Anexo 3.4

Deducción de las ecuaciones del movimiento de una viga de Timoshenko giratoria

DEDUCCIÓN DE LAS ECUACIONES DEL MOVIMIENTO DE UNA VIGA DE TIMOSHENKO GIRATORIA

Hipótesis principales: viga Timoshenko, se considera la fuerza centrífuga, se considera la fuerza de Coriolis, existe una masa en el extremo, el momento de inercia de la articulación no es despreciable, accionada mediante un actuador giratorio.

```
> restart;
> with(linalg):
Warning, the protected names norm and trace have been redefined and
unprotected
```

Determinación de la posición actual, medida en el extremo libre de la viga, del c.d.m. de la masa del extremo. Esta posición es función de la posición inicial (en reposo) de la masa, y del giro elástico en el extremo libre de la viga .

```
> psiL:=eval(psi(x,t),x=L);
```

$$psiL := \psi(L,t)$$

```
> LGx:=LGx0*cos(psiL)-LGw0*sin(psiL);
```

$$LGx := LGx0\cos(\psi(L,t)) - LGw0\sin(\psi(L,t))$$

```
> LGw:=LGx0*sin(psiL)+LGw0*cos(psiL);
```

$$LGw := LGx0\sin(\psi(L,t)) + LGw0\cos(\psi(L,t))$$

Determinación de la velocidad absoluta de un punto situado en el eje centroidal de la viga. A continuación se determina el cuadrado de esta velocidad.

```
> VX:=vector([-
Wln(x,t)*diff(theta(t),t),0,diff(Wln(x,t),t)+x*diff(theta(t),
t)]);
```

$$VX := \left[-\mathrm{Wln}(x,t)\left(\frac{\partial}{\partial t}\theta(t)\right), 0, \left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)+x\left(\frac{\partial}{\partial t}\theta(t)\right)\right]$$

```
> VX2:=expand(dotprod(VX,VX,'orthogonal'));
```

$$VX2 := \mathrm{Wln}(x,t)^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2+\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)^2+2\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)x\left(\frac{\partial}{\partial t}\theta(t)\right)+x^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2$$

Determinación de la velocidad absoluta del centro de masa de la masa en el extremo. Inmediatamente se determina el cuadrado de esta velocidad y se hace factor común para reemplazar la suma del seno al cuadrado y el coseno al cuadrado por la unidad.

```
> VG:=vector([-LGw*diff(psi(L,t),t)-LGw*diff(theta(t),t)-
Wln(L,t)*diff(theta(t),t),0,diff(Wln(L,t),t)+LGx*diff(psi(L,t
),t)+L*diff(theta(t),t)+LGx*diff(theta(t),t)]);
```

$$VG := \left[-(LGx0\sin(\psi(L,t))+LGw0\cos(\psi(L,t)))\left(\frac{\partial}{\partial t}\psi(L,t)\right)\right.$$
$$-(LGx0\sin(\psi(L,t))+LGw0\cos(\psi(L,t)))\left(\frac{\partial}{\partial t}\theta(t)\right)-\mathrm{Wln}(L,t)\left(\frac{\partial}{\partial t}\theta(t)\right), 0,$$

$$\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)+(LGx0\cos(\psi(L,t))-LGw0\sin(\psi(L,t)))\left(\frac{\partial}{\partial t}\psi(L,t)\right)$$
$$+L\left(\frac{\partial}{\partial t}\theta(t)\right)+(LGx0\cos(\psi(L,t))-LGw0\sin(\psi(L,t)))\left(\frac{\partial}{\partial t}\theta(t)\right)\Bigg]$$

```
> VG2:=expand(dotprod(VG,VG,'orthogonal'));
```

$$VG2 := 2\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)L\left(\frac{\partial}{\partial t}\theta(t)\right)+\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0^2\sin(\psi(L,t))^2$$
$$+2\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0^2\sin(\psi(L,t))^2\left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+2\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\cos(\psi(L,t))\,L\left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$-2\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\sin(\psi(L,t))\,L\left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+2\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0^2\cos(\psi(L,t))^2\left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+2\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0\sin(\psi(L,t))\,\mathrm{Wln}(L,t)$$
$$+2\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0\cos(\psi(L,t))\,\mathrm{Wln}(L,t)$$
$$+2\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\cos(\psi(L,t))$$
$$-2\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\sin(\psi(L,t))$$
$$+2\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)LGx0\cos(\psi(L,t))$$
$$-2\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)LGw0\sin(\psi(L,t))$$
$$+2\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0^2\cos(\psi(L,t))^2\left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+2\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0^2\sin(\psi(L,t))^2\left(\frac{\partial}{\partial t}\theta(t)\right)+2L\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0\cos(\psi(L,t))$$
$$-2L\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0\sin(\psi(L,t))$$
$$+2\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\sin(\psi(L,t))\,\mathrm{Wln}(L,t)\left(\frac{\partial}{\partial t}\theta(t)\right)+\mathrm{Wln}(L,t)^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2$$
$$+\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)^2+L^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2$$

$$+2\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\cos(\psi(L,t))\,\mathrm{Wln}(L,t)\left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+\left(\frac{\partial}{\partial t}\psi(L,t)\right)^2 LGw0^2\sin(\psi(L,t))^2+\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0^2\cos(\psi(L,t))^2$$
$$+\left(\frac{\partial}{\partial t}\psi(L,t)\right)^2 LGw0^2\cos(\psi(L,t))^2+\left(\frac{\partial}{\partial t}\psi(L,t)\right)^2 LGx0^2\sin(\psi(L,t))^2$$
$$+\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0^2\cos(\psi(L,t))^2+\left(\frac{\partial}{\partial t}\psi(L,t)\right)^2 LGx0^2\cos(\psi(L,t))^2$$
$$+\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0^2\sin(\psi(L,t))^2$$

```
> VG2s:=collect(combine(VG2,trig),[LGx0^2,LGw0^2]):
> VG2a:=algsubs(LGx0^2+LGw0^2=LG^2,VG2s);
```

$$VG2a := \Bigg(2\left(\frac{\partial}{\partial t}\theta(t)\right)^2\sin(\psi(L,t))\,\mathrm{Wln}(L,t)+2\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)\cos(\psi(L,t))$$
$$+2L\left(\frac{\partial}{\partial t}\theta(t)\right)^2\cos(\psi(L,t))+2\left(\frac{\partial}{\partial t}\psi(L,t)\right)\cos(\psi(L,t))\,L\left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+2\left(\frac{\partial}{\partial t}\psi(L,t)\right)\sin(\psi(L,t))\,\mathrm{Wln}(L,t)\left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+2\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)\cos(\psi(L,t))\Bigg)LGx0+\Bigg($$
$$-2\left(\frac{\partial}{\partial t}\psi(L,t)\right)\sin(\psi(L,t))\,L\left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$-2\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)\sin(\psi(L,t))$$
$$+2\left(\frac{\partial}{\partial t}\psi(L,t)\right)\cos(\psi(L,t))\,\mathrm{Wln}(L,t)\left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+2\left(\frac{\partial}{\partial t}\theta(t)\right)^2\cos(\psi(L,t))\,\mathrm{Wln}(L,t)-2L\left(\frac{\partial}{\partial t}\theta(t)\right)^2\sin(\psi(L,t))$$
$$-2\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\psi(L,t))\Bigg)LGw0+\mathrm{Wln}(L,t)^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2$$
$$+LG^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2+2LG^2\left(\frac{\partial}{\partial t}\psi(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)+LG^2\left(\frac{\partial}{\partial t}\psi(L,t)\right)^2$$
$$+2\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)L\left(\frac{\partial}{\partial t}\theta(t)\right)+\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)^2+L^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2$$

Determinación de la velocidad relativa del c.d.m. de la masa del extremo respecto al eje giratorio.

```
> VR_G:=vector([-
LGw*diff(psi(L,t),t),0,diff(Wln(L,t),t)+LGx*diff(psi(L,t),t)]
```

```
);
```

$$VR\_G := \left[-(LGx0 \sin(\psi(L,t)) + LGw0 \cos(\psi(L,t))) \left(\frac{\partial}{\partial t}\psi(L,t)\right), 0, \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) + (LGx0 \cos(\psi(L,t)) - LGw0 \sin(\psi(L,t))) \left(\frac{\partial}{\partial t}\psi(L,t)\right) \right]$$

Determinación de la fuerzas centrífuga debida a la masa del extremo. Primero se obtiene la componente en x (FCENT_G_X) y a continuación la componente en w (FCENT_G_W).

```
> FCENT_G_X:=m_l*(L+LGx)*diff(theta(t),t)^2;
```

$$FCENT\_G\_X := m\_l\,(L + LGx0 \cos(\psi(L,t)) - LGw0 \sin(\psi(L,t))) \left(\frac{\partial}{\partial t}\theta(t)\right)^2$$

```
> FCENT_G_W:=m_l*(LGw)*diff(theta(t),t)^2;
```

$$FCENT\_G\_W := m\_l\,(LGx0 \sin(\psi(L,t)) + LGw0 \cos(\psi(L,t))) \left(\frac{\partial}{\partial t}\theta(t)\right)^2$$

Determinación de la fuerzas de Coriolis debida a la masa del extremo.
A continuación se separan la componente en x (FCOR_G_X) y la componente en w (FCOR_G_W).

```
> FCOR_G:=-2*m_l*crossprod([0,-diff(theta(t),t),0],VR_G);
```

$$FCOR\_G := -2\, m\_l \left[-\left(\frac{\partial}{\partial t}\theta(t)\right) \left(\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) + (LGx0 \cos(\psi(L,t)) - LGw0 \sin(\psi(L,t))) \left(\frac{\partial}{\partial t}\psi(L,t)\right)\right), 0, -\left(\frac{\partial}{\partial t}\theta(t)\right) (LGx0 \sin(\psi(L,t)) + LGw0 \cos(\psi(L,t))) \left(\frac{\partial}{\partial t}\psi(L,t)\right) \right]$$

```
> FCOR_G_X:=dotprod([1,0,0],FCOR_G,'orthogonal');
```

$$FCOR\_G\_X := 2\, m\_l \left(\frac{\partial}{\partial t}\theta(t)\right) \left(\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) + (LGx0 \cos(\psi(L,t)) - LGw0 \sin(\psi(L,t))) \left(\frac{\partial}{\partial t}\psi(L,t)\right)\right)$$

```
> FCOR_G_W:=dotprod([0,0,1],FCOR_G,'orthogonal');
```

$$FCOR\_G\_W := 2\, m\_l \left(\frac{\partial}{\partial t}\theta(t)\right) (LGx0 \sin(\psi(L,t)) + LGw0 \cos(\psi(L,t))) \left(\frac{\partial}{\partial t}\psi(L,t)\right)$$

Determinación de la energía cinética total.

```
>
Ec:=expand(1/2*J_h*(diff(theta(t),t)+diff(psi(x=0,t),t))^2+1/
2*A*rho*diff(theta(t),t)^2*int(Wln(x,t)^2,x=0..L)+1/2*A*rho*i
nt(diff(Wln(x,t),t)^2,x=0..L)+A*rho*diff(theta(t),t)*int(diff
(Wln(x,t),t)*x,x=0..L)+1/2*A*rho*diff(theta(t),t)^2*int(x^2,x
=0..L)+int(1/2*J*rho*(diff(psi(x,t),t)+diff(theta(t),t))^2,x=
0..L)+1/2*m_l*VG2a+1/2*J_g*(diff(psi(L,t),t)+diff(theta(t),t)
)^2);
```

$$
\begin{aligned}
Ec := {} & \frac{1}{2} m\_l\, \mathrm{Wln}(L,t)^2 \left(\frac{\partial}{\partial t}\theta(t)\right)^2 + \frac{1}{2} m\_l\, L^2 \left(\frac{\partial}{\partial t}\theta(t)\right)^2 + J\_h \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(x=0,t)\right) \\
& + \frac{1}{2} J\_h \left(\frac{\partial}{\partial t}\theta(t)\right)^2 + \frac{1}{2} J\_h \left(\frac{\partial}{\partial t}\psi(x=0,t)\right)^2 + \frac{1}{6} A\,\rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 L^3 \\
& + J\_g \left(\frac{\partial}{\partial t}\psi(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right) + \frac{1}{2} A\,\rho \int_0^L \left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)^2 dx \\
& + \frac{1}{2} J\,\rho \int_0^L \left(\frac{\partial}{\partial t}\psi(x,t)\right)^2 + 2\left(\frac{\partial}{\partial t}\psi(x,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right) + \left(\frac{\partial}{\partial t}\theta(t)\right)^2 dx \\
& + m\_l\, LG^2 \left(\frac{\partial}{\partial t}\psi(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right) + m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) L \left(\frac{\partial}{\partial t}\theta(t)\right) \\
& + \frac{1}{2} m\_l\, LG^2 \left(\frac{\partial}{\partial t}\theta(t)\right)^2 + A\,\rho \left(\frac{\partial}{\partial t}\theta(t)\right) \int_0^L \left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right) x\, dx \\
& + \frac{1}{2} A\,\rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 \int_0^L \mathrm{Wln}(x,t)^2\, dx - m\_l\, L \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0 \sin(\psi(L,t)) \\
& - m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right) LGw0 \sin(\psi(L,t)) \\
& + m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right) LGx0 \cos(\psi(L,t)) \\
& + m\_l\, L \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0 \cos(\psi(L,t)) \\
& + m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \cos(\psi(L,t))\, L \left(\frac{\partial}{\partial t}\theta(t)\right) \\
& + m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \sin(\psi(L,t))\, \mathrm{Wln}(L,t) \left(\frac{\partial}{\partial t}\theta(t)\right) \\
& + m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \cos(\psi(L,t)) \\
& - m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \sin(\psi(L,t))\, L \left(\frac{\partial}{\partial t}\theta(t)\right) \\
& + m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0 \sin(\psi(L,t))\, \mathrm{Wln}(L,t) \\
& - m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \sin(\psi(L,t)) + \frac{1}{2} J\_g \left(\frac{\partial}{\partial t}\theta(t)\right)^2
\end{aligned}
$$

$$+\frac{1}{2} J\_g \left(\frac{\partial}{\partial t}\psi(L,t)\right)^2+\frac{1}{2} m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)^2$$

$$+ m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \cos(\psi(L,t)) \mathrm{Wln}(L,t) \left(\frac{\partial}{\partial t}\theta(t)\right)$$

$$+ m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0 \cos(\psi(L,t)) \mathrm{Wln}(L,t)+\frac{1}{2} m\_l\, LG^2 \left(\frac{\partial}{\partial t}\psi(L,t)\right)^2$$

A continuación se calcula la energía potencial debida a la deformación elástica.
Se ha optado por incluir en la energía potencial el trabajo que producen las componentes x de la fuerza centrífuga y de Coriolis al deformar la viga. Nota: las componentes w de esas fuerzas no deben ser incluidas aquí, pues aparecen naturalmente al derivar el Lagrangiano.

```
> fc:=FCENT_G_X+int(A*rho*x*diff(theta(t),t)^2,x=x..L);
```

$$fc := m\_l\,(L+LGx0\cos(\psi(L,t))-LGw0\sin(\psi(L,t)))\left(\frac{\partial}{\partial t}\theta(t)\right)^2$$

$$+\frac{1}{2} A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 (L^2-x^2)$$

```
>
fcor:=FCOR_G_X+int(2*A*rho*diff(Wln(x,t),t)*diff(theta(t),t),
x=x..L);
```

$$fcor := 2\, m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)$$

$$\left(\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)+(LGx0\cos(\psi(L,t))-LGw0\sin(\psi(L,t)))\left(\frac{\partial}{\partial t}\psi(L,t)\right)\right)$$

$$+\int_x^L 2\,A\,\rho\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)dx$$

Determinación de la energía potencial.

```
> Ep:=1/2*E*J*int(diff(psi(x,t),x)^2,x = 0 ..
L)+1/2*k*G*A*int(diff(Wln(x,t),x)^2,x = 0 ..
L)+1/2*k*G*A*int((psi(x,t))^2,x = 0 .. L)-
k*G*A*int(diff(Wln(x,t),x)*psi(x,t),x = 0 ..
L)+1/2*m_l*(L+LGx)*diff(theta(t),t)^2*int(diff(Wln(x,t),x)^2,
x = 0 ..
L)+1/4*rho*A*L^2*diff(theta(t),t)^2*int(diff(Wln(x,t),x)^2,x
= 0 .. L)-
1/4*rho*A*diff(theta(t),t)^2*int(diff(Wln(x,t),x)^2*x^2,x = 0
.. L)+1/2*int(fcor*diff(Wln(x,t),x)^2,x = 0 .. L);
>
```

$$Ep := \frac{1}{2} E\,J\int_0^L\left(\frac{\partial}{\partial x}\psi(x,t)\right)^2 dx+\frac{1}{2} k\,G\,A\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx+\frac{1}{2} k\,G\,A\int_0^L \psi(x,t)^2\,dx$$

$$-k\,G\,A\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\psi(x,t)\,dx+\frac{1}{2}\,m\_l$$

$$(L+LGx0\cos(\psi(L,t))-LGw0\sin(\psi(L,t)))\left(\frac{\partial}{\partial t}\theta(t)\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx$$

$$+\frac{1}{4}\rho\,A\,L^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx$$

$$-\frac{1}{4}\rho\,A\left(\frac{\partial}{\partial t}\theta(t)\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2x^2\,dx+\frac{1}{2}\int_0^L\left(\left(2\,m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\right.\right.$$

$$\left(\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)+(LGx0\cos(\psi(L,t))-LGw0\sin(\psi(L,t)))\left(\frac{\partial}{\partial t}\psi(L,t)\right)\right)$$

$$\left.+\int_x^L 2\,A\,\rho\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)dx\right)\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx$$

Aplicación de las ecuaciones de Lagrange para encontrar una ecuación global.

Determinación del Lagrangiano.

```
> e0:=expand(Ec-Ep);
```

$$e0:=\frac{1}{2}\,m\_l\,\mathrm{Wln}(L,t)^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2+\frac{1}{2}\,m\_l\,L^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2+J\_h\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(x=0,t)\right)$$

$$-\frac{1}{2}\int_0^L 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)$$

$$+2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\cos(\psi(L,t))$$

$$-2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\sin(\psi(L,t))$$

$$+2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)\int_x^L\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\,dx\,dx+\frac{1}{2}J\_h\left(\frac{\partial}{\partial t}\theta(t)\right)^2$$

$$+\frac{1}{2}J\_h\left(\frac{\partial}{\partial t}\psi(x=0,t)\right)^2+\frac{1}{6}A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2L^3+k\,G\,A\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\psi(x,t)\,dx$$

$$-\frac{1}{2}k\,G\,A\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx-\frac{1}{2}k\,G\,A\int_0^L\psi(x,t)^2\,dx$$

$$+\frac{1}{4}\rho\,A\left(\frac{\partial}{\partial t}\theta(t)\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2x^2\,dx$$

$$-\frac{1}{4}\rho\,A\,L^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx+J\_g\left(\frac{\partial}{\partial t}\psi(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)$$

$$-\frac{1}{2}m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx\,L+\frac{1}{2}A\,\rho\int_0^L\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)^2dx$$

$$+\frac{1}{2}J\,\rho\int_0^L\left(\frac{\partial}{\partial t}\psi(x,t)\right)^2+2\left(\frac{\partial}{\partial t}\psi(x,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)+\left(\frac{\partial}{\partial t}\theta(t)\right)^2dx$$

$$-\frac{1}{2}m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx\,LGx0\cos(\psi(L,t))$$

$$+\frac{1}{2}m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx\,LGw0\sin(\psi(L,t))$$

$$+m\_l\,LG^2\left(\frac{\partial}{\partial t}\psi(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)+m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)L\left(\frac{\partial}{\partial t}\theta(t)\right)$$

$$+\frac{1}{2}m\_l\,LG^2\left(\frac{\partial}{\partial t}\theta(t)\right)^2-\frac{1}{2}E\,J\int_0^L\left(\frac{\partial}{\partial x}\psi(x,t)\right)^2dx$$

$$+A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)\int_0^L\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)x\,dx+\frac{1}{2}A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2\int_0^L\mathrm{Wln}(x,t)^2\,dx$$

$$- m\_l\, L \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0 \sin(\psi(L,t))$$
$$- m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right) LGw0 \sin(\psi(L,t))$$
$$+ m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right) LGx0 \cos(\psi(L,t))$$
$$+ m\_l\, L \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0 \cos(\psi(L,t))$$
$$+ m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \cos(\psi(L,t))\, L \left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+ m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \sin(\psi(L,t))\, \mathrm{Wln}(L,t) \left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+ m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \cos(\psi(L,t))$$
$$- m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \sin(\psi(L,t))\, L \left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+ m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0 \sin(\psi(L,t))\, \mathrm{Wln}(L,t)$$
$$- m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \sin(\psi(L,t)) + \frac{1}{2} J\_g \left(\frac{\partial}{\partial t}\theta(t)\right)^2$$
$$+ \frac{1}{2} J\_g \left(\frac{\partial}{\partial t}\psi(L,t)\right)^2 + \frac{1}{2} m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)^2$$
$$+ m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \cos(\psi(L,t))\, \mathrm{Wln}(L,t) \left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+ m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0 \cos(\psi(L,t))\, \mathrm{Wln}(L,t) + \frac{1}{2} m\_l\, LG^2 \left(\frac{\partial}{\partial t}\psi(L,t)\right)^2$$

La siguiente sustitución es una estrategia para poder derivar el Lagrangiano con MAPLE. Este programa no soporta derivar respecto de una función por lo que se ha de substituir la función velocidad angular por una variable (thetapunto), derivar respecto thetapunto y deshacer la substitución.

```
> e1:=subs(diff(theta(t),t)=thetapunto,e0);
```

$$e1 := \frac{1}{2} m\_l\, L^2\, thetapunto^2$$
$$+ \frac{1}{2} J\,\rho \int_0^L \left(\frac{\partial}{\partial t}\psi(x,t)\right)^2 + 2\left(\frac{\partial}{\partial t}\psi(x,t)\right) thetapunto + thetapunto^2\, dx$$
$$+ \frac{1}{2} m\_l\, \mathrm{Wln}(L,t)^2\, thetapunto^2 + \frac{1}{2} J\_h \left(\frac{\partial}{\partial t}\psi(x=0,t)\right)^2$$

$$-\frac{1}{4}\rho\, A\, L^2\, thetapunto^2 \int_0^L \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx + m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) L\, thetapunto$$

$$+ A\,\rho\, thetapunto \int_0^L \left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right) x\, dx + \frac{1}{2} A\,\rho\, thetapunto^2 \int_0^L \mathrm{Wln}(x,t)^2\, dx$$

$$+\frac{1}{2} m\_l\, LG^2\, thetapunto^2 - \frac{1}{2}\int_0^L 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\, thetapunto \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)$$

$$+ 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\, thetapunto \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0\cos(\psi(L,t))$$

$$- 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\, thetapunto \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0\sin(\psi(L,t))$$

$$+ 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho\, thetapunto \int_x^L \frac{\partial}{\partial t}\mathrm{Wln}(x,t)\, dx\, dx + \frac{1}{2} J\_h\, thetapunto^2$$

$$+\frac{1}{2} J\_g\, thetapunto^2 + m\_l\, thetapunto^2\, LGw0\cos(\psi(L,t))\,\mathrm{Wln}(L,t)$$

$$- m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0\sin(\psi(L,t))\, L\, thetapunto$$

$$+ m\_l\, thetapunto^2\, LGx0\sin(\psi(L,t))\,\mathrm{Wln}(L,t)$$

$$- m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) thetapunto\, LGw0\sin(\psi(L,t))$$

$$+ m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) thetapunto\, LGx0\cos(\psi(L,t))$$

$$+ m\_l\, L\, thetapunto^2\, LGx0\cos(\psi(L,t))$$

$$+ m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0\cos(\psi(L,t))\, L\, thetapunto$$

$$- m\_l\, L\, thetapunto^2\, LGw0\sin(\psi(L,t))$$

$$+ m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0\sin(\psi(L,t))\,\mathrm{Wln}(L,t)\, thetapunto$$

$$+ m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0\cos(\psi(L,t))\,\mathrm{Wln}(L,t)\, thetapunto$$

$$-\frac{1}{2} m\_l\, thetapunto^2 \int_0^L \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx\, LGx0\cos(\psi(L,t))$$

$$+\frac{1}{2} m\_l\, thetapunto^2 \int_0^L \left(\frac{\partial}{\partial x} \mathrm{Wln}(x,t)\right)^2 dx\, LGw0 \sin(\psi(L,t))$$

$$+ k\, G\, A \int_0^L \left(\frac{\partial}{\partial x} \mathrm{Wln}(x,t)\right) \psi(x,t)\, dx - \frac{1}{2} k\, G\, A \int_0^L \left(\frac{\partial}{\partial x} \mathrm{Wln}(x,t)\right)^2 dx$$

$$-\frac{1}{2} k\, G\, A \int_0^L \psi(x,t)^2\, dx + \frac{1}{2} A\, \rho \int_0^L \left(\frac{\partial}{\partial t} \mathrm{Wln}(x,t)\right)^2 dx$$

$$+ J\_h\, thetapunto \left(\frac{\partial}{\partial t} \psi(x=0,t)\right) - \frac{1}{2} E\, J \int_0^L \left(\frac{\partial}{\partial x} \psi(x,t)\right)^2 dx$$

$$+ m\_l \left(\frac{\partial}{\partial t} \mathrm{Wln}(L,t)\right) \left(\frac{\partial}{\partial t} \psi(L,t)\right) LGx0 \cos(\psi(L,t))$$

$$- m\_l \left(\frac{\partial}{\partial t} \mathrm{Wln}(L,t)\right) \left(\frac{\partial}{\partial t} \psi(L,t)\right) LGw0 \sin(\psi(L,t)) + \frac{1}{2} J\_g \left(\frac{\partial}{\partial t} \psi(L,t)\right)^2$$

$$+\frac{1}{2} m\_l \left(\frac{\partial}{\partial t} \mathrm{Wln}(L,t)\right)^2 + J\_g \left(\frac{\partial}{\partial t} \psi(L,t)\right) thetapunto + \frac{1}{2} m\_l\, LG^2 \left(\frac{\partial}{\partial t} \psi(L,t)\right)^2$$

$$+\frac{1}{6} A\, \rho\, thetapunto^2 L^3 + m\_l\, LG^2 \left(\frac{\partial}{\partial t} \psi(L,t)\right) thetapunto$$

$$-\frac{1}{2} m\_l\, thetapunto^2 \int_0^L \left(\frac{\partial}{\partial x} \mathrm{Wln}(x,t)\right)^2 dx\, L$$

$$+\frac{1}{4} \rho\, A\, thetapunto^2 \int_0^L \left(\frac{\partial}{\partial x} \mathrm{Wln}(x,t)\right)^2 x^2\, dx$$

\>

```
> e2:=diff(e1,thetapunto);
```

$$e2 := \frac{1}{2} J\, \rho \int_0^L 2 \left(\frac{\partial}{\partial t} \psi(x,t)\right) + 2\, thetapunto\, dx + m\_l\, LG^2\, thetapunto$$

$$+ m\_l \left(\frac{\partial}{\partial t} \psi(L,t)\right) LGw0 \cos(\psi(L,t))\, \mathrm{Wln}(L,t)$$

$$
- m\_l\ thetapunto \int_0^L \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx\ LGx0 \cos(\psi(L,t))
$$

$$
+ m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \cos(\psi(L,t))\, L - 2\, m\_l\, L\ thetapunto\ LGw0 \sin(\psi(L,t))
$$

$$
+ \frac{1}{3} A\, \rho\ thetapunto\ L^3 - m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \sin(\psi(L,t))\, L
$$

$$
+ 2\, m\_l\ thetapunto\ LGx0 \sin(\psi(L,t))\, \mathrm{Wln}(L,t)
$$

$$
- m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) LGw0 \sin(\psi(L,t)) + m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) LGx0 \cos(\psi(L,t))
$$

$$
+ J\_h\left(\frac{\partial}{\partial t}\psi(x=0,t)\right) + J\_h\ thetapunto
$$

$$
+ 2\, m\_l\ thetapunto\ LGw0 \cos(\psi(L,t))\, \mathrm{Wln}(L,t)
$$

$$
+ 2\, m\_l\, L\ thetapunto\ LGx0 \cos(\psi(L,t))
$$

$$
+ m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \sin(\psi(L,t))\, \mathrm{Wln}(L,t)
$$

$$
+ m\_l\ thetapunto \int_0^L \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx\ LGw0 \sin(\psi(L,t)) + J\_g\ thetapunto - \frac{1}{2}\int_0^L
$$

$$
2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)
$$

$$
+ 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \cos(\psi(L,t))
$$

$$
- 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \sin(\psi(L,t))
$$

$$
+ 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\, \rho \int_x^L \frac{\partial}{\partial t}\mathrm{Wln}(x,t)\, dx\, dx + m\_l\, \mathrm{Wln}(L,t)^2\ thetapunto
$$

$$
+ m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) L + m\_l\, L^2\ thetapunto + J\_g\left(\frac{\partial}{\partial t}\psi(L,t)\right)
$$

$$
- \frac{1}{2}\rho\, A\, L^2\ thetapunto \int_0^L \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx + A\, \rho\ thetapunto \int_0^L \mathrm{Wln}(x,t)^2\, dx
$$

$$
+ A\, \rho \int_0^L \left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right) x\, dx + m\_l\, LG^2\left(\frac{\partial}{\partial t}\psi(L,t)\right)
$$

$$+\frac{1}{2}\rho\, A\ thetapunto \int_0^L \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 x^2\,dx$$

$$-\ m\_l\ thetapunto \int_0^L \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx\, L$$

>

```
> e3:=subs(thetapunto=diff(theta(t),t),e2);
```

$$e3 := m\_l\, L^2 \left(\frac{\partial}{\partial t}\theta(t)\right) + m\_l\, \mathrm{Wln}(L,t)^2 \left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+ m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \cos(\psi(L,t))\, \mathrm{Wln}(L,t)$$
$$+ m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \cos(\psi(L,t))\, L - m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \sin(\psi(L,t))\, L$$
$$- m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) LGw0 \sin(\psi(L,t)) + m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) LGx0 \cos(\psi(L,t))$$
$$+ J\_h \left(\frac{\partial}{\partial t}\psi(x=0,t)\right) + m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \sin(\psi(L,t))\, \mathrm{Wln}(L,t) - \frac{1}{2}\int_0^L$$
$$2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)$$
$$+ 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \cos(\psi(L,t))$$
$$- 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \sin(\psi(L,t))$$
$$+ 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho \int_x^L \frac{\partial}{\partial t}\mathrm{Wln}(x,t)\,dx\,dx + J\_g \left(\frac{\partial}{\partial t}\theta(t)\right) + m\_l\, LG^2 \left(\frac{\partial}{\partial t}\theta(t)\right)$$
$$+ 2\, m\_l\, L \left(\frac{\partial}{\partial t}\theta(t)\right) LGx0 \cos(\psi(L,t))$$
$$+ m\_l \left(\frac{\partial}{\partial t}\theta(t)\right) \int_0^L \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx\, LGw0 \sin(\psi(L,t))$$

$$-\frac{1}{2}\rho\,A\,L^2\left(\frac{\partial}{\partial t}\theta(t)\right)\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx$$

$$+2\,m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)LGw0\cos(\psi(L,t))\,\mathrm{Wln}(L,t)$$

$$-m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx\,LGx0\cos(\psi(L,t))$$

$$-2\,m\_l\,L\left(\frac{\partial}{\partial t}\theta(t)\right)LGw0\sin(\psi(L,t))$$

$$+2\,m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)LGx0\sin(\psi(L,t))\,\mathrm{Wln}(L,t)+J\_h\left(\frac{\partial}{\partial t}\theta(t)\right)$$

$$+m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)L+J\_g\left(\frac{\partial}{\partial t}\psi(L,t)\right)+\frac{1}{2}J\rho\int_0^L 2\left(\frac{\partial}{\partial t}\psi(x,t)\right)+2\left(\frac{\partial}{\partial t}\theta(t)\right)dx$$

$$+A\,\rho\int_0^L\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)x\,dx+m\_l\,LG^2\left(\frac{\partial}{\partial t}\psi(L,t)\right)$$

$$+\frac{1}{2}\rho\,A\left(\frac{\partial}{\partial t}\theta(t)\right)\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 x^2\,dx+A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)\int_0^L\mathrm{Wln}(x,t)^2\,dx$$

$$+\frac{1}{3}A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)L^3-m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx\,L$$

Otra vez la misma estrategia para poder derivar respecto del ángulo theta.

```
> e4:=subs(theta(t)=theta,e0);
```

$$e4:=\frac{1}{2}m\_l\,\mathrm{Wln}(L,t)^2\left(\frac{\partial}{\partial t}\theta\right)^2+\frac{1}{2}m\_l\,LG^2\left(\frac{\partial}{\partial t}\theta\right)^2$$

$$+\frac{1}{2}J\rho\int_0^L\left(\frac{\partial}{\partial t}\psi(x,t)\right)^2+2\left(\frac{\partial}{\partial t}\psi(x,t)\right)\left(\frac{\partial}{\partial t}\theta\right)+\left(\frac{\partial}{\partial t}\theta\right)^2 dx$$

$$+J\_h\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\psi(x=0,t)\right)+\frac{1}{2}m\_l\,L^2\left(\frac{\partial}{\partial t}\theta\right)^2+\frac{1}{2}J\_h\left(\frac{\partial}{\partial t}\psi(x=0,t)\right)^2$$

$$+k\,G\,A\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\psi(x,t)\,dx-\frac{1}{2}k\,G\,A\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx$$

$$-\frac{1}{2}k\,G\,A\int_0^L\psi(x,t)^2\,dx+\frac{1}{6}A\,\rho\left(\frac{\partial}{\partial t}\theta\right)^2L^3+\frac{1}{2}J\_h\left(\frac{\partial}{\partial t}\theta\right)^2+\frac{1}{2}J\_g\left(\frac{\partial}{\partial t}\theta\right)^2$$

$$+m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\sin(\psi(L,t))\,\mathrm{Wln}(L,t)\left(\frac{\partial}{\partial t}\theta\right)$$

$$-m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\sin(\psi(L,t))\,L\left(\frac{\partial}{\partial t}\theta\right)$$

$$+m\_l\left(\frac{\partial}{\partial t}\theta\right)^2LGx0\sin(\psi(L,t))\,\mathrm{Wln}(L,t)$$

$$+m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\cos(\psi(L,t))\,\mathrm{Wln}(L,t)\left(\frac{\partial}{\partial t}\theta\right)$$

$$+m\_l\left(\frac{\partial}{\partial t}\theta\right)^2LGw0\cos(\psi(L,t))\,\mathrm{Wln}(L,t)$$

$$+\frac{1}{2}m\_l\left(\frac{\partial}{\partial t}\theta\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx\,LGw0\sin(\psi(L,t))$$

$$-m\_l\,L\left(\frac{\partial}{\partial t}\theta\right)^2LGw0\sin(\psi(L,t))-m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\theta\right)LGw0\sin(\psi(L,t))$$

$$+m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\theta\right)LGx0\cos(\psi(L,t))+m\_l\,L\left(\frac{\partial}{\partial t}\theta\right)^2LGx0\cos(\psi(L,t))$$

$$-\frac{1}{4}\rho\,A\,L^2\left(\frac{\partial}{\partial t}\theta\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx$$

$$-\frac{1}{2}m\_l\left(\frac{\partial}{\partial t}\theta\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx\,LGx0\cos(\psi(L,t))$$

$$+m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\cos(\psi(L,t))\,L\left(\frac{\partial}{\partial t}\theta\right)$$

$$+\frac{1}{4}\rho\,A\left(\frac{\partial}{\partial t}\theta\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2x^2\,dx-\frac{1}{2}m\_l\left(\frac{\partial}{\partial t}\theta\right)^2\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx\,L$$

$$+m\_l\,LG^2\left(\frac{\partial}{\partial t}\psi(L,t)\right)\left(\frac{\partial}{\partial t}\theta\right)+m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)L\left(\frac{\partial}{\partial t}\theta\right)$$

$$+A\,\rho\left(\frac{\partial}{\partial t}\theta\right)\int_0^L\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)x\,dx+\frac{1}{2}A\,\rho\left(\frac{\partial}{\partial t}\theta\right)^2\int_0^L\mathrm{Wln}(x,t)^2\,dx$$

$$+J\_g\left(\frac{\partial}{\partial t}\psi(L,t)\right)\left(\frac{\partial}{\partial t}\theta\right)-\frac{1}{2}\left|\int_0^L 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\right.$$

$$+2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\cos(\psi(L,t))$$

$$-2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\sin(\psi(L,t))$$

$$+2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta\right)\int_x^L\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\,dx\,dx+\frac{1}{2}A\,\rho\left|\int_0^L\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)^2 dx\right.$$

$$-\frac{1}{2}E\,J\left|\int_0^L\left(\frac{\partial}{\partial x}\psi(x,t)\right)^2 dx+m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\cos(\psi(L,t))\right.$$

$$-m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\sin(\psi(L,t))+\frac{1}{2}J\_g\left(\frac{\partial}{\partial t}\psi(L,t)\right)^2$$

$$+\frac{1}{2}m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)^2+\frac{1}{2}m\_l\,LG^2\left(\frac{\partial}{\partial t}\psi(L,t)\right)^2$$

```
> e5:=diff(e4,theta);
```

$$e5:=0$$

Construcción de la ecuación de Lagrange para el ángulo theta.

```
> e6:=diff(e3,t)-e5-M_theta=0;
```

$$e6:=J\_g\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)+\frac{1}{2}J\,\rho\left|\int_0^L 2\left(\frac{\partial^2}{\partial t^2}\psi(x,t)\right)+2\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)dx\right.$$

$$-m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\int_0^L 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)dx\,LGx0\cos(\psi(L,t))$$

$$-\frac{1}{2}\rho\,A\,L^2\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 dx$$

$$-\frac{1}{2}\rho A L^2\left(\frac{\partial}{\partial t}\theta(t)\right)\int_0^L 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)dx$$

$$+2\,m\_l\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)LGx0\sin(\psi(L,t))\,\mathrm{Wln}(L,t)$$

$$-m\_l\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx\,LGx0\cos(\psi(L,t))$$

$$-2\,m\_l\,L\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)LGw0\sin(\psi(L,t))$$

$$+2\,m\_l\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)LGw0\cos(\psi(L,t))\,\mathrm{Wln}(L,t)+J\_h\left(\frac{\partial^2}{\partial t^2}\psi(x=0,t)\right)$$

$$+J\_h\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)+J\_g\left(\frac{\partial^2}{\partial t^2}\psi(L,t)\right)$$

$$+m\_l\left(\frac{\partial^2}{\partial t^2}\psi(L,t)\right)LGw0\cos(\psi(L,t))\,\mathrm{Wln}(L,t)$$

$$-m\_l\left(\frac{\partial^2}{\partial t^2}\psi(L,t)\right)LGw0\sin(\psi(L,t))\,L$$

$$+m\_l\left(\frac{\partial^2}{\partial t^2}\psi(L,t)\right)LGx0\sin(\psi(L,t))\,\mathrm{Wln}(L,t)$$

$$+m\_l\left(\frac{\partial^2}{\partial t^2}\psi(L,t)\right)LGx0\cos(\psi(L,t))\,L$$

$$+m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\int_0^L 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)dx\,LGw0\sin(\psi(L,t))$$

$$+2\,m\_l\,L\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)LGx0\cos(\psi(L,t))$$

$$+m\_l\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2dx\,LGw0\sin(\psi(L,t))-\frac{1}{2}\int_0^L$$

$$4\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)m\_l\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)$$

$$+2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2m\_l\left(\frac{\partial^2}{\partial t^2}\mathrm{Wln}(L,t)\right)$$

$$+4\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\cos(\psi(L,t))\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)$$

$$+2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial^2}{\partial t^2}\psi(L,t)\right)LGx0\cos(\psi(L,t))$$

$$-2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right)^2 LGx0\sin(\psi(L,t))$$

$$-4\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\sin(\psi(L,t))\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)$$

$$-2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial^2}{\partial t^2}\psi(L,t)\right)LGw0\sin(\psi(L,t))$$

$$-2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\psi(L,t)\right)^2 LGw0\cos(\psi(L,t))$$

$$+4\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)A\,\rho\int_x^L\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\,dx\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)$$

$$+2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho\int_x^L\frac{\partial^2}{\partial t^2}\mathrm{Wln}(x,t)\,dx\,dx+m\_l\,L^2\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)$$

$$+A\,\rho\int_0^L\left(\frac{\partial^2}{\partial t^2}\mathrm{Wln}(x,t)\right)x\,dx+m\_l\,LG^2\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)+m\_l\left(\frac{\partial^2}{\partial t^2}\mathrm{Wln}(L,t)\right)L$$

$$+2\,m\_l\,\mathrm{Wln}(L,t)\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)+m\_l\,\mathrm{Wln}(L,t)^2\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)$$

$$+A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)\int_0^L 2\,\mathrm{Wln}(x,t)\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)dx$$

$$+\frac{1}{2}\rho\,A\left(\frac{\partial}{\partial t}\theta(t)\right)\int_0^L 2\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)x^2\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)dx$$

$$+A\,\rho\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\int_0^L\mathrm{Wln}(x,t)^2\,dx+\frac{1}{2}\rho\,A\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\int_0^L\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 x^2\,dx$$

$$+m\_l\left(\frac{\partial^2}{\partial t^2}\mathrm{Wln}(L,t)\right)LGx0\cos(\psi(L,t))-m\_l\left(\frac{\partial^2}{\partial t^2}\mathrm{Wln}(L,t)\right)LGw0\sin(\psi(L,t))$$

$$- m\_l \left(\frac{\partial^2}{\partial t^2} \theta(t) \right) \int_0^L \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t) \right)^2 dx\, L$$

$$- m\_l \left(\frac{\partial}{\partial t} \theta(t) \right) \int_0^L 2 \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t) \right) \left(\frac{\partial^2}{\partial x\, \partial t} \mathrm{Wln}(x, t) \right) dx\, L + \frac{1}{3} A\, \rho \left(\frac{\partial^2}{\partial t^2} \theta(t) \right) L^3$$

$$- M\_theta - 2\, m\_l\, L \left(\frac{\partial}{\partial t} \theta(t) \right) LGx0 \sin(\psi(L, t)) \left(\frac{\partial}{\partial t} \psi(L, t) \right)$$

$$+ m\_l \left(\frac{\partial}{\partial t} \psi(L, t) \right)^2 LGx0 \cos(\psi(L, t))\, \mathrm{Wln}(L, t)$$

$$- m\_l \left(\frac{\partial}{\partial t} \psi(L, t) \right)^2 LGw0 \sin(\psi(L, t))\, \mathrm{Wln}(L, t)$$

$$- m\_l \left(\frac{\partial}{\partial t} \psi(L, t) \right)^2 LGx0 \sin(\psi(L, t))\, L - m\_l \left(\frac{\partial}{\partial t} \psi(L, t) \right)^2 LGw0 \cos(\psi(L, t))\, L$$

$$+ m\_l \left(\frac{\partial}{\partial t} \theta(t) \right) \int_0^L \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t) \right)^2 dx\, LGx0 \sin(\psi(L, t)) \left(\frac{\partial}{\partial t} \psi(L, t) \right)$$

$$- 2\, m\_l\, L \left(\frac{\partial}{\partial t} \theta(t) \right) LGw0 \cos(\psi(L, t)) \left(\frac{\partial}{\partial t} \psi(L, t) \right)$$

$$+ 2\, m\_l \left(\frac{\partial}{\partial t} \theta(t) \right) LGx0 \sin(\psi(L, t)) \left(\frac{\partial}{\partial t} \mathrm{Wln}(L, t) \right)$$

$$+ 2\, m\_l \left(\frac{\partial}{\partial t} \theta(t) \right) LGx0 \cos(\psi(L, t)) \left(\frac{\partial}{\partial t} \psi(L, t) \right) \mathrm{Wln}(L, t)$$

$$+ 2\, m\_l \left(\frac{\partial}{\partial t} \theta(t) \right) LGw0 \cos(\psi(L, t)) \left(\frac{\partial}{\partial t} \mathrm{Wln}(L, t) \right)$$

$$- 2\, m\_l \left(\frac{\partial}{\partial t} \theta(t) \right) LGw0 \sin(\psi(L, t)) \left(\frac{\partial}{\partial t} \psi(L, t) \right) \mathrm{Wln}(L, t)$$

$$+ m\_l \left(\frac{\partial}{\partial t} \theta(t) \right) \int_0^L \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t) \right)^2 dx\, LGw0 \cos(\psi(L, t)) \left(\frac{\partial}{\partial t} \psi(L, t) \right)$$

$$+ m\_l\, LG^2 \left(\frac{\partial^2}{\partial t^2} \psi(L, t) \right) = 0$$

Aplicación de las ecuaciones de Lagrange para encontrar las ecuaciones locales.
Determinación de la energía cinética de una rebanada de la viga.

```
> dEc:=(1/2*A*rho*diff(theta(t),t)^2*Wln(x,t)^2+1/2*A*rho*diff(Wln(x,t),t)^2+A*rho*diff(theta(t),t)*diff(Wln(x,t),t)*x+1/2*A*rho*diff(theta(t),t)^2*x^2+1/2*J*rho*(diff(psi(x,t),t)+diff(theta(t),t))^2)*dx;
```

$$dEc := \left(\frac{1}{2} A\,\rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 \mathrm{Wln}(x,t)^2 + \frac{1}{2} A\,\rho \left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)^2 + A\,\rho \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right) x + \frac{1}{2} A\,\rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2 + \frac{1}{2} J\,\rho \left(\left(\frac{\partial}{\partial t}\psi(x,t)\right) + \left(\frac{\partial}{\partial t}\theta(t)\right)\right)^2\right) dx$$

Determinación de la energía potencial.

```
> dEp:=(1/2*E*J*diff(psi(x,t),x)^2+1/2*k*G*A*diff(Wln(x,t),x)^2+1/2*k*G*A*(psi(x,t))^2-k*G*A*diff(Wln(x,t),x)*psi(x,t)+1/2*fc*diff(Wln(x,t),x)^2+1/2*fcor*diff(Wln(x,t),x)^2)*dx;
```

$$dEp := \Bigg(\frac{1}{2} E\,J \left(\frac{\partial}{\partial x}\psi(x,t)\right)^2 + \frac{1}{2} k\,G\,A \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 + \frac{1}{2} k\,G\,A\,\psi(x,t)^2 - k\,G\,A \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\psi(x,t) + \frac{1}{2}\Bigg(m\_l\,(L + LGx0\cos(\psi(L,t)) - LGw0\sin(\psi(L,t)))\left(\frac{\partial}{\partial t}\theta(t)\right)^2 + \frac{1}{2} A\,\rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 (L^2 - x^2)\Bigg)\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 + \frac{1}{2}\Bigg(2\,m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right) + (LGx0\cos(\psi(L,t)) - LGw0\sin(\psi(L,t)))\left(\frac{\partial}{\partial t}\psi(L,t)\right)\right) + \int_x^L 2\,A\,\rho\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right) dx\Bigg)\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2\Bigg) dx$$

Determinación del Lagrangiano.

```
> e0:=expand(dEc-dEp);
```

$$e0 := \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 \mathrm{Wln}(x,t)^2 + \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)^2$$

$$+ dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right) x + \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2 + \frac{1}{2} dx\, J\, \rho \left(\frac{\partial}{\partial t}\psi(x,t)\right)^2$$

$$+ dx\, J\, \rho \left(\frac{\partial}{\partial t}\psi(x,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right) + \frac{1}{2} dx\, J\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 - \frac{1}{2} dx\, E\, J \left(\frac{\partial}{\partial x}\psi(x,t)\right)^2$$

$$- \frac{1}{2} dx\, k\, G\, A \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 - \frac{1}{2} dx\, k\, G\, A\, \psi(x,t)^2 + dx\, k\, G\, A \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\psi(x,t)$$

$$- \frac{1}{2} dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 L$$

$$- \frac{1}{2} dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0 \cos(\psi(L,t))$$

$$+ \frac{1}{2} dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0 \sin(\psi(L,t))$$

$$- \frac{1}{4} dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 L^2 + \frac{1}{4} dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2$$

$$- dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)$$

$$- dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \cos(\psi(L,t))$$

$$+ dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \sin(\psi(L,t))$$

$$- dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right) \int_x^L \frac{\partial}{\partial t}\mathrm{Wln}(x,t)\, dx$$

Otra vez la misma estrategia para poder derivar respecto de la velocidad relativa de la rebanada vista desde el sistema de referencia giratorio. Como MAPLE considera x y L como variables independientes, es necesario aplicar esta estrategia dos veces, substituyendo la primera derivada respecto del tiempo de Wln(x,t) y Wln(L,t).

```
> e11:=subs(diff(Wln(x,t),t)=Wlnpunto,e0);
```

$$e11 := \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 \mathrm{Wln}(x,t)^2 + \frac{1}{2} dx\, A\, \rho\, Wlnpunto^2 + dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right) Wlnpunto\, x$$

$$+ \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2 + \frac{1}{2} dx\, J\, \rho \left(\frac{\partial}{\partial t}\psi(x,t)\right)^2 + dx\, J\, \rho \left(\frac{\partial}{\partial t}\psi(x,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)$$

$$+\frac{1}{2}\,dx\,J\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2-\frac{1}{2}\,dx\,E\,J\left(\frac{\partial}{\partial x}\psi(x,t)\right)^2-\frac{1}{2}\,dx\,k\,G\,A\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2$$
$$-\frac{1}{2}\,dx\,k\,G\,A\,\psi(x,t)^2+dx\,k\,G\,A\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\psi(x,t)$$
$$-\frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 L$$
$$-\frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0\cos(\psi(L,t))$$
$$+\frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0\sin(\psi(L,t))$$
$$-\frac{1}{4}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 L^2+\frac{1}{4}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2$$
$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)$$
$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\cos(\psi(L,t))$$
$$+dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\sin(\psi(L,t))$$
$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)\int_x^L Wlnpunto\,dx$$

```
> e11a:=subs(diff(Wln(L,t),t)=WlnpuntoA,e0);
```

$$e11a:=\frac{1}{2}\,dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2\mathrm{Wln}(x,t)^2+\frac{1}{2}\,dx\,A\,\rho\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)^2$$
$$+dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)x+\frac{1}{2}\,dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2+\frac{1}{2}\,dx\,J\,\rho\left(\frac{\partial}{\partial t}\psi(x,t)\right)^2$$
$$+dx\,J\,\rho\left(\frac{\partial}{\partial t}\psi(x,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)+\frac{1}{2}\,dx\,J\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2-\frac{1}{2}\,dx\,E\,J\left(\frac{\partial}{\partial x}\psi(x,t)\right)^2$$
$$-\frac{1}{2}\,dx\,k\,G\,A\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2-\frac{1}{2}\,dx\,k\,G\,A\,\psi(x,t)^2+dx\,k\,G\,A\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\psi(x,t)$$
$$-\frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 L$$
$$-\frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0\cos(\psi(L,t))$$

$$+\frac{1}{2}dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0\sin(\psi(L,t))$$

$$-\frac{1}{4}dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 L^2+\frac{1}{4}dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2$$

$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right) WlnpuntoA$$

$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0\cos(\psi(L,t))$$

$$+dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0\sin(\psi(L,t))$$

$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)\int_x^L\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\,dx$$

```
> e12:=diff(e11,Wlnpunto);
> e12a:=diff(e11a,WlnpuntoA);
```

$$e12 := dx\,A\,\rho\,Wlnpunto+dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)x-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)(L-x)$$

$$e12a := -dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)$$

```
> e13:=subs(Wlnpunto=diff(Wln(x,t),t),e12);
```

$$e13 := dx\,A\,\rho\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)+dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)x$$

$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)(L-x)$$

```
> e13a:=subs(WlnpuntoA=diff(Wln(x,t),t),e12a);
```

$$e13a := -dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)$$

Otra vez la misma estrategia para poder derivar respecto de la posición relativa de la rebanada vista desde el sistema de referencia giratorio.

```
> e14:=subs(Wln(x,t)=Wln,e0);
```

$$e14 := \frac{1}{2}dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 Wln^2+\frac{1}{2}dx\,A\,\rho\left(\frac{\partial}{\partial t}Wln\right)^2+dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}Wln\right)x$$

$$+\frac{1}{2}dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2+\frac{1}{2}dx\,J\,\rho\left(\frac{\partial}{\partial t}\psi(x,t)\right)^2+dx\,J\,\rho\left(\frac{\partial}{\partial t}\psi(x,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)$$

$$+\frac{1}{2}\,dx\,J\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2-\frac{1}{2}\,dx\,E\,J\left(\frac{\partial}{\partial x}\psi(x,t)\right)^2-\frac{1}{2}\,dx\,k\,G\,A\left(\frac{\partial}{\partial x}Wln\right)^2$$

$$-\frac{1}{2}\,dx\,k\,G\,A\,\psi(x,t)^2+dx\,k\,G\,A\left(\frac{\partial}{\partial x}Wln\right)\psi(x,t)-\frac{1}{2}\,dx\left(\frac{\partial}{\partial x}Wln\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 L$$

$$-\frac{1}{2}\,dx\left(\frac{\partial}{\partial x}Wln\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0\cos(\psi(L,t))$$

$$+\frac{1}{2}\,dx\left(\frac{\partial}{\partial x}Wln\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0\sin(\psi(L,t))$$

$$-\frac{1}{4}\,dx\left(\frac{\partial}{\partial x}Wln\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 L^2+\frac{1}{4}\,dx\left(\frac{\partial}{\partial x}Wln\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2$$

$$-dx\left(\frac{\partial}{\partial x}Wln\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)$$

$$-dx\left(\frac{\partial}{\partial x}Wln\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\cos(\psi(L,t))$$

$$+dx\left(\frac{\partial}{\partial x}Wln\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\sin(\psi(L,t))$$

$$-dx\left(\frac{\partial}{\partial x}Wln\right)^2 A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)\int_x^L\frac{\partial}{\partial t}Wln\,dx$$

```
> e14a:=subs(Wln(L,t)=WlnA,e0);
```

$$e14a:=\frac{1}{2}\,dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2\mathrm{Wln}(x,t)^2+\frac{1}{2}\,dx\,A\,\rho\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)^2$$

$$+dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)x+\frac{1}{2}\,dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2x^2+\frac{1}{2}\,dx\,J\,\rho\left(\frac{\partial}{\partial t}\psi(x,t)\right)^2$$

$$+dx\,J\,\rho\left(\frac{\partial}{\partial t}\psi(x,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right)+\frac{1}{2}\,dx\,J\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2-\frac{1}{2}\,dx\,E\,J\left(\frac{\partial}{\partial x}\psi(x,t)\right)^2$$

$$-\frac{1}{2}\,dx\,k\,G\,A\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2-\frac{1}{2}\,dx\,k\,G\,A\,\psi(x,t)^2+dx\,k\,G\,A\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\psi(x,t)$$

$$-\frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 L$$

$$-\frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0\cos(\psi(L,t))$$

$$+\frac{1}{2}dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0\sin(\psi(L,t))$$

$$-\frac{1}{4}dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 L^2+\frac{1}{4}dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2$$

$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}WlnA\right)$$

$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGx0\cos(\psi(L,t))$$

$$+dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right)LGw0\sin(\psi(L,t))$$

$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\rho\left(\frac{\partial}{\partial t}\theta(t)\right)\int_x^L\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\,dx$$

```
> e15:=diff(e14,Wln)+diff(e14a,WlnA);
```

$$e15:=dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 Wln$$

```
> e16:=subs(Wln=Wln(x,t),e15);
```

$$e16:=dx\,A\,\rho\left(\frac{\partial}{\partial t}\theta(t)\right)^2 \mathrm{Wln}(x,t)$$

El diferencial de cortante es una fuerza externa que actua en la rebanada. Aunque ha sido incluido el término de energía potencial debida a la deformación por cortante, al derivar respecto de la coordenada Wln no aparece.

```
> dQ:=-k*G*A*(diff(Wln(x,t),`$`(x,2))-diff(psi(x,t),x))*dx;
```

$$dQ:=-k\,G\,A\left(\left(\frac{\partial^2}{\partial x^2}\mathrm{Wln}(x,t)\right)-\left(\frac{\partial}{\partial x}\psi(x,t)\right)\right)dx$$

Construcción de la ecuación de Lagrange correspondiente a Wln.

```
> e17:=diff(e13+e13a,t)-e16-dQ=0;
```

$$e17:=dx\,A\,\rho\left(\frac{\partial^2}{\partial t^2}\mathrm{Wln}(x,t)\right)+dx\,A\,\rho\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)x$$

$$-2\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)A\rho\left(\frac{\partial}{\partial t}\theta(t)\right)(L-x)\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)$$

$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\rho\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)(L-x)$$

$$-2\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)$$

$$- dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial^2}{\partial t^2}\theta(t)\right) - dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 \mathrm{Wln}(x,t)$$
$$+ k\, G\, A \left(\left(\frac{\partial^2}{\partial x^2}\mathrm{Wln}(x,t)\right) - \left(\frac{\partial}{\partial x}\psi(x,t)\right)\right) dx = 0$$

Otra vez la misma estrategia para poder derivar respecto de la velocidad angular relativa de la rebanada vista desde el sistema de referencia giratorio. Como MAPLE considera x y L como variables independientes, es necesario aplicar esta estrategia dos veces, substituyendo la primera derivada respecto del tiempo de psi(x,t) y psi(L,t).

```
> e21:=subs(diff(psi(x,t),t)=psipunto,e0);
```

$$e21 := \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 \mathrm{Wln}(x,t)^2 + \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)^2$$
$$+ dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right) x + \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2 + \frac{1}{2} dx\, J\, \rho\, psipunto^2$$
$$+ dx\, J\, \rho\, psipunto \left(\frac{\partial}{\partial t}\theta(t)\right) + \frac{1}{2} dx\, J\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 - \frac{1}{2} dx\, E\, J \left(\frac{\partial}{\partial x}\psi(x,t)\right)^2$$
$$- \frac{1}{2} dx\, k\, G\, A \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 - \frac{1}{2} dx\, k\, G\, A\, \psi(x,t)^2 + dx\, k\, G\, A \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\psi(x,t)$$
$$- \frac{1}{2} dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 L$$
$$- \frac{1}{2} dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0 \cos(\psi(L,t))$$
$$+ \frac{1}{2} dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0 \sin(\psi(L,t))$$
$$- \frac{1}{4} dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 L^2 + \frac{1}{4} dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2$$
$$- dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)$$
$$- dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0 \cos(\psi(L,t))$$
$$+ dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0 \sin(\psi(L,t))$$
$$- dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right) \int_x^L \frac{\partial}{\partial t}\mathrm{Wln}(x,t)\, dx$$

```
> e21a:=subs(diff(psi(L,t),t)=psipuntoA,e0);
```

$$e21a := \frac{1}{2}\, dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 \mathrm{Wln}(x,t)^2 + \frac{1}{2}\, dx\, A\, \rho \left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right)^2$$

$$+ dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(x,t)\right) x + \frac{1}{2}\, dx\, A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2 + \frac{1}{2}\, dx\, J\, \rho \left(\frac{\partial}{\partial t}\psi(x,t)\right)^2$$

$$+ dx\, J\, \rho \left(\frac{\partial}{\partial t}\psi(x,t)\right)\left(\frac{\partial}{\partial t}\theta(t)\right) + \frac{1}{2}\, dx\, J\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 - \frac{1}{2}\, dx\, E\, J \left(\frac{\partial}{\partial x}\psi(x,t)\right)^2$$

$$- \frac{1}{2}\, dx\, k\, G\, A \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 - \frac{1}{2}\, dx\, k\, G\, A\, \psi(x,t)^2 + dx\, k\, G\, A \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)\psi(x,t)$$

$$- \frac{1}{2}\, dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 L$$

$$- \frac{1}{2}\, dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0 \cos(\psi(L,t))$$

$$+ \frac{1}{2}\, dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0 \sin(\psi(L,t))$$

$$- \frac{1}{4}\, dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 L^2 + \frac{1}{4}\, dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)^2 x^2$$

$$- dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\mathrm{Wln}(L,t)\right)$$

$$- dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right) psipuntoA\ LGx0 \cos(\psi(L,t))$$

$$+ dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right) psipuntoA\ LGw0 \sin(\psi(L,t))$$

$$- dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right) \int_x^L \frac{\partial}{\partial t}\mathrm{Wln}(x,t)\, dx$$

```
> e22:=diff(e21,psipunto)+diff(e21a,psipuntoA);
```

$$e22 := dx\, J\, \rho\, psipunto + dx\, J\, \rho \left(\frac{\partial}{\partial t}\theta(t)\right)$$

$$- dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right) LGx0 \cos(\psi(L,t))$$

$$+ dx \left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l \left(\frac{\partial}{\partial t}\theta(t)\right) LGw0 \sin(\psi(L,t))$$

```
> e23:=subs(psipunto=diff(psi(x,t),t),e22);
```

$$e23 := dx\, J\, \rho \left(\frac{\partial}{\partial t} \psi(x, t)\right) + dx\, J\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right) - dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right) LGx0 \cos(\psi(L, t)) + dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right) LGw0 \sin(\psi(L, t))$$

```
> e24:=subs(psi(x,t)=psi,e0);
```

$$e24 := \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)^2 \mathrm{Wln}(x, t)^2 + \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t} \mathrm{Wln}(x, t)\right)^2 + dx\, A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)\left(\frac{\partial}{\partial t} \mathrm{Wln}(x, t)\right) x + \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)^2 x^2 + \frac{1}{2} dx\, J\, \rho \left(\frac{\partial}{\partial t} \psi\right)^2 + dx\, J\, \rho \left(\frac{\partial}{\partial t} \psi\right)\left(\frac{\partial}{\partial t} \theta(t)\right) + \frac{1}{2} dx\, J\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)^2 - \frac{1}{2} dx\, E\, J \left(\frac{\partial}{\partial x} \psi\right)^2 - \frac{1}{2} dx\, k\, G\, A \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 - \frac{1}{2} dx\, k\, G\, A\, \psi^2 + dx\, k\, G\, A \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right) \psi - \frac{1}{2} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)^2 L - \frac{1}{2} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)^2 LGx0 \cos(\psi(L, t)) + \frac{1}{2} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)^2 LGw0 \sin(\psi(L, t)) - \frac{1}{4} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)^2 L^2 + \frac{1}{4} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)^2 x^2 - dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)\left(\frac{\partial}{\partial t} \mathrm{Wln}(L, t)\right) - dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)\left(\frac{\partial}{\partial t} \psi(L, t)\right) LGx0 \cos(\psi(L, t)) + dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)\left(\frac{\partial}{\partial t} \psi(L, t)\right) LGw0 \sin(\psi(L, t)) - dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right) \int_x^L \frac{\partial}{\partial t} \mathrm{Wln}(x, t)\, dx$$

```
> e24a:=subs(psi(L,t)=psiA,e21a);
```

$$e24a := \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)^2 \mathrm{Wln}(x, t)^2 + \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t} \mathrm{Wln}(x, t)\right)^2$$
$$+ dx\, A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)\left(\frac{\partial}{\partial t} \mathrm{Wln}(x, t)\right) x + \frac{1}{2} dx\, A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)^2 x^2 + \frac{1}{2} dx\, J\, \rho \left(\frac{\partial}{\partial t} \psi(x, t)\right)^2$$
$$+ dx\, J\, \rho \left(\frac{\partial}{\partial t} \psi(x, t)\right)\left(\frac{\partial}{\partial t} \theta(t)\right) + \frac{1}{2} dx\, J\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)^2 - \frac{1}{2} dx\, E\, J \left(\frac{\partial}{\partial x} \psi(x, t)\right)^2$$
$$- \frac{1}{2} dx\, k\, G\, A \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 - \frac{1}{2} dx\, k\, G\, A\, \psi(x, t)^2 + dx\, k\, G\, A \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right) \psi(x, t)$$
$$- \frac{1}{2} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)^2 L$$
$$- \frac{1}{2} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)^2 LGx0 \cos(psiA)$$
$$+ \frac{1}{2} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)^2 LGw0 \sin(psiA)$$
$$- \frac{1}{4} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)^2 L^2 + \frac{1}{4} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right)^2 x^2$$
$$- dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)\left(\frac{\partial}{\partial t} \mathrm{Wln}(L, t)\right)$$
$$- dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right) psipuntoA\, LGx0 \cos(psiA)$$
$$+ dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right) psipuntoA\, LGw0 \sin(psiA)$$
$$- dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 A\, \rho \left(\frac{\partial}{\partial t} \theta(t)\right) \int_x^L \frac{\partial}{\partial t} \mathrm{Wln}(x, t)\, dx$$

```
> e25:=diff(e24,psi)+diff(e24a,psiA);
```

$$e25 := -dx\, k\, G\, A\, \psi + dx\, k\, G\, A \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)$$
$$+ \frac{1}{2} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)^2 LGx0 \sin(psiA)$$
$$+ \frac{1}{2} dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right)^2 LGw0 \cos(psiA)$$
$$+ dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right) psipuntoA\, LGx0 \sin(psiA)$$
$$+ dx \left(\frac{\partial}{\partial x} \mathrm{Wln}(x, t)\right)^2 m\_l \left(\frac{\partial}{\partial t} \theta(t)\right) psipuntoA\, LGw0 \cos(psiA)$$

```
> e26:=subs(psi=psi(x,t),e25);
```

$$e26 := -dx\,k\,G\,A\,\psi(x,t) + dx\,k\,G\,A\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right) + \frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0\,\sin(psiA) + \frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0\,\cos(psiA) + dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right) psipuntoA\,LGx0\,\sin(psiA) + dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right) psipuntoA\,LGw0\,\cos(psiA)$$

```
> e27:=subs(psiA=psi(L,t),e26);
```

$$e27 := -dx\,k\,G\,A\,\psi(x,t) + dx\,k\,G\,A\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right) + \frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0\,\sin(\psi(L,t)) + \frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0\,\cos(\psi(L,t)) + dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right) psipuntoA\,LGx0\,\sin(\psi(L,t)) + dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right) psipuntoA\,LGw0\,\cos(\psi(L,t))$$

```
> e28:=subs(psipuntoA=diff(psi(L,t),t),e27);
```

$$e28 := -dx\,k\,G\,A\,\psi(x,t) + dx\,k\,G\,A\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right) + \frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0\,\sin(\psi(L,t)) + \frac{1}{2}\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0\,\cos(\psi(L,t)) + dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGx0\,\sin(\psi(L,t)) + dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\psi(L,t)\right) LGw0\,\cos(\psi(L,t))$$

Existe un momento (por unidad de longitud) aplicado a la rebanada. Como es función de la derivad segunda de psi(x,t) respecto de x, no aparece al derivar la energía potencial y es necesario tenerlo en cuenta.

```
> dM:=E*J*diff(psi(x,t),`$`(x,2))*dx;
```

$$dM := E\,J\left(\frac{\partial^2}{\partial x^2}\psi(x,t)\right)dx$$

Construcción de la ecuación correspondiente al ángulo psi.

```
> e29:=diff(e23,t)-e28-dM=0;
```

$$e29 := dx\,J\,\rho\left(\frac{\partial^2}{\partial t^2}\psi(x,t)\right)+dx\,J\,\rho\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)$$
$$-2\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)LGx0\cos(\psi(L,t))\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)$$
$$-dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)LGx0\cos(\psi(L,t))$$
$$+2\,dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)LGw0\sin(\psi(L,t))\left(\frac{\partial^2}{\partial x\,\partial t}\mathrm{Wln}(x,t)\right)$$
$$+dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)LGw0\sin(\psi(L,t))+dx\,k\,G\,A\,\psi(x,t)$$
$$-dx\,k\,G\,A\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)-\frac{1}{2}dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGx0\sin(\psi(L,t))$$
$$-\frac{1}{2}dx\left(\frac{\partial}{\partial x}\mathrm{Wln}(x,t)\right)^2 m\_l\left(\frac{\partial}{\partial t}\theta(t)\right)^2 LGw0\cos(\psi(L,t))-E\,J\left(\frac{\partial^2}{\partial x^2}\psi(x,t)\right)dx=0$$

```
> with(PDEtools);
```

[*PDEplot*, *build*, *casesplit*, *charstrip*, *dchange*, *dcoeffs*, *declare*, *difforder*, *dpolyform*, *dsubs*, *mapde*, *separability*, *splitstrip*, *splitsys*, *undeclare*]

```
>
answ:=pdsolve([e6,e17,e29],[psi(x,t),Wln(x,t),theta(t)],sings
ol=false);
Error, (in pdsolve/sys) found functions depending on different variables
in the given DE system: [Wln(x,t), Wln(L,t), psi(L,t), psi(x,t)]
```

Vemos que MAPLE no puede separar el sistema de ecuaciones diferenciales. Causas:
1) hay funciones de x y funciones de x=L (este problema se puede resolver reescribiendo las ecuaciones utilizando otra notación).
2) en la ecuación global hay términos integrales. El algoritmo que utiliza MAPLE no es capaz de separar todas las ecuaciones integro-diferenciales en derivadas parciales. Mas adelante volverá a aparecer este problema.

```
>
>
>
```