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Rectifiability of Radon measures

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PhD thesis of
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Preface

It is a truth universally acknowledged, that a PhD student lacking proper guidance is just a confused young person, running in circles and flailing their arms in despair. Luckily for me, I had many excellent teachers across the years, and this thesis could never have been written if it were not for them.

First and foremost, my advisor, Xavier Tolsa. He showed me the world of quantitative rectifiability, he suggested the problems to work on, he revealed the ways to move forward when I could see none. His endless patience and support were invaluable. Simply put, I am forever in his debt.

I am very grateful to Jonas Azzam for hosting me in Edinburgh in the spring of 2020. The visit did not go quite as planned – due to the coronavirus pandemic I spent most of the time locked in my apartment. Nevertheless, I learned a lot from Jonas, and I was always looking forward to our Skype calls.

Many thanks are due to Katrin Fässler and Tuomas Orponen for introducing me to the Heisenberg group during the Simons semester in Warsaw in October 2019. I am also grateful to some of my peers for many stimulating conversations on GMT, most notably to Michele Villa, Carmelo Puliatti and Alan Chang.

Of course, my mathematical education did not begin with my PhD studies. It goes without saying that I learned a ton during my mathematics degree at the University of Warsaw, and I am very thankful to all my instructors there. I should especially mention my bachelor advisor Marta Szumańska, who first showed me the beauty of GMT, and my master advisor Paweł Strzelecki. It was Paweł who suggested that I apply for the PhD programme in Barcelona.

Let's not stop here. I wouldn't have dreamed of studying mathematics if it weren't for my amazing teachers back at school – Krzysztof Kowalczyk in high school, Mariola Rucińska and Lilla Pokuszyńska in primary school. Thank you.

Finally, there are the first and most important teachers, my parents. I remember when I was 11 and my dad tried to explain what a system of two equations is – how difficult that was! Or some years before that, when we spent hours learning the multiplication table. I owe everything to their continuous and unwavering support.

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Abstract

This thesis is dedicated to the study of rectifiable measures, quantitative rectifiability, and to a lesser degree, the boundedness of singular integral operators defined with respect to measures with polynomial growth. It consists of seven chapters. The first chapter is a general introduction to the area of quantitative rectifiability, and the second contains various preliminary lemmas used throughout the thesis. The remaining five chapters are largely self-contained, as they are based on articles written by the author during his PhD studies: [Dąb19b, Dąb19a, Dąb20a, Dąb20b, AD20, DV20] (the last two were co-authored by Jonas Azzam and Michele Villa, respectively).

In Chapters III and IV we show that a Radon measure μ is n-rectifiable if and only if

$$\int_0^1 \alpha_{\mu,2}(x,r)^2\,\frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d,$$

where $\alpha_{\mu,2}(x,r)$ are coefficients quantifying local flatness of μ using the Wasserstein distance W_2. This provides an α_2 counterpart to recent results of Azzam-Tolsa and Azzam-Tolsa-Toro, where similar characterizations where shown in terms of other coefficients, the so-called β_2 and α numbers. Contrary to their results, the α_2 characterization requires no additional assumptions on densities or doubling properties of μ.

In Chapter V we introduce conical energies, which can be seen as a quantification of the notion of approximate tangent plane. We then use these energies to prove several results: a characterization of rectifiable measures, a characterization of sets containing big pieces of Lipschitz graps, and finally, a sufficient condition for boundedness of SIOs valid for measures with polynomial growth.

In Chapter VI we use a square function involving α numbers to characterize L^p functions defined on uniformly rectifiable sets. This can be seen as an extension of Tolsa's characterization of uniformly rectifiable sets in terms of the same square function.

Finally, in Chapter VII we prove a Heisenberg group counterpart of a lemma due to Guy David which asserts that non-atomic measures that define L^2 bounded Riesz transform have polynomial growth.

Contents

Introduction I

The main goal of this introduction is to sketch out the history of the quantitative rectifiability area, as well as provide background and motivation for results obtained in the thesis. A brief overview of the new results is given in Section 8.

1 Rectifiability

At its very core, this thesis is dedicated to the study of rectifiable sets and measures.

Definition 1.1. Let $1 \leq n < d$. We say that a Borel set $E \subset \mathbb{R}^d$ is *n-rectifiable* if there exists a countable number of Lipschitz maps $g_i : \mathbb{R}^n \to \mathbb{R}^d$ such that

$$\mathcal{H}^n\Big(E \setminus \bigcup_i g_i(\mathbb{R}^n)\Big) = 0,$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure.

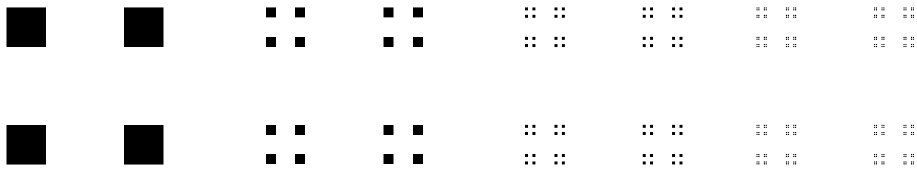
More generally, we say that a Radon measure μ on $\mathbb{R}^d$ is n-rectifiable if $\mu \ll \mathcal{H}^n$ and there exists an n-rectifiable set $E \subset \mathbb{R}^d$ such that $\mu(\mathbb{R}^d \setminus E) = 0$. Throughout most of the thesis we will be working with n-dimensional objects in $\mathbb{R}^d$, and so we will usually write "rectifiable" instead of "n-rectifiable".

The polar opposite of rectifiable sets are purely unrectifiable sets.

Definition 1.2. We say that a Borel set $F \subset \mathbb{R}^d$ is *purely n-unrectifiable* if for any Lipschitz map $g : \mathbb{R}^n \to \mathbb{R}^d$ we have

$$\mathcal{H}^n(F \cap g(\mathbb{R}^n)) = 0.$$

The history of these objects goes back almost a hundred years. The foundation stone for the study of rectifiability was laid by Besicovitch in his



F_1 F_2 F_3 F_4

Figure I.1: The first four steps of the construction of the four-corner Cantor set

1928 paper "On the fundamental geometrical properties of linearly measurable plane sets of points" [Bes28]. In the article, Besicovitch defined 1-rectifiable and purely 1-unrectifiable sets (in his terminology, "regular" and "irregular" sets) and proved their characterizations using densities and approximate tangents. A new discipline was born, one that would eventually come to be known as geometric measure theory.

One could think of rectifiable sets as a *very* weak measure-theoretic counterpart of C^1-manifolds. Compared to smooth surfaces they are very rough, and they may contain complex singularities. However, they still possess some crucial regularity properties that make them very useful.

Things go south once we lose rectifiability. It is purely unrectifiable sets that are the villains of this story. They exhibit numerous pathological behaviours: for example, suppose that F is a purely unrectifiable set with $\mathcal{H}^n(F) > 0$. Then, for almost all n-dimensional planes V, the projection of F onto V is $\mathcal{H}^n$-null. This is truly baffling, and at first rather hard to imagine, since the set we started with had positive $\mathcal{H}^n$ measure! To get an idea of how this can be, let us take a look at the most classical example of a purely 1-unrectifiable set, the four-corner Cantor set in the plane.

Example 1.3. The four-corner Cantor set $F \subset \mathbb{R}^2$ is defined as $F := \bigcap_{k\geq 1} F_k$, where the sets F_k are defined as follows (see also Figure I.1). We start with a set F_1 consisting of four squares, all of sidelength 4^{-1}, located in the corners of a unit square. In the next step, we replace each of the squares by a copy of F_1, rescaled by a factor of 4^{-1}, so that we get a set F_2 consisting of 4^2 squares of sidelength 4^{-2}. In general, to construct F_{k+1} we replace all the 4^k squares comprising F_k by copies of F_1, rescaled by a factor of 4^{-k}.

It is relatively easy to show that $0 < \mathcal{H}^1(F) < \infty$, see e.g. [Tol14, p. 35] *. At the same time, it can be shown that for almost all lines V, the projection of F onto V has zero length. See [Mat15, Chapter 10] for two different proofs of this fact. It then follows by the Besicovitch-Federer projection theorem (see Theorem 2.5) that F is purely 1-unrectifiable.

*With some more effort, one may actually prove that $\mathcal{H}^1(F) = \sqrt{2}$, see [XZ05].

The example above gives us a good idea of how purely unrectifiable sets look. They are very non-flat, very sparse, like a mist. Now, it is easy to show that for any Borel set $E \subset \mathbb{R}^d$ with $0 < \mathcal{H}^n(E) < \infty$ one can decompose it into two parts E_r and E_{pu} such that $E = E_r \cup E_{pu}$, E_r is rectifiable, and E_{pu} is purely unrectifiable. For the proof see [Mat95, Theorem 15.6]. In other words, each set as above has a "nice", rectifiable part, and an "ugly", purely unrectifiable part. It is then important to be able to distinguish between these two parts, or to verify whether the entire set is rectifiable or purely unrectifiable. To do that, many criteria have been developed throughout the years. Before we review some of them, let us say a few words about *why* rectifiable sets are useful.

There are at least two big, overarching motivations to study rectifiability. The first one comes from the calculus of variations. Suppose we wish to minimize a functional $F(\Sigma)$ among a class of competitors $\Sigma \in C$ satisfying some additional constraints. For example, in the classical Plateau problem F would be the area, while C would be a class of surfaces with a given fixed boundary. Of course, one has to be more precise when defining C, and it turns out that for F as above (but also for many other important geometric functionals) the class of smooth manifolds is too restrictive. There are two main reasons: firstly, the solutions to some problems may contain singularities. Secondly, in calculus of variations one often wishes to pass to the limit, in which case it is desirable for the class of objects we are working with to have good compactness properties. Note that both reasons are reminiscent of the motivation for introducing Sobolev functions when studying PDEs!

An incredibly rich theory has been developed to propose alternative classes of "generalised surfaces", better suited for variational problems. Perhaps the most important are the sets of finite perimeter, rectifiable currents, and rectifiable varifolds. All three classes are closely connected to rectifiable sets discussed before, essentially using them as building blocks. For an introduction to geometric measure theory oriented at calculus of variations see for example [Mor16], [Mag12], or [Sim14].

The second big motivation for the study of rectifiability comes from its connection to singular integral operators (often abbreviated as "SIOs"). This connection will be explored more in depth later on, for now let us just say that, due to omnipresence and importance of SIOs, rectifiability also plays a role in the study of removable sets for bounded analytic function, L^p solvability of the Dirichlet problem in rough domains, and the study of harmonic measure.

2 Classical rectifiability criteria

The key intuition necessary to understand rectifiability is the following: n-rectifiable sets are precisely those that resemble n-dimensional planes as you zoom in on them. Similarly, n-rectifiable measures should behave like (a constant times) n-dimensional Lebesgue measure on $\mathbb{R}^n$, on infinitesimal scales. These flatness properties are made more precise by the four classical characterizations of rectifiability: in terms of densities, approximate tangents, projections, and tangent measures. We will briefly overview them below, for a more in-depth discussion and proofs we refer the reader to [Mat95, Chapters 15–18].

Densities

Definition 2.1. Given a Radon measure μ on $\mathbb{R}^d$ and $x \in \operatorname{supp}\mu$, the lower and upper n-dimensional densities of μ at x are defined as

$$\Theta^n_*(\mu, x) = \liminf_{r\to 0} \frac{\mu(B(x,r))}{r^n} \quad \text{and} \quad \Theta^{n,*}(\mu, x) = \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^n}.$$

If at some point x the upper and lower densities are equal, we say that the n-dimensional density exists at x, and we denote it by $\Theta^n(\mu, x) := \Theta^n_*(\mu, x) = \Theta^{n,*}(\mu, x)$. In the special case $\mu = \mathcal{H}^n|_E$, we will write $\Theta^n(E, x)$ instead of $\Theta^n(\mu, x)$, and similarly for upper and lower densities.

The idea behind densities is the following: we are comparing the μ-measure of infinitesimal balls with the Lebesgue measure of n-dimensional balls of the same radius. If at many points the two quantities agree (that is, the n-dimensional density of μ exists), then one may hope that μ behaves like Lebesgue measure on infinitesimal scales, and so it is rectifiable. This is indeed the case.

Theorem 2.2. *Let μ be a finite Radon measure. Then, μ is n-rectifiable if and only if for μ-a.e. $x \in \operatorname{supp}\mu$ the density $\Theta^n(\mu, x)$ exists, and is positive and finite.*

First result of this type was obtained by Besicovitch in [Bes38], in the case $n = 1$, $d = 2$, and $\mu = \mathcal{H}^n|_E$. Morse and Randolph [MR44] obtained the result for general measures μ, still under the assumption $n = 1$, $d = 2$. The case $n = 1$ and arbitrary d is due to Moore [Moo50]. The theorem in its full generality remained an open problem for many years. It was finally solved by Preiss in his famous paper [Pre87]. An accessible version of Preiss' proof can also be found in the lecture notes of De Lellis [DL08].

Approximate tangent planes

Let $V \in G(d, m)$, where $G(d, m)$ denotes the Grassmanian space of m-dimensional linear subspaces of $\mathbb{R}^d$ (we will always consider either $m = n$ or $m = d - n$). Given a point $x \in \mathbb{R}^d$, and $\alpha \in (0, 1)$, we define

$$K(x, V, \alpha) = \{y \in \mathbb{R}^d \ : \ \operatorname{dist}(y, V + x) < \alpha|x - y|\}.$$

That is, $K(x,V,\alpha)$ is an open cone centered at x, with direction V, and aperture α. For $r>0$ we define also the truncated cone

$$K(x,V,\alpha,r)=K(x,V,\alpha)\cap B(x,r),$$

Recall that an n-plane $W\in G(d,n)$ is a tangent plane to a set E if for every $\alpha\in(0,1)$ there exists some $r=r(\alpha)>0$ such that $E\cap K(x,W^{\perp},\alpha,r)=\varnothing$. While this notion is very useful if E is a smooth manifold, in the context of general rectifiable sets it makes more sense to consider a relaxed definition.

Definition 2.3. We say that an n-plane $W\in G(d,n)$ is an *approximate tangent plane* to a Radon measure μ at $x\in\operatorname{supp}\mu$ if $\Theta^{n,*}(\mu,x)>0$ and for every $\alpha\in(0,1)$

$$\lim_{r\to 0}\frac{\mu(K(x,W^{\perp},\alpha,r))}{r^n}=0. \tag{2.1}$$

Clearly, the existence of approximate tangents is a form of local flatness. Besicovitch used this property to characterize rectifiability in the case of $n=1$, $d=2$ [Bes28], while the remaining cases are due to Federer [Fed47].

Theorem 2.4. *Let μ be finite Radon measure on $\mathbb{R}^d$ satisfying $0<\Theta^{n,*}(\mu,x)<\infty$ for μ-a.e. $x\in\mathbb{R}^d$. Then, the following are equivalent:*

(i) μ is n-rectifiable,

(ii) for μ-a.e. $x\in\mathbb{R}^d$ there is a unique approximate tangent plane to μ at x,

(iii) for μ-a.e. $x\in\mathbb{R}^d$ there is $W_x\in G(d,n)$ and $\alpha_x\in(0,1)$ such that

$$\limsup_{r\to 0}\frac{\mu(K(x,W_x^{\perp},\alpha_x,r))}{r^n}<(\alpha_x)^n\,\varepsilon(n)\,\Theta^{n,*}(\mu,x), \tag{2.2}$$

where $\varepsilon(n)$ is a small dimensional constant.

Projections

Given an n-dimensional plane V, we denote by $\pi_V:\mathbb{R}^d\to V$ the orthogonal projection onto V. Consider some n-plane W, and a set $A\subset W$ with $0<\mathcal{H}^n(A)<\infty$. It is trivial to see that for such a perfectly flat set we have $\mathcal{H}^n(\pi_V(A))>0$ for $\gamma_{d,n}$-a.e. $V\in G(d,n)$, where $\gamma_{d,n}$ denotes the Haar measure on $G(d,n)$. It is easy to see that the same is true also for subsets of C^1 surfaces, or subsets of Lipschitz graphs. The celebrated Besicovitch-Federer projections theorem asserts that this property characterizes rectifiable sets of finite measure.

Theorem 2.5. *Let $E\subset\mathbb{R}^d$ be a Borel set satisfying $0<\mathcal{H}^n(E)<\infty$. Then,*

- *E is n-rectifiable if and only if every Borel subset $A \subset E$ with $\mathcal{H}^n(A) > 0$ satisfies*
$$\mathcal{H}^n(\pi_V(A)) > 0 \qquad \text{for } \gamma_{d,n}\text{-a.e. } V \in G(d,n).$$
- *E is purely n-unrectifiable if and only if*
$$\mathcal{H}^n(\pi_V(A)) = 0 \qquad \text{for } \gamma_{d,n}\text{-a.e. } V \in G(d,n).$$

The case $n = 1, d = 2$ was shown by Besicovitch [Bes39], and the general theorem is due to Federer [Fed47].

Tangent measures

Perhaps the most literal way of understanding the expression "asymptotically flat" is the one given by tangent measures.

Definition 2.6. Let μ be a Radon measure on $\mathbb{R}^d$. Given $x \in \mathbb{R}^d$ and $r > 0$ define $T_{x,r}(y) = (y - x)/r$. Denote by $(T_{x,r})_*\mu$ the image measure of μ by $T_{x,r}$, so that
$$(T_{x,r})_*\mu(A) = \mu(rA + x), \quad A \subset \mathbb{R}^d.$$
We will say that a non-zero Radon measure ν is a tangent measure to μ at x if there exists sequences $r_k \to 0$ and c_k of positive numbers such that
$$c_k (T_{x,r_k})_*\mu \xrightarrow{w} \nu,$$
where the convergence is understood in the sense of weak convergence of measures. The set of all tangent measures as above will be denoted by $\mathrm{Tan}(\mu, x)$.

The idea is the following: the maps $T_{x,r}$ zoom in on the measure around the point x, so that when passing to the limit (along some subsequence r_k, and with c_k acting as normalizing factors) we get information about the local behaviour of μ around x. This notion of tangent measures was introduced by Preiss in [Pre87], where he also proved the following characterization of rectifiability.

Theorem 2.7. *Suppose that μ is a Radon measure satisfying $0 < \Theta^n_*(\mu, x) \leq \Theta^{n,*}(\mu, x) < \infty$ for μ-a.e. x. Then μ is n-rectifiable if and only if for μ-a.e. x all $\nu \in \mathrm{Tan}(\mu, x)$ are of the form $\nu = c\mathcal{H}^n|_V$ for some $c > 0$ and $V \in G(d,n)$.*

The four characterizations of rectifiability from above are nowadays considered classical. Let us move on to more recent results, and the field of *quantitative rectifiability.*

3 The Analyst's Traveling Salesman Theorem

Recall that the classical Traveling Salesman Problem consists of finding the shortest path connecting a finite number of points in the plane. An analyst's variant of the problem would be the following: given some set $E \subset \mathbb{R}^2$, not necessarily finite, what is the shortest curve containing E? Here, by curve we mean a Lipschitz image of an interval. Obviously, if $\mathcal{H}^1(E) = \infty$ there can be no finite length curve containing E, and so the second question is: what are the conditions ensuring that such a curve exists? In the language of GMT, this can be recast as a problem of finding a characterization of 1-rectifiable sets of finite length, along with some quantitative length estimates. This problem was solved by Peter Jones in [Jon90]. Along the way Jones laid the first building blocks for the quantitative rectifiability area. To state his result, we need to introduce his famous β numers.

Definition 3.1. Let $E \subset \mathbb{R}^d$ be a Borel set, $x \in \mathbb{R}^d$ and $r > 0$. If $B(x,r) \cap E \neq \varnothing$ we define

$$\beta_{E,\infty}(x,r) = \inf_L \sup_{y \in E \cap B(x,r)} \frac{\operatorname{dist}(y,L)}{r},$$

where the infimum is taken over all lines L intersecting $B(x,r)$. For $B(x,r) \cap E = \varnothing$ we set $\beta_{E,\infty}(x,r) = 0$. If $B = B(x,r)$, we will also write $\beta_{E,\infty}(B) := \beta_{E,\infty}(x,r)$.

In other words, $\beta_{E,\infty}(x,r) \cdot r$ is the radius of the thinnest tube containing $E \cap B(x,r)$, see Figure I.2. Hence, $\beta_{E,\infty}(x,r)$ measures how flat the set E is inside the ball $B(x,r)$. The normalization by r ensures that β numbers are scale invariant: if $E' = (E-x)/r$, then we have $\beta_{E,\infty}(x,r) = \beta_{E',\infty}(0,1)$.

Let $\mathcal{D}$ denote the standard dyadic grid on $\mathbb{R}^2$, and for $Q \in \mathcal{D}$ let B_Q be the ball with the same center as Q and of radius $5\ell(Q)$, where $\ell(Q)$ is the sidelength of Q. Define

$$\beta^2(E) = \sum_{Q \in \mathcal{D}} \beta_{E,\infty}(B_Q)^2 \ell(Q).$$

Summing over all dyadic cubes gives us information about flatness of E at all scales and locations. The main result of [Jon90] is the following Analyst's Traveling Salesman Theorem (abbreviated as TST).

Theorem 3.2 (TST). *Let $E \subset \mathbb{R}^2$ be Borel. If $\beta^2(E) < \infty$, then there exists a curve Γ such that $E \subset \Gamma$, and*

$$\mathcal{H}^1(\Gamma) \lesssim \operatorname{diam}(E) + \beta^2(E).$$

Conversely, if Γ is a curve satisfying $\mathcal{H}^1(\Gamma) < \infty$, then

$$\beta^2(\Gamma) \lesssim \mathcal{H}^1(\Gamma). \tag{3.1}$$

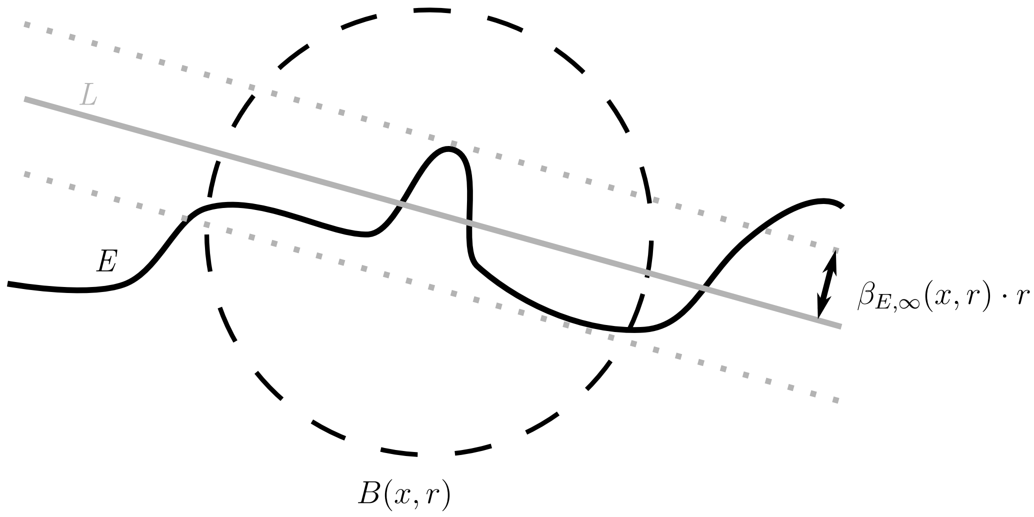


Figure I.2: The definition of $\beta_{E,\infty}(x,r)$.

Two remarks are in order.

Remark 3.3. Note that the curve Γ given by the theorem is, up to a constant, optimal, in the sense that for any curve Γ_0 containing E we have

$$\mathrm{diam}(E) + \beta^2(E) \leq \mathrm{diam}(E) + \beta^2(\Gamma_0) \overset{(3.1)}{\lesssim} \mathcal{H}^1(\Gamma_0).$$

Hence, if Γ_0 is a shortest curve containing E, we have $\mathcal{H}^1(\Gamma) \approx \mathcal{H}^1(\Gamma_0)$.

Remark 3.4. Observe that if E is a bounded subset of a line, then $\beta^2(E) = 0$, and the shortest curve containing E is a segment of length $\mathrm{diam}(E)$. Hence, the sum $\beta^2(E)$ captures the information about the curvature of set E. The reason for using squares of β numbers in $\beta^2(E)$ is, roughly speaking, Pythagorean theorem. To see that, suppose $E \subset \mathbb{R}^2$ is the union of segments $[(-1,0),(0,\varepsilon)]$ and $[(0,\varepsilon),(1,0)]$ for some small $\varepsilon > 0$. By Pythagorean theorem,

$$\mathcal{H}^1(E) = 2\sqrt{1+\varepsilon^2} = 2 + \varepsilon^2 + o(\varepsilon^2).$$

Note that $\beta_{E,\infty}(0,2) = \varepsilon/2$, and so $\mathcal{H}^1(E) \leq \mathrm{diam}(E) + 6\beta_{E,\infty}(0,2)^2$, assuming ε is small enough. As we see, compared to the line segment $[(-1,0),(1,0)]$, the increase in length related to the curvature at a given scale is controlled by the sum of squares of β numbers of that same scale.

Theorem 3.2 has found many application, for example in [BJ90], [BJ94], [BJ97], see also [Jon91]. It is natural then that much effort has been put into generalizing it. By "generalizing" one might understand two things: either proving a similar statement about curves in some metric space X, or considering coverings by higher dimensional objects instead of curves. A lot of progress has been made in both directions. In [Oki92] Okikiolu proved TST for curves in $\mathbb{R}^d$, and in [Sch07b] Schul further generalized it to the Hilbert space setting. Some results are also available for the Heisenberg group [FSSC06, LS16a, LS16b], Carnot groups [CLZ19], graph inverse limits spaces [DS17], ℓ_p spaces [BM20], and general metric spaces [Hah05, Sch07a, DS19]. In the other direction, i.e. finding a TST for higher dimensional sets, there are results related to covering sets by Hölder curves [BNV19] or by so-called topologically stable surfaces, see [AS18, AV19, Vil19a, Hyd20].

The original motivation for the Traveling Salesman Theorem came from the study of the Cauchy transform on Lipschitz curves, see [Jon90, p. 4]. This connection between geometry of sets and singular integral operators (abbreviated as SIOs) has been explored in great depth by Guy David and Stephen Semmes in their theory of uniform rectifiability.

4 Singular integral operators

To motivate the definition of uniform rectifiability, let us first make a brief detour into the world of singular integral operators. As the name suggests, they are operators given by integration against a kernel possessing some singularity. The most archetypical example is the Hilbert transform on $\mathbb{R}$, formally defined as

$$Hf(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}\, dy.$$

The higher dimensional analogue is the (vector valued) n-dimensional Riesz transform

$$Rf(x) = \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n+1}} f(y)\, dy.$$

Observe that the kernels above are not integrable (even locally), and so the definitions as stated make little sense, even for very nice functions. There are several standard ways of dealing with this problem, either by considering principal values, or by using truncated operators (see Definition 4.1). In any case, due to the antisymmetry of kernels, a lot of cancellations take place. In consequence, the modified definitions make sense for smooth and compactly supported f, and the operators can be extended to bounded operators on L^p for $1 < p < \infty$. Operators of this type naturally arise in many different contexts, including the study of convergence of Fourier series, partial differential equations, and others. For the introduction to the singular integral operators theory in this standard setting we refer the reader to [Ste70], [Duo01] or [Gra14a, Gra14b].

Observe that in the examples above the singularity of the kernel is of the same order as the dimension of the space. One could say that they are n-dimensional SIOs defined with respect to the Lebesgue measure on $\mathbb{R}^n$. In the sequel we will be concerned with the study of n-dimensional SIOs in

d-dimensional spaces defined with respect to more general measures (think of $\mathcal{H}^n$ restricted to n-dimensional sets). Let us fix some notation.

We are intersted in n-dimensional singular integral operators of convolution type, with odd C^2 kernels $k : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ satisfying for some constant $C_k > 0$

$$|\nabla^j k(x)| \le \frac{C_k}{|x|^{n+j}} \quad \text{for } x \neq 0 \quad \text{and} \quad j \in \{0, 1, 2\}. \tag{4.1}$$

We will denote the class of all such kernels by $\mathcal{K}^n(\mathbb{R}^d)$.

Definition 4.1. Given a kernel $k \in \mathcal{K}^n(\mathbb{R}^d)$, a constant $\varepsilon > 0$, and a (possibly complex) Radon measure ν, we set

$$T_\varepsilon \nu(x) = \int_{|x-y|>\varepsilon} k(y-x)\, d\nu(y), \quad x \in \mathbb{R}^d.$$

For a fixed positive Radon measure μ and all functions $f \in L^1_{loc}(\mu)$ we define

$$T_{\mu,\varepsilon} f(x) = T_\varepsilon(f\mu)(x).$$

We say that T_μ is bounded in $L^2(\mu)$ if all $T_{\mu,\varepsilon}$ are bounded in $L^2(\mu)$, uniformly in $\varepsilon > 0$. Let $M(\mathbb{R}^d)$ denote the space of all finite real Borel measures on $\mathbb{R}^d$. When endowed with total variation norm $\|\cdot\|_{TV}$, this is a Banach space. We say that T is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\mu)$ if there exists a constant C such that for all $\nu \in M(\mathbb{R}^d)$ and all $\lambda > 0$

$$\mu(\{x \in \mathbb{R}^d \ : \ |T_\varepsilon \nu(x)| > \lambda\}) \le \frac{C\|\nu\|_{TV}}{\lambda},$$

uniformly in $\varepsilon > 0$.

To motivate our interest in SIOs defined with respect to general measures, we give two applications.

Removable sets

Other than Hilbert transform and Riesz transform, perhaps the most classical SIO is the Cauchy transform. Given a finite complex valued Radon measure μ on $\mathbb{C}$, and $z \notin \operatorname{supp}\mu$ we define

$$\mathcal{C}\mu(z) = \int_{\mathbb{C}} \frac{d\mu(w)}{w-z}.$$

The importance of Cauchy transform in complex analysis comes from the fact that $\mathcal{C}\mu$ defines an analytic function on $\mathbb{C} \setminus \operatorname{supp}\mu$. This fact made Cauchy transform a perfect tool for the study of removable sets for bounded analytic functions.

We say that a compact set $E \subset \mathbb{C}$ is removable for bounded analytic functions if for every open $U \supset E$ and any bounded analytic function f :

$U \setminus E \to \mathbb{C}$, f can be extended to an analytic function on U. The Painlevé problem consists of finding geometric criteria for removability. It is not too difficult to show that if $\mathcal{H}^1(E) = 0$, then E is removable. Conversely, if the Hausdorff dimension of E is larger than 1, then E is not removable. The case of 1-dimensional sets E with $\mathcal{H}^1(E) > 0$ is much more delicate, and is closely related to the so-called analytic capacity, introduced by Ahlfors [Ahl47]. After decades of collective effort from many mathematicians it was finally settled by Mattila, Melnikov, Verdera [MMV96] and David [Dav98] that if we additionally assume that $\mathcal{H}^1(E) < \infty$, then E is removable if and only if E is purely 1-unrectifiable. We refer the reader to books [Paj02] and [Tol14] for the complete story and proofs of these beautiful results.

The results mentioned above rely deeply on identifying the measures μ on $\mathbb{C}$ such that the Cauchy transform with respect to μ is bounded on $L^2(\mu)$, in the sense that the truncated operators

$$\mathcal{C}_{\mu,\varepsilon} f(z) = \int_{|w-z|>\varepsilon} \frac{f(w)}{w-z}\, d\mu(w) \tag{4.2}$$

are bounded on $L^2(\mu)$ uniformly in $\varepsilon > 0$. Without delving into the proofs of the previous results, the connection between removability and Cauchy transform becomes evident thanks to a theorem of Xavier Tolsa. In [Tol99] and [Tol03] he showed that a set $E \subset \mathbb{C}$ is non-removable for bounded anaylytic functions if and only if there exists a (non-atomic) measure μ with $\operatorname{supp} \mu \subset E$ such that $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$ (in fact, he showed a quantitative version of this result involving analytic capacity, see also [Tol14, Theorems 4.14, 6.1]). This is essentially the only solution to the Painlevé problem available for 1-dimensional sets with $\mathcal{H}^1(E) = \infty$.

Method of layer potentials

Suppose a domain $\Omega \subset \mathbb{R}^{n+1}$ is given, and we are interested in solving the Laplace equation $\Delta u = 0$ in Ω, with either Dirichlet or Neumann L^p boundary condition on $\partial\Omega$. One of the ways to do it is by using the so-called method of layer potentials. Without going into details, let us just say that it consists of studying integral operators of the form

$$\mathcal{S}f(x) = C_n \int_{\partial\Omega} \frac{1}{|x-y|^{n-1}} f(y)\, d\mathcal{H}^n(y), \quad x \in \mathbb{R}^{n+1} \setminus \partial\Omega$$
$$\mathcal{D}f(x) = C_n \int_{\partial\Omega} \frac{\nu(y)\cdot(x-y)}{|x-y|^{n+1}} f(y)\, d\mathcal{H}^n(y), \quad x \in \mathbb{R}^{n+1} \setminus \partial\Omega,$$

where $f \in L^p(\mathcal{H}^n|_{\partial\Omega})$ and $\nu(y)$ denotes the inward unit normal of $\partial\Omega$ at y. These operators are the so called single and double layer potentials, and their kernels originate from the fundamental solution for the Laplace equation. An elementary computation shows that the functions $\mathcal{S}f$ and $\mathcal{D}f$ are harmonic in $\mathbb{R}^d \setminus \partial\Omega$. Hence, they may be treated as candidates for the solutions of the problem – $\mathcal{D}f$ is used in the Dirichlet problem, and $\mathcal{S}f$ in the Neumann problem. Of course, to solve the boundary value problem one needs to study the behaviour of $\mathcal{D}f(x)$ and $\nabla\mathcal{S}f(x)$ as x approaches $\partial\Omega$, which inevitably leads to the study of n-dimensional singular integral operators defined with respect to the surface measure on $\partial\Omega$. For an introduction to the method of layer potentials see [DK96], [Ken94], or [Van14].

In the case of domains with $C^{1,\alpha}$-regular boundaries, the scheme sketched above can be implemented rather easily. In the case of C^1 domains, it was first achieved in [FJR78]. A few years later a generalization to Lipschitz domains was obtained [Ver84]. The method of layer potentials has also been applied to a variety of other problems, including more general elliptic equation [HMT10], the heat equation [FR79, Bro89, LM95, HL96, Wat97], the Stokes systems [MMS09], or the sub-elliptic Kohn-Laplace equation [OV20]. All these results rely on a careful analysis of certain singular integral operators, whose definition depends on the problem.

With the hope that the two applications above were enough to stoke reader's curiosity and enthusiasm for the study of SIOs defined with respect to general measures, the natural question that comes to mind is the following: what are the measures μ such that reasonable n-dimensional SIOs (say, with kernels in $\mathcal{K}^n(\mathbb{R}^d)$) are bounded on $L^2(\mu)$?

5 Uniform rectifiability

First, let us look at the Cauchy transform (4.2). In the case of μ being the arclength measure on a $C^{1,\alpha}$ curve, the $L^2(\mu)$ boundedness of $\mathcal{C}_\mu$ can be easily derived from the boundedness of Hilbert transform on $\mathbb{R}$. The reason for that is the following: $C^{1,\alpha}$ curves can be very well approximated by lines, we have uniform control over the errors made by the approximation, and the Cauchy transform over a straight line is essentially the Hilbert transform. As it turns out, this idea of approximating measures by lines (or planes) is crucial for the understanding of singular integral operators with respect to general measures. The question is, just how good the approximation has to be?

Contrary to the $C^{1,\alpha}$ case, proving L^2 boundedness of Cauchy transform over Lipschitz graphs is a delicate matter. It was first obtained by Calderón in the case of graphs with small Lipschitz constants [Cal77], and the general case was solved by Coifman, McIntosh and Meyer [CMM82]. Since then, many other proofs have been found [Dav84, Mur88, CJS89, Chr90, MV95]. In fact, in [Dav84] David showed that the Cauchy transform is bounded on any curve Γ satisfying

$$\mathcal{H}^1(\Gamma \cap B(x,r)) \leq Cr, \quad x \in \Gamma, r > 0.$$

He called such curves "regular"†. A few years later it was shown that regular curves provide just the right framework for the study of 1-dimensional SIOs. To explain this, let us introduce more definitions.

Definition 5.1. We say that a Radon measure μ on $\mathbb{R}^d$ is n-Ahlfors-David regular (n-ADR) if there exists some constant $A > 0$ such that for all $x \in \operatorname{supp}\mu$ and $0 < r < \operatorname{diam}(\operatorname{supp}\mu)$ we have

$$A^{-1}r^n \leq \mu(B(x,r)) \leq Ar^n.$$

We say that a Borel set E is n-ADR if the measure $\mathcal{H}^n|_E$ is n-ADR.

It is easy to show (using e.g. [Mat95, Theorem 6.9]) that any n-ADR measure μ can be represented as $\mu = g\mathcal{H}^n|_E$, where $A^{-1} \lesssim g(x) \lesssim A$ and E is n-ADR. Hence, it usually does not make much difference whether one studies ADR sets or ADR measures. The ADR property should be thought of as "quantitative n-dimensionality". Note that regular curves are 1-ADR.

The ADR condition alone cannot imply boundedness of Cauchy transform (or SIOs in general). Note that the four-corner Cantor set F from Example 1.3 is 1-ADR, but it has been known for a long time that Cauchy transform is not L^2 bounded on F (this essentially follows from [Gar70], see also [Tol14, Section 4.7]). The reader may recall, however, that the set F is purely 1-unrectifiable. Together with our earlier remarks on the kinds of sets for which the Cauchy transform is bounded (Lipschitz graphs, regular curves), one could hope that rectifiability together with ADR condition suffice for the boundedness of SIOs. That is not the case. Observe that rectifiability, as defined in Definition 1.1, is a *qualitative* condition, while the boundedness of SIOs is a *quantitative* property. For this reason, one could construct a rectifiable, 1-ADR set E that approximates the four-corner Cantor set arbitrarily well, and in consequence the Cauchy transform would not be L^2 bounded on E.

The appropriate quantitative notion of rectifiability has been defined and studied by Guy David and Stephen Semmes in their monumental monographs [DS91] and [DS93a].

Definition 5.2. Suppose $E \subset \mathbb{R}^d$ is n-ADR. We say that E is *uniformly n-rectifiable* (abbreviated as UR) if there exist constants $\kappa > 0$ and $L > 0$ such that for every $x \in E$ and $0 < r < \operatorname{diam}(E)$ there exists a Lipchitz map $g : \mathbb{R}^n \to \mathbb{R}^d$ with $\operatorname{Lip}(g) \leq L$ such that

$$\mathcal{H}^n(E \cap B(x,r) \cap g(B^n(0,r))) \geq \kappa r^n,$$

where $B^n(0,r)$ is the n-dimensional ball in $\mathbb{R}^n$. In other words, E is UR if and only if every ball B centered at E contains a "big piece of Lipschitz image" (BPLI).

†It was communicated to the author by Guy David that Ahlfors already called such curves "regular" in the 30s.

Analogously, we will say that μ is uniformly n-rectifiable if μ is n-ADR and $\mu(\mathbb{R}^d \setminus E) = 0$ for some UR set E.

This somewhat technical definition becomes much simpler in the case of $n = 1$: uniformly 1-rectifiable sets are precisely 1-ADR subsets of regular curves. Observe also that any UR set is rectifiable, but the converse is not true.

David and Semmes proved in [DS91] the following fundamental result.

Theorem 5.3. *Suppose E is n-ADR. Then, it is uniformly n-rectifiable if and only if for all kernels $k \in \mathcal{K}^n(\mathbb{R}^d)$*[‡] *the singular integral operator $T_{\mathcal{H}^n|_E}$ associated to k is bounded on $L^2(\mathcal{H}^n|_E)$.*

Thus, David and Semmes gave an almost complete answer to the problem of characterizing measures for which the SIOs are L^2 bounded. They also proved in [DS91] and [DS93a] a dazzling number of geometric and analytic characterizations of UR sets; throughout the years the list of criteria for uniform rectifiability has been further expanded by many authors, and to this day it is an active area of research. Later on we will mention some of the characterizations, but now let us make a few remarks concerning the sharpness of the theorem above.

Remark 5.4. How restrictive is the ADR assumption in this context? Not too much. Recall that the (truncated) n-dimensional Riesz transform R_μ is given by

$$R_{\mu,\varepsilon} f(x) = \int_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{n+1}} f(y)\, d\mu(y).$$

David has shown in [Dav91, Part III, Proposition 1.4] that if R_μ is bounded on $L^2(\mu)$ (in the sense of Definition 4.1), and μ is atomless (all singletons have zero μ-measure), then μ satisfies the so-called n-polynomial growth condition, i.e. there exists some constant C such that

$$\mu(B(x,r)) \leq C r^n, \quad x \in \mathbb{R}^d,\ r > 0. \tag{5.1}$$

In other words, if we disregard measures containing atoms, the upper bound from the ADR condition is necessary for the boundedness of reasonable SIOs.

Concerning the lower bound from the ADR condition, it can be seen as a sort of non-degeneracy condition. It ensures that the measure is n-dimensional in a strong sense. If we consider a measure μ that does not satisfy the lower bound, the SIOs may still be bounded on $L^2(\mu)$, but it may be not too interesting. That is for example the case for μ equal to Lebesgue measure on a compact subset of $\mathbb{R}^d$ – if $n < d$, then of course all SIOs with kernels in $\mathcal{K}^n(\mathbb{R}^d)$ are

[‡] Originally David and Semmes assumed that kernels are C^∞, with appropriate estimates on the derivatives. The assumptions were relaxed to C^2, as in the definition of $\mathcal{K}^n(\mathbb{R}^d)$, by Tolsa [Tol09].

bounded on $L^2(\mu)$, but this is simply because the order of singularity is smaller than dimension of μ, and so the kernel is integrable.

However, there *are* interesting measures that are not ADR but that define bounded SIOs. For example, certain probability measures on Cantor-type sets, such as the one described in Section V.12[§]. See also the discussion in Subsection V.1.3 for available results related to L^2 boundedness of SIOs in non-ADR setting.

Remark 5.5. One of the implications of Theorem 5.3 is the following: if *all* SIOs with kernels in $\mathcal{K}^n(\mathbb{R}^d)$ are bounded, and E is AD regular, then E is UR. However, David and Semmes conjectured that it should be enough to assume boundedness of a single (vectorial) SIO - the Riesz transform. The David-Semmes conjecture is one of the most famous problems in the field, and this far it has been shown to be true only for $n = 1$ by Mattila, Melnikov and Verdera [MMV96] and for $n = d - 1$ by Nazarov, Tolsa and Volberg [NTV14a].

6 Quantifying flatness

In this section we finally introduce the whole menagerie of flatness quantifying coefficients which play a central role in this thesis. Recall that in Section 3 we defined Jones' β numbers that quantified local flatness of sets in a scale-invariant way. In fact, even before proving the Traveling Salesman Theorem, Jones showed the following.

Theorem 6.1 ([Jon89])**.** *Suppose that $\Gamma \subset \mathbb{R}^2$ is a 1-dimensional Lipschitz graph. Then, there exists $C > 0$ such that for any $z \in \Gamma$ and $R > 0$*

$$\int_{B(z,R)} \int_0^R \beta_{\Gamma,\infty}(x,r)^2 \, \frac{dr}{r} d\mathcal{H}^1\big|_{\Gamma}(x) \le CR.$$

We will call an estimate as above a *Carleson condition.* Jones used the theorem above in his proof of the L^2 boundedness of Cauchy integral over Lipschitz graphs. Interestingly, a more general version of Theorem 6.1 was proved earlier by Dorronsoro [Dor85] while studying affine approximations of Sobolev functions. However, Dorronsoro did not relate it to geometry or SIOs.

Note that the definition of $\beta_{\Gamma,\infty}(x,r)$ makes perfect sense also for n-dimensional sets, as long as we replace in Definition 3.1 lines by n-planes. However, the estimate above is not true for n-dimensional Lipschitz graphs if $n > 1$, and the counterexample is due to Fang [Fan90]. One can fix this problem by considering a modified version of β numbers.

[§]That is, Section 12 in Chapter V. We explain the cross-referencing system used throughout the thesis on p. 25.

6.1 β_p numbers

Definition 6.2. For $1 \le p < \infty$ and a Radon measure μ on $\mathbb{R}^d$ set[¶]

$$\beta_{\mu,p}(x,r) = \inf_L \left(\frac{1}{r^n} \int_{B(x,r)} \left(\frac{\operatorname{dist}(y,L)}{r} \right)^p d\mu(y) \right)^{1/p}, \tag{6.1}$$

where the infimum runs over all n-planes L intersecting $B(x,r)$. If $\mu = \mathcal{H}^n|_E$ for some set E, we will write $\beta_{E,p}(x,r)$ instead of $\beta_{\mu,p}(x,r)$. Furthermore, given a ball $B = B(x,r)$, we set $\beta_{\mu,p}(B) := \beta_{\mu,p}(x,r)$, and the same convention will be used with all the other coefficients.

Thus, β_p numbers can be seen as L^p variants of Jones' β_∞ numbers. It follows immediately by Hölder inequality that if $p < q$ then

$$\beta_{\mu,p}(x,r) \le \left(\frac{\mu(B(x,r))}{r^n} \right)^{1/p-1/q} \beta_{\mu,q}(x,r).$$

Hence, for measures with polynomial growth (5.1) we have $\beta_{\mu,p}(x,r) \le C\beta_{\mu,q}(x,r)$.

Fang used β_p numbers to prove a modification of Theorem 6.1 valid for n-dimensional Lipschitz graphs (though again, it follows from [Dor85]). This result was soon extended by the following theorem of David and Semmes.

Theorem 6.3 ([DS91])**.** *Let μ be n-AD regular. If $n = 1$ let $1 \le p < \infty$, and if $n \ge 2$ assume that $1 \le p < \frac{2n}{n-2}$. Then, μ is uniformly n-rectifiable if and only if there exists $C > 0$ such that for any ball $B = B(z,R)$ with $z \in \operatorname{supp}\mu$ and $R > 0$*

$$\int_B \int_0^R \beta_{\mu,p}(x,r)^2 \frac{dr}{r} d\mu(x) \le C\mu(B).$$

Together with Theorem 5.3 this answers the question we posed at the beginning of Section 5: how well should a set E be approximated by planes in order for SIOs to be L^2 bounded on E.

Due to their natural definition, coefficients β_p found many more applications. In [Tol15] Tolsa showed that for a rectifiable measure μ we have

$$\int_0^1 \beta_{\mu,2}(x,r)^2 \frac{dr}{r} < \infty \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{6.2}$$

On the other hand, Azzam and Tolsa proved in [AT15] that if a Radon measure μ satisfies (6.2) and

$$0 < \Theta^{n,*}(x,\mu) < \infty \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d, \tag{6.3}$$

[¶] The definitions of $\beta_{\mu,p}$ and other coefficients may vary slightly between different chapters, see Remark II.3.1

then μ is n-rectifiable. More recently, Edelen, Naber and Valtorta [ENV16] managed to weaken the assumption (6.3) to

$$\Theta^{n,*}(x,\mu) > 0 \quad \text{and} \quad \Theta^n_*(x,\mu) < \infty \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{6.4}$$

Theorem 6.4 ([Tol15, AT15, ENV16])**.** *Let μ be a Radon measure on $\mathbb{R}^d$. Then, μ is n-rectifiable if and only if (6.2) and (6.4) hold for μ-a.e. $x \in \mathbb{R}^d$.*

Further generalization of this result to Hilbert and Banach spaces was achieved in [ENV19].

Remark 6.5. Measures of the form $\mu = \mathcal{H}^n|_E$ for $E \subset \mathbb{R}^d$ with $0 < \mathcal{H}^n(E) < \infty$ automatically satisfy (6.3) (see [Mat95, Theorem 6.2]), and so in this special case, by the results of Azzam and Tolsa, we get a particularly clean characterization: E is n-rectifiable if and only if (6.2) holds.

Going back to general measures, it is well known that (6.3) implies $\mu \ll \mathcal{H}^n$, which is included in our definition of rectifiable measures. However, (6.4) does not imply $\mu \ll \mathcal{H}^n$ on its own, and so it is remarkable that together with (6.2) it gives rectifiability. An alternative proof of this fact is also given in [Tol19].

Remark 6.6. Note that in Theorem 6.3 we have some liberty when choosing p in β_p numbers. In the case of qualitative rectifiability, the choice of $p = 2$ is the best possible. Condition (6.2) with $\beta_{\mu,2}(x,r)$ replaced by $\beta_{\mu,p}(x,r)$ is necessary for rectifiability only for $1 \le p \le 2$. On the other hand, (6.2) together with (6.3) imply rectifiability only for $p \ge 2$. See [Tol19] for relevant counterexamples. Still, if instead of (6.3) we assume that $\Theta^n_*(\mu,x) > 0$ and $\Theta^{n,*}(\mu,x) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$, then the finiteness of β_p square function for certain $p < 2$ becomes sufficient for rectifiability, see [Paj97, BS16].

Let us mention that modified versions of β numbers are also used to study a competing notion of rectifiability for measures, the so-called *Federer rectifiability*. We say that a measure μ is n-rectifiable in the sense of Federer if there exists a countable number of Lipschitz images of $\mathbb{R}^n$, denoted by Γ_i, such that $\mu(\mathbb{R}^d \setminus \bigcup_i \Gamma_i) = 0$. No absolute continuity with respect to $\mathcal{H}^n$ is required. Dropping the absolute continuity assumption makes such measures very difficult to characterize: a surprising example of a doubling, Federer 1-rectifiable measure supported on the whole plane was found by Garnett, Killip and Schul [GKS10]. Nevertheless, for $n = 1$ significant progress has been achieved in [Ler03, BS15, BS16, AM16, BS17, MO18a, Nap20]. See also a recent survey of Badger [Bad19].

Finally, let us remark that if a set $E \subset \mathbb{R}^d$ satisfies faster decay of $\beta_{E,p}$ numbers than (6.2), then it is actually $C^{1,\alpha}$ rectifiable, in the sense that it can be covered $\mathcal{H}^n$-a.e. by $C^{1,\alpha}$ surfaces. See [Ghi20] and [DNI19] for details.

6.2 α numbers

We would like to stress that β numbers were originally introduced to study sets, and they do have some limitations when applied to general measures. They capture the *shape* of the support of measures, but they do not see the *distribution* of mass within the support. Observe that any measure with support contained in an n-dimensional plane has all β numbers equal to 0, but of course such measure may be very far from being rectifiable - think of Dirac deltas. For this reason, some assumptions on densities in Theorem 6.4 are unavoidable.

Tolsa's α numbers, introduced in [Tol09], offer a way to solve the issue mentioned above. To define them, we need a distance on the space of measures. Given Radon measures μ and ν, and an open ball B, we set

$$F_B(\mu,\nu) = \sup\left\{\left|\int \phi\ d\mu - \int \phi\ d\nu\right|\ :\ \phi \in \mathrm{Lip}_1(B)\right\},$$

where

$$\mathrm{Lip}_1(B) = \{\phi\ :\ \mathrm{Lip}(\phi) \le 1,\ \operatorname{supp}\phi \subset B\}.$$

Note that $F_B(\mu,\nu)$ measures the distance between μ and ν *inside the ball* B. See [Mat95, Chapter 14] for more information about this distance.

Definition 6.7. Given a Radon measure μ and a ball $B = B(x,r)$ we define

$$\alpha_\mu(x,r) = \inf_{c,L} \frac{1}{r\mu(B)} F_B(\mu, c\mathcal{H}^n|_L),$$

where the infimum runs over all $c \ge 0$ and all n-planes L.

The idea is the following: $\alpha_\mu(B)$ quantifies how far μ is from flat measures (i.e. measures of the form $c\mathcal{H}^n|_L$, L an n-plane) inside B. Tolsa characterized uniform rectifiability in terms of a Carleson condition imposed on α numbers.

Theorem 6.8 ([Tol09])**.** *Let μ be n-AD regular. Then, μ is uniformly n-rectifiable if and only if there exists $C > 0$ such that for any ball $B = B(z,R)$ with $z \in \operatorname{supp}\mu$ and $R > 0$*

$$\int_B \int_0^R \alpha_\mu(x,r)^2\ \frac{dr}{r} d\mu(x) \le C\mu(B). \tag{6.5}$$

Concerning the qualitative notion of rectifiability, one might expect that a condition of the form

$$\int_0^1 \alpha_\mu(x,r)^2\ \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d \tag{6.6}$$

could characterize rectifiable measures. Tolsa showed in [Tol15] that (6.6) is necessary for rectifiability. But is it sufficient? Azzam, David, and Toro proved in [ADT16] that if μ is doubling, then some condition related to (6.6) is sufficient for rectifiability. In [Orp18a] Orponen showed that for $n = d = 1$ a variant of (6.6) is sufficient for rectifiability (which in this case is equivalent to absolute continuity with respect to $\mathcal{H}^1$). Finally, Azzam, Tolsa and Toro [ATT20] proved that a measure μ satisfying (6.6) which is also pointwise doubling, i.e. such that

$$\limsup_{r\to 0^+} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d, \tag{6.7}$$

is rectifiable.

Theorem 6.9 ([Tol15, ATT20])**.** *Let μ be a Radon measure on $\mathbb{R}^d$. Then, μ is n-rectifiable if and only if (6.6) and (6.7) hold for μ-a.e. $x \in \mathbb{R}^d$.*

Also in [ATT20], the authors construct a purely 1-unrectifiable measure on $\mathbb{R}^2$ satisfying (6.6). This shows that, for general n and d, (6.6) on its own is *not* a sufficient condition for rectifiability.

To mention a few other applications of α numbers, in [Tol08] they are used to characterize rectifiability of sets of finite measure in terms of existence of principal values for the Riesz transform, and in [DEM18, Fen20, DM20] they are used to study higher co-dimensional analogues of harmonic measure.

6.3 α_p numbers

Coefficients α_p were introduced by Tolsa in [Tol12]. They can be thought of as a generalization of α numbers – in fact, under relatively mild assumptions, one has $\alpha_\mu(B) \approx \alpha_{\mu,1}(B)$, see [Tol12, Lemma 5.1]. As in the case of α coefficients, in order to define α_p numbers we need a metric on the space of measures.

Let $1 \le p < \infty$, and let μ, ν be two probability Borel measures on $\mathbb{R}^d$ satisfying $\int |x|^p \, d\mu < \infty$, $\int |x|^p \, d\nu < \infty$. The Wasserstein distance W_p between μ and ν is defined as

$$W_p(\mu,\nu) = \left(\inf_\pi \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p \, d\pi(x,y) \right)^{1/p},$$

where the infimum is taken over all transport plans between μ and ν, i.e. Borel probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$ satisfying $\pi(A \times \mathbb{R}^d) = \mu(A)$ and $\pi(\mathbb{R}^d \times A) = \nu(A)$ for all measurable $A \subset \mathbb{R}^d$. The same definition makes sense if instead of probability measures we consider μ, ν, and π of the same total mass. For more information on Wasserstein distance see for example [Vil03, Chapter 7] or [Vil08, Chapter 6].

Similarly as α numbers, α_p numbers quantify how far is a given measure from being a flat measure, that is, from being of the form $c\mathcal{H}^n|_L$ for some constant $c > 0$ and some n-plane L. In order to measure it locally (say, in a ball B), we introduce the following auxiliary function.

Let $\varphi : \mathbb{R}^d \to [0,1]$ be a radial Lipschitz function satisfying $\varphi \equiv 1$ in $B(0,2)$, $\operatorname{supp}\varphi \subset B(0,3)$, and for all $x \in B(0,3)$

$$c^{-1}\operatorname{dist}(x, \partial B(0,3))^2 \le \varphi(x) \le c\operatorname{dist}(x, \partial B(0,3))^2,$$
$$|\nabla\varphi(x)| \le c\operatorname{dist}(x, \partial B(0,3)),$$

for some constant $c > 0$. For example, one could take $\varphi(x) = \phi(|x|)$ where $\phi : [0,\infty) \to [0,1]$ is such that $\phi(r) = 1$ for $0 \le r \le 2$, $\phi(r) = 0$ for $r \ge 3$, and $\phi(r) = (3-r)^2$ for $2 < r < 3$. Given a ball $B = B(x,r) \subset \mathbb{R}^d$ we set

$$\varphi_B(y) = \varphi\left(\frac{y-x}{r}\right). \tag{6.8}$$

φ_B should be thought as a regularized characteristic function of B.

Definition 6.10. For $1 \le p < \infty$, a Radon measure μ on $\mathbb{R}^d$, and a ball $B = B(x,r)$, we define

$$\alpha_{\mu,p}(x,r) = \inf_L \frac{1}{r\mu(B)^{1/p}} W_p(\varphi_B\mu, a_{B,L}\varphi_B\mathcal{H}^n|_L),$$

where the infimum is taken over all n-planes L intersecting B, and

$$a_{B,L} = \frac{\int \varphi_B \, d\mu}{\int \varphi_B \, d\mathcal{H}^n|_L}.$$

Even though their definition is more involved than that of α numbers, α_p numbers have some advantages. Under mild assumptions on the measure, one can show that, on the one hand,

$$\beta_{\mu,p}(B) \lesssim \alpha_{\mu,p}(B),$$

and on the other hand, if $p < q$, then

$$\alpha_{\mu,p}(B) \lesssim \alpha_{\mu,q}(B),$$

see Lemma II.3.2. Thus, recalling that $\alpha_\mu \approx \alpha_{\mu,1}$, coefficients α_p simultaneously capture information given by β_p and α numbers.

Tolsa introduced α_p numbers in [Tol12] with the aim of characterizing uniformly rectifiable measures.

Theorem 6.11 ([Tol12])**.** *Let μ be an n-AD regular measure on $\mathbb{R}^d$, and suppose that $1 \le p \le 2$. Then, μ is uniformly n-rectifiable if and only if there exists $C > 0$ such that for any ball $B = B(z,R)$ with $z \in \operatorname{supp}\mu$ and $R > 0$*

$$\int_B \int_0^R \alpha_{\mu,p}(x,r)^2 \, \frac{dr}{r} d\mu(x) \le C\mu(B).$$

6.4 Other coefficients

Finally, let us briefly mention that a few other kinds of coefficients have been used in the study of rectifiability.

Menger curvature (and its higher dimensional counterparts) was studied in [Lég99, LW09, LW11, Kol17, Meu18, Goe18, GG20]. A coefficient involving center of mass is developed in [Vil19b]. In [TT15, Tol17] rectifiable sets and measures are characterized using Δ numbers, defined as $\Delta_\mu(x,r) = |\frac{\mu(B(x,r))}{r^n} - \frac{\mu(B(x,2r))}{(2r)^n}|$.

7 Recent trends

Despite our best efforts to make this introduction broad and inclusive, there are many important developments in quantitative rectifiability and related topics that we were not able to describe, simply due to the vast size of the subject matter. Nevertheless, we feel obliged to at least hint at some of them, to give the reader an idea of how diverse and active this research area is. The references below are by no means complete, they should be seen merely as an invitation to explore the topic further.

Firstly, there is the connection between rectifiability and harmonic measure. Given a domain $\Omega \subset \mathbb{R}^{n+1}$ and a continuous function $f \in C(\partial\Omega)$, let u_f denote the harmonic function on Ω with boundary values f. Fixing some $X \in \Omega$, $f \mapsto u_f(X)$ becomes a positive functional on $C(\partial\Omega)$, and so by the Riesz representation theorem, it defines a measure on $\partial\Omega$. We denote it by ω^X, and we call it the harmonic measure on $\partial\Omega$ with a pole at X. As it turns out, there is a deep connection between rectifiability of $\partial\Omega$, the relation between ω^X and $\mathcal{H}^n|_{\partial\Omega}$, and the L^p solvability of the Dirichlet problem on Ω. This has been explored in depth by many authors and by now it is very well understood. See e.g. [HMUT14, AHM+16, AHM+20].

The harmonic measure was defined using the Laplace operator Δ. More generally, given a suitable elliptic operators L we may define the elliptic measure ω_L. A lot of effort has been put into replicating the results obtained for harmonic measure to this more general setting, see e.g. [KP01, PPT18, HMM+20]. On the other hand, one can also study the caloric measure related to the heat equation. This introduces further complications due to the parabolic geometry: one has to define parabolic counterparts to uniform rectifiability and other GMT notions. See e.g. [HLN04, NS17, MP20].

As demonstrated above, and also by our remarks from the end of Section 1, one of the main motivations for the study of rectifiability are PDEs and calculus of variations. However, the natural setup for certain problems is not the Euclidean space, but another metric space (e.g. $\mathbb{R}^{n+1}$ with parabolic metric in the case of heat equation, the Heisenberg group in the case of Kohn-Laplace operator). This led to a flurry of activity aiming at generalizing classical notions and results of (Euclidean) GMT to this metric space setting. See e.g. [AK00, FO19, Bat20, AM20].

Finally, we would like to mention a trend of "quantification" of well-known qualitative results. A nice example, lying at the intersection of fractal geometry and GMT, is obtaining bounds for the Favard length of the four-corner Cantor set. Recall that in Example 1.3 we defined the four-corner Cantor set F as an intersection of sets F_k, which can be seen as better and better approximations of F. Recall that by Theorem 2.5 $\mathcal{H}^1(\pi_V(F)) = 0$ for $\gamma_{2,1}$-a.e. line V. Thus, if we define the Favard length of F as

$$\mathrm{Fav}(F) = \int_{G(2,1)} \mathcal{H}^1(\pi_V(F))\; d\gamma_{2,1}(V),$$

we have $\mathrm{Fav}(F) = 0$. In particular, $\mathrm{Fav}(F_k) \to 0$ as $k \to \infty$. That is a qualitative result. Its quantitative counterpart is: what is the rate of decay of $\mathrm{Fav}(F_k)$ as $k \to \infty$? See e.g. [Tao09, NPV11, BŁV14, CDT20].

8 New results and structure of the thesis

In this section we give a short overview of the results obtained in the thesis. Full presentations are given in the introduction to each chapter, together with an explanation of how they relate to previously known results.

Chapter II is dedicated to some preliminary definitions and estimates used throughout the thesis. We recall the definition of David-Mattila cubes, used in Chapters III and V, and we prove some basic estimates of α and β numbers.

In Chapters III and IV we show that a Radon measure μ is n-rectifiable if and only if

$$\int_0^1 \alpha_{\mu,2}(x,r)^2\, \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d,$$

see Theorem III.1.4 and Theorem IV.1.1. Thus, we provide an α_2 counterpart to Theorem 6.4 and Theorem 6.9. This characterization is especially satisfying due to no additional assumptions on densities or doubling properties of μ. Along the way we show a sufficient condition for rectifiability in terms of α and β_2 numbers, see Theorem III.1.2. These chapters contain the results from [Dąb19b, Dąb19a].

In Chapter V we introduce conical energies, which can be seen as a quantification of the notion of approximate tangent plane. We then use these energies to prove several results: a characterization of rectifiable measures Theorem V.1.3, a characterization of sets containing big pieces of Lipschitz graps (which is a stronger condition than UR) in Theorem V.1.9, and finally, a sufficient condition for the boundedness of SIOs valid for measures with polynomial growth (not necessarily ADR), see Theorem V.1.14. This chapter is based on [Dąb20a, Dąb20b].

In Chapter VI we use a square function involving α numbers, similar to that from (6.5), to characterize L^p functions defined on uniformly rectifiable sets, see Theorem VI.1.3. Based on joint work with Jonas Azzam [AD20].

Finally, recall that in Remark 5.4 we mentioned David's lemma which asserted that non-atomic measures defining L^2 bounded Riesz transform have polynomial growth. In Chapter VII we prove a counterpart of this result for Heisenberg groups, see Theorem VII.1.1. This chapter is based on [DV20], co-authored by Michele Villa.

Preliminaries II

1 Notation

The notation given below will be used in Chapters III–VI. We may use slightly different notation in Chapter VII due to the non-Euclidean, Heisenberg group setting.

Cross-referencing

Since the chapters are mostly self-contained, we decided to adapt the following system for cross-references: each object (theorem, lemma etc.) is assigned only two numbers, the first standing for section. When referencing content withing the same chapter, only those two numbers are used; when referencing an object from another chapter, three numbers are used, with the number of the chapter given at the beginning. For example, Lemma 2.1 references a lemma from the second section of the current chapter, but Lemma VI.2.2 denotes Lemma 2.2 from the second section of Chapter VI.

Estimates

Throughout the paper we will write $A \lesssim B$ whenever $A \leq CB$ for some constant C, the so-called "implicit constant". All such implicit constants may depend on dimensions n, d, and we will not track this dependence. If the implicit constant depends also on some other parameter t, we will write $A \lesssim_t B$. The notation $A \approx B$ means $A \lesssim B \lesssim A$, and $A \approx_t B$ means $A \lesssim_t B \lesssim_t A$. Moreover, if symbols $\lesssim$ or $\approx$ appear in the assumptions of a lemma, then the implicit constant of the proven estimate will depend on the implicit constants from the assumptions (see Lemma 3.4 for example).

Balls

We denote by $B(z,r) \subset \mathbb{R}^d$ an open ball with center at $z \in \mathbb{R}^d$ and radius $r > 0$. Given a ball B, its center and radius are denoted by $z(B)$ and $r(B)$, respectively. If $\lambda > 0$, then λB is defined as a ball centered at $z(B)$ of radius $\lambda r(B)$.

For a ball B and measure μ, we define the n-dimensional density of μ at B as

$$\Theta_\mu(B) = \frac{\mu(B)}{r(B)^n}.$$

For a ball $B = B(x,r)$, we write $\Theta_\mu(x,r) := \Theta_\mu(B)$.

Planes

Given two n-planes L_1, L_2, let L_1' and L_2' be the respective parallel n-planes passing through 0. Then,

$$\measuredangle(L_1, L_2) = \mathrm{dist}_H(L_1' \cap B(0,1),\ L_2' \cap B(0,1)),$$

where dist_H stands for Hausdorff distance between two sets. Clearly, we always have $\measuredangle(L_1, L_2) \in [0,1]$, and $\measuredangle(L_1, L_2) = 0$ if and only if L_1 and L_2 are parallel. Note that if L_1 and L_2 are lines in the plane, then $\measuredangle(L_1, L_2)$ is the sine of the angle between L_1 and L_2.

Given an affine subspace $L \subset \mathbb{R}^d$, we will denote the orthogonal projection onto L by π_L. The orthogonal projection onto $L^\perp$ will be denoted by $\pi_L^\perp$.

Sets

Given a set $A \subset \mathbb{R}^d$, we denote by $\mathbb{1}_A : \mathbb{R}^d \to \{0,1\}$ the characteristic function of A, and by $\#A$ the cardinality of A. If $f : \mathbb{R}^d \to \mathbb{R}$ is a function, then $f|_A$ denotes its restriction to A. Similarly, $\mu|_A$ will denote the measure μ restricted to A.

For sets $A, B \subset \mathbb{R}^d$ we define

$$\mathrm{dist}(A,B) = \inf_{a\in A}\inf_{b \in B} |a-b|,$$

while $\mathrm{dist}_H(A,B)$ will stand for the Hausdorff distance between A and B.

Dyadic lattices

Throughout all of the thesis, dyadic techniques are heavily used. However, usually we won't be able to work with "true" dyadic cubes, relying instead on certain "generalized dyadic cubes". The most classical constructions of this kind are due to Chirst [Chr90] and David [Dav88a]. Since then many other constructions of this type has been done, and depending on the context it is convenient to use different kinds of cubes. To avoid confusion, we use different fonts to distinguish between them:

- $\mathcal{D}$ denotes the David-Mattila lattice [DM00], defined in Section 2 below, and used in Chapters III and V.
- $\mathbb{D}_{\mathbb{R}^n}$ and $\mathbb{D}_{\mathbb{R}^d}$ denote the true dyadic grids on $\mathbb{R}^n$ and $\mathbb{R}^d$ respectively, as defined in Subsection IV.2.2 and used in Chapter IV. In the same subsection a few other grids are derived from them, e.g. $\mathbb{D}_\Gamma, \mathbb{D}_\Gamma^e, \widetilde{\mathbb{D}}_\Gamma$.
- $\mathscr{D}(\omega)$ denote the adjacent systems of cubes of Hytönen and Tapiola [HT14], see VI.2.2. They are used in Chapter VI.
- $\mathfrak{D}$ denotes the cubes of Käenmäki, Rajala and Suomala [KRS12], used in Chapter VII. See Subsection VII.2.3.

2 David-Mattila cubes

In Chapters III and V we will use the lattice of "dyadic cubes" constructed by David and Mattila [DM00].

Lemma 2.1 ([DM00, Theorem 3.2])**.** *Let μ be a Radon measure on $\mathbb{R}^d$, $E = \operatorname{supp}\mu$. For any constants $C_0 > 1$, $A_0 > 5000C_0$ there exists a sequence of partitions of E into Borel subsets Q, $Q \in \mathcal{D}_k$, with the following properties:*

(a) For each integer $k \geq 0$, E is the disjoint union of the "cubes" Q, $Q \in \mathcal{D}_k$, and if $k < l$, $Q \in \mathcal{D}_l$, and $R \in \mathcal{D}_k$, then either $Q \cap R = \varnothing$ or else $R \subset Q$.

(b) The general position of the cubes Q can be described as follows. For each $k \geq 0$ and each cube $Q \in \mathcal{D}_k$, there is a ball $B(Q) = B(z_Q, r(Q))$, such that

$$z_Q \in Q, \quad A_0^{-k} \leq r(Q) \leq C_0 A_0^{-k},$$
$$E \cap B(Q) \subset Q \subset E \cap 28B(Q) = E \cap B(z_Q, 28r(Q)),$$

and the balls $5B(Q), Q \in \mathcal{D}_k$, are disjoint.

Remark 2.2. The cubes of David and Mattila have many other useful properties, most notably the so-called *small boundaries*. We will not need them, however.

For any $Q \in \mathcal{D} := \bigcup_{k\geq 0} \mathcal{D}_k$ we denote by $\mathcal{D}(Q)$ the family of $P \in \mathcal{D}$ such that $P \subset Q$. Given $Q \in \mathcal{D}_k$ we set $J(Q) = k$ and $\ell(Q) = 56C_0A_0^{-k}$. Note that $r(Q) \approx \ell(Q)$. We define also $B_Q = 28B(Q) = B(z_Q, 28\, r(Q))$, so that

$$E \cap \tfrac{1}{28} B_Q \subset Q \subset B_Q.$$

Denote by $\mathcal{D}^{db}$ the family of doubling cubes, i.e. $Q \in \mathcal{D}$ satisfying

$$\mu(100B(Q)) \leq C_0 \mu(B(Q)). \tag{2.1}$$

One of the most useful properties of the David-Mattila lattice is that it provides a lot of information about doubling cubes. If the constants C_0, A_0 in Lemma 2.1 are chosen of the form $A_0 = C(C_0)^{100}$, and $C_0 = C_0(n,d)$ large enough, then it follows from the construction of the lattice that the following lemmas hold.

Lemma 2.3 ([DM00, Lemma 5.28]). *For any $R \in \mathcal{D}$ there exists a family $\{Q_i\}_{i\in I} \subset \mathcal{D}^{db}$ such that $Q_i \subset R$ and $\mu(R \setminus \bigcup_i Q_i) = 0$.*

Lemma 2.4 ([DM00, Lemma 5.31]). *Let $R \in \mathcal{D}$ and $Q \subset R$ be cubes such that all the intermediate cubes S, $Q \subsetneq S \subsetneq R$, are non-doubling, i.e. $S \in \mathcal{D} \setminus \mathcal{D}^{db}$. Then,*

$$\mu(100B(Q)) \le A_0^{-10d(J(Q)-J(R)-1)} \mu(100B(R)). \tag{2.2}$$

Remark 2.5. The constant $10d$ in (2.2) can be replaced by any positive constant if C_0 is chosen big enough. See [DM00, (5.30)] for details.

As a simple corollary we get the following:

Lemma 2.6 ([AT15, Lemma 2.4]). *Suppose the cubes $Q \in \mathcal{D}$, $R \in \mathcal{D}$, $Q \subset R$, are such that all the intermediate cubes $Q \subsetneq S \subsetneq R$ are non-doubling, i.e. $S \notin \mathcal{D}^{db}$. Then*

$$\Theta_\mu(100B(Q)) \le (C_0)^n A_0^{-9d(J(Q)-J(R)-1)} \Theta_\mu(100B(R)), \tag{2.3}$$

and

$$\sum_{S\in\mathcal{D}:Q\subset S\subset R} \Theta_\mu(100B(S)) \lesssim \Theta_\mu(100B(R)).$$

In Chapter V we will use the following lemma.

Lemma 2.7 ([CT17, Lemma 4.5]). *Let $R \in \mathcal{D}^{db}$. Then, there exists another doubling cube $Q \subsetneq R$, $Q \in \mathcal{D}^{db}$, such that*

$$\mu(Q) \approx \mu(R) \quad \textit{and} \quad \ell(Q) \approx \ell(R).$$

In Chapter III it will be convenient for us to work with cubes satisfying a doubling condition stronger than (2.1). To introduce them we need a version of [Tol14, Lemma 2.8]. For reader's convenience, we provide the proof below.

Lemma 2.8. *Let μ be a Radon measure on $\mathbb{R}^d$ and $\alpha > 1$ be some constant. Then, for μ-a.e. $x \in \mathbb{R}^d$ there exists a sequence $r_j \to 0$ such that for every j we have*

$$\mu(B(x, \alpha r_j)) \le 2\,\alpha^d \mu(B(x, r_j)). \tag{2.4}$$

Proof. Consider the set $Z \subset \operatorname{supp}\mu$ of points such that for $x \in Z$ there does not exist a sequence of radii $r_j \to 0$ satisfying (2.4). We want to show that $\mu(Z) = 0$. Let

$$Z_j = \{x \in \operatorname{supp}\mu \,:\, \mu(B(x, \alpha\, r)) > 2\,\alpha^d \mu(B(x, r)) \text{ for all } r \le 2^{-j}\}.$$

Clearly $Z = \bigcup_j Z_j$, and so it suffices to prove $\mu(Z_j) = 0$ for all $j \ge 0$.

Let B_0 be an arbitrary ball of radius 2^{-j} centered at Z_j, and choose some integer $k \ge d$. For each $x \in Z_j \cap B_0$ we set $B_x = B(x, \alpha^{-k} 2^{-j})$. Observe that, by the definition of Z_j, for $h = 0, \dots, k-1$ we have

$$\mu(\alpha^{h+1} B_x) > 2\,\alpha^d \mu(\alpha^h B_x).$$

Thus,

$$\mu(B_x) < (2\,\alpha^d)^{-1} \mu(\alpha B_x) < \cdots < (2\,\alpha^d)^{-k} \mu(\alpha^k B_x) = (2\,\alpha^d)^{-k} \mu(B(x, 2^{-j})) \le (2\,\alpha^d)^{-k} \mu(2B_0). \quad (2.5)$$

Now, we use Besicovitch covering theorem to choose points $\{x_m\} \subset Z_j \cap B_0$ such that $\bigcup_m B_{x_m}$ covers $Z_j \cap B_0$, and moreover $\sum_m \mathbb{1}_{B_{x_m}} \le C_d$. The bounded intersection property implies that $N := \#\{x_m\} < \infty$, and more precisely

$$N \omega_d (\alpha^{-k} 2^{-j})^d = \sum_m \mathcal{H}^d(B_{x_m}) \le C_d \mathcal{H}^d(2B_0) = C_d 2^d \omega_d 2^{-jd},$$

where ω_d stands for the volume of a d-dimensional ball. Hence,

$$N \le C_d 2^d \alpha^{kd}.$$

Consequently, we may use (2.5) and the fact that $\bigcup_m B_{x_m} \supset Z_j \cap B_0$ to obtain

$$\mu(Z_j \cap B_0) \le \sum_m \mu(B_{x_m}) \le N(2\,\alpha^d)^{-k} \mu(2B_0) \le C_d 2^{d-k} \mu(2B_0).$$

Since k can be chosen as large as we wish, this gives $\mu(Z_j \cap B_0) = 0$. But B_0 was an arbitrary ball, and so $\mu(Z_j) = 0$. □

We may use the lemma above to show the following.

Lemma 2.9. *There exists a constant $C = C(d, C_0, A_0)$ such that for μ-a.e. $x \in \mathbb{R}^d$ there exists a sequence of cubes $Q_j \in \mathcal{D}^{db}$ satisfying $x \in Q_j$, $\ell(Q_j) \to 0$, and*

$$\mu(100 B_{Q_j}) \le C\, \mu(B(Q_j)). \quad (2.6)$$

Proof. Let $\alpha = 2\, C_0^2\, A_0^{k+1}$, where k is a constant that will be fixed later on. Consider a sequence of balls $B(x, r_j)$ given by Lemma 2.8. Fix some j. Let Q be the smallest cube satisfying $x \in Q$ and $B(x, r_j) \subset 100B(Q)$. We have

$$72\, A_0^{-1} C_0^{-1}\, r(Q) \le r_j \le 100\, r(Q).$$

It is easy to check that, with the choice of α we made at the beginning, we have

$$B(x, \alpha\, r_j) \supset 100B(R),$$

where R is the k-th ancestor of Q, i.e. $Q \subset R$ and $J(Q) - J(R) = k$.

Now, if all the intermediate cubes S, $Q \subsetneq S \subsetneq R$, were non-doubling, then by (2.2) and Lemma 2.8 we would have

$$\begin{aligned}\mu(B(x, r_j)) \leq \mu(100B(Q)) &\overset{(2.2)}{\leq} A_0^{-10d(k-1)}\mu(100B(R)) \\ \leq A_0^{-10d(k-1)}\mu(B(x, \alpha\, r_j)) &\leq A_0^{-10d(k-1)} 2\,(2\,C_0^2\,A_0^{k+1})^d \mu(B(x, r_j)) \\ &= 2^{d+1} C_0^{2d} A_0^{-9dk+11d}\mu(B(x, r_j)).\end{aligned}$$

For $k = k(d, C_0, A_0)$ big enough the constant on the right hand side is smaller than 1, and so we reach a contradiction. It follows that one of the intermediate cubes S is doubling. Thus,

$$\begin{aligned}\mu(100B_S) \leq \mu(100B(R)) \leq \mu(B(x, \alpha\, r_j)) &\leq 2\,\alpha^d \mu(B(x, r_j)) \\ \leq 2\,\alpha^d \mu(100B(Q)) \leq 2\,\alpha^d \mu(100B(S)) &\leq 2\,C_0\,\alpha^d \mu(B(S)).\end{aligned}$$

Setting $Q_j = S$ finishes the proof. □

We will call the cubes satisfying (2.6) *strongly doubling*, and the family of all such cubes will be denoted by $\mathcal{D}^{sdb}$. We fix constants C_0 and A_0 so that all of the above holds, and from now on we will treat them as absolute constants. We will not mention dependence on them in our estimates.

3 Estimates of α and β numbers

In this section we provide some estimates of α and β coefficients used throughout the thesis.

Remark 3.1. In different parts of the thesis the definitions of α and β numbers vary slightly – the coefficients $\alpha_\mu(B)$, $\alpha_{\mu,p}(B)$, and $\beta_{\mu,p}(B)$ are normalized either using $\mu(B)$, $\mu(3B)$, or $r(B)^n$. However, within each chapter the normalizing factor is the same for all coefficients. Since the choice of normalization does not typically alter the proofs of lemmas below, we chose not to specify the normalizing factor at this point. Instead, we will simply denote it by "$\mathfrak{n}(B)$", so that

$$\beta_{\mu,p}(B)^p = \inf_L \frac{1}{\mathfrak{n}(B)} \int_B \left(\frac{\operatorname{dist}(x, L)}{r(B)}\right)^p d\mu(x),$$

and so on. Unless stated otherwise, the lemmas hold for $\mathfrak{n}(B) = \mu(B)$, $\mathfrak{n}(B) = \mu(3B)$, or $\mathfrak{n}(B) = r(B)^n$.

We begin by showing that α_2 numbers bound from above α and β_2 numbers.

Lemma 3.2. *Suppose that μ is a Radon measure, and B is a ball intersecting* $\operatorname{supp}\mu$*. Then*

$$\beta_{\mu,2}(B) \leq \alpha_{\mu,2}(B), \tag{3.1}$$

and

$$\alpha_{\mu}(B) \leq \alpha_{\mu,1}(B). \tag{3.2}$$

Furthermore, if $\mathfrak{n}(B) = \mu(3B)$*, then*

$$\alpha_{\mu,1}(B) \leq \alpha_{\mu,2}(B). \tag{3.3}$$

Remark 3.3. The fact that (3.3) only holds for $\mathfrak{n}(B) = \mu(3B)$ is precisely the reason why we choose this normalization in Chapter III.

Proof. To see $\beta_{\mu,2}(B) \leq \alpha_{\mu,2}(B)$, let L be a minimizing plane for $\alpha_{\mu,2}(B)$ and π be a minimizing transport plan between $\varphi_B\mu$ and $a_{B,L}\varphi_B\mathcal{H}^n|_L$, where $a_{B,L} = (\int \varphi_B \, d\mu)/(\int \varphi_B \, d\mathcal{H}^n|_L)$ is as in the definition of $\alpha_{\mu,2}(B)$. Then, by the definition of a transport plan, and the fact that $\varphi_B \equiv 1$ on B,

$$\begin{aligned}\alpha_{\mu,2}(B)^2 r(B)^2\,\mathfrak{n}(B) &= \int |x-y|^2 \, d\pi(x,y) \\ &\geq \int_B \operatorname{dist}(x,L)^2 \, d\mu \geq \beta_{\mu,2}(B)^2\,\mathfrak{n}(B) r(B)^2.\end{aligned}$$

For the estimate (3.2) we will use the so-called Kantorovich duality for W_1 Wasserstein distance. It states that

$$W_1(\mu,\nu) = \sup_{\operatorname{Lip}(f)\leq 1} \left| \int f \, d\mu - \int f \, d\nu \right|,$$

see [Vil08, Remark 6.5] for more information.

Let L be a minimizing plane for $\alpha_{\mu,1}(B)$, and let $a_{B,L}$ be as in the definition of $\alpha_{\mu,1}(B)$. Since $\varphi_B \equiv 1$ in B, it follows from the definition of α_μ that

$$\begin{aligned}\alpha_\mu(B) r(B)\,\mathfrak{n}(B) \leq F_B(\mu, a_{B,L}\mathcal{H}^n|_L) &= \sup_{\substack{\operatorname{Lip}(f)\leq 1 \\ \operatorname{supp}(f)\subset B}} \left| \int f \, d\mu - \int f \, a_{B,L} d\mathcal{H}^n|_L \right| \\ &= \sup_{\substack{\operatorname{Lip}(f)\leq 1 \\ \operatorname{supp}(f)\subset B}} \left| \int f\varphi_B \, d\mu - \int f\varphi_B \, a_{B,L} d\mathcal{H}^n|_L \right| \\ &\leq \sup_{\operatorname{Lip}(f)\leq 1} \left| \int f\varphi_B \, d\mu - \int f\varphi_B \, a_{B,L} d\mathcal{H}^n|_L \right| \\ &= W_1(\varphi_B\mu, a_{B,L}\varphi_B\mathcal{H}^n|_L) = \alpha_{\mu,1}(B) r(B)\,\mathfrak{n}(B).\end{aligned}$$

Finally, suppose that $\mathfrak{n}(B) = \mu(3B)$. In that case the estimate $\alpha_{\mu,1}(B) \leq \alpha_{\mu,2}(B)$ follows immediately by the Cauchy-Schwarz inequality and the fact that $\int \varphi_B \, d\mu \leq \mu(3B)$. ∎

If we assume more on the ball B, then we can improve (3.1) to an estimate of the bilateral β numbers, defined as

$$b\beta_{\mu,2}(x,r)^2 = \inf_L \frac{1}{r^n}\int_{B(x,r)} \left(\frac{\operatorname{dist}(y,L)}{r}\right)^2 \, d\mu(y) + \frac{1}{r^n}\int_{B(x,r)} \left(\frac{\operatorname{dist}(y,\operatorname{supp}\mu)}{r}\right)^2 \, d\mathcal{H}^n|_L(y).$$

Lemma 3.4. *Suppose that μ is a Radon measure, B is a ball satisfying $\mu(B) \approx r(B)^n \approx \mathfrak{n}(B)$, and L is a plane minimizing $\alpha_{\mu,2}(B)$. Then*

$$b\beta_{\mu,2}(B)^2 \lesssim r(B)^{-n-2}\int_B \operatorname{dist}(x,L)^2 \, d\mu \lesssim \alpha_{\mu,2}(B).$$

Proof. Let π be a minimizing transport plan between $\varphi_B\mu$ and $a_{B,L}\varphi_B\mathcal{H}^n|_L$ (where $a_{B,L}$ is as in the definition of $\alpha_{\mu,2}(B)$; note that $a_{B,L} \gtrsim 1$ since $\mu(B) \approx r(B)^n$). Then, by the definition of a transport plan, and the fact that $\varphi_B \equiv 1$ on B,

$$\begin{aligned}\alpha_{\mu,2}(B)^2 r(B)^2\,\mathfrak{n}(B) &= \int |x-y|^2 \, d\pi(x,y)\\ &\geq \frac{1}{2}\int_B \operatorname{dist}(x,L)^2 \, d\mu(x) + \frac{a_{B,L}}{2}\int_B \operatorname{dist}(y,\operatorname{supp}\mu)^2 \, d\mathcal{H}^n|_L(y)\\ &\gtrsim b\beta_{\mu,2}(B)^2 r(B)^2\,\mathfrak{n}(B).\end{aligned}$$

□

The following lemma allows us to control β_1 numbers in terms of β_2 and α numbers. We also show that if $B_1 \subset B_2$ and they have comparable radii, then the coefficients of B_2 bound those of B_1.

Lemma 3.5. *Suppose that μ is a Radon measure on $\mathbb{R}^d$, and that $B \subset \mathbb{R}^d$ is a ball satisfying $\mathfrak{n}(B) \approx \mathfrak{n}(2B)$. Then*

$$\beta_{\mu,1}(B) \leq \beta_{\mu,2}(B), \tag{3.4}$$

and

$$\beta_{\mu,1}(B) \lesssim \alpha_\mu(2B). \tag{3.5}$$

Moreover, given balls $B_1 \subset B_2$ such that $r(B_1) \approx r(B_2)$ and $\mathfrak{n}(B_1) \approx \mathfrak{n}(B_2)$ we have

$$\beta_{\mu,2}(B_1) \lesssim \beta_{\mu,2}(B_2), \tag{3.6}$$

$$\alpha_\mu(B_1) \lesssim \alpha_\mu(B_2). \tag{3.7}$$

Proof. The first estimate is a direct consequence of the Cauchy-Schwarz inequality.

In order to prove the second estimate, let L_B be the minimizing plane for $\beta_{\mu,1}(B)$. The estimate follows if we consider the 1-Lipschitz function $\phi(x) = \psi(x)\operatorname{dist}(x, L_B)$, where ψ is $r(B)^{-1}$-Lipschitz, $\psi \equiv 1$ on B, and $\operatorname{supp}(\psi) \subset 2B$.

The last two inequalities follow immediately from the definitions of $\beta_{\mu,2}$ and α_μ. □

Remark 3.6. Under suitable assumptions, an analogue of (3.6) and (3.7) is true also for α_2. However, the proof is much more involved, and we will only use it in Chapter IV in a very specific context. See Lemma IV.3.3, or [Tol12, Lemma 5.4].

Lemma 3.7. *Suppose that μ is a Radon measure, B is a ball with $\mu(B) > 0$, L an n-plane intersecting $0.9B$, and assume that c minimizes $F_B(\mu, c\mathcal{H}^n|_L)$. Then*

$$c \lesssim \frac{\mu(B)}{r(B)^n}. \tag{3.8}$$

Furthermore, there exists $\varepsilon > 0$ such that if $\mu(0.9B) \approx \mu(B)$, and $F_B(\mu, c\mathcal{H}^n|_L) \le \varepsilon\mu(B)r(B)$, then

$$c \gtrsim \frac{\mu(B)}{r(B)^n}. \tag{3.9}$$

Proof. Let $r = r(B)$ and consider $\Phi(x) = (r - |x - z(B)|)_+ \in \operatorname{Lip}_1(B)$. It is not difficult to see that on a significant portion (say, a half) of the n-dimensional ball $L \cap B$ we have $\Phi(x) \approx r$, and so

$$c \int \Phi(x)\ d\mathcal{H}^n|_L(x) \approx c r^{n+1}.$$

If we had $c \ge M\mu(B)r^{-n}$ for some large $M > 100$, then

$$\begin{aligned} F_B(\mu, c\mathcal{H}^n|_L) \ge c\int \Phi(x)\ d\mathcal{H}^n|_L(x) - \int \Phi(x)\ d\mu(x) &\ge Ccr^{n+1} - \mu(B)r \\ &\ge (MC-1)\mu(B)r. \end{aligned}$$

But in that case, if $M \ge 3C^{-1}$, the constant $\tilde{c} = 0$ would be better than c, since we always have $F_B(\mu, 0) \le \mu(B)r$, and thus we reach a contradiction with optimality of c. Hence, $c \le M\mu(B)r^{-n}$.

Now, assume further that $F_B(\mu, c\mathcal{H}^n|_L) \le \varepsilon\mu(B)r$, and $\mu(0.9B) \approx \mu(B)$, so that $\int \Phi(x)\ d\mu(x) \approx \mu(B)r$. If we had $c \le M^{-1}\mu(B)r^{-n}$, then

$$\begin{aligned} F_B(\mu, c\mathcal{H}^n|_L) \ge \int \Phi(x)\ d\mu(x) - c\int \Phi(x)\ d\mathcal{H}^n|_L(x) &\ge C\mu(3B)r - \widetilde{C}cr^{n+1} \\ &\ge C\mu(B)r - \frac{\widetilde{C}}{M}\mu(B)r \ge \frac{C}{2}\mu(B)r, \end{aligned}$$

assuming M large enough. This contradicts the assumption $F_B(\mu,\, c\mathcal{H}^n|_L) \le \varepsilon\mu(B)r$. □

The following lemma shows that if α and β_2 numbers are simultaneously small, then the minimizing planes for both of them are close to each other and can be used interchangeably.

Lemma 3.8. *Suppose that μ is a Radon measure on $\mathbb{R}^d$, and that $B_1, B_2 \subset \mathbb{R}^d$ are concentric balls satisfying $B_1 \subset 0.9B_2$ and*

$$\mu(B_1) \approx \mu(B_2) \approx r(B_1)^n \approx r(B_2)^n \approx \mathfrak{n}(B_2).$$

Let L_β be the n-plane minimizing $\beta_{\mu,2}(B_2)$, and L_α, $c > 0$, be the n-plane and constant minimizing $\alpha_\mu(B_2)$. Suppose further that L_α, L_β intersect $0.9B_1$. Then

$$\frac{1}{\mu(B_1)r(B_1)} F_{B_1}(\mu, c\mathcal{H}^n|_{L_\beta}) \lesssim \beta_{\mu,2}(B_2) + \alpha_\mu(B_2). \tag{3.10}$$

Proof. Set $r = r(B_1)$. It follows easily by (3.8) that $c \lesssim \mu(B_2)r(B_2)^{-n} \approx 1$, and so $F_{B_1}(\mu, c\mathcal{H}^n|_{L_\beta}) \lesssim r\mu(B_1)$. Thus, without loss of generality, we may assume that $\beta_{\mu,2}(B_2) + \alpha_\mu(B_2) < \varepsilon$ for some small $\varepsilon > 0$.

By the triangle inequality, we have

$$\begin{aligned} F_{B_1}(\mu, c\mathcal{H}^n|_{L_\beta}) &\le F_{B_1}(\mu, c\mathcal{H}^n|_{L_\alpha}) + F_{B_1}(c\mathcal{H}^n|_{L_\alpha}, c\mathcal{H}^n|_{L_\beta}) \\ &\le F_{B_2}(\mu, c\mathcal{H}^n|_{L_\alpha}) + F_{B_1}(c\mathcal{H}^n|_{L_\alpha}, c\mathcal{H}^n|_{L_\beta}). \end{aligned}$$

The first term on the right hand side is precisely $\alpha_\mu(B_2)\,\mathfrak{n}(B_2)r(B_2) \approx \alpha_\mu(B_2)\mu(B_1)r$, and so what remains to show is that

$$\frac{1}{\mu(B_1)r} F_{B_1}(c\mathcal{H}^n|_{L_\alpha}, c\mathcal{H}^n|_{L_\beta}) \lesssim \beta_{\mu,2}(B_2) + \alpha_\mu(B_2).$$

Let $x_\alpha \in L_\alpha \cap \bar{B}_1$ and $x_\beta \in L_\beta$ be such that

$$|x_\alpha - x_\beta| = \operatorname{dist}(x_\alpha, L_\beta) = \inf_{x \in L_\alpha \cap B_1} \operatorname{dist}(x, L_\beta).$$

Without loss of generality we may assume that $x_\alpha = 0$, so that L_α is a linear subspace. Denote $L'_\beta = L_\beta - x_\beta$. It follows by basic linear algebra that for $x \in L_\alpha \cap B_1$

$$\begin{aligned} \operatorname{dist}(x, L_\beta) = |x - \Pi_{L_\beta}(x)| &= |x - x_\beta - \Pi_{L'_\beta}(x - x_\beta)| \\ &= |\Pi^\perp_{L'_\beta}(x - x_\beta)| = |\Pi^\perp_{L'_\beta}(x) - x_\beta|. \end{aligned} \tag{3.11}$$

Note that by the above and the triangle inequality

$$\operatorname{dist}(x, L_\beta) = |\Pi^\perp_{L'_\beta}(x) - x_\beta| \le |x_\beta| + |\Pi^\perp_{L'_\beta}(x)|.$$

On the other hand, by our choice of x_β, $|x_\beta| \le \operatorname{dist}(x, L_\beta)$ for all $x \in L_\alpha \cap B_1$. Together with the triangle inequality and the identity (3.11) this gives

$$|x_\beta| + |\Pi^\perp_{L'_\beta}(x)| \le 2|x_\beta| + |\Pi^\perp_{L'_\beta}(x) - x_\beta| \le 3 \operatorname{dist}(x, L_\beta).$$

We put the two estimates above together to get

$$|x_\beta| + |\Pi^\perp_{L'_\beta}(x)| \approx \operatorname{dist}(x, L_\beta). \tag{3.12}$$

Now, observe that, by the definition of $\measuredangle(L_\alpha, L_\beta)$, for every $x \in L_\alpha$ we have $|\Pi^\perp_{L'_\beta}(x)| \le |x|\measuredangle(L_\alpha, L_\beta)$. Moreover, there exists a subspace $\ell \subset L_\alpha$ on which the equality is achieved, i.e. for all $x \in \ell$ we have $|\Pi^\perp_{L'_\beta}(x)| = |x|\measuredangle(L_\alpha, L_\beta)$. Consider a cone around ℓ:

$$K = \left\{ x \in \mathbb{R}^d \ : \ |\Pi_\ell(x)| \ge \frac{4}{5}|x| \right\}.$$

Since $0 \in B_1 \cap K \cap L_\alpha$, it is easy to see that $\mathcal{H}^n(B_1 \cap K \cap L_\alpha) \gtrsim r^n$, which in turn implies that for some small constant $0 < \delta \ll 1$ (depending on the implicit constant in the previous inequality and dimension) we have

$$\mathcal{H}^n(B_1 \cap K \cap L_\alpha \setminus B(0, \delta r)) \gtrsim r^n. \tag{3.13}$$

Moreover, for $x \in B_1 \cap K \cap L_\alpha \setminus B(0, \delta r)$ we have

$$\begin{aligned}
|\Pi^\perp_{L'_\beta}(x)| &= |\Pi^\perp_{L'_\beta}(\Pi_\ell(x)) + \Pi^\perp_{L'_\beta}(\Pi^\perp_\ell(x))| \ge |\Pi^\perp_{L'_\beta}(\Pi_\ell(x))| - |\Pi^\perp_{L'_\beta}(\Pi^\perp_\ell(x))| \\
&\ge |\Pi_\ell(x)|\measuredangle(L_\alpha, L_\beta) - |\Pi^\perp_\ell(x)|\measuredangle(L_\alpha, L_\beta) \overset{x\in K}{\ge} \frac{4}{5}|x|\measuredangle(L_\alpha, L_\beta) - \frac{3}{5}|x|\measuredangle(L_\alpha, L_\beta) \\
&= \frac{1}{5}|x|\measuredangle(L_\alpha, L_\beta) \approx r\measuredangle(L_\alpha, L_\beta).
\end{aligned}$$

Hence, using the above, (3.13), and (3.12) yields

$$\begin{aligned}
|x_\beta| r^{n-1} + r^n \measuredangle(L_\alpha, L_\beta) &\lesssim \int_{B_1} \frac{\operatorname{dist}(x, L_\beta)}{r} \, d\mathcal{H}^n|_{L_\alpha}(x) \\
&\overset{(3.9)}{\lesssim} c \int_{B_1} \frac{\operatorname{dist}(x, L_\beta)}{r} \, d\mathcal{H}^n|_{L_\alpha}(x).
\end{aligned} \tag{3.14}$$

Now, consider $\phi \in \operatorname{Lip}_1(B_2)$ such that $\phi(x) \approx \operatorname{dist}(x, L_\beta)$ in B_1, and $\phi(x) \lesssim \operatorname{dist}(x, L_\beta)$ in B_2. Then,

$$\begin{aligned}
&c \int_{B_1} \frac{\operatorname{dist}(x, L_\beta)}{r} \, d\mathcal{H}^n|_{L_\alpha}(x) \lesssim c \int_{B_2} \frac{\phi(x)}{r} \, d\mathcal{H}^n|_{L_\alpha}(x) \\
&\lesssim \int_{B_2} \frac{\phi(x)}{r} \, d\mu(x) + r^{-1} F_{B_2}(\mu, c\mathcal{H}^n|_{L_\alpha}) \lesssim (\beta_{\mu,2}(B_2) + \alpha_\mu(B_2))\, \mathfrak{n}(B_2).
\end{aligned} \tag{3.15}$$

The estimate (3.14) and the calculation above give $\measuredangle(L_\alpha, L_\beta) \lesssim \beta_{\mu,2}(B_2) + \alpha_\mu(B_2) < \varepsilon$. Let $\Pi : L_\alpha \to \mathbb{R}^d$ be the orthogonal projection onto L'_β, and $i : L_\alpha \to \mathbb{R}^d$ an embedding. We have

$$\|\Pi - i\|_{op} = \|\Pi - i\|_{L^\infty(L_\alpha \cap B(0,1))} \lesssim \beta_{\mu,2}(B_2) + \alpha_\mu(B_2) < \varepsilon. \tag{3.16}$$

Thus, Π is a linear isomorphism onto L'_β, with a bound on Jacobian

$$|1 - |J\Pi|| \lesssim \beta_{\mu,2}(B_2) + \alpha_\mu(B_2). \tag{3.17}$$

It follows that for any $f \in \mathrm{Lip}_1(B_1)$ we have

$$\begin{aligned}
&\left|\int f(x)\, d\mathcal{H}^n|_{L_\alpha}(x) - \int f(y)\, d\mathcal{H}^n|_{L_\beta}(y)\right| \\
&\quad = \left|\int f(x)\, d\mathcal{H}^n|_{L_\alpha}(x) - \int f(x_\beta + \Pi(x))|J\Pi(x)|\, d\mathcal{H}^n|_{L_\alpha}(x)\right| \\
&\leq \int |f(x) - f(x_\beta + \Pi(x))|\, d\mathcal{H}^n|_{L_\alpha}(x) + \int |f(x_\beta + \Pi(x))|\, |1 - |J\Pi(x)||\, d\mathcal{H}^n|_{L_\alpha}(x) \\
&\quad \leq \int_{B_1 \cup \Pi^{-1}(B_1 - x_\beta)} |x_\beta| + |x - \Pi(x)|\, d\mathcal{H}^n|_{L_\alpha}(x) \\
&\quad + \int_{\Pi^{-1}(B_1 - x_\beta)} \|f\|_{L^\infty} |1 - |J\Pi(x)||\, d\mathcal{H}^n|_{L_\alpha}(x) \\
&\quad \overset{(3.16),(3.17)}{\lesssim} |x_\beta| r^n + (\beta_{\mu,2}(B_2) + \alpha_\mu(B_2)) r^{n+1}.
\end{aligned}$$

Taking supremum over all $f \in \mathrm{Lip}_1(B_1)$, dividing by r^{n+1}, using (3.14), (3.15), the fact that $\mu(B_1) \approx r^n$, and that $c \lesssim 1$, yields the desired inequality:

$$\frac{1}{\mu(B_1) r} F_{B_1}(c\mathcal{H}^n|_{L_\alpha}, c\mathcal{H}^n|_{L_\beta}) \lesssim \beta_{\mu,2}(B_2) + \alpha_\mu(B_2).$$

□

We finish this section by showing that, for rectifiable measures, the planes minimizing $\beta_{\mu,2}(x,r)$ converge to approximate tangent planes as $r \to 0$. Since the choice of normalization does not affect the minimizing planes, without loss of generality we may assume $\mathfrak{n}(B) = r(B)^n$.

Lemma 3.9. *Let μ be a n-rectifiable measure. For $x \in \operatorname{supp}\mu$ and $r > 0$ let $L_{x,r}$ denote a minimizing plane for $\beta_{\mu,2}(x,r)$, let W'_x be the approximate tangent plane to μ at x, whenever it exists, and let $W_x = W'_x + x$. Then for μ-a.e. $x \in \operatorname{supp}\mu$ we have*

$$\frac{\mathrm{dist}_H(L_{x,r} \cap B(x,r), W_x \cap B(x,r))}{r} \xrightarrow{r \to 0} 0. \tag{3.18}$$

Proof. Recall that since μ is n-rectifiable, the density $\Theta^n(\mu,x)$ exists and satisfies $0 < \Theta^n(\mu,x) < \infty$ for μ-a.e. x. Let $M \geq 100$ be some big constant. Define

$$E_M := \{x \in \operatorname{supp}\mu \ :\ M^{-1} \leq \Theta^n(\mu,x) \leq M\}.$$

Note that for any $M_0 \geq 1$ we have $\mu(\mathbb{R}^d \setminus \bigcup_{M\geq M_0} E_M) = 0$, and so it suffices to show that for all sufficiently large M (3.18) holds for μ-a.e. $x \in E_M$. Fix some big M, and set $\nu = \mu|_{E_M}$. It is well-known that

$$M^{-1} \leq \Theta^n(\nu, x) = \Theta^n(\mu, x) \leq M \quad \text{for } \nu\text{-a.e. } x \in \operatorname{supp}\nu, \tag{3.19}$$

which can be shown e.g. using [Mat95, Corollary 6.3] in conjunction with the Lebesgue differentiation theorem. For ν-a.e. x the plane W_x is well defined by Theorem I.2.4, and also by Theorem I.6.4

$$\int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{3.20}$$

Fix $x \in E_M$ such that (3.20) and (3.19) hold, and such that W_x is well-defined. Once we show that (3.18) holds at x, the proof will be finished. From now on we will suppress the subscript x, so that $L_r =: L_{x,r}$, $W := W_x$. By applying an appropriate translation, we may assume that $x = 0$.

Given some small $r > 0$, let $A_r(y) = \frac{y}{r}$, so that $A_r(B(0,r)) = B(0,1)$. Set $L_r' = A_r(L_r)$. It is easy to see that (3.18) is equivalent to showing

$$\operatorname{dist}_H(L_r' \cap B(0,1), W \cap B(0,1)) \xrightarrow{r\to 0} 0.$$

We will prove that the convergence above holds by contradiction. Suppose it is not true, so that there is $\varepsilon > 0$ and a sequence $r_k \to 0$ such that for all k we have

$$\operatorname{dist}_H(L_{r_k}' \cap B(0,1), W \cap B(0,1)) \geq \varepsilon. \tag{3.21}$$

Let $\eta > 0$ be some tiny constant. Observe that by (3.20) for $k \geq k_0(\eta, M)$ large enough we have

$$\beta_{\mu,2}(0, r_k)^2 \leq \frac{\eta^3}{M}. \tag{3.22}$$

Indeed, otherwise one could use the fact that $\beta_{\mu,2}(0,r) \lesssim \beta_{\mu,2}(0,2r)$ (by (3.6)) to conclude that $\int_0^1 \beta_{\mu,2}(0,r)^2 \, dr/r = \infty$. Moreover, let us remark that for every $0 < \delta < 1/2$, if $k = k(\delta)$ is large enough, then we have $L_{r_k}' \cap B(0,\delta) \neq \varnothing$. This can be shown easily using the fact that $\Theta^n(\mu, x) \geq M^{-1}$, that L_{r_k} are minimizers of $\beta_{\mu,2}(0,r_k)$, and the fact that $\beta_{\mu,2}(0,r_k) \to 0$. We leave checking the details to the reader.

Now, we use the fact that for k large enough $L_{r_k}' \cap B(0,\delta) \neq \varnothing$ and the compactness properties of the Hausdorff distance to conclude that there exists some subsequence (again denoted by r_k) such that $L_{r_k}' \cap \overline{B(0,1)}$ converges in Hausdorff distance to a compact set of the form $V \cap \overline{B(0,1)}$, where V is an n-plane intersecting $B(0,\delta)$. Since $\delta > 0$ can be chosen arbitrarily small, we get that V passes through 0. Note that by (3.21)

$$\operatorname{dist}_H(V \cap B(0,1), W \cap B(0,1)) \geq \varepsilon. \tag{3.23}$$

Let $B_{\eta r_k}(V)$ denote the ηr_k-neighbourhood of V. We will show now that a large portion of measure ν in $B(0,r_k)$ is concentrated at the intersection of $B_{\eta r_k}(V)$ and $B_{\eta r_k}(W)$.

Since V passes through 0, for every $r>0$ we have $A_r^{-1}(V)=V$. Thus,

$$\frac{\operatorname{dist}_H(L_{r_k}\cap B(0,r_k), V\cap B(0,r_k))}{r_k} \xrightarrow{k\to\infty} 0. \tag{3.24}$$

Note that for k big enough

$$\begin{aligned}
\frac{1}{\nu(B(0,r_k))}&\int_{B(0,r_k)}\left(\frac{\operatorname{dist}(y,V)}{r_k}\right)^2 d\nu(y)\\
&\le \frac{1}{\nu(B(0,r_k))}\int_{B(0,r_k)}\left(\frac{\operatorname{dist}(y,L_{r_k})}{r_k}\right)^2 d\nu(y)\\
&\quad+\left(\frac{\operatorname{dist}_H(L_{r_k}\cap B(0,2r_k),V\cap B(0,2r_k))}{r_k}\right)^2\\
&\overset{(3.24)}{\le} \frac{r_k^n}{\nu(B(0,r_k))}\beta_{\mu,2}(0,r_k)^2+\eta^3 \overset{(3.19)}{\le} 2M\beta_{\mu,2}(0,r_k)^2+\eta^3 \overset{(3.22)}{\le} 3\eta^3.
\end{aligned}$$

It follows from Chebyshev's inequality and the estimate above that

$$\nu(B(0,r_k)\setminus B_{\eta r_k}(V))\le \eta^{-2}\int_{B(0,r_k)}\left(\frac{\operatorname{dist}(y,V)}{r_k}\right)^2 d\nu(y)\le 3\eta\nu(B(0,r)).$$

Hence, $\nu(B(0,r_k)\cap B_{\eta r_k}(V))\ge(1-3\eta r_k)\nu(B(0,r_k))$. On the other hand, by the definition of the approximate tangent plane W and (3.19), for any $0<\alpha<1$ we have

$$\begin{aligned}
\nu(K(0,W,\alpha,r_k)) &= \nu(B(0,r_k))-\nu(K(0,W^\perp,\sqrt{1-\alpha^2},r_k))\\
&\ge \nu(B(0,r_k))-\frac{\eta}{2M}r_k^n\ge(1-\eta)\nu(B(0,r_k)),
\end{aligned}$$

if k is large enough (depending on α, η and M). Note that $K(0,W,\alpha,r_k)\subset B_{\alpha r_k}(W)\cap B(0,r_k)$. Thus, choosing $\alpha=\eta$, if we define

$$S=S(k,\eta)=B(0,r_k)\cap B_{\eta r_k}(V)\cap B_{\eta r_k}(W),$$

then by the two previous estimates we have

$$\nu(S)\ge(1-4\eta)\nu(B(0,r_k))\ge\frac{1}{2M}r_k^n, \tag{3.25}$$

where in the second inequality we used (3.19).

We will show that if η is chosen small enough (depending on ε, the constant from (3.23)), then the estimate above leads to a contradiction. Roughly speaking, (3.25) means that a lot of measure is concentrated in the intersection of $B_{\eta r_k}(V)$ and $B_{\eta r_k}(W)$, but since V and W are somewhat well-separated by (3.23), this intersection behaves approximately like an $(n-1)$-dimensional set.

Let us start by exploiting (3.23). By the definition of Hausdorff distance and the fact that V and W are n-planes, it follows from easy linear algebra that there exists some $w \in W^{\perp}$ with $|w| = 1$ and $|\pi_V(w)| \geq \varepsilon$. Let $v_1 = \pi_V(w)/|\pi_V(w)|$, and let $V_0 \subset V$ be the orthogonal complement of $\operatorname{span}(v_1)$ in V.

We define $T = T(k,\eta)$ to be a tube-like set defined as

$$T = T(k,\eta) = \{z \in \mathbb{R}^d \ : \ |z \cdot v_1| \leq 2\eta\varepsilon^{-1} r_k,\ |\pi_{V_0}(z)| \leq r_k,\ |\pi_V^{\perp}(z)| \leq \eta r_k\}.$$

We claim that $S(k,\eta) \subset T(k,\eta)$. Indeed, let $z \in S$. The estimate $|\pi_{V_0}(z)| \leq r_k$ is trivial since $S \subset B(0, r_k)$. The estimate $|\pi_V^{\perp}(z)| \leq \eta r_k$ follows from the fact that $z \in B_{\eta r_k}(V)$. Concerning $|z \cdot v_1|$, note that since $z \in B_{\eta r_k}(W)$ and $w \in W^{\perp}$, we have $|z \cdot w| \leq \eta r_k$. We can use our choice of w and $v_1 = \pi_V(w)/|\pi_V(w)|$ to get

$$\begin{aligned}\eta r_k \geq |z \cdot w| &= |z \cdot \pi_V(w) + z \cdot \pi_V^{\perp}(w)| \\ &\geq |z \cdot \pi_V(w)| - |z \cdot \pi_V^{\perp}(w)| = |z \cdot v_1||\pi_V(w)| - |\pi_V^{\perp}(z) \cdot \pi_V^{\perp}(w)| \\ &\geq |z \cdot v_1|\varepsilon - |\pi_V^{\perp}(z)||\pi_V^{\perp}(w)| \geq |z \cdot v_1|\varepsilon - \eta r_k,\end{aligned}$$

where in the last inequality we used again $z \in B_{\eta r_k}(V)$. Thus, we have $|z \cdot v_1| \leq 2\eta\varepsilon^{-1} r_k$, and the proof of $S(k,\eta) \subset T(k,\eta)$ is finished.

Choose $\eta = \gamma\varepsilon$ for some tiny $\gamma = \gamma(M) > 0$, and let k be large enough for (3.25) to hold. It follows from the definition of T that we can cover T with a family of balls $\{B_i\}_{i\in I}$ such that $r(B_i) = \eta r_k$ and $\#I \lesssim \varepsilon^{-1}\eta^{-(n-1)}$. It is well-known that (3.19) implies that for all $y \in \mathbb{R}^d$ and $r > 0$ we have $\nu(B(y,r)) \leq M r^n$. In particular, for each $i \in I$ we have $\nu(B_i) \leq M(\eta r_k)^n$. Thus,

$$\begin{aligned}\frac{1}{2M} r_k^n \overset{(3.25)}{\leq} \nu(S) \leq \nu(T) \leq \sum_{i\in I} \nu(B_i) &\leq \#I M(\eta r_k)^n \\ &\lesssim \varepsilon^{-1}\eta^{-(n-1)} M(\eta r_k)^n = \varepsilon^{-1}\eta M r_k^n.\end{aligned}$$

That is,

$$M^{-2} \lesssim \varepsilon^{-1}\eta = \gamma.$$

This is a contradiction for $\gamma = \gamma(M)$ small enough. Hence, (3.21) is false, and so (3.18) holds for μ-a.e. $x \in E_M$. Taking $M \to \infty$ finishes the proof. □

Sufficient condition for rectifiability involving Wasserstein distance W_2 III

1 Introduction

In this chapter we prove a sufficient condition for rectifiability involving the α_2 coefficients. In fact, we will show a bit more: a sufficient condition for rectifiability involving α and β_2 numbers. Let us recall some definitions.

For $1 \leq p < \infty$ and a Radon measure μ on $\mathbb{R}^d$ we define

$$\beta_{\mu,p}(x,r) = \inf_L \left(\frac{1}{\mu(B(x,3r))} \int_{B(x,r)} \left(\frac{\operatorname{dist}(y,L)}{r} \right)^p \, d\mu(y) \right)^{1/p}.$$

Remark 1.1. Note that in this chapter we use $\mu(B(x,3r))$ as the normalizing factor. This choice is explained in Remark 1.6 and Remark II.3.3.

Let us also recall the definition of α numbers. Given Radon measures μ and ν, and an open ball B, we set

$$F_B(\mu,\nu) = \sup \left\{ \left| \int \phi \, d\mu - \int \phi \, d\nu \right| \ : \ \phi \in \operatorname{Lip}_1(B) \right\},$$

where

$$\operatorname{Lip}_1(B) = \{\phi \ : \ \operatorname{Lip}(\phi) \leq 1, \ \operatorname{supp} \phi \subset B\}.$$

The coefficient α of a measure μ in B is defined as

$$\alpha_\mu(B) = \inf_{c,L} \frac{1}{r(B)\mu(3B)} F_B(\mu, c\mathcal{H}^n|_L),$$

where the infimum runs over all $c \geq 0$ and all n-planes L.

We prove the following sufficient condition for rectifiability in terms of α and β_2 square functions.

Theorem 1.2. *Let μ be a Radon measure on $\mathbb{R}^d$. Suppose that*

$$\int_0^1 \alpha_\mu(x,r)^2 \, \frac{dr}{r} < \infty \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d, \tag{1.1}$$

and

$$\int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{1.2}$$

Then μ is n-rectifiable.

Since Tolsa has shown in [Tol15] that (1.1) and (1.2) are also necessary conditions for rectifiability, we immediately get the following characterization.

Corollary 1.3. *Let μ be a Radon measure on $\mathbb{R}^d$. Then, μ is n-rectifiable if and only if (1.1) and (1.2) hold for μ-a.e. $x \in \mathbb{R}^d$.*

Our main motivation for proving Theorem 1.2 was to get a sufficient condition for rectifiability in terms of α_2 numbers. Recall that α_p numbers were defined in Subsection I.6.3 using the Wasserstein distance W_p. Just as a quick reminder, given $1 \leq p < \infty$, a Radon measure μ on $\mathbb{R}^d$, and a ball $B \subset \mathbb{R}^d$, we defined

$$\alpha_{\mu,p}(B) = \inf_L \frac{1}{r(B)\mu(3B)^{1/p}} W_p(\varphi_B \mu, a_{B,L} \varphi_B \mathcal{H}^n|_L),$$

where the infimum is taken over all n-planes L intersecting B, φ_B is a "regularized characteristic function of B", and

$$a_{B,L} = \frac{\int \varphi_B \, d\mu}{\int \varphi_B \, d\mathcal{H}^n|_L}.$$

Since α_2 numbers bound from above both α and β_2 numbers (see Lemma II.3.2), Theorem 1.2 implies the following.

Theorem 1.4. *Let μ be a Radon measure on $\mathbb{R}^d$. Suppose that*

$$\int_0^1 \alpha_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d. \tag{1.3}$$

Then μ is n-rectifiable.

In Chapter IV we show that (1.3) is also a necessary condition for rectifiability, and so we get the following characterization.

Corollary 1.5. *Let μ be a Radon measure on $\mathbb{R}^d$. Then, μ is n-rectifiable if and only if for μ-a.e. $x \in \mathbb{R}^d$ we have*

$$\int_0^1 \alpha_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty.$$

We would like to stress that, compared to Theorem I.6.4 and Theorem I.6.9, the characterization above does not make any additional assumptions on densities or on doubling properties of the measure.

The organization of the paper, as well as the general strategy of the proof, are outlined in Section 2. For now, let us just say that Lemma 3.1, our main lemma, can be seen as a technical, more quantitative version of Theorem 1.2.

Remark 1.6. Suppose one prefers to work with homogeneous coefficients $\beta^h_{\mu,2}$ and α^h_μ, that is coefficients where the normalizing factor is r^{-n} (i.e. $\beta^h_{\mu,2}(x,r) = \frac{\mu(B(x,3r))}{r^n}\beta_{\mu,2}(x,r)$ and $\alpha^h_\mu(x,r) = \frac{\mu(B(x,3r))}{r^n}\alpha_\mu(x,r)$). Then, a possible "homogenized" modification of Lemma 3.1 is discussed in Remark 3.4. However, it is clear that "homogenized" (i.e. with α and β_2 numbers replaced by their homogeneous counterparts) versions of Theorem 1.2 and Theorem 1.4 are not true (unless we assume more about densities) – think of Lebesgue measure on $\mathbb{R}^d$.

2 Sketch of the proof

The proof of Theorem 1.2 is organized as follows. In Section 3 we formulate the main lemma. Given an appropriate David-Mattila cube R_0, the main lemma provides us with a Lipschitz graph Γ such that we have $\mu \ll \mathcal{H}^n|_\Gamma$ on a large chunk of $\Gamma \cap R_0$, and $\mu(\Gamma \cap R_0) \geq \frac{1}{2}\mu(R_0)$. In the same section we show how to use the main lemma to prove Theorem 1.2. Everything that follows is dedicated to proving the main lemma.

In Section 4 we perform the usual stopping time argument. We define the family of stopping cubes Stop, comprising high density cubes HD, low density cubes LD, big angle cubes BA (cubes whose best approximating planes form a big angle with L_0, the best approximating plane of R_0), big square function cubes BS (cubes with a big portion of points for which the square functions are larger than a certain threshold), and far cubes F (cubes with a big portion of R_{Far}, points that are far from certain best approximating planes). Cubes not contained in any of the stopping cubes form the Tree. Next, we show various good properties of cubes from the Tree, as well as estimate the measure of cubes from BS and F (it is easy).

Section 5 is devoted to constructing the Lipschitz graph Γ. One possible way to do it would be to use the tools from [DT12] – this was done for example in [AT15, ATT20]. In this paper we decided to use another well-known method, dating back at least to [DS91] and [Lég99]. We follow the way it was applied in [CMT18] and [Tol14]. It consists of showing that $R_0 \setminus \bigcup_{Q\in\mathsf{Stop}} Q$ forms a graph of a Lipschitz map F defined on a subset of L_0, and then carefully extending F to the whole L_0. The remaining part of the paper is dedicated to showing that the measure of stopping cubes is small.

In Section 6 we first show that cubes from Tree lie close to Γ (the graph of F), and then use this property to estimate the measure of low density cubes. Roughly speaking, we may cover (almost all) LD cubes with a family of (almost) disjoint balls satisfying $B \cap \Gamma \approx r(B)^n$, and such that the densities $\Theta_\mu(B)$ are low. Small measure of LD easily follows. It is crucial that we have the finiteness of the β_2 square function (1.2), as it lets us estimate the size of R_{Far} (see Lemma 4.6). This approach to bounding the measure of low density cubes comes from [AT15].

In Section 7 we define a measure ν supported on Γ. We show that ν is very close to μ for the distance $F_B(\mu, \nu)$, so that the α_ν numbers are close to α_μ. The measure ν is then used in Section 8 to estimate the size of the high density set. The general idea is to consider f – the density of ν with respect to $\mathcal{H}^n|_\Gamma$, and then to bound the L^2 norm of $|f - c_0|$, where c_0 is a certain constant. We do it using the smallness of α_μ square function (1.1), the fact that ν approximates μ well, and an appropriate type of Paley-Littlewood result (see (8.8)). Estimating $\|f - c_0\|_{L^2}$ requires a lot of work, but once we have it, it is not very difficult to bound the measure of HD cubes. Roughly speaking, high density cubes correspond to big values of f, and those we can control since $\|f - c_0\|_{L^2}$ is small. This method of estimating HD is due to [ATT20], where a similar approach from [Tol17] was refined and simplified.

Finally, in Section 9 we bound the size of big angle cubes BA. First, we show that this amounts to estimating $\|\nabla F\|_{L^2}$ (recall that F is the Lipschitz map whose graph is Γ). Using Dorronsoro's theorem, this reduces to estimating the $\beta_{\sigma,1}$ square function, where σ is the surface measure on Γ. This could be done using the smallness of either $\beta_{\mu,2}$ or α_μ square functions. For us it was easier to deal with α_μ, due to all the estimates from Section 7.

Thus, having estimated the measure of the stopping region, the proof of the main lemma is finished.

3 Main Lemma

Given $\varepsilon > 0$ and $r > 0$ let us define the set of "good points":

$$G_r^\varepsilon = \left\{ x \in \operatorname{supp} \mu \ : \ \int_0^{1000r} \left(\alpha_\mu(x,s)^2 + \beta_{\mu,2}(x,s)^2 \right) \frac{ds}{s} < \varepsilon^2 \right\}. \tag{3.1}$$

Let $\mathcal{D}$ denote the David-Mattila lattice corresponding to measure μ, as in §II.2. Recall that $\mathcal{D}^{sdb}$ is the family of strongly doubling cubes satisfying

$$\mu(100B_Q) \le C\mu(B(Q)),$$

with C as in Lemma II.2.9. Using this notation we may formulate our main lemma.

Lemma 3.1. *Let μ be a finite Radon measure on $\mathbb{R}^d$. There exists a small dimensional constant $\varepsilon_0 > 0$ such that the following holds: suppose that $R_0 \in \mathcal{D}^{sdb}$ satisfies*

$$\mu\left(R_0 \setminus G^{\varepsilon_0}_{r(R_0)}\right) \leq \varepsilon_0\, \mu(3B_{R_0}). \tag{3.2}$$

Then, there exists a set $R_G \subset R_0$, and a Lipschitz map $F : L_{R_0} \to L_{R_0}^{\perp}$ (recall that L_{R_0} denotes the n-dimensional plane minimizing $\beta_{\mu,2}(3B_{R_0})$), such that for

$$\Gamma = \left\{(x, F(x)) \ : \ x \in L_{R_0}\right\}$$

we have $R_G \subset \Gamma$,

$$\mu(R_G) \geq \frac{\mu(R_0)}{2}, \tag{3.3}$$

and $\mu|_{R_G}$ is absolutely continuous with respect to $\mathcal{H}^n$.

Several remarks are in order.

Remark 3.2. Assumption (3.2) is implied by a somewhat more natural condition

$$\int_{R_0}\int_0^{1000r(R_0)} \left(\alpha_\mu(x,s)^2 + \beta_{\mu,2}(x,s)^2\right)\, \frac{ds}{s} d\mu(x) < \varepsilon_0^3\, \mu(3B_{R_0}).$$

Remark 3.3. The constant $\frac{1}{2}$ in (3.3) can be replaced by any $\delta \in (0,1)$, as long as we allow ε_0 to depend on δ. Naturally, $\varepsilon_0(\delta) \to 0$ as $\delta \to 1$.

Remark 3.4. Recall that we defined homogeneous β and α numbers in Remark 1.6. A careful inspection of the proof of Lemma 3.1 (see Remark 3.6) shows the following. If instead of (3.1) we define for $Q \in \mathcal{D}$

$$G^\varepsilon_Q = \left\{x \in Q \ : \ \int_0^{1000r(Q)} \alpha^h_\mu(x,s)^2\, \frac{ds}{s} < \varepsilon^2\, \Theta_\mu(3B_Q)^2 \quad \text{and} \quad \int_0^{1000r(Q)} \beta^h_{\mu,2}(x,s)^2\, \frac{ds}{s} < \varepsilon^2\, \Theta_\mu(3B_Q)\right\},$$

and we replace the assumption (3.2) by $\mu\left(R_0 \setminus G^{\varepsilon_0}_{R_0}\right) \leq \varepsilon_0\, \mu(3B_{R_0})$, then the conclusion of Lemma 3.1 still holds. In other words, if the homogeneous square functions in some initial cube R_0 are small *relative to density of μ in the initial cube*, then μ is rectifiable on a large chunk of R_0.

Let us show how Lemma 3.1 may be used to prove Theorem 1.2.

Proof of Theorem 1.2 using Lemma 3.1. To show that μ is n-rectifiable it suffices to show that for any $E \subset \mathbb{R}^d$ satisfying $\mu(E) > 0$ there exists $F \subset E$ with $\mu(F) > 0$ and such that $\mu|_F$ is rectifiable. Let us fix $E \subset \mathbb{R}^d$ with $\mu(E) > 0$.

Let $\varepsilon_0 > 0$ be so small that Lemma 3.1 holds. Note that by the assumption on the finiteness of α and β square functions (1.1), (1.2), we have

$$\mu(\mathbb{R}^d \setminus G_r^{\varepsilon_0}) \xrightarrow{r\to 0} 0.$$

In particular, μ-almost all of E is contained in $\bigcup_{r>0} G_r^{\varepsilon_0}$. By the Lebesgue differentiation theorem, for μ-almost every $x \in E \cap G_r^{\varepsilon_0}$

$$\frac{\mu(B(x,s)\cap E\cap G_r^{\varepsilon_0})}{\mu(B(x,s))} \xrightarrow{s\to 0} 1.$$

Taking into account that for $s < r$ we have $G_s^{\varepsilon_0} \supset G_r^{\varepsilon_0}$, it follows that for μ-almost every $x \in E$

$$\frac{\mu(B(x,r)\cap E\cap G_r^{\varepsilon_0})}{\mu(B(x,r))} \xrightarrow{r\to 0} 1.$$

Choose some $x \in E$ such that the above and the property of Lemma II.2.9 hold. Let $r_0 > 0$ be so small that $\mu(B(x,r)\cap E\cap G_r^{\varepsilon_0}) > (1-\varepsilon_0)\mu(B(x,r))$ for all $r < r_0$.

Using Lemma II.2.9 we may choose $R_0 \in \mathcal{D}^{sdb}$ such that $x \in R_0$ and $\tilde{r} := 2r(B_{R_0}) < r_0$. We have $R_0 \subset B(x,\tilde{r}) \subset 3B_{R_0}$, and so

$$\mu(R_0 \setminus G_{r(R_0)}^{\varepsilon_0}) \le \mu(R_0 \setminus G_{\tilde{r}}^{\varepsilon_0}) \le \mu(B(x,\tilde{r}) \setminus G_{\tilde{r}}^{\varepsilon_0}) \le \varepsilon_0\mu(B(x,\tilde{r})) \le \varepsilon_0\mu(3B_{R_0}).$$

Hence, R_0 satisfies the assumptions of Lemma 3.1. We obtain a Lipschitz graph Γ and a set $R_G \subset R_0 \cap \Gamma$ such that $\mu(R_G) \ge 0.5\mu(R_0)$, and $\mu|_{R_G}$ is absolutely continuous with respect to $\mathcal{H}^n$. On the other hand, arguing as above, and using the fact that R_0 is doubling, we see that $\mu(R_0 \setminus E) \le \varepsilon_0\mu(3B_{R_0}) \le C_0\varepsilon_0\mu(R_0) < 0.5\mu(R_0)$, assuming $\varepsilon_0 < 0.5\, C_0^{-1}$.

It follows that $\mu(R_G \cap E) \ge \mu(R_G) - \mu(R_0 \setminus E) > 0$, and $\mu|_{R_G\cap E}$ is n-rectifiable. Setting $F = R_G \cap E$ concludes the proof.

□

The rest of the paper is dedicated to proving Lemma 3.1. We fix $R_0 \in \mathcal{D}^{sdb}$ satisfying (3.2). The constant ε_0 will be chosen later on. To simplify notation, we set $G = G_{r(R_0)}^{\varepsilon_0}$, $B_0 = B_{R_0}$, $r_0 = r(B_0)$, $z_0 = z_{R_0}$, $c_0 = c_{R_0}$ (where c_{R_0} is a constant minimizing $\alpha_\mu(3B_0)$), $L_0 = L_{R_0}$, (where L_{R_0} is an n-plane minimizing $\beta_{\mu,2}(3B_0)$), and $\Pi_0 = \Pi_{L_0}$.

Remark 3.5. Without loss of generality we may (and will) assume that

$$\Theta_\mu(3B_0) = 1,$$

so that (using the strong doubling property of R_0 (II.2.6))

$$\mu(100B_0) \approx \mu(R_0) \approx r_0^n \approx \ell(R_0)^n. \tag{3.4}$$

Indeed, if we consider the normalized measure $\nu = \mu/\Theta_\mu(3B_0)$, then: $\Theta_\nu(3B_0) = 1$; for any ball B with $\mu(B) > 0$ we have $\alpha_\mu(B) = \alpha_\nu(B)$, $\beta_{\mu,2}(B) = \beta_{\nu,2}(B)$; and if the assumptions of Lemma 3.1 were satisfied for μ, then they are also satisfied for ν. Sets Γ and R_G constructed for ν will also have all the desired properties when applied to μ.

Remark 3.6. The reduction to case $\Theta_\mu(3B_0) = 1$ performed above is one of the main reasons why we decided to work with non-homogeneous (i.e. normalized by $\mu(3B)$) α and β coefficients. If we assumed *a priori* that $\Theta_\mu(3B_0) = 1$, then we could replace α_μ and $\beta_{\mu,2}$ numbers in (3.1) by α_μ^h and $\beta_{\mu,2}^h$, and then carry on with the proof without making *any* changes. Roughly speaking, throughout most of the proof we work with cubes Q satisfying $\mu(3B_Q) \approx \ell(Q)^n \Theta_\mu(3B_0)$, so that $\alpha_\mu^h(3B_Q) \approx \alpha_\mu(3B_Q)\Theta_\mu(3B_0)$ and $\beta_{\mu,2}^h(3B_Q) \approx \beta_{\mu,2}(3B_Q)\Theta_\mu(3B_0)^{1/2}$ – see Remark 4.2.

Now, the claim we made in Remark 3.4 follows because the modified assumption (involving G_Q^ε) allows us to make the reduction $\Theta_\mu(3B_0) = 1$.

4 Stopping cubes

This section is dedicated to performing the stopping time argument. We will show basic properties of the resulting tree of cubes, and estimate the size of two families of stopping cubes.

The stopping conditions involve parameters $A \gg 1$, $\tau \ll 1$, $\theta \ll 1$, which depend on dimension and which will be fixed later on. The constant ε_0 is fixed at the very end of the proof, and depends on A, τ, θ.

We define the following subfamilies of $\mathcal{D}(R_0)$:

- HD_0 ("high density"), which contains cubes $Q \in \mathcal{D}(R_0)$ satisfying
$$\mu(3B_Q) > A\ell(Q)^n,$$
- LD_0 ("low density"), which contains cubes $Q \in \mathcal{D}(R_0)$ satisfying
$$\mu(1.5B_Q) < \tau\ell(Q)^n,$$
- BS_0 ("big square functions"), which contains cubes $Q \in \mathcal{D}(R_0) \setminus (\mathsf{LD}_0 \cup \mathsf{HD}_0)$ satisfying
$$\mu(Q \setminus G) > \frac{1}{2}\mu(Q). \tag{4.1}$$

Let Stop_0 be the family of maximal (and thus disjoint) cubes from $\mathsf{HD}_0 \cup \mathsf{LD}_0 \cup \mathsf{BS}_0$, and let $\mathsf{Tree}_0 \subset \mathcal{D}(R_0)$ be the family of cubes that are not contained in any $Q \in \mathsf{Stop}_0$. In particular, $\mathsf{Stop}_0 \not\subset \mathsf{Tree}_0$.

Recall that L_Q is an n-plane minimizing $\beta_{\mu,2}(3B_Q)$. We define
$$R_{\mathsf{Far}} = \{x \in 3B_0 \,:\, \operatorname{dist}(x, L_Q) \geq \sqrt{\varepsilon_0}\ell(Q) \quad \text{for some } Q \in \mathsf{Tree}_0 \text{ s.t. } x \in 3B_Q\}.$$
We introduce two more families of stopping cubes:

- BA_0 ("big angles"), which contains cubes $Q \in \mathcal{D}(R_0) \setminus \mathsf{Stop}_0$ satisfying

$$\measuredangle(L_Q, L_0) > \theta, \tag{4.2}$$

- F_0 ("far"), which consists of $Q \in \mathcal{D}(R_0) \setminus (\mathsf{Stop}_0 \cup \mathsf{BA}_0)$ satisfying

$$\mu(3B_Q \cap R_{\mathsf{Far}}) > \varepsilon_0^{1/4}\mu(3B_Q). \tag{4.3}$$

Let $\mathsf{Stop} \subset \mathcal{D}(R_0)$ be the family of maximal (and thus disjoint) cubes from $\mathsf{Stop}_0 \cup \mathsf{BA}_0 \cup \mathsf{F}_0$. Set $\mathsf{HD} = \mathsf{HD}_0 \cap \mathsf{Stop}$, $\mathsf{LD} = \mathsf{LD}_0 \cap \mathsf{Stop}$, $\mathsf{BS} = \mathsf{BS}_0 \cap \mathsf{Stop}$, $\mathsf{BA} = \mathsf{BA}_0 \cap \mathsf{Stop}$, $\mathsf{F} = \mathsf{F}_0 \cap \mathsf{Stop}$. We define $\mathsf{Tree} \subset \mathsf{Tree}_0$ as the family of cubes that are not contained in any $Q \in \mathsf{Stop}$. Note that $\mathsf{Stop} \not\subset \mathsf{Tree}$. For $P \in \mathcal{D}$ we set $\mathsf{Tree}_0(P) = \mathsf{Tree}_0 \cap \mathcal{D}(P)$, $\mathsf{Tree}(P) = \mathsf{Tree} \cap \mathcal{D}(P)$.

4.1 Properties of cubes in Tree

Lemma 4.1. *The following estimates hold:*

$$\mu(1.5B_Q) \geq \tau\ell(Q)^n \qquad \forall\, Q \in \mathsf{Tree}_0 \cup \mathsf{Stop}_0 \setminus \mathsf{LD}_0, \tag{4.4}$$
$$\mu(100B_Q) \lesssim A\ell(Q)^n \qquad \forall\, Q \in \mathsf{Tree}_0 \cup \mathsf{Stop}_0, \tag{4.5}$$
$$\mu(Q \setminus G) \leq \frac{1}{2}\mu(Q) \qquad \forall\, Q \in \mathsf{Tree}_0, \tag{4.6}$$
$$\measuredangle(L_Q, L_0) \leq \theta \qquad \forall\, Q \in \mathsf{Tree}, \tag{4.7}$$
$$\mu(3B_Q \cap R_{\mathsf{Far}}) \leq \varepsilon_0^{1/4}\,\mu(3B_Q) \qquad \forall\, Q \in \mathsf{Tree}. \tag{4.8}$$

Proof. All estimates except for (4.5) follow immediately from the stopping time conditions. (4.5) holds for R_0 because $R_0 \in \mathcal{D}^{sdb}$. To see it for $Q \in \mathsf{Tree}_0 \cup \mathsf{Stop}_0$, $Q \neq R_0$, note that the parent of Q, denoted by R, satisfies $R \in \mathsf{Tree}_0$, and so $\mu(100B_Q) \leq \mu(3B_R) \leq A\ell(R)^n \approx A\ell(Q)^n$. □

Remark 4.2. Note that, by (4.4) and (4.5), for $Q \in \mathsf{Tree}_0 \cup \mathsf{Stop}_0 \setminus \mathsf{LD}_0$ we have $\beta_{\mu,2}(3B_Q) \approx_{A,\tau} \beta^h_{\mu,2}(3B_Q)$ and $\alpha_\mu(3B_Q) \approx_{A,\tau} \alpha^h_\mu(3B_Q)$.

Lemma 4.3. *Let $R \in \mathsf{Tree}_0$. Then*

$$\sum_{Q \in \mathsf{Tree}_0(R)} \alpha_\mu(3B_Q)^2\ell(Q)^n \lesssim_{A,\tau} \varepsilon_0^2\ell(R)^n, \tag{4.9}$$

$$\sum_{Q \in \mathsf{Tree}_0(R)} \beta_{\mu,2}(3B_Q)^2\ell(Q)^n \lesssim_{A,\tau} \varepsilon_0^2\ell(R)^n. \tag{4.10}$$

Moreover, for any $x \in 3B_0$

$$\sum_{\substack{Q \in \mathsf{Tree}_0 \\ x \in 3B_Q}} \alpha_\mu(3B_Q)^2 \lesssim_{A,\tau} \varepsilon_0^2, \tag{4.11}$$

$$\sum_{\substack{Q \in \mathsf{Tree}_0 \\ x \in 3B_Q}} \beta_{\mu,2}(3B_Q)^2 \lesssim_{A,\tau} \varepsilon_0^2. \tag{4.12}$$

Proof. Let $Q \in \mathsf{Tree}_0(R)$. By the definition of G, for any $z \in 4B_Q \cap G$ we have

$$\int_0^{1000r(R_0)} \alpha_\mu(z,r)^2 \, \frac{dr}{r} < \varepsilon_0^2. \tag{4.13}$$

It is easy to see that for $300r(Q) \le r \le 400r(Q)$ we have $3B_Q \subset B(z,r) \subset 25B_Q$, and that $\mu(9B_Q) \approx_{A,\tau} \mu(B(z,3r)) \approx_{A,\tau} \mu(100B_Q)$. Using (II.3.7) with $B_1 = 3B_Q$ and $B_2 = B(z,r)$ yields

$$\alpha_\mu(3B_Q) \lesssim_{A,\tau} \alpha_\mu(B(z,r)).$$

Integrating with respect to r gives us for every $z \in 4B_Q \cap G$

$$\int_{300r(Q)}^{400r(Q)} \alpha_\mu(z,r)^2 \, \frac{dr}{r} \gtrsim_{A,\tau} \alpha_\mu(3B_Q)^2. \tag{4.14}$$

To see (4.11), let $x \in 3B_0$ and choose some $P \in \mathsf{Tree}_0$ satisfying $x \in 3B_P$. By (4.6) we may pick $z \in P \cap G$. It is clear that for all cubes $Q \in \mathsf{Tree}_0$ such that $\ell(Q) > \ell(P)$ and $x \in 3B_Q$ we have $z \in 4B_Q \cap G$. Thus, summing (4.14) over all such $Q \subset R_0$, and noticing that for any fixed sidelength $\ell(Q_0) > \ell(P)$ there are only boundedly many Q with $\ell(Q) = \ell(Q_0)$ and $3B_Q \ni x$, yields

$$\sum_{\substack{Q \in \mathsf{Tree}_0 \\ x \in 3B_Q,\ \ell(Q) > \ell(P)}} \alpha_\mu(3B_Q)^2 \lesssim_{A,\tau} \int_0^{1000r(R_0)} \alpha_{\mu,2}(z,r)^2 \, \frac{dr}{r} \lesssim \varepsilon_0^2.$$

Since the estimate holds for arbitrary $P \in \mathsf{Tree}_0$ with $x \in 3B_P$, (4.11) follows.

To see (4.9), we integrate (4.11) over $x \in 3B_R$ to get

$$\begin{aligned} \varepsilon_0^2 \ell(R)^n &\gtrsim_{A,\tau} \int_{3B_R} \sum_{Q \in \mathsf{Tree}_0} \alpha_\mu(3B_Q)^2 \, \mathbb{1}_{3B_Q}(x) \, d\mu(x) \\ &= \sum_{Q \in \mathsf{Tree}_0} \alpha_\mu(3B_Q)^2 \mu(3B_Q \cap 3B_R) \gtrsim_{A,\tau} \sum_{Q \in \mathsf{Tree}_0(R)} \alpha_\mu(3B_Q)^2 \ell(Q)^n. \end{aligned}$$

The estimates for $\beta_{\mu,2}(3B_Q)$ can be shown in the same way. □

Corollary 4.4. *We have*

$$\sum_{Q \in \mathsf{Tree}_0(R)} F_{2.5B_Q}(\mu, c_Q \mathcal{H}^n|_{L_Q})^2 \ell(Q)^{-(n+2)} \lesssim_{A,\tau} \varepsilon_0^2 \ell(R)^n, \tag{4.15}$$

$$\sum_{\substack{Q \in \mathsf{Tree}_0 \\ x \in 3B_Q}} F_{2.5B_Q}(\mu, c_Q \mathcal{H}^n|_{L_Q})^2 \ell(Q)^{-(2n+2)} \lesssim_{A,\tau} \varepsilon_0^2. \tag{4.16}$$

Proof. Let $Q \in \mathsf{Tree}_0$. Recall that by (4.4), (4.5), we have $\mu(2.5B_Q) \approx_{A,\tau} \mu(3B_Q) \approx_{A,\tau} \ell(Q)^n$. Moreover, it follows easily by (4.4) and the smallness of α and β numbers (4.11),(4.12), that the best approximating planes for $\beta_{\mu,2}(3B_Q)$ and $\alpha_\mu(3B_Q)$ intersect $2B_Q$.

Hence, by Lemma II.3.8 applied to $B_1 = 2.5B_Q$ and $B_2 = 3B_Q$, and by Lemma 4.3, we get the desired estimates. □

Corollary 4.5. *For every $Q \in \mathsf{Tree}_0$*

$$c_Q \approx_{A,\tau} 1. \tag{4.17}$$

Proof. By (4.4), (4.5), we have $\mu(1.5B_Q) \approx_{A,\tau} \mu(9B_Q) \approx_{A,\tau} \ell(Q)^n$. Together with the smallness of $\alpha_\mu(3B_Q)$ (4.11), this implies that the best approximating plane for $\alpha_\mu(3B_Q)$ intersects $2B_Q$. Thus, Lemma II.3.7 yields

$$c_Q \approx_{A,\tau} 1.$$

□

Lemma 4.6. *We have*

$$\mu(R_{\mathsf{Far}}) \lesssim_{A,\tau} \sqrt{\varepsilon_0}\mu(R_0)^n. \tag{4.18}$$

Proof. We begin by using the Chebyshev and Cauchy-Schwarz inequalities to obtain

$$\begin{aligned}
\sqrt{\varepsilon_0}\mu(R_{\mathsf{Far}}) &\leq \int_{3B_0} \left(\sum_{\substack{Q\in\mathsf{Tree}_0 \\ x\in 3B_Q}} \left(\frac{\operatorname{dist}(x, L_Q)}{\ell(Q)} \right)^2 \right)^{1/2} d\mu(x) \\
&\leq \left(\int_{3B_0} \sum_{\substack{Q\in\mathsf{Tree}_0 \\ x\in 3B_Q}} \left(\frac{\operatorname{dist}(x, L_Q)}{\ell(Q)} \right)^2 d\mu(x) \right)^{1/2} \mu(3B_0)^{1/2}.
\end{aligned}$$

By Fubini, the right hand side is equal to

$$\begin{aligned}
&\left(\sum_{Q\in\mathsf{Tree}_0} \int_{3B_Q} \left(\frac{\operatorname{dist}(x, L_Q)}{\ell(Q)} \right)^2 d\mu(x) \right)^{1/2} \mu(3B_0)^{1/2} \\
&\qquad \lesssim_{A,\tau} \left(\sum_{Q\in\mathsf{Tree}_0} \beta_{\mu,2}(3B_Q)^2 \ell(Q)^n \right)^{1/2} \mu(R_0)^{n/2}.
\end{aligned}$$

We can estimate this using the smallness of β-numbers (4.10), and thus

$$\sqrt{\varepsilon_0}\mu(R_{\mathsf{Far}}) \lesssim_{A,\tau} \varepsilon_0 \mu(R_0)^n.$$

□

4.2 Balanced balls

Lemma 4.7 ([AT15, Lemma 3.1, Remark 3.2])**.** *Let μ be a Radon measure on $\mathbb{R}^d$, and let $B \subset \mathbb{R}^d$ be some ball with radius $r > 0$ such that $\mu(B) > 0$. Let $0 < \gamma < 1$. Then there exist constants $\rho_1 = \rho_1(\gamma) > 0$ and $\rho_2 = \rho_2(\gamma) > 0$ such that one of the following alternatives holds:*

(a) There are points $x_0, \dots, x_n \in B$ *such that*

$$\mu(B(x_k, \rho_1 r) \cap B) \geq \rho_2 \mu(B) \quad \text{for } 0 \leq k \leq n,$$

and for any $y_k \in B(x_k, \rho_1 r)$, $k = 1, \dots, n$, *if we denote by* L_k^y *the* k*-plane passing through* $y_0, \dots, y_k$, *then we have*

$$\operatorname{dist}(y_k, L_{k-1}^y) \geq \gamma r. \tag{4.19}$$

(b) There exists a family of balls $\{B_i\}_{i \in I_B}$, *with radii* $r(B_i) = 4\gamma r$, *centered on* B, *so that the balls* $\{10B_i\}_{i \in I_B}$ *are pairwise disjoint,*

$$\sum_{i \in I_B} \mu(B_i) \gtrsim \mu(B), \tag{4.20}$$

and

$$\Theta_\mu(B_i) \gtrsim \gamma^{-1} \Theta_\mu(B). \tag{4.21}$$

We will say that a ball B is γ-balanced if the alternative (a) holds.

Lemma 4.8. *Let* μ *be a Radon measure on* $\mathbb{R}^d$, $B \subset \mathbb{R}^d$ *be a ball such that* $\mu(B) \approx \mu(1.1B) > 0$. *Suppose* L *is the* n*-plane minimizing* $\alpha_\mu(1.1B)$ *and that* L *intersects* $0.9B$. *There exist* $C = C(n,d) < 1$, $\gamma = \gamma(n,d) < 1$ *such that if* $\alpha_\mu(1.1B) \leq C\gamma$, *then* B *is* γ*-balanced.*

Proof. Proof by contradiction. Suppose that B is not γ-balanced, i.e. that the alternative (b) in Lemma 4.7 holds.

We will estimate $\alpha_\mu(1.1B)$ from below. Let c be the constant minimizing $\alpha_\mu(1.1B)$, so that by (II.3.8)

$$c \lesssim \Theta_\mu(1.1B) \approx \Theta_\mu(B).$$

Let balls $\{B_i\}_{i \in I_B}$ be as in Lemma 4.7 (b), with $r(B_i) = r_i = 4\gamma r(B)$. Let $f \in \operatorname{Lip}_1(1.1B)$ be defined in such a way that $f \equiv r_i$ on each B_i and $\operatorname{supp} f \subset \bigcup_{i \in I_B} 2B_i \subset 1.1B$. Then,

$$\int f \, d\mu \geq \sum_{i \in I_B} \mu(B_i) r_i \overset{(4.20)}{\gtrsim} \gamma r(B) \mu(B).$$

On the other hand,

$$c \int f \, d\mathcal{H}^n|_L \overset{(\text{II}.3.8)}{\lesssim} \Theta_\mu(B) \sum_{i \in I_B} r_i^{n+1} = \Theta_\mu(B) \sum_{i \in I_B} \Theta_\mu(B_i)^{-1} \mu(B_i) r_i$$
$$\overset{(4.21)}{\lesssim} \gamma \sum_{i \in I_B} \mu(B_i) r_i \lesssim \gamma^2 r(B) \mu(B).$$

The two estimates above imply that for some dimensional constants C_1, C_2

$$\alpha_\mu(1.1B) \geq C_1 \gamma - C_2 \gamma^2 > C\gamma,$$

if we take γ and $C = C(C_1, C_2)$ small enough. We reach a contradiction with the assumption $\alpha_\mu(1.1B) \leq C\gamma$. □

Corollary 4.9. *Let $Q \in \mathsf{Tree}_0$. Then $2.5B_Q$ is γ-balanced, where $\gamma = \gamma(n,d)$.*

Proof. We know that $\mu(1.5B_Q) \approx_{A,\tau} \mu(9B_Q)$, and that

$$\alpha_\mu(3B_Q) \overset{(4.11)}{\lesssim_{A,\tau}} \varepsilon_0,$$

which implies (for ε_0 small enough) that the best approximating plane for $3B_Q$ intersects $2B_Q$. Applying Lemma 4.8 to $B = 2.5B_Q$ finishes the proof. □

4.3 Small measure of cubes from BS and F

Lemma 4.10. *We have*

$$\sum_{Q \in \mathsf{BS}} \mu(Q) \lesssim \varepsilon_0 \mu(R_0),$$
$$\sum_{Q \in \mathsf{F}} \mu(Q) \lesssim_{A,\tau} {\varepsilon_0}^{1/4} \mu(R_0).$$

Proof. We start by estimating the measure of cubes from BS. We use the definition of BS (4.1) to get

$$\sum_{Q \in \mathsf{BS}} \mu(Q) \le 2 \sum_{Q \in \mathsf{BS}} \mu(Q \setminus G) \le 2\mu(R_0 \setminus G) \overset{(3.2)}{\le} 2\varepsilon_0 \mu(3B_0) \approx \varepsilon_0 \mu(R_0).$$

Concerning F, we use the $5R$-covering lemma to get a countable family of pairwise disjoint balls $B_i := 3B_{Q_i}$, $Q_i \in \mathsf{F}$, such that $\bigcup_i 5B_i \supset \bigcup_{Q \in \mathsf{F}} Q$. For every i we have

$$\mu(5B_i) = \mu(15B_{Q_i}) \overset{(4.5)}{\lesssim} A\ell(Q_i)^n \overset{(4.4)}{\le} \frac{A}{\tau}\mu(B_i).$$

Then

$$\sum_{Q \in \mathsf{F}} \mu(Q) \lesssim \sum_i \mu(5B_i) \lesssim_{A,\tau} \sum_i \mu(B_i)$$
$$\overset{(4.3)}{\le} \frac{1}{\varepsilon_0^{1/4}} \sum_i \mu(B_i \cap R_{\mathsf{Far}}) \le \frac{1}{\varepsilon_0^{1/4}} \mu(R_{\mathsf{Far}}) \overset{(4.18)}{\lesssim_{A,\tau}} \varepsilon_0^{1/4} \mu(R_0).$$

□

5 Construction of the Lipschitz graph

In this section we construct the Lipschitz graph Γ. At the beginning of Subsection 5.2 we define also the good set $R_G \subset \Gamma \cap R_0$, and we show that $\mu|_{R_G} \ll \mathcal{H}^n$. We start by proving some auxiliary estimates.

5.1 Estimates involving best approximating planes

Lemma 5.1 ([AT15, Lemma 6.4]). *Suppose P_1, P_2 are n-planes in $\mathbb{R}^d$, $X = \{x_0, \dots, x_n\}$ is a collection of n points, and*

$$d_1 = d_1(X) = \frac{1}{\operatorname{diam}(X)} \min_i \Big\{ \operatorname{dist}\Big(x_i, \operatorname{span}(X \setminus \{x_i\})\Big)\Big\} \in (0,1), \tag{a}$$

$$\operatorname{dist}(x_i, P_j) < d_2 \operatorname{diam}(X) \quad \textit{for} \quad i = 0, \dots, n \quad \textit{and} \quad j = 1, 2, \tag{b}$$

where $d_2 < d_1/(2d)$. Then for $y \in P_2$

$$\operatorname{dist}(y, P_1) \le d_2 \left(\frac{2d}{d_1} \operatorname{dist}(y, X) + \operatorname{diam}(X) \right). \tag{5.1}$$

Lemma 5.2. *Suppose $Q_1, Q_2 \in \mathsf{Tree}_0$ are such that $\operatorname{dist}(Q_1, Q_2) \lesssim \ell(Q_1) \approx \ell(Q_2)$. Let $P \in \mathsf{Tree}_0$ be the smallest cube such that $3B_P \supset 3B_{Q_1} \cup 3B_{Q_2}$. Then $\ell(P) \approx \ell(Q_1)$, and for all $y \in L_{Q_2}$*

$$\operatorname{dist}(y, L_{Q_1}) \lesssim_{A,\tau} \beta_{\mu,2}(3B_P)(\operatorname{dist}(y, Q_2) + \ell(Q_2)).$$

In particular,

$$\measuredangle(L_{Q_1}, L_{Q_2}) \lesssim_{A,\tau} \beta_{\mu,2}(3B_P) \lesssim_{A,\tau} \varepsilon_0. \tag{5.2}$$

Proof. Since $3B_0 \supset 3B_{Q_1} \cup 3B_{Q_2}$ and $R_0 \in \mathsf{Tree}_0$, the cube P is well-defined. The comparability $\ell(P) \approx \ell(Q_2)$ holds due to the assumption $\operatorname{dist}(Q_1, Q_2) \lesssim \ell(Q_1) \approx \ell(Q_2)$.

Since $Q_1 \in \mathsf{Tree}_0$, Corollary 4.9 tells us that $2.5B_{Q_1}$ is γ-balanced. Let $x_0, \dots, x_n \in 2.5B_{Q_1}$ be the points from alternative (a) in Lemma 4.7. Thus, we have a family of balls $\{B_k := B(x_k, \rho_1 r(2.5B_{Q_1}))\}_{k=0,\dots,n}$, such that $\mu(B_k \cap 2.5B_{Q_1}) \ge \rho_2 \mu(2.5B_{Q_1}) \approx_{A,\tau} \rho_2 \ell(Q_1)$.

Since $r(B_k) = \rho_1 r(2.5B_{Q_1}) \approx \ell(P)$, and $B_k \subset 3B_{Q_1} \subset 3B_P$, it is clear that

$$\frac{1}{\mu(B_k)} \int_{B_k} \left(\frac{\operatorname{dist}(x, L_{Q_1})}{r(B_k)} \right)^2 d\mu(x) \lesssim_{\rho_2, A, \tau} \beta_{\mu,2}(3B_{Q_1})^2 \lesssim_{A,\tau} \beta_{\mu,2}(3B_P)^2,$$

and

$$\frac{1}{\mu(B_k)} \int_{B_k} \left(\frac{\operatorname{dist}(x, L_P)}{r(B_k)} \right)^2 d\mu(x) \lesssim_{\rho_2, A, \tau} \beta_{\mu,2}(3B_P)^2.$$

Keeping in mind that ρ_2 is a dimensional constant, we will not signal dependence on it in further computations. We use the above estimates and the Chebyshev inequality to find points $y_k \in B_k$ such that

$$\operatorname{dist}(y_k, L_{Q_1}) \lesssim_{A,\tau} \beta_{\mu,2}(3B_P)\ell(P),$$
$$\operatorname{dist}(y_k, L_P) \lesssim_{A,\tau} \beta_{\mu,2}(3B_P)\ell(P).$$

We would like to apply Lemma 5.1 to n-planes L_{Q_1}, L_P and points $X = \{y_0, \dots, y_n\}$. We have $d_1 \gtrsim \gamma$ thanks to (4.19). Furthermore, due to estimate

(4.12) we know that $\beta_{\mu,2}(3B_P) \lesssim_{A,\tau} \varepsilon_0$, and so $\beta_{\mu,2}(3B_P) \approx_{A,\tau} d_2 < d_1/(2d)$ for ε_0 small enough. Thus,

$$\begin{aligned} \operatorname{dist}(y, L_{Q_1}) &\lesssim_{A,\tau} \beta_{\mu,2}(3B_P)(\operatorname{dist}(y,Q_1) + \ell(Q_1)) \quad \text{for } y \in L_P, \\ \operatorname{dist}(y, L_P) &\lesssim_{A,\tau} \beta_{\mu,2}(3B_P)(\operatorname{dist}(y,Q_1) + \ell(Q_1)) \quad \text{for } y \in L_{Q_1}. \end{aligned} \tag{5.3}$$

Since the assumptions about cubes Q_1 and Q_2 are identical, it turns out that the estimates above are also valid if we replace Q_1 with Q_2, i.e.

$$\begin{aligned} \operatorname{dist}(y, L_{Q_2}) &\lesssim_{A,\tau} \beta_{\mu,2}(3B_P)(\operatorname{dist}(y,Q_2) + \ell(Q_2)) \quad \text{for } y \in L_P, \\ \operatorname{dist}(y, L_P) &\lesssim_{A,\tau} \beta_{\mu,2}(3B_P)(\operatorname{dist}(y,Q_2) + \ell(Q_2)) \quad \text{for } y \in L_{Q_2}. \end{aligned} \tag{5.4}$$

Using the triangle inequality, estimates (5.4), (5.3), and the fact that $(\operatorname{dist}(y,Q_1)+\ell(Q_1)) \approx (\operatorname{dist}(y,Q_2)+\ell(Q_2))$ we finally reach the desired inequality

$$\operatorname{dist}(y, L_{Q_1}) \lesssim_{A,\tau} \beta_{\mu,2}(3B_P)(\operatorname{dist}(y,Q_2) + \ell(Q_2)) \quad \text{for } y \in L_{Q_2}.$$

□

Lemma 5.3. *Let $Q, P \in \mathsf{Tree}$ be such that $\ell(Q) \lesssim \ell(P)$ and $\operatorname{dist}(Q,P) \lesssim \ell(P)$. Then for any $x \in L_Q \cap CB_Q$ we have*

$$\operatorname{dist}(x, L_P) \lesssim_{A,\tau,C} \sqrt{\varepsilon_0}\,\ell(P).$$

Proof. Consider first the special case $Q \subset P$.

By Corollary 4.9, there exist balls $B_k = B(x_k, \rho_1 r(Q)),\ k = 0, \dots, n$, such that $\mu(B_k \cap 2.5B_Q) \geq \rho_2 \mu(2.5B_Q)$, and $\operatorname{dist}(y_k, L^y_{k-1}) \gtrsim \gamma \ell(Q)$ for $y_k \in B_k$ (see (4.19)).

It follows by (4.8) that, for ε_0 small enough, $B_i \setminus R_{\mathsf{Far}} \neq \varnothing$. Fix some $y_i \in B_i \setminus R_{\mathsf{Far}}$ for every $i = 0, \dots, n$, so that

$$\begin{aligned} \operatorname{dist}(y_i, L_Q) &\lesssim \sqrt{\varepsilon_0}\,\ell(Q), \\ \operatorname{dist}(y_i, L_P) &\lesssim \sqrt{\varepsilon_0}\,\ell(P). \end{aligned}$$

Let z_i be the orthogonal projection of y_i onto L_Q. Since $\ell(Q) \lesssim \ell(P)$, the triangle inequality yields

$$\operatorname{dist}(z_i, L_P) \leq |y_i - z_i| + \operatorname{dist}(y_i, L_P) \lesssim \sqrt{\varepsilon_0}\,\ell(P). \tag{5.5}$$

Furthermore, if ε_0 is small enough, $|y_i - z_i| \lesssim \sqrt{\varepsilon_0}\,\ell(Q)$ and $\operatorname{dist}(y_k, L^y_{k-1}) \gtrsim \ell(Q)$ imply that $\operatorname{dist}(z_k, L^z_{k-1}) \gtrsim \ell(Q)$, and that $z_i \in 3B_Q$. Since $L_Q = \operatorname{span}(z_0, \dots, z_n)$, it follows by elementary geometry and (5.5) that for any $x \in L_Q \cap CB_Q$

$$\operatorname{dist}(x, L_P) \lesssim_C \sqrt{\varepsilon_0}\,\ell(P),$$

which concludes the proof in the case $Q \subset P$.

Now, the general case follows by the above and Lemma 5.2. Indeed, take a cube $R \in \mathsf{Tree}$ such that $R \supset Q$ and $\ell(R) = \ell(P)$. The assumption $\operatorname{dist}(Q, P) \lesssim \ell(P)$ gives us $\operatorname{dist}(R, P) \lesssim \ell(P)$, and so we can apply Lemma 5.2 to get

$$\operatorname{dist}(y, L_P) \lesssim_{A,\tau,C} \varepsilon_0 \ell(P), \qquad y \in L_R \cap CB_R.$$

On the other hand, since $Q \subset R$, we already know that for $x \in L_Q \cap CB_Q$ we have

$$\operatorname{dist}(x, L_R) \lesssim_C \sqrt{\varepsilon_0}\ell(R) = \sqrt{\varepsilon_0}\ell(P).$$

Putting together the two inequalities above yields the desired result. $\square$

Lemma 5.4. *Suppose the cubes* $Q_1, Q_2 \in \mathsf{Tree}_0$ *satisfy* $2.5B_{Q_1} \subset 2.5B_{Q_2}$, $\ell(Q_1) \approx \ell(Q_2)$. *Then*

$$|c_{Q_1} - c_{Q_2}| \lesssim_{A,\tau} \varepsilon_0.$$

Proof. Set $B_i = 2.5B_{Q_i}, r_i = r(B_i), z_i = z(B_i), c_i = c_{Q_i}, L_i = L_{Q_i}$ for $i = 1, 2$. Let $\phi(z) = (r_1 - |z_1 - z|)_+ \in \operatorname{Lip}_1(B_1)$. Then

$$\begin{aligned}
r_1^n|c_1 - c_2| &\lesssim \left|\int \phi\ c_1 d\mathcal{H}^n|_{L_1} - \int \phi\ c_2 d\mathcal{H}^n|_{L_1}\right| \\
&\le \left|\int \phi\ c_1 d\mathcal{H}^n|_{L_1} - \int \phi\ d\mu\right| + \left|\int \phi\ d\mu - \int \phi\ c_2 d\mathcal{H}^n|_{L_2}\right| \\
&\qquad\qquad + c_2\left|\int \phi\ d\mathcal{H}^n|_{L_2} - \int \phi\ d\mathcal{H}^n|_{L_1}\right| \\
&\le F_{B_1}(\mu, c_1\mathcal{H}^n|_{L_1}) + F_{B_2}(\mu, c_2\mathcal{H}^n|_{L_2}) + c_2\left|\int \phi\ d\mathcal{H}^n|_{L_2} - \int \phi\ d\mathcal{H}^n|_{L_1}\right| \\
&\overset{(4.16),(4.17)}{\lesssim_{A,\tau}} \varepsilon_0 r_1^n + \left|\int \phi\ d\mathcal{H}^n|_{L_2} - \int \phi\ d\mathcal{H}^n|_{L_1}\right|.
\end{aligned}$$

The fact that the last term above can also be estimated by $\varepsilon_0 r_1^n$ follows easily by the fact that L_1 and L_2 are close to each other, see Lemma 5.2. $\square$

5.2 Lipschitz function F corresponding to the good part of R_0

Consider an auxiliary function

$$d(x) = \inf_{Q \in \mathsf{Tree}} \Big(\operatorname{dist}(x, Q) + \operatorname{diam}(B_Q)\Big), \quad x \in \mathbb{R}^d. \tag{5.6}$$

Let

$$R_G = \{x \in \mathbb{R}^d \ :\ d(x) = 0\}.$$

Observe that, by the definition of function d, we have $R_0 \setminus \bigcup_{Q \in \mathsf{Stop}} Q \subset R_G$.

Lemma 5.5. *We have* $\mu|_{R_G} \ll \mathcal{H}^n$, *and for* $x \in R_G$

$$\Theta^n_*(\mu, x) \approx_{A,\tau} \Theta^{*n}(\mu, x) \approx_{A,\tau} 1.$$

In consequence, $d\mu|_{R_G} = g\, d\mathcal{H}^n|_{R_G}$ *with* $g \approx_{A,\tau} 1$.

Proof. Let $x \in R_G$. Given some small $h > 0$ we use the fact that $d(x) = 0$ to find $Q \in \mathsf{Tree}$ such that $B(x,h) \subset 3B_Q$ and $\ell(Q) \approx h$. Then

$$\mu(B(x,h)) \le \mu(3B_Q) \overset{(4.5)}{\lesssim_A} \ell(Q)^n \approx h^n.$$

Now, let $P \in \mathsf{Tree}$ be such that $3B_P \subset B(x,h)$ and $\ell(P) \approx h$. Then

$$\mu(B(x,h)) \ge \mu(3B_P) \overset{(4.4)}{\gtrsim_\tau} \ell(P)^n \approx h^n.$$

Letting $h \to 0$ we get $1 \lesssim_\tau \Theta^n_*(\mu,x) \le \Theta^{*n}(\mu,x) \lesssim_A 1$ for $x \in R_G$. The upper density estimate and [Mat95, Theorem 6.9 (1)] imply $\mu|_{R_G} \ll \mathcal{H}^n|_{R_G}$ and $\mu|_{R_G}(B) \lesssim_A \mathcal{H}^n|_{R_G}(B)$ for all $B \subset \mathbb{R}^d$ Borel. The lower density estimate together with [Mat95, Theorem 6.9 (2)] give $\mathcal{H}^n|_{R_G} \ll \mu|_{R_G}$ and $\mathcal{H}^n|_{R_G}(B) \lesssim_\tau \mu|_{R_G}(B)$ (in particular, $\mathcal{H}^n|_{R_G}$ is a finite Radon measure). Putting it all together, we use Radon-Nikodym theorem to get $d\mu|_{R_G} = g\, d\mathcal{H}^n|_{R_G}$, with $g \approx_{A,\tau} 1$. □

In this subsection we will define $F(x)$ for $x \in \Pi_0(R_G) \subset L_0$.

Lemma 5.6. *If ε_0 and θ are small enough, then for any $x_1, x_2 \in \mathbb{R}^d$*

$$|\Pi_0^\perp(x_1) - \Pi_0^\perp(x_2)| \lesssim \theta|\Pi_0(x_1) - \Pi_0(x_2)| + d(x_1) + d(x_2). \tag{5.7}$$

Proof. Fix some small $h > 0$. Let $Q_1, Q_2 \in \mathsf{Tree}$ be such that

$$\operatorname{dist}(x_i, Q_i) + \operatorname{diam}(B_{Q_i}) \le d(x_i) + h, \quad i = 1,2.$$

Take any $y_i \in Q_i$. Note that $|x_i - y_i| \le d(x_i) + h$. The triangle inequality gives us

$$\begin{aligned}|\Pi_0^\perp(x_1) - \Pi_0^\perp(x_2)| &\\ \le |\Pi_0^\perp(y_1) - \Pi_0^\perp(y_2)| &+ |\Pi_0^\perp(x_1) - \Pi_0^\perp(y_1)| + |\Pi_0^\perp(x_2) - \Pi_0^\perp(y_2)| \\ &\le |\Pi_0^\perp(y_1) - \Pi_0^\perp(y_2)| + d(x_1) + d(x_2) + 2h,\end{aligned}$$

and similarly

$$|\Pi_0(y_1) - \Pi_0(y_2)| \le |\Pi_0(x_1) - \Pi_0(x_2)| + d(x_1) + d(x_2) + 2h.$$

Hence, if we show that

$$|\Pi_0^\perp(y_1) - \Pi_0^\perp(y_2)| \lesssim \theta|\Pi_0(y_1) - \Pi_0(y_2)| + d(x_1) + d(x_2) + 2h, \tag{5.8}$$

use the two former inequalities, and let $h \to 0$, we will get (5.7).

Let $P_i \in \mathsf{Tree}$ be the smallest cubes such that $3B_{P_i} \supset B_{Q_i}$ and

$$\ell(P_i) \approx \varepsilon_0^{1/n}|y_1 - y_2| + \sum_i \ell(Q_i).$$

We also take the smallest cube $R \in \mathsf{Tree}$ such that $3B_R \supset 3B_{P_1} \cup 3B_{P_2}$ and

$$\ell(R) \approx |y_1 - y_2| + \sum_i \ell(Q_i). \tag{5.9}$$

We use the fact that $3B_R \supset 3B_{P_1} \cup 3B_{P_2}$, the estimates (4.4), (4.5), the smallness of β numbers (4.12), and the bound $\varepsilon_0 \ell(R)^n \lesssim \ell(P_i)^n$, to get

$$\frac{1}{\mu(9B_{P_i})} \int_{3B_{P_i}} \left(\frac{\operatorname{dist}(w, L_R)}{\ell(R)} \right)^2 d\mu(w) \lesssim_{A,\tau} \frac{\ell(R)^n \beta_{\mu,2}(3B_R)^2}{\ell(P_i)^n} \lesssim_{A,\tau} \frac{\ell(R)^n \varepsilon_0^2}{\ell(P_i)^n} \lesssim \varepsilon_0.$$

Hence, by Chebyshev's inequality, there exist some $z_i \in 3B_{P_i}$ such that

$$\operatorname{dist}(z_i, L_R) = |z_i - \pi(z_i)| \lesssim_{A,\tau} \sqrt{\varepsilon_0}\ell(R) \lesssim \sqrt{\varepsilon_0}(|y_1 - y_2| + d(x_1) + d(x_2) + 2h), \tag{5.10}$$

where π denotes orthogonal projection onto L_R, and the second inequality is due to (5.9). Note also that, since $y_i, z_i \in 3B_{P_i}$, we have

$$|y_i - z_i| \lesssim \ell(P_i) \lesssim \varepsilon_0^{1/n}|y_1 - y_2| + d(x_1) + d(x_2) + 2h. \tag{5.11}$$

Now, the triangle inequality and 1-Lipschitz property of $\Pi_0^\perp$ give us

$$|\Pi_0^\perp(y_1) - \Pi_0^\perp(y_2)| \le |\Pi_0^\perp(\pi(z_1)) - \Pi_0^\perp(\pi(z_2))| + \sum_{i=1}^2 \Big(|z_i - \pi(z_i)| + |y_i - z_i| \Big).$$

To estimate the first term from the right hand side we use the fact that projections onto L_R and L_0 are close to each other (4.7), the triangle inequality, and 1-Lipschitz property of Π:

$$\begin{aligned} |\Pi_0^\perp(\pi(z_1)) - \Pi_0^\perp(\pi(z_2))| &\lesssim \theta|\pi(z_1)) - \pi(z_2)| \lesssim \theta|\Pi_0(\pi(z_1)) - \Pi_0(\pi(z_2))| \\ &\le \theta\Big(|\Pi_0(y_1) - \Pi_0(y_2)| + \sum_{i=1}^2 \Big(|z_i - \pi(z_i)| + |y_i - z_i| \Big) \Big). \end{aligned}$$

Putting together the two estimates above, as well as (5.10), (5.11), yields

$$\begin{aligned} |\Pi_0^\perp(y_1) - \Pi_0^\perp(y_2)| &\lesssim \theta|\Pi_0(y_1) - \Pi_0(y_2)| + \sum_{i=1}^2 \Big(|z_i - \pi(z_i)| + |y_i - z_i| \Big) \\ &\lesssim \theta|\Pi_0(y_1) - \Pi_0(y_2)| + C(A,\tau)\sqrt{\varepsilon_0}\Big(|y_1 - y_2| + d(x_1) + d(x_2) + 2h \Big) \\ &\qquad + \varepsilon_0^{1/n}|y_1 - y_2| + d(x_1) + d(x_2) + 2h. \end{aligned}$$

Since $|y_1 - y_2| \approx |\Pi_0(y_1) - \Pi_0(y_2)| + |\Pi_0^\perp(y_1) - \Pi_0^\perp(y_2)|$, we may take $\varepsilon_0 = \varepsilon_0(A, \tau, \theta)$ so small that

$$\Big(C(A,\tau)\sqrt{\varepsilon_0} + \varepsilon_0^{1/n} \Big)|y_1 - y_2| \le \theta\Big(|\Pi_0(y_1) - \Pi_0(y_2)| + |\Pi_0^\perp(y_1) - \Pi_0^\perp(y_2)| \Big).$$

Then, for θ small enough, we obtain the desired inequality (5.8):

$$|\Pi_0^\perp(y_1) - \Pi_0^\perp(y_2)| \lesssim \theta|\Pi_0(y_1) - \Pi_0(y_2)| + d(x_1) + d(x_2) + 2h.$$

□

The lemma above gives us for any $x, y \in R_G$

$$|\Pi_0^\perp(x) - \Pi_0^\perp(y)| \lesssim \theta |\Pi_0(x) - \Pi_0(y)|.$$

This allows us to define a function F on $\Pi_0(R_G) \subset L_0$ as

$$F(\Pi_0(x)) = \Pi_0^\perp(x), \qquad x \in R_G, \tag{5.12}$$

with $\mathrm{Lip}(F) \lesssim \theta$. Note that the graph of such F is precisely R_G.

5.3 Extension of F to the whole L_0

For any $z \in L_0$ let us define

$$D(z) = \inf_{x \in \Pi_0^{-1}(z)} d(x) = \inf_{Q \in \mathsf{Tree}} \Big(\mathrm{dist}(z, \Pi_0(Q)) + \mathrm{diam}(B_Q) \Big). \tag{5.13}$$

For each $z \in L_0$ with $D(z) > 0$, i.e. $z \in L_0 \setminus \Pi_0(R_G)$, we define J_z as the largest dyadic cube from L_0 such that $z \in J_z$ and

$$\mathrm{diam}(J_z) \le \frac{1}{20} \inf_{u \in J_z} D(u).$$

Let $J_i, i \in I$, be a relabeling of the set of all such cubes J_z, without repetition.

Lemma 5.7. *The cubes $\{J_i\}_{i \in I}$ are disjoint and satisfy the following:*

(a) If $z \in 15J_i$, then $5\,\mathrm{diam}(J_i) \le D(z) \le 50\,\mathrm{diam}(J_i)$.

(b) If $15J_i \cap 15J_{i'} \neq \varnothing$, then

$$\ell(J_i) \approx \ell(J_{i'}).$$

(c) For each interval J_i there are at most N intervals $J_{i'}$ such that $15J_i \cap 15J_{i'} \neq \varnothing$.

(d) $L_0 \setminus \Pi_0(R_G) = \bigcup_{i \in I} J_i = \bigcup_{i \in I} 15J_i$.

The proof is straightforward and follows directly from the definition of J_i, see [Tol14, Lemma 7.20].

Note that, since $\beta_{\mu,2}(3B_0)$ is very small (4.12) and R_0 is doubling, we have $\mathrm{dist}(z_0, L_0) \le 2r(R_0) = \frac{1}{14} r_0$. It follows that

$$\Pi_0(R_0) \subset \Pi_0(B_0) \subset \Pi_0(1.01B_0) \subset 1.1B_0 \cap L_0. \tag{5.14}$$

We define the set of indices

$$I_0 = \{ i \in I \ : \ J_i \cap 1.5B_0 \neq \varnothing \}. \tag{5.15}$$

Lemma 5.8. *The following holds:*

(a) If $i \in I_0$, then $\operatorname{diam}(J_i) \le 0.2 r_0$, *and* $3J_i \subset L_0 \cap 1.9B_0$.

(b) If $J_i \cap 1.4B_0 = \varnothing$ (in particular if $i \notin I_0$), then

$$\ell(J_i) \approx \operatorname{dist}(z_0, J_i) \approx |z_0 - z| \gtrsim \ell(R_0) \quad \textit{for all } z \in J_i.$$

Proof. We begin by proving (a). Suppose $i \in I_0$. Then $J_i \cap 1.5B_0 \neq \varnothing$ and

$$3J_i \subset L_0 \cap B(z_0, 1.5r_0 + 2\operatorname{diam}(J_i)).$$

We need to estimate $\operatorname{diam}(J_i)$. By the definition of J_i, we have

$$\operatorname{diam}(J_i) \le \frac{1}{20} \inf_{u \in J_i} D(u).$$

Since $J_i \cap 1.5B_0 \neq \varnothing$ we have $\inf_{u\in J_i} D(u) \le \max_{u \in L_0 \cap 1.5B_0} D(u)$, and so it suffices to estimate the latter quantity. Note that the definition of d (5.6) gives for $x \in 1.5B_0$

$$d(x) \le \operatorname{dist}(x, R_0) + \operatorname{diam}(B_0) \le 1.5r_0 + 2r_0 = 3.5r_0.$$

Hence, by the definition of D (5.13)

$$\max_{u \in L_0 \cap 1.5B_0} D(u) \le \max_{x \in 1.5B_0} d(x) \le 3.5r_0.$$

It follows that $\operatorname{diam}(J_i) \le \frac{7}{40} r_0$, and

$$3J_i \subset L_0 \cap B(z_0, 1.85r_0).$$

Now, let us prove (b). Suppose $J_i \cap 1.4B_0 = \varnothing$ and $z \in J_i$. Clearly, $|z_0 - z| \ge 1.4r_0$. Together with the definition of D (5.13) this gives

$$D(z) \le |\Pi_0(z_0) - z| + \operatorname{diam}(B_0) \le 3|z_0 - z|.$$

On the other hand, by (5.14) we have

$$D(z) \ge \operatorname{dist}(z, \Pi_0(R_0)) \ge \operatorname{dist}(z, 1.1B_0) = |z_0 - z| - 1.1r_0 \ge \frac{3}{14}|z_0 - z|.$$

Putting together the two estimates above gives for $z \in J_i$

$$\frac{1}{5}|z_0 - z| \le D(z) \le 3|z_0 - z|.$$

Applying Lemma 5.7 (a) yields

$$\frac{5}{3}\operatorname{diam}(J_i) \le |z_0 - z| \le 250 \operatorname{diam}(J_i).$$

Moreover, since

$$|z_0 - z| - \operatorname{diam}(J_i) \le \operatorname{dist}(z_0, J_i) \le |z_0 - z|,$$

we finally obtain

$$\frac{2}{3}\operatorname{diam}(J_i) \le \operatorname{dist}(z_0, J_i) \le 250\operatorname{diam}(J_i).$$

□

Lemma 5.9. *Given $i \in I_0$, there exists a cube $Q_i \in \mathsf{Tree}$ such that*

$$\ell(J_i) \approx \ell(Q_i),$$
$$\operatorname{dist}(J_i, \Pi_0(Q_i)) \lesssim \ell(J_i).$$

Proof. Let $i \in I_0$ and $z \in J_i$. We know by Lemma 5.7 (a) that $D(z) \approx \ell(J_i)$. Thus, by the definition of D (5.13) we may find $Q \in \mathsf{Tree}$ such that

$$\operatorname{dist}(z, \Pi_0(Q)) + \operatorname{diam}(B_Q) \approx \ell(J_i).$$

Clearly, $\ell(Q) \lesssim \ell(J_i)$, and $\operatorname{dist}(J_i, \Pi_0(Q)) \lesssim \ell(J_i)$. If $\ell(Q) \gtrsim \ell(J_i)$, we set $Q_i = Q$ and we are done. If that is not the case, then we define Q_i as the ancestor $P \supset Q$ satisfying $\ell(P) \gtrsim \ell(J_i)$ (we can always do that because $\ell(J_i) \lesssim \ell(R_0)$ by Lemma 5.8 (a)). □

For all $i \in I_0$ we define $F_i : L_0 \to L_0^\perp$ as the affine function whose graph is the n-plane L_{Q_i}. Since $\measuredangle(L_{Q_i}, L_0) \le \theta$ by (4.7), we have $\operatorname{Lip}(F_i) \lesssim \theta$. For $i \notin I_0$ set $F_i \equiv 0$, so that the graph of F_i is the plane L_0.

Lemma 5.10. *Suppose $10J_i \cap 10J_{i'} \neq \varnothing$. We have:*

(a) if $i, i' \in I_0$, then
$$\operatorname{dist}(Q_i, Q_{i'}) \lesssim \ell(J_i),$$

(b) for $x \in 100J_i$
$$|F_i(x) - F_{i'}(x)| \lesssim \sqrt{\varepsilon_0}\,\ell(J_i),$$

(c) $\|\nabla F_i - \nabla F_{i'}\|_\infty \lesssim \sqrt{\varepsilon_0}$.

Proof. Let us start with (a). We know by Lemma 5.7(b) and Lemma 5.9 that $\ell(Q_i) \approx \ell(Q_{i'}) \approx \ell(J_i) \approx \ell(J_{i'})$. Let $z_1 \in Q_i, z_2 \in Q_{i'}$ be such that $|\Pi_0(z_1) - \Pi_0(z_2)| \approx \operatorname{dist}(\Pi_0(Q_i), \Pi_0(Q_{i'}))$. Note that $d(z_1) \lesssim \ell(Q_i)$, $d(z_2) \lesssim \ell(Q_{i'})$. It follows that

$$\begin{aligned}\operatorname{dist}(Q_i, Q_{i'}) &\le |z_1 - z_2| \le |\Pi_0^\perp(z_1) - \Pi_0^\perp(z_2)| + |\Pi_0(z_1) - \Pi_0(z_2)| \\ &\overset{(5.7)}{\lesssim} |\Pi_0(z_1) - \Pi_0(z_2)| + d(z_1) + d(z_2) \lesssim \operatorname{dist}(\Pi_0(Q_i), \Pi_0(Q_{i'})) + \ell(J_i).\end{aligned}$$

On the other hand, we have by Lemma 5.9

$$\operatorname{dist}(\Pi_0(Q_i),\Pi_0(Q_{i'})) \le \operatorname{dist}(\Pi_0(Q_i),J_i)) + \operatorname{dist}(J_i,J_{i'}) + \operatorname{dist}(J_{i'},\Pi_0(Q_{i'})) + \operatorname{diam}(J_i) + \operatorname{diam}(J_{i'}) \lesssim \ell(J_i).$$

The two estimates together give us (a).

Now, (b) and (c) for $i,\ i' \in I_0$ follow immediately because we can apply Lemma 5.2 to Q_i and $Q_{i'}$. If $i,\ i' \notin I_0$, then (b) and (c) are trivially true, since $F_i = F_{i'} \equiv 0$. The only remaining case is $i \in I_0,\ i' \notin I_0$.

Since $10J_i \cap 10J_{i'} \neq \varnothing$, we know by Lemma 5.7 (b) and Lemma 5.8 that $\ell(J_i) \approx \ell(J_{i'}) \approx \ell(R_0)$. We apply Lemma 5.2 to Q_i and R_0, and the result follows.. □

Now, to define function F on $L_0 \backslash \Pi_0(R_G)$ we consider the following partition of unity: for each $i \in I$ let $\widetilde{\varphi}_i \in C^\infty(L_0)$ be such that $\widetilde{\varphi}_i \equiv 1$ on $2J_i$, $\operatorname{supp}\widetilde{\varphi}_i \subset 3J_i$, and

$$\|\nabla\widetilde{\varphi}_i\|_\infty \lesssim \ell(J_i)^{-1},$$
$$\|D^2\widetilde{\varphi}_i\|_\infty \lesssim \ell(J_i)^{-2}.$$

Now, we set

$$\varphi_i = \frac{\widetilde{\varphi}_i}{\sum_{j\in I}\widetilde{\varphi}_j}.$$

Clearly, the family $\{\varphi_i\}_{i\in I}$ is a partition of unity subordinated to sets $\{3J_i\}_{i\in I}$. Moreover, the inequalities above together with Lemma 5.7 imply that each φ_i satisfies

$$\|\nabla\varphi_i\|_\infty \lesssim \ell(J_i)^{-1},$$
$$\|D^2\varphi_i\|_\infty \lesssim \ell(J_i)^{-2}.$$

Recall that in (5.12) we defined $F(z)$ for $z \in \Pi_0(R_G)$. Concerning $L_0 \setminus \Pi_0(R_G)$, by Lemma 5.7 (d) we have $L_0 \setminus \Pi_0(R_G) = \bigcup_{i\in I} J_i = \bigcup_{i\in I} 3J_i$. Thus, for $z \in L_0 \setminus \Pi_0(R_G)$ we may set

$$F(z) = \sum_{i\in I_0}\varphi_i(z)F_i(z). \tag{5.16}$$

Using Lemmas 5.7–5.10, one may follow the proofs of [Tol14, Lemma 7.24, Remark 7.26, Lemma 7.27] to get the following.

Lemma 5.11. *The function $F : L_0 \to L_0^\perp$ is supported on $L_0 \cap 1.9B_0$ and is $C\theta$-Lipschitz, where $C > 0$ is an absolute constant. Furthermore, for $z \in 15J_i, i \in I$,*

$$|\nabla F(z) - \nabla F_i(z)| \lesssim \sqrt{\varepsilon_0}, \tag{5.17}$$

and

$$|D^2F(z)| \lesssim \frac{\sqrt{\varepsilon_0}}{\ell(J_i)}.$$

We denote the graph of F as Γ, and we define a function $f : L_0 \to \Gamma$ as

$$f(x) = (x, F(x)).$$

We set also

$$\sigma = \mathcal{H}^n|_\Gamma.$$

Lemma 5.12. *Let $i \in I_0$. Then $B(f(z_{J_i}), 2\operatorname{diam}(J_i)) \subset 2.3B_0$.*

Proof. By the definition of I_0 we have $J_i \cap 1.5B_0 \neq \varnothing$. We know by Lemma 5.8 that $\operatorname{diam}(J_i) \le 0.2r_0$, and so $z_{J_i} \in 1.7B_0$. Moreover, since F is supported on $L_0 \cap 1.9B_0$ and is Lipschitz continuous with constant comparable to θ, we have $\operatorname{dist}(f(z_{J_i}), z_{J_i}) = |F(z_{J_i})| \lesssim \theta r_0$.

It follows easily that $B(f(z_{J_i}), 2\operatorname{diam}(J_i)) \subset 2.3B_0$. □

We have defined a Lipschitz graph Γ, and a set $R_G \subset \Gamma \cap R_0$ such that $\mu|_{R_G} \ll \mathcal{H}^n$. Clearly, the measure $\mu|_{R_G}$ is n-rectifiable. What remains to be shown is that $\mu(R_G) \ge 0.5\mu(R_0)$. Since R_G contains $R_0 \setminus \bigcup_{Q\in\mathsf{Stop}} Q$, it is enough to estimate the measure of the stopping cubes – this is what we will do in the remaining part of the article.

6 Small measure of cubes from LD

In this section we will bound the measure of low density cubes. First, let us prove some additional estimates.

6.1 Γ lies close to R_0

Lemma 6.1. *There exists a constant C_1 such that for any $x \in 3B_0$*

$$\operatorname{dist}(x, \Gamma) \le C_1\, d(x).$$

Proof. First, notice that if $x \in 3B_0 \setminus 1.01B_0$, then $d(x) \gtrsim r_0$, and so the estimate $\operatorname{dist}(x, \Gamma) \le C_1\, d(x)$ is trivial. Now, assume $x \in 1.01B_0$.

Let $\xi = \Pi_0(x) \in L_0,\ y = (\xi, F(\xi)) \in \Gamma$. Lemma 5.6 gives us

$$\operatorname{dist}(x, \Gamma) \le |x - y| = |\Pi_0^\perp(x) - \Pi_0^\perp(y)| \lesssim d(x) + d(y). \tag{6.1}$$

If $\xi \in \Pi_0(R_G)$, then $y \in R_G$, which means that $d(y) = 0$ and we get $\operatorname{dist}(x, \Gamma) \lesssim d(x)$.

Now suppose $\xi \notin \Pi_0(R_G)$. Let $i \in I$ be such that $\xi \in J_i$. Note that since $x \in 1.01B_0$, then by (5.14) $\xi \in 1.1B_0$, and so $J_i \cap 1.5B_0 \neq \varnothing$. Hence, $i \in I_0$. Let $Q_i \in \mathsf{Tree}$ be the cube from Lemma 5.9 corresponding to J_i. It follows that

$$d(y) \le \operatorname{dist}(y, Q_i) + \ell(Q_i) \lesssim \operatorname{dist}(y, Q_i) + \ell(J_i). \tag{6.2}$$

Now we will estimate $\mathrm{dist}(y, Q_i)$. Let $z = (\xi, F_i(\xi)) \in L_{Q_i}$. We have

$$|y-z| = |F(\xi)-F_i(\xi)| = \Big|\sum_{j\in I_0}\varphi_j(\xi)F_j(\xi)-F_i(\xi)\Big| = \Big|\sum_{j\in I_0}\varphi_j(\xi)(F_j(\xi)-F_i(\xi))\Big| \\ \leq \sum_{j\in I_0}\varphi_j(\xi)\Big|F_j(\xi)-F_i(\xi)\Big|.$$

Since $\varphi_j(\xi) \neq 0$ only for $j \in I_0$ such that $\xi \in 3J_j$, we get from Lemma 5.10 (b) that $|F_j(\xi) - F_i(\xi)| \lesssim \ell(J_i)$. Hence,

$$|y - z| \lesssim \ell(J_i).$$

We use the smallness of $\beta_{\mu,2}(3B_{Q_i})$ and Chebyshev inequality to find $p \in 2B_{Q_i}$, $q \in L_{Q_i}$ such that $|p - q| \lesssim \ell(J_i)$. We know from Lemma 5.9 (b) that $|\Pi_0(p)-\xi| \lesssim \ell(J_i)$, and so $|\Pi_0(q)-\xi| \lesssim \ell(J_i)$. Together with the fact that both q and z belong to L_{Q_i}, and that $\measuredangle(L_0, L_{Q_i}) \leq \theta$ by (4.7), this implies

$$|z - q| \lesssim \ell(J_i).$$

Thus,

$$\mathrm{dist}(y, Q_i) \leq |y - z| + |z - q| + |q - p| \lesssim \ell(J_i).$$

From this, (6.2), Lemma 5.7 (a), and the definition od D, we get

$$d(y) \lesssim \ell(J_i) \approx D(\xi) \leq d(x).$$

The estimate above together with (6.1) conclude the proof. □

Corollary 6.2. *For every $Q \in \mathsf{Tree}$ we have*

$$\mathrm{dist}(Q, \Gamma) \lesssim \ell(Q).$$

Moreover, for $i \in I_0$ we have

$$\mathrm{dist}(Q_i, f(J_i)) \lesssim \ell(Q_i). \tag{6.3}$$

Proof. Since $Q \subset R_0 \subset B_0$, the first inequality follows immediately by Lemma 6.1 and the definition of function d.

The second inequality is implied by the first one, the fact that $\mathrm{dist}(\Pi_0(Q_i), J_i) \lesssim \ell(Q_i)$ by Lemma 5.9, and that Γ is a Lipschitz graph with a small Lipschitz constant. □

Lemma 6.3. *Let $C > 0$. If ε_0 is chosen small enough, then for each $Q \in \mathsf{Tree}$ and $x \in \Gamma \cap CB_Q$*

$$\mathrm{dist}(x, L_Q) \lesssim_{A,\tau,C} \sqrt{\varepsilon_0}\,\ell(Q).$$

Proof. There are three cases to consider.

Case 1. $x \in R_G$, i.e. $d(x) = 0$.

Fix some small $h > 0$. Let $P \in \mathsf{Tree}$ be such that $(\mathrm{dist}(x, P) + \mathrm{diam}(B_P)) \leq h \ll \ell(Q)$. Since $x \in \Gamma \cap CB_Q$, we have $\mathrm{dist}(P, Q) \lesssim \ell(Q)$. Setting $y = \Pi_{L_P}(x)$, we clearly have $|x - y| \lesssim h$, and in consequence $y \in L_P \cap C'B_Q$ with $C' \approx C$. Thus, we may apply Lemma 5.3 to get

$$\mathrm{dist}(y, L_Q) \lesssim_{A,\tau,C} \sqrt{\varepsilon_0}\ell(Q).$$

Thus, $\mathrm{dist}(x, L_Q) \lesssim_{A,\tau,C} \sqrt{\varepsilon_0}\ell(Q) + h$. Letting $h \to 0$ ends the proof in this case.

Case 2. $x = (\zeta, F(\zeta))$ for $\zeta \in L_0 \setminus \Pi_0(R_G)$, and

$$\sum_{i \in I_0} \varphi_i(\zeta) = 1.$$

Since $F(\zeta) = \sum_i \varphi_i(\zeta) F_i(\zeta)$, we get that x is a convex combination of points $\{(\zeta, F_i(\zeta))\}_{i \in I_1}$, where $I_1 \subset I_0$ consists of indices i such that $\varphi_i(\zeta) \neq 0$. Thus, it suffices to show that for each $i \in I_1$

$$\mathrm{dist}\Big((\zeta, F_i(\zeta)), L_Q\Big) \lesssim_{A,\tau,C} \sqrt{\varepsilon_0}\ell(Q).$$

First, note that since $x \in CB_Q$,

$$D(\zeta) \leq d(x) \lesssim_C \ell(Q).$$

Let $J_{i'}$ be the dyadic cube containing ζ, $i' \in I_1$. Then

$$\mathrm{diam}(J_{i'}) \leq \frac{1}{20} D(\zeta) \lesssim_C \ell(Q). \tag{6.4}$$

Moreover, as each φ_i is supported in $3J_i$, we necessarily have $3J_i \cap J_{i'} \neq \varnothing$ for $i \in I_1$. Thus, by Lemma 5.7 (b) and by Lemma 5.9,

$$\ell(Q_{i'}) \approx \ell(J_{i'}) \approx \ell(J_i) \approx \ell(Q_i) \overset{(6.4)}{\lesssim_C} \ell(Q). \tag{6.5}$$

Furthermore, Lemma 5.10 (a) implies

$$\mathrm{dist}(\Pi_0(Q_i), \Pi_0(Q_{i'})) \leq \mathrm{dist}(Q_i, Q_{i'}) \lesssim \ell(J_i).$$

Taking into account Lemma 5.9 and the fact that $\zeta \in J_{i'} \cap \Pi_0(CB_Q)$ we obtain

$$\begin{aligned}\mathrm{dist}(\Pi_0(Q_{i'}), \Pi_0(Q)) &\leq \mathrm{dist}(\Pi_0(Q_{i'}), J_{i'}) + \mathrm{diam}(J_{i'}) + \mathrm{dist}(\Pi_0(Q), J_{i'}) \\ &\lesssim_C \ell(J_{i'}) + \ell(Q).\end{aligned}$$

The three estimates above yield

$$\operatorname{dist}(\Pi_0(Q_i),\Pi_0(Q)) \le \operatorname{dist}(\Pi_0(Q_i),\Pi_0(Q_{i'})) + \operatorname{diam}(\Pi_0(Q_{i'})) \\ + \operatorname{dist}(\Pi_0(Q_{i'}),\Pi_0(Q)) \lesssim_C \ell(Q).$$

Applying Lemma 5.6 to any $y_1 \in Q_i, y_2 \in Q$ gives us

$$\operatorname{dist}(Q_i, Q) \lesssim_{A,\tau} \operatorname{dist}(\Pi_0(Q_i),\Pi_0(Q)) + \ell(Q) + \ell(Q_i) \lesssim_C \ell(Q). \tag{6.6}$$

Note that $(\zeta, F_i(\zeta)) \in L_{Q_i} \cap C' B_{Q_i}$ for some $C' = C'(n,d) > 0$. Indeed: $(\zeta, F_i(\zeta)) \in L_{Q_i}$ by the definition of F_i; to see that $(\zeta, F_i(\zeta)) \in C' B_{Q_i}$ observe that $\varphi_i(\zeta) \neq 0$, and so $\zeta \in 3J_i$, which together with Lemma 5.9 gives $(\zeta, F_i(\zeta)) \in C' B_{Q_i}$.

Due to the observation above and (6.5), (6.6), we can use Lemma 5.3 to get the desired inequality:

$$\operatorname{dist}\Big((\zeta, F_i(\zeta)), L_Q\Big) \lesssim_{A,\tau,C} \sqrt{\varepsilon_0}\,\ell(Q).$$

Case 3. $x = (\zeta, F(\zeta))$ for $\zeta \in L_0 \setminus \Pi_0(R_G)$, and

$$\sum_{i\in I_0} \varphi_i(\zeta) < 1.$$

It follows that there exists some $k \notin I_0$ such that $\zeta \in 3J_k$. Hence, by Lemma 5.8 (b)

$$\ell(J_k) \approx \operatorname{dist}(\Pi_0(z_0), J_k) \gtrsim \ell(R_0).$$

Furthermore, if $J_{i'}$ is the cube containing $\zeta = \Pi_0(x)$, then using the definition of functions d and D yields

$$\ell(J_{i'}) \lesssim D(\Pi_0(x)) \le d(x) \le \operatorname{dist}(x, Q) + \operatorname{diam}(B_Q) \lesssim \ell(Q) \le \ell(R_0).$$

Since $J_{i'} \cap 3J_k \neq \varnothing$, Lemma 5.7 (b) gives us $\ell(J_{i'}) \approx \ell(J_k)$. Thus,

$$\ell(J_{i'}) \approx \ell(Q) \approx \ell(R_0),$$

and again using Lemma 5.7 (b) we get that $\ell(J_i) \approx \ell(R_0)$ for all $i \in I_1$, where $I_1 \subset I_0$ are indices such that $\zeta \in 3J_i$. By the definition of cubes Q_i in Lemma 5.9, we also have $\ell(Q_i) \approx \ell(R_0)$.

It is clear that $\operatorname{dist}(Q_i, R_0) = 0$, and so the assumptions of Lemma 5.2 are satisfied for Q_i and R_0. Since $\operatorname{dist}((\zeta, F_i(\zeta)), Q_i) \lesssim \ell(R_0) \approx \ell(Q_i)$, we get that

$$|F_i(\zeta)| = \operatorname{dist}\Big((\zeta, F_i(\zeta)), L_0\Big) \lesssim_{A,\tau} \varepsilon_0 \ell(R_0) \approx \varepsilon_0 \ell(Q)$$

for $i \in I_1$. Hence,

$$\operatorname{dist}\Big((\zeta, F(\zeta)), L_0\Big) = |F(\zeta)| \le \sum_{i\in I_1} \varphi_i(\zeta) |F_i(\zeta)| \lesssim_{A,\tau} \varepsilon_0 \ell(Q) \sum_{i \in I_1} \varphi_i(\zeta) \le \varepsilon_0 \ell(Q).$$

At the same time, the planes L_Q and L_0 are close to each other due to Lemma 5.2, and so

$$\operatorname{dist}\Big((\zeta, F(\zeta)), L_Q\Big) \lesssim_{A,\tau} \varepsilon_0 \ell(Q).$$

□

Corollary 6.4. *Let θ and ε_0 be small enough. Suppose $Q \in \mathsf{Tree}$ satisfies $10B_Q \cap \Gamma \neq \varnothing$. Then for $y \in L_Q \cap 10B_Q$*

$$\operatorname{dist}(y, \Gamma) \lesssim_{A,\tau} \sqrt{\varepsilon_0}\ell(Q).$$

Proof. Let $\widetilde{F} : L_Q \to L_Q^{\perp}$ be defined in such a way that Γ is the graph of $\widetilde{F}$. This definition makes sense because $\measuredangle(L_Q, L_0) \le \theta$. Moreover, $\operatorname{Lip}(F) \lesssim \theta$ implies that $\operatorname{Lip}(\widetilde{F}) \lesssim \theta$.

Let $x \in L_Q$ be such that $(x, \widetilde{F}(x)) \in 10B_Q \cap \Gamma$. By the triangle inequality and Lemma 6.3 we have for $y \in L_Q \cap 10B_Q$

$$|\widetilde{F}(y)| \le |\widetilde{F}(y) - \widetilde{F}(x)| + |\widetilde{F}(x)| \lesssim \theta\ell(Q) + C(A,\tau)\sqrt{\varepsilon_0}\ell(Q).$$

Thus, for θ and ε_0 small enough, we have $(y, \widetilde{F}(y)) \in 11B_Q \cap \Gamma$ and we may use Lemma 6.3 once again to conclude that

$$\operatorname{dist}(y, \Gamma) \le |\widetilde{F}(y)| = \operatorname{dist}\Big((y, \widetilde{F}(y)), L_Q\Big) \lesssim_{A,\tau} \sqrt{\varepsilon_0}\ell(Q).$$

□

Recall that

$$R_{\mathsf{Far}} = \{x \in 3B_0 \,:\, \operatorname{dist}(x, L_Q) \ge \sqrt{\varepsilon_0}\ell(Q) \quad \text{for some } Q \in \mathsf{Tree}_0 \text{ s.t. } x \in 3B_Q\}.$$

Lemma 6.5. *For all $x \in 3B_0 \setminus R_{\mathsf{Far}}$*

$$\operatorname{dist}(x, \Gamma) \lesssim_{A,\tau} \sqrt{\varepsilon_0}\, d(x).$$

Proof. If $d(x) = 0$, then $x \in R_G \subset \Gamma$ and we are done. Suppose that $d(x) > 0$. Let $Q \in \mathsf{Tree}$ be such that

$$\operatorname{dist}(x, Q) + \operatorname{diam}(B_Q) \le 2d(x).$$

Fix some $z \in Q$ and note that $|z - x| \le 2d(x)$.

Let C_1 be the constant from Lemma 6.1. If we have $B(z, 2(C_1 + 2)d(x)) \subset 3B_0$, then let $P \in \mathsf{Tree}$ be the smallest cube satisfying $B(z, 2(C_1 + 2)d(x)) \subset 3B_P$; otherwise, set $P = R_0$.

In both cases we have $\ell(P) \approx d(x)$, as well as $x \in 3B_P$. Moreover, we know from Lemma 6.1 that

$$\operatorname{dist}(z, \Gamma) \le |z - x| + \operatorname{dist}(x, \Gamma) \le (2 + C_1)d(x).$$

Hence, $3B_P \cap \Gamma \neq \varnothing$ (for $P = R_0$ this is obvious, and for $P \subsetneq R_0$ it follows from the fact that $B(z, 2(C+2)d(x)) \subset 3B_P$).

The assumption $x \notin R_{\mathsf{Far}}$ gives us

$$|x - \Pi_{L_P}(x)| \leq \sqrt{\varepsilon_0}\ell(P),$$

and so $\Pi_{L_P}(x) \in 4B_P \cap L_P$. We apply Corollary 6.4 to $\Pi_{L_P}(x)$ to get

$$\operatorname{dist}(\Pi_{L_P}(x), \Gamma) \lesssim_{A,\tau} \sqrt{\varepsilon_0}\ell(P).$$

The two inequalities above and the fact that $\ell(P) \approx d(x)$ imply

$$\operatorname{dist}(x, \Gamma) \lesssim_{A,\tau} \sqrt{\varepsilon_0}\, d(x).$$

□

Lemma 6.6. *For every $x \in \Gamma$ we have $D(\Pi_0(x)) \leq d(x) \lesssim D(\Pi_0(x))$.*

Proof. The inequality $D(\Pi_0(x)) \leq d(x)$ follows directly from the definition of D (5.13).

To see that $d(x) \lesssim D(\Pi_0(x))$, let $Q \in \mathsf{Tree}$ be such that

$$\operatorname{diam}(B_Q) + \operatorname{dist}(\Pi_0(Q), \Pi_0(x)) \leq D(\Pi_0(x)) + h \tag{6.7}$$

for some small $h > 0$. Take any $y \in 3B_Q \setminus R_{\mathsf{Far}}$, then by Lemma 6.5 we have some $z \in \Gamma$ such that

$$\operatorname{dist}(y, \Gamma) = |y - z| \lesssim_{A,\tau} \sqrt{\varepsilon_0} d(y) \lesssim \sqrt{\varepsilon_0}\operatorname{diam}(B_Q) \overset{(6.7)}{\leq} D(\Pi_0(x)) + h.$$

Using the fact that $x, z \in \Gamma$, that $y \in 3B_Q$, and the inequality above, we have

$$|x-z| \leq 2|\Pi_0(x)-\Pi_0(z)| \leq 2|\Pi_0(x)-\Pi_0(y)|+2|\Pi_0(y)-\Pi_0(z)| \overset{(6.7)}{\lesssim} D(\Pi_0(x))+h,$$

and so

$$|x - y| \leq |x - z| + |z - y| \lesssim D(\Pi_0(x)) + h.$$

It follows that

$$d(x) \leq d(y) + |x - y| \lesssim \operatorname{diam}(B_Q) + D(\Pi_0(x)) + h \overset{(6.7)}{\lesssim} D(\Pi_0(x)) + h.$$

Letting $h \to 0$ ends the proof. □

6.2 Estimating the measure of LD

Lemma 6.7. *If ε_0 and τ are small enough, with $\varepsilon_0 = \varepsilon_0(\tau) \ll \tau$, then*

$$\sum_{Q\in\mathsf{LD}} \mu(Q) \lesssim \tau\mu(R_0). \tag{6.8}$$

Proof. Recall that by Lemma 4.6 we have $\mu(R_{\mathsf{Far}}) \lesssim_{A,\tau} \sqrt{\varepsilon_0}\mu(R_0)$. Hence, for ε_0 small enough we get $\mu(R_{\mathsf{Far}}) \leq \tau\mu(R_0)$, and so to show (6.8) it suffices to prove

$$\mu(R_{\mathsf{LD}}) \lesssim \tau\mu(R_0),$$

where $R_{\mathsf{LD}} = \bigcup_{Q\in\mathsf{LD}} Q \setminus R_{\mathsf{Far}}$.

We use Besicovitch covering theorem to find a countable collection of points $x_i \in R_{\mathsf{LD}}$ such that $x_i \in Q_i \setminus R_{\mathsf{Far}},\ Q_i \in \mathsf{LD}$, and

$$R_{\mathsf{LD}} \subset \bigcup_i B(x_i, r(Q_i)),$$

$$\sum_i \mathbb{1}_{B(x_i,r(Q_i))} \leq N,$$

where N is a dimensional constant.

Observe that $B(x_i, r(Q_i)) \subset 1.5B_{Q_i}$. It follows that

$$\mu(R_{\mathsf{LD}}) \leq \sum_i \mu(B(x_i, r(Q_i))) \leq \sum_i \mu(1.5B_{Q_i}) \lesssim \tau \sum_i r(Q_i)^n,$$

where the last inequality was obtained using the fact that $Q_i \in \mathsf{LD}$. Furthermore, since $x_i \notin R_{\mathsf{Far}}$ we may use Lemma 6.5 to get $\operatorname{dist}(x_i, \Gamma) \lesssim_{A,\tau} \sqrt{\varepsilon_0}\, d(x_i)$. Note also that $d(x_i) \lesssim r(Q_i)$. Hence,

$$\operatorname{dist}(x_i, \Gamma) \lesssim_{A,\tau} \sqrt{\varepsilon_0} r(Q_i).$$

So, if ε_0 is small enough, Γ passes close to the center of $B(x_i, r(Q_i))$. Since Γ is a Lipschitz graph with small Lipschitz constant we get

$$r(Q_i)^n \lesssim \mathcal{H}^n(\Gamma \cap B(x_i, r(Q_i))).$$

Thus,

$$\begin{aligned}\mu(R_{\mathsf{LD}}) &\lesssim \tau \sum_i r(Q_i)^n \lesssim \tau \sum_i \mathcal{H}^n(\Gamma \cap B(x_i, r(Q_i))) \\ &\lesssim \tau\mathcal{H}^n\Big(\Gamma \cap \bigcup_i B(x_i, r(Q_i))\Big) \leq \tau\mathcal{H}^n(\Gamma \cap 1.5B_0) \approx \tau\ell(R_0)^n.\end{aligned}$$

We have $\ell(R_0)^n \approx \mu(3B_0) \approx \mu(R_0)$ because $\Theta_\mu(3B_0) = 1$, see Remark 3.5, and R_0 is doubling. Hence,

$$\mu(R_{\mathsf{LD}}) \lesssim \tau\mu(R_0).$$

□

7 Approximating measure ν

In order to estimate the measure of high density cubes, we need to introduce a measure ν supported on Γ which will approximate μ.

7.1 Definition and properties of ν

Let $\eta < 1/1000$ be a small dimensional constant which will be fixed in the proof of Lemma 7.1 (c). For every $i \in I$ (the set of indices from Section 5.3) consider a finite collection of points $\{z_k'\}_{k\in K_i} \subset J_i$, $\#K_i \lesssim_n 1$, such that the balls $B(z_k', 0.5\eta\ell(J_i))$ cover the whole J_i. We set $K = \bigcup_i K_i$, $K_0 = \bigcup_{i\in I_0} K_i$.

For $k \in K_i$ we define

$$\begin{aligned} z_k &= f(z_k') \in \Gamma, \\ r_k &= \eta\ell(J_i), \\ B_k &= B(z_k, r_k). \end{aligned}$$

The following lemma collects basic properties of B_k.

Lemma 7.1. *We have the following:*

(a) For $k \in K_i$

$$\Pi_0(3B_k) \subset 2J_i. \tag{7.1}$$

(b) For $k \in K$ there exist at most $C = C(n)$ indices $k' \in K$ such that $\Pi_0(3B_k) \cap \Pi_0(3B_{k'}) \neq \varnothing$ (in particular, there are at most C indices $k' \in K$ such that $3B_k \cap 3B_{k'} \neq \varnothing$). Moreover, for all such k' we have

$$r_k \approx r_{k'}. \tag{7.2}$$

(c) For $k \in K$ and $x \in 3B_k$ we have

$$r_k \le d(x) \le \eta^{-3/2} r_k. \tag{7.3}$$

(d) For $k \in K_0$

$$3B_k \subset 2.3B_0. \tag{7.4}$$

(e) For $k \notin K_0$

$$r_k \approx |z_k - z_0| \gtrsim \ell(R_0). \tag{7.5}$$

If additionally $3B_k \cap 3B_0 \neq \varnothing$, then

$$r_k \approx \ell(R_0). \tag{7.6}$$

(f) Finally,

$$\bigcup_{k\in K} B_k \cap \Gamma = \bigcup_{k\in K} 3B_k \cap \Gamma = \Gamma \setminus R_G. \tag{7.7}$$

Proof. (a) follows immediately by the definition of B_k.

Concerning (b), suppose $k \in K_i$ and $\Pi_0(3B_k) \cap \Pi_0(3B_{k'}) \neq \varnothing$ for some $k' \in K_{i'}$. By (a) we know that $2J_i \cap 2J_{i'} \neq \varnothing$, and there are at most N such indices i', see Lemma 5.7) (c). Since $\# K_i \lesssim_n 1$ by the definition, we get that there are at most $C(n,N)$ indices k' satisfying $\Pi_0(3B_k) \cap \Pi_0(3B_{k'}) \neq \varnothing$. The estimate $r_k \approx r_{k'}$ follows by Lemma 5.7 (b).

To prove (c), recall that z_k is the center of B_k. By the definition, $\Pi_0(z_k) \in J_i$ for $i \in I$ such that $r_k = \eta \ell(J_i)$. Lemma 5.7 (a) gives us $D(\Pi_0(z_k)) \approx_n \ell(J_i)$. Hence, by Lemma 6.6 we get

$$d(z_k) \approx \ell(J_i) = \eta^{-1} r_k.$$

Now, for an arbitrary $x \in 3B_k$ we have by the 1-Lipschitz property of function d that

$$|d(x) - d(z_k)| \leq |x - z_k| \leq 3r_k,$$

Since $d(z_k) \approx \eta^{-1} r_k$, choosing η small enough we arrive at $d(x) \approx \eta^{-1} r_k$, and so for η small enough $r_k \leq d(x) \leq \eta^{-3/2} r_k$.

Concerning (d), let $i \in I_0$ be such that $\Pi_0(z_k) \in J_i$. We know by Lemma 5.12 that $B(f(z_{J_i}), 2\,\mathrm{diam}(J_i)) \subset 2.3B_0$. Since $3B_k \subset B(f(z_{J_i}), 2\,\mathrm{diam}(J_i))$, we get $3B_k \subset 2.3B_0$.

To show (e), let $k \in K \setminus K_0$. Let $i \in I \setminus I_0$ be such that $k \in K_i$, i.e. $\Pi_0(z_k) \in J_i$. By (c) and Lemma 6.6 we have $d(z_k) \approx D(\Pi_0(z_k)) \approx r_k$. At the same time, $|\Pi_0(z_k) - z_0| \approx \ell(J_i) \gtrsim \ell(R_0)$ by Lemma 5.8 (b). Recall also that $\|F\|_\infty \lesssim \theta \ell(R_0)$ due to Lipschitz continuity and the fact that $\mathrm{supp}(F) \subset 1.9B_0$, see Lemma 5.11. It follows that

$$|z_k - z_0| \leq |z_k - \Pi_0(z_k)| + |\Pi_0(z_k) - z_0| \lesssim |F(z_k)| + \ell(J_i) \lesssim \theta \ell(R_0) + \ell(J_i) \lesssim \ell(J_i),$$

and on the other hand

$$\begin{aligned} |z_k - z_0| \geq |\Pi_0(z_k) - z_0| - |z_k - \Pi_0(z_k)| &\geq C\,\ell(J_i) - |F(z_k)| \geq C\,\ell(J_i) - C'\,\theta\ell(R_0) \\ &\geq C\,\ell(J_i) - C''\,\theta\ell(J_i) \gtrsim \ell(J_i), \end{aligned}$$

for θ small enough. Hence, $|z_k - z_0| \approx \ell(J_i) \approx r_k \gtrsim \ell(R_0)$.

Now, assume also $3B_k \cap 3B_0 \neq \varnothing$, and suppose $x \in 3B_0 \cap 3B_k$. We have $\Pi_0(x) \in 2J_i$ by (7.1). Clearly, $D(\Pi_0(x)) \leq d(x) \lesssim \ell(R_0)$, and so $r_k \approx \ell(J_i) \approx D(\Pi_0(x)) \lesssim \ell(R_0)$ by Lemma 5.7 (a).

Finally, to see (7.7) note that by the definition of B_k and by (a) we have

$$f(J_i) \subset \bigcup_{k \in K_i} B_k \cap \Gamma \subset \bigcup_{k \in K_i} 3B_k \cap \Gamma \subset f(2J_i).$$

Together with Lemma 5.7 (d) this implies (7.7). □

Since η is a dimensional constant, we will usually not mention dependence on it in our further estimates.

Due to bounded superposition of $3B_k$ (Lemma 7.1 (b)) we may define a partition of unity $\{h_k\}_{k\in K}$ such that $0 \le h_k \le 1$, $\operatorname{supp} h_k \subset 3B_k$, $\operatorname{Lip}(h_k) \approx \ell(J_i)^{-1}$, and

$$h = \sum_{k\in K} h_k \equiv 1 \quad \text{on} \ \bigcup_{k\in K} 2B_k. \tag{7.8}$$

Again, by the bounded superposition of $3B_k$ we may assume

$$h_k(x) \approx 1, \quad x \in B_k. \tag{7.9}$$

Recall that $\sigma = \mathcal{H}^n|_{\Gamma}$, and that c_0 is a constant minimizing $\alpha_\mu(3B_0)$. We set

$$c_k = \begin{cases} \frac{\int h_k \ d\mu}{\int h_k \ d\sigma} & \text{for } k \in K_0, \\ c_0 & \text{for } k \notin K_0. \end{cases} \tag{7.10}$$

We define the approximating measure as

$$\nu = \mu|_{R_G} + \sum_k c_k h_k \sigma. \tag{7.11}$$

Note that, since $\mu|_{R_G} \ll \sigma$ by Lemma 5.5, we also have $\nu \ll \sigma$. To simplify the notation, we introduce

$$\begin{aligned} \mu_G &= \mu|_{R_G}, \\ \mu_B &= \mu - \mu_G, \\ \nu_B &= \nu - \mu_G = \sum_k c_k h_k d\sigma. \end{aligned}$$

Note that by Lemma 7.1 (d), (7.7), and the fact that $R_G \subset B_0$, we get

$$\Gamma \setminus (2.3B_0) = L_0 \setminus (2.3B_0) \subset \Gamma \cap \bigcup_{k\notin K_0} B_k,$$

and so by the definition of ν we have

$$\nu|_{(2.3B_0)^c} = c_0 \mathcal{H}^n|_{L_0\setminus(2.3B_0)}. \tag{7.12}$$

Lemma 7.2. *For each $k \in K_0$ there exists $P_k \in \mathsf{Tree}$ such that $3B_k \subset 2.5B_{P_k}$, and $\ell(P_k) \approx r_k$.*

Proof. We know by (7.4) that $3B_k \subset 2.3B_0$. Thus, we may define P_k as the smallest cube in Tree such that $3B_k \subset 2.5B_{P_k}$. We have $\ell(P_k) \approx r_k$ due to (7.3). □

We will write for $k \in K_0$

$$\begin{aligned} \widetilde{B}_k &= 2.5 B_{P_k}, \\ \widetilde{c}_k &= c_{P_k}, \\ L_k &= L_{P_k}, \end{aligned} \tag{7.13}$$

and for $k \notin K_0$ set $\widetilde{B}_k = 2.5B_0$, $\widetilde{c}_k = c_0$, and $L_k = L_0$.

Note that for every $k \in K$

$$\operatorname{dist}(z_k, L_k) \lesssim_{A,\tau} \sqrt{\varepsilon_0} r_k. \tag{7.14}$$

Indeed, for $k \in K_0$, it follows by Lemma 6.3 applied to z_k and P_k. For $k \notin K_0$, but such that $z_k \in 1.9B_0$, again it follows by Lemma 6.3 applied to z_k and R_0. Finally, for $k \notin K_0$ such that $z_k \notin 1.9B_0$ this is trivially true because $\Gamma \setminus (1.9B_0) = L_0 \setminus (1.9B_0)$, and so $\operatorname{dist}(z_k, L_0) = 0$.

Lemma 7.3. *For $k \in K$ the set $\Gamma \cap 3B_k$ is a Lipschitz graph over L_k, with a Lipschitz constant at most $C\sqrt{\varepsilon_0}$.*

Proof. Suppose $k \in K_0$, i.e. that $k \in K_i$ for some $i \in I_0$. We know by (5.17) that $f(15J_i)$ is a $C\sqrt{\varepsilon_0}$-Lipschitz graph over L_{Q_i} (recall that F_i is an affine function whose graph is L_{Q_i}).

At the same time, since P_k satisfies $\operatorname{dist}(P_k, Q_i) \lesssim \ell(Q_i)$ (see (6.3) and the definition of P_k) and $\ell(P_k) \approx r_k \approx \ell(Q_i)$, we can apply Lemma 5.2 to get $\measuredangle(L_{Q_i}, L_k) \lesssim_{A,\tau} \varepsilon_0$. It follows that $f(15J_i)$ is a $C\sqrt{\varepsilon_0}$-Lipschitz graph over L_k. The same is true for $k \notin K_0$: since $L_k = L_0 = \operatorname{graph}(F_i)$, it follows immediately by (5.17). We conclude by noting that

$$\Gamma \cap 3B_k \overset{(7.1)}{\subset} f(15J_i).$$

$\square$

Lemma 7.3 and (7.14) imply that for every $k \in K$

$$F_{3B_k}(\sigma, \mathcal{H}^n|_{L_k}) \lesssim_{A,\tau} \sqrt{\varepsilon_0} r_k^{n+1}. \tag{7.15}$$

Furthermore, by (4.16) we have

$$F_{\widetilde{B}_k}(\mu, \widetilde{c}_k \mathcal{H}^n|_{L_k}) \lesssim_{A,\tau} \varepsilon_0 r_k^{n+1}. \tag{7.16}$$

Lemma 7.4. *For $k \in K$ we have*

$$|c_k - \widetilde{c}_k| \lesssim_{A,\tau} \sqrt{\varepsilon_0}. \tag{7.17}$$

Proof. For $k \notin K_0$ we have $c_k = c_0 = \widetilde{c}_k$, so the claim is trivially true. Suppose $k \in K_0$. Recall that $h_k \approx 1$ in B_k (7.9), $\mathrm{Lip}(h_k) \approx r_k^{-1}$, and $\widetilde{c}_k \approx_{A,\tau} 1$ by (4.17). It follows that

$$\begin{aligned}|c_k - \widetilde{c}_k|\, r_k^n &\overset{(7.9)}{\approx} |c_k - \widetilde{c}_k| \int h_k \, d\sigma = \left| \int h_k \, d\mu - \int h_k \widetilde{c}_k \, d\sigma \right| \\ &\leq \left| \int h_k \, d\mu - \int h_k \widetilde{c}_k \, d\mathcal{H}^n|_{L_k} \right| + \widetilde{c}_k \left| \int h_k \, d\mathcal{H}^n|_{L_k} - \int h_k \, d\sigma \right| \\ &\leq F_{\widetilde{B}_k}(\mu, \widetilde{c}_k \mathcal{H}^n|_{L_k}) r_k^{-1} + \widetilde{c}_k F_{3B_k}(\sigma, \mathcal{H}^n|_{L_k}) r_k^{-1} \overset{(7.15),(7.16)}{\lesssim_{A,\tau}} \sqrt{\varepsilon_0}\, r_k^n.\end{aligned}$$

□

An immediate corollary of (4.17) and the lemma above is that for $k \in K$

$$c_k \approx_{A,\tau} 1. \tag{7.18}$$

Lemma 7.5. *The measure ν is n-AD-regular, that is, for $x \in \Gamma,\ r > 0$*

$$\nu(B(x,r)) \approx_{A,\tau} r^n$$

Proof. We know by (7.7), the definition of h (7.8), and (7.18) that

$$d\sigma|_{\Gamma \setminus R_G} = \sum_k h_k d\sigma \approx_{A,\tau} \sum_k c_k h_k d\sigma.$$

Together with Lemma 5.5 this gives

$$d\nu = d\mu_G + \sum_k c_k h_k d\sigma \approx_{A,\tau} d\sigma.$$

□

Lemma 7.6. *If $k, j \in K$ satisfy $3B_k \cap 3B_j \neq \varnothing$, then*

$$|c_k - c_j| \lesssim_{A,\tau} \sqrt{\varepsilon_0}.$$

Proof. If $3B_k \cap 3B_j \neq \varnothing$, then by (7.1) and Lemma 5.7 (b) it follows that

$$r_k \approx r_j.$$

Now, since $3B_k \cap 3B_j \neq \varnothing$ and $r_k \approx r_j$, we get that there exists $R \in \mathsf{Tree}$ such that $2.5B_R \supset \widetilde{B}_k \cup \widetilde{B}_j$ and $\ell(R) \approx r_k$. Hence, we may use Lemma 5.4 and Lemma 7.4 to obtain

$$|c_k - c_j| \leq |c_k - \widetilde{c}_k| + |\widetilde{c}_k - c_R| + |c_R - \widetilde{c}_j| + |\widetilde{c}_j - c_j| \lesssim_{A,\tau} \sqrt{\varepsilon_0}.$$

□

7.2 ν approximates μ well

Lemma 7.7. *We have*

$$3B_0 \setminus (R_G \cup R_{\mathsf{Far}}) \subset \bigcup_{k\in K} 2B_k.$$

In consequence, for every $x \in 3B_0 \setminus (R_G \cup R_{\mathsf{Far}})$ we have $h(x) = 1$.

Proof. Let $x \in 3B_0 \setminus (R_G \cup R_{\mathsf{Far}})$. We will find $k \in K$ such that $x \in 2B_k$.

By Lemma 6.5 we have $y \in \Gamma$ such that

$$|x - y| \lesssim_{A,\tau} \sqrt{\varepsilon_0}\, d(x). \tag{7.19}$$

Since $x \notin R_G$, we have $d(x) > 0$. Moreover, since $d(x) \le d(y) + |x-y| \le d(y) + 0.5d(x)$, we get that $0 < d(x) \le 2d(y)$. In particular, $y \notin R_G$ and by (7.7) there exists $k \in K$ such that $y \in B_k \cap \Gamma$. It follows by Lemma 7.1 (c) that

$$d(x) \le 2d(y) \approx r_k.$$

Together with (7.19) this gives $|x-y| \le r_k/2$, for ε_0 small enough. Since $y \in B_k$, we get that $x \in 2B_k$.

□

Lemma 7.8. *Suppose that $x \in 2.5B_0$, $r \ge Cd(x)$ for some $C > 0$, and that $B(x,r) \subset 3B_0$. Then,*

$$F_{B(x,r)}(\mu_B, h\mu) \lesssim_{A,C} \varepsilon_0^{1/4} r^{n+1}.$$

Proof. Since $B(x,r) \subset 3B_0$, and $r \ge Cd(x)$, there exists a cube $Q \in \mathsf{Tree}$ such that $B(x,r) \subset 3B_Q$ and $\ell(Q) \approx_C r$. In consequence, using the properties of Tree yields

$$\mu(B(x,r) \cap R_{\mathsf{Far}}) \le \mu(3B_Q \cap R_{\mathsf{Far}}) \overset{(4.8)}{\le} \varepsilon_0^{1/4}\mu(3B_Q) \overset{(4.5)}{\lesssim_{A,C}} \varepsilon_0^{1/4} r^n.$$

Thus, given any $\phi \in \mathrm{Lip}_1(B(x,r))$ we have

$$\left| \int \phi \, d\mu_B - \int \phi \, d\mu_B|_{(R_{\mathsf{Far}})^c} \right| \le r\mu(B(x,r) \cap R_{\mathsf{Far}}) \lesssim_{A,C} \varepsilon_0^{1/4} r^{n+1},$$

and so $F_{B(x,r)}(\mu_B, \mu_B|_{(R_{\mathsf{Far}})^c}) \lesssim_{A,C} \varepsilon_0^{1/4} r^{n+1}$. Similarly, $F_{B(x,r)}(h\mu, h\mu|_{(R_{\mathsf{Far}})^c}) \lesssim_{A,C} \varepsilon_0^{1/4} r^{n+1}$.

Now, observe that $h\mu = h\mu_B$ by the definition of h. Moreover, inside $B(x,r)$ we have

$$h\mu_B|_{(R_{\mathsf{Far}})^c} = \mu_B|_{(R_{\mathsf{Far}})^c}$$

because $h \equiv 1$ on $3B_0 \setminus (R_G \cup R_{\mathsf{Far}})$ by Lemma 7.7. Thus, the triangle inequality yields

$$F_{B(x,r)}(\mu_B, h\mu) \le F_{B(x,r)}(\mu_B, h\mu|_{(R_{\mathsf{Far}})^c}) + F_{B(x,r)}(h\mu, h\mu|_{(R_{\mathsf{Far}})^c}) \lesssim_{A,C} \varepsilon_0^{1/4} r^{n+1}.$$

□

Lemma 7.9. *If $x \in 2.5B_0$ and $r > 0$ satisfy $B(x,r) \subset 2.5B_0$, then*

$$F_{B(x,r)}(\nu_B, h\mu) \lesssim_{A,\tau} \sqrt{\varepsilon_0} \sum_{3B_k \cap B(x,r) \neq \varnothing} r_k^{n+1}. \tag{7.20}$$

Proof. Since $\nu_B = \sum_k c_k h_k \sigma$, our aim is estimating $F_{B(x,r)}(\sum_k c_k h_k \sigma, h\mu)$. Set

$$K(x,r) = \{k \in K \ : \ 3B_k \cap B(x,r) \neq \varnothing\}.$$

First, we will deal with $k \in K(x,r) \setminus K_0$. For such k by (7.6) we have

$$r_k \approx \ell(R_0). \tag{7.21}$$

In particular, $r \lesssim r_k$, and so given $\phi \in \mathrm{Lip}_1(B(x,r))$ we have $\mathrm{Lip}(\phi h_k) \lesssim 1$, $\mathrm{supp}(\phi h_k) \subset B(x,r) \cap 3B_k \subset 2.5B_0$.
Moreover, recall that

$$F_{2.5B_0}(\mu, c_0 \mathcal{H}^n|_{L_0}) \overset{(4.16),(4.5)}{\lesssim_A} \varepsilon_0 \ell(R_0)^{n+1} \overset{(7.21)}{\approx} \varepsilon_0 r_k^{n+1}. \tag{7.22}$$

In consequence, since $c_k = c_0$ by (7.10), we have for any $\phi \in \mathrm{Lip}_1(B(x,r))$

$$\begin{aligned}
&\left| \sum_{k \in K(x,r) \setminus K_0} \left(\int \phi h_k c_0 \, d\sigma - \int \phi h_k \, d\mu \right) \right| \\
&\le \sum_{k \in K(x,r) \setminus K_0} \left(\left| \int \phi h_k c_0 \, d\mathcal{H}^n|_{L_0} - \int \phi h_k \, d\mu \right| + c_0 \left| \int \phi h_k \, d\mathcal{H}^n|_{L_0} - \int \phi h_k \, d\sigma \right| \right) \\
&\qquad \le \sum_{k \in K(x,r) \setminus K_0} \left(F_{2.5B_0}(\mu, c_0 \mathcal{H}^n|_{L_0}) + c_0 F_{3B_k}(\sigma, \mathcal{H}^n|_{L_0}) \right) \\
&\qquad\qquad \overset{(7.22),(7.15),(4.17)}{\lesssim_{A,\tau}} \sum_{k \in K(x,r) \setminus K_0} \sqrt{\varepsilon_0} r_k^{n+1}.
\end{aligned}$$

Now, we turn our attention to $k \in K_0(x,r) = K(x,r) \cap K_0$. For any $\phi \in \mathrm{Lip}_1(B(x,r))$ we have

$$\begin{aligned}
&\left| \sum_{k \in K_0(x,r)} \left(\int \phi c_k h_k \, d\sigma - \int \phi h_k \, d\mu \right) \right| \\
&\qquad \le \left| \sum_{k \in K_0(x,r)} \left(\int (\phi - \phi(z_k)) c_k h_k \, d\sigma - \int (\phi - \phi(z_k)) h_k \, d\mu \right) \right| \\
&\qquad\qquad + \left| \sum_{k \in K_0(x,r)} \phi(z_k) \left(\int c_k h_k \, d\sigma - \int h_k \, d\mu \right) \right| =: I_1 + I_2.
\end{aligned}$$

We start by estimating I_1. Observe that setting $\Phi_k = (\phi - \phi(z_k))h_k$ we have $\mathrm{Lip}(\Phi_k) \lesssim 1$ and $\mathrm{supp}\,\Phi_k \subset 3B_k$. Hence,

$$\begin{aligned} I_1 &= \left| \sum_{k\in K_0(x,r)} \left(\int c_k \Phi_k \, d\sigma - \int \Phi_k \, d\mu \right) \right| \\ &\overset{(7.15),(7.18)}{\le} \sum_{k\in K_0(x,r)} \left(\left| \int c_k \Phi_k d\mathcal{H}^n|_{L_k} - \int \Phi_k \, d\mu \right| + C(A,\tau)\sqrt{\varepsilon_0} r_k^{n+1} \right) \\ &\overset{(7.17)}{\le} \sum_{k\in K_0(x,r)} \left(\left| \int \widetilde{c}_k \Phi_k d\mathcal{H}^n|_{L_k} - \int \Phi_k \, d\mu \right| + C(A,\tau)\sqrt{\varepsilon_0} r_k^{n+1} \right) \\ &\overset{(7.16)}{\lesssim_{A,\tau}} \sum_{k\in K_0(x,r)} \sqrt{\varepsilon_0} r_k^{n+1}. \end{aligned}$$

Concerning I_2, note that for $k \in K_0(x,r)$ we have by the definition of c_k (7.10)

$$\int c_k h_k \, d\sigma - \int h_k \, d\mu = 0,$$

and so

$$I_2 = 0.$$

Putting together the estimates for $k \in K(x,r) \setminus K_0$ and for $k \in K_0(x,r)$, and taking supremum over $\phi \in \mathrm{Lip}_1(B(x,r))$, we finally get

$$F_{B(x,r)}(\nu_B, h\mu) \lesssim_{A,\tau} \sqrt{\varepsilon_0} \sum_{k\in K(x,r)} r_k^{n+1}.$$

□

The previous two lemmas, and the fact that $F_B(\nu,\mu) = F_B(\nu_B,\mu_B)$, imply the following:

Lemma 7.10. *For $x \in 2.5B_0$ and $r \gtrsim d(x)$ such that $B(x,r) \subset 2.5B_0$ we have*

$$F_{B(x,r)}(\nu,\mu) \lesssim_{A,\tau} \varepsilon_0^{1/4} r^{n+1} + \sqrt{\varepsilon_0} \sum_{3B_k\cap B(x,r)\neq\varnothing} r_k^{n+1}. \tag{7.23}$$

In particular, we have

$$F_{2.5B_0}(\nu,\mu) \lesssim_{A,\tau} \varepsilon_0^{1/4} \ell(R_0)^{n+1}. \tag{7.24}$$

Lemma 7.11. *For $x \in \Gamma$ and $r \gtrsim \ell(R_0)$ such that $B(x,r) \cap 3B_0 \neq \varnothing$ we have*

$$F_{B(x,r)}(\nu, c_0\mathcal{H}^n|_{L_0}) \lesssim_{A,\tau} \varepsilon_0^{1/4} r \ell(R_0)^n. \tag{7.25}$$

Proof. Recall that by (7.12) we have

$$\nu|_{(2.3B_0)^c} = c_0\mathcal{H}^n|_{L_0\cap(2.3B_0)^c}.$$

To take advantage of this equality, we define an auxiliary function ψ such that $\psi \equiv 1$ on $2.3B_0$, $\mathrm{supp}(\psi) \subset 2.5B_0$, and $\mathrm{Lip}(\psi) \lesssim \ell(R_0)^{-1}$. Then,

$$\begin{aligned}
&\left|\int \psi\, d\nu - c_0 \int \psi\, d\mathcal{H}^n|_{L_0}\right| \\
&\qquad \overset{(7.24)}{\lesssim} \left|\int \psi\, d\mu - c_0 \int \psi\, d\mathcal{H}^n|_{L_0}\right| + \varepsilon_0^{1/4}\ell(R_0)^n \\
&\qquad\quad \overset{(4.16)}{\le} C(A,\tau)\varepsilon_0\ell(R_0)^n + \varepsilon_0^{1/4}\ell(R_0)^n \lesssim \varepsilon_0^{1/4}\ell(R_0)^n. \qquad (7.26)
\end{aligned}$$

Recall that $z_0 = z_{R_0}$. It follows that for $\phi \in \mathrm{Lip}_1(B(x,r))$ we have

$$\begin{aligned}
&\left|\int \phi\; (d\nu - c_0 d\mathcal{H}^n|_{L_0})\right| \\
&\quad = \left|\int ((\phi-\phi(z_0))\psi + \phi(z_0)\psi + \phi(1-\psi))\; (d\nu - c_0 d\mathcal{H}^n|_{L_0})\right| \\
&\overset{(7.12)}{\le} F_{2.5B_0}(\nu,\mu) + F_{2.5B_0}(c_0\mathcal{H}^n|_{L_0},\mu) + |\phi(z_0)| \left|\int \psi\; (d\nu - c_0 d\mathcal{H}^n|_{L_0})\right| + 0 \\
&\overset{(7.24),(4.16)}{\lesssim_{A,\tau}} \varepsilon_0^{1/4}\ell(R_0)^{n+1} + \varepsilon_0\ell(R_0)^{n+1} + |\phi(z_0)| \left|\int \psi\; (d\nu - c_0 d\mathcal{H}^n|_{L_0})\right| \\
&\qquad\qquad \overset{(7.26)}{\lesssim} \varepsilon_0^{1/4}\ell(R_0)^{n+1} + r\varepsilon_0^{1/4}\ell(R_0)^n \lesssim \varepsilon_0^{1/4} r\ell(R_0)^n.
\end{aligned}$$

□

8 Small measure of cubes from HD

For brevity of notation let us denote by $\Pi_*\nu$ the image measure of ν by Π_0, that is the measure such that $\Pi_*\nu(A) = \nu(\Pi_0^{-1}(A))$. Set

$$f = \frac{d\Pi_*\nu}{d\mathcal{H}^n|_{L_0}}.$$

The key estimate necessary to bound the measure of high density cubes is the following.

Lemma 8.1. *We have*

$$\|f - c_0\|^2_{L^2(\mathcal{H}^n|_{L_0})} \lesssim_{A,\tau} \varepsilon_0^{1/8}\mu(R_0). \qquad (8.1)$$

We postpone the proof of the above lemma to the next subsection. Let us show now how we can use it to estimate the measure of cubes in HD.

Lemma 8.2. *We have*

$$\sum_{Q\in\mathsf{HD}} \mu(Q) \lesssim_{A,\tau} \varepsilon_0^{1/8}\mu(R_0). \tag{8.2}$$

Proof. Recall that by Lemma 4.6 we have $\mu(R_{\mathsf{Far}}) \lesssim_{A,\tau} \sqrt{\varepsilon_0}\mu(R_0)$. Thus, to show (8.2) it suffices to prove

$$\mu(R_{\mathsf{HD}}) \lesssim \varepsilon_0^{1/8}\mu(R_0),$$

where $R_{\mathsf{HD}} = \bigcup_{Q\in\mathsf{HD}} Q \setminus R_{\mathsf{Far}}$.

For every $x \in R_{\mathsf{HD}}$ we define $B_x = B(x, r(Q_x)/100)$, where $Q_x \in \mathsf{HD}$ is such that $x \in Q_x$. We use the $5r$-covering theorem to choose $\{x_j\}_{j\in J}$ such that all B_{x_j} are pairwise disjoint and $\bigcup_j 5B_{x_j}$ covers $\bigcup_{x\in R_{\mathsf{HD}}} B_x$. Observe that $5B_{x_j} \subset 3B_{Q_{x_j}}$, and so by (4.5)

$$\mu(5B_{x_j}) \lesssim_A r(B_{x_j})^n. \tag{8.3}$$

For every j set $B_j = \frac{1}{2}B_{x_j}, Q_j = Q_{x_j}$, and let $P_j \in \mathsf{Tree}$ be the parent of Q_j. We have $\ell(P_j) \approx \ell(Q_j) \approx r(B_j)$. Since $x_j \notin R_{\mathsf{Far}}$, we can use Lemma 6.5 to obtain

$$\operatorname{dist}(x_j, \Gamma) \lesssim_{A,\tau} \sqrt{\varepsilon_0}d(x_j) \lesssim \sqrt{\varepsilon_0}\ell(P_j) \approx_{A,\tau} \sqrt{\varepsilon_0}r(B_j).$$

Since $2B_j$ are disjoint, the centers of B_j are close to Γ, and Γ is a graph of function F with $\mathrm{Lip}(F) \lesssim \theta \ll 1$, it follows that $\Pi_0(B_j)$ are disjoint as well.

We use the above to get

$$\mu(R_{\mathsf{HD}}) \le \sum_{j\in J} \mu(5B_{x_j}) \overset{(8.3)}{\lesssim_A} \sum_{j\in J} r(B_{x_j})^n \approx \sum_{j\in J} \mathcal{H}^n(\Pi_0(B_j)) = \mathcal{H}^n\Big(\bigcup_{j\in J} \Pi_0(B_j)\Big). \tag{8.4}$$

We claim that

$$\bigcup_{j\in J} \Pi_0(B_j) \subset \mathcal{BM}, \tag{8.5}$$

where

$$\mathcal{BM} = \{x \in L_0 \ :\ \mathcal{M}(f - c_0) > 1\},$$

and $\mathcal{M}$ is the Hardy-Littlewood maximal function on L_0. $\mathcal{BM}$ stands for "big $\mathcal{M}$". Before we prove (8.5), note that due to the weak type $(2,2)$ estimate for $\mathcal{M}$ we have

$$\mathcal{H}^n(\mathcal{BM}) \lesssim \|f - c_0\|^2_{L^2(\mathcal{H}^n|_{L_0})}.$$

Putting this together with (8.4), (8.5), and our key estimate from Lemma 8.1, we get that

$$\mu(R_{\mathsf{HD}}) \lesssim_{A,\tau} \varepsilon_0^{1/8}\mu(R_0).$$

Therefore, all that remains is to show (8.5).

Let $j \in J,\ y \in \Pi_0(B_j)$. Since $|y - \Pi_0(x_j)| \le r(B_j) \le r(B_{Q_j})$ and $\Pi_0(x_j) \in \Pi_0(B_{Q_j})$, we have $B(y, 25r(B_{Q_j})) \supset \Pi_0(10B_{Q_j})$. Clearly, for some $C = C(n) > 0$

$$\begin{aligned}\mathcal{M}(f - c_0)(y) &\ge \frac{C}{r(B_{Q_j})^n}\Pi_*\nu(B(y, 25r(B_{Q_j}))) - c_0 \\ &\ge \frac{C}{r(B_{Q_j})^n}\Pi_*\nu(\Pi_0(10B_{Q_j})) - c_0 \ge \frac{C}{r(B_{Q_j})^n}\nu(10B_{Q_j}) - c_0. \quad (8.6)\end{aligned}$$

Recall that by (II.3.8) and Remark 3.5 we have

$$c_0 \lesssim 1.$$

Thus, if we show that $\nu(10B_{Q_j}) \gtrsim Ar(B_{Q_j})^n$, for A big enough we will have $\mathcal{M}(f - c_0)(y) > 1$, and so we will be done.

Let us define

$$\lambda(z) = (r(10B_{Q_j}) - |z - z_{Q_j}|)_+.$$

Note that λ is 1-Lipschitz and that $\mathrm{supp}(\lambda) \subset 10B_{Q_j} \subset 2.5B_0$. Moreover,

$$7r(B_{Q_j})\mathbb{1}_{3B_{Q_j}} \le \lambda \le 10r(B_{Q_j})\mathbb{1}_{10B_{Q_j}}.$$

Note that $r(B_{Q_j}) \gtrsim d(z_{Q_j})$. We get that

$$\begin{aligned}r(B_{Q_j})\nu(10B_{Q_j}) &\gtrsim \int \lambda(z)\, d\nu(z) \\ &\overset{(7.23)}{\ge} \int \lambda(z)\, d\mu(z) - C(A,\tau)\Big(\varepsilon_0^{1/4} r(B_{Q_j})^{n+1} + \varepsilon_0^{1/2} \sum_{3B_k \cap 10B_{Q_j} \neq \varnothing} r_k^{n+1}\Big) \\ &\ge\ 7r(B_{Q_j})\mu(3B_{Q_j}) - C(A,\tau)\Big(\varepsilon_0^{1/4} r(B_{Q_j})^{n+1} + \varepsilon_0^{1/2} \sum_{3B_k \cap 10B_{Q_j} \neq \varnothing} r_k^{n+1}\Big).\end{aligned}$$

Note that for all k such that $3B_k \cap 10B_{Q_j} \neq \varnothing$ we have $r_k \lesssim_{A,\tau} r(B_{Q_j})$. Indeed, for $x \in 10B_{Q_j}$ it holds that $d(x) \lesssim_{A,\tau} r(B_{Q_j})$, and for $x \in 3B_k$ we have $r_k \le d(x)$ by Lemma 7.1 (c). Moreover, since the balls $\Pi_0(3B_k)$ are of bounded intersection by Lemma 7.1 (b), we get

$$\sum_{3B_k \cap 10B_{Q_j} \neq \varnothing} r_k^n \le \sum_{\Pi_0(3B_k) \cap \Pi_0(10B_{Q_j}) \neq \varnothing} r_k^n \lesssim_{A,\tau} r(B_{Q_j})^n.$$

Hence, using the above and the fact that $Q_j \in \mathsf{HD}$

$$\begin{aligned}r(B_{Q_j})\nu(10B_{Q_j}) &\gtrsim r(B_{Q_j})\mu(3B_{Q_j}) - C(A,\tau)\varepsilon_0^{1/4} r(B_{Q_j})^{n+1} \\ &\gtrsim Ar(B_{Q_j})^{n+1} - C(A,\tau)\varepsilon_0^{1/4} r(B_{Q_j})^{n+1} \gtrsim Ar(B_{Q_j})^{n+1},\end{aligned}$$

for ε_0 small enough. Thus, $\nu(10B_{Q_j}) \gtrsim Ar(B_{Q_j})^n$ and by (8.6) we get

$$\mathcal{M}(f - c_0)(y) > 1.$$

□

8.1 Λ-estimates

The aim of this section is to prove the crucial estimate from Lemma 8.1, i.e.

$$\|f - c_0\|^2_{L^2(\mathcal{H}^n|_{L_0})} \lesssim_{A,\tau} \varepsilon_0^{1/8}\mu(R_0). \tag{8.7}$$

From now on we will denote by $\phi : \mathbb{R}^d \to \mathbb{R}$ a radial C^∞ function such that $\phi \equiv 1$ on $B(0, 1/2)$, $\mathrm{supp}(\phi) \subset B(0, 1)$, and

$$\phi_r(x) = r^{-n}\phi\left(\frac{x}{r}\right).$$

We also set

$$\psi_r(x) = \phi_r(x) - \phi_{2r}(x).$$

A classical result of harmonic analysis (see [Ste93, Sections I.6.3, I.8.23]) states that

$$\|f - c_0\|^2_{L^2(\mathcal{H}^n|_{L_0})} \approx \int_{L_0}\int_0^\infty |\psi_r * \Pi_*\nu(z)|^2 \frac{dr}{r}\, d\mathcal{H}^n(z), \tag{8.8}$$

and so we will work with the latter expression.

For $r > 0$ and $x \in \mathbb{R}^d$ let us define

$$\widetilde{\psi}_r(x) = \psi_r \circ \Pi_0(x) \cdot \phi\left(\frac{x}{5r}\right).$$

Given a measure λ on $\mathbb{R}^d$ we set

$$\Lambda_\lambda(x, r) = \left|\widetilde{\psi}_r * \lambda(x)\right|.$$

Lemma 8.3. *We have for all $x \in \Gamma$*

$$\Lambda_\nu(x, r) = |\psi_r * \Pi_*\nu(\Pi_0(x))|. \tag{8.9}$$

Proof. By the definition of Λ_ν, it suffices to show that for all $x, y \in \Gamma$ we have

$$\widetilde{\psi}_r(x - y) = \psi_r(\Pi_0(x) - \Pi_0(y)).$$

Hence, by the definition of $\widetilde{\psi}_r$, we need to check that $\phi((5r)^{-1}(x - y)) = 1$ whenever $\psi_r(\Pi_0(x) - \Pi_0(y)) \neq 0$.

Since $\mathrm{supp}(\psi_r) \subset B(0, 2r)$, we get that $|\Pi_0(x) - \Pi_0(y)| \leq 2r$. Thus, due to the fact that Γ is a $C\theta$-Lipschitz graph, we have

$$|x - y| \leq 2(1 + C\theta)r \leq \frac{5}{2}r.$$

Hence, $y \in B(x, 5r/2)$, which gives $\phi((5r)^{-1}(x - y)) = 1$. □

The lemma above and the fact that Π_0 is bilipschitz between Γ and L_0 imply that

$$\int_{L_0}\int_0^\infty |\psi_r * \Pi_*\nu(z)|^2 \frac{dr}{r}\, d\mathcal{H}^n(z) \approx \int_{L_0}\int_0^\infty |\psi_r * \Pi_*\nu(z)|^2 \frac{dr}{r}\, d\Pi_*\sigma(z)$$
$$= \int_\Gamma\int_0^\infty \Lambda_\nu(x,r)^2 \frac{dr}{r}\, d\sigma(x).$$

In consequence of (8.8) and the above, to prove Lemma 8.1 it suffices to show that

$$\int_\Gamma\int_0^\infty \Lambda_\nu(x,r)^2 \frac{dr}{r}\, d\sigma(x) \lesssim_{A,\tau} \varepsilon_0^{1/8}\mu(R_0). \tag{8.10}$$

We start with the following simple calculation.

Lemma 8.4. *For $x \in \Gamma$ we have*

$$\Lambda_\nu(x,r) \lesssim_{A,\tau} \alpha_\nu(x,2r). \tag{8.11}$$

Moreover, for $x \in \Gamma \cap 2.5B_0$ and $d(x) \lesssim r < \eta r_0$ we have

$$\Lambda_\mu(x,r) \lesssim_{A,\tau} \alpha_\mu(3B_Q), \tag{8.12}$$

for some $Q \in \mathsf{Tree}$ such that $B(x,5r) \subset 3B_Q$ and $r \approx \ell(Q)$.

Proof. First, we will prove (8.11). Let $B = B(x,2r)$, and L_B, c_B be the minimizing plane and constant for $\alpha_\nu(B)$. Using the fact that $\Lambda_\nu(x,r) = |\psi_r * \Pi_*\nu(\Pi_0(x))|$ we get

$$\Lambda_\nu(x,r) = \left|\int \psi_r(\Pi_0(x) - \Pi_0(y))\, d\nu(y)\right|$$
$$\le \left|\int \psi_r(\Pi_0(x) - \Pi_0(y))\, d(\nu - c_B\mathcal{H}^n|_{L_B})(y)\right| + \left|\int \psi_r(\Pi_0(x) - \Pi_0(y))\, d(c_B\mathcal{H}^n|_{L_B})(y)\right|$$
$$\lesssim r^{-(n+1)} F_B(\nu, c_B\mathcal{H}^n|_{L_B}) + 0.$$

Hence, by n-AD-regularity of ν we arrive at

$$\Lambda_\nu(x,r) \lesssim_{A,\tau} \alpha_\nu(x,2r).$$

Now, let us look at (8.12). Since $x \in \Gamma \cap 2.5B_0$ and $d(x) \lesssim r < \eta r_0$, we may find $Q \in \mathsf{Tree}$ such that $B(x,5r) \subset 3B_Q$ and $\ell(Q) \approx_{A,\tau} r$. We use the fact that $|\nabla\widetilde{\psi}_r| \lesssim r^{-n-1}$ and $\operatorname{supp}\widetilde{\psi}_r \subset B(x,5r)$ to get

$$\Lambda_\mu(x,r) \le \left|\int \widetilde{\psi}_r(x-y) d(\mu - c_Q\mathcal{H}^n|_{L_Q})\right| + c_Q\left|\int \widetilde{\psi}_r(x-y) d\mathcal{H}^n|_{L_Q}\right|$$
$$\le r^{-(n+1)} F_{B(x,5r)}(\mu, c_Q\mathcal{H}^n|_{L_Q}) + c_Q\left|\int \widetilde{\psi}_r(x-y) d\mathcal{H}^n|_{L_Q}\right|$$
$$\lesssim_{A,\tau} \alpha_\mu(3B_Q) + c_Q\left|\int \widetilde{\psi}_r(x-y) d\mathcal{H}^n|_{L_Q}\right|.$$

We claim that the last integral above is equal to 0. To prove this, it suffices to show that for $x \in \Gamma$ and $y \in L_Q$ we have $\widetilde{\psi}_r(x-y) = \psi_r(\Pi_0(x) - \Pi_0(y))$, because

$$\int \psi_r(\Pi_0(y) - \Pi_0(x)) \, d(\mathcal{H}^n|_{L_Q})(y) = 0.$$

Since $\widetilde{\psi}_r(x-y) = \psi_r(\Pi_0(y) - \Pi_0(x))\phi((5r)^{-1}(x-y))$, we only have to check that $\phi((5r)^{-1}(x-y)) = 1$ for $\Pi_0(y) - \Pi_0(x) \in \operatorname{supp} \psi_r$. In other words, knowing that $|\Pi_0(y) - \Pi_0(x)| \leq 2r$, we expect that $|x-y| \leq \frac{5}{2}r$.

Indeed, the fact that Γ is a $C\theta$-Lipschitz graph, that $\measuredangle(L_Q, L_0) \leq \theta$, $|\Pi_0(y) - \Pi_0(x)| \leq 2r$, and Lemma 6.3, imply

$$|\Pi_0^\perp(y) - \Pi_0^\perp(x)| \lesssim_{A,\tau} \theta r.$$

Hence,

$$|x-y| \leq 2r + C(A,\tau)\theta r \leq \frac{5}{2}r,$$

as expected. □

Before we proceed, let us state the following auxiliary result. Recall that given a ball B, $z(B)$ denotes the center of B.

Lemma 8.5 ([ATT20, Lemma 6.11])**.** *Let B be a ball centered on an ε-Lipschitz graph Γ, and f a function such that*

$$\|f - f(z(B))\|_{L^\infty(3B \cap \Gamma)} \lesssim \varepsilon,$$

and $f(x) \approx 1$ uniformly for $x \in 3B \cap \Gamma$. Then

$$\int_B \int_0^{r(B)} \alpha_{f\sigma}(x,r)^2 \frac{dr}{r} d\sigma(x) \lesssim \varepsilon^2 r(B)^n,$$

where σ denotes the surface measure on Γ.

We split the area of integration from (8.10) into several pieces. We will estimate each of them separately.

Lemma 8.6. *For every $k \in K$ we have*

$$\int_{B_k} \int_0^{\eta^2 d(x)} |\Lambda_\nu(x,r)|^2 \, \frac{dr}{r} d\sigma(x) \lesssim_{A,\tau} \varepsilon_0 r_k^n.$$

Proof. By Lemma 7.1 (c) we know that for $x \in B_k$ we have $\eta^2 d(x) \leq \eta^{1/2} r_k$. Hence,

$$\int_{B_k} \int_0^{\eta^2 d(x)} |\Lambda_\nu(x,r)|^2 \, \frac{dr}{r} d\sigma(x) \leq \int_{B_k} \int_0^{\eta^{1/2} r_k} |\Lambda_\nu(x,r)|^2 \, \frac{dr}{r} d\sigma(x).$$

Let $g(x) = \sum_{j\in K} c_j h_j(x)$. Note that for $x \in 3B_k \cap \Gamma$ we have $h(x) = 1$, due to (7.7) and the definition of h (7.8). Thus, by Lemma 7.6,

$$|g(x) - c_k| = \Big| \sum_{j\in K} (c_j - c_k) h_j(x) \Big| \lesssim_{A,\tau} \sqrt{\varepsilon_0} \sum_{j\in K} h_j(x) = \sqrt{\varepsilon_0}.$$

Hence, by (7.18), $g(x) \approx_{A,\tau} 1$. Since $\nu|_{3B_k} = g\sigma|_{3B_k}$, and $\Gamma \cap 3B_k$ is a $C\sqrt{\varepsilon_0}$-Lipschitz graph by Lemma 7.3, we can apply Lemma 8.5 and get

$$\int_{B_k} \int_0^{\eta^{1/2} r_k} |\Lambda_\nu(x,r)|^2 \frac{dr}{r} d\sigma(x) \overset{(8.11)}{\lesssim_{A,\tau}} \int_{B_k} \int_0^{\eta^{1/2} r_k} |\alpha_\nu(x,2r)|^2 \frac{dr}{r} d\sigma(x) \lesssim_{A,\tau} \varepsilon_0 r_k^n. \tag{8.13}$$

□

Let $M(\mathbb{R}^d)$ denote the space of finite Borel measures on $\mathbb{R}^d$.

Lemma 8.7 ([ATT20, Lemma 8.2]). *For $\lambda \in M(\mathbb{R}^d)$ we define*

$$T\lambda(x) = \left(\int_0^\infty \Lambda_\lambda(x,r)^2 \frac{dr}{r} \right)^{1/2},$$

and for $f \in L^2(\sigma)$ set $T_\sigma f = T(f\sigma)$. Then T_σ is bounded in $L^p(\sigma)$ for $1 < p < \infty$, and T is bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\sigma)$. Furthermore, the norms $\|T_\sigma\|_{L^p(\sigma)\to L^p(\sigma)}$ and $\|T\|_{M(\mathbb{R}^d)\to L^{1,\infty}(\sigma)}$ are bounded above by some absolute constants depending only on p, n and d.

Lemma 8.8. *We have*

$$\int_{\Gamma\cap 2.4B_0} \int_{\eta^2 d(x)}^{\eta r_0} \Lambda_\nu(x,r)^2 \frac{dr}{r} d\sigma(x) \lesssim_{A,\tau} \varepsilon_0^{1/8} \ell(R_0)^n.$$

Proof. Since $\nu = (\nu_B - h\mu) + (h\mu - \mu_B) + \mu$, for each $x \in \Gamma \cap 2.4B_0$ we split

$$\begin{aligned} &\int_{\eta^2 d(x)}^{\eta r_0} \Lambda_\nu(x,r)^2 \frac{dr}{r} \\ &\quad \lesssim \int_{\eta^2 d(x)}^{\eta r_0} (\Lambda_{\nu_B}(x,r) - \Lambda_{h\mu}(x,r))^2 \frac{dr}{r} + \int_{\eta^2 d(x)}^{\eta r_0} \Lambda_{\mu_B - h\mu}(x,r)^2 \frac{dr}{r} \\ &\qquad + \int_{\eta^2 d(x)}^{\eta r_0} \Lambda_\mu(x,r)^2 \frac{dr}{r}. \end{aligned} \tag{8.14}$$

Let

$$H = \left\{ x \in \Gamma \cap 2.4B_0 \ : \ \int_{\eta^2 d(x)}^{\eta r_0} \Lambda_{\mu_B - h\mu}(x,r)^2 \frac{dr}{r} > \varepsilon_0^{1/4} \right\}.$$

We divide our area of integration into two parts:

$$\begin{aligned} &\int_{\Gamma\cap 2.4B_0} \int_{\eta^2 d(x)}^{\eta r_0} \Lambda_\nu(x,r)^2 \frac{dr}{r} d\sigma(x) = \\ &\int_H \int_{\eta^2 d(x)}^{\eta r_0} \Lambda_\nu(x,r)^2 \frac{dr}{r} d\sigma(x) + \int_{\Gamma\cap 2.4B_0\setminus H} \int_{\eta^2 d(x)}^{\eta r_0} \Lambda_\nu(x,r)^2 \frac{dr}{r} d\sigma(x) =: I_1 + I_2. \end{aligned} \tag{8.15}$$

In order to estimate I_1, note that for $x \in 2.4B_0$ and $r < \eta r_0$ we have $B(x,5r) \subset 2.5B_0$. Since $\operatorname{supp}\widetilde{\psi}_r \subset 5B(0,r)$ we get $\Lambda_{\mu_B - h\mu}(x,r) = \Lambda_{(\mu_B-h\mu)|_{2.5B_0}}(x,r)$. Hence, by Lemma 8.7 applied to $\lambda = (\mu_B - h\mu)|_{2.5B_0}$

$$\sigma(H) \le \sigma\Big(\Big\{x\in\Gamma \,:\, T\Big((\mu_B-h\mu)|_{2.5B_0}\Big) > \varepsilon_0^{1/8}\Big\}\Big) \lesssim \varepsilon_0^{-1/8}(\mu_B-h\mu)(2.5B_0).$$

Since $h=1$ on $3B_0\setminus(R_{\mathsf{Far}}\cup R_G)$ by Lemma 7.7, $\mu_B(R_G) = (h\mu)(R_G) = 0$ by their definition and (7.7), and $\mu(R_{\mathsf{Far}})$ is small by Lemma 4.6, we have

$$(\mu_B - h\mu)(2.5B_0) \le \mu_B(R_{\mathsf{Far}}) = \mu(R_{\mathsf{Far}}) \lesssim_{A,\tau} \varepsilon_0^{1/2}\ell(R_0)^n.$$

Thus, for ε_0 small enough

$$\sigma(H) \le C(A,\tau)\varepsilon_0^{-1/8}\varepsilon_0^{1/2}\ell(R_0)^n \le \varepsilon_0^{1/4}\ell(R_0)^n.$$

Now, consider the density $q = \frac{d\nu|_{2.5B_0}}{d\sigma}$. Arguing as before we see that for $x\in 2.4B_0$ and $r<\eta r_0$ we have $\Lambda_\nu(x,r) = \Lambda_{q\sigma}(x,r)$. By n-AD-regularity of ν (Lemma 7.5) we get $\|q\|^4_{L^4(\sigma)} \lesssim_{A,\tau} \sigma(2.5B_0)\approx \ell(R_0)^n$. Using the $L^4(\sigma)$ boundedness of T_σ yields

$$I_1 \le \int_H |T_\sigma q(x)|^2\, d\sigma(x) \le \sigma(H)^{1/2}\|T_\sigma(q)\|^2_{L^4(\sigma)} \lesssim_{A,\tau} \varepsilon_0^{1/8}\ell(R_0)^n. \tag{8.16}$$

We move on to estimating I_2. Observe that by the definition of H we have

$$\int_{\Gamma\cap 2.4B_0\setminus H}\int_{\eta^2 d(x)}^{\eta r_0} \Lambda_{\mu_B - h\mu}(x,r)^2\,\frac{dr}{r}d\sigma(x) \lesssim \varepsilon_0^{1/4}\ell(R_0)^n.$$

Thus, by (8.14),

$$\begin{aligned} I_2 &= \int_{\Gamma\cap 2.4B_0\setminus H}\int_{\eta^2 d(x)}^{\eta r_0} \Lambda_\nu(x,r)^2\,\frac{dr}{r}d\sigma(x)\\ &\lesssim \int_{\Gamma\cap 2.4B_0}\int_{\eta^2 d(x)}^{\eta r_0} (\Lambda_{\nu_B}(x,r)-\Lambda_{h\mu}(x,r))^2\,\frac{dr}{r}d\sigma(x) + \varepsilon_0^{1/4}\ell(R_0)^n\\ &\quad+\int_{\Gamma\cap 2.4B_0}\int_{\eta^2 d(x)}^{\eta r_0} \Lambda_\mu(x,r)^2\,\frac{dr}{r}d\sigma(x) =: I_{21} + \varepsilon_0^{1/4}\ell(R_0)^n + I_{22}. \end{aligned}\tag{8.17}$$

To handle I_{22}, we use (8.12) to get for $x\in\Gamma\cap 2.4B_0$.

$$\int_{\eta^2 d(x)}^{\eta r_0} \Lambda_\mu(x,r)^2\,\frac{dr}{r} \lesssim_\eta \sum_{\substack{Q\in\mathsf{Tree}\\ x\in 3B_Q}} \alpha_\mu(3B_Q)^2 \overset{(4.11)}{\lesssim_{A,\tau}} \varepsilon_0^2.$$

Hence,

$$I_{22} = \int_{\Gamma\cap 2.4B_0}\int_{\eta^2 d(x)}^{\eta r_0} \Lambda_\mu(x,r)^2\,\frac{dr}{r}d\sigma(x) \lesssim \varepsilon_0^2\ell(R_0)^n. \tag{8.18}$$

Finally, we deal with the integral I_{21}. Observe that, since $\Lambda_{\nu_B}(x,r) - \Lambda_{h\mu}(x,r) = \widetilde{\psi}_r * \nu_B(x) - \widetilde{\psi}_r * h\mu(x)$, and $|\nabla\widetilde{\psi}_r| \lesssim r^{-n-1}$, we have

$$|\Lambda_{\nu_B}(x,r)-\Lambda_{h\mu}(x,r)|^2 \lesssim \left(r^{-n-1}F_{B(x,5r)}(\nu_B,h\mu)\right)^2 \overset{(7.20)}{\lesssim_{A,\tau}} \varepsilon_0 \left(\sum_{3B_k\cap B(x,5r)\neq\varnothing} \frac{r_k^{n+1}}{r^{n+1}}\right)^2.$$

Note that for $k\in K$ such that $3B_k\cap B(x,5r)\neq\varnothing$, for $\eta^2 d(x) < r < \eta r_0$, and for any $y\in 3B_k\cap B(x,5r)$, we have

$$r_k \overset{(7.3)}{\leq} d(y) \leq d(x) + 5r \leq (\eta^{-2}+5)r. \tag{8.19}$$

Thus, $r_k \leq \eta^{-3}r$, and for some big $C' = C'(A,\tau)$ we have $B_k \subset C'B_0$. It follows by the Cauchy-Schwarz inequality, the fact that B_k are centered on Γ, and that they are of bounded intersection, that

$$\left(\sum_{3B_k\cap B(x,5r)\neq\varnothing} \frac{r_k^{n+1}}{r^{n+1}}\right)^2 \leq \left(\sum_{3B_k\cap B(x,5r)\neq\varnothing} \frac{r_k^{n+2}}{r^{n+2}}\right)\left(\sum_{3B_k\cap B(x,5r)\neq\varnothing} \frac{r_k^{n}}{r^{n}}\right) \tag{8.20}$$

$$\lesssim \sum_{3B_k\cap B(x,5r)\neq\varnothing} \frac{r_k^{n+2}}{r^{n+2}}. \tag{8.21}$$

Together with the fact that $r_k \leq \eta^{-3}r$ this implies

$$\begin{aligned}
I_{21} &= \int_{\Gamma\cap 2.4B_0}\int_{\eta^2 d(x)}^{\eta r_0} (\Lambda_{\nu_B}(x,r)-\Lambda_{h\mu}(x,r))^2\,\frac{dr}{r}d\sigma(x)\\
&\lesssim \varepsilon_0 \int_{\Gamma\cap 2.4B_0}\int_{\eta^2 d(x)}^{\eta r_0} \sum_{3B_k\cap B(x,5r)\neq\varnothing} r_k^{n+2}\frac{dr}{r^{n+3}}d\sigma(x)\\
&= \varepsilon_0 \sum_{k\in K} r_k^{n+2}\int_{\Gamma\cap 2.4B_0}\int_{\eta^2 d(x)}^{\eta r_0} \mathbb{1}_{B(x,5r)\cap 3B_k\neq\varnothing}(x)\frac{dr}{r^{n+3}}d\sigma(x)\\
&\leq \varepsilon_0 \sum_{B_k\subset C'B_0} r_k^{n+2}\int_{\Gamma\cap 2.4B_0}\int_{\eta^3 r_k}^{\eta r_0} \mathbb{1}_{B(x,5r)\cap 3B_k\neq\varnothing}(x)\frac{dr}{r^{n+3}}d\sigma(x).
\end{aligned}$$

Now, note that if $B(x,5r)\cap 3B_k\neq\varnothing$, then

$$x\in B(z_k, 5r+3r_k) \overset{(8.19)}{\subset} B(z_k,\eta^{-3}r).$$

Hence,

$$\begin{aligned}
I_{21} &\lesssim \varepsilon_0 \sum_{B_k\subset C'B_0} r_k^{n+2}\int_{\eta^3 r_k}^{\eta r_0}\int_{B(z_k,\eta^{-3}r)} d\sigma(x)\frac{dr}{r^{n+3}}\\
&\lesssim \varepsilon_0 \sum_{B_k\subset C'B_0} r_k^{n+2}\int_{\eta^3 r_k}^{\eta r_0}\frac{dr}{r^3}\\
&\lesssim \varepsilon_0 \sum_{B_k\subset C'B_0} r_k^{n} \lesssim \varepsilon_0\sigma(C'B_0) \lesssim \varepsilon_0\ell(R_0)^n.
\end{aligned}$$

Together with (8.15), (8.16), (8.17), and (8.18), this concludes the proof. □

We are finally ready to complete the proof of (8.10). Let us split the area of integration into four subsets:

$$A_1 = \{(x,r) \;:\; B(x,2r) \cap 2.3B_0 = \varnothing\},$$
$$A_2 = \{(x,r) \;:\; B(x,2r) \cap 2.3B_0 \neq \varnothing,\; r > \eta r_0\},$$
$$A_3 = \{(x,r) \;:\; B(x,2r) \cap 2.3B_0 \neq \varnothing,\; \eta^2 d(x) < r \leq \eta r_0\},$$
$$A_4 = \{(x,r) \;:\; B(x,2r) \cap 2.3B_0 \neq \varnothing,\; 0 < r \leq \min(\eta^2 d(x),\, \eta r_0)\},$$

we also set

$$I_i = \iint_{A_i} \Lambda_\nu(x,r)^2 \,\frac{dr}{r} d\sigma(x).$$

Since $\nu|_{(2.3B_0)^c} = c_0 \mathcal{H}^n|_{L_0 \cap (2.3B_0)^c}$ by (7.12), for $(x,r) \in A_1$ we have

$$\Lambda_\nu(x,r) = c_0 \Lambda_{\mathcal{H}^n|_{L_0}}(x,r) = 0,$$

and so $I_1 = 0$.

Now let $(x,r) \in A_2$. Since $B(x,2r) \cap 2.3B_0 \neq \varnothing,\; r > \eta r_0$, we have

$$|x - z_0| \leq 2r + 2.3r_0 < \eta^{-2} r,$$

so that $r \geq \max(\eta r_0,\, \eta^2|x - z_0|)$. It follows that

$$\begin{aligned}
I_2 &\leq \int_\Gamma \int_{\max(\eta r_0,\, \eta^2|x-z_0|)}^\infty \Lambda_\nu(x,r)^2 \,\frac{dr}{r} d\sigma(x) \\
&\overset{(8.11)}{\lesssim_{A,\tau}} \int_\Gamma \int_{\max(\eta r_0,\, \eta^2|x-z_0|)}^\infty \alpha_\nu(x,2r)^2 \,\frac{dr}{r} d\sigma(x) \\
&\overset{(7.25)}{\lesssim_{A,\tau}} \varepsilon_0^{1/2} \ell(R_0)^{2n} \int_\Gamma \int_{\max(\eta r_0,\, \eta^2|x-z_0|)}^\infty \frac{dr}{r^{2n+1}} d\sigma(x) \\
&\approx \varepsilon_0^{1/2} \ell(R_0)^{2n} \int_\Gamma \frac{1}{\max(r_0,\, \eta|x - z_0|)^{2n}} \, d\sigma(x) \\
&\approx \varepsilon_0^{1/2} \ell(R_0)^{2n} \left(\int_{\Gamma \cap 1.9B_0} \frac{1}{r_0^{2n}} \, d\sigma(x) + \int_{\Gamma \setminus 1.9B_0} \frac{1}{|x - z_0|^{2n}} \, d\sigma(x) \right) \approx \varepsilon_0^{1/2} \ell(R_0)^n,
\end{aligned}$$

where we used in the last line that $\Gamma \setminus 1.9B_0 = L_0 \setminus 1.9B_0$, see Lemma 5.11.

Concerning $(x,r) \in A_3$, note that necessarily $x \in 2.4B_0$, and so by Lemma 8.8

$$I_3 \leq \int_{\Gamma \cap 2.4B_0} \int_{\eta^2 d(x)}^{\eta r_0} \Lambda_\nu(x,r)^2 \,\frac{dr}{r} d\sigma(x) \lesssim_{A,\tau} \varepsilon_0^{1/8} \ell(R_0)^n.$$

Finally, for $(x,r) \in A_4$, we only need to consider x such that $d(x) > 0$ and $x \in 2.4B_0 \cap \Gamma$, and since all such x are contained in some B_k we get

$$\begin{aligned}
I_4 &\leq \int_{\Gamma \cap 2.4B_0} \int_0^{\eta^2 d(x)} \Lambda_\nu(x,r)^2 \,\frac{dr}{r} d\sigma(x) \leq \sum_{B_k \cap 2.4B_0 \neq \varnothing} \int_{B_k} \int_0^{\eta^2 d(x)} \Lambda_\nu(x,r)^2 \,\frac{dr}{r} d\sigma(x) \\
&\overset{\text{Lemma 8.6}}{\lesssim_{A,\tau}} \sum_{B_k \cap 2.4B_0 \neq \varnothing} \varepsilon_0 r_k^n \approx \sum_{B_k \cap 2.4B_0 \neq \varnothing} \varepsilon_0 \sigma(B_k) \overset{\text{Lemma 7.1}}{\lesssim} \varepsilon_0 \sigma(CB_0) \approx \varepsilon_0 \ell(R_0)^n.
\end{aligned}$$

Putting together all the estimates above finishes the proof of (8.10) and Lemma 8.1.

9 Small measure of cubes from BA

We know by Lemma 4.6 that $\mu(R_{\mathsf{Far}}) \lesssim_{A,\tau} \sqrt{\varepsilon_0}\mu(R_0)$. Thus, in order to estimate the measure of $\bigcup_{Q\in\mathsf{BA}} Q$, it suffices to bound the measure of

$$R_{\mathsf{BA}} = \bigcup_{Q\in\mathsf{BA}} Q \setminus R_{\mathsf{Far}}.$$

Lemma 9.1. *We have*

$$\mu(R_{\mathsf{BA}}) \lesssim_A \theta^{-2}\|\nabla F\|_{L^2}^2.$$

Proof. For every $x \in R_{\mathsf{BA}}$ we define $B_x = B(x, r(Q_x)/100)$, where $Q_x \in \mathsf{BA}$ is such that $x \in Q_x$. We use the $5r$-covering theorem to choose $\{x_i\}_{i\in J}$ such that all B_{x_i} are pairwise disjoint and $\bigcup_i 5B_{x_i}$ covers $\bigcup_{x\in R_{\mathsf{BA}}} B_x$. Observe that

$$5B_{x_i} \subset 3B_{Q_{x_i}}. \tag{9.1}$$

Set $B_i = \frac{1}{2}B_{x_i}, Q_i = Q_{x_i}$, and let $P_i \in \mathsf{Tree}$ be the parent of Q_i. We have $\ell(P_i) \approx \ell(Q_i) \approx r(B_i)$. Since $x_i \notin R_{\mathsf{Far}}$, we can use Lemma 6.5 to obtain

$$\mathrm{dist}(x_i, \Gamma) \lesssim_{A,\tau} \sqrt{\varepsilon_0}d(x_i) \lesssim \sqrt{\varepsilon_0}\ell(P_i) \approx \sqrt{\varepsilon_0}r(B_i).$$

Hence, for small ε_0, we get that $\frac{1}{4}B_i \cap \Gamma \neq \varnothing$. It follows that for each $i \in J$ we can choose balls $B_{i,1}$, $B_{i,2} \subset B_i$ centered at Γ, with $r(B_{i,1}) \approx r(B_{i,2}) \approx r(B_i)$, and such that $\mathrm{dist}(B_{i,1}, B_{i,2}) \gtrsim r(B_i)$. Then, for any points $y_k \in B_{i,k} \cap \Gamma$, $k = 1, 2$, we have

$$r(B_i) \lesssim |y_1 - y_2| \lesssim |\Pi_0(y_1) - \Pi_0(y_2)|. \tag{9.2}$$

Since $y_1, y_2 \in \Gamma \cap B_i \subset \Gamma \cap B_{P_i}$, we have by Lemma 6.3

$$\mathrm{dist}(y_k, L_{P_i}) \lesssim_{A,\tau} \sqrt{\varepsilon_0}\ell(P_i), \qquad k = 1, 2.$$

Let $w_k = \Pi_{L_{P_i}}(y_k)$. By the estimate above we have $|y_k - w_k| \lesssim_{A,\tau} \sqrt{\varepsilon_0}\ell(P_i)$. Moreover, it is easy to see that $w_k \in B_{P_i}$.

Since $\measuredangle(L_{Q_i}, L_0) > \theta$ and $Q_i \in \mathsf{Tree}_0$ by the definition of BA (4.2), $\ell(Q_i) \approx \ell(P_i)$, and $\mathrm{dist}(Q_i, P_i) = 0$, we may use Lemma 5.2 with Q_i, P_i to get

$$\measuredangle(L_{P_i}, L_0) \geq \measuredangle(L_{Q_i}, L_0) - \measuredangle(L_{P_i}, L_{Q_i}) \geq \theta - C(A,\tau)\varepsilon_0 \gtrsim \theta.$$

Thus,

$$\begin{aligned}
&|F(\Pi_0(y_1)) - F(\Pi_0(y_2))| = |\Pi_0^\perp(y_1) - \Pi_0^\perp(y_2)| \\
&\geq |\Pi_0^\perp(w_1) - \Pi_0^\perp(w_2)| - \sum_{k=1}^2 |y_k - w_k| \gtrsim \theta|\Pi_0(w_1) - \Pi_0(w_2)| - \sum_{k=1}^2 |y_k - w_k| \\
&\qquad \geq \theta|\Pi_0(y_1) - \Pi_0(y_2)| - 2\sum_{k=1}^2 |y_k - w_k| \\
&\qquad \overset{(9.2)}{\gtrsim} \theta r(B_i) - c(A,\tau)\sqrt{\varepsilon_0}r(B_i) \gtrsim \theta r(B_i),
\end{aligned}$$

for ε_0 small enough.

Now, denoting by m_i the mean of F over the ball $\Pi_0(B_i)$, we have

$$\begin{aligned}|F(\Pi_0(y_1)) - F(\Pi_0(y_2))| &\le |F(\Pi_0(y_1)) - m_i| + |F(\Pi_0(y_2)) - m_i| \\ &\le 2\max_{k=1,2} |F(\Pi_0(y_k)) - m_i|.\end{aligned}$$

Hence, the estimates above give us for some $k \in \{1, 2\}$

$$|F(\Pi_0(y_k)) - m_i| \gtrsim \theta r(B_i). \tag{9.3}$$

Since the estimate above holds for all points $y_k \in B_{i,k} \cap \Gamma$, and $\Pi_0(B_{i,k} \cap \Gamma) \approx r(B_i)^n$, we can use Poincaré's inequality to get

$$r(B_i)^2 \int_{\Pi_0(B_i)} |\nabla F(\xi)|^2 \, d\mathcal{H}^n(\xi) \gtrsim \int_{\Pi_0(B_i)} |F(\xi) - m_i|^2 \, d\mathcal{H}^n(\xi) \gtrsim \theta^2 r(B_i)^{n+2}$$

for all $i \in J$.

We claim that the n-dimensional balls $\{\Pi_0(B_i)\}_{i\in J}$ are pairwise disjoint. This follows easily by the fact that $2B_i = B_{x_i}$ are pairwise disjoint, $\frac{1}{4}B_i \cap \Gamma \neq \varnothing$, and Γ is a graph of a Lipschitz function with a small Lipschitz constant.

Hence, we may sum the inequality above over all $i \in J$ to finally get

$$\begin{aligned}\|\nabla F\|_{L^2}^2 &\ge \sum_{i\in J} \int_{\Pi_0(B_i)} |\nabla F|^2 \, d\mathcal{H}^n \gtrsim \sum_{i\in J} \theta^2 r(B_i)^n \\ &\overset{(4.5)}{\gtrsim} A^{-1}\theta^2 \sum_{i\in J} \mu(3B_{Q_i}) \overset{(9.1)}{\gtrsim_A} \theta^2 \sum_{i\in J} \mu(5B_{x_i}) \ge \theta^2 \mu(R_{\mathsf{BA}}).\end{aligned}$$

□

To estimate $\|\nabla F\|_{L^2}$ we will use a well-known theorem due to Dorronsoro. We reformulate it slightly for the sake of convenience.

Theorem 9.2 ([Dor85, Theorem 2]). *Let $F : \mathbb{R}^n \to \mathbb{R}^{d-n}$ be an L-Lipschitz function, with L sufficiently small, and let $\Gamma \subset \mathbb{R}^d$ be the graph of F, and $\sigma = \mathcal{H}^n|_\Gamma$. Then*

$$\int_\Gamma \int_0^\infty \beta_{\sigma,1}(x,r)^2 \, \frac{dr}{r} d\sigma \approx \|\nabla F\|_{L^2}^2.$$

To estimate the integral above we split the area of integration into four subfamilies:

$$\begin{aligned}A_1 &= \{(x,r) \;:\; B(x,r) \cap 1.9B_0 = \varnothing\},\\ A_2 &= \{(x,r) \;:\; B(x,r) \cap 1.9B_0 \neq \varnothing,\; r > 0.1r_0\},\\ A_3 &= \{(x,r) \;:\; B(x,r) \cap 1.9B_0 \neq \varnothing,\; \eta^2 d(x) \le r < 0.1r_0\},\\ A_4 &= \{(x,r) \;:\; B(x,r) \cap 1.9B_0 \neq \varnothing,\; r < \min(\eta^2 d(x), 0.1r_0)\},\end{aligned}$$

we also set

$$I_i = \iint_{A_i} \beta_{\sigma,1}(x,r)^2 \, \frac{dr}{r} d\sigma(x).$$

Firstly, note that for $(x,r) \in A_1$ we have $B(x,r) \cap \Gamma = B(x,r) \cap L_0$ because $\mathrm{supp}(F) \subset 1.9B_0$, and so

$$I_1 = 0. \tag{9.4}$$

Lemma 9.3. *We have*

$$I_2 \lesssim_{A,\tau} \varepsilon_0^{1/2} \ell(R_0)^n.$$

Proof. Let $(x,r) \in A_2$. Observe that since $\sigma \approx_{A,\tau} \nu$, we have

$$\beta_{\sigma,1}(x,r) \approx_{A,\tau} \beta_{\nu,1}(x,r) \overset{(II.3.5)}{\lesssim} \alpha_\nu(x,2r) \overset{(7.25)}{\lesssim_{A,\tau}} \varepsilon_0^{1/4} \frac{\ell(R_0)^n}{r^n}.$$

Note that if $B(x,r) \cap 1.9B_0 \neq \varnothing$, then necessarily $x \in B(z_0, 1.9r_0 + r) \subset B(z_0, 20r)$. Hence,

$$\begin{aligned} I_2 \le \int_{0.1r_0}^{\infty} \int_{B(z_0,20r)} \beta_{\sigma,1}(x,r)^2 \, d\sigma \frac{dr}{r} &\lesssim_{A,\tau} \varepsilon_0^{1/2} \int_{0.1r_0}^{\infty} \int_{B(z_0,20r)} \frac{\ell(R_0)^{2n}}{r^{2n+1}} \, d\sigma dr \\ &\lesssim \varepsilon_0^{1/2} \int_{0.1r_0}^{\infty} \frac{\ell(R_0)^{2n}}{r^{n+1}} \, dr \approx \varepsilon_0^{1/2} \ell(R_0)^n. \end{aligned}$$

□

Lemma 9.4. *We have*

$$I_3 \lesssim_{A,\tau} \varepsilon_0 \ell(R_0)^n.$$

Proof. Let $(x,r) \in A_3$. Since $B(x,r) \cap 1.9B_0 \neq \varnothing$ and $\eta^2 d(x) \le r < 0.1r_0$, it is clear that $B(x,2r) \subset 2.1B_0$ and we may find a cube $P = P(x,r) \in \mathsf{Tree}$ such that $B(x,2r) \subset 3B_P$ and $r \approx \ell(P)$. We will estimate the average distance of $B(x,r) \cap \Gamma$ to L_P.

Bounding the part corresponding to $B(x,r) \cap R_G \subset 3B_P \cap R_G$ is straightforward: Lemma 5.5 states that $d\mu|_{R_G} = g \, d\mathcal{H}^n|_{R_G}$ with $g \approx_{A,\tau} 1$, and so

$$\begin{aligned} &\int_{B(x,r)\cap R_G} \frac{\mathrm{dist}(y,L_P)}{r} \, d\sigma(y) \lesssim_{A,\tau} \int_{3B_P \cap R_G} \frac{\mathrm{dist}(y,L_P)}{\ell(P)} \, d\mu(y) \\ &\lesssim_{A,\tau} \left(\int_{3B_P \cap R_G} \left(\frac{\mathrm{dist}(y,L_P)}{\ell(P)} \right)^2 \, d\mu(y) \right)^{1/2} \ell(P)^{n/2} \lesssim_\tau \beta_{\mu,2}(3B_P) \ell(P)^n. \end{aligned} \tag{9.5}$$

Dealing with the part outside of R_G is a bit more delicate. By (7.7) and the definition of functions h_k (7.8),

$$\begin{aligned} \int_{B(x,r)\setminus R_G} \frac{\mathrm{dist}(y,L_P)}{r} \, d\sigma(y) &= \sum_{k\in K} \int_{B(x,r)} \frac{\mathrm{dist}(y,L_P)}{r} \, h_k(y) \, d\sigma(y) \\ &\overset{(7.18)}{\approx_{A,\tau}} \sum_{k\in K} \int_{B(x,r)} \frac{\mathrm{dist}(y,L_P)}{r} \, c_k h_k(y) \, d\sigma(y) = \int_{B(x,r)} \frac{\mathrm{dist}(y,L_P)}{r} \, d\nu_B(y). \end{aligned}$$

Consider the 1-Lipschitz function $\Phi(y) = \psi(y)\,\mathrm{dist}(y, L_P)$, where ψ is r^{-1}-Lipschitz, $\psi \equiv 1$ on $B(x,r)$, $|\psi| \le 1$, and $\mathrm{supp}(\psi) \subset B(x, 2r)$.

$$\int_{B(x,r)} \frac{\mathrm{dist}(y, L_P)}{r}\, d\nu_B(y) \le \int_{B(x,2r)} \frac{\psi(y)\,\mathrm{dist}(y, L_P)}{r}\, d\nu_B(y)$$
$$\le \int_{B(x,2r)} \frac{\psi(y)\,\mathrm{dist}(y, L_P)}{r}\, h(y)\, d\mu(y) + r^{-1} \left| \int_{B(x,2r)} \Phi(y)\, d(\nu_B - h\mu)(y) \right|$$

Since $|\psi|, |h| \le 1$, the first term on the right hand side above can be bounded by $\beta_{\mu,2}(3B_P)\ell(P)^n$, just as in (9.5). Concerning the second term,

$$r^{-1} \left| \int_{B(x,2r)} \Phi(y)\, d(\nu_B - h\mu)(y) \right| \overset{(7.20)}{\lesssim_{A,\tau}} \sqrt{\varepsilon_0}\, r^{-1} \sum_{3B_k \cap B(x,2r) \neq \varnothing} r_k^{n+1}$$

Gathering all the calculations above we get that

$$\beta_{\sigma,1}(x,r)^2 \lesssim_{A,\tau} \beta_{\mu,2}(3B_P)^2 + \varepsilon_0 \left(\sum_{3B_k \cap B(x,2r) \neq \varnothing} \frac{r_k^{n+1}}{r^{n+1}} \right)^2. \tag{9.6}$$

Integrating the first term over A_3, since each $P(x,r)$ has sidelength comparable to r and $\mathrm{dist}(P(x,r), x) \lesssim_{A,\tau} r$, it is easy to see that

$$\iint_{A_3} \beta_{\mu,2}(3B_{P(x,r)})^2\, \frac{dr}{r} d\sigma \lesssim_{A,\tau} \sum_{P \in \mathsf{Tree}} \beta_{\mu,2}(3B_P)^2 \ell(P)^n \overset{(4.10)}{\lesssim_{A,\tau}} \varepsilon_0^2 \ell(R_0)^n.$$

Moving on to the second term from (9.6), note that if $y \in 3B_k \cap B(x, 2r) \neq \varnothing$, then by (7.3) we have $r_k \le d(y) \le 2r + d(x) \le (2 + \eta^2)r$. Thus, following calculations from the proof of Lemma 8.8 (more precisely (8.19) and onwards), we get that

$$\varepsilon_0 \iint_{A_3} \left(\sum_{3B_k \cap B(x,2r) \neq \varnothing} \frac{r_k^{n+1}}{r^{n+1}} \right)^2 \frac{dr}{r} d\sigma \lesssim_{A,\tau} \varepsilon_0 \ell(R_0)^n.$$

Hence, $I_3 \lesssim_{A,\tau} \varepsilon_0 \ell(R_0)^n$. □

Lemma 9.5. *We have*

$$I_4 \lesssim \varepsilon_0 \ell(R_0)^n.$$

Proof. Let $(x, r) \in A_4$, so that $\eta^2 d(x) \ge r > 0$. It follows by (7.7) that $x \in B_k$ for some $k \in K$. Then,

$$r \le \eta^2 d(x) \overset{(7.3)}{\le} \eta^{1/2} r_k.$$

Note also that $x \in 2B_0$. Thus,

$$I_3 \leq \sum_{B_k \cap 2B_0 \neq \varnothing} \int_{B_k} \int_0^{\eta^{1/2} r_k} \beta_{\sigma,1}(x,r)^2 \, \frac{dr}{r} d\sigma(x)$$
$$\overset{\text{Lemma 7.5}}{\approx_{A,\tau}} \sum_{B_k \cap 2B_0 \neq \varnothing} \int_{B_k} \int_0^{\eta^{1/2} r_k} \beta_{\nu,1}(x,r)^2 \, \frac{dr}{r} d\sigma(x)$$
$$\overset{\text{(II.3.5)}}{\lesssim} \sum_{B_k \cap 2B_0 \neq \varnothing} \int_{B_k} \int_0^{\eta^{1/2} r_k} \alpha_{\nu}(x,2r)^2 \, \frac{dr}{r} d\sigma(x) \overset{(8.13)}{\lesssim_{A,\tau}} \sum_{B_k \cap 2B_0 \neq \varnothing} \varepsilon_0 r_k^n \lesssim \varepsilon_0 \ell(R_0)^n.$$

□

Putting together the estimates for I_1, I_2, I_3 and I_4, we get that

$$\int_\Gamma \int_0^\infty \beta_{\sigma,1}(x,r)^2 \, \frac{dr}{r} d\sigma \lesssim_{A,\tau} \sqrt{\varepsilon_0} \ell(R_0)^n \approx \sqrt{\varepsilon_0} \mu(R_0).$$

Thus, Lemma 9.1 and Theorem 9.2 give us

$$\mu(R_{\mathsf{BA}}) \lesssim_{A,\tau,\theta} \sqrt{\varepsilon_0} \mu(R_0).$$

Taking into account the estimates for other stopping cubes, we arrive at

$$\mu\left(\bigcup_{Q \in \mathsf{Stop}} Q \right) < \frac{\mu(R_0)}{2}.$$

Thus, $\mu(R_G) \geq 0.5\mu(R_0)$, and since R_G is a subset of the Lipschitz graph Γ and $\mu|_{R_G}$ is n-rectifiable, the proof of Lemma 3.1 is finished.

Necessary condition for rectifiability involving Wasserstein distance W_2 IV

1 Introduction

The aim of this chapter is to prove a necessary condition for rectifiability involving the α_2 coefficients. The complete definition was given in Subsection I.6.3, now let us fix the notation specific to this chapter: for $1 \leq p < \infty$, a Radon measure μ on $\mathbb{R}^d$, a ball $B = B(x,r) \subset \mathbb{R}^d$ with $\mu(B) > 0$, and an n-plane L intersecting B, we define

$$\alpha_{\mu,p,L}(B) = \frac{1}{r\,\mu(B)^{1/p}} W_p(\varphi_B \mu, a_{B,L}\varphi_B \mathcal{H}^n|_L), \tag{1.1}$$

where φ_B is a "regularized characteristic function", and

$$a_{B,L} = \frac{\int \varphi_B \; d\mu}{\int \varphi_B \; d\mathcal{H}^n|_L}.$$

We will usually omit the subscripts and just write a. We define also

$$\alpha_{\mu,p}(B) = \inf_L \alpha_{\mu,p,L}(B),$$

where the infimum is taken over all n-planes L intersecting B. For a ball $B = B(x,r)$ we will sometimes write $\alpha_{\mu,p}(x,r)$ instead of $\alpha_{\mu,p}(B)$.

Our goal is to show the following.

Theorem 1.1. *Let μ be an n-rectifiable measure on $\mathbb{R}^d$. Then for μ-a.e. $x \in \mathbb{R}^d$*

$$\int_0^1 \alpha_{\mu,2}(x,r)^2 \; \frac{dr}{r} < \infty. \tag{1.2}$$

Remark 1.2. Note that in this chapter we chose the normalizing factor $\mu(B)$. However, in this case it is not really important: for rectifiable measure μ the density $\Theta^n(\mu, x)$ exists, and is positive and finite, for μ-a.e. x. Thus, the condition (1.2) satisfied by α_2 numbers normalized by $\mu(B)$ is equivalent to that same condition satisfied by r^{-n} or $\mu(3B)$-normalized α_2 numbers.

In Theorem III.1.4 we showed that (1.2) is also a sufficient condition for rectifiability (we used a different normalization of α_2, but it does not matter, see Remark 1.5). Putting the two results together, we get the following characterization.

Corollary 1.3. *Let μ be a Radon measure on $\mathbb{R}^d$. Then μ is n-rectifiable if and only if for μ-a.e. $x \in \mathbb{R}^d$ we have*

$$\int_0^1 \alpha_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty.$$

Remark 1.4. The characterization above is sharp in the following sense. Suppose $1 \le p \le q < \infty$. Then it follows easily by Hölder's inequality, definition of α_p numbers, and the fact that $\operatorname{supp} \varphi_B \subset 3B$, that

$$\alpha_{\mu,p}(B) \le \left(\frac{\mu(3B)}{\mu(B)} \right)^{1/p-1/q} \alpha_{\mu,q}(B).$$

Hence, for doubling measures, α_p numbers are increasing in p. It is well known that rectifiable measures are pointwise doubling, i.e.

$$\limsup_{r\to 0^+} \frac{\mu(B(x,2r))}{\mu(B(x,r))} < \infty \qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^d,$$

and so the finiteness of α_2 square function (1.2) implies finiteness of α_p square function for any $1 \le p \le 2$. However, in general one cannot expect finiteness of α_p square function for $p > 2$, see Remark 1.6. In other words, Theorem 1.1 cannot be improved.

Remark 1.5. For technical reasons, in Chapter III we defined α_p numbers normalizing by $\mu(3B)$ (i.e. in (1.1) we replace $\mu(B)$ with $\mu(3B)$). Of course, the $3B$-normalized coefficients are *smaller* than the B-normalized variant used here. Hence, if (1.2) is finite for B-normalized α_2 numbers, then it is finite for $3B$-normalized α_2 numbers, and so Theorem III.1.4 may be applied to get Corollary 1.3.

Remark 1.6. The example from [Tol19] shows that one cannot expect finiteness of the α_p square function when $p > 2$. Indeed, it is easy to see that α_p numbers bound from above β_p numbers (see Lemma II.3.2, the same proof works with arbitrary $1 \le p < \infty$). Tolsa gave an example of a rectifiable measure such that for all $p > 2$ the square function involving β_p in infinite almost everywhere. Hence, the α_p square function of that measure is also infinite almost everywhere.

Theorem 1.1 yields an easy corollary involving *bilateral* β numbers. Set

$$b\beta_{\mu,2}(x,r)^2 = \inf_L \frac{1}{r^n}\int_{B(x,r)} \left(\frac{\mathrm{dist}(y,L)}{r}\right)^2 d\mu(y) \\ + \frac{1}{r^n}\int_{B(x,r)} \left(\frac{\mathrm{dist}(y,\mathrm{supp}\,\mu)}{r}\right)^2 d\mathcal{H}^n|_L(y).$$

As shown in Lemma II.3.4, if a ball $B(x,r)$ satisfies $\mu(B(x,r)) \approx r^n$, then $\alpha_{\mu,2}(x,r)$ bound from above $b\beta_{\mu,2}(x,r)$. Since for n-rectifiable measure μ we have $0 < \Theta^n(\mu,x) < \infty$ μ-almost everywhere, we immediately get the following.

Corollary 1.7. *Let μ be an n-rectifiable measure on $\mathbb{R}^d$. Then for μ-a.e. $x \in \mathbb{R}^d$ we have*

$$\int_0^1 b\beta_{\mu,2}(x,r)^2 \,\frac{dr}{r} < \infty.$$

1.1 Localizing Theorem 1.1 and Organization of the Paper

Theorem 1.1 follows easily from the following lemma.

Lemma 1.8. *Let μ be an n-rectifiable measure on $\mathbb{R}^d$, and let $\Gamma \subset \mathbb{R}^d$ be an n-dimensional 1-Lipschitz graph. Suppose $R \in \mathbb{D}_\Gamma$ with $\ell(R) = 1$ (see (2.2) for the defintion of $\mathbb{D}_\Gamma$). Then, for any $0 < \varepsilon < 1$, there exists a set $R' \subset R$ such that $\mu(R') \geq (1-\varepsilon)\mu(R)$ and*

$$\int_{R'}\int_0^1 \alpha_{\mu,2}(x,r)^2 \,\frac{dr}{r}\, d\mu(x) < \infty. \tag{1.3}$$

Proof of Theorem 1.1 using Lemma 1.8. Let μ be n-rectifiable. It is well known that in the definition of rectifiability (Definition I.1.1) one may replace Lipschitz images by Lipschitz graphs, or by C^1 manifolds, see e.g. [Mat95, Theorem 15.21]. Each C^1 manifold is contained in a countable union of (possibly rotated) Lipschitz graphs Γ with $\mathrm{Lip}(\Gamma) \leq 1$. Hence, there exists a countable family of n-dimensional 1-Lipschitz graphs Γ_i such that

$$\mu\Big(\mathbb{R}^d \setminus \bigcup_i \Gamma_i\Big) = 0.$$

Each Γ_i is a countable union of dyadic Γ_i-cubes $R_i^j \in \mathbb{D}_{\Gamma_i}$ satisfying $\ell(R_i^j) = 1$. Clearly, $\mu(\mathbb{R}^d \setminus \bigcup_{i,j} R_i^j) = 0$.

Now, denote the set of x where (1.2) does *not* hold by $\mathcal{B}$, and suppose that $\mu(\mathcal{B}) > 0$. Then, there exists R_i^j such that $\mu(\mathcal{B} \cap R_i^j) > 0$. Let $\varepsilon > 0$ be such that $\mu(\mathcal{B} \cap R_i^j) > 2\varepsilon\mu(R_i^j)$. Applying Lemma 1.8 to R_i^j and ε as above we reach a contradiction. Thus, $\mu(\mathcal{B}) = 0$. □

The rest of the article is dedicated to proving Lemma 1.8. Let us give a brief outline of the proof.

We introduce the necessary tools in Section 2. In Section 3 we show various estimates of α_2 coefficients, usually relying heavily on the results from [Tol12]. In Section 4 we define a family of measures $\{\nu_Q\}_{Q\in\mathbb{D}_\Gamma}$, where $\nu_Q \ll \mathcal{H}^n|_\Gamma$, and each ν_Q approximates μ in some ball around Q. Roughly speaking, ν_Q is defined by projecting the measure of Whitney cubes onto the graph Γ – but only those Whitney cubes whose sidelength is not much bigger than $\ell(Q)$. Then, we construct a tree of good cubes satisfying

$$\sum_{Q\in\mathsf{Tree}} \alpha_{\nu_Q,2}(\widetilde{B}_Q)^2\ell(Q)^n < \infty,$$

where $\widetilde{B}_Q$ are balls with the same center as the corresponding cube Q. The stopping region of the tree of good cubes is small. In Section 5 we use the estimate above to show that actually

$$\sum_{Q\in\mathsf{Tree}} \alpha_{\mu,2}(\widetilde{B}_Q)^2\ell(Q)^n < \infty.$$

Using the inequality above, we prove (1.3) with $R' = R \setminus \bigcup_{Q\in\mathsf{Stop}(\mathsf{Tree})} Q$. This finishes the proof of Lemma 1.8.

2 Preliminaries

2.1 Notation

For a Borel measure ν on $\mathbb{R}^d$ and a Borel map $T : \mathbb{R}^d \to \mathbb{R}^d$, we denote by $T_*\nu$ the pushforward of ν, that is, a measure on $\mathbb{R}^d$ such that for all Borel $A \subset \mathbb{R}^d$

$$T_*\nu(A) = \nu(T^{-1}(A)).$$

In expressions of the form $W_p(\mu_1, a\mu_2)$, the letter a will always mean the unique constant for which the total mass of $a\mu_2$ is equal to that of μ_1. In other words,

$$a = \frac{\mu_1(\mathbb{R}^d)}{\mu_2(\mathbb{R}^d)}.$$

It may happen that a appears in the same line several times, and every time refers to a different quantity. We hope that this will not cause too much confusion.

Let us once and for all fix a measure μ, an n-dimensional 1-Lipschitz graph Γ, and a constant $0 < \varepsilon < 1$ for which we are proving Lemma 1.8. We fix also a coordinate system such that $\Gamma = \{(x, A(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^d$, where $A : \mathbb{R}^n \to \mathbb{R}^{d-n}$ is a 1-Lipschitz map.

We will denote by L_0 the subspace of $\mathbb{R}^d$ formed by the points whose last $d-n$ coordinates are zeros, so that Γ is a graph over L_0. We will write Π_0 and Π_Γ to denote projections onto L_0 and Γ, respectively, orthogonal to L_0. For the sake of convenience, instead of dealing with the usual surface measure on Γ we will work with

$$\sigma = (\Pi_\Gamma)_* \mathcal{H}^n|_{L_0},$$

which is comparable to $\mathcal{H}^n|_\Gamma$ (note that for $x \in \Gamma$ we have $\sigma(B(x,r)) \approx r^n$).

Given a ball $B \subset \mathbb{R}^d$ centered at Γ denote by L_B an n-plane minimizing $\alpha_{\sigma,2}(B)$ (note that for an open ball B, it could happen that $L_B \cap B = \varnothing$). Concerning the existence of minimizers, it follows easily from the fact that W_2 metrizes weak convergence of measures (see e.g. [Vil08, Theorem 6.9]), from good compactness properties of weak convergence, and from the fact that the minimizing sequence is of the special form $\varphi_B a_{B,L_k} \mathcal{H}^n|_{L_k}$. There may be more than one minimizing plane; if that happens, we simply choose one of them.

For any Radon measure ν such that $\nu(B) > 0$ we set

$$\widehat{\alpha}_{\nu,2}(B) = \alpha_{\nu,2,L_B}(B).$$

Clearly, $\widehat{\alpha}_{\nu,2}(B) \geq \alpha_{\nu,2}(B)$. We will show that

$$\int_{R'} \int_0^1 \widehat{\alpha}_{\mu,2}(x,r)^2 \, \frac{dr}{r} \, d\mu(x) < \infty, \tag{2.1}$$

which implies (1.3).

2.2 Γ-cubes

We denote by $\mathbb{D}_{\mathbb{R}^n}, \mathbb{D}_{\mathbb{R}^d}$ the dyadic lattices on L_0 and $\mathbb{R}^d$, respectively. We assume the cubes to be half open-closed, i.e. of the form

$$Q = \left[\frac{k_1}{2^j}, \frac{k_1+1}{2^j}\right) \times \cdots \times \left[\frac{k_i}{2^j}, \frac{k_i+1}{2^j}\right),$$

where $i = n$ for $\mathbb{D}_{\mathbb{R}^n}$, $i = d$ for $\mathbb{D}_{\mathbb{R}^d}$, and $k_1, \ldots, k_i, j$, are arbitrary integers. The sidelength of Q as above will be denoted by $\ell(Q) = 2^{-j}$.

The dyadic lattice on Γ is defined as

$$\mathbb{D}_\Gamma = \{\Pi_\Gamma(Q_0) : Q_0 \in \mathbb{D}_{\mathbb{R}^n}\}. \tag{2.2}$$

The elements of $\mathbb{D}_\Gamma$ will be called Γ-cubes, or just cubes. For every $Q \in \mathbb{D}_\Gamma$ and the corresponding $Q_0 \in \mathbb{D}_{\mathbb{R}^n}$ we define the sidelength of Q as $\ell(Q) = \ell(Q_0)$, and the center of Q as $z_Q = \Pi_\Gamma(z_{Q_0})$, where z_{Q_0} is the center of Q_0. We set

$$B_Q = B(z_Q, 3\,\mathrm{diam}(Q)),$$
$$\widetilde{B}_Q = \Lambda B_Q,$$

where $\Lambda = \Lambda(n) > 1$ is a constant fixed during the proof. We define also

$$\varphi_Q = \varphi_{B_Q},$$
$$L_Q = L_{B_Q},$$
$$V(Q) = \{x \in \mathbb{R}^d : \Pi_\Gamma(x) \in Q\}.$$

Recall that L_{B_Q} is the n-plane minimizing $\alpha_{\sigma,2}(B_Q)$, and that φ_{B_Q} was defined in (I.6.8). The "V" in $V(Q)$ stands for "vertical", since $V(Q)$ is a sort of vertical cube. Note also that $Q \subset B_Q \subset \widetilde{B}_Q$ and $r(B_Q) \approx \ell(Q)$.

Given $P \in \mathbb{D}_\Gamma$, we will write $\mathbb{D}_\Gamma(P)$ to denote the family of $Q \in \mathbb{D}_\Gamma$ such that $Q \subset P$.

Remark 2.1. Let us fix $R \in \mathbb{D}_\Gamma$ with $\ell(R) = 1$ for which we are proving Lemma 1.8. Note that for $x \in R$ and $0 < r < 1$ computing $\alpha_{\mu,2}(x,r)$ involves only $\mu|_B$, where B is some ball containing R. Thus, when proving (2.1), we may and will assume that μ is a finite, compactly supported measure.

For every $e \in \{0,1\}^n$ consider the translated dyadic grid on L_0

$$\mathbb{D}^e_{\mathbb{R}^n} = \frac{1}{3}(e, 0\dots, 0) + \mathbb{D}_{\mathbb{R}^n},$$

and the corresponding translated dyadic grid on Γ

$$\mathbb{D}^e_\Gamma = \{\Pi_\Gamma(Q) : Q \in \mathbb{D}^e_{\mathbb{R}^n}\}.$$

Let us also define the translated dyadic lattice on $\mathbb{R}^d$

$$\mathbb{D}^e_{\mathbb{R}^d} = \frac{1}{3}(e, 0, \dots, 0) + \mathbb{D}_{\mathbb{R}^d}.$$

The union of all translated dyadic grids on Γ will be called an extended grid on Γ:

$$\widetilde{\mathbb{D}}_\Gamma = \bigcup_{e \in \{0,1\}^n} \mathbb{D}^e_\Gamma.$$

For each $Q \in \widetilde{\mathbb{D}}_\Gamma$ we define B_Q, φ_Q etc. in the same way as for $Q \in \mathbb{D}_\Gamma$.

The main reason for introducing the extended grid is to use a variant of the well-known one-third trick, which was already used in this context by Okikiolu [Oki92].

Lemma 2.2. *There exists $k_0 = k_0(n, \Lambda) > 0$ such that for every $Q \in \mathbb{D}_\Gamma$ with $\ell(Q) \leq 2^{-k_0}$ there exists $P_Q \in \widetilde{\mathbb{D}}_\Gamma$ satisfying $\ell(P_Q) = 2^{k_0}\ell(Q)$ and $3\widetilde{B}_Q \subset V(P_Q)$.*

Proof. First, we remark that for every $j \geq 0$ and for every $x \in L_0$ there exists $e \in \{0,1\}^n$ and $P \in \mathbb{D}^e_{\mathbb{R}^n}$ with $\ell(P) = 2^{-j}$ and $x \in \frac{2}{3}P$. For a nice proof of this fact see [Ler03, Section 3].

Now, consider the point $\Pi_0(z_Q)$. If we take $P \in \mathbb{D}^e_{\mathbb{R}^n}$ with $\ell(P) = 2^{k_0}\ell(Q)$ such that $\Pi_0(z_Q) \in \frac{2}{3}P$, we see that the n-dimensional ball $B^n(\Pi_0(z_Q), 9\Lambda\,\mathrm{diam}(Q))$ is contained in P as soon as $\frac{2^{k_0}}{3}\ell(Q) \geq 9\Lambda\,\mathrm{diam}(Q)$.

It follows that for $P_Q \in \mathbb{D}^e_\Gamma$ such that $\Pi_0(P_Q) = P$ we have $3\widetilde{B}_Q \subset V(P_Q)$. □

It may happen that the cube $P_Q \in \widetilde{\mathbb{D}}_\Gamma$ from the lemma above is not unique, so let us just fix one for each $Q \in \mathbb{D}_\Gamma$. The direction $e \in \{0,1\}^n$ such that $P_Q \in \mathbb{D}^e_\Gamma$ will be denoted by $e(Q)$, and the integer k such that $\ell(P_Q) = 2^{k_0}\ell(Q) = 2^{-k}$ will be denoted by $k(Q)$.

We will use later on the fact that

$$9\,\mathrm{diam}(Q) \leq 2^{k_0}\ell(Q) = 2^{-k(Q)}. \tag{2.3}$$

2.3 Whitney cubes

A very useful tool for approximating the measure μ close to Γ are Whitney cubes. For each $e \in \{0,1\}^n$ we consider the decomposition of $\mathbb{R}^d \setminus \Gamma$ into a family $\mathcal{W}^e$ of Whitney dyadic cubes from $\mathbb{D}^e_{\mathbb{R}^d}$. That is, the elements of $\mathcal{W}^e \subset \mathbb{D}^e_{\mathbb{R}^d}$ are pairwise disjoint, their union equals $\mathbb{R}^d \setminus \Gamma$, and there exist dimensional constants $K > 20, D_0 \geq 1$ such that for every $Q \in \mathcal{W}^e$

a) $10Q \subset \mathbb{R}^d \setminus \Gamma$,

b) $KQ \cap \Gamma \neq \varnothing$,

c) there are at most D_0 cubes $Q' \in \mathcal{W}^e$ such that $10Q \cap 10Q' \neq \varnothing$. Furthermore, for such cubes Q' we have $\ell(Q') \approx \ell(Q)$.

For the proof see [Ste70, Chapter VI, §1] or [Gra14a, Appendix J]. Moreover, it is not difficult to construct Whitney cubes in such a way that if $y \in \Gamma,\ Q \in \mathcal{W}^e$ and $B(y,r) \cap Q \neq \varnothing$, then

$$\begin{aligned} &\mathrm{diam}(Q) \leq r, \\ &Q \subset B(y, 3r), \end{aligned} \tag{2.4}$$

see [Tol15, Section 2.3] for details. We set

$$\mathcal{W}^e_k = \{Q \in \mathcal{W}^e : \ell(Q) \leq 2^{-k}\},$$

and also, for every $Q \in \mathbb{D}_\Gamma$ satisfying $\ell(Q) \leq 2^{-k_0}$,

$$\mathcal{W}_Q = \mathcal{W}^{e(Q)}_{k(Q)}.$$

Remark 2.3. It follows immediately from the definition of $k(Q)$ that if $P \in \mathcal{W}_Q$, then

$$\ell(P) \leq 2^{-k(Q)} = 2^{k_0}\ell(Q).$$

2.4 Constants and Parameters

For reader's convenience, we collect here all the constants that appear in the proof. We indicate what depends on what, and when each constant gets fixed. As usually, the notation "$C_1 = C_1(C_2)$" means that C_1 is a constant whose precise value depends on some parameter C_2. An absolute constant is a constant that does not depend on any other parameter.

Recall that the measure μ, the Lipschitz graph Γ, and the constant $0 < \varepsilon < 1$ were fixed at the very beginning, in Subsection 2.1, and also that $\mathrm{Lip}(\Gamma) \leq 1$. Moreover, in Remark 2.1 we fixed $R \in \mathbb{D}_\Gamma$ with $\ell(R) = 1$, and without loss of generality we assumed that μ is finite and compactly supported.

- Λ is an absolute constant from the definition of $\widetilde{B}_Q = \Lambda B_Q$, it is fixed in (5.2) (actually, one can take $\Lambda = 9\sqrt{2}$);
- $k_0 = k_0(n, \Lambda)$ is an integer from Lemma 2.2;
- $\varepsilon_0 = \varepsilon_0(n)$ is the constant from Lemma 3.1;
- K and D_0 are dimensional constants from the definition of Whitney cubes;
- $\lambda = \lambda(k_0, K, n, d) > 3$ is fixed in Lemma 5.1, more precisely in equation (5.1) (one can choose e.g. $\lambda = C(n,d)\, K\, 2^{k_0}$);
- $M = M(\varepsilon, \lambda, \Lambda, n, d, \mu) > 100$ is chosen in Lemma 4.2.

3 Estimates of α_2 coefficients

Recall that Γ is an n-dimensional 1-Lipschitz graph that was fixed in Subsection 2.1, $\sigma = (\Pi_\Gamma)_* \mathcal{H}^n|_{L_0}$, and that L_Q is the plane minimizing $\alpha_{\sigma,2}(B_Q)$. The next lemma states that Γ-cubes Q whose best approximating planes L_Q form big angle with L_0 have large α_2 numbers. In consequence, there are very few cubes of this kind (in fact, they form a Carleson family).

Lemma 3.1. *There exists $\varepsilon_0 = \varepsilon_0(n) > 0$ such that for every $Q \in \widetilde{\mathbb{D}}_\Gamma$ with $\measuredangle(L_Q, L_0) > 1 - \varepsilon_0$ we have*

$$\alpha_{\sigma,2}(B_Q) \gtrsim 1.$$

Proof. Suppose $Q \in \widetilde{\mathbb{D}}_\Gamma$. Take $x_k \in 0.5B_Q \cap \Gamma$, $k = 1, \dots, n$, such that $|x_k - z_Q| = 0.5r(B_Q)$, and the vectors $\{\Pi_0(x_k - z_Q)\}_k$ form an orthogonal basis of L_0. Set $B_0 = B(z_Q, \eta r(B_Q))$, $B_k = B(x_k, \eta r(B_Q))$, where $\eta = \eta(n) < 0.01$ is a small dimensional constant that will be chosen later. Clearly, for all $k = 0, \dots, n$ we have $B_k \subset B_Q$.

If L_Q does not intersect one of the balls, say B_k, then by Lemma II.3.2

$$\alpha_{\sigma,2}(B_Q)^2 r(B_Q)^{n+2} \gtrsim \int_{B_Q} \operatorname{dist}(x, L_Q)^2 \, d\sigma \\ \geq \int_{\frac{1}{2}B_k} \operatorname{dist}(x, L_Q)^2 \, d\sigma \gtrsim \eta^{n+2} r(B_Q)^{n+2}.$$

Now suppose that L_Q intersects all B_k. Then, since B_k are all centered at Γ, Γ is 1-Lipschitz, and x_k were chosen appropriately, it is easy to see that for $\eta = \eta(n)$ and $\varepsilon_0 = \varepsilon_0(n)$ small enough we have $\measuredangle(L_Q, L_0) \leq 1 - \varepsilon_0$. □

The following two lemmas will let us compare α_2 coefficients at similar scales, so that we can pass from the integral form of α_2 square function (1.2) to its dyadic variant.

Lemma 3.2 ([Tol12, Lemma 5.3])**.** *Let ν be a finite measure supported inside the ball $B' \subset \mathbb{R}^d$. Let $B \subset \mathbb{R}^d$ be another ball such that $3B \subset B'$, with $r(B) \approx r(B')$ and $\nu(B) \approx \nu(B') \approx r(B)^n$. Let L be an n-plane which intersects B and let $f : L \to [0,1]$ be a function such that $f \equiv 1$ on $3B$, $f \equiv 0$ on $L \setminus B'$. Then*

$$W_2(\varphi_B \nu, a\varphi_B \mathcal{H}^n|_L) \lesssim W_2(\nu, af\mathcal{H}^n|_L).$$

Recall that $\widehat{\alpha}_{\mu,2}(B) = \alpha_{\mu,2,L_B}(B)$.

Lemma 3.3. *Let ν be a Radon measure on $\mathbb{R}^d$, $B_1, B_2 \subset \mathbb{R}^d$ be balls centered at Γ with $3B_1 \subset B_2$, $r(B_1) \approx r(B_2)$, $\nu(B_1) \approx \nu(3B_2) \approx r(B_2)^n$. Then we have*

$$\widehat{\alpha}_{\nu,2}(B_1) \lesssim \widehat{\alpha}_{\nu,2}(B_2) + \alpha_{\sigma,2}(B_2). \tag{3.1}$$

Proof. We begin by noting that since $\nu(3B_1) \lesssim \nu(B_1)$, we have $\widehat{\alpha}_{\nu,2}(B_1) \lesssim 1$. As a result, it suffices to prove the lemma under the assumption $\alpha_{\sigma,2}(B_2) \leq \delta$ for some small constant $\delta > 0$ which will be fixed later on.

For brevity of notation set $\varphi_i = \varphi_{B_i}$, $L_i = L_{B_i}$ for $i = 1, 2$. We want to apply Lemma 3.2 with $B = B_1$, $B' = 3B_2$, $\nu = \varphi_2 \nu$, $L = L_2$, $f = \varphi_2|_L$. What needs to be checked is that $B_1 \cap L_2 \neq \varnothing$. If this intersection were empty, we would have by Lemma II.3.4

$$\alpha_{\sigma,2}(B_2)^2 r(B_2)^{n+2} \gtrsim \int_{B_2} \operatorname{dist}(x, L_2)^2 \, d\sigma \geq \int_{B_1} \operatorname{dist}(x, L_2)^2 \, d\sigma \\ \geq \int_{\frac{1}{2}B_1} \frac{1}{2} r(B_1)^2 \, d\sigma \approx r(B_1)^{n+2} \approx r(B_2)^{n+2}.$$

Thus, if $B_1 \cap L_2 = \varnothing$, then $\alpha_{\sigma,2}(B_2) \gtrsim 1$ and we arrive at a contradiction with $\alpha_{\sigma,2}(B_2) \leq \delta$ for δ small enough.

So the assumptions of Lemma 3.2 are met and we get

$$W_2(\varphi_1 \nu, a\varphi_1 \mathcal{H}^n|_{L_2}) \lesssim W_2(\varphi_2 \nu, a\varphi_2 \mathcal{H}^n|_{L_2}). \tag{3.2}$$

Similarly, taking $\nu = \varphi_2\sigma$ and $B = B_1$, $B' = 3B_2$, $L = L_2$, $f = \varphi_2|_L$ it follows that

$$W_2(\varphi_1\sigma, a\varphi_1\mathcal{H}^n|_{L_2}) \lesssim W_2(\varphi_2\sigma, a\varphi_2\mathcal{H}^n|_{L_2}). \tag{3.3}$$

Using the triangle inequality, the scaling of W_2, the fact that L_1 minimizes $\alpha_{\sigma,2}(B_1)$, and the inequalities above, we arrive at

$$\begin{aligned}
W_2(\varphi_1\nu, a\varphi_1\mathcal{H}^n|_{L_1}) &\le W_2(\varphi_1\nu, a\varphi_1\mathcal{H}^n|_{L_2}) \\
&\quad + \left(\frac{\int \varphi_1\, d\nu}{\int \varphi_1\, d\sigma}\right)^{1/2} \Big(W_2(\varphi_1\sigma, a\varphi_1\mathcal{H}^n|_{L_1}) + W_2(\varphi_1\sigma, a\varphi_1\mathcal{H}^n|_{L_2})\Big) \\
&\overset{L_1 \text{ minimizer}}{\lesssim} W_2(\varphi_1\nu, a\varphi_1\mathcal{H}^n|_{L_2}) + \left(\frac{\nu(3B_1)}{r(B_1)^n}\right)^{1/2} W_2(\varphi_1\sigma, a\varphi_1\mathcal{H}^n|_{L_2}) \\
&\lesssim W_2(\varphi_1\nu, a\varphi_1\mathcal{H}^n|_{L_2}) + W_2(\varphi_1\sigma, a\varphi_1\mathcal{H}^n|_{L_2}) \\
&\overset{(3.2),(3.3)}{\lesssim} W_2(\varphi_2\nu, a\varphi_2\mathcal{H}^n|_{L_2}) + W_2(\varphi_2\sigma, a\varphi_2\mathcal{H}^n|_{L_2}).
\end{aligned} \tag{3.4}$$

Dividing both sides by $r(B_1)^{1+n/2}$ yields

$$\widehat{\alpha}_{\nu,2}(B_1) \lesssim \widehat{\alpha}_{\nu,2}(B_2) + \alpha_{\sigma,2}(B_2).$$

□

For technical reasons we define a modified version of α_2 coefficients. For any $Q \in \widetilde{\mathbb{D}}_\Gamma$ set

$$\widetilde{\alpha}_{\nu,2}(Q) = \begin{cases} 1 & \text{if } \measuredangle(L_Q, L_0) > 1 - \varepsilon_0, \\ \ell(Q)^{-(1+\frac{n}{2})} W_2(\psi_Q\nu, a\psi_Q\mathcal{H}^n|_{L_Q}) & \text{otherwise,} \end{cases}$$

where ε_0 is as in Lemma 3.1, and

$$\psi_Q = \mathbb{1}_{V(Q)},$$
$$a = \frac{\int \psi_Q\, d\nu}{\int \psi_Q\, d\mathcal{H}^n|_{L_Q}}.$$

Recall that $\sigma = (\Pi_\Gamma)_*\mathcal{H}^n|_{L_0} \approx \mathcal{H}^n|_\Gamma$.

Lemma 3.4. *Let $\nu \ll \sigma$, $B \subset \mathbb{R}^d$ be a ball, $Q \in \widetilde{\mathbb{D}}_\Gamma$. Suppose they satisfy $3B \subset V(Q) \cap B_Q$, $r(B) \approx \ell(Q)$, $\nu(B) \approx \nu(Q) \approx \ell(Q)^n$. Then*

$$\widehat{\alpha}_{\nu,2}(B) \lesssim_{\varepsilon_0} \widetilde{\alpha}_{\nu,2}(Q) + \alpha_{\sigma,2}(B_Q).$$

Proof. Since $\nu(B) > 0$ and $\operatorname{supp}\nu \subset \Gamma$, we certainly have $\sigma(3B) \approx r(B)^n$. Moreover, our assumptions imply that $\nu(3B) \approx \nu(B)$, and so $\widehat{\alpha}_{\nu,2}(B) \lesssim 1$. Thus, we may argue in the same way as in the beginning of the proof of Lemma 3.3 to conclude that, without loss of generality, $L_Q \cap B \neq \varnothing$. Similarly, we may assume that $\measuredangle(L_Q, L_0) \leq 1 - \varepsilon_0$, because otherwise it would follow from Lemma 3.1 that $\alpha_{\sigma,2}(B_Q)$ is big.

Now, since $\measuredangle(L_Q, L_0) \leq 1 - \varepsilon_0$, we get that $V(Q) \cap L_Q \subset \kappa B_Q$ for some constant κ depending on ε_0; we may assume $\kappa > 10$.

We use Lemma 3.2 twice, first with $B = B,\ B' = \kappa B_Q,\ \nu = \psi_Q \nu,\ L = L_Q,\ f = \psi_Q\big|_L$, and then with $B = B,\ B' = \kappa B_Q,\ \nu = \varphi_Q \sigma,\ L = L_Q,\ f = \varphi_Q\big|_L$, to obtain

$$\begin{aligned} W_2(\varphi_B \nu, a\varphi_B \mathcal{H}^n|_{L_Q}) &\lesssim_\kappa W_2(\psi_Q \nu, a\psi_Q \mathcal{H}^n|_{L_Q}), \\ W_2(\varphi_B \sigma, a\varphi_B \mathcal{H}^n|_{L_Q}) &\lesssim_\kappa W_2(\varphi_Q \sigma, a\varphi_Q \mathcal{H}^n|_{L_Q}). \end{aligned}$$

By the triangle inequality, the scaling of W_2, the fact that L_B minimizes $\alpha_{\sigma,2}(B)$, and the estimates above we get

$$\begin{aligned} W_2(\varphi_B \nu, a\varphi_B \mathcal{H}^n|_{L_B}) &\leq W_2(\varphi_B \nu, a\varphi_B \mathcal{H}^n|_{L_Q}) \\ &\quad + \left(\frac{\int \varphi_B \, d\nu}{\int \varphi_B \, d\sigma}\right)^{1/2} \Big(W_2(\varphi_B \sigma, a\varphi_B \mathcal{H}^n|_{L_B}) + W_2(\varphi_B \sigma, a\varphi_B \mathcal{H}^n|_{L_Q}) \Big) \\ &\lesssim W_2(\varphi_B \nu, a\varphi_B \mathcal{H}^n|_{L_Q}) + \left(\frac{\nu(3B)}{r(B)^n}\right)^{1/2} W_2(\varphi_B \sigma, a\varphi_B \mathcal{H}^n|_{L_Q}) \\ &\lesssim_\kappa W_2(\psi_Q \nu, a\psi_Q \mathcal{H}^n|_{L_Q}) + W_2(\varphi_Q \sigma, a\varphi_Q \mathcal{H}^n|_{L_Q}). \end{aligned}$$

Dividing both sides by $r(B)^{1+n/2}$ yields the desired result. □

We will need an estimate which is a slight modification of [Tol12, Lemma 6.2]. In order to formulate it, let us introduce the usual martingale difference operator. Recall that if $P \in \mathbb{D}_\Gamma^e$ for some $e \in \{0,1\}^n$, then $P' \in \mathbb{D}_\Gamma^e$ is a child of P if $P' \subset P$ and $\ell(P') = \frac{1}{2}\ell(P)$. Children of $P \in \mathbb{D}_{\mathbb{R}^n}^e$ are defined analogously.

Given $g \in L^1_{loc}(\sigma)$ and $P \in \mathbb{D}_\Gamma^e$ we set

$$\Delta_P^\sigma g(x) = \begin{cases} \frac{\int_{P'} g \, d\sigma}{\sigma(P')} - \frac{\int_P g \, d\sigma}{\sigma(P)} & : x \in P',\ P' \text{ a child of } P, \\ 0 & : x \notin P. \end{cases}$$

Given $h \in L^1_{loc}(\mathcal{H}^n|_{L_0})$ and $P \in \mathbb{D}_{\mathbb{R}^n}^e$ we define analogously $\Delta_P h(x)$:

$$\Delta_P h(x) = \begin{cases} \frac{\int_{P'} h \, d\mathcal{H}^n}{\ell(P')^n} - \frac{\int_P h \, d\mathcal{H}^n}{\ell(P)^n} & : x \in P',\ P' \text{ a child of } P, \\ 0 & : x \notin P. \end{cases}$$

Recall that for $g \in L^2(\sigma)$ we have

$$g = \sum_{P \in \mathbb{D}_\Gamma^e} \Delta_P^\sigma g,$$

in the sense of $L^2(\sigma)$, and

$$\|g\|^2_{L^2(\sigma)} = \sum_{P\in\mathbb{D}^e_\Gamma} \|\Delta^\sigma_P g\|^2_{L^2(\sigma)},$$

for details see e.g. [Dav91, Part I] or [Gra14a, Section 6.4.2].

Let us introduce also some additional vocabulary. We will say that a family of cubes $\mathsf{Tree} \subset \mathbb{D}^e_\Gamma$ is a tree with root R_0 if it satisfies:

(T1) $R_0 \in \mathsf{Tree}$, and for every $Q \in \mathsf{Tree}$ we have $Q \subset R_0$,

(T2) for every $Q \in \mathsf{Tree}$ such that $Q \neq R_0$, the parent of Q also belongs to Tree.

By iterating (T2), we can actually see that if $Q \in \mathsf{Tree}$, then all the intermediate cubes $Q \subset P \subset R_0$ also belong to Tree.

The stopping region of Tree, denoted by $\mathsf{Stop}(\mathsf{Tree})$, is the family of all the cubes $P \in \mathbb{D}^e_\Gamma(R_0)$ satisfying:

(S) $P \notin \mathsf{Tree}$, but the parent of P belongs to Tree.

It is easy to see that the cubes from $\mathsf{Stop}(\mathsf{Tree})$ are pairwise disjoint, and that they are maximal descendants of R_0 not belonging to Tree. Moreover, for every $x \in R_0$ we have either $x \in P$ for some $P \in \mathsf{Stop}(\mathsf{Tree})$, or $x \in Q_k$ for a sequence of cubes $\{Q_k\}_k \subset \mathsf{Tree}$ satisfying $\ell(Q_k) \xrightarrow{k\to\infty} 0$.

The following lemma is a modified version of [Tol12, Lemma 6.2].

Lemma 3.5. *Let ν be a Radon measure on Γ of the form $\nu = g\sigma$, with $g \in L^1(\sigma)$, $0 \le g \le C$ for some $C > 1$. Consider a cube $Q \in \widetilde{\mathbb{D}}_\Gamma$ and a tree Tree with root Q. Suppose that for all $P \in \mathsf{Tree}$ we have $C^{-1}\ell(P)^n \le \nu(P) \le C\ell(P)^n$. Then, we have*

$$\widetilde{\alpha}_{\nu,2}(Q)^2 \lesssim_{\varepsilon_0,C} \alpha_{\sigma,2}(B_Q)^2 + \sum_{P\in\mathsf{Tree}} \|\Delta^\sigma_P g\|^2_{L^2(\sigma)} \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{S\in\mathsf{Stop}(\mathsf{Tree})} \frac{\ell(S)^2}{\ell(Q)^{n+2}}\nu(S), \tag{3.5}$$

and

$$\sum_{P\in\mathsf{Tree}} \|\Delta^\sigma_P g\|^2_{L^2(\sigma)} \le C\|g\|_{L^1(\sigma)} = C\nu(\Gamma). \tag{3.6}$$

In the proof we will use [Tol12, Remark 3.14]. It can be thought of as a flat counterpart of Lemma 3.5 – it is valid for more general measures ν (even more general then what we state below), but at the price of assuming $\Gamma = L_0 \simeq \mathbb{R}^n$.

Lemma 3.6 (simplified [Tol12, Remark 3.14])**.** *Suppose $Q \in \mathbb{D}_{\mathbb{R}^n}$ is a dyadic cube in $\mathbb{R}^n$ and Tree is a tree with root Q. Consider a measure $\nu = g\mathcal{H}^n|_Q$ such that $\nu(P) \approx \ell(P)^n$ for $P \in \mathsf{Tree}$. Then,*

$$W_2(\nu, a\mathcal{H}^n|_Q) \lesssim \sum_{P\in\mathsf{Tree}} \|\Delta_P g\|^2_{L^2(\mathcal{H}^n)}\ell(P)\ell(Q) + \sum_{S\in\mathsf{Stop}(\mathsf{Tree})} \ell(S)^2\nu(S).$$

Remark 3.7. The definition of a tree of dyadic cubes in [Tol12, p. 492] is slightly more restrictive than the one we adopted. Apart from conditions (T1) and (T2), they also satisfy

(T3) if $Q \in \mathsf{Tree}$, then either all the children of Q belong to Tree, or none of them.

Equivalently, if $Q \in \mathsf{Tree}$, and Q is not the root, then all the brothers of Q also belong to Tree. To underline the difference between the two notions, sometimes the terms *coherent* and *semicoherent* family of cubes are used. The former refers to trees satisfying (T1–T3), the latter to those satisfying (T1–T2).

Nevertheless, [Tol12, Remark 3.14] cited above is true for both coherent and semicoherent families of cubes. That is, property (T3) is never used in the proof of either [Tol12, Remark 3.14] or the preceding "key lemma" [Tol12, Lemma 3.13].

We are finally ready to prove Lemma 3.5.

Proof of Lemma 3.5. Let $L = L_Q$. If $\measuredangle(L, L_0) > 1 - \varepsilon_0$, then by Lemma 3.1 and the definition of $\widetilde{\alpha}_{\nu,2}(Q)$

$$\widetilde{\alpha}_{\nu,2}(Q)^2 = 1 \lesssim \alpha_{\sigma,2}(B_Q)^2,$$

and we are done. Now assume that $\measuredangle(L, L_0) \le 1 - \varepsilon_0$.

Let $\tilde{\Pi}_L$ be the projection from $\mathbb{R}^d$ onto L, orthogonal to L_0. We also consider the flat measure $\sigma_L = (\tilde{\Pi}_L)_*\sigma = (\tilde{\Pi}_L)_*\mathcal{H}^n|_{L_0} = c_L\mathcal{H}^n|_L$ (recall that Π_Γ is a projection orthogonal to L_0, so that $\tilde{\Pi}_L \circ \Pi_\Gamma = \tilde{\Pi}_L$). Define $g_0 : L_0 \to \mathbb{R}$ as $g_0 = g \circ \Pi_\Gamma$.

By triangle inequality

$$\begin{aligned} W_2(\psi_Q\nu, a\psi_Q\mathcal{H}^n|_L) &= W_2(\psi_Q\nu, a\psi_Q\sigma_L) \\ &\le W_2(\psi_Q\nu, \psi_Q(\tilde{\Pi}_L)_*\nu) + W_2(\psi_Q(\tilde{\Pi}_L)_*\nu, a\psi_Q\sigma_L). \quad (3.7) \end{aligned}$$

The first term from the right hand side is estimated by $\alpha_{\sigma,2}(B_Q)$:

$$\begin{aligned} W_2(\psi_Q\nu, \psi_Q(\tilde{\Pi}_L)_*\nu)^2 &\le \int_Q |x - \tilde{\Pi}_L(x)|^2 \, d\nu(x) \approx_{\varepsilon_0} \int_Q \operatorname{dist}(x, L)^2 \, d\nu(x) \\ &\lesssim_C \int_Q \operatorname{dist}(x, L)^2 \, d\sigma(x) \lesssim \alpha_{\sigma,2}(B_Q)^2\ell(Q)^{n+2}. \end{aligned}$$

We estimate the second term from the right hand side of (3.7) using the fact that $\Pi_0|_{L\cap V(Q)} : L \cap V(Q) \to L_0 \cap V(Q)$ is bilipschitz, with a constant depending on ε_0 (because $\measuredangle(L, L_0) \le 1 - \varepsilon_0$):

$$\begin{aligned} W_2(\psi_Q(\tilde{\Pi}_L)_*\nu, a\psi_Q\sigma_L) &\approx_{\varepsilon_0} W_2(\psi_Q(\Pi_0)_*((\tilde{\Pi}_L)_*\nu), a\psi_Q(\Pi_0)_*\sigma_L) \\ &= W_2(\psi_Q g_0\mathcal{H}^n|_{L_0}, a\psi_Q\mathcal{H}^n|_{L_0}). \end{aligned}$$

By Lemma 3.6 we have

$$W_2(\psi_Q g_0 \mathcal{H}^n|_{L_0}, a\psi_Q \mathcal{H}^n|_{L_0})^2 \lesssim \sum_{P' \in \mathsf{Tree}_{\mathbb{R}^n}} \|\Delta_{P'} g_0\|^2_{L^2(L_0)} \ell(P')\ell(Q) + \sum_{S \in \mathsf{Stop}(\mathsf{Tree})} \ell(S)^2 \nu(S),$$

where $\mathsf{Tree}_{\mathbb{R}^n} \subset \mathbb{D}_{\mathbb{R}^n}$ is the tree formed by cubes $P' = \Pi_0(P)$, $P \in \mathsf{Tree}$, and $L^2(L_0) = L^2(\mathcal{H}^n|_{L_0})$.

Using (3.7) and the estimates above we get

$$W_2(\psi_Q \nu, a\psi_Q \mathcal{H}^n|_L)^2 \lesssim_{\varepsilon_0} \alpha_{\sigma,2}(B_Q)^2 \ell(Q)^{n+2} + \sum_{P' \in \mathsf{Tree}_{\mathbb{R}^n}} \|\Delta_{P'} g_0\|^2_{L^2(L_0)} \ell(P')\ell(Q) + \sum_{S \in \mathsf{Stop}(\mathsf{Tree})} \ell(S)^2 \nu(S).$$

We conclude the proof of (3.5) by noting that for each $P \in \mathsf{Tree}$

$$\|\Delta_P^\sigma g\|_{L^2(\sigma)} = \|\Delta_{\Pi_0(P)} g_0\|_{L^2(L_0)}.$$

The estimate (3.6) follows trivially from the fact that if $e \in \{0,1\}^n$ is such that $Q \in \mathbb{D}_\Gamma^e$, then

$$\sum_{P \in \mathsf{Tree}} \|\Delta_P^\sigma g\|^2_{L^2(\sigma)} \le \sum_{P \in \mathbb{D}_\Gamma^e} \|\Delta_P^\sigma g\|^2_{L^2(\sigma)} = \|g\|^2_{L^2(\sigma)} \le C\|g\|_{L^1(\sigma)}.$$

□

We would like to use Lemma 3.5 also on measures with unbounded density. An approximation argument allows us to get rid of the boundedness assumption, at least if we assume additionally that $\nu(B_P) \le C\ell(P)^n$ for $P \in \mathsf{Tree}$.

Lemma 3.8. *Let $\nu = g\sigma$ with $g \in L^1(\sigma)$, $g \ge 0$. Consider a cube $Q \in \widetilde{\mathbb{D}}_\Gamma$ and a tree Tree with root Q. Suppose there exists $C > 1$ such that for all $P \in \mathsf{Tree}$ we have $C^{-1}\ell(P)^n \le \nu(P) \le \nu(B_P) \le C\ell(P)^n$. Then, we have*

$$\widetilde{\alpha}_{\nu,2}(Q)^2 \lesssim_{\varepsilon_0, C} \alpha_{\sigma,2}(B_Q)^2 + \sum_{P \in \mathsf{Tree}} \|\Delta_P^\sigma g\|^2_{L^2(\sigma)} \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{S \in \mathsf{Stop}(\mathsf{Tree})} \frac{\ell(S)^2}{\ell(Q)^{n+2}} \nu(S), \tag{3.8}$$

and

$$\sum_{P \in \mathsf{Tree}} \|\Delta_P^\sigma g\|^2_{L^2(\sigma)} \le C\|g\|_{L^1(\sigma)} = C\nu(\Gamma). \tag{3.9}$$

We divide the proof into smaller pieces. Let $\mathsf{Stop} = \mathsf{Stop}(\mathsf{Tree})$. First, we define the set of good points as

$$G = Q \setminus \bigcup_{P \in \mathsf{Stop}} P.$$

Note that the points from $x \in G$ are not contained in any stopping cube, and so there are arbitrarily small cubes $P \in \mathsf{Tree}$ containing x. We introduce the following approximating measure:

$$\tilde{\nu} = \nu|_G + \sum_{S \in \mathsf{Stop}} \frac{\nu(S)}{\sigma(S)} \sigma|_S.$$

It is clear that for $Q \in \mathsf{Tree} \cup \mathsf{Stop}$ we have $\tilde{\nu}(Q) = \nu(Q)$. Moreover, for $Q \in \mathsf{Tree}$

$$C^{-1}\ell(Q)^n \leq \tilde{\nu}(Q) = \nu(Q) \leq C\ell(Q)^n. \tag{3.10}$$

On the other hand, each $S \in \mathsf{Stop}$ is a child of some $Q \in \mathsf{Tree}$, so that

$$\tilde{\nu}(S) = \nu(S) \leq \nu(Q) \leq C\ell(Q)^n = 2^n C\ell(S)^n. \tag{3.11}$$

Lemma 3.9. *We have*

$$\left\| \frac{d\tilde{\nu}}{d\sigma} \right\|_{L^\infty(\sigma)} \lesssim C.$$

Proof. It is trivial that for $x \in S \in \mathsf{Stop}$ the density is constant and

$$\frac{d\tilde{\nu}}{d\sigma}(x) = \frac{\nu(S)}{\sigma(S)} = \frac{\nu(S)}{\ell(S)^n} \overset{(3.11)}{\leq} 2^n\, C.$$

On the other hand, by the definition of $\tilde{\nu}$, for σ-a.e. $x \in G$ we have $\frac{d\tilde{\nu}}{d\sigma}(x) = \frac{d\nu}{d\sigma}(x) = g(x)$. Moreover, for σ-a.e. $x \in G$ we have a sequence of cubes $Q_j \in \mathsf{Tree}$ such that $\ell(Q_j) = 2^{-j}$ and $x \in Q_j$. Note that there exists some integer $j_0 > 0$ (depending on dimension) such that

$$Q_{j+j_0} \subset B(x, 2^{-j}) \subset B_{Q_j}.$$

It follows that

$$\frac{d\tilde{\nu}}{d\sigma}(x) = \frac{d\nu}{d\sigma}(x) = \lim_{j\to\infty} \frac{\nu(B(x,2^{-j}))}{\sigma(B(x,2^{-j}))} \leq \lim_{j\to\infty} \frac{\nu(B_{Q_j})}{\sigma(Q_{j+j_0})} \leq \lim_{j\to\infty} \frac{C\ell(Q_j)^n}{\ell(Q_{j+j_0})^n} = C\, 2^{nj_0}.$$

Thus,

$$\left\| \frac{d\tilde{\nu}}{d\sigma} \right\|_{L^\infty(\sigma)} \lesssim C.$$

□

Let $\tilde{g} \in L^1(\sigma) \cap L^\infty(\sigma)$ be such that $\tilde{\nu} = \tilde{g}\sigma$. Applying Lemma 3.5 to $\tilde{\nu}$ yields

$$\widetilde{\alpha}_{\tilde{\nu},2}(Q)^2 \lesssim_{\varepsilon_0, C} \alpha_{\sigma,2}(B_Q)^2 + \sum_{P \in \mathsf{Tree}} \|\Delta_P^\sigma \tilde{g}\|^2_{L^2(\sigma)} \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{S \in \mathsf{Stop}} \frac{\ell(S)^2}{\ell(Q)^{n+2}} \tilde{\nu}(S), \tag{3.12}$$

and

$$\sum_{P\in\mathsf{Tree}} \|\Delta_P^\sigma \tilde{g}\|_{L^2(\sigma)}^2 \leq C\|\tilde{g}\|_{L^1(\sigma)} = C\tilde{\nu}(\Gamma) = C\nu(\Gamma). \tag{3.13}$$

Observe that for $P\in\mathsf{Tree}$ we have

$$\Delta_P^\sigma \tilde{g} = \Delta_P^\sigma g. \tag{3.14}$$

Indeed, for $x\notin P$ both quantities are equal to zero. For $x\in P'\subset P$, where P' is a child of P, we have $P'\in\mathsf{Tree}\cup\mathsf{Stop}$, and so

$$\Delta_P^\sigma \tilde{g}(x) = \frac{\int_{P'} \tilde{g}\,d\sigma}{\sigma(P')} - \frac{\int_P \tilde{g}\,d\sigma}{\sigma(P)} = \frac{\tilde{\nu}(P')}{\sigma(P')} - \frac{\tilde{\nu}(P)}{\sigma(P)} = \frac{\nu(P')}{\sigma(P')} - \frac{\nu(P)}{\sigma(P)} = \Delta_P^\sigma g.$$

Hence, (3.9) follows immediately from (3.13).

Since for $S\in\mathsf{Stop}$ we have $\tilde{\nu}(S)=\nu(S)$, we can use (3.14) to transform (3.12) into

$$\widetilde{\alpha}_{\tilde{\nu},2}(Q)^2 \lesssim_{\varepsilon_0,C} \alpha_{\sigma,2}(B_Q)^2 + \sum_{P\in\mathsf{Tree}} \|\Delta_P^\sigma g\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{S\in\mathsf{Stop}} \frac{\ell(S)^2}{\ell(Q)^{n+2}}\nu(S). \tag{3.15}$$

In order to reach (3.8) and finish the proof of Lemma 3.8, we only need to show how to pass from the estimate on $\widetilde{\alpha}_{\tilde{\nu},2}(Q)$ (3.15) to one on $\widetilde{\alpha}_{\nu,2}(Q)$.

Proof of Lemma 3.8. Recall that if $\measuredangle(L_Q,L_0)>1-\varepsilon_0$, then $\widetilde{\alpha}_{\nu,2}(Q)=1$, but at the same time $\alpha_{\sigma,2}(B_Q)\gtrsim 1$ by Lemma 3.1, so this case is trivial. Suppose $\measuredangle(L_Q,L_0)\leq 1-\varepsilon_0$. We define a transport plan between $\psi_Q\tilde{\nu}$ and $\psi_Q\nu$:

$$d\pi(x,y) = \mathbb{1}_{Q\cap G}(x)d\nu(x)d\delta_x(y) + \sum_{S\in\mathsf{Stop}} \frac{\mathbb{1}_S(x)\mathbb{1}_S(y)}{\sigma(S)} d\nu(x)d\sigma(y),$$

and we estimate

$$W_2(\psi_Q\tilde{\nu},\psi_Q\nu)^2 \leq \int |x-y|^2\,d\pi(x,y) \lesssim \sum_{S\in\mathsf{Stop}} \ell(S)^2\nu(S).$$

From the triangle inequality, the bound above, and (3.15), we get that

$$\begin{aligned}
\widetilde{\alpha}_{\nu,2}(Q)^2 &\approx \ell(Q)^{-(n+2)} W_2(\psi_Q\nu, a\psi_Q\mathcal{H}^n|_{L_Q})^2 \\
&\lesssim \ell(Q)^{-(n+2)}\Big(W_2(\psi_Q\tilde{\nu},\psi_Q\nu)^2 + W_2(\psi_Q\tilde{\nu}, a\psi_Q\mathcal{H}^n|_{L_Q})^2\Big) \\
&\lesssim_{\varepsilon_0,C} \alpha_{\sigma,2}(B_Q)^2 + \sum_{P\in\mathsf{Tree}} \|\Delta_P^\sigma g\|_{L^2(\sigma)}^2 \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{S\in\mathsf{Stop}} \frac{\ell(S)^2}{\ell(Q)^{n+2}}\nu(S).
\end{aligned}$$

□

4 Approximating measures

We will construct a family of measures on Γ that will approximate μ. For every Whitney cube $P \in \mathcal{W}^e$ we define $g_P : \Gamma \to \mathbb{R}$ as

$$g_P(x) = \frac{\mu(P)}{\ell(P)^n} \mathbb{1}_{\Pi_\Gamma(P)}(x).$$

Note that $\int g_P \, d\sigma = \mu(P)$.

Given $e \in \{0,1\}^n,\ k \in \mathbb{Z}$, we define the following measures supported on Γ:

$$\nu^e = \mu|_\Gamma + \left(\sum_{P \in \mathcal{W}^e} g_P \right) \sigma,$$

$$\nu_k^e = \mu|_\Gamma + \left(\sum_{P \in \mathcal{W}_k^e} g_P \right) \sigma.$$

Moreover, for every $Q \in \mathbb{D}_\Gamma$ with $\ell(Q) \le 2^{-k_0}$ we set

$$\nu_Q = \nu_{k(Q)}^{e(Q)} = \mu|_\Gamma + \left(\sum_{P \in \mathcal{W}_Q} g_P \right) \sigma.$$

Note that, since we assume μ is finite and compactly supported (see Remark 2.1), all the measures $\nu^e,\ \nu_k^e$, are also finite and compactly supported.

We defined ν_Q in such a way that, for "good" $Q \in \mathbb{D}_\Gamma$, the measures $\mu|_{B_Q}$ and $\nu_Q\Big|_{B_Q}$ are close in the W_2 distance. This will be shown in Section 5. The rest of this section is dedicated to the construction of a tree of "good cubes".

Recall that $R \in \mathbb{D}_\Gamma$ is a Γ-cube fixed in Remark 2.1, and $0 < \varepsilon \ll 1$ is a small constant fixed in Subsection 2.1.

Lemma 4.1. *Let $\lambda > 3$. Then, there exist a big constant $M = M(\varepsilon, \lambda, \Lambda, n, d, \mu) \gg 1$ and a tree of good cubes $\mathsf{Tree} = \mathsf{Tree}(\lambda, \varepsilon, M) \subset \mathbb{D}_\Gamma(R)$ with root R, such that for every $Q \in \mathsf{Tree}$ we have*

$$\mu(\lambda \widetilde{B}_Q) \le M \ell(Q)^n,$$
$$\mu(Q) \ge M^{-1} \ell(Q)^n,$$

the stopping region $\mathsf{Stop} = \mathsf{Stop}(\mathsf{Tree})$ is small:

$$\mu\left(\bigcup_{Q \in \mathsf{Stop}} Q \right) < \varepsilon,$$

and $\widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2$ satisfy the packing condition:

$$\sum_{Q \in \mathsf{Tree}} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2 \ell(Q)^n < \infty. \tag{4.1}$$

We split the proof into several small lemmas. First, we define auxiliary families of good cubes in $\mathbb{D}_\Gamma^e$ using a standard stopping time argument.

For each $e \in \{0,1\}^n$ there exists a finite collection of cubes $\{R_i^e\} \subset \mathbb{D}_\Gamma^e$ such that $\ell(R_i^e) = 1$, $R_i^e \cap R \neq \varnothing$. Set $R^e = \bigcup_i R_i^e$. Let $M \gg 1$ be constant to be fixed later on, and set

$$\begin{aligned}\mathcal{H}D_{\nu,0}^e &= \{Q \in \mathbb{D}_\Gamma^e : \ Q \subset R^e, \ \nu^e(\lambda \widetilde{B}_Q) > M\ell(Q)^n\},\\ \mathcal{H}D_{\mu,0}^e &= \{Q \in \mathbb{D}_\Gamma^e : \ Q \subset R^e, \ \mu(\lambda \widetilde{B}_Q) > M\ell(Q)^n\},\\ \mathsf{LD}_0^e &= \{Q \in \mathbb{D}_\Gamma^e : \ Q \subset R^e, \ \mu(Q) < M^{-1}\ell(Q)^n\}.\end{aligned}$$

$\mathcal{H}D$ and LD stand for "high density" and "low density". Let $\mathsf{Stop}^e \subset \mathbb{D}_\Gamma^e$ be the family of maximal with respect to inclusion cubes from $\mathcal{H}D_{\nu,0}^e \cup \mathcal{H}D_{\mu,0}^e \cup \mathsf{LD}_0^e$, and set $\mathcal{H}D_\nu^e = \mathcal{H}D_{\nu,0}^e \cap \mathsf{Stop}^e$, $\mathcal{H}D_\mu^e = \mathcal{H}D_{\mu,0}^e \cap \mathsf{Stop}^e$, $\mathsf{LD}^e = \mathsf{LD}_0^e \cap \mathsf{Stop}^e$. Note that cubes from Stop^e are pairwise disjoint. We define Tree^e as the family of those cubes from $\bigcup_i \mathbb{D}_\Gamma^e(R_i^e)$ which are not contained in any cube from Stop^e. Actually, this might not be a tree, but it is a finite collection of trees with roots R_i^e.

Lemma 4.2. *For $M = M(\varepsilon, \lambda, \Lambda, n, d, \mu)$ big enough, we have for all $e \in \{0,1\}^n$*

$$\mu\Bigg(\bigcup_{Q \in \mathsf{Stop}^e} Q\Bigg) < \frac{\varepsilon}{2^n}. \tag{4.2}$$

Proof. Let $e \in \{0,1\}^n$. It is easy to see that the measure of LD^e is small: for every $Q \in \mathsf{LD}^e$ we have $\mu(Q) \le M^{-1}\sigma(Q)$, so

$$\mu\Bigg(\bigcup_{Q \in \mathsf{LD}^e} Q\Bigg) \le M^{-1}\sigma(R^e) \approx M^{-1}. \tag{4.3}$$

To estimate the measure of $\mathcal{H}D_\mu^e$, define for some big $N \gg 1$

$$H_N = \{x \in \mathbb{R}^d : \mu(B(x,r)) > Nr^n \text{ for some } r \in (0,1)\}.$$

Since μ is n-rectifiable, the density $\Theta^n(x,\mu)$ exists, and is positive and finite μ-a.e. Moreover, recall that $\mu(\mathbb{R}^d)$ is finite. This implies that for $N = N(\mu, \varepsilon, n)$ big enough

$$\mu(H_N) \le \frac{\varepsilon}{2^{n+2}}.$$

We will show that, if M is chosen big enough, then for all $Q \in \mathcal{H}D_\mu^e$ we have $Q \subset H_N$. Indeed, let $x \in Q \in \mathcal{H}D_\mu^e$. Then $B(x, 2\lambda r(\widetilde{B}_Q)) \supset \lambda \widetilde{B}_Q$, and so

$$\mu(B(x, 2\lambda r(\widetilde{B}_Q))) \ge \mu(\lambda \widetilde{B}_Q) > M\ell(Q)^n > N(6\lambda\Lambda \operatorname{diam}(Q))^n = N(2\lambda r(\widetilde{B}_Q))^n,$$

for M big enough with respect to N, λ, Λ, n. Moreover, note that for $Q \in \mathcal{H}D_\mu^e$ we have

$$\frac{\mu(\mathbb{R}^d)}{M} > \ell(Q)^n \approx_\Lambda r(\widetilde{B}_Q)^n,$$

and so taking M big enough (depending on $\mu(\mathbb{R}^d), \lambda, \Lambda, n$) we can ensure that all $Q \in \mathcal{H}D^e_\mu$ satisfy $2\lambda r(\widetilde{B}_Q) < 1$. Thus, $x \in H_N$, and we conclude that

$$\mu\Bigg(\bigcup_{Q\in\mathcal{H}D^e_\mu} Q\Bigg) \leq \mu(H_N) \leq \frac{\varepsilon}{2^{n+2}}. \tag{4.4}$$

Since ν^e is a finite n-rectifiable measure, we can argue in the same way as above to get

$$\nu^e\Bigg(\bigcup_{Q\in\mathcal{H}D^e_\nu} Q\Bigg) \leq \frac{\varepsilon}{2^{n+2}}.$$

Smallness of $\mu(\bigcup_{Q\in\mathcal{H}D^e_\nu} Q)$ follows from the fact that $\mu|_\Gamma \leq \nu^e$. Putting this together with (4.3) and (4.4) we get

$$\mu\Bigg(\bigcup_{Q\in\mathsf{Stop}^e} Q\Bigg) < \frac{\varepsilon}{2^n}.$$

We take M so big that the above holds for all $e \in \{0,1\}^n$, and the proof is finished. □

For each $e \in \{0,1\}^n,\ k = 0,1,2,\dots$, let g^e_k be the density of ν^e_k with respect to σ. Note that, due to the definition of Tree^e, for any $Q \in \mathsf{Tree}^e$ we have

$$M^{-1}\,\ell(Q)^n \leq \nu^e_k(Q) \leq \nu^e_k(B_Q) \leq M\,\ell(Q)^n.$$

Hence, given a cube $Q \in \mathsf{Tree}^e$ with $\ell(Q) = 2^{-k}$, we can estimate $\widetilde{\alpha}_{\nu^e_k,2}(Q)^2$ using Lemma 3.8 (applied to ν^e_k and $\mathsf{Tree} = \{P \in \mathsf{Tree}^e\ :\ P \subset Q\}$) to get

$$\widetilde{\alpha}_{\nu^e_k,2}(Q)^2 \lesssim_{\varepsilon_0,M} \alpha_{\sigma,2}(B_Q)^2 + \sum_{\substack{P\in\mathsf{Tree}^e\\ P\subset Q}} \|\Delta^\sigma_P g^e_k\|^2_{L^2(\sigma)} \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{\substack{S\in\mathsf{Stop}^e\\ S\subset Q}} \frac{\ell(S)^2}{\ell(Q)^{n+2}} \nu^e_k(S). \tag{4.5}$$

The following lemma states that the right hand side of this estimate can be made independent of k.

Lemma 4.3. *For all $Q \in \mathsf{Tree}^e$ with $\ell(Q) = 2^{-k},\ k \geq 0$, we have*

$$\widetilde{\alpha}_{\nu^e_k,2}(Q)^2 \lesssim_{\varepsilon_0,M} \alpha_{\sigma,2}(B_Q)^2 + \sum_{\substack{P\in\mathsf{Tree}^e\\ P\subset Q}} \|\Delta^\sigma_P g^e_0\|^2_{L^2(\sigma)} \frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{\substack{S\in\mathsf{Stop}^e\\ S\subset Q}} \frac{\ell(S)^2}{\ell(Q)^{n+2}} \nu^e(S). \tag{4.6}$$

Moreover,

$$\sum_{P\in\mathsf{Tree}^e} \|\Delta^\sigma_P g^e_0\|^2_{L^2(\sigma)} \lesssim M\|g^e_0\|_{L^1(\sigma)} = M\nu^e_0(\Gamma) \leq M\mu(\mathbb{R}^d). \tag{4.7}$$

Proof. We claim that for $P \in \mathsf{Tree}^e$ with $\ell(P) \le 2^{-k}$ (in particular, for $P \in \mathsf{Tree}^e$ such that $P \subset Q$) we have

$$\Delta^\sigma_P g^e_k = \Delta^\sigma_P g^e_0. \tag{4.8}$$

Indeed, for $x \notin P$ both sides of (4.8) are zero. For $x \in P' \subset P$, where $P' \in \mathsf{Tree}^e \cup \mathsf{Stop}^e$ is a child of P, we have

$$\begin{aligned}\Delta^\sigma_P g^e_0(x) - \Delta^\sigma_P g^e_k(x) &= \frac{\nu^e_0(P') - \nu^e_k(P')}{\ell(P')^n} - \frac{\nu^e_0(P) - \nu^e_k(P)}{\ell(P)^n}\\ &= \ell(P')^{-n}\left(\sum_{S\in\mathcal{W}^e_0\setminus\mathcal{W}^e_k} \frac{\mu(S)}{\ell(S)^n}\sigma(P'\cap\Pi_\Gamma(S))\right)\\ &\qquad - \ell(P)^{-n}\left(\sum_{S\in\mathcal{W}^e_0\setminus\mathcal{W}^e_k} \frac{\mu(S)}{\ell(S)^n}\sigma(P\cap\Pi_\Gamma(S))\right).\end{aligned}$$

The Whitney cubes S in the sums above above satisfy $\ell(S) > 2^{-k} \ge \ell(P)$, and moreover we have $\Pi_\Gamma(S) \in \mathbb{D}^e_\Gamma$. Hence, we either have $P \cap \Pi_\Gamma(S) = P$ or $P \cap \Pi_\Gamma(S) = \varnothing$. The same is true for P'. Moreover, we have $P\cap\Pi_\Gamma(S) \ne \varnothing$ if and only if $P' \cap \Pi_\Gamma(S) \ne \varnothing$. It follows that the right hand side above is equal to

$$\sum_{\substack{S\in\mathcal{W}^e_0\setminus\mathcal{W}^e_k\\ P'\cap\Pi_\Gamma(S)\ne\varnothing}} \frac{\mu(S)}{\ell(S)^n} - \sum_{\substack{S\in\mathcal{W}^e_0\setminus\mathcal{W}^e_k\\ P\cap\Pi_\Gamma(S)\ne\varnothing}} \frac{\mu(S)}{\ell(S)^n} = 0.$$

Thus $\Delta^\sigma_P g^e_k = \Delta^\sigma_P g^e_0$. Using this equality, and also the fact that $\nu^e_k \le \nu^e$, we transform (4.5) into

$$\widetilde{\alpha}_{\nu^e_k,2}(Q)^2 \lesssim_{\varepsilon_0,M} \alpha_{\sigma,2}(B_Q)^2 + \sum_{\substack{P\in\mathsf{Tree}^e\\ P\subset Q}} \|\Delta^\sigma_P g^e_0\|^2_{L^2(\sigma)}\frac{\ell(P)}{\ell(Q)^{n+1}} + \sum_{\substack{P\in\mathsf{Stop}^e\\ P\subset Q}} \frac{\ell(P)^2}{\ell(Q)^{n+2}}\nu^e(P). \tag{4.9}$$

Concerning (4.7), it is an immediate consequence of (3.9) when we apply Lemma 3.8 to ν^e_0 and the trees $\{Q \in \mathsf{Tree}^e : Q \subset R^e_i\}$ (recall that the union of such trees gives the entire Tree^e). □

We finally define Tree as the collection of cubes $Q \in \mathbb{D}_\Gamma$ such that for every $e \in \{0,1\}^n$ there exists $P \in \mathsf{Tree}^e$ satsfying $\ell(P) = \ell(Q)$ and $P \cap Q \ne \varnothing$. It is easy to check that Tree is indeed a tree, and that the stopping cubes $\mathsf{Stop} = \mathsf{Stop}(\mathsf{Tree})$ satisfy $\bigcup_{Q\in\mathsf{Stop}} Q \subset \bigcup_e \bigcup_{Q\in\mathsf{Stop}^e} Q$. Thus,

$$\mu\left(\bigcup_{Q\in\mathsf{Stop}} Q\right) \le \sum_{e\in\{0,1\}^n} \mu\left(\bigcup_{Q\in\mathsf{Stop}^e} Q\right) \overset{(4.2)}{\le} \varepsilon.$$

Moreover, $\mathsf{Tree} \subset \mathsf{Tree}_{(0,\dots,0)}$, so for all $Q \in \mathsf{Tree}$

$$\mu(\lambda \widetilde{B}_Q) \le M\ell(Q)^n,$$
$$\mu(Q) \ge M^{-1}\ell(Q)^n.$$

The only thing that remains to be shown is the packing condition (4.1).

Lemma 4.4. *We have*

$$\sum_{Q\in\mathsf{Tree}} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2\ell(Q)^n < \infty.$$

Proof. Recall that in Lemma 2.2 we defined a constant $k_0 > 0$ such that for any $Q \in \mathbb{D}_\Gamma$, $\ell(Q) \le 2^{-k_0}$, there exists a cube $P_Q \in \widetilde{\mathbb{D}}_\Gamma$ satisfying $3\widetilde{B}_Q \subset V(P_Q)$, $\ell(P_Q) = 2^{k_0}\ell(Q)$. Since there are only finitely many $Q \in \mathsf{Tree}$ with $\ell(Q) > 2^{-k_0}$, we may ignore them in the estimates that follow.

Suppose $Q \in \mathsf{Tree}$ and $\ell(Q) \le 2^{-k_0}$, let P_Q be as above. Recall that $\nu_Q = \nu_{k(Q)}^{e(Q)}$, where $e = e(Q)$, $k = k(Q)$ are such that $P_Q \in \mathbb{D}_\Gamma^e$ and $\ell(P_Q) = 2^{-k}$.

We defined Tree in such a way that necessarily $P_Q \in \mathsf{Tree}^e$. It follows from Lemma 3.4 applied with $\nu = \nu_Q$, $B = \widetilde{B}_Q$, $Q = P_Q$, that

$$\widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q) \lesssim_{\varepsilon_0,M,k_0} \widetilde{\alpha}_{\nu_Q,2}(P_Q) + \alpha_{\sigma,2}(B_{P_Q}).$$

We use (4.6) and the inequality above to obtain

$$\widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2$$
$$\lesssim_{\varepsilon_0,M,k_0} \alpha_{\sigma,2}(B_{P_Q})^2 + \sum_{\substack{P\in\mathsf{Tree}^e\\ P\subset P_Q}} \|\Delta_P^\sigma g_0^e\|^2_{L^2(\sigma)} \frac{\ell(P)}{\ell(P_Q)^{n+1}} + \sum_{\substack{S\in\mathsf{Stop}^e\\ S\subset P_Q}} \frac{\ell(S)^2}{\ell(P_Q)^{n+2}}\nu^e(S).$$

Taking into account that each $P_Q \in \mathsf{Tree}^e$ may correspond to only a bounded number of $Q \in \mathsf{Tree}$, and that $\ell(Q) \approx_{k_0} \ell(P_Q)$, we get

$$\sum_{Q\in\mathsf{Tree}:P_Q\in\mathsf{Tree}^e} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2\ell(Q)^n \lesssim_{\varepsilon_0,M,k_0} \sum_{Q'\in\mathsf{Tree}^e} \alpha_{\sigma,2}(B_{Q'})^2\ell(Q')^n$$
$$+ \sum_{Q'\in\mathsf{Tree}^e} \sum_{\substack{P\in\mathsf{Tree}^e\\ P\subset Q'}} \|\Delta_P^\sigma g_0^e\|^2_{L^2(\sigma)} \frac{\ell(P)}{\ell(Q')} + \sum_{Q'\in\mathsf{Tree}^e} \sum_{\substack{S\in\mathsf{Stop}^e\\ S\subset Q'}} \frac{\ell(S)^2}{\ell(Q')^2}\nu^e(S).$$

The first sum from the right hand side is finite because σ is uniformly rectifiable, see Theorem I.6.8. We estimate the second sum by changing the order of summation:

$$\sum_{Q'\in\mathsf{Tree}^e} \sum_{\substack{P\in\mathsf{Tree}^e\\ P\subset Q'}} \|\Delta_P^\sigma g_0^e\|^2_{L^2(\sigma)} \frac{\ell(P)}{\ell(Q')} = \sum_{P\in\mathsf{Tree}^e} \|\Delta_P^\sigma g_0^e\|^2_{L^2(\sigma)} \sum_{\substack{Q'\in\mathsf{Tree}^e\\ Q'\supset P}} \frac{\ell(P)}{\ell(Q')}$$
$$\lesssim \sum_{P\in\mathsf{Tree}^e} \|\Delta_P^\sigma g_0^e\|^2_{L^2(\sigma)} \overset{(4.7)}{\lesssim} M\mu(\mathbb{R}^d) < \infty.$$

The third sum is treated similarly:

$$\sum_{Q'\in\mathsf{Tree}^e}\sum_{\substack{S\in\mathsf{Stop}^e\\ S\subset Q'}}\frac{\ell(S)^2}{\ell(Q')^2}\nu^e(S)=\sum_{S\in\mathsf{Stop}^e}\nu^e(S)\sum_{\substack{Q'\in\mathsf{Tree}^e\\ Q'\supset S}}\frac{\ell(S)^2}{\ell(Q')^2}\lesssim\sum_{S\in\mathsf{Stop}^e}\nu^e(S)<\infty.$$

Thus,

$$\sum_{Q\in\mathsf{Tree}}\widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2\ell(Q)^n=\sum_{e\in\{0,1\}^n}\sum_{Q\in\mathsf{Tree}:P_Q\in\mathsf{Tree}^e}\widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2\ell(Q)^n<\infty.$$

□

5 From approximating measures to μ

To prove Lemma 1.8 we need to pass from the estimates on $\widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)$ shown in Lemma 4.1 to estimates on $\widehat{\alpha}_{\mu,2}(B_Q)$.

Recall that $K>20$ is the constant such that for all Whitney cubes $Q\in\mathcal{W}^e$ we have $KQ\cap\Gamma\neq\varnothing$, and $k_0=k_0(n,\Lambda)$ is an integer from Lemma 2.2.

Lemma 5.1. *There exists $\lambda=\lambda(k_0,K,n,d)>3$ such that if $M=M(\varepsilon,\lambda,\Lambda,n,d,\mu)$ and $\mathsf{Tree}=\mathsf{Tree}(\lambda,M,\varepsilon)$ are as in Lemma 4.1, then for all $Q\in\mathsf{Tree}$ with $\ell(Q)\le 2^{-k_0}$*

$$\widehat{\alpha}_{\mu,2}(B_Q)^2\lesssim_{M,\lambda,\Lambda}\widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2+\alpha_{\sigma,2}(\widetilde{B}_Q)^2+\frac{1}{\ell(Q)^{n+2}}\sum_{\substack{P\in\mathcal{W}_Q\\ P\subset\lambda\widetilde{B}_Q}}\mu(P)\ell(P)^2.$$

Proof. Let $Q\in\mathsf{Tree}$ with $\ell(Q)\le 2^{-k_0}$. We will define an auxiliary measure μ_Q. Set

$$I_Q=\{P\in\mathcal{W}_Q:\Pi_\Gamma(P)\cap 3\widetilde{B}_Q\neq\varnothing\}.$$

It is easy to check that

$$\bigcup_{P\in I_Q}P\subset\lambda\widetilde{B}_Q, \tag{5.1}$$

for $\lambda=\lambda(k_0,K,n,d)$ big enough (e.g. $\lambda=C(n,d)K2^{k_0}$ works). It is crucial that all cubes in I_Q have sidelength bounded by $2^{k_0}\ell(Q)$, otherwise no such λ would exist.

Recall that the functions $g_P(x)=\frac{\mu(P)}{\ell(P)^n}\mathbb{1}_{\Pi_\Gamma(P)}(x),\ P\in\mathcal{W}_Q$, were used to define ν_Q at the beginning of Section 4. Let

$$a_P=\frac{\int\varphi_{\widetilde{B}_Q}g_P\,d\sigma}{\mu(P)}.$$

Note that for $P\in\mathcal{W}_Q\setminus I_Q$ we have $a_P=0$. The measure μ_Q is defined as

$$\mu_Q=\varphi_{\widetilde{B}_Q}\mu|_\Gamma+\sum_{P\in I_Q}a_P\mu|_P.$$

First, let us show that if Λ (the constant from the definition of $\widetilde{B}_Q = \Lambda B_Q$) is big enough, then $\mu|_{3B_Q} = \mu_Q\big|_{3B_Q}$. We need to check the following: if $P \in \mathcal{W}^{e(Q)}$ is such that $P \cap 3B_Q \neq \varnothing$, then $P \in I_Q$ and $a_P = 1$.

Note that for all such P we have

$$\ell(P) \leq \operatorname{diam}(P) \overset{(2.4)}{\leq} r(3B_Q) = 9\operatorname{diam}(Q) \overset{(2.3)}{\leq} 2^{-k(Q)},$$

and so $P \in \mathcal{W}_Q$. Furthermore, the fact that $P \cap 3B_Q \neq \varnothing$ and (2.4) imply that $P \subset 9B_Q$. Since Π_Γ is $\sqrt{2}$-Lipschitz continuous, and B_Q is centered at Γ, we get that for Λ big enough (e.g. $\Lambda = 9\sqrt{2}$)

$$\Pi_\Gamma(P) \subset \Lambda B_Q = \widetilde{B}_Q. \tag{5.2}$$

We conclude that $P \in I_Q$ and $a_P = 1$, and so,

$$\mu|_{3B_Q} = \mu_Q\big|_{3B_Q}. \tag{5.3}$$

Set $L = L_{\widetilde{B}_Q}$. We will apply Lemma 3.2 with $\nu = \mu_Q$, $B_1 = B_Q$, $B_2 = \lambda\widetilde{B}_Q$, $L = L$, and $f = \varphi_{\widetilde{B}_Q}$. Notice that $\operatorname{supp}\mu_Q \subset \lambda\widetilde{B}_Q$ by (5.1). Moreover, using the same trick as in the beginning of the proof of Lemma 3.3, we may assume that $L \cap B_Q \neq \varnothing$. Since $\mu_Q(B_Q) \approx_M \mu_Q(\lambda\widetilde{B}_Q) \approx_M \ell(Q)^n$ by Lemma 4.1, and $r(\lambda\widetilde{B}_Q) = \lambda\Lambda r(B_Q)$, the assumptions of Lemma 3.2 are met, and we get that

$$W_2(\varphi_Q\mu_Q, a\varphi_Q\mathcal{H}^n|_L) \lesssim_{M,\lambda,\Lambda} W_2(\mu_Q, a\varphi_{\widetilde{B}_Q}\mathcal{H}^n|_L). \tag{5.4}$$

Applying the triangle inequality yields

$$\begin{aligned} W_2(\mu_Q, a\varphi_{\widetilde{B}_Q}\mathcal{H}^n|_L)^2 &\lesssim W_2(\mu_Q, \varphi_{\widetilde{B}_Q}\nu_Q)^2 + W_2(\varphi_{\widetilde{B}_Q}\nu_Q, a\varphi_{\widetilde{B}_Q}\mathcal{H}^n|_L)^2 \\ &\approx_M W_2(\mu_Q, \varphi_{\widetilde{B}_Q}\nu_Q)^2 + \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2\ell(Q)^{n+2}. \end{aligned} \tag{5.5}$$

To estimate $W_2(\mu_Q, \varphi_{\widetilde{B}_Q}\nu_Q)$ we define the following transport plan:

$$d\pi(x,y) = \varphi_{\widetilde{B}_Q}(x)d\mu|_\Gamma(x)d\delta_x(y) + \sum_{P\in I_Q}\frac{1}{\mu_Q(P)}d\mu_Q\big|_P(x)\varphi_{\widetilde{B}_Q}(y)g_P(y)d\sigma(y).$$

Then,

$$\begin{aligned} W_2(\mu_Q, \varphi_{\widetilde{B}_Q}\nu_Q)^2 &\leq \int |x-y|^2\, d\pi(x,y) \lesssim \sum_{P\in I_Q}\ell(P)^2\int\varphi_{\widetilde{B}_Q}(y)g_P(y)d\sigma(y). \\ &\leq \sum_{P\in I_Q}\mu(P)\ell(P)^2 \overset{(5.1)}{\leq} \sum_{\substack{P\in\mathcal{W}_Q \\ P\subset\lambda\widetilde{B}_Q}}\mu(P)\ell(P)^2. \end{aligned}$$

Putting together (5.3), (5.4), (5.5), and the estimate above, we get

$$W_2(\varphi_Q\mu, a\varphi_Q\mathcal{H}^n|_L) \lesssim_{M,\lambda,\Lambda} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2\ell(Q)^{n+2} + \sum_{\substack{P\in\mathcal{W}_Q\\ P\subset\lambda\widetilde{B}_Q}} \mu(P)\ell(P)^2.$$

Finally, we use the triangle inequality, the estimate $\mu(3B_Q) \approx_M \sigma(B_Q) \approx r(B_Q)^n$, and the fact that L_Q minimizes $\alpha_{\sigma,2}(B_Q)$, to get

$$\begin{aligned}\widehat{\alpha}_{\mu,2}(B_Q)^2\ell(Q)^{n+2} &\approx_M W_2(\varphi_Q\mu, a\varphi_Q\mathcal{H}^n|_{L_Q}) \le W_2(\varphi_Q\mu, a\varphi_Q\mathcal{H}^n|_L)\\ &+ \left(\frac{\int\varphi_Q\, d\mu}{\int\varphi_Q\, d\sigma}\right)^{1/2}\Big(W_2(\varphi_Q\sigma, a\varphi_Q\mathcal{H}^n|_{L_Q}) + W_2(\varphi_Q\sigma, a\varphi_Q\mathcal{H}^n|_L)\Big)\\ &\lesssim_M W_2(\varphi_Q\mu, a\varphi_Q\mathcal{H}^n|_L) + W_2(\varphi_Q\sigma, a\varphi_Q\mathcal{H}^n|_L)\\ &\lesssim W_2(\varphi_Q\mu, a\varphi_Q\mathcal{H}^n|_L) + \alpha_{\sigma,2}(\widetilde{B}_Q)^2\ell(Q)^{n+2},\end{aligned}$$

and so the proof is complete. □

We are ready to finish the proof of Lemma 1.8.

Proof of Lemma 1.8. Recall that R is a Γ-cube with $\ell(R) = 1$, and $\varepsilon > 0$ is an arbitrary small constant, and that they were both fixed in Subsection 2.1. Let λ, M, Tree, and Stop be as in Lemma 5.1 and Lemma 4.1. Set

$$R' = R \setminus \bigcup_{P\in\mathsf{Stop}} P.$$

By Lemma 4.1, we have $\mu(R') \ge (1-\varepsilon)\mu(R)$. Our aim is to show that

$$\int_{R'}\int_0^1 \alpha_{\mu,2}(x,r)^2\,\frac{dr}{r}\,d\mu(x) < \infty.$$

For any $x \in R'$ we have arbitrarily small cubes from Tree containing x. Hence, for any $k \ge k_0 + 3$, $r \in (2^{-k}, 2^{-k+1}]$, we have $3B(x,r) \subset B_Q$ for the cube $Q \in \mathsf{Tree}$ containing x and satisfying $\ell(Q) = 2^{-k+3}$. Thus, by Lemma 3.3,

$$\widehat{\alpha}_{\mu,2}(B(x,r))^2 \lesssim_M \widehat{\alpha}_{\mu,2}(B_Q)^2 + \alpha_{\sigma,2}(B_Q)^2.$$

Integrating both sides with respect to r yields

$$\begin{aligned}\int_{2^{-k}}^{2^{-k+1}} \widehat{\alpha}_{\mu,2}(B(x,r))^2\,\frac{dr}{r} &\lesssim_M \int_{2^{-k}}^{2^{-k+1}} (\widehat{\alpha}_{\mu,2}(B_Q)^2 + \alpha_{\sigma,2}(B_Q)^2)\,\frac{dr}{r}\\ &\approx \widehat{\alpha}_{\mu,2}(B_Q)^2 + \alpha_{\sigma,2}(B_Q)^2.\end{aligned}$$

The inequality above holds for all $x \in Q \cap R'$, so

$$\begin{aligned}\int_{Q\cap R'}\int_{2^{-k}}^{2^{-k+1}} \widehat{\alpha}_{\mu,2}(B(x,r))^2\,\frac{dr}{r}\,d\mu(x) &\lesssim_M (\widehat{\alpha}_{\mu,2}(B_Q)^2 + \alpha_{\sigma,2}(B_Q)^2)\mu(Q)\\ &\approx_M (\widehat{\alpha}_{\mu,2}(B_Q)^2 + \alpha_{\sigma,2}(B_Q)^2)\ell(Q)^n.\end{aligned}$$

Summing over all $Q \in \mathsf{Tree}$ with $\ell(Q) = 2^{-k+3}$, and then over all $k \geq k_0 + 3$, we get

$$\int_{R'} \int_0^{2^{-k_0-2}} \widehat{\alpha}_{\mu,2}(B(x,r))^2 \, \frac{dr}{r} \, d\mu(x) \lesssim_M \sum_{\substack{Q\in\mathsf{Tree} \\ \ell(Q)\leq 2^{-k_0}}} \widehat{\alpha}_{\mu,2}(B_Q)^2 \ell(Q)^n + \sum_{\substack{Q\in\mathsf{Tree} \\ \ell(Q)\leq 2^{-k_0}}} \alpha_{\sigma,2}(B_Q)^2 \ell(Q)^n. \quad (5.6)$$

On the other hand, for any $r > 0$ we have

$$\widehat{\alpha}_{\mu,2}(B(x,r))^2 \lesssim \frac{\mu(\mathbb{R}^d)}{r^n},$$

so

$$\int_{R'} \int_{2^{-k_0-2}}^1 \widehat{\alpha}_{\mu,2}(B(x,r))^2 \, \frac{dr}{r} \, d\mu(x) < \infty.$$

Thus, in order to prove Lemma 1.8, it suffices to show that the sums on the right hand side of (5.6) are finite.

The finiteness of

$$\sum_{Q\in\mathbb{D}_\Gamma,\ Q\subset R} \alpha_{\sigma,2}(B_Q)^2 \ell(Q)^n$$

follows by Theorem I.6.8. To estimate the other sum we apply Lemma 5.1:

$$\sum_{\substack{Q\in\mathsf{Tree} \\ \ell(Q)\leq 2^{-k_0}}} \widehat{\alpha}_{\mu,2}(B_Q)^2 \ell(Q)^n \lesssim \sum_{\substack{Q\in\mathsf{Tree} \\ \ell(Q)\leq 2^{-k_0}}} \widehat{\alpha}_{\nu_Q,2}(\widetilde{B}_Q)^2 \ell(Q)^n + \sum_{\substack{Q\in\mathsf{Tree} \\ \ell(Q)\leq 2^{-k_0}}} \alpha_{\sigma,2}(\widetilde{B}_Q)^2 \ell(Q)^n + \sum_{\substack{Q\in\mathsf{Tree} \\ \ell(Q)\leq 2^{-k_0}}} \sum_{\substack{P\in\mathcal{W}_Q \\ P\subset\lambda\tilde{B}_Q}} \mu(P) \frac{\ell(P)^2}{\ell(Q)^2}.$$

The first sum is finite by Lemma 4.1, the second by Theorem I.6.8. Concerning the last sum, we may estimate it in the following way:

$$\sum_{\substack{Q\in\mathsf{Tree} \\ \ell(Q)\leq 2^{-k_0}}} \sum_{\substack{P\in\mathcal{W}_Q \\ P\subset\lambda\tilde{B}_Q}} \mu(P) \frac{\ell(P)^2}{\ell(Q)^2} \lesssim \sum_{e\in\{0,1\}^n} \sum_{\substack{P\in\mathcal{W}^e \\ P\subset\lambda\tilde{B}_R}} \mu(P) \sum_{\substack{Q\in\mathsf{Tree} \\ \lambda\tilde{B}_Q\supset P}} \frac{\ell(P)^2}{\ell(Q)^2} \lesssim \sum_{e\in\{0,1\}^n} \sum_{\substack{P\in\mathcal{W}^e \\ P\subset\lambda\tilde{B}_R}} \mu(P) \leq \sum_{e\in\{0,1\}^n} \mu(\lambda\widetilde{B}_R) = 2^n \mu(\lambda\widetilde{B}_R) < \infty.$$

Thus,

$$\sum_{Q\in\mathsf{Tree}} \widehat{\alpha}_{\mu,2}(B_Q)^2 \ell(Q)^n < \infty.$$

□

Cones, rectifiability, and singular integral operators V

1 Introduction

Let $m < d$ be positive integers. Given an m-plane $V \in G(d,m)$, a point $x \in \mathbb{R}^d$, and $\alpha \in (0,1)$, we define

$$K(x,V,\alpha) = \{y \in \mathbb{R}^d \ : \ \operatorname{dist}(y, V+x) < \alpha|x-y|\}.$$

That is, $K(x,V,\alpha)$ is an open cone centered at x, with direction V, and aperture α.

Let $0 < n < d$. It is well-known that if a set $E \subset \mathbb{R}^d$ satisfies for some $V \in G(d, d-n),\ \alpha \in (0,1)$, the condition

$$x \in E \qquad \Rightarrow \qquad E \cap K(x,V,\alpha) = \varnothing, \tag{1.1}$$

then E is contained in some n-dimensional Lipschitz graph Γ, and $\operatorname{Lip}(\Gamma) \leq \frac{1}{\alpha}$, see e.g. [Mat95, Proof of Lemma 15.13].

To what extent can we weaken the condition (1.1) and still get meaningful information about the geometry of E? It depends on what we mean by "meaningful information", naturally. One could ask for the rectifiability of E, or if E contains big pieces of Lipschitz graphs, or whether nice singular integral operators are bounded on $L^2(E)$. In this chapter we answer these three questions.

1.1 Rectifiability

A measure-theoretic analogue of (1.1), well-suited to the study of rectifiability, is that of an approximate tangent plane from Section I.2. For reader's convenience we recall the definition below.

For $r > 0$ we define the truncated cone

$$K(x, V, \alpha, r) = K(x, V, \alpha) \cap B(x, r),$$

and for $0 < r < R$ we define the doubly truncated cone

$$K(x, V, \alpha, r, R) = K(x, V, \alpha, R) \setminus K(x, V, \alpha, r).$$

Definition 1.1. We say that an n-plane $W \in G(d, n)$ is an *approximate tangent plane* to a Radon measure μ at $x \in \operatorname{supp}\mu$ if $\Theta^{n,*}(\mu, x) > 0$ and for every $\alpha \in (0, 1)$

$$\lim_{r\to 0} \frac{\mu(K(x, W^\perp, \alpha, r))}{r^n} = 0. \tag{1.2}$$

Recall that a classical result of Federer characterizes rectifiable measures in terms of existence of approximate tangent planes, see Theorem I.2.4.

The results we prove in this paper are of similar nature. More precisely, we introduce and study *conical energies*.

Definition 1.2. Suppose μ is a Radon measure on $\mathbb{R}^d$, and $x \in \operatorname{supp}\mu$. Let $V \in G(d, d-n)$, $\alpha \in (0, 1)$, $1 \leq p < \infty$ and $R > 0$. We define the (V, α, p)-*conical energy of μ at x up to scale R* as

$$\mathcal{E}_{\mu,p}(x, V, \alpha, R) = \int_0^R \left(\frac{\mu(K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r}.$$

For $E \subset \mathbb{R}^d$ we set also $\mathcal{E}_{E,p}(x, V, \alpha, R) = \mathcal{E}_{\mathcal{H}^n|_E,p}(x, V, \alpha, R)$.

The conical energies can be seen as a “quantification” of the notion of approximate tangent plane. We are ready to state our first result.

Theorem 1.3. *Let $1 \leq p < \infty$. Suppose μ is a Radon measure on $\mathbb{R}^d$ satisfying $\Theta^{n,*}(\mu, x) > 0$ and $\Theta^n_*(\mu, x) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$. Assume that for μ-a.e. $x \in \mathbb{R}^d$ there exists some $V_x \in G(d, d-n)$ and $\alpha_x \in (0, 1)$ such that*

$$\mathcal{E}_{\mu,p}(x, V_x, \alpha_x, 1) < \infty, \tag{1.3}$$

and the mapping $x \mapsto (V_x, \alpha_x)$ is measurable. Then, μ is n-rectifiable.

Conversely, if μ is n-rectifiable, then for μ-a.e. $x \in \mathbb{R}^d$ there exists $V_x \in G(d, d-n)$ such that for all $\alpha \in (0, 1)$ we have

$$\mathcal{E}_{\mu,p}(x, V_x, \alpha, 1) < \infty. \tag{1.4}$$

Remark 1.4. The “necessary” part of Theorem 1.3 improves on Theorem I.2.4 in the following way. Existence of approximate tangents means that the conical density simply converges to 0, while (1.4) means that the conical density satisfies a Dini-type condition, and converges to 0 rather fast.

Remark 1.5. Concerning the "sufficient" part of Theorem 1.3: clearly, condition (I.2.2) is weaker than (1.3). However, Theorem 1.3 has the following advantage over Theorem I.2.4: we only require $\Theta^{n,*}(\mu, x) > 0$ and $\Theta^n_*(\mu, x) < \infty$ for our criterion to hold. In particular, we do not assume $\mu \ll \mathcal{H}^n$. It is not clear to the author how to show a criterion involving (I.2.2) or (1.2) without assuming a priori $\mu \ll \mathcal{H}^n$.

Question 1.6. Suppose μ is a Radon measure on $\mathbb{R}^d$ satisfying $\Theta^{n,*}(\mu, x) > 0$ and $\Theta^n_*(\mu, x) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$. Assume that for μ-a.e. $x \in \mathbb{R}^d$ there is an approximate tangent plane to μ at x. Does this imply that μ is n-rectifiable?

Let us mention related results. The behaviour of conical densities on purely unrectifiable sets is studied in [CKRS10] and [Käe10, §5]. In [Mat88, KS08, CKRS10, KS11] the relation between conical densities for higher dimensional sets and their porosity is investigated.

Higher order rectifiability in terms of approximate differentiability of sets is studied in [San19]. In [DNI19] the authors characterize $C^{1,\alpha}$ rectifiable sets using approximate tangents paraboloids, essentially obtaining a $C^{1,\alpha}$ counterpart of Theorem I.2.4. See also [Ghi20] and [GG20] for related results.

We would also like to mention recent results of Badger and Naples that nicely complement Theorem 1.3. In [Nap20, Theorem D] Naples showed that a modified version of (1.2) can be used to characterize pointwise doubling measures *carried by Lipschitz graphs*, that is measures vanishing outside of a countable union of n-dimensional Lipschitz graphs. In an even more recent paper [BN20] the authors completely describe measures carried by n-dimensional Lipschitz graphs on $\mathbb{R}^d$. They use a Dini condition imposed on the so-called *conical defect*, and their condition is closely related to (1.3). Note the absence of densities in the assumptions (and conclusion) of their results.

1.2 Big pieces of Lipschitz graphs

Before stating our next theorem, we need to recall some definitions.

Definition 1.7. We say that an n-ADR set $E \subset \mathbb{R}^d$ has *big pieces of Lipschitz graphs* (BPLG) if there exist constants $\kappa, L > 0$, such that the following holds.

For all balls B centered at E, $0 < r(B) < \operatorname{diam}(E)$, there exists a Lipschitz graph Γ_B with $\operatorname{Lip}(\Gamma_B) \le L$, such that

$$\mathcal{H}^n(E \cap B \cap \Gamma_B) \ge \kappa\, r(B)^n.$$

Sets with BPLG were studied e.g. in [Dav88b, DS93a, DS93b] as one of the possible quantitative counterparts of rectifiability. Let us point out that the class of sets with BPLG is strictly smaller than the class of uniformly rectifiable sets – sets containing BPLG are uniformly rectifiable, but the converse is not true. An example of a uniformly rectifiable set that does not contain BPLG is due to Hrycak, although he never wrote it down, see [Azz19, Appendix].

While many characterizations of uniformly rectifiable sets are available, the sets containing BPLG are not as well understood. David and Semmes showed in [DS93b] that a set contains BPLG if and only if it has *big projections* and satisfies the *weak geometric lemma*. We refer the reader to [DS93b] or [DS93a, §I.1.5] for details. In a very recent paper [Orp20] Orponen characterized the BPLG property in terms of having *plenty of big projections*, which settled a problem going back to [DS93b].

In another recent paper, Martikainen and Orponen [MO18b] managed to characterize sets with BPLG in terms of L^2 norms of their projections. Interestingly, the authors use the information about projections of an n-ADR set E to draw conclusions about intersections with cones of some subset $E' \subset E$ with $\mathcal{H}^n(E') \approx \mathcal{H}^n(E)$. This in turn allows them to find a Lipschitz graph intersecting an ample portion of E'. We will use some of their techniques to prove a characterization of sets containing BPLG in terms of the following property.

Definition 1.8. Let $1 \le p < \infty$. We say that a measure μ has *big pieces of bounded energy for p*, abbreviated as BPBE(p), if there exist constants $\alpha, \kappa, M_0 > 0$ such that the following holds.

For all balls B centered at $\operatorname{supp}\mu$, $0 < r(B) < \operatorname{diam}(\operatorname{supp}\mu)$, there exist a set $G_B \subset B$ with $\mu(G_B) \ge \kappa\, \mu(B)$, and a direction $V_B \in G(d, d-n)$, such that for all $x \in G_B$

$$\mathcal{E}_{\mu,p}(x, V_B, \alpha, r(B)) = \int_0^{r(B)} \left(\frac{\mu(K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r} \le M_0. \tag{1.5}$$

Theorem 1.9. *Let $1 \le p < \infty$. Suppose $E \subset \mathbb{R}^d$ is n-ADR. Then E has BPLG if and only if $\mathcal{H}^n|_E$ has BPBE(p).*

Remark 1.10. In particular, for n-ADR sets, the condition BPBE(p) is equivalent to BPBE(q) for all $1 \le p, q < \infty$.

Remark 1.11. In fact, one can show that an a priori slightly weaker condition than BPBE is already sufficient for BPLG. To be more precise, in (1.5) replace $K(x, V, \alpha, r)$ with $K(x, V, \alpha, r) \cap G_B$, so that we get

$$\int_0^{r(B)} \left(\frac{\mathcal{H}^n(K(x, V, \alpha, r) \cap E \cap G_B)}{r^n} \right)^p \frac{dr}{r} \le M_0. \tag{1.6}$$

We show that this "weak" BPBE is sufficient for BPLG in Proposition 9.1. It is obvious that (1.6) is also necessary for BPLG: if E contains BPLG, then choosing $G_B = \Gamma_B$ as in Definition 1.7, one can pick the corresponding V and α so that $K(x, V, \alpha, r) \cap \Gamma_B = \varnothing$.

It is tempting to consider also the following definition.

Definition 1.12. Let $1 \le p < \infty$. We say that a measure μ has *bounded mean energy* (BME) for p if there exist constants $\alpha, M_0 > 0$, and for every $x \in \operatorname{supp}\mu$ there exists a direction $V_x \in G(d, d-n)$, such that the following holds.

For all balls B centered at $\operatorname{supp}\mu$, $0 < r(B) < \operatorname{diam}(\operatorname{supp}\mu)$, we have

$$\int_B \mathcal{E}_{\mu,p}(x, V_x, \alpha, r(B))\, d\mu(x) = \int_B \int_0^{r(B)} \left(\frac{\mu(K(x, V_x, \alpha, r))}{r^n}\right)^p \frac{dr}{r} d\mu(x) \le M_0\, \mu(B).$$

In other words we require $\mu(K(x, V_x, \alpha, r))^p r^{-np} \frac{dr}{r} d\mu(x)$ to be a Carleson measure. This condition looks quite natural due to many similar characterizations of uniform rectifiability, e.g. Theorem I.6.3, Theorem I.6.8 or Theorem I.6.11.

It is easy to see, using the compactness of $G(d, d-n)$ and Chebyshev's inequality, that BME for p implies BPBE(p). However, the reverse implication does not hold. In Section 11 we give an example of a set containing BPLG that does not satisfy BME. The problem is the following. In the definition above, the plane V_x is fixed for every $x \in \operatorname{supp}\mu$ once and for all, and we do not allow it to change between different scales. This is too rigid.

Question 1.13. Can one modify the definition of BME, allowing the planes V_x to depend on r, but with some additional control on the oscillation of $V_{x,r}$, so that the modified BME could be used to characterize BPLG, or uniform rectifiability?

1.3 Boundedness of SIOs

Recall that in Section I.4 we introduced $\mathcal{K}^n(\mathbb{R}^d)$, the class of odd C^2 kernels $k : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ satisfying for some constant $C_k > 0$

$$|\nabla^j k(x)| \le \frac{C_k}{|x|^{n+j}} \quad \text{for } x \neq 0 \quad \text{and} \quad j \in \{0, 1, 2\}. \tag{1.7}$$

A singular integral operator applied to a Radon measure ν, with a kernel $k \in \mathcal{K}^n(\mathbb{R}^d)$, and a truncation parameter $\varepsilon > 0$, was defined as

$$T_\varepsilon \nu(x) = \int_{|x-y|>\varepsilon} k(y-x)\, d\nu(y), \quad x \in \mathbb{R}^d.$$

For a fixed positive Radon measure μ and a function $f \in L^1_{loc}(\mu)$ we also set

$$T_{\mu,\varepsilon} f(x) = T_\varepsilon(f\mu)(x).$$

If μ is n-ADR, the necessary and sufficient conditions for $L^2(\mu)$ boundedness of T_μ were discussed in Section I.5. In the non-ADR setting less is known. A necessary condition for the boundedness of SIOs in $L^2(\mu)$, where μ is Radon and non-atomic, is the polynomial growth condition:

$$\mu(B(x,r)) \le C_1 r^n \qquad \text{for all } x \in \operatorname{supp}\mu,\ r>0, \tag{1.8}$$

see [Dav91, Proposition 1.4 in Part III]. Eiderman, Nazarov and Volberg showed in [ENV14] that if μ has vanishing lower density, then the Riesz transform is unbounded. Their result was generalized to SIOs associated to gradients of single layer potentials in [CAMT19]. Nazarov, Tolsa and Volberg proved in [NTV14b] that if $E \subset \mathbb{R}^{n+1}$ satisfies $\mathcal{H}^n(E) < \infty$ and the n-dimensional Riesz transform is bounded in $L^2(\mathcal{H}^n|_E)$, then E is n-rectifiable. That the same is true for gradients of single layer potentials was shown by Prat, Puliatti and Tolsa in [PPT18].

Concerning sufficient conditions for boundedness of SIOs, in [AT15] Azzam and Tolsa estimated the Cauchy transform of a measure using its β numbers. Their method was further developed by Girela-Sarrión [GS19]. He gives a sufficient condition for boundedness of singular integral operators with kernels in $\mathcal{K}^n(\mathbb{R}^d)$ in terms of β numbers. We use the main lemma from [GS19] to prove the following criterion involving 2-conical energy.

Theorem 1.14. *Let μ be a Radon measure on $\mathbb{R}^d$ satisfying the polynomial growth condition (1.8). Suppose that μ has BPBE(2) Then, all singular integral operators T_μ with kernels $k \in \mathcal{K}^n(\mathbb{R}^d)$ are bounded in $L^2(\mu)$, with norm depending only on BPBE constants, the polynomial growth constant C_1, and the constant C_k from (1.7).*

Remark 1.15. A similar result, with BPBE(2) condition replaced by BPBE(1) condition, has already been shown in [CT17, Theorem 10.2]. It is easy to see that for measures satisfying polynomial growth (1.8) we have

$$\mathcal{E}_{\mu,2}(x,V,\alpha,R) \le C_1\, \mathcal{E}_{\mu,1}(x,V,\alpha,R),$$

and so BPBE(2) is a weaker assumption than BPBE(1). Moreover, in Section 12 we show that the measure constructed in [JM00] does not satisfy BPBE(1), but it trivially satisfies BPBE(2). Hence, Theorem 1.14 really does improve on [CT17, Theorem 10.2].

Remark 1.16. Recall that for n-ADR sets the condition BPBE(p) was equivalent to BPLG, regardless of p. By the remark above, it is clear that if we replace the n-ADR condition with polynomial growth (i.e. if we drop the lower regularity assumption), then the condition BPBE(p) is no longer independent of p. In general we only have one implication: for $1 \le p < q < \infty$

$$BPBE(p) \quad \Rightarrow \quad BPBE(q).$$

Remark 1.17. Theorem 1.14 is sharp in the following sense. If one tried to weaken the assumption BPBE(2) to BPBE(p) for some $p > 2$, then the theorem would no longer hold. The reason is that for any $p > 2$ one may construct a Cantor-like probability measure μ, say on a unit square in $\mathbb{R}^2$, that has linear growth and such that for all $x \in \operatorname{supp}\mu$

$$\int_0^1 \left(\frac{\mu(B(x,r))}{r}\right)^p \frac{dr}{r} \lesssim 1,$$

(that is, a much stronger version of BPBE(p) holds), but nevertheless, the Cauchy transform is not bounded on $L^2(\mu)$. See [Tol14, Chapter 4.7].

Sadly, the implication of Theorem 1.14 cannot be reversed. Let $E \subset \mathbb{R}^2$ be the previously mentioned example of a 1-ADR uniformly rectifiable set that does not contain BPLG. In particular, by Theorem 1.9 E does not satisfy BPBE(p) for any p. Nevertheless, by the results of David and Semmes Theorem I.5.3, all nice singular integral operators are bounded on $L^2(E)$.

1.4 Cones and projections

Let us note that [CT17, Theorem 10.2] was merely a tool to prove the main result of [CT17]: a lower bound on analytic capacity involving L^2 norms of projections. Chang and Tolsa proved also an interesting inequality showing the connection between 1-conical energy and L^2 norms of projections. We introduce additional notation before stating their result.

Definition 1.18. Suppose $V \in G(d, d-n)$, $\alpha \in (0,1)$, and $1 \le p < \infty$. Let B be a ball. The *(V,α,p)-conical energy of μ in B* is

$$\mathcal{E}_{\mu,p}(B,V,\alpha) = \int_B \int_0^{r(B)} \left(\frac{\mu(K(x,V,\alpha,r))}{r^n}\right)^p \frac{dr}{r} d\mu(x).$$

We define also

$$\mathcal{E}_{\mu,p}(\mathbb{R}^d,V,\alpha) = \int_{\mathbb{R}^d} \int_0^{\infty} \left(\frac{\mu(K(x,V,\alpha,r))}{r^n}\right)^p \frac{dr}{r} d\mu(x).$$

We will often suppress the arguments V, α, and write simply $\mathcal{E}_{\mu,p}(B)$, $\mathcal{E}_{\mu,p}(\mathbb{R}^d)$.

Remark 1.19. For $p = 1$ we have

$$\int_0^\infty \frac{\mu(K(x,V,\alpha,r))}{r^n} \frac{dr}{r} = \int_{K(x,V,\alpha)} \int_{|x-y|}^\infty \frac{dr}{r^{n+1}}\, d\mu(y) = n^{-1} \int_{K(x,V,\alpha)} \frac{1}{|x-y|^n}\, d\mu(y), \quad (1.9)$$

and so

$$\mathcal{E}_{\mu,1}(\mathbb{R}^d, V, \alpha) = n^{-1}\int_{\mathbb{R}^d}\int_{K(x,V,\alpha)} \frac{1}{|x-y|^n}\, d\mu(y)d\mu(x). \tag{1.10}$$

In their paper Chang and Tolsa were working with the expression from the right hand side above.

Given $V \in G(d,m)$ we will denote by $\pi_V : \mathbb{R}^d \to V$ the orthogonal projection onto V, and by $\pi_V^\perp : \mathbb{R}^d \to V^\perp$ the orthogonal projection onto $V^\perp$. We endow $G(d,m)$ with the natural probability measure $\gamma_{d,m}$, see [Mat95, Chapter 3], and with a metric $d(V,W) = \|\pi_V - \pi_W\|_{op}$, where $\|\cdot\|_{op}$ is the operator norm. We write $\pi_V\mu$ to denote the image measure of μ by the projection π_V. If $\pi_V\mu \ll \mathcal{H}^n|_V$, then we identify $\pi_V\mu$ with its density with respect to $\mathcal{H}^n|_V$, and $\|\pi_V\mu\|_{L^2(V)}$ denotes the L^2 norm of this density. Otherwise, we set $\|\pi_V\mu\|_{L^2(V)} = \infty$.

Proposition 1.20 ([CT17, Corollary 3.11])**.** *Let $V_0 \in G(d,n)$ and $\alpha > 0$. Then, there exist constants $\lambda, C > 1$ such that for any finite Borel measure μ in $\mathbb{R}^d$,*

$$\mathcal{E}_{\mu,1}(\mathbb{R}^d, V_0^\perp, \alpha) \overset{(1.10)}{\approx} \int_{\mathbb{R}^d}\int_{K(x,V_0^\perp,\alpha)} \frac{1}{|x-y|^n}\, d\mu(y)d\mu(x)$$
$$\leq C\int_{B(V_0,\lambda\alpha)} \|\pi_V\mu\|^2_{L^2(V)}\, d\gamma_{d,n}(V).$$

Let us note that a variant of this estimate was also proved in [MO18b], for a measure of the form $\mu = \mathcal{H}^n|_E$, with E a suitable set.

The inequality converse to that of Proposition 1.20 in general is not true, but it is not far off. Additional assumptions on μ are necessary, and one has to add another term to the left hand side. See [CT17, Remark 3.12, Appendix A].

In the light of results mentioned above, as well as the characterization of sets with BPLG from [MO18b], the connection between L^2 norms of projections and cones is quite striking. Note that the proof of the Besicovitch-Federer projection theorem also involves careful analysis of measure in cones, see [Mat95, Chapter 18]. Exploring further the relationship between cones and projections would be very interesting.

Question 1.21. Is it possible to obtain an inequality similar to that of Proposition 1.20, but with $\mathcal{E}_{\mu,2}$ on the left hand side, and some quantity involving $\pi_V\mu$ on the right hand side?

1.5 Organization of the chapter

In Section 2 we state our main lemma, a corona decomposition-like result. Roughly speaking, it says that if a measure μ has polynomial growth, and for some $V \in G(d, d-n)$, $\alpha \in (0,1)$ we have $\mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha) < \infty$, then we can decompose $\mathcal{D}$ into a family of trees such that:

- for every tree, μ is "well-behaved" at the scales and locations of the tree,
- we have a good control on the number of trees (see (2.2)).

We prove the main lemma in Sections 3–5. Let us point out that in the case $p = 1$ an analogous corona decomposition was already shown in [CT17, Lemma 5.1]. Our proof follows the same general strategy, but some key estimates had to be done differently (most notably the estimates in Section 4).

In Section 6 we show how to use the main lemma and results from [GS19] to get Theorem 1.14. Sections 7 and 8 are dedicated to the proof of Theorem 1.3. The "sufficient part" follows from our main lemma, while the "necessary part" is deduced from the corresponding β_2 result of Tolsa [Tol15]. We prove Theorem 1.9 in Sections 9 and 10. To show the "sufficient part" we use the results from [MO18b], whereas the "necessary part" follows from a simple geometric argument. Finally, in Section 11 we construct a set with BPLG that does not satisfy BME condition, and in Section 12 we show that the measure from [JM00] satisfies BPBE(2), but not BPBE(1).

2 Main lemma

In order to formulate our main lemma we need to introduce some additional notation.

Let μ be a compactly supported Radon measure with polynomial growth (1.8). Suppose $\mathcal{D}$ is the associated David-Mattila lattice, as in Lemma II.2.1, and assume that

$$R_0 = \operatorname{supp} \mu \in \mathcal{D}$$

is the biggest cube.

Given a family of cubes $\mathsf{Top} \subset \mathcal{D}^{db}$ satisfying $R_0 \in \mathsf{Top}$ we define the following families associated to each $R \in \mathsf{Top}$:

- $\mathsf{Next}(R)$ is the family of maximal cubes $Q \in \mathsf{Top}$ strictly contained in R,
- $\mathsf{Tr}(R)$ is the family of cubes $Q \in \mathcal{D}$ contained in R, but *not* contained in any $P \in \mathsf{Next}(R)$.

Clearly, $\mathcal{D} = \bigcup_{R \in \mathsf{Top}} \mathsf{Tr}(R)$. Define

$$\mathsf{Good}(R) = R \setminus \bigcup_{Q \in \mathsf{Next}(R)} Q.$$

Lemma 2.1 (main lemma)**.** *Let μ be a compactly supported Radon measure on $\mathbb{R}^d$. Suppose there exists $r_0 > 0$ such that for all $x \in \operatorname{supp}\mu$, $0 < r \leq r_0$, we have*

$$\mu(B(x,r)) \leq C_1 r^n. \tag{2.1}$$

Assume further that for some $V \in G(d, d-n)$, $\alpha \in (0,1)$, and $1 \leq p < \infty$, we have $\mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha) < \infty$.

Then, there exists a family of cubes $\mathsf{Top} \subset \mathcal{D}^{db}$, and a corresponding family of Lipschitz graphs $\{\Gamma_R\}_{R\in\mathsf{Top}}$, satisfying:

(i) the Lipschitz constants of Γ_R are uniformly bounded by a constant depending on α,

(ii) μ-almost all $\mathsf{Good}(R)$ is contained in Γ_R,

(iii) for all $Q \in \mathsf{Tr}(R)$ we have $\Theta_\mu(2B_Q) \lesssim \Theta_\mu(2B_R)$.

Moreover, the following packing condition holds:

$$\sum_{R\in\mathsf{Top}} \Theta_\mu(2B_R)^p \mu(R) \lesssim_\alpha (C_1)^p \mu(\mathbb{R}^d) + \mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha). \tag{2.2}$$

The implicit constant does not depend on r_0.

We prove the lemma above in Sections 3–5. From this point on, until the end of Section 5, we assume that μ is a compactly supported Radon measure satisfying the growth condition (2.1), and that there exist $V \in G(d, d-n)$, $\alpha \in (0,1)$, $1 \leq p < \infty$, such that

$$\mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha) < \infty.$$

For simplicity, in our notation we will suppress the parameters V and α. That is, we will write $\mathcal{E}_{\mu,p}(\mathbb{R}^d) = \mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha)$, as well as $K = K(0, V, \alpha)$, $K(x) = K(x, V, \alpha)$, and $K(x,r) = K(x, V, \alpha, r)$. Finally, given $0 < r < R$, set

$$K(x, r, R) = K(x, R) \setminus K(x, r).$$

Parameters

In the proof of Lemma 2.1 we will use a number of parameters. To make it easier to keep track of what depends on what, and at which point the parameters get fixed, we list them below. Recall that "$C_1 = C_1(C_2)$" means that "the value of C_1 depends the value of C_2."

- $A = A(p) > 1$ is the "HD" constant, it is fixed in Lemma 5.1.
- $\tau = \tau(\alpha, t)$ is the "LD" constant, it is fixed in (4.1).
- $M = M(\alpha) > 1$ is the "key estimate" constant, it is chosen in Lemma 3.3.

- $\eta = \eta(M,t) \in (0,1)$ is the constant from the definition of $\mathcal{E}_{\mu,p}(Q)$ in (3.1), it is fixed in the proof of Lemma 4.4.
- $t = t(M,\alpha) > M$ is the "t-neighbour" constant, see Section 3.3. It is fixed just below (4.7), but depends also on Lemma 3.5 and Lemma 3.7.
- $\Lambda = \Lambda(M) > 2M$ is the constant from Lemma 3.8.
- $\varepsilon = \varepsilon(\tau,\alpha,\eta) \in (0,1)$ is the "BCE" constant, it is fixed in Lemma 4.4.

3 Construction of a Lipschitz graph Γ_R

Suppose $R \in \mathcal{D}^{db}$. In this section we will construct a corresponding tree of cubes $\mathsf{Tree}(R)$, and a Lipschitz graph Γ_R that "approximates μ at scales and locations from $\mathsf{Tree}(R)$"; see Lemma 3.8.

3.1 Stopping cubes

Consider constants $A \gg 1$, $0 < \varepsilon \ll \tau \ll 1$, and $0 < \eta \ll 1$, which will be fixed later on. Given $Q \in \mathcal{D}$ we set

$$\mathcal{E}_{\mu,p}(Q) = \frac{1}{\mu(Q)} \int_{2B_Q} \int_{\eta r(Q)}^{\eta^{-1} r(Q)} \left(\frac{\mu(K(x,r))}{r^n} \right)^p \frac{dr}{r} d\mu(x). \tag{3.1}$$

For any $R \in \mathcal{D}^{db}$ we define the following families of cubes:

- $\mathsf{BCE}_0(R)$, the family of big conical energy cubes, consisting of $Q \in \mathcal{D}(R)$ such that
$$\sum_{Q \subset P \subset R} \mathcal{E}_{\mu,p}(P) > \varepsilon\, \Theta_\mu(2B_R)^p.$$
- $\mathsf{HD}_0(R)$, the high density family, consisting of $Q \in \mathcal{D}^{db}(R) \setminus \mathsf{BCE}_0(R)$ such that
$$\Theta_\mu(2B_Q) > A\, \Theta_\mu(2B_R).$$
- $\mathsf{LD}_0(R)$, the low density family, consisting of $Q \in \mathcal{D}(R) \setminus \mathsf{BCE}_0(R)$ such that
$$\Theta_\mu(2B_Q) < \tau\, \Theta_\mu(2B_R).$$

We denote by $\mathsf{Stop}(R)$ the family of maximal (hence, disjoint) cubes from $\mathsf{BCE}_0(R) \cup \mathsf{HD}_0(R) \cup \mathsf{LD}_0(R)$, and we set $\mathsf{BCE}(R) = \mathsf{BCE}_0(R) \cap \mathsf{Stop}(R)$, $\mathsf{HD}(R) = \mathsf{HD}_0(R) \cap \mathsf{Stop}(R)$, $\mathsf{LD}(R) = \mathsf{LD}_0(R) \cap \mathsf{Stop}(R)$.

Note that the cubes in $\mathsf{HD}(R)$ are doubling (by the definition), while the cubes from $\mathsf{LD}(R)$ and $\mathsf{BCE}(R)$ may be non-doubling.

We define $\mathsf{Tree}(R)$ as the family of cubes from $\mathcal{D}(R)$ which are not strictly contained in any cube from $\mathsf{Stop}(R)$ (in particular, $\mathsf{Stop}(R) \subset \mathsf{Tree}(R)$). Note that it may happen that $R \in \mathsf{BCE}(R)$, in which case $\mathsf{Tree}(R) = \{R\}$.

Basic properties of cubes in $\mathsf{Tree}(R)$ are collected in the lemma below.

Lemma 3.1. *Suppose $Q \in \mathsf{Tree}(R)$. Then,*

$$\Theta_\mu(2B_Q) \lesssim A\,\Theta_\mu(2B_R). \tag{3.2}$$

Moreover, for $Q \in \mathsf{Tree}(R) \setminus \mathsf{Stop}(R)$

$$\tau\,\Theta_\mu(2B_R) \le \Theta_\mu(2B_Q), \tag{3.3}$$

$$\sum_{Q\subset P\subset R} \mathcal{E}_{\mu,p}(P) \le \varepsilon\,\Theta_\mu(2B_R)^p. \tag{3.4}$$

Finally, for every $Q \in \mathsf{Tree}(R)$ there exists a doubling cube $P(Q) \in \mathsf{Tree}(R) \cap \mathcal{D}^{db}$ such that $Q \subset P(Q)$ and $\ell(P(Q)) \lesssim_{A,\tau} \ell(Q)$. If $R \notin \mathsf{Stop}(R)$, we have $P(Q) \in \mathsf{Tree}(R) \cap \mathcal{D}^{db} \setminus \mathsf{Stop}(R)$.

Proof. First, note that if $R \in \mathsf{Stop}(R)$, then $\mathsf{Tree}(R) = \{R\}$ and the lemma above is trivial. Assume that $R \notin \mathsf{Stop}(R)$.

Inequalities (3.3) and (3.4) are obvious by the definition $\mathsf{LD}(R)$ and $\mathsf{BCE}(R)$.

Concerning (3.2), note that for $Q \in \mathsf{Tree}(R) \cap \mathcal{D}^{db} \setminus \mathsf{Stop}(R)$ we have $\Theta_\mu(2B_Q) \le A\,\Theta_\mu(2B_R)$ by the high density stopping condition. In general, given $Q \in \mathsf{Tree}(R)$, let $P(Q)$ be the smallest doubling cube containing Q, other than Q. Since $R \in \mathcal{D}^{db}$ and $R \notin \mathsf{Stop}(Q)$, we certainly have $P(Q) \in \mathsf{Tree}(R) \cap \mathcal{D}^{db} \setminus \mathsf{Stop}(R)$, and so $\Theta_\mu(2B_{P(Q)}) \le A\,\Theta_\mu(2B_R)$.

Denote by $P_1, P_2, \dots, P_k$ all the intermediate cubes, so that $Q \subset P_1 \subset \dots \subset P_k \subset P(Q)$. Since P_j are non-doubling, we have by Lemma II.2.6

$$\begin{aligned}\Theta_\mu(2B_Q) \lesssim \Theta_\mu(2B_{P_1}) \lesssim \Theta_\mu(100B(P_1)) \le (C_0A_0)^d\,A_0^{-9d(k-1)}\Theta_\mu(100B(P(Q)))\\ \lesssim \Theta_\mu(2B_{P(Q)}) \le A\,\Theta_\mu(2B_R),\end{aligned}$$

which proves (3.2).

Finally, to see that $\ell(P(Q)) \lesssim_{A,\tau} \ell(Q)$, note that $P_1 \in \mathsf{Tree}(R) \setminus \mathsf{Stop}(R)$, and so $\tau\,\Theta_\mu(2B_R) \le \Theta_\mu(2B_{P_1})$. On the other hand, a minor modification of the computation above shows that

$$\Theta_\mu(2B_{P_1}) \lesssim_{C_0,A_0} A_0^{-9d(k-1)} A\,\Theta_\mu(2B_R).$$

It follows that $k \lesssim_{A,\tau} 1$. □

The following estimate of the measure of cubes in $\mathsf{BCE}(R)$ will be used later on in the proof of the packing estimate (2.2).

Lemma 3.2. *We have*

$$\sum_{Q\in\mathsf{BCE}(R)} \mu(Q) \le \frac{1}{\varepsilon\,\Theta_\mu(2B_R)^p} \sum_{P\in\mathsf{Tree}(R)} \mathcal{E}_{\mu,p}(P)\mu(P). \tag{3.5}$$

Proof. We use the fact that for $Q \in \mathsf{BCE}(R)$ we have

$$\sum_{Q\subset P\subset R} \mathcal{E}_{\mu,p}(P) > \varepsilon\, \Theta_\mu(2B_R)^p$$

to conclude that

$$\begin{aligned}\Theta_\mu(2B_R)^p \sum_{Q\in\mathsf{BCE}(R)} \mu(Q) &\le \frac{1}{\varepsilon} \sum_{Q\in\mathsf{BCE}(R)} \mu(Q) \sum_{\substack{P\in\mathcal{D}\\ Q\subset P\subset R}} \mathcal{E}_{\mu,p}(P)\\ &= \frac{1}{\varepsilon} \sum_{P\in\mathsf{Tree}(R)} \mathcal{E}_{\mu,p}(P) \sum_{\substack{Q\in\mathsf{BCE}(R)\\ Q\subset P}} \mu(Q) \le \frac{1}{\varepsilon} \sum_{P\in\mathsf{Tree}(R)} \mathcal{E}_{\mu,p}(P)\mu(P).\end{aligned}$$

□

3.2 Key estimate

We introduce some additional notation. Given $x \in \mathbb{R}^d$ and $\lambda > 0$ set

$$K^\lambda(x) = K(x, V, \lambda\alpha).$$

For $Q \in \mathcal{D}$, we denote

$$K^\lambda_Q = \bigcup_{x\in Q} K^\lambda(x).$$

If $\lambda = 1$, we will write K_Q instead of K^1_Q.

Lemma 3.3. *There exists a constant $M = M(\alpha) > 1$ such that, if $Q \in \mathsf{Tree}(R)$ and $P \in \mathcal{D}(R)$ satisfy*

$$P \cap K_Q^{1/2} \setminus MB_Q \neq \varnothing \tag{3.6}$$

and

$$\operatorname{dist}(Q, P) \ge Mr(P),$$

then $P \notin \mathsf{Tree}(R)$.

Proof. Taking $M = M(\alpha) > 1$ big enough, we can choose cubes $P',\ Q' \in \mathcal{D}(R)$ such that

- $P \subsetneq P' \subset R,\ P' \subset K_Q^{3/4}$, and $\ell(P') \approx \operatorname{dist}(P', Q)$,
- $Q \subsetneq Q' \subset R,\ \ell(Q') \approx M^{-1}\ell(P')$, and $\operatorname{dist}(P', Q') \approx \ell(P')$.

Moreover, if M is taken big enough, we have for all $x \in 2B_{Q'}$

$$2B_{P'} \subset K(x).$$

Thus, if η is taken small enough (say, $\eta \ll M^{-1}$), we have

$$\left(\frac{\mu(2B_{P'})}{\ell(P')^n}\right)^p \mu(2B_{Q'}) \lesssim_\eta \int_{2B_{Q'}} \int_{\eta r(Q')}^{\eta^{-1}r(Q')} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r} d\mu(x)$$
$$= \mathcal{E}_{\mu,p}(Q')\mu(Q'). \quad (3.7)$$

Since $Q \in \mathsf{Tree}(R)$ and $Q \subsetneq Q'$, we have $Q' \in \mathsf{Tree}(R) \setminus \mathsf{Stop}(R)$, and so

$$\Theta_\mu(2B_{P'})^p \approx \left(\frac{\mu(2B_{P'})}{\ell(P')^n}\right)^p \overset{(3.7)}{\lesssim_\eta} \frac{\mu(Q')}{\mu(2B_{Q'})}\mathcal{E}_{\mu,p}(Q') \le \mathcal{E}_{\mu,p}(Q') \overset{(3.4)}{\le} \varepsilon\, \Theta_\mu(2B_R)^p.$$

It follows that, for ε small enough, $P' \in \mathsf{LD}_0(R)$. Since $P \subsetneq P'$, we get that $P \notin \mathsf{Tree}(R)$. □

We set

$$G_R = R \setminus \bigcup_{Q\in\mathsf{Stop}(R)} Q \qquad \text{and} \qquad \widetilde{G}_R = \bigcap_{k=1}^{\infty} \bigcup_{\substack{Q\in\mathsf{Tree}(R)\\ r(Q)\le A_0^{-k}}} 2MB_Q. \qquad (3.8)$$

Note that $G_R \subset \widetilde{G}_R$.

Lemma 3.4. *For all $x, y \in \widetilde{G}_R$ we have $y \notin K^{1/2}(x)$. Thus, $\widetilde{G}_R$ is contained in an n-dimensional Lipschitz graph with Lipschitz constant depending only on α.*

Proof. Proof by contradiction. Suppose that $x, y \in \widetilde{G}_R$ and $x - y \in K^{1/2}$. Let $Q, P \in \mathsf{Tree}(R)$ be such that $x \in 2MB_Q,\ y \in 2MB_P$, with sidelength so small that $P \cap (K_Q^{1/2} \setminus MB_Q) \neq \varnothing$ and $\operatorname{dist}(Q,P) \ge Mr(P)$. It follows by Lemma 3.3 that $P \notin \mathsf{Tree}(R)$, and so we reach a contradiction. □

3.3 Construction of Γ_R

The Lipschitz graph from Lemma 3.4 can be thought of as a first approximation of Γ_R. It contains the "good set" $\widetilde{G}_R$, but we would also like for Γ_R to lie close to cubes from $\mathsf{Tree}(R)$. In this subsection we show how to do it.

Given $t > 1$, we say that cubes $Q, P \in \mathcal{D}$ are *t-neighbours* if they satisfy

$$t^{-1}\, r(Q) \le r(P) \le t\, r(Q) \qquad (3.9)$$

and

$$\operatorname{dist}(Q,P) \le t(r(Q) + r(P)). \qquad (3.10)$$

If at least one of the conditions above does not hold, we say that Q and P are *t-separated.* We will also say that a family of cubes is t-separated if the cubes from that family are pairwise t-separated.

Consider a big constant $t = t(M, \alpha) > M$ which will be fixed later on. We denote by $\mathsf{Sep}(R)$ a maximal t-separated subfamily of $\mathsf{Stop}(R)$ (it exists by Zorn's lemma). Clearly, for every $Q \in \mathsf{Stop}(R)$ there exists some $P \in \mathsf{Sep}(R)$ which is a t-neighbour of Q.

Furthermore, we define $\mathsf{Sep}^*(R)$ as the family of all cubes $Q \in \mathsf{Sep}(R)$ satisfying the following two conditions:

$$2MB_Q \cap \widetilde{G}_R = \varnothing, \tag{3.11}$$

and for all $P \in \mathsf{Sep}(R),\ P \neq Q$, we have

$$2MB_P \not\subset 2MB_Q. \tag{3.12}$$

Lemma 3.5. *Suppose $t = t(M)$ is big enough. Then, for all $Q, P \in \mathsf{Sep}^*(R)$, $Q \neq P$, we have $Q \not\subset 1.5MB_P$.*

Proof. Suppose $Q \in \mathsf{Sep}^*(R)$, and $Q \subset 1.5MB_P$. We will show that $P \notin \mathsf{Sep}^*(R)$.

Firstly, if $r(Q) > t^{-1}r(P)$, then $Q \subset 1.5MB_P$ implies that Q and P are t-neighbours (for t big enough), and so $P \notin \mathsf{Sep}^*(R)$. On the other hand, if $r(Q) \leq t^{-1}r(P)$, then (if t is big enough) $Q \subset 1.5MB_P$ implies $2MB_Q \subset 2MB_P$, contradicting (3.12). □

Lemma 3.6. *For every $Q \in \mathsf{Sep}(R)$ at least one of the following is true:*

(a) $2MB_Q \cap \widetilde{G}_R \neq \varnothing$,

(b) there exists $P \in \mathsf{Sep}^*(R)$ such that $2MB_P \subset 2MB_Q$.

Proof. If $Q \in \mathsf{Sep}^*(R)$, then of course (b) holds (with $P = Q$). Suppose that $Q \notin \mathsf{Sep}^*(R)$, and that (a) does not hold (i.e. $2MB_Q \cap \widetilde{G}_R = \varnothing$). We will find $P \in \mathsf{Sep}^*(R)$ such that $2MB_P \subset 2MB_Q$.

Since $Q \notin \mathsf{Sep}^*(R)$ and (3.11) holds, condition (3.12) must be false. Thus, we get a cube $Q_1 \in \mathsf{Sep}(R)$ such that $2MB_{Q_1} \subset 2MB_Q$. If $Q_1 \in \mathsf{Sep}^*(R)$, we get (b) with $P = Q_1$. Otherwise, we continue as follows.

Reasoning as before, $Q_1 \in \mathsf{Sep}(R) \setminus \mathsf{Sep}^*(R)$ and $2MB_{Q_1} \cap \widetilde{G}_R = \varnothing$ ensures that there exists a cube $Q_2 \in \mathsf{Sep}(R)$ such that $2MB_{Q_2} \subset 2MB_{Q_1}$. Iterating this process, we get a (perhaps infinite) sequence of cubes $Q_0 := Q, Q_1, Q_2, \dots$ satisfying $2MB_{Q_{j+1}} \subset 2MB_{Q_j}$.

If the algorithm never stops, then $\bigcap_{j=0}^{\infty} 2MB_{Q_j} \neq \varnothing$. But, by the definition of $\widetilde{G}_R$ (3.8) we have $\bigcap_{j=0}^{\infty} 2MB_{Q_j} \subset \widetilde{G}_R$, and so we get a contradiction with $2MB_Q \cap \widetilde{G}_R = \varnothing$. Thus, the algorithm stops at some cube Q_m, which means that $Q_m \in \mathsf{Sep}^*(R)$. Setting $P = Q_m$ finishes the proof. □

Lemma 3.7. *Suppose $t = t(M)$ is big enough. Then:*

(a) for all $Q, P \in \mathsf{Sep}^*(R)$, $Q \neq P$, *we have*

$$Q \cap K_P^{1/2} = P \cap K_Q^{1/2} = \varnothing, \tag{3.13}$$

(b) for all $x \in \widetilde{G}_R$ *and for all* $Q \in \mathsf{Sep}^*(R)$ *we have*

$$x \notin K_Q^{1/2} \quad \textit{and} \quad Q \cap K^{1/2}(x) = \varnothing. \tag{3.14}$$

Proof of (a). Proof by contradiction. Suppose $Q \cap K_P^{1/2} \neq \varnothing$ (which by symmetry of cones implies $P \cap K_Q^{1/2} \neq \varnothing$). Without loss of generality, assume $r(Q) \leq r(P)$. Since Q and P are t-separated, at least one of the conditions (3.9), (3.10) fails, i.e.

$$r(Q) \leq t^{-1} r(P) \quad \text{or} \quad \operatorname{dist}(Q, P) > t(r(Q) + r(P)).$$

We know by Lemma 3.5 that $Q \not\subset 1.5MB_P$. It is easy to see that in either of the cases considered above, this implies $Q \cap 1.2MB_P = \varnothing$. It follows that $Q \cap (K_P^{1/2} \setminus MB_P) \neq \varnothing$ and $r(Q) \leq r(P) \leq M^{-1} \operatorname{dist}(Q, P)$. Hence, we can use Lemma 3.3 to conclude that $Q \notin \mathsf{Tree}(R)$. This contradicts $Q \in \mathsf{Sep}^*(R)$. □

Proof of (b). Proof by contradiction. Suppose $x \in K_Q^{1/2}$. We have $x \notin 2MB_Q$ by (3.11). Since $x \in \widetilde{G}_R$, we can find an arbitrarily small cube $P \in \mathsf{Tree}(R)$ such that $x \in 2MB_P$. Taking $r(P)$ small enough we will have $r(P) \leq M^{-1} \operatorname{dist}(Q, P)$ and $P \cap K_Q^{1/2} \setminus MB_Q \neq \varnothing$ (because $x \in K_Q^{1/2} \setminus 2MB_Q$). Lemma 3.3 yields $P \notin \mathsf{Tree}$, a contradiction. □

Lemma 3.8. *There exists a Lipschitz graph* Γ_R, *with Lipschitz constant depending only on* α, *such that*

$$\widetilde{G}_R \subset \Gamma_R.$$

Moreover, there exists a big constant $\Lambda = \Lambda(M, t) > 1$ *such that for every* $Q \in \mathsf{Tree}(R)$ *we have*

$$\Lambda B_Q \cap \Gamma_R \neq \varnothing. \tag{3.15}$$

Proof. Recall that for each cube $Q \in \mathcal{D}$ we have a "center" denoted by $x_Q \in Q$. Set $F = \{x_Q : Q \in \mathsf{Sep}^*(R)\} \cup \widetilde{G}_R$. It follows by Lemma 3.4 and Lemma 3.7 that for any $x, y \in F$ we have $x - y \notin K^{1/2}$. Thus, there exists a Lipschitz graph Γ_R, with slope depending only on α, such that $F \subset \Gamma_R$.

Concerning the second statement, it is clearly true for $Q \in \mathsf{Sep}^*(R)$ (even with $\Lambda = 1$). For $Q \in \mathsf{Sep}(R)$, we have by Lemma 3.6 that either $2MB_Q \cap \widetilde{G}_R \neq \varnothing$ or there exists $P \in \mathsf{Sep}^*(R)$ with $2MB_P \subset 2MB_Q$. Thus, (3.15) holds if $\Lambda \geq 2M$.

If $Q \in \mathsf{Stop}(R)$, there exists some $P \in \mathsf{Sep}(R)$ which is a t-neighbour of Q, so that for some $\Lambda = \Lambda(t, M) > 1$ we have $\Lambda B_Q \supset 2MB_P$, and $2MB_P$ intersects Γ_R. Finally, for a general $Q \in \mathsf{Tree}(R)$, either Q contains some cube from $\mathsf{Stop}(R)$, or $Q \subset \widetilde{G}_R$. In any case, $\Lambda B_Q \cap \Gamma_R \neq \varnothing$. □

Remark 3.9. Note that while for a general cube $Q \in \mathsf{Tree}(R)$ we only have $\Lambda B_Q \cap \Gamma_R \neq \varnothing$, we have a better estimate for the root R:

$$B_R \cap \Gamma_R \neq \varnothing. \tag{3.16}$$

Indeed, (3.16) is clear if the set $\widetilde{G}_R$ is non-empty. If $\widetilde{G}_R = \varnothing$, then $\mathsf{Sep}^*(R) \neq \varnothing$, so that for some $P \in \mathsf{Sep}^*(R)$ we have $x_P \in \Gamma_R \cap B_R$.

4 Small measure of cubes from $\mathsf{LD}(R)$

In the proof of the packing estimate (2.2) it will be crucial to have a bound on the measure of low density cubes.

Lemma 4.1. *We have*

$$\sum_{Q \in \mathsf{LD}(R)} \mu(Q) \lesssim_{t,\alpha} \tau \mu(R).$$

In particular, for τ small enough we have

$$\sum_{Q \in \mathsf{LD}(R)} \mu(Q) \leq \tau^{1/2} \mu(R). \tag{4.1}$$

We begin by defining some auxiliary subfamilies of $\mathsf{LD}(R)$.

Lemma 4.2. *There exists a t-separated family $\mathsf{LD}_{\mathsf{Sep}}(R) \subset \mathsf{LD}(R)$ such that*

$$\sum_{Q \in \mathsf{LD}(R)} \mu(Q) \lesssim_t \sum_{Q \in \mathsf{LD}_{\mathsf{Sep}}(R)} \mu(Q).$$

Proof. We construct the family $\mathsf{LD}_{\mathsf{Sep}}(R)$ in the following way. Define $\mathsf{LD}_1(R)$ as a maximal t-separated subfamily of $\mathsf{LD}(R)$. Next, define $\mathsf{LD}_2(R)$ as a maximal t-separated subfamily of $\mathsf{LD}(R) \setminus \mathsf{LD}_1(R)$. In general, having defined $\mathsf{LD}_j(R)$, we define $\mathsf{LD}_{j+1}(R)$ to be a maximal t-separated subfamily of $\mathsf{LD}(R) \setminus (\mathsf{LD}_1(R) \cup \cdots \cup \mathsf{LD}_j(R))$.

We claim that there is only a bounded number of non-empty families $\mathsf{LD}_j(R)$, with the bound depending on t. Indeed, if $Q \in \mathsf{LD}_j(R)$, then Q has at least one t-neighbour in each family $\mathsf{LD}_k(R)$, $k \leq j$. It follows easily from the definition of t-neighbours that the number of t-neighbours of any given cube is bounded by a constant $C(t)$. Hence, $j \leq C(t)$.

Set $\mathsf{LD}_{\mathsf{Sep}}(R)$ to be the family $\mathsf{LD}_j(R)$ maximizing $\sum_{Q \in \mathsf{LD}_j(R)} \mu(Q)$. Then,

$$\sum_{Q \in \mathsf{LD}(R)} \mu(Q) \leq C(t) \sum_{Q \in \mathsf{LD}_{\mathsf{Sep}}(R)} \mu(Q).$$

$\square$

We define also a family $\mathsf{LD}^*_{\mathsf{Sep}}(R) \subset \mathsf{LD}_{\mathsf{Sep}}(R)$ in the following way: we remove from $\mathsf{LD}_{\mathsf{Sep}}(R)$ all the cubes P for which there exists some $Q \in \mathsf{LD}_{\mathsf{Sep}}(R)$ such that

$$1.1B_Q \cap 1.1B_P \neq \varnothing \qquad \text{and} \qquad r(Q) < r(P). \tag{4.2}$$

Lemma 4.3. *For each $Q \in \mathsf{LD}_{\mathsf{Sep}}(R)$ at least one of the following is true:*

(a) $1.2B_Q \cap \widetilde{G}_R \neq \varnothing$

(b) There exists some $P \in \mathsf{LD}^*_{\mathsf{Sep}}(R)$ such that $1.2B_P \subset 1.2B_Q$.

Proof. Suppose $Q \in \mathsf{LD}_{\mathsf{Sep}}(R)$, and that (a) does not hold. We will find P such that (b) is satisfied.

If $Q \notin \mathsf{LD}^*_{\mathsf{Sep}}(R)$, then there exists some cube $Q_1 \in \mathsf{LD}_{\mathsf{Sep}}(R)$ such that

$$1.1B_Q \cap 1.1B_{Q_1} \neq \varnothing \qquad \text{and} \qquad r(Q_1) < r(Q). \tag{4.3}$$

Since Q and Q_1 are t-separated, and (3.10) holds, it follows that $t\,r(Q_1) < r(Q)$. Thus, Q_1 is tiny compared to Q and we have $1.2B_{Q_1} \subset 1.2B_Q$. If $Q_1 \in \mathsf{LD}^*_{\mathsf{Sep}}(R)$, we set $P = Q_1$ and we are done. Otherwise, we iterate as in Lemma 3.6 (with $2M$ replaced by 1.2) to find a finite sequence $Q_1,\ Q_2, \dots,\ Q_m$ satisfying $1.2B_{Q_{j+1}} \subset 1.2B_{Q_j}$, and such that $Q_m \in \mathsf{LD}^*_{\mathsf{Sep}}(R)$. □

Lemma 4.4. *For each $Q \in \mathsf{LD}^*_{\mathsf{Sep}}(R)$ we have*

$$\mu\Big(Q \cap \bigcup_{P \in \mathsf{LD}^*_{\mathsf{Sep}}(R)} (K_P^{1/2} \setminus MB_P)\Big) \lesssim_{\tau,\alpha,\eta} \varepsilon\mu(Q). \tag{4.4}$$

In particular, if ε is small enough, then for each $Q \in \mathsf{LD}^*_{\mathsf{Sep}}(R)$ we can choose a point

$$w_Q \in Q \setminus \bigcup_{P \in \mathsf{LD}^*_{\mathsf{Sep}}(R)} (K_P^{1/2} \setminus MB_P). \tag{4.5}$$

Proof. Suppose $Q \in \mathsf{LD}^*_{\mathsf{Sep}}(R)$ and that we have $Q \cap K_P^{1/2} \setminus MB_P \neq \varnothing$ for some $P \in \mathsf{LD}^*_{\mathsf{Sep}}(R)$. Note that if we had $Mr(Q) \le \operatorname{dist}(Q,P)$, then the assumptions of Lemma 3.3 would be satisfied, and we would arrive at $Q \notin \mathsf{Tree}(R)$, a contradiction. Thus,

$$\operatorname{dist}(Q,P) \le Mr(Q) < t\,r(Q). \tag{4.6}$$

It follows that (3.10) – one of the t-neigbourhood conditions – is satisfied. Since Q and P are t-separated, we necessarily have $t\,r(Q) \le r(P)$ or $t\,r(P) \le r(Q)$.

If we had $t\,r(Q) \le r(P)$, then (4.6) implies $\operatorname{dist}(Q,P) \le r(P)$. Hence, $1.1B_Q \cap 1.1B_P \neq \varnothing$. But this cannot be true, by the definition of $\mathsf{LD}^*_{\mathsf{Sep}}(R)$. It follows that

$$t\,r(P) \le r(Q). \tag{4.7}$$

Let $S \supset P$ be the biggest ancestor of P satisfying $r(S) \leq \delta r(Q)$ for some small constant $\delta = \delta(\alpha)$ which will be fixed in a few lines. If t is big enough, then $S \neq P$. Thus, $r(S) \approx_\delta r(Q)$, and $S \in \mathsf{Tree}(R) \setminus \mathsf{Stop}(R)$. Recall that by the definition of $\mathsf{LD}^*_{\mathsf{Sep}}(R)$ we have $1.1B_Q \cap 1.1B_P = \varnothing$. It follows that if $\delta < 0.001$, then $4B_S \cap 1.05B_Q = \varnothing$. Now, using this separation, it is not difficult to check that for $\delta = \delta(\alpha)$ small enough, for any $x \in K_P^{1/2} \cap Q$ we have

$$2B_S \subset K(x).$$

Observe also that, due to (4.6) and the fact that $r(S) \leq \delta r(Q)$, we have

$$2B_S \subset B(x,r) \qquad \text{for } r \in \left(\frac{\eta^{-1}}{2} r(Q),\ \eta^{-1} r(Q)\right),$$

provided that η is small enough (say, $\eta^{-1} \gg t$). Putting together the two estimates above, we get that

$$\mu(2B_S) \leq \mu(K(x,r))$$

for any $x \in K_P^{1/2} \cap Q \supset Q \cap K_P^{1/2} \setminus MB_P$ and all $r \in (\eta^{-1} r(Q)/2,\ \eta^{-1} r(Q))$.

Integrating the above over all $x \in A$, where $A \subset Q \cap K_P^{1/2} \setminus MB_P$ is an arbitrary measurable subset, yields

$$\begin{aligned}\mu(A)\Theta_\mu(2B_R)^p &\overset{(3.3)}{\leq} \tau^{-1}\mu(A)\Theta_\mu(2B_S)^p \approx_{\tau,\alpha} \mu(A)\left(\frac{\mu(2B_S)}{r(Q)^n}\right)^p \\ &\lesssim_\eta \int_A \int_{\eta r(Q)}^{\eta^{-1} r(Q)} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r} d\mu(x). \qquad (4.8)\end{aligned}$$

Now, let P_i be some ordering of cubes $P \in \mathsf{LD}^*_{\mathsf{Sep}}(R)$ satisfying $Q \cap K_P^{1/2} \setminus MB_P \neq \varnothing$. We define $A_1 = Q \cap K_{P_1}^{1/2} \setminus MB_{P_1}$, and for $i > 1$

$$A_i = Q \cap K_{P_i}^{1/2} \setminus \Big(MB_{P_i} \cup \bigcup_{j=1}^{i-1} A_j\Big).$$

Observe that A_i are pairwise disjoint and their union is $Q \cap \bigcup_{P \in \mathsf{LD}^*_{\mathsf{Sep}}(R)} (K_P^{1/2} \setminus MB_P)$. Thus,

$$\begin{aligned}\mu\Big(Q \cap \bigcup_{P \in \mathsf{LD}^*_{\mathsf{Sep}}(R)} K_P^{1/2} \setminus MB_P\Big)\Theta_\mu(2B_R)^p &= \sum_i \mu(A_i)\Theta_\mu(2B_R)^p \\ &\overset{(4.8)}{\lesssim}_{\tau,\alpha,\eta} \int_{\bigcup_i A_i} \int_{\eta r(Q)}^{\eta^{-1} r(Q)} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r} d\mu(x) \leq \mathcal{E}_{\mu,p}(Q)\mu(Q).\end{aligned}$$

Note that since $Q \notin \mathsf{BCE}(R)$, we have $\mathcal{E}_{\mu,p}(Q)\mu(Q) \leq \varepsilon\Theta_\mu(2B_R)^p\mu(Q)$. So the estimate (4.4) holds. $\square$

Lemma 4.5. *There exists an n-dimensional Lipschitz graph Γ_{LD} passing through all the points w_P, $P \in \mathsf{LD}^*_{\mathsf{Sep}}(R)$. The Lipschitz constant of Γ_{LD} depends only on α.*

Proof. It suffices to show that for any $Q, P \in \mathsf{LD}^*_{\mathsf{Sep}}(R)$, $Q \neq P$, we have

$$w_Q - w_P \notin K^{1/2}. \tag{4.9}$$

Without loss of generality assume $r(P) \leq r(Q)$. By (4.5) we have

$$w_Q \notin K_P^{1/2} \setminus MB_P.$$

In particular,

$$w_Q \notin K^{1/2}(w_P) \setminus MB_P.$$

So, to prove (4.9), it is enough to show that

$$w_Q \notin MB_P. \tag{4.10}$$

Assume the contrary, i.e. $w_Q \in MB_P$. Then,

$$\operatorname{dist}(Q, P) \leq CMr(P) \leq t(r(Q) + r(P)).$$

That is, (3.10) holds. But Q and P are t-separated, and so (3.9) must fail. Hence,

$$r(P) \leq t^{-1} r(Q).$$

Q and P belong to $\mathsf{LD}^*_{\mathsf{Sep}}(R)$, so by (4.2) we have $1.1B_Q \cap 1.1B_P = \varnothing$. Thus,

$$\operatorname{dist}(w_Q, B_P) \geq 0.1 r(B_Q) \geq Ct\, r(B_P) > Mr(B_P).$$

So (4.10) holds. □

We can finally finish the proof of Lemma 4.1.

Proof of Lemma 4.1. By Lemma 4.2 it suffices to estimate the measure of cubes from $\mathsf{LD}_{\mathsf{Sep}}(R)$. Let $\mathcal{G}$ denote an arbitrary finite subfamily of $\mathsf{LD}_{\mathsf{Sep}}(R)$. We use the covering lemma [Tol14, Theorem 9.31] to choose a subfamily $\mathcal{F} \subset \mathcal{G}$ such that

$$\bigcup_{Q \in \mathcal{G}} 1.5B_Q \subset \bigcup_{Q \in \mathcal{F}} 2B_Q,$$

and the balls $\{1.5B_Q\}_{Q\in\mathcal{F}}$ are of bounded superposition.

The above and the LD stopping condition give

$$\sum_{Q\in\mathcal{G}} \mu(Q) \leq \sum_{Q\in\mathcal{F}} \mu(2B_Q) \lesssim \tau\, \Theta_\mu(2B_R) \sum_{Q\in\mathcal{F}} r(B_Q)^n. \tag{4.11}$$

Now, it follows from Lemma 4.3 and Lemma 4.5 that for each $Q \in \mathcal{G} \subset \mathsf{LD}_{\mathsf{Sep}}(R)$ there exists either $w_Q \in \Gamma_{\mathsf{LD}} \cap 1.2B_Q$ or $x \in \widetilde{G}_R \cap 1.2B_Q \subset \Gamma_R \cap 1.2B_Q$. Hence,

$$\mathcal{H}^n(1.5B_Q \cap (\Gamma_{\mathsf{LD}} \cup \Gamma_R)) \approx_\alpha r(B_Q)^n.$$

Now, using the bounded superposition property of $\mathcal{F}$ we get

$$\sum_{Q\in\mathcal{F}} r(B_Q)^n \approx_\alpha \sum_{Q\in\mathcal{F}} \mathcal{H}^n(1.5B_Q\cap(\Gamma_{\mathsf{LD}}\cup\Gamma_R)) \lesssim \mathcal{H}^n\Big(\bigcup_{Q\in\mathcal{F}} 1.5B_Q\cap(\Gamma_{\mathsf{LD}}\cup\Gamma_R)\Big)$$

$$\le \mathcal{H}^n(2B_R\cap(\Gamma_{\mathsf{LD}}\cup\Gamma_R)) \approx_\alpha r(R)^n \approx \mu(2B_R)\Theta_\mu(2B_R)^{-1} \overset{R\in\mathcal{D}^{db}}{\approx} \mu(R)\Theta_\mu(2B_R)^{-1}.$$

Together with (4.11), this gives

$$\sum_{Q\in\mathcal{G}} \mu(Q) \lesssim_\alpha \tau\mu(R).$$

Since $\mathcal{G}$ was an arbitrary finite subfamily of $\mathsf{LD}_{\mathsf{Sep}}(R)$, we finally arrive at

$$\sum_{Q\in\mathsf{LD}_{\mathsf{Sep}}(R)} \mu(Q) \lesssim_\alpha \tau\mu(R).$$

□

5 Top cubes and packing estimate

5.1 Definition of Top

In order to define the Top family, we need to introduce some additional notation. Given $Q \in \mathcal{D}$, let $\mathcal{MD}(Q)$ denote the family of maximal cubes from $\mathcal{D}^{db}(Q)\setminus\{Q\}$. It follows from Lemma II.2.3 that the cubes from $\mathcal{MD}(Q)$ cover μ-almost all of Q.

Given $R \in \mathcal{D}^{db}$ set

$$\mathsf{Next}(R) = \bigcup_{Q\in\mathsf{Stop}(R)} \mathcal{MD}(Q).$$

Since we always have $\mathcal{MD}(Q) \neq \{Q\}$, it is clear that $\mathsf{Next}(R) \neq \{R\}$.

Observe that if $P \in \mathsf{Next}(R)$, then by Lemma 3.1 and Lemma II.2.6 we have for all intermediate cubes $S \in \mathcal{D},\ P \subset S \subset R$,

$$\Theta_\mu(2B_S) \lesssim_A \Theta_\mu(2B_R). \tag{5.1}$$

We are finally ready to define Top. It is defined inductively as $\mathsf{Top} = \bigcup_{k\ge 0}\mathsf{Top}_k$. First, set

$$\mathsf{Top}_0 = \{R_0\},$$

where R_0 was defined as $\operatorname{supp}\mu$. Having defined Top_k, we set

$$\mathsf{Top}_{k+1} = \bigcup_{R\in\mathsf{Top}_k} \mathsf{Next}(R).$$

Note that for each $k \ge 0$ the cubes from Top_k are pairwise disjoint.

5.2 Definition of ID

We distinguish a special type of Top cubes. We say that $R \in \mathsf{Top}$ is increasing density, $R \in \mathsf{ID}$, if

$$\mu\Big(\bigcup_{Q\in \mathsf{HD}(R)} Q\Big) \geq \frac{1}{2}\mu(R).$$

Lemma 5.1. *If A is big enough, then for all $R \in \mathsf{ID}$*

$$\Theta_\mu(2B_R)^p\mu(R) \leq \frac{1}{2}\sum_{Q\in \mathsf{Next}(R)} \Theta_\mu(2B_Q)^p\mu(Q). \tag{5.2}$$

Proof. The definition of ID and the HD stopping condition imply that for any $R \in \mathsf{ID}$

$$\Theta_\mu(2B_R)^p\mu(R) \leq 2\,\Theta_\mu(2B_R)^p \sum_{Q\in \mathsf{HD}(R)} \mu(Q) \leq 2A^{-p}\sum_{Q\in \mathsf{HD}(R)} \Theta_\mu(2B_Q)^p\mu(Q).$$

Note that all $Q \in \mathsf{HD}(R)$ are doubling, and so by Lemma II.2.7

$$\Theta_\mu(2B_Q)^p\mu(Q) \lesssim \sum_{P\in \mathcal{MD}(Q)} \Theta_\mu(2B_P)^p\mu(P) = \sum_{\substack{P\in \mathsf{Next}(R)\\ P\subset Q}} \Theta_\mu(2B_P)^p\mu(P).$$

If A is taken big enough, then the estimates above yield (5.2). $\square$

5.3 Packing condition

We will now establish the packing condition (2.2). For $S \in \mathsf{Top}$ set $\mathsf{Top}(S) = \mathsf{Top} \cap \mathcal{D}(S)$ and $\mathsf{Top}_j(S) = \mathsf{Top}_j \cap \mathcal{D}(S)$. For $k \geq 0$ we also define

$$\mathsf{Top}_0^k(S) = \bigcup_{0\leq j\leq k} \mathsf{Top}_j(S),$$
$$\mathsf{ID}_0^k(S) = \mathsf{ID} \cap \mathsf{Top}_0^k(S).$$

Recall that μ satisfies the following polynomial growth condition: there exist $C_1 > 0$ and $r_0 > 0$ such that for all $x \in \operatorname{supp}\mu,\ 0 < r \leq r_0$, we have

$$\mu(B(x,r)) \leq C_1 r^n. \tag{5.3}$$

Lemma 5.2. *For all $S \in \mathsf{Top}$ we have*

$$\sum_{R\in \mathsf{Top}(S)} \Theta_\mu(2B_R)^p\mu(R)$$
$$\lesssim_{\varepsilon,\eta,\tau} (C_1)^p\mu(S) + \int_{2B_S}\int_0^{\eta^{-1}C_0 r(S)} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r}\,d\mu(x). \tag{5.4}$$

The implicit constant does not depend on r_0.

Proof. First, we deal with ID cubes. Note that

$$\sum_{R\in\mathsf{ID}_0^k(S)}\Theta_\mu(2B_R)^p\mu(R)\overset{(5.2)}{\leq}\frac{1}{2}\sum_{R\in\mathsf{ID}_0^k(S)}\sum_{Q\in\mathsf{Next}(R)}\Theta_\mu(2B_Q)^p\mu(Q)$$
$$\leq\frac{1}{2}\sum_{Q\in\mathsf{Top}_0^{k+1}(S)}\Theta_\mu(2B_Q)^p\mu(Q),$$

where the last inequality follows from the fact that $\bigcup_{R\in\mathsf{Top}_0^k}\mathsf{Next}(R)=\mathsf{Top}_0^{k+1}$. Now, observe that for $Q\in\mathsf{Top}_{k+1}$ we have $r(Q)\leq C_0A_0^{-k}r(R_0)$, and so if k is big enough, then $r(2B_Q)\leq r_0$. Thus, by (5.3)

$$\Theta_\mu(2B_Q)\leq C_1. \tag{5.5}$$

Hence,

$$\sum_{R\in\mathsf{Top}_0^k(S)}\Theta_\mu(2B_R)^p\mu(R)=\sum_{R\in\mathsf{Top}_0^k(S)\setminus\mathsf{ID}}\Theta_\mu(2B_R)^p\mu(R)+\sum_{R\in\mathsf{ID}_0^k(S)}\Theta_\mu(2B_R)^p\mu(R)$$
$$\leq\sum_{R\in\mathsf{Top}_0^k(S)\setminus\mathsf{ID}}\Theta_\mu(2B_R)^p\mu(R)+\frac{1}{2}\sum_{R\in\mathsf{Top}_0^{k+1}(S)}\Theta_\mu(2B_R)^p\mu(R)$$
$$\leq\sum_{R\in\mathsf{Top}_0^k(S)\setminus\mathsf{ID}}\Theta_\mu(2B_R)^p\mu(R)+\frac{1}{2}\sum_{R\in\mathsf{Top}_0^k(S)}\Theta_\mu(2B_R)^p\mu(R)+\frac{(C_1)^p}{2}\mu(S). \tag{5.6}$$

Note that for small cubes $Q\in\mathsf{Top}_0^k(S)$ (i.e. satisfying $r(2B_Q)\leq r_0$) we have (5.5), while for big cubes the trivial estimate $\Theta_\mu(2B_Q)\leq\mu(2B_S)r_0^{-n}$ holds. It follows that

$$\sum_{R\in\mathsf{Top}_0^k(S)}\Theta_\mu(2B_R)^p\mu(R)\leq(k+1)\left((C_1)^p+\mu(2B_S)^pr_0^{-np}\right)\mu(S)<\infty,$$

and so we may deduce from (5.6) that

$$\sum_{R\in\mathsf{Top}_0^k(S)}\Theta_\mu(2B_R)^p\mu(R)\leq 2\sum_{R\in\mathsf{Top}_0^k(S)\setminus\mathsf{ID}}\Theta_\mu(2B_R)^p\mu(R)+(C_1)^p\mu(S).$$

Letting $k\to\infty$ we arrive at

$$\sum_{R\in\mathsf{Top}(S)}\Theta_\mu(2B_R)^p\mu(R)\leq 2\sum_{R\in\mathsf{Top}(S)\setminus\mathsf{ID}}\Theta_\mu(2B_R)^p\mu(R)+(C_1)^p\mu(S). \tag{5.7}$$

Now, we need to estimate the sum from the right hand side. By the definition of ID we have for all $R\in\mathsf{Top}(S)\setminus\mathsf{ID}$

$$\mu\Bigg(R\setminus\bigcup_{Q\in\mathsf{HD}(R)}Q\Bigg)\geq\frac{1}{2}\,\mu(R),$$

and so by Lemma II.2.1 (c) we get

$$\mu(R) \le 2\,\mu\Big(R \setminus \bigcup_{Q\in\mathsf{Stop}(R)} Q\Big) + 2\,\mu\Big(\bigcup_{Q\in\mathsf{Stop}(R)\setminus\mathsf{HD}(R)} Q\Big)$$
$$= 2\,\mu\Big(R \setminus \bigcup_{Q\in\mathsf{Next}(R)} Q\Big) + 2\sum_{Q\in\mathsf{LD}(R)} \mu(Q) + 2\sum_{Q\in\mathsf{BCE}(R)} \mu(Q).$$

The measure of low density cubes is small due to (4.1), and so for τ small enough we have

$$\mu(R) \le 3\,\mu\Big(R \setminus \bigcup_{Q\in\mathsf{Next}(R)} Q\Big) + 3\sum_{Q\in\mathsf{BCE}(R)} \mu(Q).$$

Thus,

$$\sum_{R\in\mathsf{Top}(S)\setminus\mathsf{ID}} \Theta_\mu(2B_R)^p\mu(R) \le 3\sum_{R\in\mathsf{Top}(S)} \Theta_\mu(2B_R)^p\mu\Big(R\setminus\bigcup_{Q\in\mathsf{Next}(R)} Q\Big)$$
$$+\,3\sum_{R\in\mathsf{Top}(S)\setminus\mathsf{ID}} \Theta_\mu(2B_R)^p \sum_{Q\in\mathsf{BCE}(R)} \mu(Q). \qquad (5.8)$$

Concerning the first sum, notice that if $\mu\big(R\setminus\bigcup_{Q\in\mathsf{Next}(R)} Q\big) > 0$, then we have arbitrarily small cubes P belonging to $\mathsf{Tree}(R)$. In particular, by (3.3) and (5.3), we have $\Theta_\mu(2B_R) \le \tau^{-1}\Theta_\mu(2B_P) \le \tau^{-1}C_1$, taking $P\in\mathsf{Tree}(R)\setminus\mathsf{Stop}(R)$ small enough. Recall also that for $R\in\mathsf{Top}(S)$, the sets $R\setminus\bigcup_{Q\in\mathsf{Next}(R)} Q$ are pairwise disjoint. Hence,

$$\sum_{R\in\mathsf{Top}(S)} \Theta_\mu(2B_R)^p\mu\Big(R\setminus\bigcup_{Q\in\mathsf{Next}(R)} Q\Big) \le (\tau^{-1}C_1)^p\mu(S). \qquad (5.9)$$

To estimate the second sum from (5.8), we apply (3.5) to get

$$\sum_{R\in\mathsf{Top}(S)} \Theta_\mu(2B_R)^p \sum_{Q\in\mathsf{BCE}(R)} \mu(Q) \le \frac{1}{\varepsilon}\sum_{R\in\mathsf{Top}(S)}\sum_{P\in\mathsf{Tree}(R)} \mathcal{E}_{\mu,p}(P)\mu(P)$$
$$\le \frac{1}{\varepsilon}\sum_{P\in\mathcal{D}(S)} \mathcal{E}_{\mu,p}(P)\mu(P)$$

By the definition of $\mathcal{E}_{\mu,p}(P)$, and the bounded intersection property of the balls $2B_P$ for cubes P of the same generation, we have

$$\sum_{P\in\mathcal{D}(S)} \mathcal{E}_{\mu,p}(P)\mu(P) = \sum_k \sum_{P\in\mathcal{D}_{\mu,k}(S)} \int_{2B_P}\int_{\eta r(P)}^{\eta^{-1}r(P)} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r}$$
$$\lesssim \sum_k \int_{2B_S}\int_{\eta A_0^{-k}}^{\eta^{-1}C_0A_0^{-k}} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r}$$
$$\lesssim_\eta \int_{2B_S}\int_0^{\eta^{-1}C_0 r(S)} \left(\frac{\mu(K(x,r))}{r^n}\right)^p \frac{dr}{r}d\mu(x).$$

Consequently,

$$\sum_{R\in\mathsf{Top}(S)}\Theta_\mu(2B_R)^p\sum_{Q\in\mathsf{BCE}(R)}\mu(Q)\lesssim_{\varepsilon,\eta}\int_{2B_S}\int_0^{\eta^{-1}C_0r(S)}\left(\frac{\mu(K(x,r))}{r^n}\right)^p\frac{dr}{r}d\mu(x).$$

Together with (5.7), (5.8), and (5.9), this gives (5.4). □

Let us put together all the ingredients of the proof of the main lemma.

Proof of Lemma 2.1. Let $\mathsf{Top}\subset\mathcal{D}^{db}$ be as above, and $\{\Gamma_R\}_{R\in\mathsf{Top}}$ be as in Lemma 3.8. Then, properties (i) and (ii) are ensured by Lemma 3.8. Property (iii) follows from (5.1). We get the packing estimate (2.2) from (5.4) by taking $S=R_0$. □

6 Application to singular integral operators

To prove Theorem 1.14, we will use geometric characterizations of boundedness of operators from $\mathcal{K}^n(\mathbb{R}^d)$ shown in [GS19, Sections 4, 5, 9]. For $n=1,\ d=2$, a variant of this characterization valid for the Cauchy transform was already proved in [Tol05].

For $Q,S\in\mathcal{D}$, $Q\subset S$, we set

$$\delta_\mu(Q,S)=\int_{2B_S\setminus 2B_Q}\frac{1}{|y-x_Q|^n}\,d\mu(y).$$

The notation $\mathsf{Good}(R),\ \mathsf{Tr}(R),\ \mathsf{Next}(R)$ used below was introduced in Section 2.

Lemma 6.1 ([GS19])**.** *Let μ be a compactly supported Radon measure on $\mathbb{R}^d$ satisfying the growth condition (1.8). Assume there exists a family of cubes $\mathsf{Top}\subset\mathcal{D}^{db}$, and a corresponding family of Lipschitz graphs $\{\Gamma_R\}_{R\in\mathsf{Top}}$, satisfying:*

(i) Lipschitz constants of Γ_R are uniformly bounded by some absolute constant,

(ii) μ-almost all $\mathsf{Good}(R)$ is contained in Γ_R,

(iii) for all $Q\in\mathsf{Tr}(R)$ we have $\Theta_\mu(2B_Q)\lesssim\Theta_\mu(2B_R)$.

(iv) for all $Q\in\mathsf{Next}(R)$ there exists $S\in\mathcal{D}$, $Q\subset S$, such that $\delta_\mu(Q,S)\lesssim\Theta_\mu(2B_R)$, and $2B_S\cap\Gamma_R\neq\varnothing$.

Then, for every singular integral operator T with kernel $k\in\mathcal{K}^n(\mathbb{R}^d)$ we have

$$\sup_{\varepsilon>0}\|T_\varepsilon\mu\|^2_{L^2(\mu)}\lesssim\sum_{R\in\mathsf{Top}}\Theta_\mu(2B_R)^2\mu(R),$$

with the implicit constant depending on C_1 and the constant C_k from (1.7).

We are going to use Lemma 2.1 together with Lemma 3.8 and Lemma 6.1 to get the following.

Lemma 6.2. *Let μ be a compactly supported Radon measure on $\mathbb{R}^d$ satisfying the growth condition (1.8). Assume further that for some $V \in G(d, d-n)$, $\alpha \in (0,1)$, we have $\mathcal{E}_{\mu,2}(\mathbb{R}^d, V, \alpha) < \infty$.*

Then, for every singular integral operator T with kernel $k \in \mathcal{K}^n(\mathbb{R}^d)$ we have

$$\sup_{\varepsilon>0} \|T_\varepsilon \mu\|^2_{L^2(\mu)} \lesssim \mu(\mathbb{R}^d) + \mathcal{E}_{\mu,2}(\mathbb{R}^d, V, \alpha), \tag{6.1}$$

with the implicit constant depending on C_1, α and the constant C_k from (1.7).

Proof. Using Lemma 2.1 (with $p = 2$), it is clear that the assumptions (i)-(iii) of Lemma 6.1 are satisfied. We still have to check if (iv) holds. Once we do that, the packing estimate (2.2) together with Lemma 6.1 will ensure that (6.1) holds.

Suppose $R \in \mathsf{Top}$, $Q \in \mathsf{Next}(R)$. We are looking for $S \in \mathcal{D}$ such that $\delta_\mu(Q,S) \lesssim \Theta_\mu(2B_R)$, and $2B_S \cap \Gamma_R \neq \varnothing$. Let $P \in \mathsf{Stop}(R)$ be such that $Q \subset P$. By Lemma 3.8 we have some constant Λ such that

$$\Lambda B_P \cap \Gamma_R \neq \varnothing.$$

Together with (3.16), this implies that there exists $S \in \mathsf{Tree}(R)$ such that $P \subset S$, $r(S) \approx_\Lambda r(P)$, and

$$2B_S \cap \Gamma_R \neq \varnothing.$$

We split

$$\delta_\mu(Q,S) = \int_{2B_S \setminus 2B_P} \frac{1}{|y-x_Q|^n}\, d\mu(y) + \int_{2B_P \setminus 2B_Q} \frac{1}{|y-x_Q|^n}\, d\mu(y).$$

Concerning the first integral, for $y \in 2B_S \setminus 2B_P$ we have $|y - x_Q| \approx r(S) \approx_\Lambda r(P)$, and so

$$\int_{2B_S \setminus 2B_P} \frac{1}{|y-x_Q|^n}\, d\mu(y) \lesssim \Theta_\mu(2B_S) \overset{(3.2)}{\lesssim_A} \Theta_\mu(2B_R).$$

To deal with the second integral, observe that there are no doubling cubes between Q and P. Then, it follows from Lemma II.2.6 that

$$\int_{2B_P \setminus 2B_Q} \frac{1}{|y-x_Q|^n}\, d\mu(y) \lesssim \Theta_\mu(100B(P)).$$

If $P = R$, then P is doubling and we have $\Theta_\mu(100B(P)) \lesssim \Theta_\mu(2B_R)$. Otherwise, the parent of P, denoted by P', belongs to $\mathsf{Tree}(R) \setminus \mathsf{Stop}(R)$. Since $100B(P) \subset 2B_{P'}$, we get

$$\Theta_\mu(100B(P)) \lesssim \Theta_\mu(2B_{P'}) \overset{(3.2)}{\lesssim_A} \Theta_\mu(2B_R).$$

Either way, we get that $\delta_\mu(Q,S) \lesssim_A \Theta_\mu(2B_R)$, and so the assumption (iv) of Lemma 6.1 is satisfied. □

Lemma 6.2 allows us to use the non-homogeneous $T1$ theorem of Nazarov, Treil and Volberg [NTV97] to prove a version of Theorem 1.14 in the case of a fixed direction V, i.e. if for all $x \in \operatorname{supp} \mu$ we have $V_x \equiv V$.

Lemma 6.3. *Let μ be a Radon measure on $\mathbb{R}^d$ satisfying the polynomial growth condition (1.8). Suppose that there exist $M_0 > 1$, $\alpha \in (0,1)$, $V \in G(d, d-n)$, such that for every ball B we have*

$$\mathcal{E}_{\mu,2}(B, V, \alpha) \leq M_0 \mu(B). \tag{6.2}$$

Then, all singular integral operators T_μ with kernels in $\mathcal{K}^n(\mathbb{R}^d)$ are bounded in $L^2(\mu)$. The bound on the operator norm of T_μ depends only on C_1, α, M_0, and the constant C_k from (1.7).

Proof. We apply Lemma 6.2 to $\mu|_B$, where B is an arbitrary ball, and get that

$$\sup_{\varepsilon>0} \|T_\varepsilon(\mu|_B)\|^2_{L^2(\mu|_B)} \lesssim_{C_1,\alpha,C_k} \mu(B) + \mathcal{E}_{\mu|_B,2}(\mathbb{R}^d, V, \alpha).$$

It is easy to see that, using the assumptions (1.8) and (6.2), we have

$$\mathcal{E}_{\mu|_B,2}(\mathbb{R}^d, V, \alpha) \lesssim \mathcal{E}_{\mu,2}(B, V, \alpha) + C_1^2 \mu(B) \leq (1 + C_1^2)\mu(B).$$

Hence,

$$\sup_{\varepsilon>0} \|T_\varepsilon(\mu|_B)\|^2_{L^2(\mu|_B)} \lesssim_{C_1,\alpha,C_k,M_0} \mu(B). \tag{6.3}$$

The L^2 boundedness of T_μ follows by the non-homogeneous $T1$ theorem from [NTV97]. The condition (6.3) is slightly weaker than the original assumption in [NTV97], but this is not a problem, see the discussion in [Tol14, §3.7.2]. □

We are ready to finish the proof of Theorem 1.14.

Proof of Theorem 1.14. Let B be an arbitrary ball intersecting $\operatorname{supp} \mu$. Recall that, by the definition of BPBE(2), there exist $M_0 > 1$, $\kappa > 0$, $V_B \in G(d, d-n)$, and $G_B \subset B$ such that $\mu(G_B) \geq \kappa \mu(B)$ and for all $x \in G_B$

$$\int_0^{r(B)} \left(\frac{\mu(K(x, V_B, \alpha, r))}{r^n} \right)^2 \frac{dr}{r} \leq M_0.$$

By the polynomial growth condition (1.8) we also have

$$\int_{r(B)}^\infty \left(\frac{\mu(K(x, V_B, \alpha, r))}{r^n} \right)^2 \frac{dr}{r} \leq \int_{r(B)}^\infty \frac{\mu(B)^2}{r^{2n+1}}\, dr \lesssim \frac{\mu(B)^2}{r(B)^{2n}} \leq C_1^2.$$

Hence, for all $x \in G_B$

$$\int_0^\infty \left(\frac{\mu(K(x, V_B, \alpha, r))}{r^n} \right)^2 \frac{dr}{r} \lesssim_{C_1,M_0} 1.$$

Set $\nu = \mu|_{G_B}$. The estimate above implies that for all balls $B' \subset \mathbb{R}^d$ we have

$$\mathcal{E}_{\nu,2}(B', V_B, \alpha) = \int_{B'} \int_0^{r(B')} \left(\frac{\nu(K(x, V_B, \alpha, r))}{r^n} \right)^2 \frac{dr}{r} d\nu(x) \lesssim_{C_1, M_0} \nu(B').$$

Clearly, $\nu = \mu|_{G_B}$ has polynomial growth, and so we may apply Lemma 6.3 to conclude that all singular integral operators T_ν with kernels in $\mathcal{K}^n(\mathbb{R}^d)$ are bounded in $L^2(\nu)$. Thus, the corresponding maximal operators T_*, defined as

$$T_*\nu(x) = \sup_{\varepsilon>0} |T_\varepsilon \nu(x)| \qquad \text{for } \nu \in M(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$

are bounded from $M(\mathbb{R}^d)$ to $L^{1,\infty}(\nu)$, see [Tol14, Theorem 2.21].

Recall that for all balls B we have $\mu(G_B) \approx_\kappa \mu(B)$. For any fixed T, the operator norm of $T_{\mu|_{G_B},\varepsilon} : L^2(\mu|_{G_B}) \to L^2(\mu|_{G_B})$ is bounded uniformly in B and ε, and so the same is true for the operator norm of $T_* : M(\mathbb{R}^d) \to L^{1,\infty}(\mu|_{G_B})$. Hence, we may use the good lambda method [Tol14, Theorem 2.22] to conclude that T_μ is bounded in $L^2(\mu)$. □

7 Sufficient condition for rectifiability

The aim of this section is to prove the following sufficient condition for rectifiability.

Proposition 7.1. *Suppose μ is a Radon measure on $\mathbb{R}^d$ satisfying $\Theta^{n,*}(\mu, x) > 0$ and $\Theta^n_*(\mu, x) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$. Assume further that for μ-a.e. $x \in \mathbb{R}^d$ there exists some $V_x \in G(d, d-n)$ and $\alpha_x \in (0,1)$ such that*

$$\int_0^1 \left(\frac{\mu(K(x, V_x, \alpha_x, r))}{r^n} \right)^p \frac{dr}{r} < \infty, \tag{7.1}$$

and the mapping $x \mapsto (V_x, \alpha_x)$ is μ-measurable. Then, μ is n-rectifiable.

We reduce the proposition above to the following lemma.

Lemma 7.2. *Suppose μ is a Radon measure on $B(0,1) \subset \mathbb{R}^d$, and assume that there exists a constant $C_* > 0$ such that $\Theta^n_*(\mu, x) \le C_*$ and $\Theta^{n,*}(\mu, x) > 0$ for μ-a.e. $x \in \mathbb{R}^d$. Assume further that there exist $M_0 > 0$, $V \in G(d, d-n)$ and $\alpha \in (0,1)$ such that for μ-a.e. $x \in \mathbb{R}^d$*

$$\int_0^1 \left(\frac{\mu(K(x, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r} \le M_0. \tag{7.2}$$

Then, μ is n-rectifiable.

Proof of Proposition 7.1 using Lemma 7.2. To show that μ is rectifiable, it suffices to prove that for any bounded $E \subset \operatorname{supp}\mu$ of positive measure there exists $F \subset E$, $\mu(F) > 0$, such that $\mu|_F$ is rectifiable. Given any such E we may rescale it and translate it, so without loss of generality $E \subset B(0,1)$.

Since $0 < \Theta^{n,*}(\mu, x)$ and $\Theta^n_*(\mu, x) < \infty$ for μ-a.e. $x \in E$, choosing $C_* > 1$ big enough, we get that the set

$$E' = \{x \in E \ : \ \Theta^{n,*}(\mu, x) > 0, \ \Theta^n_*(\mu, x) \le C_*\} \tag{7.3}$$

has positive μ-measure.

Let $\alpha_j \to 0$ be a decreasing sequence of numbers, and let $\{V_m\}_{m\in\mathbb{N}}$ be a countable and dense subset of $G(d, d-n)$. It is clear that for any $\alpha \in (0,1)$, $V \in G(d, d-n)$, there exist α_j, V_m, such that $K(0, V_m, \alpha_j) \subset K(0, V, \alpha)$.

We consider all the pairs $\{(\alpha_j, V_m)\}_{j,m}$, and relabel them to get a sequence (α_k, V_k), $k \in \mathbb{N}$. For μ-a.e. $x \in \operatorname{supp}\mu$ let α_x, V_x be the angle and $(d-n)$-plane for which (7.1) holds. Set

$$E_k = \{x \in E' \ : \ K(x, V_k, \alpha_k) \subset K(x, V_x, \alpha_x)\},$$

Observe that E_k are measurable due to measurability of $x \mapsto (V_x, \alpha_x)$, and that $\mu(E' \setminus \bigcup_{k=0}^\infty E_k) = 0$. Pick any $k \in \mathbb{N}$ such that $\mu(E_k) > 0$. For μ-a.e. $x \in E_k$ we have

$$\int_0^1 \left(\frac{\mu(K(x, V_k, \alpha_k, r))}{r^n} \right)^p \frac{dr}{r} < \infty.$$

Thus, choosing $M_0 \gg 1$ big enough, we get that the set $F \subset E_k$ of points such that

$$\int_0^1 \left(\frac{\mu(K(x, V_k, \alpha_k, r))}{r^n} \right)^p \frac{dr}{r} \le M_0$$

satisfies $\mu(F) > 0$. Finally, using the Lebesgue differentiation theorem and (7.3), it is easy to see that for μ-a.e. $x \in F$ we have $\Theta^{n,*}(\mu|_F, x) = \Theta^{n,*}(\mu, x) > 0$ and $\Theta^n_*(\mu|_F, x) = \Theta^n_*(\mu, x) \le C_*$. Hence, $\mu|_F$ satisfies the assumptions of Lemma 7.2, and so it is n-rectifiable. □

7.1 Proof of Lemma 7.2 for $\mu \ll \mathcal{H}^n$

First, we will prove Lemma 7.2 under the additional assumption $\Theta^{n,*}(\mu, x) < \infty$ for μ-a.e. $x \in \mathbb{R}^d$ (which is equivalent to $\mu \ll \mathcal{H}^n$).

Using similar tricks as in the proof of Proposition 7.1, it is easy to see that we may actually replace $\Theta^{n,*}(\mu, x) < \infty$ by a stronger condition: without loss of generality, we can assume that there exist $C_1 > 0$ and $r_0 > 0$ such that for all $x \in \operatorname{supp}\mu$ and all $0 < r \le r_0$ we have

$$\mu(B(x, r)) \le C_1 r^n. \tag{7.4}$$

Then, the assumptions of Lemma 2.1 are satisfied, and we get a family of cubes $\mathsf{Top} \subset \mathcal{D}^{db}$ and an associated family of Lipschitz graphs Γ_R, $R \in \mathsf{Top}$. The cubes from Top satisfy the packing condition

$$\sum_{R\in\mathsf{Top}} \Theta_\mu(2B_R)^p \mu(R) \lesssim \mu(\mathbb{R}^d) + \mathcal{E}_{\mu,p}(\mathbb{R}^d, V, \alpha) \le (1+M_0)\mu(B(0,1)).$$

It follows that for μ-a.e. $x \in \mathbb{R}^d$ we have

$$\sum_{R\in\mathsf{Top}:\, R\ni x} \Theta_\mu(2B_R)^p < \infty.$$

Fix some x for which the above holds. Denote by $R_0 \supset R_1 \supset \dots$ the sequence of cubes from Top containing x. We claim that for μ-a.e. x this sequence is finite.

Indeed, if the sequence is infinite, we have $\Theta_\mu(2B_{R_i}) \to 0$. On the other hand, let $i \ge 0$ and $r(R_{i+1}) \le r \le r(R_i)$. Since $R_{i+1} \in \mathsf{Next}(R_i)$, we get from (5.1)

$$\Theta_\mu(x,r) \lesssim_A \Theta_\mu(2B_{R_i}).$$

In consequence,

$$\Theta^{n,*}(\mu,x) \lesssim_A \limsup_{i\to\infty} \Theta_\mu(2B_{R_i}) = 0,$$

which may happen only on a set of μ-measure 0 because $\Theta^{n,*}(\mu,x) > 0$ for μ-a.e. $x \in \mathbb{R}^d$.

Hence, for μ-a.e. $x \in \mathbb{R}^d$ the sequence $\{R_i\}$ is finite. This means that if R_k denotes the smallest Top cube containing x, then $x \in \mathsf{Good}(R_k)$. It follows that

$$\mu\Big(\mathbb{R}^d \setminus \bigcup_{R\in\mathsf{Top}} \mathsf{Good}(R)\Big) = 0.$$

By Lemma 2.1 (ii) we have $\mu(\mathsf{Good}(R_k) \setminus \Gamma_{R_k}) = 0$. Hence,

$$\mu\Big(\mathbb{R}^d \setminus \bigcup_{R\in\mathsf{Top}} \Gamma_R\Big) = 0,$$

and so μ is n-rectifiable.

7.2 Proof of Lemma 7.2 in full generality

Thanks to the partial result from the preceding subsection, it is clear that to prove Lemma 7.2 in full generality, it suffices to show that for μ satisfying the assumptions of Lemma 7.2 we have

$$M_n\mu(x) = \sup_{r>0} \frac{\mu(B(x,r))}{r^n} < \infty \quad \text{for } \mu\text{-a.e. } x \in B(0,1).$$

To do that, we will use techniques from [Tol19, Section 5].

Lemma 7.3 ([Tol19, Lemma 5.1])**.** *Let $C > 2$. Suppose that μ is a Radon measure on $\mathbb{R}^d$, and that $\Theta^n_*(\mu, x) \leq C_*$ for μ-a.e. $x \in \mathbb{R}^d$. Then, for μ-a.e. $x \in \mathbb{R}^d$ there exists a sequence of radii $r_k \to 0$ such that*

$$\mu(B(x, Cr_k)) \leq 2C^d \mu(B(x, r_k)) \leq 20\, C_* C^{n+d} r_k^n. \tag{7.5}$$

Let $\lambda < \frac{1}{2}$ be a small constant depending on α, to be chosen later. By the lemma above (used with $C = \lambda^{-1}$) and Vitali's covering theorem (see [Mat95, Theorem 2.8]), there exists a family of pairwise disjoint closed balls B_i, $i \in I$, centered at $x_i \in \operatorname{supp}\mu \subset B(0,1)$, which cover μ-almost all of $B(0,1)$, and which satisfy

$$\mu(B_i) \leq 2\lambda^{-d}\mu(\lambda B_i) \leq 20\, C_* \lambda^{-d}\, r(B_i)^n,$$

and

$$r(B_i) \leq \rho$$

for some arbitrary fixed $\rho > 0$. We may assume that (7.2) holds for all the centers x_i. Choose $I_0 \subset I$ a finite subfamily such that

$$\mu(B(0,1) \setminus \bigcup_{i \in I_0} B_i) \leq \varepsilon \mu(B(0,1)),$$

where $\varepsilon > 0$ is some small constant. Clearly, $I_0 = I_0(\rho, \varepsilon)$.

For each $i \in I_0$ we consider an n-dimensional disk D_i, centered at x_i, parallel to $V^\perp \in G(d,n)$, with radius $\lambda r(B_i)$. We define an approximating measure

$$\nu = \sum_{i \in I_0} \frac{\mu(B_i)}{\mathcal{H}^n(D_i)} \mathcal{H}^n|_{D_i}.$$

Note that

$$\nu(D_i) = \mu(B_i) \approx_\lambda \mu(\lambda B_i) \lesssim_\lambda C_* r(B_i)^n. \tag{7.6}$$

Moreover, since I_0 is a finite family, the definition of ν and (7.6) imply that ν satisfies the polynomial growth condition (2.1) with $r_0 = \min_{i \in I_0} r(B_i)/2$ and $C_1 = C(\lambda) C_*$, i.e. for $0 < r < r_0$ and $x \in \operatorname{supp}\nu$

$$\nu(B(x,r)) \leq C(\lambda) C_* r^n. \tag{7.7}$$

Lemma 7.4. *For $\lambda = \lambda(\alpha) < \frac{1}{2}$ small enough, we have*

$$\mathcal{E}_{\nu,p}(\mathbb{R}^d, V, \tfrac{1}{2}\alpha) \lesssim_{\lambda,p} (M_0 + \mu(B(0,1))^p)\mu(B(0,1)).$$

The implicit constant does not depend on ρ, ε.

Proof. Let $i \in I_0$ and $x \in D_i$. We will estimate the ν-measure of $K(x, V, \frac{1}{2}\alpha, r)$.

First, note that $\nu(K(x, V, \frac{1}{2}\alpha, r)) = \nu(K(x, V, \frac{1}{2}\alpha, r) \setminus B_i)$. Indeed, $B_i \cap \operatorname{supp}\nu = D_i$, and $D_i \cap K(x, V, \frac{1}{2}\alpha) = \varnothing$ because D_i is parallel to $V^\perp$. Thus, $\nu(K(x, V, \frac{1}{2}\alpha, r) \cap B_i) = 0$. It follows immediately that for $r \leq (1-\lambda) r(B_i)$ we have $\nu(K(x, V, \frac{1}{2}\alpha, r)) = 0$.

Concerning $r > (1-\lambda)r(B_i)$, if $\lambda = \lambda(\alpha)$ is small enough, then

$$K(x, V, \tfrac{1}{2}\alpha, r) \setminus B_i \subset K(x_i, V, \tfrac{3}{4}\alpha, 2r) \setminus B_i$$

because $x \in \lambda B_i$. Thus, it suffices to estimate $\nu(K(x_i, V, \frac{3}{4}\alpha, 2r) \setminus B_i)$.

Suppose $r > (1-\lambda)r(B_i)$ and $j \in I_0$ is such that $D_j \cap K(x_i, V, \frac{3}{4}\alpha, 2r) \setminus B_i \neq \varnothing$. Since B_i and B_j are disjoint, we have

$$r(B_j) + r(B_i) + \mathrm{dist}(B_i, B_j) \le 3r \quad \text{and} \quad \mathrm{dist}(D_i, D_j) \ge \frac{r(B_i)}{2} + \frac{r(B_j)}{2}.$$

It follows easily that, for $\lambda = \lambda(\alpha)$ small enough, we get $\lambda B_j \subset K(x_i, V, \alpha, 4r)$. Thus,

$$\begin{aligned}\nu(K(x_i, V, \tfrac{3}{4}\alpha, 2r)) = \nu(K(x_i, V, \tfrac{3}{4}\alpha, 2r) \setminus B_i) &\le \sum_{j \in I_0 : \lambda B_j \subset K(x_i, V, \alpha, 4r)} \nu(D_j) \\ &\overset{(7.6)}{\approx_\lambda} \sum_{j \in I_0 : \lambda B_j \subset K(x_i, V, \alpha, 4r)} \mu(\lambda B_j) \le \mu(K(x_i, V, \alpha, 4r)).\end{aligned}$$

Hence,

$$\int_0^{1/4} \left(\frac{\nu(K(x_i, V, \frac{3}{4}\alpha, 2r))}{r^n} \right)^p \frac{dr}{r} \lesssim_\lambda \int_0^1 \left(\frac{\mu(K(x_i, V, \alpha, r))}{r^n} \right)^p \frac{dr}{r} \overset{(7.2)}{\le} M_0.$$

This gives

$$\begin{aligned}&\int_{D_i} \int_0^\infty \left(\frac{\nu(K(x, V, \frac{1}{2}\alpha, r))}{r^n} \right)^p \frac{dr}{r} d\nu(x) \\ &\qquad \le \int_{D_i} \int_0^\infty \left(\frac{\nu(K(x_i, V, \frac{3}{4}\alpha, r))}{r^n} \right)^p \frac{dr}{r} d\nu(x) \\ &\qquad \le C(\lambda) M_0 \nu(D_i) + \int_{D_i} \int_{1/4}^\infty \left(\frac{\nu(\mathbb{R}^d)}{r^n} \right)^p \frac{dr}{r} d\nu(x) \\ &\qquad \lesssim_{\lambda,p} M_0 \nu(D_i) + \nu(\mathbb{R}^d)^p \nu(D_i) \le M_0 \mu(B_i) + \mu(B(0,1))^p \mu(B_i).\end{aligned}$$

Summing over $i \in I_0$ yields

$$\mathcal{E}_{\nu,p}(\mathbb{R}^d, V, \tfrac{1}{2}\alpha) \lesssim_{\lambda,p} (M_0 + \mu(B(0,1))^p) \mu(B(0,1)).$$

□

Lemma 7.5. *For $\lambda = \lambda(\alpha) < \frac{1}{2}$ small enough, we have*

$$\int M_n \nu(x)^p \, d\nu(x) \lesssim_{\alpha,\lambda,p} \Big((C_*)^p + M_0 + \mu(B(0,1))^p \Big) \mu(B(0,1)).$$

The constants on the right hand side do not depend on ρ, ε.

Proof. By (7.7) and Lemma 7.4, we may use Lemma 2.1 to get a family of cubes Top_ν satisfying properties (i)-(iii) of Lemma 2.1, and such that

$$\sum_{R\in\mathsf{Top}_\nu} \Theta_\nu(2B_R)^p \nu(R) \lesssim_{\alpha,\lambda} (C_*)^p \nu(\mathbb{R}^d) + C(p)(M_0 + \mu(B(0,1))^p)\mu(B(0,1))$$
$$\lesssim_{\alpha,\lambda,p} \Big((C_*)^p + M_0 + \mu(B(0,1))^p\Big)\mu(B(0,1)). \quad (7.8)$$

Now, the property (iii) of Lemma 2.1 lets us estimate $M_n\nu(x)$. Indeed, suppose $x \in \operatorname{supp}\nu$, and let $r_1 > 0$ be such that

$$M_n\nu(x) \le 2\frac{\nu(B(x,r_1))}{r_1^n}.$$

Since $\operatorname{supp}\nu \subset B(0,2)$, we have $r_1 \le 4$. Let $Q \in \mathcal{D}_\nu$ be the smallest cube satisfying $x \in Q$ and $B(x,r_1) \cap \operatorname{supp}\nu \subset 2B_Q$ (such a cube exists because the largest cube $Q_0 := \operatorname{supp}\nu$ clearly satsfies $\operatorname{supp}\nu \subset 2B_{Q_0}$). Let $R \in \mathsf{Top}_\nu$ be the top cube such that $Q \in \mathsf{Tr}(R)$. Clearly, $\ell(Q) \approx r_1$. By Lemma 2.1 (iii), we have

$$\frac{\nu(B(x,r_1))}{r_1^n} \lesssim \Theta_\nu(2B_Q) \lesssim \Theta_\nu(2B_R).$$

Thus, $M_n\nu(x)^p \lesssim \sum_{R\in\mathsf{Top}_\nu} \mathbb{1}_R(x)\Theta_\nu(2B_R)^p$. Integrating with respect to ν and applying (7.8) yields the desired estimate. □

Lemma 7.6. *We have*

$$\int M_n\mu(x)^p \, d\mu(x) \lesssim_{\alpha,\lambda,p} \Big((C_*)^p + M_0 + \mu(B(0,1))^p\Big)\mu(B(0,1)).$$

In particular, $M_n\mu(x) < \infty$ *for* μ*-a.e.* $x \in B(0,1)$.

Proof. Denote

$$M_{n,\rho}\mu(x) = \sup_{r\ge\rho} \frac{\mu(B(x,r))}{r^n}.$$

Recall that $I_0 = I_0(\rho,\varepsilon)$ and set

$$E_{\varepsilon,\rho} = \operatorname{supp}\mu \cap \bigcup_{i\in I_0} B_i.$$

We claim that

$$\int_{E_{\varepsilon,\rho}} M_{n,\rho}(\mathbb{1}_{E_{\varepsilon,\rho}}\mu)(x)^p \, d\mu(x) \lesssim \int M_{n,\rho}\nu(x)^p \, d\nu(x). \quad (7.9)$$

Indeed, let $x, x' \in B_j$, $j \in I_0$, and $r \ge \rho$. Then, using repeatedly the fact that $r(B_i) \le \rho \le r$ for $i \in I_0$,

$$\mu(B(x,r) \cap E_{\varepsilon,\rho}) \le \mu(B(x',3r) \cap E_{\varepsilon,\rho}) \le \sum_{i\in I_0 : B_i \cap B(x',3r) \neq \varnothing} \mu(B_i)$$
$$= \sum_{i\in I_0 : B_i \cap B(x',3r) \neq \varnothing} \nu(D_i) \le \nu(B(x',5r)).$$

Hence, for all $x \in B_j,\ j \in I_0$,

$$M_{n,\rho}(\mathbb{1}_{E_{\varepsilon,\rho}}\mu)(x) \leq 5^n \inf_{x' \in B_j} M_{n,\rho}\nu(x').$$

Integrating both sides of the inequality with respect to μ in $E_{\varepsilon,\rho}$ yields (7.9).

Lemma 7.5 and (7.9) give

$$\int_{E_{\varepsilon,\rho}} M_{n,\rho}(\mathbb{1}_{E_{\varepsilon,\rho}}\mu)(x)^p \, d\mu(x) \\ \leq C(\alpha,\lambda,p)\Big((C_*)^p + M_0 + \mu(B(0,1))^p\Big)\mu(B(0,1)) =: K,$$

where K is independent of ρ and ε.

Set $\varepsilon_k = 2^{-k}$. Observe that, for a fixed $\rho > 0$, we have $\mu(\mathbb{R}^d \setminus \liminf_k E_{\varepsilon_k,\rho}) = 0$, where

$$\liminf_k E_{\varepsilon_k,\rho} = \bigcup_{j=1}^{\infty} G_j \quad \text{and} \quad G_j = \bigcap_{k=j}^{\infty} E_{\varepsilon_k,\rho}.$$

The inclusion $G_j \subset E_{\varepsilon_j,\rho}$ gives

$$\int_{G_j} M_{n,\rho}(\mathbb{1}_{G_j}\mu)(x)^p \, d\mu(x) \leq \int_{E_{\varepsilon_j,\rho}} M_{n,\rho}(\mathbb{1}_{E_{\varepsilon_j,\rho}}\mu)(x)^p \, d\mu(x) \leq K.$$

Since the sequence of sets G_j is increasing, we easily get that for μ-a.e. $x \in B(0,1)$

$$\mathbb{1}_{G_j}(x)\, M_{n,\rho}(\mathbb{1}_{G_j}\mu)(x) \xrightarrow{j\to\infty} M_{n,\rho}\mu(x),$$

and the convergence is monotone. Hence, by monotone convergence theorem,

$$\int M_{n,\rho}\mu(x)^p \, d\mu(x) \leq K.$$

The estimate is uniform in ρ, and so once again monotone convergence gives

$$\int M_n\mu(x)^p \, d\mu(x) \leq K.$$

□

Taking into account Lemma 7.6 and Section 7.1, the proof of Lemma 7.2 is finished.

8 Necessary condition for rectifiability

In this section we will prove the following.

Proposition 8.1. *Suppose μ is an n-rectifiable measure on $\mathbb{R}^d$ and $1 \leq p < \infty$. Then, for μ-a.e. $x \in \mathbb{R}^d$ there exists $V_x \in G(d, d-n)$ such that for any $\alpha \in (0,1)$ we have*

$$\int_0^1 \left(\frac{\mu(K(x,V_x,\alpha,r))}{r^n}\right)^p \frac{dr}{r} < \infty.$$

First, we recall the definition of β_2 numbers, as defined by David and Semmes [DS91].

Definition 8.2. Given a Radon measure μ, $x \in \operatorname{supp} \mu$, $r > 0$, and an n-plane L, define

$$\beta_{\mu,2}(x,r) = \inf_L \left(\frac{1}{r^n} \int_{B(x,r)} \left(\frac{\operatorname{dist}(y,L)}{r} \right)^2 d\mu(y) \right)^{1/2},$$

where the infimum is taken over all n-planes intersecting $B(x,r)$.

Tolsa showed the following necessary condition for rectifiability in terms of β_2 numbers.

Theorem 8.3 ([Tol15]). *Suppose μ is an n-rectifiable measure on $\mathbb{R}^d$. Then, for μ-a.e. $x \in \mathbb{R}^d$ we have*

$$\int_0^1 \beta_{\mu,2}(x,r)^2 \, \frac{dr}{r} < \infty. \tag{8.1}$$

Remark 8.4. When showing that rectifiable sets have approximate tangents almost everywhere one uses the so-called *linear approximation* properties, see [Mat95, Theorems 15.11 and 15.19]. The theorem of Tolsa improves on the linear approximation property, and that allows us to improve on the classical approximate tangent plane result.

Before proving Proposition 8.1 we need one more lemma. Recall that if $\alpha > 0$, W is an n-plane, and $0 < r < R$, then $K(x, W^\perp, \alpha, r, R) = K(x, W^\perp, \alpha, R) \setminus K(x, W^\perp, \alpha, r)$.

Lemma 8.5. *Let $\alpha, \varepsilon \in (0,1)$ be some constants satisfying $\eta := 1 - \alpha - 3\varepsilon > 0$. Let $x \in \mathbb{R}^d$, $r > 0$, and suppose that W and L are n-planes satisfying $x \in W$ and*

$$\operatorname{dist}_H(L \cap B(x,r), W \cap B(x,r)) \le \varepsilon r. \tag{8.2}$$

Then

$$K(x, W^\perp, \alpha, r, 2r) \subset B(x, 2r) \setminus B_{\eta r}(L).$$

Proof. Suppose $y \in K(x, W^\perp, \alpha, r, 2r)$, so that $r < |x - y| < 2r$ and $|x - \pi_W(y)| < \alpha |x - y|$. We need to show that $\operatorname{dist}(y, L) > \eta r$.

Set $y' = \pi_L(y)$, $x' = \pi_L(x)$. Then

$$\begin{aligned}\operatorname{dist}(y,L) = |y - y'| &\ge |x - y| - |x' - y'| - |x - x'| \\ &= |x - y| - |x' - y'| - \operatorname{dist}(x, L) \overset{(8.2)}{\ge} |x - y| - |x' - y'| - \varepsilon r.\end{aligned}$$

Let $\tilde{\pi}_W$ and $\tilde{\pi}_L$ denote the orthogonal projections onto the n-planes parallel to W and L passing through the origin. It follows from (8.2) that $\|\tilde{\pi}_W - \tilde{\pi}_L\|_{op} \le \varepsilon$. Thus,

$$|x' - y'| = |\tilde{\pi}_L(x-y)| \le |\tilde{\pi}_W(x-y)| + \|\tilde{\pi}_W - \tilde{\pi}_L\|_{op}\,|x-y| \le |\tilde{\pi}_W(x-y)| + 2\varepsilon r.$$

Hence, using the fact that $|\tilde{\pi}_W(x-y)| = |x - \pi_W(y)| < \alpha|x-y|$, we get from the two estimates above

$$\mathrm{dist}(y,L) \ge |x-y| - |\tilde{\pi}_W(x-y)| - 3\varepsilon r \ge (1-\alpha)|x-y| - 3\varepsilon r \ge (1-\alpha-3\varepsilon)r = \eta r.$$

□

Proof of Proposition 8.1. Let μ be n-rectifiable. For $r>0$ and $x \in \operatorname{supp}\mu$ let $L_{x,r}$ be the n-plane minimizing $\beta_{\mu,2}(x,r)$. We know that for μ-a.e. $x \in \operatorname{supp}\mu$ we have (8.1) and (II.3.18) (in particular, the approximate tangent plane W_x exists). Fix such x. Set $V_x = W_x^{\perp}$, let $\alpha \in (0,1)$ be arbitrary, and for $0<r<R$ set $K(r) = K(x, V_x, \alpha, r)$, $K(r,R) = K(x,V_x,\alpha,r,R)$. We will show that

$$\int_0^1 \left(\frac{\mu(K(r))}{r^n}\right)^p \frac{dr}{r} < \infty. \tag{8.3}$$

Let $\varepsilon > 0$ be a constant so small that $\eta := 1-\alpha-3\varepsilon > 0$. Use Lemma II.3.9 to find $r_0 > 0$ such that for $0 < r \le r_0$ we have

$$\mathrm{dist}_H(L_{x,r} \cap B(x,r), W_x \cap B(x,r)) \le \varepsilon r.$$

Then, it follows from Lemma 8.5 that for all $0 < r \le r_0$

$$K(r/2, r) \subset B(x,r) \setminus B_{\eta r}(L_{x,r}).$$

Note that by Chebyshev's inequality

$$\mu(B(x,r)\setminus B_{\eta r}(L_{x,r})) \le \eta^{-2}\int_{B(x,r)} \left(\frac{\mathrm{dist}(y, L_{x,r})}{r}\right)^2 d\mu(y) = \eta^{-2} r^n \beta_{\mu,2}(x,r)^2.$$

Hence, for $0 < r \le r_0$ we have

$$\frac{\mu(K(r/2,r))}{r^n} \lesssim_\eta \beta_{\mu,2}(x,r)^2,$$

and so

$$\int_0^{r_0} \frac{\mu(K(r/2,r))}{r^n}\frac{dr}{r} \lesssim_\eta \int_0^{r_0} \beta_{\mu,2}(x,r)^2 \frac{dr}{r} \overset{(8.1)}{<} \infty. \tag{8.4}$$

Now, observe that for any integer $N > 0$

$$
\begin{aligned}
\int_{2^{-N}r_0}^{r_0/2} \frac{\mu(K(r))}{r^n}\,\frac{dr}{r} &\lesssim (r_0)^{-n}\sum_{k=1}^{N}\mu(K(2^{-k}r_0))2^{kn}\\
&\leq 2^n(r_0)^{-n}\sum_{k=1}^{N}\mu(K(2^{-k}r_0))2^{kn} - (r_0)^{-n}\sum_{k=1}^{N}\mu(K(2^{-k}r_0))2^{kn}\\
&= (r_0)^{-n}\sum_{k=1}^{N}\mu(K(2^{-k}r_0))2^{(k+1)n} - (r_0)^{-n}\sum_{k=1}^{N}\mu(K(2^{-k}r_0))2^{kn}\\
&\leq (r_0)^{-n}\sum_{k=2}^{N+1}\Big(\mu(K(2^{-k+1}r_0)) - \mu(K(2^{-k}r_0))\Big)2^{kn} + \frac{\mu(K(2^{-(N+1)}r_0))}{(2^{-(N+1)}r_0)^n}\\
&\lesssim \int_0^{r_0}\frac{\mu(K(r/2,r))}{r^n}\,\frac{dr}{r} + \Theta_\mu(x, 2^{-(N+1)}r_0).
\end{aligned}
$$

Letting $N \to \infty$, we get from the above and (8.4) that

$$
\int_0^{r_0}\frac{\mu(K(r))}{r^n}\,\frac{dr}{r} \lesssim_\eta \int_0^{r_0}\beta_{\mu,2}(x,r)^2\,\frac{dr}{r} + \Theta^{n,*}(\mu,x) < \infty,
$$

for μ-a.e. $x \in \operatorname{supp}\mu$, where we also used the fact that $\Theta^{n,*}(\mu,x) < \infty$ μ-almost everywhere (because μ is n-rectifiable). The integral $\int_{r_0}^1 \frac{\mu(K(r))}{r^n}\,\frac{dr}{r}$ is obviously finite, and so we get that

$$
\int_0^1 \frac{\mu(K(r))}{r^n}\,\frac{dr}{r} < \infty,
$$

which is precisely (8.3) with $p = 1$. To get the same with $p > 1$, note that since $\Theta^{n,*}(\mu,x) < \infty$ for μ-a.e. x, we have

$$
\begin{aligned}
\int_0^1 \left(\frac{\mu(K(r))}{r^n}\right)^p \frac{dr}{r} &\leq \int_0^1 \frac{\mu(K(r))}{r^n}\Theta_\mu(x,r)^{p-1}\,\frac{dr}{r}\\
&\leq \sup_{0<r<1}\Theta_\mu(x,r)^{p-1}\int_0^1\frac{\mu(K(r))}{r^n}\,\frac{dr}{r} < \infty.
\end{aligned}
$$

□

9 Sufficient condition for BPLG

In this section we prove the "sufficient part" of Theorem 1.9. After a suitable translation and rescaling, it suffices to show the following:

Proposition 9.1. *Suppose $p \geq 1$, $E \subset \mathbb{R}^d$ is n-AD-regular, and $0 \in E$. Let $\alpha > 0$, $M_0 > 1$, $\kappa > 0$, and assume that there exist $F \subset E \cap B(0,1)$ and $V \in G(d, d-n)$, such that $\mathcal{H}^n(F) \geq \kappa$, and for all $x \in F$*

$$
\int_0^1 \left(\frac{\mathcal{H}^n(K(x,V,\alpha,r)\cap F)}{r^n}\right)^p \frac{dr}{r} \leq M_0. \tag{9.1}
$$

Then there exists a Lipschitz graph Γ, with Lipschitz constant depending on α, n, d, such that

$$\mathcal{H}^n(F \cap \Gamma) \gtrsim \kappa, \tag{9.2}$$

with the implicit constant depending on p, M_0, α, n, d, and the AD-regularity constants of E.

To prove the above we will use techniques developed in [MO18b]. Fix $V \in G(d, d-n)$. Let $\theta > 0$ and $M \in \{0, 1, 2 \dots\}$. In the language of Martikainen and Orponen, a set $E \subset \mathbb{R}^d$ has the *n-dimensional (θ, M)-property* if for all $x \in E$

$$\#\{j \in \mathbb{Z} \ : \ K(x, V, \theta, 2^{-j}, 2^{-j+1}) \cap E \neq \varnothing\} \leq M.$$

It is easy to see that if E has the n-dimensional $(\theta, 0)$-property, then E is contained in a Lipschitz graph with Lipschitz constant bounded by $1/\theta$, see [MO18b, Remark 1.11].

The main proposition of [MO18b] reads as follows.

Proposition 9.2 ([MO18b, Proposition 1.12])**.** *Assume that E is n-AD-regular, and assume that $F_1 \subset E \cap B(0,1)$ is an $\mathcal{H}^n$-measurable subset with $\mathcal{H}^n(F_1) \approx_C 1$. Suppose further that F_1 satisfies the n-dimensional (θ, M)-property for some $\theta > 0$, $M \geq 0$. Then there exists and $\mathcal{H}^n$-measurable subset $F_2 \subset F_1$ with $\mathcal{H}^n(F_2) \approx_{C,\theta,M} 1$ which satisfies the $(\theta/b, 0)$-property. Here $b \geq 1$ is a constant depending only on d.*

Remark 9.3. It follows immediately from the proposition above that if we construct $F_1 \subset E \cap B(0,1)$ with $\mathcal{H}^n(F_1) \approx \kappa$ satisfying the n-dimensional $(\alpha/2, M)$-property, then we will get a Lipschitz graph Γ such that (9.2) holds. Hence, we will be done with the proof of Proposition 9.1.

To construct F_1 we will use another lemma from [MO18b].

Lemma 9.4 ([MO18b, Lemma 2.1])**.** *Let E be an n-AD-regular set with $\mathcal{H}^n(E) \geq C > 0$, let $F \subset E \cap B(0.1)$ be an $\mathcal{H}^n$-measurable subset, and let*

$$F_\varepsilon = \{x \in F \ : \ \mathcal{H}^n(F \cap B(x, r_x)) \leq \varepsilon r_x^n \text{ for some radius } 0 < r_x \leq 1\}.$$

Then $\mathcal{H}^n(F_\varepsilon) \lesssim \varepsilon$ with the bound depending only on C and the AD-regularity constant of E.

Note that the set $F \setminus F_\varepsilon$ does not have to be AD-regular. Nevertheless, we gain some extra regularity that will prove useful.

Now, let E and $F \subset E \cap B(0,1)$ be as in the assumptions of Proposition 9.1. We apply Lemma 9.4 to conclude that for some ε, depending on κ and the AD-regularity constant of E, we have

$$\mathcal{H}^n(F \setminus F_\varepsilon) \geq \frac{\kappa}{2}.$$

Set $F_1 = F \setminus F_\varepsilon$.

Lemma 9.5. *There exists* $M = M(M_0, \varepsilon, \alpha, n)$ *such that* F_1 *satisfies the* n*-dimensional* $(\alpha/2, M)$*-property.*

Proof. Denote by $F_{\mathsf{Bad}} \subset F_1$ the set of $x \in F_1$ such that

$$\#\{j \in \mathbb{Z} \ : \ K(x, V, \alpha/2, 2^{-j}, 2^{-j+1}) \cap F_1 \neq \varnothing\} > M. \tag{9.3}$$

We will show that, if M is chosen big enough, the set F_{Bad} is empty.

Let $x \in F_{\mathsf{Bad}}$ and $j \in \mathbb{Z}$ be such that there exists $x_j \in K(x, V, \alpha/2, 2^{-j}, 2^{-j+1}) \cap F_1$. It is easy to see that for some $\lambda = \lambda(\alpha)$, independent of j, we have

$$B(x_j, \lambda 2^{-j}) \subset K(x, V, \alpha, 2^{-j-1}, 2^{-j+2}).$$

Since $x_j \in F_1 = F \setminus F_\varepsilon$, it follows that

$$\mathcal{H}^n(F \cap B(x_j, \lambda 2^{-j})) > \varepsilon(\lambda 2^{-j})^n.$$

The two observations above give

$$\frac{\mathcal{H}^n(F \cap K(x, V, \alpha, 2^{-j+2}))}{(2^{-j+2})^n} \geq \frac{\mathcal{H}^n(F \cap K(x, V, \alpha, 2^{-j-1}, 2^{-j+2}))}{(2^{-j+2})^n} \gtrsim_{\alpha,\lambda} \varepsilon.$$

By (9.3), there are more than M different scales (i.e. j's) for which the above holds. Thus, for $x \in F_{\mathsf{Bad}}$ we have

$$\int_0^1 \left(\frac{\mathcal{H}^n(K(x, V, \alpha, r) \cap F)}{r^n} \right)^p \frac{dr}{r} \gtrsim_{\alpha,\lambda} M\varepsilon^p.$$

Taking $M = M(M_0, \varepsilon, \alpha, n, p)$ big enough we get a contradiction with (9.1). Thus, F_{Bad} is empty. Now, it follows trivially by the definition of F_{Bad} that F_1 satisfies the n-dimensional $(\alpha/2, M)$-property. □

By Remark 9.3, this finishes the proof of Proposition 9.1.

10 Necessary condition for BPLG

In this section we prove the "necessary part" of Theorem 1.9. After rescaling, translating, and using the BPLG property, it is clear that it suffices to show the following:

Proposition 10.1. *Suppose* $E \subset \mathbb{R}^d$ *is* n*-AD-regular, and* $0 \in E$*. Let* $p \geq 1$*. Assume there exists a Lipschitz graph* Γ *such that* $\mathcal{H}^n(\Gamma \cap E \cap B(0,1)) \geq \kappa$*. Then there exists* $\alpha = \alpha(\mathrm{Lip}(\Gamma)) > 0$*,* $V \in G(d, d-n)$*, and a set* $F \subset \Gamma \cap E \cap B(0,1)$*, such that* $\mathcal{H}^n(F) \gtrsim \kappa$*, and for* $x \in F$

$$\int_0^1 \left(\frac{\mathcal{H}^n(K(x, V, \alpha, r) \cap E)}{r^n} \right)^p \frac{dr}{r} \leq M_0, \tag{10.1}$$

where $M_0 > 1$ *is a constant depending on* p, $\mathrm{Lip}(\Gamma)$, κ *and the AD-regularity constant of* E*.*

We begin by fixing some additional notation. Set $\mu = \mathcal{H}^n|_E$. We will denote the AD-regularity constant of E by C_0, so that for every $x \in E,\ 0 < r < \operatorname{diam}(E)$,

$$C_0^{-1} r^n \le \mu(B(x,r)) \le C_0 r^n.$$

Remark 10.2. Since we assume that E is AD-regular, the exponent p in (10.1) does not really matter. For any $p > 1$ we have

$$\left(\frac{\mathcal{H}^n(K(x,V,\alpha,r)\cap E)}{r^n}\right)^p \le C_0^{p-1}\frac{\mathcal{H}^n(K(x,V,\alpha,r)\cap E)}{r^n},$$

and so it is enough to prove (10.1) for $p = 1$.

Set $L = \operatorname{Lip}(\Gamma)$. Let $V \in G(d, d-n)$ be such that Γ is an L-Lipschitz graph over $V^\perp$, and let $\theta = \theta(L) > 0$ be such that

$$K(x,V,\theta)\cap\Gamma = \varnothing \qquad \text{for all } x \in \Gamma.$$

Set $\alpha = \min(\frac{\theta}{2}, 0.1, \frac{1}{4L})$.

For every $x \in E \cap B(0,1)\setminus\Gamma$ consider the ball $B_x = B(x, 0.01\operatorname{dist}(x,\Gamma))$. We use the $5r$-covering lemma to choose a countable subfamily of pairwise disjoint balls $B_j = B(x_j, r_j),\ r_j = 0.01\operatorname{dist}(x_j,\Gamma),\ j \in \mathbb{Z}$, such that

$$E\cap B(0,1)\setminus\Gamma \subset \bigcup_{j\in\mathbb{Z}} 5B_j.$$

Observe that

$$\sum_{j\in\mathbb{Z}} r_j^n \le C_0 \sum_{j\in\mathbb{Z}} \mu(B_j) = C_0\mu\Big(\bigcup_{j\in\mathbb{Z}} B_j\Big) \le C_0\mu(B(0,2)) \lesssim C_0^2. \tag{10.2}$$

For each $j \in \mathbb{Z}$ set

$$K_j = \bigcup_{y\in 5B_j} K(y,V,\alpha), \qquad K_j(r) = \bigcup_{y\in 5B_j} K(y,V,\alpha,r).$$

Lemma 10.3. *For each $j \in \mathbb{Z}$ we have*

$$\mathcal{H}^n(K_j\cap\Gamma) \lesssim_L r_j^n. \tag{10.3}$$

Moreover,

$$K_j(r)\cap\Gamma = \varnothing \qquad \textit{for } r < r_j. \tag{10.4}$$

Proof. (10.4) is very easy – observe that for $r < r_j$ we have $K_j(r) \subset 6B_j$, and so for $y \in K_j(r)$

$$\operatorname{dist}(y,\Gamma) \ge \operatorname{dist}(x_j,\Gamma) - 6r_j = (1-0.06)\operatorname{dist}(x_j,\Gamma) > 0.$$

Concerning (10.3), we claim that since $\Gamma = \operatorname{graph}(F)$ for some L-Lipschitz function $F : V^\perp \to V$, and since α is sufficiently small, for all $x \in \mathbb{R}^d$ we have

$$K(x, V, \alpha) \cap \Gamma \subset B(x, C \operatorname{dist}(x, \Gamma)), \tag{10.5}$$

where $C = C(L) > 1$. Indeed, if $\operatorname{dist}(x, \Gamma) = 0$, then $K(x, V, \alpha) \cap \Gamma = \varnothing$ and there is nothing to prove. Suppose $\operatorname{dist}(x, \Gamma) > 0$, $y \in K(x, V, \alpha) \cap \Gamma$, and let $z \in \Gamma$ be the image of x under the projection onto Γ orthogonal to $V^\perp$, i.e. $z = \pi_V^\perp(x) + F(\pi_V^\perp(x))$.

Observe that, since Γ is a Lipschitz graph,

$$|x - z| \lesssim_L \operatorname{dist}(x, \Gamma),$$

and also $\pi_V^\perp(x) = \pi_V^\perp(z)$. By the definition of a cone, $y \in K(x, V, \alpha)$ gives

$$|\pi_V^\perp(z - y)| = |\pi_V^\perp(x - y)| < \alpha |x - y|.$$

On the other hand, $y \in \Gamma$ and the above imply

$$|\pi_V(z - y)| \le L |\pi_V^\perp(z - y)| < L\alpha |x - y|.$$

The three estimates above yield

$$\begin{aligned} |x - y| \le |x - z| + |z - y| &\le C(L) \operatorname{dist}(x, \Gamma) + |\pi_V^\perp(z - y)| + |\pi_V(z - y)| \\ &\le C(L) \operatorname{dist}(x, \Gamma) + \alpha|x - y| + L\alpha|x - y| \le C(L) \operatorname{dist}(x, \Gamma) + \frac{1}{2}|x - y|. \end{aligned}$$

Hence, $|x - y| \lesssim_L \operatorname{dist}(x, \Gamma)$ and (10.5) follows.

Now, going back to (10.3), note that for $y \in 5B_j$ we have $\operatorname{dist}(y, \Gamma) \approx r_j$, so that $K(y, V, \alpha) \cap \Gamma \subset B(y, Cr_j)$ for some $C = C(L)$. Moreover, $B(y, Cr_j) \subset B(x_j, 10Cr_j)$. Therefore, $K_j \cap \Gamma \subset B(x_j, 10Cr_j) \cap \Gamma$, and (10.3) easily follows. □

Proof of Proposition 10.1. Let $x \in \Gamma \cap B(0, 1)$ and $0 < r < 1$. Since $\{5B_j\}_{j\in\mathbb{Z}}$ cover $E \cap B(0, 1) \setminus \Gamma$, and $K(x, V, \alpha, r) \cap \Gamma = \varnothing$, we have

$$\mu(K(x, V, \alpha, r)) \le \sum_{j\in\mathbb{Z}\,:\, 5B_j \cap K(x,V,\alpha,r) \neq \varnothing} \mu(5B_j) \lesssim C_0 \sum_{j\in\mathbb{Z}\,:\, 5B_j \cap K(x,V,\alpha,r) \neq \varnothing} r_j^n.$$

Notice that $5B_j \cap K(x, V, \alpha, r) \neq \varnothing$ if and only if $x \in K_j(r)$. Hence, using the above and Lemma 10.3 yields

$$\begin{aligned} &\int_{\Gamma\cap B(0,1)} \int_0^1 \frac{\mu(K(x, V, \alpha, r))}{r^n} \frac{dr}{r} d\mathcal{H}^n(x) \\ &\qquad \lesssim_{C_0} \int_{\Gamma\cap B(0,1)} \int_0^1 \frac{1}{r^n} \sum_{j\in\mathbb{Z}} r_j^n \mathbb{1}_{K_j(r)}(x) \frac{dr}{r} d\mathcal{H}^n(x) \\ &= \sum_{j\in\mathbb{Z}} r_j^n \int_{\Gamma\cap B(0,1)} \int_0^1 \frac{1}{r^n} \mathbb{1}_{K_j(r)}(x) \frac{dr}{r} d\mathcal{H}^n(x) \overset{(10.4)}{\le} \sum_{j\in\mathbb{Z}} r_j^n \int_{K_j\cap\Gamma} \int_{r_j} \frac{1}{r^n} \frac{dr}{r} d\mathcal{H}^n(x) \\ &\qquad \lesssim \sum_{j\in\mathbb{Z}} r_j^n \int_{K_j\cap\Gamma} r_j^{-n} d\mathcal{H}^n(x) \overset{(10.3)}{\lesssim_L} \sum_{j\in\mathbb{Z}} r_j^n \overset{(10.2)}{\lesssim_{C_0}} 1. \end{aligned}$$

We know that $\mathcal{H}^n(\Gamma\cap B(0,1)\cap E) \geq \kappa$, and so we can use Chebyshev's inequality to conclude that there exist $M_0 = M_0(L, C_0, \kappa) > 1$ and $F \subset \Gamma \cap B(0,1) \cap E$ with $\mathcal{H}^n(F) \geq \frac{\kappa}{2}$ such that for all $x \in F$

$$\int_0^1 \frac{\mu(K(x, V, \alpha, r))}{r^n} \frac{dr}{r} \leq M_0.$$

□

11 Set with BPLG but no BME

We will show the following.

Proposition 11.1. *Fix an aperture parameter $\alpha \in (0,1)$. There exists a sequence of 1-ADR sets $E_N = E_N(\alpha) \subset B(0,1) \subset \mathbb{R}^2$, $N \geq 100(1+\log_2(\alpha^{-1}))$, with the following properties:*

(i) they all contain BPLG in a uniform way, that is, they are 1-ADR with the same constans C_0, and they all satisfy the BPLG condition (see Definition 1.7) with $L = 1$ and some uniform $\kappa > 0$.

(ii) regardless of the choice of directions $V_x \in G(2,1)$, they all have big conical energies:

$$\int_{E_N} \mathcal{E}_{E_N,1}(x, V_x, \alpha, 1)\, d\mathcal{H}^1(x) = \int_{E_N}\int_0^1 \frac{\mathcal{H}^1(K(x, V_x, \alpha, r)\cap E_N)}{r} \frac{dr}{r} d\mathcal{H}^1(x) \gtrsim_\alpha N. \quad (11.1)$$

Let $\alpha_k \to 0$. Now, a disjoint union of appropriately rescaled sets $E_N(\alpha_k)$ would contain BPLG and would not satisfy the BME condition (Definition 1.12) for any M_0 and $\alpha > 0$. We omit the details.

Remark 11.2. In this section, the notation $\measuredangle(L_1, L_2)$ will denote the "true" angle between two lines, and not its sine, as it was used in other chapters.

Let $M = 100\lceil\alpha^{-1}\rceil$, so that $M \approx \alpha^{-1}$. In the lemma below we construct a Lipschitz graph $\Gamma = \Gamma(N, M)$ that can be seen as the first approximation of the set E_N. There exists a fixed direction V_0, such that for all directions V close to V_0 ($\measuredangle(V, V_0) \leq \pi/8$), the conical energy $\mathcal{E}_{\Gamma,1}(x, V, \alpha, 1)$ is bigger than N for all x belonging to a neighbourhood of a large portion of Γ. Rescaled and rotated copies of Γ will be then used as building blocks in the construction of E_N.

Let Δ be the usual dyadic grid of open intervals on $(-1, 1)$, and let Δ_k denote the dyadic intervals of length 2^{-k}.

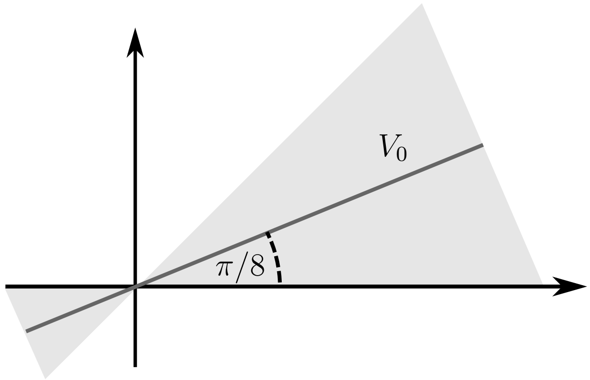


FIGURE V.1: V_0 is the line forming angle $\pi/8$ with the x axis. The lines V for which (11.2) holds are lying in the grey region.

Lemma 11.3. *Let $N \geq 100(1+\log_2(\alpha^{-1}))$ be an integer. There exists a piecewise linear 1-Lipschitz function $g : [-1,1] \to [-M^{-1}, M^{-1}]$, and a collection of disjoint dyadic intervals $\mathcal{I} \subset \Delta$ with the following properties:*

(P1) $g(-1) = g(1) = 0$.

(P2) For every $I \in \mathcal{I}$ we have $I \subset [-1/2, 1/2]$, the function $g|_I$ is increasing, and for $t \in I$ we have $g'(t) = 1$.

(P3) $\#\mathcal{I} = 2^{-M}\, 2^{N(M+1)}$ and $\mathcal{I} \subset \Delta_{N(M+1)}$. Hence,

$$\mathcal{H}^1\left(\bigcup_{I \in \mathcal{I}} I\right) = 2^{-M} \approx_\alpha 1.$$

(P4) Let $\Gamma = \mathrm{graph}(g)$, $G : [-1,1] \to \Gamma$ be the graph map $G(t) = (t, g(t))$, and let $V_0 = \{(x,y) : y = \tan(\pi/8)x\} \in G(2,1)$. For any $I \in \mathcal{I}$, any $x \in \mathbb{R}^2$ with $\mathrm{dist}(x, G(I)) < 2^{-N(M+1)}$, and all $V \in G(2,1)$ satisfying $\measuredangle(V, V_0) \leq \pi/8$, we have

$$\int_0^1 \frac{\mathcal{H}^1(K(x,V,\alpha,r)\cap\Gamma)}{r}\,\frac{dr}{r} \gtrsim N. \tag{11.2}$$

See Figure V.1.

For an idea of what Γ looks like, see the graph at the bottom of Figure V.3. Before we prove Lemma 11.3, let us show how it can be used to prove Proposition 11.1.

11.1 Construction of E_N

Let $\Gamma = \Gamma(M, N)$ be the 1-Lipschitz graph from Lemma 11.3. The set E_N will consist of one "big" Lipschitz graph $\Gamma_0 = \Gamma$, and three layers of much smaller Lipschitz graphs stacked on top of the big one. The small graphs will be rescaled and rotated versions of Γ. Another way to see E_N is as a union of four bilipschitz curves $\Gamma_0, \dots, \Gamma_3$, and this is how we are going to define it. Roughly speaking, if Γ_i is already defined, Γ_{i+1} will be constructed by replacing some of the segments comprising Γ_i with rescaled and rotated copies of Γ.

First, let $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by $\pi/4$. Set $L_0 = \{(x, 0) \ : \ x \in \mathbb{R}\}$ and for $k \geq 1$ set $L_k = \rho^k(L_0) \in G(2,1)$ (here ρ^k denotes k compositions of ρ, and the same notation is used for δ defined below).

Define also $r_k = 2^{-kN(M+1)-k/2}$, and let $\delta : \mathbb{R}^2 \to \mathbb{R}^2$ be the dilation by factor r_1, i.e. $\delta(x) = r_1 x$. Note that $r_k = (r_1)^k$, so that δ^k is the dilation by factor r_k. The constant r_1 was chosen in such a way that for an interval $I \in \mathcal{I} \subset \Delta_{N(M+1)}$ we have $\mathcal{H}^1(G(I)) = 2r_1$ by (P2) (where G is the graph map of g).

We will abuse the notation and identify the segment $S_0 := [-1,1] \times \{0\}$ with $[-1,1] \subset \mathbb{R}$.

Set $\Gamma_0 = \Gamma$, and let $\gamma_0 = \sigma_0 : S_0 \to \Gamma_0$ be defined as the natural graph map $\gamma_0(t) = \sigma_0(t) = G(t) = (t, g(t))$.

Lemma 11.4. *Let $k \in \{1,2,3\}$. There exist maps $\gamma_k : S_0 \to \mathbb{R}^2$ such that:*

a) the sets $\Gamma_k := \gamma_k(S_0)$ are of the form

$$\Gamma_k = \left(\Gamma_{k-1} \setminus \bigcup_{I \in \mathcal{I}^k} S_{k,I}\right) \cup \bigcup_{I \in \mathcal{I}^k} \Gamma_{k,I},$$

where $I = (I_1, \dots, I_k) \in \mathcal{I}^k$ and $\mathcal{I}$ is the family of intervals from Lemma 11.3,

b) the sets $S_{k,I}$ are segments, with $S_{k,I} = G_{k,I}(S_0)$ and $G_{k,I} := \tau_I \circ \rho^k \circ \delta^k$ for some translation τ_I (in particular, $\mathcal{H}^1(S_{k,I}) = 2\, r_k$ and $S_{k,I}$ are parallel to L_k),

c) the $\Gamma_{k,I}$ are rescaled and rotated copies of Γ, with $\Gamma_{k,I} = G_{k,I}(\Gamma_0)$ (in particular, since the endpoints of Γ_0 and S_0 coincide, the same is true for $\Gamma_{k,I}$ and $S_{k,I}$),

d) for $k = 1$, $J \in \mathcal{I}$, we have $S_{1,J} = \sigma_0(J) \subset \Gamma_0$, and for $k > 1$, if $I = (I', J) \in \mathcal{I}^{k-1} \times \mathcal{I}$, then $S_{k,I} = G_{k-1,I'}(S_{1,J}) \subset \Gamma_{k-1,I'} \subset \Gamma_{k-1}$,

e) if $I = (I', J)$, a_1, a_2 are the endpoints of $S_{k,I}$, and b_1, b_2 are the endpoints of $\Gamma_{k-1,I'}$, then

$$|a_i - b_j| \gtrsim r_{k-1}$$

for $i, j \in \{1,2\}$ (i.e. $S_{k,I}$ is "deep inside" $\Gamma_{k-1,I'}$), *f) the maps γ_k are of the form $\gamma_k = \sigma_k \circ \cdots \circ \sigma_0$, where $\sigma_k : \Gamma_{k-1} \to \Gamma_k$ is defined as*

$$\sigma_k(x) = \begin{cases} x, & \text{for } x \in \Gamma_{k-1} \setminus \bigcup_{I \in \mathcal{I}^k} S_{k,I} \\ G_{k,I}(x) \circ \sigma_0 \circ G_{k,I}^{-1}(x), & \text{for } x \in S_{k,I},\ I \in \mathcal{I}^k. \end{cases}$$

In particular, $\sigma_k(S_{k,I}) = \Gamma_{k,I}$.

g) $\|\sigma_k - id\|_{L^\infty(\Gamma_{k-1})} \leq 2M^{-1} r_k$,

Proof of Lemma 11.4. We will define σ_k inductively.

First, for any $I \in \mathcal{I}$ set $S_{1,I} := \sigma_0(I) \subset \Gamma_0$. Observe that by (P2) $S_{1,I}$ is a segment parallel to L_1. Moreover, since $\mathcal{H}^1(I) = 2^{1/2} r_1$, we have $\mathcal{H}^1(S_{1,I}) = 2 r_1$. It follows that $S_{1,I} = \tau_I \circ \rho \circ \delta(S_0)$ for some translation τ_I. Define $G_{1,I} : \mathbb{R}^2 \to \mathbb{R}^2$ as $G_{1,I} = \tau_I \circ \rho \circ \delta$, and $\Gamma_{1,I} = G_{1,I}(\Gamma_0)$.

We define $\sigma_1 : \Gamma_0 \to \mathbb{R}^2$ as in f). In other words, $\sigma_1|_{S_{1,I}}$ can be seen as a graph map parametrizing the Lipschitz graph $\Gamma_{1,I}$. It is very easy to see that $S_{1,I}$, $\Gamma_{1,I}$, and σ_1 defined in this way satisfy all the conditions except for e) and g), which we will prove later on.

Now, suppose that σ_{k-1}, γ_{k-1}, etc. have already been defined, and that they satisfy a) – d), f).

For any $I = (I', J) \in \mathcal{I}^{k-1} \times \mathcal{I}$ set $S_{k,I} := G_{k-1,I'}(S_{1,J}) \subset \Gamma_{k-1,I'}$. Since $S_{1,J}$ is parallel to L_1 and $G_{k-1,I'} = \tau_{I'} \circ \rho^{k-1} \circ \delta^{k-1}$, $S_{k,I}$ is a segment parallel to L_k. Moreover, since $\mathcal{H}^1(S_{1,J}) = 2 r_1$, we have $\mathcal{H}^1(S_{k,I}) = 2 r_1 r_{k-1} = 2 r_k$. It follows that $S_{k,I} = \tau_I \circ \rho^k \circ \delta^k(S_0)$ for some translation τ_I.

We define $\sigma_k : \Gamma_0 \to \mathbb{R}^2$ as in f), so that $\sigma_k|_{S_{k,I}}$ can be seen as a graph map parametrizing the Lipschitz graph $\Gamma_{k,I}$. It is easy to see that σ_k, Γ_k, etc. defined this way satisfy a) – d), f).

Proof of e). Let $k = 1$. Recall that for all $I \in \mathcal{I}$ we have $I \subset [-1/2, 1/2]$ by (P2). Hence, $S_{1,I} = \sigma_0(I) \subset \sigma_0([-1/2, 1/2]) \subset \Gamma_0$. If $x \in \sigma_0([-1/2, 1/2])$ is arbitrary and if $y \in \Gamma_0$ is one of the endpoints of Γ_0, we have $|x - y| \gtrsim 1 = r_0$. So e) holds for $k = 1$. For $k \in \{2, 3\}$ the claim follows from the fact that if $I = (I', J) \in \mathcal{I}^{k-1} \times \mathcal{I}$, then $S_{k,I} = G_{k-1,I'}(S_{1,J})$ and $\Gamma_{k-1,I'} = G_{k-1,I'}(\Gamma_0)$.

Proof of g). We have $\sigma_k = id$ on $\Gamma_{k-1} \setminus \bigcup_{I \in \mathcal{I}^k} S_{k,I}$, and for $x \in S_{k,I}$

$$\begin{aligned} |\sigma_k(x) - x| &= \left| G_{k,I} \circ \sigma_0 \circ G_{k,I}^{-1}(x) - G_{k,I} \circ G_{k,I}^{-1}(x) \right| \\ &= r_k \left| \sigma_0 \circ G_{k,I}^{-1}(x) - G_{k,I}^{-1}(x) \right| \leq r_k \|g\|_\infty \leq 2M^{-1} r_k, \end{aligned}$$

where we used the fact that $\sigma_0(t) = (t, g(t))$, and that $\|g\|_\infty \leq 2M^{-1}$ by Lemma 11.3. □

Lemma 11.5. *The maps γ_k and σ_k from Lemma 11.4 are bilipschitz, with bilipschitz constants independent of N.*

Proof. It suffices to show that σ_k is bilipschitz with $\mathrm{Lip}(\sigma_k)$ and $\mathrm{Lip}(\sigma_k^{-1})$ independent of N, and then the same will be true for γ_k by Lemma 11.4 f).

Suppose that σ_j are already known to be bilipschitz for $0 \le j \le k-1$, with $\mathrm{Lip}(\sigma_j)$ and $\mathrm{Lip}(\sigma_j^{-1})$ independent of N (clearly, the condition holds for σ_0). Let $x, y \in \Gamma_{k-1}$. Our aim is to show that $|\sigma_k(x) - \sigma_k(y)| \approx |x-y|$.

Case 1. $|x-y| > 6M^{-1}r_k$. It follows from Lemma 11.4 g) that

$$|\sigma_k(x)-\sigma_k(y)| \le |x-y|+|\sigma_k(x)-x|+|\sigma_k(y)-y| \le |x-y|+4M^{-1}r_k \le 2|x-y|,$$

and

$$|\sigma_k(x)-\sigma_k(y)| \ge |x-y|-|\sigma_k(x)-x|-|\sigma_k(y)-y| \ge |x-y|-4M^{-1}r_k \ge \frac{1}{3}|x-y|.$$

Case 2. $x, y \in \Gamma_{k-1} \setminus \bigcup_{I\in\mathcal{I}^k} S_{k,I}$. This case is trivial, because $|\sigma_k(x) - \sigma_k(y)| = |x-y|$.

Case 3. $|x-y| \le 6M^{-1}r_k$, and $x, y \in \overline{S_{k,I}}$ for some $I \in \mathcal{I}^k$. Using the fact that σ_0 is bilipschitz we get

$$\begin{aligned}|\sigma_k(x) - \sigma_k(y)| &= \left|G_{k,I} \circ \sigma_0 \circ G_{k,I}^{-1}(x) - G_{k,I} \circ \sigma_0 \circ G_{k,I}^{-1}(y)\right| \\ &= r_k\left|\sigma_0 \circ G_{k,I}^{-1}(x) - \sigma_0 \circ G_{k,I}^{-1}(y)\right| \approx r_k\left|G_{k,I}^{-1}(x) - G_{k,I}^{-1}(y)\right| = |x-y|.\end{aligned}$$

Case 4. $|x-y| \le 6M^{-1}r_k$, $x \in S_{k,I}$ for some $I \in \mathcal{I}^k$, and $y \in \Gamma_{k-1} \setminus \overline{S_{k,I}}$.

We claim that
$$y \in \Gamma_{k-1,I'}, \tag{11.3}$$
where $I = (I', J) \in \mathcal{I}^{k-1} \times \mathcal{I}$ and $\Gamma_{k-1,I'}$ is the Lipschitz graph containing $S_{k,I}$. Indeed, by the induction assumption, the map $\gamma_{k-1}^{-1} : \Gamma_{k-1} \to S_0$ is bilipschitz with $\mathrm{Lip}(\gamma_{k-1})$, $\mathrm{Lip}(\gamma_{k-1}^{-1})$ independent of N. Since $\mathcal{H}^1(S_{k,I}) = 2\,r_k$ and $\mathcal{H}^1(\Gamma_{k-1,I'}) \approx r_{k-1}$, we get that $\mathcal{H}^1(\gamma_{k-1}^{-1}(S_{k,I})) \approx r_k$ and $\mathcal{H}^1(\gamma_{k-1}^{-1}(\Gamma_{k-1,I'})) \approx r_{k-1}$. Moreover, we have

$$\gamma_{k-1}^{-1}(S_{k,I}) \subset \gamma_{k-1}^{-1}(\Gamma_{k-1,I'}) \subset S_0, \tag{11.4}$$

where all three sets are segments. If a_1, a_2 and b_1, b_2 are the endpoints of $\gamma_{k-1}^{-1}(S_{k,I})$ and $\gamma_{k-1}^{-1}(\Gamma_{k-1,I'})$, respectively, then it follows from Lemma 11.4 e) and from the bilipschitz property of γ_{k-1} that for $i, j \in \{1, 2\}$ we have

$$|a_i - b_j| \gtrsim r_{k-1}. \tag{11.5}$$

Recall that $x \in S_{k,I}$ and $|x-y| \lesssim M^{-1}r_k$, so that $\mathrm{dist}(y, S_{k,I}) \lesssim M^{-1}r_k$. Hence,
$$\mathrm{dist}(\gamma_{k-1}^{-1}(y), \gamma_{k-1}^{-1}(S_{k,I})) \lesssim M^{-1}r_k.$$
Putting this together with (11.4) and (11.5), and assuming that $M \ge M_0$ for some absolute constant $M_0 > 10$, we get that $\gamma_{k-1}^{-1}(y) \in \gamma_{k-1}^{-1}(\Gamma_{k-1,I'})$, which is equivalent to $y \in \Gamma_{k-1,I'}$.

Now, let $z \in \overline{S_{k,I}}$ be an endpoint of $S_{k,I}$ minimizing the distance to x. Observe that $x - z \in L_k$ and $\sigma_k(x) - x \in L_k^\perp$, so

$$|\sigma_k(x) - z|^2 = |\sigma_k(x) - x|^2 + |x - z|^2. \tag{11.6}$$

Moreover, since z is an endpoint of $S_{k,I}$, the point $G_{k,I}^{-1}(z)$ is an endpoint of S_0, and so by (P1) $g(G_{k,I}^{-1}(z)) = 0$. Together with the fact that g is 1-Lipschitz this gives

$$\begin{aligned}|\sigma_k(x) - x| &= \left|G_{k,I} \circ \sigma_0 \circ G_{k,I}^{-1}(x) - G_{k,I} \circ G_{k,I}^{-1}(x)\right| \\ &= r_k\left|\sigma_0 \circ G_{k,I}^{-1}(x) - G_{k,I}^{-1}(x)\right| = r_k\left|g(G_{k,I}^{-1}(x))\right| = r_k\left|g(G_{k,I}^{-1}(x)) - g(G_{k,I}^{-1}(z))\right| \\ &\le r_k\left|G_{k,I}^{-1}(x) - G_{k,I}^{-1}(z)\right| = |x - z|. \end{aligned}\tag{11.7}$$

Furthermore, observe that since $y \in \Gamma_{k-1,I'} \setminus \overline{S_{k,I}}$, $z \in \overline{S_{k,I}}$ is an endpoint of $S_{k,I}$, $\mathcal{H}^1(S_{k,I}) = 2\,r_k$, and $|x - y| \lesssim M^{-1} r_k$, we get that the point $\gamma_{k-1}^{-1}(z) \in S_0$ lies between the points $\gamma_{k-1}^{-1}(x)$ and $\gamma_{k-1}^{-1}(y)$. We already know that γ_{k-1} is bilipschitz, and so

$$\begin{aligned}|x - z| + |z - y| &\approx |\gamma_{k-1}^{-1}(x) - \gamma_{k-1}^{-1}(z)| + |\gamma_{k-1}^{-1}(z) - \gamma_{k-1}^{-1}(y)| \\ &= |\gamma_{k-1}^{-1}(x) - \gamma_{k-1}^{-1}(y)| \approx |x - y|. \end{aligned}\tag{11.8}$$

Now, we need to further differentiate between two subcases.

Subcase 4a. $|x-y| \le 6M^{-1} r_k$, $x \in S_{k,I}$, and $y \in S_{k,Y}$ for some $Y \in \mathcal{I}^k$, $I \neq Y$.

We claim that the point z is a common endpoint of $S_{k,Y}$ and $S_{k,I}$. Indeed, since $y \in \Gamma_{k-1,I'}$ by (11.3), we have $Y = (I', Z) \in \mathcal{I}^{k-1} \times \mathcal{I}$ and $S_{k,Y} \subset \Gamma_{k-1,I'}$. By Lemma 11.4 d) $S_{k,Y} = G_{k-1,I'}(S_{1,Z}) = G_{k-1,I'} \circ \sigma_0(Z)$, and $S_{k,I} = G_{k-1,I'} \circ \sigma_0(J)$. Recall that $|x - y| \le 6M^{-1} r_k$, which implies $\operatorname{dist}(S_{k,I}, S_{k,Y}) \le 6M^{-1} r_k$, and so $\operatorname{dist}(Z, J) \lesssim M^{-1} r_{k-1}^{-1} r_k = M^{-1} r_1$. By (P3) J and Z are dyadic intervals of length $\sqrt{2}\,r_1$, which implies that $\operatorname{dist}(Z, J) = 0$. Hence, the point z is a common endpoint of $S_{k,I}$ and $S_{k,Y}$, and the estimates (11.6), (11.7) are also valid with x replaced by y.

The Lipschitz property of σ_k follows easily:

$$|\sigma_k(x) - \sigma_k(y)| \le |\sigma_k(x) - z| + |z - \sigma_k(y)| \overset{(11.6),(11.7)}{\lesssim} |x - z| + |z - y| \overset{(11.8)}{\approx} |x - y|.$$

The converse inequality is a consequence of the fact that $S_{k,I}$ and $S_{k,Y}$ are co-linear, $x - y \in L_k$, $\sigma_k(x) - x \in L_k^\perp$, and $\sigma_k(y) - y \in L_k^\perp$:

$$\begin{aligned}|\sigma_k(x) - \sigma_k(y)|^2 &= |\sigma_k(x) - x + x - y + y - \sigma_k(y)|^2 \\ &= |x - y|^2 + |\sigma_k(x) - x + y - \sigma_k(y)|^2 \ge |x - y|^2. \end{aligned}$$

Subcase 4b. $|x - y| \le 6M^{-1} r_k$, $x \in S_{k,I}$ for some $I \in \mathcal{I}^k$, and $y \in \Gamma_{k-1} \setminus \bigcup_{Y \in \mathcal{I}^k} S_{k,Y}$.

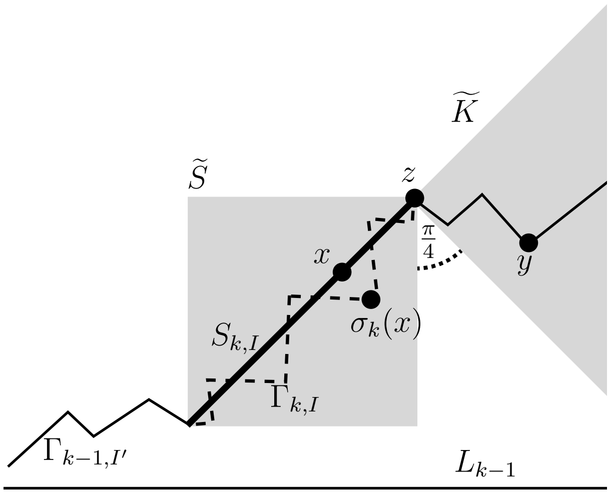


Figure V.2: Points x, y, z lie on $\Gamma_{k-1,I'}$ (continuous curve above), which is a 1-Lipschitz graph over the line L_{k-1}. x belongs to the segment $S_{k,I} \subset \Gamma_{k-1,I'}$ (thick segment above), and z is an endpoint of $S_{k,I}$. $\sigma_k(x)$ lies on $\Gamma_{k,I}$ (dashed curve above), a 1-Lipschitz graph over $S_{k,I}$ with the same endpoints as $S_{k,I}$. The 1-Lipschitz property implies that $\Gamma_{k,I} \subset \widetilde{S}$, where $\widetilde{S}$ is a square having $S_{k,I}$ as diagonal. On the other hand, the 1-Lipschitz property of $\Gamma_{k-1,I'}$ implies that $\Gamma_{k-1,I'} \subset K_0 := \overline{K(z, L_{k-1}, \sin(\pi/4))}$, i.e. it lies in the two-sided version of cone $\widetilde{K}$ above. In particular, $y \in K_0$. However, in Subcase 4b we assume that $|x-y| \leq 6M^{-1}r_k$ and $y \notin S_{k,I}$, and so y must lie in $\widetilde{K}$, and not in the other one-sided cone comprising K_0. Since L_{k-1} and $S_{k,I}$ form an angle $\pi/4$, the observations above imply $\pi/4 \leq \measuredangle(\sigma_k(x), z, y) \leq \pi$ (see the dotted angle).

In this case we have $\sigma_k(y) = y$. The upper bound follows from previous estimates:

$$|\sigma_k(x) - y| \leq |\sigma_k(x) - z| + |z-y| \overset{(11.6),(11.7)}{\lesssim} |x-z| + |z-y| \overset{(11.8)}{\approx} |x-y|.$$

Concerning the lower bound, it follows by elementary geometry and properties of our construction that $\pi/4 \leq \measuredangle(\sigma_k(x), z, y) \leq \pi$, see Figure V.2. Thus, using the law of cosines

$$\begin{aligned}|\sigma_k(x) - y|^2 &= |\sigma_k(x) - z|^2 + |z-y|^2 - 2|\sigma_k(x) - z||z-y|\cos(\measuredangle(\sigma_k(x), z, y))\\ &\geq |\sigma_k(x) - z|^2 + |z-y|^2 - \sqrt{2}|\sigma_k(x) - z||z-y|\\ &\geq \left(1 - \frac{\sqrt{2}}{2}\right)(|\sigma_k(x) - z|^2 + |z-y|^2) \overset{(11.6)}{\gtrsim} |x-z|^2 + |z-y|^2 \gtrsim |x-y|^2.\end{aligned}$$

Since this was the last case we had to check, we get that σ_k is bilipschitz, as claimed.

□

Finally, we set

$$E_N = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.$$

Note that due to Lemma 11.4 a)

$$E_N = \Gamma_0 \cup \bigcup_{I\in\mathcal{I}} \Gamma_{1,I} \cup \bigcup_{I\in\mathcal{I}^2} \Gamma_{2,I} \cup \bigcup_{I\in\mathcal{I}^3} \Gamma_{3,I}. \tag{11.9}$$

That is, E_N is a union of a single big Lipschitz graph, and three layers of smaller Lipschitz graphs.

11.2 E_N has BPLG

In this section we show that E_N has big pieces of Lipschitz graphs, with constants independent of N.

Observe that E_N is AD-regular because it is a union of four bilipschitz curves. The ADR constants do not depend on N due to Lemma 11.5.

Lemma 11.6. *For any $x \in E_N$ and any $0 < r < \operatorname{diam}(E_N)$ we can find a Lipschitz graph Σ (depending on x and r) such that*

$$\mathcal{H}^1(E_N \cap B(x,r) \cap \Sigma) \gtrsim r, \tag{11.10}$$

with the implicit constant independent of N.

First, we prove an auxiliary estimate. Given integers $i, l \in \{0,1,2,3\}$ define $\gamma_{l,i} : \Gamma_l \to \Gamma_i$ as $\gamma_{l,i} = \gamma_i \circ \gamma_l^{-1}$.

Lemma 11.7. *Let $i, l \in \{0,1,2,3\}$ and $k = \min(i,l)$. Then*

$$\|\gamma_{l,i} - id\|_{L^\infty(\Gamma_l)} \le 6M^{-1} r_{k+1}. \tag{11.11}$$

Proof. If $i = l$ the result is clear because $\gamma_{l,i} = id$. Assume $l > i$. Applying $(l-i)$-many times Lemma 11.4 g) we get that

$$|x - \gamma_{l,i}(x)| \le \sum_{j=i+1}^{l} |\gamma_{l,j-1}(x) - \gamma_{l,j}(x)| = \sum_{j=i+1}^{l} |\gamma_{l,j-1}(x) - \sigma_j(\gamma_{l,j-1}(x))|$$
$$\le \sum_{j=i+1}^{l} 2M^{-1} r_j \le 2(j-k)M^{-1} r_{i+1} \le 6M^{-1} r_{i+1}.$$

On the other hand, if $l < i$, then applying the estimate above to $y = \gamma_{l,i}(x)$ we get

$$|x - \gamma_{l,i}(x)| = |\gamma_{i,l}(y) - y| \le 6M^{-1} r_{l+1}.$$

□

Proof of Lemma 11.6. Let $x \in E_N$ and $0 < r < \operatorname{diam}(E_N)$. By (11.9) there exist $j \in \{0,1,2,3\}$ and $I \in \mathcal{I}^j$ such that $x \in \Gamma_{j,I}$.

Suppose $r < r_j$. Since $\Gamma_{j,I}$ is a Lipschitz graph satisfying $\mathcal{H}^1(\Gamma_{j,I}) \geq r_j > r$, we have

$$\mathcal{H}^1(E_N \cap B(x,r) \cap \Gamma_{j,I}) = \mathcal{H}^1(B(x,r) \cap \Gamma_{j,I}) \gtrsim r.$$

That is, we may choose $\Sigma = \Gamma_{j,I}$.

Now assume $r_j \leq r < r_0 = 1$. Let $k \in \{0,1,2\}$ be such that $r_{k+1} \leq r < r_k$ (of course, $k+1 \leq j$). Let $y = \gamma_{j,k}(x)$. Observe that, by Lemma 11.4 a), since $y \in \Gamma_k$, there exists some $k' \in \{0,\dots,k\}$ such that $y \in \Gamma_{k',I'}$ for some $I' \in \mathcal{I}^{k'}$. Since $k' \leq k$, we have $\mathcal{H}^1(\Gamma_{k',I'}) \approx r_{k'} \geq r_k > r$. Moreover, assuming $M \geq 12$, (11.11) gives

$$\operatorname{dist}(x, \Gamma_{k',I'}) \leq |x-y| = |x - \gamma_{j,k}(x)| \leq \frac{r_{k+1}}{2} \leq \frac{r}{2},$$

and so

$$\mathcal{H}^1(E_N \cap B(x,r) \cap \Gamma_{k',I'}) = \mathcal{H}^1(B(x,r) \cap \Gamma_{k',I'}) \gtrsim r.$$

Hence, we may choose $\Sigma = \Gamma_{k',I'}$.

Finally, for $1 < r < \operatorname{diam}(E_N) \approx 1$, the condition (11.10) is satisfied with $\Sigma = \Gamma_0$. □

11.3 E_N has big conical energy

In this section we show that E_N satisfies (11.1).

We introduce additional notation. Analogously to the definition of $S_{k,I}$ for $k \in \{0,1,2,3\}$, for $I = (I', J) \in \mathcal{I}^3 \times \mathcal{I}$ we define $S_{4,I} = G_{3,I'}(S_{1,J})$.

If $I \in \mathcal{I}^{k+j}$ is of the form $I = (I', I'') \in \mathcal{I}^k \times \mathcal{I}^j$, we will write

$$S_{k,I} := S_{k,I'}, \qquad \Gamma_{k,I} := \Gamma_{k,I'}, \qquad G_{k,I} := G_{k,I'}.$$

Lemma 11.8. *Let $I = (I_1, I_2, I_3, I_4) \in \mathcal{I}^4$, and let $x \in S_{4,I} \subset \Gamma_{3,I} \subset E_N$. Then, for any $V \in G(2,1)$ we have*

$$\int_0^1 \frac{\mathcal{H}^1(K(x,V,\alpha,r) \cap E_N)}{r} \frac{dr}{r} \gtrsim N. \tag{11.12}$$

Proof. Let $I \in \mathcal{I}^4$, $x \in S_{4,I}$ and $V \in G(2,1)$ be as above. Recall that $L_0 = \{(x,0) : x \in \mathbb{R}\}$, ρ is the counterclockwise rotation by $\pi/4$, and $L_k = \rho^k(L_0) \in G(2,1)$. Recall also that $V_0 = \{(x,y) : y = \tan(\pi/8)x\}$ is the line from (P4) in Lemma 11.3. Observe that there exists some $k \in \{0,1,2,3\}$ such that $\measuredangle(\rho^{-k}(V), V_0) \leq \pi/8$. Fix such k. We are going to use (P4) with respect to $\Gamma_{k,I}$ to arrive at (11.12).

Recall that $S_{k+1,I} = G_{k,I}(S_{1,I_{k+1}})$, where $G_{k,I} = \tau \circ \rho^k \circ \delta^k$ for some translation τ. Recall also that $G_{k,I}(\Gamma) = \Gamma_{k,I}$. Let $x' = G_{k,I}^{-1}(x)$, and $V' =$

$\rho^{-k}(V)$. Then, using the fact that $G_{k,I}$ is a similarity with stretching factor r_k, we get

$$\int_0^1 \frac{\mathcal{H}^1(K(x,V,\alpha,r)\cap E_N)}{r}\,\frac{dr}{r} \geq \int_0^1 \frac{\mathcal{H}^1(K(x,V,\alpha,r)\cap \Gamma_{k,I})}{r}\,\frac{dr}{r}$$
$$= \int_0^1 r_k\,\frac{\mathcal{H}^1(K(x',V',\alpha,r_k^{-1}r)\cap \Gamma)}{r}\,\frac{dr}{r} = \int_0^{r_k^{-1}} \frac{\mathcal{H}^1(K(x',V',\alpha,s)\cap \Gamma)}{s}\,\frac{ds}{s}. \tag{11.13}$$

Recall that k was chosen in such a way that $V' = \rho^{-k}(V)$ satisfies $\measuredangle(V',V_0) \leq \pi/8$. In order to use (P4), it only remains to show that $\operatorname{dist}(x', G(I')) \leq 2^{-N(M+1)}$ for some $I' \in \mathcal{I}$.

Observe that $\gamma_{3,k}(S_{4,I}) \subset S_{k+1,I}$. We know from (11.11) that if $M \geq 6$, then

$$\operatorname{dist}(x, S_{k+1,I}) \leq \operatorname{dist}(x, \gamma_{3,k}(S_{4,I})) \leq |x - \gamma_{3,k}(x)| \leq r_{k+1}. \tag{11.14}$$

Thus,

$$\operatorname{dist}(x', S_{1,I_{k+1}}) = \operatorname{dist}(G_{k,I}^{-1}(x), G_{k,I}^{-1}(S_{k+1,I})) = r_k^{-1}\operatorname{dist}(x, S_{k+1,I})$$
$$\leq r_k^{-1} r_{k+1} = r_1 = 2^{-N(M+1)-1/2} \leq 2^{-N(M+1)}.$$

$S_{1,I_{k+1}}$ was defined as $\sigma_0(I_{k+1}) = G(I_{k+1})$, and so it follows from (P4) that the last term in (11.13) is greater than CN for some absolute constant C. Thus, (11.12) holds. □

Now we can finish the proof of Proposition 11.1. Observe that

$$\mathcal{H}^1\Big(\bigcup_{I\in\mathcal{I}^4} S_{4,I}\Big) = \sum_{I'\in\mathcal{I}^3,\,J\in\mathcal{I}} \mathcal{H}^1(G_{3,I'}(S_{1,J}))$$
$$= (\#\mathcal{I})^3\, r_3 \sum_{J\in\mathcal{I}} \mathcal{H}^1(S_{1,J}) \geq (\#\mathcal{I})^3\, r_3 \sum_{J\in\mathcal{I}} \mathcal{H}^1(J)$$
$$\overset{(P3)}{=} 2^{-3M}\, 2^{3N(M+1)}\, 2^{-3N(M+1)-3/2}\, 2^{-M+1} = 2^{-4M-1/2} \approx_\alpha 1,$$

where we also used that M is a constant depending only on α. Together with Lemma 11.8, this shows that the set E_N has the desired property (11.1), i.e.

$$\int_{E_N}\int_0^1 \frac{\mathcal{H}^1(K(x,V_x,\alpha,r)\cap E_N)}{r}\,\frac{dr}{r}\,d\mathcal{H}^1(x)$$
$$\geq \sum_{I\in\mathcal{I}4} \int_{S_{4,I}}\int_0^1 \frac{\mathcal{H}^1(K(x,V_x,\alpha,r)\cap E_N)}{r}\,\frac{dr}{r}\,d\mathcal{H}^1(x) \gtrsim_\alpha N.$$

Thus, the proof of Proposition 11.1 is complete. All that remains to prove is Lemma 11.3. We do that in the following two subsections.

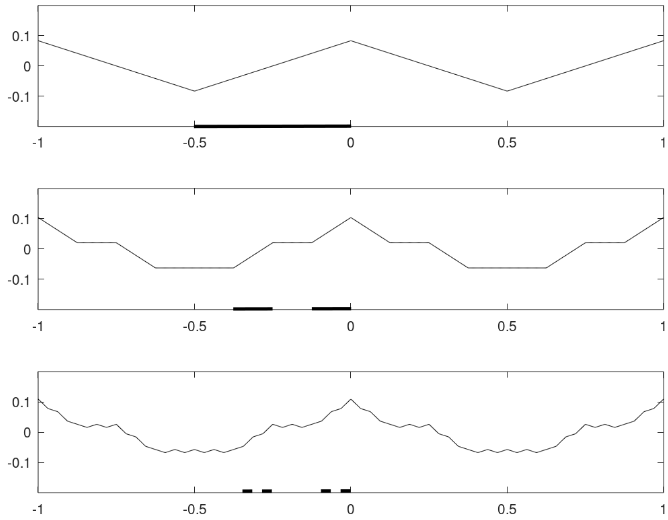


Figure V.3: Top to bottom: graphs of g_1, g_2, and $g_3 = g$ when $N = 2$ and $M = 3$. The thick segments denote intervals in $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$, respectively.

11.4 Construction of g

In this subsection we construct a function g and a family of dyadic intervals $\mathcal{I}$ that satisfy (P1), (P2), and (P3).

First, we define a family of auxiliary functions. For $j = 1, \ldots, M$ we define $f_j : [-1, 1] \to [-M^{-1}2^{-jN}, M^{-1}2^{-jN}]$ as

$$f_j(t) = \frac{h(2^{jN}t)}{M2^{jN}},$$

where $h(t) : \mathbb{R} \to [-1, 1]$ is the 1-Lipschitz triangle wave:

$$h(t) = |t \bmod 4 - 2| - 1.$$

In the above $t \bmod 4$ denotes the unique number $s \in [0, 4)$ such that $t = 4k + s$ for some $k \in \mathbb{Z}$.

Note that for all j we have $\mathrm{Lip}(f_j) = M^{-1}$. For $j = 1, \ldots, M$ we define also $g_j : [-1, 1] \to [-M^{-1}2^{-N+1}, M^{-1}2^{-N+1}]$ as

$$g_j(t) = \sum_{i=1}^{j} f_i(t),$$

and we set $\Gamma_j = \mathrm{graph}(g_j) \subset B(0, 1) \subset \mathbb{R}^2$, $g = g_M$, $\Gamma = \Gamma_M$. See Figure V.3. Observe that g is 1-Lipschitz.

Proof of (P1). We want to show that $g(1) = g(-1) = 0$. Since h is an even function, the functions f_j and g_j are also even. Hence, $g(1) = g(-1)$. Note also that if we have some function $\tilde{g}$ satisfying properties (P2) and (P4), then for any constant $C \in \mathbb{R}$ the function $\tilde{g} + C$ will also satisfy (P2) and (P4). In other words, these properties are invariant under adding constants. It follows that we can work with the function g as defined above, prove (P2) and (P4), and at the end replace g by $g - g(1)$. So the property (P1) is not an issue. □

We proceed to define the family $\mathcal{I} \subset \Delta_{(M+1)N}$.

Recall that Δ_k denotes the open dyadic intervals of length 2^{-k}. Observe that for any j the functions f_j and g_j are linear on each interval from Δ_{jN}, and we have $f_j' = M^{-1}$ on every second interval, and $f_j' = -M^{-1}$ on the rest.

Set $\mathcal{G}_j \subset \Delta_{jN}$ to be the family of dyadic intervals I contained in $[-1/2, 1/2]$ such that for all $1 \le i \le j$ we have $f_i' = M^{-1}$ on I. It is easy to see that each $\mathcal{G}_j$ consists of 2^{jN-j} disjoint intervals of length 2^{-jN}, see Figure V.3. We define also $\mathcal{I} \subset \Delta_{(M+1)N}$ as the family of dyadic intervals of length $2^{-(M+1)N}$ contained in $\bigcup_{I \in \mathcal{G}_M} I$.

Proof of (P3). By the definition above we have

$$\#\mathcal{I} = 2^N \cdot \#\mathcal{G}_M = 2^{(M+1)N-M}, \tag{11.15}$$

so the property (P3) holds. □

Proof of (P2). We have defined $\mathcal{G}_j$ in such a way that if $t \in I \in \mathcal{G}_j$ then $g_j'(t) = jM^{-1}$. It follows that if $t \in I \in \mathcal{I}$, then $t \in J$ for some $J \in \mathcal{G}_M$, and so $g' = 1$. Thus, (P2) holds. □

11.5 Γ has big conical energy

This subsection is dedicated to proving (P4). We recall the statement for reader's convenience:

(P4) Let $\Gamma = \text{graph}(g)$, $G : [-1, 1] \to \Gamma$ be the graph map $G(t) = (t, g(t))$, and let $V_0 = \{(x, y) : y = \tan(\pi/8)x\} \in G(2, 1)$. For any $I \in \mathcal{I}$, any $x \in \mathbb{R}^2$ with $\text{dist}(x, G(I)) < 2^{-N(M+1)}$, and all $V \in G(2, 1)$ satisfying $\measuredangle(V, V_0) \le \pi/8$, we have

$$\int_0^1 \frac{\mathcal{H}^1(K(x, V, \alpha, r) \cap \Gamma)}{r} \frac{dr}{r} \gtrsim N. \tag{11.16}$$

Fix x, I, and V as above. We will show (11.16).

Since $\text{dist}(x, G(I)) < 2^{-N(M+1)}$, there exists $t_0 \in I$ such that $|x - G(t_0)| \le 2^{-N(M+1)}$. Fix such t_0.

For every $j = 1, \dots, M$ define $G_j(t) = (t, g_j(t))$. For every $t \in \bigcup_{I \in \Delta_{jN}} I$ set $L_j(t) \subset \mathbb{R}^2$ to be the line tangent to Γ_j at $G_j(t)$. We define also $I_j(t)$ as the unique interval from Δ_{jN} containing t. Note that, since g_j is linear on intervals from Δ_{jN}, we have $L_j(t) = L_j(t')$ whenever $t' \in I_j(t)$. Denote by L_0 the x-axis.

Observe that if $I_M(t) \in \mathcal{G}_M$, then for each $1 \le j \le M$ we have $g_j'(t) = jM^{-1}$. Thus,

$$\measuredangle(L_j(t), L_0) = \arctan(jM^{-1}), \quad \text{and} \quad \measuredangle(L_j(t), V_0) \le \pi/8. \tag{11.17}$$

Set $L_j = L_j(t) - (t, g_j(t))$. Note that $(0,0) \in L_j$, and that the definition of L_j does not depend on t, as long as $I_M(t) \in \mathcal{G}_M$. Since $\measuredangle(V, V_0) \le \pi/8$, it follows from (11.17) that there exists some $1 \le j \le M$ such that

$$\measuredangle(V, L_j) \le \max_{1\le i\le M} \Big(\arctan(iM^{-1}) - \arctan((i-1)M^{-1})\Big) = \arctan(M^{-1}) \le M^{-1}. \tag{11.18}$$

Fix such j. Recall that $M = 100\lceil \alpha^{-1} \rceil$, and so

$$\measuredangle(V, L_j) \le M^{-1} \le \frac{\alpha}{10}. \tag{11.19}$$

Hence, for any $r > 0$

$$K(x, V, \alpha, r) \supset K(x, L_j, \alpha/2, r). \tag{11.20}$$

Lemma 11.9. *For $t \in [-1, 1]$ we have $|G(t) - G_j(t)| = |g(t) - g_j(t)| \le 2M^{-1}\, 2^{-N(j+1)}$.*

Proof. The estimate follows immediately from the definition of g and g_j:

$$|g(t) - g_j(t)| = \left| \sum_{i=j+1}^{M} f_i(t) \right| \le \sum_{i=j+1}^{M} |f_i(t)| \le \sum_{i=j+1}^{\infty} \frac{1}{M} 2^{-iN} \le 2M^{-1}\, 2^{-N(j+1)}.$$

□

Recall that $t_0 \in I \in \mathcal{I}$ was such that $|x - G(t_0)| \le 2^{-N(M+1)}$. Set $x' = G_j(t_0)$. Then, by the lemma above, we have

$$|x - x'| \le |x - G(t_0)| + |G(t_0) - G_j(t_0)| \le 2^{-N(M+1)} + 2^{-N(j+1)} \le 2^{-N(j+1)+1}. \tag{11.21}$$

Let $I' \in \mathcal{G}_j$ be the unique dyadic interval in Δ_{jN} containing I. That is, $I' = I_j(t_0)$.

Recall that for any $0 < r < R$ the notation $K(x, V, \alpha, r, R)$ stands for a twice truncated cone $K(x, V, \alpha, R) \setminus B(x, r)$. In the lemma below we show that for all the scales between $2^{-N(j+1)}$ and 2^{-Nj}, $G(I')$ has large intersection with the the twice truncated cone centered at x' with direction L_j corresponding to that scale.

Lemma 11.10. *For $t \in I'$ such that $|G(t) - x'| \geq 2^{-N(j+1)}$ we have $G(t) \in K(x', L_j, \alpha/8)$. Moreover, for integers k satisfying $Nj \leq k \leq N(j+1) - 1$ we have*

$$\mathcal{H}^1(G(I') \cap K(x', L_j, \alpha/8, 2^{-k-1}, 2^{-k+2})) \gtrsim 2^{-k}. \tag{11.22}$$

Proof. Let $t \in I'$ satisfy $|G(t) - x'| \geq 2^{-N(j+1)}$. Recall that, since $I' \in \mathcal{G}_j$, the set $G_j(I')$ is a segment parallel to L_j. We also know that $x' = G_j(t_0) \in G_j(I')$, and so by Lemma 11.9

$$\operatorname{dist}(G(t), L_j + x') \leq |G(t) - G_j(t)| \leq 2M^{-1}\, 2^{-N(j+1)} \leq \frac{\alpha}{8}|G(t) - x'|,$$

where we also used that $M = 100\lceil \alpha^{-1} \rceil$. Thus, $G(t) \in K(x', L_j, \alpha/8)$.

Now, let k be an integer such that $Nj \leq k \leq N(j+1) - 1$. Let $t \in I'$ be such that $2^{-k} < |t - t_0|$, so that

$$\begin{aligned}|G(t) - x'| \geq |G_j(t) - G_j(t_0)| - |G(t) - G_j(t)| &\geq |t - t_0| - 2M^{-1}\, 2^{-N(j+1)} \\ &\geq 2^{-k} - 2M^{-1}\, 2^{-N(j+1)} \geq 2^{-N(j+1)}.\end{aligned}$$

Hence, by our previous result, $G(t) \in K(x', L_j, \alpha/8)$. At the same time, the calculation above shows that $|G(t) - x'| \geq 2^{-k-1}$. Similarly,

$$|G(t) - x'| \leq |G_j(t) - G_j(t_0)| + |G(t) - G_j(t)| \leq \sqrt{2}|t - t_0| + 2M^{-1}\, 2^{-N(j+1)}.$$

Hence, for $t \in I'$ such that $2^{-k} \leq |t - t_0| \leq 2^{-k+1}$ we have

$$2^{-k-1} \leq |G(t) - x'| \leq 2^{-k+2}.$$

That is, for $t \in I'$ with $2^{-k} \leq |t - t_0| \leq 2^{-k+1}$ we have

$$G(t) \in K(x', L_j, \alpha/8, 2^{-k-1}, 2^{-k+2}).$$

Since G is bilipschitz, (11.22) follows. □

Later on we will need the following simple lemma about the inclusions of twice truncated cones.

Lemma 11.11. *Let $x_1, x_2 \in \mathbb{R}^2$, $L \in G(2,1)$, $r > 0$ and $\alpha_0 \in (0, 1/4)$. Suppose that $|x_1 - x_2| \leq \alpha_0 r$. Then*

$$K(x_1, L, \alpha_0, \alpha_0^{-1}|x_1 - x_2|, r) \subset K(x_2, L, 4\alpha_0, 2r).$$

Proof. Let $y \in K(x_1, L, \alpha_0, \alpha_0^{-1}|x_1 - x_2|, r)$, so that $\alpha_0^{-1}|x_1 - x_2| < |y - x_1| \leq r$ and $\operatorname{dist}(y, L + x_1) \leq \alpha_0|y - x_1|$. It is clear that for any $p \in L + x_1$ we have $\operatorname{dist}(p, L + x_2) = |x_1 - x_2|$, and so

$$\begin{aligned}\operatorname{dist}(y, L + x_2) \leq \operatorname{dist}(y, L + x_1) + |x_1 - x_2| &\leq \alpha_0|y - x_1| + \alpha_0|y - x_1| \\ &\leq 2\alpha_0|y - x_2| + 2\alpha_0|x_1 - x_2|.\end{aligned}$$

At the same time, we have

$$|y - x_2| \geq |y - x_1| - |x_1 - x_2| \geq (\alpha_0^{-1} - 1)|x_1 - x_2| \geq |x_1 - x_2|.$$

Putting the two estimates together gives $y \in K(x_2, L, 4\alpha_0)$. To see that $y \in B(x_2, 2r)$, note that $|y - x_2| \leq |y - x_1| + |x_1 - x_2| \leq 2r$. □

Recall that in (11.22) we showed a lower bound on the length of intersection of $G(I')$ with a cone centered at x'. However, to prove (11.16) we need information about the intersections with cones centered at x. We use (11.22) and Lemma 11.11 to get the following.

Lemma 11.12. *Let k be an integer such that $\alpha^{-1} 2^{-N(j+1)+8} < 2^{-k} \leq 2^{-Nj-3}$. Then, we have*

$$\mathcal{H}^1(G(I') \cap K(x, L_j, \alpha/2, 2^{-k})) \gtrsim 2^{-k}, \tag{11.23}$$

Proof. First, recall that $x' = G_j(t_0)$ and $|x - x'| \leq 2^{-N(j+1)+1}$ by (11.21). By our assumptions on k we have

$$8\alpha^{-1}|x - x'| \leq \alpha^{-1} 2^{-N(j+1)+4} \leq 2^{-k-4} < 2^{-k-1}. \tag{11.24}$$

Hence, we may apply Lemma 11.11 with $x_1 = x'$, $x_2 = x$, $L = L_j$, $\alpha_0 = \alpha/8$, $r = 2^{-k-1}$, to get

$$K(x', L_j, \alpha/8, 8\alpha^{-1}|x - x'|, 2^{-k-1}) \subset K(x, L_j, \alpha/2, 2^{-k}).$$

Since $8\alpha^{-1}|x - x'| \leq 2^{k-4}$ by (11.24), it follows from the above that

$$K(x', L_j, \alpha/8, 2^{-k-4}, 2^{-k-1}) \subset K(x, L_j, \alpha/2, 2^{-k}). \tag{11.25}$$

Note that we have $Nj \leq k - 3 \leq N(j+1) - 1$ due to our assumptions on k. Thus, we may use (11.22) to get

$$\mathcal{H}^1(G(I') \cap K(x, L_j, \alpha/8, 2^{-k-4}, 2^{-k-1})) \gtrsim 2^{-k}.$$

Together with (11.25), this concludes the proof. □

We are ready to finish the proof of Lemma 11.3.

Proof of (P4). We want to show that

$$\int_0^1 \frac{\mathcal{H}^1(K(x, V, \alpha, r) \cap \Gamma)}{r} \frac{dr}{r} \gtrsim N. \tag{11.26}$$

We use (11.20) to write

$$\int_0^1 \frac{\mathcal{H}^1(K(x, V, \alpha, r) \cap \Gamma)}{r} \frac{dr}{r} \geq \int_0^1 \frac{\mathcal{H}^1(K(x, L_j, \alpha/2, r) \cap \Gamma)}{r} \frac{dr}{r}$$
$$\geq \int_{\alpha^{-1} 2^{-N(j+1)+9}}^{2^{-Nj-3}} \frac{\mathcal{H}^1(K(x, L_j, \alpha/2, r) \cap \Gamma)}{r} \frac{dr}{r}. \tag{11.27}$$

Note that $\alpha^{-1}2^{-N(j+1)+9} < 2^{-Nj-3}$ due to the assumption $N \geq 100(1 + \log_2(\alpha^{-1}))$. Now let $\alpha^{-1}2^{-N(j+1)+9} \leq r < 2^{-Nj-3}$, and let k be the unique integer such that $2^{-k} \leq r < 2^{-k+1}$. Then, k satisfies the assumptions of Lemma 11.12, and we get

$$\mathcal{H}^1(K(x, L_j, \alpha/2, r) \cap \Gamma) \geq \mathcal{H}^1(K(x, L_j, \alpha/2, 2^{-k}) \cap \Gamma) \gtrsim 2^{-k} \approx r.$$

It follows from (11.27) and the above that

$$\int_0^1 \frac{\mathcal{H}^1(K(x, V, \alpha, r) \cap \Gamma)}{r} \frac{dr}{r} \gtrsim \int_{\alpha^{-1}2^{-N(j+1)+9}}^{2^{-Nj-3}} 1 \, \frac{dr}{r}$$
$$= \log(2)\,(N(j+1)-9-\log_2(\alpha^{-1})-Nj-3) = \log(2)(N-\log_2(\alpha^{-1})-12) \geq \frac{N}{100},$$

where we used the assumption $N \geq 100(1 + \log_2(\alpha^{-1}))$ in the last inequality. Thus, the proof of (11.26) is finished. □

12 Example of Joyce and Mörters

In this section we will show that the measure μ constructed in [JM00] satisfies the assumptions of Theorem 1.14, but does not satisfy BPBE(1). Hence, Theorem 1.14 is a true improvement on its $\mathcal{E}_{\mu,1}$ analogue [CT17, Theorem 10.2].

12.1 Construction of μ

For reader's convenience, we sketch out the construction of Joyce and Mörters below.

Let $M \geq 3$ be a large constant, and $1/2 < \beta_k < 1$ be a sequence of numbers converging to 1. For $k \geq 1$ we define $m_k = Mk$, $m(k) = m_1 \dots m_k = M^k\, k!$, and

$$\sigma_k = \left(\frac{k+1}{k}\right)^{\beta_k}.$$

We set also $\alpha_j = 2^{-n}\pi$ for all $2^n \leq j < 2^{n+1},\ n \geq 0$.

We proceed to define a compact set $E \subset \mathbb{R}^2$ on which the measure μ will be supported. First, let E_0 be a closed ball of diameter 1. We place m_1 closed balls of diameter $2r_1 := \sigma_1/m_1$ inside E_0. We do it in such a way, that

- their centers lie on the diameter of E_0 forming angle α_1 with the x axis,
- the boundaries of the first and the last ball touch the boundary of E_0,
- they overlap as little as possible, i.e. the distance between the centers of two neighbouring balls is $(1 - \sigma_1/m_1)/(m_1 - 1)$.

We call these balls the *balls of generation 1*, we denote their family by $\mathcal{B}_1$, and we set $E_1 = \bigcup_{B\in\mathcal{B}_1} B$.

Now suppose that E_k has already been defined as a union of balls $\bigcup_{B\in\mathcal{B}_k} B$, and that $\#\mathcal{B}_k = m(k)$. Inside every ball $B \in \mathcal{B}_k$ we place m_{k+1} closed balls of diameter $2r_{k+1} := \sigma_1 \dots \sigma_{k+1}/m(k+1)$. We do it in such a way, that

- their centers lie on the diameter of B forming angle $\sum_{i=1}^{k+1} \alpha_i$ with the x axis,
- the boundaries of the first and the last ball touch touch the boundary of B,
- they overlap as little as possible, i.e. the distance between the centers of two neighbouring balls is

$$d_{k+1} := \frac{\sigma_1 \dots \sigma_k}{m(k)} \cdot \frac{1 - \sigma_{k+1}/m_{k+1}}{m_{k+1} - 1}.$$

The balls defined above are called the *balls of generation (k+1)*, and their family is denoted by $\mathcal{B}_{k+1}$. Clearly, $\#\mathcal{B}_{k+1} = m_{k+1} \cdot m(k) = m(k+1)$. We set $E_{k+1} = \bigcup_{B\in\mathcal{B}_{k+1}} B$, and $E = \bigcap_{k\geq 0} E_k$.

It is shown in [JM00, §2.1] that if M is chosen appropriately, then two balls of generation $(k+1)$ may intersect only if they are contained in the same ball of generation k. It follows that there exists a natural probability measure μ supported on E defined by

$$\mu(B) = m(k)^{-1} \quad \text{for } B \in \mathcal{B}_k,\ k \geq 1. \tag{12.1}$$

If the sequence β_k is chosen properly, the set E has the following curious property: it is of non-σ-finite length, but all the projections of E onto lines are of zero length. Moreover, the Menger curvature of E is finite. However, we will not use those properties.

12.2 The assumptions of Theorem 1.14 are satisfied

In [JM00, §2.1] Joyce and Mörters construct a function $\varphi : [0, d_1) \to \mathbb{R}$ satisfying $\varphi(r) < r$ and

$$\int_0^{d_1} \frac{\varphi(r)^2}{r^3}\, dr < \infty.$$

They also show that for $0 < r < d_1$ the measure μ satisfies $\mu(B(x,r)) \leq 84\,\varphi(r)$. It follows easily that $\mu(B(x,r)) \leq C_1 r$ for $C_1 = \max(84, 1/d_1)$ and all $r > 0$. Furthermore, by the observations above and the fact that $\mu(\mathbb{R}^2) = 1$, for all $x \in E = \operatorname{supp}\mu$ we have

$$\int_0^\infty \left(\frac{\mu(B(x,r))}{r}\right)^2 \frac{dr}{r} \lesssim \int_0^{d_1} \frac{\varphi(r)^2}{r^3}\, dr + \int_{d_1}^\infty \frac{1}{r^3}\, dr \leq M_0 \tag{12.2}$$

for some M_0 depending only on d_1 and φ.

Obviously, for any $V \in G(2,1),\ \alpha \in (0,1), R > 0$, we have

$$\mathcal{E}_{\mu,2}(x,V,\alpha,R) = \int_0^R \left(\frac{\mu(K(x,V,\alpha,r))}{r}\right)^2 \frac{dr}{r} \leq \int_0^\infty \left(\frac{\mu(B(x,r))}{r}\right)^2 \frac{dr}{r},$$

and so the assumptions of Theorem 1.14 are trivially satisfied.

Let us note that the boundedness of nice singular integral operators on $L^2(\mu)$ for this particular measure μ is not a new result. It is well known that measures satisfying (12.2) behave well with respect to SIOs. For example, one can use (12.2) and [Mat96, Theorem 2.2] to prove local curvature condition for μ, and then boundedness of Cauchy transform follows from [Tol99, Theorem 1.1].

12.3 $\mathcal{E}_{\mu,1}$ is not bounded

Let $x \in E,\ V \in G(2,1)$, and $\alpha \in (0,1)$ be given. We will show that

$$\mathcal{E}_{\mu,1}(x,V,\alpha,1) = \int_0^1 \frac{\mu(K(x,V,\alpha,r))}{r} \frac{dr}{r} = \infty. \tag{12.3}$$

First, we identify the lines $W \in G(2,1)$ with the angle $\theta_W \in [0,\pi)$ they form with the x axis. We will abuse notation by writing $K(x,\theta_W,\alpha,R)$ to denote $K(x,W,\alpha,R)$. Set $\theta := \theta_V$.

Definition 12.1. We will say that an integer k is a *good index* if

$$\left| \Big(\sum_{j=1}^k \alpha_j - N\pi \Big) - \theta \right| \leq \frac{\alpha}{8}, \tag{12.4}$$

where N is the integer satisfying $2^N \leq k < 2^{N+1}$. By the definition of α_j, this is equivalent to

$$\left|(k-2^N+1)\frac{\pi}{2^N} - \theta\right| \leq \frac{\alpha}{8}. \tag{12.5}$$

Our strategy is the following: first, we show that there are many good indices. Then, we prove that if k is a good index, then $\mu(K(x,\theta,\alpha,2\,r_k))r_k^{-1}$ is large. Put together, the two facts will imply (12.3).

We define $N_0 = N_0(\alpha)$ to be a large integer, to be fixed in Lemmas 12.2 and 12.3.

Lemma 12.2. *If $N_0 = N_0(\alpha)$ is large enough, then for all $N > N_0$ we have a large portion of good indices satisfying $2^N \leq k < 2^{N+1}$, that is,*

$$\#\{2^N \leq k < 2^{N+1} \ :\ k \text{ is a good index}\} \gtrsim 2^N \alpha.$$

Proof. Let N_0 be so big that $2^{-N_0}\pi < \alpha/100$, and let $N > N_0$. Let $2^N \le k_0 < 2^{N+1}$ be the index minimizing $\left|(k_0 - 2^N + 1)\pi 2^{-N} - \theta\right|$. It is clear that

$$\left|(k_0 - 2^N + 1)2^{-N}\pi - \theta\right| \le 2^{-N}\pi,$$

and so it follows from (12.5) that all integers k such that $2^N \le k < 2^{N+1}$ and $|(k-k_0)2^{-N}\pi| \le \alpha/10$ are good indices. It is easy to see that there are at least $C2^N\alpha$ such integers, where C is some absolute constant. □

Recall that r_k was the radius of balls of k-th generation, and $x \in E$ is arbitrary. For $k \ge 1$ let $B_k \in \mathcal{B}_k$ be a ball of generation k containing x (there may be two such balls, in which case we just choose one).

Lemma 12.3. *If $N_0 = N_0(\alpha)$ is large enough, then for all good indices $k \ge 2^{N_0}$ we have*

$$\mu(K(x,\theta,\alpha,2\,r_k)) \gtrsim \mu(B_k).$$

Proof. Let y be the center of B_{k+1}, so that $|x-y| \le r_{k+1}$. By construction,

$$r_{k+1} = r_k\,\sigma_{k+1}(Mk)^{-1} \le r_k\,k^{-1}. \tag{12.6}$$

Since $k \ge 2^{N_0}$, for N_0 big enough we get

$$|x-y| \le r_{k+1} \le \frac{\alpha}{50} r_k. \tag{12.7}$$

Then, it follows from Lemma 11.11 that

$$K(y,\theta,\alpha/4, 4\alpha^{-1}|x-y|, r_k) \subset K(x,\theta,\alpha,2\,r_k).$$

Since $4\alpha^{-1}|x-y| \le r_k/2$ by (12.7), we get

$$K(y,\theta,\alpha/4, r_k/2, r_k) \subset K(x,\theta,\alpha,2\,r_k). \tag{12.8}$$

On the other hand, using the definition of good index (12.4) we arrive at

$$K(y, \textstyle\sum_{j=1}^k \alpha_j - N\pi, \alpha/20, r_k/2, r_k) \subset K(y,\theta,\alpha/4, r_k/2, r_k). \tag{12.9}$$

For brevity, set K to be the cone from the left hand side above, and let L be the axis of K. Recall that the diameter of B_k (let us call it D) forms angle $\sum_{j=1}^k \alpha_j - N\pi$ with the x axis; that is, D is parallel to L. Since y is the center of B_{k+1}, it follows from the construction of E that $y \in D$. Hence, $D \subset L$.

We claim that the balls of generation $(k+1)$ contained in $B_k \cap B(y,r_k) \setminus B(y,r_k/2)$, are in fact contained in K. Indeed, suppose z belongs to such ball, so that

$$\operatorname{dist}(z,L) = \operatorname{dist}(z,D) \le r_{k+1} \overset{(12.7)}{\le} \frac{\alpha}{50} r_k \le \frac{\alpha}{25}|z-y|.$$

Thus, $z \in \mathsf{K}$.

Since $y \in D$ and B_k is a ball of radius r_k, it follows that a large portion of balls of generation $(k+1)$ contained in B_k is also contained in $B(y, r_k) \setminus B(y, r_k/2)$. That is, they are of the type considered above. Hence,

$$\mu(\mathsf{K}) \gtrsim \mu(B_k).$$

By (12.9) and (12.8) we have $\mathsf{K} \subset K(x, \theta, \alpha, 2\,r_k)$, and so the proof is finished. □

Lemma 12.4. *For $k \geq 2$*

$$\frac{\mu(B_k)}{2\,r_k} \gtrsim \frac{1}{k}.$$

Proof. By the definition of μ (12.1), r_k, and σ_k we have

$$\frac{\mu(B_k)}{2\,r_k} = m(k)^{-1}\frac{m(k)}{\sigma_1 \dots \sigma_k} = \frac{1}{\sigma_1 \dots \sigma_k} = \left(\frac{1}{2}\right)^{\beta_1} \cdots \left(\frac{k}{k+1}\right)^{\beta_k} \geq \frac{1}{2} \cdots \frac{k}{k+1} = \frac{1}{k+1},$$

where in the last inequality we used the fact that $1/2 < \beta_k < 1$. □

We are ready to finish the proof of the estimate (12.3).

Proof of (12.3). Observe that if $k > N_0$ is a good index, then by Lemma 12.3 and Lemma 12.4 for $r \in (2\,r_k, 4\,r_k)$

$$\frac{\mu(K(x, \theta, \alpha, r))}{r} \gtrsim \frac{1}{k},$$

and so

$$\int_{2\,r_k}^{4\,r_k} \frac{\mu(K(x, \theta, \alpha, r))}{r}\frac{dr}{r} \gtrsim \frac{1}{k}. \tag{12.10}$$

Recall that $r_{k+1} \leq k^{-1}\,r_k$ by (12.6). Hence,

$$\begin{aligned}
\int_0^1 \frac{\mu(K(x, \theta, \alpha, r))}{r}\frac{dr}{r} &\geq \sum_{k \geq 2^{N_0}} \int_{2\,r_k}^{4\,r_k} \frac{\mu(K(x, \theta, \alpha, r))}{r}\frac{dr}{r} \\
&\geq \sum_{N=N_0}^{\infty} \sum_{\substack{2^N \leq k < 2^{N+1} \\ k \text{ is good}}} \int_{2\,r_k}^{4\,r_k} \frac{\mu(K(x, \theta, \alpha, r))}{r}\frac{dr}{r} \overset{(12.10)}{\gtrsim} \sum_{N=N_0}^{\infty} \sum_{\substack{2^N \leq k < 2^{N+1} \\ k \text{ is good}}} \frac{1}{k} \\
&\approx \sum_{N=N_0}^{\infty} \sum_{\substack{2^N \leq k < 2^{N+1} \\ k \text{ is good}}} 2^{-N} \overset{\textit{Lemma } 12.2}{\gtrsim} \sum_{N=N_0}^{\infty} 2^{-N} 2^N \alpha = \infty.
\end{aligned}$$

□

An α number characterization of L^p spaces on uniformly rectifiable sets VI

1 Introduction

Recall that in Section I.6.2 we defined α numbers, the flatness quantifying coefficients introduced in [Tol09]. In this chapter we will use a slightly modified definition compared to that of Definition I.6.7. For a (possibly real-valued) measure μ and $n \in \mathbb{N}$, if $B = B(x,r)$ we define

$$\alpha_\mu^n(x,r) = \alpha_\mu^n(B) := \frac{1}{r^{n+1}} \inf_{c\in\mathbb{R},L} F_B(\mu,\, c\mathcal{H}^n|_L), \tag{1.1}$$

where the infimum is taken over all $c \in \mathbb{R}$ and all n-planes L that intersect B. We will often omit the superscript n, as it will be fixed throughout.

Remark 1.1. Note that in this chapter we normalize α numbers by r^{-n}. Furthermore, since we will be working with real-valued measures, the infimum above is taken over $c \in \mathbb{R}$ (if μ is a positive measure then it does not make any difference).

For an extended discussion of α numbers, see Section I.6.2. Recall that α numbers can be used to characterize uniformly rectifiable measures by the following result of Tolsa.

Theorem 1.2 ([Tol09, Theorem 1.2])**.** *An n-ADR measure σ is UR if and only if the measure $\alpha_\sigma(x,r)^2 d\sigma(x)\frac{dr}{r}$ is a Carleson measure, meaning that for all balls B centered on* $\operatorname{supp}\sigma$ *with* $0 < r(B) < \operatorname{diam}(\operatorname{supp}\sigma)$,

$$\int_0^{r_B}\int_B \alpha_\sigma(x,r)^2\, d\sigma(x)\frac{dr}{r} \le C_0\,\sigma(B)$$

for some fixed $C_0 > 0$.

The purpose of this chapter is to extend Tolsa's result to measures that are not AD-regular, but are given by L^p functions defined on UR sets.

Given a Radon measure σ, $f \in L^1_{loc}(\sigma)$, and a ball $B = B(x,r)$ with $\sigma(B(x,r)) > 0$ set

$$f_B = f_{x,r} = \frac{\int_B f \, d\sigma}{\sigma(B)}.$$

Theorem 1.3. *Let σ be a UR measure and $f \in L^p(\sigma)$ where $1 < p < \infty$. Then*

$$\|f\|_{L^p(\sigma)} \approx \left\| \left(\int_0^\infty \left(\alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r) \right)^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)}, \tag{1.2}$$

with the implicit constant depending on p and σ.

Sharpness of the result

An interesting aspect of our result is the presence of two terms that comprise our square function. We don't know whether the result holds for general UR sets without the second term. Neither of the terms bounds the other in the pointwise sense: one could be zero while the other is nonzero. On the other hand, we don't know whether the norm of the square function involving only $\alpha_{f\sigma}$ dominates the one involving only $|f|_{x,r}\alpha_\sigma$. The reverse inequality is certainly not true, as the latter square function vanishes if σ is the Lebesgue measure on $\Sigma = \mathbb{R}^n$.

Question. Let σ be a UR measure and $f \in L^p(\sigma)$ where $1 < p < \infty$. Do we have

$$\|f\|_{L^p(\sigma)} \lesssim \left\| \left(\int_0^\infty \alpha_{f\sigma}(x,r)^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} ? \tag{1.3}$$

Equivalently, is it true that

$$\left\| \left(\int_0^\infty (|f|_{x,r}\alpha_\sigma(x,r))^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} \lesssim \left\| \left(\int_0^\infty \alpha_{f\sigma}(x,r)^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} ?$$

The answer to the question above is obviously affirmative in the flat case, i.e. $\sigma = \mathcal{H}^n|_L$ for L a n-dimensional plane. It is also positive if σ is an AD-regular measure on a n-dimensional plane L, i.e. $\sigma = g\mathcal{H}^n|_L$ for some function g satisfying $A^{-1} \leq g(x) \leq A$. Indeed, let $\tilde{\sigma} = \mathcal{H}^n|_L$, so that $f\sigma = fg\tilde{\sigma}$. In that case, by Theorem 1.3

$$\|f\|_{L^p(\sigma)} \approx_A \|fg\|_{L^p(\tilde\sigma)} \approx_p \left\| \left(\int_0^\infty \alpha_{fg\tilde\sigma}(x,r)^2 \, \frac{dr}{r} \right)^{\frac12} \right\|_{L^p(\tilde\sigma)}$$
$$\approx_A \left\| \left(\int_0^\infty \alpha_{f\sigma}(x,r)^2 \, \frac{dr}{r} \right)^{\frac12} \right\|_{L^p(\sigma)} .$$

Finally, one could show that (1.3) is true for "sufficiently flat" UR measures σ. What we mean by this is that if the constant C_0 from Theorem 1.2 is sufficiently small, then some variant of Carleson's embedding theorem can be used* to show that

$$\left\| \left(\int_0^\infty \left(|f|_{x,r} \alpha_\sigma(x,r) \right)^2 \, \frac{dr}{r} \right)^{\frac12} \right\|_{L^p(\sigma)} \lesssim_p C_0 \|f\|_{L^p(\sigma)} .$$

Together with (1.2) this gives

$$\|f\|_{L^p(\sigma)} \le C(p,\sigma) \left\| \left(\int_0^\infty \alpha_{f\sigma}(x,r)^2 \, \frac{dr}{r} \right)^{\frac12} \right\|_{L^p(\sigma)} + C(p,\sigma) C_0 \|f\|_{L^p(\sigma)} .$$

Assuming the Carleson constant C_0 to be small enough, we get that the last term can be absorbed by the left hand side. To make this more rigorous, one should perhaps track the dependence of $C(p,\sigma)$ (the implicit constant from (1.2)) on the UR constants of σ with more diligence than we did. However, the implicit constants can only get *better* as σ becomes flatter, and they certainly cannot blow-up as the Carleson constant C_0 goes to 0: if σ satisfies the Carleson condition of Theorem 1.2 with some C_0, then it also satisfies it with constant C_0' for every $C_0' \ge C_0$.

Organization of the chapter

In Section 2 we introduce the necessary tools and make some initial reductions. We define also Jf, a dyadic variant of the square function from Theorem 1.3, see (2.4).

We show that $\|Jf\|_2 \lesssim \|f\|_2$ in Section 3. The proof uses martingale difference operators, and it is inspired by how Theorem 1.2 was originally proved, see [Tol09, Section 4]. In Section 4 we use the estimate $\|Jf\|_2 \lesssim \|f\|_2$

*For $p = 2$ use e.g. [Tol14, Theorem 5.8], for $p \neq 2$ one can show a corresponding statement by proving an appropriate good-lambda inequality, in the spirit of what we do in Section 4 (but simpler).

and an appropriate good-lambda inequality to conclude that $\|Jf\|_p \lesssim \|f\|_p$ for general $1 < p < \infty$.

Finally, in Section 5 we prove $\|f\|_p \lesssim \|Jf\|_p$. To do that we use the Littlewood-Paley theory of David, Journé and Semmes [DJS85].

2 Preliminaries

2.1 Notation

In our estimates we will write $f \lesssim g$ to denote $f \le Cg$ for some constant C (the so-called "implicit constant"). If the implicit constant depends on a parameter t, i.e. $C = C(t)$, we will write $f \lesssim_t g$. The notation $f \approx g$ and $f \approx_t g$ stands for $g \lesssim f \lesssim g$ and $g \lesssim_t f \lesssim_t f$, respectively. To make the notation lighter, we will usually not track the dependence of C on dimensions n, d, on the ADR constant of σ, or the parameter $1 < p < \infty$.

For simplicity, we will sometimes write

$$\|f\|_p := \|f\|_{L^p(\sigma)}.$$

Recall that we introduced the notation f_B to signify the average of f over a ball B with respect to σ. For general Borel sets $E \subset \mathbb{R}^d$ with $\sigma(E) > 0$ and $f \in L^1_{loc}(\sigma)$ we will write

$$\langle f \rangle_E = \frac{\int_E f \, d\sigma}{\sigma(E)}.$$

If $v, w \in \mathbb{R}^d$, then $v \cdot w$ denotes their scalar product.

2.2 Adjacent systems of cubes

As usual, we will work with a family of subsets of $\operatorname{supp}\sigma =: \Sigma$ that in many ways resemble the family of dyadic cubes on $\mathbb{R}^n$. For this reason we will call these sets "cubes". Many different systems of cubes have been constructed throughout the years, beginning with the work of David [Dav88a] and Christ [Chr90]. In our proof it will be convenient to use adjacent systems of cubes constructed by Hytönen and Tapiola [HT14]. One should think of them as a generalization of the translated dyadic grids in $\mathbb{R}^n$, widely used to perform the "1/3 trick".

First, we will say that a family $\mathscr{D}$ of Borel subsets of Σ satisfies *the usual properties of David-Christ cubes* if $\mathscr{D} = \bigcup_{k\in\mathbb{Z}} \mathscr{D}_k$, and for each $k \in \mathbb{Z}$:

(a) for $P,\ Q \in \mathscr{D}_k,\ P \neq Q$, we have $\sigma(P \cap Q) = \varnothing$,

(b) the sets in $\mathscr{D}_k$ cover Σ:

$$\Sigma = \bigcup_{Q\in\mathscr{D}_k} Q,$$

(c) for each $Q \in \mathscr{D}_k$ and each $l \geq k$

$$Q = \bigcup_{P \in \mathscr{D}_l : P \subset Q} P,$$

(d) there exists $0 < \delta < 1$ (independent of k) such that each $Q \in \mathscr{D}_k$ has a center $z(Q) \in Q$ satisfying

$$B\Big(z(Q), \frac{\delta^k}{5}\Big) \cap \Sigma \subset Q \subset B(z(Q), 3\delta^k) \cap \Sigma. \tag{2.1}$$

Consequently, as long as $\delta^k \lesssim \operatorname{diam}(\Sigma)$, we have $\sigma(Q) \approx \delta^{kn}$. Set $\ell(Q) := \delta^k$.

(e) the cubes $Q \in \mathscr{D}_k$ have thin boundaries, that is, there exists $\gamma \in (0,1)$ such that for $\eta \in (0, 0.1)$ we have

$$\sigma(\{x \in \Sigma \ : \ \operatorname{dist}(x, Q) + \operatorname{dist}(x, \Sigma \setminus Q) < \eta\ell(Q)\}) \leq \eta^\gamma \sigma(Q). \tag{2.2}$$

Remark 2.1. Note that in the above we assume $\mathscr{D}_k$ to be defined for all $k \in \mathbb{Z}$. In the case of unbounded Σ, this translates to having arbitrarily large cubes as $k \to -\infty$. In the case of compact Σ, there exists some k_0 such that for all $k \leq k_0$ we have $\mathscr{D}_k = \{\Sigma\}$. However, in our proof we will assume that Σ is unbounded, see Lemma 2.5.

In our setting, the results [HT14, Theorem 2.9, Theorem 5.9] can be summarized as follows.

Lemma 2.2. *Let σ be a n-AD regular measure on $\mathbb{R}^d$. Then, there exist $1 \leq N < \infty$ and a small constant $0 < \delta < 0.01$, depending only on the ADR constant of σ, such that the following holds. Let $\Omega = \{1, \ldots, N\}$. For each $\omega \in \Omega$ we have a system of cubes $\mathscr{D}(\omega)$ satisfying the usual properties of David-Christ cubes, and additionally, for all $x \in \Sigma$ and $0 < r < \operatorname{diam}(\Sigma)$ there are $\omega \in \Omega$, $k \in \mathbb{Z}$ and $Q \in \mathscr{D}_k(\omega)$ with*

$$B(x, r) \cap \Sigma \subset Q$$

and

$$\ell(Q) = \delta^k \approx_\delta r.$$

Remark 2.3. The construction in [HT14] is valid for general (geometrically) doubling metric spaces, possibly with no underlying measure space structure. The constants N and δ from Lemma 2.2 depend on the doubling constant of the metric space. Hytönen and Tapiola construct two different kinds of cubes, which they call open and closed cubes, see [HT14, Theorem 2.9]. In the above we consider closed cubes, so that properties (b), (c) and (d) follow immediately from [HT14, Theorem 2.9]. To get the property (a) one uses the fact that interiors of P and Q are disjoint by [HT14, (2.11)], and then $\sigma(\partial P) = \sigma(\partial Q) = 0$ follows from (e). To prove the thin boundaries property (e) one may adapt the proof of Christ [Chr90, pp. 610–612] together with AD-regularity of σ. We omit the details.

From now on, let us fix a uniformly rectifiable measure σ, with $\Sigma = \operatorname{supp}\sigma$. Let Ω, δ and $\mathscr{D}(\omega)$ be as in Lemma 2.2. For simplicity, in our estimates we will not track the dependence of implicit constants on δ.

For all $\omega \in \Omega$ and $Q \in \mathscr{D}_k(\omega)$ we will write

$$\mathscr{D}(Q) := \{P \in \mathscr{D}(\omega) \ : \ P \subset Q\},$$
$$\mathsf{Ch}(Q) := \mathscr{D}(Q) \cap \mathscr{D}_{k+1}(\omega).$$

The elements of $\mathsf{Ch}(Q)$ will be called children of Q, and Q will be called their parent.

Set

$$B_Q := B(z(Q), 4\ell(Q)),$$

so that $Q \subset B_Q \cap \Sigma$, and whenever $P \in \mathscr{D}(Q)$ we also have $B_P \subset B_Q$.

Fix some $\omega_0 \in \Omega$, and set

$$\mathscr{D} := \mathscr{D}(\omega_0).$$

This will be our system of reference. Given $Q \in \mathscr{D}$ we define $\omega(Q) \in \Omega$ to be the index such that there exists $R(Q) \in \mathscr{D}(\omega(Q))$ satisfying $B_Q \cap \operatorname{supp}\sigma \subset R(Q)$ and $\ell(R(Q)) \approx \ell(Q)$. If there is more than one such ω, we simply choose one. We define also $\mathcal{G}(\omega) \subset \mathscr{D}$ as the family of cubes $Q \in \mathscr{D}$ such that $\omega(Q) = \omega$. Clearly,

$$\bigcup_{\omega\in\Omega} \mathcal{G}(\omega) = \mathscr{D}.$$

2.3 α-numbers

In proving the main theorem, it will be more convenient to work with dyadic versions of the α-numbers. Below we will introduce the notation needed for this framework. Given a Radon measure μ we denote by $L^\mu_{x,r}$ a minimizing n-plane for $\alpha_\mu(x,r)$, and by $c^\mu_{x,r}$ the corresponding constant. They may be non-unique, in which case we just choose one of the minimizers. Set $\mathcal{P}^\mu_{x,r} = \mathcal{H}^n|_{L^\mu_{x,r}}$ and $\mathcal{L}^\mu_{x,r} = c^\mu_{x,r}\mathcal{P}^\mu_{x,r}$. If $B = B(x,r)$ we will also write L^μ_B, c^μ_B etc.

For $Q \in \mathscr{D}$ and a Radon measure μ we set

$$\alpha_\mu(Q) := \alpha_\mu(B_Q).$$

We will write $L^\mu_Q := L^\mu_{B_Q}$, $c^\mu_Q := c^\mu_{B_Q}$ etc.

Observe that whenever $B_1 \subset B_2$ are balls, we have $\mathrm{Lip}_1(B_1) \subset \mathrm{Lip}_1(B_2)$, and so if $r(B_1) \geq Cr(B_2)$, then

$$\alpha_\mu(B_1) = \frac{1}{r(B_1)^{n+1}} \inf_{c\in\mathbb{R},L} F_{B_1}(\mu, c\mathcal{H}^n|_L) \\ \leq \frac{1}{r(B_1)^{n+1}} F_{B_1}(\mu, \mathcal{L}^\mu_{B_2}) \leq \frac{1}{r(B_1)^{n+1}} F_{B_2}(\mu, \mathcal{L}^\mu_{B_2}) \approx_C \alpha_\mu(B_2). \quad (2.3)$$

Consider the following square function:

$$J(x) = \left(\sum_{x\in Q\in\mathscr{D}} \alpha_{f\sigma}(Q)^2 + |f|^2_{B_Q} \alpha_\sigma(Q)^2 \right)^{1/2}. \quad (2.4)$$

Theorem 1.3 will follow from the following dyadic version:

Theorem 2.4. *Let σ be a uniformly rectifiable measure with unbounded support, and let $f \in L^p(\sigma)$ for some $1 < p < \infty$. Then*

$$\|Jf\|_{L^p(\sigma)} \approx \|f\|_{L^p(\sigma)}.$$

First, let us show why we may assume that $\operatorname{supp}\sigma$ is unbounded.

Lemma 2.5. *It suffices to only prove Theorem 1.3 in the case that* $\operatorname{supp}\sigma$ *is unbounded.*

Proof. Suppose σ did have compact support. Without loss of generality, we may assume $\operatorname{diam}(\operatorname{supp}\sigma) = 1$, $\operatorname{supp}\sigma \subseteq \mathbb{B} = B(0,1)$, and $L^\sigma_{\mathbb{B}} = \mathbb{R}^n$. Let

$$\mu = \sigma + \mathcal{P}^\sigma_{\mathbb{B}}|_{\mathbb{R}^n\setminus 4\mathbb{B}}.$$

It is not hard to show that μ is also UR. If Theorem 1.3 holds for UR measures of unbounded support, then it holds for μ. Let $f \in L^p(\sigma) \subseteq L^p(\mu)$ and let

$$\theta^f_\sigma(x,r) := \alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r),$$

so that, by the Theorem 1.3,

$$\left\| \left(\int_0^\infty \theta^f_\mu(x,r)^2 \, \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\mu)} \approx \|f\|_{L^p(\mu)} = \|f\|_{L^p(\sigma)}. \quad (2.5)$$

Observe that

$$\theta^f_\sigma(x,r) = \theta^f_\mu(x,r) \text{ for } x \in \operatorname{supp}\sigma \text{ and } 0 < r < 2.$$

Thus,

$$\left\|\left(\int_0^2 \theta_\sigma^f(x,r)^2 \,\frac{dr}{r}\right)^{1/2}\right\|_{L^p(\sigma)} = \left\|\left(\int_0^2 \theta_\mu^f(x,r)^2 \,\frac{dr}{r}\right)^{1/2}\right\|_{L^p(\sigma)}$$
$$\leq \left\|\left(\int_0^\infty \theta_\mu^f(x,r)^2 \,\frac{dr}{r}\right)^{1/2}\right\|_{L^p(\mu)} \overset{(2.5)}{\lesssim} \|f\|_{L^p(\sigma)}.$$

Furthermore, we claim that for any $x \in \operatorname{supp}\sigma$ and $r > 2$ we have

$$\theta_\sigma^f(x,r) \lesssim r^{-n}|f|_{\mathbb{B}}. \tag{2.6}$$

Indeed, since $\operatorname{supp} f \subset \operatorname{supp}\sigma \subset \mathbb{B}$,

$$\alpha_{f\sigma}(x,r) \leq \frac{1}{r^{n+1}} F_{B(x,r)}(f\sigma, 0) \leq \frac{1}{r^n}\int_{\mathbb{B}} |f|\; d\sigma \approx \frac{1}{r^n}|f|_{\mathbb{B}}, \tag{2.7}$$

and also

$$|f|_{x,r}\alpha_\sigma(x,r) \lesssim \frac{1}{r^n}\int_{\mathbb{B}} |f|\; d\sigma \approx \frac{1}{r^n}|f|_{\mathbb{B}}.$$

It follows from (2.6) that

$$\int_2^\infty \theta_\sigma^f(x,r)^2 \frac{dr}{r} \lesssim \int_2^\infty |f|_{\mathbb{B}}^2 \frac{dr}{r^{2n+1}} \lesssim |f|_{\mathbb{B}}^2,$$

and so

$$\left\|\left(\int_0^\infty \theta_\sigma^f(x,r)^2 \,\frac{dr}{r}\right)^{1/2}\right\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\sigma)} + \int_{\mathbb{B}} |f| d\sigma \lesssim \|f\|_{L^p(\sigma)}.$$

To finish the proof we now need to show the reverse inequality. Notice that since f is supported on $\operatorname{supp}\sigma$, $\alpha_{f\mu}(x,r) = \alpha_{f\sigma}(x,r)$ for all $x \in \operatorname{supp}\mu$ and $r > 0$. We can argue just as in (2.7) to get that for $x \in \operatorname{supp}\mu$ and $r \geq 2$,

$$\alpha_{f\mu}(x,r) = \alpha_{f\sigma}(x,r) \leq \frac{1}{r^{n+1}} F_{B(x,r)}(f\sigma, 0) \lesssim \frac{|f|_{\mathbb{B}}}{r^n} \lesssim \frac{|f|_{2\mathbb{B}}}{r^n}\alpha_\sigma(2\mathbb{B}),$$

where we also used $\alpha_\sigma(2\mathbb{B}) \approx 1$.

Hence,

$$\begin{aligned}
&\int_{\mathbb{B}}\left(\int_0^\infty \alpha_{f\mu}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}} d\mu(x)\\
&\quad\lesssim \int_{\mathbb{B}}\left(\int_0^2 \alpha_{f\mu}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}} d\mu(x) + \int_{\mathbb{B}}\left(\int_2^\infty \alpha_{f\mu}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}} d\mu(x)\\
&\quad\lesssim \int_{\mathbb{B}}\left(\int_0^2 \alpha_{f\sigma}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}} d\mu(x) + \int_{\mathbb{B}}\left(\int_2^\infty (|f|_{2\mathbb{B}}\alpha_\sigma(2\mathbb{B}))^2\frac{dr}{r^{2n+1}}\right)^{\frac{p}{2}} d\mu(x)\\
&\quad\lesssim \int_{\mathbb{B}}\left(\int_0^2 \alpha_{f\sigma}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}} d\mu(x) + (|f|_{2\mathbb{B}}\alpha_\sigma(2\mathbb{B}))^p\\
&\quad\lesssim \int_{\mathbb{B}}\left(\int_0^\infty \theta_\sigma^f(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}} d\mu(x)
\end{aligned}$$

where we used (2.3) in the final inequality.

Furthermore, for $x\in\mathbb{R}^n\backslash 4\mathbb{B}$, if $\alpha_{f\mu}(x,r)\neq 0$, then $r\geq |x|/2$ and so

$$\begin{aligned}
&\int_{\mathbb{R}^n\backslash 4\mathbb{B}}\left(\int_0^\infty \alpha_{f\mu}(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}} d\mu(x)\\
&\quad= \sum_{j=2}^\infty \int_{\mathbb{R}^n\cap(2^{j+1}\mathbb{B}\backslash 2^j\mathbb{B})}\left(\int_{|x|/2}^\infty (|f|_{2\mathbb{B}}\alpha_\sigma(2\mathbb{B}))^2\frac{dr}{r^{2n+1}}\right)^{\frac{p}{2}} d\mu(x)\\
&\quad\lesssim (|f|_{2\mathbb{B}}\alpha_\sigma(2\mathbb{B}))^p\sum_{j=2}^\infty \int_{\mathbb{R}^n\cap(2^{j+1}\mathbb{B}\backslash 2^j\mathbb{B})}|x|^{-pd}d\mu(x)\\
&\quad\lesssim (|f|_{2\mathbb{B}}\alpha_\sigma(2\mathbb{B}))^p \lesssim \int_{\mathbb{B}}\left(\int_0^\infty |f|_{x,r}\alpha_\sigma(x,r)^2\frac{dr}{r}\right)^{\frac{p}{2}} d\mu(x)
\end{aligned}$$

again using (2.3). These two estimates imply

$$\left\|\left(\int_0^\infty \alpha_{f\mu}(x,r)^2\frac{dr}{r}\right)^{1/2}\right\|_{L^p(\mu)} \lesssim \left\|\left(\int_0^\infty \theta_\sigma^f(x,r)^2\frac{dr}{r}\right)^{1/2}\right\|_{L^p(\sigma)}. \tag{2.8}$$

Note that for $x\in\operatorname{supp}\sigma$ and $r<2$, we have $\mathcal{P}^\sigma_{x,r}=\mathcal{P}^\mu_{x,r}$. For $r\geq 2$, notice that $\alpha_\sigma(2\mathbb{B})\approx 1$, and so

$$\alpha_\mu(x,r)\leq \frac{1}{r^{n+1}}F_{B(x,r)}(\mu,\mathcal{P}^\sigma_{\mathbb{B}}) = \frac{1}{r^{n+1}}F_{B(x,r)}(\sigma,\mathcal{P}^\sigma_{\mathbb{B}}\llcorner 4B)\lesssim r^{-n}\lesssim \frac{\alpha_\sigma(2\mathbb{B})}{r^n},$$

hence

$$|f|^\mu_{x,r}\alpha_\mu(x,r)\leq |f|_{2\mathbb{B}}\frac{\alpha_\sigma(2\mathbb{B})}{r^{2n}},$$

where $|f|^\mu_{x,r}=\int_{B(x,r)} f\ d\mu/\mu(B(x,r))$. Thus, just as how we proved (2.8), we can show

$$\left\|\int_0^\infty (|f|^\mu_{x,r})^2\alpha_\mu(x,r)^2\frac{dr}{r}\right\|_{L^p(\mu)} \lesssim \left\|\int_0^\infty |f|^2_{x,r}\alpha_\sigma(x,r)^2\frac{dr}{r}\right\|_{L^p(\sigma)}.$$

This, (2.8) and (2.5) imply the desired estimate:

$$\|f\|_{L^p(\sigma)} \lesssim \left\| \left(\int_0^\infty \theta_\sigma^f(x,r)^2 \frac{dr}{r} \right)^{1/2} \right\|_{L^p(\sigma)}.$$

□

Proof of the Theorem 1.3 using Theorem 2.4. By Lemma 2.5, we may assume that $\operatorname{supp}\sigma = \Sigma$ is unbounded, so that Theorem 2.4 holds.

Let $x \in \Sigma$, $r > 0$. Let $k \in \mathbb{Z}$ be such that $\delta^{k+1} < r \le \delta^k$, and let Q be a cube in $\mathscr{D}_k$ containing x. Recall that $Q \subset B(z(Q), 3\ell(Q))$. Since $r \le \ell(Q)$, we have

$$B(x,r) \subset B(z(Q), 3\ell(Q) + r) \subset B(z(Q), 4\ell(Q)) = B_Q.$$

Hence, by (2.3),

$$\alpha_{f\sigma}(x,r) \lesssim \alpha_{f\sigma}(Q).$$

We also have $|f|_{x,r} \lesssim |f|_{B_Q}$, and so

$$|f|_{x,r}\alpha_\sigma(x,r) \lesssim |f|_{B_Q}\alpha_\sigma(Q).$$

Consequently,

$$\int_{\delta^{k+1}}^{\delta^k} (\alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r))^2 \, \frac{dr}{r} \lesssim \alpha_{f\sigma}(Q)^2 + |f|_{B_Q}^2\alpha_\sigma(Q))^2.$$

Summing over $k \in \mathbb{Z}$ yields

$$\int_0^\infty (\alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r))^2 \, \frac{dr}{r} \lesssim \sum_{x\in Q\in\mathscr{D}} \alpha_{f\sigma}(Q)^2 + |f|_{B_Q}^2\alpha_\sigma(Q))^2.$$

Similarly, for $x \in \Sigma$, $r > 0, \delta^{k+1} < r \le \delta^k$, we may consider a cube $Q \in \mathscr{D}_{k+2}$ such that $x \in Q \subset B_Q \subset B(x,r)$. Mimicking the estimates above, one gets

$$\sum_{x\in Q\in\mathscr{D}} \alpha_{f\sigma}(Q)^2 + |f|_{B_Q}^2\alpha_\sigma(Q))^2 \lesssim \int_0^\infty (\alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r))^2 \, \frac{dr}{r}.$$

Putting the two estimates together, we get the comparability of the dyadic and continuous variants of the square function:

$$Jf(x)^2 = \sum_{x\in Q\in\mathscr{D}} \alpha_{f\sigma}(Q)^2 + |f|_{B_Q}^2\alpha_\sigma(Q))^2 \approx \int_0^\infty (\alpha_{f\sigma}(x,r) + |f|_{x,r}\alpha_\sigma(x,r))^2 \, \frac{dr}{r}.$$

□

Theorem 2.4 will follow from the results from the next three sections. From now on we assume that σ is a uniformly rectifiable measure with unbounded support.

3 $\|Jf\|_2 \lesssim \|f\|_2$

First, we prove the estimate $\|Jf\|_p \lesssim \|f\|_p$ in the case $p = 2$.

Proposition 3.1. *Let $f \in L^2(\sigma)$. Then*

$$\sum_{Q\in\mathscr{D}} (\alpha_{f\sigma}(Q)^2 + |f|^2_{B_Q}\alpha_\sigma(Q)^2)\ell(Q)^n \lesssim \|f\|^2_{L^2(\sigma)}.$$

Our main tool in the proof of Proposition 3.1 are the martingale difference operators associated to systems of cubes $\mathscr{D}(\omega)$.

Given $\omega \in \Omega$, $Q \in \mathscr{D}(\omega)$, and $f \in L^1_{loc}(\sigma)$ we set

$$\Delta_Q f = \sum_{P\in \mathsf{Ch}(Q)} \langle f\rangle_P \mathbb{1}_P - \langle f\rangle_Q \mathbb{1}_Q.$$

Observe that all $\Delta_Q f$ have zero mean, i.e. $\int \Delta_Q f \, d\sigma = 0$.

It is well known (see e.g. [Gra14a, Chapter 6.4]) that given $f \in L^2(\sigma)$ and some system of cubes $\mathscr{D}(\omega)$ we have

$$f = \sum_{Q\in\mathscr{D}(\omega)} \Delta_Q f$$

with the convergence understood in the L^2 sense. It is crucial that $\sigma(\Sigma) = \infty$, so that $f + C \in L^2(\sigma)$ if and only if $C = 0$ (in the case $\sigma(\Sigma) < \infty$ one would have to subtract from the left hand side above the average of f).

Note that $\Delta_Q f$ are mutually orthogonal in $L^2(\sigma)$, so that

$$\|f\|^2_{L^2(\sigma)} = \sum_{Q\in\mathscr{D}(\omega)} \left\|\Delta_Q f\right\|^2_{L^2(\sigma)} \tag{3.1}$$

Moreover, if $Q \in \mathscr{D}(\omega)$, then for σ-a.e. $x \in Q$

$$f(x) = \langle f\rangle_Q + \sum_{P\in\mathscr{D}(Q)} \Delta_P f(x). \tag{3.2}$$

Lemma 3.2. *Suppose $Q \in \mathscr{D}$, and let $R = R(Q) \in \mathscr{D}(\omega(Q))$ be as in Section 2.1. Then, for $f \in L^2(\sigma)$ we have*

$$\alpha_{f\sigma}(Q) \lesssim |\langle f\rangle_R|\alpha_\sigma(R) + \sum_{P\in\mathscr{D}(R)} \frac{\ell(P)^{1+n/2}}{\ell(Q)^{1+n}} \|\Delta_P f\|_2. \tag{3.3}$$

Proof. Let $\varphi \in \mathrm{Lip}_1(B_Q)$ and consider a candidate for $\mathcal{L}_Q^{f\sigma}$ of the form $\langle f\rangle_R \mathcal{L}_Q^\sigma$. For all $x \in B_Q \cap \operatorname{supp}\sigma$ we have $x \in R$, so that using (3.2)

$$\begin{aligned}
&\left|\int \varphi(x) f(x)\, d\sigma(x) - \langle f\rangle_R \int \varphi(x)\, d\mathcal{L}_Q^\sigma(x)\right| \\
&\quad= \left|\int \varphi(x)\langle f\rangle_R + \sum_{P\in\mathscr{D}(R)} \varphi(x)\Delta_P f(x)\, d\sigma(x) - \int \varphi(x)\langle f\rangle_R\, d\mathcal{L}_Q^\sigma(x)\right| \\
&\quad\le |\langle f\rangle_R| \left|\int \varphi(x)\, d\sigma(x) - \int \varphi(x)\, d\mathcal{L}_Q^\sigma(x)\right| \\
&\qquad + \sum_{P\in\mathscr{D}(R)} \left|\int \varphi(x)\Delta_P f(x)\, d\sigma(x)\right| =: I_1 + I_2.
\end{aligned}$$

It is clear that

$$I_1 \le |\langle f\rangle_R|\, \alpha_\sigma(Q)\ell(Q)^{n+1} \overset{(2.3)}{\lesssim} |\langle f\rangle_R|\, \alpha_\sigma(R)\ell(Q)^{n+1},$$

which gives rise to the first term on the right hand side of (3.3).

Concerning I_2, we use the zero mean property of martingale difference operators, and the fact that $\varphi \in \mathrm{Lip}_1(B_Q)$, to get

$$\begin{aligned}
I_2 &= \sum_{P\in\mathscr{D}(R)} \left|\int \big(\varphi(x) - \varphi(z(P))\big)\Delta_P f(x)\, d\sigma(x)\right| \\
&\le \sum_{P\in\mathscr{D}(R)} \int |\varphi(x) - \varphi(z(P))||\Delta_P f(x)|\, d\sigma(x) \\
&\lesssim \sum_{P\in\mathscr{D}(R)} \ell(P)\|\Delta_P f\|_1 \overset{\text{Hölder}}{\lesssim} \sum_{P\in\mathscr{D}(R)} \ell(P)^{1+n/2}\|\Delta_P f\|_2\,.
\end{aligned}$$

Dividing by $\ell(Q)^{n+1}$ and taking supremum over $\varphi \in \mathrm{Lip}_1(B_Q)$ yields (3.3). □

Proof of Proposition 3.1. We begin by noting that, since σ is uniformly rectifiable, $\alpha_\sigma(Q)^2\ell(Q)^n$ is a Carleson measure by the results from [Tol09], see Theorem 1.2. Therefore, the estimate

$$\sum_{Q\in\mathscr{D}} |f|_{B_Q}^2 \alpha_\sigma(Q)^2\ell(Q)^n \lesssim \|f\|_{L^2(\sigma)}^2$$

follows immediately from Carleson's embedding theorem, see e.g. [Tol14, Theorem 5.8], and we only need to estimate the sum involving $\alpha_{f\sigma}(Q)$.

Observe that for each $\omega \in \Omega$ and $R \in \mathscr{D}(\omega)$ there is at most a bounded number of cubes $Q \in \mathscr{D}$ such that $R(Q) = R$.

Fix some $\omega \in \Omega$. Recall that $\mathcal{G}(\omega)$ is the family of cubes $Q \in \mathscr{D}$ such that $\omega(Q) = \omega$. We apply (3.3) and the observation above to get

$$\begin{aligned}&\sum_{Q\in\mathcal{G}(\omega)} \alpha_{f\sigma}(Q)^2\ell(Q)^n \\ &\quad\lesssim \sum_{R\in\mathscr{D}(\omega)} |\langle f\rangle_R|^2\alpha_\sigma(R)^2\ell(R)^n + \sum_{R\in\mathscr{D}(\omega)}\Bigg(\sum_{P\in\mathscr{D}(R)} \frac{\ell(P)^{1+n/2}}{\ell(R)^{1+n/2}}\|\Delta_P f\|_2\Bigg)^2 \\ &\quad=: S_1 + S_2.\end{aligned}$$

Concerning S_1, we may use Carleson's embedding theorem again to estimate $S_1 \lesssim \|f\|_2^2$.

Moving on to S_2, we apply the Cauchy-Schwarz inequality to get

$$S_2 \le \sum_{R\in\mathscr{D}(\omega)}\Bigg(\sum_{P\in\mathscr{D}(R)}\frac{\ell(P)}{\ell(R)}\|\Delta_P f\|_2^2\Bigg)\Bigg(\sum_{P\in\mathscr{D}(R)}\frac{\ell(P)^{n+1}}{\ell(R)^{n+1}}\Bigg).$$

It is easy to see that, due to AD-regularity of σ, $\sum_{P\in\mathscr{D}(R)}\frac{\ell(P)^{n+1}}{\ell(R)^{n+1}} \lesssim 1$. Thus,

$$\begin{aligned}S_2 \le \sum_{R\in\mathscr{D}(\omega)}\sum_{P\in\mathscr{D}(R)}\frac{\ell(P)}{\ell(R)}\|\Delta_P f\|_2^2 &= \sum_{P\in\mathscr{D}(\omega)}\|\Delta_P f\|_2^2 \sum_{\substack{R\in\mathscr{D}(\omega)\\ R\supset P}}\frac{\ell(P)}{\ell(R)} \\ &\lesssim \sum_{P\in\mathscr{D}(\omega)}\|\Delta_P f\|_2^2 \overset{(3.1)}{\lesssim} \|f\|_2^2.\end{aligned}$$

Putting the estimates above together we arrive at

$$\sum_{Q\in\mathcal{G}(\omega)}\alpha_{f\sigma}(Q)^2\ell(Q)^n \lesssim \|f\|_2^2.$$

Summing over all $\omega \in \Omega$ (recall that $\#\Omega$ is bounded) we get the desired estimate. □

4 $\|Jf\|_p \lesssim \|f\|_p$ for $1 < p < \infty$

In this section we use the estimate $||Jf||_2 \lesssim \|f\|_2$ to prove $||Jf||_p \lesssim \|f\|_p$ for general $1 < p < \infty$. More precisely, we will show a localized version of the estimate, which implies the global estimate via a limiting argument.

Fix an arbitrary $Q_0 \in \mathscr{D}$ and set

$$J_0 f(x) := \Bigg(\sum_{x\in Q\in\mathscr{D}(Q_0)}\alpha_{f\sigma}(Q)^2 + |f|_{B_Q}^2\alpha_\sigma(Q)^2\Bigg)^{1/2}.$$

Proposition 4.1. *Let $1 < p < \infty$ and $f \in L^p(\sigma)$. Then,*

$$\|J_0 f\|_{L^p(Q_0)} \lesssim_p \|f\|_{L^p(B_{Q_0})}.$$

The proposition follows easily from a good-lambda inequality stated below. Let M denote the non-centered maximal Hardy-Littlewood operator with respect to σ, i.e.

$$Mf(x) = \sup\{|f|_B \ : \ x \in B,\ B \text{ is a ball}\}.$$

Since σ is AD-regular, the operator M is bounded on $L^p(\sigma)$ for $p > 1$, see e.g. [Tol14, Theorem 2.6, Remark 2.7].

Lemma 4.2. *Let $f \in L^1_{loc}(\sigma)$. For any $\alpha > 1$ there exists $\varepsilon = \varepsilon(\alpha) > 0$ such that for all $\lambda > 0$*

$$\sigma(\{x \in Q_0 \ : \ J_0 f(x) > \alpha\lambda,\ Mf(x) \le \varepsilon\lambda\}) \le \frac{9}{10}\sigma(\{x \in Q_0 \ : \ J_0 f(x) > \lambda\}). \tag{4.1}$$

Let us show how to use the above to prove Proposition 4.1.

Proof of Proposition 4.1. Note that $J_0 f = J_0(f\mathbb{1}_{B_{Q_0}})$, so without loss of generality we may assume that $\operatorname{supp} f \subset B_{Q_0}$. Let $\alpha = \alpha(p) > 1$ be so close to 1 that $0.9\alpha^p < 0.95$, and let $\varepsilon = \varepsilon(\alpha)$ be as in Lemma 4.2. We use the layer cake representation to get

$$\begin{aligned}
\int_{Q_0} J_0 f(x)^p \, d\sigma(x) &= p\int_0^\infty \lambda^{p-1}\sigma(\{x \in Q_0 \ : \ J_0 f(x) > \lambda\})\, d\lambda \\
&= p\alpha^p \int_0^\infty \lambda^{p-1}\sigma(\{x \in Q_0 \ : \ J_0 f(x) > \alpha\lambda\})\, d\lambda \\
&\le p\alpha^p \int_0^\infty \lambda^{p-1}\sigma(\{x \in Q_0 \ : \ J_0 f(x) > \alpha\lambda,\ Mf(x) \le \varepsilon\lambda\})\, d\lambda \\
&\quad + p\alpha^p \int_0^\infty \lambda^{p-1}\sigma(\{x \in Q_0 \ : \ Mf(x) > \varepsilon\lambda\})\, d\lambda \\
&\overset{(4.1)}{\le} \frac{9}{10}p\alpha^p \int_0^\infty \lambda^{p-1}\sigma(\{x \in Q_0 \ : \ J_0 f(x) > \lambda\})\, d\lambda + \alpha^p\varepsilon^{-p}\int_{Q_0} Mf(x)^p \, d\sigma(x) \\
&\le \frac{19}{20}p\int_0^\infty \lambda^{p-1}\sigma(\{x \in Q_0 \ : \ J_0 f(x) > \lambda\})\, d\lambda + \alpha^p\varepsilon^{-p}\int_{Q_0} Mf(x)^p \, d\sigma(x) \\
&= \frac{19}{20}\int_{Q_0} J_0 f(x)^p \, d\sigma(x) + \alpha^p\varepsilon^{-p}\int_{Q_0} Mf(x)^p \, d\sigma(x).
\end{aligned}$$

Absorbing the first term from the right hand side into the left hand side, we arrive at

$$\int_{Q_0} J_0 f(x)^p \, d\sigma(x) \le 20\alpha^p\varepsilon^{-p}\int Mf(x)^p \, d\sigma(x).$$

We use the L^p boundedness of M and the assumption $\operatorname{supp} f \subset B_{Q_0}$ to conclude

$$\int_{Q_0} J_0 f(x)^p \, d\sigma(x) \lesssim_{\alpha,\varepsilon} \int_{B_{Q_0}} f(x)^p \, d\sigma(x).$$

□

The remainder of this section is dedicated to proving Lemma 4.2.

4.1 Preliminaries

Fix $\alpha > 1$ and $\lambda > 0$. First, we set

$$E_\lambda = \{x \in Q_0 \ : \ J_0 f(x) > \lambda\}.$$

Consider the covering of E_λ with a family of cubes $\mathscr{C}_\lambda \subset \mathscr{D}(Q_0)$ such that for every $S \in \mathscr{C}_\lambda$ we have

$$\sigma(S \cap E_\lambda) \geq 0.99\sigma(S)$$

and S is the maximal cube with this property. Since the cubes from $\mathscr{C}_\lambda$ are pairwise disjoint, to get (4.1) it is enough to find $\varepsilon = \varepsilon(\alpha)$ such that for each $S \in \mathscr{C}_\lambda$ we have

$$\sigma(\{x \in S \ : \ J_0 f(x) > \alpha\lambda, \ Mf(x) \leq \varepsilon\lambda\}) \leq \frac{8}{10}\sigma(S). \tag{4.2}$$

Fix $S \in \mathscr{C}_\lambda$. Without loss of generality assume that

$$\sigma(\{x \in S \ : \ Mf(x) \leq \varepsilon\lambda\}) > \frac{8}{10}\sigma(S), \tag{4.3}$$

otherwise there is nothing to prove.

Given $x \in S$, we split the sum from the definition of $J_0 f(x)$ into two parts:

$$\begin{aligned} &J_0 f(x)^2 \\ &= \sum_{x \in Q \in \mathscr{D}(S)} \Big(\alpha_{f\sigma}(Q)^2 + |f|^2_{B_Q}\alpha_\sigma(Q)^2\Big) + \sum_{S \subsetneq Q \in \mathscr{D}(Q_0)} \Big(\alpha_{f\sigma}(Q)^2 + |f|^2_{B_Q}\alpha_\sigma(Q)^2\Big) \\ &\qquad =: J_1 f(x)^2 + J_2 f(x)^2. \end{aligned} \tag{4.4}$$

Clearly, $J_2 f(x) \equiv J_2 f$ is just a constant. By the definition of $\mathscr{C}_\lambda$ there exists $y \in \hat{S}$ (where $\hat{S}$ is the parent of S) such that $y \notin E_\lambda$. By the definition of E_λ, we get that

$$J_2 f \leq J_0 f(y) \leq \lambda.$$

We will show the following.

Lemma 4.3. *There exists a set $S_1 \subset S$ such that $\sigma(S_1) \geq 0.5\sigma(S)$ and*

$$\int_{S_1} J_1 f(x)^2 \, d\sigma(x) \lesssim \varepsilon^2\lambda^2\sigma(S_1)$$

The estimate (4.2) follows from the above easily. Indeed, using Chebyshev, we can find $S_2 \subset S_1$ such that for all $x \in S_2$ we have $J_1 f(x) \lesssim \varepsilon\lambda$ and $\sigma(S_2) \geq 0.5\sigma(S_1) \geq 0.2\sigma(S)$. Then, choosing $\varepsilon = \varepsilon(\alpha)$ small enough, (4.4) gives $J_0 f(x)^2 \leq \lambda^2 + C\varepsilon^2\lambda^2 \leq \alpha^2\lambda^2$ on S_2, so that

$$\sigma(\{x \in S \ : \ J_0 f(x) > \alpha\lambda, \ Mf(x) \leq \varepsilon\lambda\}) \leq \sigma(S \setminus S_2) \leq \frac{8}{10}\sigma(S).$$

So our goal is to prove Lemma 4.3.

4.2 Calderón-Zygmund decomposition

Let $R = R(S)$ be as in Section 2.2, so that $B_S \cap \operatorname{supp}\sigma \subset R$. We consider a variant of the Calderón-Zygmund decomposition of $f\mathbb{1}_R$ with respect to $\mathscr{D}(R)$ at the level $2\varepsilon\lambda$.

First, let $\{Q_j\}_j \subset \mathscr{D}(R)$ be maximal cubes satisfying $|f|_{B_{Q_j}} \geq 2\varepsilon\lambda$. Note that for all $x \in Q_j$ (and recalling that M is the non-centered maximal function) we have

$$Mf(x) \geq |f|_{B_{Q_j}} \geq 2\varepsilon\lambda.$$

Hence, $\bigcup_j Q_j \subset \{x \in S \ : \ Mf(x) \geq 2\varepsilon\lambda\}$, and so

$$\sigma(R \setminus \bigcup_j Q_j) \geq \sigma(S \setminus \bigcup_j Q_j) \geq \sigma(\{x \in S \ : \ Mf(x) \leq \varepsilon\lambda\}) \overset{(4.3)}{\geq} \frac{8}{10}\sigma(S) \approx \ell(S)^n \approx \ell(R)^n. \quad (4.5)$$

In particular, $Q_j \neq R$ for all j. Thus, by the maximality of Q_j we get easily

$$|f|_{B_{Q_j}} \approx \varepsilon\lambda. \quad (4.6)$$

We define $g \in L^\infty(\sigma)$ by

$$g(x) = f(x)\mathbb{1}_{R\setminus\bigcup_j Q_j}(x) + \sum_j \langle f\rangle_{Q_j}\mathbb{1}_{Q_j}(x).$$

From the definition of Q_j and (4.6) it follows that $\|g\|_\infty \lesssim \varepsilon\lambda$. We define also $b \in L^1(\sigma)$ as

$$b(x) = \sum_j (f(x) - \langle f\rangle_{Q_j})\mathbb{1}_{Q_j}(x) =: \sum_j b_j(x).$$

Note that $f = g + b$ and for all j we have $\int b_j \, d\sigma = 0$.

4.3 Definition of S_1

We set $S_1 = S \setminus N_\eta$, where N_η is some small neighbourhood of $\bigcup_j Q_j$. To make this more precise, given a small $\eta > 0$ we define $N_\eta = \bigcup_j N_{\eta,j}$, where

$$N_{\eta,j} = \{x \in \operatorname{supp}\sigma \ : \ \operatorname{dist}(x, Q_j) < \eta\ell(Q_j)\}.$$

The thin boundaries property of $\mathscr{D}$ (2.2) gives

$$\sigma(N_{\eta,j} \setminus Q_j) \leq \eta^\gamma \sigma(Q_j)$$

for some $\gamma \in (0,1)$. From (4.5) and the fact that $\sigma(S) \approx \sigma(R)$ we get

$$\sigma(S \setminus N_\eta) \geq \sigma(S \setminus \bigcup_j Q_j) - \sum_j \sigma(N_{\eta,j} \setminus Q_j) \overset{(4.5)}{\geq} \frac{8}{10}\sigma(S) - \sum_j \eta^\gamma \sigma(Q_j)$$
$$\geq \frac{8}{10}\sigma(S) - \eta^\gamma\sigma(R) \geq \frac{8}{10}\sigma(S) - C\eta^\gamma\sigma(S) = \left(\frac{8}{10} - C\eta^\gamma\right)\sigma(S).$$

Here C depends only on the implicit constant in $\sigma(S) \approx \sigma(R)$, which in turn depends on the ADR constant of σ and on the parameters from the definition of the system $\mathscr{D}$.

Choosing η so small that $C\eta^\gamma < 0.1$, we get that $S_1 = S \setminus N_\eta$ satisfies

$$\sigma(S_1) \geq \frac{7}{10}\sigma(S).$$

4.4 Estimating $J_1 f$

Now, we will show that

$$\int_{S_1} J_1 f(x)^2 \, d\sigma(x) \lesssim \varepsilon^2 \lambda^2 \sigma(S_1) \tag{4.7}$$

Recall that

$$J_1 f(x)^2 = \sum_{x \in Q \in \mathscr{D}(S)} \alpha_{f\sigma}(Q)^2 + \sum_{x \in Q \in \mathscr{D}(S)} |f|_{B_Q}^2 \alpha_\sigma(Q)^2 =: J_1' f(x)^2 + J_1'' f(x)^2.$$

First we deal with $J_1'' f$. Observe that for all $Q \in \mathscr{D}(S)$ intersecting S_1 we have

$$|f|_{B_Q} \lesssim \varepsilon\lambda. \tag{4.8}$$

Indeed, let $y \in Q \cap S_1$, and let $P \in \mathscr{D}(R)$ be such that $y \in P$, $\ell(Q) \approx \ell(P)$, and $B_Q \subset B_P$. By the maximality of Q_j and the fact that $P \setminus \bigcup_j Q_j \neq \varnothing$ we get $|f|_{B_P} \leq 2\varepsilon\lambda$. Estimate (4.8) follows from the inclusion $B_Q \subset B_P$.

Using (4.8) as well as Theorem 1.2 we get

$$\begin{aligned}\int_{S_1} \sum_{x \in Q \in \mathscr{D}(S)} |f|_{B_Q}^2 \alpha_\sigma(Q)^2 \, d\sigma(x) &\lesssim \varepsilon^2 \lambda^2 \sum_{Q \in \mathscr{D}(S)} \alpha_\sigma(Q)^2 \sigma(Q \cap S_1) \\ &\lesssim \varepsilon^2 \lambda^2 \sum_{Q \in \mathscr{D}(S)} \alpha_\sigma(Q)^2 \sigma(Q) \lesssim \varepsilon^2 \lambda^2 \sigma(S) \approx \varepsilon^2 \lambda^2 \sigma(S_1).\end{aligned}$$

Thus, we are only left with showing

$$\int_{S_1} J_1' f(x)^2 \, d\sigma(x) = \int_{S_1} \sum_{x \in Q \in \mathscr{D}(S)} \alpha_{f\sigma}(Q)^2 \, d\sigma(x) \lesssim \varepsilon^2 \lambda^2 \sigma(S_1). \tag{4.9}$$

Lemma 4.4. *For $Q \in \mathscr{D}(S)$ we have*

$$\alpha_{f\sigma}(Q) \lesssim \alpha_{g\sigma}(Q) + \varepsilon\lambda \sum_{j : Q_j \cap B_Q \neq \varnothing} \frac{\ell(Q_j)^{n+1}}{\ell(Q)^{n+1}}.$$

Proof. Let $\varphi \in \mathrm{Lip}_1(B_Q)$. Then, using the decomposition $f(y) = g(y) + b(y)$ valid for all $y \in R \supset B_S \cap \operatorname{supp}\sigma \supset B_Q \cap \operatorname{supp}\sigma$,

$$\begin{aligned}
&\left|\int \varphi(y) f(y)\, d\sigma(y) - \int \varphi(y)\, d\mathcal{L}_Q^{g\sigma}(y)\right| \\
&\qquad \leq \left|\int \varphi(y) g(y)\, d\sigma(y) - \int \varphi(y)\, d\mathcal{L}_Q^{g\sigma}(y)\right| + \left|\int \varphi(y) b(y)\, d\sigma(y)\right| \\
&\qquad\qquad \lesssim \ell(Q)^{n+1}\alpha_{g\sigma}(Q) + \sum_j \left|\int \varphi(y) b_j(y)\, d\sigma(y)\right|.
\end{aligned}$$

Concerning the second term on the right hand side, recall that $\int b_j\, d\sigma = 0$ and that $\operatorname{supp} b_j \subset Q_j$. Keeping that in mind, denoting by x_j the center of Q_j, we estimate in the following way:

$$\begin{aligned}
&\sum_j \left|\int \varphi(y) b_j(y)\, d\sigma(y)\right| = \sum_j \left|\int (\varphi(y) - \varphi(x_j)) b_j(y)\, d\sigma(y)\right| \\
&\quad \leq \sum_j \int \left|(\varphi(y) - \varphi(x_j)) b_j(y)\right|\, d\sigma(y) \lesssim \sum_{j : Q_j \cap B_Q \neq \varnothing} \ell(Q_j) \int \left|b_j(y)\right|\, d\sigma(y) \\
&\quad = \sum_{j : Q_j \cap B_Q \neq \varnothing} \ell(Q_j) \int_{Q_j} \left|f(y) - \langle f\rangle_{Q_j}\right|\, d\sigma(y) \lesssim \sum_{j : Q_j \cap B_Q \neq \varnothing} \ell(Q_j)^{n+1} \langle |f|\rangle_{Q_j} \\
&\quad \overset{(4.6)}{\lesssim} \varepsilon\lambda \sum_{j : Q_j \cap B_Q \neq \varnothing} \ell(Q_j)^{n+1}.
\end{aligned}$$

Together with the previous string of estimates, taking supremum over all $\varphi \in \mathrm{Lip}_1(B_Q)$, we get

$$\alpha_{f\sigma}(Q) \lesssim \alpha_{g\sigma}(Q) + \varepsilon\lambda \sum_{j : Q_j \cap B_Q \neq \varnothing} \frac{\ell(Q_j)^{n+1}}{\ell(Q)^{n+1}}.$$

□

An immediate consequence of Lemma 4.4 is the estimate

$$\begin{aligned}
&\int_{S_1} J_1' f(x)^2\, d\sigma(x) \\
&\quad \lesssim \int_{S_1} J_1' g(x)^2\, d\sigma(x) + \varepsilon^2\lambda^2 \int_{S_1} \sum_{x \in Q \in \mathscr{D}(S)} \left(\sum_{j : Q_j \cap B_Q \neq \varnothing} \frac{\ell(Q_j)^{n+1}}{\ell(Q)^{n+1}} \right)^2 d\sigma(x).
\end{aligned} \tag{4.10}$$

Using Proposition 3.1 and the fact that $\|g\|_\infty \lesssim \varepsilon\lambda$, $\operatorname{supp} g \subset R$, we get

$$\int_{S_1} J_1' g(x)^2\, d\sigma(x) \leq \|Jg\|_2^2 \lesssim \|g\|_2^2 \leq \|g\|_\infty^2\, \sigma(R) \lesssim \varepsilon^2\lambda^2\sigma(R) \approx \varepsilon^2\lambda^2\sigma(S_1). \tag{4.11}$$

Moving on to the second term from the right hand side of (4.10), denote by $\mathsf{Tree} \subset \mathscr{D}(S)$ the family of cubes contained in S that intersect S_1. We have

$$\int_{S_1} \sum_{x\in Q\in\mathscr{D}(S)} \left(\sum_{j:Q_j\cap B_Q\neq\varnothing} \frac{\ell(Q_j)^{n+1}}{\ell(Q)^{n+1}} \right)^2 d\sigma(x)$$
$$\leq \sum_{Q\in\mathsf{Tree}} \sigma(Q) \left(\sum_{j:Q_j\cap B_Q\neq\varnothing} \frac{\ell(Q_j)^{n+1}}{\ell(Q)^{n+1}} \right)^2$$
$$\overset{\text{Cauchy-Schwarz}}{\lesssim} \sum_{Q\in\mathsf{Tree}} \ell(Q)^{-n-2} \left(\sum_{j:Q_j\cap B_Q\neq\varnothing} \ell(Q_j)^{n+2} \right) \left(\sum_{j:Q_j\cap B_Q\neq\varnothing} \ell(Q_j)^{n} \right). \tag{4.12}$$

Note that since $Q \in \mathsf{Tree}$, we have $Q \cap S_1 \neq \varnothing$. By the definition of S_1, this implies that for all j such that $Q_j \cap B_Q \neq \varnothing$ we have $\ell(Q) \gtrsim_\eta \ell(Q_j)$. Indeed, if $\ell(Q) \ll \eta\ell(Q_j)$, then $B_Q \cap Q_j \neq \varnothing$ implies $Q \subset N_{\eta,j}$, which would contradict $Q \cap S_1 \neq \varnothing$.

By the observation above, we have some $C = C(\eta)$ such that if $B_Q \cap Q_j \neq \varnothing$, then $Q_j \subset CB_Q$. Consequently,

$$\sum_{j:Q_j\cap B_Q\neq\varnothing} \ell(Q_j)^n \lesssim \sum_{j:Q_j\subset CB_Q} \sigma(Q_j) \leq \sigma(CB_Q) \approx_\eta \ell(Q)^n.$$

Thus, the right hand side of (4.12) can be estimated by

$$\sum_{Q\in\mathsf{Tree}} \ell(Q)^{-2} \sum_{j:Q_j\cap B_Q\neq\varnothing} \ell(Q_j)^{n+2} = \sum_j \ell(Q_j)^{n+2} \sum_{Q\in\mathsf{Tree}:Q_j\cap B_Q\neq\varnothing} \ell(Q)^{-2}. \tag{4.13}$$

As noted above, $Q_j \cap B_Q \neq \varnothing$ implies $\ell(Q) \gtrsim_\eta \ell(Q_j)$. Hence,

$$\sum_{Q\in\mathsf{Tree}:Q_j\cap B_Q\neq\varnothing} \ell(Q)^{-2} \lesssim_\eta \ell(Q_j)^{-2},$$

where we used the fact that the sum above is essentially a geometric series. Putting this together with (4.13) and (4.12), we get

$$\int_{S_1} \sum_{x\in Q\in\mathscr{D}(S)} \left(\sum_{j:Q_j\cap B_Q\neq\varnothing} \frac{\ell(Q_j)^{n+1}}{\ell(Q)^{n+1}} \right)^2 d\sigma(x) \lesssim_\eta \sum_j \ell(Q_j)^n \lesssim \ell(R)^n \approx \sigma(S_1).$$

Together with (4.10) and (4.11) this gives the desired estimate (4.9):

$$\int_{S_1} J_1' f(x)^2 \, d\sigma(x) \lesssim_\eta \varepsilon^2\lambda^2\sigma(S_1).$$

This finishes the proof of Lemma 4.3.

5 $\|f\|_p \lesssim \|Jf\|_p$ for $1 < p < \infty$

In this section we show the second inequality of Theorem 2.4.

Proposition 5.1. *Let $f \in L^p(\sigma)$ for some $1 < p < \infty$. Then*

$$\|f\|_{L^p(\sigma)} \lesssim \|Jf\|_{L^p(\sigma)} . \tag{5.1}$$

5.1 Littlewood-Paley theory

Our main tool will be the Littlewood-Paley theory for spaces of homogeneous type developed by David, Journé and Semmes in [DJS85]. We follow the way it was paraphrased (in English) in [Tol17, Section 15].

For $r > 0$, $x \in \Sigma$, and $g \in L^1_{loc}(\sigma)$, let

$$D_r g(x) = \frac{\phi_r * (g\sigma)(x)}{\phi_r * \sigma(x)} - \frac{\phi_{2r} * (g\sigma)(x)}{\phi_{2r} * \sigma(x)}$$

where $\phi_r(y) = r^{-n}\phi(y/r)$ and ϕ is a radially symmetric smooth nonnegative function supported in $B(0,1)$ with $\int_{\mathbb{R}^d} \phi = 1$.

For a function $g \in L^1_{loc}(\sigma)$ and $r > 0$, we denote

$$S_r g(x) = \frac{\phi_r * (g\sigma)(x)}{\phi_r * \sigma(x)},$$

so that

$$D_r g = S_r g - S_{2r} g.$$

Let W_r be the operator of multiplication by $1/S_r^* 1$. We consider the operators

$$\tilde{S}_r = S_r W_r S_r^* \quad \text{and} \quad \tilde{D}_r = \tilde{S}_r - \tilde{S}_{2r}.$$

Note that $\tilde{S}_r$, and thus $\tilde{D}_r$, are self-adjoint and $\tilde{S}_r 1 \equiv 1$, so that

$$\tilde{D}_r 1 = \tilde{D}_r^* 1 = 0. \tag{5.2}$$

Let $s_r(x, y)$ the kernel of S_r with respect to σ, that is, so we can write

$$S_r g(x) = \int s_r(x, y)\, g(y)\, d\sigma(y).$$

Observe that

$$s_r(x, y) = \frac{1}{\phi_r * \sigma(x)}\, \phi_r(x - y)$$

and the kernel of $\tilde{S}_r$ is

$$\tilde{s}_r(x, y) = \int s_r(x, z)\, \frac{1}{S_r^* 1(z)}\, s_r(y, z)\, d\sigma(z).$$

We claim that the kernel $\tilde{d}_r(x, \cdot)$ for the operator $\tilde{D}_r$ is supported in $B(x, 4r)$ and satisfies the Lipschitz bounds

$$|\tilde{d}_r(x,y) - \tilde{d}_r(x,z)| \lesssim |y-z| r^{-n-1}. \tag{5.3}$$

Indeed, let $x, x' \in \operatorname{supp} \sigma$. Since ϕ_r is Cr^{-n-1}-Lipschitz and σ is AD-regular,

$$\begin{aligned}|\phi_r * \sigma(x) - \phi_r * \sigma(x')| &= \left| \int (\phi_r(x-y) - \phi_r(x'-y)) d\sigma(y) \right| \\ &\lesssim \frac{|x-x'|}{r^{n+1}} \sigma(B(x,r) \cup B(x',r)) \lesssim \frac{|x-x'|}{r}.\end{aligned}$$

Thus, for $y \in \operatorname{supp} \sigma$,

$$\begin{aligned}&|s_r(x,y) - s_r(x',y)| \\ &\quad \leq \frac{|\phi_r(x-y) - \phi_r(x'-y)|}{\phi_r * \sigma(x)} + \phi_r(x'-y) \left| \frac{1}{\phi_r * \sigma(x)} - \frac{1}{\phi_r * \sigma(x')} \right| \\ &\quad \lesssim \frac{|x-x'|}{r^{n+1}} + r^{-n} \frac{|\phi_r * \sigma(x) - \phi_r * \sigma(x')|}{\phi_r * \sigma(x)^2} \approx \frac{|x-x'|}{r^{n+1}}.\end{aligned}$$

Hence,

$$\begin{aligned}|\tilde{s}_r(x,y) - \tilde{s}_r(x',y) &= \left| \int (s_r(x,z) - s_r(x',z)) \frac{1}{S_r^* 1(z)}\, s_r(y,z)\, d\sigma(z) \right| \\ &\lesssim \frac{|x-x'|}{r^{n+1}} \left| \int \frac{1}{S_r^* 1(z)}\, s_r(y,z)\, d\sigma(z) \right| \lesssim \frac{|x-x'|}{r^{n+1}}\end{aligned}$$

where in the last line we used the fact that $\int s_r(y,z) d\sigma(z) = 1$ and

$$S_r^* 1(z) = \int \frac{\phi_r(x-z)}{\phi_r * \sigma(x)} d\sigma(x) \geq \int_{B(z,r/2)} \frac{r^{-n}}{\phi_r * \sigma(x)} d\sigma(x) \approx 1.$$

Since $\tilde{d}_r = \tilde{s}_r - \tilde{s}_{2r}$ and is symmetric, this proves (5.3). Moreover, notice that if $x \in \operatorname{supp} \sigma$, $\operatorname{supp} s_r(x, \cdot) \subseteq B(x,r)$, and so the integrand of $\tilde{s}_r$ is nonzero only when $z \in B(x,r) \cap B(y,r)$, meaning $|x-y| \leq 2r$, and so $\operatorname{supp} \tilde{s}_r \subseteq B(x,2r)$, hence $\operatorname{supp} \tilde{d}_r \subseteq B(x,4r)$, which proves our claim.

Theorem 5.2. *[DJS85] Let $r_k = 2^{-k}$, and $g \in L^p(\sigma)$, $1 < p < \infty$, we have*

$$\|g\|_{L^p(\sigma)} \approx \left\| \left(\sum_{k \in \mathbb{Z}} |\tilde{D}_{r_k} g|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)}. \tag{5.4}$$

The original result is stated for $p = 2$, but this case implies the other cases (see for example the proof of [Tol01, Corollary 6.1]).

Let $\tilde{D}_k := \tilde{D}_{r_k},\ \tilde{d}_k := \tilde{d}_{r_k}$. By (5.4), it is clear that to prove (5.1), it suffices to show that

$$\left\| \left(\sum_{k\in\mathbb{Z}} |\tilde{D}_k f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\sigma)} \lesssim \|Jf\|_{L^p(\sigma)}.$$

In fact, we will show a stronger, pointwise inequality which immediately implies the one above.

Lemma 5.3. *Let $x \in \Sigma$, $k \in \mathbb{Z}$, and let $Q \in \mathscr{D}$ be the smallest cube containing x and such that* $\operatorname{supp} \tilde{d}_k(x,\cdot) \subset 0.5B_Q$. *Then, $\ell(Q) \approx r_k$ and*

$$|\tilde{D}_k f(x)| \lesssim \alpha_{f\sigma}(Q) + |f|_{B_Q}\alpha_\sigma(Q). \tag{5.5}$$

The remainder of this section is devoted to the proof of this lemma.

5.2 Preliminaries

Fix $x \in \Sigma,\ k \in \mathbb{Z}$, and let Q be as above. As noted just above (5.3), we have $\tilde{d}_k(x,\cdot) \subset B(x, 4r_k)$, and so $\ell(Q) \approx r_k$ follows immediately.

We make a few simple reductions.

Remark 5.4. Without loss of generality we may assume that $\alpha_\sigma(Q) \le \varepsilon$ for some small ε. Indeed, if we had $\alpha_\sigma(Q) \ge \varepsilon$, then using (5.3) and the fact that $\operatorname{supp} \tilde{d}_k(x,\cdot) \subset B_Q$

$$\begin{aligned} |\tilde{D}_k f(x)| = \left| \int \tilde{d}_k(x,y) f(y)\ d\sigma(y) \right| &\le \left\| \tilde{d}_k(x,\cdot) \right\|_\infty \int_{B_Q} |f(y)|\ d\sigma(y) \\ &\lesssim \ell(Q)^{-n} \int_{B_Q} |f(y)|\ d\sigma(y) \approx |f|_{B_Q} \lesssim_\varepsilon |f|_{B_Q}\alpha_\sigma(Q), \end{aligned}$$

and so in this case (5.5) holds. From now on we assume $\alpha_\sigma(Q) \le \varepsilon$.

Remark 5.5. Similarly, without loss of generality we may assume that $L_Q^{f\sigma} \cap 0.5B_Q \neq \varnothing$. If we had $L_Q^{f\sigma} \cap 0.5B_Q = \varnothing$, then $L_Q^{f\sigma} \cap \operatorname{supp} \tilde{d}_r(x,\cdot) = \varnothing$ so that

$$\int \tilde{d}_k(x,y)\ d\mathcal{L}_Q^{f\sigma}(y) = 0.$$

This implies

$$|\tilde{D}_k f(x)| = \left| \int \tilde{d}_k(x,y) f(y)\ d\sigma(y) \right| \lesssim \alpha_{f\sigma}(B),$$

and so (5.5) is true also in this case.

Recall that $c_Q^{f\sigma}$, c_Q^{σ} are the constants minimizing $\alpha_{f\sigma}(Q)$, $\alpha_\sigma(Q)$, respectively. Since σ is AD-regular and $\alpha_\sigma(Q) \le \varepsilon$, we get by [ATT20, Lemma 3.3]

$$c_Q^\sigma \approx 1. \tag{5.6}$$

To show (5.5) we begin by using (5.2) and the triangle inequality:

$$\begin{aligned}
|\tilde{D}_k f(x)| &= \left|\int_\Sigma \tilde{d}_k(x,y) f(y)\, d\sigma(y)\right| \\
&\overset{(5.2)}{=} \left|\int_\Sigma \tilde{d}_k(x,y) f(y)\, d\sigma(y) - \frac{c_Q^{f\sigma}}{c_Q^\sigma}\int_\Sigma \tilde{d}_k(x,y)\, d\sigma(y)\right| \\
&\le \left|\int_\Sigma \tilde{d}_k(x,y) f(y)\, d\sigma(y) - \int_{L_Q^{f\sigma}} \tilde{d}_k(x,y)\, d\mathcal{L}_Q^{f\sigma}(y)\right| \\
&+ \left|\int_{L_Q^{f\sigma}} \tilde{d}_k(x,y)\, d\mathcal{L}_Q^{f\sigma}(y) - \frac{c_Q^{f\sigma}}{c_Q^\sigma}\int_{L_Q^\sigma} \tilde{d}_k(x,y)\, d\mathcal{L}_Q^{\sigma}(y)\right| \\
&+ \left|\frac{c_Q^{f\sigma}}{c_Q^\sigma}\right| \left|\int_{L_Q^\sigma} \tilde{d}_k(x,y)\, d\mathcal{L}_Q^{\sigma}(y) - \int_\Sigma \tilde{d}_k(x,y)\, d\sigma(y)\right| =: (I) + (II) + (III).
\end{aligned} \tag{5.7}$$

Using the Lipschitz property of $\tilde{d}_k$ (5.3) we immediately get that $(I) \lesssim \alpha_{f\sigma}(Q)$, and that

$$(III) \lesssim \left|\frac{c_Q^{f\sigma}}{c_Q^\sigma}\right| \alpha_\sigma(Q) \overset{(5.6)}{\approx} \left|c_Q^{f\sigma}\right| \alpha_\sigma(Q). \tag{5.8}$$

Lemma 5.6. *We have* $\left|c_Q^{f\sigma}\right| \lesssim |f|_{B_Q}$.

Proof. Indeed, if we had $\left|c_Q^{f\sigma}\right| \ge \Lambda |f|_{B_Q}$ for some big $\Lambda > 10$, then $\tilde{c}_Q^{f\sigma} = 0$ would be a better competitor for a constant minimizing $\alpha_{f\sigma}(Q)$. To see that, note that for any $\varphi \in \mathrm{Lip}_1(B_Q)$

$$\left|\int \varphi f\, d\sigma - 0\right| \le C\ell(Q)^{n+1}|f|_{B_Q}.$$

That is, $F_{B_Q}(f\sigma, 0) \le C\ell(Q)^{n+1}|f|_{B_Q}$. On the other hand, taking a positive $\psi \in \mathrm{Lip}_1(B_Q)$ such that $\psi(x) = \ell(Q)$ for $x \in 0.7B_Q$, and using the assumption $L_Q^{f\sigma} \cap 0.5B_Q \neq \varnothing$ we get

$$\begin{aligned}
\alpha_{f\sigma}(Q)\ell(Q)^{n+1} &\gtrsim \left|\int \psi f\, d\sigma - c_Q^{f\sigma}\int_{L_Q^{f\sigma}} \psi\, d\mathcal{H}^n\right| \\
&\ge \left|c_Q^{f\sigma}\right| \ell(Q)\mathcal{H}^n(0.7B_Q \cap L_Q^{f\sigma}) - \left|\int \psi f\, d\sigma\right| \\
&\ge \tilde{C}\Lambda |f|_{B_Q}\ell(Q)^{n+1} - C\ell(Q)^{n+1}|f|_{B_Q} \ge \frac{\tilde{C}\Lambda}{2}|f|_{B_Q}\ell(Q)^{n+1} > F_{B_Q}(f\sigma, 0),
\end{aligned}$$

assuming Λ big enough. This contradicts the optimality of $c_Q^{f\sigma}$. □

Using the lemma above and (5.8) we get

$$(III) \lesssim |f|_{B_Q}\alpha_\sigma(Q).$$

Hence, by (5.7), to finish the proof of (5.5) it remains to show that

$$(II) = \left|c_Q^{f\sigma}\right| \left| \int_{L_Q^{f\sigma}} \tilde{d}_k(x,y)\, d\mathcal{H}^n(y) - \int_{L_Q^\sigma} \tilde{d}_k(x,y)\, d\mathcal{H}^n(y) \right| \lesssim \alpha_{f\sigma}(Q) + |f|_{B_Q}\alpha_\sigma(Q).$$

This can be seen as an estimate of how far from each other the planes $L_Q^{f\sigma}$ and L_Q^σ are.

The inequality above follows immediately from Proposition 5.7 proven in the next subsection, together with the already established estimate $\left|c_Q^{f\sigma}\right| \lesssim |f|_{B_Q}$.

5.3 Angles between planes approximating $f\sigma$ and σ

In the following proposition we do not use uniform rectifiability in any way, and so we state it for a general AD-regular measure μ. Recall that given a ball B we defined $\mathcal{P}_B^\mu = \mathcal{H}^n \llcorner L_B^\mu$.

Proposition 5.7. *Let μ be an n-AD-regular measure on $\mathbb{R}^d$, and let $f \in L^1_{loc}(\mu)$. Let $x \in \operatorname{supp}\mu$, $r > 0$, $B = B(x,r)$, and suppose that $L_B^{f\mu} \cap 0.5B \neq \varnothing$. Then,*

$$|c_B^{f\mu}| \frac{1}{r^{n+1}} F_B(\mathcal{P}_B^\mu, \mathcal{P}_B^{f\mu}) \lesssim \alpha_{f\mu}(B) + |c_B^{f\mu}|\alpha_\mu(B). \tag{5.9}$$

In the proof of Proposition 5.7 we will use the following lemma.

Lemma 5.8. *Let $B = B(x,r)$ and let L_1, L_2 be two n-planes intersecting $0.5B$. Set $\mathcal{P}_1 = \mathcal{H}^n|_{L_1}$, $\mathcal{P}_2 = \mathcal{H}^n|_{L_2}$. Then,*

$$\frac{1}{r^n} F_B(\mathcal{P}_1, \mathcal{P}_2) \lesssim \operatorname{dist}_H(L_1 \cap B, L_2 \cap B). \tag{5.10}$$

Proof. First, set

$$D = \frac{\operatorname{dist}_H(L_1 \cap B, L_2 \cap B)}{r}.$$

Note that we always have $F_B(\mathcal{P}_1, \mathcal{P}_2) \lesssim r^{n+1}$ so that if $D \gtrsim 1$, then (5.10) follows trivially. Hence, without loss of generality we may assume that $D \le \varepsilon$ for some $\varepsilon > 0$ to be fixed later.

We claim that if ε is chosen small enough (depending only on n, d), then there exists an isometry $A : L_1 \to L_2$ such that for $y \in B \cap L_1$ we have $|y - A(y)| \lesssim Dr$. To see that, let $y_1 \in L_1 \cap B$ be arbitrary. Set $y_2 = \pi_{L_2}(y_1)$. Clearly,

$$|y_1 - y_2| \le Dr \le \varepsilon r.$$

Let $v_1, \ldots, v_n$ be an orthonormal basis of the linear plane $L_1' := L_1 - y_1$. For $i = 1, \ldots, d$ define

$$w_i := \pi_{L_2}(y_1 + v_i) - y_2 \in L_2 - y_2 =: L_2'.$$

In fact, since $y_2 = \pi_{L_2}(y_1)$, we have $w_i = \pi_{L_2'}(v_i)$. It is easy to see that for all $v \in L_1'$ we have

$$|\pi_{L_2'}(v) - v| \lesssim D|v|.$$

Hence, $|w_i - v_i| \lesssim D \le \varepsilon$ and for $i \neq j$

$$|w_i \cdot w_j| = |(w_i - v_i) \cdot (w_j - v_j) + (w_i - v_i) \cdot v_j + v_i \cdot (w_j - v_j)| \lesssim D \le \varepsilon.$$

Choosing ε small enough (depending only on dimensions), we get easily that $\{w_i\}$ is a basis of L_2'. Moreover, if $\{\hat{w}_i\}$ is the orthonormal basis of L_2' constructed from $\{w_i\}$ using the Gram-Schmidt process, then it follows from the estimates above that for all $i = 1, \ldots, n$

$$|\hat{w}_i - v_i| \lesssim D.$$

We define the map $A : L_1 \to L_2$ as the unique isometry such that $A(y_1) = y_2$ and $A(y_1 + v_i) = y_2 + \hat{w}_i$. It follows immediately from basic linear algebra that for $y \in L_1 \cap B$ we have $|y - A(y)| \lesssim Dr$.

Now, let $\varphi \in \mathrm{Lip}_1(B)$. We have

$$\left| \int_{L_1} \varphi(y)\, d\mathcal{H}^n(y) - \int_{L_2} \varphi(y)\, d\mathcal{H}^n(y) \right| = \left| \int_{L_1} \varphi(y)\, d\mathcal{H}^n(y) - \int_{L_1} \varphi(A(y))\, d\mathcal{H}^n(y) \right|$$
$$\le \int_{L_1} |\varphi(y) - \varphi(A(y))|\, d\mathcal{H}^n(y) \lesssim \int_{L_1 \cap B} Dr\, d\mathcal{H}^n(y) \lesssim Dr^{n+1}.$$

Taking supremum over $\varphi \in \mathrm{Lip}_1(B)$ finishes the proof. □

Proof of Proposition 5.7. For simplicity of notation we will usually omit the subscript B, i.e. we will write $L^\mu := L^\mu_B$, $c^{f\mu} := c^{f\mu}_B$, and so on.

Without loss of generality we can assume that $c^{f\mu} \ge 0$. Indeed, if that was not the case we could consider $g = -f$. Then the plane and constant $L^{g\mu} = L^{f\mu}$, $c^{g\mu} = -c^{f\mu} \ge 0$ are minimizing for $\alpha_{g\mu}(B)$, and we have $\alpha_{g\mu}(B) = \alpha_{f\mu}(B)$. Thus, proving (5.9) for g is equivalent to proving it for f, and $c^{g\mu} \ge 0$.

Note that we always have $F_B(\mathcal{P}^\mu, \mathcal{P}^{f\mu}) \lesssim r^{n+1}$ so that if $\alpha_\mu(B) \gtrsim 1$, then (5.9) is trivial. Assume that $\alpha_\mu(B) \le \varepsilon$ for some small $\varepsilon > 0$ (depending on dimensions and AD-regularity constants), to be fixed later.

Note that if ε is small enough, then one can use AD-regularity of μ to conclude that $L^\mu \cap 0.5B \neq \varnothing$ (see for example [Tol09, Lemma 3.1]). We use this observation, the assumption $L^{f\mu} \cap 0.5B \neq \varnothing$ and (5.10) to estimate

$$c^{f\mu} \frac{1}{r^{n+1}} F_B(\mathcal{P}^\mu, \mathcal{P}^{f\mu}) \lesssim c^{f\mu} \frac{\mathrm{dist}_H(L^\mu \cap B, L^{f\mu} \cap B)}{r} =: c^{f\mu} D.$$

Our aim is to show that

$$c^{f\mu} D \lesssim c^{f\mu} \alpha_\mu(B) + \alpha_{f\mu}(B). \tag{5.11}$$

Let $0 < \eta < 0.01$ be some dimensional constant. Note that, since $L^{f\mu} \cap 0.5B \neq \varnothing$, the set $L^{f\mu} \cap 0.9B$ is a n-dimensional ball with $\mathcal{H}^n(L^{f\mu} \cap 0.9B) \approx r^n$. We claim that we can find a n-dimensional ball B_0 contained in $L^{f\mu} \cap 0.9B$, of radius ηr (in particular $r(\mathsf{B}_0) \approx_\eta r(B)$), and such that

$$\operatorname{dist}(z, L^\mu) \geq 10\eta D r \qquad \text{for all } z \in \mathsf{B}_0. \tag{5.12}$$

Indeed, if there was no such ball, i.e. if for all n-dimensional balls $\mathsf{B}_0 \subset L^{f\mu} \cap 0.9B$ of radius ηr there was some $z \in \mathsf{B}_0$ with $\operatorname{dist}(z, L^\mu) \leq 10\eta D r$, then it would follow easily from the definition of Hausdorff distance, and from the fact that L^μ and $L^{f\mu}$ are n-planes intersecting $0.5B$, that

$$\operatorname{dist}_H(L^\mu \cap B, L^{f\mu} \cap B) \lesssim \eta D r = \eta \operatorname{dist}_H(L^\mu \cap B, L^{f\mu} \cap B).$$

For η small enough, this is a contradiction. We omit the details, which can be readily filled in e.g. using [AT15, Lemma 6.4].

Consider an open neighbourhood of B_0 given by

$$U := \{y \in \mathbb{R}^n \ : \ \operatorname{dist}(y, \mathsf{B}_0) < \eta D r\},$$

and also for $\lambda > 0$ set

$$\lambda U := \{y \in \mathbb{R}^n \ : \ \operatorname{dist}(y, \mathsf{B}_0) < \lambda \eta D r\}.$$

Since $D \leq 1$, one should think of U as a d-dimensional pancake around B_0 of thickness $\eta D r$, so that the smaller D, the flatter the pancake. Note that by (5.12) for all $0 < \lambda < 10$ we have $\lambda U \cap L^\mu = \varnothing$, and also $\lambda U \subset B$ because $\mathsf{B}_0 \subset 0.9B$.

Let $\varphi : \mathbb{R}^d \to [0, \eta D r]$ be a function satisfying $\varphi \equiv \eta D r$ in U, $\operatorname{supp} \varphi \subset 2U$, and $\operatorname{Lip}(\varphi) \leq 1$. Clearly, $\varphi \in \operatorname{Lip}_1(B)$, and so

$$\left| \int \varphi f \, d\mu - \int \varphi \, d\mathcal{L}^{f\mu} \right| \leq \alpha_{f\mu}(B) r^{n+1}. \tag{5.13}$$

Furthermore, note that $\varphi \equiv \eta D r$ on B_0, so that

$$\int \varphi \, d\mathcal{L}^{f\mu} = c^{f\mu} \int_{L^{f\mu}} \varphi \, d\mathcal{H}^n \geq c^{f\mu} \eta D r \mathcal{H}^n(\mathsf{B}_0) = C(d) c^{f\mu} D \eta^{n+1} r^{n+1}.$$

Together with (5.13) this implies

$$\int \varphi f \, d\mu \geq C(\eta, d) c^{f\mu} D r^{n+1} - \alpha_{f\mu}(B) r^{n+1}. \tag{5.14}$$

Recall that we are trying to prove $c^{f\mu}D \lesssim c^{f\mu}\alpha_\mu(B) + \alpha_{f\mu}(B)$. If we had $c^{f\mu}D \le \Lambda\alpha_{f\mu}(B)$ for some $\Lambda = \Lambda(\eta, d) > 100$, then there is nothing to prove. So without loss of generality assume that $c^{f\mu}D \ge \Lambda\alpha_{f\mu}(B)$. In that case (5.14) gives

$$\int \varphi f \, d\mu \gtrsim_\eta c^{f\mu}Dr^{n+1}. \tag{5.15}$$

Now we define a modified version of φ. Recall that $\operatorname{supp}\varphi \subset 2U$. For all $y \in \operatorname{supp}\mu \cap 2U$ let $B_y = B(y, \eta Dr/5)$. We use the $5r$ covering theorem to extract from $\{B_y\}_{y\in\operatorname{supp}\mu\cap 2U}$ a subfamily of pairwise disjoint balls $\{B_i\}_{i\in I}$ such that $\operatorname{supp}\mu \cap 2U \subset \bigcup_i 5B_i$. Note that $\bigcup_i 10B_i \subset 4U$, and in particular, $\bigcup_i 10B_i \cap L^\mu = \varnothing$. Moreover, the balls $10B_i$ have bounded intersection. Thus, we may consider a partition of unity

$$\Psi = \sum_{i\in I} \psi_i,$$

such that $\operatorname{supp}\psi_i \subset 10B_i$ for each $i \in I$, $\Psi \equiv 1$ on $\bigcup_i 5B_i$, and $\operatorname{Lip}\Psi \lesssim (\eta Dr)^{-1}$.

Consider $\Phi = \varphi\Psi$. We have

$$\|\nabla\Phi\|_\infty \le \|\nabla\varphi\|_\infty\|\Psi\|_\infty + \|\varphi\|_\infty\|\nabla\Psi\|_\infty \lesssim 1 + \eta Dr(\eta Dr)^{-1} = 1.$$

Hence, $C\Phi \in \operatorname{Lip}_1(B)$ for some $C \approx 1$, so that

$$\left| \int \Phi f \, d\mu - \int \Phi \, d\mathcal{L}^{f\mu} \right| \le C^{-1}\alpha_{f\mu}(B)r^{n+1}. \tag{5.16}$$

On the other hand, observe that $\Psi \equiv 1$ on $\operatorname{supp}\varphi \cap \operatorname{supp}\mu$. By (5.15)

$$\int \Phi f \, d\mu = \int \varphi f \, d\mu \gtrsim_\eta c^{f\mu}Dr^{n+1}.$$

Together with (5.16) this gives

$$\int \Phi \, d\mathcal{L}^{f\mu} \ge C(\eta)c^{f\mu}Dr^{n+1} - C^{-1}\alpha_{f\mu}(B)r^{n+1} \gtrsim_\eta c^{f\mu}Dr^{n+1}, \tag{5.17}$$

where we used once again the additional assumption $c^{f\mu}D \ge \Lambda\alpha_{f\mu}(B)$ we made along the way (and choosing Λ large).

Now we will show that

$$\int_{L^{f\mu}} \Phi \, d\mathcal{H}^n \lesssim_\eta \alpha_\mu(B)r^{n+1}. \tag{5.18}$$

Since $\mathcal{L}^{f\mu} = c^{f\mu}\mathcal{H}^n|_{L^{f\mu}}$, together with (5.17) this will give $c^{f\mu}D \lesssim_\eta c^{f\mu}\alpha_\mu(B)$, and so the proof of (5.11) will be finished.

Recall that $\operatorname{supp}\Phi \subset \operatorname{supp}\Psi \subset \bigcup_i 10B_i$, and that $\|\Phi\|_\infty \le \|\varphi\|_\infty = \eta Dr$. Hence,

$$\int_{L^{f\mu}} \Phi \, d\mathcal{H}^n \lesssim_\eta Dr \sum_{i\in I} \mathcal{H}^n(L^{f\mu} \cap 10B_i) \lesssim_\eta \#I(Dr)^{n+1}.$$

To estimate $\#I$ we will use AD-regularity of μ. Recall that $\{B_i\}_{i\in I}$ are pairwise disjoint, they are centered at points from $\operatorname{supp}\mu \cap 2U$, and $r(B_i) = \eta r D/5$. Thus,

$$\#I(rD)^n \approx_\eta \sum_{i\in I} \mu(B_i) = \mu\Big(\bigcup_{i\in I} B_i\Big).$$

On the other hand, since the balls $\{B_i\}$ are centered at points from $2U$, we have $\bigcup_{i\in I} B_i \subset 3U$ and

$$\mu\Big(\bigcup_{i\in I} B_i\Big) \le \mu(3U).$$

To bound $\mu(3U)$ consider $\tilde{\varphi} \in \operatorname{Lip}_1(B)$ such that $\tilde{\varphi} \ge 0$, $\tilde{\varphi} \equiv \eta r D$ on $3U$ and $\operatorname{supp}\tilde{\varphi} \subset 4U$. Recalling that $4U \cap L^\mu = \varnothing$, we arrive at

$$rD\mu(3U) \lesssim_\eta \int \tilde{\varphi}\, d\mu = \left| \int \tilde{\varphi}\, d\mu - \int \tilde{\varphi}\, d\mathcal{L}^\mu \right| \le \alpha_\mu(B) r^{n+1}.$$

Putting all the estimates above together we get (5.18):

$$\int_{L^{f\mu}} \Phi\, d\mathcal{H}^n \lesssim_\eta \#I(Dr)^{n+1} \lesssim_\eta rD\mu(3U) \lesssim_\eta \alpha_\mu(B) r^{n+1}.$$

□

A necessary condition for the L^2 boundedness of the Riesz transform on Heisenberg groups VII

1 Introduction

The motivation behind this note is the following question: what are the measures μ on the Heisenberg group $\mathbb{H}^n$ which guarantee that the (correct notion of) Riesz transform is bounded from $L^2(\mu)$ to itself? This question (or some variant of it) with $\mathbb{R}^n$ instead of $\mathbb{H}^n$, was one of the major starting points of quantitative rectifiability, as described in Chapter I. We described some motivation for developing GMT in more general settings (including the Heisenberg group) in Section I.7. We should mention that the study of Heisenberg geometry can be approached from different perspectives and with different applications in mind; for example, see [NY18] for a connection with theoretical computer science.

In the last couple of years, there has been some progress towards an answer to our initial question; see for example [CFO19], [FO19] and [Orp18b]. In this chapter we give a necessary condition to be imposed on a Radon measure μ on $\mathbb{H}^n$ for the Riesz transform to be $L^2(\mu)$ bounded. Here R_μ is the singular integral operator whose kernel is the horizontal gradient of the fundamental solution of the Heisenberg sub-Laplacian, as defined in [CM12]. Note that due to the non-Euclidean setting, we will use different notation than in previous chapters (e.g. $B(p,r)$ will denote the ball with respect to the Korányi metric, and not the Euclidean distance). See Section 2 for precise definitions.

Theorem 1.1. *Let μ be a Radon measure on $\mathbb{H}^n$ such that R_μ is bounded on*

$L^2(\mu)$ with norm C_1, and such that $\mu(F) = 0$ whenever $\dim_H(F) \leq 2$[*]*. Then there exists a constant C_2 such that for all balls $B(x,r) \subset \mathbb{H}^n$, we have*

$$\mu(B(x,r)) \leq C_2 r^{2n+1}. \tag{1.1}$$

Here C_2 depends only on n and C_1, and the ball $B(x,r)$ is defined with respect to the Korányi metric, see Section 2.

A corresponding statement holds in the Euclidean setting, and is a result of David, [Dav91, Part III, Proposition 1.4]. See also [Orp17], Proposition 6.9 for a more detailed proof of the same result. Let $\mathcal{R}_\mu^n$ denote the standard n-dimensional Riesz transform in $\mathbb{R}^d$.

Theorem 1.2. *Assume that μ is a non-atomic Radon measure on $\mathbb{R}^d$ such that $\mathcal{R}_\mu^n$ is bounded on $L^2(\mu)$ with norm C_1. Then, for all Euclidean balls $B_{\mathbb{R}^d}(x,r) \subset \mathbb{R}^d$ we have*

$$\mu(B_{\mathbb{R}^d}(x,r)) \leq C_2 r^n \tag{1.2}$$

Here C_2 depends only on C_1, n, and d.

A measure satisfying (1.2) (or (1.1)) is said to have *polynomial growth*. Let us give a couple of remarks.

Remark 1.3. Although the result itself (both in the Euclidean and Heisenberg case) is not very hard, it is nevertheless very useful. For example, most tools developed in the last two decades that take quantitative rectifiability beyond AD-regular measures still need polynomial growth[†] (see for example the book by Tolsa [Tol14]). Thus, we expect that our result will be quite useful, too.

Remark 1.4. While the two results above look similar, there is actually a difference, in the sense that, in the Heisenberg case, there actually exist lower dimensional measures which give a bounded Riesz transform, but are not atomic.

This is *not* a byproduct of the proof, but rather a fact of the Heisenberg geometry. Indeed, the 2-dimensional t-axis (or any Heisenberg translate of it) gives a bounded $(2n+1)$-dimensional Riesz transform; this is simply because on these sets the kernel vanishes identically, see (2.4).

One can construct a more interesting example in the vertical plane of the one dimensional Heisenberg group $\mathbb{H}$, say. Consider a tube of height 1 and radius ε_1^2 around the t-axis, and take the intersection with the vertical plane. Call the resulting rectangle $R_{1,1}$. Cut out from R_1 two smaller rectangles $R_{2,1}$ and $R_{2,2}$, one in the top right corner and one in the bottom left corner, both

[*]This assumption comes from the fact that the kernel of R_μ vanishes on the vertical lines, which have dimension 2. See also Remark 1.4 below.

[†]With some exceptions, see for example [AS18], or [BS15].

of height ε_2 and width ε_2^2, for some $\varepsilon_2 \leq \varepsilon_1/4$. We proceed in this way, so that after k steps we have 2^{k-1} disjoint rectangles $\{R_{k,i}\}_i$ of height ε_k and width ε_k^2. Consider the natural probability measure μ on the Cantor-like set $C = \bigcap_k \bigcup_i R_{k,i}$. It is not difficult to show that, if $\varepsilon_k \to 0$ are small enough, the Heisenberg Riesz transform is bounded on $L^2(\mu)$; the idea is that the set is concentrated along the t-axis, and thus the kernel is very small (see (2.4) below). Depending on the choice of (ε_k) we have $\dim_H(C) \in [0, 2]$.

Plan of the chapter

In Section 2 we briefly recall basic facts about Heisenberg groups and the Riesz transform. We also introduce a family of "dyadic cubes" suitable to our setting.

Section 3 is dedicated to Lemma 3.1, our main technical lemma. Roughly speaking, we show that if a measure μ is such that R_μ is bounded on $L^2(\mu)$, and there is some cube Q_0 with a very high concentration of μ (i.e. $\mu(Q_0) \gg \ell(Q_0)^{2n+1}$), then we can find a family $\mathsf{HD}(Q_0)$ of much smaller cubes, contained in Q_0, such that

a) a very large portion of measure μ on Q_0 is concentrated on the cubes from $\mathsf{HD}(Q_0)$,

b) the family $\mathsf{HD}(Q_0)$ is relatively small, in the sense that it consists of few cubes.

In Section 4 we show that if the polynomial growth condition (1.1) is not satisfied, then we can find a cube satisfying the assumptions of our main lemma. This in turn allows us to start an iteration algorithm, consisting of using the main lemma countably many times, that results in constructing a set Z with $\mu(Z) > 0$ and $\dim_H(Z) \leq 2$. This finishes the proof of Theorem 1.1.

2 Preliminaries

In our estimates we will often use the notation $f \lesssim g$ which means that there exists some absolute constant C for which $f \leq Cg$. If the constant C depends on some parameter t, we will write $f \lesssim_t g$. Notation $f \approx g$ will stand for $f \lesssim g \lesssim f$, and $f \approx_t g$ is defined analogously. For simplicity, in our estimates we will suppress the dependence on dimension n and on absolute constants λ, Λ (see (2.7)).

2.1 Heisenberg group

In this paper we consider the n-th Heisenberg group with exponential coordinates (see [CDPT07] or [Fäs19] for a swift introduction to the Heisenberg group in a context close to ours). In practice, we will denote a point $p \in \mathbb{H}^n$ as $(z,t) \in \mathbb{R}^{2n} \times \mathbb{R}$, and $z = (x_1, ..., x_n, y_1, ..., y_n)$. In these coordinates the group law in $\mathbb{H}^n$ takes the form

$$p \cdot q = \left(z + z', t + t' + \frac{1}{2} \sum_{i=1}^{n} (x_i y_i' - y_i x_i') \right),$$

where $p = (z,t)$ and $q = (z',t')$. Note that the group operation is not commutative. The identity element is the origin $(0,0)$ and the inverse is given by $p^{-1} = (-z,-t)$. We make $\mathbb{H}^n$ into a metric space by setting $d(p,q) := \|q^{-1} \cdot p\|_{\mathbb{H}}$, where

$$\|p\|_{\mathbb{H}}^4 := |z|^4 + 16t^2, \tag{2.1}$$

and $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^{2n}$.

Note that $\|\cdot\|_{\mathbb{H}}$ is 1-homogeneous with respect to the anisotropic dilation $p \mapsto \lambda p = (\lambda z, \lambda^2 t)$, $\lambda > 0$. The metric d is sometimes called the Korányi metric.

Given $p \in \mathbb{H}^n$ and $r > 0$ we set

$$B(p,r) = \{q \mid d(p,q) \le r\}, \quad U(p,r) = \{q \mid d(p,q) < r\}.$$

For $\alpha > 0$ we will write $\mathcal{H}^\alpha$ to denote the usual α-dimensional Hausdorff measure with respect to metric d. For $A \subset \mathbb{H}^n$ we set $\dim_H(A)$ to be the Hausdorff dimension of A.

It follows easily from the definition of the Korányi metric that for all $p \in \mathbb{H}^n$ and $r > 0$ we have

$$\mathcal{H}^{2n+2}(B(p,r)) = \mathcal{H}^{2n+2}(B(0,1))\, r^{2n+2}. \tag{2.2}$$

Thus, even though the topological dimension of $\mathbb{H}^n$ is $2n+1$, the Hausdorff dimension of $\mathbb{H}^n$ is equal to $2n+2$. For the sake of brevity we set $D := 2n+2$. Usually one denotes the Hausdorff dimension of $\mathbb{H}^n$ by Q, but we have decided to save that letter for cubes; hence the non-standard notation.

It is also easy to check that if $\mathcal{L}^{2n+1}$ denotes the usual Lebesgue measure on $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$, then we have a constant $C > 0$ such that

$$\mathcal{L}^{2n+1} = C\mathcal{H}^D. \tag{2.3}$$

2.2 Heisenberg Riesz transform

Recall that, for a function $u : \mathbb{H}^n \to \mathbb{R}$, the horizontal gradient of u is given by

$$\nabla_{\mathbb{H}} u := (X_1 u, ..., X_n u, Y_1 u, ..., Y_n u),$$

where the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$ and $\frac{\partial}{\partial t}$ represent the left invariant translates of the canonical basis at the identity. In particular, $X_1, \dots, X_n, Y_1, \dots, Y_n$ span the horizontal distribution in $\mathbb{H}^n$.

The Heisenberg sublaplacian $\Delta_{\mathbb{H}}$ is given by $\sum_{i=1}^{n} X_i^2 + Y_i^2$, and its fundamental solution is

$$G(p) := c_n \|p\|_{\mathbb{H}}^{2-D}.$$

The $(D-1)$-dimensional Riesz kernel in $\mathbb{H}^n$, first considered in [CM12], is given by $K(p) = \nabla_{\mathbb{H}} G(p)$. The Riesz transform is formally defined as

$$R_\mu f(p) = \int_{\mathbb{H}^n} K(q^{-1} \cdot p) f(q)\, d\mu(q).$$

Since it is not clear whether the integral above converges, one considers the truncated Riesz transform given by the formula

$$R_{\mu,\delta} f(p) = \int_{\mathbb{H}^n \setminus B(p,\delta)} K(q^{-1} \cdot p) f(q)\ d\mu(q),$$

for $\delta > 0$. We say that R_μ is bounded on $L^2(\mu)$ if the truncated operators $R_{\mu,\delta}$ are bounded on $L^2(\mu)$ uniformly in $\delta > 0$.

One can easily check that the Riesz kernel is actually equal to

$$K(z,t) = n \left(\frac{-2x_1|z|^2 + 8y_1 t}{\|(z,t)\|_{\mathbb{H}}^{2n+4}}, \dots, \frac{-2x_n|z|^2 + 8y_n t}{\|(z,t)\|_{\mathbb{H}}^{2n+4}}, \right. \\ \left. \frac{-2y_1|z|^2 - 8x_1 t}{\|(z,t)\|_{\mathbb{H}}^{2n+4}}, \dots, \frac{-2y_n|z|^2 - 8x_n t}{\|(z,t)\|_{\mathbb{H}}^{2n+4}} \right).$$

Hence,

$$|K(z,t)|^2 = n^2 \frac{4|z|^2}{(|z|^4 + 16t^2)^{n+1}}. \tag{2.4}$$

This implies the curious fact that $|K(z,t)| \le C$ whenever

$$|z| \le\ 16|t|^{n+1}, \tag{2.5}$$

which is a 'paraboloidal' double cone around t-axis with vertex at the origin. This fact will play a key role in the subsequent analysis.

Chousionis and Mattila showed in [CM12, Proposition 3.11] that the Riesz kernel is a standard kernel. In particular, it satisfies the following continuity property: whenever $q_1, q_2 \neq p \in \mathbb{H}^n$, we have

$$|K(p^{-1} \cdot q_1) - K(p^{-1} \cdot q_2)| \lesssim \max \left\{ \frac{d(q_1,q_2)}{d(p,q_1)^D}, \frac{d(q_1,q_2)}{d(p,q_2)^D} \right\}.$$

Taking $p = 0$ and $q_1 = \tilde{q}_1{}^{-1} \cdot \tilde{p},\ q_2 = \tilde{q}_2{}^{-1} \cdot \tilde{p}$, one gets immediately that for all $\tilde{q}_1, \tilde{q}_2 \neq \tilde{p} \in \mathbb{H}^n$

$$|K(\tilde{q}_1{}^{-1} \cdot \tilde{p}) - K(\tilde{q}_2{}^{-1} \cdot \tilde{p})| \lesssim \max \left\{ \frac{d(\tilde{q}_1,\tilde{q}_2)}{d(\tilde{p},\tilde{q}_1)^D}, \frac{d(\tilde{q}_1,\tilde{q}_2)}{d(\tilde{p},\tilde{q}_2)^D} \right\}. \tag{2.6}$$

2.3 Dyadic cubes

We are going to use a family of decompositions of $\mathbb{H}^n$ into subsets that share many properties with the standard dyadic cubes from $\mathbb{R}^n$. The most classical constructions of this kind are due to Chirst [Chr90] and David [Dav88a], but for us it will be more convenient to use the "cubes" constructed in [KRS12].

First, note that given any ball $B(p, 2r)$, one may use the $5r$-covering lemma and the property (2.2) to conclude that there exists some absolute constant m such that $B(p, 2r)$ may be covered by m balls $B(p_i, r)$, where $\{p_i\}_{i=1}^m$ are points in $B(p, 2r)$. That is, $\mathbb{H}^n$ is geometrically doubling. In particular, we can use [KRS12, Theorem 2.1, Remark 2.2].

Lemma 2.1 ([KRS12]). *For all $k \in \mathbb{Z}$ there exists a family of subsets of $\mathbb{H}^n$, denoted by $\mathfrak{D}_k$, such that*

(i) $\mathbb{H}^n = \bigcup_{Q \in \mathfrak{D}_k} Q$,

(ii) if $k \geq l$, and $Q \in \mathfrak{D}_k$, $P \in \mathfrak{D}_l$, then either $Q \cap P = \varnothing$ or $Q \subset P$,

(iii) for every $Q \in \mathfrak{D}_k$ there exists $p_Q \in Q$ such that

$$U(p_Q, \lambda 2^{-k}) \subset Q \subset B(p_Q, \Lambda 2^{-k}) \tag{2.7}$$

for some absolute constants $\lambda, \Lambda > 0$.

Let us stress once more that we will not keep track of how various parameters appearing in the proof depend on λ and Λ.

We set $\mathfrak{D} = \bigcup_k \mathfrak{D}_k$. For $Q \in \mathfrak{D}_k$ we define the sidelength of Q as $\ell(Q) = 2^{-k}$. Clearly, by (2.2) and (2.7), for $Q \in \mathfrak{D}$ we have

$$\mathcal{H}^D(Q) \approx \ell(Q)^D.$$

It follows that if $Q \in \mathfrak{D}$, then for $k \geq 0$

$$\#\left\{P \in \mathfrak{D} \mid P \subset Q,\ \ell(P) = 2^{-k}\ell(Q)\right\} \approx 2^{kD}. \tag{2.8}$$

Given a Radon measure μ and $Q \in \mathfrak{D}$ we will denote the $(D-1)$-dimensional density of μ in Q by

$$\Theta_\mu(Q) = \frac{\mu(Q)}{\ell(Q)^{D-1}}.$$

For simplicity, we will suppress the dependence on μ and simply write $\Theta(Q)$.

3 Main lemma

Our main tool in the proof of Theorem 1.1 is the following lemma.

Lemma 3.1. *Let μ be a Radon measure on $\mathbb{H}^n$ such that R_μ is bounded on $L^2(\mu)$ with norm C_1. There exist constants $A = A(n) > 1$, $s = s(A,n) \in (0,1/2)$ and $M = M(C_1,n) > 100$ such that the following holds.*

Suppose that $Q_0 \in \mathfrak{D}$ satisfies $\Theta(Q_0) \geq M$. Set $N = \left\lfloor A^{-2}\log(\Theta(Q_0))\right\rfloor$. Then, the family of high density cubes

$$\mathsf{HD}(Q_0) = \left\{Q \in \mathfrak{D} \mid Q \subset Q_0,\ \ell(Q) = 2^{-N}\ell(Q_0),\ \Theta(Q) > 2\,\Theta(Q_0)\right\}$$

satisfies

$$\sum_{Q\in\mathsf{HD}(Q_0)} \mu(Q) \geq (1-\Theta(Q_0)^{-s})\mu(Q_0). \tag{3.1}$$

Moreover, we have

$$\sum_{Q\in\mathsf{HD}(Q_0)} \ell(Q)^2 \leq C_p\,\ell(Q_0)^2 \tag{3.2}$$

for some dimensional constant C_p (“p” stands for “packing”).

The rest of this section is dedicated to proving the lemma above. For brevity of notation, we set $\Theta_0 = \Theta(Q_0)$. Observe that the integer N was chosen in such a way that

$$2^{A^2N} \approx \Theta_0 \geq M. \tag{3.3}$$

In particular, we have $N \geq N_0$ for some very big N_0 depending on M and A.

We split the proof of Lemma 3.1 into several steps.

First, note that by the pigeonhole principle and (2.8), we can find a cube $Q_1 \in \mathfrak{D}$ with sidelength $\ell(Q_1) = 2^{-AN}\ell(Q_0)$ such that

$$\mu(Q_1) \gtrsim \frac{\mu(Q_0)}{2^{AND}}. \tag{3.4}$$

Without loss of generality, by applying the appropriate translation, we can assume that Q_1 is centred at the origin, i.e. $p_{Q_1} = 0$. Set

$$T := \left\{(z,t) \in Q_0 \,|\, |z| \leq 2^{-N}\ell(Q_0)\right\}$$

and for any $\kappa > 0$ set

$$T_\kappa := \left\{(z,t) \in Q_0 \,|\, |z| \leq \kappa\, 2^{-N}\ell(Q_0)\right\}.$$

Observe that $Q_1 \subset T$. In a sense, T can be seen as a tube with vertical axis passing through $p_{Q_1} = 0$. Note also that for any cube $Q \subset Q_0 \setminus T$ we have $\operatorname{dist}(Q,Q_1) \gtrsim 2^{-N}\ell(Q_0)$.

We start by proving a few preliminary results.

Lemma 3.2. *There are at most $C(\kappa)\,2^{2N}$ cubes of sidelength $2^{-N}\ell(Q_0)$ contained in T_κ.*

Proof. Observe that since $0 \in Q_0$, and by (2.7) $Q_0 \subset B(p_{Q_0}, \Lambda\ell(Q_0))$, we have $Q_0 \subset B(0, 2\Lambda\ell(Q_0))$. Hence,

$$\begin{aligned} T_\kappa &\subset \left\{(z,t) \in B(0, 2\Lambda\ell(Q_0)) \mid |z| \le \kappa\, 2^{-N}\ell(Q_0)\right\} \\ &\subset \left\{(z,t) \in \mathbb{H}^n \mid |z| \le \kappa\, 2^{-N}\ell(Q_0),\ 16|t|^2 \le (2\Lambda\ell(Q_0))^4\right\} =: \widetilde{T}_\kappa. \end{aligned}$$

By (2.3),

$$\mathcal{H}^D(\widetilde{T}_\kappa) = C\mathcal{L}^{2n+1}(\widetilde{T}_\kappa) \approx (\kappa 2^{-N}\ell(Q_0))^{2n}(2\Lambda\ell(Q_0))^2 \approx_\kappa 2^{-2nN}\ell(Q_0)^D.$$

It follows that $\mathcal{H}^D(T_\kappa) \lesssim_\kappa 2^{-2nN}\ell(Q_0)^D$. On the other hand, recall that for any cube Q with sidelength $\ell(Q) = 2^{-N}\ell(Q_0)$ we have $\mathcal{H}^D(Q) \approx 2^{-ND}\ell(Q_0)^D$. Since all such cubes are pairwise disjoint, we get

$$\begin{aligned} \#\left\{Q \in \mathcal{D} \mid \ell(Q) = 2^{-N}\ell(Q_0),\ Q \subset T_\kappa\right\} &\lesssim \frac{\mathcal{H}^D(T_\kappa)}{2^{-ND}\ell(Q_0)^D} \\ &\lesssim_\kappa \frac{2^{-2nN}\ell(Q_0)^D}{2^{-N(2n+2)}\ell(Q_0)^D} = 2^{2N}. \end{aligned}$$

□

Lemma 3.3. *Let $Q \in \mathfrak{D}$ satisfy $Q \subset Q_0 \setminus T$ and $\ell(Q) = \ell(Q_1) = 2^{-AN}\ell(Q_0)$. Then*

$$\mu(Q) \le \frac{\mu(Q_0)}{\Theta_0\, 2^{AND}}.$$

Proof. Suppose the claim above is false. Then we can find a cube $Q_2 \subset Q_0 \setminus T$ with $\ell(Q_2) = 2^{-AN}\ell(Q_0)$ such that

$$\mu(Q_2) \ge \frac{\mu(Q_0)}{\Theta_0\, 2^{AND}}. \tag{3.5}$$

Let $0 < \delta < \operatorname{dist}(Q_1, Q_2)$, let $p \in Q_2$ be arbitrary, and consider

$$R_{\mu,\delta}(\mathbb{1}_{Q_1})(p) = \int_{Q_1} K(q^{-1}\cdot p)\, d\mu(q).$$

By triangle inequality,

$$|R_{\mu,\delta}(\mathbb{1}_{Q_1})(p)| \ge \left|\int_{Q_1} K(p)\, d\mu(q)\right| - \left|\int_{Q_1} K(q^{-1}\cdot p) - K(p)\, d\mu(q)\right|. \tag{3.6}$$

We estimate the first term as follows. Note that, since $p \in Q_2$ and Q_2 lies outside T, then, writing $p = (z,t)$ and using (2.4), we have

$$|K(p)|^2 \approx \frac{|z|^2}{(|z|^4 + 16t^2)^{n+1}} \gtrsim \frac{|z|^2}{\ell(Q_0)^{4(n+1)}} \ge 2^{-2N}\ell(Q_0)^{-4n-2} = 2^{-2N}\ell(Q_0)^{-2D+2}.$$

And thus we also have

$$\left|\int_{Q_1} K(p)\,d\mu(q)\right| = |K(p)|\,\mu(Q_1) \gtrsim 2^{-N}\,\frac{\mu(Q_1)}{\ell(Q_0)^{D-1}}. \tag{3.7}$$

For the second term in (3.6) we use the continuity of the kernel K (2.6) and the fact that $d(p,q) \approx \|p\|_{\mathbb{H}} \geq 2^{-N}\ell(Q_0)$ (because $p \in Q_2 \subset Q_0 \setminus T$):

$$|K(q^{-1}\cdot p) - K(p)| \lesssim \frac{\|q\|_{\mathbb{H}}}{\min(\|p\|_{\mathbb{H}}, d(p,q))^D} \lesssim \frac{2^{-AN}\ell(Q_0)}{(2^{-N}\ell(Q_0))^D} = \frac{2^{-AN+DN}}{\ell(Q_0)^{D-1}}. \tag{3.8}$$

Taking $A \geq 2D$ we get

$$\left|\int_{Q_1} K(q^{-1}\cdot p) - K(p)\,d\mu(q)\right| \lesssim 2^{-AN/2}\frac{\mu(Q_1)}{\ell(Q_0)^{D-1}}.$$

Together with (3.7) and (3.6), assuming N_0 bigger than some absolute constant (recall that $N \geq N_0$), this gives

$$|R_{\mu,\delta}(\mathbb{1}_{Q_1})(p)| \gtrsim 2^{-N}\,\frac{\mu(Q_1)}{\ell(Q_0)^{D-1}}$$

for all $p \in Q_2$.

Now, we use the estimate above and the $L^2(\mu)$ boundedness of R_μ to get

$$2^{-N}\frac{\mu(Q_1)}{\ell(Q_0)^{D-1}}\mu(Q_2)^{\frac{1}{2}} \lesssim \left(\int |R_{\mu,\delta}(\mathbb{1}_{Q_1})(p)|^2\,d\mu(p)\right)^{\frac{1}{2}} \leq C_1\mu(Q_1)^{\frac{1}{2}}.$$

Our assumptions on Q_1 (3.4) and Q_2 (3.5) yield

$$\begin{aligned} C_1 &\gtrsim 2^{-N}\frac{\mu(Q_1)^{\frac{1}{2}}\mu(Q_2)^{\frac{1}{2}}}{\ell(Q_0)^{D-1}} \gtrsim 2^{-N}\frac{\mu(Q_0)}{2^{AND}\ell(Q_0)^{D-1}}\Theta_0^{-1/2} = 2^{-AND-N}\Theta_0^{1/2} \\ &\overset{(3.3)}{\approx} 2^{-AND-N}\,2^{A^2N/2}. \end{aligned}$$

Taking $A \geq 5D$ we can bound the last term from below in the following way:

$$2^{-AND-N+A^2N/2} \geq 2^{A^2N/4} \overset{(3.3)}{\gtrsim} M^{1/4}.$$

Putting together the estimates above gives $C_1 \gtrsim M^{1/4}$, which is a contradiction for $M = M(C_1, n)$ big enough. □

We immediately get the following cor.

Corollary 3.4. *We have*

$$\mu(T_2) \geq (1 - \Theta_0^{-1})\mu(Q_0). \tag{3.9}$$

Proof. Observe that if $Q \in \mathcal{D}$ satisfies $\ell(Q) = \ell(Q_1) = 2^{-AN}\ell(Q_0)$ and $Q \not\subset T_2$, then we have $Q \cap T = \varnothing$ (assuming A large enough with respect to Λ). It follows that Q satisfies the assumptions of Lemma 3.3, and so

$$\mu(Q) \leq 2^{-AND}\Theta_0^{-1}\mu(Q_0).$$

Summing over all such Q and using (2.8) yields

$$\mu(Q_0 \setminus T_2) \leq \Theta_0^{-1}\mu(Q_0).$$

□

Recall that

$$\mathsf{HD}(Q_0) = \left\{Q \in \mathfrak{D} \mid Q \subset Q_0,\ \ell(Q) = 2^{-N}\ell(Q_0),\ \Theta(Q) > 2\Theta_0\right\},$$

and that Λ is the absolute constant such that $Q \subset B(p_Q, \Lambda\ell(Q))$. Without loss of generality, we may assume $\Lambda > 2$.

We are ready to prove the first part of Lemma 3.1, the estimate (3.1).

Lemma 3.5. *There exists $s = s(A, n) \in (0, 1/2)$ such that*

$$\sum_{Q \in \mathsf{HD}(Q_0)} \mu(Q) \geq (1 - \Theta_0^{-s})\mu(Q_0). \tag{3.10}$$

Proof. We will prove (3.10) by contradiction. Suppose that

$$\sum_{Q \in \mathsf{HD}(Q_0)} \mu(Q) < (1 - \Theta_0^{-s})\mu(Q_0). \tag{3.11}$$

Set

$$\mathsf{LD}(Q_0) = \left\{Q \in \mathfrak{D} \mid Q \subset T_{2\Lambda},\ \ell(Q) = 2^{-N}\ell(Q_0),\ \Theta(Q) \leq 2\Theta_0\right\}.$$

It is easy to see that the cubes from $\mathsf{HD}(Q_0) \cup \mathsf{LD}(Q_0)$ cover T_2. If we assume $\Theta_0 \geq M > 100$, and $s < 1/2$, then $\Theta_0^{-s}/2 \geq \Theta_0^{-1}$, and so by (3.9) and (3.11) we get

$$\sum_{Q \in \mathsf{LD}(Q_0)} \mu(Q) \geq \frac{\Theta_0^{-s}}{2}\mu(Q_0). \tag{3.12}$$

On the other hand, recall from Lemma 3.2 that there are at most $C2^{2N}$ cubes of sidelength $2^{-N}\ell(Q_0)$ contained in $T_{2\Lambda}$, where $C = C(\Lambda, n)$. Moreover, for any $Q \in \mathsf{LD}(Q_0)$ we have

$$\mu(Q) \leq 2\Theta_0\ell(Q)^{D-1} = 2\,\mu(Q_0)\frac{\ell(Q)^{D-1}}{\ell(Q_0)^{D-1}} = 2^{-N(D-1)+1}\mu(Q_0).$$

In consequence,

$$\sum_{Q \in \mathsf{LD}(Q_0)} \mu(Q) \leq C2^{2N}2^{-N(D-1)+1}\mu(Q_0).$$

This contradicts (3.12) because

$$C\,2^{-ND+3N+1} = 2\,C\,(2^{-A^2N})^{(-D+3)A^{-2}} \overset{(3.3)}{\leq} \widetilde{C}(n)\Theta_0^{(-D+3)A^{-2}} \leq \frac{\Theta_0^{-s}}{2},$$

choosing $s = s(A,n)$ small enough. □

We move on to the second part of Lemma 3.1, i.e. the packing estimate (3.2).

Lemma 3.6. *We have*

$$\bigcup_{Q\in \mathsf{HD}(Q_0)} Q \subset T_{2\Lambda}. \tag{3.13}$$

In consequence,

$$\sum_{Q\in \mathsf{HD}(Q_0)} \ell(Q)^2 \lesssim \ell(Q_0)^2. \tag{3.14}$$

Proof. We will prove that for $Q\in \mathsf{HD}(Q_0)$ we have $Q\cap T_2 \neq \varnothing$. Then, since $\ell(Q) = 2^{-N}\ell(Q_0)$, it follows easily from (2.7) that indeed $Q\subset T_{\Lambda+2}(Q_0)\subset T_{2\Lambda}(Q_0)$.

We argue by contradiction. Suppose that $Q\in \mathsf{HD}(Q_0)$ and $Q\cap T_2 = \varnothing$. Consider the cubes $\{P_i\}_{i\in I}$ with $\ell(P_i) = 2^{-AN}\ell(Q_0) = 2^{-(A-1)N}\ell(Q)$ and $P_i\subset Q$. Then, $Q = \bigcup_i P_i$, for all $i\in I$ we have $P_i\cap T_2 = \varnothing$, and $\#I\approx 2^{(A-1)ND}$ by (2.8).

We use Lemma 3.3 to conclude that for all $i\in I$

$$\mu(P_i) \leq \frac{\mu(Q_0)}{\Theta_0\, 2^{AND}}.$$

Summing over $i\in I$ yields

$$\mu(Q) = \sum_{i\in I}\mu(P_i) \leq \#I\cdot \frac{\mu(Q_0)}{\Theta_0\,2^{AND}} \approx 2^{(A-1)ND}\frac{\mu(Q_0)}{\Theta_0\,2^{AND}} = \frac{\mu(Q_0)}{\Theta_0\,2^{ND}},$$

so that

$$\Theta(Q) = \frac{\mu(Q)}{(2^{-N}\ell(Q_0))^{D-1}} \lesssim \frac{\mu(Q_0)}{\Theta_0\,2^{ND}}\cdot\frac{1}{2^{-N(D-1)}\ell(Q_0)^{D-1}} = \frac{\Theta_0}{\Theta_0\,2^N} = 2^{-N}\leq 1.$$

But this contradicts the assumption $Q\in \mathsf{HD}(Q_0)$:

$$\Theta(Q)\geq 2\Theta_0 \geq 2M > 1,$$

and so the proof of (3.13) is finished.

Concerning (3.14), note that by (3.13) and Lemma 3.2 we have

$$\#\mathsf{HD}(Q_0) \lesssim 2^{2N}. \tag{3.15}$$

Hence,

$$\sum_{Q\in \mathsf{HD}(Q_0)} \ell(Q)^2 = \ell(Q_0)^2\,2^{-2N}\sum_{Q\in \mathsf{HD}(Q_0)} 1 \lesssim \ell(Q_0)^2.$$

□

4 Iteration argument

To complete the proof of Theorem 1.1, we assume that the measure μ does not satisfy the polynomial growth condition (1.1). Then we will use Lemma 3.1 countably many times to construct a set Z with positive μ-measure and with Hausdorff dimension at most 2.

Suppose that there exists a ball $B(x,r)$ with $\mu(B(x,r)) \geq C_2 r^{2n+1}$; if C_2 is big enough, we can find a cube $Q_0 \in \mathfrak{D},\ Q \subset B(x,r)$ such that

$$\Theta(Q_0) \geq M,$$

where M is the constant from Lemma 3.1.

Let $A > 1$ be as in Lemma 3.1. Following the notation of Lemma 3.1, for an arbitrary cube $Q \in \mathfrak{D}$ with $\Theta(Q) \geq M$, set

$$N(Q) := \left\lfloor A^{-2} \log(\Theta(Q)) \right\rfloor$$

and

$$\mathsf{HD}(Q) := \left\{ P \in \mathfrak{D} \,|\, P \subset Q,\, \ell(P) = 2^{-N(Q)}\ell(Q),\, \Theta(P) > 2\Theta(Q) \right\}.$$

Put $Z_0 := Q_0,\ \mathsf{HD}_0 := \{Q_0\},\ \mathsf{HD}_1 := \mathsf{HD}(Q_0)$, and $Z_1 := \bigcup_{Q \in \mathsf{HD}_1} Q$. Proceeding inductively, for all $j \geq 2$ we define

$$\mathsf{HD}_j := \bigcup_{Q \in \mathsf{HD}_{j-1}} \mathsf{HD}(Q),$$

$$Z_j := \bigcup_{Q \in \mathsf{HD}_j} Q.$$

Note that for each j the cubes in HD_j form a disjoint family. Moreover, $\{Z_j\}_{j \geq 0}$ form a decreasing sequence of sets, that is $Z_{j+1} \subset Z_j$. Define

$$Z := \bigcap_{j \geq 0} Z_j.$$

Lemma 4.1. *We have*

$$\mu(Z) \gtrsim_{M,s} \mu(Q_0).$$

Proof. Observe that for $Q \in \mathsf{HD}_j$ we have

$$\Theta(Q) \geq 2^j \Theta(Q_0) \geq 2^j M. \tag{4.1}$$

In particular, $\Theta(Q) \geq M$ and so we may apply Lemma 3.1 to Q. It follows that for any $j \geq 0$ we have

$$\begin{aligned}\mu(Z_{j+1}) = \sum_{Q \in \mathsf{HD}_{j+1}} \mu(Q) = \sum_{Q \in \mathsf{HD}_j} \sum_{P \in \mathsf{HD}(Q)} \mu(P) &\overset{(3.1)}{\geq} \sum_{Q \in \mathsf{HD}_j} (1 - \Theta(Q)^{-s})\mu(Q) \\ &\overset{(4.1)}{\geq} \sum_{Q \in \mathsf{HD}_j} (1 - 2^{-js}M^{-s})\mu(Q) = (1 - 2^{-js}M^{-s})\mu(Z_j).\end{aligned}$$

Using this estimate $(j+1)$ times we arrive at

$$\mu(Z_{j+1}) \geq \prod_{i=0}^{j}(1-2^{-is}M^{-s})\mu(Q_0). \tag{4.2}$$

Since Z_j form a sequence of decreasing sets, we get by the continuity of measure

$$\mu(Z) = \lim_{j\to\infty}\mu(Z_j) \geq \prod_{i=0}^{\infty}(1-2^{-is}M^{-s})\mu(Q_0) = C(s,M)\mu(Q_0),$$

where $C(s,M)$ is positive and finite because $\sum_{i=0}^{\infty} 2^{-is} < \infty$. □

Lemma 4.2. *We have*

$$\dim_H(Z) \leq 2.$$

Proof. Recall that $N(Q) = \lfloor A^{-2}\log(\Theta(Q))\rfloor$. It follows from (4.1) that for $Q \in \mathsf{HD}_j$ we have $N(Q) \geq C_3 j A^{-2}$ for some absolute constant $C_3 > 0$. Thus, for $Q \in \mathsf{HD}_j$ and $P \in \mathsf{HD}(Q)$

$$\ell(P) = 2^{-N(Q)}\ell(Q) \leq 2^{-C_3 j A^{-2}}\ell(Q).$$

Using this observation j times we get that for $P \in \mathsf{HD}_{j+1}$

$$\ell(P) \leq 2^{-C_4 j(j+1)A^{-2}}\ell(Q_0),$$

where $C_4 = C_3/2$. Hence, the cubes from HD_j form coverings of Z with decreasing diameters, well suited for estimating the Hausdorff measure of Z.

Let $0<\varepsilon<1$, $0<\delta<1$ be small. Let $j\geq 0$ be so big that for $Q\in\mathsf{HD}_j$ we have $\operatorname{diam}(Q)\leq \Lambda\ell(Q)\leq\delta$. Then,

$$\mathcal{H}^{2+\varepsilon}_\delta(Z) \leq \Lambda^{2+\varepsilon}\sum_{Q\in\mathsf{HD}_j}\ell(Q)^{2+\varepsilon} \leq \Lambda^{2+\varepsilon}(2^{-C_4 j(j-1)A^{-2}}\ell(Q_0))^{\varepsilon}\sum_{Q\in\mathsf{HD}_j}\ell(Q)^2. \tag{4.3}$$

It follows by (3.2) that

$$\sum_{Q\in\mathsf{HD}_j}\ell(Q)^2 = \sum_{P\in\mathsf{HD}_{j-1}}\sum_{Q\in\mathsf{HD}(P)}\ell(Q)^2 \leq C_p\sum_{P\in\mathsf{HD}_{j-1}}\ell(P)^2.$$

Using the estimate above j times, and putting it together with (4.3) we arrive at

$$\mathcal{H}^{2+\varepsilon}_\delta(Z) \leq \Lambda^{2+\varepsilon}(C_p)^j\, 2^{-\varepsilon C_4 j(j-1)A^{-2}}\,\ell(Q_0)^{2+\varepsilon}.$$

The right hand side above converges to 0 as $j\to\infty$ (just note that the exponent at C_p is linear in j while the exponent at 2 is quadratic in j). Hence, $\mathcal{H}^{2+\varepsilon}_\delta(Z) = 0$. Letting $\delta\to 0$ we get $\mathcal{H}^{2+\varepsilon}(Z)=0$. Since this is true for arbitrarily small $\varepsilon>0$, it follows that

$$\dim_H(Z) = \inf\{t\geq 0 \ :\ \mathcal{H}^t(Z)=0\} \leq 2.$$

□

Proof of Theorem 1.1. We have found a set $Z \subset \mathbb{H}^n$ of dimension smaller than or equal to 2 (Lemma 4.2) but which nevertheless has positive μ-measure (Lemma 4.1). This contradicts the assumptions of Theorem 1.1. Thus, there exists $C_2 = C_2(n, C_1)$ such that $\mu(B(x,r)) \leq C_2 r^{2n+1}$ for all $x \in \mathbb{H}^n$ and $r > 0$. □

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