos \alpha_{k_1} \frac{R_{k_1+1}}{\ell(z_1, z_2)}\Big) \ell(z_1, z_2),$$

so that, using the same argument as at the end of the proof of Lemma 1.5.3 together with the Taylor expansion for the cosine, the proof is completed. □

The two previous lemmas show that the arc-length parametrization γ of Γ is such that the estimate

$$\mathcal{C}_* f(z) \lesssim M^2(\mathcal{C}f)(z)$$

holds for every $z \in \Gamma$.

Final remarks on the curve Γ.

The curve Γ that we studied in this section can be considered as an example of a critical curve for which the main theorem holds. Indeed, another look at the estimates we got tells that most of those concerning the geometry of the curve are close to being sharp. Moreover, the finite second difference $|\gamma(x+\varepsilon)+\gamma(x-\varepsilon)-2\gamma(x)|$ has the right decay we need; the choice of a slower decay for the angles α_j causes worse estimates for $|\alpha(x,\varepsilon)|$ and, hence, the finite second difference estimate to fail. Let us notice that the spiraling of Γ close to the point z_0 also gives an idea of how the critical curves may look like.

Asymptotically smooth curves that are not C^1 may also be defined by means of complex analysis (exploiting, for example, the results in [Pom78]) but we found a constructive approach more convenient to our purposes.

Chapter 2

Measures that define a compact Cauchy transform

2.1 Introduction

In what follows we will identify the plane with the complex field $\mathbb{C}$. Let μ be a positive Radon measure on $\mathbb{C}$ with compact support and without atoms. For $\varepsilon > 0$, $f \in L^1_{loc}(\mu)$ and $z \in \mathbb{C}$ we set

$$\mathcal{C}_{\mu,\varepsilon} f(z) := \int_{|z-w|>\varepsilon} \frac{f(w)}{z-w} d\mu(w).$$

We define the Cauchy transform operator $\mathcal{C}_\mu$ in a principal value sense, i.e., as the limit

$$\mathcal{C}_\mu f(z) := \lim_{\varepsilon \to 0} \mathcal{C}_{\mu,\varepsilon} f(z)$$

for every z such that the above limit exists. We say that the Cauchy transform is bounded from $L^2(\mu)$ to $L^2(\mu)$ if the truncated operators $\mathcal{C}_{\mu,\varepsilon} \colon L^2(\mu) \to L^2(\mu)$ are bounded uniformly in ε.

As a consequence of the work of Mattila and Verdera (see [MV09] or the book by Tolsa [Tol14, Chapter 8]), the Cauchy transform is bounded from $L^2(\mu)$ to $L^2(\mu)$ if and only if the truncated operators $\{\mathcal{C}_{\mu,\varepsilon}\}_\varepsilon$ converge as ε tends to 0 in the weak operator topology of the space of bounded linear operators from $L^2(\mu)$ to $L^2(\mu)$. Moreover, if we denote as $\mathcal{C}^w_\mu$ the limit of the aforementioned net, for all $f \in L^2(\mu)$ and for μ-almost every z, the principal value $\mathcal{C}_\mu f(z)$ exists and it coincides with $\mathcal{C}^w_\mu f(z)$. This is a peculiarity of the Cauchy transform and it does not hold for every singular integral operator. Now, it makes sense to introduce the following definition.

Definition 2.1.1. We say that the Cauchy transform is compact from $L^2(\mu)$ to $L^2(\mu)$ if it is bounded in $L^2(\mu)$ and $\mathcal{C}^w_\mu$ is compact as an operator from $L^2(\mu)$ to $L^2(\mu)$.

As a consequence of the results we cited, one may replace $\mathcal{C}^w_\mu$ in Definition 2.1.1 with the principal value $\mathcal{C}_\mu$. A useful tool to study the Cauchy transform of a measure μ is the so-called Menger curvature $c(\mu)$, that was first related to the Cauchy transform in [Mel95] and [MV95]. Denoting by $R(z,w,\zeta)$ the radius of the circumference passing though z, w and ζ, and defining

$$c^2_\mu(z) := \iint \frac{1}{R(z,w,\zeta)^2} d\mu(w) d\mu(\zeta),$$

the Menger curvature of μ is defined as

$$c^2(\mu) := \int c^2_\mu(z) d\mu(z). \tag{2.1.1}$$

Let $d, n \in \mathbb{N}$ with $n \leq d$. Given a cube Q in $\mathbb{R}^d$, we denote by $\ell(Q)$ its side length and by

$$\Theta_\mu^n(Q) := \frac{\mu(Q)}{\ell(Q)^n} \tag{2.1.2}$$

its n-dimensional density. If $z \in \mathbb{R}^d$, we define the upper density of μ at z as

$$\Theta_\mu^{n,*}(z) := \limsup_{\ell(Q)\to 0} \Theta_\mu^n(Q), \tag{2.1.3}$$

where Q spans over the cubes centered at z. Replacing the superior limit with the inferior limit we get the definition of the lower density $\Theta_{*,\mu}^n(z)$. If $\Theta_\mu^{n,*}(z) = \Theta_{*,\mu}^n(z)$, we denote that common value as $\Theta_\mu^n(z)$ and call it "density of μ at the point z". In the case $d = 2$ and $n = 1$, for brevity we write $\Theta_\mu(Q) := \Theta_\mu^1(Q)$ and we omit the index n from the notation for the upper and lower densities at any point.

The aim of the present work is to characterize the measures μ on the plane such that its associated Cauchy transform defines a compact operator from $L^2(\mu)$ into $L^2(\mu)$. Not much literature is available concerning compactness for singular integral operators in the context of Euclidean spaces equipped with a measure different from the Lebesgue measure. We point out that a $T(1)$-like criterion for the compactness of Calderón-Zygmund operators in Euclidean spaces is available due to the work of Villarroya [Vil15].

We denote by $K(L^2(\mu), L^2(\mu))$ the space of compact linear operators from $L^2(\mu)$ to $L^2(\mu)$. We will see that a crucial condition to get a compact Cauchy transform is to require that

$$\Theta_\mu^*(z) = 0$$

for every $z \in \mathbb{C}$. Our main result is the following.

Theorem 2.1. *Let μ be a compactly supported positive Radon measure on $\mathbb{C}$ without atoms. The following conditions are equivalent:*

(a) $\mathcal{C}_\mu$ is compact from $L^2(\mu)$ to $L^2(\mu)$.

(b) the two following properties hold:

(1) $\Theta_\mu^*(z) = 0$ uniformly, which means that the limit in (2.1.3) is 0 uniformly in $z \in \mathbb{C}$.

(2) $c^2(\mu|_Q)/\mu(Q) \to 0$ as $\ell(Q) \to 0$, where $\mu|_Q$ stands for the restriction of μ to the cube Q.

(c) the truncated operators $\mathcal{C}_{\mu,\varepsilon}$ converge as $\varepsilon \to 0$ in the operator norm of the space of bounded linear operators from $L^2(\mu)$ to $L^2(\mu)$.

We remark that the proof of the theorem relies on the $T(1)$-theorem for the Cauchy transform (see [Tol14]) and that one could replace the cubes with balls in condition (b), as well as in (2.1.2).

Theorem 2.1 can be generalized to higher dimensions taking into consideration the n-Riesz transform $\mathcal{R}_\mu^n$ on $\mathbb{R}^d$ for $n \leq d$ in place of the Cauchy transform. If μ is a compactly supported positive Radon measure on $\mathbb{R}^d$ without atoms, $\varepsilon > 0$, $f \in L^1_{loc}(\mu)$ and $z \in \mathbb{R}^d$, the truncated Riesz transform is defined as

$$\mathcal{R}_{\mu,\varepsilon}^n f(z) := \int_{|z-w|>\varepsilon} \frac{x-y}{|x-y|^{n+1}} f(y) d\mu(y).$$

As in the case of the Cauchy transform, thanks again to the result in [MV09], the weak limit $\mathcal{R}_\mu^{n,w}$ of $\mathcal{R}_{\mu,\varepsilon}^n$ as $\varepsilon \to 0$ exists provided the $\mathcal{R}_{\mu,\varepsilon}^n$ are uniformly bounded on $L^2(\mu)$, and we can understand the compactness of the Riesz transform as in Definition

2.1.1. The main difference with the Cauchy transform is that the only case in which boundedness is known to imply that the principal value exists is for $n = d - 1$. This is a consequence of [NTV14a].

In this more general context, Theorem 2.1 reads as follows.

Theorem 2.2. *Let μ be a compactly supported positive Radon measure on $\mathbb{R}^d$ without atoms. The following conditions are equivalent:*

(a) $\mathcal{R}^n_\mu$ is compact from $L^2(\mu)$ to $L^2(\mu)$.

(b) the two following properties hold:

(1) $\Theta^{n-1,*}_\mu(z) = 0$ uniformly in $z \in \mathbb{R}^d$.

(2) $\|\mathcal{R}^n_\mu \chi_Q\|^2_{L^2(\mu|_Q)}/\mu(Q) \to 0$ as $\ell(Q) \to 0$.

(c) the truncated operators $\mathcal{R}^n_{\mu,\varepsilon}$ converge as $\varepsilon \to 0$ in the operator norm of the space of bounded linear operators from $L^2(\mu)$ to $L^2(\mu)$.

Theorem 2.2 can be proved with minor changes of the proof that we will discuss for the case of the Cauchy transform. Combining condition (b) in Theorem 2.2 with [MV09, Theorem 1.6], we can infer that if $\mathcal{R}^n_\mu$ is compact then the principal value $\mathcal{R}^n_\mu(x)$ exists for μ-almost every x.

The work is structured as follows. In Section 2.2 we deal with two toy models: first we show a direct proof of the non-compactness of the Cauchy transform of the one dimensional Lebesgue measure on a segment. Then, we prove that the Cauchy transform of a disc endowed with the planar Lebesgue measure is compact. We remark that these two cases may be seen as direct consequences of Theorem 2.1 and they do not enter its proof, at least directly. However, we think that studying them separately may serve as a further motivation of the main theorem and it may help the reader in understanding the reason why we drove our attention to conditions on the density of the measure. In Section 2.3 we prove Theorem 2.1. As an application of this result, Section 2.4 is devoted to the discussion of the case of the general planar Cantor sets. We conclude the exposition with a remark on the generalization of the main theorem to other singular integral operators.

2.2 The Cauchy transform on a segment and on the disc

It may be worth recalling the following property of compact operators: if X and Y are Banach spaces, $T\colon X \to Y$ is a compact operator and $\{u_k\}_k$ is a sequence in X such that $u_k \rightharpoonup u$ for some $u \in X$ (weak convergence), then $Tu_k \to Tu$ (strongly) in Y. We will use this property both for the proof of the following proposition and for the proof of the main theorem. Let us start by considering the Cauchy transform on a segment. Given an interval I on the real line, we denote by $\mathcal{H}^1$ the 1-dimensional Hausdorff measure and use the notation $L^2(I) := L^2\big(\mathcal{H}^1|_{(I\times\{0\})}\big)$. Without loss of generality, we analyze the case $I = [0, 1]$.

Proposition 2.2.1. *Let $\mu := \mathcal{H}^1|_{([0,1]\times\{0\})}$. The Cauchy transform $\mathcal{C}_\mu$ is not a compact operator from $L^2(\mu)$ into $L^2(\mu)$.*

Proof. Let $\mathcal{C}_\mu$ be the Cauchy transform of the measure $\mu := \mathcal{H}^1|_{([0,1]\times\{0\})}$, acting on functions belonging to $L^2([0, 1])$.

For $k \in \mathbb{N}$, let us define the function $f_k\colon \mathbb{R} \to \mathbb{R}$ as

$$f_k(x) := 2^{(k-1)/2}\big(\chi_{[1/2-2^{-k},1/2]}(x) - \chi_{[1/2,1/2+2^{-k}]}(x)\big).$$

Notice that $\|f_k\|_{L^2([0,1])} = \|f_k\|_{L^2(\mathbb{R})} = 1$ and that $\{f_k\}_k$ converges to 0 in the weak topology of $L^2([0,1])$. However, $\{f_k\}_k$ does not converge in the strong topology of $L^2([0,1])$.

Let us denote by Hf_k the Hilbert transform of f_k

$$Hf_k(x) := \text{p.v}\int \frac{f_k(y)}{x-y}dy$$

for $x \in \mathbb{R}$. We claim that Hf_k does not converge to 0 in the strong topology of $L^2([0,1])$. Hence $\mathcal{C}_\mu = H$ is not compact in $L^2(\mu)$.

A well known fact regarding Hilbert transform (see e.g. [Ste70]) is that

$$\|Hf\|_{L^2(\mathbb{R})} = \pi\|f\|_{L^2(\mathbb{R})}$$

for every $f \in L^2(\mathbb{R})$.

The following argument proves that $\|\mathcal{C}_\mu f_k\|_{L^2([0,1])} = \|Hf_k\|_{L^2([0,1])}$ tends to π for $k \to \infty$.

It is enough to show that

$$\|Hf_k\|^2_{L^2([1,+\infty))} \to 0 \qquad \text{for} \qquad k \to \infty \tag{2.2.1}$$

and

$$\|Hf_k\|^2_{L^2((-\infty,0])} \to 0 \qquad \text{for} \qquad k \to \infty. \tag{2.2.2}$$

To prove (2.2.1), first notice that for $y \in \operatorname{supp} f_k$ and $x \geq 1$, it holds that $|x-y| \geq |x-3/4|$. Then

$$\begin{aligned}\|Hf_k\|^2_{L^2(1,+\infty)} &= \int_1^{+\infty}\Big|\int_{1/2-2^{-k}}^{1/2+2^{-k}} \frac{f_k(y)}{x-y}dy\Big|^2 dx \\ &\leq \int_1^{+\infty} \frac{1}{|x-\frac{3}{4}|^2}\Big(\int_{1/2-2^{-k}}^{1/2+2^{-k}} |f_k(y)|dy\Big)^2 dx \\ &\leq 2^{-k+1}\int_1^{+\infty}\frac{1}{|x-\frac{3}{4}|^2}dx \lesssim 2^{-k},\end{aligned}$$

which gives (2.2.1). The proof of (2.2.2) is analogous. □

Now we turn to analyze the Cauchy transform on the disc. Let $D := D(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ and let $\varepsilon > 0$. Let $\mu = dA$ be the 2-dimensional Lebesgue measure restricted to D.

Lemma 2.2.1. *The operator* $\mathcal{C}_{\mu,\varepsilon} \colon L^2(dA) \to L^2(dA)$ *is compact for every* $\varepsilon > 0$.

Proof. Let $z, w \in \mathbb{C}$ and let $K_\varepsilon(z,w) := \chi_{D(z,\varepsilon)^c}(w)/(z-w)$. By the Hilbert-Schmidt's Theorem (see [Bre11, Theorem 6.12]), to prove the lemma it is enough to show that the integral

$$\int_D |K_\varepsilon(z,w)|^2 dA(z)$$

converges. This occurs because

$$\int_D |K_\varepsilon(z,w)|^2 dA(z) \leq \frac{A(D)}{\varepsilon^2} = \frac{\pi}{\varepsilon^2},$$

so the proof is complete. □

For $f \in L^2(dA)$ let us define

$$\mathcal{C}_\mu^\varepsilon f(z) := \mathcal{C}_\mu f(z) - \mathcal{C}_{\mu,\varepsilon} f(z).$$

By Lemma 2.2.1, to prove that $\mathcal{C}_\mu$ belongs to $K(L^2(dA), L^2(dA))$ it suffices to prove that $\|\mathcal{C}_\mu^\varepsilon\|_{L^2(dA)\to L^2(dA)} \to 0$ as $\varepsilon \to 0$. Indeed, this implies that $\{\mathcal{C}_{\mu,\varepsilon}\}_{\varepsilon>0}$ converges in operator norm to the Cauchy transform, which proves that it is compact.

For $f \in L^2(dA)$, a direct computation using polar coordinates gives

$$\begin{aligned}
\int_D |\mathcal{C}_\mu^\varepsilon f(z)|^2 dA(z) &= \int_D \Big| \int_{|z-w|<\varepsilon} \frac{f(w)}{z-w} dA(w) \Big|^2 dA(z) \\
&= \int_D \Big| \int_0^{2\pi} \int_0^\varepsilon e^{-i\theta} f(z+re^{i\theta}) \chi_D(z+re^{i\theta}) dr d\theta \Big|^2 dA(z) \\
&\leq \int_D \int_0^{2\pi} \int_0^\varepsilon |f(z+re^{i\theta}) \chi_D(z+re^{i\theta})|^2 dr d\theta dA(z) \\
&\leq 2\pi\varepsilon \|f\|^2_{L^2(dA)},
\end{aligned}$$

where in the last inequality we used Fubini's Theorem. Hence

$$\|\mathcal{C}_\mu^\varepsilon f\|_{L^2(dA)\to L^2(dA)} \leq (2\pi\varepsilon)^{1/2},$$

so $\mathcal{C}_\mu \in K(L^2(dA), L^2(dA))$.

Remark 3. The integral

$$\int_{B(0,1)} \frac{1}{|z|} dA(z) \tag{2.2.3}$$

plays a crucial role in the proof of the compactness of the Cauchy transform of a disc. When focusing on the general case in which dA is replaced by a measure μ, one may be tempted to guess that we need a density condition which gives that the analogue of (2.2.3) converges. This drives our attention to measures with zero linear density, which we will prove to be a necessary condition for the Cauchy transform to be compact.

2.3 The proof of Theorem 2.1

2.3.1 Necessary conditions for the compactness.

In order to prove the necessity of the conditions in Theorem 2.1, we argue by contradiction: assuming that there exists a sequence of cubes $\{Q_j\}_j$ such that $\ell(Q_j) \to 0$ but $\limsup \Theta^1_\mu(Q_j) > 0$, we will prove that the Cauchy transform does not define a compact operator on $L^2(\mu)$.

We recall that a necessary condition to have the $L^2(\mu)$-boundedness of $\mathcal{C}_\mu$ is that μ has linear growth (see [Dav91]). In particular we choose to denote by C_0 a positive constant such that

$$\mu(Q) \leq C_0 \ell(Q) \tag{2.3.1}$$

for every cube in $\mathbb{R}^2$.

Suppose that we can find $\Theta > 0$ and a sequence of cubes $\{Q_j\}_j$ with $\ell(Q_j) \to 0$ such that

$$\limsup_j \Theta^1_\mu(Q_j) = \Theta.$$

In particular, for every $\delta > 0$ we can find a cube Q with $\ell(Q) \leq \delta$ such that

$$\Theta^1_\mu(Q) \geq \frac{\Theta}{2}. \tag{2.3.2}$$

This cube contains two disjoint cubes with the properties stated in the following lemma. The proof of the lemma is a variant of the one in [Lég99, Lemma 2.3].

Lemma 2.3.1. *Let Q be a cube of side length $\ell(Q)$ such that $\Theta^1_\mu(Q) \geq \Theta/2$. There exist $C_1, C_1' \in \mathbb{N}$, both greater than 1 and depending on Θ and C_0, such that we can find two cubes Q' and Q'' with side lengths $\ell(Q') = \ell(Q'') = \ell(Q)/C_1$ and with the following properties*

1. $\ell(Q') \leq \operatorname{dist}(Q', Q'') \lesssim C_1 \ell(Q')$.
2. $\min(\mu(Q'), \mu(Q'')) \geq \ell(Q)/C_1'$.

For example, one can choose C_1 and C_1' as $C_1 = 12C_0\Theta^{-1}$ and $C_1' = 12^3 C_0^2 \Theta^{-3}$. In particular we remark that C_1 and C_1' are independent on $\ell(Q)$.

Proof. Let us argue by contradiction. We split Q into a grid of C_1^2 equal cubes of side length $\ell(Q)/C_1$ whose sides are parallel to the sides of Q; we denote this collection of cubes as $\mathcal{D}$. Let us assume that each couple of cubes $Q', Q'' \in \mathcal{D}$, is such that either they touch (so that $\operatorname{dist}(Q', Q'') = 0$) or $\min(\mu(Q'), \mu(Q'')) \leq \ell(Q)/C_1'$.

By construction we have that

$$\sum_{\widetilde{Q} \in \mathcal{D}} \mu(\widetilde{Q}) = \mu(Q) = \Theta(Q)\ell(Q). \tag{2.3.3}$$

Now let us consider the family

$$\mathcal{G} := \Big\{ \widetilde{Q} \in \mathcal{D} : \mu(\widetilde{Q}) \geq \frac{\ell(Q)}{C_1'} \Big\}.$$

By hypothesis, all the cubes in $\mathcal{G}$ must be contained in a single cube of side length $3\ell(Q)/C_1$ that we denote as P. The growth condition (2.3.1) gives

$$\mu(P) \leq C_0 \ell(P) = 3C_0 \ell(Q)/C_1,$$

so that

$$\sum_{\widetilde{Q} \in \mathcal{G}} \mu(\widetilde{Q}) \leq 3\frac{C_0}{C_1}\ell(Q). \tag{2.3.4}$$

For those cubes of $\mathcal{D}$ not belonging to $\mathcal{G}$ we can write

$$\sum_{\widetilde{Q} \in \mathcal{D}\setminus\mathcal{G}} \mu(\widetilde{Q}) \leq \frac{C_1^2}{C_1'}\ell(Q). \tag{2.3.5}$$

By hypothesis we have that $\Theta(Q) \geq \Theta/2$. Then, gathering (2.3.3), (2.3.4) and (2.3.5) we get the inequality

$$\frac{C_1^2}{C_1'} + 3\frac{C_0}{C_1} \geq \frac{\Theta}{2}. \tag{2.3.6}$$

Choosing C_1 and C_1' big enough, (2.3.6) gives a contradiction. A possible choice is the one reported below the statement of the lemma. □

Remark 4. Choose C_1 and C_1' as specified after Lemma 2.3.1. Using the linear growth of the measure μ, the condition (2) of Lemma 2.3.1 actually implies that Q' and Q''

are such that

$$\left(\frac{\Theta}{12C_0}\right)^3 \mu(Q) \leq \mu(Q') \leq \mu(Q) \tag{2.3.7}$$

and

$$\left(\frac{\Theta}{12C_0}\right)^3 \mu(Q') \leq \mu(Q'') \leq \left(\frac{12C_0}{\Theta}\right)^3 \mu(Q').$$

The same inequalities hold reversing the roles of Q' and Q'', so that $\mu(Q')$, $\mu(Q'')$ and $\mu(Q)$ are all comparable with implicit constants depending on C_0 and Θ.

As we have already pointed out, the values of C_1 and C_1' do not depend on $\ell(Q)$. So, we can apply Lemma 2.3.1 to cubes Q with $\ell(Q) \leq \delta$ that verify (2.3.2) and, for every $\delta > 0$, we can find a couple of cubes as in the lemma. In particular, notice that $\ell(Q') = \ell(Q'') \to 0$ for $\delta \to 0$. This will lead to a contradiction.

Given a cube P, we define the function $\varphi_P = \chi_P/\mu(P)^{1/2}$. We have that $\|\varphi_P\|_{L^2(\mu)} = 1$ for every cube P and that

$$\varphi_{Q_j} \rightharpoonup 0$$

weakly in $L^2(\mu)$ for every sequence of cubes $\{Q_j\}_j$ such that $\ell(Q_j) \to 0$.

Now, taking Q, Q' and Q'' as in Lemma 2.3.1, we can write

$$|\langle \mathcal{C}_\mu \varphi_{Q'}, \varphi_{Q''}\rangle| \leq \|\mathcal{C}_\mu \varphi_{Q'}\|_{L^2(\mu)} \|\varphi_{Q''}\|_{L^2(\mu)} = \|\mathcal{C}_\mu \varphi_{Q'}\|_{L^2(\mu)}. \tag{2.3.8}$$

The proof of the necessity of the density condition of Theorem 2.1 follows from (2.3.8) if we can prove that $|\langle \mathcal{C}_\mu \varphi_{Q'}, \varphi_{Q''}\rangle|$ is bounded from below by a positive constant which does not depend on $\ell(Q')$; indeed, this would imply that $\|\mathcal{C}_\mu \varphi_{Q'}\|_{L^2(\mu)}$ does not converge to 0 for $\ell(Q') \to 0$, which contradicts the compactness of the Cauchy transform.

Lemma 2.3.2. *Let Q, Q' and Q'' be as in Lemma 2.3.1. There exists a constant $c > 0$, which depends only on Θ and C_0, such that*

$$c\frac{\mu(Q)}{\ell(Q')} \leq |\langle \mathcal{C}_\mu \varphi_{Q'}, \varphi_{Q''}\rangle|. \tag{2.3.9}$$

Proof. Suppose without loss of generality that the centers of the cubes Q' and Q'' are aligned with the real axis. Since Q' and Q'' are contained in Q we have that

$$\begin{aligned} \left|\operatorname{Re}\langle \mathcal{C}_\mu \varphi_{Q'}, \varphi_{Q''}\rangle\right| &= \frac{1}{\left(\mu(Q')\mu(Q'')\right)^{1/2}} \left|\operatorname{Re}\langle \mathcal{C}_\mu \chi_{Q'}, \chi_{Q''}\rangle\right| \\ &\geq \frac{1}{\mu(Q)} \left|\operatorname{Re}\langle \mathcal{C}_\mu \chi_{Q'}, \chi_{Q''}\rangle\right|. \end{aligned} \tag{2.3.10}$$

Suppose that $\operatorname{Re}(z-w) > 0$ for every $z \in Q''$ and $w \in Q'$. Then

$$\begin{aligned} \left|\operatorname{Re}\langle \mathcal{C}_\mu \chi_{Q'}, \chi_{Q''}\rangle\right| &= \left|\operatorname{Re}\int_{Q''} \mathcal{C}_\mu \chi_{Q'}(z)\,d\mu(z)\right| \\ &= \int_{Q''}\int_{Q'} \frac{\operatorname{Re}(z-w)}{|z-w|^2}\,d\mu(w)\,d\mu(z). \end{aligned} \tag{2.3.11}$$

Lemma 2.3.1 ensures that, if $z \in Q''$ and $w \in Q'$, we have that $\operatorname{Re}(z-w) \geq \operatorname{dist}(Q', Q'') \geq \ell(Q')$ and

$$|z-w| \leq 2\ell(Q') + 2\ell(Q'') + 2\operatorname{dist}(Q', Q'') \leq 4\left(1 + \frac{6C_0}{\Theta}\right)\ell(Q')$$

so that, using (2.3.10), (2.3.11) and (2.3.7) we have

$$|\operatorname{Re}\langle \mathcal{C}_\mu \chi_{Q'}, \chi_{Q''}\rangle| \geq \frac{1}{16\left(1+\frac{6C_0}{\Theta}\right)^2}\frac{\mu(Q')\mu(Q'')}{\ell(Q')} \geq c(\Theta, C_0)\frac{\mu(Q)^2}{\ell(Q')}, \tag{2.3.12}$$

where $c = c(\Theta, C_0) = \Theta^8(12C_0)^{-6}(4\Theta + 24C_0)^{-2}$. The Lemma follows from (2.3.12) and (2.3.10). □

The inequality (2.3.9) together with the condition (2) in Lemma 2.3.1 implies

$$|\langle \mathcal{C}_\mu \varphi_{Q'}, \varphi_{Q''}\rangle| \geq \frac{c(\Theta, C_0)}{C_1'},$$

which is the bound from below we are looking for.

The following lemma gives other necessary conditions for the Cauchy transform of a measure to be compact.

Lemma 2.3.3. *Let μ be a compactly supported positive Radon measure on $\mathbb{C}$ without atoms. Suppose that $\mathcal{C}_\mu$ defines a compact operator from $L^2(\mu)$ to $L^2(\mu)$. Then*

$$\frac{\|\mathcal{C}_\mu \chi_Q\|^2_{L^2(\mu|_Q)}}{\mu(Q)} \to 0 \qquad \text{as} \qquad \ell(Q) \to 0 \tag{2.3.13}$$

and

$$\frac{c^2(\mu|_Q)}{\mu(Q)} \to 0 \qquad \text{as} \qquad \ell(Q) \to 0. \tag{2.3.14}$$

Proof. Let us first prove (2.3.13). Consider a sequence of cubes $\{Q_j\}_j$ such that $\ell(Q_j) \to 0$ as $j \to \infty$. As before, if we define $\varphi_j := \chi_{Q_j}/\mu(Q_j)^{1/2}$, we have that

$$\varphi_j \rightharpoonup 0$$

weakly in $L^2(\mu)$. Then, since we suppose the Cauchy transform to be compact, we have that

$$\|\mathcal{C}_\mu \varphi_j\|^2_{L^2(\mu)} \to 0$$

for $j \to \infty$. The inequalities

$$\|\mathcal{C}_\mu \chi_{Q_j}\|^2_{L^2(\mu|_{Q_j})} \leq \|\mathcal{C}_\mu \chi_{Q_j}\|^2_{L^2(\mu)} \leq \mu(Q_j)\|\mathcal{C}_\mu \varphi_j\|^2_{L^2(\mu)},$$

conclude the proof of (2.3.13).

Let Q be an arbitrary cube in $\mathbb{R}^2$. From a formula due to Tolsa and Verdera (see [TV18b], Theorem 2) applied to the measure $\mu|_Q$, we have that

$$\|\mathcal{C}_\mu \chi_Q\|^2_{L^2\left(\mu|_Q\right)} = \frac{\pi^2}{3}\int_Q \theta_\mu(z)^2 d\mu(z) + \frac{1}{6}c^2(\mu|_Q). \tag{2.3.15}$$

Since we suppose $\mathcal{C}_\mu$ to be compact, we proved that $\theta_\mu(z) = 0$ for every $z \in \mathbb{R}^2$, so that the integral in the right hand side of (2.3.15) vanishes and, using (2.3.13), we get (2.3.14). □

2.3.2 Sufficient conditions for the compactness.

The proof that we present now relies on the $T(1)$-theorem of David and Journe. More specifically, we prove that proper truncates of the Cauchy transform are compact operators and, then, we estimate the operator norm of the difference between $\mathcal{C}$ and those truncates.

Let μ be a positive Radon measure with compact support in $\mathbb{C}$. Let $z \in \operatorname{supp}\mu$ and let Q_z be a square containing the support of μ and centered at z. Let $\ell(Q_z)$ denote its side length. For $j \in \mathbb{N}$ we denote as $Q_j(z)$ the square centered at z and with side-length $2^{-j}\ell(Q_z)$. Moreover, we define

$$\Delta_j(z) := Q_j(z) \setminus Q_{j+1}(z).$$

Exploiting Hilbert-Schmidt's Theorem, a proof analogous to the one of Lemma 2.2.1 shows that the truncated operator

$$T_j f(z) := \int_{\Delta_j(z)} \frac{f(w)}{z-w} d\mu(w)$$

is a compact operator from $L^2(\mu)$ to $L^2(\mu)$. Let us define

$$\mathcal{C}_\mu^N f(w) := \sum_{j=0}^{N-1} T_j f(w)$$

and show that, under the hypothesis on the measure reported in the statement of Theorem 2.1, it converges in the $L^2(\mu)$–$L^2(\mu)$ operator norm to the Cauchy transform. This will prove that $\mathcal{C}_\mu \in K(L^2(\mu), L^2(\mu))$.

The kernel of $\mathcal{C}_\mu - \mathcal{C}_\mu^{N-1}$ is localized, so in order to estimate the $L^2(\mu)$-norm of this operator it suffices to apply the $T(1)$-Theorem (see [Tol14, Chapter 3]) to testing cubes contained in $Q_N(z)$ with $z \in \operatorname{supp}(\mu)$. More precisely, we can write

$$\begin{aligned}\|\mathcal{C}_\mu - \mathcal{C}_\mu^{N-1}\|_{L^2(\mu)\to L^2(\mu)} &\lesssim \sup_{z\in\operatorname{supp}\mu}\ \sup_{\widetilde{Q}\subseteq Q_N(z)} \Theta(\widetilde{Q}) + \sup_{z\in\operatorname{supp}\mu}\ \sup_{\widetilde{Q}\subseteq Q_N(z)} \frac{\|\mathcal{C}\chi_{\widetilde{Q}}\|_{L^2(\mu|_{\widetilde{Q}})}}{\mu(\widetilde{Q})^{1/2}} \\ &\equiv I_N + II_N.\end{aligned} \tag{2.3.16}$$

First, $I_N \to 0$ as $N \to \infty$ by the hypothesis (2) of Theorem 2.1 on the density of μ.

To show that $II_N \to 0$ as $N \to \infty$, it suffices to recall formula (2.3.15), which yields

$$\|\mathcal{C}_\mu \chi_{\widetilde{Q}}\|^2_{L^2(\mu|_{\widetilde{Q}})} \lesssim c^2(\mu|_{\widetilde{Q}}).$$

The ratio $c^2(\mu|_{\widetilde{Q}})/\mu(\widetilde{Q})$ has the correct behavior due to the condition (2) of Theorem 2.1. This concludes the proof of the equivalence of the conditions (a) and (b). In order to complete the proof of the theorem, it suffices to observe that the equivalence of (b) and (c) follows from (2.3.16).

2.4 An example: a generalized planar Cantor set

As an application of Theorem 2.1 we analyze the particular case of the planar Cantor sets (see e.g. [Gar72, p. 87]). Let $Q^0 := [0,1]^2$ be the unit square and let $\lambda := \{\lambda_n\}_{n=1}^\infty$ be a sequence of non-negative numbers such that $0 \le \lambda_n \le 1/2$ for every $n = 1, 2, \ldots$. The Cantor set is defined by means of an inductive construction:

- define 4 squares $\{Q_j^1\}_{j=1}^4$ of side length λ_1 such that each one of them contains a distinct vertex of Q_0 and call $E_1 := \cup_{j=1}^4 Q_j^1$.

- iterate the first step for each of the 4 cubes but using λ_2 as a scaling factor. As a result we get $2^4 = 16$ squares of side length $\sigma_2 = \lambda_1\lambda_2$. We denote those squares as $\{Q_j^2\}_j$. Then, define the second-step approximation of the Cantor set as $E_2 := \cup_{j=1}^{2^4} Q_j^2$.
- as a result of n analogous iterations, at the n-th step we get a collection of 4^n cubes $\{Q_j^n\}_j$ whose side length is $\sigma_n := \prod_{j=1}^n \lambda_j$ and a set $E_n := \cup_{j=1}^{4^n} Q_j^n$.

The planar Cantor set is defined as

$$E = E(\lambda) := \bigcap_{n=1}^{\infty} E_n.$$

We denote by p the canonical probability measure associated with $E(\lambda)$. In particular, p is uniquely identified by imposing that $p(Q) = 4^{-n}$ for every square that composes E_n. We denote by $\mathcal{C}_p$ the Cauchy transform associated with the measure p.

Let $\theta_k := 2^{-k}\sigma_k^{-1}$. It is known (see e.g. [Tol14], Lemma 4.29) that for the probability measure on the Cantor set, it holds that

$$c_p^2(x) \approx \sum_{k=0}^{\infty} \theta_k^2$$

for every $x \in E(\lambda)$.

As a consequence of Theorem 2.1, $\mathcal{C}_p$ is compact from $L^2(p)$ to $L^2(p)$ if and only if $\sum_{k=0}^{\infty} \theta_k^2$ converges. This condition holds if and only if $\mathcal{C}_p$ is bounded from $L^2(p)$ to $L^2(p)$ (see [MTV03]).

2.5 A counterexample to Theorem 2.1 for other kernels

A natural question is to ask if any analogue of Theorem 2.1 holds also for other singular integral operators of the form

$$Tf(z) = \int_{\mathbb{C}} K(z,w) f(w) d\mu(w),$$

where K is a kernel in a proper class and the singular integral operator has to be understood in the usual sense. For a kernel good enough so that the T(1)-theorem applies, similar considerations as the ones for the sufficiency in the proof of Theorem 2.1 apply. In particular, in order to have T is compact from $L^2(\mu)$ to $L^2(\mu)$ it suffices to require

1. $\Theta_\mu^*(z) = 0$ for every $z \in \mathbb{C}$.
2. $\|T\chi_Q\|^2_{L^2(\mu|_Q)}/\mu(Q) \to 0$ as $\ell(Q) \to 0$.
3. $\|T^*\chi_Q\|^2_{L^2(\mu|_Q)}/\mu(Q) \to 0$ as $\ell(Q) \to 0$.

However, these conditions turn out not to be necessary even in easy cases. An immediate example that shows that the density condition (1) is not necessary is the operator with kernel

$$K(z,w) = \frac{\mathrm{Im}(z-w)}{|z-w|^2}$$

and the measure $\mu = \mathcal{H}^1|_{((0,1)\times\{0\})}$.

This operator (trivially) belongs to $K(L^2(\mu), L^2(\mu))$ even though μ has positive linear density at each point of $(0,1) \times \{0\}$.

Chapter 3

L^2-boundedness of Gradients of Single Layer Potentials and Uniform Rectifiability

3.1 Introduction

The purpose of this chapter is to extend the solution of the codimension 1 David-Semmes problem for the Riesz transform to operators defined by gradients of single layer potentials associated with elliptic PDE's in divergence form with Hölder continuous coefficients. The single layer potential and its gradient play an important role in the solvability of this type of equations and also in the study of the corresponding elliptic measure. Recall that the David-Semmes problem deals with the connection between the Riesz transforms and rectifiability. This was solved in 1996 for the 1-dimensional Riesz transform (or equivalently, for the Cauchy transform) by Mattila, Melnikov and Verdera in [MMV96] by using the connection between Menger curvature and the Cauchy kernel. The case of codimension 1 was solved more recentely by Nazarov, Tolsa and Volberg in [NTV14a] by different methods, relying on the harmonicity of the codimension 1 Riesz kernel. The David-Semmes problem is still open in the remaining dimensions $n \in [2, d-2]$ in $\mathbb{R}^d$.

Given a Borel measure μ in $\mathbb{R}^d$ (from now on we assume all measures to be Borel in the present chapter), recall that its n-dimensional Riesz transform is defined by

$$\mathcal{R}^n\mu(x) = \int \frac{x-y}{|x-y|^{n+1}}\, d\mu(y),$$

whenever the integral makes sense. Also, for a function $f \in L^1_{loc}(\mu)$, we write $\mathcal{R}^n_\mu f(x) = \mathcal{R}^n(f\mu)(x)$.

The n-dimensional Hausdorff measure is denoted by $\mathcal{H}^n$. A set $E \subset \mathbb{R}^d$ is called n-rectifiable if there are Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$, $i = 1, 2, \dots$, such that

$$\mathcal{H}^n\Big(E \setminus \bigcup_i f_i(\mathbb{R}^n)\Big) = 0.$$

A set F is called purely n-unrectifiable if $\mathcal{H}^n(F \cap E) = 0$ for every n-rectifiable set E. As for sets, one can define a notion of rectifiabilty also for measures: a measure μ is said to be n-rectifiable if it vanishes outside an n-rectifiable set $E \subset \mathbb{R}^d$ and, moreover, it is absolutely continuous with respect to $\mathcal{H}^n|_E$.

In most of this work we deal with measures that present a certain degree of regularity. A measure μ in $\mathbb{R}^d$ is called n-AD-regular (or just AD-regular or Ahlfors-David regular) if there exists some constant $C_0 > 0$ such that

$$C_0^{-1} r^n \leq \mu(B(x,r)) \leq C_0\, r^n \quad \text{for all } x \in \operatorname{supp}(\mu) \text{ and } 0 < r \leq \operatorname{diam}(\operatorname{supp}(\mu)).$$

A set $E \subset \mathbb{R}^d$ is n-AD-regular if the measure $\mathcal{H}^n|_E$ is n-AD-regular.

The set E is called uniformly n-rectifiable if it is n-AD-regular and there exist $\theta, M > 0$ such that for all $x \in E$ and all $r > 0$ there is a Lipschitz mapping g from the ball $B_n(0,r)$ in $\mathbb{R}^n$ to $\mathbb{R}^d$ with $\operatorname{Lip}(g) \leq M$ such that

$$\mathcal{H}^n(E \cap B(x,r) \cap g(B_n(0,r))) \geq \theta r^n.$$

A measure μ is called uniformly n-rectifiable if it is n-AD-regular and its support is uniformly n-rectifiable.

It is easy to check that if a set (or a measure) is uniformly n-rectifiable, then it is also n-rectifiable. The converse implication is false. In fact, uniform n-rectifiability is a quantitative version of the notion of n-rectifiability introduced by David and Semmes [DS93]. One of their motivations to introduce this notion was the desire to find a good framework where one can study the $L^2(\mu)$ boundedness of singular integral operators. Indeed, they showed that if μ is n-AD-regular, the fact that μ is uniformly n-rectifiable is equivalent to the $L^2(\mu)$-boundedness of a sufficiently big class of singular integral operators with an odd and smooth enough Calderón-Zygmund kernel. In particular, if μ is uniformly n-rectifiable, then the n-dimensional Riesz transform $\mathcal{R}^n_\mu$ is bounded in $L^2(\mu)$.

The David-Semmes problem consists in proving that the converse statement holds. That is, that under the background assumption of n-AD-regularity on the measure μ, the $L^2(\mu)$ boundedness of the Riesz transform $\mathcal{R}^n_\mu$ implies the uniform n-rectifiability of μ. As mentioned above, the answer is only known (and positive) in the cases $n = 1$ and $n = d - 1$ in $\mathbb{R}^d$, by [MMV96] and [NTV14a], respectively.

The solution of the David-Semmes problem has had important applications to the solution of other relevant questions. In the dimension 1 case in the plane, this has played an essential role in the geometric characterization of removable singularities for bounded analytic functions, and in particular in the solution of Vitushkin's conjecture for sets with finite length by David [Dav98]. In the codimension 1 case, the analogous result involving the removable singularities for Lipschitz harmonic functions has been solved in [NTV14b]. Other remarkable applications of the solution of the David-Semmes problem in codimension 1 deal with the metric and geometric properties of harmonic measure. In particular, this is a key ingredient in the recent solution of two problems about harmonic measure raised by Christopher Bishop in the early 1990's [Bis92]. The first one is the fact that the mutual absolute continuity of harmonic measure for an open set $\Omega \subset \mathbb{R}^{n+1}$ with respect to the surface measure $\mathcal{H}^n$ in a subset of $\partial\Omega$ implies the rectifiability of that subset [Azz+16c]. The second one is the solution of the so called two-phase problem in the works [AMT17b] and [Azz+16d].

The results just mentioned also make sense for solutions of elliptic equations and for the elliptic measure. So in view of potential applications, it is natural to try to extend the solution of the David-Semmes problem to gradients of singular layer potentials, which are the analogues of the Riesz transform in the context of elliptic PDE's.

Next we introduce the precise ellipic PDE's in which we are interested. Let $A = (a_{ij})_{1\leq i,j\leq n+1}$ be an $(n+1)\times(n+1)$ matrix whose entries $a_{ij}\colon \mathbb{R}^{n+1} \to \mathbb{R}$ are measurable functions in $L^\infty(\mathbb{R}^{n+1})$. Assume also that there exists $\Lambda > 0$ such that

$$\Lambda^{-1}|\xi|^2 \leq \langle A(x)\xi, \xi\rangle, \quad \text{for all } \xi \in \mathbb{R}^{n+1} \text{ and a.e. } x \in \mathbb{R}^{n+1}, \tag{3.1.1}$$

$$\langle A(x)\xi, \eta\rangle \leq \Lambda|\xi||\eta|, \quad \text{for all } \xi, \eta \in \mathbb{R}^{n+1} \text{ and a.e. } x \in \mathbb{R}^{n+1}. \tag{3.1.2}$$

We consider the elliptic equation

$$L_A u(x) := -\text{div}\,(A(\cdot)\nabla u(\cdot))\,(x) = 0, \tag{3.1.3}$$

which should be understood in the distributional sense. We say that a function $u \in W^{1,2}_{\text{loc}}(\Omega)$ is a *solution* of (4.1.3) or L_A*-harmonic* in an open set $\Omega \subset \mathbb{R}^{n+1}$ if

$$\int A\nabla u \cdot \nabla\varphi = 0, \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

We denote by $\mathcal{E}_A(x, y)$, or just by $\mathcal{E}(x, y)$ when the matrix A is clear from the context, the *fundamental solution* for L_A in $\mathbb{R}^{n+1}$, so that $L_A\mathcal{E}_A(\cdot, y) = \delta_y$ in the distributional sense, where δ_y is the Dirac mass at the point $y \in \mathbb{R}^{n+1}$. For a construction of the fundamental solution under the assumption (4.1.1) and (4.1.2) on the matrix A we refer to [HK07]. For a measure μ, the function $f(x) = \int \mathcal{E}_A(x, y)\, d\mu(y)$ is usually known as the *single layer potential* of μ. We consider the singular integral operator T whose kernel is

$$K(x, y) = \nabla_1\mathcal{E}_A(x, y) \tag{3.1.4}$$

(the subscript 1 means that we take the gradient with respect to the first variable), so that

$$T\mu(x) = \int K(x, y)\, d\mu(y) \tag{3.1.5}$$

when x is away from $\text{supp}(\mu)$. That is, $T\mu$ is the gradient of the single layer potential of μ.

Given a function $f \in L^1_{loc}(\mu)$, we set also

$$T_\mu f(x) = T(f\,\mu)(x) = \int K(x, y)f(y)\, d\mu(y), \tag{3.1.6}$$

and, for $\varepsilon > 0$, we consider the ε-truncated version

$$T_\varepsilon\mu(x) = \int_{|x-y|>\varepsilon} K(x, y)\, d\mu(y).$$

We also write $T_{\mu,\varepsilon}f(x) = T_\varepsilon(f\mu)(x)$. We say that the operator T_μ is bounded in $L^2(\mu)$ if the operators $T_{\mu,\varepsilon}$ are bounded in $L^2(\mu)$ uniformly on $\varepsilon > 0$.

In the special case when A is the identity matrix, $-L_A$ is the Laplacian and T is the n-dimensional Riesz transform up to a constant factor depending only on the dimension n.

Without any hypothesis on the smoothness of the coefficients of the matrix A, one cannot expect the kernel $K(\cdot, \cdot)$ in (4.1.4) to be of Calderón-Zygmund type, and thus we need to impose some regularity condition on A. We say that the matrix A is Hölder continuous with exponent α (or briefly C^α continuous), if there exists $\alpha > 0$ and $C_h > 0$ such that

$$|a_{ij}(x) - a_{ij}(y)| \leq C_h|x - y|^\alpha \quad \text{for all } x, y \in \mathbb{R}^{n+1} \text{ and } 1 \leq i, j \leq n + 1. \tag{3.1.7}$$

Under this assumption on the coefficients, the kernel $K(\cdot, \cdot)$ turns out to be locally of Calderón-Zygmund type (see Lemma 4.2.1 for more details). However, we remark that in general $K(\cdot,\cdot)$ is neither homogeneous (of degree $-n$) nor antisymmetric (even locally).

Our main result is the following.

Theorem 3.1. *Let μ be a compactly supported n-AD-regular measure in $\mathbb{R}^{n+1}$. Let A be an elliptic matrix satisfying (4.1.1), (4.1.2) and (4.1.5), and let T_μ be the associated operator given by (4.1). The operator T_μ is bounded in $L^2(\mu)$ if and only if μ is uniformly n-rectifiable.*

The assumption that μ is compactly supported in the theorem above is necessary and it is due to the fact that the C^α continuity of the matrix A is a property which is not scale invariant. We also remark that it is already known that T_μ is bounded in $L^2(\mu)$ if μ is uniformly n-rectifiable (see Theorem 2.5 from [CMT19]). Our contribution is the converse statement.

Theorem 3.1 should be compared to a recent result obtained by Conde-Alonso, Mourgoglou and Tolsa in [CMT19], which in a sense complements our theorem. The precise result is the following.

Theorem 3.2 ([CMT19])**.** *Let μ be a non-zero Borel measure in $\mathbb{R}^{n+1}$. Let A be an elliptic matrix satisfying (4.1.1), (4.1.2) and (4.1.5), and let T_μ be the associated operator. Suppose that the upper density $\limsup_{r\to 0}\frac{\mu(B(x,r))}{(2r)^n}$ is positive μ-a.e. in $\mathbb{R}^{n+1}$, and the lower density $\liminf_{r\to 0}\frac{\mu(B(x,r))}{(2r)^n}$ vanishes μ-a.e. in $\mathbb{R}^{n+1}$. Then T_μ is not bounded in $L^2(\mu)$.*

Notice that, in the case $\mu=\mathcal{H}^n|_E$, the assumptions on the upper and lower densities in the theorem above imply that E is purely n-unrectifiable. This theorem extends an analogous result proved previously by Eiderman, Nazarov and Volberg [ENV14] for the n-dimensional Riesz transform.

Our proof of Theorem 3.1 follows the same scheme as the proof of the corresponding result for the Riesz transform in [NTV14a]. In particular, it also relies on a variational argument which uses the fact that L_A-harmonic functions satisfy a maximum principle. It also uses the so-called BAUP criterion of David and Semmes [DS93, p. 139]. However, there are some important differences between our arguments and the ones in [NTV14a]. An important one is that we use a martingale difference decomposition in terms of the David-Semmes lattice, instead of the quasiorthogonality arguments in [NTV14a]. We think that using a martingale decomposition makes the whole construction much more transparent. Further, the quasiorthogonality arguments seem to require the antisymmetry of the kernel, which does not hold in our case. On the other hand, the fact that the matrix A is non-constant makes our arguments and estimates more involved and technical. For example, the reflection trick required to apply later the variational argument is more delicate, as well as the approximation techniques used to transfer estimates among different measures (see Section 3.8 below). The reader can find the scheme of the proof of Theorem 3.1 at the end of Section **??**.

By combining Theorem 3.1 and Theorem A from [CMT19], we are also able to derive the following rectifiability result for general sets.

Theorem 3.3. *Let $E\subset\mathbb{R}^{n+1}$ be a compact set with $\mathcal{H}^n(E)<\infty$. Let A and T be as in Theorem 3.1. If $T_{\mathcal{H}^n|_E}$ is bounded in $L^2(\mathcal{H}^n|_E)$, then E is n-rectifiable.*

The analogous result in case that A is the identity and T is the n-dimensional Riesz transform (modulo some constant factor) has been proved in [NTV14b]. Theorem

3.3 is proved almost in the same way as in [NTV14b]: by an argument inspired by a covering theorem of Pajot, one decomposes $\mu = \mathcal{H}^n|_E$ into a measure μ_0 with vanishing lower density and a countable collection of measures μ_k such that each μ_k can be extended to another n-AD-regular measure $\widetilde{\mu}_k$ such that $T_{\widetilde{\mu}_k}$ is bounded in $L^2(\widetilde{\mu}_k)$. Theorem A implies that $\mu_0 \equiv 0$, and Theorem 3.1 implies that each measure $\widetilde{\mu}_k$ is uniformly n-rectifiable. The only specific feature of the Riesz kernel that is used in [NTV14b] is its antisymmetry. As mentioned above, we cannot ensure that the kernel $K(\cdot,\cdot)$ is anti°symmetric. However, this is not a problem in our case because by Lemma 3.2.5 below it turns out that, for any measure μ with growth of degree n (see (3.2.1) for the definition), T_μ is bounded in $L^2(\mu)$ if and only if the operator $T_\mu^{(a)}$ associated with the antisymmetric part of $K(\cdot,\cdot)$ is bounded in $L^2(\mu)$. Then, in order to prove Theorem 3.3 we just apply the same arguments as in [NTV14b] to $T_\mu^{(a)}$ instead of the n-dimensional Riesz transform.

An important application of Theorem 3.3 deals with elliptic measure. Given a Wiener regular open set $\Omega \subset \mathbb{R}^{n+1}$, the elliptic measure (or L_A-harmonic measure) for Ω with pole at $p \in \Omega$ is the probability measure $\omega^p_{L_A}$ supported on $\partial\Omega$ such that, for every $f \in C_0(\partial\Omega)$, $\int f\, d\omega^p_{L_A}$ equals the value at p of the L_A-harmonic extension of f to Ω. For a basic reference on elliptic measure, see [HMT92], and for some additional background see [Azz+16a, Section 2.4], for example. Analogously to harmonic measure, the connection between the metric properties of elliptic measure and the geometric properties of Ω (in particular, the rectifiability of $\partial\Omega$) has been a subject of intense investigation in the last years. See for example the works [Akm+17], [Azz+16a], [Hof+15], [HMT10], [HMT], [Ken+16]. Our result in connection with elliptic measure is the following.

Theorem 3.4. *Let $n \geq 2$ and let A be an elliptic matrix satisfying (4.1.1), (4.1.2) and (4.1.5). Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be a bounded open connected Wiener regular set, let $p \in \Omega$, and let $\omega^p_{L_A}$ be the elliptic measure in Ω associated with L_A, with pole p. Suppose that there exists a set $E \subset \partial\Omega$ such that $0 < \mathcal{H}^n(E) < \infty$ and that the elliptic measure $\omega^p_{L_A}|_E$ is absolutely continuous with respect to $\mathcal{H}^n|_E$. Then $\omega^p_{L_A}|_E$ is n-rectifiable.*

Remark that $\omega^p_{L_A}|_E$ being n-rectifiable means that it is concentrated on an n-rectifiable set and it is absolutely continuous with respect to $\mathcal{H}^n|_E$. In the case of $-L_A$ being the Laplacian and ω_{L_A} the harmonic measure, the same result has been proved in [Azz+16c], and it can be considered as a kind of converse of the famous Riesz brothers theorem on harmonic measure in planar simply connected domains. The preceding result follows from Theorem 3.3 by essentially the same arguments as the ones for harmonic measure in [Azz+16c]. Nevertheless, for the reader's convenience the arguments are sketched in the final Section 3.12.

3.2 Preliminaries

3.2.1 General notation

We use the standard notation $a \lesssim b$ if there is a fixed constant $C > 0$ (depending on other fixed parameters, such as the ambient dimension) such that $a \leq Cb$. To make the dependence of the constant on a parameter t explicit, we will also write $a \lesssim_t b$. We will also write $b \gtrsim a$ if $a \lesssim b$ and $a \approx b$ if both $a \lesssim b$ and $b \lesssim a$.

We use the notation $B(x,r)$ for the open ball in $\mathbb{R}^{n+1}$ centered at x of radius r. For a ball $B = B(x,r)$ and $a > 0$ we write $aB = B(x,ar)$ for the centered rescaling of the ball. For $0 < r < R$, we denote by

$$A(x, r, R) := \{y \in \mathbb{R}^{n+1} : r < |x - y| < R\}$$

the open annulus centered at x with radii r and R. Also, given $t > 0$ and a set E, we write

$$\mathcal{U}_t(E) := \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, E) \leq t\}$$

for the closed t-neighborhood of E.

Given a measure μ, we write $\langle \cdot, \cdot \rangle_\mu$ for the scalar product in $L^2(\mu)$ and $m_{\mu,E} f := \mu(E)^{-1} \int_E f d\mu$ for the μ-average of a measurable function f on a set E.

We denote by $AD(C_0, \mathbb{R}^d)$ the set of n-AD-regular measures on $\mathbb{R}^d$ with constant C_0. We say that μ has growth of degree n (or n-growth) if

$$\mu(B(x, r)) \leq C\, r^n \quad \text{for all } x \in \mathbb{R}^{n+1}. \tag{3.2.1}$$

We denote the Lebesgue measure in $\mathbb{R}^{n+1}$ by $\mathcal{L}^{n+1}$. Quite often we will also use the standard notations dx or dy when integrating against this measure.

Given a matrix $A(\cdot)$ with variable coefficients, we denote by $A^T(\cdot)$ its transpose.

3.2.2 David-Semmes dyadic cubes

In this section we collect some standard definitions and results that we need throughout the rest of the chapter. Let us start by introducing a dyadic system of (so-called) cubes associated with an AD-regular measure μ. They were introduced by David (see [Dav88], [Dav91, Appendix 1] and also the work of Christ [Chr90]). We remark that in the general case they are not euclidean cubes, so that in case of ambiguity we also refer to them as David-Semmes cubes or μ-cubes.

Definition 3.2.1 (David-Semmes lattice $\mathcal{D}_\mu$)**.** Let $\mu \in AD(C_0, \mathbb{R}^{n+1})$. The David and Semmes' lattice $\mathcal{D}_\mu$ associated with μ is a countable disjoint union of families of Borel sets, that we denote as $\mathcal{D}_\mu^j$. The elements of $\mathcal{D}_\mu^j$ are called dyadic μ-cubes (or just cubes) of the j-th generation and satisfy the following properties:

1. $\mathcal{D}_\mu^j$ is a partition of $\operatorname{supp}\mu$. This means that $\operatorname{supp}\mu = \bigcup_{Q \in \mathcal{D}_\mu^j} Q$ and $Q \cap Q' = \varnothing$ for every $Q, Q' \in \mathcal{D}_\mu^j$ with $Q \neq Q'$.
2. If $Q \in \mathcal{D}_\mu^j$ and $Q' \in \mathcal{D}_\mu^k$ for $k \geq j$, then either $Q' \subset Q$ or $Q \cap Q' = \varnothing$.
3. For every k and $Q \in \mathcal{D}_\mu^k$ we have
$$2^{-k} \lesssim \operatorname{diam} Q \leq 2^{-k}$$
and
$$\mu(Q) \approx 2^{-kn}.$$
4. The cubes have thin boundary, i.e. there exist two constants $C, \gamma_0 > 0$ depending on C_0 and the dimension n such that for every $\varepsilon > 0$ and $Q \in \mathcal{D}_\mu^k$ we have
$$\begin{aligned} \mu\{x \in Q \colon \operatorname{dist}(x, \operatorname{supp}\mu \setminus Q) < \varepsilon 2^{-k}\} \\ + \mu\{x \in \operatorname{supp}\mu \setminus Q \colon \operatorname{dist}(x, Q) < \varepsilon 2^{-k}\} \leq C \varepsilon^{\gamma_0} \mu(Q). \end{aligned} \tag{3.2.2}$$
5. For $Q \in \mathcal{D}_\mu^k$ there exists a point $x_Q \in Q$, also called *center* of Q, such that
$$\operatorname{dist}(x_Q, \operatorname{supp}\mu \setminus Q) \gtrsim 2^{-k}.$$

We need to associate a typical side length to each cube. For $Q \in \mathcal{D}_\mu^k$, the natural temptation is to define $\ell(Q) := 2^{-k}$. However, we have to take into account that a cube may belong to $\mathcal{D}_\mu^j \cap \mathcal{D}_\mu^k$ for some $j \neq k$. A solution to this problem is to think about a cube as a couple (Q, k), so that the side length is now well defined. Bearing this in mind, in what follows we decide to omit this occurrence and simply indicate a cube by Q. We also associate the ball $B_Q := B(x_Q, \ell(Q))$ with Q.

For $Q \in \mathcal{D}_\mu^k$, we denote by

$$\mathsf{Ch}(Q) := \{P \in \mathcal{D}_\mu^{k+1} \colon P \subset Q\}$$

the family of children of Q.

3.2.3 β and α-numbers

Let us consider a ball $B = B(x, r) \subset \mathbb{R}^{n+1}$ and a Radon measure μ. For a hyperplane L in $\mathbb{R}^{n+1}$, we set

$$\beta_\mu^L(B) := \sup_{x \in \operatorname{supp}\mu \cap B} \frac{\operatorname{dist}(x, L)}{r}, \qquad \beta_{\mu,1}^L(B) := \frac{1}{r^n} \int_B \frac{\operatorname{dist}(x, L)}{r}\, d\mu(x),$$

and taking the infimum over all the hyperplanes L in $\mathbb{R}^{n+1}$, we define

$$\beta_\mu(B) := \inf_L \beta_\mu^L(B), \qquad \beta_{\mu,1}(B) := \inf_L \beta_{\mu,1}^L(B).$$

Let μ, ν be two Radon measures on $\mathbb{R}^{n+1}$. We define the distance

$$d_B(\mu, \nu) = \sup_f \int f d(\mu - \nu),$$

where the supremum is taken over all 1-Lipschitz functions whose support is contained in B. Given a hyperplane L, we define

$$\alpha_\mu^L(B) := \frac{1}{r^{n+1}} \inf_{c \geq 0} d_B(\mu, c\mathcal{H}^n|_L)$$

and

$$\alpha_\mu(B) := \inf_L \alpha_\mu^L(B),$$

where the infimum is taken over all hyperplanes.

For an n-AD-regular measure μ and a ball B such that $\frac{1}{2}B \cap \operatorname{supp}\mu \neq \varnothing$, the following inequalities are standard (see [DS93, p. 27] and [Tol09]):

$$\beta_\mu^L(B)^{n+1} \lesssim \beta_{\mu,1}^L\left(\tfrac{3}{2}B\right) \lesssim \alpha_\mu^L(2B).$$

Given a hyperplane H through the origin, we also denote

$$\beta_\mu^{(H)}(B) = \inf_L \beta_\mu^L(B), \qquad \alpha_\mu^{(H)}(B) = \inf_L \alpha_\mu^L(B),$$

where in both cases the infimum is taken over all hyperplanes L which are parallel to H.

3.2.4 Carleson packing condition and Riesz families

The following are standard definitions.

Definition 3.2.2 (Carleson packing condition)**.** We say that $\mathcal{F} \subset \mathcal{D}_\mu$ is a Carleson family if there exists a constant $C > 0$ such that for every $P \in \mathcal{D}_\mu$ we have

$$\sum_{Q \in \mathcal{F}, Q \subset P} \mu(Q) \leq C\mu(P).$$

Definition 3.2.3 (Riesz families and Riesz systems)**.** Let $\{\psi_Q\}_{Q\in\mathcal{D}_\mu}$ be a family of functions in $L^2(\mu)$. We say that $\{\psi_Q\}_{Q\in\mathcal{D}_\mu}$ forms a Riesz family with constant $C > 0$ if

$$\Big\| \sum_{Q\in\mathcal{D}_\mu} a_Q \psi_Q \Big\|^2_{L^2(\mu)} \leq C \sum_{Q\in\mathcal{D}_\mu} a_Q^2$$

for any sequence $\{a_Q\}_Q$ of real numbers with finitely many non-zero terms. The family $\{\Psi_Q\}_{Q\in\mathcal{D}_\mu}$ of sets of functions is said to be a Riesz system with constant $C > 0$ if $\{\psi_Q\}_{Q\in\mathcal{D}_\mu}$ is a Riesz family with constant C for every choice of $\psi_Q \in \Psi_Q$.

A particular Riesz system that is useful for our purposes is the so-called *Haar system.* Let N be a positive integer. Given $Q \in \mathcal{D}_\mu$ and $C > 0$, we define $\Psi_Q^{Haar}(N)$ as the set of functions ψ such that

1. $\operatorname{supp}\psi \subset Q$.
2. ψ is constant on every μ-cube Q' which is N levels down from Q.
3. $\int \psi d\mu = 0$ and $\int \psi^2 d\mu \leq C$.

The set of functions $\Psi_Q^{Haar}(N)$ forms a Riesz family with constant C.

Let $\{\Psi_Q\}_{Q\in\mathcal{D}_\mu}$ be a Riesz system. For any $Q \in \mathcal{D}_\mu$ and $\widetilde{M} > 1$ we define

$$\xi_{\widetilde{M}}(Q) \coloneqq \inf_{\substack{E: E\supset \widetilde{M}B_Q,\\ \mu(E)<+\infty}} \sup_{\psi\in\Psi_Q} \mu(Q)^{-1/2} |\langle T_\mu \chi_E, \psi\rangle_\mu|.$$

Lemma 3.2.1. *Let $\delta > 0$ and $\widetilde{M} > 1$. If T_μ is bounded in $L^2(\mu)$, then the family*

$$\mathcal{F}_\delta \coloneqq \{Q \in \mathcal{D}_\mu : \xi_{\widetilde{M}}(Q) > \delta\}$$

is Carleson.

Proof. See [NTV14a, Section 14]. There the proof is presented in the case of the Riesz transform, but it works without any difference in our framework. □

3.2.5 Partial Differential Equations

For any uniformly elliptic matrix A with Hölder continuous coefficients, one can show that $K(x, y) = \nabla_1 \mathcal{E}(x, y)$ is locally a Calderón-Zygmund kernel:

Lemma 3.2.2. *Let A be an elliptic matrix with Hölder continuous coefficients satisfying* (4.1.1)*,* (4.1.2) *and* (4.1.5)*. If $K(\cdot,\cdot)$ is given by* (4.1.4)*, then it is locally a Calderón-Zygmund kernel. That is, for any given $R > 0$,*

(a) $|K(x, y)| \lesssim |x - y|^{-n}$ *for all $x, y \in \mathbb{R}^{n+1}$ with $x \neq y$ and $|x - y| \leq R$.*
(b) $|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \lesssim |y - y'|^\alpha |x - y|^{-n-\alpha}$ *for all $y, y' \in B(x, R)$ with $2|y - y'| \leq |x - y|$.*
(c) $|K(x, y)| \lesssim |x - y|^{(1-n)/2}$ *for all $x, y \in \mathbb{R}^{n+1}$ with $|x - y| \geq 1$.*

All the implicit constants in (a), (b) and (c) depend on Λ and C_h, while the ones in (a) and (b) depend also on R.

The statements above are rather standard. For more details, see Lemma 2.1 from [CMT19].

Let ω_n denote the surface measure of the unit sphere of $\mathbb{R}^{n+1}$. For any elliptic matrix A_0 with constant coefficients, we have an explicit expression for the fundamental solution of L_{A_0}, which we denote by $\Theta(x, y; A_0)$. More precisely, $\Theta(x, y; A_0) = \Theta(x - y; A_0)$ with

$$\Theta(z; A_0) = \Theta(z; A_{0,s}) = \begin{cases} \dfrac{-1}{(n-1)\omega_n \sqrt{\det A_{0,s}}} \dfrac{1}{(A_{0,s}^{-1} z \cdot z)^{(n-1)/2}} & \text{for } n \geq 3, \\[2ex] \dfrac{1}{4\pi \sqrt{\det A_{0,s}}} \log\left(A_{0,s}^{-1} z \cdot z\right) & \text{for } n = 2, \end{cases} \tag{3.2.3}$$

where $A_{0,s}$ is the symmetric part of A_0, that is, $A_{0,s} = \frac{1}{2}(A + A^T)$.

As a consequence of (4.2.3), we have

$$\nabla\Theta(z; A_0) = \frac{1}{\omega_n \sqrt{\det A_{0,s}}} \frac{A_{0,s}^{-1} z}{(A_{0,s}^{-1} z \cdot z)^{(n+1)/2}}. \tag{3.2.4}$$

The next result is proven in Lemma 2.2 of [KS11].

Lemma 3.2.3. *Let A be an elliptic matrix with Hölder continuous coefficients satisfying (4.1.1), (4.1.2) and (4.1.5). Let also $\Theta(\cdot, \cdot; \cdot)$ be given by (4.2.3). Then, for all $x, y \in \mathbb{R}^{n+1}$ with $x \neq y$ and $|x - y| \leq R$,*

1. $|\mathcal{E}_A(x, y) - \Theta(x, y; A(x))| \lesssim |x - y|^{\alpha - n + 1}$,
2. $|\nabla_1 \mathcal{E}_A(x, y) - \nabla_1 \Theta(x, y; A(x))| \lesssim |x - y|^{\alpha - n}$,
3. $|\nabla_1 \mathcal{E}_A(x, y) - \nabla_1 \Theta(x, y; A(y))| \lesssim |x - y|^{\alpha - n}$.

Similar inequalities hold if we reverse the roles of x and y and we replace ∇_1 by ∇_2. All the implicit constants depend on Λ, C_h, and R.

The following lemma is an easy consequence of the preceding result.

Lemma 3.2.4. *Let μ be a compactly supported n-AD-regular measure in $\mathbb{R}^{n+1}$. Let A be an elliptic matrix satisfying (4.1.1), (4.1.2) and (4.1.5), and let T_μ be the associated operator given by (4.1). Let $A_s = \frac{1}{2}(A + A^T)$ be the symmetric part of A. Consider the operator*

$$T_\mu^s f(x) = \int \nabla_1 \mathcal{E}_{A_s}(x, y)\, f(y)\, d\mu(y).$$

Then, $T_\mu - T_\mu^s$ is compact in $L^p(\mu)$, for $1 < p < \infty$. In particular, T_μ is bounded in $L^2(\mu)$ if and only if T_μ^s is bounded in $L^2(\mu)$.

Recall that $\mathcal{E}_{A_s}$ stands for the fundamental solution of $L_{A_s} u := -\operatorname{div}(A_s \nabla u)$.

Proof. For any function $f \in L^p(\mu)$, we have

$$T_\mu f(x) - T_\mu^s f(x) = \int \big(\nabla_1 \mathcal{E}_A(x, y) - \nabla_1 \mathcal{E}_{A_s}(x, y)\big) f(y)\, d\mu(y).$$

By (4.2.3) $\Theta(x, y; A(x)) = \Theta(x, y; A_s(x))$ and thus, by Lemma 4.2.2, the kernel of $T_\mu - T_\mu^s$ satisfies, for all $x, y \in \mathbb{R}^{n+1}$ with $x \neq y$ and $|x - y| \leq R$,

$$\begin{aligned} |\nabla_1 \mathcal{E}_A(x, y) - \nabla_1 \mathcal{E}_{A_s}(x, y)| &\leq |\nabla_1 \mathcal{E}_A(x, y) - \nabla_1 \Theta(x, y; A(x))| \\ &\quad + |\nabla_1 \Theta(x, y; A_s(x)) - \nabla_1 \mathcal{E}_{A_s}(x, y)| \\ &\lesssim \frac{1}{|x - y|^{n - \alpha}}. \end{aligned}$$

By standard arguments, using the AD-regularity of μ, this implies that $T_\mu - T^s_\mu$ is compact, and thus bounded in $L^p(\mu)$. □

Because of the preceding lemma, it is clear that to prove Theorem 3.1 we can assume that the matrix A is symmetric. So in the rest of the chapter *we will assume A to be symmetric.*

By almost the same arguments as above we derive that

$$\big|\nabla_1\mathcal{E}_A(x,y)+\nabla_1\mathcal{E}_A(y,x)\big| \lesssim \frac{1}{|x-y|^{n-\alpha}} \quad \text{for all } x,y\in\mathbb{R}^{n+1},\ x\neq y, \text{ and } |x-y|\leq R. \tag{3.2.5}$$

So, modulo the regularizing kernel $|x-y|^{-(n-\alpha)}$, $\nabla_1\mathcal{E}_A(x,y)$ behaves as if it were antisymmetric. In particular, we have the following result.

Lemma 3.2.5. *Let μ be a compactly supported n-AD-regular measure in $\mathbb{R}^{n+1}$. Let A be an elliptic matrix satisfying (4.1.1), (4.1.2) and (4.1.5), and let T_μ be the associated operator given by (4.1), with kernel $K(x,y)=\nabla_1\mathcal{E}_A(x,y)$. Consider the antisymmetric operator $T^{(a)}_\mu$ and the symmetric operator $T^{(s)}_\mu$ associated with the kernels*

$$K^{(a)}(x,y)=\frac{1}{2}\big(K(x,y)-K(y,x)\big) \qquad \text{and} \qquad K^{(s)}(x,y)=\frac{1}{2}\big(K(x,y)+K(y,x)\big)$$

respectively, so that $T_\mu = T^{(a)}_\mu + T^{(s)}_\mu$. Then the operator $T^{(s)}_\mu$ is compact in $L^p(\mu)$, for $1<p<\infty$. In particular, T_μ is bounded in $L^2(\mu)$ if and only if $T^{(a)}_\mu$ is bounded in $L^2(\mu)$.

Contrarily to the natural temptation at this point, in the rest of the chapter we *do not* assume the kernel to be antisymmetic. This is because our proof heavily relies on a maximum principle (see, for example, Lemma 3.11.2), which cannot be ensured to hold if we work just with the antisymmetric part.

From Lemma 3.2.5 we derive the existence of a “weak limit operator”.

Proposition 3.2.1. *Let μ be a compactly supported n-AD-regular measure in $\mathbb{R}^{n+1}$. Let A be an elliptic matrix satisfying (4.1.1), (4.1.2) and (4.1.5), and let T_μ be the associated operator given by (4.1). Suppose that T_μ is bounded in $L^2(\mu)$. Then, for all $1<p<\infty$ and $f\in L^p(\mu)$, $T_{\mu,\varepsilon}f$ has a weak limit in $L^p(\mu)$ as $\varepsilon\to 0$. Further, denoting by $T^w_\mu f$ such a weak limit, the operator T^w_μ is bounded in $L^p(\mu)$ for $1<p<\infty$ and, for all $f\in L^p(\mu)$,*

$$T_\mu f(x) = T^w_\mu f(x) \quad \text{for } \mu\text{-a.e. } x\in\operatorname{supp}\mu\setminus\operatorname{supp} f.$$

Recall that saying that $T_{\mu,\varepsilon}f$ has a weak limit $T^w_\mu f$ in $L^p(\mu)$ as $\varepsilon\to 0$ means that for all $g\in L^{p'}(\mu)$,

$$\lim_{\varepsilon\to 0}\int T_{\mu,\varepsilon}f\, g\, d\mu = \int T^w_\mu f\, g\, d\mu.$$

Proof. Consider the antisymmetric and symmetric operators $T^{(a)}_\mu$, $T^{(s)}_\mu$ from Lemma 3.2.5, so that, for all $\varepsilon>0$,

$$T_{\mu,\varepsilon}f = T^{(a)}_{\mu,\varepsilon}f + T^{(s)}_{\mu,\varepsilon}f.$$

Since $T^{(a)}_\mu$ is antisymmetric, for all $f\in L^p(\mu)$, $1<p<\infty$, the functions $T^{(a)}_{\mu,\varepsilon}f$ converge weakly in $L^p(\mu)$ as $\varepsilon\to 0$. This was shown by Mattila and Verdera in [MV95] and an alternative argument is provided in [NTV14a].

Concerning the symmetric operator $T^{(s)}_\mu$, from the estimate (3.2.5) it easily follows that $T^{(s)}_{\mu,\varepsilon} f$ converges to $T^{(s)}_\mu f = T^{(s)}_{\mu,0} f$ strongly in $L^p(\mu)$, and thus also weakly in $L^p(\mu)$. Hence, $T_{\mu,\varepsilon} f$ admits a weak limit in $L^p(\mu)$ as $\varepsilon \to 0$.

The last statement in the lemma follows by standard arguments. $\square$

From now on, for μ and T_μ as above, when T_μ is bounded in $L^2(\mu)$ we will identify T_μ with the weak limit operator T^w_μ, so that for any function $f \in L^p(\mu)$, $T_\mu f$ makes sense as a function in $L^p(\mu)$.

3.3 The flattening lemmas and the alternating layers

From this section until the end of Section 3.11 we assume that μ is an n-AD-regular measure with compact support and that T_μ is bounded in $L^2(\mu)$. In order to prove Theorem 3.1 we have to show that μ is uniformly n-rectifiable.

3.3.1 Existence of balls with small β-number

We want to prove that in any ball centered at a point of $\operatorname{supp}\mu$ either we can find a ball, which is not too small, in which the measure is very flat or we have a lower bound for a regularized two-sided truncation of $T\mu$ at some point and at proper scales.

Let $\psi_0\colon [0,+\infty) \to [0,1]$ be a continuous function such that $\psi_0(x) = 1$ for $x \leq 1$ and $\psi_0(x) = 0$ for $x \geq 2$. For $z \in \mathbb{R}^{n+1}$ and $0 < r_1 < r_2$, we define

$$\psi_{z,r_1,r_2}(x) := \psi_0\Big(\frac{|z-x|}{r_2}\Big) - \psi_0\Big(\frac{|z-x|}{r_1}\Big).$$

We have that $\operatorname{supp}\psi_{z,r_1,r_2} \subset B(z,r_2) \setminus B(z,r_1)$ and $0 \leq \psi_{z,r_1,r_2} \leq 1$. The proof of the following lemma relies on a touching point argument and it is based on the scheme of the proof of [Tol15, Lemma 3.3]. We remark that this can also be proved via a variation on the blow-up argument in [NTV14a, Lemma 5].

Lemma 3.3.1. *Let $\mu \in AD(C_0, \mathbb{R}^{n+1})$, $R \leq 4$ and let $B = B(x,R)$ be a ball centered at $\operatorname{supp}\mu$. Let $K, \varepsilon > 0$. There is $\rho = \rho(K,\varepsilon,C_0)$ small enough such that at least one of the two following conditions is verified:*

1. *There exists a ball $B(x',r) \subset B(x,R)$ centered at $\operatorname{supp}\mu$ with $r \in [\rho R, R]$ such that*
$$\beta_\mu(B(x',r)) \leq \varepsilon.$$
2. *There is a point $z \in \operatorname{supp}\mu \cap B(x,R/4)$ and $r \in [\rho R, R]$, such that*
$$|T(\psi_{z,\rho R,r}\mu)(z)| > K.$$

Before reporting the proof, we remark that the assumption $R \leq 4$ in the statement of the lemma is justified by the fact that we are interested in applying this result to the balls associated with David-Semmes cubes with small enough side length.

Proof. Suppose that the alternative (1) in the statement of the lemma does not hold. Then

$$\beta_\mu(B(x',r)) > \varepsilon \tag{3.3.1}$$

for every $x' \in \operatorname{supp}\mu \cap B$ and $r \in [\rho R, R]$ such that $B(x',r) \subset B$.

Being the measure μ n-AD-regular, by standard arguments it follows that there exists an open ball B' contained in $\frac{1}{4}B$ such that $B' \cap \operatorname{supp}\mu = \varnothing$ and $r(B') \geq c_1 R$ with $c_1 = c_1(n, C_0)$. Possibly by taking a dilation of this ball, we can suppose that

$B' \cap \operatorname{supp}\mu = \varnothing$ but there is at least a point $z \in \partial B' \cap \operatorname{supp}\mu$. Without loss of generality, let $z = 0$ and suppose that $\vec{n} := (0, \dots, 0, 1)$ is the outer normal vector to $\partial B'$ at z. Since $B' \subset \frac{1}{4}B$, we also have that $r(B') \leq R/4$.

We denote by L the hyperplane $\{x\colon x\cdot\vec{n} = 0\}$, by U the upper half space $\{x\colon x\cdot\vec{n} > 0\}$ and by D the lower one $D := \mathbb{R}^{n+1} \setminus (U \cup L)$. For $0 < \rho \ll 1$ to be chosen later and for $j \geq 0$, we denote by B_j the ball centered at 0 and with radius

$$r(B_j) := \Big(\frac{2}{\varepsilon}\Big)^j \rho\, R.$$

Let j be such that $r(B_j) \leq r(B')$. Short geometric computations prove the inequality

$$\operatorname{dist}(y, L) \leq \frac{1}{2}\frac{r(B_j)^2}{r(B')} \quad \text{for every } y \in D \cap B_j \setminus B'. \tag{3.3.2}$$

We denote $\vec{v} := A(0)^T\vec{n}$. Using the definition of $\vec{v}$ and (4.2), we get that there exists $c_2 > 0$ such that

$$\vec{v} \cdot \nabla_1\Theta(0, y; A(0)) \geq c_2 \frac{\vec{v} \cdot A(0)^{-1}y}{|y|^{n+1}} = c_2 \frac{\vec{n} \cdot y}{|y|^{n+1}} > 0 \quad \text{for every } y \in U. \tag{3.3.3}$$

Choose now an integer $N > 1$ such that $r := r(B_N) \leq r(B')$. As a direct application of Lemma 4.2.2 and the growth of μ, we can find two constants $c_3, c_3' > 0$ such that

$$\begin{aligned} \Big| \int_{D\cap B(0,r)} &\vec{v} \cdot \big(\nabla_1 \mathcal{E}(0, y) - \nabla_1\Theta(0, y; A(0))\big)\psi_{0,\rho R,r}(y)d\mu(y)\Big| \\ &\leq c_3' \int_{D\cap B} \frac{1}{|y|^{n-\alpha}} d\mu(y) \leq c_3 R^\alpha. \end{aligned} \tag{3.3.4}$$

Let χ_{0,r_1,r_2} be the characteristic function of the annulus centered at 0 with inner and outer radius r_1 and r_2 respectively. Then, choosing ρ small enough to get $r > 2\rho R$ and using (3.3.3), we have that

$$\begin{aligned} \int_{U\cap B(0,r)} &\vec{v} \cdot \nabla_1\Theta(0, y; A(0))\psi_{0,\rho R,r}(y)d\mu(y) \\ &\geq \int_{U\cap B(0,r)} \vec{v} \cdot \nabla_1\Theta(0, y; A(0))\chi_{0,2\rho R,r}(y)d\mu(y). \end{aligned} \tag{3.3.5}$$

Since $\beta_\mu(B_j) \geq \varepsilon$ by hypothesis (3.3.1), we have that there exists $y \in \operatorname{supp}\mu \cap B_j$ whose distance from L is greater than $\varepsilon r(B_j)$. As a consequence of (3.3.2), the point y cannot belong to D if

$$\varepsilon\, r(B_j) \geq \frac{1}{2}\frac{r(B_j)^2}{r(B')},$$

which implies that $y \in U \cap B_j$ for every $r(B_j) \leq 2\varepsilon r(B')$. Since $\mu \in AD(n, C_0, \mathbb{R}^{n+1})$, assuming ε small enough if necessary, it follows that

$$\mu\big(U \cap B_{j+1} \setminus (B_{j-1} \cup \mathcal{U}_{\varepsilon r(B_j)/2}(L))\big) \geq C_0^{-1}c(\varepsilon)r(B_j)^n,$$

for some constant $c(\varepsilon) > 0$, where $\mathcal{U}_{\varepsilon r(B_j)/2}(L)$ stands for the $\varepsilon r(B_j)/2$-neighborhood of L. Taking into account (3.3.3), for $j \geq 0$ we deduce that

$$\int_{U\cap B_{j+1}\setminus B_{j-1}} \vec{v}\cdot\nabla_1\Theta(0,y;A(0))d\mu(y)$$
$$\geq \mu\big(U\cap B_{j+1}\setminus(B_{j-1}\cup\mathcal{U}_{\varepsilon r(B_j)/2}(L))\big)\frac{\varepsilon r(B_j)}{2r(B_{j-1})^{n+1}}$$
$$\geq C_0^{-1}c(\varepsilon).$$

for some constant $c(\varepsilon)$. Therefore

$$\int_{U\cap B(0,r)} \vec{v}\cdot\nabla_1\Theta(0,y;A(0))\chi_{0,2\rho R,r}(y)d\mu(y)$$
$$=\sum_{j=1}^{N}\int_{U\cap B_j\setminus B_{j-1}} \vec{v}\cdot\nabla_1\Theta(0,y;A(0))d\mu(y)\geq C_0^{-1}\sum_{j=2}^{N-1}c(\varepsilon)=C_0^{-1}c(\varepsilon)(N-2). \tag{3.3.6}$$

Now we need to study the analogous integrals for the lower half-space. As in (3.3.4), we have

$$\Big|\int_{U\cap B(0,r)} \vec{v}\cdot\big(\nabla_1\mathcal{E}(0,y)-\nabla_1\Theta(0,y;A(0))\big)\psi_{0,\rho R,r}(y)d\mu(y)\Big|\leq c_4R^\alpha \tag{3.3.7}$$

for some $c_4>0$. Moreover, by (3.3.2) and the growth of μ,

$$\sum_{j=1}^{N}\Big|\int_{D\cap B_j\setminus B_{j-1}} \vec{v}\cdot\nabla_1\Theta(0,y;A(0))d\mu(y)\Big|$$
$$\leq c_5\sum_{j=1}^{N}\Big|\int_{D\cap B_j\setminus B_{j-1}}\frac{\operatorname{dist}(y,L)}{|y|^{n+1}}d\mu(y)\Big|\leq c_5C_0c(\varepsilon)\sum_{j=1}^{N}\frac{r(B_j)}{r(B')}\leq C_0c_6(\varepsilon). \tag{3.3.8}$$

Gathering (3.3.4), (3.3.5), (3.3.6), (3.3.7) and (3.3.8) we get

$$\vec{v}\cdot T(\psi_{z,\rho R,r}\mu)(z)\geq C_0^{-1}c(\varepsilon)(N-2)-(c_3+c_4)R^\alpha-C_0c_6(\varepsilon),$$

which gives the desired estimate for N big enough (which forces ρ to be small enough).
□

3.3.2 Existence of balls and cubes with small α-number

For a euclidean cube Q of side length $\ell(Q)$ and given an hyperplane L, we define

$$\alpha_\mu^L(Q):=\frac{1}{\ell(Q)^{n+1}}\inf_{c\geq 0}d_Q(\mu,c\mathcal{H}^n_{|L}),$$

where d_Q is defined as in (3.2.3) and

$$\alpha_\mu(Q):=\inf_L\alpha_\mu^L(Q),$$

where the infimum is taken over all hyperplanes.

Lemma 3.3.2 (Existence of α-flat euclidean cubes)**.** *Let $\mu\in AD(C_0,\mathbb{R}^{n+1})$. For all $M'>10$ and $\varepsilon'>0$, there is ε small enough such that the following holds. If the ball*

$B = B(x,r)$ is such that $\beta_\mu^L(B) < \varepsilon$ for some affine hyperplane L passing through x, then there exists a euclidean cube Q such that

- *$3M'Q \subset B$.*
- *there is a constant $\rho_0 = \rho_0(\varepsilon')$ such that $\rho_0 r \le \ell(Q) \le r/M'$.*
- *there exists a positive constant $C_0' = C_0'(C_0, n)$ such that $\mu(Q) \ge C_0'^{-1}\ell(Q)^n$ and Q has C_0'-thin boundary, i.e.*

$$\mu\big(\{x \in 2Q : \mathrm{dist}(x, \partial Q) \le \lambda\ell(Q)\}\big) \le C_0'\lambda\mu(2Q) \qquad \textit{for all} \qquad \lambda > 0.$$

- *$\alpha_\mu^L(3M'Q) \le \varepsilon'$.*

Proof. See [GT18, Lemma 3.2]. We remark that this lemma was originally stated in a more general setting than the one of AD-regular measures. □

As a consequence of Lemma 3.3.2 we have the following.

Lemma 3.3.3 (Existence of α-flat balls)**.** *Let $\mu \in AD(C_0, \mathbb{R}^{n+1})$. Let $B = B(x,r)$ be a ball centered at $x \in \operatorname{supp}\mu$ with $\beta_\mu^L(B) < \varepsilon$ for some hyperplane L and some ε small enough. For every $M > 10$ and $\tilde\varepsilon > 0$, if ε is small enough there exists $\widetilde B = B(\tilde x, \tilde r)$ with $\tilde x \in \operatorname{supp}\mu$ such that:*

1. *$M\widetilde B \subset B(x,r)$.*
2. *$\tilde r > \sigma r$ for some constant σ depending on $\widetilde\varepsilon$.*
3. *$\alpha_\mu^L(M\widetilde B) \le \widetilde\varepsilon$.*

Proof. Let $M' \ge \max\{M, 10\}$ and let $\varepsilon' > 0$ to be chosen later. Let Q, C_0' and ρ_0 be as in Lemma 3.3.2. Since Q has C_0'-thin boundary, the measure μ cannot be concentrated in a too small neighborhood of ∂Q. Indeed, suppose that $\mu(Q \setminus Q_\lambda) = 0$. For $\lambda > 0$, using the AD-regularity and Lemma 3.3.2, we have

$$\mu(Q) \le \mu(Q_\lambda) \le \lambda C_0'\mu(2Q) \lesssim \lambda C_0 C_0'\ell(Q)^n$$

and

$$C_0'^{-1}\ell(Q)^n \le \mu(Q),$$

so that $\lambda \gtrsim C_0^{-1}C_0'^{-2}$. This leads to a contradiction for λ small enough, depending just on C_0 and C_0'.

Thus, there exists a point $\widetilde z \in \operatorname{supp}\mu$ such that $\mathrm{dist}(\widetilde z, \partial Q) > \sigma'\ell(Q)$ for some constant $\sigma' > 0$ depending on C_0 and n.

Define $\widetilde r := \mathrm{dist}(\widetilde z, \partial Q)$ and $\widetilde B := B(\widetilde z, \widetilde r)$. The first point of the lemma is satisfied with $\sigma = \sigma'\rho_0$. The point (2) is also true by construction, since $B(\tilde x, M\widetilde r) \subset 3M'Q \subset B(x,r)$.

We are left to prove that we can choose ε' and M' such that (3) is verified, too. This follows easily after observing that

$$\begin{aligned}\alpha_\mu^L(B(\tilde x, M\widetilde r)) &= \frac{1}{(M\widetilde r)^{n+1}} \inf_{c \ge 0} d_B(\mu, c\mathcal{H}^n_{|L}) \\ &\le \Big(\frac{3M'\ell(Q)}{M\widetilde r}\Big)^{n+1} \frac{1}{(3M'\ell(Q))^{n+1}} \inf_{c\ge 0} d_{3M'Q}(\mu, c\mathcal{H}^n_{|L}) < \Big(\frac{3M'}{M\sigma}\Big)^{n+1}\varepsilon'.\end{aligned}$$

The proof is completed by choosing ε' and M' such that $(3M'/M\sigma)^{n+1}\varepsilon' \le \widetilde\varepsilon$. □

Our proof of the existence of balls and cubes with small α-number relies on the following result by Girela-Sarrión and Tolsa.

Note that the $L^2(\mu)$-boundedness of any singular integral operator is not required in the previous lemma, so the statement is purely geometric.

The scheme of the proof of the next lemma resembles that of [NTV14a, Section 15].

Lemma 3.3.4. *For every $M > 1$ and $\bar{\varepsilon} > 0$ there exist an integer N, a finite set $\mathcal{H}$ of hyperplanes through the origin and a Carleson family $\mathcal{F} \subset \mathcal{D}_\mu$ with the following property. If $P \in \mathcal{D}_\mu \setminus \mathcal{F}$, there exist $H \in \mathcal{H}$ and a cube $Q \subset P$ at most N levels down from P for which*

$$\alpha_\mu^{(H)}(MB_Q) \leq \bar{\varepsilon}. \tag{3.3.9}$$

Proof. The idea is to combine Lemma 3.3.1 and Lemma 3.3.3. We fix a cube and we show that either the condition (1) in Lemma 3.3.1 is verified, so that we can find a ball with small β-number and apply Lemma 3.3.3, or the cube belongs to a Carleson family. We define the family $\mathcal{F}$ as the collection of cubes for which condition (1) in Lemma 3.3.1 does not apply.

Let $P \in \mathcal{D}_\mu$ and let $R := \ell(P)$. Let ε and K be as in Lemma 3.3.1, to be chosen later. We analyze the two different cases starting from the “flat" one.

Case (1).

Suppose that there is $\rho > 0$ such that $r > \rho R$ and we can find a ball $B(z, r) \subset B(x_P, R)$ with

$$\beta_\mu^L(B(z, r)) \leq \varepsilon$$

for some hyperplane L. Let H be a hyperplane through the origin whose normal spans an angle at most ε with the normal to L. Elementary geometric considerations lead to

$$\beta_\mu^{(H)}(B(z, r)) \leq 2\varepsilon.$$

It is possible to suppose that H belongs to a finite family $\mathcal{H}$ of hyperplanes: it suffices to define $\mathcal{H}$ as the family of hyperplanes whose normal vectors form an ε-net on the unit sphere $\mathbb{S}^n$.

By Lemma 3.3.3 for every $\widetilde{\varepsilon} > 0$ to be chosen later and ε small enough (depending on $\widetilde{\varepsilon}$) there are $\sigma > 0$ and a ball $B(\widetilde{z}, 2(M+2)\widetilde{r})$ such that $\widetilde{r} > \sigma r$ and

$$\alpha_\mu^{(H)}\big(B(\widetilde{z}, 2(M+2)\widetilde{r})\big) \leq \widetilde{\varepsilon}. \tag{3.3.10}$$

Take a point $z' \in \operatorname{supp}\mu$ such that $|\widetilde{z} - z'| < \widetilde{\varepsilon}\widetilde{r}$. We choose the cube $Q \in \mathcal{D}_\mu$ as the one such that $z' \in Q$ and $\widetilde{r} \leq \ell(Q) \leq 2\widetilde{r}$. For $\widetilde{\varepsilon} < 1$ we have

$$|\tilde{z} - x_Q| \leq |z' - x_Q| + |z' - \tilde{z}| < \ell(Q) + \widetilde{\varepsilon}\widetilde{r} < 2\ell(Q).$$

Now we use the stability of the α-number under small shifts and proper rescalings to compare $\alpha_\mu^{(H)}(MB_Q)$ to $\alpha_\mu^{(H)}(B(\widetilde{z}, 2(M+2)\widetilde{r}))$ and, hence, to prove that it is small. Being $M > 1$, we have $(M+2)/3 < M$. So, using the inclusions

$$MB_Q \subset B(x_Q, 2M\widetilde{r}) \subset B(\tilde{z}, 2(M+2)\widetilde{r}),$$

for some plane L parallel to H we can write

$$\begin{aligned}\alpha_\mu^{(H)}(MB_Q) &= \alpha_\mu^L(MB_Q)\\ &= \frac{1}{(M\ell(Q))^{n+1}}\inf_{c\geq 0} d_{MB_Q}(\mu, c\mathcal{H}^n|_L)\\ &\leq \frac{2^{n+1}}{(2M\widetilde{r})^{n+1}}\inf_{c\geq 0} d_{B(x_Q,2M\widetilde{r})}(\mu, c\mathcal{H}^n|_L)\\ &\leq \Big(\frac{2(M+2)}{M}\Big)^{n+1}\frac{1}{(2(M+2)\widetilde{r})^{n+1}}\inf_{c\geq 0} d_{B(\widetilde{z},2(M+2)\widetilde{r})}(\mu, c\mathcal{H}^n|_L)\\ &\leq 6^{n+1}\alpha_\mu^{(H)}\big(B(\widetilde{z},2(M+2)\widetilde{r})\big).\end{aligned}$$

Then, recalling (3.3.10) we have

$$\alpha_\mu^{(H)}(MB_Q) \leq 6^{n+1}\widetilde{\varepsilon}.$$

The proof of (3.3.9) is completed by choosing ε such that $\bar{\varepsilon} = 6^{n+1}\widetilde{\varepsilon}$, where $\bar{\varepsilon}$ is as in the statement of the lemma. The cube Q is at most N levels down from P for some N that, being $\ell(Q) \geq \ell(P)\sigma\rho/2$, satisfies

$$N \leq \log_2 \frac{\ell(P)}{\ell(Q)} \leq 1 - \log_2\rho - \log_2\sigma. \tag{3.3.11}$$

Again, we remark that the estimate in the right hand side of (3.3.11) depends just on M and $\bar{\varepsilon}$.

Case (2).

Let z be a point in $\operatorname{supp}\mu \cap B(x, R/4)$, such that

$$|T(\psi_{z,\rho R,r}\mu)(z)| > K.$$

Let Q be the largest μ-cube containing z with $\ell(Q) < r/32$ and let Q' be the largest μ-cube containing z with $\ell(Q') < \rho R/32$. Then $Q' \subset Q \subset P$.

The idea of this part of the proof is to apply Lemma 3.2.1 to prove that the family $\mathcal{F}$ of μ-cubes P for which case (2) applies is Carleson. To this purpose, consider the set $E = 10B_P$, which contains $B(z, 2R)$. We claim that there is a constant $\widetilde{C}$ such that

$$|m_{\mu,Q}(T_\mu\chi_E) - m_{\mu,Q'}(T_\mu\chi_E)| \geq K - \widetilde{C}. \tag{3.3.12}$$

To prove this, we consider two continuous functions f_1 and f_2 with $|f_1|, |f_2| \leq 1$ and such that

$$\chi_E = f_1 + \psi_{z,\rho R,r} + f_2,$$

$\operatorname{supp} f_1 \subset B(z, 2\rho r)$ and $\operatorname{supp} f_2 \cap B(z, r) = \varnothing$.

Using the $L^2(\mu)$-boundedness of T_μ, the regularity of the measure and the fact that $Q' \subset Q$, we have

$$\begin{aligned}\int |T_\mu f_1|^2 d\mu \lesssim \int |f_1|^2 d\mu \leq \mu(\operatorname{supp} f_1)\\ \leq \mu(B(z, 2\rho R)) \lesssim (\rho R)^n \lesssim \ell(Q')^n \lesssim \mu(Q') \leq \mu(Q),\end{aligned}$$

which yields that there exists a constant $C_1 > 0$ such that

$$|m_{\mu,Q}(T_\mu f_1) - m_{\mu,Q'}(T_\mu f_1)| \leq |m_{\mu,Q}(T_\mu f_1)| + |m_{\mu,Q'}(T_\mu f_1)| \leq C_1. \tag{3.3.13}$$

Using $L^2(\mu)$-boundedness again we have

$$\|T_\mu \psi_{z,\rho R,r}\|_{L^2(\mu)} \lesssim \|\psi_{z,\rho R,r}\|_{L^2(\mu)} \leq \mu(B(z,2r))^{1/2} \lesssim r^{n/2} \lesssim \ell(Q)^{n/2} \lesssim \mu(Q)^{1/2},$$

which implies that there exists a constant $C_2 > 0$ such that

$$|m_{\mu,Q}(T_\mu \psi_{z,\rho R,r})| \leq C_2. \tag{3.3.14}$$

By the choice of Q', we have that $Q' \subset B(z, \rho R/2)$. Indeed

$$Q' \subset B(z', 8\ell(Q')) \subset B(z', \rho R/4) \subset B(z, \rho R/2). \tag{3.3.15}$$

Being $B(z,\rho R) \cap \operatorname{supp} \mu = \varnothing$, we have the following estimate for the Hölder norm:

$$\|T_\mu \psi_{z,\rho R,r}\|_{C^\alpha(B(z,\rho R/2))} \lesssim (\rho R)^{-\alpha}, \tag{3.3.16}$$

so there exists a constant $C_3 > 0$ such that for every $y \in Q'$

$$\begin{aligned} |m_{\mu,Q'}(T_\mu \psi_{z,\rho R,r})| &\geq |T_\mu(\psi_{z,\rho R,r})(y)| - |m_{\mu,Q'}(T_\mu \psi_{z,\rho R,r}) - T_\mu(\psi_{z,\rho R,r})(y)| \\ &\geq K - \|T_\mu \psi_{z,\rho R,r}\|_{C^\alpha(B(z,\rho R/2))} \operatorname{dist}(Q', \operatorname{supp} \psi_{z,\rho R,r})^\alpha \geq K - C_3. \end{aligned} \tag{3.3.17}$$

Gathering (3.3.14) and (3.3.17) we get

$$|m_{\mu,Q'}(T_\mu \psi_{z,\rho R,r}) - m_{\mu,Q}(T_\mu \psi_{z,\rho R,r})| \geq K - C_2 - C_3. \tag{3.3.18}$$

Let us estimate the difference between the averages of $T_\mu f_2$ over the μ-cubes Q and Q'. Arguing as in (3.3.15) and (3.3.16), we have that $Q \subset B(z, r/2)$ and

$$\|T_\mu f_2\|_{C^\alpha(B(z,r/2))} \lesssim \ell(Q)^{-\alpha},$$

so there exists a constant $C_4 > 0$ such that

$$|m_{\mu,Q}(T_\mu f_2) - m_{\mu,Q'}(T_\mu f_2)| \leq C_4. \tag{3.3.19}$$

Gathering (3.3.13), (3.3.18) and (3.3.19), we prove the claim (3.3.12). Now, if we choose $\theta > 0$ and we define

$$\psi_P := \big(\theta \ell(P)\big)^{n/2} \Big(\frac{\chi_Q}{\mu(Q)} - \frac{\chi_{Q'}}{\mu(Q')} \Big),$$

as a consequence of (3.3.12) we get

$$\begin{aligned} &\mu(P)^{-1/2} |\langle T_\mu \chi_E, \psi_P \rangle_\mu| \\ &\qquad = \mu(P)^{-1/2} \big(\theta \ell(P)\big)^{n/2} |m_{\mu,Q}(T_\mu \chi_E) - m_{\mu,Q'}(T_\mu \chi_E)| \geq C_0^{-1/2} \theta^{n/2} (K - \widetilde{C}). \end{aligned} \tag{3.3.20}$$

We remark that θ serves as a normalizing factor in order to get a bound on the $L^2(\mu)$ norm of ψ_P. In this way, we have that ψ_P belongs to the Haar system $\Psi_P^{Haar}(N)$ of depth

$$N = \log_2(\ell(P)/\ell(Q)) \leq \log_2 \theta^{-1} + \widetilde{C}$$

so that we can combine (3.3.20) and Lemma 3.2.1. Indeed, recalling the definition of $\xi_{\widetilde{M}}(P)$ provided in (3.2.4), (3.3.20) proves that $\xi_5(P) \geq C_0^{-1/2} \theta^{n/2} (K - \widetilde{C})$, which implies that $\mathcal{F}$ is a Carleson family for K big enough. □

As an immediate corollary of the preceding lemma we get the following.

Lemma 3.3.5. *For every $M > 1$ and $\bar{\varepsilon} > 0$ there exist an integer N' and a finite set $\mathcal{H}$ of hyperplanes through the origin with the following property: for every $P \in \mathcal{D}_\mu$, there exist $H \in \mathcal{H}$ and a cube $Q \subset P$ at most N' levels down from P for which $\alpha_\mu^{(H)}(MB_Q) \leq \bar{\varepsilon}$.*

Proof. Consider the family $\mathcal{F}$ in the preceding lemma. Since this is a Carleson family, for any $P \in \mathcal{D}_\mu$ there exists some $P' \in \mathcal{D}_\mu \setminus \mathcal{F}$ contained in P with $\ell(P') \approx \ell(P)$. Then, by definition, there exists a cube $Q \subset P'$, with $\ell(Q) \approx \ell(P') \approx \ell(P)$ and such that $\alpha_\mu^{(H)}(MB_Q) \leq \bar{\varepsilon}$ for some hyperplane $H \in \mathcal{H}$. □

3.3.3 The alternating layers

A general feature of non-Carleson families is that, for every positive integer K_0, it is possible to find a μ-cube and $(K_0 + 1)$ layers of finitely many cubes so that each of them tiles up the initial cube up to a set of small measure (for the details see [NTV14a, Lemma 7]). This result can be refined by finding intermediate layers of very flat cubes using Lemma 3.3.5. For the proof of the following lemma we refer to [NTV14a, Section 16].

Lemma 3.3.6. *Let $\varepsilon > 0$, $M > 1$ and let H be a hyperplane through the origin in $\mathbb{R}^{n+1}$. Let $\mathcal{A} \subset \mathcal{D}_\mu$ be a non-Carleson family such that each $Q \in \mathcal{A}$ contains a cube $Q' \in \mathcal{D}_\mu$ at most N' levels down from Q such that $\alpha_\mu^{(H)}(MB_{Q'}) < \varepsilon$. Then, for every positive integer K and every $\eta > 0$ there exist a cube $R_0 \in \mathcal{A}$ and $(K+1)$ alternating pairs of finite layers $\mathcal{NB}_k$ and $\mathcal{FL}_k$ in $\mathcal{D}_\mu$ with $k = 0, 1, \ldots, K$ such that the following properties hold*

1. *$\mathcal{NB}_0 = \{R_0\}$.*
2. *$\mathcal{NB}_k \subset \{Q \in \mathcal{D}_\mu \colon Q \subset R_0\} \cap \mathcal{A}$ for any $k = 0, \ldots, K$.*
3. *for every $k = 0, \ldots, K$ and $Q \in \mathcal{FL}_k$ we have*
$$\alpha_\mu^{(H)}(MB_Q) < \varepsilon.$$
4. *for every $k = 0, \ldots, K$ and $Q \in \mathcal{FL}_k$ there exists a cube $P \in \mathcal{NB}_k$, $P \supset Q$.*
5. *for every $k = 1, \ldots, K$ and $P' \in \mathcal{NB}_k$ there exists a cube $Q \in \mathcal{FL}_{k-1}$, $P' \subset Q$.*
6. *$\sum_{Q \in \mathcal{FL}_K} \mu(Q) \geq (1 - \eta)\mu(R_0)$.*

We will apply Lemma 3.3.6 to the study of non-BAUP cubes (see the next section for the definition); this explains the choice of the notation '$\mathcal{NB}_k$' for some layers. The other layers are denoted as '$\mathcal{FL}_k$' to indicate that they consist of quite flat cubes (i.e. with a small α-number).

Remark 5. The property 6 in the lemma says that $\mathcal{FL}_K$ tiles up R_0 up to a set of negligible measure. If follows that the same holds for any $\mathcal{FL}_k$ for every $k = 0, \ldots, K$. Moreover, as a consequence of the inductive construction in [NTV14a], the lattice

$$\mathcal{FL} = \bigcup_k \mathcal{FL}_k$$

has only finitely many elements.[1] This is useful for technical purposes.

[1] Each of the so-called *non-Carleson layers* $\{\mathcal{L}_m\}_{m=0}^M$ appearing in [NTV14a, Section 13] is finite.

3.4 The non-BAUP cubes and the martingale difference decomposition

The acronym BAUP referred to a μ-cube literally stands for *Bilaterally Approximable by a Union of Planes*. Being more suitable to our purposes, in what follows we prefer to formulate the equivalent definition of non-BAUP cubes as in [NTV14a, Section 22], instead of the original definition of David and Semmes in [DS93].

Definition 3.4.1 (Non-BAUP cube)**.** A cube $Q \in \mathcal{D}_\mu$ is said to be non-BAUP with parameter $\delta > 0$ (or non-δ-BAUP) if there exists a point $z_Q^a \in Q \cap \operatorname{supp} \mu$ such that for every affine hyperplane L passing through z_Q^a we can find a point $z_Q^b \in L \cap B(z_Q^a, \ell(Q))$ such that $B(z_Q^b, \delta\ell(Q)) \cap \operatorname{supp} \mu = \varnothing$.

A geometric criterion for uniform rectifiability provided by David and Semmes (see [DS93]) asserts that if, for any parameter $\delta > 0$, the cubes which are non-δ-BAUP form a Carleson family, then μ is uniformly rectifiable.

To prove Theorem 3.1 we will use the BAUP criterion. *We will assume that, for some $\delta > 0$, the family of non-BAUP cubes with parameter δ is non-Carleson and we will get a contradiction.* Our assumption implies that, for some $H \in \mathcal{H}$ and all $\varepsilon > 0$, $M > 1$ (to be chosen below), the family $\mathcal{A} = \mathcal{A}(M, \varepsilon, H, N')$ of cubes $Q \in \mathcal{D}_\mu$ which are non-BAUP with parameter δ and contain a cube $Q' \in \mathcal{D}_\mu$ at most N' levels down from Q such that $\alpha_\mu^{(H)}(MB_{Q'}) < \varepsilon$ is also non-Carleson. So we can apply Lemma 3.3.6 with this family $\mathcal{A}$ to construct the layers of cubes $\mathcal{NB}_k$ and $\mathcal{FL}_k$ with the parameters η and K in the lemma to be chosen below.

We remark now a property that will be used later on: for $R \in \mathcal{FL}_k$ and $Q \subset R$ such that $Q \in \mathcal{NB}_{k+1}$ for some k, we have

$$\ell(Q) \leq C\varepsilon\delta^{-1}\ell(R). \tag{3.4.1}$$

In particular, for any $\Delta > 0$, choosing $\varepsilon\delta^{-1} \ll \Delta$, one has $\ell(Q) \ll \Delta\ell(R)$.

Let $R_0 \in \mathcal{D}_\mu$ be as in Lemma 3.3.6. We are interested in partitioning the collection of cubes contained in R_0 and below a suitable subfamily of cubes that we denote Top_1 (see (3.4.2) for its definition) into subfamilies (the so-called *trees*) with intermediate layers of non-δ-BAUP cubes like in [NTV14a]. We proceed via a stopping time argument.

A collection $\mathcal{T} \subset \mathcal{D}_\mu$ is a *tree* if the following properties hold:

- $\mathcal{T}$ has a maximal element (with respect to inclusion) $Q(\mathcal{T})$ which contains all the other elements of $\mathcal{T}$ as subsets of $\mathbb{R}^{n+1}$. The cube $Q(\mathcal{T})$ is called the *root* of $\mathcal{T}$.
- If Q, Q_0 belong to $\mathcal{T}$ and $Q \subset Q_0$, then any cube $Q' \in \mathcal{D}_\mu$ such that $Q \subset Q' \subset Q_0$ also belongs to $\mathcal{T}$.
- If $Q \in \mathcal{T}$, then either all the sons belong to $\mathcal{T}$ or none of them do.

Now we proceed to build the trees. For $1 \leq k \leq K - 1$, we denote

$$\mathsf{Top}_k := \{Q \in \mathsf{Ch}(Q') : Q' \in \mathcal{FL}_k\} \tag{3.4.2}$$

and, for $Q \in \mathsf{Top}_k$,

$$\mathcal{NB}(Q) := \{Q' \in \mathcal{NB}_{k+1} : Q' \subset Q\} \quad \text{and} \quad \mathsf{Stop}(Q) := \{Q' \in \mathcal{FL}_{k+1} : Q' \subset Q\}.$$

Note that $\mathcal{NB}(Q)$ and $\mathsf{Stop}(Q)$ are finite families because $\mathcal{NB}_{k+1}$ and $\mathcal{FL}_{k+1}$ are finite.

We write $\mathsf{Top} := \bigcup_{k=1}^{K-1} \mathsf{Top}_k$. Now, for every $Q \in \mathsf{Top}$ we let $\mathsf{Tree}(Q)$ be the collection of μ-cubes which are contained in Q and are not strictly contained in any cube from $\mathsf{Stop}(Q)$. Clearly Q is the root of $\mathsf{Tree}(Q)$.

For $f \in L^2(\mu)$ and $Q \in \mathcal{D}_\mu$ we denote

$$\Delta_Q f = \sum_{S \in \mathsf{Ch}(Q)} m_{\mu,S}(f)\chi_S - m_{\mu,Q}(f)\chi_Q, \tag{3.4.3}$$

so that we have the orthogonal expansion

$$\chi_{R_0}\big(f - m_{\mu,R_0}(f)\big) = \sum_{Q \in \mathcal{D}_\mu : Q \subset R_0} \Delta_Q f,$$

in the $L^2(\mu)$-sense. Then, taking $f = T\mu$ (recall that this function makes sense because of Proposition 3.2.1) and using the notation $T_R\mu := \sum_{Q \in \mathsf{Tree}(R)} \Delta_Q T\mu$ for $R \in \mathsf{Top}$, we can write

$$\int_{R_0} |T\mu - m_{\mu,R_0}(T\mu)|^2 \, d\mu = \sum_{Q \in \mathcal{D}_\mu : Q \subset R_0} \|\Delta_Q T\mu\|^2_{L^2(\mu)} \geq \sum_{R \in \mathsf{Top}} \|T_R\mu\|^2_{L^2(\mu)}.$$

Since T_μ is bounded from $L^\infty(\mu)$ to $BMO(\mu)$, the left hand side is bounded above by $\mu(R_0)$, and thus we get

$$\sum_{R \in \mathsf{Top}} \|T_R\mu\|^2_{L^2(\mu)} \leq C\, \mu(R_0). \tag{3.4.4}$$

Let $0 < \eta \ll 1$ (to be chosen later) be the parameter defining the lattice of alternating layers from Lemma 3.3.6. Denote by Nice the subfamily of the cubes $R \in \mathsf{Top}$ such that

$$\sum_{Q \in \mathsf{Stop}(R)} \mu(Q) \geq (1 - \eta^{1/2})\, \mu(R).$$

The following easy lemma concerns the abundance of Nice cubes.

Lemma 3.4.1. *We have*

$$\sum_{R \in \mathsf{Top} \setminus \mathsf{Nice}} \mu(R) \leq (K-1)\eta^{1/2}\mu(R_0).$$

Proof. By construction, the cubes $R \in \mathsf{Top} \setminus \mathsf{Nice}$ satisfy

$$\mu(R) \leq \frac{1}{\eta^{1/2}}\, \mu\Bigg(R \setminus \bigcup_{Q \in \mathsf{Stop}(R)} Q\Bigg) \leq \frac{1}{\eta^{1/2}}\, \mu\Bigg(R \setminus \bigcup_{Q \in \mathcal{FL}_K} Q\Bigg)$$

Thus, recalling that $\sum_{Q \in \mathcal{FL}_K} \mu(Q) \geq (1-\eta)\mu(R_0)$ and that there are $K-1$ layers of cubes in the family Top, we get

$$\begin{aligned}\sum_{R \in \mathsf{Top} \setminus \mathsf{Nice}} \mu(R) &\leq \frac{1}{\eta^{1/2}} \sum_{R \in \mathsf{Top} \setminus \mathsf{Nice}} \mu\Bigg(R \setminus \bigcup_{Q \in \mathcal{FL}_K} Q\Bigg) \\ &\leq \frac{K-1}{\eta^{1/2}} \mu\Bigg(R_0 \setminus \bigcup_{Q \in \mathcal{FL}_K} Q\Bigg) \leq \frac{(K-1)\eta}{\eta^{1/2}}\, \mu(R_0) = (K-1)\eta^{1/2}\mu(R_0).\end{aligned}$$

$\square$

The main ingredient for the proof of Theorem 3.1 is the following result.

Proposition 3.4.1. *Assume that ε and η are chosen small enough in the construction of the alternating layers in Lemma 3.3.6, depending on δ. Then there is $c_1 > 0$ depending also on δ such that for every $R \in \mathsf{Nice}$ with $\ell(R)$ small enough we have*

$$\|T_R\mu\|^2_{L^2(\mu)} \geq c_1\mu(R). \tag{3.4.5}$$

We remark that the smallness condition on the Nice cubes in the proposition depends just on δ, the Hölder and elliptic conditions on the matrix A and the AD-regularity of μ.

Proof of Theorem 3.1 using Proposition 3.4.1. By Lemma 3.4.1 and the property (6) in Lemma 3.3.6, assuming $\eta \leq 1/4$, we have

$$\begin{aligned}\sum_{R\in\mathsf{Nice}} \mu(R) &\geq \sum_{R\in\mathsf{Top}} \mu(R) - (K-1)\eta^{1/2}\mu(R_0)\\ &\geq \sum_{k=1}^{K-1}\sum_{Q\in\mathcal{FL}_k} \mu(Q) - (K-1)\eta^{1/2}\,\mu(R_0)\\ &\geq (K-1)(1-\eta-\eta^{1/2})\mu(R_0) \geq \frac{1}{4}(K-1)\,\mu(R_0).\end{aligned}$$

Denote by Nice' the family of Nice cubes R which are small enough so that (3.4.5) holds for them. Clearly

$$\sum_{R\in\mathsf{Nice}} \mu(R) \leq \sum_{R\in\mathsf{Nice}'} \mu(R) + C'\,\mu(R_0),$$

with C' depending on the smallness condition for R and on $\operatorname{diam}(\operatorname{supp}\mu)$. By (3.4.4), we have

$$\sum_{R\in\mathsf{Nice}'} \mu(R) \leq c_1(\delta)^{-1} \sum_{R\in\mathsf{Nice}'} \|T_R\mu\|^2_{L^2(\mu)} \leq c_1(\delta)^{-1}C\,\mu(R_0).$$

Thus

$$\frac{1}{4}(K-1)\,\mu(R_0) \leq C'\,\mu(R_0) + c_1(\delta)^{-1}C\,\mu(R_0).$$

So we get a contradiction if K is chosen big enough. Hence, the initial assumption that the family of non-δ-BAUP cubes is not Carleson cannot be true. □

Proposition 3.4.1 will be proved along the next Sections 3.5-3.11.

3.4.1 Scheme of the proof of Proposition 3.4.1

We argue by contradiction, assuming that

$$\|T_R\mu\|^2_{L^2(\mu)} \ll \mu(R). \tag{3.4.6}$$

First, it is important to determine how L_A and its associated objects transform under a change of variable. For this reason, we include the relevant formulas in *Section 3.5*.

Then in *Section 3.6* we show that it suffices to prove the proposition with the additional assumption $A(x_R) = Id$ and H equal to the horizontal hyperplane through the origin; this puts us in a simpler geometric situation and makes the other technicalities in the rest of the proof more transparent.

A measure σ supported on hyperplanes which approximate μ at the level of the children of cubes from $\mathsf{Stop}(R)$ is introduced in *Section 3.7*.

In *Section 3.8* we construct the auxiliary matrix $\widehat{A}$, that we define via reflections with respect to a suitable hyperplane, and we study the gradient of its associated single layer potential $\widehat{T}_\mu$. We assume the hyperplane to be horizontal. In particular, we prove that the horizontal component of $\widehat{T}\sigma(x)$ is very close, in some $L^2(\sigma)$ sense, to that of $\widehat{T}\sigma(x^*)$, x^* denoting the reflection of x with respect to the horizontal plane. This proof relies on R belonging to Nice, the properties of $\widehat{A}$, and the contradiction hypothesis (3.4.6).

Section 3.9 and *Section 3.10* contain the definitions and the properties of a new approximating measure ν, a vector field Ψ, and other mathematical objects important for the conclusion of the proof. In particular, we highlight that Section 3.10 uses the intermediate non-BAUP layers.

Section 3.11 concludes the proof of Proposition 3.4.1 via a variational argument. This method produces a pointwise inequality that, integrated against the vector field Ψ constructed in Lemma 3.10.2, gives the desired contradiction.

3.5 The change of variable

The fact that we are considering a matrix A which is uniformly elliptic and symmetric allows to perform a particular change of variables. The following lemma and its corollary are standard. For the proofs we refer to [AM17, Lemma 4.8].

Lemma 3.5.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and assume that A is a uniformly elliptic matrix in Ω with real entries and $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a bi-Lipschitz map. If we set*

$$A_\phi := |\det D(\phi)|\, D(\phi^{-1})(A\circ\phi)D(\phi^{-1})^T,$$

where D denotes the differential matrix, then A_ϕ is a uniformly elliptic matrix in $\phi^{-1}(\Omega)$ and $u : \Omega \to \mathbb{R}$ is a weak solution of $L_A u = 0$ in Ω if and only if $\widetilde{u} = u \circ \phi$ is a weak solution of $L_{A_\phi}\widetilde{u} = 0$ in $\phi^{-1}(\Omega)$.

Corollary 3.5.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and assume that A is a uniformly elliptic symmetric matrix in Ω with real entries. Let $O : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a rotation. For a fixed point $y_0 \in \Omega$ define $S = \sqrt{A(y_0)}\, O$. If*

$$A_S(\cdot) = S^{-1}(A \circ S)(\cdot)(S^{-1})^T,$$

then A_S is uniformly elliptic in $S^{-1}(\Omega)$ and $A_S(z_0) = Id$ for $z_0 = S^{-1}y_0$. Further, u is a weak solution of $L_A u = 0$ in Ω if and only if $\widetilde{u} = u \circ S$ is a weak solution of $L_{A_S}\widetilde{u} = 0$ in $S^{-1}(\Omega)$.

In Corollary 3.5.1 we identified S with its associated linear map. The matrix S is well defined because A is symmetric and uniformly elliptic, so that it admits a unique square root with the property of being symmetric, uniformly elliptic and having real entries. Further, we have

$$A_S(z_0) = (\sqrt{A(y_0)}\, O)^{-1}A(S(z_0))((\sqrt{A(y_0)}\, O)^{-1})^T = Id.$$

Some standard linear algebra gives that S^{-1} is a special bi-Lipschitz change of variables that takes balls to ellipsoids and its eigenvalues determine lengths of semi-axes. Denoting by $\lambda_{\max}$ and $\lambda_{\min}$ respectively the maximal and the minimal eigenvalues of

S^{-1}, the maximum eccentricity of the image of a ball is $\sqrt{\lambda_{\max}/\lambda_{\min}}$. The ellipticity allows to bound it from below by $\sqrt{\Lambda}^{-1}$ and above by $\sqrt{\Lambda}$.

It follows that $\Lambda^{-1/2} \leq \|S^{-1}\| \leq \Lambda^{1/2}$, so that S^{-1} distorts distances by at most a constant depending on ellipticity. The collection $\widetilde{\mathcal{D}}_\mu := \{S^{-1}(Q)\}_{Q\in\mathcal{D}_\mu}$ forms a dyadic grid on $S^{-1}(\operatorname{supp}\mu) = \operatorname{supp}(S^{-1}_\sharp \mu)$ of cubes of David-Semmes type, where the involved constants depend on the ones in $\mathcal{D}_\mu$ and ellipticity.

The next easy lemma shows how the fundamental solution and the gradient of the single layer potential transform after a change of variable.

Lemma 3.5.2. *Let $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a locally bilipschitz map and let $\mathcal{E}_A$ be the fundamental solution of $L_A = -\operatorname{div}(A\nabla\cdot)$. Set $A_\phi = |\det D(\phi)|\, D(\phi^{-1})(A\circ\phi)D(\phi^{-1})^T$. Then*

$$\mathcal{E}_{A_\phi}(x,y) = \mathcal{E}_A(\phi(x),\phi(y))$$

and

$$\nabla_1 \mathcal{E}_{A_\phi}(x,y) = D(\phi)^T(x)\, \nabla_1 \mathcal{E}_A(\phi(x),\phi(y)) \quad \text{for } x,\ y \in \mathbb{R}^{n+1}.$$

Proof. The proof is an application of the change of variable formula for the integral. Let $f \in C_c^\infty(\mathbb{R}^{n+1})$. For every $x \in \mathbb{R}^{n+1}$, the definition of fundamental solution gives

$$f(\phi(x)) = \int A(y)\nabla_2 \mathcal{E}_A(\phi(x),y)\cdot \nabla f(y)\, dy.$$

Set $E(x,y) := \mathcal{E}_A(\phi(x),\phi(y))$. If we denote $y' := \phi^{-1}(y)$ and use the standard change of variable formula together with the chain rule, we get

$$\begin{aligned} f(\phi(x)) &= \int |\det D(\phi)(y')|\, A(\phi(y'))\nabla_2 \mathcal{E}_A(\phi(x),\phi(y'))\cdot\nabla f(\phi(y'))\, dy' \\ &= \int |\det D(\phi)(y')\, |A(\phi(y'))D(\phi^{-1})^T(y')\nabla_2 E(x,y')\cdot D(\phi^{-1})^T(y')\nabla(f\circ\phi)(y')\, dy' \\ &= \int A_\phi(y')\nabla_2 E(x,y')\cdot\nabla(f\circ\phi)(y')\, dy', \end{aligned}$$

which proves the first identity in the lemma. The second identity follows from the chain rule. □

Define

$$T_\phi\nu(x) = \int \nabla_1 \mathcal{E}_{A_\phi}(x,y)\, d\nu(y). \tag{3.5.1}$$

Analogously, define the operator $T_{\phi,\nu}$ as in (4.1). Then, by the previous lemma we have:

Lemma 3.5.3. *Let $\phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a bilipschitz map, ν a Radon measure, and $\phi_\sharp\nu$ its image measure. Then,*

$$T_\phi\nu(x) = D(\phi)^T(x)\, T\phi_\sharp\nu(\phi(x)).$$

Proof. The proof is an immediate application of Lemma 3.5.2 and the change of variable formula. Indeed

$$\begin{aligned} T_\phi\nu(x) &= \int \nabla_1 \mathcal{E}_{A_\phi}(x,y)\, d\nu(y) = \int D(\phi)^T(x)\, \nabla_1 \mathcal{E}_A\big(\phi(x),\phi(y)\big) d\nu(y) \\ &= D(\phi)^T(x)\int \nabla_1 \mathcal{E}_A(\phi(x),z)\, d(\phi_\sharp\nu)(z) = D(\phi)^T(x)\, T\phi_\sharp\nu(\phi(x)). \end{aligned}$$ □

3.6 Reduction to the case $A(x_R) = Id$ and H horizontal

From now on, unless specified, we will denote by R a given cube in Nice. In this section we will show that to prove Proposition 3.4.1 we may assume that $A(x_R) = Id$ and the hyperplane H in Lemma 3.3.6 to be horizontal. Indeed, let $O : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a rotation which transforms the horizontal hyperplane (through the origin) H' into $(\sqrt{A(x_R)})^{-1}H$. Consider the linear map $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ associated with the matrix $S = \sqrt{A(x_R)}\,O$ and, as in Corollary 3.5.1, set

$$A_\phi(\cdot) = S^{-1}(A \circ \phi)(\cdot)(S^{-1})^T,$$

so that A_ϕ is uniformly elliptic and $A_\phi(y_R) = Id$ for $y_R = S^{-1}x_R$. Consider also the measure $\nu = (\phi^{-1})_\sharp\mu$ and the operator T_ϕ defined in (3.5.1). By Lemma 3.5.3,

$$T_\phi\nu(x) = S \cdot T\phi_\sharp\nu(\phi(x)) = S \cdot T\mu(\phi(x)). \tag{3.6.1}$$

Also, for any function f,

$$T_\phi(f\nu)(x) = S \cdot T(\phi_\sharp(f\nu))(\phi(x)) = S \cdot T\big((f \circ \phi^{-1})\mu\big)(\phi(x)).$$

Therefore, by the $L^2(\mu)$-boundedness of T_μ,

$$\begin{aligned}\int \big|T_\phi(f\nu)(x)\big|^2\, d\nu(x) &\approx \int \big|T((f\circ\phi^{-1})\mu)(\phi(x))\big|^2\, d\phi_\sharp^{-1}\mu(x)\\ &= \int \big|T((f\circ\phi^{-1})\mu)(y)\big|^2\, d\mu(y)\\ &\leq C\int |f\circ\phi^{-1}|^2\, d\mu = C\int |f|^2\, d\nu.\end{aligned}$$

So $T_{\phi,\nu}$ is bounded in $L^2(\nu)$.

Let $\mathcal{D}_\nu$ be the lattice $\mathcal{D}_\nu = \{\phi^{-1}(Q) : Q \in \mathcal{D}_\mu\}$. Momentarily, use the notation Δ_Q^μ instead of Δ_Q, which we used in (3.4.3), and define $\Delta_{Q'}^\nu$ analogously for $Q' \in \mathcal{D}_\nu$. Write also

$$T_{\phi,\phi^{-1}(R)}\nu = \sum_{Q\in\mathsf{Tree}(R)} \Delta^\nu_{\phi^{-1}(Q)} T_\phi\nu.$$

Assuming Proposition 3.4.1 to hold in the case $A_\phi(y_R) = Id$ (applied to ν and T_ϕ), we deduce that

$$\|T_{\phi,\phi^{-1}(R)}\nu\|^2_{L^2(\nu)} \geq c\,\nu(\phi^{-1}(R)) = c\,\mu(R), \tag{3.6.2}$$

taking into account that the BAUP property is stable by homothecies, as well as the smallness of the α-numbers for the stopping cubes and the root of the tree.

We claim that

$$\|T_{\phi,\phi^{-1}(R)}\nu\|^2_{L^2(\nu)} \approx \|T_R\mu\|^2_{L^2(\mu)} \tag{3.6.3}$$

with the implicit constant in (3.6.3) independent of the cube R.

Together with (3.6.2) this implies that $\|T_R\mu\|^2_{L^2(\mu)} \gtrsim \mu(R)$ and proves Proposition 3.4.1 in full generality. The proof of (3.6.3) is a routine task which we show now for the reader's convenience. Observe that for any cube $Q \in \mathcal{D}_\mu$, by (3.6.1),

$$\begin{aligned} m_{\nu,\phi^{-1}(Q)}(T_\phi \nu) &= \frac{1}{\nu(\phi^{-1}(Q))} \int_{\phi^{-1}(Q)} T_\phi \nu \, d\nu \\ &= \frac{1}{\mu(Q)} \int_Q T_\phi \nu(\phi^{-1}(x)) \, d\mu(x) \\ &= \frac{1}{\mu(Q)} \int_Q S \cdot T\mu(x) \, d\mu(x) = S \cdot m_{\mu,Q}(T\mu). \end{aligned}$$

Denote by $\mathsf{Ch}_{\mathsf{Stop}}(R)$ the family of all children of cubes from $\mathsf{Stop}(R)$. By the preceding identity, we obtain

$$\begin{aligned} \|T_{\phi,\phi^{-1}(R)}\nu\|^2_{L^2(\nu)} &= \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \left| m_{\nu,\phi^{-1}(Q)}(T_\phi\nu) \; - m_{\nu,\phi^{-1}(R)}(T_\phi\nu)\right|^2 \nu(\phi^{-1}(Q)) \\ &= \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \left| S\cdot \left(m_{\mu,Q}(T\mu) - m_{\mu,R}(T\mu)\right)\right|^2 \mu(Q) \\ &\approx \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \left| m_{\mu,Q}(T\mu) - m_{\mu,R}(T\mu)\right|^2 \mu(Q) \\ &= \|T_R\mu\|^2_{L^2(\mu)}, \end{aligned}$$

as claimed.

Remark also that if μ is well approximated in some cube $Q \in \mathcal{D}_\mu$ by some measure of the form $c\mathcal{H}^n|_L$, where L is some hyperplane parallel to H, then it follows that $\nu = (\phi^{-1})_\sharp\mu$ is well approximated in $\phi^{-1}(Q)$ by a measure of the form

$$\phi^{-1}_\sharp(c\mathcal{H}^n|_L) = c'\mathcal{H}^n|_{\phi^{-1}(L)}.$$

Observe that $\phi^{-1}(L)$ is a hyperplane parallel to the horizontal hyperplane H', by the definition of O. Using this fact, the reader can check that if $\alpha_\mu^{(H)}(MB_Q) < \varepsilon$, then $\alpha_\mu^{(H')}(\phi^{-1}(MB_Q)) < c''\varepsilon$.

3.7 The approximating measure

From now on, in order to prove Proposition 3.4.1 for a given $R \in \mathsf{Nice}$, we assume that $A(x_R) = Id$. Recall also that we assume A to be symmetric. In this section we will construct a measure σ which should be considered as an approximation of μ, in a sense.

For every $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$, $R \in \mathsf{Nice}$, let L_Q be an n-plane parallel to $H = \{x \in \mathbb{R}^{n+1} \colon x_{n+1} = 0\}$ such that $\alpha_\mu^{L_Q}(MB_Q) \le C\varepsilon$. Let $\tilde\varepsilon, t > 0$ be some parameters to be chosen later, with $\varepsilon \ll \tilde\varepsilon \ll t \ll 1$ and such that $\beta^{L_Q}_{\infty,\mu}(MB_Q) + \beta^{L_Q}_{\infty,\mu}(B_Q) \le \tilde\varepsilon/10$ for all $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$, $R \in \mathsf{Nice}$.

Denote

$$Q_{(t)} = \{x \in Q : \operatorname{dist}(x, \operatorname{supp}\mu \setminus Q) \ge t\,\ell(Q)\}.$$

Now for $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$ with $R \in \mathsf{Nice}$, set $\tilde\mu_Q = \mu|_{Q_{(t)}}$ and $\tilde\mu = \sum_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)} \tilde\mu_Q$. Let φ be some C^∞ radial function supported on $B(0,1)$ such that $\int \varphi(x) d\mathcal{H}^n|_H(x) = 1$ and, for $r > 0$, set $\varphi_r(x) = r^{-n}\varphi(x/r)$. Denote by Π_{L_Q} the orthogonal projection on L_Q, define

$$\tilde\sigma_Q = \Pi_{L_Q\sharp}\mu|_{Q_{(t)}} \quad \text{and} \quad \sigma_Q = (\tilde\sigma_Q * \varphi_{2\tilde\varepsilon\ell(Q)})\mathcal{H}^n|_{L_Q},$$

and then set

$$\sigma = \sum_{Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)} \sigma_Q.$$

Observe that $\|\sigma_Q\| = \|\tilde{\sigma}_Q\| = \|\mu_{Q_{(t)}}\|$ for every $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$, so

$$\|\sigma\| = \|\tilde{\mu}\|.$$

Moreover, using the thin boundary condition and the abundance parameter η,

$$\begin{aligned} \|\tilde{\mu} - \mu|_R\| &\leq \mu\Big(R \setminus \bigcup_{Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)} Q\Big) + \sum_{Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)} \mu(Q \setminus Q_{(t)}) \\ &\leq \eta^{1/2}\mu(R) + Ct^{\gamma_0}\mu(R) \lesssim t^{\gamma_0}\mu(R), \end{aligned} \tag{3.7.1}$$

taking $\eta \ll t$.

Note also that, for each $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$, by the definition of σ_Q,

$$\operatorname{supp}\sigma_Q \subset \mathcal{U}_{3\tilde{\varepsilon}\ell(Q)}(\operatorname{supp}\Pi_{L_Q}\sharp\mu|_{Q_{(t)}}) \subset \mathcal{U}_{3\tilde{\varepsilon}\ell(Q)}(\mathcal{U}_{3\tilde{\varepsilon}\ell(Q)}(Q_{(t)})) = \mathcal{U}_{6\tilde{\varepsilon}\ell(Q)}(Q_{(t)}). \tag{3.7.2}$$

As a consequence, for $P, Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$ with $P \neq Q$, we have

$$\begin{aligned} \operatorname{dist}(\operatorname{supp}\sigma_P, \operatorname{supp}\sigma_Q) &\geq \operatorname{dist}\big(\mathcal{U}_{6\tilde{\varepsilon}\ell(P)}(P_{(t)}), \mathcal{U}_{6\tilde{\varepsilon}\ell(Q)}(Q_{(t)})\big) \\ &\geq \operatorname{dist}(P_{(t)}, Q_{(t)}) - 6\tilde{\varepsilon}\,(\ell(P) + \ell(Q)) \\ &\geq t\,\max(\ell(P), \ell(Q)) - 6\tilde{\varepsilon}\,(\ell(P) + \ell(Q)) \\ &\geq \frac{t}{2}\,\max(\ell(P), \ell(Q)). \end{aligned} \tag{3.7.3}$$

We will need the following lemma:

Lemma 3.7.1. *Let $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$. If $f \in \operatorname{Lip}_\alpha\Big(\mathcal{U}_{10\tilde{\varepsilon}\ell(Q)}(Q_{(t)})\Big)$, then*

$$\left|\int f(x)d(\sigma_Q - \tilde{\mu}_Q)(x)\right| \lesssim M^\alpha \operatorname{Lip}_\alpha(f)\tilde{\varepsilon}^\alpha \ell(Q)^\alpha \mu(Q).$$

Proof. Write

$$\left|\int fd(\sigma_Q - \tilde{\mu}_Q)\right| \leq \left|\int fd(\sigma_Q - \tilde{\sigma}_Q)\right| + \left|\int fd(\tilde{\sigma}_Q - \tilde{\mu}_Q)\right| = T_1 + T_2.$$

By the definition of $\tilde{\sigma}_Q$ and the fact that $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$ and therefore $\beta_{\infty,\mu}(MB_Q) \leq \tilde{\varepsilon}$,

$$T_2 = \left|\int \big(f(\Pi_{L_Q}(x)) - f(x)\big)d\tilde{\mu}_Q(x)\right| \lesssim M^\alpha \operatorname{Lip}_\alpha(f)\tilde{\varepsilon}^\alpha \ell(Q)^\alpha \mu(Q).$$

For the other term, by Fubini

$$\begin{aligned} T_1 &= \left|\int f(y)d\tilde{\sigma}_Q(y) - \int f(y)d(\tilde{\sigma}_Q * \varphi_{2\tilde{\varepsilon}\ell(Q)})\mathcal{H}^n|_{L_Q}(y)\right| \\ &= \left|\int \big(f(y) - f * (\varphi_{2\tilde{\varepsilon}\ell(Q)}\mathcal{H}^n|_{L_Q})(y)\big)d\tilde{\sigma}_Q(y)\right| \\ &\leq \int \sup_{|z| \leq 2\tilde{\varepsilon}\ell(Q)} \big|f(y) - f(y+z)\big|d\tilde{\sigma}_Q(y) \lesssim M^\alpha \operatorname{Lip}_\alpha(f)\tilde{\varepsilon}^\alpha \ell(Q)^\alpha \mu(Q). \end{aligned}$$

□

Next we show that σ has n-growth.

Lemma 3.7.2. *The measure σ has polynomial growth of degree n. That is,*

$$\sigma(B(x,r)) \leq C\, r^n \quad \text{for all } x \in \mathbb{R}^{n+1},\ r>0.$$

Proof. First we will check that σ_Q has n-growth for each $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$. Denoting $g_Q = \tilde\sigma_Q * \varphi_{2\tilde\varepsilon\ell(Q)}$ and since $\sigma_Q = g_Q\,\mathcal{H}^n|_{L_Q}$, this is equivalent to showing that $\|g_Q\|_\infty \lesssim 1$. To prove this, for $x \in L_Q$, using that $\Pi_{L_Q}(x) = x$, we write

$$\begin{aligned} g_Q(x) &= \int \varphi_{2\tilde\varepsilon\ell(Q)}(x-y)\, d\Pi_{L_Q\sharp}\mu|_{Q_{(t)}}(y) = \int \varphi_{2\tilde\varepsilon\ell(Q)}(x-\Pi_{L_Q}(y))\, d\mu|_{Q_{(t)}}(y) \\ &= \int_{Q_{(t)}} \big(\varphi_{2\tilde\varepsilon\ell(Q)} \circ \Pi_{L_Q}\big)(x-y)\, d\mu(y) \lesssim \frac{1}{(\tilde\varepsilon\ell(Q))^n}\, \mu\Big(Q \cap \Pi_{L_Q}^{-1}\big(B(x, 2\tilde\varepsilon\ell(Q))\big)\Big). \end{aligned}$$

Since $\beta_{\infty,\mu}^{L_Q}(B_Q) \leq \tilde\varepsilon/10$, there is some constant C depending at most on n such that

$$\mu\Big(Q \cap \Pi_{L_Q}^{-1}\big(B(x,2\tilde\varepsilon\ell(Q))\big)\Big) \leq \mu(B(x, C\,\tilde\varepsilon\ell(Q))) \lesssim (\tilde\varepsilon\ell(Q))^n,$$

which ensures that $\|g_Q\|_\infty \lesssim 1$, as wished.

Next, for a fixed ball $B(x,r)$, let I be the family of cubes $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$ such that $2B_Q \cap B(x,r) \neq \varnothing$. We split $I = I_1 \cup I_2$, where I_1 is the subfamily of the cubes from I with side length at most r and $I_2 = I \setminus I_1$. Then we have

$$\sigma(B(x,r)) \leq \sum_{Q\in I_1} \|\sigma_Q\| + \sum_{Q \in I_2} \sigma_Q(B(x,r)).$$

For each $Q \in I_1$, we have $\operatorname{supp}\sigma_Q \subset 2B_Q \subset B(x,4r)$, and thus

$$\sum_{Q\in I_1} \|\sigma_Q\| \leq C \sum_{Q\in I_1} \mu(Q) \leq C\mu(B(x,4r)) \leq C\, r^n.$$

On the other hand, it is immediate to check that there is a bounded number of cubes $Q \in I_2$, with the bound depending on the parameters of the lattice $\mathcal{D}_\mu$ and thus on the AD-regularity constant of μ. Hence, using also the n-growth of σ_Q,

$$\sum_{Q\in I_2} \sigma_Q(B(x,r)) \leq C \sum_{Q \in I_2} r^n \leq C\, r^n,$$

which completes the proof of the lemma. □

3.8 Approximation argument and reflection

3.8.1 The matrix $\widehat{A}$ and its associated operators $\widehat{T}$ and S

Recall that we assume that A is a symmetric matrix such that $A(x_R) = Id$. Given a parameter $\Delta \in (0, 1/10)$ to be chosen below, we set $d = \Delta\ell(R)$ and we assume that a "good" approximating n-plane for $\operatorname{supp}\mu \cap B_R$ is $L_R = \{x \in \mathbb{R}^{n+1} \colon x_{n+1} = 2d\}$. That is, $\alpha_\mu^{L_R}(MB_R) \leq \varepsilon$. We also take $H = \{x \in \mathbb{R}^{n+1} \colon x_{n+1} = 0\}$, so that L_R is a translation of H along the $(n+1)$-th direction. Further, we suppose that $B_R \subset B(0, 2\ell(R))$.

Given $x \in \mathbb{R}^{n+1}$ we denote by x^* the reflection of x with respect to H, that is $x^* = (x_1, x_2, \ldots, x_n, -x_{n+1})$. Now we define a matrix $\widehat{A}$ which satisfies some kind of invariance under this reflection. First, we consider an auxiliary matrix B defined on

$\{x : x_{n+1} \geq 0\}$ by

$$B(x) = \begin{cases} A(x) & \text{if } x_{n+1} \geq d \\ \\ A(x)\frac{x_{n+1}}{d} + Id\left(1 - \frac{x_{n+1}}{d}\right) & \text{if } 0 \leq x_{n+1} \leq d. \end{cases}$$

Notice that $B(0) = Id$. For $x_{n+1} < 0$ we set

$$B(x) = \begin{pmatrix} b_{1,1}(x^*) & \cdots & b_{1,n}(x^*) & -b_{1,n+1}(x^*) \\ b_{2,1}(x^*) & \cdots & b_{2,n}(x^*) & -b_{2,n+1}(x^*) \\ \vdots & \ddots & \vdots & \vdots \\ b_{n,1}(x^*) & \cdots & b_{n,n}(x^*) & -b_{n,n+1}(x^*) \\ -b_{n+1,1}(x^*) & \cdots & -b_{n+1,n}(x^*) & b_{n+1,n+1}(x^*) \end{pmatrix},$$

where $b_{ij}(x^*)$ are the coefficients of $B(x^*)$. In this way, for $\phi(x) = x^*$, it holds

$$B = |\det D(\phi)|\, D(\phi^{-1})(B\circ\phi)D(\phi^{-1})^T.$$

Observe that

$$D(\phi^{-1}) = D(\phi^{-1})^T = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Next we define

$$\widehat{A}(x) = \begin{cases} B(x) & \text{if } |x| \leq 100\ell(R), \\ \\ \left(2 - \frac{|x|}{100\ell(R)}\right) B(x) + \left(\frac{|x|}{100\ell(R)} - 1\right) Id & \text{if } 100\ell(R) \leq |x| \leq 200\ell(R) \\ \\ Id & \text{if } |x| \geq 200\ell(R) \end{cases}$$

Note that, for $\phi(x) = x^*$, we still have

$$\widehat{A} = |\det D(\phi)|\, D(\phi^{-1})(\widehat{A}\circ\phi)D(\phi^{-1})^T.$$

So, denoting $D = D(\phi^{-1}) = D(\phi^{-1})^T$, we have

$$\widehat{A}(x) = D\,\widehat{A}(x^*)\,D. \tag{3.8.1}$$

Lemma 3.8.1. *For $\ell(R)$ small enough, the matrix $\widehat{A}(x)$ just defined is Hölder continuous with exponent $\alpha/2$ in R.*

Proof. As a first step, we prove that the auxiliary matrix B defined above is $C^{\alpha/2}$ inside the ball $B(0, 200\ell(R))$. Because of the definition of B, it suffices to check the

Hölder regularity condition for $0 \le x_{n+1}, y_{n+1} \le d$. In this case

$$\begin{aligned}\big|B(x)-B(y)\big| &= \Big|A(x)\frac{x_{n+1}}{d} - A(y)\frac{y_{n+1}}{d} + Id\Big(1-\frac{x_{n+1}}{d}\Big) - Id\Big(1-\frac{y_{n+1}}{d}\Big)\Big| \\ &= \Big|(A(x)-Id)\frac{x_{n+1}}{d} - (A(y)-Id)\frac{y_{n+1}}{d}\Big| \\ &\le \frac{|x_{n+1}-y_{n+1}|}{d}\,|A(x)-Id| + \frac{y_{n+1}}{d}\,|A(x)-A(y)| \\ &\le C\frac{|x_{n+1}-y_{n+1}|}{d}\,\ell(R)^\alpha + C\,|x-y|^\alpha,\end{aligned}$$

where we took into account that

$$|A(x)-Id| = |A(x)-A(x_R)| \le C\,\ell(R)^\alpha.$$

Now we write

$$\begin{aligned}\frac{|x_{n+1}-y_{n+1}|}{d}\,\ell(R)^\alpha &\le \frac{|x_{n+1}-y_{n+1}|^\alpha}{d^\alpha}\,\ell(R)^\alpha \\ &= \frac{1}{\Delta^\alpha}\,|x_{n+1}-y_{n+1}|^\alpha \le \frac{\ell(R)^{\alpha/2}}{\Delta^\alpha}\,|x_{n+1}-y_{n+1}|^{\alpha/2}.\end{aligned}$$

Thus for $\ell(R)$ small enough, we have $\ell(R)^{\alpha/2}/\Delta^\alpha \le 1$ and we get

$$\big|B(x)-B(y)\big| \le C\,|x-y|^{\alpha/2} + C\,|x-y|^\alpha \le C\,|x-y|^{\alpha/2},$$

since $|x-y| \lesssim \ell(R) \lesssim 1$. This proves the $(\alpha/2)$-Hölder regularity in the ball $B(0, 200\ell(R))$.

The next step is to prove that the matrix $\widehat{A}$ is $C^{\alpha/2}$ inside the ball $B(0,200\ell(R))$. The regularity inside $B(0,100\ell(R))$ follows from the regularity of B. Consider $x, y \in B(0,200\ell(R)) \setminus B(0,100\ell(R))$. Exploiting the definition of $\widehat{A}$ together with the Hölder regularity of the matrix B inside $B(0,200\ell(R))$ we have

$$\begin{aligned}\big|\widehat{A}(x)-\widehat{A}(y)\big| &= \Big|\Big(2-\frac{|x|}{100\ell(R)}\Big)B(x) - \Big(2-\frac{|y|}{100\ell(R)}\Big)B(y) + \Big(\frac{|x|-|y|}{100\ell(R)}\Big)Id\Big| \\ &\le 2|B(x)-B(y)| + \Big|(B(x)-Id)\,\frac{|x|}{100\ell(R)} - (B(y)-Id)\,\frac{|y|}{100\ell(R)}\Big| \\ &\le 2|B(x)-B(y)| + |B(x)-Id|\,\frac{\big||x|-|y|\big|}{100\ell(R)} + |B(x)-B(y)|\,\frac{|y|}{100\ell(R)} \\ &\le C|x-y|^{\alpha/2} + |B(x)-Id|\,\frac{|x-y|}{100\ell(R)} + C|x-y|^{\alpha/2}\frac{|y|}{100\ell(R)}\end{aligned}$$

so that, being $x, y \in B(0,200\ell(R))$ and $B(0) = Id$, we can write

$$\begin{aligned}\big|\widehat{A}(x)-\widehat{A}(y)\big| &\le C|x-y|^{\alpha/2} + C|x|^{\alpha/2}\frac{|x-y|}{100\ell(R)} + C|x-y|^{\alpha/2}\frac{|y|}{100\ell(R)} \\ &\le C|x-y|^{\alpha/2} + C\ell(R)^{\alpha/2}\,\frac{|x-y|^{\alpha/2}}{\ell(R)^{\alpha/2}} + C|x-y|^{\alpha/2} \le C|x-y|^{\alpha/2}.\end{aligned}$$

The matrix $\widehat{A}$ is trivially $C^{\alpha/2}$ in $\mathbb{R}^{n+1} \setminus B(0,200\ell(R))$. To finish the proof, take x with $|x| \le 200\ell(R)$, y with $|y| \ge 200\ell(R)$ and choose a point $\tilde{y}$ with $|\tilde{y}| = 200\ell(R)$

and $|x-\tilde{y}| \leq |x-y|$. Then write

$$\big|\widehat{A}(x) - \widehat{A}(y)\big| = \big|\widehat{A}(x) - Id\big| = \big|\widehat{A}(x) - \widehat{A}(\tilde{y})\big| \leq C|x-\tilde{y}|^{\alpha/2} \leq C|x-y|^{\alpha/2}. \qquad \square$$

From now on we assume that $\ell(R) \leq 1$ so that the estimates in Lemma 4.2.1 hold for all $x, y \in R$. Also, the estimates in Lemma 4.2.1 and Lemma 4.2.2 hold for $\widehat{A}$ with $\alpha/2$ replacing α. Further, we will take $0 < \varepsilon \ll \Delta \ll 1$, so that $A(x) = \widehat{A}(x)$ for all x in a neighborhood of R.

Let $\mathcal{E}_{\widehat{A}}$ be the fundamental solution associated with $L_{\widehat{A}}$, set $\widehat{K}(x,y) = \nabla_1 \mathcal{E}_{\widehat{A}}(x,y)$, and define

$$\widehat{T}\mu(x) = \int \widehat{K}(x,y)d\mu(y).$$

Note that, by (3.8.1) and Lemma 3.5.2, for $x,\ y \in \mathbb{R}^{n+1}$ we have

$$\mathcal{E}_{\widehat{A}}(x,y) = \mathcal{E}_{\widehat{A}}(x^*,y^*) \text{ and } \widehat{K}(x,y) = \nabla_1 \mathcal{E}_{\widehat{A}}(x,y) = D\,\nabla_1 \mathcal{E}_{\widehat{A}}(x^*,y^*) = D\,\widehat{K}(x^*,y^*). \tag{3.8.2}$$

Define now the operator

$$S\mu(x) = \int K_S(x,y)d\mu(y),$$

associated with the kernel

$$K_S(x,y) = \widehat{K}(x,y) - \widehat{K}(x^*,y),$$

so that

$$S\mu(x) = \widehat{T}\mu(x) - \widehat{T}\mu(x^*).$$

Remark 6. The operators $\widehat{T}_\mu$ and S_μ are bounded in $L^2(\mu|_R)$. Indeed, the $L^2(\mu|_R)$ boundedness of $\widehat{T}_\mu$ follows from the one of T_μ and the fact that the difference between their kernels is bounded in modulus by $1/|x-y|^{n-\alpha/2}$, by a freezing argument using Lemma 4.2.2. Then to prove the $L^2(\mu|_R)$ boundedness of S_μ it suffices to show that the operator U_μ defined by

$$U_\mu f(x) = \widehat{T}_\mu f(x^*)$$

is bounded in $L^2(\mu|_R)$. To show this, write

$$\int_R |U_\mu f(x)|^2\, d\mu(x) = \int_R |\widehat{T}_\mu f(x^*)|^2\, d\mu(x) = \int |\widehat{T}_\mu f(y)|^2\, d\phi_\sharp\mu(y),$$

where $\phi_\sharp\mu$ is the image measure of $\mu|_R$ by the reflection $\phi\colon x \mapsto x^*$. Since $\widehat{T}_\mu$ is bounded in $L^2(\mu|_R)$ and $\phi_\sharp\mu$ has n-polynomial growth, it follows that $\widehat{T}_\mu$ is bounded from $L^2(\mu|_R)$ to $L^2(\phi_\sharp\mu)$, which implies that U_μ is bounded in $L^2(\mu|_R)$, as wished.

Recall that $H = \{x\colon x_{n+1} = 0\}$. We denote by Π_H the orthogonal projection on H, we set

$$\widehat{T}^H\mu(x) = \Pi_H(\widehat{T}\mu(x)), \quad S^H\mu(x) = \Pi_H(S\mu(x)),$$

and we define similarly $\widehat{T}^H_\mu$, S^H_μ, etc. The kernel of $\widehat{T}^H$ is $\widehat{K}^H(x,y) := \Pi_H(\widehat{K}(x,y))$ and the one of S^H is $K^H_S(x,y) := \Pi_H(K_S(x,y))$. Note that, from the second identity in (3.8.2), we get

$$\widehat{K}^H(x,y) = \widehat{K}^H(x^*,y^*) \quad \text{for all } x,y \in \mathbb{R}^{n+1} \text{ with } x \neq y. \tag{3.8.3}$$

3.8.2 The approximation lemmas

This section is devoted to announce some technical approximation lemmas.

Lemma 3.8.2 (First Approximation Lemma). *For every $R \in \mathsf{Nice}$ we have*

$$\|\widehat{T}\sigma\|^2_{L^2(\sigma)} \leq C\,\mu(R) \tag{3.8.4}$$

and

$$\|S\sigma\|^2_{L^2(\sigma)} \leq C\,\mu(R). \tag{3.8.5}$$

For the horizontal operator S^H we have a much better estimate:

Lemma 3.8.3 (Second Approximation Lemma). *Let $R \in \mathsf{Nice}$. Let $\varepsilon_1, \varepsilon_2 > 0$ and suppose that $\|T_R\mu\|^2_{L^2(\mu)} \leq \varepsilon_1\,\mu(R)$. Then*

$$\|S^H\sigma\|^2_{L^2(\sigma)} \leq \varepsilon_2\,\mu(R), \tag{3.8.6}$$

if ε_1, $\ell(R)$, t, and Δ are small enough and M is big enough.

Essentially, the estimates in the above lemmas hold because σ is a very good approximation of the measure μ at the scales and location of $\mathsf{Tree}(R)$. Further, in the case of Lemma 3.8.3 the reflection involved in the definition of S plays an essential role in the localization that allows to transfer the estimates from the measure μ to the compactly supported measure σ with a small error.

The proof of Lemmas 3.8.2 and 3.8.3 follows from the next three auxiliary lemmas.

Lemma 3.8.4. *Let $R \in \mathsf{Nice}$ and let*

$$f = \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S\mu)\,\chi_Q \quad \text{and} \quad f^H = \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S^H\mu)\,\chi_Q.$$

Then,

$$\|T_R\mu - f\|^2_{L^2(\mu)} \lesssim \mu(R),$$

and, for any $\varepsilon_3 > 0$,

$$\|T_R^H\mu - f^H\|^2_{L^2(\mu)} \leq \varepsilon_3\,\mu(R), \tag{3.8.7}$$

if ε, $\tilde{\varepsilon}$, and $\ell(R)$ are small enough and M is big enough.

Lemma 3.8.5. *Let $R \in \mathsf{Nice}$, denote*

$$\tilde{f} = \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S\tilde{\mu})\,\chi_Q \quad \text{and} \quad \tilde{f}^H = \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S^H\tilde{\mu})\,\chi_Q,$$

and let f, f^H be as in Lemma 3.8.4. Then, for any $\varepsilon_4 > 0$, if t and Δ are small enough,

$$\|\tilde{f} - f\|^2_{L^2(\mu)} + \|\tilde{f}^H - f^H\|^2_{L^2(\mu)} \leq \varepsilon_4\,\mu(R).$$

Lemma 3.8.6. *Let $R \in \mathsf{Nice}$ and $\tilde{f}, \tilde{f}^H$ be as in Lemmas 3.8.4 and 3.8.5. Also, set*

$$\tilde{h} = \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(\widehat{T}\tilde{\mu})\,\chi_Q.$$

Then, for any $\varepsilon_5 > 0$ we have

$$\|\widehat{T}\sigma\|^2_{L^2(\sigma)} \leq C\,\|\tilde{h}\|^2_{L^2(\mu)} + \varepsilon_5\,\mu(R), \tag{3.8.8}$$

$$\|S\sigma\|^2_{L^2(\sigma)} \le C\,\|\tilde f\|^2_{L^2(\mu)} + \varepsilon_5\,\mu(R), \tag{3.8.9}$$

and

$$\|S^H\sigma\|^2_{L^2(\sigma)} \le C\,\|\tilde f^H\|^2_{L^2(\mu)} + \varepsilon_5\,\mu(R) \tag{3.8.10}$$

if ε, $\tilde\varepsilon$, t and $\ell(R)$ are small enough.

Proof of the Approximation Lemmas 3.8.2, 3.8.3 using Lemmas 3.8.4, 3.8.5, 3.8.6. The estimates (3.8.5) and (3.8.6) follow just by an immediate application of the three auxiliary lemmas and the triangle inequality. For example, to show (3.8.6), assume $\|T_R\mu\|^2_{L^2(\mu)} \le \varepsilon_1\,\mu(R)$ and then by (3.8.10), (3.8.7), and Lemma 3.8.5,

$$\begin{aligned}\|S^H\sigma\|^2_{L^2(\sigma)} &\le C\|\tilde f^H\|^2_{L^2(\mu)} + \varepsilon_5\mu(R)\\ &\le C\|T^H_R\mu\|^2_{L^2(\mu)} + C\|T^H_R\mu - f^H\|^2_{L^2(\mu)} + C\|f^H - \tilde f^H\|^2_{L^2(\mu)} + \varepsilon_5\mu(R)\\ &\lesssim (\varepsilon_1+\varepsilon_3+\varepsilon_4+\varepsilon_5)\,\mu(R).\end{aligned}$$

The proof of (3.8.5) is analogous.

To show (3.8.4) we just apply (3.8.8) and use the fact that $\widehat T_\mu$ is bounded in $L^2(\mu|_R)$:

$$\begin{aligned}\|\widehat T\sigma\|^2_{L^2(\sigma)} &\le C\,\|\tilde h\|^2_{L^2(\mu)} + \varepsilon_5\,\mu(R) = C\,\Big\|\sum_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(\widehat T\tilde\mu)\,\chi_Q\Big\|^2_{L^2(\mu)} + \varepsilon_5\,\mu(R)\\ &\le C\,\|\widehat T\tilde\mu\|^2_{L^2(\mu|_R)} + \varepsilon_5\,\mu(R) \lesssim \mu(R).\end{aligned}$$

□

3.8.3 Proof of Lemma 3.8.4

First we set $\widehat T^\phi\mu(x) = \widehat T\mu(\phi(x)) = \widehat T\mu(x^*)$ and $\widehat T^{\phi,H}\mu(x) = \widehat T^H\mu(\phi(x)) = \widehat T^H\mu(x^*)$, so that $S\mu(x) = \widehat T\mu(x) - \widehat T^\phi\mu(x)$ and $S^H\mu(x) = \widehat T^H\mu(x) - \widehat T^{\phi,H}\mu(x)$. In what follows we write $m_Q(f) = m_{\mu,Q}(f)$ to simplify the notation. Denote by x'_R the orthogonal projection of x_R on L_R. Notice that

$$|x'_R - x_R| \lesssim \alpha_\mu^{L_R}(2B_R)^{1/(n+1)}\ell(R) \lesssim (M^{n+1}\varepsilon)^{1/(n+1)}\ell(R) \ll \ell(R).$$

Consider a C^1 function $\widetilde\chi_{M,R}$, radial with respect to x'_R, and such that $\chi_{B(x'_R, M\ell(R)/2)} \le \widetilde\chi_{M,R} \le \chi_{B\left(x'_R,\frac34 M\ell(R)\right)}$ and $\|\nabla\widetilde\chi_{M,R}\|_\infty \lesssim (M\ell(R))^{-1}$. For $x\in Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)$ and $M>1$, we split the difference $T_R\mu(x) - f(x)$ as follows:

$$\begin{aligned}T_R\mu(x) - f(x) &= m_Q(T\mu) - m_R(T\mu) - m_Q(S\mu)\\ &= m_Q(T\mu) - m_R(T\mu) - m_Q(\widehat T\mu) + m_Q(\widehat T^\phi\mu)\\ &= m_Q(T_\mu\widetilde\chi_{M,R}) + m_Q(T_\mu(1-\widetilde\chi_{M,R})) - m_R(T_\mu\widetilde\chi_{M,R})\\ &\quad - m_R(T_\mu(1-\widetilde\chi_{M,R})) - m_Q(\widehat T_\mu\widetilde\chi_{M,R}) - m_Q(\widehat T_\mu(1-\widetilde\chi_{M,R}))\\ &\quad + m_Q(\widehat T^\phi_\mu\widetilde\chi_{M,R}) + m_Q(\widehat T^\phi_\mu(1-\widetilde\chi_{M,R})),\end{aligned}$$

so that we have

$$\begin{aligned}\left|T_R\mu(x) - f(x)\right| &\le |m_Q(T_\mu\widetilde{\chi}_{M,R}) - m_Q(\widehat{T}_\mu\widetilde{\chi}_{M,R})| \\ &\quad + |m_Q(T_\mu(1-\widetilde{\chi}_{M,R})) - m_R(T_\mu(1-\widetilde{\chi}_{M,R}))| \\ &\quad + |m_Q(\widehat{T}_\mu^\phi(1-\widetilde{\chi}_{M,R})) - m_Q(\widehat{T}_\mu(1-\widetilde{\chi}_{M,R}))| \\ &\quad + |m_R(T_\mu\widetilde{\chi}_{M,R})| + |m_Q(\widehat{T}_\mu^\phi\widetilde{\chi}_{M,R})| \\ &=: I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{3.8.11}$$

We perform the analogous splitting for $\left|T_R^H\mu(x) - f^H(x)\right|$, so that we have

$$\left|T_R^H\mu(x) - f^H(x)\right| \le I_1^H + I_2^H + I_3^H + I_4^H + I_5^H,$$

with

$$\begin{aligned} I_1^H &= |m_Q(T_\mu^H\widetilde{\chi}_{M,R}) - m_Q(\widehat{T}_\mu^H\widetilde{\chi}_{M,R})|, \\ I_2^H &= |m_Q(T_\mu^H(1-\widetilde{\chi}_{M,R})) - m_R(T_\mu^H(1-\widetilde{\chi}_{M,R}))|, \\ I_3^H &= |m_Q(\widehat{T}_\mu^{\phi,H}(1-\widetilde{\chi}_{M,R})) - m_Q(\widehat{T}_\mu^H(1-\widetilde{\chi}_{M,R}))|, \\ I_4^H &= |m_R(T_\mu^H\widetilde{\chi}_{M,R})|, \\ I_5^H &= |m_Q(\widehat{T}_\mu^{\phi,H}\widetilde{\chi}_{M,R})|. \end{aligned}$$

Obviously, $I_i^H \le I_i$ for each i.

Estimate of I_2. Notice that for $x' \in Q$,

$$\begin{aligned} &|m_Q(T_\mu(1-\widetilde{\chi}_{M,R})) - m_R(T_\mu(1-\widetilde{\chi}_{M,R}))| \\ &\qquad \le \frac{1}{\mu(Q)}\int_Q \left|T_\mu(1-\widetilde{\chi}_{M,R})(x) - T_\mu(1-\widetilde{\chi}_{M,R})(x')\right| d\mu(x) \\ &\qquad\quad + \frac{1}{\mu(R)}\int_R \left|T_\mu(1-\widetilde{\chi}_{M,R})(x) - T_\mu(1-\widetilde{\chi}_{M,R})(x')\right| d\mu(x) \\ &\qquad \le 2\sup_{y,y'\in R} |T_\mu(1-\widetilde{\chi}_{M,R})(y) - T_\mu(1-\widetilde{\chi}_{M,R})(y')| \end{aligned}$$

and to estimate this supremum, observe that for $y, y' \in R$, by Lemma 4.2.1,

$$\begin{aligned} |T_\mu(1-\widetilde{\chi}_{M,R})(y) - T_\mu(1-\widetilde{\chi}_{M,R})(y')| &\le \int_{\left(\frac{1}{2}MB_R\right)^c} |K(y,z) - K(y',z)| d\mu(z) \\ &\lesssim \int_{\left(\frac{1}{2}MB_R\right)^c} \frac{\ell(R)^\alpha}{|x_R - z|^{n+\alpha}} d\mu(z) \lesssim M^{-\alpha}, \end{aligned}$$

where the last inequality follows by standard estimates using the growth of the measure μ. Therefore

$$I_2 = |m_Q(T_\mu(1-\widetilde{\chi}_{M,R})) - m_R(T_\mu(1-\widetilde{\chi}_{M,R}))| \lesssim M^{-\alpha}.$$

Estimate of I_3. By Lemma 4.2.1 and standard arguments,

$$\begin{aligned} I_3 &= \frac{1}{\mu(Q)} \left| \int_Q \Big(\widehat{T}_\mu(1-\widetilde{\chi}_{M,R})(x^*) - \widehat{T}_\mu(1-\widetilde{\chi}_{M,R})(x) \Big)\, d\mu(x) \right| \\ &\leq \sup_{x\in Q} \left| \widehat{T}_\mu(1-\widetilde{\chi}_{M,R})(x^*) - \widehat{T}_\mu(1-\widetilde{\chi}_{M,R})(x) \right| \\ &\leq \sup_{x\in Q} \int_{\left(\frac{1}{2}MB_R\right)^c} |\widehat{K}(x,y) - \widehat{K}(x^*,y)| d\mu(y) \\ &\lesssim \int_{\left(\frac{1}{2}MB_R\right)^c} \frac{\Delta^{\alpha/2}\ell(R)^{\alpha/2}}{|x_Q - y|^{n+\alpha/2}} d\mu(y) \lesssim \frac{\Delta^{\alpha/2}}{M^{\alpha/2}}. \end{aligned}$$

Estimate of I_1. This term is estimated by a freezing argument. Indeed, recalling that $\varepsilon \ll \Delta$, we have $\widehat{A}(x) = A(x)$ for all $x \in Q$, and thus

$$\begin{aligned} |\nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \mathcal{E}_{\widehat{A}}(x,y)| &\leq |\nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \Theta(x,y;A(x))| \\ &\quad + |\nabla_1 \mathcal{E}_{\widehat{A}}(x,y) - \nabla_1 \Theta(x,y;\widehat{A}(x))| \lesssim \frac{1}{|x-y|^{n-\alpha/2}}, \end{aligned}$$

Integrating with respect to $y \in MB_R$ we derive

$$|T_\mu \widetilde{\chi}_{M,R}(x) - \widehat{T}_\mu \widetilde{\chi}_{M,R}(x)| \lesssim \int_{MB_R} \frac{d\mu(y)}{|x-y|^{n-\alpha/2}} \lesssim (M\ell(R))^{\alpha/2},$$

and so

$$I_1 = |m_Q(T_\mu \widetilde{\chi}_{M,R}) - m_Q(\widehat{T}_\mu \widetilde{\chi}_{M,R})| \lesssim (M\ell(R))^{\alpha/2} \leq \varepsilon_3,$$

for $\ell(R)$ small enough (depending on M).

Estimate of I_4. We write

$$m_R(T_\mu \widetilde{\chi}_{M,R}) = m_R(T_\mu(\widetilde{\chi}_{M,R} - \chi_R)) + m_R(T_\mu \chi_R).$$

Concerning the term $m_R(T_\mu \chi_R)$, the antisymmetric part of the kernel of T_μ does not contribute to the average, hence we can write

$$m_R(T_\mu \chi_R) = \frac{1}{2\mu(R)} \iint_{R\times R} (K(y,z) + K(z,y))\, d\mu(y) d\mu(z).$$

Using now the estimate (3.2.5) and the n-growth of the measure μ, for any $y \in R$ we get

$$\int_R |K(y,z) + K(z,y)| d\mu(z) \lesssim \int_R \frac{d\mu(z)}{|y-z|^{n-\alpha}} \lesssim \ell(R)^\alpha,$$

and so

$$|m_R(T_\mu \chi_R)| \lesssim \ell(R)^\alpha.$$

To conclude with I_4 it remains to estimate $|m_R(T_\mu(\widetilde{\chi}_{M,R} - \chi_R))|$. Given some small constant $\kappa \in (0, 1/10)$ to be chosen below, let $\widetilde{\chi}_{\kappa,R}$ be a C^1 function which equals 1 on $\mathcal{U}_{\kappa\ell(R)}(R)$, vanishes out of $\mathcal{U}_{\kappa 2\ell(R)}(R)$ and satisfies $\|\nabla \widetilde{\chi}_{\kappa,R}\|_\infty \lesssim (\kappa\ell(R))^{-1}$, and denote $\varphi = \widetilde{\chi}_{M,R} - \widetilde{\chi}_{\kappa,R}$. In particular we have

$$\widetilde{\chi}_{M,R} - \chi_R = \varphi + \widetilde{\chi}_{\kappa,R} - \chi_R.$$

Then we split as follows:

$$\left|\int_R T_\mu(\widetilde{\chi}_{M,R}-\chi_R)d\mu\right| \leq \left|\int_R T_\mu\varphi\, d\mu\right| + \int_R |T_\mu(\widetilde{\chi}_{\kappa,R}-\chi_R)|\, d\mu =: A+B. \quad (3.8.12)$$

The Cauchy-Schwarz inequality and the thin boundary condition (3.2.2) of R give us the estimate

$$\begin{aligned} B &\leq \|T_\mu(\widetilde{\chi}_{\kappa,R}-\chi_R)\|_{L^2(\mu|_R)}\, \mu(R)^{1/2} \lesssim \|\widetilde{\chi}_{\kappa,R}-\chi_R\|_{L^2(\mu)}\, \mu(R)^{1/2} \\ &\leq \mu\left(\mathcal{U}_{2\kappa\ell(R)}(R)\setminus R\right)^{1/2} \mu(R)^{1/2} \lesssim \kappa^{\gamma_0/2}\mu(R). \end{aligned} \quad (3.8.13)$$

Now it remains to estimate the term A. We consider another auxiliary function $\tilde{\varphi}$ supported on $\mathcal{U}_{\kappa\ell(R)/4}(R)$ such that $\tilde{\varphi}\equiv 1$ on $\mathcal{U}_{\kappa\ell(R)/8}(R)$ and $\|\nabla\tilde{\varphi}\|_\infty \lesssim (\kappa\ell(R))^{-1}$. Write

$$A = \left|\int_R T_\mu\varphi\, d\mu\right| \leq \left|\int \tilde{\varphi}\, T_\mu\varphi\, d\mu\right| + \left|\int (\chi_R-\tilde{\varphi})T_\mu\varphi\, d\mu\right| \quad (3.8.14)$$

For the second term above, notice that the definition of $\tilde{\varphi}$ and the thin boundary condition imply that $\|\chi_R-\tilde{\varphi}\|_{L^2(\mu)} \lesssim \kappa^{\gamma_0/2}\mu(R)^{1/2}$. Therefore,

$$\begin{aligned} \left|\int (\chi_R-\tilde{\varphi})T_\mu\varphi\, d\mu\right| &\lesssim \|\varphi\|_{L^2(\mu)}\, \|\chi_R-\tilde{\varphi}\|_{L^2(\mu)} \\ &\lesssim \mu\big(B(x_R', \tfrac{3}{4}M\ell(R))\big)^{1/2}\, \kappa^{\gamma_0/2}\mu(R)^{1/2} \\ &\lesssim M^{1/2}\kappa^{\gamma_0/2}\mu(R). \end{aligned} \quad (3.8.15)$$

To treat the first term in (3.8.14), taking $c_R \geq 0$, split it as follows

$$\begin{aligned} \left|\int \tilde{\varphi}\, T_\mu\varphi\, d\mu\right| &\leq \left|\int \tilde{\varphi}\, T_\mu\varphi\, d\left(\mu - c_R\mathcal{H}^n|_{L_R}\right)\right| + \left|c_R \int \tilde{\varphi}\, T_{\mu - c_R\mathcal{H}^n|_{L_R}}\varphi\, d\mathcal{H}^n|_{L_R}\right| \\ &\quad + \left|c_R\int \tilde{\varphi}\, T_{c_R\mathcal{H}^n|_{L_R}}\varphi\, d\mathcal{H}^n|_{L_R}\right| =: A_1 + A_2 + A_3. \end{aligned} \quad (3.8.16)$$

To estimate A_1 we would like to use the α-numbers. However, we can only guarantee that $T_\mu\varphi$ is Hölder continuous on $\operatorname{supp}\tilde{\varphi}$. So we convolve this function with a non-negative, radial, C^∞ function θ supported on $B(0,\hat{\kappa}\ell(R))$, and such that $\int \theta\, d\mathcal{L}^{n+1} = 0$ and $\|\nabla\theta\|_\infty \lesssim (\hat{\kappa}\ell(R))^{-n+2}$, with $\hat{\kappa}\in(0,\kappa/20)$ to be chosen. Then we write

$$\begin{aligned} A_1 &\leq \left|\int \tilde{\varphi}\, [\theta * T_\mu\varphi]\; d\left(\mu - c_R\mathcal{H}^n|_{L_R}\right)\right| + \left|\int \tilde{\varphi}\, [T_\mu\varphi - \theta * T_\mu\varphi]\; d\left(\mu - c_R\mathcal{H}^n|_{L_R}\right)\right| \\ &=: A_{1,1} + A_{1,2}. \end{aligned}$$

We turn first our attention to $A_{1,1}$:

$$A_{1,1} \leq \left\|\nabla\left(\tilde{\varphi}\, [\theta * T_\mu\varphi]\right)\right\|_\infty M^{n+1}\ell(R)^{n+1}\alpha_\mu^{L_R}(MB_R). \quad (3.8.17)$$

Notice that

$$\left\|\nabla\left(\tilde{\varphi}\, [\theta * T_\mu\varphi]\right)\right\|_\infty \leq \left\|\nabla\left(\theta * T_\mu\varphi\right)]\right\|_{\infty,\operatorname{supp}\tilde{\varphi}} + \|\nabla\tilde{\varphi}\|_\infty \|\theta * T_\mu\varphi\|_{\infty,\operatorname{supp}\tilde{\varphi}}.$$

Since $\operatorname{dist}(\operatorname{supp}\varphi, \operatorname{supp}\tilde\varphi) \geq \kappa\ell(R)/4$ and $\operatorname{supp}\theta \subset B(0,\kappa\ell(R)/20)$, we derive

$$\|\theta * T_\mu\varphi\|_{\infty,\operatorname{supp}\tilde\varphi} \lesssim \frac{\mu\big(B(x_R', M\ell(R))\big)}{(\kappa\,\ell(R))^n} \lesssim \frac{M^n}{\kappa^n}$$

and

$$\big\|\nabla\big(\theta * T_\mu\varphi\big)\big\|_{\infty,\operatorname{supp}\tilde\varphi} \lesssim \frac{M^n}{\kappa^n}\,\|\nabla\theta\|_1 \lesssim \frac{M^n}{\kappa^n\,\hat\kappa\,\ell(R)}.$$

Hence, using also that $\|\nabla\tilde\varphi\|_\infty \lesssim (\kappa\ell(R))^{-1}$,

$$\big\|\nabla\big(\tilde\varphi\,[\theta * T_\mu\varphi]\big)\big\|_\infty \lesssim \frac{M^n}{\kappa^n\,\hat\kappa\,\ell(R)} + \frac{M^n}{\kappa^{n+1}\ell(R)} \lesssim \frac{M^n}{\kappa^n\,\hat\kappa\,\ell(R)}.$$

Plugging this estimate into (3.8.17), we obtain

$$A_{1,1} \lesssim \varepsilon\frac{M^{2n+1}}{\hat\kappa\kappa^n}\mu(R).$$

Concerning the term $A_{1,2}$, we have

$$A_{1,2} \leq \int \tilde\varphi\,\big|T_\mu\varphi - \theta * T_\mu\varphi\big|\; d\big|\mu - c_R\mathcal{H}^n|_{L_R}\big| \lesssim \big\|T_\mu\varphi - \theta * T_\mu\varphi\big\|_{\infty,\operatorname{supp}\tilde\varphi}\,\ell(R)^n.$$

For each $x\in\operatorname{supp}\tilde\varphi$, we write

$$\begin{aligned}|T_\mu\varphi(x) - \theta * T_\mu\varphi(x)| &\leq \sup_{y\in B(x,\hat\kappa\ell(R))} |T_\mu\varphi(x) - T_\mu\varphi(y)|\\ &\leq \sup_{y\in B(x,\hat\kappa\ell(R))} \int_{\operatorname{supp}\varphi} |K(x,z) - K(y,z)|\,d\mu(z).\end{aligned}$$

Using the fact that $\operatorname{dist}(x,\operatorname{supp}\varphi)\geq\kappa\ell(R)$ and the Hölder continuity of K, for x and y as above we get

$$\int_{\operatorname{supp}\varphi} |K(x,z) - K(y,z)|\,d\mu(z) \lesssim \int_{|x-z|\geq\kappa\ell(R)} \frac{(\hat\kappa\ell(R))^\alpha}{|x-z|^{n+\alpha}}\,d\mu(z) \lesssim \frac{\hat\kappa^\alpha}{\kappa^\alpha},$$

and thus

$$A_{1,2} \lesssim \frac{\hat\kappa^\alpha}{\kappa^\alpha}\,\ell(R)^n.$$

Together with the estimates for $A_{1,1}$, choosing $\hat\kappa = \kappa^2$, this gives

$$A_1 \leq A_{1,1} + A_{1,2} \lesssim \mu(R)\left(\varepsilon\frac{M^{2n+1}}{\kappa^{n+2}} + \kappa^\alpha\right).$$

To deal with A_2, we write

$$A_2 = \left|c_R\int\tilde\varphi(x)T_{\mu - c_R\mathcal{H}^n|_{L_R}}\varphi(x)\;d\mathcal{H}^n|_{L_R}(x)\right| \approx \left|\int T^*_{\mathcal{H}^n|_{L_R}}\tilde\varphi(x)\,d(\mu - c_R\mathcal{H}^n|_{L_R})(x)\right|,$$

where T^* denotes the transpose of the gradient of the single layer potential. Arguing as for the term A_1, essentially reversing the roles of φ and $\tilde\varphi$, we get

$$A_2 \lesssim \mu(R)\left(\varepsilon\frac{M^{n+1}}{\kappa^{n+2}} + M^n\,\kappa^\alpha\right).$$

We leave the details for the reader.

Now we will estimate the term A_3 in (3.8.16). To this end, first we take into account that

$$\left|\int \tilde\varphi\, T_{\mathcal{H}^n|_{L_R}} \tilde\varphi \, d\mathcal{H}^n|_{L_R}\right| \lesssim \ell(R)^{n+\alpha}.$$

This follows by the same argument used to prove that $|m_R(T_\mu\chi_R)| \lesssim \ell(R)^\alpha$ in (3.8.3). Then we have

$$\begin{aligned}
A_3 &\approx \left|\int \tilde\varphi\, T_{\mathcal{H}^n|_{L_R}} \varphi \, d\mathcal{H}^n|_{L_R}\right| \\
&\lesssim \ell(R)^{n+\alpha} + \left|\int \tilde\varphi\, T_{\mathcal{H}^n|_{L_R}} (\varphi+\tilde\varphi) \, d\mathcal{H}^n|_{L_R}\right| \\
&\leq \ell(R)^{n+\alpha} + \left|\int \tilde\varphi\, T_{\mathcal{H}^n|_{L_R}} \widetilde\chi_{M,R} \, d\mathcal{H}^n|_{L_R}\right| + \left|\int \tilde\varphi\, T_{\mathcal{H}^n|_{L_R}} (\widetilde\chi_{M,R}-\varphi-\tilde\varphi) \, d\mathcal{H}^n|_{L_R}\right| \\
&= \ell(R)^{n+\alpha} + A_{3,1} + A_{3,2}.
\end{aligned}$$

The Cauchy-Schwarz inequality and the $L^2(\mathcal{H}^n|_{L_R})$-boundedness of $T_{\mathcal{H}^n|_{L_R}}$ imply

$$A_{3,2} \lesssim \|\widetilde\chi_{M,R}-\varphi-\tilde\varphi\|_{L^2(\mathcal{H}^n|_{L_R})}\, \ell(R)^{n/2} = \|\widetilde\chi_{\kappa,R}-\tilde\varphi\|_{L^2(\mathcal{H}^n|_{L_R})}\, \ell(R)^{n/2}.$$

To estimate $\|\widetilde\chi_{\kappa,R}-\tilde\varphi\|_{L^2(\mathcal{H}^n|_{L_R})}$ we use the α-numbers and the thin boundary condition of R with respect to μ:

$$\begin{aligned}
\|\widetilde\chi_{\kappa,R}-\tilde\varphi\|^2_{L^2(\mathcal{H}^n|_{L_R})} &\leq \int |\widetilde\chi_{\kappa,R}-\tilde\varphi|^2\, d\mu + \left|\int |\widetilde\chi_{\kappa,R}-\tilde\varphi|^2\, d(\mu-\mathcal{H}^n|_{L_R})\right| \\
&\lesssim \mu(\mathcal{U}_{2\kappa\ell(R)}(R)\setminus R) \\
&\qquad\qquad + \alpha_\mu^{L_R}(MB_R)\,(M\ell(B_R))^{n+1}\,\|\nabla(|\widetilde\chi_{\kappa,R}-\tilde\varphi|^2)\|_\infty \\
&\lesssim \kappa^{\gamma_0}\,\mu(R) + \varepsilon M^{n+1}\,\kappa^{-1}\,\ell(R)^n.
\end{aligned}$$

where we took into account that $\|\nabla(|\widetilde\chi_{\kappa,R}-\tilde\varphi|^2)\|_\infty \lesssim (\kappa\ell(R))^{-1}$. Thus,

$$A_{3,2} \lesssim \left(\kappa^{\gamma_0/2} + \varepsilon^{1/2} M^{(n+1)/2}\,\kappa^{-1/2}\right)\,\mu(R).$$

Next we deal with $A_{3,1}$. To this end, we write

$$\begin{aligned}
A_{3,1} &\lesssim \sup_{x\in L_R\cap B(x_R',2\ell(R))} |T_{\mathcal{H}^n|_{L_R}}\widetilde\chi_{M,R}(x)|\;\mu(R) \\
&\lesssim \sup_{x\in L_R\cap B(x_R',2\ell(R))} |T_{x,\mathcal{H}^n|_{L_R}}\widetilde\chi_{M,R}(x)|\;\mu(R) + M^{n+\alpha}\ell(R)^\alpha\mu(R),
\end{aligned} \tag{3.8.18}$$

where T_x denotes the frozen operator. To simplify notation we denote by $K_x(\cdot) = \nabla_1\Theta(\cdot,0;A(x))$ its kernel. For any $x\in L_R\cap B(x_R',2\ell(R))$, by the change of variable $z=2x-y$,

$$\begin{aligned}
T_{x,\mathcal{H}^n|_{L_R}}\widetilde\chi_{M,R}(x) &= \int K_x(x-y)\,\widetilde\chi_{M,R}(y)\,d\mathcal{H}^n|_{L_R}(y) \\
&= \int K_x(z-x)\,\widetilde\chi_{M,R}(2x-z)\,d\mathcal{H}^n|_{L_R}(z).
\end{aligned} \tag{3.8.19}$$

Hence,

$$
\begin{aligned}
2\,T_{x,\mathcal{H}^n|_{L_R}}\widetilde{\chi}_{M,R}(x) &= \int K_x(x-y)\,\widetilde{\chi}_{M,R}(y)\,d\mathcal{H}^n|_{L_R}(y) \\
&\quad + \int K_x(y-x)\,\widetilde{\chi}_{M,R}(2x-y)\,d\mathcal{H}^n|_{L_R}(y) \\
&= \int K_x(x-y)\,\big(\widetilde{\chi}_{M,R}(y)-\widetilde{\chi}_{M,R}(2x-y)\big)\,d\mathcal{H}^n|_{L_R}(y).
\end{aligned}
\tag{3.8.20}
$$

To estimate the last integral, recall that $\widetilde{\chi}_{M,R}$ is radial with respect to x'_R, and hence

$$\widetilde{\chi}_{M,R}(2x-y)=\widetilde{\chi}_{M,R}(2x'_R-(2x-y))=\widetilde{\chi}_{M,R}(y+2(x'_R-x)).$$

Thus, for all $x\in L_R\cap B(x'_R,2\ell(R))$,

$$\operatorname{supp}\big(\widetilde{\chi}_{M,R}-\widetilde{\chi}_{M,R}(2x-\cdot)\big)\subset A\big(x'_R,\tfrac12 M\ell(R),2M\ell(R)\big).$$

Also, for all $y\in L_R$, since $\widetilde{\chi}_{M,R}$ is Lipschitz with constant $c/(M\ell(R))$,

$$\big|\widetilde{\chi}_{M,R}(y)-\widetilde{\chi}_{M,R}(2x-y)\big|=\big|\widetilde{\chi}_{M,R}(y)-\widetilde{\chi}_{M,R}(y+2(x'_R-x))\big|\lesssim\frac{|x'_R-x|}{M\,\ell(R)}\lesssim\frac{1}{M}.$$

So we get

$$|T_{x,\mathcal{H}^n|_{L_R}}\widetilde{\chi}_{M,R}(x)|\lesssim\frac{1}{M}\int_{A\left(x'_R,\frac12 M\ell(R),2M\ell(R)\right)}|K_x(x-y)|\,d\mathcal{H}^n|_{L_R}(y)\lesssim\frac{1}{M}.\tag{3.8.21}$$

Together with (3.8.18), this gives

$$A_{3,1}\lesssim\big(M^{n+\alpha}\ell(R)^\alpha+M^{-1}\big)\,\mu(R).$$

Now, gathering this estimate with the one of $A_{3,2}$, we get

$$A_3\lesssim\Big(\kappa^{\gamma_0/2}+\varepsilon^{1/2}M^{(n+1)/2}\,\kappa^{-1/2}+M^{n+\alpha}\ell(R)^\alpha+M^{-1}+\ell(R)^\alpha\Big)\;\mu(R),$$

and then, by (3.8.15),

$$
\begin{aligned}
A &\lesssim M^{1/2}\kappa^{\gamma_0/2}\mu(R)+A_1+A_2+A_3\\
&\lesssim M^{1/2}\kappa^{\gamma_0/2}\mu(R)+\left(\varepsilon\frac{M^{2n+1}}{\kappa^{n+2}}+\kappa^\alpha\right)\mu(R)+\left(\varepsilon\frac{M^{n+1}}{\kappa^{n+2}}+M^n\,\kappa^\alpha\right)\mu(R)\\
&\quad+\Big(\kappa^{\gamma_0/2}+\varepsilon^{1/2}M^{(n+1)/2}\,\kappa^{-1/2}+M^{n+\alpha}\ell(R)^\alpha+M^{-1}+\ell(R)^\alpha\Big)\;\mu(R).
\end{aligned}
$$

Note that if M is chosen big enough, then κ and $\ell(R)$ small enough, and finally ε small enough (in this order), we get

$$A\le\varepsilon_3\,\mu(R).$$

We can now conclude the estimate of the term I_4 in (3.8.11). From (3.8.3), the last estimate, and (3.8.13), we obtain

$$
\begin{aligned}
I_4 &\le|m_R(T_\mu(\widetilde{\chi}_{M,R}-\chi_R))|+|m_R(T_\mu\chi_R)|\\
&\lesssim\ell(R)^\alpha+\frac{1}{\mu(R)}\,(A+B)\lesssim\ell(R)^\alpha+\kappa^{\gamma_0/2}+\varepsilon_3\lesssim\varepsilon_3,
\end{aligned}
$$

assuming again $\ell(R)$ and κ to be small enough.

Estimate of I_5 and I_5^H. Recall that $I_5 = |m_Q(\widehat{T}_\mu^\phi \widetilde{\chi}_{M,R})|$ and that, for $x \in Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$, by definition we have $\widehat{T}_\mu^\phi \widetilde{\chi}_{M,R}(x) = \widehat{T}_\mu \widetilde{\chi}_{M,R}(x^*)$. We split it as follows

$$\begin{aligned} \left|\widehat{T}\left(\widetilde{\chi}_{M,R}\mu\right)(x^*)\right| \leq{}& \left|\widehat{T}(\widetilde{\chi}_{M,R}\mu)(x^*) - \widehat{T}(\widetilde{\chi}_{M,R} c_R \mathcal{H}^n|_{L_R})(x^*)\right| \\ &+ \left|c_R \widehat{T}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x^*)\right|. \end{aligned} \tag{3.8.22}$$

We consider now the first term in the right hand side of inequality (3.8.22). Let $\widehat{T}_{x^*}$ be the frozen operator associated with the kernel $\widehat{K}_{x^*}(\cdot) := \nabla_1 \Theta(\cdot, 0; \widehat{A}(x^*))$. Notice that by Lemma 3.8.1 and Lemma 4.2.2,

$$\begin{aligned} &\left|\widehat{T}_\mu(\widetilde{\chi}_{M,R})(x^*) - \widehat{T}_{c_R\mathcal{H}^n|_{L_R}}(\widetilde{\chi}_{M,R})(x^*)\right| \\ &\qquad \lesssim \left|\widehat{T}_{x^*, c_R\mathcal{H}^n|_{L_R}}(\widetilde{\chi}_{M,R})(x^*) - \widehat{T}_{c_R\mathcal{H}^n|_{L_R}}(\widetilde{\chi}_{M,R})(x^*)\right| \\ &\qquad\quad + \left|\widehat{T}_\mu(\widetilde{\chi}_{M,R})(x^*) - \widehat{T}_{x^*,\mu}(\widetilde{\chi}_{M,R})(x^*)\right| \\ &\qquad\quad + \left|\widehat{T}_{x^*,\mu}(\widetilde{\chi}_{M,R})(x^*) - \widehat{T}_{x^*, c_R\mathcal{H}^n|_{L_R}}(\widetilde{\chi}_{M,R})(x^*)\right| \\ &\qquad \lesssim \left|\int \widetilde{\chi}_{M,R}(y)\, \widehat{K}_{x^*}(x^* - y) d(\mu - c_R\mathcal{H}^n|_{L_R})(y)\right| \\ &\qquad\quad + M^{\alpha/2}\ell(R)^{\alpha/2}. \end{aligned} \tag{3.8.23}$$

To estimate the remaining term in the last inequality, we will use the α-numbers. To this end we consider an auxiliary smooth function ψ which equals 1 on $\mathbb{R}^{n+1} \setminus B(x^*, \Delta\ell(R)/2)$ and vanishes in $B(x^*, \Delta\ell(R)/4)$, with $\|\nabla\psi\|_\infty \leq 1/(\Delta\ell(R))$. Then taking into account that $\psi \equiv 1$ on $MB_R \cap \operatorname{supp}\mu$, the remaining term in the inequality above equals

$$\begin{aligned} &\left|\int \widetilde{\chi}_{M,R}(y)\, \psi(y)\, \widehat{K}_{x^*}(x^* - y) d(\mu - c_R\mathcal{H}^n|_{L_R})(y)\right| \\ &\qquad\qquad \leq \alpha_\mu^{L_R}(MB_R)\, (M\ell(R))^{n+1} \left\|\nabla\left(\widetilde{\chi}_{M,R}\, \psi\, \widehat{K}_{x^*}(x^* - \cdot)\right)\right\|_\infty. \end{aligned}$$

It is easy to check that $\left\|\nabla\left(\widetilde{\chi}_{M,R}\, \psi\, \widehat{K}_{x^*}(x^* - \cdot)\right)\right\|_\infty \lesssim C(M,\Delta)\, \ell(R)^{-n-1}$. Thus, the integral on the right hand side of (3.8.23) does not exceed $C(M,\Delta)\,\varepsilon$, and so

$$\left|\widehat{T}_\mu(\widetilde{\chi}_{M,R})(x^*) - \widehat{T}_{c_R\mathcal{H}^n|_{L_R}}(\widetilde{\chi}_{M,R})(x^*)\right| \lesssim C(M,\Delta)\,\varepsilon + (M\ell(R))^{\alpha/2}. \tag{3.8.24}$$

To estimate the second term on the right hand side of inequality (3.8.22), we denote by w the orthogonal projection of x on L_R (recall that $x \in Q$), by w^* the reflection of w with respect to H, and we split

$$\begin{aligned} \left|\widehat{T}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x^*)\right| \leq{}& \left|\widehat{T}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x^*) - \widehat{T}_{x^*}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x^*)\right| \\ &+ \left|\widehat{T}_{x^*}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x^*) - \widehat{T}_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x^*)\right| \\ &+ \left|\widehat{T}_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x^*) - \widehat{T}_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(w^*)\right| \\ &+ \left|\widehat{T}_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(w^*)\right|, \end{aligned} \tag{3.8.25}$$

where $\widehat{T}_{x_R}$ is the frozen operator associated with the kernel $\widehat{K}_{x_R}(\cdot) := \nabla_1\Theta(\cdot, 0; \widehat{A}(x_R))$. Using Lemma 4.2.2, it is easy to check that the first term on the right hand side does not exceed $C\,(M\ell(R))^{\alpha/2}$. For the third term, since

$$\mathrm{dist}(x^*, \operatorname{supp}\mu \cap 2MB_R) \approx \Delta\ell(R) \ll |x^* - w^*| = |x - w| \lesssim \varepsilon^{1/(n+1)}\ell(R),$$

by standard arguments we derive

$$\begin{aligned}&\big|\widehat{T}_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x^*) - \widehat{T}_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(w^*)\big|\\ &\qquad\lesssim \frac{|x^* - w^*|}{(\Delta\ell(R))^{n+1}}\,\mathcal{H}^n|_{L_R}\big(B(x_R', M\ell(R))\big) \lesssim \varepsilon^{1/(n+1)}\Delta^{-n}M^n.\end{aligned}$$

Next we estimate the second term on the right hand side of (3.8.25). We have

$$\begin{aligned}&\big|\widehat{T}_{x^*}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x^*) - \widehat{T}_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x^*)\big|\\ &\qquad\qquad \le \int_{2MB_R} |\widehat{K}_{x^*}(x^* - y) - \widehat{K}_{x_R}(x^* - y)|\, d\mathcal{H}^n|_{L_R}(y).\end{aligned}$$

By (4.2), we have

$$\begin{aligned}&\widehat{K}_{x^*}(z) - \widehat{K}_{x_R}(z)\\ &\quad= \frac{\omega_n^{-1}}{\sqrt{\det \widehat{A}(x^*)}}\frac{\widehat{A}(x^*)^{-1}z}{(\widehat{A}(x^*)^{-1}z\cdot z)^{(n+1)/2}} - \frac{\omega_n^{-1}}{\sqrt{\det \widehat{A}(x_R)}}\frac{\widehat{A}(x_R)^{-1}z}{(\widehat{A}(x_R)^{-1}z\cdot z)^{(n+1)/2}}.\end{aligned}$$

By standard estimates and the Hölder continuity of $\widehat{A}$ it follows that, for any $z \in \mathbb{R}^{n+1}$,

$$\big|\widehat{K}_{x^*}(z) - \widehat{K}_{x_R}(z)\big| \lesssim \frac{|x^* - x_R|^{\alpha/2}}{|z|^n} \lesssim \frac{\ell(R)^{\alpha/2}}{|z|^n}. \tag{3.8.26}$$

Since, for any $x \in R$, $\mathrm{dist}(x^*, L_R) \approx \Delta\,\ell(R)$, we deduce

$$\begin{aligned}\int_{2MB_R}\big|\widehat{K}_{x^*}(x^* - y) - \widehat{K}_{x_R}(x^* - y)\big|\, d\mathcal{H}^n|_{L_R}(y) &\lesssim \frac{\ell(R)^{\alpha/2}}{(\Delta\ell(R))^n}\,\mathcal{H}^n(2MB_R \cap L_R)\\ &\approx M^n\Delta^{-n}\ell(R)^{\alpha/2}.\end{aligned}$$

Therefore, plugging all these estimates in (3.8.25), we get

$$\begin{aligned}\big|\widehat{T}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x^*)\big| &\lesssim M^{\alpha/2}\ell(R)^{\alpha/2} + \varepsilon^{1/(n+1)}\Delta^{-n}M^n\\ &\qquad + M^n\Delta^{-n}\ell(R)^{\alpha/2} + \big|\widehat{T}_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(w^*)\big|.\end{aligned}$$

To deal with the last term on the right hand side of (3.8.25), we distinguish between the vertical and the horizontal components, so we set

$$\big|\widehat{T}_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(w^*)\big| \le \big|\widehat{T}^V_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(w^*)\big| + \big|\widehat{T}^H_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(w^*)\big|.$$

Being $\widehat{A}(x_R) = Id$, $\widehat{T}_{x_R}$ coincides with the Riesz transform modulo some constant factor. Hence its vertical component coincides with the Poisson transform modulo some constant factor, so that

$$\big|\widehat{T}^V_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(w^*)\big| \lesssim 1. \tag{3.8.27}$$

The horizontal component is estimated like the term $T^H_x(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(x)$ in (3.8.19). The reader can check that the same estimates hold just replacing x by either w^* or

x_R appropriately, and T_x by $\widehat{T}_{x_R}$. A key point is that, for the kernel $\widehat{K}^H_{x_R}$ of $\widehat{T}^H_{x_R}$, the change of variable $z = 2w - y$ gives us

$$\begin{aligned}\widehat{T}^H_{x_R,\mathcal{H}^n|_{L_R}}\widetilde{\chi}_{M,R}(w^*) &= \int \widehat{K}^H_{x_R}(w^* - y)\,\widetilde{\chi}_{M,R}(y)\,d\mathcal{H}^n|_{L_R}(y)\\ &= \int \widehat{K}^H_{x_R}(z - (2w - x^*))\,\widetilde{\chi}_{M,R}(2w - z)\,d\mathcal{H}^n|_{L_R}(z)\\ &= \int \widehat{K}^H_{x_R}(z - w^*)\,\widetilde{\chi}_{M,R}(2w - z)\,d\mathcal{H}^n|_{L_R}(z),\end{aligned}$$

which is analogous to (3.8.19). Notice that the last identity is only valid for the horizontal component of the kernel $\widehat{K}_{x_R}$ (taking into account that $\widehat{K}_{x_R}$ is the kernel of the Riesz transform modulo some constant factor, since $\widehat{A}(x_R) = Id$). Then, as in (3.8.20), we can write

$$2\,\widehat{T}^H_{x_R,\mathcal{H}^n|_{L_R}}\widetilde{\chi}_{M,R}(w^*) = \int \widehat{K}^H_{x_R}(w^* - y)\,\big(\widetilde{\chi}_{M,R}(y) - \widetilde{\chi}_{M,R}(2w - y)\big)\,d\mathcal{H}^n|_{L_R}(y).$$

Thus, as in (3.8.21), we get

$$\big|\widehat{T}^H_{x_R}(\widetilde{\chi}_{M,R}\mathcal{H}^n|_{L_R})(w^*)\big| \lesssim \frac{1}{M}.$$

Together with (3.8.24), this yields

$$\begin{aligned}I_5 &\le \sup_{x\in R}\big|\widehat{T}_\mu\widetilde{\chi}_{M,R}(x^*)\big|\\ &\lesssim C(M,\Delta)\,\varepsilon + M^{\alpha/2}\ell(R)^{\alpha/2}\\ &\qquad + M^{\alpha/2}\ell(R)^{\alpha/2} + \varepsilon^{1/(n+1)}\Delta^{-n}M^n + 1 + \frac{1}{M} + M^n\Delta^{-n}\ell(R)^{\alpha/2}.\end{aligned}$$

For I_5^H we get almost the same estimate. The only difference is that we do not have to estimate the vertical term in (3.8.27), and thus the summand 1 does not appear in the last inequality. So we have

$$I_5^H \lesssim C(M,\Delta)\,\varepsilon + M^{\alpha/2}\ell(R)^{\alpha/2} + M^{\alpha/2}\ell(R)^{\alpha/2} + \varepsilon^{1/(n+1)}\Delta^{-n}M^n + \frac{1}{M} + M^n\Delta^{-n}\ell(R)^{\alpha/2}.$$

Thus, for M big enough, $\ell(R)$ small enough and ε small enough, we get

$$I_5 \lesssim 1 \quad\text{and}\quad I_5^H \lesssim \varepsilon_3.$$

Recall that we showed that $I_i^H \le I_i \lesssim \varepsilon_3$ for $i = 1, \dots, 4$, by choosing the parameters M and κ properly and assuming ε and $\ell(R)$ small enough. Then, gathering the estimates obtained for $I_1, \dots, I_5$ and I_5^H, the lemma follows.

3.8.4 Proof of Lemma 3.8.5

Recall that

$$f - \tilde{f} = \sum_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S(\mu - \tilde{\mu}))\,\chi_Q$$

and

$$f^H - \tilde{f}^H = \sum_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S^H(\mu - \tilde{\mu}))\,\chi_Q.$$

So we have

$$\begin{aligned}\|f^H - \tilde f^H\|_{L^2(\mu)} &\le \|f-\tilde f\|_{L^2(\mu)} \le \|S(\mu-\tilde\mu)\|_{L^2(\mu|_R)}\\ &\le \|S(\mu|_R-\tilde\mu)\|_{L^2(\mu|_R)} + \|S(\mu|_{R^c})\|_{L^2(\mu|_R)}.\end{aligned}$$

To estimate the first term on the right hand side we use the $L^2(\mu|_R)$ boundedness of S_μ and (3.7.1):

$$\|S(\mu|_R-\tilde\mu)\|_{L^2(\mu|_R)} \lesssim \|\tilde\mu-\mu|_R\| \lesssim t^{\gamma_0}\mu(R).$$

To deal with the second term we split R^c in two regions:

$$D_1 = \mathcal{U}_{\Delta^{1/2}\ell(R)}(R)\setminus R, \qquad D_2 = \mathbb{R}^{n+1}\setminus \mathcal{U}_{\Delta^{1/2}\ell(R)}(R).$$

Then we have

$$\|S(\mu|_{R^c})\|_{L^2(\mu|_R)} \le \|S(\chi_{D_1}\mu)\|_{L^2(\mu|_R)} + \|S(\chi_{D_2}\mu)\|_{L^2(\mu|_R)}.$$

By the $L^2(\mu|_R)$ boundedness of S_μ and the thin boundary property, we have

$$\|S(\chi_{D_1}\mu)\|^2_{L^2(\mu|_R)} \lesssim \mu(D_1) \lesssim \Delta^{\gamma_0/2}\,\mu(R).$$

To estimate $\|S(\chi_{D_2}\mu)\|_{L^2(\mu|_R)}$, recall that

$$S(\chi_{D_2}\mu)(x) = \int_{D_2}\big(\widehat K(x,y) - \widehat K(x^*,y)\big)\,d\mu(y)$$

For $x\in R$, $y\in D_2$, we have

$$|x-x^*| \le 2\Delta\,\ell(R) \ll \frac12\,\Delta^{1/2}\ell(R) \le \frac12\,|x-y|,$$

and thus

$$\big|\widehat K(x,y)-\widehat K(x^*,y)\big| \lesssim \frac{\Delta^{\alpha/2}\ell(R)^{\alpha/2}}{|x-y|^{n+\alpha/2}}.$$

Therefore, by standard estimates using the n-growth of μ, for $x\in R$,

$$\begin{aligned}\big|S(\chi_{D_2}\mu)(x)\big| &\lesssim \int_{|x-y|>\frac12\Delta^{1/2}\ell(R)} \frac{\Delta^{\alpha/2}\ell(R)^{\alpha/2}}{|x-y|^{n+\alpha/2}}\,d\mu(y)\\ &\lesssim \frac{\Delta^{\alpha/2}\ell(R)^{\alpha/2}}{(\Delta^{1/2}\ell(R))^{\alpha/2}} \lesssim \Delta^{\alpha/4}.\end{aligned}$$

Hence,

$$\|S(\chi_{D_2}\mu)\|^2_{L^2(\mu|_R)} \lesssim \Delta^{\alpha/2}\mu(R).$$

Together with the previous estimates, this yields

$$\|f^H-\tilde f^H\|_{L^2(\mu)} \le \|f-\tilde f\|_{L^2(\mu)} \lesssim \big(t^{\gamma_0} + \Delta^{\min(\alpha/2,\gamma_0/2)}\big)\,\mu(R),$$

which proves the lemma.

3.8.5 Proof of Lemma 3.8.6

We will just prove (3.8.10). The arguments for the other inequalities (3.8.8) and (3.8.9) are totally analogous. Indeed, the reader can easily check that the operators

$\widehat{T}$, S, and S^H are essentially interchangeable in the estimates below.

Recall that the measure σ was defined in Section 3.7, and that $\tilde{\varepsilon}$ is such that $\beta_{\infty,\mu}(MB_Q) \le \tilde{\varepsilon}$ for all $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$, $R \in \mathsf{Nice}$.

Let τ be a small number to be chosen below, with $\varepsilon \ll \tau \ll \min(t, \Delta) \ll 1$. For a fixed $Q \in \mathsf{Ch}_{\mathsf{Stop}}(Q)$ and $x \in \mathbb{R}^{n+1}$, let $\tilde{\chi}_1(x)$ be a smooth radial function such that $\operatorname{supp} \tilde{\chi}_1 \subset B(0, \tau\ell(Q))$ and $\tilde{\chi}_1(x) \equiv 1$ in $B(0, \frac{1}{2}\tau\ell(Q))$. Let also $\tilde{\chi}_2$ be a smooth radial function supported on the annulus $A\big(0, \frac{1}{2}\tau\ell(Q), \frac{1}{2}M\ell(Q)\big)$ and such that $\tilde{\chi}_2 \equiv 1$ in $A\big(0, \tau\ell(Q), \frac{1}{4}M\ell(Q)\big)$. Finally, set $\tilde{\chi}_3$ a smooth radial function supported on $B\big(0, \frac{1}{4}M\ell(Q)\big)^c$, such that $\tilde{\chi}_3 \equiv 1$ in $B\big(0, \frac{1}{2}M\ell(Q)\big)^c$. We construct the functions $\tilde{\chi}_i$ so that $\tilde{\chi}_1 + \tilde{\chi}_2 + \tilde{\chi}_3 = 1$. Notice that they depend on the cube Q. Now denote $K_{S^H_{i,Q}}(x,y) = K_{S^H}(x,y)\tilde{\chi}_i(|x-y|)$ for $i = 1,2,3$, so that $K_{S^H}(x,y) = \sum_{i=1}^3 K_{S^H_{i,Q}}(x,y)$, and denoting by $S^H_{i,Q}$ the operator associated with the truncated kernel $K_{S^H_{i,Q}}$, we also have $S^H = \sum_{i=1}^3 S^H_{i,Q}$. Further, we can write

$$S^H\sigma = \sum_{i=1}^3 \sum_{Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)} \chi_{\operatorname{supp}\sigma_Q} \cdot S^H_{i,Q}\sigma =: \sum_{i=1}^3 S^H_i \sigma \quad \text{in } L^2(\sigma)$$

and

$$S^H\mu = \sum_{i=1}^3 \sum_{Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)} \chi_Q \cdot S^H_{i,Q}\mu =: \sum_{i=1}^3 S^H_i \mu \quad \text{in } L^2(\mu|_R)$$

To shorten the notation, we will write $K_{S^H_i}(x,y)$ instead of $K_{S^H_{i,Q}}(x,y)$ when Q is clear from the context. We split

$$\|S^H\sigma\|_{L^2(\sigma)} \le \|S^H_1\sigma\|_{L^2(\sigma)} + \|S^H_2\sigma\|_{L^2(\sigma)} + \|S^H_3\sigma\|_{L^2(\sigma)}. \tag{3.8.28}$$

Estimate of $\|S^H_3\sigma\|_{L^2(\sigma)}$. For $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$ and $x, x' \in \mathcal{U}_{10\tilde{\varepsilon}\ell(Q)}(Q_{(t)})$, we have

$$\begin{aligned} &|S^H_{3,Q}\sigma(x) - S^H_{3,Q}\tilde{\mu}(x')| \\ &\qquad \le |S^H_{3,Q}\sigma(x) - S^H_{3,Q}\tilde{\mu}(x)| + |S^H_{3,Q}\tilde{\mu}(x) - S^H_{3,Q}\tilde{\mu}(x')| = S_{31} + S_{32}. \end{aligned} \tag{3.8.29}$$

Notice that for $x \in \mathcal{U}_{10\tilde{\varepsilon}\ell(Q)}(Q_{(t)})$ and $y,\ y' \in \mathcal{U}_{10\tilde{\varepsilon}\ell(P)}(P_{(t)})$, $P \in \mathsf{Ch}_{\mathsf{Stop}}(R)$, by Lemma 4.2.1,

$$\begin{aligned} \big|K_{S^H_{3,Q}}(x,y) - K_{S^H_{3,Q}}(x,y')\big| &\lesssim \frac{|y-y'|^{\alpha/2}}{C(t)(\ell(Q) + \ell(P) + \operatorname{dist}(P,Q))^{n+\alpha/2}} \\ &\lesssim \frac{\ell(P)^{\alpha/2}}{C(t)D(P,Q)^{n+\alpha/2}}, \end{aligned} \tag{3.8.30}$$

where $D(P,Q) = \ell(Q) + \ell(P) + \operatorname{dist}(P,Q)$ and the t-dependence of $C(t)$ comes from the comparability $|x-y| \approx |x-y'|$, which depends on t (due to $\tilde{\varepsilon} \ll t$ being very small). Applying now Lemma 3.7.1, with the Lip_α-constant coming from (3.8.30),

$$\begin{aligned} S_{31} = |S^H_{3,Q}\sigma(x) - S^H_{3,Q}\tilde{\mu}(x)| &\le \sum_{P \in \mathsf{Ch}_{\mathsf{Stop}}(R)} \left| \int K_{S^H_{3,Q}}(x,y)\, d(\sigma_P - \tilde{\mu}_P)(y) \right| \\ &\lesssim M^{\alpha/2}\tilde{\varepsilon}^{\alpha/2} C(t) \sum_{P \in \mathsf{Ch}_{\mathsf{Stop}}(R)} \frac{\ell(P)^{\alpha/2}\mu(P)}{D(P,Q)^{n+\alpha/2}}. \end{aligned} \tag{3.8.31}$$

Concerning S_{32}, by standard estimates, one gets

$$S_{32} \leq \int |K_{S^H_{3,Q}}(x,y) - K_{S^H_{3,Q}}(x',y)|d\tilde{\mu}(y) \lesssim \int_{|x-y|\geq \frac{1}{8}M\ell(Q)} \frac{|x-x'|^{\alpha/2}}{|x_Q - y|^{n+\alpha/2}} d\tilde{\mu}(y) \lesssim \frac{1}{M^{\alpha/2}}. \tag{3.8.32}$$

As a consequence of (3.8.31) and (3.8.32),

$$|S^H_{3,Q}\sigma(x) - m_{\mu,Q}(S^H_{3,Q}\tilde{\mu})| \leq M^{\alpha/2}\tilde{\varepsilon}^{\alpha/2}C(t) \sum_{P\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \frac{\ell(P)^{\alpha/2}\mu(P)}{D(P,Q)^{n+\alpha/2}} + \frac{C}{M^{\alpha/2}}.$$

This implies that for $x \in \operatorname{supp} \sigma_Q$, $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$,

$$|S^H_3\sigma(x)| \lesssim |m_{\mu,Q}(S^H_3\tilde{\mu})| + M^{\alpha/2}\tilde{\varepsilon}^{\alpha/2}C(t) \sum_{P\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \frac{\ell(P)^{\alpha/2}\mu(P)}{D(P,Q)^{n+\alpha/2}} + \frac{C}{M^{\alpha/2}}. \tag{3.8.33}$$

Denote

$$g(x) = \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \sum_{P\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \frac{\ell(P)^{\alpha/2}\mu(P)}{D(P,Q)^{n+\alpha/2}} \chi_Q(x). \tag{3.8.34}$$

Since $\mu(Q) \approx \sigma(Q)$ for each Q, squaring and integrating (3.8.33) with respect to σ, we obtain

$$\begin{aligned}\|S^H_3\sigma\|^2_{L^2(\sigma)} &\lesssim \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S^H_3\tilde{\mu})^2\mu(Q) \\ &\quad + M^{\alpha}\tilde{\varepsilon}^{\alpha}C(t)\|g\|^2_{L^2(\mu)} + M^{-\alpha} \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \mu(Q) \\ &\approx \Big\| \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S^H_3\tilde{\mu})\chi_Q \Big\|^2_{L^2(\mu|_R)} + M^{\alpha}\tilde{\varepsilon}^{\alpha}C(t)\|g\|^2_{L^2(\mu)} + M^{-\alpha}\mu(R).\end{aligned} \tag{3.8.35}$$

We will estimate $\|g\|_{L^2(\mu)}$ by duality: for any non-negative funtion $h \in L^2(\mu)$ write

$$\begin{aligned}\int gh\, d\mu &= \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \sum_{P\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \frac{\ell(P)^{\alpha/2}\mu(P)}{D(P,Q)^{n+\alpha/2}} \int_Q h d\mu \\ &= \sum_{P\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \mu(P) \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \frac{\ell(P)^{\alpha/2}}{D(P,Q)^{n+\alpha/2}} \int_Q h d\mu.\end{aligned} \tag{3.8.36}$$

Notice that for each $z \in P \in \mathsf{Ch}_{\mathsf{Stop}}(R)$, integrating on annuli we get

$$\begin{aligned}\sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \frac{\ell(P)^{\alpha/2}}{D(P,Q)^{n+\alpha/2}} \int_Q h d\mu \\ &\lesssim \int_Q \frac{\ell(P)^{\alpha/2} h(y)}{(\ell(P)+|z-y|)^{n+\alpha/2}} d\mu(y) \\ &= \int_{|z-y|\leq \ell(P)} \frac{\ell(P)^{\alpha/2} h(y)}{(\ell(P)+|z-y|)^{n+\alpha/2}} d\mu(y) \\ &\quad + \sum_{i=1}^{\infty} \int_{2^{i-1}\ell(P)\leq|z-y|\leq 2^i\ell(P)} \frac{\ell(P)^{\alpha/2} h(y)}{(\ell(P)+|z-y|)^{n+\alpha/2}} d\mu(y) \\ &\lesssim \sum_{i=0}^{\infty} \frac{2^{-i\alpha/2}\mu(B(z,2^i\ell(P)))}{(2^i\ell(P))^n} m_{\mu,B(z,2^i\ell(P))}(h).\end{aligned} \tag{3.8.37}$$

Now let M_μ stand for the centered maximal Hardy-Littlewood operator with respect to μ. Since $m_{\mu,B(z,2^i\ell(P))}(h) \lesssim M_\mu h(z)$ and

$$\sum_{i=0}^{\infty} \frac{2^{-i\alpha/2}\mu(B(z,2^i\ell(P)))}{(2^i\ell(P))^n} \leq C,$$

by (3.8.36) and (3.8.37),

$$\int gh\, d\mu \lesssim \sum_{P\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \inf_{z\in P} M_\mu h(z)\, \mu(P) \leq \int M_\mu h\, d\mu \lesssim \|h\|_{L^2(\mu)} \mu(R)^{1/2}.$$

Therefore,

$$\|g\|_{L^2(\mu)} \lesssim \mu(R)^{1/2}.$$

Plugging this into (3.8.35) we get

$$\begin{aligned}\|S_3^H\sigma\|_{L^2(\sigma)}^2 &\lesssim \Big\| \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S_3^H\tilde{\mu})\chi_Q \Big\|_{L^2(\mu|_R)}^2 + \big(M^{-\alpha} + M^{\alpha}\tilde{\varepsilon}^{\alpha} C(t)\big)\,\mu(R) \\ &\lesssim \|\tilde{f}^H\|_{L^2(\mu)}^2 + \Big\| \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S_1^H\tilde{\mu})\chi_Q \Big\|_{L^2(\mu|_R)}^2 \\ &\quad + \Big\| \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S_2^H\tilde{\mu})\chi_Q \Big\|_{L^2(\mu|_R)}^2 + \big(M^{-\alpha} + M^{\alpha}\tilde{\varepsilon}^{\alpha} C(t)\big)\,\mu(R).\end{aligned}$$

Estimate of $\|S_1^H\sigma\|_{L^2(\sigma)}$. Recall that, by (3.7.2), for each $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$,

$$\operatorname{supp}\sigma_Q \subset \mathcal{U}_{3\tilde{\varepsilon}\ell(Q)}(\operatorname{supp}\Pi_{L_Q\sharp}\mu|_{Q_{(t)}}) \subset \mathcal{U}_{6\tilde{\varepsilon}\ell(Q)}(Q_{(t)}),$$

and, for $P, Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$ with $P \neq Q$, by (3.7.3),

$$\operatorname{dist}(\operatorname{supp}\sigma_P, \operatorname{supp}\sigma_Q) \geq \frac{t}{2} \max(\ell(P), \ell(Q)).$$

Therefore, recalling that $\tau \ll t$,

$$S_1^H \sigma(x) = S_1^H \sigma_Q(x) \quad \text{for all } x \in \operatorname{supp} \sigma_Q. \tag{3.8.38}$$

Let J_Q be the convex hull of $\mathcal{U}_{10\tilde{\varepsilon}\ell(Q)}(Q_{(3\tau)}) \cap L_Q$. Then the following hold:

1. By the thin boundary condition, we have

$$\sigma_Q\big((J_Q)^c\big) \le \mu(Q \setminus \mathcal{U}_{20\tilde{\varepsilon}\ell(Q)}(Q_{(3\tau)})) \le \mu(Q \setminus Q_{(4\tau)}) \lesssim \tau^{\gamma_0}\mu(Q).$$

2. Let ψ_{3B_Q} be a smooth function that equals 1 in $2B_Q$, vanishes in $(3B_Q)^c$ and such that $\|\nabla\psi_{3B_Q}\|_\infty \lesssim \ell(Q)^{-1}$. Then, for each $x \in L_Q \cap \mathcal{U}_{\tau\ell(Q)}(J_Q)$,

$$\chi_{B(x,3\tilde{\varepsilon}\ell(Q))}\, \Pi_{L_Q\sharp}\mu_{|Q_{(t)}} = \chi_{B(x,3\tilde{\varepsilon}\ell(Q))}\, \Pi_{L_Q\sharp}(\psi_{3B_Q}\mu).$$

Notice now that

$$\|S_1^H\sigma\|^2_{L^2(\sigma)} = \sum_{Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \|S_1^H\sigma\|^2_{L^2(\sigma_Q)},$$

and for each $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$,

$$\|S_1^H\sigma\|^2_{L^2(\sigma_Q)} = \|S_1^H\sigma\|^2_{L^2(\sigma_Q|_{J_Q})} + \|S_1^H\sigma\|^2_{L^2(\sigma_Q|_{(J_Q)^c})} = S_{11} + S_{12}.$$

Write $S_1^H\sigma(x) = \widehat{T}_1^H\sigma(x) - \widehat{T}_1^H\sigma(x^*)$. Since $\tau\ell(Q) \ll \Delta\ell(R)$, we have $\widehat{T}_1^H\sigma(x^*) = 0$. Therefore, by (3.8.38), the Cauchy-Schwarz inequality, the property (1) of J_Q, and the n-growth of the measure σ,

$$\begin{aligned} S_{12} &= \big\|\widehat{T}_1^H\sigma\big\|^2_{L^2(\sigma_Q|_{(J_Q)^c})} = \big\|\widehat{T}_1^H\sigma_Q\big\|^2_{L^2(\sigma_Q|_{(J_Q)^c})} \\ &\le \big\|\widehat{T}_1^H\sigma_Q\big\|^2_{L^4(\mathcal{H}^n|_{L_Q})}\, \sigma_Q\big((J_Q)^c\big)^{1/2} \lesssim \|\widehat{T}_1\sigma_Q\|^2_{L^4(\mathcal{H}^n|_{L_Q})}\, \tau^{\gamma_0/2}\mu(Q)^{1/2}. \end{aligned}$$

Recall that $\sigma_Q = g_Q\, \mathcal{H}^n|_{L_Q}$ for some function g_Q such that $0 \le g_Q \lesssim \chi_{2B_Q\cap L_Q}$. Since $\widehat{T}^H_{1,\mathcal{H}^n|_{L_Q}}$ is bounded in $L^4(\mathcal{H}^n|_{L_Q})$ (by the uniform rectifiability of L_Q, for example), we have $\|\widehat{T}_1\sigma_Q\|_{L^4(\mathcal{H}^n|_{L_Q})} \lesssim \ell(Q)^{n/4}$, and thus

$$S_{12} \lesssim \tau^{\gamma_0/2}\mu(Q).$$

We treat now S_{11}. To this end, notice that for $x \in J_Q$, by (3.8.38), a freezing argument and the antisymmetry of the kernel $\nabla_1\Theta(\cdot,\cdot;\widehat{A}(x))$, we have

$$\begin{aligned} |S_1^H\sigma(x)| = |S_1^H\sigma_Q(x)| &\le \left|\int \nabla_1\Theta(x,y;\widehat{A}(x))d\sigma_Q(y)\right| + C\ell(Q)^\alpha \\ &= \left|\int_{|x-y|\le\tau\ell(Q)} \nabla_1\Theta(x,y;\widehat{A}(x))(g_Q(y)-g_Q(x))d\mathcal{H}^n|_{L_Q}(y)\right| + C\ell(Q)^\alpha \\ &\lesssim \int_{|x-y|\le\tau\ell(Q)} \frac{\operatorname{Lip}\Big(g_{Q|_{L_Q\cap\mathcal{U}_{\tau\ell(Q)}(J_Q)}}\Big)}{|x-y|^{n-1}}\, d\mathcal{H}^n|_{L_Q}(y) + \ell(Q)^\alpha \\ &\lesssim \tau\ell(Q)\operatorname{Lip}\Big(g_{Q|_{L_Q\cap\mathcal{U}_{\tau\ell(Q)}(J_Q)}}\Big) + \ell(Q)^\alpha. \end{aligned} \tag{3.8.39}$$

To estimate $\mathrm{Lip}\Big(g_Q|_{L_Q\cap\mathcal{U}_{\tau\ell(Q)}(J_Q)}\Big)$, observe that for $z\in L_Q\cap\mathcal{U}_{\tau\ell(Q)}(J_Q)$, $\Pi_{L_Q}(z)=z$. Then property (2) of J_Q implies that

$$\begin{aligned}
\big|\nabla g_Q(z)\big| &= \big|\nabla(\Pi_{L_Q\sharp}\mu|_{Q_{(t)}} * \varphi_{2\tilde\varepsilon\ell(Q)})(z)\big| = \big|(\nabla\varphi_{2\tilde\varepsilon\ell(Q)} * \Pi_{L_Q\sharp}\psi_{3B_Q}\mu)(z)\big| \\
&= \left|\int \nabla\varphi_{2\tilde\varepsilon\ell(Q)}(z-y)\,d\Pi_{L_Q\sharp}\psi_{3B_Q}\mu(y)\right| \\
&= \left|\int \nabla\varphi_{2\tilde\varepsilon\ell(Q)}\big(z-\Pi_{L_Q}(y)\big)\psi_{3B_Q}(y)\,d\mu(y)\right| \\
&= \left|\int \nabla\varphi_{2\tilde\varepsilon\ell(Q)}\big(\Pi_{L_Q}(z)-\Pi_{L_Q}(y)\big)\psi_{3B_Q}(y)\,d\mu(y)\right| \\
&= \left|\int (\nabla\varphi_{2\tilde\varepsilon\ell(Q)}\circ\Pi_{L_Q})(z-y)\psi_{3B_Q}(y)\,d(\mu-c_Q\mathcal{H}^n|_{L_Q})(y)\right| \\
&\lesssim \alpha_\mu^{L_Q}(MB_Q)M^{n+1}\ell(Q)^{n+1}\,\mathrm{Lip}\,\big((\nabla\varphi_{2\tilde\varepsilon\ell(Q)}\circ\Pi_{L_Q})(z-\cdot)\psi_{3B_Q}\big)
\end{aligned}$$

To estimate $\mathrm{Lip}\,\big((\nabla\varphi_{2\tilde\varepsilon\ell(Q)}\circ\Pi_{L_Q})(x-\cdot)\psi_{3B_Q}\big)$, we use the fact that

$$\mathrm{Lip}(\nabla(\varphi_{2\tilde\varepsilon\ell(Q)}\circ\Pi_{L_Q}))\leq \mathrm{Lip}(\nabla\varphi_{2\tilde\varepsilon\ell(Q)})+\|\nabla\varphi_{2\tilde\varepsilon\ell(Q)}\|_\infty\frac{C}{\ell(Q)}\lesssim\frac{1}{(\tilde\varepsilon\ell(Q))^{n+2}},$$

and then it follows easily also that

$$\mathrm{Lip}\,\big((\nabla\varphi_{2\tilde\varepsilon\ell(Q)}\circ\Pi_{L_Q})(x-\cdot)\psi_{3B_Q}\big)\lesssim\frac{1}{(\tilde\varepsilon\ell(Q))^{n+2}}.$$

Therefore,

$$\big|\nabla g_Q(z)\big|\lesssim\frac{\alpha_\mu^{L_Q}(MB_Q)M^{n+1}}{\tilde\varepsilon^{n+2}\ell(Q)}.$$

Plugging this estimate in (3.8.39), we get that for $x\in J_Q$,

$$|S_1^H\sigma(x)|\lesssim\alpha_\mu^{L_Q}(MB_Q)\frac{\tau M^{n+1}}{\tilde\varepsilon^{n+2}}+\ell(Q)^\alpha.$$

Thus,

$$S_{11}=\|S_1^H\sigma\|^2_{L^2(\sigma_{Q|J_Q})}\lesssim\big(M^{2n+2}\,\tau^2\,\tilde\varepsilon^{-2n-4}\,\varepsilon^2+\ell(Q)^{2\alpha}\big)\,\mu(Q).$$

Therefore, if $\ell(Q)$ and ε are small enough, we obtain

$$S_{11}=\|S_1^H\sigma\|^2_{L^2(\sigma_{Q|J_Q})}\leq\frac{\varepsilon_5}{2}\,\mu(Q),$$

and finally

$$\|S_1^H\sigma\|^2_{L^2(\sigma)}=\sum_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)}\|S_1^H\sigma\|^2_{L^2(\sigma_Q)}\leq\big(\frac{\varepsilon_5}{2}+C\tau^{\gamma_0/2}\big)\mu(R)\leq\varepsilon_5\,\mu(R),$$

for τ small enough.

Estimate of $\|S_2^H\sigma\|_{L^2(\sigma)}$. First we will estimate $\|S_2^H\sigma\|_{L^2(\sigma)}$ in terms of $\|S_2^H\sigma\|_{L^2(\tilde\mu)}$. Recall that, by definition, $\tilde\sigma_Q = \Pi_{L_Q\sharp}\mu_{|Q_{(t)}}$. By Fubini

$$\begin{aligned}
\|S_2^H\sigma\|^2_{L^2(\sigma)} = \int |S_2^H\sigma(x)|^2 d\sigma(x) &= \sum_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)} \int |S_2^H\sigma(x)|^2 d\sigma_Q(x)\\
&= \sum_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)} \int |S_2^H\sigma(x)|^2 \left(\tilde\sigma_Q * \varphi_{2\tilde\varepsilon\ell(Q)}\right)(x)\, d\mathcal{H}^n|_{L_Q}(x)\\
&= \sum_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)} \int \left(\varphi_{\tilde\varepsilon\ell(Q)} * |S_2^H\sigma|^2\mathcal{H}^n|_{L_Q}\right)(x)\, d\tilde\sigma_Q(x)\\
&\leq \sum_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)} \int \sup_{\substack{|y-x|\leq 2\tilde\varepsilon\ell(Q)\\ y\in L_Q}} |S_2^H\sigma(y)|^2 d\tilde\sigma_Q(x)\\
&= \sum_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)} \int \sup_{\substack{|y-\Pi_{L_Q}(z)|\leq 2\tilde\varepsilon\ell(Q)\\ y\in L_Q}} |S_2^H\sigma(y)|^2 d\mu_{|Q_{(t)}}(z)\\
&\leq \sum_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)} \int \sup_{\substack{|y-z|\leq 3\tilde\varepsilon\ell(Q)\\ y\in L_Q}} |S_2^H\sigma(y)|^2 d\mu_{|Q_{(t)}}(z),
\end{aligned} \tag{3.8.40}$$

since $|y-z| \leq |y-\Pi_{L_Q}(z)| + |\Pi_{L_Q}(z)-z| \leq 3\tilde\varepsilon\ell(Q)$. For such y, z, we write

$$\left|S_2^H\sigma(y)\right| \leq \left|S_2^H\sigma(z)\right| + \int \left|K_{S_2^H}(y,x) - K_{S_2^H}(z,x)\right| d\sigma(x).$$

Taking into account that $\left|K_{S_2^H}(y,\cdot) - K_{S_2^H}(z,\cdot)\right|$ is supported in

$$A\Big(y, \tfrac12\tau\ell(Q), \tfrac12 M\ell(Q)\Big) \cup A\Big(z, \tfrac12\tau\ell(Q), \tfrac12 M\ell(Q)\Big)$$

and that $\tilde\varepsilon \ll \tau$, by Lemma 4.2.1, we deduce

$$\begin{aligned}
\int \left|K_{S_2^H}(y,x) - K_{S_2^H}(z,x)\right| d\sigma(x) &\lesssim \int_{\frac14\tau\ell(Q)\leq|x-y|\leq M\ell(Q)} \frac{|y-z|^{\alpha/2}}{|x-y|^{n+\alpha/2}} d\sigma(x)\\
&\lesssim \tilde\varepsilon^{\alpha/2} M^n\, \tau^{-n-\alpha/2}.
\end{aligned}$$

Therefore, by (3.8.40),

$$\|S_2^H\sigma\|^2_{L^2(\sigma)} \lesssim \|S_2^H\sigma\|^2_{L^2(\tilde\mu)} + \tilde\varepsilon^\alpha M^{2n}\, \tau^{-2n-\alpha}\mu(R).$$

Notice that arguing as in (3.8.31), for $x \in \mathcal{U}_{10\tilde\varepsilon\ell(Q)}(Q_{(t)})$, we get

$$\begin{aligned}
|S_2^H\sigma(x)| &\leq |S_2^H\sigma(x) - S_2^H\tilde\mu(x)| + |S_2^H\tilde\mu(x)|\\
&\lesssim M^{\alpha/2}\tilde\varepsilon^{\alpha/2} C(t,\tau) \sum_{P\in\mathsf{Ch}_{\mathsf{Stop}}(R)} \frac{\ell(P)^{\alpha/2}\mu(P)}{D(P,Q)^{n+\alpha/2}} + |S_2^H\tilde\mu(x)|.
\end{aligned}$$

Define g as in (3.8.34). Arguing as in (3.8.35), (3.8.36) and (3.8.37), we get

$$\|S_2^H\sigma\|^2_{L^2(\tilde\mu)} \lesssim \tilde\varepsilon^\alpha M^\alpha C(t,\tau)\|g\|^2_{L^2(\tilde\mu)} + \|S_2^H\tilde\mu\|^2_{L^2(\tilde\mu)}.$$

Therefore, estimating $\|g\|^2_{L^2(\tilde\mu)}$ by duality as it was done in the estimate of S_3^H, we have

$$\|S_2^H\sigma\|^2_{L^2(\sigma)} \lesssim \tilde\varepsilon^\alpha C(M,t,\tau)\mu(R) + \|S_2^H\tilde\mu\|^2_{L^2(\mu_{|R})}. \tag{3.8.41}$$

Estimate of $\|S_2^H\tilde\mu\|^2_{L^2(\mu_{|R})}$. We write

$$\begin{aligned}\|S_2^H\tilde\mu\|^2_{L^2(\mu_{|R})} \lesssim & \int |S_2^H(\mu_{|R})(x)|^2 d\tilde\mu(x)\\ &+\int |S_2^H(\mu_{|R})(x)|^2 d|\mu|_R-\tilde\mu|(x) + \int_R |S_2^H\tilde\mu - S_2^H(\mu_{|R})|^2 d\mu(x).\end{aligned} \tag{3.8.42}$$

Concerning the last term on the right hand side, by (3.7.1) and the fact that the maximal operator $S^H_{*,\mu}$ is bounded in $L^2(\mu|_R)$, we derive

$$\int_R |S_2^H\tilde\mu - S_2^H(\mu_{|R})|^2 d\mu(x) \lesssim \|\tilde\mu - \mu|_R\| \lesssim t^{\gamma_0}\,\mu(R).$$

To deal with the second term on the right hand side of (3.8.42) we argue analogously, using Cauchy-Schwarz and the $L^4(\mu|_R)$ boundedness of $S^H_{*,\mu}$. Then we get

$$\int |S_2^H(\mu_{|R})(x)|^2 d\big|\mu|_R - \tilde\mu\big|(x) \lesssim t^{\gamma_0/2}\,\mu(R).$$

Finally we turn our attention to the first term. For $x\in Q\in \mathsf{Ch}_{\mathsf{Stop}}(R)$, we write $T^H_{2,x}$ for the corresponding frozen operator related to the kernel $\widehat K_{2,x}$. Taking into account that $M\ell(Q)\ll \Delta\ell(R)$, we write

$$\begin{aligned}\big|S_2^H(\mu_{|R})(x)\big| &= \big|\widehat T_2^H(\mu_{|R})(x)\big| \le \big|\widehat T^H_{2,x}(\mu_{|R})(x)\big| + \big|\widehat T_2^H(\mu_{|R})(x) - \widehat T^H_{2,x}(\mu_{|R})(x)\big|\\ &\le \big|\widehat T^H_{2,x}(\mu_{|R})(x)\big| + C\ell(R)^{\alpha/2}\\ &\le \big|\widehat T^H_{2,x}(c_Q\mathcal H^n|_{L_Q})(x)\big| + \Big|\int \widehat K^H_{2,x}(x-y)d(\mu - c_Q\mathcal H^n|_{L_Q})(y)\Big| + C\ell(R)^{\alpha/2}\\ &= S_{21}+S_{22}+C\ell(R)^{\alpha/2}.\end{aligned}$$

Notice that $\widehat T^H_{2,x}(c_Q\mathcal H^n|_{L_Q})(x')=0$ for $x'=\Pi_{L_Q}(x)$, $x\in Q$. Therefore, using the standard estimates in Lemma 4.2.1 we get

$$\begin{aligned}S_{21} &= \big|\widehat T_{2,x}(c_Q\mathcal H^n|_{L_Q}))(x) - \widehat T_{2,x}(c_Q\mathcal H^n|_{L_Q})(x')\big|\\ &\lesssim \int_{100MB_Q} |\widehat K_{2,x}(x-y) - \widehat K_{2,x}(x'-y)|\, d\mathcal H^n|_{L_Q}(y) \lesssim C(M,\tau)\tilde\varepsilon^{\alpha/2}.\end{aligned}$$

To estimate the term S_{22}, we will use the fact that $\alpha_\mu^{L_Q}(MB_Q)\le\varepsilon$, that is

$$S_{22}\le C(M)\,\mathrm{Lip}(\widehat K_2^H)\,\alpha_\mu^{L_Q}(MB_Q)\ell(Q)^{n+1}\lesssim C(M,\tau)\varepsilon.$$

Hence,

$$\int |S_2^H(\mu_{|R})(x)|^2 d\tilde\mu(x) \lesssim C(M,\tau)(\tilde\varepsilon^\alpha + \varepsilon^2 + \ell(R)^\alpha)\mu(R).$$

Gathering the estimates above we get

$$\|S_2^H\tilde\mu\|^2_{L^2(\mu_{|R})} \lesssim \Big(t^{\gamma_0/2} + C(M,\tau)(\tilde\varepsilon^\alpha+\varepsilon^2+\ell(R)^\alpha)\Big)\,\mu(R) \le \frac{\varepsilon_5}{10}\,\mu(R),$$

by choosing $t, \ell(R), \varepsilon, \tilde{\varepsilon}$ small enough. Together with (3.8.41), this implies that

$$\|S_2^H \sigma\|_{L^2(\sigma)}^2 \lesssim \tilde{\varepsilon}^\alpha M^\alpha C(t)\mu(R) + \frac{\varepsilon_5}{10}\,\mu(R)\mu(R) \leq \frac{\varepsilon_5}{5}\,\mu(R),$$

by appropriate choices of M, t and $\tilde{\varepsilon}$ again.

End of the proof of Lemma 3.8.6. Taking into account that

$$\Big\| \sum_{Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)} m_{\mu,Q}(S_2^H \tilde{\mu})\chi_Q \Big\|_{L^2(\mu|_R)}^2 \leq \|S_2^H \tilde{\mu}\|_{L^2(\mu|_R)}^2,$$

from the splitting (3.8.28) and the estimates obtained for $\|S_1^H \sigma\|_{L^2(\sigma)}$, $\|S_2^H \sigma\|_{L^2(\sigma)}$, $\|S_3^H \sigma\|_{L^2(\sigma)}$, and $\|S_2^H \tilde{\mu}\|_{L^2(\mu|_R)}^2$, we derive

$$\|S^H \sigma\|_{L^2(\sigma)}^2 \lesssim \|\tilde{f}^H\|_{L^2(\mu)}^2 + \sum_{Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)} |m_{\mu,Q}(S_1^H \tilde{\mu})|^2\, \mu(Q) + \frac{\varepsilon_5}{2}\,\mu(R). \tag{3.8.43}$$

Hence to conclude the proof of the lemma it just remains to estimate the second term on the right hand side above.

For a fixed cube $Q \in \mathsf{Ch}_{\mathsf{Stop}}(R)$, we write

$$\mu(Q)\,\big|m_{\mu,Q}(S_1^H \tilde{\mu})\big| \leq \left|\int_Q S_1^H(\chi_Q \mu)\, d\mu\right| + \left|\int_Q S_1^H(\tilde{\mu} - \chi_Q \mu)\, d\mu\right|. \tag{3.8.44}$$

To estimate the first term on the right hand side, recall that by (3.2.5), the difference between the kernel of $S_{1,Q}^H$ and its antisymmetric part satisfies

$$\left|K_{S_{1,Q}^H}(x,y) - K_{S_{1,Q}^{H,(a)}}(x,y)\right| \lesssim \frac{1}{|x-y|^{n-\alpha/2}},$$

and so

$$\left|\int_Q S_1^H(\chi_Q \mu)\, d\mu\right| \lesssim \int_Q \frac{1}{|x-y|^{n-\alpha/2}}\, d\mu \lesssim \ell(Q)^{n+\alpha/2} \lesssim \ell(R)^{\alpha/2}\, \mu(Q). \tag{3.8.45}$$

Concerning the second term on the right hand side of (3.8.44), observe that

$$\tilde{\mu} - \chi_Q \mu = \chi_{Q^c}\tilde{\mu} - \chi_{Q \setminus Q_{(t)}}\mu.$$

Then, using the fact that $\operatorname{supp} K_{S_{1,Q}^H}(x,\cdot) \subset B(x, M^{-1}\ell(Q))$ and Cauchy-Schwarz we deduce

$$\begin{aligned}\left|\int_Q S_1^H(\tilde{\mu} - \chi_Q \mu)\, d\mu\right| &= \left|\int_Q |S_1^H(\chi_{\mathcal{U}_{M^{-1}\ell(Q)}(Q)\setminus Q}\tilde{\mu} - \chi_{Q\setminus Q_{(t)}}\mu)\, d\tilde{\mu}\right| \\ &\leq \left\|S_1^H(\chi_{\mathcal{U}_{M^{-1}\ell(Q)}(Q)\setminus Q}\tilde{\mu} - \chi_{Q\setminus Q_{(t)}}\mu)\right\|_{L^2(\mu|_Q)}\, \mu(Q)^{1/2}.\end{aligned}$$

Notice that $S_{1,Q}^H(\cdot\mu)$ is bounded in $L^2(\mu|_R)$. Indeed, one can easily check that for all $g \in L^2(\mu|_R)$ and all $x \in \mathbb{R}^{n+1}$,

$$|S_{1,Q}^H(g\,\mu)(x)| \leq S_*^H(g\,\mu)(x),$$

where $S_*^H(\cdot\,\mu)$ is the maximal operator associated with $S^H(\cdot\,\mu)$ and then the claim follows from Cotlar's inequality. This fact, together with the thin boundary condition for Q yields

$$\big\|S_1^H(\chi_{\mathcal{U}_{M^{-1}\ell(Q)}(Q)\setminus Q}\tilde{\mu} - \chi_{Q\setminus Q_{(t)}}\mu)\big\|_{L^2(\mu|_Q)} \lesssim \mu\big(\mathcal{U}_{\tau\ell(Q)}(Q)\setminus Q\big)^{1/2} + \mu\big(Q\setminus Q_{(t)}\big)^{1/2}$$
$$\lesssim (\tau^{\gamma_0/2} + t^{\gamma_0/2})\,\mu(Q)^{1/2}.$$

Therefore,

$$\left|\int_Q S_1^H(\chi_{Q^c}\tilde{\mu})\,d\tilde{\mu}\right| \lesssim (\tau^{\gamma_0/2} + t^{\gamma_0/2})\,\mu(Q).$$

Together with (3.8.44) and (3.8.45), the last estimate yields

$$\big|m_{\mu,Q}(S_1^H\tilde{\mu})\big| \lesssim \ell(R)^{\alpha/2} + \tau^{\gamma_0/2} + t^{\gamma_0/2}.$$

Plugging this into (3.8.43), we get

$$\big\|S^H\sigma\big\|_{L^2(\sigma)}^2 \lesssim \|\tilde{f}^H\|_{L^2(\mu)}^2 + \big(\ell(R)^\alpha + \tau^{\gamma_0} + t^{\gamma_0}\big)\,\mu(R) + \frac{\varepsilon_5}{2}\,\mu(R),$$

which proves the lemma by choosing τ, t, and $\ell(R)$ small enough.

3.9 The continuous measure ν

We consider $R \in \mathsf{Nice}$ and σ as above. Because of technical reasons, it is convenient to replace σ by a continuous measure ν (i.e., a measure absolutely continuous with respect to Lebesgue measure). Let φ be a radial non-negative C^∞ function supported in $B(0,1)$ such that $\int \varphi\,d\mathcal{L}^{n+1} = 1$, and set

$$\nu = \sigma * \frac{1}{s^{n+1}}\,\varphi\left(\frac{\cdot}{s}\right), \tag{3.9.1}$$

where s is small enough and will be fixed below. For the moment, let us say that $s \ll \min_{Q\in\mathsf{Ch}_{\mathsf{Stop}}(R)} \ell(Q)$.

Recall that, by Lemma 3.7.2, σ has n-polynomial growth. It is immediate to check that the same holds for ν, that is

$$\nu(B(x,r)) \leq C\,r^n \quad \text{for all } x\in\mathbb{R}^{n+1},\ r>0. \tag{3.9.2}$$

The estimate in the following lemma is the analogue of (3.8.4) in Lemma 3.8.2 with ν replacing σ.

Lemma 3.9.1. *Assume $s > 0$ small enough in the definition of ν and $\ell(R) \leq 1$. We have*

$$\int |\widehat{T}\nu|^2\,d\nu \leq C\,\ell(R)^n.$$

We remark that the smallness requirement on s in this lemma may depend on the number of cubes in $\mathsf{Ch}_{\mathsf{Stop}}(R)$, thus the value of the threshold is merely qualitative.

Proof. By Fubini, Lemma 4.2.2 and the n-growth of σ,

$$\int |\widehat{T}\nu|^2\,d\nu = \int |\widehat{T}\nu|^2\,d(\varphi_s * \sigma) = \int (|\widehat{T}\nu|^2) * \varphi_s\,d\sigma$$
$$\leq \int \sup_{|x-y|\leq s} |\widehat{T}\nu(y)|^2\,d\sigma(x) \leq \int \sup_{|x-y|\leq s} |\widehat{T}_y\nu(y)|^2\,d\sigma(x) + C\,\ell(R)^{n+\alpha}. \tag{3.9.3}$$

For all $x \in \operatorname{supp} \sigma$ and y such that $|y-x| \leq s$, we write

$$|\widehat{T}_y \nu(y)| = |(\varphi_s * \widehat{T}_y \sigma)(y)| \leq |(\varphi_s * \widehat{T}_x \sigma)(y)| + |\varphi_s * (\widehat{T}_x \sigma - \widehat{T}_y \sigma)(y)|, \tag{3.9.4}$$

where $\widehat{T}_y$ stands for the frozen operator with kernel $\nabla_1 \Theta(\cdot, 0; A(y))$. To estimate the last term on the right hand side, observe that one can estimate the kernel of $\widehat{T}_x - \widehat{T}_y$ as in (3.8.26). Recall that σ is supported in a finite union of n-planes, and that it has a smooth density with respect to $\mathcal{H}^n$ on each n-plane. Then one easily gets

$$\big|\widehat{T}_x \sigma(y) - \widehat{T}_y \sigma(y)\big| \leq C(\sigma)\, |x-y|^{\alpha/2}, \tag{3.9.5}$$

with $C(\sigma)$ depending on the precise form of σ (like the number of cubes in $\mathsf{Ch}_{\mathsf{Stop}}(R)$, for example).

Concerning the first term on the right hand side of (3.9.4), we claim that, if $|x-y| \leq s$,

$$|(\varphi_s * \widehat{T}_x \sigma)(y)| \leq |\widehat{T}_{x,s} \sigma(x)| + C,$$

where $\widehat{T}_{x,s}$ stands for the s-truncated version of $\widehat{T}_x$. The arguments to show this are quite standard, but we show the details for the reader's convenience. We write

$$|(\varphi_s * \widehat{T}_x \sigma)(y)| \leq \big|\big(\varphi_s * \widehat{T}_x(\chi_{B(y,2s)}\sigma)\big)(y)\big| + \big|\big(\varphi_s * \widehat{T}_x(\chi_{B(y,2s)^c}\sigma)\big)(y)\big|.$$

We have

$$\begin{aligned}
|\varphi_s * \widehat{T}_x(\chi_{B(y,2s)}\sigma)(y)| &\lesssim \int \varphi_s(y-w) \int_{B(y,2s)} \frac{1}{|w-z|^n}\, d\sigma(z)\, d\mathcal{L}^{n+1}(w) \\
&\lesssim \frac{1}{s^{n+1}} \int_{B(y,2s)} \int_{|w-z|\leq 3s} \frac{1}{|w-z|^n}\, d\mathcal{L}^{n+1}(w)\, d\sigma(z) \\
&\lesssim \frac{1}{s^{n+1}} \int_{B(y,2s)} s\, d\sigma(z) \lesssim 1.
\end{aligned}$$

Also, by standard estimates,

$$\begin{aligned}
|\varphi_s * \widehat{T}_x(\chi_{B(y,2s)^c}\sigma)(y)| &\leq \sup_{|y-z|\leq s} |\widehat{T}_x(\chi_{B(y,2s)^c}\sigma)(z)| \\
&\leq |\widehat{T}_{x,s}\sigma(x)| + C \sup_{r>s} \frac{\sigma(B(x,r))}{r^n} \leq |\widehat{T}_{x,s}\sigma(x)| + C,
\end{aligned}$$

which concludes the proof of our claim.

By (3.9.4), (3.9.5), and the claim above, we deduce

$$|\widehat{T}_y \nu(y)| \leq C(\sigma)\, |x-y|^{\alpha/2} + |\widehat{T}_{x,s}\sigma(x)| + C \leq C(\sigma)\, s^{\alpha/2} + |\widehat{T}_{x,s}\sigma(x)| + C,$$

since $|x-y| \leq s$. Plugging this estimate into (3.9.3), we get

$$\int |\widehat{T}\nu|^2\, d\nu \lesssim \int |\widehat{T}_{x,s}\sigma(x)|^2\, d\sigma(x) + C(\sigma)\, s^\alpha\, \ell(R)^n + C\ell(R)^n + C\, \ell(R)^{n+\alpha}.$$

Taking into account that $\ell(R) \leq 1$ and using the connection between the kernels of $\widehat{T}_x$ and $\widehat{T}$ stated in Lemma 4.2.2, we derive

$$\int |\widehat{T}\nu|^2\, d\nu \lesssim \int |\widehat{T}_s \sigma|^2\, d\sigma + C(\sigma)\, s^\alpha\, \ell(R)^n + C\ell(R)^n.$$

Since $\widehat{T}$ is bounded in $L^2(\sigma)$ (with a qualitative bound on the norm, at least), by standard Calderón-Zygmund theory we deduce that

$$\int |\widehat{T}_s\sigma|^2\,d\sigma \to \int |\widehat{T}\sigma|^2\,d\sigma \qquad \text{as } s\to 0.$$

Thus, using also (3.8.4),

$$\int |\widehat{T}\nu|^2\,d\nu \lesssim \int |\widehat{T}\sigma|^2\,d\sigma + \ell(R)^n \lesssim \ell(R)^n.$$

which proves the lemma. □

Our next objective is to show that $\int |S^H\nu|^2\,d\nu$ is very small if $\int |S^H\sigma|^2\,d\sigma$ is also small. That is, we have to transfer the estimate in Lemma 3.8.3 to the measure ν. The fact that we are considering just the horizontal component H will be essential in this case. We need the following auxiliary result, proven in [NTV14a, Lemma 1].

Lemma 3.9.2. *Suppose that f is a C^2-smooth compactly supported function on an n-plane L parallel to H. Then the function $\mathcal{R}^H(f\,\mathcal{H}^n|_L)$ is a Lipschitz function in $\mathbb{R}^{n+1}$, harmonic outside* $\operatorname{supp}(f\,\mathcal{H}^n|_L)$*, and it satisfies*

$$\sup|\mathcal{R}^H(f\,\mathcal{H}^n|_L)| \le C D^2 \sup_L |\nabla_H^2 f|$$

and

$$\|\mathcal{R}^H(f\,\mathcal{H}^n|_L)\|_{\mathrm{Lip}} \le C D \sup_L |\nabla_H^2 f|,$$

where D is the diameter of $\operatorname{supp}(f\,\mathcal{H}^n|_L)$ *and ∇_H is the partial gradient involving only the derivatives in the directions parallel to H.*

Note that the second differential $\nabla_H^2 f$ and the corresponding supremum on the right hand side are considered on L only (the function f in the lemma does not even need to be defined outside L) while the H-restricted Riesz transform $\mathcal{R}^H(f\,\mathcal{H}^n|_L)$ on the left hand side is viewed as a function on the entire space $\mathbb{R}^{n+1}$ and its supremum and the Lipschitz norm are also taken in $\mathbb{R}^{n+1}$.

Remark 7. Below, we will apply Lemma 3.9.2 to the operator $\widehat{T}_x$, by means of the change of variable $\phi(y) = \widehat{A}(x)^{1/2}\,y$. Note that then the matrix A_ϕ in Corollary 3.5.1 coincides with the identity, and thus the operator T_ϕ in (3.5.1) equals the Riesz transform, modulo a universal factor. Hence, by Lemma 3.5.3, denoting $D_x = \widehat{A}(x)^{1/2}$, for any measure η we have

$$c_n\,\mathcal{R}\eta(y) = D_x\,\widehat{T}_x((D_x)_\sharp\eta)(D_x y), \tag{3.9.6}$$

for all x, y.

Lemma 3.9.3. *Assume $s > 0$ small enough in the definition of ν and let $\varepsilon' > 0$. If $\|T_R\mu\|^2_{L^2(\mu)} \le \varepsilon_1\,\mu(R)$, then*

$$\int |S^H\nu|^2\,d\nu \lesssim \varepsilon'\,\ell(R)^n,$$

assuming that ε, ε_1, $\ell(R)$, t, and Δ are small enough and M is big enough (as in Lemma 3.8.3).

Proof. Recall that

$$S^H\nu(x) = \widehat{T}^H\nu(x) - \widehat{T}^H\nu(x^*).$$

Consider the matrix $D_x = \widehat{A}(x)^{1/2}$ and the n-plane $H_x = D_x^{-1}(H)$. Then we write

$$\begin{aligned}\int |S^H\nu|^2\,d\nu \lesssim \int |\Pi_{H_x}\, D_x\,\widehat{T}_x\nu(x) - \widehat{T}^H\nu(x)|^2\,d\nu(x)\\ + \int |\Pi_{H_x}\, D_x\,\widehat{T}_x\nu(x) - \widehat{T}^H\nu(x^*)|^2\,d\nu(x).\end{aligned} \tag{3.9.7}$$

To estimate the first integral on the right hand side we claim that

$$|\Pi_{H_x}\, D_x\,\widehat{T}_x\nu(x) - \Pi_H\widehat{T}\nu(x)| \lesssim \ell(R)^{\alpha/2}\,(1+|\widehat{T}\nu(x)|) \quad \text{for all } x\in \operatorname{supp}\nu, \tag{3.9.8}$$

and also that the same estimate holds replacing ν by σ. That is,

$$|\Pi_{H_x}\, D_x\,\widehat{T}_x\sigma(x) - \Pi_H\widehat{T}\sigma(x)| \lesssim \ell(R)^{\alpha/2}\,(1+|\widehat{T}\sigma(x)|) \quad \text{for all } x\in \operatorname{supp}\sigma. \tag{3.9.9}$$

To prove (3.9.8), we fix $x\in\operatorname{supp}\nu$ and we set

$$|\Pi_{H_x}\, D_x\,\widehat{T}_x\nu(x) - \Pi_H\widehat{T}\nu(x)| \leq |\Pi_{H_x}\, D_x\,(\widehat{T}_x\nu(x) - \widehat{T}\nu(x))| + |(\Pi_{H_x}\, D_x - \Pi_H)\,\widehat{T}\nu(x)|. \tag{3.9.10}$$

Now we estimate the first summand on the right hand side:

$$|\Pi_{H_x}\, D_x\,(\widehat{T}_x\nu(x) - \widehat{T}\nu(x))| \lesssim |\widehat{T}_x\nu(x) - \widehat{T}\nu(x)| \lesssim \int \frac{1}{|x-y|^{n-\alpha/2}}\,d\nu(y) \lesssim \ell(R)^{\alpha/2}, \tag{3.9.11}$$

using (3.9.2) in the last inequality.

Concerning the last summand on the right hand side of (3.9.10), we have

$$|(\Pi_{H_x}\, D_x - \Pi_H)\,\widehat{T}\nu(x)| \leq \big(\|\Pi_{H_x}D_x - \Pi_{H_x}\| + \|\Pi_{H_x} - \Pi_H\|\big)\,|\widehat{T}\nu(x)|.$$

By the Hölder continuity of $\widehat{A}$, we have

$$\|\Pi_{H_x}D_x - \Pi_{H_x}\| \leq \|D_x - Id\| \lesssim |x-x_R|^{\alpha/2} \leq C\,\ell(R)^{\alpha/2}.$$

Also, taking into account that $H_x = D_x^{-1}(H)$, we get

$$\|\Pi_{H_x} - \Pi_H\| \lesssim \|D_x - Id\| \lesssim \ell(R)^{\alpha/2}.$$

Thus,

$$|(\Pi_{H_x}\, D_x - \Pi_H)\,\widehat{T}\nu(x))| \lesssim \ell(R)^{\alpha/2}|\widehat{T}\nu(x)|,$$

which together with (3.9.11) concludes the proof of (3.9.8). The arguments for (3.9.9) are analogous and are left for the reader.

From the claim (3.9.8) and applying Lemma 3.9.1, we derive

$$\int |\Pi_{H_x}\, D_x\,\widehat{T}_x\nu(x) - \widehat{T}^H\nu(x)|^2\,d\nu(x) \lesssim \ell(R)^\alpha\left(\ell(R)^n + \int |\widehat{T}\nu|^2\,d\nu\right) \lesssim \ell(R)^{n+\alpha}.$$

To deal with the second integral on the right hand side of (3.9.7), we write

$$\begin{aligned}\int |\Pi_{H_x} D_x \widehat{T}_x\nu(x) - \widehat{T}^H\nu(x^*)|^2\, d\nu(x)&\\ \lesssim \int |\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \widehat{T}^H\sigma(x^*)|^2\, d\sigma(x)&\\ + \left|\int |\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \widehat{T}^H\sigma(x^*)|^2\, d(\sigma-\nu)(x)\right|&\\ + \int |\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \Pi_{H_x} D_x \widehat{T}_x\nu(x)|^2\, d\nu(x)&\\ + \int |\widehat{T}^H\sigma(x^*) - \widehat{T}^H\nu(x^*)|^2\, d\nu(x)&\\ =: I_1 + I_2 + I_3 + I_4.&\end{aligned}$$

To deal with the term I_1 we apply (3.9.9) and Lemmas 3.8.3 and 3.8.2, and then we get

$$\begin{aligned}I_1 &\lesssim \int |S^H\sigma|^2\, d\sigma + \int |\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \widehat{T}^H\sigma(x)|^2\, d\sigma(x)\\ &\lesssim \int |S^H\sigma|^2\, d\sigma + \ell(R)^\alpha \left(\ell(R)^n + \int |\widehat{T}\sigma|^2\, d\sigma\right)\\ &\lesssim (\varepsilon_2 + \ell(R)^\alpha)\, \ell(R)^n.\end{aligned}$$

Next we consider the integral I_3. To this end, observe that for any given x, since $\widehat{T}_x$ is a convolution operator,

$$\Pi_{H_x} D_x \widehat{T}_x\nu(x) = \Pi_{H_x} D_x \widehat{T}_x(\varphi_s * \sigma)(x) = \varphi_s * \big(\Pi_{H_x} D_x \widehat{T}_x\sigma\big)(x).$$

Therefore,

$$\begin{aligned}|\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \Pi_{H_x} D_x \widehat{T}_x\nu(x)| &= \big|\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \varphi_s * \big(\Pi_{H_x} D_x \widehat{T}_x\sigma\big)(x)\big|\\ &\leq \sup_{|y-x|\leq s} |\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \Pi_{H_x} D_x \widehat{T}_x\sigma(y)|.\end{aligned} \tag{3.9.12}$$

Recall now that, by (3.9.6),

$$D_x \widehat{T}_x\sigma(x) = c_n\mathcal{R}(D_{x^{-1}\sharp}\sigma)(D_x^{-1}x). \tag{3.9.13}$$

Since σ is supported on a finite union of planes parallel to H, it follows that the measure $D_{x^{-1}\sharp}\sigma$ is supported on a finite union of planes which are parallel to $H_x = D_x^{-1}H$. Then, by Lemma 3.9.2 (applied with H_x instead of H), it turns out that $\Pi_{H_x} D_x \widehat{T}_x\sigma(\cdot)$ is a Lipschitz function (with the Lipschitz norm depending on the precise construction of σ, and in particular on the number of cubes in $\mathsf{Ch}_{\mathsf{Stop}}(R)$). Hence, the right hand side of (3.9.12) tends to 0 uniformly on x as $s \to 0$, so

$$|\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \Pi_{H_x} D_x \widehat{T}_x\nu(x)| \to 0 \quad \text{as } s \to 0,$$

uniformly on x too. This implies that

$$I_3 = I_3(s) = \int |\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \Pi_{H_x} D_x \widehat{T}_x\nu(x)|^2\, d\nu(x) \to 0 \quad \text{as } s \to 0.$$

To estimate I_4, note that

$$\widehat{T}^H\nu(x^*) = \int \widehat{K}^H(x^*,y)\,d\nu(y) = \int \big(\widehat{K}^H(x^*,\cdot) * \varphi_s\big)(y)\,d\sigma(y).$$

By the Hölder continuity of $\widehat{K}^H(x^*,\cdot)$ with $x \in \operatorname{supp}\sigma$, it follows easily that $\widehat{T}^H\nu(x^*) \to \widehat{T}^H\sigma(x^*)$ as $s \to 0$ uniformly for $x \in \operatorname{supp}\sigma$, taking into account also that for $x \in \operatorname{supp}\sigma \cup \operatorname{supp}\nu$,

$$\operatorname{dist}(x^*, \operatorname{supp}\sigma \cup \operatorname{supp}\nu) \gtrsim \Delta\,\ell(R) \gg s,$$

for s small enough. Then we deduce that

$$I_4 = I_4(s) \to 0 \qquad \text{as } s \to 0.$$

Finally we turn our attention to the term I_2. Observe that

$$\begin{aligned} I_2 &= \left| \int |\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \widehat{T}^H\sigma(x^*)|^2\,d(\sigma - \varphi_s * \sigma)(x) \right| \\ &\le \int \Big| |\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \widehat{T}^H\sigma(x^*)|^2 - \varphi_s * \big(|\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \widehat{T}^H\sigma(x^*)|^2\big) \Big|\,d\sigma(x) \\ &\lesssim \ell(R)^n \sup_{\substack{x\in\operatorname{supp}\sigma\\ |y-x|\le s}} \Big| |\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \widehat{T}^H\sigma(x^*)|^2 - |\Pi_{H_y} D_y \widehat{T}_y\sigma(y) - \widehat{T}^H\sigma(y^*)|^2 \Big|. \end{aligned}$$

We claim now that $\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \widehat{T}^H\sigma(x^*)$ is a Hölder continuous function of x, for x in a small neighborhood of $\operatorname{supp}\sigma$. Clearly, this implies that

$$\sup_{\substack{x\in\operatorname{supp}\sigma\\ |y-x|\le s}} \Big| |\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \widehat{T}^H\sigma(x^*)|^2 - |\Pi_{H_y} D_y \widehat{T}_y\sigma(y) - \widehat{T}^H\sigma(y^*)|^2 \Big| \to 0 \qquad \text{as } s \to 0,$$

and thus

$$I_2 = I_2(s) \to 0 \qquad \text{as } s \to 0.$$

By the same arguments used to estimate I_4, it is easy to check that $\widehat{T}^H\sigma(x^*)$ is a Hölder continuous function of x, for x in a small neighborhood of $\operatorname{supp}\sigma$. Thus, to prove our claim it suffices to show that $\Pi_{H_x} D_x \widehat{T}_x\sigma(x)$ is a Hölder continuous function of x in that neighborhood. To this end, for x, y in a small neighborhood of $\operatorname{supp}\sigma$ we write

$$\begin{aligned} \big|\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \Pi_{H_y} D_y \widehat{T}_y\sigma(y)\big| &\le \big|\Pi_{H_x} D_x \widehat{T}_x\sigma(x) - \Pi_{H_x} D_x \widehat{T}_x\sigma(y)\big| \\ &\quad + \big|\Pi_{H_x} D_x\,(\widehat{T}_x\sigma(y) - \widehat{T}_y\sigma(y))\big| \\ &\quad + \big|(\Pi_{H_x} D_x - \Pi_{H_y} D_y)\,\widehat{T}_y\sigma(y))\big| =: J_1 + J_2 + J_3. \end{aligned}$$

By (3.9.13) and Lemma 3.9.2 (applied with H_x replacing H) we have

$$\begin{aligned} J_1 &= c_n \big|\Pi_{H_x} \mathcal{R}(D_{x^{-1}\sharp}\sigma)(D_x^{-1}x) - \Pi_{H_x} \mathcal{R}(D_{x^{-1}\sharp}\sigma)(D_x^{-1}y)\big| \\ &\le C(\sigma)\,|D_x^{-1}x - D_x^{-1}y| \le C(\sigma)|x-y|. \end{aligned}$$

Regarding J_2, we have

$$J_2 \lesssim \big|\widehat{T}_x\sigma(y) - \widehat{T}_y\sigma(y)\big|.$$

Recall that $\widehat{T}_x - \widehat{T}_y$ is an odd convolution operator whose kernel $K = \widehat{K}_x - \widehat{K}_y$ is given as in (3.8.3), and it satisfies

$$|K(z)| \leq C\,|x-y|^{\alpha/2}\,\frac{1}{|z|^n} \quad \text{and} \quad |\nabla K(z)| \leq C\,|x-y|^{\alpha/2}\,\frac{1}{|z|^{n+1}}. \tag{3.9.14}$$

From this fact and the smoothness of the density of σ with respect to $\mathcal{H}^n$ on a finite union of n-planes, one easily gets

$$\big|\widehat{T}_x\sigma(y) - \widehat{T}_y\sigma(y)\big| \leq C(\sigma)|x-y|^{\alpha/2}.$$

Next we turn to J_3:

$$\begin{aligned} J_3 &\leq \|\Pi_{H_x} D_x - \Pi_{H_y} D_y\|\,|\widehat{T}_y\sigma(y)| \\ &\leq \big(\|(\Pi_{H_x} - \Pi_{H_y})\,D_x\| + \|\Pi_{H_y}\,(D_x - D_y)\|\big)\,|\widehat{T}_y\sigma(y)| \\ &\lesssim \big(\|\Pi_{H_x} - \Pi_{H_y}\| + \|D_x - D_y\|\big)\,|\widehat{T}_y\sigma(y)|. \end{aligned}$$

Recall that $D_x = \widehat{A}(x)^{1/2}$ and $H_x = D_x^{-1}(H)$. Then, by the Hölder continuity of $\widehat{A}$, we derive

$$\|\Pi_{H_x} - \Pi_{H_y}\| + \|D_x - D_y\| \lesssim_\sigma |x-y|^{\alpha/2}.$$

Taking into account that $|\widehat{T}_y\sigma(y)| \leq C(\sigma)$, we deduce that

$$J_3 \leq C(\sigma)\,|x-y|^{\alpha/2}.$$

Thus $\Pi_{H_x} D_x \widehat{T}_x\sigma(x)$ is a Hölder continuous function of x with exponent $\alpha/2$, as claimed.

The lemma follows from the estimates obtained for I_1, I_2, I_3, and I_4. □

3.10 The function h and the vector field Ψ

For each cube Q from the intermediate non-BAUP layer $\mathcal{NB}(R)$ with non-BAUPness parameter $\delta > 0$, we define a function h_Q as follows. First we consider a radial C^∞ function h_0 supported in $B(0,1)$ such that $h_0 = 1$ on $B(0,1/2)$ and $0 \leq h_0 \leq 1$. Then we set

$$h_Q(x) = h_0\left(\frac{x - z_Q^a}{\delta\,\ell(Q)}\right) - h_0\left(\frac{x - z_Q^b}{\delta\,\ell(Q)}\right),$$

where z_Q^a and z_Q^b are the points introduced in Definition 3.4.1 and such that the vector $z_Q^a - z_Q^b$ is parallel to H. This can be achieved by taking the n-plane L in Definition 3.4.1 parallel to H. Note that $\operatorname{supp} h_Q \subset 3B_Q$, and the support of the negative part of h_Q does not intersect $\operatorname{supp}\mu$. On the other hand, the support of the positive part of h_Q includes a sufficiently big portion of the measure, so that $\int h_Q\,d\mu \gtrsim c(\delta)\,\mu(Q)$.

Next, by a Vitali type covering lemma, we extract a subfamily $\mathcal{NB}'(R) \subset \mathcal{NB}(R)$ such that the balls $4B_Q$, $Q \in \mathcal{NB}'(R)$, are pairwise disjoint and so that

$$\sum_{Q\in\mathcal{NB}'(R)} \mu(Q) \geq c\,\mu(R),$$

where c depends at most on the AD-regularity constant of μ. Then we define

$$h = \sum_{Q\in\mathcal{NB}'(R)} h_Q.$$

Lemma 3.10.1. *Assume ε and the parameter s in the definition of ν in (3.9.1) small enough. Then the function h satisfies:* $\operatorname{supp} h \subset 3B_R$, $\operatorname{dist}(\operatorname{supp} h, H) \geq \Delta\,\ell(R)/2$, $h \geq 0$ *on* $\operatorname{supp}\nu$ *and*

$$\int h\,d\nu \geq c_7(\delta)\;\nu(\mathbb{R}^{n+1}),$$

with $c_7(\delta) > 0$.

The proof of this lemma is elementary and follows from the construction of h.

Our next objective consists in constructing a vector field Ψ satisfying the properties stated in the next lemma.

Lemma 3.10.2. *There exists a compactly supported Lipschitz vector field* $\Psi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *which satisfies the following:*

(i) $\Psi = \sum_{Q\in\mathcal{NB}'(R)} \Psi_Q$, $\operatorname{supp}\Psi \subset 3B_R \cap \mathbb{R}^{n+1}_+$, *and* $\operatorname{dist}(\operatorname{supp}\Psi, H) \geq \frac{\Delta}{2}\,\ell(R)$.

(ii) For each $Q \in \mathcal{NB}'(R)$, $\operatorname{supp}\Psi_Q \subset 3B_Q$ *and*

$$\int \Psi_Q\,d\mathcal{L}^{n+1} = 0, \quad \|\Psi_Q\|_\infty \lesssim \frac{1}{\delta\,\ell(Q)}, \quad \textit{and} \quad \|\Psi_Q\|_{\mathrm{Lip}} \lesssim \frac{1}{\delta^2\ell(Q)^2}.$$

(iii) $\displaystyle\int |\Psi|\,d\mathcal{L}^{n+1} \lesssim \delta^{-1}\,\ell(R)^n.$

(iv) For each $Q \in \mathcal{NB}'(R)$,

$$\widehat{T}^{H,*}(\Psi_Q\,\mathcal{L}^{n+1}) = h_Q + e_Q,$$

with the "error term" e_Q *satisfying*

$$|e_Q(x)| \lesssim \frac{C(\delta)\,\ell(R)^{\tilde\gamma}\,\ell(Q)^{n+\tilde\beta}}{(|x-x_Q| + \ell(Q))^{n+\tilde\beta}} \qquad \textit{for all } x \in 10B_R,$$

where $\tilde\beta$ *and* $\tilde\gamma$ *are some fixed positive constants depending on* n *and* α.

(v) $\|S^H(|\Psi|\mathcal{L}^{n+1})\|_{L^2(\nu)} \leq C(\delta)\,\mu(R)^{1/2}$, *assuming the parameter* s *in the definition of* ν *small enough.*

We remark that in the statement (iv) above, $\widehat{T}^{H,*}(\Psi_Q\,\mathcal{L}^{n+1})$ stands for the adjoint of $\widehat{T}^H$ applied to the vectorial measure $\Psi_Q\,\mathcal{L}^{n+1}$. That is,

$$\widehat{T}^{H,*}(\Psi_Q\,\mathcal{L}^{n+1})(x) = \int \widehat{K}^H(y,x)\cdot\Psi_Q(y)\,d\mathcal{L}^{n+1}(y),$$

where '$\cdot$' is the scalar product. Sometimes, abusing notation, we will write $\widehat{T}^{H,*}\Psi_Q$ instead of $\widehat{T}^{H,*}(\Psi_Q\,\mathcal{L}^{n+1})$. We will use analogous notations for other operators.

Proof. To construct each function Ψ_Q for $Q \in \mathcal{NB}'(R)$ we argue as in [NTV14a, Section 24]. Let v_Q be the unit vector in the direction $z_Q^a - z_Q^b$. Consider the function

$$g_Q(x) = \int_{-\infty}^0 h_Q(x + tv_Q)\,dt,$$

so that $\nabla_{v_Q} g_Q = h_Q$. Since the restriction of h_Q to any line parallel to v_Q consists of two opposite bumps, the support of h_Q is contained in the convex hull of $B(z_Q^a, \delta\ell(Q))$ and $B(z_Q^b, \delta\ell(Q))$. Also, since $\|\nabla^j h_Q\|_{L^\infty} \le C(j)[\delta\ell(Q)]^{-j}$ and since $\operatorname{supp} h_Q$ intersects any line parallel to v_Q over two intervals of total length $4\delta\ell(Q)$ or less, we have

$$|\nabla^j g_Q(x)| \le \int_{-\infty}^0 |(\nabla^j h_Q)(x + tv_Q)|\, dt \le \frac{C(j)}{[\delta\ell(Q)]^{j-1}} \tag{3.10.1}$$

for all $j \ge 0$.

We define the vector fields

$$\Psi_Q = -\Delta g_Q\, v_Q, \qquad \Psi = \sum_{Q\in\mathcal{NB}'(R)} \Psi_Q,$$

so that the properties (i) and (ii) in the lemma hold, because of (3.10.1). Indeed, the mean zero property holds because the integral of the Laplacian of a compactly supported C^∞ function over the entire space is 0 and the support property holds because the balls $B(x_Q, 3\ell(Q))$ lie deep inside $3B_R$. The property (iii) is also immediate:

$$\int |\Psi|\, d\mathcal{L}^{n+1} = \sum_{Q\in\mathcal{NB}'(R)} \int |\Psi_Q|\, d\mathcal{L}^{n+1} \lesssim \sum_{Q\in\mathcal{NB}'(R)} [\delta\ell(Q)]^{-1}\, \mathcal{L}^{n+1}(B(x_Q, 3\ell(Q))) \tag{3.10.2}$$

$$\lesssim \delta^{-1} \sum_{Q\in\mathcal{NB}'(R)} \ell(Q)^n \lesssim \delta^{-1} \sum_{Q\in\mathcal{NB}'(R)} \mu(Q) \lesssim \delta^{-1}\mu(R)\,. \tag{3.10.3}$$

Next we turn our attention to the statement (iv). Since $\widehat{A}(x_R) = Id$, the kernel of $\widehat{T}_{x_R}$ is the gradient of the fundamental solution of the Laplacian (i.e., the Riesz kernel times an absolute constant). Thus, $\widehat{T}_{x_R}(\Delta g_Q) = \nabla g_Q$ and so $\widehat{T}^H_{x_R}(\Delta g_Q) = \nabla_H g_Q$. Therefore, since $v_Q \in H$,

$$\widehat{T}^{H,*}_{x_R}\Psi_Q = \widehat{T}^{H,*}_{x_R}(-\Delta g_Q\, v_Q) = \widehat{T}^H_{x_R}(\Delta g_Q)\cdot v_Q = \widehat{T}_{x_R}(\Delta g_Q)\cdot v_Q = \nabla_{v_Q} g_Q = h_Q.$$

Hence,

$$\widehat{T}^{H,*}\Psi_Q = h_Q + \big(\widehat{T}^{H,*}\Psi_Q - \widehat{T}^{H,*}_{x_R}\Psi_Q\big) =: h_Q + e_Q.$$

We estimate e_Q as follows:

$$|e_Q(x)| \le \big|\widehat{T}^{H,*}\Psi_Q(x) - \widehat{T}^{H,*}_x\Psi_Q(x)\big| + \big|\widehat{T}^{H,*}_x\Psi_Q(x) - \widehat{T}^{H,*}_{x_R}\Psi_Q(x)\big|. \tag{3.10.4}$$

For the first summand on the right hand side we write

$$\begin{aligned}
\big|\widehat{T}^{H,*}\Psi_Q(x) - \widehat{T}^{H,*}_x\Psi_Q(x)\big| &\le \int |\widehat{K}^H(y,x) - \widehat{K}^H_x(y,x)|\, |\Psi_Q(y)|\, d\mathcal{L}^{n+1}(y)\\
&\lesssim \int \frac{1}{|x-y|^{n-\alpha/2}}\, |\Psi_Q(y)|\, d\mathcal{L}^{n+1}(y)\\
&\lesssim \frac{1}{\delta\,\ell(Q)} \int_{B(x_Q,3\ell(Q))} \frac{1}{|x-y|^{n-\alpha/2}}\, d\mathcal{L}^{n+1}(y)\\
&\lesssim \frac{1}{\delta\,\ell(Q)}\, \ell(Q)^{1+\alpha/2} \lesssim \delta^{-1}\ell(R)^{\alpha/2}.
\end{aligned}$$

Concerning the last summand in (3.10.4), we write

$$\big|\widehat{T}_x^{H,*}\Psi_Q(x) - \widehat{T}_{x_R}^{H,*}\Psi_Q(x)\big| \leq \int |\widehat{K}_x^H(y-x) - \widehat{K}_{x_R}^H(y-x)|\,|\Psi_Q(y)|\,d\mathcal{L}^{n+1}(y).$$

As in (3.9.14) we have

$$\begin{aligned}|\widehat{K}_x^H(y-x) - \widehat{K}_{x_R}^H(y-x)| \leq |\widehat{K}_x(y-x) - \widehat{K}_{x_R}(y-x)| &\lesssim \frac{|x-x_R|^{\alpha/2}}{|x-y|^n}\\ &\lesssim \frac{\ell(R)^{\alpha/2}}{|x-y|^n}\end{aligned}$$

for all $x \in 10B_R$. Hence, for such points x,

$$\begin{aligned}\big|\widehat{T}_x^{H,*}\Psi_Q(x) - \widehat{T}_{x_R}^{H,*}\Psi_Q(x)\big| &\lesssim \ell(R)^{\alpha/2}\int \frac{1}{|x-y|^n}\,|\Psi_Q(y)|\,d\mathcal{L}^{n+1}(y)\\ &\lesssim \frac{\ell(R)^{\alpha/2}}{\delta\,\ell(Q)}\int_{B(x_Q,3\ell(Q))}\frac{1}{|x-y|^n}\,d\mathcal{L}^{n+1}(y) \lesssim \delta^{-1}\ell(R)^{\alpha/2}.\end{aligned}$$

Therefore,

$$|e_Q(x)| \lesssim \delta^{-1}\ell(R)^{\alpha/2} \quad \text{for all } x \in 10B_R. \tag{3.10.5}$$

On the other hand, we also have

$$|e_Q(x)| \leq \big|\widehat{T}^{H,*}\Psi_Q(x)\big| + \big|\widehat{T}_{x_R}^{H,*}\Psi_Q(x)\big|.$$

For $x \in 6B_Q$, we have

$$\big|\widehat{T}^{H,*}\Psi_Q(x)\big| \lesssim \frac{1}{\delta\,\ell(Q)}\int_{B(x_Q,3\ell(Q))}\frac{1}{|x-y|^n}\,d\mathcal{L}^{n+1}(y) \lesssim \delta^{-1}.$$

Using that Ψ_Q has zero mean and standard estimates, for $x \in (6B_Q)^c$ we get

$$\begin{aligned}\big|\widehat{T}^{H,*}\Psi_Q(x)\big| &\leq \int |\widehat{K}^H(y-x) - \widehat{K}^H(x_Q-x)|\,|\Psi_Q(y)|\,d\mathcal{L}^{n+1}(y)\\ &\lesssim \frac{1}{\delta\,\ell(Q)}\int_{B(x_Q,3\ell(Q))}\frac{\ell(Q)^{\alpha/2}}{|x-x_Q|^{n+\alpha/2}}\,d\mathcal{L}^{n+1}(y)\\ &\lesssim \frac{\ell(Q)^{n+\alpha/2}}{\delta\,|x-x_Q|^{n+\alpha/2}}.\end{aligned}$$

So we infer that for all $x \in \mathbb{R}^{n+1}$,

$$\big|\widehat{T}^{H,*}\Psi_Q(x)\big| \lesssim \frac{\delta^{-1}\ell(Q)^{n+\alpha/2}}{(\ell(Q)+|x-x_Q|)^{n+\alpha/2}}.$$

The same estimate holds for $\big|\widehat{T}_x^{H,*}\Psi_Q(x)\big|$, and thus

$$|e_Q(x)| \lesssim \frac{\delta^{-1}\ell(Q)^{n+\alpha/2}}{(\ell(Q)+|x-x_Q|)^{n+\alpha/2}} \quad \text{for all } x \in \mathbb{R}^{n+1}. \tag{3.10.6}$$

Denote $\overline{\gamma} = \alpha/(2(2n+\alpha))$. Notice that $\overline{\gamma}\alpha/2 = \alpha^2/(4(2n+\alpha)) < 1/4$ and $(1-\overline{\gamma})(n+\gamma) = n+\alpha/4$. So, by taking a suitable weighted geometric mean of

(3.10.5) and (3.10.6), we obtain

$$|e_Q(x)| = |e_Q(x)|^{\overline{\gamma}}|e_Q(x)|^{1-\overline{\gamma}} \lesssim \frac{\delta^{-1}\ell(R)^{\alpha^2/(4(2n+\alpha))}\ell(Q)^{n+\alpha/4}}{\big(|x-x_Q|+\ell(Q)\big)^{n+\alpha/4}}$$

for all $x \in 10B_R$, which completes the proof of (iv) by choosing $\tilde{\gamma} = \alpha^2/(4(2n+\alpha))$ and $\tilde{\beta} = \alpha/4$.

Finally we turn our attention to the estimate (v). First we will show that

$$\|S^H(|\Psi|\mathcal{L}^{n+1})\|^2_{L^2(\mu|_R)} \leq C(\delta)\,\mu(R). \tag{3.10.7}$$

We consider the auxiliary measure

$$\xi = \sum_{Q\in\mathcal{NB}'(R)} \frac{1}{\ell(Q)}\,\mathcal{L}^{n+1}|_{3B_Q}.$$

We claim that ξ has n-polynomial growth. That is,

$$\xi(B(x,r)) \lesssim r^n \qquad \text{for all } x\in\mathbb{R}^{n+1},\ r>0. \tag{3.10.8}$$

The arguments to prove this are standard, but we show the details for the reader's convenience. It suffices to prove the preceding inequality for $x \in \operatorname{supp}\xi \subset \bigcup_{Q\in\mathcal{NB}'(R)} 3B_Q$. So fix a point $x\in 3B_Q$, for some $Q\in\mathcal{NB}'(R)$. Since the balls $4B_P$, $P\in\mathcal{NB}'(R)$, are pairwise disjoint, it is clear that the condition (3.10.8) holds for $r<\ell(Q)$. In the case $r\geq \ell(Q)$, let $I(x,r)$ denote the family of cubes $P\in\mathcal{NB}'(R)$ such that $3B_P\cap B(x,r)\neq\varnothing$. Taking into account again that the balls $4B_S$, $S\in\mathcal{NB}'(R)$, are pairwise disjoint, it follows that, for any $P\in I(x,r)$, $r\geq\ell(P)$ and then $B_P\subset B(x,7r)$. Therefore,

$$\xi(B(x,r)) \leq \sum_{P\in I(x,r)} \xi(3B_P) \approx \sum_{P\in I(x,r)} \ell(P)^n \leq \sum_{P\in I(x,r)} \mu(P) \leq \mu(B(x,7r)) \lesssim r^n.$$

Recall now that $\mu|_R$ is n-AD-regular and $\widehat{T}_{\mu|_R}$ is bounded in $L^2(\mu|_R)$. As a consequence, the maximal operator

$$\widehat{T}_{\xi,*}f(x) = \sup_{\varepsilon>0}|\widehat{T}_{\xi,\varepsilon}f(x)| = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon}\widehat{K}(x,y)\,f(y)\,d\xi(y)\right|$$

is bounded from $L^2(\xi)$ to $L^2(\mu|_R)$ (see Proposition 5 from [Dav84]).

Consider the vector field $\widetilde{\Psi}$ defined by

$$\widetilde{\Psi} = \sum_{Q\in\mathcal{NB}'(R)} \ell(Q)\,\Psi_Q,$$

so that $|\Psi|\,\mathcal{L}^{n+1} = |\widetilde{\Psi}|\,\xi$. Observe that, by (ii),

$$\|\widetilde{\Psi}\|_{L^\infty(\xi)} \lesssim \delta^{-1},$$

and thus

$$\|\widetilde{\Psi}\|^2_{L^2(\xi)} \lesssim \delta^{-2}\sum_{Q\in\mathcal{NB}'(R)}\ell(Q)^n \lesssim \delta^{-2}\sum_{Q\in\mathcal{NB}'(R)}\mu(Q) \lesssim \delta^{-2}\,\mu(R).$$

For each $x \in R$, we split

$$|S^H(|\Psi|\,\mathcal{L}^{n+1})(x)| = |S^H(|\widetilde{\Psi}|\,\xi)(x)| \le |\widehat{T}(|\widetilde{\Psi}|\xi)(x)| + |\widehat{T}(|\widetilde{\Psi}|\xi)(x^*)|.$$

By standard estimates, it is also immediate to check that

$$|\widehat{T}(|\widetilde{\Psi}|\xi)(x^*)| \le |\widehat{T}_*(|\widetilde{\Psi}|\xi)(x)| + M_n(|\widetilde{\Psi}|\xi)(x),$$

where M_n is the maximal radial operator

$$M_n\tau(x) = \sup_{r>0} \frac{|\tau|(B(x,r))}{r^n}, \tag{3.10.9}$$

for any signed measure τ. So we deduce that

$$\|S^H(|\Psi|\,\mathcal{L}^{n+1})\|^2_{L^2(\mu|_R)} \lesssim \|\widehat{T}_{\xi,*}(|\widetilde{\Psi}|)\|^2_{L^2(\mu|_R)} + \|M_n(|\widetilde{\Psi}|\xi)\|^2_{L^2(\mu|_R)}.$$

Analogously to $\widehat{T}_{\xi,*}$, the operator $M_n(\cdot\,\xi)$ is also bounded from $L^2(\xi)$ to $L^2(\mu|_R)$ (see [Dav84] again). Hence,

$$\|S^H(|\Psi|\,\mathcal{L}^{n+1})\|^2_{L^2(\mu|_R)} \lesssim \|\widetilde{\Psi}\|^2_{L^2(\xi)} \lesssim \delta^{-2}\,\mu(R). \tag{3.10.10}$$

Our next objective is to prove the analogous estimate in $L^2(\sigma)$, that is,

$$\|S^H(|\Psi|\mathcal{L}^{n+1})\|^2_{L^2(\sigma)} \le C(\delta)\,\mu(R).$$

Recall that $\sigma = \sum_{P\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \sigma_P$, where $\sigma_P = g_P\,\mathcal{H}^n|_{L_P}$, with $g_P \lesssim \chi_{2B_P}$. So we have

$$\|S^H(|\Psi|\mathcal{L}^{n+1})\|^2_{L^2(\sigma)} = \sum_{P\in \mathsf{Ch}_{\mathsf{Stop}}(R)} \|S^H(|\Psi|\mathcal{L}^{n+1})\|^2_{L^2(\sigma_P)}.$$

For each $P \in \mathsf{Ch}_{\mathsf{Stop}}(R)$ we split

$$\begin{aligned}\|S^H(|\Psi|\mathcal{L}^{n+1})\|^2_{L^2(\sigma_P)} \le 2\int |S^H(\chi_{3B_P}|\Psi|\mathcal{L}^{n+1})|^2\,d\sigma_P \\ + 2\int |S^H(\chi_{(3B_P)^c}|\Psi|\mathcal{L}^{n+1})|^2\,d\sigma_P.\end{aligned} \tag{3.10.11}$$

Concerning the first summand on the right hand side, we have

$$\int |S^H(\chi_{3B_P}|\Psi|\mathcal{L}^{n+1})|^2\,d\sigma_P \lesssim \int |S^H(\chi_{3B_P}|\Psi|\mathcal{L}^{n+1})|^2\,d\mathcal{H}^n|_{L_P}. \tag{3.10.12}$$

Since $\widehat{T}_{\mathcal{H}^n|_{L_P}}$ is bounded in $L^2(\mathcal{H}^n|_{L_P})$, the same argument as in (3.10.10) shows that

$$\|S^H(\chi_{3B_P}|\Psi|\,\mathcal{L}^{n+1})\|^2_{L^2(\mathcal{H}^n|_{L_P})} \lesssim \|\chi_{3B_P}\widetilde{\Psi}\|^2_{L^2(\xi)} \lesssim \delta^{-2}\,\ell(P)^n, \tag{3.10.13}$$

taking into account that $\|\widetilde{\Psi}\|_{L^\infty(\xi)} \lesssim \delta^{-1}$ and the polynomial growth of ξ for the last inequality.

To estimate the last integral on the right hand side of (3.10.11) we will show first that

$$\big|S^H(\chi_{(3B_P)^c}|\Psi|\mathcal{L}^{n+1})(x) - S^H(\chi_{(3B_P)^c}|\Psi|\mathcal{L}^{n+1})(y)\big| \lesssim \delta^{-1} \quad \text{for all } x,y\in 2B_P. \tag{3.10.14}$$

To this end, note that the left hand side above equals

$$\begin{aligned}\big|S^H(\chi_{(3B_P)^c}|\widetilde\Psi|\xi)(x) - S^H(\chi_{(3B_P)^c}|\widetilde\Psi|\xi)(y)\big| \\ \leq \big|\widehat T^H_\xi(\chi_{(3B_P)^c}|\widetilde\Psi|)(x) - \widehat T^H_\xi(\chi_{(3B_P)^c}|\widetilde\Psi|)(y)\big| \\ + \big|\widehat T^H_\xi(\chi_{(3B_P)^c}|\widetilde\Psi|)(x^*) - \widehat T^H_\xi(\chi_{(3B_P)^c}|\widetilde\Psi|)(y^*)\big|.\end{aligned}$$

Taking into account that both x and y are far from the $\operatorname{supp}(\chi_{(3B_P)^c}|\widetilde\Psi|)$, more precisely, $|x-y| \lesssim \ell(P) \lesssim \min(\operatorname{dist}(x,(3B_P)^c), \operatorname{dist}(y,(3B_P)^c))$, by standard estimates from Calderón-Zygmund theory it follows that

$$\big|\widehat T^H_\xi(\chi_{(3B_P)^c}|\widetilde\Psi|)(x) - \widehat T^H_\xi(\chi_{(3B_P)^c}|\widetilde\Psi|)(y)\big| \lesssim \big\|\chi_{(3B_P)^c}|\widetilde\Psi|\big\|_{L^\infty(\xi)} \lesssim \delta^{-1}.$$

By analogous reasons, the same estimate holds replacing x by x^* and y by y^*. Hence, (3.10.14) is proven.

From (3.10.14) we infer that

$$\begin{aligned}\big\|S^H(\chi_{(3B_P)^c}|\Psi|\mathcal{L}^{n+1})\big\|_{\infty,2B_Q} &\leq \big|m_{\mu,P}(S^H(\chi_{(3B_P)^c}|\Psi|\mathcal{L}^{n+1}))\big| + C\delta^{-1} \\ &\leq \big|m_{\mu,P}(S^H(|\Psi|\mathcal{L}^{n+1}))\big| \\ &\quad + \big|m_{\mu,P}(S^H(\chi_{3B_P}|\Psi|\mathcal{L}^{n+1}))\big| + C\delta^{-1}.\end{aligned}$$

Arguing again as in (3.10.10), we obtain

$$\begin{aligned}\big|m_{\mu,P}(S^H(\chi_{3B_P}|\Psi|\mathcal{L}^{n+1}))\big|^2 &\leq m_{\mu,P}\big(|S^H(\chi_{3B_P}|\Psi|\mathcal{L}^{n+1})|^2\big) \\ &\lesssim \frac{1}{\mu(P)}\,\|\chi_{3B_P}|\widetilde\Psi|\|^2_{L^2(\xi)} \lesssim \delta^{-2}.\end{aligned}$$

Therefore,

$$\|S^H(\chi_{(3B_P)^c}|\Psi|\mathcal{L}^{n+1})\|_{\infty,2B_P} \leq \big|m_{\mu,P}(S^H(|\Psi|\mathcal{L}^{n+1}))\big| + C\delta^{-1}.$$

As a consequence,

$$\int |S^H(\chi_{(3B_P)^c}|\Psi|\mathcal{L}^{n+1})|^2\,d\sigma_P \lesssim \big|m_{\mu,P}(S^H(|\Psi|\mathcal{L}^{n+1}))\big|^2\,\ell(P)^n + \delta^{-2}\ell(P)^n.$$

Together with (3.10.12) and (3.10.13), this yields

$$\begin{aligned}\int |S^H(|\Psi|\mathcal{L}^{n+1})|^2\,d\sigma_P &\lesssim \big|m_{\mu,P}(S^H(|\Psi|\mathcal{L}^{n+1}))\big|^2\,\ell(P)^n + \delta^{-2}\ell(P)^n \\ &\lesssim \int_P |S^H(|\Psi|\mathcal{L}^{n+1})|^2\,d\mu + \delta^{-2}\ell(P)^n.\end{aligned}$$

Summing on $P \in \mathsf{Ch}_{\mathsf{Stop}}(R)$ and using (3.10.7), we obtain

$$\|S^H(|\Psi|\mathcal{L}^{n+1})\|^2_{L^2(\sigma)} \lesssim \|S^H(|\Psi|\mathcal{L}^{n+1})\|^2_{L^2(\mu|_R)} + \delta^{-2}\ell(R)^n \leq C(\delta)\,\ell(R)^n.$$

To prove the final estimate in (v) we just use the preceding inequality and take into account that

$$\int |S^H(|\Psi|\mathcal{L}^{n+1})|^2\,d\nu = \int |S^H(|\Psi|\mathcal{L}^{n+1})|^2\,d(\varphi_s * \sigma)$$
$$= \int (|S^H(|\Psi|\mathcal{L}^{n+1})|^2) * \varphi_s\,d\sigma \to \int |S^H(|\Psi|\mathcal{L}^{n+1})|^2\,d\sigma$$

as $s \to 0$, since $|S^H(|\Psi|\mathcal{L}^{n+1})|^2$ is a continuous function. □

3.11 The variational argument

In this section we will prove the following:

Proposition 3.11.1. *Let $R \in \mathsf{Nice}$ and ν be as in Section 3.9. Suppose that ε and $\ell(R)$ are small enough, depending on the non-BAUPness parameter δ. Then we have*

$$\|S^H\nu\|^2_{L^2(\sigma)} \geq c_8(\delta)\,\mu(R).$$

Together with Lemma 3.9.3 this shows that, for each $R \in \mathsf{Nice}$, $\|T_R\mu\|^2_{L^2(\mu)} \geq \varepsilon_1\,\mu(R)$, assuming that ε, $\ell(R)$, t, and Δ are small enough and M is big enough. This proves Proposition 3.4.1 and Theorem 3.1.

3.11.1 A pointwise inequality

The first step to prove Proposition 3.11.1 is the next one.

Lemma 3.11.1. *Suppose that for some $0 < \lambda \leq 1$ the inequality*

$$\int |S^H\nu|^2 d\nu \leq \lambda\,\nu(\mathbb{R}^{n+1})$$

holds. Let h be the function in Lemma 3.10.1 and $c_7(\delta)$ the constant in the same lemma. Then, there is some function $b \in L^\infty(\nu)$ such that

(i) $0 \leq b \leq 2$,

(ii) $\int b\,h\,d\nu \geq c_7(\delta)\,\nu(\mathbb{R}^{n+1})$,

and such that the measure $\eta = b\nu$ satisfies

$$\int |S^H\eta|^2 d\eta \leq 2\lambda\,\nu(\mathbb{R}^{n+1}) \tag{3.11.1}$$

and

$$|S^H\eta(x)|^2 + 2S^{H,*}((S^H\eta)\eta)(x) \leq 6c_7(\delta)^{-1}\lambda \quad \text{for } \eta\text{-a.e. } x \in \mathbb{R}^{n+1}. \tag{3.11.2}$$

Proof. In order to find such a function b, we consider the following class of admissible functions

$$\mathcal{A} = \left\{a \in L^\infty(\nu) :\, a \geq 0,\, \textstyle\int a\,h\,d\nu \geq c_7(\delta)\,\nu(\mathbb{R}^{n+1})\right\} \tag{3.11.3}$$

and we define a functional J on $\mathcal{A}$ by

$$J(a) = \lambda\|a\|_{L^\infty(\nu)}\,\nu(\mathbb{R}^{n+1}) + \int |S^H(a\nu)|^2 a\,d\nu.$$

Observe that $1 \in \mathcal{A}$ and

$$J(1) = \lambda\,\nu(\mathbb{R}^{n+1}) + \int |S^H\nu|^2\,d\nu \leq 2\lambda\,\nu(\mathbb{R}^{n+1}).$$

Thus

$$\inf_{a\in\mathcal{A}} J(a) \leq 2\lambda\,\nu(\mathbb{R}^{n+1}).$$

Since $J(a) \geq \lambda\|a\|_{L^\infty(\nu)}\,\nu(\mathbb{R}^{n+1})$, it is clear that

$$\inf_{a\in\mathcal{A}} J(a) = \inf_{a\in\mathcal{A}:\|a\|_{L^\infty(\nu)}\leq 2} J(a).$$

We claim that J attains a global minimum on $\mathcal{A}$, i.e. there is a function $b \in \mathcal{A}$ such that $J(b) \leq J(a)$ for all $a \in \mathcal{A}$. Indeed, by the Banach-Alaoglu theorem there exists a sequence $\{a_k\}_k \subset \mathcal{A}$, with $J(a_k) \to \inf_{a\in\mathcal{A}} J(a)$, $\|a_k\|_{L^\infty(\nu)} \leq 2$, so that a_k converges weakly $*$ in $L^\infty(\nu)$ to some function $b \in \mathcal{A}$. It is clear that b satisfies (i) and (ii). Recall that we denoted by K_S^H the kernel of S^H. Since $y \mapsto K_S^H(x,y)$ belongs to $L^1(\nu)$ (recall that ν has bounded density with respect to Lebesgue measure), it follows that for all $x \in \mathbb{R}^{n+1}$ $S^H(a_k\nu)(x) \to S^H(b\nu)(x)$ as $k \to \infty$. Taking into account that, for every k,

$$|S^H(a_k\nu)(x)| \lesssim \int \frac{1}{|x-y|^n}\,d\nu(y) < \infty$$

by the dominated convergence theorem we infer that

$$\int |S^H(a_k\nu)|^2 d\nu \to \int |S^H(b\nu)|^2 d\nu \quad \text{ as } \; k \to \infty.$$

Using also that $\|b\|_{L^\infty(\nu)} \leq \limsup_k \|a_k\|_{L^\infty(\nu)}$, it follows that $J(b) \leq \limsup_k J(a_k)$, which proves the claim that $J(\cdot)$ attains a minimum at b.

The estimate (3.11.1) for $\eta = b\,\nu$ follows from the fact that $J(b) \leq J(1) \leq 2\lambda\,\nu(\mathbb{R}^{n+1})$.

To prove (3.11.2) we will apply a variational argument taking advantage of the fact that b is a minimizer for J. Let B be any ball centered in $\operatorname{supp}\eta$. Now, for every $0 \leq t < 1$, define

$$b_t = (1 - t\chi_B)b + t\,\frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})}\,b,$$

where we used the notation $(h\eta)(A) = \int_A h\,d\eta$. To make the writing easier, we will also write below just $(h\eta)(A)$. It is clear that $b_t \in \mathcal{A}$ for all $0 \leq t < 1$ and $b_0 = b$. Therefore,

$$\begin{aligned} J(b) \leq J(b_t) &= \lambda\|b_t\|_\infty \nu(\mathbb{R}^{n+1}) + \int |S^H(b_t\nu)|^2 b_t\,d\nu \\ &\leq \lambda\left(1 + t\frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})}\right)\|b\|_\infty \nu(\mathbb{R}^{n+1}) + \int |S^H(b_t\nu)|^2 b_t\,d\nu := H(t). \end{aligned}$$

Since $H(0) = J(b)$, we have that $H(0) \leq H(t)$ for $0 \leq t < 1$, thus $H'(0+) \geq 0$ (assuming that $H'(0+)$ exists). Notice that

$$\left.\frac{db_t}{dt}\right|_{t=0} = -\chi_B b + \frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})}\,b,$$

Therefore,

$$
\begin{aligned}
0 \leq H'(0+) &= \lambda \frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})}\, \|b\|_\infty \nu(\mathbb{R}^{n+1}) + \frac{d}{dt}\Big|_{t=0} \int |S^H(b_t\nu)|^2 b_t d\nu \\
&= \lambda \frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})} \|b\|_\infty \nu(\mathbb{R}^{n+1}) \\
&\quad + 2\int S^H\left(\frac{db_t}{dt}\Big|_{t=0}\nu\right)\cdot S^H\eta\, b\,d\nu + \int |S^H\eta|^2 \frac{db_t}{dt}\Big|_{t=0} d\nu \\
&= \lambda \frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})}\, \|b\|_\infty \nu(\mathbb{R}^{n+1}) \\
&\quad + 2\int S^H\left(\left(-\chi_B b + \frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})} b\right)\nu\right)\cdot S^H\eta\, b\,d\nu \\
&\quad + \int |S^H\eta|^2 \left(-\chi_B b + \frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})} b\right) d\nu \\
&= \lambda \frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})} \|b\|_\infty \nu(\mathbb{R}^{n+1}) - 2\int S^H(\chi_B\eta)\cdot S^H\eta\, d\eta \\
&\quad + 2\frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})} \int |S^H\eta|^2\, d\eta - \int_B |S^H\eta|^2\, d\eta + \frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})} \int |S^H\eta|^2\, d\eta.
\end{aligned}
$$

The fact that the derivatives above commute with the integral sign and with the operator S^H is guaranteed by the fact that b_t is an affine function of t and then one can expand the integrand $|S^H(b_t\nu)|^2 b_t$ and obtain a polynomial expression on t. Rearranging terms and using also that $\lambda \leq 1$ and that $J(b) \leq 2\lambda\,(h\eta)(\mathbb{R}^{n+1})$, we get

$$
\begin{aligned}
&\int_B |S^H\eta|^2\, d\eta + 2\int S^H(\chi_B\eta)\cdot S^H\eta\, d\eta \\
&\qquad\qquad \leq \frac{(h\eta)(B)}{(h\eta)(\mathbb{R}^{n+1})}\left[\lambda\|b\|_\infty \nu(\mathbb{R}^{n+1}) + 3\int |S^H\eta|^2\, d\eta\right] \\
&\qquad\qquad \leq 3\, c_7(\delta)^{-1} J(b)\, (h\eta)(B) \leq 6\, c_7(\delta)^{-1}\lambda\, (h\eta)(B).
\end{aligned}
$$

Dividing by $\eta(B)$, recalling that $h \leq 1$ and taking into account that

$$\int S^H(\chi_B\eta)\cdot S^H\eta\, d\eta = \int_B S^{H,*}((S^H\eta)\eta)\, d\eta,$$

we obtain

$$\frac{1}{\eta(B)}\int_B |S^H\eta|^2 d\eta + \frac{2}{\eta(B)}\int_B S^{H,*}((S^H\eta)\eta)\, d\eta \leq 6\, c_7(\delta)^{-1}\lambda.$$

Then, letting $\eta(B) \to 0$ and applying Lebesgue's differentiation theorem, we deduce that

$$|S^H\eta(x)|^2 + 2S^{H,*}((S^H\eta)\eta)(x) \leq 6\, c_7(\delta)^{-1}\lambda \quad \text{for } \eta\text{-a.e. } x \in \mathbb{R}^{n+1},$$

as desired. □

Lemma 3.11.2. *Assume that* $\int |S^H\nu|^2\, d\nu \leq \lambda\nu(\mathbb{R}^{n+1})$ *for some* $0 < \lambda \leq 1$, *and let* b *and* η *be as in Lemma 3.11.1. Then we have*

$$|S^H\eta(x)|^2 + 4S^{H,*}((S^H\eta)\eta)(x) \leq 12\, c_7(\delta)^{-1}\lambda + C\ell(R)^{\alpha/2} \quad \textit{for all } x \in \mathbb{R}^{n+1}_+. \tag{3.11.4}$$

Proof. Since η has a bounded density with respect to Lebesgue measure which is also uniformly bounded, it is immediate to check that the expression on the left hand side of (3.11.4) is a continuous function of x. Thus, by Lemma 3.11.1 and by continuity, the inequality (3.11.4) holds for all $x \in \operatorname{supp} \eta$.

For any $x \in \partial\mathbb{R}^{n+1}_+ = H$, using (3.8.3) and that $x = x^*$, we get

$$\widehat{K}^H(y^*, x) = \widehat{K}^H(y, x^*) = \widehat{K}^H(y, x),$$

and thus, for any vectorial measure $\vec{\omega}$,

$$\begin{aligned} S^{H,*}\vec{\omega}(x) &= \int K^H_S(y, x) \cdot d\vec{\omega}(y) \\ &= \int \widehat{K}^H(y, x) \cdot d\vec{\omega}(y) - \int \widehat{K}^H(y^*, x) \cdot d\vec{\omega}(y) = 0. \end{aligned}$$

Now we claim that the definition of S^H implies

$$\sup_{x\in\mathbb{R}^{n+1}_+} |S^{H,*}\vec{\omega}(x)| \leq \sup_{x\in\operatorname{supp}(\vec{\omega})} |S^{H,*}\vec{\omega}(x)|, \tag{3.11.5}$$

for each vector valued measure $\vec{\omega}$ which is compactly supported in $\mathbb{R}^{n+1}$ and absolutely continuous with respect to Lebesgue measure with a bounded density function. To show this, by the maximum principle, it is enough to show that $S^{H,*}\vec{\omega}$ is $\widehat{A}$-harmonic in $\mathbb{R}^{n+1}_+ \setminus \operatorname{supp}(\vec{\omega})$. In turn, to this end it suffices to show that for $1 \leq k \leq n$ and for any signed measure $d\omega = g\,dx$, with $g \in L^\infty$ and compactly supported in $\mathbb{R}^{n+1}_+$, the function

$$f(x) := \int \big(\partial_{y_k}\mathcal{E}_{\widehat{A}}(y, x) - \partial_{y_k}\mathcal{E}_{\widehat{A}}(y^*, x)\big)\, d\omega(y)$$

is $\widehat{A}$-harmonic in $\mathbb{R}^{n+1}_+ \setminus \operatorname{supp}(\omega)$. Given $\varphi \in C^\infty_c(\mathbb{R}^{n+1}_+ \setminus \operatorname{supp}\omega)$, by Fubini's theorem we get

$$\begin{aligned} \int \widehat{A}\,\nabla f\,\nabla\varphi\,dx &= \int \widehat{A}(x)\nabla_x\left(\int \partial_{y_k}(\mathcal{E}_{\widehat{A}}(y, x) - \mathcal{E}_{\widehat{A}}(y^*, x))\,g(y)\,dy\right) \cdot \nabla\varphi(x)\,dx \\ &= \iint \widehat{A}(x)\nabla_x\partial_{y_k}(\mathcal{E}_{\widehat{A}}(y, x) - \mathcal{E}_{\widehat{A}}(y^*, x)) \cdot \nabla\varphi(x)\,dx\,g(y)\,dy \\ &= \int \partial_{y_k} \int \widehat{A}(x)\nabla_x\mathcal{E}_{\widehat{A}}(y, x) \cdot \nabla\varphi(x)\,dx\,g(y)dy \\ &\quad - \int \partial_{y_k} \int \widehat{A}(x)\nabla_x\mathcal{E}_{\widehat{A}}(y^*, x) \cdot \nabla\varphi(x)\,dx\,g(y)dy \\ &= \int (\partial_{y_k}\varphi(y) - \partial_{y_k}\varphi(y^*))\,g(y)\,dy = 0. \end{aligned}$$

Therefore, f is $\widehat{A}$-harmonic $\mathbb{R}^{n+1}_+ \setminus \operatorname{supp}(\vec{\omega})$ and thus (3.11.5) holds.

To prove (3.11.4) we use the elementary formula

$$\frac{1}{2}|z|^2 = \sup_{\substack{\beta>0 \\ e\in\mathbb{R}^{n+1}, \|e\|=1}} \beta\,\langle e, z\rangle - \frac{1}{2}\beta^2 \quad \text{for all } z \in \mathbb{R}^{n+1}.$$

We apply it with $z = S^H\eta(x)$ and we get

$$\frac{1}{2}|S^H\eta(x)|^2 = \sup_{\substack{\beta>0 \\ e\in\mathbb{R}^{n+1}, \|e\|=1}} \beta\,\langle e, S^H\eta(x)\rangle - \frac{1}{2}\beta^2. \tag{3.11.6}$$

Now, if $e = (e_1, \dots, e_{n+1})$ and we define the vector valued measure $\eta e = (\eta e_1, \dots, \eta e_{n+1})$, for all $x \in \mathbb{R}^{n+1}_+$ we obtain

$$\begin{aligned}\langle e, S^H\eta(x)\rangle &= \int K_S^H(x,y)\cdot e\, d\eta(y) = \int K_S^H(x,y)\cdot d(\eta e)(y)\\ &= -S^{H,*}(\eta e)(x) + e\cdot \int \left[K_S^H(x,y) + K_S^H(y,x)\right]\, d\eta(y).\end{aligned}$$

Taking into account $\widehat{K}^H(y^*,x) = \widehat{K}^H(y,x^*)$ and (3.2.5) applied to $\widehat{A}$, we derive

$$\begin{aligned}|K_S^H(x,y) + K_S^H(y,x)| &= |\widehat{K}^H(x,y) - \widehat{K}^H(x^*,y) + \widehat{K}^H(y,x) - \widehat{K}^H(y^*,x)|\\ &\leq |\widehat{K}^H(x,y) + \widehat{K}^H(y,x)| + |\widehat{K}^H(y,x^*) + \widehat{K}^H(x^*,y)|\\ &\lesssim \frac{1}{|x-y|^{n-\alpha/2}} + \frac{1}{|x^*-y|^{n-\alpha/2}} \lesssim \frac{1}{|x-y|^{n-\alpha/2}},\end{aligned}$$

since $|x-y| \leq |x^*-y|$ for all $x, y \in \mathbb{R}^{n+1}_+$. So the function

$$F(x) := \int \left[K_S^H(x,y) + K_S^H(y,x)\right]\, d\eta(y)$$

satisfies

$$|F(x)| \lesssim \int \frac{1}{|x-y|^{n-\alpha/2}}\, d\eta(y) \lesssim \ell(R)^{\alpha/2}$$

if $\operatorname{dist}(x,R) \leq 1$. In the case that $\operatorname{dist}(x,Q) \geq 1$, we use the fact that $|\widehat{K}^H(x,y)| + |\widehat{K}^H(y,x)| \lesssim 1$ by Lemma 4.2.1 (c), and it also follows that

$$|F(x)| \leq \int \left|K_S^H(x,y) + K_S^H(y,x)\right| d\eta(y) \lesssim \|\eta\| \lesssim \ell(R)^n \lesssim \ell(R)^{\alpha/2},$$

So in both cases we get

$$\langle e, S^H\eta(x)\rangle = -S^{H,*}(\eta e)(x) + F(x)\cdot e, \tag{3.11.7}$$

with $|F(x)| \lesssim \ell(R)^{\alpha/2}$.

We insert the above calculation in (3.11.6) and by (3.11.5) we get, for $x \in \mathbb{R}^{n+1}_+$,

$$\begin{aligned}
&|S^H\eta(x)|^2 + 4S^{H,*}\left([S^H\eta]\eta\right)(x) \\
&\quad = \sup_{\substack{\beta>0 \\ e\in\mathbb{R}^{n+1}, \|e\|=1}} \left\{-2\beta S^{H,*}(\eta e)(x) + 2\beta F(x)\cdot e - \beta^2 + 4S^{H,*}\left([S^H\eta]\eta\right)(x)\right\} \\
&\quad = \sup_{\substack{\beta>0 \\ e\in\mathbb{R}^{n+1}, \|e\|=1}} \left\{S^{H,*}\left(-2\beta\eta e + 4[S^H\eta]\eta\right)(x) + 2\beta F(x)\cdot e - \beta^2\right\} \\
&\quad \le \sup_{\substack{\beta>0 \\ e\in\mathbb{R}^{n+1}, \|e\|=1}} \sup_{z\in\mathrm{supp}(\eta)} \left\{S^{H,*}\left(-2\beta\eta e + 4[S^H\eta]\eta\right)(z) + 2\beta F(x)\cdot e - \beta^2\right\} \\
&\quad = \sup_{z\in\mathrm{supp}(\eta)} \sup_{\substack{\beta>0 \\ e\in\mathbb{R}^{n+1}, \|e\|=1}} \left\{S^{H,*}\left(-2\beta\eta e + 4[S^H\eta]\eta\right)(z) + 2\beta F(x)\cdot e - \beta^2\right\}.
\end{aligned}$$

Now we reverse the process using again (3.11.7) to obtain

$$\begin{aligned}
&|S^H\eta(x)|^2 + 4S^{H,*}\left([S^H\eta]\eta\right)(x) \\
&\le \sup_{z\in\mathrm{supp}(\eta)} \sup_{\substack{\beta>0 \\ e\in\mathbb{R}^{n+1}, \|e\|=1}} \left\{-2\beta S^{H,*}(\eta e)(z) + 4S^{H,*}\left([S^H\eta]\eta\right)(z) + 2\beta F(x)\cdot e - \beta^2\right\} \\
&= \sup_{z\in\mathrm{supp}(\eta)} \sup_{\substack{\beta>0 \\ e\in\mathbb{R}^{n+1}, \|e\|=1}} \Big\{ -2\beta\langle S^H\eta(z), e\rangle - 2\beta F(z)\cdot e \\
&\qquad\qquad + 4S^{H,*}\left([S^H\eta]\eta\right)(z) + 2\beta F(x)\cdot e - \beta^2\Big\} \\
&= \sup_{z\in\mathrm{supp}(\eta)} \sup_{\substack{\beta>0 \\ e\in\mathbb{R}^{n+1}, \|e\|=1}} \left\{-2\beta\left\langle S^H\eta(z) + (F(x) - F(z)),\, e\right\rangle + 4S^{H,*}\left([T\eta]\eta\right)(z) - \beta^2\right\} \\
&= \sup_{z\in\mathrm{supp}(\eta)} \left\{\left|S^H\eta(z) + (F(x) + G(z))\right|^2 + 4S^{H,*}\left([S^H\eta]\eta\right)(z)\right\} \\
&\le \sup_{z\in\mathrm{supp}(\eta)} \left\{2|S^H\eta(z)|^2 + 4S^{H,*}\left([S^H\eta]\eta\right)(z)\right\} + C\,\ell(R)^{\alpha/2}.
\end{aligned}$$

Finally, we apply (3.11.2) to get

$$|S^H\eta(x)|^2 + 4S^{H,*}((S^H\eta)\eta)(x) \le 12\, c_7(\delta)^{-1}\lambda + C\ell(R)^{\alpha/2} \quad \text{for all } x \in \mathbb{R}^{n+1}_+, \tag{3.11.8}$$

as wished. □

3.11.2 Proof of Proposition 3.11.1

Let $R \in \mathsf{Nice}$ and ν be as in Section 3.9. We have to show that

$$\|S^H\nu\|^2_{L^2(\sigma)} \ge c_8(\delta)\,\mu(R),$$

with $c_8(\delta) > 0$. We assume that this does not hold and we argue by contradiction. So we suppose that $\int |S^H\nu|^2 d\nu \le \lambda\,\nu(\mathbb{R}^{n+1})$ for some small $\lambda \in (0,1)$ to be fixed below and then we will get a contradiction if λ is chosen small enough (depending on δ). By Lemma 3.11.2, our assumption implies that the measure η defined in Lemma 3.11.1 satisfies

$$|S^H\eta(x)|^2 + 4S^{H,*}((S^H\eta)\eta)(x) \le 12\, c_7(\delta)^{-1}\lambda + C\ell(R)^{\alpha/2} \quad \text{for all } x \in \mathbb{R}^{n+1}_+.$$

Consider the vector field Ψ from Lemma 3.10.1 in Section 3.10. Multiplying the preceding inequality by $|\Psi|$ and integrating with respect to Lebesgue measure, we derive

$$\begin{aligned}\int |S^H\eta|^2\,|\Psi|\,d\mathcal{L}^{n+1} \leq 4\int S^{H,*}\big((S^H\eta)\eta\big)\,|\Psi|\,d\mathcal{L}^{n+1} \\ + \big(12c_7(\delta)^{-1}\lambda + C\ell(R)^{\alpha/2}\big)\int |\Psi|\,d\mathcal{L}^{n+1}.\end{aligned} \tag{3.11.9}$$

By Lemma 3.10.2 we have

$$\big(12c_7(\delta)^{-1}\lambda + C\ell(R)^{\alpha/2}\big)\int |\Psi|\,d\mathcal{L}^{n+1} \leq C(\delta)\,\big(\lambda + \ell(R)^{\alpha/2}\big)\,\ell(R)^n.$$

Regarding the first integral on the right hand side of (3.11.9), we have

$$\begin{aligned}\int S^{H,*}\big((S^H\eta)\eta\big)\,|\Psi|\,d\mathcal{L}^{n+1} &= \int S^H\eta\cdot S^H(|\Psi|\,\mathcal{L}^{n+1})\,d\eta \\ &\leq \left(\int |S^H\eta|^2\,d\eta\right)^{1/2}\left(2\int |S^H(|\Psi|\,\mathcal{L}^{n+1})|^2\,d\nu\right)^{1/2} \\ &\leq \lambda^{1/2}\,\eta(\mathbb{R}^{n+1})^{1/2}\,C(\delta)\,\mu(R) \leq C(\delta)\,\lambda^{1/2}\,\mu(R),\end{aligned}$$

by (3.11.1) and (v) from Lemma 3.10.2. So we derive

$$\begin{aligned}\int |S^H\eta|^2\,|\Psi|\,d\mathcal{L}^{n+1} &\leq C(\delta)\,\lambda^{1/2}\mu(R) + C(\delta)\,(\lambda + \ell(R)^{\alpha/2})\,\mu(R) \\ &\leq C(\delta)\,(\lambda^{1/2} + \ell(R)^{\alpha/2})\,\mu(R).\end{aligned} \tag{3.11.10}$$

Next we will estimate from below the integral on the left hand side above. By Cauchy-Schwarz, we have

$$\begin{aligned}\int |S^H\eta|^2\,|\Psi|\,d\mathcal{L}^{n+1} &\geq \left(\int |S^H\eta|\,|\Psi|\,d\mathcal{L}^{n+1}\right)^2\left(\int |\Psi|\,d\mathcal{L}^{n+1}\right)^{-1} \\ &\geq \frac{c(\delta)}{\mu(R)}\left(\int S^H\eta\cdot\Psi\,d\mathcal{L}^{n+1}\right)^2 = \frac{c(\delta)}{\mu(R)}\left(\int S^{H,*}(\Psi\,\mathcal{L}^{n+1})\,d\eta\right)^2.\end{aligned} \tag{3.11.11}$$

By the definition of S^H and the fact that $\widehat{K}^H(y^*,x) = \widehat{K}^H(y,x^*)$ (by (3.8.3)), we get

$$\begin{aligned}S^{H,*}(\Psi\,\mathcal{L}^{n+1})(x) &= \int \widehat{K}^H(y,x)\cdot\Psi(y)\,d\mathcal{L}^{n+1}(y) - \int \widehat{K}^H(y^*,x)\cdot\Psi(y)\,d\mathcal{L}^{n+1}(y) \\ &= \widehat{T}^{H,*}(\Psi\,\mathcal{L}^{n+1})(x) - \widehat{T}^{H,*}(\Psi\,\mathcal{L}^{n+1})(x^*).\end{aligned}$$

Thus, by Lemma 3.10.2 (iv),

$$\begin{aligned}\int S^{H,*}(\Psi\,\mathcal{L}^{n+1})\,d\eta &= \int \widehat{T}^{H,*}(\Psi\,\mathcal{L}^{n+1})(x)\,d\eta(x) - \int \widehat{T}^{H,*}(\Psi\,\mathcal{L}^{n+1})(x^*)\,d\eta(x) \\ &= \sum_{Q\in\mathcal{NB}'(R)}\int (h_Q + e_Q)\,d\eta - \int \widehat{T}^{H,*}(\Psi\,\mathcal{L}^{n+1})(x^*)\,d\eta(x).\end{aligned} \tag{3.11.12}$$

By Lemma 3.10.1 and Lemma 3.10.2 (iv),

$$\sum_{Q\in\mathcal{NB}'(R)}\int (h_Q+e_Q)\,d\eta \geq c(\delta)\,\mu(R) - C(\delta)\sum_{Q\in\mathcal{NB}'(R)}\int \frac{\ell(R)^{\tilde\gamma}\,\ell(Q)^{n+\tilde\beta}}{(|x-x_Q|+\ell(Q))^{n+\tilde\beta}}\,d\eta(x).$$

Using the polynomial growth of ν (recall (3.9.2)) and standard estimates, for each $Q\in\mathcal{NB}'(R)$ we get

$$\int \frac{\ell(R)^{\tilde\gamma}\,\ell(Q)^{n+\tilde\beta}}{(|x-x_Q|+\ell(Q))^{n+\tilde\beta}}\,d\eta(x) \lesssim \ell(R)^{\tilde\gamma}\,\ell(Q)^n. \tag{3.11.13}$$

Thus

$$\begin{aligned}\sum_{Q\in\mathcal{NB}'(R)}\int (h_Q+e_Q)\,d\eta &\geq c(\delta)\,\mu(R) - C(\delta)\,\ell(R)^{\tilde\gamma}\sum_{Q\in\mathcal{NB}'(R)}\mu(Q)\\ &\geq \big(c(\delta)-C'(\delta)\,\ell(R)^{\tilde\gamma}\big)\,\mu(R).\end{aligned} \tag{3.11.14}$$

To estimate the last integral on the right hand side of (3.11.12) we take into account that, if $x\in\operatorname{supp}\eta$, then $x^*\in\mathbb{R}^{n+1}_-$, and thus

$$h_Q(x^*)=0 \quad \text{for all } Q\in\mathcal{NB}'(R),$$

since $\operatorname{supp} h_Q\subset 3B_Q\subset\mathbb{R}^{n+1}_+$ because, recalling (3.4.1) and choosing Δ as in Section 3.8, $\ell(Q)\ll\Delta\,\ell(R)$.

Therefore, for $x\in\operatorname{supp}\eta$, using again Lemma 3.10.2 (iv),

$$\begin{aligned}\big|\widehat T^{H,*}(\Psi\,\mathcal{L}^{n+1})(x^*)\big| &= \Bigg|\sum_{Q\in\mathcal{NB}'(R)} e_Q(x^*)\Bigg|\\ &\leq C(\delta)\sum_{Q\in\mathcal{NB}'(R)}\frac{\ell(R)^{\tilde\gamma}\,\ell(Q)^{n+\tilde\beta}}{(|x^*-x_Q|+\ell(Q))^{n+\tilde\beta}}\\ &\leq C(\delta)\sum_{Q\in\mathcal{NB}'(R)}\frac{\ell(R)^{\tilde\gamma}\,\ell(Q)^{n+\tilde\beta}}{(|x-x_Q|+\ell(Q))^{n+\tilde\beta}}.\end{aligned}$$

Hence, from (3.11.13) we derive

$$\left|\int \widehat T^{H,*}(\Psi\,\mathcal{L}^{n+1})(x^*)\,d\eta(x)\right| \leq C(\delta)\,\ell(R)^{\tilde\gamma}\sum_{Q\in\mathcal{NB}'(R)}\ell(Q)^n \leq C(\delta)\,\ell(R)^{\tilde\gamma}\mu(R).$$

Plugging this estimate and (3.11.14) into (3.11.12), we obtain

$$\int S^{H,*}(\Psi\,\mathcal{L}^{n+1})\,d\eta \geq \big(c(\delta)-C''(\delta)\,\ell(R)^{\tilde\gamma}\big)\,\mu(R).$$

Then, by (3.11.11),

$$\int |S^H\eta|^2\,|\Psi|\,d\mathcal{L}^{n+1} \geq \big(c(\delta)-C''(\delta)\,\ell(R)^{\tilde\gamma}\big)^2\,\mu(R).$$

Together with (3.11.10), this implies that

$$\big(c(\delta) - C''(\delta)\,\ell(R)^{\tilde\gamma}\big)^2 \mu(R) \le C(\delta)\,\big(\lambda^{1/2} + \ell(R)^{\alpha/2}\big)\,\mu(R).$$

So we get a contradiction if $\ell(R)$ and λ are small enough, depending on δ. This concludes the proof of Lemma 3.11.1, and thus of Theorem 3.1.

3.12 Proof of Theorem 3.4

The arguments are very similar to the ones in [Azz+16c] and thus we only sketch them.

To simplify notation, we will write ω^p instead of $\omega^p_{L_A}$. Recall that the Green function for the operator $L_A u = -\operatorname{div} A(\cdot)\nabla u$ satisfies, for every $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$,

$$\int_{\partial\Omega} \varphi\, d\omega^x - \varphi(x) = -\int_\Omega A^T(y)\nabla_y G(x,y)\cdot\nabla\varphi(y)\,dy, \qquad \text{for a.e. } x\in\Omega.$$

See (2.6) in [Azz+16a], for example. From this equation it easily follows that

$$G(p,x) = \mathcal{E}(p,x) - \int \mathcal{E}(z,x)\,d\omega^p(z) \quad \text{for all } p,x\in\Omega. \tag{3.12.1}$$

We assume that $G(p,x) = 0$ if $x \notin \Omega$, so that the preceding identity also holds in this case. The identity (3.12.1) provides the key connection between the gradient of the single layer potential and elliptic measure. Indeed, differentiating with respect to x, we derive

$$\nabla_2 G(p,x) = \nabla_2\mathcal{E}(p,x) - \int \nabla_2\mathcal{E}(z,x)\,d\omega^p(z).$$

Then, by Lemma 4.2.2, it follows that

$$\begin{aligned} |T\omega^p(x)| &\le \left|\int \nabla_2\mathcal{E}(z,x)\,d\omega^p(z)\right| + \left|\int \nabla_1\mathcal{E}(x,z)\,d\omega^p(z) - \int \nabla_2\mathcal{E}(z,x)\,d\omega^p(z)\right| \\ &\le |\nabla_2 G(p,x)| + \frac{C}{|x-p|^n} + \int \frac{C}{|x-z|^{n-\alpha}}\,d\omega^p(z) \\ &\le |\nabla_2 G(p,x)| + \frac{C}{|x-p|^n} + C\,M_n\omega^p(x), \end{aligned} \tag{3.12.2}$$

where M_n is the maximal radial operator defined in (3.10.9).

By almost the same arguments as in [Azz+16c, Lemma 3.3] one can prove the following:

Lemma 3.12.1. *Let $n \ge 2$ and $\Omega \subset \mathbb{R}^{n+1}$ be a bounded open connected Wiener regular set. Let $B = \bar B(x_0, r)$ be a closed ball with $x_0 \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$. Then, for all $a > 0$,*

$$\omega^x(aB) \gtrsim \inf_{z\in 2B\cap\Omega} \omega^z(aB)\,r^{n-1}\,G(x,y) \qquad \text{for all } x\in\Omega\setminus 2B \text{ and } y\in B\cap\Omega,$$

with the implicit constant independent of a.

Analogously, as in [Azz+16c, Lemma 3.4], we have:

Lemma 3.12.2. *There is $\delta_0 > 0$ depending only on $n \ge 1$ so that the following holds for $\delta \in (0, \delta_0)$. Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be a bounded Wiener regular domain, $n-1 < s \le n+1$,*

$\xi \in \partial\Omega$, $r > 0$, and $B = B(\xi, r)$. Then

$$\omega^x(B) \gtrsim_{n,s} \frac{\mathcal{H}^s_\infty(\partial\Omega \cap \delta B)}{(\delta r)^s} \quad \text{for all } x \in \delta B \cap \Omega.$$

In the statement above, $\mathcal{H}^s_\infty$ stands for the s-dimensional Hausdorff content.

The following can be proved as in [Azz+16c, Lemma 3.1]:

Lemma 3.12.3. *Let Ω be as above and let $p \in \Omega$. For $\mathcal{L}^{n+1}$-almost all $x \in \Omega^c$ we have*

$$\mathcal{E}(p, x) - \int_{\partial\Omega} \mathcal{E}(z, x)\, d\omega^p(z) = 0.$$

Then we get:

Lemma 3.12.4. *Let L_A, Ω and E be as in Theorem 3.4. Then we have*

$$M_n\omega^p(x) + T_*\omega^p(x) < \infty \quad \text{for } \omega^p\text{-a.e. } x \in E.$$

Above, M_n is the maximal radial operator defined in (3.10.9).

This result can be deduced from the preceding lemmas arguing as in [Azz+16c]. For the convenience of the reader we show the detailed proof below. Remark that, instead of the stopping time arguments from [Azz+16c], we use a simpler approach relying on the Lebesgue differentiation theorem.

Proof. For ω^p-a.e. $x \in E$, we write

$$\limsup_{r\to 0} \frac{\omega^p(B(x,r))}{r^n} \le \limsup_{r\to 0} \frac{\omega^p(B(x,r))}{\mathcal{H}^n(B(x,r)\cap E)} \; \limsup_{r\to 0} \frac{\mathcal{H}^n(B(x,r)\cap E)}{r^n}.$$

The first lim sup on the right hand side is finite ω^p-a.e. in E because of the absolute continuity of ω^p with respect to $\mathcal{H}^n$ in E, while the last one is also finite by the classical density bounds for Hausdorff measure. Hence the left hand side is also finite ω^p-a.e. in E, or equivalently,

$$M_n\omega^p(x) < \infty \quad \text{for } \omega^p\text{-a.e. } x \in E.$$

It remains to show that $T_*\omega^p(x) < \infty$ for ω^p-a.e. $x \in E$. To this end, for $k \ge 1$ we define

$$E_k = \{x \in E : M_n\omega^p(x) \le k\},$$

so that $E = \bigcup_{k\ge 1} E_k$, up to a set of ω^p-measure zero. For a fixed $k \ge 1$, let $x \in E_k$ be a density point of E_k, and let r_0 be small enough so that

$$\frac{\omega^p(B(x,r)\cap E_k)}{\omega^p(B(x,r))} \ge \frac{1}{2} \quad \text{for } 0 < r \le r_0.$$

Observe that, since $\omega^p(B(z,\rho)\cap E_k) \le k\rho^n$ for all $z \in E_k$ and all $\rho > 0$, by Frostman's Lemma we have

$$\mathcal{H}^n_\infty(B(x,r)\cap\partial\Omega) \ge \mathcal{H}^n_\infty(B(x,r)\cap E_k) \ge C(k)\,\omega^p(B(x,r)\cap E_k) \ge \frac{C(k)}{2}\,\omega^p(B(x,r)), \tag{3.12.3}$$

for $0 < r \le r_0$.

Next we consider a radial C^∞ function $\varphi : \mathbb{R}^{n+1} \to [0,1]$ which vanishes in $B(0,1)$ and equals 1 on $\mathbb{R}^{n+1} \setminus B(0,2)$, and for $r > 0$ and $z \in \mathbb{R}^{n+1}$ we denote $\varphi_r(z) = \varphi\left(\frac{z}{r}\right)$

and $\psi_r = 1 - \varphi_r$. We set

$$\widetilde{T}_r\omega^p(z) = \int \nabla_2\mathcal{E}(y,z)\,\varphi_r(z-y)\,d\omega^p(y).$$

Note that, by Lemma 4.2.2,

$$\begin{aligned} |T_r\omega^p(x)| &\leq \left|\int \varphi(x-y)\nabla_2\mathcal{E}(y,x)\,d\omega^p(y)\right| \\ &\quad + \int \big|\chi_{|x-y|>r} - \varphi(x-y)\big|\,\big|\nabla_2\mathcal{E}(y,x)\big|\,d\omega^p(y) \\ &\quad + \int_{|x-y|>r} \big|\nabla_1\mathcal{E}(x,y) - \nabla_2\mathcal{E}(y,x)\big|\,d\omega^p(y) \\ &\leq \widetilde{T}_r\omega^p(x) + C\,M_n\omega^p(x) + \int \frac{C}{|x-y|^{n-\alpha}}\,d\omega^p(y) \\ &\leq \widetilde{T}_r\omega^p(x) + C\,M_n\omega^p(x). \end{aligned} \tag{3.12.4}$$

To estimate $\widetilde{T}_r\omega^p(x)$, first we assume that

$$\omega^p(B(x, 2\delta_0^{-1}r)) \leq 2\delta_0^{-(n+1)}\omega^p(B(x,2r)), \tag{3.12.5}$$

with δ_0 as in Lemma 3.12.2. For a fixed $x \in E_k$ and $z \in \mathbb{R}^{n+1} \setminus \big[\operatorname{supp}(\varphi_r(x-\cdot)\,\omega^p) \cup \{p\}\big]$, consider the function

$$u_r(z) = \mathcal{E}(p,z) - \int \mathcal{E}(y,z)\,\varphi_r(x-y)\,d\omega^p(y), \tag{3.12.6}$$

so that, by (3.12.1) and Lemma 3.12.3,

$$G(p,z) = u_r(z) - \int \mathcal{E}(y,z)\,\psi_r(x-y)\,d\omega^p(y) \quad \text{for } \mathcal{L}^{n+1}\text{-a.e. } z \in \mathbb{R}^{n+1}. \tag{3.12.7}$$

Differentiating (3.12.6) with respect to z, we obtain

$$\nabla u_r(z) = \nabla_2\mathcal{E}(p,z) - \int \nabla_2\mathcal{E}(y,z)\,\varphi_r(x-y)\,d\omega^p(y).$$

In the particular case $z = x$ we get (using also the Hölder continuity of u_r)

$$\nabla u_r(x) = \nabla_2\mathcal{E}(p,x) - \widetilde{T}_r\omega^p(x),$$

and thus

$$\big|\widetilde{T}_r\omega^p(x)\big| \lesssim \frac{1}{\operatorname{dist}(p,\partial\Omega)^n} + |\nabla u_r(x)|. \tag{3.12.8}$$

Since u_r is L_{A^T}-harmonic in $\mathbb{R}^{n+1} \setminus \big[\operatorname{supp}(\varphi_r(x-\cdot)\,\omega^p) \cup \{p\}\big]$ (and so in $B(x,r)$) and A is Hölder continuous, using Moser's Harnack inequality, we have

$$|\nabla u_r(x)| \lesssim \frac{1}{r}\left(\fint_{B(x,r/2)} |u_r(z)|^2\,dz\right)^{1/2} \lesssim \frac{1}{r}\fint_{B(x,r)} |u_r(z)|\,dz. \tag{3.12.9}$$

From the identity (3.12.7) we deduce that

$$|\nabla u_r(x)| \lesssim \frac{1}{r} \fint_{B(x,r)} G(p,z)\,dz + \frac{1}{r} \fint_{B(x,r)} \int \mathcal{E}(y,z)\,\psi_r(x-y)\,d\omega^p(y)\,dz$$
$$=: I + II.$$

To estimate the term II we use Fubini and the fact that $\operatorname{supp}\psi_r \subset B(x,2r)$:

$$II \lesssim \frac{1}{r^{n+2}} \int_{y\in B(x,2r)} \int_{z\in B(x,r)} \frac{1}{|z-y|^{n-1}}\,dz\,d\omega^p(y) \tag{3.12.10}$$
$$\lesssim \frac{\omega^p(B(x,2r))}{r^n} \lesssim M_n\omega^p(x). \tag{3.12.11}$$

We want to show now that $I \lesssim_k 1$. Clearly it is enough to show that

$$\frac{1}{r}\,|G(p,y)| \lesssim_k 1 \qquad \text{for all } y \in B(x,r)\cap\Omega \tag{3.12.12}$$

(still under the assumptions $x \in E_k$, $0 < r \le r_0/2$, and (3.12.5)). To prove this, observe that by Lemma 3.12.1 (with $B = B(x,r)$, $a = 2\delta_0^{-1}$), for all $y \in B(x,r)\cap\Omega$, we have

$$\omega^p(B(x,2\delta_0^{-1}r)) \gtrsim \inf_{z\in B(x,2r)\cap\Omega} \omega^z(B(x,2\delta_0^{-1}r))\,r^{n-1}\,|G(p,y)|.$$

On the other hand, by Lemma 3.12.2 and (3.12.3), for any $z \in B(x,2r)\cap\Omega$ and $0 < r \le r_0/2$,

$$\omega^z(B(x,2\delta_0^{-1}r)) \gtrsim \frac{\mathcal{H}^n_\infty(B(x,2r)\cap\partial\Omega)}{r^n} \gtrsim C(k)\frac{\omega^p(B(x,2r))}{r^n}.$$

Therefore we have

$$\omega^p(B(x,2\delta_0^{-1}r)) \gtrsim C(k)\frac{\omega^p(B(x,2r))}{r^n}\,r^{n-1}\,|G(p,y)|,$$

and thus, by (3.12.5),

$$\frac{1}{r}\,|G(p,y)| \lesssim_k \frac{\omega^p(B(x,2\delta_0^{-1}r))}{\omega^p(B(x,2r))} \lesssim_k 1,$$

which proves (3.12.12). So we deduce that

$$|\widetilde{T}_r\omega^p(x)| \lesssim_k \frac{1}{\operatorname{dist}(p,\partial\Omega)^n} + 1 \tag{3.12.13}$$

for $x \in E_k$ and $0 < r \le r_0/2$ satisfying (3.12.5).

In the case where (3.12.5) does not hold, we consider the largest $s > 0$ of the form $s = 2\delta_0^j r$, $j > 0$, such that (3.12.5) holds with s replacing r. By standard methods from non-doubling Calderón-Zygmund theory, it follows that such s exists for ω^p-a.e. $x \in E_k$ and moreover

$$|\widetilde{T}_r\omega^p(x)| \le |\widetilde{T}_s\omega^p(x)| + C\,M_n\omega^p(x).$$

See, for example, Lemmas 2.8 and 2.20 from [Tol14]. Then, applying (3.12.13) with $r = s$, we infer that

$$|\widetilde{T}_r\omega^p(x)| \lesssim_k \frac{1}{\text{dist}(p,\partial\Omega)^n} + 1 + M_n\omega^p(x) \lesssim_k \frac{1}{\text{dist}(p,\partial\Omega)^n} + 1.$$

So in any case we deduce that $|\widetilde{T}_r\omega^p(x)|$ is bounded uniformly for ω^p-a.e. $x \in E_k$ and r small enough. By (3.12.4), this implies that the same holds for $|T_r\omega^p(x)|$, and thus it follows that $T_*\omega^p(x) < \infty$ for ω^p-a.e. $x \in E_k$, and so for ω^p-a.e. $x \in E$, as wished. □

From this lemma and (3.2.5) we deduce that the antisymmetric operator $T^{(a)}$ satisfies

$$T_*^{(a)}\omega^p(x) \le M_n\omega^p(x) + T_*\omega^p(x) < \infty.$$

Next we apply the following *Tb* type theorem due to Nazarov, Treil and Volberg [NTV02], [Vol03] in combination with the methods in [Tol00]. For the detailed proof in the case of the Cauchy transform, see [Tol14, Theorem 8.13]. The same arguments with very minor modifications work for antisymmetric operators.

Theorem 3.5. *Let σ be a Radon measure with compact support on $\mathbb{R}^{n+1}$ and consider a σ-measurable set G with $\sigma(G) > 0$ such that*

$$G \subset \{x \in \mathbb{R}^{n+1} : M_n\sigma(x) < \infty \text{ and } T_*^{(a)}\sigma(x) < \infty\}.$$

Then there exists a Borel subset $G_0 \subset G$ with $\sigma(G_0) > 0$ such that $\sup_{x\in G_0} M_n\sigma|_{G_0}(x) < \infty$ and $T^{(a)}_{\sigma|_{G_0}}$ is bounded in $L^2(\sigma|_{G_0})$.

Applying this theorem to the measure $\sigma = \omega^p$ and the set $G = E$, we infer that there exists a subset $G_0 \subset E$ with $\omega^p(E) > 0$ such that $T^{(a)}_{\omega^p|_{G_0}}$ is bounded in $L^2(\omega^p|_{G_0})$. Then, by Lemma 3.2.5 it turns out that $T_{\omega^p|_{G_0}}$ is also bounded in $L^2(\omega^p|_{G_0})$. Since ω^p is absolutely continuous with respect to $\mathcal{H}^n$ on G_0, by applying Theorem 3.3 we deduce that G_0 is n-rectifiable. Now, by a standard exhausting argument we deduce that ω^p is concentrated in an n-rectifiable set and thus ω^p is n-rectifiable.

Chapter 4

Gradient of the single layer potential and quantitative rectifiability for general Radon measures

4.1 Introduction

In the work [PPT18] reported in the previous chapter, Laura Prat, Xavier Tolsa and the author dealt with the connection between rectifiability and the boundedness of the gradient of the single layer potential. This operator plays a central role in the study of partial differential equations. Our goal is to investigate the nature of the gradient of the single layer potential for certain elliptic operators and apply the results to the study of elliptic measure.

An elliptic equivalent of the so-called David-Semmes problem in codimension 1 was considered in [PPT18], under the assumption of Hölder continuity of the coefficients of the matrix defining a differential operator in divergence form. The case of the codimension 1 Riesz transform was studied in the deep works of Mattila, Melnikov and Verdera in the plane and by Nazarov, Tolsa and Volberg for higher dimensions (see [MMV96] and [NTV14a]). We remark that the David-Semmes problem for higher codimensions is still unsolved.

In the same spirit of [PPT18], the aim of the present article is to establish an elliptic equivalent of a quantitative rectifiability theorem that Girela-Sarrión and Tolsa proved for the Riesz transfom in [GT18].

Let μ be a Radon measure on $\mathbb{R}^{n+1}$. Its associated n-dimensional Riesz transform is

$$\mathcal{R}^n_\mu f(x) = \int \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mu(y), \qquad f \in L^1_{\mathrm{loc}}(\mu),$$

whenever the integral makes sense. Given $x \in \mathbb{R}^{n+1}$ and $r > 0$, we denote by $B(x,r)$ the open ball of center x and radius r. A Radon measure μ has growth of degree n if there exists a constant $C > 0$ such that

$$\mu\big(B(x,r)\big) \leq C r^n \qquad \text{for all } x \in \mathbb{R}^{n+1},\ r > 0.$$

We call μ n-Ahlfors-David regular (also abbreviated by n-AD-regular or just AD-regular) if there exists some constant $C > 0$, also referred to as an AD-regularity constant, such that

$$C^{-1} r^n \leq \mu\big(B(x,r)\big) \leq C r^n \qquad \text{for all } x \in \operatorname{supp}\mu,\ 0 < r < \operatorname{diam}(\operatorname{supp}\mu).$$

A set $E \subset \mathbb{R}^{n+1}$ is said n-AD-regular if $\mathcal{H}^n|_E$ is a n-AD-regular measure, $\mathcal{H}^n$ denoting the n-dimensional Hausdorff measure in $\mathbb{R}^{n+1}$. Note that the support of an n-AD-regular measure is n-AD-regular.

A set $E \subset \mathbb{R}^{n+1}$ is called n-rectifiable if there exists a countable family of Lipschitz functions $f_j \colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ such that

$$\mathcal{H}^n\Big(E \setminus \bigcup_j f_j(\mathbb{R}^n)\Big) = 0.$$

A measure μ is rectifiable if it vanishes outside a rectifiable set E and, moreover, it is absolutely continuous with respect to $\mathcal{H}^n|_E$.

David and Semmes introduced the quantitative version of the notion of rectifiability, which is important because of its relations with singular integrals. A set E is called n-uniformly rectifiable (or just uniformly rectifiable) if it is n-AD regular and there exist $\theta, M > 0$ such that for all $x \in E$ and all $r > 0$ there is a Lipschitz mapping g from the ball $B_n(0,r) \subset \mathbb{R}^n$ to $\mathbb{R}^{n+1}$ with $\operatorname{Lip}(g) \leq M$ such that

$$\mathcal{H}^n\big(E \cap B(x,r) \cap g(B_n(0,r))\big) \geq \theta r^n.$$

We say that a measure μ is n-uniformly rectifiable if it is n-AD-regular and it vanishes out of a n-uniformly rectifiable set.

Many characterizations of uniformly rectifiable measures are present in the literature. In particular, if the measure is n-AD-regular, then it is n-uniformly rectifiable if and only if its associated n-Riesz transform is bounded on L^2 (see [DS91], [MMV96] and [NTV14a]).

This fact plays a crucial role in the study of the geometric properties of harmonic measure. In particular, it was used in [Azz+16b] to prove that the mutual absolute continuity of the the harmonic measure for an open set $\Omega \subset \mathbb{R}^{n+1}$ with respect to surface measure $\mathcal{H}^n$ in a subset of $\partial\Omega$ implies the n-rectifiability of that subset. This answered a problem raised by Bishop (see [Bis92]).

The analogous result for elliptic measure has been proved in [PPT18], following the ideas of [Azz+16b], as an application of the characterization of uniform rectifiability via the boundedness of the gradient of single layer potential.

Another question proposed by Bishop asks whether, given two disjoint domains $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$, mutual absolute continuity of their respective harmonic measures implies absolute continuity with respect to surface measure in $\partial\Omega_1 \cap \partial\Omega_2$ and rectifiability.

This is a so-called *two phase problem* for harmonic measure and was eventually solved in its full generality in [Azz+16d]. This work relies on three main tools: a blow-up argument for harmonic measure (see also [KPT09] and [TV18b]), a monotonicity formula ([ACF84]) and a quantitative rectifiability criterion (see [GT18]).

In particular, we point out that the theorem by Girela-Sarrión and Tolsa served to overcome some intrinsic technical issue in the formulation of the problem and it can be interpreted as an adapted version of previous results by David and Léger, which were formulated in terms of the so-called Menger curvature of a measure (see [Dav98] and [Lég99]). Their theorem is of fundamental importance also in other two-phase problems examined in [AMT17a] and the very recent work [PT19]. The goal of the present chapter is to encounter an analogue criterion in the context of elliptic PDE's in divergence form with Hölder coefficients.

Let $A = (a_{ij})_{1\leq i,j\leq n+1}$ be an $(n+1)\times(n+1)$ matrix whose entries $a_{ij}\colon \mathbb{R}^{n+1}\to\mathbb{R}$ are measurable functions in $L^\infty(\mathbb{R}^{n+1})$. Assume also that there exists $\Lambda>0$ such that

$$\Lambda^{-1}|\xi|^2 \leq \langle A(x)\xi,\xi\rangle, \quad \text{for all } \xi\in\mathbb{R}^{n+1} \text{ and a.e. } x\in\mathbb{R}^{n+1}, \tag{4.1.1}$$
$$\langle A(x)\xi,\eta\rangle \leq \Lambda|\xi||\eta|, \quad \text{for all } \xi,\eta\in\mathbb{R}^{n+1} \text{ and a.e. } x\in\mathbb{R}^{n+1}. \tag{4.1.2}$$

We consider the elliptic equation

$$L_A u(x) := -\mathrm{div}\,(A(\cdot)\nabla u(\cdot))\,(x) = 0, \tag{4.1.3}$$

which should be understood in the distributional sense. We say that a function $u\in W^{1,2}_{\mathrm{loc}}(\Omega)$ is a *solution* of (4.1.3), or L_A-*harmonic*, in an open set $\Omega\subset\mathbb{R}^{n+1}$ if

$$\int A\nabla u\cdot\nabla\varphi = 0, \quad \text{for all } \varphi\in C_c^\infty(\Omega).$$

We denote by $\mathcal{E}_A(x,y)$, or just by $\mathcal{E}(x,y)$ when the matrix A is clear from the context, the *fundamental solution* for L_A in $\mathbb{R}^{n+1}$, so that $L_x\mathcal{E}_A(x,y)=\delta_y$ in the distributional sense, where δ_y is the Dirac mass at the point $y\in\mathbb{R}^{n+1}$. For a construction of the fundamental solution under the assumptions (4.1.1) and (4.1.2) on the matrix A we refer to [HK07]. Given a measure μ, the function $f(x)=\int\mathcal{E}_A(x,y)\,d\mu(y)$ is usually known as the *single layer potential* of μ. We define

$$K(x,y) = \nabla_1\mathcal{E}_A(x,y), \tag{4.1.4}$$

the subscript 1 indicating that we take the gradient with respect to the first variable, and we consider (4.1.4) as the kernel of the singular integral operator

$$T\mu(x) = \int K(x,y)\,d\mu(y),$$

for x away from $\mathrm{supp}(\mu)$. Observe that $T\mu$ is the gradient of the single layer potential of μ.

Given a function $f\in L^1_{loc}(\mu)$, we set also

$$T_\mu f(x) = T(f\,\mu)(x) = \int K(x,y)f(y)\,d\mu(y),$$

and, for $\varepsilon>0$, we consider the ε-truncated version

$$T_\varepsilon\mu(x) = \int_{|x-y|>\varepsilon} K(x,y)\,d\mu(y).$$

We also write $T_{\mu,\varepsilon}f(x) = T_\varepsilon(f\mu)(x)$. We say that the operator T_μ is bounded on $L^2(\mu)$ if the operators $T_{\mu,\varepsilon}$ are bounded on $L^2(\mu)$ uniformly on $\varepsilon>0$.

In the specific case when A is the identity matrix, $-L_A=\Delta$ and T is the n-dimensional Riesz transform up to a dimensional constant factor. We say that the matrix A is Hölder continuous with exponent $\alpha\in(0,1)$ (or briefly C^α continuous), if there exists $C_h>0$ such that

$$|a_{ij}(x)-a_{ij}(y)| \leq C_h|x-y|^\alpha \quad \text{for all } x,y\in\mathbb{R}^{n+1} \text{ and } 1\leq i,j\leq n+1. \tag{4.1.5}$$

Under this assumption on the coefficients, the kernel $K(\cdot,\cdot)$ turns out to be locally of Calderón-Zygmund type (see Lemma 4.2.1 for more details). However we remark that, contrarily to what happens in the case of the kernel of the Riesz transform, in general $K(\cdot,\cdot)$ is neither homogeneous nor antisymmetric (not even locally).

For our applications, it is useful to determine whether $T_{\mu,\varepsilon} f$ converges pointwise μ-almost everywhere for $\varepsilon \to 0$. In case it does, we denote the limit as

$$\mathrm{p.\,v}\, T_\mu f(x) = \lim_{\varepsilon\to 0} T_{\mu,\varepsilon} f(x)$$

and we call it the *principal value* of the integral $T_\mu f(x)$. One can prove the existence of the principal values for general Radon measures with compact support under the additional assumption of $L^2(\mu)$-boundedness of T_μ. In particular, our first result is the following.

Theorem 4.1. *Let μ be a Radon measure on $\mathbb{R}^{n+1}$ with compact support and with growth of degree n, i.e. suppose that there is $C > 0$ such that*

$$\mu(B(x,r)) \le C r^n \quad \textit{for all} \;\; x \in \mathbb{R}^{n+1}.$$

Let A be a matrix that satisfies (4.1.1), (4.1.2) and (4.1.5) and assume, moreover, that the gradient of the single layer potential T_μ associated with L_A is bounded on $L^2(\mu)$. Then:

1. *for $1 \le p < \infty$ and all $f \in L^p(\mu)$, $\mathrm{p.\,v}\, T_\mu f(x)$ exists for μ-a.e. $x \in \mathbb{R}^{n+1}$;*
2. *for all $\nu \in M(\mathbb{R}^{n+1})$, $\mathrm{p.\,v}\, T\nu(x)$ exists for μ-a.e. $x \in \mathbb{R}^{n+1}$.*

If $A \equiv Id$, Theorem 4.1 reduces to its analogous for the Riesz transform (see for example [Tol14, Chapter 8]). In light of this result, in the rest of the chapter we will often denote the principal value operator simply as $T\nu$ with abuse of notation.

Given a ball $B = B(x,r) \subset \mathbb{R}^{n+1}$, we denote by $r(B)$ its radius and, for $a > 0$, by aB its dilation $B(x,ar)$. Multiple notions of density come into play in this chapter. For a ball B, we denote

$$\Theta_\mu(B) = \frac{\mu(B)}{r(B)^n}$$

and, for $\gamma > 0$, its smoothened version

$$P_{\mu,\gamma}(B) := \sum_{j\ge 0} 2^{-j\gamma}\Theta_\mu(2^j B). \tag{4.1.6}$$

We remark that if $\gamma_1 \le \gamma_2$, then

$$P_{\mu,\gamma_2}(B) = \sum_{j\ge 0} 2^{-j\gamma_2}\Theta_\mu(2^j B) \le \sum_{j\ge 0} 2^{-j\gamma_1}\Theta_\mu(2^j B) = P_{\mu,\gamma_1}(B).$$

Another notion of density that we need is the pointwise one. In particular, we denote the upper and lower n-densities of μ at x respectively as

$$\Theta^*_\mu(x) := \limsup_{r\to 0} \frac{\mu\big(B(x,r)\big)}{(2r)^n} \quad \text{and} \quad \Theta_{*,\mu}(x) := \liminf_{r\to 0} \frac{\mu\big(B(x,r)\big)}{(2r)^n}.$$

A way to quantify the flatness of a measure at the level of a ball B is in terms of the β-coefficients. For an n-plane L we denote

$$\beta^L_{\mu,1}(B) = \frac{1}{r(B)^n}\int_B \frac{\mathrm{dist}(x,L)}{r(B)}\,d\mu(x) \qquad \text{and} \qquad \beta_{\mu,1}(B) = \inf_L \beta^L_{\mu,1}(B),$$

the infimum being taken over all hyperplanes in $\mathbb{R}^{n+1}$. Using a standard notation, given $E \subset \mathbb{R}^{n+1}$ with $\mu(E) > 0$ and $f \in L^1_{loc}(\mu)$ we write

$$m_{\mu,E}(f) = \frac{1}{\mu(E)} \int_E f d\mu$$

for the mean of f with respect to the measure μ on the set E. The main result of the chapter is the following.

Theorem 4.2. *Let $n > 1$, let μ be a Radon measure on $\mathbb{R}^{n+1}$ with compact support and consider an open ball $B \subset \mathbb{R}^{n+1}$. Let $C_0, C_1 > 0$ and let A be a matrix satisfying (4.1.1), (4.1.2) and (4.1.5). Denote by T_μ the gradient of the single layer potential associated with L_A and μ. Suppose that μ and B are such that, for some positive λ, δ and ϵ and some $\tilde{\alpha} \in (0,1)$, the following properties hold*

1. *$r(B) \leq \lambda$.*
2. *$C_0^{-1} r(B)^n \leq \mu(B) \leq C_0 r(B)^n$.*
3. *$P_{\mu,\tilde{\alpha}}(B) \leq C_0$ and for all $x \in B$ and $0 < r \leq r(B)$ we have $\mu\big(B(x,r)\big) \leq C_0 r^n$.*
4. *$T_{\mu|_B}$ is bounded on $L^2(\mu|_B)$ with $\|T_{\mu|_B}\|_{L^2(\mu|_B)\to L^2(\mu|_B)} \leq C_1$ and $T(\chi_{2B}\mu) \in L^2(\mu|_B)$.*
5. *$\beta_{\mu,1}(B) \leq \delta$.*
6. *We have*
$$\int_B \big|T\mu(x) - m_{\mu,B}(T\mu)\big|^2 d\mu(x) \leq \epsilon\mu(B).$$

There exists a choice of λ, δ and ϵ small enough and a proper choice of $\tilde{\alpha} = \tilde{\alpha}(\alpha, n)$, all possibly depending on C_0 and C_1, such that if μ satisfies (1)–$\cdots$–(6), there exists a n-uniformly rectifiable set Γ that covers a big portion of the support of μ inside B. That is to say, there exists $\tau > 0$ such that

$$\mu(B \cap \Gamma) \geq \tau\mu(B).$$

Notice that Theorem 4.2 immediately implies that a big piece of $\mu|_B$ is mutually absolutely continuous with a big piece of $\mathcal{H}^n|_\Gamma$. This is a relevant feature in light of possible applications, in particular to elliptic measure.

Our proof of the theorem shows that a good choice for $\tilde{\alpha}$ is $\tilde{\alpha} = \alpha/2^{n+1}$. It is not clear whether Theorem 4.2 holds with a condition on $P_{\mu,\alpha}(B)$, that would be a more natural homogeneity to assume. We remark that the integral in the left hand side of the assumption (6) makes sense because of the existence of principal values ensured by Theorem 4.1 and the hypothesis $P_{\mu,\alpha}(B) < +\infty$. For a sketch of the argument we refer to the end of Section 4.3.

The main conceptual difference with respect to the analogous theorem for the Riesz transform in [GT18] is that we need to require the ball B to be small enough. The locality of our result reflects the non-scale invariant character of the Hölder regularity assumption for the coefficients of the matrix A. This issue is evident also in [PPT18] and it is not clear how to overcome this difficulty without making further assumptions on the matrix.

Another difference is that we could not formulate the theorem in terms of $P_{\mu,1}$. The proofs of the rectifiability results for the harmonic measure in [AMT17b] and [Azz+16d] actually rely on the fact that the theorem of Girela-Sarrión and Tolsa holds for $\tilde{\alpha} = 1$. However, a slight variation on their arguments allows to overcome this technical obstacle. We close the introduction by presenting an application of Theorem 4.2, which is, in fact, its main motivation.

Before stating it, recall that if Ω is a Wiener regular set, the elliptic measure $\omega^p_{L_A}$ with pole at p associated with the elliptic operator L_A is the probability measure supported on $\partial\Omega$ such that, for $f \in C_0(\partial\Omega)$,

$$\int f d\omega^p_{L_A} = \tilde{f}(p),$$

where $\tilde{f}$ denotes the L_A-harmonic extension of f. A large literature is available on the subject. For example, we refer to [HKM06] and [Ken92] for its definition and basic properties.

Theorem 4.3. *Let $n \geq 2$ and let A be an elliptic matrix satisfying (4.1.1), (4.1.2) and (4.1.5). Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ be two Wiener-regular domains and, for $p_i \in \Omega_i$, $i \in \{1,2\}$, let $\omega^{p_i}_{L_A,i}$ be the respective elliptic measures in Ω_i associated with L_A and with pole p_i. Suppose that E is a Borel set such that $\omega^{p_1}_{L_A,1}|_E \ll \omega^{p_2}_{L_A,2}|_E \ll \omega^{p_1}_{L_A,1}|_E$. Then there exists an n-rectifiable set $F \subset E$ with $\omega^{p_1}_{L_A,1}(E \setminus F) = 0$ such that $\omega^{p_1}_{L_A,1}|_F$ and $\omega^{p_2}_{L_A,2}|_F$ are mutually absolutely continuous with respect to $\mathcal{H}^n|_F$.*

We remark that the generalization of the blow-up methods for the harmonic measure to our elliptic context is contained in the work [AM18]. Also, the proof of Theorem 4.3 follows closely the path of the work [Azz+16d]. However, some variations are needed so that we decided to sketch the proof at the end of the chapter, where we also provide precise references for the reader's convenience.

We finally remark that recently several studies have appeared concerning the connection between the geometry of a domain and the properties of its associated elliptic measure, among which we list [Akm+17], [Azz+16a], [Hof+15], [HMT10], [HMT] and [Ken+16].

The structure of the chapter

Section 4.2 is devoted to settle our notation and to make an overview of the results in PDE's relevant for our work. In particular, we need some estimate for the gradient of the fundamental solution coming from homogenization theory.

In *Section 4.3* we prove Theorem 4.1.

Section 4.4 contains the statement of the Main Lemma that we use to prove Theorem 4.2. The biggest advantage of the formulation of this lemma with respect to the one of the main theorem is that the flatness condition on the β_1-number is replaced by an hypothesis on the α-numbers. The latter are more powerful when trying to transfer the flatness estimates to the integrals.

In *Section 4.5* we discuss an equivalent formulation of the Main Lemma in terms of an auxiliary elliptic operator which shares more symmetries than L_A. This is a novelty of the elliptic case, this issue not being present in the work of Girela-Sarrión and Tolsa.

The *Sections 4.6, 4.7, 4.8 and 4.9* follow the path of the original work for the Riesz transform, with some minor variations. They are necessary for expository reasons; indeed, they present the core of the contradiction argument for the proof of the Main Lemma and the construction of a periodic auxiliary measure.

Section 4.10 consists of the proof of two crucial results: the existence of the limit of proper smooth truncates of the potential of bounded periodic functions and a localization estimate for the potential close to a cube. We emphasize that these proofs rely on the periodicity of the modification of the elliptic matrix.

In *Section 4.11* we complete the proof of the Main Lemma via a variational technique. We highlight that one of the most delicate point consists in finding an appropriate variant of a maximum principle in an infinite strip in our elliptic setting. Our argument heavily exploits the additional symmetries provided by the modified matrix.

In the final *Section 4.12*, we present the application of the main rectifiability theorem to the study of elliptic measure, sketching the proof of Theorem 4.3.

4.2 Preliminaries and notation

It is useful to write $a \lesssim b$ to denote that there is a constant $C > 0$ such that $a \leq Cb$. To make the dependence of the constant on a parameter t explicit, we will write $a \lesssim_t b$. Also, we say that $b \gtrsim a$ if $a \lesssim b$ and $a \approx b$ if both $a \lesssim b$ and $b \lesssim a$.

All the cubes, unless specified, will be considered with their sides parallel to the coordinate axes. Given a cube Q, we denote its side length as $\ell(Q)$ and, for $a > 0$, we understand aQ as the cube with side length $a\ell(Q)$ and sharing the center with Q.

We say that a cube Q has t-thin boundary if

$$\mu\big\{x \in 2Q : \operatorname{dist}(x, \partial Q) \leq \lambda\ell(Q)\big\} \leq t\lambda\mu(2Q)$$

for every $\lambda > 0$. Analogously to (4.1.6), we define

$$P_{\mu,\gamma}(Q) = \sum_{j\geq 0} 2^{-j\gamma}\Theta_\mu(2^jQ) = \sum_{j\geq 0} 2^{-j\gamma}\frac{\mu(2^jQ)}{\ell(2^jQ)^n}.$$

Given a measure μ and a measurable set E, we denote as $\mu|_E$ the restriction of μ to E and, for $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, we use the notation $\phi_\sharp\mu(E) := \mu(\phi^{-1}(E))$. An important tool in the study of rectifiabilty is the so-called α-number introduced by Tolsa in [Tol09]. Let us fix a cube $Q \subset \mathbb{R}^{n+1}$ and consider two Radon measures μ and ν on $\mathbb{R}^{n+1}$. A natural way to define a distance between μ and ν is to consider the supremum

$$d_Q(\mu,\nu) := \sup_f \int f\,d(\mu-\nu), \tag{4.2.1}$$

where $f \in \operatorname{Lip}(\mathbb{R}^{n+1})$, $\|f\|_{\operatorname{Lip}} \leq 1$ and $\operatorname{supp} f \subseteq Q$. The distance d_Q offers a way of quantifying the "flatness" of a measure alternative to that via β_1-numbers. More precisely, if we consider a n-plane L in $\mathbb{R}^{n+1}$, we can define

$$\alpha_\mu^L(Q) := \frac{1}{\ell(Q)^{n+1}} \inf_{c\geq 0} d_Q(\mu, c\mathcal{H}^n|_L). \tag{4.2.2}$$

Given a matrix $A(\cdot)$, possibly with variable coefficients, we use the notation $A^T(\cdot)$ to indicate its transpose. Also, we write $\mathcal{L}^{n+1}$ for the Lebesgue measure on $\mathbb{R}^{n+1}$.

Partial Differential Equations. For any uniformly elliptic matrix A with Hölder continuous coefficients, one can show that $K(x,y) = \nabla_1\mathcal{E}(x,y)$ is locally a Calderón-Zygmund kernel.

Lemma 4.2.1. *Let A be an elliptic matrix with Hölder continuous coefficients satisfying (4.1.1), (4.1.2) and (4.1.5). If $K(\cdot,\cdot)$ is given by (4.1.4), then it is locally a Calderón-Zygmund kernel. That is, for any given $R > 0$,*

(a) $|K(x,y)| \lesssim |x-y|^{-n}$ for all $x, y \in \mathbb{R}^{n+1}$ with $x \neq y$ and $|x-y| \leq R$.

(b) $|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \lesssim |y-y'|^\alpha|x-y|^{-n-\alpha}$ for all $y, y' \in B(x,R)$ with $2|y-y'| \leq |x-y|$.

(c) $|K(x,y)| \lesssim |x-y|^{(1-n)/2}$ *for all* $x,y \in \mathbb{R}^{n+1}$ *with* $|x-y| \geq 1$.
All the implicit constants in (a), (b) and (c) depend on Λ *and* $\|A\|_\alpha$*, while the ones in (a) and (b) depend also on* R.

The statements above are rather standard. For more details, see Lemma 2.1 from [CMT19].

Let ω_n denote the surface measure of the unit sphere of $\mathbb{R}^{n+1}$. For any elliptic matrix A_0 with constant coefficients, we have an explicit expression for the fundamental solution of L_{A_0}, which we denote by $\Theta(x,y;A_0)$. More precisely, $\Theta(x,y;A_0) = \Theta(x-y;A_0)$ with

$$\Theta(z;A_0) = \Theta(z;A_{0,s}) = \begin{cases} \dfrac{-1}{(n-1)\omega_n\sqrt{\det A_{0,s}}}\dfrac{1}{(A_{0,s}^{-1}z\cdot z)^{(n-1)/2}} & \text{for } n \geq 2, \\[2ex] \dfrac{1}{4\pi\sqrt{\det A_{0,s}}}\log\left(A_{0,s}^{-1}z\cdot z\right) & \text{for } n=1, \end{cases} \tag{4.2.3}$$

where $A_{0,s}$ is the symmetric part of A_0, that is, $A_{0,s} = \frac{1}{2}(A_0 + A_0^T)$.

The reason why only the symmetric part of A_0 enters (4.2.3) it that, using Schwarz's theorem to exchange the order of partial derivatives writing $A_0 = \{a_{ij}\}_{i,j}$, for every appropriate function u we have

$$\begin{aligned} L_{A_0}u &= -\sum_{i,j}\partial_i(a_{ij}\partial_j u) \\ &= -\frac{1}{2}\sum_{i,j}a_{ij}\partial_i\partial_j u - \frac{1}{2}\sum_{i,j}a_{ij}\partial_j\partial_i u \\ &= -\sum_{i,j}\frac{a_{ij}+a_{ji}}{2}\partial_i\partial_j u = L_{A_{0,s}}u. \end{aligned} \tag{4.2.4}$$

These formal considerations can be made rigorous by standard arguments.

Differentiating (4.2.3) we have

$$\nabla\Theta(z;A_0) = \frac{1}{\omega_n\sqrt{\det A_{0,s}}}\frac{A_{0,s}^{-1}z}{(A_{0,s}^{-1}z\cdot z)^{(n+1)/2}}.$$

The next result is proven in [KS11, Lemma 2.2].

Lemma 4.2.2. *Let* A *be an elliptic matrix with Hölder continuous coefficients satisfying (4.1.1), (4.1.2) and (4.1.5). Let also* $\Theta(\cdot,\cdot;\cdot)$ *be given by (4.2.3). Then, for* $x,y \in \mathbb{R}^{n+1}$, $0 < |x-y| \leq R$,

1. $|\mathcal{E}_A(x,y) - \Theta(x,y;A(x))| \lesssim |x-y|^{\alpha-n+1}$,
2. $|\nabla_1\mathcal{E}_A(x,y) - \nabla_1\Theta(x,y;A(x))| \lesssim |x-y|^{\alpha-n}$,
3. $|\nabla_1\mathcal{E}_A(x,y) - \nabla_1\Theta(x,y;A(y))| \lesssim |x-y|^{\alpha-n}$.

Similar inequalities hold if we reverse the roles of x *and* y *and we replace* ∇_1 *by* ∇_2. *All the implicit constants depend on* Λ, $\|A\|_\alpha$, *and* R.

The gradient of the fundamental solution in the periodic case. We denote as Λ_α the set of matrices such that (4.1.1), (4.1.2) hold and with α-Hölder coefficients. We say that the matrix $A \in \Lambda_\alpha$ is ℓ-periodic, $\ell > 0$, if

$$A(x+\ell z) = A(x) \quad \text{for every } z \in \mathbb{Z}^{n+1}.$$

For periodic matrices the estimates in Lemma 4.2.1 turn out to be global.

Lemma 4.2.3 ([KS11])**.** *Let $A \in \Lambda_\alpha$ be 1-periodic and let $\mathcal{E}_A$ be the fundamental solution of L_A. Let $K(\cdot,\cdot)$ is given by (4.1.4). Then*

1. $|\nabla_1 \mathcal{E}_A(x,y)| \le c_1 |x-y|^{-n}$ *for every $x, y \in \mathbb{R}^{n+1}$ with $x \neq y$.*
2. $|\nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \mathcal{E}_A(x',y)| + |\nabla_1 \mathcal{E}_A(y,x) - \nabla_1 \mathcal{E}_A(y,x')| \le c_2 |x-x'|^\alpha |x-y|^{-(n+\alpha)}$ *for every $x, x', y \in \mathbb{R}^{n+1}$ such that $2|x-x'| \le |x-y|$.*

The constants appearing in (1) and (2) are such that $c_1 \approx_{n,\Lambda} c_2 \approx_{n,\Lambda} \|A\|_\alpha$.

The period of the matrix plays an important role in our construction, so it is useful to rephrase the previous lemma for matrices with a period different from 1. We are interested in studying matrices with small period, so we only consider the case in which it is strictly smaller than 1.

Lemma 4.2.4. *Let $0 < \ell < 1$. Let $A \in \Lambda_\alpha$ be ℓ-periodic and let $\mathcal{E}_A$ be the fundamental solution associated with L_A. Then*

1. $|\nabla_1 \mathcal{E}_A(x,y)| \le c_1' |x-y|^{-n}$ *for every $x, y \in \mathbb{R}^{n+1}$ with $x \neq y$.*
2. $|\nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \mathcal{E}_A(x',y)| + |\nabla_1 \mathcal{E}_A(y,x) - \nabla_1 \mathcal{E}_A(y,x')| \le c_2' |x-x'|^\alpha |x-y|^{-n-\alpha}$ *for every $x, x', y \in \mathbb{R}^{n+1}$ such that $2|x-x'| \le |x-y|$.*

The constants appearing in (1) and (2) are such that $c_1' \approx_{n,\Lambda} c_2' \approx_{n,\Lambda} \|A\|_\alpha$.

Proof. For $\ell \in (0,1)$ and all $x \in \mathbb{R}^{n+1}$ we define the rescaled matrix

$$\tilde{A}(x) := A(\ell x)$$

and we denote by $\tilde{\mathcal{E}}$ the fundamental solution of $L_{\tilde{A}}$. By the definition of fundamental solution, it is not difficult to see that

$$\nabla_1 \tilde{\mathcal{E}}(x,y) = \ell^n \nabla_1 \mathcal{E}_A(\ell x, \ell y) \qquad \text{for } x, y \in \mathbb{R}^{n+1}. \tag{4.2.5}$$

Moreover,

$$|\tilde{A}(x) - \tilde{A}(y)| = |A(\ell x) - A(\ell y)| \le \ell^\alpha \|A\|_\alpha |x-y|^\alpha \le \|A\|_\alpha |x-y|^\alpha,$$

so that $\|\tilde{A}\|_\alpha \le \|A\|_\alpha$. Applying Lemma 4.2.3 together with (4.2.5) we get

$$|\nabla_1 \mathcal{E}_A(x,y)| = \ell^{-n} |\nabla_1 \tilde{\mathcal{E}}(\ell^{-1}x, \ell^{-1}y)| \lesssim \ell^{-n} |\ell^{-1}x - \ell^{-1}y|^{-n} = |x-y|^{-n}$$

for any x, y and

$$\begin{aligned} |\nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \mathcal{E}_A(x',y)| &= \ell^{-n} |\nabla_1 \tilde{\mathcal{E}}(\ell^{-1}x, \ell^{-1}y) - \nabla_1 \tilde{\mathcal{E}}(\ell^{-1}x', \ell^{-1}y)| \\ &\lesssim \ell^{-n} \frac{|\ell^{-1}x - \ell^{-1}x'|^\alpha}{|\ell^{-1}x - \ell^{-1}y|^{n+\alpha}} = \frac{|x-x'|^\alpha}{|x-y|^{n+\alpha}}. \end{aligned}$$

for $2|x-x'| \le |x-y|$. The same estimate holds for $|\nabla_1 \mathcal{E}_A(y,x) - \nabla_1 \mathcal{E}_A(y,x')|$. □

The following is the (global) analogue of Lemma 4.2.2 in the 1-periodic setting.

Lemma 4.2.5. *Let $A \in \Lambda_\alpha$ be 1-periodic. Then for every $x, y \in \mathbb{R}^{n+1}$, $x \neq y$, we have*

$$\begin{aligned} \big| \mathcal{E}_A(x,y) - \Theta(x,y;A(x)) \big| &\lesssim |x-y|^{\alpha-n+1} \\ \big| \nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \Theta(x,y;A(x)) \big| &\lesssim |x-y|^{\alpha-n} \\ \big| \nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \Theta(x,y;A(y)) \big| &\lesssim |x-y|^{\alpha-n}, \end{aligned}$$

the implicit constants depending on $\|A\|_\alpha$ *and* Λ*. Similar estimates hold if we replace* ∇_1 *by* ∇_2*.*

Let us now recall some result from elliptic homogenization. For more details we refer to the work by Avellaneda and Lin [AL91]. For this purpose, we need to recall the definition of vector of correctors χ and homogenized matrix A_0. Let $\ell > 0$ and let $A \in \Lambda_\alpha$ be a 1-periodic matrix. We will denote by $\chi(x) = (\chi^i(x))$, for $i \in \{1, \dots, n+1\}$ the vector of correctors, which is defined as the solution of the following cell problem

$$\begin{cases} L\chi = \operatorname{div} A, \\ \chi \text{ is 1-periodic,} \\ \int_{[0,1]^{n+1}} \chi(x)dx = 0, \end{cases} \tag{4.2.6}$$

where the first condition in (4.2.6) has to be understood in coordinates as

$$\sum_{i,j} \partial_{x^i} \big[a_{ij}\partial_{x^j}\chi^h\big](x) = -\sum_i \partial_{x^i} a_{ih}(x),$$

$(a_{ij})_{i,j}$ being the coefficients of the matrix A. An important fact is that that

$$\|\nabla\chi\|_\infty \le C,$$

the bound C depending only on n, α and $\|A\|_{C^\alpha}$. We remark that $\nabla\chi$ denotes the matrix with variable coefficients whose entries are $\partial_i \chi^j$ for $i, j = 1, \dots, n+1$. Now, if we consider the following family of elliptic operators

$$L_\epsilon \coloneqq \operatorname{div}\big(A(x/\epsilon)\nabla\,\cdot\big)$$

depending on the parameter $\epsilon > 0$, it can be proved that for any $f \in L^2(\mathbb{R}^{n+1})$, the solutions $u_\epsilon \in W^{1,2}(\mathbb{R}^{n+1})$ of

$$L_\epsilon u_\epsilon = \operatorname{div} f$$

converge weakly in $W^{1,2}(\mathbb{R}^{n+1})$ to a function u_0 as $\varepsilon \to 0$. This function solves the equation

$$L_0 u_0 \coloneqq \operatorname{div}(A_0 \nabla u_0) = \operatorname{div} f,$$

where A_0 is an elliptic matrix with constant coefficients usually called *homogenized matrix* (see, for example, [She18]).

Homogenization is a powerful tool to study the fundamental solution of an elliptic equation in divergence form whose associated matrix is periodic and has C^α coefficients. The main result that we will use is the following (see [AL91, Lemma 2] and [KS11, Lemma 2.5]).

Lemma 4.2.6. *Let* $A \in \Lambda_\alpha$*. Let us assume that* A *is* 1*-periodic. Then there exists* $\gamma \in (0, 1)$ *depending on* $\alpha, \|A\|_{C^\alpha}$ *and* n *such that*

$$\big|\,\mathcal{E}_A(x, y) - (Id + \nabla\chi(x))\Theta(x, y; A_0)\big| \lesssim \frac{c}{|x - y|^{n+\gamma-1}} \tag{4.2.7}$$

and

$$\big|\nabla_1\,\mathcal{E}_A(x, y) - (Id + \nabla\chi(x))\nabla_1\Theta(x, y; A_0)\big| \lesssim \frac{c}{|x - y|^{n+\gamma}}, \tag{4.2.8}$$

where Id *denotes the identity matrix and the implicit constants in* (4.2.7) *and* (4.2.8) *depend just on* n, α *and* $\|A\|_\alpha$*.*

The period of the coefficients of A plays a crucial role in these estimates. We will be dealing with matrices with periodicity different from 1, so we need a suitably adapted version of the previous lemma. Let $A \in \Lambda_\alpha$ be a ℓ-periodic matrix. Let us define the 1-periodic matrix

$$\tilde{A}(x) := A(\ell x)$$

for $x \in \mathbb{R}^{n+1}$ and let $\tilde{c}hi$ denote the vector of correctors associated with $\tilde{A}$ defined according to (4.2.6). For $\ell > 0$ we define

$$\chi_\ell(x) := \ell\,\tilde{\chi}\Big(\frac{x}{\ell}\Big).$$

Lemma 4.2.7. *Let $0 < \ell < 1$. Let $A \in \Lambda_\alpha$ be an ℓ-periodic matrix. Then there exists $\gamma \in (0,1)$ and $c > 0$, both depending just on n, α and $\|A\|_\alpha$ such that*

$$\big|\nabla_1 \mathcal{E}_A(x,y) - \nabla_1 \Theta(x,y;A(x))\big| \leq c\ell^\alpha |x-y|^{\alpha-n}, \tag{4.2.9}$$
$$\big|\nabla_2 \mathcal{E}_A(x,y) - \nabla_2 \Theta(x,y;A(y))\big| \leq c\ell^\alpha |x-y|^{\alpha-n}. \tag{4.2.10}$$
$$\big|\nabla_1 \mathcal{E}_A(x,y) - (Id + \nabla\chi_\ell(x))\nabla_1 \Theta(x,y;A_0)\big| \leq c\ell^\gamma |x-y|^{-n-\gamma}. \tag{4.2.11}$$

for every $x \neq y$.

Proof. Let $\tilde{\mathcal{E}}$ denote the fundamental solution of the operator $L_{\tilde{A}}$. As in (4.2.5), we have

$$\nabla_1 \mathcal{E}_A(x,y) = \ell^{-n}\nabla_1\tilde{\mathcal{E}}(x/\ell, y/\ell), \tag{4.2.12}$$

so an application of Lemma 4.2.5 gives

$$\begin{aligned}\big|\nabla_1 \mathcal{E}_A(x,y) &- \nabla_1 \Theta(x,y;A(x))\big| \\ &= \ell^{-n}|\nabla_1 \mathcal{E}_{\tilde{A}}(\ell^{-1}x, \ell^{-1}y) - \nabla_1\Theta(x,y;\tilde{A}(\ell^{-1}x))| \leq c\ell^\alpha |x-y|^{\alpha-n}.\end{aligned}$$

Using (4.2.8) and (4.2.12), we get

$$\begin{aligned}|\nabla_1 \mathcal{E}_A(x,y) &- (Id + \nabla\chi_\ell(x))\nabla_1\Theta(x,y;A_0)| \\ &= \ell^{-n}|\nabla_1\tilde{\mathcal{E}}(x/\ell, y/\ell) - (Id + \nabla\tilde{\chi}(x/\ell))\nabla_1\Theta(x/\ell, y/\ell; A_0)| \\ &\lesssim \frac{c\ell^{n+\gamma}}{\ell^n|x-y|^{n+\gamma}} = \frac{c\ell^\gamma}{|x-y|^{n+\gamma}},\end{aligned}$$

where c depends on n, α and $\|\tilde{A}\|_\alpha$, $\|\tilde{A}\|_\alpha \leq \|A\|_\alpha$. Inequality (4.2.10) follows as (4.2.9). □

4.3 The existence of principal values

The purpose of the present section is to prove Theorem 4.1. The proof of the existence of principal values can be divided into the study of two different cases: the case in which μ is a rectifiable measure and the one in which μ has zero n-density, i.e.

$$\lim_{r\to 0} \frac{\mu(B(x,r))}{r^n} = 0 \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^{n+1}. \tag{4.3.1}$$

Indeed, without providing the detailed argument, we recall that by means of [PPT18, Theorem 2] we can decompose a measure μ for which T_μ is bounded on $L^2(\mu)$ into the sum of a rectifiable measure and a measure with zero n-density almost everywhere.

4.3.1 Principal values for rectifiable measures with compact support

This subsection follows the scheme of [CMT19, Section 2.2]. The proof of the existence of principal values for T_μ if the measure μ is rectifiable and has compact support relies on the following result.

Theorem 4.4. *Let μ be a rectifiable measure. Let $K \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ be an odd kernel and homogeneous of degree $-n$, i.e. $K(x) = -K(-x)$ and $K(\lambda x) = \lambda^{-n} K(x)$. Assume, for some $M = M(n)$, the further regularity condition*

$$|\nabla_j K(x)| \lesssim_n C(j)|x|^{-n-j} \quad \text{for all } 0 \le j \le M \quad \text{and } x \in \mathbb{R}^{n+1} \setminus \{0\}. \tag{4.3.2}$$

Then the operator $T_{K,\mu}$ is bounded on $L^2(\mu)$ with operator norm

$$\|T_{K,\mu}\|_{L^2(\mu)\to L^2(\mu)} \lesssim_n \|K|_{\mathbb{S}^n}\|_{C^M(\mathbb{R}^{n+1})}. \tag{4.3.3}$$

Moreover, the principal value

$$T_{K,\mu} f(x) = \lim_{\varepsilon\to 0} \int_{|x-y|\ge\varepsilon} K(x-y) f(y) d\mu(y)$$

exists μ-almost everywhere.

The proof of the boundedness of $T_{K,\mu}$ is due to David and Semmes. The result on principal values was first proved imposing an analogous condition for all $j = 0, 1, 2, \dots$ (for a more detailed exposition we refer, for example, to [Mat95, Chapter 20]). We remark that it has been recently improved by Mas (see [Mas13, Corollary 1.6]).

The previous theorem together with a spherical harmonics expansion of the kernel is the key tool to prove the following result.

Lemma 4.3.1. *Let μ be an n-rectifiable measure. There exists $M = M(n)$ such that the following holds. Let $b(x,z)$ be odd in z and homogeneous of degree $-n$ in z, and assume $D^\alpha_z b(x,z)$ is continuous and bounded on $\mathbb{R}^{n+1} \times \mathbb{S}^n$, for any multi-index $|\alpha| \le M$. Then for every $f \in L^2(\mu)$, the limit*

$$Bf(x) = \lim_{\varepsilon\to\varepsilon} \int_{|x-y|>0} b(x, x-y) f(y) d\mu(y)$$

exists for μ-almost every x.

Proof. Let $\{\varphi_{j,l}\}_{j\ge1, 1\le l\le N_j}$ be an orthonormal basis of $L^2(\mathbb{S}^n)$ consisting of surface spherical harmonics of degree j. Recall that (see [AH12, (2.12)])

$$N_j = O(j^{n-1}), \qquad \text{for } j \gg 1. \tag{4.3.4}$$

Using the homogeneity assumption for $b(x,\cdot)$ and the orthonormal expansion, we write

$$\begin{aligned} b(x,z) = b\Big(x, \frac{z}{|z|}\Big)|z|^{-n} &= \sum_{j\ge1} \sum_{l=1}^{N_j} \langle b(x,\cdot), \varphi_{j,l}\rangle_{L^2(\mathbb{S}^n)} \varphi_{j,l}\Big(\frac{z}{|z|}\Big)|z|^{-n} \\ &= \sum_{j,l} b_{j,l}(x) \varphi_{j,l}\Big(\frac{z}{|z|}\Big)|z|^{-n}, \end{aligned} \tag{4.3.5}$$

where $b_{j,l}(x) := \langle b(x,\cdot), \varphi_{j,l}\rangle_{L^2(\mathbb{S}^n)}$. Since $b(x,\cdot)$ is an odd function and $\varphi_{2j,l}$ is even for every j, $b_{j,l}(x) \equiv 0$ for j even. Being b in $L^\infty(\mathbb{R}^{n+1} \times \mathbb{S}^n)$ by hypothesis and Hölder's inequality, we have

$$|b_{j,l}(x)| \leq C(n)\|b(x,\cdot)\|_{L^\infty(\mathbb{S}^n)}\|\varphi_{j,l}\|_{L^2(\mathbb{S}^n)} \leq C(n)\|b\|_{L^\infty(\mathbb{R}^{n+1}\times\mathbb{S}^n)} \leq C(n). \tag{4.3.6}$$

Moreover, recalling that we can suppose j odd, the function $\tilde{K}_{j,l}(z) := \varphi_{j,l}\big(z/|z|\big)|z|^{-n}$ satisfies the hypothesis in Theorem 4.4: there exists an harmonic polynomial $P_{j,l}$ of odd degree j such that $\varphi_{j,l}(z/|z|) = P_{j,l}(z)/|z|^j$, so

$$\Big|\nabla\varphi_{j,l}\Big(\frac{z}{|z|}\Big)\Big| \lesssim \frac{1}{|z|}$$

and

$$\big|\nabla\tilde{K}_{j,l}(z)\big| \lesssim \Big|\nabla\varphi_{j,l}\Big(\frac{z}{|z|}\Big)\Big|\frac{1}{|z|^n} + \Big|\varphi_{j,l}\Big(\frac{z}{|z|}\Big)\Big|\frac{1}{|z|^{n+1}} \lesssim \frac{1}{|z|^{n+1}}.$$

Analogous estimates hold for higher order derivatives. So, Theorem 4.4 ensures that

$$T_{\tilde{K}_{j,l},\mu}f(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\tilde{K}_{j,l}(x-y)f(y)d\mu(y) \equiv \lim_{\varepsilon\to 0}T_{\tilde{K}_{j,l},\mu,\varepsilon}f(x) \tag{4.3.7}$$

exists for μ-a.e x. Recall also that by the Theorem 4.4 there exists $M = M(n)$ such that $T_{\tilde{K}_{j,l},\mu}$ is bounded on $L^2(\mu)$ with operator norm

$$\|T_{\tilde{K}_{j,l},\mu}\|_{L^2(\mu)\to L^2(\mu)} \lesssim \|\tilde{K}_{j,l}|_{\mathbb{S}^n}\|_{C^M(\mathbb{S}^n)} = \|\varphi_{j,l}\|_{C^M(\mathbb{S}^n)}. \tag{4.3.8}$$

Gathering (4.3.5), (4.3.6) and (4.3.7), to prove the lemma it is enough to show that the dominated convergence theorem applies and, in particular, that

$$\sum_{j,l}\big|b_j(x)T_{\tilde{K}_{j,l},\mu,\varepsilon}f(x)\big| \leq C(x) < \infty, \tag{4.3.9}$$

where $C(x)$ does not depend on ε. By Lebesgue differentiation theorem, to prove (4.3.9) it suffices to show that for every ball $B_0 \subset \mathbb{R}^{n+1}$ we have

$$\begin{aligned}\sum_{j,l}\int_{B_0}\big|b_{j,l}(x)T_{\tilde{K}_{j,l},\mu,\varepsilon}f(x)\big|d\mu(x) &\lesssim_{B_0,n}\sum_{j,l,m}\|b_{j,l}\|_\infty\|\varphi_{j,l}\|_{C^m(\mathbb{S}^n)}\|f\|_{L^2(\mu)}\\ &\leq C\|f\|_{L^2(\mu)}\end{aligned}$$

for some $C > 0$, where the first inequality above uses the L^2-boundedness (4.3.8).

The smoothness of b implies that (see [Ste70, p. 3.1.5])

$$\|b_{j,l}\|_\infty \lesssim \frac{1}{j^{\frac{3}{2}n+1+M}},$$

where the exponent on the right hand side is chosen accordingly to what we need next. Now, recall that the Sobolev space $H^s(\mathbb{S}^n)$, $s \in \mathbb{R}$ can be defined via spherical harmonics expansion. In particular, it is the completion of $C^\infty(\mathbb{S}^n)$ with respect to the norm

$$\|v\|_{H^s(\mathbb{S}^n)} := \Big(\sum_{j,l}\Big(j + \frac{n-1}{2}\Big)^{2s}|v_{j,l}|^2\Big)^{1/2}, \tag{4.3.10}$$

where $v_{j,l} = \langle v, \varphi_{j,l}\rangle_{L^2(\mathbb{S}^n)}$. For the definition and the properties of this space, we refer for example to [AH12, Section 3.8] and to [AH12, Section 6.3] for the relation of (4.3.10) with that via the restriction of the gradient to the unit sphere. By Sobolev embedding theorem, $H^s(\mathbb{S}^n)$ continuously embeds into $C(\mathbb{S}^n)$ for $s > n/2$. So, choosing $s = n/2$ and using (4.3.10) can estimate

$$\|D^m \varphi_{j,l}\|_{C(\mathbb{S}^n)} \lesssim_n \|\varphi_{j,l}\|_{H^{s+m}(\mathbb{S}^n)} = \Big(\frac{2j+n-1}{2}\Big)^{\frac{n}{2}+m}.$$

Hence, using (4.3.4)

$$\sum_{j,l} \|b_{j,l}\| \|\varphi_{j,l}\|_{C^M(\mathbb{S}^n)} \lesssim_n \sum_{m=0}^{M} \sum_{j\geq 1} N_j j^{-\frac{3}{2}n-1-M} j^{\frac{n}{2}+m} \lesssim \sum_{j\geq 1} \frac{1}{j^2} < \infty,$$

which concludes the proof. □

Theorem 4.5. *Let μ be an n-rectifiable measure on $\mathbb{R}^{n+1}$ with compact support. Let A be a matrix having the properties (4.1.1), (4.1.2) and (4.1.5). Then for every $f \in L^2(\mu)$ the principal value*

$$T_\mu f(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \nabla_1 \mathcal{E}(x,y) f(y) d\mu(y)$$

exists for μ-almost every x.

Proof. Let $\varepsilon > 0$ and denote $b(x,z) := \nabla_1 \Theta(z, 0; A(x))$. As a consequence of the explicit formula (4.2.3), it is not difficult to see that each component of b verifies the hypothesis of Lemma 4.3.1. So, split $T_{\mu,\varepsilon}$ as

$$\begin{aligned} T_{\mu,\varepsilon} f(x) = &\int_{|x-y|>\varepsilon} b(x, x-y) f(y) d\mu(y) \\ &+ \int_{|x-y|>\varepsilon} \Big(\nabla_1 \mathcal{E}(x,y) - \nabla_1 \Theta(x,y;A(x))\Big) f(y) d\mu(y). \end{aligned} \tag{4.3.11}$$

The limit for $\varepsilon \to 0$ of the first integral in the right hand side of (4.3.11) exists μ-a.e. because of Lemma 4.3.1. On the other hand, $\nabla_1 \mathcal{E}(x,y) - \nabla_1 \Theta(x,y;A(x))$ defines an operator which is compact on $L^p(\mu)$ because of Lemma 4.2.2, which guarantees that the limit for $\varepsilon \to 0$ exists for μ-a.e. x and concludes the proof. □

4.3.2 Principal values for measures with zero density

We argue as in [Tol14, Chapter 8], proving the existence of the principal values passing through the existence of the weak limit and following the approach of Mattila and Verdera [MV95]. Again, we suppose that μ has compact support.

A combination of the proof of [MV95, Theorem 1.4] (see also [Tol14, Theorem 8.10]) and Lemma 4.2.2 makes possible to prove that if μ is a Radon measure in $\mathbb{R}^{n+1}$ with growth of degree n, then for every $1 < p < \infty$ and $f \in L^p(\mu)$, $\{T_{\mu,\epsilon} f\}_\epsilon$ admits a weak limit $T^w_\mu f$ in $L^p(\mu)$ as $\epsilon \to 0$. Moreover, the representation formula

$$T^w_\mu f(x) = \lim_{r\to 0} \fint_{B(x,r)} T_\mu\big(f\chi_{B(x,r)^c}\big)(y) d\mu(y) \tag{4.3.12}$$

holds for μ-almost every $x \in \mathbb{R}^{n+1}$, giving an explicit way of computing the weak limit. We remark that, in general, we can only infer that formula (4.3.12) holds if T_μ has an antisymmetric kernel.

Let us recall the following theorem by Mattila and Verdera (see [MV95]), here reported in the formulation of [Tol14, Theorem 8.11].

Theorem 4.6. *Let μ be a Radon measure in $\mathbb{R}^d$ that has growth of degree n and zero n-dimensional density μ-a.e. Let $\mathcal{T}_\mu$ be an n-dimensional antisymmetric Calderón-Zygmund operator. Then, for all $1 < p < \infty$ and $f \in L^p(\mu)$, $\mathrm{p.v}\,\mathcal{T}_\mu f(x)$ exists for μ-a.e. $x \in \mathbb{R}^d$ and coincides with $\mathcal{T}^w_\mu f(x)$. Also, for all $\nu \in M(\mathbb{C})$, $\mathrm{p.v}\,\mathcal{T}\nu(x)$ exists for μ-a.e. $x \in \mathbb{R}^d$.*

This result can be transferred to the gradients of the single layer potential T_μ.

Theorem 4.7. *Let μ be a Radon measure in $\mathbb{R}^{n+1}$ that has growth of degree n, zero n-dimensional density and compact support. Suppose that T_μ is a bounded operator from $L^2(\mu)$ to $L^2(\mu)$. Then, for all $1 < p < \infty$ and $f \in L^p(\mu)$, $\mathrm{p.v}\,T_\mu f(x)$ exists for μ-a.e. $x \in \mathbb{R}^{n+1}$ and coincides with $T^w_\mu f(x)$. Also, for all $\nu \in M(\mathbb{C})$, $\mathrm{p.v}\,T\nu(x)$ exists for μ-a.e. $x \in \mathbb{R}^{n+1}$.*

Proof. Let $1 < p < \infty$ and $f \in L^p(\mu)$. We decompose $T_\mu f$ into its symmetric and antisymmetric part. That is to say,

$$T_\mu f(x) = T^{(a)}_\mu f(x) + T^{(s)}_\mu f(x),$$

where $T^{(a)}_\mu$ is the integral operator with kernel $(\nabla_1 \mathcal{E}(x,y) - \nabla_1 \mathcal{E}(y,x))/2$ and $T^{(s)}_\mu$ whose kernel is $(\nabla_1 \mathcal{E}(x,y) + \nabla_1 \mathcal{E}(y,x))/2$. We can apply Theorem 4.6 to antisymmetric part $T^{(a)}_\mu$, obtaining that $\mathrm{p.v}\,T^{(a)}_\mu f(x)$ exists for μ-a.e. x.

On the other hand, $T^{(s)}_\mu$ defines a compact operator on $L^p(\mu)$ since

$$\int |\nabla_1 \mathcal{E}(x,y) + \nabla_1 \mathcal{E}(y,x)| d\mu(y) \lesssim \mathrm{diam}(\mathrm{supp}\,\mu)^\alpha,$$

so that the principal values exist.

The fact that $T^w_\mu f$ coincides with $\mathrm{p.v}\,T_\mu f$ a.e. follows from the definition of weak limit together with dominated convergence theorem:

$$\int T^w_\mu f g \, d\mu = \lim_{\varepsilon \to 0} \int T_{\mu,\varepsilon} f g \, d\mu = \int \mathrm{p.v}\,T_\mu f g \, d\mu \quad \text{for all} \quad g \in L^{p'}(\mu),$$

p' being the Hölder conjugate exponent of p. □

A remark on the well-posedness of the assumption (6) of Theorem 4.2. Let T, μ and B be as in Theorem 4.2. Let $x, y \in B$ and $\varepsilon > 0$ and write

$$T_\epsilon \mu(x) - T_\epsilon \mu(y) = T_{\mu,\epsilon}\chi_{2B}(x) - T_{\mu,\epsilon}\chi_{2B}(y) + \big[T_{\mu,\epsilon}\chi_{\mathbb{R}^{n+1}\setminus 2B}(x) - T_{\mu,\epsilon}\chi_{\mathbb{R}^{n+1}\setminus 2B}(y)\big].$$

Now observe that, being the operator $T_{\mu|_B}$ bounded on $L^2(\mu|_B)$, Theorem 4.1 (2) applies with $\nu = \chi_{2B}\mu$. So, the first two summands on the right hand side of (4.3.2) admit a limit as $\varepsilon \to 0$ for almost every $x, y \in B$. The limit for $\varepsilon \to 0$ of the last summand exists, too. Indeed, since x, y do not belong to $\mathbb{R}^{n+1} \setminus 2B$, for $\varepsilon < r(B)$,

$$T_{\mu,\epsilon}\chi_{\mathbb{R}^{n+1}\setminus 2B}(x) - T_{\mu,\epsilon}\chi_{\mathbb{R}^{n+1}\setminus 2B}(y) = \int_{\mathbb{R}^{n+1}\setminus 2B} \big(\nabla_1 \mathcal{E}(x,z) - \nabla_1 \mathcal{E}(y,z)\big) d\mu(y). \tag{4.3.13}$$

If we assume $\tilde{\alpha} \leq \alpha$ in the statement of the main theorem, an application of the Calderón-Zygmund property of the kernel combined with a dyadic decomposition of the domain of integration gives

$$\Big|\int_{\mathbb{R}^{n+1}\setminus 2B}\big(\nabla_1\,\mathcal{E}(x,z)-\nabla_1\,\mathcal{E}(y,z)\big)d\mu(z)\Big| \lesssim |x-y|^{\alpha}\sum_{j=1}^{+\infty}\int_{2^{j+1}B\setminus 2^jB}\frac{1}{|x-z|^{n+\alpha}}d\mu(z)$$
$$\leq P_{\mu,\alpha}(B)\leq P_{\mu,\tilde{\alpha}}(B)<+\infty. \tag{4.3.14}$$

In particular, this tells that $T\mu(x)-T\mu(y)$ exists in the principal value sense for almost every $x,y\in B$.

We also want to point out that $T\mu - m_{\mu,B}(T\mu)$ defines an $L^2(\mu|_B)$-function. Indeed, for $x\in B$ and using (4.3.14),

$$|T\mu(x)-m_{\mu,B}(T\mu)|\leq \frac{1}{\mu(B)}\int_B|T\mu(x)-T\mu(y)|d\mu(y)$$
$$\leq |T(\chi_{2B}\mu)(x)|+\big(m_{\mu,B}|T(\chi_{2B}\mu)|^2\big)^{1/2}+P_{\mu,\tilde{\alpha}}(B).$$

The right hand side of the previous majorization defines an $L^2(\mu|_B)$ function because of the assumptions $T(\chi_{2B}\mu)\in L^2(\mu|_B)$ and $P_{\mu,\tilde{\alpha}}(B)<+\infty$ in Theorem 4.2.

4.4 The Main Lemma

A careful read of [GT18] shows that the same arguments as the ones for the Riesz transform give that, in order to prove Theorem 4.2, it suffices to prove the following result.

Lemma 4.4.1 (Main Lemma). *Let $n>1$ and let $C_0,C_1>0$ be some arbitrary constants. There exist $M=M(C_0,C_1,n)>0$ big enough, $\lambda(C_0,C_1,n)>0$ and $\epsilon=\epsilon(C_0,C_1,M,n)>0$ small enough such that if $\delta=\delta(M,C_0,C_1,n)>0$ is sufficiently small, then the following holds. Let μ be a Radon measure in $\mathbb{R}^{n+1}$ with compact support and $Q_0\subset\mathbb{R}^{n+1}$ a cube centered at the origin satisfying the properties:*

1. *$\ell(MQ_0)\leq\lambda$.*
2. *$\mu(Q_0)=\ell(Q_0)^n$.*
3. *$P_{\mu,\tilde{\alpha}}(MQ_0)\leq C_0$.*
4. *For all $x\in 2Q_0$ and $0<r\leq\ell(Q_0)$, $\Theta_\mu(B(x,r))\leq C_0$.*
5. *Q_0 has C_0-thin boundary.*
6. *$\alpha_\mu^L(3MQ_0)\leq\delta$, for some hyperplane L through the origin.*
7. *$T_{\mu|_{2Q_0}}$ is bounded on $L^2(\mu|_{2Q_0})$ with $\|T_{\mu|_{2Q_0}}\|_{L^2(\mu|_{2Q_0})\to L^2(\mu|_{2Q_0})}\leq C_1$.*
8. *We have*
$$\int_{Q_0}|T\mu(x)-m_{\mu,Q_0}(T\mu)|^2d\mu(x)\leq\epsilon\mu(Q_0). \tag{4.4.1}$$

Then there exists some constant $\tau>0$ and a uniformly n-rectifiable set $\Gamma\subset\mathbb{R}^{n+1}$ such that
$$\mu(Q_0\cap\Gamma)\geq\tau\mu(Q_0),$$
where the constant τ and the uniform rectifiability constants of Γ depend on all the constants above.

The matrix A may have a very general form. In particular, we need some additional argument to overcome the lack of "symmetries" of the matrix with respect to reflections and to periodization (the exact meaning of this sentence will be clear after the reading of Section 4.5, where we recall how second order PDE's in divergence form are affected by a change of variable). Indeed, this is a crucial point for our proof to work. A similar problem has been faced in [PPT18]. First, in order to be able to argue via a change of variables, we have to show that we can assume the matrix A to be symmetric.

We recall Schur's lemma for integral operators with a reproducing kernel. The proof is a standard application of Cauchy-Schwarz's inequality.

Lemma 4.4.2. *Let $K\colon \mathbb{R}^{n+1}\times\mathbb{R}^{n+1}\to\mathbb{R}^{n+1}$ be a function such that, for a constant $C>0$, we have*

$$\int |K(x,y)|d\mu(x)\le C \tag{4.4.2}$$

and

$$\int |K(x,y)|d\mu(y)\le C. \tag{4.4.3}$$

Then the operator $Tf=K*f$ is a continuous operator from $L^2(\mu)$ to $L^2(\mu)$ and

$$\|T\|_{L^2(\mu)\to L^2(\mu)}\le C. \tag{4.4.4}$$

Proof. Splitting $|K(x,y)f(y)|=|K(x,y)|^{1/2}\big(|K(x,y)|^{1/2}|f(y)|\big)$ and applying Hölder's inequality together with (4.4.3), we get

$$\begin{aligned}\Big|\int K(x,y)f(y)d\mu(y)\Big|^2 &\le \Big(\int |K(x,y)|d\mu(y)\Big)\Big(\int |K(x,y)||f(y)|^2d\mu(y)\Big)\\ &\le C\Big(\int |K(x,y)||f(y)|^2d\mu(y)\Big).\end{aligned} \tag{4.4.5}$$

So, applying (4.4.5), (4.4.2) and Fubini's theorem, we get

$$\begin{aligned}\int\Big|\int K(x,y)d\mu(y)\Big|^2d\mu(x) &\le C\iint |K(x,y)||f(y)|^2d\mu(y)d\mu(x)\\ &\le C^2\int |f(y)|^2d\mu(y),\end{aligned} \tag{4.4.6}$$

which gives (4.4.4). □

Let A be a matrix as before. We denote by $A_s=(A+A^T)/2$ its symmetric part and by $T_\mu^{A_s}$ its correspondent gradient of the single layer potential.

Recalling that, for any matrix A_0 with constant coefficients we have $\Theta(\cdot,\cdot;A_0)=\Theta(\cdot,\cdot;A_{0,s})$, we can formulate the following lemma.

Lemma 4.4.3. *Let Q be a cube in $\mathbb{R}^{n+1}$ such that, for $M>1$, $P_{\mu,\alpha}(MQ)\le C_1$. The operator $T^{(s)}_{\mu|_{2Q}}$ is bounded on $L^2(\mu|_{2Q})$ if and only if $T_{\mu|_{2Q}}$ is bounded on $L^2(\mu|_{2Q})$. In particular*

$$\|T_{\mu|_{2Q}}\|_{L^2(\mu|_{2Q})\to L^2(\mu|_{2Q})}=\|T^{A_s}_{\mu|_{2Q}}\|_{L^2(\mu|_{2Q})\to L^2(\mu|_{2Q})}+O(\ell(Q)^\alpha). \tag{4.4.7}$$

Moreover

$$\begin{aligned}&\int_Q \big|T^{A_s}\mu(x)-m_{\mu,Q}(T^{A_s}\mu)\big|^2d\mu(x)\\ &\qquad\lesssim_{\Lambda,\|A\|_\alpha}\int_Q\big|T\mu(x)-m_{\mu,Q}(T\mu)\big|^2d\mu(x)+\big(M^\alpha\ell(Q)^\alpha+M^{-\alpha}\big)^2\mu(Q).\end{aligned} \tag{4.4.8}$$

Proof. Let us first prove (4.4.7). The identity (4.2.4) for matrices with constant coefficients leads to

$$
\begin{aligned}
T^{A_s}_{\mu|_{2Q}} f(x) &= \int_{2Q} \nabla_1 \mathcal{E}_{A_s}(x,y) f(y) d\mu(y) \\
&= \int_{2Q} \big(\nabla_1 \mathcal{E}_{A_s}(x,y) - \nabla_1 \Theta(x,y;A_s(x))\big) f(y) d\mu(y) \\
&+ \int_{2Q} \big(\nabla_1 \Theta(x,y;A(x)) - \nabla_1 \mathcal{E}(x,y)\big) f(y) d\mu(y) + \int_{2Q} \nabla_1 \mathcal{E}(x,y) f(y) d\mu(y) \\
&\equiv I + II + T_{\mu|_{2Q}} f(x).
\end{aligned} \tag{4.4.9}
$$

To estimate I and II in (4.4.9) it suffices, then, to invoke Lemma 4.2.7 and Schur's Lemma. This finishes the proof of the first part of the lemma.

Let us now prove (4.4.8). We split

$$
\begin{aligned}
&T\mu(x) - m_{\mu,Q}(T\mu) \\
&\quad = \Big(T(\chi_{MQ}\mu)(x) - m_{\mu,Q}\big(T(\chi_{MQ}\mu)\big)\Big) + \Big(T(\chi_{(MQ)^c}\mu)(x) - m_{\mu,Q}\big(T(\chi_{(MQ)^c}\mu)\big)\Big).
\end{aligned} \tag{4.4.10}
$$

Let us estimate the two terms in the right hand side separately. Again, as a consequence of (4.4.9) and Lemma 4.2.2 we can write

$$
\Big| T(\chi_{MQ}\mu) - m_{\mu,Q}(T(\chi_{MQ}\mu)) - \Big(T^{A_s}(\chi_{MQ}\mu) + m_{\mu,Q}\big(T^{A_s}(\chi_{MQ}\mu)\big)\Big)\Big| \lesssim M^\alpha \ell(Q)^\alpha. \tag{4.4.11}
$$

To bound the second term in the right hand side of (4.4.10), notice that for $x, y \in Q$ standard estimates together with Lemma 4.2.7 give

$$
\begin{aligned}
\big|T_\mu \chi_{(MQ)^c}(x) - T_\mu \chi_{(MQ)^c}(y)\big| &\lesssim \int_{(MQ)^c} \frac{|x-y|^\alpha}{|x-z|^{n+\alpha}} d\mu(z) \\
&\lesssim \frac{|x-y|^\alpha}{\ell(MQ)^\alpha} P_{\mu,\alpha}(MQ) \lesssim \frac{1}{M^\alpha} P_{\mu,\alpha}(MQ) \lesssim \frac{1}{M^\alpha},
\end{aligned}
$$

so that, averaging over y in Q we have

$$
\big|T(\chi_{(MQ)^c}\mu)(x) - m_{\mu,Q}\big(T(\chi_{(MQ)^c}\mu)\big)\big| \lesssim M^{-\alpha}
$$

The same calculations lead to

$$
\Big|T^{A_s}(\chi_{(MQ)^c}\mu)(x) - m_{\mu,Q}\big(T^{A_s}(\chi_{(MQ)^c}\mu)\big)\Big| \lesssim M^{-\alpha},
$$

so the inequality (4.4.8) in the statement of the lemma follows by gathering all the previous considerations. □

A gathering of Lemma 4.4.1 and Lemma 4.4.3 shows that it suffices to prove Theorem 4.2 under the additional assumption that the matrix A is symmetric. Indeed, proving Lemma 4.4.1 with $A = A_s$ gives it in the non-symmetric case with worse assumptions on the parameters involved. We omit further details.

Remark 8. Arguing as in Lemma 4.4.3, one could prove that

$$\|T_{\mu|_{2Q}}\|_{L^2(\mu|_{2Q})\to L^2(\mu|_{2Q})} = \|T^a_{\mu|_{2Q}}\|_{L^2(\mu|_{2Q})\to L^2(\mu|_{2Q})} + O(\ell(Q)^\alpha),$$

where T^a is the operator corresponding to the antisymmetric part of the kernel $K(\cdot,\cdot)$, that is to say $K^a(x,y) = (K(x,y)-K(y,x))/2$. However, as in [PPT18] and [CMT19], we prefer not to make this reduction because it would create problems later on in the proof. In particular, it would be an obstacle to the application of the maximum principle, which is a crucial tool in Section 4.11.

4.5 The modification of the matrix

4.5.1 The change of variable

The following lemma deals with how the fundamental solution and its gradient are affected by a change of variable.

Lemma 4.5.1 (see [PPT18], Lemma 13)**.** *Let $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a locally bilipschitz map and let $A \in \Lambda_\alpha$. Let $\mathcal{E}_A$ be the fundamental solution of $L_A = -\operatorname{div}(A\nabla\cdot)$. Set $A_\phi := |\det\phi| D(\phi^{-1})(A\circ\phi)D(\phi^{-1})^T$. Then*

$$\mathcal{E}_{A_\phi}(x,y) = \mathcal{E}_A(\phi(x),\phi(y)) \text{ for } x,y \in \mathbb{R}^{n+1},$$

and

$$\nabla_1 \mathcal{E}_{A_\phi}(x,y) = D(\phi)^T(x)\,\nabla_1 \mathcal{E}_A(\phi(x),\phi(y)) \quad \text{for} \quad x \in \mathbb{R}^{n+1}.$$

Let us state a lemma concerning how the gradient of the fundamental solution transforms under a change of variable ϕ as in Lemma 4.5.1. We use the notation

$$T_\phi\mu(x) := \int \nabla_1 \mathcal{E}_{A_\phi}(x,y)d\mu(y).$$

Lemma 4.5.2 (see [PPT18], Lemma 14)**.** *Let $\phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a bilipschitz change of variables. For every $x \in \mathbb{R}^{n+1}$ we have*

$$T_\phi\mu(x) = D(\phi)^T(x) T\phi_\sharp\mu(\phi(x)).$$

A particularly useful change of variable is the one that turns the symmetric part of the matrix at a given point into the identity. For the following statement we refer to [Azz+16a].

Lemma 4.5.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and assume that A is a uniformly elliptic matrix with real entries. Let $A_s = (A + A^T)/2$ be the symmetric part of A and for a fixed point $y_0 \in \Omega$ define $S = \sqrt{A_s(y_0)}$. If*

$$\tilde{A}(\cdot) = S^{-1}(A\circ S)(\cdot)S^{-1},$$

then $\tilde{A}$ is uniformly elliptic, $\tilde{A}_s(z_0) = Id$ if $z_0 = S^{-1}y_0$ and u is a weak solution of $L_A u = 0$ in Ω if and only if $\tilde{u} = u\circ S$ is a weak solution of $L_{\tilde{A}}\tilde{u} = 0$ in $S^{-1}(\Omega)$.

As a remark, we want to point out that the change of variables defined in Lemma 4.5.3 is a linear map and, in particular, a bilipschitz map of $\mathbb{R}^{n+1}$ to itself. Its bilipschitz constant depends on the ellipticity of the matrix A.

We need the notion of flatness for images of cubes via maps of the aforementioned type. For a set $E \subset \mathbb{R}^{n+1}$, we define the α-number in an analogous ways as for cubes.

In particular, for any hyperplane L and any measure ν, we denote

$$\alpha_\nu^L(E) := \frac{1}{\operatorname{diam}(E)^{n+1}} \inf_{c\geq 0} d_E(\nu, c\,\mathcal{H}^n\,|_L).$$

This particular notation will be used only in this section.

Lemma 4.5.4. *Let φ be an affine, bilipschitz change of variables of $\mathbb{R}^{n+1}$. Let L be a hyperplane in $\mathbb{R}^{n+1}$. Let $J_\varphi > 0$ be the Jacobian of φ. Then, for any Radon measure ν, for any cube $Q \subset \mathbb{R}^{n+1}$ and any constant $c \geq 0$ we have that*

$$d_Q(\nu, c\,\mathcal{H}^n\,|_L) \approx_{n,C} d_{\varphi(Q)}\big(\varphi_\sharp\nu, c\,\mathcal{H}^n\,|_{\varphi(L)}\big). \tag{4.5.1}$$

Hence,

$$\alpha_\nu^L(Q) \approx_{n,C} \alpha_{\varphi_\sharp\nu}^{\varphi(L)}(\varphi(Q)). \tag{4.5.2}$$

Proof. Formula (4.5.2) is an immediate consequence of (4.5.1) and the fact that $\ell(Q) \approx_C \operatorname{diam}(\varphi(Q))$.

Let us prove (4.5.1). For every $c \geq 0$

$$\varphi_\sharp\big(c\,\mathcal{H}^n\,|_L\big) = c(\varphi_\sharp\,\mathcal{H}^n)|_{\varphi(L)}.$$

Indeed for any $\varphi_\sharp\,\mathcal{H}^n\,|_L$-measurable set E we have

$$\varphi_\sharp(c\,\mathcal{H}^n\,|_L)(E) = c\,\mathcal{H}^n\left(\varphi^{-1}(E)\cap L\right) = c\,\mathcal{H}^n\left(\varphi^{-1}(E\cap\varphi(L))\right) = c(\varphi_\sharp\,\mathcal{H}^n)|_{\varphi(L)}(E).$$

Moreover, as a consequence of the Radon-Nikodym differentiation theorem (see [EG92, Lemma 1, p. 92]), we have

$$\mathcal{H}^n\left(\varphi^{-1}(E)\right) = J_\varphi\,\mathcal{H}^n(E).$$

So,

$$d_Q(\nu, c\,\mathcal{H}^n\,|_L) \approx_C d_{\varphi(Q)}\big(\varphi_\sharp\nu, \varphi_\sharp c\,\mathcal{H}^n\,|_L\big) \approx_{n,C} d_{\varphi(Q)}\big(\varphi_\sharp\nu, c\,\mathcal{H}^n\,|_{\varphi(L)}\big),$$

which proves the lemma. □

4.5.2 Reduction of the Main Lemma to the case $A(0) = Id$

Recall that by Lemma 4.4.3 we can assume A to be a symmetric matrix.

Let us begin with a preliminary observation. Let $Q_0 \subset \mathbb{R}^{n+1}$ be a cube as in the Main Lemma and let us denote $S := A_s(z_{Q_0})^{1/2}$, where z_{Q_0} is the center of Q_0. We choose the map φ so that $\varphi(x) = Sx$. By Lemma 4.5.3 we have that $A_\varphi(\varphi^{-1}(z_{Q_0})) = Id$. Denoting $\nu = \varphi^{-1}{}_\sharp\mu$ and arguing as in [PPT18, Section 6], Lemma 4.5.2 gives

$$\int_{Q_0} \left|T\mu(x) - m_{\mu,Q_0}(T\mu)\right|^2 d\mu(x) \approx \int_{\varphi^{-1}(Q_0)} \left|T_\varphi\nu(x) - m_{\nu,\varphi^{-1}(Q_0)}(T_\varphi\nu)\right|^2 d\nu(x)$$

and

$$\|T_\varphi\nu\|_{L^2\left(\nu|_{\varphi^{-1}(2Q_0)}\right)} \approx \|T\mu\|_{L^2\left(\mu|_{(2Q_0)}\right)},$$

the implicit constants in the formulas above depending only on φ and, hence, on the ellipticity of the matrix A.

Using these facts and Lemma 4.5.4, in order to prove Lemma 4.4.1 it suffices to study the variant stated below.

Lemma 4.5.5. *Let $n > 1$ and let $C_0, C_1 > 0$ be some arbitrary constants. There exists $M = M(C_0, C_1, n) > 1$ big enough, $\lambda(C_0, C_1, n) > 0$ small enough and $\tilde{\epsilon} = \tilde{\epsilon}(C_0, C_1, M, n) > 0$ small enough such that if $\delta = \delta(M, C_0, C_1, n) > 0$ is small enough, then the following holds. Let μ be a Radon measure in $\mathbb{R}^{n+1}$, $Q_0 \subset \mathbb{R}^{n+1}$ a cube centered at the origin and $\nu := \varphi^{-1}{}_\sharp\mu$, φ being as in the comments preceding the lemma, satisfying the following properties:*

1. $A_\varphi\big(\varphi^{-1}(0)\big) = Id$.
2. $\ell(MQ_0) \leq \lambda$.
3. $\nu\big(\varphi^{-1}(Q_0)\big) = \ell(Q_0)^n$.
4. $P_{\nu,\tilde{\alpha}}\big(\varphi^{-1}(MQ_0)\big) \leq C_0$.
5. *For all $x \in 2Q_0$ and $0 < r \leq \ell(\widetilde{Q})$, $\Theta_\mu(B(x,r)) \leq C_0$.*
6. *Q_0 has C_0-thin boundary.*
7. $\alpha_\nu^{\varphi^{-1}(H)}\big(\varphi^{-1}(3MQ_0)\big) \leq \delta$, *where* $H = \{x \in \mathbb{R}^{n+1} \colon x_{n+1} = 0\}$.
8. $T_{\varphi,\nu|_{\varphi^{-1}(2Q_0)}}$ *is bounded on* $L^2(\nu|_{\varphi^{-1}(2Q_0)})$ *with*
$$\big\|T_{\varphi,\nu|_{\varphi^{-1}(2Q_0)}}\big\|_{L^2(\nu|_{\varphi^{-1}(2Q_0)})\to L^2(\nu|_{\varphi^{-1}(2Q_0)})} \leq C_1.$$
9. *we have*
$$\int_{\varphi^{-1}(Q_0)} \big|T_\varphi\nu(x) - m_{\nu,\varphi^{-1}(Q_0)}(T_\varphi\nu)\big|^2 d\nu(x) \leq \tilde{\epsilon}\nu\big(\varphi^{-1}(Q_0)\big).$$

Then there exists some constant $\tau > 0$ and a uniformly n-rectifiable set $\Gamma \subset \mathbb{R}^{n+1}$ such that
$$\mu(Q_0 \cap \Gamma) \geq \tau\mu(Q_0),$$
where the constant τ and the UR constants of Γ depend on all the constants above.

The aim of most of the rest of the chapter is to provide the proof of this result.

In what follows, for the sake of simplicity of the notation, we will assume that $A(0) = A(z_{Q_0}) = Id$, which in particular gives that $\varphi = Id$, $\mu = \nu$ and $T_{\varphi,\mu} = T_\mu$. Indeed, if this is not the case, we should carry the following proofs for the image of cubes via φ^{-1}, periodize with respect to the image of a lattice of standard cubes and work with T_φ instead of T. This would be a merely notational complication that we prefer to avoid to make the arguments more accessible.

Reduction to a periodic matrix. The forthcoming lemma shows, roughly speaking, that the local structure of the matrix A close to Q_0 is what matters to the purposes of Lemma 4.4.1. An immediate consequence of this fact is that, without loss of generality, we can replace A with a periodic matrix, provided that the new matrix coincides with A in a suitable neighborhood of the cube Q_0.

In what follows, we assume the matrix $\bar{A}$ to have Hölder continuous coefficients of exponent $\alpha/2$ for technical reasons that will result clearer later on.

Lemma 4.5.6. *Let $\bar{A} \in \Lambda_{\alpha/2}$ be such that $\bar{A}(x) = A(x)$ for every $x \in 2Q_0$. Let $\bar{T}$ denote the gradient of the single layer potential associated with $\bar{A}$. The operator $T_{\mu|_{2Q_0}}$ is bounded on $L^2(\mu|_{2Q_0})$ if and only if $\bar{T}_{\mu|_{2Q_0}}$ is bounded on $L^2(\mu|_{2Q_0})$ and*
$$\|T_{\mu|_{2Q_0}}\|_{L^2(\mu|_{2Q_0})\to L^2(\mu|_{2Q_0})} = \|\bar{T}_{\mu|_{2Q_0}}\|_{L^2(\mu|_{2Q_0})\to L^2(\mu|_{2Q_0})} + O\big(\ell(Q_0)^{\alpha/2}\big).$$

Moreover we have

$$\int_{Q_0} |T\mu(x) - m_{\mu,Q_0}(T\mu)|^2 d\mu(x) \lesssim \int_{Q_0} |\bar{T}\mu(x) - m_{\mu,Q_0}(\bar{T}\mu)|^2 d\mu(x) + \big(\ell(MQ_0)^\alpha + M^{-\alpha/2}\big)^2 \mu(Q_0), \tag{4.5.3}$$

where M is as in the statement of Lemma 4.4.1 and the implicit constant in (4.5.3) depends on $\operatorname{diam}(\operatorname{supp}\mu)$.

The proof of Lemma 4.5.6 relies on the fact that $\Theta(\cdot,\cdot;A(x)) = \Theta(\cdot,\cdot;\bar{A}(x))$ for every $x \in 2Q_0$ and it is very similar to the one of Lemma 4.4.3, so that we omit it.

In the rest of the chapter, without additional specifications, we will deal with a matrix $\bar{A}$ periodic with period ℓ, $2\ell(Q_0) < \ell \lesssim \ell(Q_0)$.

The definition of the matrix $\bar{A}$. The construction in the present subsection is dictated by the necessity of having an auxiliary matrix which agrees with A on $2Q_0$ and has the further properties of being periodic (which is crucial to use the estimates of the theory of homogenization) and of presenting 'additional simmetries' with respect to reflections (see the forthcoming Lemma 4.5.8). For a scheme of this construction we also refer to Figure 1.

Let e_j denote the j-th element of the canonical basis of $\mathbb{R}^{n+1}$. We denote by $\psi_j\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ the map

$$\psi_j(x) := x + (3\ell(Q_0) - 2x_j)e_j, \tag{4.5.4}$$

which corresponds to the reflection across the hyperplane P_j orthogonal to e_j and which passes through the point $\frac{3}{2}\ell(Q_0)e_j$. Let $0 < \delta < 1/10$. Given a matrix $B(x)$

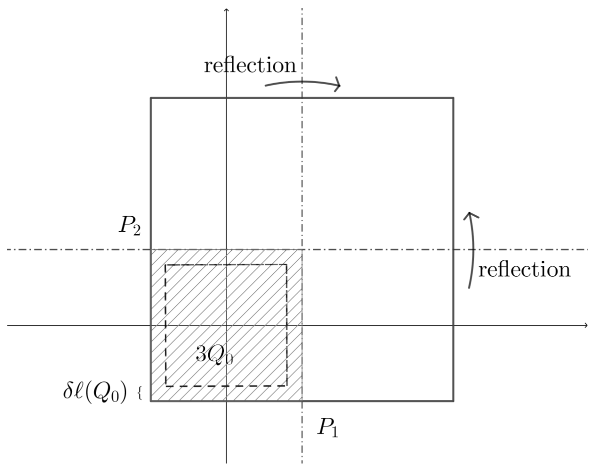

FIGURE 4.1: A schematization of the construction of $\bar{A}$ at the level of the periodic unit.

with variable coefficients, we define B_j as

$$B_j = B_{\psi_j} = D(\psi_j^{-1})(B \circ \psi_j) D(\psi_j^{-1})^T. \tag{4.5.5}$$

Moreover, we define the matrix $\widetilde{B}$ as

$$\widetilde{B}(x) = \begin{cases} B(x) & \text{for } \operatorname{dist}\big(x, \partial(3Q_0)\big) \geq \delta\ell(Q_0), \\ \frac{\operatorname{dist}(x,\partial(3Q_0))}{\delta\ell(Q_0)} B(x) + \Big(1 - \frac{\operatorname{dist}(x,\partial(3Q_0))}{\delta\ell(Q_0)}\Big) Id & \text{for } \operatorname{dist}\big(x, \partial(3Q_0)\big) < \delta\ell(Q_0). \end{cases} \tag{4.5.6}$$

It is also useful to introduce the notation

$$\widehat{B}_j(x) = \begin{cases} B(x) & \text{for } x_j \leq \frac{3}{2}\ell(Q_0), \\ B_j(x) & \text{for } x_j > \frac{3}{2}\ell(Q_0). \end{cases} \tag{4.5.7}$$

Let us apply the previous constructions to the matrix A. First, observe that the matrix $\hat{A}_j$ is not necessarily continuous. However, $\widehat{(\widetilde{A})}_j$ is continuous because $Id_j = Id$ and $\widetilde{A}|_{\partial(3Q_0)} \equiv Id$. Our aim, now, is to define the final auxiliary matrix $\bar{A}$ by an iteration of the construction in (4.5.7) along every direction and which is followed by a periodization. Before doing so, let us observe that for $i, j \in \{1, \ldots, n+1\}$,

$$(\widetilde{A}_i)_j(x) = (\widetilde{A}_j)_i(x), \qquad x \in \mathbb{R}^{n+1}.$$

This follows directly from (4.5.5) using the facts that $\psi_i\big(\psi_j(x)\big) = \psi_j\big(\psi_i(x)\big)$ and that the matrices $D(\psi_i^{-1}), D(\psi_j^{-1})$ are diagonal. Thus by the linearity of the interpolation in (4.5.6) we have that

$$\widehat{(\widehat{(\widetilde{A})}_i)}_j = \widehat{(\widehat{(\widetilde{A})}_j)}_i =: \widehat{(\widetilde{A})}_{i,j}, \tag{4.5.8}$$

so the order of the modifications is not relevant.

Let us now construct the matrix $\bar{A}$ in two steps:

- For x belonging to the cube of side length $6\ell(Q_0)$ centered at the point with coordinates $\frac{3}{2}\ell(Q_0)(1, \ldots, 1)$ we define

  $$\bar{A}(x) := \widehat{(\widetilde{A})}_{1,\ldots,n+1}.$$

- By (4.5.6), the matrix $\bar{A}$ defined in the first step coincide with Id for x belonging to the boundary of the cube with side length $6\ell(Q_0)$ and centered at $\frac{3}{2}\ell(Q_0)(1, \ldots, 1)$. Hence, $\bar{A}$ admits a continuous and $6\ell(Q_0)$-periodic extension to $\mathbb{R}^{n+1}$ so that

  $$\bar{A}(x) = \bar{A}\big(x + 6\vec{k}\ell(Q_0)\big)$$

  for every $\vec{k} \in \mathbb{Z}^{n+1}$.

The following holds.

Lemma 4.5.7. *The matrix $\bar{A}$ is well-defined, Hölder continuous with exponent $\alpha/2^{n+1}$ and periodic with period $6\ell(Q_0)$.*

The well-definition of $\bar{A}$ follows from (4.5.8). The proof of the Hölder regularity is a minor variation of that of [PPT18, Lemma 8.1], where a similar modification of the matrix was involved. In particular, the exponent $\alpha/2^{n+1}$ is given by the fact that every reflection of the matrix across a hyperplane halves the order of the Hölder regularity. We also point out that, being $\bar{A}$ periodic, there is no need to introduce a radial cut-off for the matrix as in [PPT18].

For the rest of the paper we use the notation $\tilde{\alpha} := \alpha/2^{n+1}$.

Properties of $\mathcal{E}_{\bar{A}}$. As a consequence of the definition of $\bar{A}$ and, more specifically, of its periodicity and the fact that by construction

$$\bar{A}_j(x) = \bar{A}(x)$$

for every $x \in \mathbb{R}^{n+1}$ and $j = 1, \dots, n+1$, we have the following.

Lemma 4.5.8.

$$\mathcal{E}_{\bar{A}}(x,y) = \mathcal{E}_{\bar{A}}(\psi_j(x), \psi_j(y)) \qquad \text{for} \qquad j = 1, \dots, n+1 \tag{4.5.9}$$

and

$$\mathcal{E}_{\bar{A}}(x,y) = \mathcal{E}_{\bar{A}}\left(x + 6\vec{k}\ell(Q_0), y + 6\vec{k}\ell(Q_0)\right) \qquad \text{for} \qquad \vec{k} \in \mathbb{Z}^{n+1}. \tag{4.5.10}$$

By Lemma 4.2.3, the function $\bar{K} = \nabla_1 \mathcal{E}_{\bar{A}}(\cdot,\cdot)$ is (globally) a Calderón-Zygmund kernel. In particular

(a) $|\bar{K}(x,y)| \lesssim |x-y|^{-n}$ for all $x, y \in \mathbb{R}^{n+1}$ with $x \neq y$.

(b) $|\bar{K}(x,y) - \bar{K}(x,y')| + |\bar{K}(y,x) - \bar{K}(y',x)| \lesssim |y-y'|^{\tilde{\alpha}}|x-y|^{-n-\tilde{\alpha}}$ for $2|y-y'| \leq |x-y|$.

Let $\bar{T}_\mu$ denote the singular integral operator associated with $\bar{K}$,

$$\bar{T}_\mu f(x) = \int \bar{K}(x,y) f(y) d\mu(y).$$

Lemma 4.5.6 tells that we can prove the Main Lemma for $\bar{T}$ instead of T, possibly by slightly worsening the parameters involved.

4.6 A first localization lemma

It is useful to provide a local analogue of the BMO-type estimate (4.4.1). This is possible because of the smallness of the α-number and the bound for the $P_{\mu,\tilde{\alpha}}$-density. Also, recall that because of the assumptions in Lemma 4.4.1, we have $\mu(MQ_0) \lesssim M^n \mu(Q_0)$. In what follows we sketch the proof of the localization of (4.4.1) for $\bar{T}_\mu$, highlighting the differences with respect to the case of the Riesz transform (see [GT18, Lemma 4.2]).

In the rest of the chapter we omit to indicate the dependence of the implicit constants in our estimates on C_0 and C_1.

Lemma 4.6.1. *For δ small enough depending on M, the following inequality holds*

$$\int_{Q_0} |\bar{T}_\mu \chi_{MQ_0}|^2 d\mu \lesssim \left(\epsilon + \frac{1}{M^{2\tilde{\alpha}}} + M^{4n+2}\delta^{1/(4n+4)} + (M\ell(Q_0))^{2\tilde{\alpha}}\right)\mu(Q_0). \tag{4.6.1}$$

Proof. First, observe that

$$\int_{Q_0} |\bar{T}_\mu(\chi_{MQ_0})|^2 d\mu \leq 2\int_{Q_0} |\bar{T}_\mu(\chi_{MQ_0}) - m_{\mu,Q_0}(\bar{T}_\mu \chi_{MQ_0})|^2 d\mu + 2|m_{\mu,Q_0}(\bar{T}_\mu \chi_{MQ_0})|^2 \mu(Q_0). \tag{4.6.2}$$

Let us estimate the two summands on the right hand side of (4.6.2) separately. To study the first one, we write

$$\begin{aligned}\int_{Q_0} &\left|\bar{T}_\mu \chi_{MQ_0} - m_{\mu,Q_0}(\bar{T}_\mu \chi_{MQ_0})\right|^2 d\mu \\ &\leq 2\int_{Q_0} \left|\bar{T}_\mu \chi_{(MQ_0)^c}(x) - m_{\mu,Q_0}(\bar{T}_\mu \chi_{(MQ_0)^c})\right|^2 d\mu(x) + 2\int_{Q_0} |\bar{T}\mu - m_{\mu,Q_0}(\bar{T}\mu)|^2 d\mu.\end{aligned} \tag{4.6.3}$$

Applying Lemma 4.2.1, it follows that for $x, y \in Q_0$

$$\begin{aligned}|\bar{T}_\mu \chi_{(MQ_0)^c}(x) - \bar{T}_\mu \chi_{(MQ_0)^c}(y)| &\leq \int_{(MQ_0)^c} |\bar{K}(x,z) - \bar{K}(y,z)| d\mu(z) \\ &\lesssim |x-y|^{\tilde{\alpha}} \int_{(MQ_0)^c} \frac{1}{|x-z|^{n+\tilde{\alpha}}} d\mu(z) \\ &\lesssim |x-y|^{\tilde{\alpha}} \sum_{j=1}^{\infty} \int_{2^{j+1}MQ_0 \setminus 2^j MQ_0} \frac{1}{|x-z|^{n+\tilde{\alpha}}} d\mu(z) \lesssim \frac{|x-y|^{\tilde{\alpha}}}{\ell(MQ_0)^{\tilde{\alpha}}} P_{\mu,\tilde{\alpha}}(MQ_0) \lesssim \frac{1}{M^{\tilde{\alpha}}},\end{aligned}$$

being $P_{\mu,\tilde{\alpha}}(MQ_0) \lesssim 1$. Then, averaging the previous inequality over the variable y, we get

$$\left|\bar{T}_\mu \chi_{(MQ_0)^c}(x) - m_{\mu,Q_0}(\bar{T}_\mu \chi_{(MQ_0)^c})\right| \lesssim \frac{1}{M^{\tilde{\alpha}}}$$

and

$$\int_{Q_0} |\bar{T}_\mu \chi_{(MQ_0)^c}(x) - m_{\mu,Q_0}(\bar{T}_\mu \chi_{(MQ_0)^c})|^2 d\mu(x) \lesssim \frac{1}{M^{2\tilde{\alpha}}} \mu(Q_0).$$

Recalling that by hypothesis we have

$$\int_{Q_0} \left|\bar{T}\mu - m_{\mu,Q_0}(\bar{T}\mu)\right|^2 d\mu \leq \epsilon \mu(Q_0),$$

we can estimate (4.6.3) as

$$\int_{Q_0} \left|\bar{T}_\mu(\chi_{(MQ_0)^c}) - m_{\mu,Q_0}(\bar{T}_\mu \chi_{MQ_0})\right|^2 d\mu \lesssim \Big(\epsilon + \frac{1}{M^{2\tilde{\alpha}}}\Big)\mu(Q_0). \tag{4.6.4}$$

An application of Lemma 4.2.7 together with the antisimmetry of $\nabla_1 \Theta(\cdot,\cdot;\bar{A}(x))$ also gives

$$\left|m_{\mu,Q_0}(\bar{T}_\mu \chi_{Q_0})\right| \lesssim \frac{1}{\mu(Q_0)} \int_{Q_0}\int_{Q_0} |x-y|^{-n+\tilde{\alpha}} d\mu(x) d\mu(y) \lesssim \ell(Q_0)^{\tilde{\alpha}}. \tag{4.6.5}$$

Minor variations of the arguments which prove [GT18, (4.2)] show that that

$$\begin{aligned}|m_{\mu,Q_0}(\bar{T}_\mu \chi_{MQ_0})| &\overset{(4.6.5)}{\lesssim} |m_{\mu,Q_0}(\bar{T}_\mu \chi_{MQ_0 \setminus Q_0})| + \ell(Q_0)^{\tilde{\alpha}} \\ &\lesssim M^{2n+1}\delta^{1/8(n+1)} + \big(M\ell(Q_0)\big)^{\tilde{\alpha}} + \ell(Q_0)^{\tilde{\alpha}} \\ &\lesssim M^{2n+1}\delta^{1/8(n+1)} + \big(M\ell(Q_0)\big)^{\tilde{\alpha}}.\end{aligned} \tag{4.6.6}$$

For the sake of brevity we omit the details and we just point out that the presence of the second summand on the right hand side comes from the estimate

$$\Big| \int_{Q_0} \bar{T}\big(\varphi \mathcal{H}^n|_H\big)\, d\mathcal{H}^n|_H \Big| \lesssim \big(M\ell(Q_0)\big)^{\tilde{\alpha}} \ell(Q_0)^n, \tag{4.6.7}$$

where φ is a proper even C^1 function with $0 \le \varphi \le 1$ and supported on $MQ_0 \setminus Q_0$. To get the estimate (4.6.7), we just write

$$\begin{aligned}
&\Big| \int_{Q_0} \bar{T}\big(\varphi \mathcal{H}^n|_H\big)\, d\mathcal{H}^n|_H \Big| \\
&\quad \le \int_{Q_0} \int_{MQ_0} \Big| \frac{1}{2}\bar{K}(x,y) - \frac{1}{2}\nabla_1 \Theta(x,y;\bar{A}(x)) \Big| \, d\mathcal{H}^n|_H(x) d\mathcal{H}^n|_H(y) \\
&\qquad + \int_{Q_0} \int_{MQ_0} \Big| \frac{1}{2}\bar{K}(x,y) - \frac{1}{2}\nabla_1 \Theta(x,y;\bar{A}(y)) \Big| \, d\mathcal{H}^n|_H(x) d\mathcal{H}^n|_H(y) \\
&\qquad + \frac{1}{2} \Big| \int_{Q_0} \int_{MQ_0} \big(\nabla_1 \Theta(x,y;\bar{A}(x)) + \nabla_1 \Theta(x,y;\bar{A}(y))\big)\, d\mathcal{H}^n|_H(x) d\mathcal{H}^n|_H(y) \Big|.
\end{aligned}$$

Then, the third summand is null because of the antisymmetry of its integrand and the first two terms can be estimated via Lemma 4.2.2.

Gathering (4.6.2), (4.6.4) and (4.6.6) we are able to conclude the proof of the lemma. □

4.7 The David and Mattila lattice associated with μ and its properties

The dyadic lattice constructed by David and Mattila [DM00, Theorem 3.2] is a powerful tool in the study of the geometry of Radon measures. Its main properties are listed in the following lemma, that we state for a general Radon measure with compact support.

Lemma 4.7.1 (David and Mattila). *Let σ be a compactly supported Radon measure in $\mathbb{R}^{n+1}$. Consider two constants $K_0 > 1$ and $A_0 > 5000K_0$ and denote $W = \operatorname{supp} \sigma$. Then there exists a sequence of partitions of W into Borel subsets Q, $Q \in \mathcal{D}_{\sigma,k}$, which we will refer to as cells, with the following properties:*

- *For each integer $k \ge 0$, W is the disjoint union of the cells Q, $Q \in \mathcal{D}_{\sigma,k}$. If $k < l$, $Q \in \mathcal{D}_{\sigma,l}$, and $R \in \mathcal{D}_{\sigma,k}$, then either $Q \cap R = \emptyset$ or $Q \subset R$.*
- *For each $k \ge 0$ and each cell $Q \in \mathcal{D}_{\sigma,k}$, there is a ball $B(Q) = B\big(z_Q, r(Q)\big)$ such that*

$$\begin{gathered}
z_Q \in W,\ A_0^{-k} \le r(Q) \le K_0 A_0^{-k} \\
W \cap B(Q) \subset Q \subset W \cap 28B(Q) = W \cap B\big(z_Q, 28r(Q)\big),
\end{gathered}$$

and the balls $5B(Q), Q \in \mathcal{D}_{\sigma,k}$ are disjoint.
- *The cells $Q \in \mathcal{D}_{\sigma,k}$ have small boundaries. By this, we mean that for each $Q \in \mathcal{D}_{\sigma,k}$ and each integer $l \ge 0$, if we set*

$$\begin{aligned}
N_l^{\text{int}} &:= \big\{x \in Q : \operatorname{dist}(x, W \setminus Q) < A_0^{-k-l}\big\} \\
N_l^{\text{ext}}(Q) &:= \big\{x \in W \setminus Q : \operatorname{dist}(x, Q) < A_0^{-k-l}\big\}
\end{aligned}$$

and

$$N_l(Q) := N_l^{\text{int}}(Q) \cup N_l^{\text{ext}}(Q),$$

we get

$$\sigma\big(N_l(Q)\big) \leq \big(C^{-1}K_0^{-3(n+1)-1}A_0\big)^{-l}\sigma\big(90B(Q)\big)$$

- *Denote by* $\mathcal{D}_{\sigma,k}^{\text{db}}$ *the family of cells* $Q \in \mathcal{D}_{\sigma,k}$ *for which*

$$\sigma\big(100B(Q)\big) \leq K_0\sigma\big(B(Q)\big).$$

We have that $r(Q) = A_0^{-k}$ *when* $Q \in \mathcal{D}_{\sigma,k} \setminus \mathcal{D}_{\sigma,k}^{\text{db}}$ *and*

$$\sigma\big(100B(Q)\big) \leq K_0^{-1}\sigma\big(100^{l+1}B(Q)\big) \tag{4.7.1}$$

for all $l \geq 1$ *with* $100^l \leq K_0$ *and* $Q \in \mathcal{D}_{\sigma,k} \setminus \mathcal{D}_{\sigma,k}^{\text{db}}$.

Let us denote $\mathcal{D}_\sigma := \bigcup_k \mathcal{D}_{\sigma,k}$. Let us choose A_0 big enough so that

$$C^{-1}K_0^{-3(n+1)-1}A_0 > A_0^{1/2} > 10. \tag{4.7.2}$$

Here we list some useful quantities associated with each cell $Q \in \mathcal{D}_{\sigma,k}$:

- $J(Q) := k$, which may be interpreted as the *generation* of Q.
- $\ell(Q) := 56K_0A_0^{-k}$, that we also call *side length*. Notice that

$$\frac{1}{28}K_0^{-1}\ell(Q) \leq \operatorname{diam}(28B(Q)) \leq \ell(Q)$$

and $r(Q) \approx \operatorname{diam}(Q) \approx \ell(Q)$.
- calling z_Q the *center* of Q, we denote $B_Q := 28B(Q) = B(z_Q, 28r(Q))$, which in particular gives

$$Q \cap \frac{1}{28}B_Q \subset Q \subset B_Q.$$

We recall, now, some of the properties of the cells in the David and Mattila lattice.

The choice in (4.7.2) implies, for $0 < \lambda \leq 1$, the estimate

$$\sigma\big(\{x \in Q : \operatorname{dist}(x, W \setminus Q) \leq \lambda\ell(Q)\}\big) + \sigma\big(\{x \in 3.5B_Q \setminus Q : \operatorname{dist}(x, Q) \leq \lambda\ell(Q)\}\big) \\ \leq c\lambda^{1/2}\sigma(3.5B_Q).$$

We denote $\mathcal{D}_\sigma^{\text{db}} := \bigcup_{k\geq 0}\mathcal{D}_{\sigma,k}^{\text{db}}$ and we say that it is the lattice of *doubling cells*. This notation is justified by the fact that, for $Q \in \mathcal{D}_\sigma^{\text{db}}$, we have

$$\sigma(3.5B_Q) \leq \sigma(100B(Q)) \leq K_0\sigma(B(Q)) \leq K_0\sigma(Q).$$

An important feature of the David and Mattila lattice is that every cell $Q \in \mathcal{D}_\sigma$ can be covered by doubling cells up to a set of σ-measure zero ([DM00, Lemma 5.28]). Moreover, if we have two cells $R, Q \in D_\sigma$ with $Q \subset R$ and such that every intermediate cell $Q \subsetneq S \subsetneq R$ belongs to $\mathcal{D}_\sigma \setminus \mathcal{D}_\sigma^{\text{db}}$, we have the control

$$\sigma(100B(Q)) \leq A_0^{-10n(J(Q)-J(R)-1)}\sigma(100B(R)) \tag{4.7.3}$$

on the decay of the measure. The estimate (4.7.3) is proved via an iterated application of the inequality

$$\sigma(100B(Q)) \leq A_0^{-10n}\sigma(100B(\hat{Q})), \tag{4.7.4}$$

where $\hat{Q}$ is the cell from $\mathcal{D}_{\sigma,J(Q)-1}$ containing Q (also called *parent* of Q). We remark that (4.7.4) follows by (4.7.1) and a proper choice of A_0 and K_0 (see [DM00, Lemma 5.31]).

For $Q \in \mathcal{D}_\sigma$, we denote by $\mathcal{D}_\sigma(Q)$ the cells in $\mathcal{D}_\sigma$ which are contained in Q and $\mathcal{D}_\sigma^{\rm db}(Q) := \mathcal{D}_\sigma(Q) \cap \mathcal{D}_\sigma^{\rm db}$.

4.8 The Key Lemma, the stopping time condition and a first modification of the measure

The hearth of the proof of Lemma 4.5.5 is to provide a control on the abundance of cells with low density (in some sense that we clarify below). The whole construction that we are about to discuss depends on some auxiliary parameter to be chosen properly later in the proof.

Definition 4.8.1 (Low density cells)**.** Let $0 < \theta_0 \ll 1$. A cell $Q \in \mathcal{D}_\mu$ is said to be of low density if

$$\Theta_\sigma(3.5B_Q) \le \theta_0$$

and it has maximal side length. We denote by LD the family of low density cells.

Most of the rest of the chapter deals with the proof of the fact that the low density cells fail to cover a significant portion of Q_0.

Lemma 4.8.1 (Key Lemma)**.** *Let ϵ, δ and M be as in Lemma 4.4.1. There exists $\epsilon_0 > 0$ such that if M is big enough and θ_0, δ and ϵ are small enough, then*

$$\mu\left(Q_0 \setminus \bigcup_{Q \in \mathsf{LD}} Q\right) \ge \epsilon_0 \mu(Q_0). \tag{4.8.1}$$

To prove the main Lemma 4.5.5 using the results in the Key Lemma, it suffices to refer to the construction in [GT18, Section 10], which relies on a subtle covering argument together with the connection between uniform rectifiability and the Riesz transform, and invoke [PPT18, Theorem 1.1 and Theorem 1.2] in place of the results of Nazarov, Tolsa and Volberg. So, the rest of the present article (a part from the last section) is devoted to the proof of Lemma 4.8.1.

We argue by contradiction: assume that (4.8.1) does not hold, that is to say

$$\mu\left(\bigcup_{Q \in \mathsf{LD}} Q\right) > (1-\epsilon_0)\mu(Q_0). \tag{4.8.2}$$

More specifically, we want to show that a choice of ϵ_0 small enough leads to an absurd. The proof is based on a stopping time argument. Roughly speaking, for $Q \in \mathsf{LD}$, we say that a cell R belongs to its associated stopping family if it is a descendant of Q (i.e. $R \subset Q$) and it is sufficiently small. The definition of stopping cells depends on a parameter t, which has to be thought small and that will be appropriately chosen later.

Definition 4.8.2 (Stopping cells)**.** Let $Q \in \mathsf{LD}$. Let $0 < t < 1$. We say that $R \in \mathsf{Stop}(Q)$ if the following conditions are verified and it has maximal side length

- $R \in \mathcal{D}_\mu^{\rm db}$, $R \subset Q$.
- $\ell(R) \le t\ell(Q)$.

We also denote $\mathsf{Stop} := \bigcup_{Q\in\mathsf{LD}} \mathsf{Stop}(Q)$ the family of all the stopping cells.

Assuming that the stopping cells in $\mathsf{Stop}(Q)$ are doubling makes sense in light of the fact that doubling cells cover Q up to a set of μ-measure zero. In particular, this implies that (4.8.2) is equivalent to

$$\mu\Bigg(\bigcup_{Q\in\mathsf{Stop}} Q\Bigg) > (1-\epsilon_0)\mu(Q_0).$$

The proof of the Key Lemma 4.8.1 involves a periodization of the measure μ, which is essentially carried out by replicating $\mu|Q_0$ on the horizontal plan according to the periodicity of the matrix $\bar{A}$.

The cells close to the boundary of Q_0 may give problems, so that our first temptation would be to try not to incorporate them into the contruction. This is possible just in the case their contribution to the measure of Q_0 is negligible. So, we say that $P \in \mathsf{Bad}$ if $P \in \mathsf{Stop}$ and $1.1B_P \cap \partial Q_0 \neq \emptyset$.

Another technical problem is that Stop may contain infinitely many cells. This second difficulty can be easily overcome considering a finite family of cells, named Stop_0, which contains a big portion of the measure of Stop, e.g.

$$\mu\Bigg(\bigcup_{Q\in\mathsf{Stop}_0} Q\Bigg) > (1-2\epsilon_0)\mu(Q_0). \tag{4.8.3}$$

The rest of the section is devoted to a justification of the last affirmations concerning Bad and the first modification of the measure μ. It is essentially a rewriting of [GT18, Lemma 6.2, Lemma 6.3, Lemma 6.4] in our context, in which we highlight the right homogeneities coming from our elliptic setting.

The following lemma contains an estimate of the density $P_{\mu,\tilde{\alpha}}$ of the stopping cells in terms of the low density parameter θ_0.

Lemma 4.8.2. *Let $Q \in \mathsf{Stop}$ and let $t = \theta_0^{1/(n+\tilde{\alpha})}$. We have*

$$\Theta_\mu(2B_Q) \le P_{\mu,\tilde{\alpha}}(2B_Q) \lesssim \theta_0^{\frac{\tilde{\alpha}}{n+\tilde{\alpha}}}.$$

Proof. The first inequality is an immediate consequence of the definition of $P_{\mu,\tilde{\alpha}}$. To prove the second inequality, we consider the maximal cell $R' \in \mathcal{D}_\mu$ such that $Q \subset R' \subset R$ and $\ell(R') \le t\ell(R)$ and write

$$\begin{aligned} P_{\mu,\tilde{\alpha}}(2B_Q) \lesssim & \sum_{P\in\mathcal{D}_\sigma: Q\subset P\subset R'} \Theta_\mu(2B_P)\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\tilde{\alpha}} + \sum_{P\in\mathcal{D}_\sigma: R'\subset P\subset R} \Theta_\mu(2B_P)\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\tilde{\alpha}} \\ & + \sum_{P\in\mathcal{D}_\sigma: R\subset P\subset Q_0} \Theta_\mu(2B_P)\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\tilde{\alpha}} + \sum_{k\ge1} 2^{-k\tilde{\alpha}}\Theta_\mu(2^kB_P) \\ =& I + II + III + IV. \end{aligned}$$

Then, the estimates work as in the case of the Riesz transform. In particular, the same arguments prove

$$I + II \lesssim \frac{\theta_0}{t^n}$$

and

$$III + IV \lesssim t^{\tilde{\alpha}},$$

which justifies the choice of t in the statement of the lemma. □

For the rest of the chapter we assume $t = \theta_0^{1/(n+\tilde{\alpha})}$.

Using the estimates in Lemma 4.8.2, one can prove (see [GT18, Lemma 6.3] that

$$\mu\Big(\bigcup_{\mathsf{Bad}} Q\Big) \lesssim \theta_0^{\frac{\tilde{\alpha}}{n+\tilde{\alpha}}} \mu(Q_0). \tag{4.8.4}$$

First modification of the measure. As already mentioned, for technical purposes it is useful to modify the measure inside Q_0 by taking just finitely many stopping cells and getting rid of the cells in Bad. To make the previous statement rigorous, we choose a small parameter $0 < \kappa_0 \ll 1$ to be fixed later and, after denoting

$$I_{\kappa_0}(Q) \coloneqq \{x \in Q : \operatorname{dist}(x, \operatorname{supp}\sigma \setminus Q) \geq \kappa_0 \ell(Q)\},$$

we define the modified measure

$$\mu_0 \coloneqq \mu|_{Q_0^c} + \sum_{Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}} \mu|_{I_{\kappa_0}(Q)}.$$

Using (4.8.3) and (4.8.4), it is not difficult to prove that μ_0 differs from μ, in the sense of the total mass, possibly by a very small quantity. Indeed,

$$\|\mu - \mu_0\| \leq \Big(2\epsilon_0 + C\theta_0^{\tilde{\alpha}/(n+\tilde{\alpha})} + \kappa_0^{1/2}\Big)\mu(Q_0). \tag{4.8.5}$$

For this modification to be useful to our purposes, we need the gradient of the single layer potential associated with this measure to satisfy a localization estimate analogue to (4.6.1). This is easily proved by gathering the $L^2(\mu|_{Q_0})$-boundedness of $\bar{T}_{\mu|_{Q_0}}$, the estimate (4.8.5) and the localization estimate (4.6.1) for μ (see [GT18, Lemma 6.4]).

Lemma 4.8.3. *If δ is chosen small enough (depending on M), then*

$$\int_{Q_0} |\bar{T}(\chi_{MQ_0}\mu_0)|^2 d\mu_0 \lesssim \Big(\epsilon + \frac{1}{M^{2\tilde{\alpha}}} + M^{4n+2}\delta^{1/(4n+4)} + (M\ell(Q_0))^{2\tilde{\alpha}} + \epsilon_0 + \theta_0^{\tilde{\alpha}/(n+\tilde{\alpha})} + \kappa_0^{1/2}\Big)\mu(Q_0).$$

4.9 Periodization and smoothing of the measure

The periodization. We want to get rid of the truncation at the level of $M\ell(Q_0)$ present in Lemma 4.8.3. This can be done replicating the measure periodically by means of horizontal translations. The localization of the gradient of the single layer potential associated with this auxiliary measure will make us able to implement a variational argument in Section 4.11.

We denote by

$$\mathcal{M} \coloneqq \big\{Q_0 + z_P : z_P \in 6\ell(Q_0)\mathbb{Z}^n \times \{0\}\big\}$$

the family of disjoint cubes covering H and obtained translating Q_0 along the coordinate (horizontal) axes. The factor 6 is chosen in order for this periodization to be coherent with the period of the matrix $\bar{A}$. Given $P \in \mathcal{M}$ we denote by z_P its center and by $T_P \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ the translation

$$T_P(x) \coloneqq x + z_P,$$

so that the periodization of the measure reads

$$\widetilde{\mu} := \sum_{P\in\mathcal{M}} T_{P\sharp}\mu_0|_{Q_0}.$$

Observe that $\mu_0(\partial Q_0) = 0$, which implies $\chi_{Q_0}\widetilde{\mu} = \mu_0$.

As for the first modification of the measure, we have to prove the equivalent of the localization Lemma 4.8.3. This can be done as for the Riesz transform (see [GT18, Lemma 7.2]) because $\widetilde{\mu}$ is very flat at the level of $3MQ_0$.

Lemma 4.9.1. *Let κ_0, θ_0 and ϵ_0 be as in Section 4.8 and δ as in the Main Lemma. Letting*

$$\tilde{\delta} := M^{n+1}\Big(\epsilon_0 + \theta_0^{\tilde{\alpha}/(n+\tilde{\alpha})} + \kappa_0^{1/2} + \delta^{1/2}\Big),$$

we have

$$\alpha^H_{\widetilde{\mu}}(3MQ_0) \lesssim \tilde{\delta}.$$

Moreover, for

$$\tilde{\epsilon} := \epsilon + \frac{1}{M^{2\tilde{\alpha}}} + M^{4n+2}\delta^{1/(4n+4)} + \epsilon_0 + \theta_0^{\tilde{\alpha}/(n+\tilde{\alpha})} + \kappa_0^{1/2} + M^{2n+2}\tilde{\delta}^{1/(4n+5)} + (M\ell(Q_0))^{2\tilde{\alpha}}$$

we have

$$\int_{Q_0} \left|\bar{T}(\chi_{MQ_0}\widetilde{\mu})\right|^2 d\widetilde{\mu} \lesssim \tilde{\epsilon}\widetilde{\mu}(Q_0).$$

It is not difficult to see that the measure $\widetilde{\mu}$ has polynomial growth:

$$\widetilde{\mu}(B(x,r)) \lesssim r^n \quad \text{for every } x\in\mathbb{R}^{n+1} \text{ and } r>0.$$

The following lemma contains a technical estimate for a suitably modified version of the density $P_{\widetilde{\mu},\tilde{\alpha}}(2B_Q)$.

Lemma 4.9.2. *For every $Q \in \mathsf{Stop}_0 \setminus \mathsf{Bad}$ the inequality*

$$\int_{1.1B_Q\setminus Q}\int_Q \frac{1}{|x-y|^n}\, d\widetilde{\mu}(x)\, d\widetilde{\mu}(y) \lesssim \theta_0^{\frac{\tilde{\alpha}}{(n+\tilde{\alpha})(1+2n)}}\widetilde{\mu}(Q)$$

holds. Moreover, the function

$$p_{\widetilde{\mu},\tilde{\alpha}}(x) := \sum_{Q\in\mathsf{Stop}_0\setminus\mathsf{Bad}} \chi_Q P_{\widetilde{\mu},\tilde{\alpha}}(2B_Q)$$

satisfies

$$\int_{Q_0} p^2_{\widetilde{\mu},\tilde{\alpha}}\, d\widetilde{\mu} \lesssim \theta_0^{\frac{2\tilde{\alpha}}{(n+\tilde{\alpha})(1+2\tilde{\alpha})}}\widetilde{\mu}(Q_0). \tag{4.9.1}$$

Remark on the proof. In order to prove (4.9.1) it suffices to follow the path of [GT18, Lemma 7.4] taking into consideration the right homogeneity given by α, which leads to

$$\int_{Q_0} p^2_{\widetilde{\mu},\tilde{\alpha}}\, d\widetilde{\mu} \lesssim \Big(\kappa + \frac{\theta_0^{\frac{2\tilde{\alpha}}{(n+\tilde{\alpha})}}}{\kappa^{2\tilde{\alpha}}} + \theta_0^{\frac{\tilde{\alpha}}{n+\tilde{\alpha}}}\Big)\widetilde{\mu}(Q_0), \tag{4.9.2}$$

where $0<\kappa<1$ is a small constant. Inequality (4.9.2) gives the desired estimate after making the choice $\kappa = \theta_0^{2\tilde{\alpha}/[(n+\tilde{\alpha})(1+2\tilde{\alpha})]}$. □

The smoothing. A priori, the measure μ_0 may not be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^{n+1}$. This would constitute a problem when trying to implement the variational techniques. For this reason, it it useful to consider the following further modification of the measure

$$\eta_0 := \sum_{Q\in \mathsf{Stop}_0 \setminus \mathsf{Bad}} \frac{\mu_0(Q)}{\mathcal{H}^{n+1}(\frac{1}{4}B(Q))}\mathcal{H}^{n+1}|_{\frac{1}{4}B(Q)}$$

and its periodization

$$\eta := \sum_{P\in\mathcal{M}} T_{P\sharp}\eta_0.$$

We remark that, being Stop_0 a finite family, the measures η_0 and η both have bounded density with respect to $\mathcal{H}^{n+1}$. A specific control on the density is not relevant to the purposes of our proof. The following lemma contains a localization estimate for the potential associated with η.

Lemma 4.9.3. *Denoting*

$$\epsilon' := \widetilde{\epsilon} + \ell(Q_0)^{2\tilde{\alpha}} + M^n\kappa_0^{-2n-2\tilde{\alpha}}\theta_0^{\frac{2\tilde{\alpha}}{(n+\tilde{\alpha})(1+2\tilde{\alpha})}} + \theta_0^{\frac{2\tilde{\alpha}}{(n+\tilde{\alpha})(1+2n)}},$$

we have

$$\int_{Q_0} |\bar{T}(\chi_{MQ_0}\eta)|^2 d\eta \lesssim \epsilon'\eta(Q_0).$$

The presence of the summand $\ell(Q_0)^{2\tilde{\alpha}}$ in ϵ' (already taken into account in $\widetilde{\epsilon}$) to point out that, as in (4.6.6), the lack of antisimmetry of $\bar{K}(\cdot,\cdot)$ gives the error term

$$\left|m_{\widetilde{\mu},Q}(\bar{T}_{\widetilde{\mu}}\chi_Q)\right| \lesssim \ell(Q)^{\tilde{\alpha}} \lesssim \ell(Q_0)^{\tilde{\alpha}}$$

for every $Q\in \mathsf{Stop}_0 \setminus \mathsf{Bad}$. This contribution is not present in the case of an elliptic matrix with constant coefficients. The rest of the proof is analogous to the one of [GT18, Lemma 8.1] and all is needed is a careful check that Lemma 4.9.2 applies and the new homogeneity does not affect the final result. We omit further details.

Remark 9. Observe that the expressions of $\widetilde{\delta}, \widetilde{\epsilon}$ and ϵ' all include a summand which depends on ϵ_0. In particular, the quantities in question are small if ϵ_0 is chosen small enough. Then, the choice $\varepsilon_0 \ll 1$ (which is possible because we assumed (4.8.2) to hold) gives the localization for the potentials associated with the auxiliary measures.

4.10 The localization of $\bar{T}\eta$

Let $L^\infty_{\mathcal{M}}$ denote the set of functions $f\in L^\infty(\eta)$ such that

$$f(x+z_P) = f(x)$$

for every $x\in\mathbb{R}^{n+1}$ and $P\in\mathcal{M}$.

Let $\varphi\in C^1(\mathbb{R}^{n+1})$ be a non-negative function whose support is contained in $B(0,2)$ and that equals 1 on $B(0,1)$. For $r>0$ and $x\in\mathbb{R}^{n+1}$ let us set $\varphi_r(x) := \varphi(x/r)$. Observe that $\|\nabla\varphi\|_\infty \lesssim 1$. For $x,y\in\mathbb{R}^{n+1}$ we define the regularized kernel

$$\tilde{K}_r(x,y) = \bar{K}(x,y)\varphi_r(x-y)$$

and its associated operator

$$\tilde{T}_r(f\eta)(x) := \int \tilde{K}_r(x,y)f(y)d\eta(y), \qquad \text{for } f \in L^\infty_{\mathcal{M}}(\eta),$$

where the integral above is absolutely convergent. We are interested in getting an existence result for the limit

$$\mathrm{p.v}\,\bar{T}(f\eta)(x) = \lim_{r\to\infty} \tilde{T}_r(f\eta)(x). \tag{4.10.1}$$

For simplicity, we denote the principal value in (4.10.1) just as $\bar{T}(f\eta)(x)$.

Lemma 4.10.1. *Let $f \in L^\infty_{\mathcal{M}}$. The principal value $\bar{T}(f\eta)(x)$ exists for every $x \in \mathbb{R}^{n+1}$. Moreover, given any compact set $F \subset \mathbb{R}^{n+1}$, there exist $r_0 = r_0(F) > 0$ and a constant c_F depending on F such that for $s > r \geq r_0$*

$$\big\|\tilde{T}_r(f\eta) - \tilde{T}_s(f\eta)\big\|_{\infty,F} \lesssim \frac{c_F}{r^\gamma}\|f\|_\infty,$$

where $\gamma \in (0,1)$ is as in Lemma 4.2.7.

Remark 10. Lemma 4.10.1 implies that the limit in (4.10.1) converges uniformly on compact sets and in $\operatorname{supp}\eta$.

Proof. Recall that we can assume $\ell(Q_0) < 1$. Let $s > r$. Let us denote $\nu := f\eta$ and $\varphi_{r,s}(x) := \varphi_r(x) - \varphi_s(x)$ for every $x \in \mathbb{R}^{n+1}$ and $\tilde{K}(x,y) := \bar{K}(x,y)\varphi_{r,s}(x-y)$. Because of the periodicity of f and the definition of η, we have

$$\nu = \sum_{P\in\mathcal{M}} (T_P)_\sharp(\chi_{Q_0}\nu)$$

so that

$$\begin{aligned}\tilde{T}_r(f\eta)(x) - \tilde{T}_s(f\eta)(x) &= \int \tilde{K}_{r,s}(x,y)d\Big(\sum_{P\in\mathcal{M}} (T_P)_\sharp(\chi_{Q_0}\nu)\Big)(y)\\ &= \sum_{P\in\mathcal{M}} \int_{Q_0} \tilde{K}_{r,s}(x,y+z_p)d\nu(y),\end{aligned} \tag{4.10.2}$$

the last equality being a consequence of $\tilde{K}_{r,s}$ having compact support, which implies that the sum has only finitely many non-zero terms.

Let A_0 be the homogenized matrix associated with $\{\mathcal{L}_\epsilon\}_{\epsilon>0}$ and χ_ℓ be as in Section 4.2, with $\ell = 6\ell(Q_0)$. Recall that

$$\|\nabla\chi_\ell\|_\infty \lesssim 1.$$

The matrix A_0 is an elliptic matrix whose coefficients are constant and can be expressed in terms of χ and those of A. We denote by $\Theta(\cdot,\cdot;A_0)$ the fundamental solution of the operator $\mathcal{L}_0 = -\operatorname{div}(A_0\nabla)$. We decompose the right hand side of

(4.10.2) as

$$\begin{aligned}
&\sum_{P\in\mathcal{M}}\int_{Q_0}\tilde K_{r,s}(x,y+z_p)d\nu(y)\\
&=\sum_{P\in\mathcal{M}}\int_{Q_0}\big(\bar K(x,y+z_P)-(Id+\nabla\chi_\ell(x))\nabla_1\Theta(x,y+z_P;A_0)\big)\varphi_{r,s}(x-y-z_P)d\nu(y)\\
&\qquad+\sum_{P\in\mathcal{M}}\int_{Q_0}\big(Id+\nabla\chi_\ell(x)\big)\nabla_1\Theta(x,y+z_P;A_0)\varphi_{r,s}(x-y-z_P)d\nu(y)\\
&\equiv I_{r,s}(x)+II_{r,s}(x).
\end{aligned}$$

Let us observe that since F is compact and $y\in Q_0$, there exists a compact set $\tilde F$ such that $\pm(x-y)\in\tilde F$, so that if we choose $r_0\geq 2\operatorname{diam}(\tilde F)$, both $\varphi_{r,s}(x-y-z_P)$ and $\varphi_{r,s}(x-y+z_P)$ vanish for $|z_P|<r$. Moreover, $|x-y|\leq\operatorname{diam}(\tilde F)\leq r/2\leq|z_P|$ and

$$|(x-y)-z_P|\approx|(x-y)+z_P|\approx|z_P|.$$

Let us now estimate $I_{r,s}(x)$. As stated in Lemma 4.2.7, there exist $C>0$ and $\gamma\in(0,1)$ depending only on n and α such that

$$\big|\nabla_1\mathcal{E}_{\bar A}(x,y+z_P)-\big(Id+\nabla\chi_\ell(x)\big)\nabla_1\Theta(x,y+z_P;A_0)\big|\leq C\ell(Q_0)^\gamma|x-y-z_P|^{-(n+\gamma)}$$

for every $x,y\in\mathbb{R}^{n+1}$. Then, exploiting the linear growth of η and the considerations on the support of $\varphi_{r,s}$, we get

$$\begin{aligned}
|I_{r,s}(x)|&\lesssim\sum_{P\in\mathcal{M},|z_P|\geq r}\int_{Q_0}\frac{\ell(Q_0)^\gamma d|\nu|(y)}{|x-y-z_P|^{n+\gamma}}\\
&\lesssim\|f\|_\infty\sum_{P\in\mathcal{M},|z_P|\geq r}\frac{\ell(P)^{n+\gamma}}{|z_P|^{n+\gamma}}\\
&\lesssim\frac{\|f\|_\infty\ell(Q_0)^\gamma}{r^\gamma}.
\end{aligned}\tag{4.10.3}$$

In the last inequality of (4.10.3) we used the convergence of $\sum_{P\in\mathcal{M}}\ell(P)^n|z_P|^{-n}$.

We are left with the estimate of $II_{r,s}(x)$. Using the antisymmetry of $\nabla_1\Theta(\cdot,\cdot;A_0)$ and the properties of standard Calderón-Zygmund kernels, the same argument of [GT18, Lemma 8.2] proves that there exists a constant $c_F>0$ such that

$$\begin{aligned}
|II_{r,s}(x)|&\lesssim\|Id+\nabla\chi_\ell\|_\infty\sum_{P\in\tilde{\mathcal{M}}}\int_{Q_0}\nabla_1\Theta(x,y+z_P;A_0)\varphi_r(x-y-z_P)d\nu(y)\\
&\lesssim\sum_{P\in\tilde{\mathcal{M}}}\int_{Q_0}\nabla_1\Theta(x,y+z_P;A_0)\varphi_r(x-y-z_P)d\nu(y)\\
&\lesssim\frac{c_F\|f\|_\infty}{r}.
\end{aligned}\tag{4.10.4}$$

We conclude the proof of the lemma gathering (4.10.3), (4.10.4) and observing that, being $\gamma\in(0,1)$ and $r>1$, $r^{-1}<r^{-\gamma}$. □

The measure η is $\mathcal{M}$-periodic and the matrix $\bar A$, by construction, is $6\ell(Q_0)$-periodic. This implies that for every $f\in L^\infty_{\mathcal{M}}(\eta)$ and $r>0$, the function $\widetilde{T}_r(f\eta)$

is $\mathcal{M}$-periodic, too. The same holds for $\mathrm{p.v}\, T(f\eta)$. Using Lemma 4.10.1, the following result is immediate.

Corollary 4.10.1. *$\bar{T}_\eta$ is a bounded operator from $L^\infty_{\mathcal{M}}$ to $L^\infty_{\mathcal{M}}$. For $r>0$ big enough and for every $f \in L^\infty_{\mathcal{M}}(\eta)$ we have*

$$\left\|\bar{T}(f\eta) - \tilde{T}_r(f\eta)\right\|_{\infty,F} \lesssim_F \frac{\|f\|_\infty}{r^\gamma}.$$

Our next intent is to prove the final localization estimate

$$\int_{Q_0} \left|\bar{T}\eta\right|^2 d\eta \ll \eta(Q_0). \tag{4.10.5}$$

We have already proved that for M big enough there exists $\epsilon' \ll 1$ such that

$$\int_{Q_0} \left|\bar{T}(\chi_{MQ_0}\eta)\right|^2 d\eta \lesssim \epsilon' \eta(Q_0). \tag{4.10.6}$$

Then, in order to prove (4.10.5), it suffices to use the estimate in the following lemma.

Lemma 4.10.2. *Let $f \in L^1_{loc}(\eta)$ be a $\mathcal{M}$-periodic function and let $\tilde{M} = 6\tilde{N}$, where $\tilde{N} \geq 3$ is an odd number. For all $x \in 2Q_0$ we have*

$$\left|\bar{T}\big(\chi_{(\tilde{M}Q_0)^c} f\eta\big)(x)\right| \lesssim \frac{1}{\tilde{M}^\gamma \ell(Q_0)^n} \int_{Q_0} |f| d\eta. \tag{4.10.7}$$

Proof. Being $\widetilde{N}$ odd, there exists a subfamily $\widetilde{\mathcal{M}} \subset \mathcal{M}$ such that

$$\chi_{(\widetilde{M}Q_0)^c}\eta = \sum_{P \in \widetilde{\mathcal{M}}} T_{P\sharp}\eta$$

and whose elements $P \in \widetilde{\mathcal{M}}$ satisfy $|z_P| \gtrsim \widetilde{M}\ell(Q_0)$. In particular

$$|x - y - z_P| \approx |z_p| \quad \text{for } x, y \in 2Q_0. \tag{4.10.8}$$

Let $r>0$ and $x \in 2Q_0$. Denote $\nu := f\eta$ and observe that there are just finitely many cubes $P \in \widetilde{\mathcal{M}}$ such that $|z_P| < r$. Arguing as in the proof of Lemma 4.10.1,

$$\begin{aligned}
\tilde{T}_r\big(\chi_{(\tilde{M}Q_0)^c} f\eta\big)(x) &= \int \bar{K}(x,y)\varphi_r(x-y)d\nu(y) \\
&= \sum_{P \in \widetilde{M}} \int_{Q_0} \bar{K}(x, y+z_P)\varphi_r(x-y-z_P)d\nu(y) \\
&= \sum_{P \in \tilde{\mathcal{M}}} \int_{Q_0} \big(\bar{K}(x,y+z_P) - (Id + \nabla\chi_\ell(x))\nabla_1\Theta(x, y+z_P; A_0)\big)\varphi_r(x-y-z_P)d\nu(y) \\
&\quad + \sum_{P \in \tilde{\mathcal{M}}} \int_{Q_0} \big(Id + \nabla\chi_\ell(x)\big)\nabla_1\Theta(x, y+z_P; A_0)\varphi_r(x-y-z_P)d\nu(y) \\
&\equiv I_r(x) + II_r(x)
\end{aligned}$$

Let us estimate $I_r(x)$. Using (4.10.8) together with Lemma 4.2.7 and the estimate $|z_P| \gtrsim \tilde{M}\ell(Q_0)$ for $P \in \tilde{\mathcal{M}}$, we can write

$$|I_r(x)| \lesssim \sum_{P\in\tilde{\mathcal{M}}} \int_{Q_0} \frac{\ell(Q_0)^\gamma}{|x-y-z_P|^{n+\gamma}} d\nu(y) \approx \sum_{P\in\tilde{\mathcal{M}}} \int_{Q_0} \frac{\ell(Q_0)^\gamma}{|z_P|^{n+\gamma}} d\nu(y)$$
$$= \sum_{P\in\tilde{\mathcal{M}}} \frac{\ell(Q_0)^\gamma}{|z_P|^{n+\gamma}} |\nu|(Q_0) \lesssim \frac{|\nu|(Q_0)}{\tilde{M}^\gamma \ell(Q_0)^n} \Big(\sum_{P\in\tilde{\mathcal{M}}} \frac{\ell(Q_0)^n}{|z_P|^n} \Big) \lesssim \frac{1}{\tilde{M}^\gamma \ell(Q_0)^n} \int_{Q_0} |f| d\eta. \tag{4.10.9}$$

We claim that

$$|II_r(x)| \lesssim \frac{1}{\tilde{M}\ell(Q_0)^n} \int_{Q_0} |f| d\eta. \tag{4.10.10}$$

The calculations to prove (4.10.10) exploit the fact that $\|\nabla\chi_\ell\|_\infty \lesssim 1$ and the anti-simmetry of $\nabla_1 \Theta(\cdot,\cdot;A_0)$ and resemble those of [GT18, Lemma 8.4], so that we leave the verification to the reader.
The estimates (4.10.9) and (4.10.10), together with the observation that $\tilde{M}^{-1} \leq \tilde{M}^{-\gamma}$, conclude the proof of the lemma after taking the limit for $r \to \infty$. □

Corollary 4.10.2 (Final localization estimate). *We have*

$$\int_{Q_0} |\bar{T}\eta|^2 d\eta \lesssim \Big(\frac{1}{M^{2\gamma}} + \epsilon'\Big)\eta(Q_0).$$

Proof. Inequality (4.10.7) in the case $f \equiv 1$ reads

$$|\bar{T}(\chi_{MQ_0}\eta)| \lesssim \frac{1}{M^\gamma},$$

so that applying it together with (4.10.6), we have

$$\int_{Q_0} |\bar{T}\eta|^2 d\eta \lesssim \int_{Q_0} |\bar{T}(\chi_{MQ_0}\eta)|^2 d\eta + \int_{Q_0} |\bar{T}(\chi_{(MQ_0)^c}\eta)|^2 d\eta \lesssim \Big(\frac{1}{M^{2\gamma}} + \epsilon'\Big)\eta(Q_0),$$

which finishes the proof. □

4.11 A pointwise inequality and the conclusion of the proof

The following lemma implements a variational technique inspired by potential theory that allows to obtain a pointwise inequality for the potential of a proper auxiliary measure. We denote as $\bar{T}^*\vec{\xi}$ the operator that, given a vector-valued measure $\vec{\xi}$, is defined by

$$\bar{T}^*\vec{\xi}(x) = \int \nabla_1 \bar{\mathcal{E}}(y,x) \cdot d\vec{\xi}(y)$$

and which corresponds to the adjoint of $\bar{T}$.

Lemma 4.11.1. *Suppose that for some $0 < \lambda \leq 1$ the inequality*

$$\int_{Q_0} |\bar{T}\eta|^2 d\eta \leq \lambda\eta(Q_0)$$

holds. Then there is a function $b \in L^\infty(\eta)$ such that

- $0 \leq b \leq 2$.
- *b is $\mathcal{M}$-periodic.*

- $\int_{Q_0} b\, d\eta = \eta(Q_0)$.

and such that the measure $\nu = b\eta$ satisfies

$$\int_{Q_0} |\bar{T}\nu|^2 d\nu \leq \lambda \nu(Q_0) \tag{4.11.1}$$

and

$$|\bar{T}\nu|^2(x) + 2\bar{T}^*\big((\bar{T}\nu)\nu\big)(x) \leq 6\lambda \text{ for } \nu\text{-a.e. } x \in \mathbb{R}^{n+1}. \tag{4.11.2}$$

Proof. The proof is a minor variation of the proof of [GT18, Lemma 9.1]. In particular, we recall that the way to prove (4.11.2) consists in defining an adapted energy functional

$$J(a) = \lambda \|a\|_{L^\infty(\eta)} \eta(Q_0) + \int_{Q_0} |\bar{T}(a\eta)|^2 d\eta,$$

where a ranges in

$$\mathcal{A} = \Big\{ a \in L^\infty(\eta) : a \geq 0, a \text{ is } \mathcal{M}\text{-periodic, and } \int_{Q_0} a\, d\eta = \eta(Q_0) \Big\}.$$

Then, one proves that J admits a minimizer in $\mathcal{A}$ and gets (4.11.2) by taking proper competitors. The proof does not use the antisymmetry of the kernel of T but just its $\mathcal{M}$-periodicity which follows by the construction of $\bar{A}$. □

4.11.1 A maximum principle

Let λ, b and ν be as in Lemma 4.11.1. In order to perform the final argument to get the contradiction, we need to extend the inequality (4.11.2) out of the support of ν. More precisely, the next step consists in proving that a inequality similar to that provided by Lemma 4.11.1 holds in a suitable strip. To this purpose, some version of the maximum principle is needed. The elliptic setting of the problem makes this procedure slightly more technical than the one adopted by Girela-Sarrión and Tolsa in the case of the Riesz transform.

Before presenting the main result of the section, we introduce some notation. We denote by $\tilde{H}$ the hyperplane

$$\tilde{H} := \{x \in \mathbb{R}^{n+1} : x_{n+1} = 3\ell(Q_0)/2\},$$

which corresponds to the translate of H that contains the upper face of $3Q_0$. Let $K_S \gg 1$ to be chosen later and let S denote the strip

$$S := \{x \in \mathbb{R}^{n+1} : \operatorname{dist}(x, \tilde{H}) < K_S \ell(Q_0)\}.$$

Its boundary ∂S is given by the union of two hyperplanes ∂S_+ and ∂S_- which lay in the upper and lower half spaces respectively. Let

$$x_{S\pm} = \frac{3}{2}\ell(Q_0)(1, \ldots, 1, 1) \pm \big(0, \ldots, 0, K_S \ell(Q_0)\big). \tag{4.11.3}$$

For the proof of our next lemma we need to invoke a result on elliptic measure. Suppose that $\Omega \subsetneq \mathbb{R}^{n+1}$ is an open set with n-AD-regular boundary and consider a point $p \in \Omega$. Let ω_Ω^p denote the elliptic measure on $\partial\Omega$ associated with the operator $L_{\bar{A}}$ with pole at p. For the proof of the following standard result we refer to [AM17, Lemma 2.3].

Lemma 4.11.2. *Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be open with n-AD-regular boundary with constant C_{AD}. There exists $\vartheta = \vartheta(n, A, C_{AD}) \in (0,1)$ such that for every $x \in \partial\Omega$ and $0 < r <$ $\operatorname{diam}\Omega$, we have*

$$\omega_\Omega^y\big(B(x,r)^c\big) \le C\Big(\frac{|x-y|}{r}\Big)^\vartheta \qquad \text{for } y \in \Omega \cap B(x,r). \tag{4.11.4}$$

An application of (4.11.4) gives a boundary regularity result for $L_{\bar{A}}$-harmonic functions, see e.g Lemma 2.10 in [Azz+16a].

Lemma 4.11.3. *Let $\Omega \subsetneq \mathbb{R}^{n+1}$ be open with n-AD-regular boundary with constant C_{AD}. Let $u \ge 0$ be $L_{\bar{A}}$-harmonic function in $B(x,4r)\cap\Omega$ and continuous in $B(x,4r)\cap \bar{\Omega}$. Suppose, moreover, that $u \equiv 0$ in $\partial\Omega \cap B(x,4r)$. Then, extending u by zero in $B(x,4r)\setminus\bar{\Omega}$, there exists $\vartheta = \vartheta(n, A, C_{AD}) \in (0,1)$ such that u is ϑ-Hölder continuous in $B(x,4r)$ and, in particular,*

$$u(y) \lesssim_{n,A,C_{AD}} \Big(\frac{\operatorname{dist}(y,\partial\Omega)}{r}\Big)^\vartheta \sup_{B(x,2r)} u \qquad \text{for all } y \in B(x,r).$$

Lemma 4.11.4 (Maximum principle on the strip)**.** *Let S be the strip as before and let f be a bounded continuous $L_{\bar{A}}$-harmonic function on S so that $f|_{\partial S} \equiv 0$. Then $f \equiv 0$ on S.*

Proof. Choose $R > 100K_S$ and set $S_R := S \cap [-R,R]^{n+1}$. For $p \in S$, denote $h_p :=$ $\operatorname{dist}(p, \partial S)$ and let x_p be a point that realizes the distance. We choose p far from the "vertical" parts $\partial S_R \setminus (\partial S_+ \cup \partial S_-)$ of ∂S_R, in particular such that $B(x_p, R/10) \cap (\partial S_R \setminus \partial S) = \emptyset$. Let ω_R^p denote the elliptic measure with pole at p associated with $L_{\bar{A}}$ on S_R. The family $\{S_R\}_R$ is a collection of AD-regular sets whose AD-regularity constants do not depend on R. Then inequality (4.11.4) implies that there exit two constants C and ϑ, both independent on R, such that

$$\omega_R^p(\partial S_R \setminus \partial S) \le \omega_R^p\big(B(x_p, R/10)^c\big) \le C\Big(\frac{h_p}{R}\Big)^\vartheta.$$

By hypothesis we may assume $f \le 1$ on $\partial S_R \setminus \partial S$. Thus, we have

$$|f(p)| = \Big|\int f\, d\omega_R^p\Big| \le \|f|_{\partial S_R\setminus\partial S}\|_\infty \omega_R^p(\partial S_R \setminus \partial S) \le C\Big(\frac{h_P}{R}\Big)^\vartheta. \tag{4.11.5}$$

The results stated in the lemma follows by passing to the limit in (4.11.5) for $R \to \infty$. □

Now, we prove an existence result on the infinite strip S.

Lemma 4.11.5. *There exists a function $f_S \colon \bar{S} \to \mathbb{R}$ such that:*

1. *f_S is $L_{\bar{A}}$-harmonic in the strip S and continuous in $\bar{S}$.*
2. *f_S is $\mathcal{M}$-periodic.*
3. *$f_S(x) = \pm 1$ on $\partial S_\pm$ and $f_S(x) = 0$ for $x \in \tilde{H}$.*

Proof. Let $k \in \mathbb{N}$, $k \ge 100K_S$ and denote $S_k = S \cap [-k,k]^{n+1}$. We define the continuous functions f_k on ∂S_k as

$$f_k(x) = \frac{x_{n+1} - \frac{3}{2}\ell(Q_0)}{K_S \ell(Q_0)}.$$

In particular, observe that $f_k(x) = \pm 1$ for $x \in \partial S_\pm$ and

$$f(x) = -f\Big((x_1, \dots, x_n, -x_{n+1} + 3\ell(Q_0))\Big),$$

i.e. it is antisymmetric with respect to $\tilde{H}$.

Define u_k be the $L_{\bar{A}}$-harmonic function such that $u_k|_{\partial S_k} = f_k$, whose existence is guaranteed by the continuity of f_k and the AD-regularity of S_k. Our aim is to prove that, a part from possibly considering a proper subsequence, u_j converges uniformly in the compact subsets of S_k, for every k to an $L_{\bar{A}}$-harmonic function in S.

We claim that there exist $\gamma \in (0,1)$ and $C_k > 0$ such that

$$|u_j(x) - u_j(y)| \leq C_k|x-y|^\gamma \qquad \text{for} \qquad x, y \in \bar{S}_k, \quad j \geq k+2. \tag{4.11.6}$$

Assume that (4.11.6) holds. As a consequence of Ascoli-Arzelà's theorem together with standard a diagonalization argument, there is a function f_S so that u_k converges to f_S uniformly on the compact subsets of S. The $L_{\bar{A}}$-harmonicity of f_S is a consequence of Caccioppoli's esimate (cfr. [HKM06, Theorem 3.77]).

To prove (2), define $\vec{v} = (6\ell(Q_0), 0, \dots, 0)$ and observe that, being the matrix $\bar{A}$ $\mathcal{M}$-periodic and since f_S is constant on $\partial S_\pm$, the function $f(x) = f_S(x) - f_S(x + \vec{v})$ satisfies the hypotesis of Lemma 4.11.4. So, $f \equiv 0$ and f_S is $\mathcal{M}$-periodic.

To prove (3), first observe that $A(x) = A_\phi\Big((x_1, \dots, x_n, -x_{n+1} + 3\ell(Q_0))\Big)$, where ϕ is the function that maps a point to its reflected with respect to $\tilde{H}$ and A_ϕ is defined as in (4.5.5). Then we can apply again Lemma 4.11.4 to

$$\tilde{f}(x) = f_S(x) + f_S\Big((x_1, \dots, x_n, -x_{n+1} + 3\ell(Q_0))\Big),$$

which is $L_{\bar{A}}$-harmonic and vanishes on ∂S.

We are left with the proof of the claim (4.11.6). By Lemma 4.11.3, there exists $\vartheta \in (0,1)$ depending only on n, $\bar{A}$ and the AD-regularity of $\partial\Omega$ (hence independent both on j and k) such that u_j is ϑ-Hölder continuous in the set $\{x \in \bar{\Omega} : \operatorname{dist}(x, \partial\Omega) \leq 2\ell(Q_0)\}$. Being $\|u_j\|_\infty \leq 2$ for every j, by De Giorgi-Nash interior estimates we can infer that there exists γ_k independent on j such that, for every $j \geq k+2$, u_j is γ_k-Hölder continuous in $\{x \in \Omega_{k+1} : \operatorname{dist}(x, \partial\Omega) > \ell(Q_0)\}$. Gathering the interior and the boundary regularity of u_j proves (4.11.6). □

By the previous lemma, Lemma 4.11.3 and the fact that $f_S \equiv 0$ on $\tilde{H}$, we have the estimate

$$|f_S(y)| \lesssim \left(\frac{\operatorname{dist}(y, \tilde{H})}{K_S \ell(Q_0)}\right)^\vartheta, \qquad \text{for} \quad y \in S \text{ with } \operatorname{dist}(y, \tilde{H}) \leq 10\ell(Q_0).$$

Let us define the auxiliary function

$$F_S(x) := f_S(x)\bar{T}\nu(x_{S+}).$$

Observe that $F_S|_{\partial S_\pm} \equiv \pm \bar{T}\nu(x_{S+})$. The rest of the present section is devoted to the proof of the following, which is an approximated maximum principle on S.

Lemma 4.11.6 (Pointwise bound for the potential on the strip)**.** *For $x \in S$ we have*

$$|\bar{T}\nu(x) - F_S(x)|^2 + 4\bar{T}^*\big((\bar{T}\nu)\nu\big)(x) \lesssim \lambda^{1/2} + \frac{1}{K_S^{2\tilde{\alpha}}} + \frac{1}{K_S^{\vartheta}} + (C_S\ell(Q_0))^{\tilde{\alpha}} + \left(\frac{K_S}{C_S}\right)^{\tilde{\alpha}},$$

where C_S is a constant chosen so that $C_S \gg K_S$.

Before proving this lemma, we need some auxiliary result.

Lemma 4.11.7. *Let x_{S+} and x_{S-} as in (4.11.3). Then:*

1. *For $x \in \partial S_+$, $\operatorname{dist}(x, x_{S+}) \lesssim \ell(Q_0)$ we have the estimate*

$$|\bar{T}\nu(x) - \bar{T}\nu(x_{S+})| \lesssim \frac{1}{K_S^{\tilde{\alpha}}}. \tag{4.11.7}$$

The analogous estimate holds for $x \in \partial S_-$, replacing x_{S+} with x_{S-}.

2. *The difference of $-\bar{T}\nu(x_{S+})$ and $\bar{T}\nu(x_{S-})$ can be estimated as*

$$|\bar{T}\nu(x_+) + \bar{T}\nu(x_{S-})| \lesssim \frac{1}{K_S^{\tilde{\alpha}}}.$$

3. *For x with $\operatorname{dist}(x, \tilde{H}) \geq 2\ell(Q_0)$ we have*

$$\bar{T}^*\big((\bar{T}\nu)\nu\big)(x) \lesssim \lambda^{1/2}. \tag{4.11.8}$$

Proof. Let us begin with the proof of (1). Because of the $\mathcal{M}$-periodicity of $\bar{T}\nu$, we can assume without loss of generality that $x_H \in [-3\ell(Q_0), 3\ell(Q_0)]^n \times \{0\}$, x_H denoting the projection of x on H. We claim that for $P \in \mathcal{M}$ and $y \in Q_0$ we have

$$|\bar{K}(x, y + z_P) - \bar{K}(x_{S+}, y + z_P)| \lesssim \frac{\ell(Q_0)^{\tilde{\alpha}}}{(K_S\ell(Q_0))^{n+\tilde{\alpha}} + |z_P|^{n+\tilde{\alpha}}}.$$

This follows from the (global) Calderón-Zygmund estimates for $\bar{K}(\cdot,\cdot)$ once we observe that $|x - x_{S+}| \lesssim |x - y - z_P| \approx K_S\ell(Q_0) + |z_P|$. So, for $r > 0$, standard calculations give

$$\begin{aligned}\big|\tilde{T}_r\nu(x) - \tilde{T}_r\nu(x_{S+})\big| &\lesssim \sum_{P\in\mathcal{M}} \int_{Q_0} \frac{\ell(Q_0)^{\tilde{\alpha}}}{(K_S\ell(Q_0))^{n+\tilde{\alpha}} + |z_P|^{n+\tilde{\alpha}}} d\nu(y) \\ &= \sum_{P\in\mathcal{M}} \frac{\ell(Q_0)^{n+\tilde{\alpha}}}{(K_S\ell(Q_0))^{n+\tilde{\alpha}} + |z_P|^{n+\tilde{\alpha}}} \lesssim \frac{\ell(Q_0)^{n+\tilde{\alpha}}}{(K_S\ell(Q_0))^{n+\tilde{\alpha}}} = \frac{1}{K_S^{\tilde{\alpha}}}.\end{aligned}$$

Being this estimate independent on the choice of r, in the limit for $r \to 0$ we have (4.11.7). The proof of the analogous estimate for x_{S-} is identical, so we omit it and go to the proof of (2).

Denote by x^* the reflection of the point x across $x_0 = \frac{3}{2}\ell(Q_0)(1, \dots, 1)$, i.e.

$$x^* = 2x_0 - x.$$

By the specific choice of x_0, this transformation can be obtained via a composition of the reflections ψ_j's with respect to the hyperplanes passing through x_0 which we defined in (4.5.4):

$$x^* = \psi_1 \circ \dots \circ \psi_{n+1}(x). \tag{4.11.9}$$

Moreover,

$$(x_{S+})^* = 3\ell(Q_0)(1, \dots, 1) - \frac{3}{2}\ell(Q_0)(1, \dots, 1) - \big(0, \dots, 0, K_S\ell(Q_0)\big) = x_{S\_}. \tag{4.11.10}$$

Thus, an immediate application of Lemma 4.5.1 gives that, for $y \in Q_0$,

$$\bar{K}(x_{S_-}, y + z_P) = -\bar{K}(x_{S_+}, y^* + z_P^*), \qquad P \in \mathcal{M}. \tag{4.11.11}$$

Observe that

$$|y + z_P - (y^* - z_P^*)| \le |z_P - (-z_P^*)| + |y - y^*| \lesssim \ell(Q_0),$$

which, combined with Lemma 4.2.1 and (4.11.11) (applied with $-z_P = z_{-P}$ replacing z_P), gives

$$\begin{aligned} &\big|\bar{K}(x_{S_+}, y + z_P) + \bar{K}(x_{S_-}, y - z_P)\big| \\ &\quad = |\bar{K}(x_{S_+}, y + z_P) - \bar{K}(x_{S_+}, y^* - z_P^*)| \lesssim \frac{\ell(Q_0)^{\tilde{\alpha}}}{(K_S \ell(Q_0))^{n+\tilde{\alpha}} + |z_P|^{n+\tilde{\alpha}}}. \end{aligned} \tag{4.11.12}$$

Taking $r > 0$ and using (4.11.12), we have

$$\big|\tilde{T}_r\nu(x_{S+}) - \tilde{T}_r\nu(x_{S-})\big| \lesssim \sum_{P \in \mathcal{M}, |z_P| > r} \frac{\ell(Q_0)^{n+\tilde{\alpha}}}{K_S \ell(Q_0)^{n+\tilde{\alpha}} + |z_P|^{n+\tilde{\alpha}}} \lesssim \frac{1}{K_S^{\tilde{\alpha}}}$$

which, taking the limit for $r \to \infty$, proves (2).

We are left with the proof of (3). Set $\sigma = (\bar{T}\nu)\nu$ and observe that this measure is $\mathcal{M}$-periodic. So, without loss of generality, we can assume that $x_H \in [-3\ell(Q_0), 3\ell(Q_0)]^n \times \{0\}$. Let $r > 0$ and, denoting by A_0 the homogenized matrix associated with $\bar{A}$, by χ the vector of correctors and $\ell = 6\ell(Q_0)$, write

$$\begin{aligned} \tilde{T}_r\sigma(x) &= \sum_{P \in \mathcal{M}} \int_{Q_0} \tilde{K}_r(y + z_P, x) d\sigma(y) \\ &= \sum_{P \in \mathcal{M}} \int_{Q_0} \Big(\bar{K}(y + z_P, x) - \big(Id + \nabla\chi_\ell(y + z_P)\big)\nabla_1\Theta(y + z_P, x; A_0)\Big) \\ &\qquad\qquad \times \varphi_r(x - y - z_P) d\sigma(y) \\ &\quad + \sum_{P \in \mathcal{M}} \int_{Q_0} \big(Id + \nabla\chi_\ell(y + z_P)\big)\nabla_1\Theta(y + z_P, x; A_0)\varphi_r(x - y - z_P) d\sigma(y) \\ &\equiv I_r + II_r. \end{aligned}$$

Recalling that $\|\nabla\chi_\ell\|_\infty \lesssim 1$ and using Lemma 4.2.7, we can proceed with the following estimates

$$\begin{aligned} |I_r| &\lesssim \sum_{P \in \mathcal{M}} \int_{Q_0} \frac{\ell(Q_0)^\gamma}{|x - y - z_P|^{n+\gamma}} d|\sigma|(y) \\ &\lesssim \sum_{P \in \mathcal{M}} \int_{Q_0} \frac{\ell(Q_0)^\gamma}{(\operatorname{dist}(x, \tilde{H}) + |z_P|)^{n+\gamma}} d|\sigma|(y) \\ &\lesssim \frac{\ell(Q_0)^{n+\gamma}}{(\operatorname{dist}(x, \tilde{H}) + |z_P|)^{n+\gamma}} \frac{|\sigma|(Q_0)}{\ell(Q_0)^n} \\ &\lesssim \frac{\ell(Q_0)^{n+\gamma}}{\operatorname{dist}(x, \tilde{H})^\gamma} \frac{|\sigma|(Q_0)}{\ell(Q_0)^n} \lesssim \frac{|\sigma|(Q_0)}{\ell(Q_0)^n}, \end{aligned} \tag{4.11.13}$$

where the last inequality holds because we assumed $\operatorname{dist}(x, \tilde{H}) \geq 2\ell(Q_0)$. We claim that
$$|II_r| \lesssim \frac{|\sigma|(Q_0)}{\ell(Q_0)}. \tag{4.11.14}$$
It is possible to prove this estimate analogously to the case of the Riesz transform. We omit its proof in order not to make the presentation too lengthy. We remark that the calculations that lead to (4.11.14) solely relies on the Calderón-Zygmung property of the kernel and some geometric considerations that are independent on its specific expression. We refer to [GT18, (8.20)] for more details. Gathering (4.11.13), (4.11.14) and passing to the limit on r, we get
$$\bar{T}^*\big((\bar{T}\nu)\nu\big)(x) \lesssim \frac{1}{\ell(Q_0)^n}\int_{Q_0} |\bar{T}\nu| d\nu.$$
Then, recalling (4.11.1), the growth of ν and using Cauchy-Schwarz's inequality,
$$\bar{T}^*\big((\bar{T}\nu)\nu\big)(x) \lesssim \frac{1}{\ell(Q_0)^n}\Big(\int_{Q_0} |\bar{T}\nu|^2 d\nu\Big)^{1/2}\nu(Q_0) \lesssim \lambda^{1/2},$$
which finishes the proof of (3). □

The following result is a direct consequence of Lemma 4.11.7.

Corollary 4.11.1. *For $x \in \partial S$*
$$|\bar{T}\nu(x) - F_S(x)|^2 \lesssim \frac{1}{K_S^{2\tilde{\alpha}}}, \tag{4.11.15}$$
where the implicit constant does not depend on S.

Another result which is needed for the application of the maximum principle is the estimate of $|F_S(x)|$ for x close to the support of the measure ν.

Lemma 4.11.8. *For $x \in \mathbb{R}^{n+1}$ with $\operatorname{dist}(x, \tilde{H}) \leq 10\ell(Q_0)$ we have*
$$|F_S(x)| \lesssim \frac{1}{K_S^{\vartheta+\gamma}}.$$

Proof. Because of the Hölder continuity of f_S, we can write
$$|F_S(x)| \lesssim \Big(\frac{\operatorname{dist}(x, \tilde{H})}{K_S\ell(Q_0)}\Big)^{\vartheta}|T\nu(x_{S+})| \lesssim \frac{|T\nu(x_{S+})|}{K_S^{\vartheta}}.$$
So, to prove the lemma, it suffices to show that
$$|\bar{T}\nu(x_{S+})| \leq C$$
for some constant $C > 0$ not depending on K_S. Recall now that $\nu = b\,\eta$. Applying Lemma 4.10.2 with $\tilde{M} = 6K_S$ and $f = b$ to the point $0 \in 2Q_0$, we have the estimate
$$\big|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(0)\big| \lesssim \frac{1}{(6K_S)^{\gamma}\ell(Q_0)^n}\int_{Q_0} |b|\, d\eta \lesssim 1, \tag{4.11.16}$$
where the implicit constant in the last inequality does not depend on K_S. Now, we observe that the (global) Calderón-Zygmund properties of $\bar{K}$ and the fact that

$|x_{S+}| \lesssim K_S \ell(Q_0)$ imply

$$\left|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(0) - \bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(x_{S+})\right| \lesssim \int_{(6K_SQ_0)^c} \left|\bar{K}(0,y) - \bar{K}(x_{S+},y)\right| d\nu(y)$$
$$\lesssim \int_{(6K_SQ_0)^c} \frac{|x_{S+}|^{\tilde{\alpha}}}{(|y| + |x_{S+}|)^{n+\tilde{\alpha}}} d\nu(y) \lesssim 1. \tag{4.11.17}$$

Then, by (4.11.16), (4.11.17) and the triangle inequality, we have

$$\left|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu(x_{S+})\big)\right|$$
$$\leq \left|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(0)\right| + \left|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(0) - \bar{T}\big(\chi_{(6K_SQ_0)^c}\nu\big)(x_{S+})\right| \lesssim 1. \tag{4.11.18}$$

Moreover, since $\operatorname{dist}(x_{S+}, \operatorname{supp}\nu)^n \gtrsim K_S \ell(Q_0)$ and estimating the kernel via Lemma 4.2.4,

$$\left|\bar{T}\big(\chi_{6K_SQ_0}\nu\big)(x_{S+})\right| \leq \int_{6K_SQ_0} |\bar{K}(x_{S+},y)| d\nu(y) \lesssim \int_{6K_SQ_0} \frac{1}{|x_{S+} - y|^n} d\nu(y)$$
$$\lesssim \frac{\nu(6K_SQ_0)}{\operatorname{dist}(x_{S+}, \operatorname{supp}\nu)^n} \lesssim \frac{K_S^n \ell(Q_0)^n}{\operatorname{dist}(x_{S+}, \operatorname{supp}\nu)^n} \lesssim 1. \tag{4.11.19}$$

Thus, gathering (4.11.18) and (4.11.19) we obtain

$$|\bar{T}\nu(x_{S+})| \leq \left|\bar{T}\big(\chi_{6K_SQ_0}\nu\big)(x_{S+})\right| + \left|\bar{T}\big(\chi_{(6K_SQ_0)^c}\nu(x_{S+})\big)\right| \lesssim 1,$$

which proves the lemma. □

In order to be able to use the previous lemma, from now on we will assume without loss of generality $K_S \geq 3$ and we suppose it to be an odd number. Observe that for $x \in \operatorname{supp}\nu$, Lemma 4.11.8 and (4.11.2) give

$$\begin{aligned} &\sup_{x\in\operatorname{supp}\nu} \left|\bar{T}\nu(x) - F_S(x)\right|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(x) \\ &\quad\leq \sup_{x\in\operatorname{supp}\nu} 2\left|\bar{T}\nu(x)\right|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(x) + 2|F_S(x)|^2 \\ &\quad\leq 12\lambda + 2|F_S(x)|^2 \lesssim \lambda + \frac{1}{K_S^{\vartheta}}. \end{aligned} \tag{4.11.20}$$

Moreover, by (4.11.8) and (4.11.15),

$$\sup_{x\in\partial S} \left|\bar{T}\nu(x) - F_S(x)\right|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(x) \lesssim \frac{1}{K_S^{2\tilde{\alpha}}} + \lambda^{1/2}$$

which, together with (4.11.20) brings us to

$$\sup_{x\in\partial S\cup\operatorname{supp}\nu} \left|\bar{T}\nu(x) - F_S(x)\right|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(x) \lesssim \lambda^{1/2} + \frac{1}{K_S^{2\tilde{\alpha}}} + \frac{1}{K_S^{\vartheta}}. \tag{4.11.21}$$

Finally, we provide the proof of Lemma 4.11.6.

Proof. We recall that $\bar{A} = \bar{A}^T$. Let $\vec{g} \in L^\infty(S;\mathbb{R}^{n+1})$. We claim that $\bar{T}^*\big(\vec{g}\mathcal{L}^{n+1}\big)$ is a $L_{\bar{A}^T}$-harmonic (vector valued) function. This would imply the maximum principle

$$\sup_{x\in S} \bar{T}^*\big(\vec{g}\mathcal{L}^{n+1}\big)(x) = \sup_{x\in \partial S\cap \operatorname{supp}\nu} \bar{T}^*\big(\vec{g}\mathcal{L}^{n+1}\big)(x). \tag{4.11.22}$$

Observe that, because of Lemma 4.11.5, the same equality holds with $F_S(x)$ in place of $\bar{T}^*(\vec{g}\mathcal{L}^{n+1})(x)$. Let $\varphi \in C_c^\infty(S \setminus \operatorname{supp}\vec{g})$ be a test function. To prove the claim, apply the definition of $\bar{T}^*$ together with Fubini's theorem together with the fact that $\bar{\mathcal{E}}(x,y) = \mathcal{E}_{\bar{A}^T}(y,x)$:

$$\begin{aligned}\int \bar{A}^T\nabla\bar{T}^*(\vec{g}\mathcal{L}^{n+1})\cdot\nabla\varphi &= \int \bar{A}^T\nabla_x\Big(\int \nabla_y\bar{\mathcal{E}}(y,x)\cdot\vec{g}(y)dy\Big)\cdot\nabla\varphi(x)dx\\ &= \int \nabla_y\Big(\int \bar{A}^T\nabla_x\bar{\mathcal{E}}(y,x)\cdot\nabla\varphi(x)dx\Big)\cdot\vec{g}(y)dy\\ &= \int \nabla_y\Big(\int \bar{A}^T\nabla_x\,\mathcal{E}_{\bar{A}^T}(x,y)\cdot\nabla\varphi(x)dx\Big)\cdot\vec{g}(y)dy\\ &= \int \nabla\varphi\cdot\vec{g} = 0.\end{aligned}$$

Notice that for every $z \in \mathbb{R}^{n+1}$ we have the elementary relation

$$|z|^2 = \sup_{\beta\geq 0, e\in\mathbb{S}^n} 2\langle e, z\rangle - \beta^2,$$

so that, choosing $z = \bar{T}\nu(x) - F_S(x)$, it reads

$$\big|\bar{T}\nu(x) - F_S(x)\big|^2 = \sup_{\beta\geq 0, e\in\mathbb{S}^n} 2\langle e, \bar{T}\nu(x)\rangle - 2\langle e, F_S(x)\rangle - \beta^2. \tag{4.11.23}$$

We want to show that the argument of the supremum in the right hand side of (4.11.23) differs from a $L_{\bar{A}}$-harmonic function possibly by a small term. This will allow to apply the maximum principle on the strip and to finish the proof.

For a fixed $e \in \mathbb{S}^n$ and $x \in \operatorname{supp}\nu$, we split

$$\langle e, \bar{T}\nu(x)\rangle = -\bar{T}^*(\nu e)(x) + \big(\bar{T}^*(\nu e)(x) + \langle e, \bar{T}\nu(x)\rangle\big)$$

and consider that, claiming that the dominated convergence theorem applies,

$$\bar{T}^*(\nu e)(x) + \langle e, \bar{T}\nu(x)\rangle = \lim_{r\to\infty}\int \big(\tilde{K}_r(x,y) + \tilde{K}_r(y,x)\big)\cdot e\, d\nu(y). \tag{4.11.24}$$

To prove that the previous identity holds, set $C_S \gg K_S$ to be chosen later. By the triangle inequality, the antisimmetry of $\nabla_1\Theta(x,y;\bar{A}(x))$ and the linear growth of ν, we have

$$\begin{aligned}&\int_{|x-y|<C_S\ell(Q_0)} \big|\tilde{K}_r(x,y) + \tilde{K}_r(y,x)\big| d\nu(y)\\ &\quad\leq \int_{|x-y|<C_S\ell(Q_0)} \big|\bar{K}(x,y) - \nabla_1\Theta(x,y;\bar{A}(x))\big| d\nu(y)\\ &\qquad + \int_{|x-y|<C_S\ell(Q_0)} \big|\bar{K}(y,x) - \nabla_1\Theta(y,x;\bar{A}(x))\big| d\nu(y)\\ &\quad\lesssim \int_{|x-y|<C_S\ell(Q_0)} \frac{1}{|x-y|^{n-\tilde{\alpha}}} d\nu(y) \lesssim \big(C_S\ell(Q_0)\big)^{\tilde{\alpha}}.\end{aligned} \tag{4.11.25}$$

So, to bound (4.11.24) we have to estimate the integral on its rights hand side for $|x-y|>C_S\ell(Q_0)$. As before, by the periodicity of M_S we can assume that $x_H \in [-3\ell(Q_0), 3\ell(Q_0)]^n \times \{0\}$. Using arguments analogous to the ones in Lemma 4.11.7, it is possible to prove that for $y \in Q_0$ and z_P such that $|x-y-z_P|>C_S\ell(Q_0)$, we have

$$|\bar{K}(x,y+z_P)+\bar{K}(x,y-z_P)| \lesssim \frac{(K_S\ell(Q_0))^{\tilde{\alpha}}}{|z_P|^{n+\tilde{\alpha}}+|x|^{n+\tilde{\alpha}}},$$

hence, calling $\mathcal{M}_S$ the subset of $P \in \mathcal{M}$ such that $|x-y-z_P|>C_S\ell(Q_0)$, we have

$$\sum_{P\in\mathcal{M}_S}\int_{Q_0}|\tilde{K}_r(x,y+z_P)-\tilde{K}_r(x,y-z_P)|d\nu(y) \lesssim \left(\frac{K_S}{C_S}\right)^{\tilde{\alpha}}. \tag{4.11.26}$$

Analogously, one can prove

$$\sum_{P\in\mathcal{M}_S}\int_{Q_0}|\tilde{K}_r(y+z_P,x)-\tilde{K}_r(y-z_P,x)|d\nu(y) \lesssim \left(\frac{K_S}{C_S}\right)^{\tilde{\alpha}}, \tag{4.11.27}$$

so, gathering (4.11.25), (4.11.26) and (4.11.27) and letting $r\to\infty$, we can estimate (4.11.24) as

$$\left|\bar{T}^*(\nu e)(x)+\langle e,\bar{T}\nu(x)\rangle\right| \lesssim (C_S\ell(Q_0))^{\tilde{\alpha}}+\left(\frac{K_S}{C_S}\right)^{\tilde{\alpha}}. \tag{4.11.28}$$

We are now ready to proceed with the calculations for the maximum principle. Indeed, taking $x\in S$, an application of (4.11.23) and (4.11.28) gives

$$\begin{aligned}
&\left|\bar{T}\nu(x)-F_S(x)\right|^2+4\bar{T}^*((\bar{T}\nu)\nu)(x)\\
&\quad=\sup_{\beta\geq 0, e\in\mathbb{S}^n} 2\langle e,\bar{T}\nu(x)\rangle-2\langle e,F_S(x)\rangle-\beta^2+\bar{T}^*((\bar{T}\nu)\nu)(x)\\
&\quad\lesssim \sup_{\beta\geq 0, e\in\mathbb{S}^n} -2\bar{T}^*(\nu e)(x)-2\langle e,F_S(x)\rangle-\beta^2+\bar{T}^*((\bar{T}\nu)\nu)(x)+(C_S\ell(Q_0))^{\tilde{\alpha}}+\left(\frac{K_S}{C_S}\right)^{\tilde{\alpha}}.
\end{aligned}$$

Thus, using the maximum principle (4.11.22) we have

$$\begin{aligned}
&|\bar{T}\nu(x)-F_S(x)|^2+4\bar{T}^*((T\nu)\nu)(x)\\
&\lesssim \sup_{\beta\geq 0, e\in\mathbb{S}^n} 2-\bar{T}^*\big(\nu e+(\bar{T}\nu)\nu\big)(x)-2\langle e,F_S(x)\rangle-\beta^2+(C_S\ell(Q_0))^{\tilde{\alpha}}+\left(\frac{K_S}{C_S}\right)^{\tilde{\alpha}}\\
&\leq \sup_{z\in\partial S\cup\operatorname{supp}\nu}\ \sup_{\beta\geq 0, e\in\mathbb{S}^n} -2\bar{T}^*\big(\nu e+(\bar{T}\nu)\nu\big)(z)-2\langle e,F_S(z)\rangle-\beta^2+(C_S\ell(Q_0))^{\tilde{\alpha}}+\left(\frac{K_S}{C_S}\right)^{\tilde{\alpha}}.
\end{aligned}$$

So, another application of (4.11.23) and (4.11.28) concludes the proof of the lemma. Indeed, recalling the estimate (4.11.21) on $\partial S \cup \operatorname{supp}\nu$,

$$
\begin{aligned}
&|\bar{T}\nu(x) - F_S(x)|^2 + 4\bar{T}^*((T\nu)\nu)(x)\\
&\lesssim \sup_{z\in\partial S\cup\operatorname{supp}\nu}\ \sup_{\beta\geq 0, e\in\mathbb{S}^n} 2\langle e, \bar{T}\nu(x)\rangle - 2\langle e, F_S(x)\rangle - \beta^2\\
&\qquad\qquad + \bar{T}^*((\bar{T}\nu)\nu)(x) + (C_S\ell(Q_0))^{\tilde{\alpha}} + \Big(\frac{K_S}{C_S}\Big)^{\tilde{\alpha}}\\
&\lesssim \sup_{z\in\partial S\cup\operatorname{supp}\nu} |\bar{T}\nu(z) - F_S(z)|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(z) + (C_S\ell(Q_0))^{\tilde{\alpha}} + \Big(\frac{K_S}{C_S}\Big)^{\tilde{\alpha}}\\
&\lesssim \lambda^{1/2} + \frac{1}{K_S^{2\tilde{\alpha}}} + \frac{1}{K_S^{\vartheta}} + (C_S\ell(Q_0))^{\tilde{\alpha}} + \Big(\frac{K_S}{C_S}\Big)^{\tilde{\alpha}}. \qquad \square
\end{aligned}
$$

4.11.2 The conclusion of the proof of the Key Lemma

To simplify the notation, set

$$\mathtt{Err}(K_S, C_S, \ell(Q_0)) := \frac{1}{K_S^{2\tilde{\alpha}}} + \frac{1}{K_S^{\vartheta}} + (C_S\ell(Q_0))^{\tilde{\alpha}} + \Big(\frac{K_S}{C_S}\Big)^{\tilde{\alpha}}.$$

Notice that if $x \in 2Q_0$, Lemma 4.11.6 together with Lemma 4.11.8 allows to majorize $|\bar{T}\nu(x)|^2$ as

$$
\begin{aligned}
\big|\bar{T}\nu(x)\big|^2 &\lesssim |\bar{T}\nu(x) - F_S(x)|^2 + |F_S(x)|^2 + 4\bar{T}^*((\bar{T}\nu)\nu)(x) - 4\bar{T}^*((\bar{T}\nu)\nu)(x)\\
&\lesssim \lambda^{1/2} + \mathtt{Err}(K_S, C_S, \ell(Q_0)) + |F_S(x)|^2 - \bar{T}^*((\bar{T}\nu)\nu)(x)\\
&\lesssim \lambda^{1/2} + \mathtt{Err}(K_S, C_S, \ell(Q_0)) - \bar{T}^*((\bar{T}\nu)\nu)(x).
\end{aligned} \tag{4.11.29}
$$

Let φ be a smooth function such that $\chi_{Q_0} \leq \varphi \leq \chi_{2Q_0}$ and $\|\nabla\varphi\|_\infty \lesssim \ell(Q_0)^{-1}$. Set $\psi := \bar{A}^T\nabla\varphi$ and observe that it verifies

$$
\begin{aligned}
\bar{T}^*[\psi\,\mathcal{L}^{n+1}](x) = \bar{T}^*[\bar{A}^T\nabla\varphi\,\mathcal{L}^{n+1}](x) &= \int \nabla_1\mathcal{E}_{\bar{A}}(y,x)\cdot\bar{A}^T(y)\nabla\varphi(y)dy\\
&= \int \bar{A}(y)\,\nabla_1\mathcal{E}_{\bar{A}}(y,x)\cdot\nabla\varphi(y)dy = \varphi(x),
\end{aligned}
$$

the last equality being a consequence of the definition of fundamental solution.

The choice of $\varphi \geq \chi_{Q_0}$, together with Cauchy-Schwarz's inequality, gives

$$
\begin{aligned}
\nu(Q_0) \leq \int\varphi d\nu = \int \bar{T}^*(\psi\mathcal{L}^{n+1})d\nu = \int \bar{T}\nu\cdot\psi d\mathcal{L}^{n+1}\\
\leq \Big(\int |\bar{T}\nu|^2|\psi|d\mathcal{L}^{n+1}\Big)^{1/2}\Big(\int|\psi|d\mathcal{L}^{n+1}\Big)^{1/2}.
\end{aligned} \tag{4.11.30}
$$

Now, observe that

$$\|\psi\|_\infty \leq \|\bar{A}^T\|_\infty\|\nabla\varphi\|_\infty \lesssim \ell(Q_0)^{-1} \tag{4.11.31}$$

and

$$\int|\psi|d\mathcal{L}^{n+1} \lesssim \frac{1}{\ell(Q_0)}\mathcal{L}^{n+1}(2Q_0) \lesssim \ell(Q_0)^n. \tag{4.11.32}$$

We claim that

$$\int|\bar{T}\nu|^2|\psi|d\mathcal{L}^{n+1} \ll \ell(Q_0)^n.$$

Applying (4.11.29) and (4.11.32), we can write

$$\begin{aligned}
&\int |\bar{T}\nu|^2|\psi|d\mathcal{L}^{n+1}\\
&\lesssim \big(\lambda^{1/2}+\texttt{Err}(K_S,C_S,\ell(Q_0))\big)\int |\psi|d\mathcal{L}^{n+1}+\Big|\int \bar{T}^*\big((\bar{T}\nu)\nu\big)|\psi|d\mathcal{L}^{n+1}\Big|\\
&\lesssim \big(\lambda^{1/2}+\texttt{Err}(K_S,C_S,\ell(Q_0))\big)\int |\psi|d\mathcal{L}^{n+1}\\
&\qquad+\Big|\int \bar{T}^*\big(\chi_{(30Q_0)^c}(\bar{T}\nu)\nu\big)|\psi|d\mathcal{L}^{n+1}\Big|+\Big|\int \bar{T}^*\big(\chi_{30Q_0}(\bar{T}\nu)\nu\big)|\psi|d\mathcal{L}^{n+1}\Big| \qquad (4.11.33)\\
&\lesssim \big(\lambda^{1/2}+\texttt{Err}(K_S,C_S,\ell(Q_0))\big)\ell(Q_0)^n+\Big|\int \bar{T}^*\big(\chi_{(30Q_0)^c}(\bar{T}\nu)\nu\big)|\psi|d\mathcal{L}^{n+1}\Big|\\
&\qquad+\Big|\int \bar{T}^*\big(\chi_{30Q_0}(\bar{T}\nu)\nu\big)|\psi|d\mathcal{L}^{n+1}\Big|\\
&=\big(\lambda^{1/2}+\texttt{Err}(K_S,C_S,\ell(Q_0))\big)\ell(Q_0)^n+I+II,
\end{aligned}$$

where I and II are defined by the last equality.
The estimate for I is an application of (4.10.7) with $\tilde{M}=30$. In particular,

$$\begin{aligned}
\big|\bar{T}^*\big(\chi_{(30Q_0)^c}(\bar{T}\nu)b\eta)\big)(x)\big| &\lesssim \frac{1}{\ell(Q_0)^n}\int_{Q_0}|(\bar{T}\nu)b|d\eta\\
&\le \frac{\nu(Q_0)^{1/2}}{\ell(Q_0)^n}\Big(\int_{Q_0}|\bar{T}\nu|^2d\nu\Big)^{1/2}\lesssim \lambda^{1/2}\frac{\nu(Q_0)}{\ell(Q_0)^n},
\end{aligned}$$

which, together with (4.11.32), implies

$$I\lesssim \lambda^{1/2}\nu(Q_0). \qquad (4.11.34)$$

For the estimate of II, recall that $|\bar{K}(x,y)|\lesssim |x-y|^{-n}$. This and (4.11.31) imply

$$\big|\bar{T}\big(|\psi|d\mathcal{L}^{n+1}\big)(x)\big|=\Big|\int \bar{K}(x,y)|\psi|(y)dy\Big|\lesssim \frac{1}{\ell(Q_0)}\int_{2Q_0}\frac{1}{|x-y|^n}dy\lesssim \frac{\ell(Q_0)}{\ell(Q_0)}=1.$$

Then, by Cauchy-Schwarz's inequality, the periodicity of $\bar{T}\nu$ and the localization (4.11.1),

$$II\le \Big|\int_{30Q_0}\bar{T}(|\psi|\mathcal{L}^{n+1})\cdot\bar{T}\nu d\nu\Big|\lesssim \nu(Q_0)^{1/2}\Big(\int_{30Q_0}|\bar{T}\nu|^2d\nu\Big)^{1/2}\lesssim \lambda^{1/2}\nu(Q_0). \qquad (4.11.35)$$

So, gathering (4.11.30), (4.11.33), (4.11.34) and (4.11.35), we have

$$\nu(Q_0)\lesssim \big(\texttt{Err}(K_S,C_S,\ell(Q_0))+\lambda^{1/2}\big)^{1/2}\nu(Q_0). \qquad (4.11.36)$$

Choosing K_S big enough, K_S/C_S small enough, $C_S\ell(Q_0)$ and λ small enough, we have

$$\texttt{Err}(K_S,C_S,\ell(Q_0))+\lambda^{1/2}\ll 1,$$

so (4.11.36) brings us to the contradiction

$$\nu(Q_0)\ll \nu(Q_0).$$

This proves the Key Lemma and, hence, completes the proof of Theorem 4.2.

4.12 The two-phase problem for the elliptic measure

To the purpose of the application to the study of the elliptic measure, it is useful to reformulate Theorem 4.2 under slightly different hypothesis. The proof of the following closely resembles that of [AMT17b, Theorem 3.3].

Theorem 4.8. *Let μ be a Radon measure in $\mathbb{R}^{n+1}$ and let $B \subset \mathbb{R}^{n+1}$ be a ball with $\mu(B) > 0$. Assume that, for some constants $C_0, C_1 > 0$ and $0 < \lambda, \delta, \tau \ll 1$ the following conditions hold:*

1. $r(B) \leq \lambda$.
2. $P_{\mu,\tilde{\alpha}}(B) \leq C_0 \Theta_\mu(B)$.
3. *There is some n-plane L through the center of B such that $\beta^L_{\mu,1}(B) \leq \delta \Theta_\mu(B)$.*
4. *There is $G_B \subset B$ such that for all $x \in G_B$*

$$\sup_{0<r\leq 2r(B)} \frac{\mu\big(B(x,r)\big)}{r^n} + T_*(\chi_{2B}\mu)(x) \leq C_1 \Theta_\mu(B).$$

5. $\int_{G_B} |T\mu(x) - m_{\mu,G_B}|^2 d\mu(x) \leq \tau \Theta_\mu(B)^2 \mu(B)$.

There exists $\vartheta > 0$ such that, if δ, τ and λ are small enough (depending on C_0 and C_1), there is a n-uniformly rectifiable set Γ such that

$$\mu(B \cap \Gamma) \geq \vartheta \mu(B).$$

The proof in the case $A \equiv Id$ is based on a *Tb* theorem for *suppressed kernels* by Nazarov, Treil and Volberg. To replicate the proof of Azzam, Mourgoglou and Tolsa in the elliptic context, we define the suppressed kernel associated with $K(\cdot,\cdot)$ as

$$\tilde{K}_\Phi(x,y) = \tilde{\chi}\Big(\frac{|x-y|^2}{\Phi(x)\Phi(y)}\Big) K(x,y),$$

where $\tilde{\chi} \colon [0,+\infty) \to [0,1]$ is a smooth, vanishes identically in $[0,1/2]$ and equals 1 in $[1,+\infty)$ and Φ is a 1-Lipschitz function to be chosen as in the proof of [AMT17b]. Then, one can split

$$K(x,y) = \frac{1}{2}\big(K(x,y) + K(y,x)\big) + \frac{1}{2}\big(K(x,y) - K(y,x)\big) = K^{(s)}(x,y) + K^{(a)}(x,y),$$

apply the *Tb* theorem for suppressed kernels (see also [Tol14, Section 5.12] and the references therein) to the antisymmetric part of K and exploit the L^2-boundedness of the symmetric part guaranteed by the freezing technique of Lemma 4.2.2. We leave to the interested reader to check that there is no further difficulty in the proof Theorem 4.8.

The rest of the present section is devoted to show how to apply Theorem 4.8 to prove the two-phase problem for the elliptic measure.

After possibly splitting the set E, we can assume $\operatorname{diam} E \leq \frac{1}{10} \min\big(\operatorname{diam}\Omega_1, \operatorname{diam}\Omega_2\big)$. We choose the poles p_i, $i = 1,2$ such that $p_i \in \Omega_i \cap 2\tilde{B} \setminus \tilde{B}$, where $\tilde{B}$ is a ball centered at E with radius $r(\tilde{B}) = 2 \operatorname{diam} E$.

We are going to apply Theorem 4.8 to the measure ω_1: we are going to prove that we can find an n-rectifiable set $F \subset E$ such that $\omega_1|_F \ll \mathcal{H}^n|_F \ll \omega_1|_F$. In particular, we can suppose that Ω_1 is such that

$$\mathcal{H}^{n+1}\big(\tilde{B} \cap \Omega_1\big) \approx r(\tilde{B}). \tag{4.12.1}$$

By the so-called Bourgain's estimates (see [PPT18, Lemma 32] for the statement in the elliptic case and [Azz+16b] for a proof in the case $A \equiv Id$) together with (4.12.1), we can infer that there exists δ_0 such that

$$\omega_1\big(2\delta^{-1}\tilde{B}\big) \approx 1, \qquad \text{for } 0 < \delta < \delta_0.$$

Let $a, \tilde{\gamma} > 0$ and $i = 1, 2$. We say that a ball B is a-$P_{\omega_i,\tilde{\gamma}}$-doubling if

$$P_{\omega_i,\tilde{\gamma}}(B) \leq a\Theta_{\omega_i}(B).$$

The following lemma is important for the applicability of the doubling condition.

Lemma 4.12.1. *Let $\tilde{\gamma} \in (0,1)$. Let Ω_1, Ω_2 be Wiener regular domains in $\mathbb{R}^{n+1}$ and let $E \subset \partial\Omega_1 \cap \partial\Omega_2$ be a set on which $\omega_1|_E \ll \omega_2|_E \ll \omega_1|_E$. Then there exists $a = a(\tilde{\gamma}, n)$ big enough such that for $\omega_1|_E$-almost every $x \in \mathbb{R}^{n+1}$ we can find a sequence of a-$P_{\omega_i,\tilde{\gamma}}$-doubling balls $B(x, r_i)$ with $r_i \to 0$ as $i \to \infty$.*

Proof. Let $i = 1, 2$. Let $m \in \mathbb{Z}, m \geq 1$ and denoting

$$Z_m := \big\{x \in \partial\Omega_i : \text{ for all } j \geq m,\ B(x, 2^{-j}) \text{ is not } a\text{-}P_{\omega_i,\tilde{\gamma}}\text{-doubling}\big\}$$

it suffices to prove that $\omega_i|_E(Z_m) = 0$ for every m. Arguing as in [Azz+16d, Lemma 6.1] we have that, for $x \in Z_m$, we can estimate the elliptic measure of $B(x, r)$ as

$$\omega_i(B(x,r)) \leq C(m) r^{n+\tilde{\gamma}} \qquad \text{for } r \leq 2^{-m}.$$

Then

$$\omega|_E(A) \leq \omega(A) \leq C(m)\mathcal{H}^{n+\tilde{\gamma}}(A) \qquad \text{for any } A \subset Z_m.$$

We recall that the dimension of $\omega|_E$ can be defined as

$$\dim \omega|_E := \inf\big\{s \colon \exists\, F \subset \partial\Omega \text{ s.t. } \mathcal{H}^s(F) = 0 \\ \text{and } \omega|_E(F \cap K) = \omega|_E(\partial\Omega \cap K)\, \forall K \subset \mathbb{R}^{n+1} \text{ compact}\big\}$$

First let us bound $\dim \omega|_E$ from below. Let $F \subset \partial\Omega$ be such that $\mathcal{H}^{n+\tilde{\gamma}}(F) = 0$. For $K \subset Z_m$ compact and such that $\omega|_E(K) > 0$, we have $\omega|_E(F \cap K) \leq C(m)\mathcal{H}^{n+\tilde{\gamma}}(F \cap K) = 0$. This in turn implies

$$\dim \omega|_E \geq n + \tilde{\gamma}. \tag{4.12.2}$$

Conversely, [AM17] gives that $\dim \omega|_E = n$, which gathered with (4.12.2) tells that

$$n \geq n + \tilde{\gamma}.$$

Being $\tilde{\gamma} > 0$, this brings to a contradiction and, in particular, this proves that $\omega(Z_m) = 0$ for every m. □

Let $i = 1, 2$. Denote by $u_i(\cdot) = G_i(p_i, \cdot)$ the Green function associated with Ω_i with pole at p_i. We understand that u_i is extended by zero to Ω_i^c. As a corollary of [Azz+16a, Theorem 1.5], which was formulated under weaker assumptions on the regularity of the matrix A, we can state the following monotonicity formula.

Lemma 4.12.2 (Monotonicity formula)**.** *Let Ω_i and u_i be as above and let $R > 0$. Suppose that that $A_s(\xi) = Id$ for $\xi \in \partial\Omega_1 \cap \partial\Omega_2$. Then, setting*

$$\gamma(\xi, r) = \Big(\frac{1}{r^2}\int_{B(\xi,2r)} \frac{|\nabla u_1(y)|^2}{|y-\xi|^{n-1}} dy\Big) \cdot \Big(\frac{1}{r^2}\int_{B(\xi,2r)} \frac{|\nabla u_2(y)|^2}{|y-\xi|^{n-1}} dy\Big),$$

we have that, for some $c > 0$,

$$\gamma(\xi, r) \leq \gamma(\xi, s) e^{c(s^\alpha - r^\alpha)} < \infty \quad \text{for } 0 < r \leq s < R.$$

We remark that Azzam, Garnett, Mourgoglou and Tolsa proved their result under the hypothesis $A(\xi) = Id$. However, the same proof works under our assumption.[1]

The following lemma is crucial to prove the elliptic variant version of the blowups.

Lemma 4.12.3. *Let Ω_1 be a Wiener regular domain and denote by $\omega_1 = \omega_1^{p_1}$ its associated elliptic measure with pole at $p_1 \in \Omega_1$. Let B be a ball centered at $\partial\Omega_1$ and such that $p_1 \notin 10B$. Assuming that $\omega_1(8B) \leq C\omega_1(\delta_0 B)$ and $\mathcal{H}^{n+1}(B \setminus \Omega_1) \geq C^{-1} r(B)^{-1}$, we have*

$$\mathcal{H}^{n+1}(\Omega_1 \cap 2\delta_0 B) \gtrsim r(B)^{n+1}. \tag{4.12.3}$$

Moreover

$$\mathcal{H}^{n+1}(2\delta_0 B \setminus \Omega_1) \approx \mathcal{H}^{n+1}(2\delta_0 B \setminus \Omega_2) \approx r(B)^{n+1}. \tag{4.12.4}$$

Proof. Denote $r = r(B)$. Let us first prove (4.12.3). Consider a smooth function $\varphi \geq 0$ such that $\varphi \equiv 1$ on $\delta_0 B$ and $\operatorname{supp}\varphi \subset 2\delta_0 B$. In particular, suppose that $\|\varphi\|_\infty \lesssim (\delta_0 r)^{-1}$. Then, recalling that, by the properties of Green's function and being x_1 outside of the support of φ,

$$\int \varphi \, d\omega_1 = -\int A^T \nabla u_1 \cdot \nabla \varphi,$$

we use the ellipticity of the matrix A and write

$$\begin{aligned} \omega_1(2\delta_0 B) &\leq \int \varphi \, d\omega_1 \leq \int |\nabla u_1 \cdot A \nabla \varphi| \\ &\lesssim \int |\nabla u_1||\nabla \varphi| = \int_{\Omega_1 \cap 2\delta_0 B} |\nabla u_1||\nabla \varphi| \lesssim \frac{1}{\delta_0 r} \int_{\Omega_1 \cap 2\delta_0 B} |\nabla u_1|. \end{aligned}$$

Then applying, in order, Hölder's and Caccioppoli's inequalities,

$$\begin{aligned} \frac{1}{\delta_0 r} \int_{\Omega_1 \cap 2\delta_0 B} |\nabla u_1| &\leq \frac{\mathcal{H}^{n+1}(\Omega_1 \cap 2\delta_0 B)^{1/2}}{\delta_0 r} \Big(\int_{2\delta_0 B} |\nabla u_1|^2 \Big)^{1/2} \\ &\lesssim \frac{\mathcal{H}^{n+1}(\Omega_1 \cap 2\delta_0 B)^{1/2}}{\delta_0 r} \frac{1}{\delta_0 r} \Big(\int_{4\delta_0 B} |u_1|^2 \Big)^{1/2}, \end{aligned}$$

so

$$\omega_1(2\delta_0 B) \lesssim \mathcal{H}^{n+1}(\Omega_1 \cap 2\delta_0 B)^{1/2} \frac{(\delta_0 r)^{(n+1)/2}}{(\delta_0 r)^2} \sup_{4\delta_0 B} |u_1|.$$

At this point, recalling that (see [PPT18, Lemma 32])

$$\sup_{y \in 4\delta_0 B} u_1(y) \lesssim \frac{\omega_1(8B)}{r^{n-1}},$$

we have

$$\omega_1(\delta_0 B) \lesssim \mathcal{H}^{n+1}(\Omega_1 \cap 2\delta_0 B)^{1/2} \frac{(\delta_0 r)^{(n-3)/2}}{(\delta_0 r)^{n-1}} \omega_1(8B)$$

which, since we suppose $\omega_1(8B) \leq C\omega_1(\delta_0 B)$, concludes the proof of (4.12.3).

[1] It suffices to define the matrix D in [Azz+16a, Appendix A.1] as $D = A(\xi) - A$ and observe that $L_{A(\xi)} = L_{A_s(\xi)} = Id$.

The second estimate in the statement of the lemma is a direct application of the first one (see also [Azz+16d, Lemma 3.4]). □

The following lemma provides the connection between the function γ in Lemma 4.12.2 and elliptic measure.

Lemma 4.12.4. *Let $i = 1, 2$ and Ω_i, p_i be as above. Let $0 < R < \min_i \operatorname{dist}(p_i, \partial\Omega_i)$. Then, for $0 < r < R/4$ and $\xi \in \partial\Omega_1 \cap \partial\Omega_2$ we have*

$$\frac{\omega_1(B(\xi, r))}{r^n} \frac{\omega_2(B(\xi, r)}{r^n} \lesssim \gamma(\xi, 2r)^{1/2}. \tag{4.12.5}$$

Moreover, if $r < \delta_0 R/8$ and $\omega_i(B(\xi, 8r)) \lesssim \omega_i(B(\xi, \delta_o r))$,

$$\gamma(\xi, r)^{1/2} \lesssim \frac{\omega_1(B(\xi, 16\delta_0^{-1}))}{r^n} \cdot \frac{\omega_2(B(\xi, 16\delta_0^{-1}))}{r^n}. \tag{4.12.6}$$

The proof of (4.12.5) is analogous to that for the harmonic measure in [KPT09]. The proof of (4.12.6) is an application of Caccioppoli's inequality together with Lemma 4.12.3 (see also [Azz+16d, Lemma 3.5]).

The blowup technique for the elliptic measure developed in [AM17] is crucial to prove the next lemma. We remark that the authors formulated this result under more general assumptions on the matrix A then the ones of the present work.

Lemma 4.12.5. *Let Ω_1, Ω_2 and E be as above. Let $\varepsilon < 1/100$ and, for $m \geq 1$, define E_m as the set of $\xi \in E$ such that for all $\xi \in E$, $0 < r < 1/m$ and $i = 1, 2$ the following properties hold:*
(E1) $\omega_i(B(\xi, 2r)) \leq m\, \omega_i(B(\xi, r))$.
(E2) $\mathcal{H}^{n+1}(B(\xi, r) \cap \Omega_i) \geq \frac{1}{m} r^{n+1}$.
(E3) $\beta_{\omega_1,1}(B(\xi, r)) < \varepsilon r^{-n} \omega_1(B(\xi, r))$.
The sets E_m cover E up to a set of ω_1-measure 0, i.e.

$$\omega_1\Big(E \setminus \bigcup_{m\geq 1} E_m\Big) = 0.$$

The proof follows by known results in the literature. However, we think that it may be useful to the reader to dispose of precise references.

Sketch of the proof. Set

$$E^* = \Big\{\xi \in E : \lim_{r\to 0} \frac{\omega_1(E \cap B(\xi, r))}{\omega_1(B(\xi, r))} = \lim_{r\to 0} \frac{\omega_2(E \cap B(\xi, r))}{\omega_2(B(\xi, r))} = 1\Big\}.$$

One can see that $\omega_i(E \setminus E^*) = 0$, $i = 1, 2$. Now, for $\xi \in E^*$, set $h(\xi) = \frac{d\omega_1}{d\omega_2}(\xi)$,

$$\Lambda = \big\{\xi \in E^* : 0 < h(\xi) < \infty\big\}$$

and

$$\Gamma = \{\xi \in \Lambda : \xi \text{ is a Lebesgue point for } h \text{ with respect to } \omega_1\}.$$

By Lebesgue differentiantion theorem, $\omega_i(E \setminus \Gamma) = \omega_i(E^* \setminus \Gamma)$ for $i = 1, 2$. Then, in order to prove the lemma it suffices to show that for ω_1-almost every $\xi \in \Gamma$:
(P1) ω_1 is locally doubling, i.e.

$$\limsup_{r\to 0} \frac{\omega_1(B(\xi, 2r))}{\omega_1(B(\xi, r))} < \infty.$$

(P2) For $i = 1, 2$

$$\liminf_{r\to 0} \frac{\mathcal{H}^{n+1}(B(\xi,r)\cap\Omega_i)}{r^{n+1}} > 0$$

(P3) We have the flatness estimate

$$\lim_{r\to 0} \beta_{\omega_1,1}(B(\xi,r))\frac{r^n}{\omega_1(B(\xi,r))} = 0.$$

The condition (P1) holds because of the flatness of the tangents $\mathrm{Tan}(\omega_i,\xi)$, see [AM17, Theorem 1.3], which is known to imply the locally doubling condition ([Pre87, Corollary 2.7]).

The property (P2) follows by the arguments in [Azz+16d] together with (4.12.4). To prove (P3), it suffices to argue as in the end of [Azz+16d, Section 5]. □

Now consider $m \geq 1$ such that $\omega_i(E_m)$.

Lemma 4.12.6. *Let $\delta > 0$. For ω_1-almost every $x \in E_m$ there is $r_x > 0$ such that, given an a-$P_{\tilde\gamma,\omega_1}$-doubling ball $B(x,r)$ with $r \leq r_x$, there exits a set $G_m(x,r) \subset E_m \cap B(x,r)$ such that*

$$\frac{\omega_1(B(z,t))}{t^n} \lesssim \frac{\omega_1(B(x,r))}{r^n} \quad \textit{for every } z \in G_m(x,r),\ 0 < t \leq 2r.$$

In particular,

$$\omega_1(B(x,r)\setminus G_m(x,r)) \leq \delta\omega_1(B(x,r)). \tag{4.12.7}$$

and, if we denote by $\tilde E_{m\delta}$ the set of points where (4.12.7) is verified, we have

$$\omega_1\big(E_m \setminus \tilde E_{m,\delta}\big) = 0.$$

This lemma can be proved arguing as in [Azz+16d, Lemma 6.2] and more precisely combining the locally doubling property of the elliptic measure ensured by the blowup argument together with Lemma 4.12.4.

We also point out that their argument relies on the monotonicity formula of Alt, Caffarelli and Friedman. So, to prove it in the elliptic case we have to invoke Lemma 4.12.2, whose hypothesis include the assumption $A_s(x) = Id$. This, of course, is not true in general. However, one can argue via the change of variable in Lemma 4.5.3 to achieve this property. For a more detailed treatment of how the elliptic measure varies under that transformation we refer to [Azz+16a, Corollary 2.5]. We omit further details.

From now on fix $\tilde\gamma = \tilde\alpha$. The following lemma contains an estimate of the potential of ω_1 which is needed to recollect the property (4) in Theorem 4.8.

Lemma 4.12.7 (cfr. [Azz+16d, Lemma 6.3])**.** *Let $0 < c \ll 1$ to be chosen small enough. For $m \geq 1$ and $\delta > 0$, let $\tilde E_{m,\delta}$ and r_{x_0} be as in the previous lemma. Consider $x_0 \in \tilde E_{m,\delta}$ and take*

$$0 < r_0 < \min\big(r_{x_0}, 1/m, \mathrm{dist}(p_1, \partial\Omega_1)\big).$$

Assume, moreover, that $B_0 = B(x_0, r)$ is an a-$P_{\omega_1,\tilde\alpha}$-doubling ball. Then, for all $x \in G_m(x_0,r_0)$ we have

$$T_*(\chi_{2B_0}\omega_1)(x) \lesssim \Theta_{\omega_1}(B_0).$$

Proof. Suppose $A_s(x_0) = Id$. Indeed, if this is not the case, one can argue via a change of variable as mentioned before. Also, without loss of generality, we can consider only the case $r \leq r_0/4$.

Let $\varepsilon > 0$. The proof relies on the estimates for the smoothened potential

$$\tilde{T}_\varepsilon \omega_1(z) := \int K(z,y)\varphi_\varepsilon(z-y)d\omega_1(y), \qquad z \in \mathbb{R}^{n+1},$$

where $\varphi\colon \mathbb{R}^{n+1} \to [0,1]$ is a smooth radial function whose support is contained in $\mathbb{R}^{n+1} \setminus B(0,1)$, equals 1 on $\mathbb{R}^{n+1} \setminus B(0,2)$ and φ_ϵ denotes the dilate $\varphi_\varepsilon(z) = \varphi(\epsilon^{-1}z)$.

Now take $x \in G_m(x_0, r_0)$, consider $r \le r_0/4$ and define

$$v_r(z) = \mathcal{E}(p_1, z) - \int \mathcal{E}(z,y)\varphi_r(x-y)d\omega_1(y), \qquad z \in \mathbb{R}^{n+1} \setminus [\operatorname{supp}(\varphi_r(x-\cdot)\omega_1) \cup \{p_1\}]. \tag{4.12.8}$$

Recall that $A_s(x_0) = Id$ and that $\Theta(\cdot; A(x_0)) = \Theta(\cdot; A_s(x_0))$. On the same range of z of (4.12.8) we consider

$$\bar{v}_r(z) = \Theta(p_1 - z; Id) - \int \Theta(z-y; Id)\varphi_r(x-y)d\omega_1(y).$$

As in [Azz+16d, Lemma 6.3], to prove the lemma it suffices to show the validity of the estimate

$$|\tilde{T}_r\omega_1(x) - \tilde{T}_{r_0/4}\omega_1(x)| \lesssim \Theta_{\omega_1}(B_0).$$

To this purpose, observe that

$$\begin{aligned} |\tilde{T}_r\omega_1(x) - \tilde{T}_{r_0/4}\omega_1(x)| &= |\nabla v_r(x) - \nabla v_{r_0/4}(x)| \\ &= \Big| \int \nabla_1 \mathcal{E}(x,y)\big(\varphi_r(x-y) - \varphi_{r_0/4}(x-y)\big)d\omega_1(y)\Big|. \end{aligned}$$

Now, using Lemma 4.2.2 and the Hölder continuity of A, it is not difficult (recall that $r_0 \le 1$) to prove that

$$|\nabla_1 \mathcal{E}(x,y) - \nabla_1 \Theta(x-y; Id)| \lesssim \frac{r_0^{\tilde{\alpha}}}{|x-y|^n} \le \frac{1}{|x-y|^n},$$

which in turn implies

$$\begin{aligned} &|\tilde{T}_r\omega_1(x) - \tilde{T}_{r_0/4}\omega_1(x)| \\ &\qquad \lesssim \Theta_{\omega_1}(B_0) + \Big| \int \nabla_1 \Theta(x-y; Id)\big(\varphi_r(x-y) - \varphi_{r_0/4}(x-y)\big)d\omega_1(y)\Big| \\ &\qquad = |\nabla \bar{v}_r(x) - \nabla \bar{v}_{r_0/4}(x)| + \Theta_{\omega_1}(B_0). \end{aligned} \tag{4.12.9}$$

We claim that $|\bar{v}_r(x) - \bar{v}_{r_0/4}(x)| \lesssim \Theta_{\omega_1}(B_0)$, which would conclude the proof. To show this, notice that functions $\bar{v}_r$ and $\bar{v}_{r_0/4}$ are harmonic outside $\operatorname{supp}(\varphi_r(x-\cdot)\omega_1) \cup \{p_1\}$, hence in particular in $B(x,r)$. Then, an application of the mean value property gives

$$|\nabla \bar{v}_r(x) - \nabla \bar{v}_{r_0/4}(x)| \lesssim \frac{1}{r} \fint_{B(x,r)} |\bar{v}_r(z) - \bar{v}_{r_0/4}(z)| dz. \tag{4.12.10}$$

Another application of the freezing argument together with the $C^{\tilde{\alpha}}$-continuity of A proves

$$|\bar{v}_r(z) - \bar{v}_{r_0/4}(z) - v_r(z) - v_{r_0/4}(z)| \lesssim r_0^{\tilde{\alpha}} r\, \Theta_{\omega_1}(B_0), \qquad z \in B(x,r)$$

that, gathered with (4.12.9) and (4.12.10) gives

$$\begin{aligned}|\tilde{T}_r\omega_1(x) - \tilde{T}_{r_0/4}\omega_1(x)| &\lesssim \Theta_{\omega_1}(B_0) + \frac{1}{r}\fint_{B(x,r)} |v_r(z) - v_{r_0/4}(z)|dz \\ &\leq \Theta_{\omega_1}(B_0) + \frac{1}{r}\fint_{B(x,r)} |v_r(z)|dz + \frac{1}{r}\fint_{B(x,r)} |v_{r_0/4}(z)|dz.\end{aligned}$$

From this point on, the proof is analogous to that in [Azz+16d]. □

The proof of the Theorem 4.3 follows the footprints of that of [AMT17b] and [Azz+16d]. More precisely, taking $x_0 \in \tilde{E}_{m,\delta}$ and r_0 as in Lemma 4.12.7, we split the set $G_m(x_0, r_0)$ as a union of

$$G_m^{zd}(x_0, r_0) = \left\{x \in G_m(x_0, r_0) : \lim_{r\to 0} \Theta_{\omega_1}\big(B(x,r)\big) = 0\right\}$$

and

$$G_m^{pd}(x_0, r_0) = G_m(x_0, r_0) \setminus G_m^{zd}(x_0, r_0).$$

Then, using Lemma 4.12.7, the elliptic analogue of [Azz+16d, Lemma 6.5] and the rectifiability Theorem 4.8 that we proved in the present chapter, it is possible to infer that

$$\omega_1(G_m^{zd}(x_0, r_0)) = 0.$$

On the other side, [PPT18, Theorem 3] ensures the existence of an n-rectifiable set $F(x_0, r_0) \subset G_m^{pd}(x_0, r_0)$ of mutual absolute continuity of the elliptic measure $\omega_1|_{F(x_0,r_0)}$ and the Hausdorff measure $\mathcal{H}^n|_{F(x_0,r_0)}$ that covers $G_m(x_0, r_0)$ up to a ω_1-null set. This concludes the proof of Theorem 4.3.

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