Generalized exterior–algebraic electromagnetism in (k, n)-dimensional space–time

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PhD thesis / year 2022

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I have no special talent. I am only passionately curious.

Albert Einstein

Acknowledgments

I would like to thank the people to whom I am grateful, not only for their help in preparing this thesis, but also for their continuous support and uninterrupted guidance throughout the last years.

First and foremost, I wish to express my deepest gratitude to my supervisors, Alfonso Martinez and Josep Font-Segura, for encouraging me to undertake this research project and for their daily support, with a friendly and insightful guidance throughout my graduate studies.

I wish also to express my gratitude to the academics agreeing to be part of my doctoral committee, as well as who accepted to serve as substitute examiners.

I would like to thank my collaborators and colleagues. Andrea Giusti, for his friendship and technical help, and Francesco Mainardi, for his mentorship since I was undergraduate. Thanks to Arezou Rezazadeh, who supported me in difficult moments and shared a part her experience with me. Mehdi Dabirnia and Giuseppe Cocco, for having put up with my presence in the office. My gratitude goes also to Marc Tudela-Pi and Aracelys Garcia Moreno, for having listened to me and shared a part of our path. Moreover, thank to Albert Guillén i Fàbregas, for leading the research group I had the pleasure and honour to join. In general, many thanks to the faculty members and staff of the Department of Information and Communication Technologies, in particular to Lydia, for having continuously solved all my bureaucratic problems and doubts.

In conclusion, I would like to extend my appreciation to those who have not directly been part of my academic life. First of all, I wish to thank my mum Paola and my dad Fulvio, for their unconditional support and for teaching me the value of things. Besides, a special acknowledge to my sisters, Diana and Linda, who have been dealing with my oddities since the first days. Thanks also to my uncle Davide, for his closeness and help. And to my two nephews, Paolo and Rocco, bringing joy in my life.

I would also like to thank some of my dearest friends, in particular Nicholas and Alessandro, for having not missed the opportunity to come and meet me in this experience.

Last but not least, I wish to dedicate the last few words to Natalia. I owe her my deepest gratitude for her constant inspiration and daily support, going through difficulties and amazing times.

Abstract

This doctoral thesis aims to find a connection between two different disciplines. On the one hand, the electromagnetic theory, one of the most known and most applied theories in physics, properly described by the famous Maxwell equations. On the other hand, the theory of information and communication, provided of a mathematical structure which mainly includes the concepts of probability and statistics. In order to establish a contact point between the two, we first decided to develop a suitable mathematical framework, which could accomodate the two theories in the appropriate context.

We therefore chose to use the mathematical theory of exterior algebra, because it allows to combine a simple and intuitive method coming from a classical vector conception, with more advanced but equally effective mathematical tools. Having to build a theory from the beginning, we have opted to consider a procedure as general as possible and, therefore, we proceed in a space-time with arbitrary dimensions, both as regards time and as regards space. We formulate our theory in this space-time, utilizing multivector fields, also of arbitrary degree, in order to broaden the classical concept of vector field.

The electromagnetic theory is thus generalized through these multivectors and the fact of having several free parameters, such as the dimensions of space-time and the grade of the multivectoral field, allows to identify various models, obtaining the known ones and opening the doors to new horizons.

In practice, to build our theory we can follow two distinct but complementary approaches. In the first place, making an analogy with the classical theory, we can directly use the generalized definitions of exterior algebra to postulate a natural extension of the electromagnetic theory in arbitrary dimensions. Secondly, we have developed in parallel a dynamical theory, so-called Lagrangian, purposely built for multivector fields of arbitrary grade.

Regardless of the path chosen, we have obtained a consistent theory that presents its equations of motion, corresponding to the generalized Maxwell equations, and all the equivalent physical quantities resulting from new conservation laws, which identify the quantities of the system that remain unchanged, such as energy and momentum.

The connection point with the theory of communication emerges in dealing with the electromagnetic waves coming from the solutions of Maxwell equations. Studied from the multivectoral point of view of exterior algebra, these waves might open the doors to a new interpretation of the transmission of signals from a different perspective.

Sommario

La presente tesi di dottorato ha l'obiettivo di trovare un punto di congiunzione tra due diverse discipline. Da un lato la teoria elettromagnetica, una delle teorie più conosciute ed applicate in fisica e propriamente descritta dalle celeberrime equazioni di Maxwell. Dall'altro, la teoria della informazione e comunicazione, dotata di una struttura matematica che comprende maggiormente i concetti di probabilità e statistica. Per stabilire un contatto tra le due, abbiamo deciso, in primo luogo, di sviluppare una struttura matematica appropriata, che potesse conciliare le due teorie in un giusto contesto.

Abbiamo quindi deciso di utilizzare la teoria matematica dell'algebra esteriore, perché è in grado di unire un metodo semplice ed intuitivo proveniente da una concezione vettoriale classica, con strumenti matematici più avanzati ma ugualmente efficaci. Dovendo costruire una teoria dal principio, abbiamo optato per considerare una trattazione il più generale possibile e, pertanto, procediamo in uno spazio-tempo con dimensioni arbitrarie, sia per quanto riguarda il tempo, sia per quanto riguarda lo spazio. Formuliamo la nostra teoria in tale spazio-tempo, basandoci su campi multivettoriali, anch'essi di grado arbitrario, in modo da ampliare il concetto classico di campo vettoriale.

La teoria elettromagnetica viene così generalizzata attraverso questi multivettori ed il fatto di avere diversi parametri liberi, quali le dimensioni dello spazio-tempo ed il grado del campo multivettoriale, permette di identificare svariati modelli, ricavando quelli noti ed aprendo le porte verso nuovi orizzonti.

Operativamente, per costruire la nostra teoria possiamo seguire due approcci, distinti ma complementari. In primo luogo, facendo una analogia con la teoria classica, possiamo usare direttamente le definizioni generalizzate dell'algebra esteriore per postulare una estensione naturale della teoria elettromagnetica in dimensioni arbitrarie. Secondariamente, abbiamo sviluppato in parallelo una teoria dinamica, cosiddetta Lagrangiana, appositamente costruita per i campi multivettoriali di grado arbitrario.

Indipendentemente dal percorso scelto, abbiamo ricavato una teoria consistente che presenta le sue equazioni del moto, ovvero le equazioni di Maxwell generalizzate, e tutte le quantità fisiche equivalenti risultanti da nuove leggi di conservazione, che identificano le grandezze del sistema che rimangono invariate, quali l'energia e la quantità di moto.

Il punto di connessione con la teoria della comunicazione emerge nel trattare le onde elettromagnetiche provenienti dalle soluzioni delle equazioni di Maxwell. Trattate dal punto di vista multivettoriale dell'algebra esteriore, tali onde possono aprire le porte ad una nuova interpretazione della trasmissione di segnali da una differente prospettiva.

Resumen

Esta tesis doctoral tiene el objetivo de encontrar un vínculo entre dos disciplinas diferentes. Por un lado, la teoría electromagnética, una de las teorías más conocidas y aplicadas de la física y debidamente descrita por las famosas ecuaciones de Maxwell. Por otro lado, la teoría de la información y de la comunicación, dotada de una estructura matemática que comprende mayormente los conceptos de probabilidad y estadística. Para establecer un punto de encuentro entre las dos, primero decidimos desarrollar una estructura matemática apropiada, que pudiera conciliar las dos teorías en el contexto adecuado.

Por lo tanto, decidimos utilizar la teoría matemática del álgebra exterior, porque es capaz de combinar un método simple e intuitivo proveniente de una concepción vectorial clásica, con herramientas matemáticas más avanzadas pero igualmente efectivas. Al tener que construir una teoría desde el principio, hemos optado por considerar un tratamiento lo más general posible y, por tanto, procedemos en un espacio-tiempo con dimensiones arbitrarias, tanto en el tiempo como en el espacio. Construimos nuestra teoría en este espacio-tiempo, basándonos en campos multivectoriales, también de grado arbitrario, para ampliar el concepto clásico de campo vectorial.

La teoría electromagnética se generaliza a través de estos multivectores y el hecho de tener varios parámetros libres, como las dimensiones del espacio-tiempo y el grado del campo multivectorial, permite identificar varios modelos, obteniendo los ya conocidos y abriendo las puertas a nuevos horizontes.

Operacionalmente, para construir nuestra teoría podemos seguir dos enfoques distintos pero complementarios. En primer lugar, haciendo una analogía con la teoría clásica, podemos utilizar directamente las definiciones generalizadas del álgebra exterior para postular una extensión natural de la teoría electromagnética en dimensiones arbitrarias. En segundo lugar, hemos desarrollado, en paralelo, una teoría dinámica, llamada Lagrangiana, construida a propósito para campos multivectoriales de grado arbitrario.

Independientemente del enfoque elegido, hemos obtenido una teoría consistente que presenta sus ecuaciones de movimiento, es decir, las ecuaciones de Maxwell generalizadas, y todas las cantidades físicas equivalentes resultantes de las nuevas leyes de conservación, que identifican las cantidades del sistema que permanecen sin alterar, tales como energía y momento.

El punto de conexión con la teoría de la comunicación surge al estudiar las ondas electromagnéticas provenientes de las soluciones de las ecuaciones de Maxwell. Consideradas desde el punto de vista multivectorial del álgebra exterior, estas ondas pueden abrir las puertas a una nueva interpretación de la transmisión de señales desde una perspectiva diferente.

Resum

Aquesta tesi doctoral pretén enllaçar dues disciplines diferents. D'una banda, la teoria electromagnètica, una de les teories més conegudes i aplicades de la física i degudament descrita per les conegudes equacions de Maxwell. D'altra banda, la teoria de la informació i la comunicació, dotada d'una estructura matemàtica que entén millor els conceptes de probabilitat i estadística. Per tal d'establir el contacte entre ambdues, vam decidir, en primer lloc, desenvolupar un marc matemàtic adequat, que pogués conciliar les dues teories en un context adient.

Per això hem decidit utilitzar la teoria matemàtica de l'àlgebra exterior, perquè ens permet de combinar un mètode senzill i intuïtiu provinent d'una concepció vectorial clàssica, amb eines matemàtiques més avançades però igualment efectives. Havent de constituir una teoria des de l'inici, hem optat per considerar un tractament que fos el més general possible i, per tant, considerem un espai-temps amb dimensions arbitràries, tant pel que fa al temps com pel que fa a l'espai. Formulem la nostra teoria en aquest espai-temps, basant-nos en camps multivectors, també de grau arbitrari, per tal d'ampliar el concepte clàssic de camp vectorial.

La teoria electromagnètica es generalitza així a través d'aquests multivectors i el fet de tenir diversos paràmetres lliures, com les dimensions de l'espai-temps i del camp multivectorial, ens permet identificar diversos models, obtenint els coneguts i obrint les portes a nous horitzons.

Pràcticament, per construir la nostra teoria podem seguir dos enfocaments diferents però complementàries. En primer lloc, mitjançant una analogia de la teoria clàssica, podem utilitzar directament les definicions generalitzades d'àlgebra exterior per postular una extensió natural de la teoria electromagnètica en dimensions arbitràries. En segon lloc, hem constituït en paral·lel una teoria dinàmica, l'anomenada Lagrangiana, construïda expressament per a camps multivectors de grau arbitrari.

Independentment de l'opció escollida, hem obtingut una teoria coherent que presenta equacions de moviment, és a dir, equacions generalitzades de Maxwell, així com totes les magnituds físiques equivalents resultants de noves lleis de conservació, que identifiquen les magnituds del sistema que romanen inalterades, com ara l'energia i l'impuls.

El punt de connexió amb la teoria de la comunicació sorgeix en tractar les ones electromagnètiques procedents de les solucions de les equacions de Maxwell. Tractats des del punt de vista multivectoral de l'àlgebra exterior, aquestes ones permeten obrir les portes a una nova interpretació de la transmissió de senyals des d'una perspectiva diferent.

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Part I

Cover Essay

Chapter 1

Motivation and introduction

This doctoral thesis follows an interdisciplinary approach, combining two fields of study, namely the physical theory of electromagnetism and the mathematical theory of communication. First, the modern communication theory would not exist without an accurate mathematical description [Sha48]. A key ingredient thereof is signal theory, with its mathematical representation as a linear space with a set of basis elements [BC02, Ch. 1]. Second, and since electromagnetic waves are often used to transmit information by modulating signals, we consider the classical theory of electromagnetism to establish a temptative connection to physics. Electromagnetic theory was firstly formalized mathematically by J. C. Maxwell [Max03] to describe in a unified manner experimental results obtained in previous years by distinguished scientists such as H. C. Oersted, M. Faraday and A. M. Ampère [Dar00]. The original mathematical descriptions of the electromagnetic field in terms of quaternions and vector calculus were later complemented by different frameworks, such as tensor calculus or differential forms.

The idea that information theory and physics are closely connected has a long history, as evidenced in some works of R. Landauer [Lan96, Lan99], especially concerning quantum aspects. In order to find some connections between the two subjects, we have developed a generalized theory of electromagnetism based on multivectors (grade-r vectors) in exterior algebra living in a space–time with k time coordinates and n space coordinates [CFSM19, Sect. 2].

The mathematical framework of exterior calculus brings the advantage of merging the simplicity and intuitiveness of standard vector calculus, since formulas admit explicit expressions, together with powerful tools of tensorial calculus and differential forms, as formulas can be given for arbitrary value of k, n, and r [CFSM20]. According to the principles of relativity, space and time are not independent, but both are coordinates in space–time [LEMW23, pp. 111–120]. Although having multiple time coordinates leads to several physical issues [Vel12], in the mathematical construction of space–time these problems do not arise and in this work we consider only physical models with $k \leq 1$. Having generic free parameters r, k, n, allows to build a general theory which provides a mathematical foundation to future elaborations of signal theory of multivectors in such space–times. This perspective changes our way to model signals, classically written as functions of time [Gal68, Ch. 8], as explicit functions of both space and time coordinates, thereby increasing their level of sophistication.

In the framework of exterior calculus, the generalized Maxwell equations are given by [CFSM20, Eqs. (42) and (43)]

$$\boldsymbol{\partial} \lrcorner \mathbf{F} = \mathbf{J}, \tag{1.1}$$

$$\boldsymbol{\partial} \wedge \mathbf{F} = 0, \tag{1.2}$$

where $\boldsymbol{\partial}$ is the derivative vector operator, $\mathbf{F}$ represents the generalized multivector electromagnetic field and $\mathbf{J}$ corresponds to the current source density.

This process of generalization has been carried out along two complementary paths. First, we use basic notions of exterior calculus, interior and exterior derivatives, respectively written $\boldsymbol{\partial}\lrcorner$ and $\boldsymbol{\partial}\wedge$ as in (1.1) and (1.2), and the concepts of generalized flux and circulation, Stokes theorems and in order to obtain multivectorial electromagnetism in (k, n) space–time, as an analogue to the vectorial formulation. In parallel, we define a suitable Lagrangian density and develop a Lagrangian approach for r-vector fields in exterior algebra and calculus. This way, we are able to construct a Lagrangian theory compatible with multivector fields, which recovers the equations describing the dynamics of a system.

Flux and circulation provide the integral form of Maxwell equations. The connection from the integral to the differential formulation in (1.1) and (1.2) is given thanks to Stokes theorems in exterior calculus, which are formally defined linking generalized circulation to the exterior derivative and generalized flux to interior derivative [CFSM19, Sect. 3]. We also establish some links between exterior algebra and calculus and other mathematical theories. In particular, multivectors and tensors are strongly connected, since they can be defined in the same algebraic space–time, despite their different basis, and mathematical operations among them are well-defined [CFSM21b, Sect. 2.5], e. g., the divergence of a tensor is written with the interior derivative operator.

Building on the analogy between generalized and classical formulations of electromagnetism, a generalized stress–energy–momentum tensor of rank 2 is presented. This tensor is also derived from a conservation law relating the tensor with the generalized Lorentz force density [CFSM20, Sect. 4.2], defined as

$$\mathbf{f} = \mathbf{J} \lrcorner \mathbf{F} = (\boldsymbol{\partial} \lrcorner \mathbf{F}) \lrcorner \mathbf{F}. \tag{1.3}$$

The expression of the stress–energy–momentum tensor, together with the Lorentz force and the exterior-algebraic form of Maxwell equations, is also obtained within the Lagrangian formulation and the invariance of the action to infinitesimal space–time translations or to infinitesimal changes in the fields.

In the framework of exterior calculus, we have developed a Lagrangian exterior-algebraic theory as a dynamical theory of multivector fields. For a given a Lagrangian density, the principle of stationary action applied to variations of the field yields the Euler–Lagrange equations, which describe the dynamics of the system. For the generalized electromagnetic field, the Euler–Lagrange equations coincide with generalized Maxwell equations derived following the first approach, having the form seen in (1.1) and (1.2), [CFSM21b, Sect. 4].

Physical quantities such as energy, linear momentum, angular momentum and spin are identified in the Lagrangian formulation and generalized to a multivector fields. The stress–energy–momentum, including the components of energy, momentum and the equivalent Maxwell stress tensor, is derived in an elegant manner to be manifestly symmetric [MFSC21, Sect. 4.3]. Conservation laws for the energy-momentum and angular momentum are deduced by applying the calculus of variations to our exterior-algebraic Lagrangian theory, respectively to infinitesimal space–time transations and space–time rotations. Such conservation laws are fundamental because they identify conserved quantities, which are constant during the physical process that the system describes.

Having described both approaches that we developed to study our problem, we can provide a more detailed discussion about the Maxwell equations themselves. The generalized Maxwell field $\mathbf{F}$ is a grade-r multivector with $\binom{k+n}{r}$ components [CFSM20, Sect. 3.1]. To recover classical electromagnetism, we select one time coordinate and the three dimensional space, namely $k = 1$ and $n = 3$, so that the space–time is the "ordinary" four-dimensional space–time [LEMW23, pp. 75–80]. Then, the value $r = 2$ identifies a bivector $\mathbf{F}$ which suitably includes the six total components of electric and magnetic fields, as said in the previous paragraph. The components of the bivectorial Maxwell field and of the two-indices antisymmetric Faraday tensor match, and the same the components of $\mathbf{F}$ and of the 2-form identifying the electromagnetic field [CFSM19, Sect. 5]. In this Minkowski $(1, 3)$ space–time, the time-space components are identified as the electric field, while the space-space as the magnetic field.

Independently of the chosen path, purely exterior-algebraic or Lagrangian, the generalized Maxwell equations form the starting point for our work. In the absence of charged sources, the classical Maxwell equations can be written as a set of uncoupled wave equations, one for every component of both electric and magnetic field, operating some vectorial manipulations. Alternatively, the fields can be written in terms of vector and scalar potentials, and every component of the potentials satisfies the same wave equation [Jac99, Sect. 6.2], even in the presence of charged sources, having chosen the appropriate gauge. The same procedure is extended to generalized electromagnetism. Applying the properties of interior and exterior products to Maxwell equations in (1.1) and (1.2) allows us to define a generalized wave equation in arbitrary dimensions, namely

$$(\boldsymbol{\partial} \cdot \boldsymbol{\partial})\,\mathbf{F} = 0. \tag{1.4}$$

Like the classical case [Gri07, Sect. 9.1.2], the solutions of the wave equation in (1.4) would lead to the expression for electromagnetic waves, possibly written in terms of the so-called plane waves [Jac99, Ch. 7]. .

It may be instructive to express the Maxwell equations in the Fourier domain, as suggested by the decomposition of the field in plane waves. In a analogy to the expression in the usual four-dimensional space–time [CTDRG97, Sect. 1.B], the definition of the Fourier transform is extended to multivector fields as

$$\hat{\mathbf{F}}(\boldsymbol{\xi}) = \int_{\mathbb{R}^{k+n}} \mathrm{d}\boldsymbol{\xi} e^{-2\pi j \boldsymbol{\xi}\cdot\mathbf{x}} \mathbf{F}(\mathbf{x}). \tag{1.5}$$

From this definition, one could attempt to generalize the concept of propagating waves in the vacuum, i. e. in the absence of sources.

To conclude, we describe some possible future advances which may be developed thanks to the outcomes of our research. We identify three main such lines of investigation, as briefly explained in the final chapter of this cover essay.

First of all, we deal with the number of degrees of freedom. In mathematical and physical theories, one of the fundamental issues to tackle is the determination of the degrees of freedom needed to solve a specific problem. Here, some interesting results might involve the independent components of the potential, when solving Maxwell equations or looking for the components of the spin. Secondly, we dedicate a section to quantization. One might argue that our generalized electromagnetic theory should be quantized, in order to find a connection with quantum communication. The topic of quantization is not included in this thesis and it is only marginally discussed, but the fact of having introduced the concept of spin provides an opening for this new quantum path. Finally, it would be possible to establish some potential application to communication and information theory by means of signal theory, generalizing the duality between space–time and frequency/energy–momentum descriptions of signals.

1.1. Contributions

This essay presents the content of the following three main articles,

- [CFSM19]: **I. Colombaro**, J. Font-Segura, A. Martinez, An introduction to space–time exterior calculus, *Mathematics* (2019), 7(6): 564. [arXiv:2002.12604]
- [CFSM20]: **I. Colombaro**, J. Font-Segura, A. Martinez, Generalized Maxwell equations for exterior–algebra multivectors in (k,n) space–time dimensions. *Eur. Phys. Jour. Plus* (2020); 135: 305. [arXiv:2003.07709]
- [CFSM21b]: **I. Colombaro**, J. Font-Segura, A. Martinez, An exterior–algebraic derivation of the Euler-Lagrange equations from the principle of stationary action. *Mathematics* (2021); 9(18): 2178. [arXiv:2110.10514]

and also discusses the following papers that make part of the research project and complement the contents provided by the previous ones:

- [MFSC21] A. Martinez, J. Font-Segura, **I. Colombaro**, An exterior-algebraic derivation of the symmetric stress–energy–momentum tensor in flat space–time. *European Physical Journal Plus* (2021); 136: 212. [arXiv:2104.07013]
- [MCFS21] A. Martinez, **I. Colombaro**, J. Font-Segura. On the angular momentum and spin of generalized electromagnetic field for r-vectors in (k,n) space–time dimensions. *European Physical Journal Plus* (2021); 136: 1047. [arXiv:2110.10531]

The full version of the aforementioned publications is included in part II of this document.

Chapter 2

Research Overview

In this chapter, we present a synopsis of the research undertaken during the doctoral project. The papers listed in 1.1 are contextualized and connected hereinafter. As stated earlier, we deal with multivector fields in a generalized space–time with an arbitrary number of dimensions. We build a generalized theory of electromagnetism within the framework of exterior algebra and calculus, with the ultimate purpose of considering signals carried by electromagnetic radiation.

2.1. Exterior calculus, multivectors and tensors

In our research, we aim to generalize the theory of electromagnetism. To maintain the description as general as possible, we consider a framework characterized by arbitrary k time and n space dimensions. In analogy with the four-dimensional Minkowski space–time [LEMW23, pp. 118–120], this space–time is endowed with a canonical basis and an inner product. Basis vector have positive (resp. negative) unitary norm when multiplying two identical spatial (resp. temporal) components [Nab12, pp. 9–10]; non-identical components are orthogonal.

In this space–time, we deal with multivector fields of grade r, whose basis are labelled by ordered lists of r indices and they are geometrically interpreted as oriented hypersurfaces. In exterior calculus, oriented integrals may also be evaluated over oriented m-hypersurfaces $\mathcal{V}^m$. The orientation of $\mathcal{V}^m$ is given by the multivector differential basis orthogonal to it [CFSM19, Sect. 3.1].

The Maxwell field $\mathbf{F}$, which we also refer to as generalized electromagnetic field, can also be written in terms of a multivector potential $\mathbf{A}$ of grade $(r-1)$ [CFSM20, Sect. 3.2], by means of the operation of exterior derivative. As in the classical theory of electromagnetism, there exists gauge invariance of Maxwell equations also in the exterior–algebraic formulation.

The differential Maxwell equations admit an integral form in terms of the flux and circulation of the Maxwell field $\mathbf{F}$, which we also refer to as generalized electromagnetic field. In general, the flux of a generalized r-vector field across an m-dimensional hypersurface $\mathcal{V}^m$ is defined as [CFSM19, Eq. (30)]

$$\mathcal{F}(\mathbf{F}, \mathcal{V}^m) = \int_{\mathcal{V}^m} \mathrm{d}^m \mathbf{x}^{\mathcal{H}^{-1}} \lrcorner \mathbf{F}, \tag{2.1}$$

where $\mathcal{H}^{-1}$ is the inverse Hodge operation in [CFSM19, Eq. (10)]. The flux $\mathcal{F}(\mathbf{F}, \mathcal{V}^m)$ is non vanishing across the domain $\mathcal{V}^m$ for values of $m \geq k+n-r$. Similarly, the circulation of a generalized r-vector field along an m-dimensional hypersurface $\mathcal{V}^m$ is given by , [CFSM19, Eq. (27)]

$$\mathcal{C}(\mathbf{F}, \mathcal{V}^m) = \int_{\mathcal{V}^m} \mathrm{d}^m \mathbf{x} \llcorner \mathbf{F}. \tag{2.2}$$

The circulation $\mathcal{C}(\mathbf{F}, \mathcal{V}^m)$ can be defined as long as $m \geq r$.

To have an intuitive idea of the meaning of flux and circulation in multivector fields of grade r, we show in the following some plots about the geometrical interpretation of the interior products inside the integral (2.2). First, we consider a setup with $k = 0$, $n = 3$, a multivector field of grade $r = 2$ and a hypersurface $\mathcal{V}^2$ of dimension $m = 2$. Since $m = r$, the interior product in (2.2) is equivalent to a dot product [CFSM19, Sect. 2.2] so that the resulting scalar from the operation would be given by the area of the projector of the bivector on the surface differential, as represented in Figure 2.1 for a specific point $\mathbf{x}_0$ in the surface $\mathcal{V}^2$. In the figure, both $\mathrm{d}^2\mathbf{x}_0$ and $\mathbf{F}(\mathbf{x}_0)$ are *locally* represented by oriented, tangent planes, and the sign of the red area is function of the respective orientations. For example in Figure 2.1, the surface differential is oriented as $\mathbf{e}_{12}$, while the bivector field as $\mathbf{e}_{12} - \mathbf{e}_{13}$, oriented as $\mathbf{e}_2 + \mathbf{e}_3$ due to the Hodge dual. As a result, the interior product between the two leads to a positive scalar. As a remark, it is easy to see that the scalar product vanishes at the space–time points $\mathbf{x}_0$ where $\mathbf{F}(\mathbf{x}_0)$ is orthogonal to the surface. The circulation in (2.2) is the result of integration (summing) the resulting scalar products for all $\mathbf{x}$ in the surface.

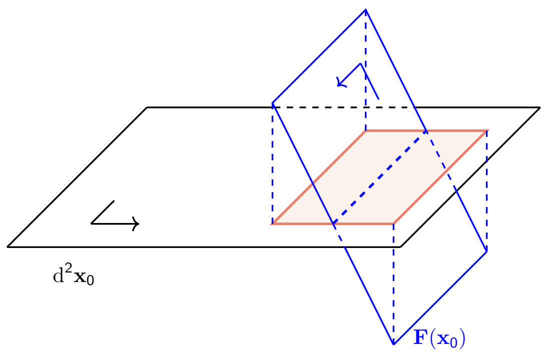

Figure 2.1: Example of $\mathrm{d}^m\mathbf{x}_0 \llcorner \mathbf{F}(\mathbf{x}_0)$ (red area) for $k = 0$, $n = 3$ and $m = r = 2$.

Our second example deals with the circulation of a bivector field $\mathbf{F}$ along a three-dimensional volume, that is $r = 2$ and $m = 3$. We find a different situation in this example, as the result of the operation is not a scalar but a vector describing the orientation of the bivector field with respect to the three-dimensional volume. As before, we show in Figure 2.2 the geometrical interpretation of $\mathrm{d}^3\mathbf{x} \llcorner \mathbf{F}(\mathbf{x})$ at a given space-time point $\mathbf{x}_0$ in the volume. Now, $\mathrm{d}^3\mathbf{x}_0$ is a three-dimensional differential that can be *locally* represented as a cube, $\mathbf{F}(\mathbf{x}_0)$ is a two-dimensional field that can be represented as an oriented surface, as before. For the example in Figure 2.2, the volume differential with basis $\mathbf{e}_{123}$ has orientation indicated by a three-dimensional arrow along the positive directions $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, while the bivector field is spanned by $-\mathbf{e}_{13}$ and oriented along $\mathbf{e}_2$. As a result, the interior product between the two leads to a vector in the $\mathbf{e}_2$ direction, whose sign defines an orientation.

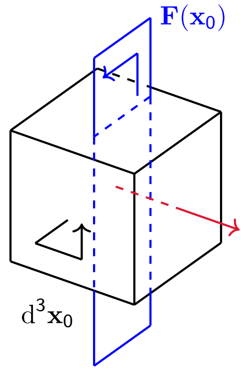

Figure 2.2: Example of $\mathrm{d}^m\mathbf{x}_0 \llcorner \mathbf{F}(\mathbf{x}_0)$ (red vector) for $k = 0$, $n = m = 3$ and $r = 2$.

Exterior and tensor calculus are closely connected. The generalized electromagnetic field components correspond to the antisymmetric Faraday tensor $F_{\mu\nu}$ [CTDRG97, pp. 17] in our generalized space–time [CFSM21b, Sect. 2.4]. Also, some fundamental operations between multivectors and tensors are available. The derivative operator $\boldsymbol{\partial}$ used in exterior calculus is expressed in vector basis [CFSM19, Eq. (35)] and it can act not only on multivectors as interior or exterior derivative, but also on tensors as tensorial contraction [CFSM21b, Eq. (19)]. Considering the symmetric stress–energy–momentum bitensor associated to the generalized electromagnetic field $\mathbf{T}_{\text{gem}}$, and evaluating its interior derivative, we find a vector as a result, corresponding to the divergence of the tensor, as presented in [MFSC21, Eq. (66)] and [CFSM21a, Eq. (29)]. The stress–energy–momentum tensor might be constructed in exterior calculus by comparing the generalized theory with the classical one and proving its role in the conservation law for the equivalent energy–momentum related to the Lorentz force in (1.3) [CFSM20, Appendix A.2], namely

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\text{gem}} + \mathbf{f} = 0. \tag{2.3}$$

Exterior calculus corresponds to working on the tangent bundle of a differential manifold, thereby establishing a direct link with differential forms [Zhd20, Sect. 4.1.6]. To give a practical example, the generalization of Stokes theorems for flux and circulation in exterior calculus are proved thanks to the formulation of the generalized Stokes theorem for differential forms [CFSM19, Sects. 3.4 and 3.5], and the same proof could also be set with vectorial or tensorial notation [Rut89, Ch. 3]. Generalized Stokes theorems can be properly defined and proved for flux and circulation of multivectors and for the divergence of tensors multivectorial valued as well. Regarding multivectors, two Stokes theorems also connect the circulation (resp. flux) with the exterior (resp. interior) derivative [CFSM19, Sect. 3.4 (resp. 3.5], that is [CFSM19, Eqs. (48)

and (60)],

$$\mathcal{F}(\mathbf{F}, \partial\mathcal{V}^m) = \mathcal{F}(\boldsymbol{\partial} \lrcorner \mathbf{F}, \mathcal{V}^m), \tag{2.4}$$
$$\mathcal{C}(\mathbf{F}, \partial\mathcal{V}^m) = \mathcal{C}(\boldsymbol{\partial} \wedge \mathbf{F}, \mathcal{V}^m), \tag{2.5}$$

where $\partial\mathcal{V}^m$ is the boundary of an m-dimensional hypersurface $\mathcal{V}^m$. In analogy with classical formulations, Stokes theorems allow to write the integral Maxwell equations in (2.1) and (2.2) in their differential form as presented in (1.1) and (1.2).

We have also related Stokes theorems with conservation laws. The flux of the stress–energy–momentum tensor is related to the conservation laws for energy and momentum components [CFSM20, Sect. 4.1] and, in general, it can be applied to every symmetric bitensor [CFSM20, Appendix A.4]. Concerning generalized electromagnetism, we connected the divergence of our symmetric stress–energy–momentum tensor, written as an interior derivative, to the conservation law for the generalized energy and momentum [CFSM20, Sects. 4.2 and 4.3], as in (2.3). For the three-indices tensors, we can also apply the Stokes theorem described in [MCFS21, Appendix A] to prove the conservation of angular momentum.

2.2. Lagrangian Dynamics

In mathematical physics, several theories are formulated following a Lagrangian approach. Examples include classical mechanics or analytical mechanics [LL76, Ch. 1] and, more interesting for our aims, field theories, from the classical theory of fields [LL87, Ch. 3] to quantum field theory [BS80, Ch. 1].

Similarly to what is presented in literature for scalar, vector and tensor fields, and for classical field theory in general [Mag05, Ch. 3], we have developed a Lagrangian formulation of generalized multivector fields in (k, n) space–time. We study a Lagrangian density that integrated over a suitable region of space–time leads a mathematical object called action [BS80, Sect. 1.3]. Then, we identify the equations of conservation by computing the variations of the action and imposing that it remains unchanged under these variations.

We consider a Lagrangian density $\mathcal{L}$ given by the linear combination of scalar products between two same graded multivector fields, properly multiplied by a suitable coefficient [CFSM21a, Eq. (4)]. The Lagrangian density should satisfy some properties [Ram01, Sect. 1.5], e. g., it should depend only on the fields and their derivative not exceeding the first order [CFSM21b, Sect. 1], in order to provide the definition of the action and that the results are physically acceptable.

Once we have an adequate definition of the Lagrangian density, we can calculate different kinds of variations. First of all, we can vary the multivector fields by adding an infinitesimal perturbation to them and we consequently apply the principle of stationary action, also known as principle of least action [LL76, Sect. 2], stating that the path followed by the fields is invariant for the action. This procedure leads to the dynamical equations of the system, known also as Euler–Lagrange equations [CTDRG97, pp. 15–16] which, considering a Lagrangian density depending on the $(r-1)$-vector potential $\mathbf{A}$ and its derivatives, are written

$$\partial_{\mathbf{A}}\mathcal{L} = (-1)^{r-1}\boldsymbol{\partial} \lrcorner \left(\partial_{\boldsymbol{\partial}\wedge\mathbf{A}}\mathcal{L}\right) + (-1)^{r}\boldsymbol{\partial} \wedge \left(\partial_{\boldsymbol{\partial}\lrcorner\mathbf{A}}\mathcal{L}\right). \tag{2.6}$$

The generalized electromagnetic field $\mathbf{F}$ is related to the potential $\mathbf{A}$ by means of the relation $\mathbf{F} = \boldsymbol{\partial} \wedge \mathbf{A}$, so that the last summand in (2.6) does not provide any contribution if we consider our generalized theory of electromagnetism. To obtain the generalized Maxwell equations as in (1.1) and (1.2), we set the Lagrangian density [CFSM21b, Sect. 4.2]

$$\mathcal{L} = \frac{(-1)^{r-1}}{2}\mathbf{F}\cdot\mathbf{F} + \mathbf{J}\cdot\mathbf{A}. \tag{2.7}$$

Secondly, another kind of variations acts on coordinates instead of changing directly the multivector fields. Perturbing the coordinates, we obtain the laws of conservation, which identify some quantities that are not changing during the evolution of the system. For $k = 1$, such behavior described by the constant of motion [LL76, Ch. II], as the evolution of the system is interpreted across the time dimension. The generalization of the constant of motion for our general case of (n, k, r) is left as an open problem.

In section 2.2.1, we analyze the two main cases studied with this latter method. If the action is invariant under infinitesimal space–time translations, then we obtain the conservation law for the stress–energy–momentum tensor, while the conservation of the generalized angular momentum is given because of invariance under rotations of the coordinates. The possibility of building an exterior–algebraic Lagrangian dynamics and find the related conserved quantities is an important topic that deserves a separate section in the sequel.

2.2.1. Conservation laws and conserved quantities

As anticipated, conservation laws have the advantage of identifying quantities that are conserved independently of the evolution of the system. In [MFSC21, CFSM21a] and in [MCFS21], the detailed procedure to find the conservation laws, respectively, for the stress–energy–momentum tensor and for the angular momentum is described.

The stress–energy–momentum bitensor, i. e. a tensor of rank 2, includes quantities as energy, linear momentum and the components equivalent to the classical Maxwell stress tensor. The evaluation of the stress–energy–momentum tensor in exterior calculus, by means of Lagrangian variations, is computed in [CFSM21a] and [MFSC21] and it allows to find a tensor which is symmetric by definition. This fact could be an advantage since, in classical field theories, the canonical stress–energy tensor is not necessarily symmetric and we should compute further operation to obtain its symmetrized version [FR04].

To find the stress–energy–momentum tensor, we start by setting a Lagrangian density function $\mathcal{L}$ as the sum of terms given by scalar products of same-graded multivector fields [CFSM21a, Eq. (4)], and we define the action $\mathcal{S}$ as the integral of the Lagrangian density $\mathcal{L}$ over a suitable integration domain, with respect to the $k + n$ space–time coordinates [MFSC21, Eq. (21)]. We then introduce a perturbation in the origin of the axis by adding an infinitesimal translation and, imposing the invariance of the action, we obtain a conservation law for a symmetric bitensor, written in terms of the divergence of the tensor [MFSC21, Sect. 3.6]. Due to the linearity of the Lagrangian density $\mathcal{L}$, we might write the tensor as the sum of different terms, as many as $\mathcal{L}$, and the divergence of the stress–energy–momentum tensor would be given by the linear combination of the divergences of its summands [MFSC21, Sect. 3.1].

The divergence of the stress–energy–momentum tensor is the quantity which vanishes for isolated systems and it describes the flux of energy and momentum across regions of space-time [FR04]. These latter facts are confirmed in exterior algebra and we can recover the classical physical results by setting $k = 1$, $n = 3$ and $r = 2$, as already explained above in the thesis.

We consider the simple case of the free generalized electromagnetism, obtained by means of the first term of Lagrangian density in (2.7) without the interaction term, namely $\mathbf{J} = 0$, written

$$\mathcal{L}_{\text{free-gem}} = -\frac{1}{2}\mathbf{F} \cdot \mathbf{F}. \tag{2.8}$$

The consequent expression of the stress–energy–momentum tensor associated to the Lagrangian density in (2.8) results particularly elegant as well as symmetric [MFSC21, Eq. (183)], and it is written

$$\mathbf{T}_{\text{free-gem}} = -\frac{1}{2}\big(\mathbf{F} \odot \mathbf{F} + \mathbf{F} \oslash \mathbf{F}\big), \tag{2.9}$$

where $\odot$ and $\oslash$ are two operations defined in [MFSC21]. The conservation law obtained for the Lagrangian density $\mathcal{L}_{\text{free-gem}}$ in (2.8), and involving the stress–energy–momentum tensor $\mathbf{T}_{\text{free-gem}}$ in (2.9), is given by the simple, coordinate-free, expression [MFSC21, Eq. (189)]

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\text{free-gem}} = (-1)^{r-1}\big(\mathbf{F} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{F}) - \mathbf{F} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{F})\big). \tag{2.10}$$

Besides electromagnetism, we have been able to prove the validity of our symmetric stress–energy–momentum tensor, obtained by means of our exterior-algebraic procedure, also for different classical theories of field. We verified that the definition works properly for the free scalar fields, given as zero-grade vector for $r = 0$ [MFSC21, Sect. 4.2] and for the electromagnetic field in (2.7), with an additional term due to the interaction with an external source [MFSC21, Sect. 4.3], where the conservation law corresponds to a natural generalization of the Lorentz force density. The addition of an appropriate mass term gives the free Proca field [MFSC21, Sect. 4.5], while the concept of trace in exterior calculus [CFSM20, Eq. (78)] allows to define the free Yang–Mills fields [MFSC21, Sect. 4.4].

Following an analogous procedure as above, but perturbing now the origin of the axis by means of space–time rotations, allows to find the conservation laws for the generalized angular momentum [MCFS21, Sect. 2]. The angular momentum density is mathematically described by a three-indices tensor and it admits a conservation law given by its divergence, for the free generalized electromagnetic field. The total angular momentum is written as sum of four terms, the orbital angular momentum, the spin, one term corresponding to the center-of-mass velocity and one last summand depending on the origin of coordinates and containing the contribution of the flux of the stress–energy–momentum tensor [MCFS21, Eq. (73)]. Each component may also be decomposed in the normal-mode representation in the Fourier domain, as explained in 2.3.1. Some considerations and details about the definition of the spin are also provided in the concluding section 3.2 on quantization, since the spin commonly arise in quantized theories.

2.3. Generalized Maxwell equations and electromagnetic waves

Maxwell equations are one of the most used tools in the scientific world. They describe all the phenomena related to electric and magnetic manifestations, and they also provide mathematical tools which are applicable to several other fields of science. In the development of the framework of exterior calculus, in Section 2.1, we have generalized the concepts of circulation and flux in (k, n) space–time. Related to these latter, we have stated the Stokes theorems to move from the integral formulation of circulation and flux of a multivector field to the differential formulation of exterior and interior derivatives of the field, respectively.

By simple comparison with the classical theory, a natural and intuitive generalization of electromagnetism in exterior calculus has been mathematically outlined for a generalized multivectorial Maxwell field $\mathbf{F}$, which includes electric and magnetic fields in its components. Generalized Maxwell equations are given by circulation and flux of $\mathbf{F}$ in integral form [CFSM19, Sect. 4.3], as written in (2.1) and (2.2), and by exterior and interior derivatives of $\mathbf{F}$ in differential form [CFSM20, Sect. 3.1], [CFSM19, Sect. 4.2], as in (1.1) and (1.2).

We may also refer to the generalized differential formulation of Maxwell equations in (1.1) and (1.2) as inhomogeneous Maxwell equations, using the same terminology of classical theory. In total analogy with vectorial electromagnetism, we can also name the equations of motion of free generalized electromagnetism as homogeneous Maxwell equations, written

$$\boldsymbol{\partial} \lrcorner \mathbf{F} = 0, \tag{2.11}$$

$$\boldsymbol{\partial} \wedge \mathbf{F} = 0. \tag{2.12}$$

These latter differential expressions, in absence of charged sources, have also been computed by means of the Lagrangian approach, by applying the principle of stationary action to the Lagrangian density in (2.8), while the inhomogeneous equations are computed thanks to (2.7).

From this setup we are able to recover the classical form of Maxwell equations by setting $k = 1$ and $n = 3$, as ordinary Minkowski space–time, and $r = 2$, with the components of the bivectorial electromagnetic field coinciding with the components of the antisymmetric two-indices Faraday tensor [CFSM20, Sect. 1], as mentioned before in Sect. 2.1. The bivector field in $(1, 3)$ space–time is written in terms of the classical electric field $\mathbf{E}$ and the Hogde dual of the classical magnetic field $\mathbf{B}$ [CFSM19, Sect. 4], namely

$$\mathbf{F} = \mathbf{e}_0 \wedge \mathbf{E} + \mathbf{B}^{\mathcal{H}}, \tag{2.13}$$

where the time-space components correspond to the electric field and the space-space components to the magnetic field.

The same analogy we did with classical vectorial electromagnetic theory can be done also to obtain the formulation of electromagnetism by means of the other mathematical descriptions, differential forms [Des81] or tensor calculus notation [Nab12, Ch. 2]. In general, there are considerable applications supporting exterior algebra as a natural setting for an abstract yet intuitive form of Maxwell equations, for any value of r, k and n [CFSM20, Sect. 1].

Writing the exterior-algebraic Maxwell equations in terms of the multivector potential $\mathbf{A}$ allows to find the number of independent components of the potential needed to solve the problem, namely $\binom{k+n-2}{r-1}$ [CFSM20, Sect. 4.1], taking into account the gauge invariance, too. The same number of independent components can also be found by making the dual formulation of generalized Maxwell equations, where the duality is obtained by swapping the roles of interior and exterior derivatives [CFSM21b, Sect. 4.4]. The independent components of the potential field, commonly discussed also in classical electrodynamics [Gri07, Sect. 2.3], is revisited in the concluding section on the degrees of freedom.

2.3.1. Electromagnetic waves and Fourier domain

The set of solutions of Maxwell equations include electromagnetic waves in vacuum. In the classical theory of electromagnetism, it is shown how the electric and magnetic vector fields components satisfy separately the wave equation, and the same behaviour is followed by the components of the scalar and vector potential, with a suitable choice of the gauge [FLS77, Sect. 18-6]. Analogously, the components of the Maxwell field $\mathbf{F}$ satisfy a generalized wave equation, as written in (1.4), characterized by an extended d'Alambert operator, which includes all the second-order time coordinates derivatives with negative sign, due to the metric, and all the second-order spatial coordinates derivatives with positive sign.

Equation (1.4) is found as a direct consequence of combining Maxwell equations by using the basic properties of exterior algebra and it is also verified for the multivector potential $\mathbf{A}$ in the generalized Lorenz gauge [CFSM20, Eq. (49)] $\boldsymbol{\partial} \lrcorner \mathbf{A} = 0$. The same wave-like behaviour is also found by employing the Lagrangian approach [CFSM21b, pp. 12–13], to validate the multivectorial formulation of electromagnetic theory in exterior calculus, independently of the approach.

In order to deal with generalized propagating waves, it is useful to introduce the Fourier transform in exterior calculus. In (k, n) dimensions, we indicate the space–time position vector as $\mathbf{x}$, having the time coordinates in the first k components and the spatial coordinates in the following n components [CFSM19, Sect. 2]. We define a vector $\boldsymbol{\xi}$, which contains the equivalent components of the frequency and wave number, respectively k and n, so that $\boldsymbol{\xi}$ and the generalized position vector $\mathbf{x}$ are conjugate variable [Jam02, Sect. 1.5].

We can compute the Fourier transform of all the quantities involved in our theory, obtaining the generalized theory of electromagnetism expressed in the Fourier domain. We can easily move from the space–time domain to the Fourier domain and vice versa, respectively, thanks to the Fourier transform in (1.5) and its inverse [CFSM20, Eq. (A70)] written, taking Maxwell field as an example,

$$\mathbf{F}(\mathbf{x}) = \int_{\mathbb{R}^{k+n}} \mathrm{d}^{k+n}\boldsymbol{\xi}\,\delta(\boldsymbol{\xi}\cdot\boldsymbol{\xi})e^{2\pi j\boldsymbol{\xi}\cdot\mathbf{x}}\hat{\mathbf{F}}(\boldsymbol{\xi}). \tag{2.14}$$

Fourier transforms are extensively applied in classical and quantum field theory, in order to find the normal mode expansion of the fields [PS95, Sec. 2.3]. Teh integral in (2.14) can be interpreted as a linear combination of complex exponentials of the form $e^{2\pi j\boldsymbol{\xi}\cdot\mathbf{x}}$, each describing a plane wave propagating towards the $\boldsymbol{\xi}$ direction. Such observation is actually the fundamental basis to express the generalized-electromagnetic-wave solutions of Maxwell equations in the normal mode decomposition [MCFS21, Sect. 3.2], playing a crucial role in describing propagation processes.

In general, Fourier transforms are also useful to give physical details about the dynamics of the system. Going beyond the topics covered by this thesis, we can find the what is known in literature as dispersion relation, as some attentive reader could have recognized from the Dirac delta in (2.14) [CFSM20, Sect. 3.3]. The relation $\boldsymbol{\xi}\cdot\boldsymbol{\xi} = 0$ may be geometrically interpreted as a constraint on wave number components and provides as such some information about the wave propagation.

The dispersion relation, which we could also note that corresponds to the wave equation in the Fourier domain, is connected to the phase velocity, whose definition implies the existence of propagating waves and, as a consequence, the propagation of signals [FLS64, Sect. 31-3]. We know also that the phase velocity depends on the frequency and wavelength, two quantities included in $\boldsymbol{\xi}$ and, using these quantities, a physical application of propagating waves in media as electromagnetic signals has been studied in [LP73].

Chapter 3

Conclusions and Outline for Future Research

In this final chapter, we discuss the main conclusions of the thesis and extend some of the key concepts. Some possible advances and a brief outline of the potential for future work are also presented.

3.1. Degrees of freedom

The first interesting conclusion regards the subject of the degrees of freedom. In our analysis, we deal with the number of independent components of the multivector potential $\mathbf{A}$ needed to find a solution of the system of generalized Maxwell equations. Despite the total number of components of $\mathbf{A}$ is $\binom{k+n}{r-1}$, a certain redundancy among them leads to have $\mathcal{D} = \binom{k+n-2}{r-1}$ independent components, as found in [CFSM20, Sect. 4.3], with basic properties of exterior calculus, and in [CFSM21b, Sect. 4.4], thanks to the Lagrangian approach. Evaluating $\mathcal{D}$ for ordinary flat $(1,3)$ space–time and considering a bivectorial electromagnetic field, namely $r = 2$, then the result would give $\mathcal{D} = 2$, corresponding to the number of polarizations associated to a classical propagating electromagnetic field.

Concerning our multivectorial formulation of Maxwell equations, due to the general nature of the theory, for a given set of values of k, n and r, we might know the number of degrees of freedom, as the number of independent components of the potential, which are necessary to find a solution [Ros80].

Moving towards a quantum development of the subject, we find an explicit expression of the spin [MCFS21, Eq. (79)], depending on the potential components in the Fourier domain. For the definition of the spin, the number of independent components of the potential results to be $\mathcal{Q} = \binom{k+n-4}{r-2}$ [MCFS21, Sect. 3.3]. Differently from the previous case, where we were looking at solutions of Maxwell equations, we have followed a different procedure concerning the spin . The number $\mathcal{Q}$ arises by imposing invariance of the free generalized electromagnetic field Lagrangian under rotation in space–time and considering the so-called Coulomb-ℓ gauge, or, from another perspective, considering the transverse normal modes of the multivector field [CTDRG97, Sec. $\mathrm{B_I}$]. Thus, as $\mathcal{D}$ may refer to the polarizations of the generalized electromagnetic field with respect to a certain direction of propagation, then $\mathcal{Q}$ may identify distinct pairs of integer unitary spin particles related to that direction, and it would surely be worthy of studying some possible connection between $\mathcal{D}$ and $\mathcal{Q}$.

To deepen this branch of investigation and to find a potential application in communication theory, some contact points could be found between generalized electromagnetism and the degrees of freedom of a communication channel. Some advances in this direction might also be helpful to give an extensive and satisfactory interpretation of how many degrees of freedom one has in a channel. A particularly appreciable connection could be done with Landau's eigenvalue theorem [Lan75], where the eigenvalues identify the number of degrees of freedom of the space–time field used for communication [Fra15].

3.2. Quantization

Together with relativity, quantization, or quantum theory, is one of the most noteworthy topics in modern physics. Although the process of quantization is only marginally mentioned by the papers considered for this defence, it is maybe the most interesting topic to be further studied. The key concept of quantum processes is that the energy is recovered in quantized levels, depending on an integer number. Historically, this fact was first observed by Planck, in 1900, in the formula describing the black body radiation [Pla00a], leading to the consequence that an oscillator with frequency ν should have levels of energies which are integer multiples of a minimum quantity $h\nu$ [Pla00b].

In 1905, the same relation between energy and frequency was applied by Einstein in the explanation of the photoelectric effect [Ein05], in the publication that earned him the Nobel prize. The constant h, called Planck constant [ZeaPDG20], corresponds dimensionally to the smallest value of the action, the quantity given by the integral of the Lagrangian density, seen in 2.2. Thus, the energy is transferred in packages which are multiples of a small minimum amount and, surprisingly, even the zero-point energy is not vanishing. In quantum theories, vacuum fluctuations are not only admitted and mathematically explained, but also physically experimented, as the example of Casimir effect [MR16].

Likewise classical Maxwell equations and the theory of electromagnetism have evolved into quantum electrodynamics [Fey85], we may also extend the process of quantization to our generalized theory of electromagnetism. The components of quantized multivectorial fields could not be represented by real valued functions, but they would be mathematically described by self-adjoin operators. As already anticipated in 2.3.1, generalized fields can be expanded in a basis formed by a complete set of orthonormal functions, such as the complex exponentials with suitable coefficients.

Propagating waves is one of the most profitable ways to model signals, from the physical point of view, but it is not the only possible method. In field theories, propagation is also well described by the mathematical tool of Green functions, which in quantum field theory identify fundamental objects called *propagators*. Green functions can also be defined in arbitrary-dimensional space–time exterior calculus, even if it is not explored in this essay.

A potential starting point to introduce the notion of quantization in exterior–algebraic generalized Maxwell equations is mentioned in [CFSM20, Sect. 4.3], where the flux of the symmetric stress–energy–momentum tensor written in the Fourier domain suggests the quantization of the multivector potential. In fact, representing multivectorial fields as inverse Fourier transforms, as explained in 2.3.1, gives an excellent expansion for the field to be quantized, since it is expressed in generalized plane waves basis [Mag05, Sect. 4.3.1].

Regarding the Lagrangian approach, in [CFSM21b, Sects. 4.2-4.3] we have presented the Lagrangian density for generalized electromagnetism. In presence of a Lagrangian, quantization can be implemented by means of the canonical procedure [Sre07, Ch. 55]. Beyond generalized electromagnetism, we might also be able to provide the quantization of Yang-Mills and Proca fields, whose Lagrangian density have been properly defined in exterior calculus.

The most precise step for advance towards quantization in exterior algebra is done in [MCFS21]. The spin term in [MCFS21, Eq. (76)] is a bridge between classical and quantum electromagnetism. As observed in the paper, to give an answer to [Jac99, Problem 7.27], this definition of spin could not be fully justified from the classical point of view, also due to its gauge dependence, even though it is a first conceptual attempt to go beyond classical electromagnetism.

3.3. Information Theory

Last, we are going to mention some potential prospects of the connection between multivectorial electromagnetism and information theory. Different models of information transmission may be designed and analysed by representing signals as generalized electromagnetic waves, as suggested in 2.3.1. Higher-frequency bands would be progressively introduced for communication, taking into consideration that, in order to select suitable frequencies, we can exploit the energy–momentum relation in Minkowski space, which establish a link between the frequency and the components of the spatial momentum and it comes from the relation $\boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0$ in (2.14). We could also thoroughly investigate some physical aspects by studying of the effect of relativistic and quantum theories on communication channels [Gor62] and by exploring their impact on the classical definition of the characteristic quantities of information transmission.

Bibliography

[BC02] Nevio Benvenuto and Giovanni Cherubini. *Algorithms for Communications Systems and their Applications*. John Wiley & Sons, 2002.

[BS80] N. N. Bogoliubov and D.V. Shirkov. *Introduction to Theory of Quantized Fields*. John Wiley & Sons, 3 edition, 1980.

[CFSM19] Ivano Colombaro, Josep Font-Segura, and Alfonso Martinez. An introduction to space–time exterior calculus. *Mathematics*, 7(6):564, June 2019.

[CFSM20] Ivano Colombaro, Josep Font-Segura, and Alfonso Martinez. Generalized Maxwell equations for exterior-algebra multivectors in (k, n) space-time dimensions. *European Physical Journal Plus*, 135(305), March 2020.

[CFSM21a] Ivano Colombaro, Josep Font-Segura, and Alfonso Martinez. Derivation of the symmetric stress–energy–momentum tensor in exterior algebra. *Journal of Physics: Conference Series*, 2090(1):012050, 2021.

[CFSM21b] Ivano Colombaro, Josep Font-Segura, and Alfonso Martinez. An exterior algebraic derivation of the Euler–Lagrange equations from the principle of stationary action. *Mathematics*, 9(18):2178, September 2021.

[CTDRG97] Claude Cohen-Tannoudji, Jacques Dupont-Roc, and Gilbert Grynberg. *Photons and Atoms: Introduction to Quantum Electrodynamics*. Wiley, 1997.

[Dar00] Olivier Darrigol. *Electrodynamics from Ampere to Einstein*. Oxford University Press, York, 2000.

[Des81] G. A. Deschamps. Electromagnetics and differential forms. *Proc. IEEE*, 69:676–696, June 1981.

[Ein05] Albert Einstein. Über einen die erzeugung und verwandlung des lichtes betreffenden heuristischen gesichtspunkt. *Ann. d. Phys.*, 17:132, 1905.

[Fey85] Richard P. Feynman. *QED: The Strange Theory of Light and Matter*. Princeton University Press, 1985.

[FLS64] Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman Lectures on Physics, Vol. I: Mainly Mechanics, Radiation, and Heat*. Addison-Wesley, 1964.

[FLS77] Richard P. Feynman, Robert B. Leighton, and Matthew Sands. *The Feynman Lectures on Physics, Vol. II: Mainly Electromagnetism and Matter*. Addison-Wesley, 1977.

[FR04] Michael Forger and Hartmann Römer. Currents and the energy-momentum tensor in classical field theory: a fresh look at an old problem. *Annals of Physics*, 309(2):306–389, Feb 2004.

[Fra15] Massimo Franceschetti. On Landau's eigenvalue theorem and information cut-sets. *IEEE Transactions on Information Theory*, 61(9):5042–5051, 2015.

[Gal68] Robert G. Gallager. *Information theory and reliable communication*. John Wiley & Sons, 1968.

[Gor62] J. P. Gordon. Quantum effects in communication systems. *Proc. IRE*, 50(9):1898–1908, 1962.

[Gri07] David J. Griffiths. *Introduction to Electrodynamics*. Dorling Kindersley, 3 edition, 2007.

[Jac99] John David Jackson. *Classical Electrodynamics*. John Wiley & Sons, 3rd edition, 1999.

[Jam02] J. F. James. *A Student's Guide to Fourier Transforms: With applications in physics and engineering*. Cambridge University Press, 2 edition, 2002.

[Lan75] H. J. Landau. On Szegö's eigenvalue distribution theorem and non-hermitian kernels. *Journal d'Analyse Mathematique*, 28:335–357, 1975.

[Lan96] Rolf Landauer. The physical nature of information. *Phys. Lett. A*, 217:188–193, 1996.

[Lan99] Rolf Landauer. Information is a physical entity. *Physica A*, 263:63–67, 1999.

[LEMW23] H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl. *The principle of relativity. Memoirs on the special and general theory of relativity*. Dover Publications, Mineola, NY, USA, 1923.

[LL76] L. D. Landau and E. M. Lifshitz. *Mechanics*, volume Volume 1 of *Course of Theoretical Physics*. Butterworth-Heinemann, 3 edition, 1976.

[LL87] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields*, volume 2 of *Course of Theoretical Physics*. Butterworth-Heinemann, 4 edition, 1987.

[LP73] R. Landauer and S. T. Peng. Velocity modulation of propagating waves. *J. Appl. Phys.*, 44(23):1156–1161, 1973.

[Mag05] Michele Maggiore. *A Modern Introduction to Quantum Field Theory*. Oxford Master Series in Statistical, Computational, and Theoretical Physics. Oxford Univ. Press, Oxford, 2005.

[Max03] James Clerk Maxwell. *A Treatise on Electricity & Magnetism*. Dover Publications, 2003.

[MCFS21] Alfonso Martinez, Ivano Colombaro, and Josep Font-Segura. On the angular momentum and spin of generalized electromagnetic field for r-vectors in (k, n) space-time dimensions. *European Physical Journal Plus*, 136(1047), Oct. 2021.

[MFSC21] Alfonso Martinez, Josep Font-Segura, and Ivano Colombaro. An exterior-algebraic derivation of the symmetric stress-energy-momentum tensor in flat space-time. *European Physical Journal Plus*, 136(212), February 2021.

[MR16] U. Mohideen and Anushree Roy. Precision measurement of the casimir force from 0.1 to 0.9 μm. *Phys. Rev. Lett.*, 81:4549, 2016.

[Nab12] Gregory L. Naber. *The Geometry of Minkowski Spacetime*. Applied Mathematical Sciences 92. Springer-Verlag, New York, 2 edition, 2012.

[Pla00a] Max Planck. *Ann. d. Phys.*, 1:69, 1900.

[Pla00b] Max Planck. *Ann. d. Phys.*, 1:719, 1900.

[PS95] Michael E. Peskin and Daniel V. Schroeder. *An introduction to quantum field theory*. Addison-Wesley, 1 edition, 1995.

[Ram01] Pierre Ramond. *Field theory: a modern primer*. Frontiers in Physics Series, Vol 74. Westview Press, 2 edition, 2001.

[Ros80] Joe Rosen. Redundancy and superfluity for electromagnetic fields and potentials. *American Journal of Physics*, 48(12):1071–1073, 1980.

[Rut89] Aris Rutherford. *Vectors, tensors, and the basic equations of fluid mechanics*. Dover books on engineering. Dover Publications, New York, 1989.

[Sha48] Claude E. Shannon. A mathematical theory of communication. *Bell Sys. Tech. J.*, 27:379–423, 623–656, Jul. and Oct. 1948.

[Sre07] Mark Allen Srednicki. *Quantum field theory*. Cambridge University Press, 2007.

[Vel12] Milen Velev. Relativistic mechanics in multiple time dimensions. *Physics Essays*, 25(3):403–438, 2012.

[ZeaPDG20] P. A. Zyla and et al. (Particle Data Group). *Review of Particle Physics*. 083C01. 2020.

[Zhd20] Michael S. Zhdanov. Chapter 4.1 - Maxwell's equations and numerical electromagnetic modeling in the context of the theory of differential forms. In Junzo Kasahara, Michael S. Zhdanov, and Hitoshi Mikada, editors, *Active Geophysical Monitoring (Second Edition)*, pages 245 – 267. Elsevier, second edition, 2020.

Part II

Publications

An Introduction to Space-Time Exterior Calculus *

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Abstract

The basic concepts of exterior calculus for space-time multivectors are presented: interior and exterior products, interior and exterior derivatives, oriented integrals over hypersurfaces, circulation and flux of multivector fields. Two Stokes theorems relating the exterior and interior derivatives with circulation and flux respectively are derived. As an application, it is shown how the exterior-calculus space-time formulation of the electromagnetic Maxwell equations and Lorentz force recovers the standard vector-calculus formulations, in both differential and integral forms.

1 Introduction

Vector calculus has, since its introduction by J. W. Gibbs [1] and Heaviside, been the tool of choice to represent many physical phenomena. In mechanics, hydrodynamics and electromagnetism, quantities such as forces, velocities and currents are modeled as vector fields in space, while flux, circulation, divergence or curl describe operations on the vector fields themselves.

With relativity theory, it was observed that space and time are not independent but just coordinates in space–time [2] (pp. 111–120). Tensors like the Faraday tensor in electromagnetism were quickly adopted as a natural representation of fields in space–time [3] (pp. 135–144). In parallel, mathematicians such as Cartan generalized the fundamental theorems of vector calculus, i.e., Gauss, Green, and Stokes, by means of differential forms [4]. Later on, differential forms were used in Hamiltonian mechanics, e. g. to calculate trajectories as vector field integrals [5] (pp. 194–198).

A third extension of vector calculus is given by geometric and Clifford algebras [6], where vectors are replaced by multivectors and operations such as the cross and the dot products subsumed in the geometric product. However, the absence of an explicit formula for the geometric product hinders its widespread use. An alternative would have been the exterior algebra developed by Grassmann which nevertheless has received little attention in the literature [7]. An early work in this direction was Sommerfeld's presentation of electromagnetism in terms of six-vectors [8].

We present a generalization of vector calculus to exterior algebra and calculus. The basic notions of space–time exterior algebra, introduced in Section 2, are extended to exterior calculus in Section 3 and applied to rederive the equations of electromagnetism in Section 4. In contrast to geometric algebra, our interior and exterior products admit explicit formulations, thereby merging the simplicity and intuitiveness of standard vector calculus with the power of tensors and differential forms.

* This work was funded in part by the Spanish Ministry of Economy and Competitiveness under grants TEC2016-78434-C3-1-R and BES-2017-081360. Published in: **Mathematics (2019), 7(6), 564. DOI:** 10.3390/math7060564.

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2 Exterior Algebra

Vector calculus is constructed around the vector space $\mathbf{R}^3$, where every point is represented by three spatial coordinates. In relativity theory the underlying vector space is $\mathbf{R}^{1+3}$ and time is treated as a coordinate in the same footing as the three spatial dimensions. We build our theory in space–time with k time dimensions and n space dimensions. The number of space–time dimensions is thus $k+n$ and we may refer to a (k,n)- or $(k+n)$-space–time, $\mathbf{R}^{k+n}$. We adopt the convention that the first k indices, i.e., $i = 0,\dots,k-1$, correspond to time components and the indices $i = k,\dots,k+n-1$ represent space components and both k and n are non-negative integers. A point or position in this space–time is denoted by $\mathbf{x}$, with components $\{x_i\}_{i=0}^{k+n-1}$ in the canonical basis $\{\mathbf{e}_i\}_{i=0}^{k+n-1}$, that is

$$\mathbf{x} = \sum_{i=0}^{k+n-1} x_i \mathbf{e}_i. \tag{1}$$

Given two arbitrary canonical basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$, then their dot product in space–time is

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} -1, & i = j,\, 0 \le i \le k-1, \\ +1, & i = j,\, k \le i \le k+n-1, \\ 0, & i \ne j. \end{cases} \tag{2}$$

For convenience, we define the symbol $\Delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ as the metric diagonal tensor in Minkowski space–time [2] (pp. 118–120), such that time unit vectors $\mathbf{e}_i$ have negative norm $\Delta_{ii} = -1$, whereas space unit vectors $\mathbf{e}_i$ have positive norm $\Delta_{ii} = +1$. The **dot product** of two vectors $\mathbf{x}$ and $\mathbf{y}$ is the extension by linearity of the product in Equation (2), namely

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=0}^{k+n-1} x_i y_i \Delta_{ii} = -\sum_{i=0}^{k-1} x_i y_i + \sum_{i=k}^{k+n-1} x_i y_i. \tag{3}$$

2.1 Grade, Multivectors, and Exterior Product

In addition to the $(k+n)$-dimensional vector space $\mathbf{R}^{k+n}$ with canonical basis vectors $\mathbf{e}_i$, there exist other natural vector spaces indexed by ordered lists $I = (i_1,\dots,i_m)$ of m non-identical space and time indices for every $m = 0,\dots,k+n$. As there are $\binom{k+n}{m}$ such lists, the dimension of this vector space is $\binom{k+n}{m}$. We shall refer to m as **grade** and to these vectors as **multivectors** or grade-m vectors if we wish to be more specific. A general multivector can be written as

$$\mathbf{v} = \sum_I v_I \mathbf{e}_I, \tag{4}$$

where the summation extends to all possible ordered lists with m indices. If $m = 0$, the list is empty and the corresponding vector space is $\mathbf{R}$. The direct sum of these vector spaces for all m is a larger vector space of dimension $\sum_{m=0}^{k+n} \binom{k+n}{m} = 2^{k+n}$, the exterior algebra. In tensor algebra, multivectors correspond to antisymmetric tensors of rank m. In this paper, we study vector fields $\mathbf{v}(\mathbf{x})$, namely multivector-valued functions $\mathbf{v}$ varying over the space–time position $\mathbf{x}$.

The basis vectors for any grade m may be constructed from the canonical basis vectors $\mathbf{e}_i$ by means of the exterior product (also known as wedge product), an operation denoted by $\wedge$ [9] (p. 2). We identify the vector $\mathbf{e}_I$ for the ordered list $I = (i_1, i_2, \dots, i_m)$ with the exterior product of $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_m}$:

$$\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \dots \wedge \mathbf{e}_{i_m}. \tag{5}$$

In general, we may compute the exterior product as follows. Let two basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ have grades $m = |I|$ and $m' = |J|$, where $|I|$ and $|J|$ are the lengths of the respective index lists. Let $(I,J) = \{i_1,\dots,i_m,j_1,\dots,j_{m'}\}$ denote the concatenation of I and J, let $\sigma(I,J)$ denote the signature of the permutation sorting the elements of this concatenated list of $m+m'$ indices, and let $\varepsilon(I,J)$ denote the resulting sorted list, which we also denote by $I+J$. Then, the exterior product $\mathbf{e}_I$ of $\mathbf{e}_J$ is defined as

$$\mathbf{e}_I \wedge \mathbf{e}_J = \sigma(I,J)\mathbf{e}_{\varepsilon(I,J)}. \tag{6}$$

The **exterior product** of vectors $\mathbf{v}$ and $\mathbf{w}$ is the bilinear extension of the product in Equation (6),

$$\mathbf{v} \wedge \mathbf{w} = \sum_{I,J} v_I w_J \, \mathbf{e}_I \wedge \mathbf{e}_J. \tag{7}$$

Since permutations with repeated indices have zero signature, the exterior product is zero if $m + m' > k + n$ or more generally if both vectors have at least one index in common. Therefore, the exterior product is either zero or a vector of grade $m+m'$. Further, the exterior product is a skew-commutative operation, as we can also write Equation (6) as $\mathbf{e}_I \wedge \mathbf{e}_J = (-1)^{|I||J|}\mathbf{e}_J \wedge \mathbf{e}_I$.

At this point, we define the dot product $\cdot$ for arbitrary grade-m basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ as

$$\mathbf{e}_I \cdot \mathbf{e}_J = \Delta_{I,J} = \Delta_{i_1,j_1}\Delta_{i_2,j_2}\cdots\Delta_{i_m,j_m}, \tag{8}$$

where I and J are the ordered lists $I = (i_1, i_2, \dots, i_m)$ and $J = (j_1, j_2, \dots, j_m)$. As before, we extend this operation to arbitrary grade-m vectors by linearity.

Finally, we define the complement of a multivector. For a unit vector $\mathbf{e}_I$ with grade m, its Grassmann or Hodge **complement** [10] (pp. 361–364), denoted by $\mathbf{e}_I^{\mathcal{H}}$, is the unit $(k+n-m)$-vector

$$\mathbf{e}_I^{\mathcal{H}} = \Delta_{I,I}\sigma(I, I^c)\mathbf{e}_{I^c}, \tag{9}$$

where I^c is the complement of the list I, namely the ordered sequence of indices not included in I. As before, $\sigma(I, I^c)$ is the signature of the permutation sorting the elements of the concatenated list (I, I^c) containing all space–time indices. In other words $\mathbf{e}_{I^c}$ is the basis vector of grade $k+n-m$ whose indices are in the complement of I. In addition, we define the inverse complement transformation as

$$\mathbf{e}_I^{\mathcal{H}^{-1}} = \Delta_{I^c,I^c}\sigma(I^c, I)\mathbf{e}_{I^c}. \tag{10}$$

We extend the complement and its inverse to general vectors in the space–time algebra by linearity.

2.2 Interior Products

While the exterior product of two multivectors is an operation that outputs a multivector whose grade is the addition of the input grades, the dot product takes two multivectors of identical grade and subtracts their grades, yielding a zero-grade multivector, i.e., a scalar. We say that the exterior product raises the grade while the dot product lowers the grade. In this section, we define the left and right interior products of two multivectors as operations that lower the grade and output a multivector whose grade is the difference of the input multivector grades.

As always, we start by defining the operation for the canonical basis vectors. Let $\mathbf{e}_I$ and $\mathbf{e}_J$ be two basis vectors of respective grades $|I|$ and $|J|$. The left interior product, denoted by $\lrcorner$, is defined as

$$\mathbf{e}_I \lrcorner \mathbf{e}_J = \Delta_{I,I}\sigma\big(\varepsilon(I, J^c)^c, I\big)\mathbf{e}_{\varepsilon(I,J^c)^c}. \tag{11}$$

If I is not a subset of J, that is when there are elements in I not present in J, e. g. for $|I| > |J|$, the signature of the permutation sorting the concatenated list $\big(\varepsilon(I, J^c)^c, I\big)$ is zero as there are repeated indices in the list to be sorted, and the left interior product is zero. Otherwise, if I is a subset of J, the permutation rearranges the indices in J in such a way that the last $|I|$ positions coincide with I and $\varepsilon(I, J^c)^c$ represents the first $|J| - |I|$ elements in the rearranged sequence, that is $\varepsilon(I, J^c)^c = J \setminus I$.

The right interior product, denoted by $\llcorner$, of two basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ is defined as

$$\mathbf{e}_I \llcorner \mathbf{e}_J = \Delta_{J,J}\sigma\big(J, \varepsilon(I^c, J)^c\big)\mathbf{e}_{\varepsilon(I^c,J)^c}. \tag{12}$$

As with the left interior product, if J is a subset of I, $\varepsilon(I^c, J)^c = I \setminus J$ then the permutation rearranges the indices in I so that the first $|J|$ positions coincide with J, otherwise the right interior product is zero.

In general, we have that $\mathbf{e}_I \lrcorner \mathbf{e}_J = \mathbf{e}_J \llcorner \mathbf{e}_I(-1)^{|I|(|J|-|I|)}$, as verified in Appendix A.1. We note that these interior products are not commutative, unless either $|J| - |I|$ or $|I|$ is an even number, e. g. when $|I| = |J|$, in which case both interior products coincide with the dot product of the two vectors. The interior products may therefore be seen as generalizations of the dot product.

As with the dot and the exterior products, the value of the interior products does not depend on the choice of basis and we may thus compute the **left interior product** of two vectors $\mathbf{v}$ and $\mathbf{w}$ as

$$\mathbf{v} \lrcorner \mathbf{w} = \sum_{I,J} v_I w_J \, \mathbf{e}_I \lrcorner \mathbf{e}_J \,, \tag{13}$$

and a similar expression holds for the **right interior product** $\mathbf{v} \llcorner \mathbf{w}$. Both are grade-lowering operations, as the left (resp. right) interior product is either zero or a multivector of grade $m' - m$ (resp. $m - m'$).

The interior products are not independent operations from the exterior product, as they can be expressed in terms of the latter, the Hodge complement and its inverse (proved in Appendix A.2):

$$\mathbf{e}_I \lrcorner \mathbf{e}_J = \left(\mathbf{e}_I \wedge \mathbf{e}_J^{\mathcal{H}}\right)^{\mathcal{H}^{-1}}, \tag{14}$$
$$\mathbf{e}_I \llcorner \mathbf{e}_J = \left(\mathbf{e}_I^{\mathcal{H}^{-1}} \wedge \mathbf{e}_J\right)^{\mathcal{H}}. \tag{15}$$

If $\mathbf{u}$ and $\mathbf{v}$ are 1-vectors and $\mathbf{w}$ is an r-vector, then we have the following expression

$$\mathbf{u} \lrcorner (\mathbf{v} \wedge \mathbf{w}) = (-1)^r (\mathbf{u} \cdot \mathbf{v})\mathbf{w} + \mathbf{v} \wedge (\mathbf{u} \lrcorner \mathbf{w}), \tag{16}$$

as proved in Appendix A.3. This expression can be seen as a generalization of the vectorial expression

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \tag{17}$$

in the vector space $\mathbf{R}^3$, i.e., a $k = 0$, $n = 3$ space–time. This fact is built of the realization that the cross product between two vectors $\mathbf{v}$ and $\mathbf{w}$ can be expressed in the following alternative ways

$$\mathbf{v} \times \mathbf{w} = (\mathbf{v} \wedge \mathbf{w})^{\mathcal{H}^{-1}} = \mathbf{v} \lrcorner \mathbf{w}^{\mathcal{H}^{-1}} = \mathbf{v} \lrcorner \mathbf{w}^{\mathcal{H}}. \tag{18}$$

Whenever it holds that $I \subseteq J$, the interior and exterior products are related by the following:

$$(\mathbf{e}_I \lrcorner \mathbf{e}_J) \wedge \mathbf{e}_I = \Delta_{I,I} \mathbf{e}_J, \tag{19}$$
$$\mathbf{e}_I \wedge (\mathbf{e}_J \llcorner \mathbf{e}_I) = \Delta_{I,I} \mathbf{e}_J. \tag{20}$$

Having introduced the basic notions of space–time exterior algebra, the next section focuses on operations with elements in the exterior algebra, namely integrals and derivatives of vector fields.

3 Integrals and Derivatives of Vector Fields: Circulation and Flux

3.1 Oriented Integrals

Integrals are, together with derivatives, the fundamental mathematical objects of **calculus**. For example, operations on vectors fields lying in exterior algebra such as the flux and the circulation are expressed in terms of integrals over high-dimensional geometric objects. The integral of an m-graded vector field $\mathbf{v}$ over a hypersurface $\mathcal{V}^m$ of the same dimension, denoted as

$$\int_{\mathcal{V}^m} \mathrm{d}^m\mathbf{x} \cdot \mathbf{v}, \tag{21}$$

is the limit of the Riemann sums for the dot product $\mathrm{d}^m\mathbf{x} \cdot \mathbf{v}$ over points in the hypersurface, where $\mathrm{d}^m\mathbf{x}$ is an m-dimensional infinitesimal vector element. For any $\ell = 0, \dots, k+n$, the infinitesimal vector element $\mathrm{d}^\ell\mathbf{x}$ is given by the sum of all possible differentials for ℓ-dimensional hypersurfaces in a (k, n) space–time, and is represented in the canonical basis as

$$\mathrm{d}^\ell\mathbf{x} = \sum_{I=(i_1,\dots,i_\ell)} \mathrm{d}x_I \mathbf{e}_I, \tag{22}$$

where for a given list $I = (i_1, \dots, i_\ell)$ each differential is given by $dx_I = \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_\ell}$.

As in traditional calculus, the integral in Equation (21) exhibits coordinate invariance, while the integrand $\mathrm{d}^m\mathbf{x}\cdot\mathbf{v}$ is regarded as an oriented object. Orientation is well defined for integrals along a curve from one point to another, or integrals over a surface oriented at the direction of the normal to the surface. Switching the extreme points of the curve, or taking the opposite direction of the normal would induce a change of sign in the line and surface integrals. In our generalization of vector calculus, a positive orientation is implicit in the ordering of the canonical basis. The skew-symmetry property of the exterior product Equation (6) may introduce sign changes to compensate an eventual change of orientation after changes of coordinates such as permutations of the space–time components.

For a given hypersurface $\mathcal{V}^m$, a convenient transformation for solving the integral in Equation (21) is one such that, at a given point $\mathbf{x}$ in the hypersurface, the infinitesimal vector element $\mathrm{d}^m\mathbf{x}$ has one component that is *tangent* to the hypersurface at that point. Let $\mathbf{e}_{\|}$ be a unit m-graded vector parallel to $\mathcal{V}^m$ at point $\mathbf{x}$, and let $\mathbf{e}'_0,\dots,\mathbf{e}'_{k+n-1}$ form an orthonormal basis of $\mathbf{R}^{k+n}$ such that $\mathbf{e}_{\|}=\mathbf{e}'_{k+n-m}\wedge\cdots\wedge\mathbf{e}'_{k+n-1}$ for the given point $\mathbf{x}$ in $\mathcal{V}^m$. This change of coordinates from the canonical basis to the new basis is described by a unitary matrix U, dependent on $\mathbf{x}$, and that satisfies

$$\mathbf{e}_0\wedge\cdots\wedge\mathbf{e}_{k+n-1}=\det(U)\,\mathbf{e}'_0\wedge\cdots\wedge\mathbf{e}'_{k+n-1}. \tag{23}$$

Being a unitary matrix, the determinant of U is ±1. Assuming an orientation-preserving change of coordinates, that is $\det(U)=1$, the infinitesimal vector element in Equation (22) for $\ell=m$ can be expressed as

$$\mathrm{d}^m\mathbf{x}=\mathrm{d}x_{\|}\mathbf{e}_{\|}+\sum_{I=(i_1,\dots,i_m)\,:\,I\cap\perp\neq\emptyset}\mathrm{d}x_I\mathbf{e}'_I, \tag{24}$$

where $\perp=\{0,\dots,k+n-m-1\}$ is the set of indices for the unit vectors in the new basis orthogonal to $\mathcal{V}^m$. Since all elements in the summation in Equation (24) have at least one differential element lying outside the integration hypersurface, their integrals vanish and therefore

$$\int_{\mathcal{V}^m}\mathrm{d}^m\mathbf{x}=\int_{\mathcal{V}^m}dx_{\|}\mathbf{e}_{\|}. \tag{25}$$

In analogy to $\mathbf{e}_{\|}$, a multivector of grade m, we define a unit $(k+n-m)$-grade vector $\mathbf{e}_{\perp}$ normal to $\mathcal{V}^m$ at point $\mathbf{x}$ such that $\mathbf{e}_{\perp}\wedge\mathbf{e}_{\|}=\mathbf{e}_0\wedge\cdots\wedge\mathbf{e}_{k+n-1}$. From Equation (10), we see that one such normal multivector with the correct orientation is

$$\mathbf{e}_{\perp}=\frac{\mathbf{e}_{\|}^{\mathcal{H}^{-1}}}{\mathbf{e}_{\|}^{\mathcal{H}^{-1}}\cdot\mathbf{e}_{\|}^{\mathcal{H}^{-1}}}. \tag{26}$$

For the common spaces considered in vector calculus, $\mathbf{R}^2$ and $\mathbf{R}^3$, and according to Equation (23), orientation-preserving changes of coordinates must respectively satisfy $\mathbf{e}_{\perp}\wedge\mathbf{e}_{\|}=\mathbf{e}_0\wedge\mathbf{e}_1$ and $\mathbf{e}_{\perp}\wedge\mathbf{e}_{\|}=\mathbf{e}_0\wedge\mathbf{e}_1\wedge\mathbf{e}_2$, where $\mathbf{e}_{\perp}$ is the basis element normal to $\mathcal{V}^m$. These two equalities turn out to describe the counterclockwise (resp. right-hand rule) orientation when $\mathbf{e}_{\perp}$ conventionally points outside an integration path for $\mathbf{R}^2$ (resp. a surface for $\mathbf{R}^3$) [5] (pp. 184–185).

Building on the concepts and operations of circulation and flux in vector calculus, the right and left interior products lead to general definitions of circulation and flux of multivector fields in exterior algebra along and across hypersurfaces of arbitrary number of dimensions.

3.2 Circulation and Flux of Multivector Fields

Definition 1. *The circulation of a vector field* $\mathbf{v}(\mathbf{x})$ *of grade* m *along an* ℓ*-dimensional hypersurface* $\mathcal{V}^\ell$*, denoted by* $\mathcal{C}(\mathbf{v},\mathcal{V}^\ell)$*, is given by*

$$\mathcal{C}(\mathbf{v},\mathcal{V}^\ell)=\int_{\mathcal{V}^\ell}\mathrm{d}^\ell\mathbf{x}\llcorner\mathbf{v}. \tag{27}$$

¶

Expressing the vector field in the canonical basis and using the definition of $\mathrm{d}^\ell\mathbf{x}$ in Equation (22), the circulation can be specified in some cases of interest. For $\ell=m$, the circulation reads

$$\int_{\mathcal{V}^m}\mathrm{d}^m\mathbf{x}\cdot\mathbf{v}=\sum_{I=(i_1,\dots,i_m)}\Delta_{I,I}\int_{\mathcal{V}^m}dx_Iv_I. \tag{28}$$

For instance, for $\ell = m = 1$ and $\mathbf{R}^n$, this formula recovers the definition the circulation of a vector field along a closed path with the appropriate orientation.

Alternatively, using Equation (25), we note that $\mathbf{v}$ is integrated along the direction of $\mathbf{e}_{||}$, tangential to the hypersurface, in an orientation-preserving change of coordinates, that is

$$\int_{\mathcal{V}^m} \mathrm{d}^m\mathbf{x} \llcorner \mathbf{v} = \int_{\mathcal{V}^m} \mathrm{d}x_{||}\, \mathbf{e}_{||} \llcorner \mathbf{v}. \tag{29}$$

Intuitively, the circulation Equation (27) measures the alignment of an m-vector field $\mathbf{v}$ with respect to $\mathcal{V}^\ell$ for any ℓ and m, with the circulation being an $(\ell - m)$-vector if $\ell \geq m$ and zero otherwise.

Definition 2. *The flux of a vector field* $\mathbf{v}(\mathbf{x})$ *of grade* m *across an* ℓ*-dimensional hypersurface* $\mathcal{V}^\ell$*, denoted by* $\mathcal{F}(\mathbf{v}, \mathcal{V}^\ell)$*, is given by*

$$\mathcal{F}(\mathbf{v}, \mathcal{V}^\ell) = \int_{\mathcal{V}^\ell} \mathrm{d}^\ell\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner \mathbf{v}. \tag{30}$$

¶

Expressing both $\mathbf{v}$ and $\mathrm{d}^\ell\mathbf{x}$ in the canonical basis, and using the inverse Hodge operation in Equation (10), the flux in the special case of $\ell = k + n - m$ can be written as

$$\int_{\mathcal{V}^\ell} \mathrm{d}^\ell\mathbf{x}^{\mathcal{H}^{-1}} \cdot \mathbf{v} = \sum_{I=(i_1,\dots,i_m)} \sigma(I, I^c) \int_{\mathcal{V}^\ell} dx_{I^c} v_I. \tag{31}$$

As an example in $\mathbf{R}^3$, the flux of a vector field $\mathbf{v}$ through a surface $\mathcal{V}^2$ reads

$$\int_{\mathcal{V}^2} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}} \cdot \mathbf{v} = \int_{\mathcal{V}^2} \sum_{I, i\notin I} \mathrm{d}x_I \sigma(i, I)\mathbf{e}_i \cdot \mathbf{v}. \tag{32}$$

The right-hand side of Equation (32) is a conventional surface integral, upon the identification of $\sum_{I,i\notin I} \mathrm{d}x_I\sigma(i, I)\mathbf{e}_i$ as an infinitesimal surface element $\mathrm{d}\mathbf{S}$.

Alternatively, using the analogous of Equation (25) for the differential vector element $\mathrm{d}^\ell\mathbf{x}^{\mathcal{H}^{-1}}$, the equivalent to Equation (29) for the flux is

$$\int_{\mathcal{V}^\ell} \mathrm{d}^\ell\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner \mathbf{v} = \int_{\mathcal{V}^\ell} \mathrm{d}x_{||}\, \mathbf{e}_{||}^{\mathcal{H}^{-1}} \lrcorner \mathbf{v}. \tag{33}$$

This equation implies that $\mathbf{v}$ is integrated along a normal component to the hypersurface since $\mathbf{e}_{||}^{\mathcal{H}^{-1}}$ is a multivector of grade $k + n - \ell$ orthogonal to $\mathcal{V}^\ell$. Intuitively, the flux Equation (30) measures the magnitude of the multivector field crossing the hypersurface. In general, the flux is a vector of grade $(m + \ell - n - k)$ if $\ell \geq k + n - m$ and zero otherwise. For instance, if $\ell = k + n$, the flux of $\mathbf{v}$ over an $(k + n)$-dimensional hypersurface $\mathcal{V}^{k+n}$ gives the integral of $\mathbf{v}$ over $\mathcal{V}^{k+n}$, an extension of the volume integral to $\mathbf{R}^{k+n}$,

$$\int_{\mathcal{V}^{k+n}} \mathrm{d}^{k+n}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner \mathbf{v} = \int_{\mathcal{V}^{k+n}} \mathrm{d}x_{i_1,\cdots,i_{k+n}}\mathbf{v}, \tag{34}$$

where we used the relation $1^{\mathcal{H}} = \mathbf{e}_{i_1,\dots,i_{k+n}}$, implying that $\mathrm{d}^{k+n}\mathbf{x}^{\mathcal{H}^{-1}} = \mathrm{d}x_{i_1,\cdots,i_{k+n}}$, and that $1 \lrcorner \mathbf{v} = \mathbf{v}$.

3.3 Exterior and Interior Derivatives

In vector calculus, extensive use is made of the nabla operator ∇, a vector operator that takes partial space derivatives. For instance, operations such as gradient, divergence or curl are expressed in terms of this operator. In our case, we need the generalization to (k, n) space–time to the differential vector operator $\boldsymbol{\partial}$, defined as $(-\partial_0, -\partial_2, \dots, -\partial_{k-1}, \partial_k, \dots, \partial_{k+n-1})$, that is

$$\boldsymbol{\partial} = \sum_{i=0}^{k+n-1} \Delta_{ii}\mathbf{e}_i\partial_i. \tag{35}$$

For a given vector field $\mathbf{v}$ of grade m, we define the **exterior derivative** of $\mathbf{v}$ as $\boldsymbol{\partial} \wedge \mathbf{v}$, namely

$$\boldsymbol{\partial} \wedge \mathbf{v} = \sum_{i=0}^{k+n-1} \sum_{I} \Delta_{ii} \partial_i v_I \sigma(i, I)\, \mathbf{e}_{\varepsilon(i,I)}. \tag{36}$$

The grade of the exterior derivative of $\mathbf{v}$ is $m+1$, unless $m = k+n$, in which case the exterior derivative is zero, as can be deduced from the fact that all signatures are zero.

In addition, we define the **interior derivative** of $\mathbf{v}$ as $\boldsymbol{\partial} \lrcorner\, \mathbf{v}$, namely

$$\boldsymbol{\partial} \lrcorner\, \mathbf{v} = \sum_{i,I:\, i \in I} \partial_i v_I \sigma(I \setminus i, i) \mathbf{e}_{I \setminus i}. \tag{37}$$

The grade of the interior derivative of $\mathbf{v}$ is $m-1$, unless $m = 0$, in which case the interior derivative is zero, as implied by the fact that the grade of $\boldsymbol{\partial}$ is larger than the grade of $\mathbf{v}$. Using Equation (16) with $\mathbf{u} = \boldsymbol{\partial}$ and assuming that $\mathbf{v}$ and $\mathbf{w}$ are 1-vectors, we obtain a generalization of Leibniz's product rule

$$\boldsymbol{\partial} \lrcorner\, (\mathbf{v} \wedge \mathbf{w}) = \mathbf{v}(\boldsymbol{\partial} \cdot \mathbf{w}) - (\boldsymbol{\partial} \cdot \mathbf{v})\mathbf{w}. \tag{38}$$

The formulas for the exterior and interior derivatives allow us express some common expressions in vector calculus. For a scalar function ϕ, its gradient is given by its exterior derivative $\nabla\phi = \boldsymbol{\partial} \wedge \phi$, while for a vector field $\mathbf{v}$, its divergence $\nabla \cdot \mathbf{v}$ is given by its interior derivative $\nabla \cdot \mathbf{v} = \boldsymbol{\partial} \lrcorner\, \mathbf{v}$. From Equation (16) we further observe that for a scalar function ϕ we recover the relation

$$\nabla \cdot (\nabla\phi) = (\nabla \cdot \nabla)\phi. \tag{39}$$

In addition, for a vector fields $\mathbf{v}$ in $\mathbf{R}^3$, taking into account Equation (18) then the curl can be variously expressed as

$$\nabla \times \mathbf{v} = (\nabla \wedge \mathbf{v})^{\mathcal{H}^{-1}} = \nabla \lrcorner\, \mathbf{v}^{\mathcal{H}^{-1}} = \nabla \lrcorner\, \mathbf{v}^{\mathcal{H}}. \tag{40}$$

This formula allows us to write the curl of a vector field $\nabla \times \mathbf{v}$ in terms of the exterior and interior products and the Hodge complement, while generalizing both the cross product and the curl to grade-m vector fields in space–time algebras with different dimensions. Moreover, from Equation (16) we can recover for $r = 1$ the well-known formula for the curl of the curl of a vector,

$$\nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2\mathbf{v}. \tag{41}$$

It is easy to verify that the exterior derivative of an exterior derivative is zero, as is the interior derivative of an interior derivative, that is for any vector field $\mathbf{v}$, we have that

$$\boldsymbol{\partial} \wedge (\boldsymbol{\partial} \wedge \mathbf{v}) = 0 \tag{42}$$
$$\boldsymbol{\partial} \lrcorner\, (\boldsymbol{\partial} \lrcorner\, \mathbf{v}) = 0. \tag{43}$$

In regard to the vector space $\mathbf{R}^3$, and using Equation (18), these expressions imply the well-known facts that the curl of the gradient and the divergence of the curl are zero:

$$\nabla \times (\nabla\phi) = (\nabla \wedge (\nabla \wedge \phi))^{\mathcal{H}^{-1}} = 0 \tag{44}$$
$$\nabla \cdot (\nabla \times \mathbf{v}) = \nabla \lrcorner\, (\nabla \lrcorner\, \mathbf{v}^{\mathcal{H}}) = 0. \tag{45}$$

3.4 Stokes Theorem for the Circulation

In vector calculus in $\mathbf{R}^3$, the Kelvin-Stokes theorem for the circulation of a vector field $\mathbf{v}$ of grade 1 along the boundary $\partial\mathcal{V}^2$ of a bidimensional surface $\mathcal{V}^2$ relates its value to that of the surface integral of the curl of the vector field over the surface itself. In the notation used in the previous section, the surface integral is the flux of the curl of the vector field across the surface and this theorem reads

$$\int_{\partial\mathcal{V}^2} \mathrm{d}\mathbf{x} \cdot \mathbf{v} = \int_{\mathcal{V}^2} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}} \cdot (\nabla \times \mathbf{v}). \tag{46}$$

Taking into account the identity $\nabla \times \mathbf{v} = (\nabla \wedge \mathbf{v})^{\mathcal{H}^{-1}}$ in Equation (40), we rewrite the right-hand side in Equation (46) as

$$\begin{aligned}\int_{\mathcal{V}^2} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}} \cdot (\nabla \times \mathbf{v}) &= \int_{\mathcal{V}^2} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}} \cdot (\nabla \wedge \mathbf{v})^{\mathcal{H}^{-1}} \\ &= \int_{\mathcal{V}^2} \mathrm{d}^2\mathbf{x} \cdot (\nabla \wedge \mathbf{v}),\end{aligned} \tag{47}$$

where we used that $\mathbf{u} \cdot \mathbf{w} = \mathbf{u}^{\mathcal{H}^{-1}} \cdot \mathbf{w}^{\mathcal{H}^{-1}} = \mathbf{u}^{\mathcal{H}} \cdot \mathbf{w}^{\mathcal{H}}$ for vectors $\mathbf{u}$, $\mathbf{w}$. The *flux* of the curl of the vector field *across* a surface is also the *circulation* of the exterior derivative of the vector field *along* that surface.

The generalized Stokes theorem for differential forms [4] (p. 80) allows us to extend the Kelvin-Stokes theorem to multivectors of any grade m as we do in the following theorem.

Theorem 1. *The circulation of a grade-m vector field $\mathbf{v}$ along the boundary $\partial\mathcal{V}^\ell$ of an ℓ-dimensional hypersurface $\mathcal{V}^\ell$ is equal to the circulation of the exterior derivative of $\mathbf{v}$ along $\mathcal{V}^\ell$:*

$$\mathcal{C}(\mathbf{v}, \partial\mathcal{V}^\ell) = \mathcal{C}(\boldsymbol{\partial} \wedge \mathbf{v}, \mathcal{V}^\ell). \tag{48}$$

¶

As hinted at above, the role of the vector curl in the right-hand side of Equation (46) is played by the exterior derivative in this generalized theorem.

Proof. We start by stating the generalized Stokes Theorem for differential forms [4] (pp. 80)

$$\int_{\partial\mathcal{V}^\ell} \omega = \int_{\mathcal{V}^\ell} \mathrm{d}\omega, \tag{49}$$

where ω is a differential form and $\mathrm{d}\omega$ its exterior derivative, represented by the operator

$$\mathrm{d} = \sum_j \mathrm{d}x_j \partial_j. \tag{50}$$

Expressing the circulations in Equation (48) by means of the integrals in Equation (27), we obtain

$$\int_{\partial\mathcal{V}^\ell} \mathrm{d}^{\ell-1}\mathbf{x} \llcorner \mathbf{v} = \int_{\mathcal{V}^\ell} \mathrm{d}^\ell\mathbf{x} \llcorner (\boldsymbol{\partial} \wedge \mathbf{v}). \tag{51}$$

In the integral in the left-hand side of Equation (51), the integrand is a differential form $\omega = \mathrm{d}^{\ell-1}\mathbf{x} \llcorner \mathbf{v}$. After expanding the interior product using the definitions of $\mathrm{d}^{\ell-1}\mathbf{x}$ and $\mathbf{v}$ we obtain

$$\omega = \left(\sum_{J_{\ell-1}} \mathrm{d}x_J\, \mathbf{e}_J\right) \llcorner \left(\sum_{I_m} v_I\, \mathbf{e}_I\right) = \sum_{J_{\ell-1}, I_m \,:\, I \subseteq J} \Delta_{I,I} \sigma(I, \varepsilon(I, J^c)^c) v_I\, \mathrm{d}x_J\, \mathbf{e}_{\varepsilon(I,J^c)^c}. \tag{52}$$

Then, computing the exterior derivative of this form with Equation (50) gives

$$\mathrm{d}\omega = \sum_{J_{\ell-1}, I_m \,:\, I \subseteq J} \sum_{j \notin J} \Delta_{I,I} \sigma(I, \varepsilon(I, J^c)^c) \partial_j v_I \sigma(j, J)\, \mathrm{d}x_{\varepsilon(j,J)}\, \mathbf{e}_{\varepsilon(I,J^c)^c}. \tag{53}$$

We next write down the integrand in the right-hand side of Equation (51), $\mathrm{d}^\ell\mathbf{x} \llcorner (\boldsymbol{\partial} \wedge \mathbf{v})$, that is

$$\begin{aligned}\mathrm{d}^\ell\mathbf{x} \llcorner (\boldsymbol{\partial} \wedge \mathbf{v}) &= \left(\sum_{K_{\ell+1}} \mathrm{d}x_K\, \mathbf{e}_K\right) \llcorner \left(\sum_{I_m} \sum_{j \notin I} \Delta_{j,j} \partial_j v_I\, \sigma(j, I)\, \mathbf{e}_{\varepsilon(j,I)}\right) \\ &= \sum_{K_\ell, I_m \,:\, \varepsilon(j,I) \subseteq K_\ell} \sum_{j \notin I} \Delta_{j,j} \partial_j v_I\, \mathrm{d}x_K\, \sigma(j, I) \Delta_{\varepsilon(j,I),\varepsilon(j,I)}\, \sigma(\varepsilon(j, I), \varepsilon(K^c, \varepsilon(j, I))^c)\, \mathbf{e}_{\varepsilon(K^c,\varepsilon(j,I))^c} \\ &= \sum_{K_\ell, I_m \,:\, \varepsilon(j,I) \subseteq K_{\ell+1}} \sum_{j \notin I} \Delta_{I,I} \partial_j v_I\, \mathrm{d}x_K\, \sigma(j, I) \sigma(\varepsilon(j, I), \varepsilon(K^c, \varepsilon(j, I))^c)\, \mathbf{e}_{\varepsilon(K^c,\varepsilon(j,I))^c},\end{aligned} \tag{54}$$

and verify that it coincides with exterior derivative in Equation (53). As the set of m indices I_m is included in the sets $J_{\ell-1}$ or K_ℓ in Equation (53) or (54), we may write $K_\ell = \varepsilon(J_{\ell-1}, j)$ for some $j \notin J_{\ell-1}$. Then, we obtain the following chain of equalities for the basis elements in Equations (53) and (54):

$$\mathbf{e}_{\varepsilon(K^c,\varepsilon(j,I))^c} = \mathbf{e}_{\varepsilon(K^c\cup\{j\}\cup I)^c} = \mathbf{e}_{\varepsilon(J^c\setminus\{j\}\cup\{j\}\cup I)^c} = \mathbf{e}_{\varepsilon(J^c\cup I)^c} = \mathbf{e}_{\varepsilon(I,J^c)^c}. \tag{55}$$

Therefore, and using that $\varepsilon(J^c \setminus \{j\}, \varepsilon(j, I))^c = J \setminus I$, we can write Equation (54) as

$$\mathrm{d}^\ell\mathbf{x} \llcorner (\boldsymbol{\partial} \wedge \mathbf{v}) = \sum_{J_{\ell-1}, I_m, j\notin I\,:\,\varepsilon(j,I)\subseteq J\cup\{j\}} \Delta_{I,I}\partial_j v_I \,\mathrm{d}x_{\varepsilon(J,j)}\,\sigma(j,I)\sigma(\varepsilon(j,I), J\setminus I)\,\mathbf{e}_{\varepsilon(I,J^c)^c}. \tag{56}$$

Comparing Equation (56) with Equation (53), the expressions coincide if this identity holds:

$$\sigma(j, J)\sigma(I, J \setminus I) = \sigma(j, I)\sigma(j + I, J \setminus I). \tag{57}$$

To prove Equation (57) we exploit that the σ are permutation signatures and that the signature of the composition of permutations is the product of the respective signatures. We proceed with the help of a visual aid in Figure 1, which depicts the identity between two different ways of sorting the concatenated list $(j, I, J \setminus I)$. On the left column we first sort the list $(I, J \setminus I)$ to obtain J and then sort the list (j, J). On the right column, we first sort the list (j, I) and then the list $(j+I, J\setminus I)$. This proves Equation (57) and the theorem.

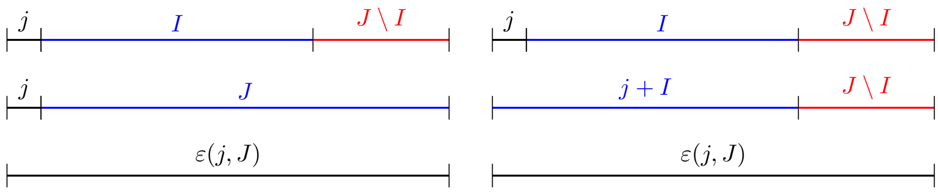

Figure 1: Visual aid for the identity among permutations in Equation (57).

Finally, we note that, had we defined the circulation with the left interior product, we would have got an incompatible relation in Equation (57), which could not be solved. □

3.5 Stokes Theorem for the Flux

In vector calculus in $\mathbf{R}^3$, the Gauss theorem relates the volume integral of the divergence of a vector field $\mathbf{v}$ over a region $\mathcal{V}^3$ to the surface integral of the vector field over the region boundary $\partial\mathcal{V}^3$. In the notation used in previous sections, and taking into account that both the surface integral and the volume integral can be expressed as fluxes for $\mathbf{R}^3$, this theorem reads

$$\int_{\partial\mathcal{V}^3} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}} \cdot \mathbf{v} = \int_{\mathcal{V}^3} \mathrm{d}^3\mathbf{x}^{\mathcal{H}^{-1}}(\nabla\cdot\mathbf{v}). \tag{58}$$

Making use of the identity $\nabla \cdot \mathbf{v} = \nabla \lrcorner \mathbf{v}$, we can rewrite the right-hand side in Equation (58) as

$$\int_{\mathcal{V}^3} \mathrm{d}^3\mathbf{x}^{\mathcal{H}^{-1}}(\nabla\cdot\mathbf{v}) = \int_{\mathcal{V}^3} \mathrm{d}^3\mathbf{x}^{\mathcal{H}^{-1}}(\nabla\lrcorner\mathbf{v}). \tag{59}$$

In other words, the Gauss theorem relates the flux of the interior derivative of a vector field $\mathbf{v}$ across a region $\mathcal{V}^3$ to the flux of the vector field itself across the region boundary $\partial\mathcal{V}^3$.

The generalized Stokes theorem for differential forms allows us to extend the Gauss theorem to multivectors of any grade m as we do in the following theorem.

Theorem 2. *The flux of a grade-m vector field $\mathbf{v}$ across the boundary $\partial\mathcal{V}^\ell$ of an ℓ-dimensional hypersurface $\mathcal{V}^\ell$ is equal to the flux of the interior derivative of $\mathbf{v}$ across $\mathcal{V}^\ell$:*

$$\mathcal{F}(\mathbf{v}, \partial\mathcal{V}^\ell) = \mathcal{F}(\boldsymbol{\partial} \lrcorner \mathbf{v}, \mathcal{V}^\ell). \tag{60}$$

¶

Proof. Expressing the fluxes in Equation (60) by means of the integrals in Equation (30), we obtain

$$\int_{\partial\mathcal{V}^\ell} \mathrm{d}^{\ell-1}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{v} = \int_{\mathcal{V}^\ell} \mathrm{d}^{\ell}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, (\boldsymbol{\partial} \lrcorner\, \mathbf{v}). \tag{61}$$

As in the proof of Theorem 1, we apply the Stokes theorem for differential forms in Equation (49) upon the identifications ω with $\mathrm{d}^{\ell-1}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{v}$ and $\mathrm{d}\omega$ with $\mathrm{d}^{\ell}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, (\boldsymbol{\partial} \lrcorner\, \mathbf{v})$. First, for ω, we get

$$\begin{aligned}\left(\sum_{J_{\ell-1}} \mathrm{d}x_J \Delta_{J^c,J^c}\sigma(J^c,J)\mathbf{e}_{J^c}\right) \lrcorner \left(\sum_{I_m} v_I\,\mathbf{e}_I\right) &= \sum_{J_{\ell-1},I_m\,:\,J^c\subseteq I} v_I\,\mathrm{d}x_J\sigma(J^c,J)\sigma(\varepsilon(J^c,I^c)^c,J^c)\,\mathbf{e}_{\varepsilon}(J^c,I^c)^c \\ &= \sum_{J_{\ell-1},I_m\,:\,J^c\subseteq I} v_I\,\mathrm{d}x_J\sigma(J^c,J)\sigma(I\setminus J^c,J^c)\,\mathbf{e}_{I\setminus J^c}. \end{aligned}\tag{62}$$

Now, taking the exterior derivative of Equation (62), we obtain

$$\mathrm{d}\omega = \sum_{J_{\ell-1},I_m\,:\,J^c\subseteq I}\ \sum_{j\notin J} \partial_j v_I\,\mathrm{d}x_{\varepsilon(j,J)}\sigma(j,J)\sigma(J^c,J)\sigma(I\setminus J^c,J^c)\,\mathbf{e}_{I\setminus J^c}. \tag{63}$$

This quantity should be equal to $\mathrm{d}^{\ell}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, (\boldsymbol{\partial} \lrcorner\, \mathbf{v})$ in the right-hand side of Equation (61), which we expand as

$$\begin{aligned}\mathrm{d}^{\ell}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, (\boldsymbol{\partial} \lrcorner\, \mathbf{v}) &= \left(\sum_{K_\ell} \mathrm{d}x_K\Delta_{K^c,K^c}\sigma(K^c,K)\mathbf{e}_{K^c}\right) \lrcorner \left(\sum_{I:j\in I}\partial_j v_I\sigma(I\setminus j,j)\mathbf{e}_{I\setminus j}\right) \\ &= \sum_{K_\ell,I_m\,:\,K^c\subseteq I\setminus j}\ \sum_{j\in I}\partial_j v_I\,\mathrm{d}x_K\sigma(K^c,K)\sigma(I\setminus j,j)\sigma(I\setminus j\setminus K^c,K^c)\,\mathbf{e}_{I\setminus j\setminus K^c}.\end{aligned}\tag{64}$$

We first consider the sets in the summations in the alternative expressions for $\mathrm{d}\omega$, Equations (63) and (64). Since J^c contains j and is a subset of I, but K^c does not contain j and is also a subset of I (with $j\in I$), then we can assert that $K = J\cup\{j\}$ so that the conditions in the summations are equivalent. The basis elements coincide and so do the differentials and derivatives, and it remains to verify the identity

$$\sigma(j,J)\sigma(J^c,J)\sigma(I\setminus J^c,J^c) = \sigma(K^c,K)\sigma(I\setminus j,j)\sigma(I\setminus j\setminus K^c,K^c). \tag{65}$$

With the definition $L = I\setminus J^c$, and expressed in terms of j, J, and L, this condition gives

$$\sigma(j,J)\sigma(J^c,J)\sigma(L,J^c) = \sigma(J^c\setminus j,J+j)\sigma(J^c\setminus j+L,j)\sigma(L,J^c\setminus j). \tag{66}$$

Multiplying both sides of the equation by $\sigma(J^c,J)$, $\sigma(J^c\setminus j,J+j)$ and $\sigma(J^c\setminus j,j)$, and taking into account that the square of a signature is $+1$, we obtain

$$\sigma(J^c\setminus j,j)\sigma(L,J^c)\sigma(j,J)\sigma(J^c\setminus j,J+j) = \sigma(L,J^c\setminus j)\sigma(J^c\setminus j+L,j)\sigma(J^c\setminus j,j)\sigma(J^c,J). \tag{67}$$

We start by simplifying Equation (67) by noting that

$$\sigma(J^c\setminus j,j)\sigma(L,J^c) = \sigma(L,J^c\setminus j)\sigma(J^c\setminus j+L,j), \tag{68}$$

with help of the visual aid in Figure 2. The permutations on the left column first merge $(J^c\setminus j)$ with j and then the resulting J^c with L. Similarly, on the right column, we start with L, $(J^c\setminus j)$ and $\{j\}$, then concatenate $(L,J^c\setminus j)$ and then add j, getting the same result as the left column.

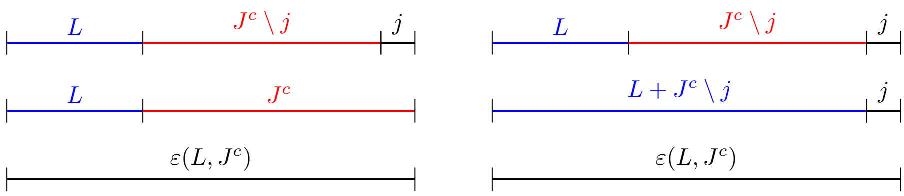

Figure 2: Visual aid for the identity $\sigma(J^c\setminus j,j)\sigma(L,J^c) = \sigma(L,J^c\setminus j)\sigma(J^c\setminus j+L,j)$.

Therefore, we have reduced Equation (67) to the simpler form

$$\sigma(j, J)\sigma(J^c \setminus j, J + j) = \sigma(J^c \setminus j, j)\sigma(J^c, J), \tag{69}$$

which we prove with the aid depicted in Figure 3. On the left column, j and J are first merged and then the concatenation $(J^c \setminus j, J + j)$ gives the sorted $\varepsilon(J^c, J)$. On the right column, after sorting $(J^c \setminus j)$ with j, merging it with J leads to the same final sequence. $\square$

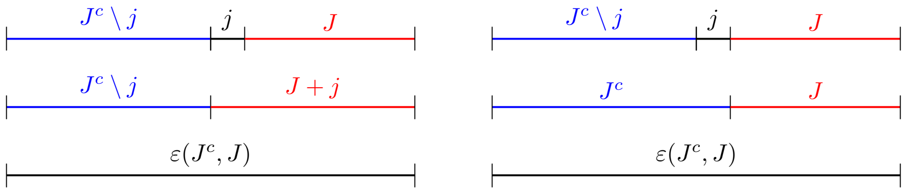

Figure 3: Visual aid for the identity $\sigma(j, J)\sigma(J^c \setminus j, J + j) = \sigma(J^c \setminus j, j)\sigma(J^c, J)$.

4 An Application to Electromagnetism in 1+3 Dimensions

In this section, we show how to recover the standard form of Maxwell equations and Lorentz force in $1 + 3$ dimensions from a formulation with *exterior calculus* involving an electromagnetic bivector field $\mathbf{F}$ and a 4-dimensional current density vector $\mathbf{J}$. In the appropriate units, the bivector field $\mathbf{F}$ can be decomposed as $\mathbf{F} = \mathbf{F_E} + \mathbf{F_B}$, where $\mathbf{F_E}$ contains the electric-field $\mathbf{E}$ time-space components and $\mathbf{F_B}$ contains the space-space components for the magnetic field $\mathbf{B}$. Similarly, the current density depends on the charge density ρ and the spatial current density $\mathbf{j}$. More specifically,

$$\mathbf{J} = \rho\mathbf{e}_0 + \mathbf{j} \tag{70}$$

$$\mathbf{F} = \mathbf{F_E} + \mathbf{F_B} = \mathbf{e}_0 \wedge \mathbf{E} + \mathbf{B}^{\varkappa}. \tag{71}$$

Here the Hodge complement acts only on the space components, and $\mathbf{B}^{\varkappa} = \mathbf{B}^{\varkappa^{-1}}$. The bivector field $\mathbf{F}$ is closely related to the Faraday tensor, a rank-2 antisymmetric tensor.

Maxwell equations, in their differential form, constrain the divergence of the electric and the magnetic field, Equations (72) and (73), respectively, and the curl of $\mathbf{E}$ and $\mathbf{B}$, namely Equations (74) and (75) [11] (p. 4-1).

$$\nabla \cdot \mathbf{E} = \rho \tag{72}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{73}$$

$$\nabla \times \mathbf{E} = -\partial_0\mathbf{B} \tag{74}$$

$$\nabla \times \mathbf{B} = \partial_0\mathbf{E} + \mathbf{j}. \tag{75}$$

We refer to Equations (73) and (74) as homogeneous Maxwell equations and to Equations (72) and (75) as inhomogeneous Maxwell equations, as they include the fields and the sources given by charge and current densities. In exterior-calculus notation, both pairs of equations can be combined into simple multivector equations,

$$\boldsymbol{\partial} \wedge \mathbf{F} = 0 \tag{76}$$

$$\boldsymbol{\partial} \lrcorner \mathbf{F} = \mathbf{J}, \tag{77}$$

where $\boldsymbol{\partial}$ is the differential operator $\boldsymbol{\partial} = -\partial_0\mathbf{e}_0 + \nabla$ for $k = 1$ and $n = 3$. As a consistency check, note that the wedge product raises the grade of $\mathbf{F}$, and the zero in Equation (76) is the zero trivector; also, as the left interior product lowers the grade of $\mathbf{F}$, both sides of Equation (77) relate space–time vectors.

Next to Maxwell equations, the Lorentz force density $\boldsymbol{f}$ characterizes, after integrating over the appropriate region, the force exerted by the electromagnetic field upon a system of charges described by the charge and current densities ρ and $\mathbf{j}$ [11] (pp. 13-1–13-3),

$$\boldsymbol{f} = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B}. \tag{78}$$

In relativistic form, the Lorentz force density becomes a four-dimensional vector $\mathbf{f}$ [2] (pp. 153–157). The time component of this vector is $\mathbf{j}\cdot\mathbf{E}$, the power dissipated per unit of volume, or after integrating over the appropriate region, the rate of work being done on the charges by the fields. In exterior-calculus notation, the Lorentz force density vector can be computed as a left interior product, namely

$$\mathbf{f} = \mathbf{J} \lrcorner \mathbf{F}. \tag{79}$$

4.1 Equivalence of the Lorentz Force Density

In this section, we prove that Equation (79) indeed recovers the relativistic Lorentz force density by verifying that its components in both vector-calculus and exterior-calculus coincide. From the definitions of $\mathbf{J}$ and $\mathbf{F}$, and using the distributive property of the interior product, we get

$$\begin{aligned}\mathbf{f} &= (\rho\mathbf{e}_0 + \mathbf{j}) \lrcorner (\mathbf{F_E} + \mathbf{F_B}) \\ &= \rho\mathbf{e}_0 \lrcorner \mathbf{F_E} + \rho\mathbf{e}_0 \lrcorner \mathbf{F_B} + \mathbf{j} \lrcorner \mathbf{F_E} + \mathbf{j} \lrcorner \mathbf{F_B} \\ &= \rho\mathbf{e}_0 \lrcorner (\mathbf{e}_0 \wedge \mathbf{E}) + \rho\mathbf{e}_0 \lrcorner \mathbf{B}^{\varkappa} + \mathbf{j} \lrcorner (\mathbf{e}_0 \wedge \mathbf{E}) + \mathbf{j} \lrcorner \mathbf{B}^{\varkappa^{-1}}.\end{aligned} \tag{80}$$

Some straightforward calculations give $\mathbf{e}_0 \lrcorner (\mathbf{e}_0 \wedge \mathbf{E}) = \mathbf{E}$, $\mathbf{e}_0 \lrcorner \mathbf{B}^{\varkappa} = 0$, and $\mathbf{j} \lrcorner (\mathbf{e}_0 \wedge \mathbf{E}) = \mathbf{e}_0\mathbf{j}\cdot\mathbf{E}$. In addition, the formula for the left interior product in Equation (18) gives $\mathbf{j} \lrcorner \mathbf{B}^{\varkappa^{-1}} = (\mathbf{j}\wedge\mathbf{B})^{\varkappa^{-1}} = \mathbf{j}\times\mathbf{B}$, where the cross product is only valid for three dimensions. With these calculations, we obtain

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{e}_0\mathbf{j}\cdot\mathbf{E} + \mathbf{j}\times\mathbf{B}, \tag{81}$$

namely, a time-component $\mathbf{j}\cdot\mathbf{E}$ and a spatial component equal to the Lorentz force density $\rho\mathbf{E}+\mathbf{j}\times\mathbf{B}$.

4.2 Equivalence of the Differential Form of Maxwell Equations

In this section, we prove that Equation (76) indeed recovers the homogeneous Maxwell equations and that Equation (77) recovers the inhomogeneous Maxwell equations.

First, we observe that the exterior derivative $\boldsymbol{\partial}\wedge\mathbf{F}$ gives a trivector with 4 components, while the homogeneous Maxwell equations are a scalar, Equation (73), and a vector, Equation (74). We shall verify that the scalar equation turns out to be given by the trivector component $\mathbf{e}_{123}$ of $\boldsymbol{\partial}\wedge\mathbf{F}$, while the vector equation is given by the trivector components $\mathbf{e}_{012}$, $\mathbf{e}_{013}$, and $\mathbf{e}_{023}$ of the exterior derivative.

We evaluate the exterior derivative $\boldsymbol{\partial}\wedge\mathbf{F}$ using the decomposition of $\mathbf{F}$ in Equation (71),

$$\begin{aligned}\boldsymbol{\partial}\wedge\mathbf{F} &= -\partial_0\mathbf{e}_0\wedge\mathbf{e}_0\wedge\mathbf{E} - \partial_0\mathbf{e}_0\wedge\mathbf{B}^{\varkappa} + \nabla\wedge\mathbf{e}_0\wedge\mathbf{E} + \nabla\wedge\mathbf{B}^{\varkappa} \\ &= -\partial_0\mathbf{e}_0\wedge\mathbf{B}^{\varkappa} - \mathbf{e}_0\wedge(\nabla\wedge\mathbf{E}) + \nabla\wedge\mathbf{B}^{\varkappa} \\ &= -\mathbf{e}_0\wedge(\partial_0\mathbf{B}^{\varkappa} + \nabla\wedge\mathbf{E}) + \nabla\wedge\mathbf{B}^{\varkappa},\end{aligned} \tag{82}$$

where we used that $\mathbf{e}_0\wedge\mathbf{e}_0 = 0$ and that $\nabla\wedge\mathbf{e}_0 = -\mathbf{e}_0\wedge\nabla$ in the second step of Equation (82). Taking advantage of Equation (40) we have the equality $\nabla\wedge\mathbf{E} = (\nabla\times\mathbf{E})^{\varkappa}$, while $\nabla\wedge\mathbf{B}^{\varkappa} = (\nabla\cdot\mathbf{B})^{\varkappa}$, and

$$\boldsymbol{\partial}\wedge\mathbf{F} = -\mathbf{e}_0\wedge(\partial_0\mathbf{B}^{\varkappa} + (\nabla\times\mathbf{E})^{\varkappa}) + (\nabla\cdot\mathbf{B})^{\varkappa}. \tag{83}$$

Indeed, the first summand vanishes when $\partial_0\mathbf{B}^{\varkappa} + (\nabla\times\mathbf{E})^{\varkappa} = 0$ or, taking the inverse Hodge complement, when Equation (74) holds. In terms of components, the spatial Hodge complement in this equation transforms a spatial vector into a bivector with components $\mathbf{e}_{12}$, $\mathbf{e}_{13}$, and $\mathbf{e}_{23}$ only and this equation recovers the homogeneous Maxwell equation in Equation (74). After taking the exterior product with $\mathbf{e}_0$, we obtain the trivector components $\mathbf{e}_{012}$, $\mathbf{e}_{013}$, and $\mathbf{e}_{023}$. Similarly, the second term vanishes for $(\nabla\cdot\mathbf{B})^{\varkappa} = 0$, recovering Equation (73). In terms of components, the spatial Hodge complement directly transforms a scalar into a trivector with a unique component $\mathbf{e}_{123}$, recovering the homogeneous Maxwell equation in Equation (73).

We move on to the inhomogeneous Maxwell equations. We compute the interior derivative $\boldsymbol{\partial}\lrcorner\mathbf{F}$,

$$\begin{aligned}\boldsymbol{\partial}\lrcorner\mathbf{F} &= (-\partial_0\mathbf{e}_0 + \nabla)\lrcorner(\mathbf{F_E} + \mathbf{F_B}) \\ &= -\partial_0\mathbf{E} - \partial_0\mathbf{e}_0\lrcorner\mathbf{B}^{\varkappa} + \mathbf{e}_0\nabla\cdot\mathbf{E} + \nabla\lrcorner\mathbf{B}^{\varkappa} \\ &= -\partial_0\mathbf{E} + \mathbf{e}_0\nabla\cdot\mathbf{E} + \nabla\lrcorner\mathbf{B}^{\varkappa},\end{aligned} \tag{84}$$

since $\mathbf{e}_0 \lrcorner \mathbf{B}^{\mathcal{H}} = 0$. The interior derivative $\boldsymbol{\partial} \lrcorner \mathbf{F}$ gives a space–time vector with 4 components, while the inhomogeneous Maxwell equations are a scalar, Equation (72), and a spatial vector, Equation (75).

We can verify that the scalar equation turns out to be given by the vector component $\mathbf{e}_0$ of $\boldsymbol{\partial} \lrcorner \mathbf{F}$, while the spatial vector equation is given by the spatial vector components $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ of $\boldsymbol{\partial} \lrcorner \mathbf{F}$. Indeed, if we match this expression with the current density vector $\mathbf{J}$, then the time component $\mathbf{e}_0$ of $\boldsymbol{\partial} \lrcorner \mathbf{F}$ gives Equation (72). Selecting the space components of $\boldsymbol{\partial} \lrcorner \mathbf{F}$, the differential equation is

$$-\partial_0 \mathbf{E} + \nabla \lrcorner \mathbf{B}^{\mathcal{H}} = \mathbf{j}, \tag{85}$$

which, using the relation $\nabla \lrcorner \mathbf{B}^{\mathcal{H}} = \nabla \times \mathbf{B}$ can be written as Equation (75).

4.3 Equivalence of the Integral Form of Maxwell Equations

After studying the exterior-calculus differential formulation of Maxwell equations, we recover the standard integral formulation. Applying the Stokes Theorem 1 to Equation (76), we find that *the circulation of the bivector field* $\mathbf{F}$ *along the boundary of any three-dimensional space–time volume* $\mathcal{V}^3$ *is zero*:

$$\int_{\partial \mathcal{V}^3} \mathrm{d}^2\mathbf{x} \cdot \mathbf{F} = \int_{\mathcal{V}^3} \mathrm{d}^3\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{F}) = 0. \tag{86}$$

At this point, Equation (86) is a scalar equation and we obtain the pair of homogeneous Maxwell equations by considering two different hypersurfaces $\mathcal{V}^3$.

First, let the domain $\mathcal{V}^3 = V$ contain only spatial coordinates. There are no tangential components to V with time indices and the contribution of $\mathbf{F_E}$ to the circulation of $\mathbf{F}$ over $\partial\mathcal{V}^3$ in Equation (86) is zero, i.e.,

$$\int_{\partial V} \mathrm{d}^2\mathbf{x} \cdot \mathbf{F} = \int_{\partial V} \mathrm{d}^2\mathbf{x} \cdot \mathbf{F_B}. \tag{87}$$

Using that $\mathbf{u} \cdot \mathbf{w} = \mathbf{u}^{\mathcal{H}^{-1}} \cdot \mathbf{w}^{\mathcal{H}^{-1}}$ for any vectors $\mathbf{u}$, $\mathbf{w}$, and therefore $\mathrm{d}^2\mathbf{x} \cdot \mathbf{F} = \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}} \cdot \mathbf{F}_{\mathbf{B}}^{\mathcal{H}^{-1}}$ and the definition $\mathbf{F_B} = \mathbf{B}^{\mathcal{H}}$, the integral in the right-hand side of Equation (87) becomes

$$\begin{aligned}\int_{\partial V} \mathrm{d}^2\mathbf{x} \cdot \mathbf{F_B} &= \int_{\partial V} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}} \cdot \mathbf{B} \\ &= \int_{\partial V} \mathrm{d}\mathbf{S} \cdot \mathbf{B},\end{aligned} \tag{88}$$

where we used Equation (32) to write the last surface integral. Substituting Equation (88) back into Equation (86) gives the Gauss law for the magnetic field [11] (pp. 1-5–1-9).

Let now $\mathcal{V}^3$ be a time-space domain $(t_0, t_1) \times S$, where S is a two-dimensional spatial surface. With no real loss of generality we assume that S lies on the $\mathbf{e}_1 \wedge \mathbf{e}_2$ plane. Its boundary $\partial\mathcal{V}^3$ is the union of the sets $(t_0, t_1) \times \partial S$, $t_0 \times S$ and $t_1 \times S$. For the first set, we choose $\mathbf{e}_\perp$ as the vector normal to ∂S pointing outwards on the plane defined by S and $\mathbf{e}_\parallel = \mathbf{e}_0 \wedge \mathbf{e}_{\partial S}$, where $\mathbf{e}_{\partial S}$ is a vector tangent to ∂S with a counterclockwise orientation, so that $\mathbf{e}_\perp \wedge \mathbf{e}_\parallel = -\mathbf{e}_{012}$. Further, since $\mathbf{e}_\parallel$ is a time-space bivector, the contribution of $\mathbf{F_B}$ to the circulation of $\mathbf{F}$ over this first set in Equation (86) is zero, and

$$\int_{(t_0,t_1)\times\partial S} \mathrm{d}^2\mathbf{x} \cdot \mathbf{F} = -\int_{(t_0,t_1)\times\partial S} \mathrm{d}^2\mathbf{x}_\parallel \cdot \mathbf{F_E}. \tag{89}$$

Writing the differential vector as $\mathrm{d}^2\mathbf{x}_\parallel = \mathrm{d}t\, \mathrm{d}x \mathbf{e}_{0x}$, parameterizing the line integral over the boundary ∂S by the variable x with unit vector $\mathbf{e}_{0x}$, and using that $\mathbf{e}_{0x} \cdot \mathbf{F_E} = -\mathbf{e}_x \cdot \mathbf{E}$ and therefore $\mathrm{d}x\, \mathbf{e}_{0x} \cdot \mathbf{F_E} = -\,\mathrm{d}\mathbf{x} \cdot \mathbf{E}$, the integral of the right-hand side of Equation (89) becomes

$$\int_{t_0}^{t_1} \mathrm{d}t \int_{\partial S} \mathrm{d}\mathbf{x} \cdot \mathbf{E}. \tag{90}$$

For the second and third sets the normal vector to the integration surface pointing outwards are $\mathbf{e}_\perp = -\mathbf{e}_0$ and $\mathbf{e}_\perp = \mathbf{e}_0$ respectively. Since $\mathbf{e}_\parallel$ is a space-space bivector in both cases, then the contribution of $\mathbf{F_E}$ to the circulation is zero. We express the circulations of $\mathbf{F_B}$ as fluxes of $\mathbf{B}$ and surface integrals as done in Equation (88).Using these observations the integral for the circulation of $\mathbf{F}$ over these two sets in Equation (86) is given by

$$\int_{t_0\times S} \mathrm{d}^2\mathbf{x}\cdot\mathbf{F} + \int_{t_1\times S} \mathrm{d}^2\mathbf{x}\cdot\mathbf{F} = -\int_S \mathrm{d}\mathbf{S}\cdot\mathbf{B}(t_0) + \int_S \mathrm{d}\mathbf{S}\cdot\mathbf{B}(t_1). \tag{91}$$

Combining Equations (90) and (91) in Equation (86) we recover the integral over time of the so called Faraday law [11] (pp. 17-1–17-2). Equivalently, taking the time derivative recovers the usual Faraday law, namely

$$\int_{\partial S} \mathrm{d}\mathbf{x}\cdot\mathbf{E} + \partial_t \int_S \mathrm{d}\mathbf{S}\cdot\mathbf{B} = 0. \tag{92}$$

In regard to the inhomogeneous Maxwell equations, applying the Stokes Theorem 2 to Equation (77), we find that *the flux of the bivector field* $\mathbf{F}$ *across the boundary of any three-dimensional space–time volume is equal to the flux of the current density* $\mathbf{J}$ *across the three-dimensional space–time volume*:

$$\int_{\partial\mathcal{V}^3} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}}\cdot\mathbf{F} = \int_{\mathcal{V}^3} \mathrm{d}^3\mathbf{x}^{\mathcal{H}^{-1}}\cdot(\boldsymbol{\partial}\lrcorner\mathbf{F}) = \int_{\mathcal{V}^3} \mathrm{d}^3\mathbf{x}^{\mathcal{H}^{-1}}\cdot\mathbf{J}. \tag{93}$$

As with the homogeneous Maxwell equations, the scalar Equation (93) yields the inhomogeneous Maxwell equations by considering two different hypersurfaces $\mathcal{V}^3$.

First, let the integration domain $\mathcal{V}^3$ be a spatial volume V. Since there are no normal components to V with space indices only, the contribution of $\mathbf{F_B}$ to the flux is zero so that Equation (93) becomes

$$\int_{\partial V} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}}\cdot\mathbf{F_E} = \int_V \mathrm{d}^3\mathbf{x}^{\mathcal{H}^{-1}}\cdot\mathbf{J}. \tag{94}$$

From the definition of inverse Hodge complement in Equation (10), we write the differential vectors

$$\mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}} = -\sum_{I,i\notin I} \mathrm{d}^2x_I\sigma(0i,I)\mathbf{e}_{0i} \tag{95}$$

$$\mathrm{d}^3\mathbf{x}^{\mathcal{H}^{-1}} = -\,\mathrm{d}V\mathbf{e}_0. \tag{96}$$

Plugging these expressions in Equation (94), using the definitions of $\mathbf{F_E}$ and $\mathbf{J}$, and computing the dot products on both sides of the equality, we obtain that Equation (94) simplifies as

$$-\int_{\partial V}\left(\sum_{I,i\notin I} \mathrm{d}^2x_I\sigma(0i,I)\mathbf{e}_{0i}\right)\cdot\left(\sum_j E_j\mathbf{e}_{0j}\right) = -\int_V \mathrm{d}V\mathbf{e}_0\cdot(\rho\mathbf{e}_0+\mathbf{j}) \tag{97}$$

$$\int_{\partial V}\sum_{I,i\notin I} \mathrm{d}^2x_I\sigma(i,I)E_i = \int_V \mathrm{d}V\rho \tag{98}$$

$$\int_{\partial V}\mathrm{d}\mathbf{S}\cdot\mathbf{E} = \int_V \mathrm{d}V\rho. \tag{99}$$

In Equation (98) we used that $\sigma(0i,I)=\sigma(i,I)$ and in Equation (99) we used that $\sum_{I,i\notin I}\mathrm{d}^2x_I\sigma(i,I)E_i = \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}}\cdot\mathbf{E}$. Since the Hodge complement is over space, the result is a surface integral with positive orientation as in Equation (32). We recovered in Equation (99) the Gauss law for the electric field [11] (pp. 4-7–4-9).

For $\mathcal{V}^3=(t_0,t_1)\times S$ where S is a two-dimensional surface lying on the $\mathbf{e}_1\wedge\mathbf{e}_2$ plane, the boundary $\partial\mathcal{V}^3$ is the union of the sets $(t_0,t_1)\times\partial S$, $t_0\times S$ and $t_1\times S$. For the first set, since $\mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}}$ has no time components, the contribution of $\mathbf{F_E}$ to this set is zero, that is

$$\int_{(t_0,t_1)\times\partial S} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}}\cdot\mathbf{F} = \int_{(t_0,t_1)\times\partial S} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}}\cdot\mathbf{F_B}. \tag{100}$$

As in the homogeneous case, we choose $\mathbf{e}_\perp$ as the vector normal to ∂S pointing outwards on the plane defined by S and $\mathbf{e}_\parallel=\mathbf{e}_0\wedge\mathbf{e}_{\partial S}$, where $\mathbf{e}_{\partial S}$ is a vector tangent to ∂S with a counterclockwise orientation, such that $\mathbf{e}_\perp\wedge\mathbf{e}_\parallel=-\mathbf{e}_{012}$ introduces a change of sign. Expressing $\mathrm{d}^2\mathbf{x}_\parallel^{\mathcal{H}^{-1}}$ and $\mathbf{F_B}$ in the canonical basis, defining $I = (0, i)$ so that I^c contains only space indexes, and using that $\mathbf{e}_{I^c} \cdot \mathbf{e}_{i^c} = 1$ and $\sigma(I^c, I) = \sigma(I^c, 0, i) = \sigma(0, I^c, i) = \sigma(I^c, i)$, we obtain that Equation (100) simplifies to

$$\begin{aligned}\int_{(t_0,t_1)\times\partial S} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}} \cdot \mathbf{F} &= \int_{(t_0,t_1)\times\partial S} \left(\sum_I \mathrm{d}x_I \sigma(I^c, I)\mathbf{e}_{I^c}\right) \cdot \left(\sum_i B_i \mathbf{e}_{i^c} \sigma(i^c, i)\right) \\ &= -\int_{t_0}^{t_1} \mathrm{d}t \int_{\partial S} \mathrm{d}x B_x \\ &= -\int_{t_0}^{t_1} \mathrm{d}t \int_{\partial S} \mathrm{d}\mathbf{x} \cdot \mathbf{B}.\end{aligned} \tag{101}$$

For the second and third sets we respectively choose $\mathbf{e}_\perp = -\mathbf{e}_0$ and $\mathbf{e}_\perp = \mathbf{e}_0$ pointing outside $\mathcal{V}^3$, implying that the contribution of $\mathbf{F_B}$ is zero for this set as the inverse Hodge complement of $\mathbf{e}_\perp$ is a space vector. Expressing $\mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}}$ in Equation (95) and using similar steps as in Equations (97)–(99), the left-hand side of Equation (93) over these two sets is given by

$$\int_{t_0\times\partial S} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}} \cdot \mathbf{F} + \int_{t_1\times\partial S} \mathrm{d}^2\mathbf{x}^{\mathcal{H}^{-1}} \cdot \mathbf{F} = -\int_S \mathrm{d}\mathbf{S} \cdot \mathbf{E}(t_0) + \int_S \mathrm{d}\mathbf{S} \cdot \mathbf{E}(t_1) \tag{102}$$

Finally, for the right-hand side of Equation (93), we choose $\mathbf{e}_\perp$ as the vector normal to $\mathcal{V}^3$ pointing outside. Since $\mathbf{e}_\parallel = \mathbf{e}_{012}$ implies that $\mathbf{e}_\perp \wedge \mathbf{e}_\parallel = -\mathbf{e}_{0123}$, we obtain that

$$\int_{\mathcal{V}^3} \mathrm{d}^3\mathbf{x}^{\mathcal{H}^{-1}} \cdot \mathbf{J} = -\int_{t_0}^{t_1} \mathrm{d}t \int_S \mathrm{d}\mathbf{S} \cdot \mathbf{j}. \tag{103}$$

We have thusly recovered the integral form of the Ampere-Maxwell equation [11] (p. 18-1–18-4) integrated over the time interval (t_0, t_1) by combining Equations (101)–(103) into Equation (93), that is

$$\int_{t_0}^{t_1} \mathrm{d}t \int_{\partial S} \mathrm{d}\mathbf{x} \cdot \mathbf{B} = \int_{t_0}^{t_1} \mathrm{d}t \int_S \mathrm{d}\mathbf{S} \cdot \mathbf{j} + \int_S \mathrm{d}\mathbf{S} \cdot \mathbf{E}(t_1) - \int_S \mathrm{d}\mathbf{S} \cdot \mathbf{E}(t_0). \tag{104}$$

5 Summary

In this paper, we aimed at showing how exterior calculus provides a tool merging the simplicity and intuitiveness of standard vector calculus with the power of tensors and differential forms. Set in the context of a general space–time algebra with multiple space and time components, we provided the basic concepts of exterior algebra and calculus, such as multivectors, wedge product and interior products, with a distinction between left and right products, Hodge complement, and exterior and interior derivatives. While a space–time with multiple time coordinates leads to several issues from the physical point of view [12], we did not deal with these problems as this paper focuses on the mathematical constructions. We also defined oriented integrals, with two important examples being the flux and circulation of grade m-vector fields as integrals of the normal and tangent components of the field to a hypersurface respectively. These operations extend the standard circulation of a vector field as a line integral and the flux of a vector field as a surface integral in three dimensions to any number of dimensions and any vector grade.

Armed with the theory of differential forms, we proved two exterior-calculus Stokes theorems, one for the circulation and one for the flux, that generalize the Kelvin-Stokes, Gauss and Green theorems. We saw that the flux of the curl of a vector field in three dimensions across a surface is also the circulation of the exterior derivative of the vector field along that surface. In exterior calculus, these Stokes theorems hold for any number of dimensions and any vector grade and are simply expressed in terms of the exterior and interior derivatives for the circulation and flux respectively.

As an application of our tools, we showed how to recover the classical laws of electromagnetism, Maxwell equations and Lorentz force, from a exterior-calculus formalism in relativistic space–time with one temporal and three spatial dimensions. The electromagnetic field is described by a bivector field with six components, closely related to Faraday's antisymmetric tensor, containing both electric and magnetic fields. The differential form of Maxwell equations relates the exterior derivative of the bivector field with the zero trivector and the interior derivative of the field with the current density vector. In the integral form, these equations correspond to the statements that the circulation of the bivector field along the boundary of any three-dimensional space–time volume is zero, and that the flux of the bivector field across the boundary of any three-dimensional space–time volume is equal to the flux of the current density across the same space–time volume.

A Proofs of product identities

In this appendix, we verify the relations about interior products introduced in Section 2.

A.1 Relation between left and right interior products

We now prove the formula

$$\mathbf{e}_I \lrcorner\, \mathbf{e}_J = \mathbf{e}_J \llcorner\, \mathbf{e}_I (-1)^{|I|(|J|-|I|)}, \tag{105}$$

relating left and right interior products. For two lists I and J, we have

$$\mathbf{e}_I \lrcorner\, \mathbf{e}_J = \Delta_{I,I}\sigma\big(J \setminus I, I\big)\mathbf{e}_{J\setminus I}, \tag{106}$$

$$\mathbf{e}_J \llcorner\, \mathbf{e}_I = \Delta_{I,I}\sigma\big(I, J \setminus I\big)\mathbf{e}_{J\setminus I}, \tag{107}$$

where we assumed that $I \subseteq J$ with no loss of generality and used that $\varepsilon(I, J^c)^c = J \setminus I$ in this case. The only difference between the expressions lies in the signatures, that are related by setting $A = J \setminus I$ and $B = I$ in the following lemma.

Lemma 1. *Given two arbitrary lists A and B, of length $|A|$ and $|B|$ respectively, then the permutations sorting the concatenated lists (A, B) and (B, A) satisfy the formula*

$$\sigma(A, B) = \sigma(B, A)(-1)^{|A||B|}. \tag{108}$$

¶

Proof. Given a list A, let $\bar{A}$ be the reversed list, namely the list where the order of all the elements is reversed. Counting the number of position jumps needed to reverse the list, we obtain the signature of this reversing operation as

$$\sigma_{\mathrm{r}}(A) = \sigma_{\mathrm{r}}(\bar{A}) = (-1)^{|A|-1+|A|-2+\ldots+1} = (-1)^{\frac{|A|(|A|-1)}{2}}. \tag{109}$$

The proof is based on the identity between two different ways of rearranging the concatenated list (A, B) into the ordered list $\varepsilon(A, B)$, as depicted in Figure 4.

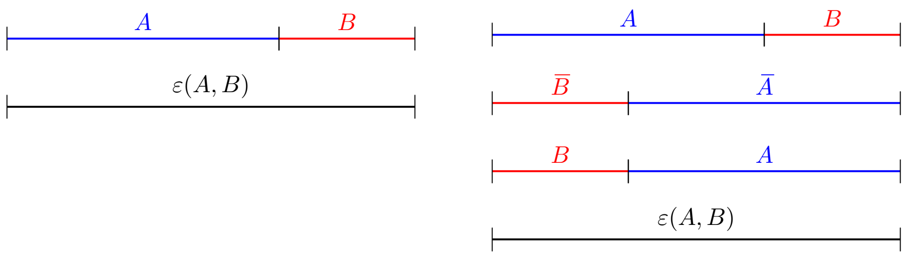

Figure 4: Visual aid for the relation between $\sigma(A, B)$ and $\sigma(B, A)$.

First, in the left column of Figure 4 we depict how a single permutation with signature $\sigma(A, B)$ orders the list (A, B). In the right column of Figure 4 we depict how a different series of permutations achieves the same result. We start by reversing the concatenated list (A, B), an operation with signature $\sigma_{\mathrm{r}}(\bar{B}, \bar{A})$. Then, we separately partially reverse the lists $\bar{B}$ and $\bar{A}$, operations with respective signatures

$\sigma_{\mathrm{r}}(\overline{B})$ and $\sigma_{\mathrm{r}}(\overline{A})$. A final permutation with signature $\sigma(B,A)$ orders the list (B,A) into $\varepsilon(A,B)$. Since the signature of a composition of permutations is the product of the signatures, we obtain that

$$\sigma(A,B) = \sigma_{\mathrm{r}}(\overline{B},\overline{A})\sigma_{\mathrm{r}}(\overline{A})\sigma_{\mathrm{r}}(\overline{B})\sigma(B,A). \tag{110}$$

Using Equation (109) in every σ_{r} in Equation (110) and carrying out some simplifications yields Equation (108). □

A.2 Relation between interior and exterior products

We start with the expression for the left interior product Equation (14). From Equations (9) and (10), we compute

$$\begin{aligned}\left(\mathbf{e}_I \wedge \mathbf{e}_J^{\mathcal{H}}\right)^{\mathcal{H}^{-1}} &= \left(\Delta_{J,J}\sigma(J,J^c)\mathbf{e}_I \wedge \mathbf{e}_{J^c}\right)^{\mathcal{H}^{-1}} \\ &= \Delta_{J,J}\Delta_{\varepsilon(I,J^c)^c,\varepsilon(I,J^c)^c}\,\sigma(J,J^c)\sigma(I,J^c)\sigma(\varepsilon(I,J^c)^c,\varepsilon(I,J^c))\,\mathbf{e}_{\varepsilon(I,J^c)^c},\end{aligned} \tag{111}$$

and since $\Delta_{\varepsilon(I,J^c)^c,\varepsilon(I,J^c)^c} = \Delta_{J\setminus I,J\setminus I}$, we can conclude that $\Delta_{J,J}\Delta_{\varepsilon(I,J^c)^c,\varepsilon(I,J^c)^c} = \Delta_{I,I}$. If we now compare the result with Equation (11), we need just to verify the identity

$$\sigma\big(\varepsilon(I,J^c)^c,I\big) = \sigma(J,J^c)\sigma(I,J^c)\sigma(\varepsilon(I,J^c)^c,\varepsilon(I,J^c))\,, \tag{112}$$

or equivalently

$$\sigma\big(\varepsilon(I,J^c)^c,I\big)\sigma(J,J^c) = \sigma(\varepsilon(I,J^c)^c,\varepsilon(I,J^c))\sigma(I,J^c)\,. \tag{113}$$

The left-hand side of Equation (113) corresponds to taking the sets $\varepsilon(I,J^c)^c = J\setminus I$, I and J^c, in this order, and then merging and sorting $J\setminus I$ with I and then merging and sorting the resulting set J with J^c, as shown in the left column of Figure 5. On the right-hand side, we start we the same three lists, but we first merge and sort I with J^c, and then we get the whole list by merging and sorting the result with $J\setminus I$, as represented in the right column of Figure 5. Thus, starting from the three sets and rearranging them in different ways, we get the same final ordered list, and since the signatures of the left-hand side and right-hand side are the same, and Equation (113) is proved. As a consequence, Equation (14) is verified.

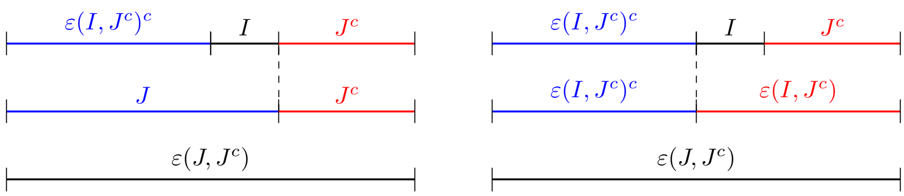

Figure 5: Visual aid for the permutations in Equation (113).

Afterwards, we prove the formula for the right interior product Equation (15). Using Equations (9) and (10), we write

$$\begin{aligned}\left(\mathbf{e}_I^{\mathcal{H}^{-1}} \wedge \mathbf{e}_J\right)^{\mathcal{H}} &= \left(\Delta_{I^c,I^c}\sigma(I^c,I)\mathbf{e}_{I^c} \wedge \mathbf{e}_J\right)^{\mathcal{H}} \\ &= \Delta_{I^c,I^c}\sigma(I^c,I)\sigma(I^c,J)\,\mathbf{e}^{\mathcal{H}}_{\varepsilon(I^c,J)} \\ &= \Delta_{I^c,I^c}\sigma(I^c,I)\sigma(I^c,J)\Delta_{\varepsilon(I^c,J),\varepsilon(I^c,J)}\sigma(\varepsilon(I^c,J),\varepsilon(I^c,J)^c)\,\mathbf{e}_{\varepsilon(I^c,J)^c}.\end{aligned} \tag{114}$$

Using that $\Delta_{\varepsilon(I^c,J),\varepsilon(I^c,J)}\Delta_{I^c,I^c} = \Delta_{J,J}$, in order to prove the validity of Equation (15), we need to prove the relation

$$\sigma(J,\varepsilon(I^c,J)^c) = \sigma(I^c,I)\sigma(I^c,J)\sigma(\varepsilon(I^c,J),\varepsilon(I^c,J)^c)\,. \tag{115}$$

We can prove it applying Lemma 1 to obtain the expression Equation (112), or by following the same procedure as before, paying attention to the difference that now list J is included in I.

A.3 Triple mixed product

Given two 1-vectors $\mathbf{u}$ and $\mathbf{v}$ and a r-vector $\mathbf{w}$, we prove the relation

$$\mathbf{u} \lrcorner (\mathbf{v} \wedge \mathbf{w}) = (-1)^r (\mathbf{u} \cdot \mathbf{v})\mathbf{w} + \mathbf{v} \wedge (\mathbf{u} \lrcorner \mathbf{w}). \tag{116}$$

Proof. We start by evaluating $\mathbf{u} \lrcorner (\mathbf{v} \wedge \mathbf{w})$ explicitly, separating terms $i = j$ and $i \neq j$, namely

$$\mathbf{u} \lrcorner (\mathbf{v} \wedge \mathbf{w}) = \sum_{\substack{i,j,I \\ j \notin I, i \in I}} \Delta_{i,i} u_i v_j w_I \sigma(j, I)\sigma(I + j \setminus i, i)\mathbf{e}_{I+j\setminus i} + \sum_{\substack{i,I \\ i \in I}} \Delta_{i,i} u_i v_i w_I \sigma(i, I)\sigma(I, i)\mathbf{e}_I, \tag{117}$$

then, using $\sigma(i, I)\sigma(I, i) = (-1)^r$ and adding and removing a term $(-1)^r \sum_{\substack{i,I \\ i \notin I}} \Delta_{i,i} u_i v_i w_I \mathbf{e}_I$, we get

$$\mathbf{u} \lrcorner (\mathbf{v} \wedge \mathbf{w}) = \sum_{\substack{i,j,I \\ i \in I, j \notin I \setminus i}} \Delta_{i,i} u_i v_j w_I \sigma(j, I)\sigma(I + j \setminus i, i)\mathbf{e}_{I+j\setminus i} + (-1)^r \sum_{i,I} \Delta_{i,i} u_i v_i w_I \mathbf{e}_I. \tag{118}$$

More concretely, the left-hand side $\mathbf{u} \lrcorner (\mathbf{v} \wedge \mathbf{w})$ is given by

$$\begin{aligned}
&\sum_{\substack{i,j,I \\ j \notin I, i \in I}} \Delta_{i,i} u_i v_j w_I \sigma(j, I)\sigma(I + j \setminus i, i)\mathbf{e}_{I+j\setminus i} - (-1)^r \sum_{\substack{i,I \\ i \notin I}} \Delta_{i,i} u_i v_i w_I \mathbf{e}_I \\
&= \sum_{\substack{i,j,I \\ j \notin I, i \in I}} \Delta_{i,i} u_i v_j w_I \sigma(j, I)\sigma(I + j \setminus i, i)\mathbf{e}_{I+j\setminus i} - \sum_{\substack{i,j,I \\ j = i, j \notin I \setminus i, i \in I}} \Delta_{i,i} u_i v_j w_I \sigma(j, I)\sigma(I + j \setminus i, i)\mathbf{e}_{I+j\setminus i} \\
&= \sum_{\substack{i,j,I \\ i \in I, j \notin I \setminus i}} \Delta_{i,i} u_i v_j w_I \sigma(j, I)\sigma(I + j \setminus i, i)\mathbf{e}_{I+j\setminus i}.
\end{aligned} \tag{119}$$

Similarly, we evaluate the right-hand side $\mathbf{v} \wedge (\mathbf{u} \lrcorner \mathbf{w})$ as

$$\mathbf{v} \wedge (\mathbf{u} \lrcorner \mathbf{w}) = \sum_{\substack{i,j,I \\ i \in I, j \notin I \setminus i}} \Delta_{i,i} u_i v_j w_I \sigma(I \setminus i, i)\sigma(j, I \setminus i)\mathbf{e}_{I+j\setminus i}. \tag{120}$$

Comparing Equations (119) and (120), it remains to prove the equality, and now we prove the equality

$$\sigma(j, I)\sigma(I + j \setminus i, i) = \sigma(I \setminus i, i)\sigma(j, I \setminus i). \tag{121}$$

We rewrite Equation (121) multiplying both sides for $\sigma(j, I)\sigma(j, I \setminus i)$ so that we obtain

$$\sigma(j, I \setminus i)\sigma(I + j \setminus i, i) = \sigma(I \setminus i, i)\sigma(j, I) \tag{122}$$

which we verify with the help of Figure 6. On the left column, we first merge j with $I \setminus i$ and then the resulting list with i. On the right columns, the permutations first join $I \setminus i$ and i and the resulting I is then merged with j, getting the same result in both sides of the relation. Thus, we can write

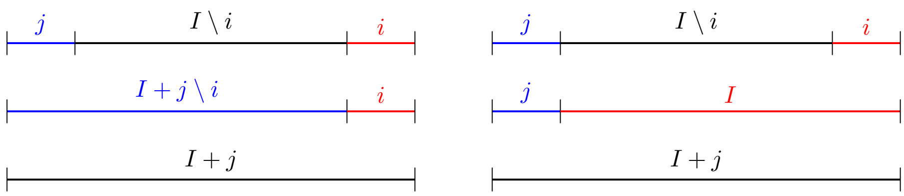

Figure 6: Visual aid for the identity $\sigma(j, I \setminus i)\sigma(I + j \setminus i, i) = \sigma(I \setminus i, i)\sigma(j, I)$.

$$\mathbf{u} \lrcorner (\mathbf{v} \wedge \mathbf{w}) = \sum_{\substack{i,j,I \\ i \in I, j \notin I \setminus i}} \Delta_{i,i} u_i v_j w_I \sigma(I \setminus i, i)\sigma(j, I \setminus i)\mathbf{e}_{I+j\setminus i} + (-1)^r \left(\sum_i \Delta_{i,i} u_i v_i\right)\left(\sum_I w_I \mathbf{e}_I\right), \tag{123}$$

where we identify the term $(-1)^r(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$, and finally conclude

$$\mathbf{u} \lrcorner (\mathbf{v}\wedge\mathbf{w}) - \mathbf{v}\wedge(\mathbf{u} \lrcorner \mathbf{w}) = (-1)^r(\mathbf{u}\cdot\mathbf{v})\mathbf{w}, \tag{124}$$

which proves our initial formula. □

Bibliography

[1] J. W. Gibbs and E. B. Wilson, *Vector Analysis.* New Haven, Yale University Press, 1929.

[2] H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The principle of relativity. Memoirs on the special and general theory of relativity.* Dover, 1923.

[3] M. M. G. Ricci and T. Levi-Civita, "Méthodes de calcul différentiel absolu et leurs applications," *Mathematische Annalen*, vol. 54, no. 1-2, pp. 125–201, 1900.

[4] E. Cartan, *Les Systemes Differentiels Exterieurs Et Leurs Applications Geometriques.* Hermann & Cie, 1945.

[5] V. I. Arnold, A. Weinstein, and K. Vogtmann, *Mathematical Methods Of Classical Mechanics.* Springer, 1989.

[6] W. K. Clifford, *Mathematical Papers.* London: Macmillan, 1882.

[7] H. Grassmann, *Extension Theory*, ser. History of Mathematics 19. American Mathematical Society, London Mathematical Society, 2000.

[8] A. Sommerfeld, "Zur relativitätstheorie. i. vierdimensionale vektoralgebra," *Annalen der Physik*, vol. 337, no. 9, pp. 749–776, 1910.

[9] S. Winitzki, *Linear Algebra via Exterior Products*, 2010.

[10] T. Frankel, *The Geometry of Physics*, 3rd ed. Cambridge University Press, 2012.

[11] R. P. Feynman, R. B. Leyton, and M. Sands, *Lectures on Physics.* AW, 1964, vol. 2.

[12] M. Velev, "Relativistic mechanics in multiple time dimensions," *Physics Essays*, vol. 25, no. 3, pp. 403–438, 2012.

Generalized Maxwell equations for exterior-algebra multivectors in (k, n) space-time dimensions *

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Abstract

This paper presents an exterior-algebra generalization of electromagnetic fields and source currents as multivectors of grades r and $r-1$ respectively in a flat space-time with n space and k time dimensions. Formulas for the Maxwell equations and the Lorentz force for arbitrary values of r, n, and k are postulated in terms of interior and exterior derivatives, in a form that closely resembles their vector-calculus analogues. These formulas lead to solutions in terms of potentials of grade $r-1$, and to conservation laws in terms of a stress-energy-momentum tensor of rank 2 for any values of r, n, and k, for which a simple explicit formula is given. As an application, an expression for the flux of the stress-energy-momentum tensor across an $(n+k-1)$-dimensional slice of space-time is given in terms of the Fourier transform of the potentials. The abstraction of Maxwell equations with exterior calculus combines the simplicity and intuitiveness of vector calculus, as the formulas admit explicit expressions, with the power of tensors and differential forms, as the formulas can be given for any values of r, n, and k.

1 Introduction

Classical electromagnetic phenomena are described by solutions of Maxwell's equations, a set of coupled partial differential equations relating the electric and magnetic fields across space and time with the charges and currents generating them. Let $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ respectively denote the electric and magnetic vector fields distributed over time t and space $\mathbf{x}$, and $\rho(t, \mathbf{x})$ and $\mathbf{j}(t, \mathbf{x})$ respectively denote the charge and current densities. In vector-calculus notation, and in units chosen such that the speed of light is set to 1 and all the quantities in the pairs time and space, electric and magnetic fields, and charge and current density have the same units, the standard microscopic Maxwell's equations take the well-known form [1, Sects. 26, 30] or [2, Sect. 6.6]

$$\nabla \cdot \mathbf{E} = \rho \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

$$\nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}. \tag{4}$$

Conventionally, Eqs. (2) and (3) (resp. (1) and (4)) are referred to as homogeneous (resp. inhomogeneous) Maxwell equations.

*This work has been funded in part by the Spanish Ministry of Science, Innovation and Universities under grants TEC2016-78434-C3-1-R and BES-2017-081360. Published in: ***Eur. Phys. J. Plus*** **135, 305 (2020). DOI:** 10.1140/epjp/s13360-020-00305-y.

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As a complement to Maxwell equations, the electromagnetic field creates in turn a force on charge and current densities described by the Lorentz force density $\boldsymbol{f}$ [1, Sect. 17] or [2, Sect. 6.7], a vector field given by

$$\boldsymbol{f} = \rho\mathbf{E} + \mathbf{j}\times\mathbf{B}. \tag{5}$$

Integrated over a region of space, the Lorentz force density $\mathbf{f}(t,\mathbf{x})$ gives the force acting upon the charges in that region. In addition, the rate of work being done on the charges by the fields is given by $\mathbf{j}\cdot\mathbf{E}$ [1, Sect. 17] or [2, Sect. 11.9].

In relativistic form, the charge and current densities are combined into a single vector $\mathbf{J}$ with four components whose time and space components are ρ and $\mathbf{j}$ respectively, while the electric and magnetic fields are combined into an antisymmetric tensor $\mathbf{F}$ of rank 2 indexed by pairs of space-time dimensions, the Faraday tensor. In the space-time metric $(-1,1,1,1)$, the components α,β of the Faraday tensor in its contravariant form, $F^{\alpha\beta}$ [1, Sect. 23] or [2, Sect. 6.7], are given by

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{6}$$

Both pairs of homogeneous and inhomogeneous equations are combined into single equations, namely [1, Sects. 26, 30] or [2, Sect. 11.9]

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{7}$$

$$\partial_\alpha F^{\alpha\beta} = J^\beta. \tag{8}$$

Also, the Lorentz force becomes a four-dimensional vector f, whose time and space components are $\mathbf{j}\cdot\mathbf{E}$ and $\boldsymbol{f}$ respectively. The component α of the Lorentz force, f^α, is given in its contravariant form by [1, Sect. 29] or [2, Sect. 11.9]

$$\mathrm{f}^\alpha = F^{\alpha\beta}J_\beta. \tag{9}$$

The basic equations of electromagnetism have a very rich history beyond the common vector and tensor formulas given above. At various moments quaternions [3], six-vectors [4, 5, 6], geometric or Clifford algebras [7, 8], or differential forms [9, 10] have all been put forward as convenient descriptions of fields, sources, and forces. Depending on whether simplicity, generality, computation ease, or intuitiveness or some combination thereof is preferred, each of these objects carries its own advantages and disadvantages. The purpose of this paper is to show how a formulation by means of exterior algebra, building on historical work on six-vectors for the electromagnetic field by Sommerfeld [4, 5], attains a good balance of simplicity in terms of notation and operations involved, with no need of covariant and contravariant representations; generality, as the equations are naturally cast in a relativistic form valid for any flat space-time with generalized electromagnetic fields and source densities given by r-vectors and $(r-1)$-vectors respectively, objects whose appearance is obscured when differential forms are used; and computation ease and intuitiveness shared with three-dimensional vector calculus.

In Sect. 2 we present the necessary notions and operations of exterior algebra and calculus, as needed for our purposes. Explicit formulas for the exterior and interior products and derivatives of multivectors, and circulation and flux of multivector fields are given. These concepts are put into use in Sect. 3, where we present the generalized Maxwell equations and Sect. 4, presenting the generalized Lorentz force and a simple expression of its associated stress-energy tensor. Both sections include several applications to support exterior algebra as a natural setting for an abstract yet intuitively simple form of Maxwell equations, Lorentz force, and stress-energy tensor for any values of the grade r and the space-time dimensions.

2 Exterior Algebra and Calculus

In this section, we present the basic notions and operations of exterior calculus; the section contents are a condensed summary of [11].

2.1 Exterior Algebra: Space-time, Multivectors, and Products

We build our theory on a flat space-time with k time dimensions and n space dimensions identified with $\mathbf{R}^{k+n}$. We represent the canonical basis of this (k,n)- or $(k+n)$-space-time by $\{\mathbf{e}_i\}_{i=0}^{k+n-1}$ and adopt the convention that its first k indices, i. e. $i = 0, \dots, k-1$, correspond to time components while the indices $i = k, \dots, k+n-1$ represent space components. A point or position in space-time is denoted by $\mathbf{x}$, with components x_i in the canonical basis $\{\mathbf{e}_i\}_{i=0}^{k+n-1}$, that is

$$\mathbf{x} = \sum_{i=0}^{k+n-1} x_i \mathbf{e}_i. \tag{10}$$

As first observed by Grassmann [12], next to the $(k+n)$-dimensional Euclidean vector space $\mathbf{R}^{k+n}$, for every value of the parameter $m = 0, \dots, k+n$ there exist other natural vector spaces with basis vectors $\mathbf{e}_I$ indexed by ordered lists $I = (i_1, \dots, i_m)$ of m non-identical space-time indices. For instance, if $m = 0$, the list is empty and the vector space is $\mathbf{R}$; for $m = 1$ we recover the standard vector space $\mathbf{R}^{k+n}$; for $m = 2$ and $m = 3$, the exterior-algebra basis vectors can be respectively identified with oriented plane and volume elements. This parameter m plays a very important role in exterior algebra and we refer to it as grade and to these vectors as multivectors or grade-m vectors if we wish to be more specific. A general multivector field $\mathbf{v}(\mathbf{x})$ of grade m, possibly a function of the position $\mathbf{x}$, with components $v_I(\mathbf{x})$ in the canonical basis $\{\mathbf{e}_I\}_{I\in\mathcal{I}_m}$ can be written as

$$\mathbf{v}(\mathbf{x}) = \sum_{I\in\mathcal{I}_m} v_I(\mathbf{x})\mathbf{e}_I, \tag{11}$$

where $\mathcal{I}_m$ denotes the set of all ordered lists with m indices. The dimension of the vector space containing all grade-m multivectors is $\binom{k+n}{m}$, the number of lists in $\mathcal{I}_m$. We denote by $|I|$ the length of a list I, e. g. $|I| = m$ for $I \in \mathcal{I}_m$ and by $\mathrm{gr}(\mathbf{v})$ the operation that returns the grade of the vector $\mathbf{v}$, e. g. $\mathrm{gr}\,\mathbf{v} = m$ in (11).

As Sommerfeld observed [4, 5], the electric and magnetic fields may be combined in a six-vector with components indexed by pairs of distinct space-time dimensions. Rather than the usual antisymmetric Faraday tensor in (6), one has a bivector, i. e. a multivector with grade 2. We extend this observation and consider a generalized electromagnetic field $\mathbf{F}(\mathbf{x})$ at every space-time point $\mathbf{x}$ given by an r-vector, of grade r, and a source density given by a vector $\mathbf{J}(\mathbf{x})$ of grade $(r-1)$. We give this generalized electromagnetic field the name of Maxwell field. The phenomenological description of the charges corresponding to this source density $\mathbf{J}(\mathbf{x})$ will be done elsewhere. As for the generalized Lorentz force density $\mathbf{f}(\mathbf{x})$, we shall see in Sect. 4 that it remains a vector of grade 1 with $k+n$ components. For convenience, we often write $\mathbf{F}$, $\mathbf{J}$, and $\mathbf{f}$, dropping the explicit dependence on $\mathbf{x}$.

Having defined multivectors and their canonical bases for various grades m, we consider the operations acting on them. While these operations may change the grades of their input objects or have different inputs with different grades, we always consider objects with a fixed grade. In other words, these objects are not sums of multivectors with different grades, as it would be in a Clifford or geometric algebra [13]. In order of presentation, and loosely following [11, Sect. 2], we define the dot or scalar product, the wedge product, the interior products, and the Hodge and inverse Hodge complements. With no real loss of generality, we define the operations only for the canonical basis vectors, the operation acting on general multivectors being a mere extension by linearity of the former.

Given two arbitrary canonical basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ in $\mathbf{R}^{k+n}$, their dot space-time product, also denoted by $\Delta_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$, is

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} -1, & i = j,\, 0 \le i \le k-1, \\ 1, & i = j,\, k \le i \le k+n-1. \\ 0, & i \neq j, \end{cases} \tag{12}$$

We adopt the convention that time unit vectors $\mathbf{e}_i$ have negative norm $\Delta_{ii} = -1$ and space unit vectors $\mathbf{e}_i$ have positive norm $\Delta_{ii} = +1$. We extend the definition of dot product $\cdot$ to arbitrary grade-m basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ as

$$\mathbf{e}_I \cdot \mathbf{e}_J = \Delta_{IJ} = \Delta_{i_1 j_1}\Delta_{i_2 j_2}\cdots\Delta_{i_m j_m}, \tag{13}$$

where I and J are the ordered lists $I = (i_1, i_2, \dots, i_m)$ and $J = (j_1, j_2, \dots, j_m)$.

The exterior algebra is defined on the direct sum of these vector spaces over all values of m, resulting in a larger vector space of dimension 2^{k+n}. In this larger space, one could define a geometric product between two multivectors, possibly of mixed grade, and thus obtain its associated geometric or Clifford algebra [13]. Using instead the tensor product of two vectors, one would obtain the tensor algebra, for which multivectors correspond to antisymmetric tensors of rank m. For the purposes of electromagnetism, there is no need of considering the full machinery of tensor or geometric algebras. In the exterior algebra, the product between two multivectors is the exterior or outer product, which we define next.

Let two basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ have grades $m = |I|$ and $m' = |J|$. Let $(I, J) = \{i_1, \dots, i_m, j_1, \dots, j_{m'}\}$ be the concatenation of I and J, let $\sigma(I, J)$ denote the signature of the permutation sorting the elements of this concatenated list of $|I| + |J|$ indices, and let $I + J$, or $\varepsilon(I, J)$ if $I + J$ is ambiguous, denote the resulting sorted list. Then, the exterior product $\mathbf{e}_I$ of $\mathbf{e}_J$ is defined as

$$\mathbf{e}_I \wedge \mathbf{e}_J = \sigma(I, J)\mathbf{e}_{I+J}. \tag{14}$$

Since permutations with repeated indices have zero signature, the exterior product is zero if $|I| + |J| > k + n$ or more generally if both vectors have at least one index in common. The exterior product is thus either zero or a vector of grade $|I| + |J|$.

The exterior product constructs the basis vectors for any grade m from the canonical basis vectors $\mathbf{e}_i$. We do so by identifying the vector $\mathbf{e}_I$ for the ordered list $I = (i_1, \dots, i_m)$ with the exterior product of $\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_m}$, that is

$$\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_m}. \tag{15}$$

Furthermore, as we can intuitively note from (14), the exterior product is an operation that adds the grades of the input multivectors, while the dot product subtracts their grades, yielding a scalar, i. e. a zero-grade multivector. We now define two generalizations of the dot product, the left and right interior products of two multivectors, as grade-lowering operations that output a multivector whose grade is the difference of the input multivector grades.

Let $\mathbf{e}_I$ and $\mathbf{e}_J$ be two basis vectors of respective grades $|I|$ and $|J|$. The left interior product, denoted by $\lrcorner$, is defined as

$$\mathbf{e}_I \lrcorner \mathbf{e}_J = \Delta_{II}\sigma(J \setminus I, I)\mathbf{e}_{J\setminus I} \tag{16}$$

if I is a subset of J and zero otherwise. The new basis $\mathbf{e}_{J\setminus I}$ is a vector of grade $|J| - |I|$ and it presents the indices of J excluding those in common with I. The right interior product, denoted by $\llcorner$, of two basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ is defined as

$$\mathbf{e}_I \llcorner \mathbf{e}_J = \Delta_{JJ}\sigma(J, I \setminus J)\mathbf{e}_{I\setminus J} \tag{17}$$

if J is a subset of I and zero otherwise. Both are grade-lowering operations, as the left (resp. right) interior product is either zero or a multivector of grade $|J| - |I|$ (resp. $|I| - |J|$).

Finally, we define the complement of a multivector, a concept needed to capture the idea of orthogonal subspace. For a unit vector $\mathbf{e}_I$ with grade $|I|$, its Grassmann or Hodge complement, denoted by $\mathbf{e}_I^{\mathcal{H}}$, is a unit $(k + n - |I|)$-vector given by

$$\mathbf{e}_I^{\mathcal{H}} = \Delta_{II}\sigma(I, I^c)\mathbf{e}_{I^c}, \tag{18}$$

where I^c is the complement of the list I, namely the ordered sequence of indices not included in I, and $\sigma(I, I^c)$ is the signature of the permutation sorting the elements of the concatenated list (I, I^c) of all space-time indices. In other words $\mathbf{e}_{I^c}$ is a basis vector of grade $k + n - |I|$ whose indices are absent from I. In addition, we define the inverse complement transformation as

$$\mathbf{e}_I^{\mathcal{H}^{-1}} = \Delta_{I^cI^c}\sigma(I^c, I)\mathbf{e}_{I^c}. \tag{19}$$

2.2 Some Properties of the Exterior and Interior Products

This section lists some relevant commutative and distributive properties of the exterior and interior products of multivectors of fixed grade; these properties will be needed later in our generalized electromagnetic sources, fields, and equations.

The exterior product (14) is a skew-commutative operation, as we have

$$\mathbf{u} \wedge \mathbf{v} = (-1)^{\operatorname{gr}\mathbf{u} \cdot \operatorname{gr}\mathbf{v}} \mathbf{v} \wedge \mathbf{u}. \tag{20}$$

Concerning the interior products, we have that

$$\mathbf{u} \rfloor \mathbf{v} = \mathbf{v} \lfloor \mathbf{u} (-1)^{\operatorname{gr}\mathbf{u} \cdot (\operatorname{gr}\mathbf{u} + \operatorname{gr}\mathbf{v})}, \tag{21}$$

and the interior products are also skew-commutative, unless $\operatorname{gr}\mathbf{u} = \operatorname{gr}\mathbf{v}$, when both are commutative and coincide with the dot product of the two vectors.

Given two vectors $\mathbf{v}$ and $\mathbf{v}'$ and two r-vectors $\mathbf{w}$ and $\mathbf{w}'$, then it holds that

$$(\mathbf{v} \wedge \mathbf{w}) \cdot (\mathbf{w}' \wedge \mathbf{v}') = (-1)^r (\mathbf{v} \cdot \mathbf{v}')(\mathbf{w} \cdot \mathbf{w}') + (\mathbf{v}' \rfloor \mathbf{w}) \cdot (\mathbf{w}' \lfloor \mathbf{v}). \tag{22}$$

Or equivalently, using (20) and (21), it holds that

$$(\mathbf{v} \cdot \mathbf{v}')(\mathbf{w} \cdot \mathbf{w}') = (\mathbf{v} \wedge \mathbf{w}) \cdot (\mathbf{v}' \wedge \mathbf{w}') + (\mathbf{v} \rfloor \mathbf{w}') \cdot (\mathbf{v}' \rfloor \mathbf{w}). \tag{23}$$

Consider now two vectors $\mathbf{u}$ and $\mathbf{v}$ and a r-vector $\mathbf{w}$, then it holds that

$$\mathbf{u} \rfloor (\mathbf{w} \lfloor \mathbf{v}) = (\mathbf{u} \rfloor \mathbf{w}) \lfloor \mathbf{v}, \tag{24}$$

or alternatively, using (21), we have that

$$\mathbf{u} \rfloor (\mathbf{v} \rfloor \mathbf{w}) = -\mathbf{v} \rfloor (\mathbf{u} \rfloor \mathbf{w}). \tag{25}$$

As already found in [11, Eq. (16)], for the same $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ it also holds that

$$\mathbf{u} \rfloor (\mathbf{v} \wedge \mathbf{w}) = (-1)^r (\mathbf{u} \cdot \mathbf{v})\mathbf{w} + \mathbf{v} \wedge (\mathbf{u} \rfloor \mathbf{w}). \tag{26}$$

Finally, for a vector $\mathbf{u}$, a $(r-1)$-vector $\mathbf{v}$ and a r-vector $\mathbf{w}$, the following equalities are true

$$(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (\mathbf{w} \lfloor \mathbf{u}) = \mathbf{u} \cdot (\mathbf{v} \rfloor \mathbf{w}), \tag{27}$$

or, alternatively, using (21), we have that

$$(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = (-1)^{r-1} (\mathbf{u} \rfloor \mathbf{w}) \cdot \mathbf{v} = (\mathbf{v} \rfloor \mathbf{w}) \cdot \mathbf{u}. \tag{28}$$

The relations (22), (24) and (27) are proved in A.1.

2.3 Exterior Calculus: Derivatives and Integrals, Circulation and Flux

In vector calculus, extensive use is made of the nabla operator ∇, a vector operator that takes partial space derivatives. For example, the divergence and curl in Maxwell equations are expressed in terms of this operator. In our case, we need its generalization to (k, n)-space-time, the differential vector operator $\boldsymbol{\partial}$ defined as $(-\partial_0, -\partial_2, \dots, -\partial_{k-1}, \partial_k, \dots, \partial_{k+n-1})$, that is

$$\boldsymbol{\partial} = \sum_{i=0}^{k+n-1} \Delta_{ii} \mathbf{e}_i \partial_i. \tag{29}$$

As done in more detail in [11, Sect. 3], we define the exterior derivative of $\mathbf{v}$, $\boldsymbol{\partial} \wedge \mathbf{v}$, of a given vector field $\mathbf{v}$ of grade m as

$$\boldsymbol{\partial} \wedge \mathbf{v} = \sum_{i=0}^{k+n-1} \sum_{I \in \mathcal{I}_m} \Delta_{ii} \sigma(i, I) \partial_i v_I \, \mathbf{e}_{i+I} = \sum_{i, I \in \mathcal{I}_m :\, i \notin I} \Delta_{ii} \sigma(i, I) \partial_i v_I \, \mathbf{e}_{i+I}. \tag{30}$$

The grade of the exterior derivative of $\mathbf{v}$ is $m+1$, unless $m = k+n$, in which case the exterior derivative is zero, as can be deduced from the fact that all signatures are zero. In addition, we define the interior derivative of $\mathbf{v}$ as $\boldsymbol{\partial} \lrcorner \mathbf{v}$, namely

$$\boldsymbol{\partial} \lrcorner \mathbf{v} = \sum_{i=0}^{k+n-1} \sum_{I \in \mathcal{I}_m} \sigma(I \setminus i, i) \partial_i v_I \mathbf{e}_{I \setminus i} = \sum_{i, I \in \mathcal{I}_m : i \in I} \sigma(I \setminus i, i) \partial_i v_I \mathbf{e}_{I \setminus i}. \tag{31}$$

The grade of the interior derivative of $\mathbf{v}$ is $m-1$, unless $m = 0$, in which case the interior derivative is zero, as can be deduced from the fact the grade of $\boldsymbol{\partial}$ is larger than the grade of $\mathbf{v}$. It is easy to verify that the exterior derivative of an exterior derivative is zero, as is the interior derivative of an interior derivative, that is for any vector field $\mathbf{v}$, we have that

$$\boldsymbol{\partial} \wedge (\boldsymbol{\partial} \wedge \mathbf{v}) = 0 \tag{32}$$
$$\boldsymbol{\partial} \lrcorner (\boldsymbol{\partial} \lrcorner \mathbf{v}) = 0. \tag{33}$$

From the identity (26), we note that the interior derivative of the exterior derivative is related to the exterior derivative of the interior derivative as

$$\boldsymbol{\partial} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{v}) = (-1)^{\mathrm{gr}(\mathbf{v})} (\boldsymbol{\partial} \cdot \boldsymbol{\partial}) \mathbf{v} + \boldsymbol{\partial} \wedge (\boldsymbol{\partial} \lrcorner \mathbf{v}). \tag{34}$$

From the chain of identities in (27), given a $(r-1)$-vector $\mathbf{v}$ and a r-vector $\mathbf{w}$, then the equalities convert into the product rule for the derivative which we write

$$\boldsymbol{\partial} \cdot (\mathbf{v} \lrcorner \mathbf{w}) = (\boldsymbol{\partial} \wedge \mathbf{v}) \cdot \mathbf{w} + (-1)^{\mathrm{gr}\,(\mathbf{v})} (\boldsymbol{\partial} \lrcorner \mathbf{w}) \cdot \mathbf{v}, \tag{35}$$

taking into account that the operator $\boldsymbol{\partial}$ is described by a vector basis $\mathbf{e}_i$ but it acts as a derivative, too.

The formulas for the exterior and interior derivatives allow us express some common expressions in vector calculus. For instance, the gradient of a scalar field ω is given by $\nabla \omega = \boldsymbol{\partial} \wedge \omega$ or the divergence of a vector field $\mathbf{v}$ is given by $\nabla \cdot \mathbf{v} = \boldsymbol{\partial} \lrcorner \mathbf{v}$. As another example, the cross product of two vector fields $\mathbf{v}$ and $\mathbf{w}$ in $\mathbf{R}^3$ can be variously expressed as

$$\mathbf{v} \times \mathbf{w} = (\mathbf{v} \wedge \mathbf{w})^{\mathcal{H}^{-1}} = \mathbf{v} \lrcorner \mathbf{w}^{\mathcal{H}^{-1}} = \mathbf{v} \lrcorner \mathbf{w}^{\mathcal{H}}. \tag{36}$$

This formula allows us to write the curl of a three-dimensional vector field in several equivalent ways in terms of the exterior and interior products and the Hodge complements, thereby providing a generalization of the cross product and the curl to grade-m vector fields in space-times with different dimensions.

Integrals are, together with derivatives, the fundamental mathematical objects of calculus. For example, operations on vectors fields such as flux and circulation are expressed in terms of integrals over high-dimensional geometric objects. For any $\ell = 0, \dots, k+n$, we define an infinitesimal vector element $\mathrm{d}^\ell \mathbf{x}$ as the sum of all possible differentials for ℓ-dimensional hypersurfaces in a (k, n)-space-time. This infinitesimal vector element is represented in the canonical basis as

$$\mathrm{d}^\ell \mathbf{x} = \sum_{I \in \mathcal{I}_\ell} \mathrm{d}x_I \mathbf{e}_I, \tag{37}$$

where for a given list $I = (i_1, \dots, i_\ell)$ the differential is given by $dx_I = \mathrm{d}x_{i_1} \cdots \mathrm{d}x_{i_\ell}$. A positive orientation is implicit in (37), as the skew-symmetry of the product (14) may introduce sign changes to compensate an eventual change of orientation after coordinates change, e. g. permutations of the space-time components.

The circulation of a vector field $\mathbf{v}(\mathbf{x})$ of grade m along an ℓ-dimensional hypersurface $\mathcal{V}^\ell$ with $\ell = m$ is defined in terms of the right interior product, or equivalently the scalar product as $\ell = m$, that is

$$\int_{\mathcal{V}^m} \mathrm{d}^m \mathbf{x} \llcorner \mathbf{v} = \int_{\mathcal{V}^m} \mathrm{d}^m \mathbf{x} \cdot \mathbf{v}. \tag{38}$$

This formula recovers for $\ell = m = 1$ and $\mathbf{R}^n$ the definition of the circulation of a vector field along a closed path with the appropriate orientation. Intuitively, the circulation measures the alignment of the vector field $\mathbf{v}$ with respect to $\mathcal{V}^m$.

The Stokes theorem for the circulation [11, Sect. 3.4] states that the circulation of a grade-m vector field $\mathbf{v}$ along the boundary $\partial\mathcal{V}^{m+1}$ of an $(m+1)$-dimensional hypersurface $\mathcal{V}^{m+1}$ is equal to the circulation of the exterior derivative of $\mathbf{v}$ along the hypersurface $\mathcal{V}^{m+1}$:

$$\int_{\partial\mathcal{V}^{m+1}} \mathrm{d}^m\mathbf{x}\cdot\mathbf{v} = \int_{\mathcal{V}^{m+1}} \mathrm{d}^{m+1}\mathbf{x}\cdot(\boldsymbol{\partial}\wedge\mathbf{v}). \tag{39}$$

The role of the vector curl in the usual form of the Kelvin-Stokes theorem is played by the exterior derivative in this version of the theorem. Recall that the Kelvin-Stokes theorem for the circulation of a vector field $\mathbf{v}$ of grade 1 along the boundary $\partial\mathcal{V}^2$ of a bidimensional surface $\mathcal{V}^2$ relates its value to that of the surface integral of the curl of the vector field over the surface itself.

The flux of a vector field $\mathbf{v}(\mathbf{x})$ of grade m across an ℓ-dimensional hypersurface $\mathcal{V}^\ell$ is defined in terms of the left interior product and the inverse Hodge complement of the infinitesimal vector element (37), that is

$$\int_{\mathcal{V}^\ell} \mathrm{d}^\ell\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{v}. \tag{40}$$

For example, this formula recovers the flux across a line on the plane or across a surface in three-dimensional space. Alternatively, if $\ell = k+n$, one can verify that the flux of $\mathbf{v}$ over an $(k+n)$-dimensional hypersurface $\mathcal{V}^{k+n}$ gives the volume integral of $\mathbf{v}$ over $\mathcal{V}^{k+n}$. Intuitively, the flux (40) measures the magnitude of the multivector field crossing the hypersurface. In general, the flux is a vector of grade $(m+\ell-n-k)$ if $\ell \geq k+n-m$ and zero otherwise.

As with the circulation, a generalized Stokes theorem [11, Sect. 3.5] states that the flux of a grade-m vector field $\mathbf{v}$ across the boundary $\partial\mathcal{V}^\ell$ of an ℓ-dimensional hypersurface $\mathcal{V}^\ell$ is equal to the flux of the interior derivative of $\mathbf{v}$ across $\mathcal{V}^\ell$:

$$\int_{\partial\mathcal{V}^\ell} \mathrm{d}^{\ell-1}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{v} = \int_{\mathcal{V}^\ell} \mathrm{d}^\ell\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, (\boldsymbol{\partial}\lrcorner\,\mathbf{v}). \tag{41}$$

Armed with the relevant definitions of space-time algebra, the products defined on multivectors, and the derivatives and integrals in exterior calculus, we postulate in the next section the generalized Maxwell equations for arbitrary r, k, and n.

3 Generalized Maxwell Equations

3.1 Differential Form of Maxwell Equations

For a given r, we consider a Maxwell field $\mathbf{F}(\mathbf{x})$ and a generalized source density $\mathbf{J}(\mathbf{x})$ at every point $\mathbf{x}$ of the flat (k,n)-space-time. The Maxwell field $\mathbf{F}(\mathbf{x})$ is a multivector field of grade r and the source density $\mathbf{J}(\mathbf{x})$ a multivector field of grade $(r-1)$. For convenience, we often write $\mathbf{F}$ and $\mathbf{J}$, dropping the explicit dependence on $\mathbf{x}$. Going directly to the heart of the matter, we postulate the generalized Maxwell equations for arbitrary r, k, and n to be the following two coupled differential equations relating $\mathbf{F}(\mathbf{x})$ and $\mathbf{J}(\mathbf{x})$:

$$\boldsymbol{\partial}\lrcorner\,\mathbf{F} = \mathbf{J}, \tag{42}$$

$$\boldsymbol{\partial}\wedge\mathbf{F} = 0. \tag{43}$$

The wedge product (14) in the exterior derivative (30) raises the grade of $\mathbf{F}$ and the zero in (43) is the null $(r+1)$-vector; similarly, as the left interior product (16) in the interior derivative (31) lowers the grade of $\mathbf{F}$, both sides of (42) are $(r-1)$-vectors. A source density conservation law, $\boldsymbol{\partial}\lrcorner\,\mathbf{J} = 0$, follows from (42) and (33).

For given r, the Maxwell field and the source density have respectively $\binom{k+n}{r}$ and $\binom{k+n}{r-1}$ components at each point of space-time. For $r=2$, $k=1$, and $n=3$, the field is the usual electromagnetic field, in bivector form rather than in the tensor form (6), whose three components with space-space indices $F_{ij}(\mathbf{x})$, $i,j=1,2,3$, $i<j$, represent the magnetic field, and whose remaining three components with space-time indices $F_{ij}(\mathbf{x})$, $i=0$, $j=1,2,3$, represent the electric field. More precisely, we have

$\mathbf{F} = \mathbf{e}_0 \wedge \mathbf{E} + \mathbf{B}^{\varkappa}$, with the Hodge complement defined in (18). Similarly, the first component of the source density represents the density of charge $\rho(\mathbf{x})$ and the last three represent the space current density $\mathbf{j}(\mathbf{x})$, namely $\mathbf{J} = \rho\mathbf{e}_0 + \mathbf{j}$. It can be verified rather easily that these bivector equations are equivalent to the differential form of the vector Maxwell equations in (1)–(4) [11, Sect. 4.2].

Two other less obvious examples are $r = 1$, $k = 0$, and $n = 3$, and $r = 2$, $k = 0$, and $n = 3$. In the first case, the field $\mathbf{F}$ is a vector field that we can identify with $\mathbf{E}$, the source density is a scalar, namely ρ and there is no time, and we recover the equations of electrostatics. Indeed, as $\boldsymbol{\partial} = \nabla$ and $\boldsymbol{\partial}\lrcorner = \nabla\cdot$, and using (36) to write $\boldsymbol{\partial}\wedge = \nabla\times$ in three dimensions, we obtain

$$\nabla \cdot \mathbf{E} = \rho, \tag{44}$$

$$\nabla \times \mathbf{E} = 0. \tag{45}$$

In the second case, the field $\mathbf{F}$ is a bivector field that we can identify with the Hodge complement of $\mathbf{B}$, that is $\mathbf{F} = \mathbf{B}^{\varkappa}$, the source density is a vector, namely $\mathbf{j}$, and we recover the equations of magnetostatics. Indeed, using (36) we obtain

$$\nabla \times \mathbf{B} = \mathbf{j}, \tag{46}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{47}$$

Polar vectors, such as the electric field, are naturally represented by a vector. Although axial vectors, such as the magnetic fied, can be represented as vectors in three dimensions, it might be more natural to represent them as bivectors.

3.2 Vector Potential

As in standard electromagnetism, one can introduce a potential field $\mathbf{A}(\mathbf{x})$, now a multivector field of grade $r-1$ with $\binom{k+n}{r-1}$ components. According to Poincaré's Lemma [14, Sect. 36.G], stated in exterior-algebra notation, whenever the exterior derivative of an r-vector field $\mathbf{F}$ vanishes on a contractible domain, then $\mathbf{F}$ must be the exterior derivative of an $(r-1)$ vector. In a suitably contractible domain, this Lemma implies that the homogenous Maxwell equation (43) is equivalent to

$$\mathbf{F} = \boldsymbol{\partial} \wedge \mathbf{A}. \tag{48}$$

For the simple examples listed in the previous section, the potential is respectively the usual relativistic 4-potential, the scalar potential in electrostatics and the vector potential in magnetostatics. If we replace the potential $\mathbf{A}$ by a new field $\mathbf{A}' = \mathbf{A} + \boldsymbol{\partial} \wedge \mathbf{G}$, where $\mathbf{G}$ is an $(r-2)$-vector gauge field, the homogenous Maxwell equation (43) is unchanged thanks to (32). There is therefore some unavoidable ambiguity on the value of the vector potential if $r \geq 2$.

A possibly useful example is the Lorenz gauge, for which we set $\boldsymbol{\partial} \lrcorner \mathbf{A} = 0$. In this case, the exterior-calculus identity (34) allows us to write the inhomogeneous Maxwell equation (42) as

$$\boldsymbol{\partial} \lrcorner \mathbf{F} = (-1)^{r-1}(\boldsymbol{\partial} \cdot \boldsymbol{\partial})\mathbf{A} = \mathbf{J}. \tag{49}$$

In this gauge, the Maxwell equations become $\binom{k+n}{r-1}$ uncoupled ultrahyperbolic wave equations for the separate components of $\mathbf{J}$. Similarly, we may choose a transverse gauge where not only $\boldsymbol{\partial} \lrcorner \mathbf{A} = 0$ is satisfied, but also $\boldsymbol{\partial}_{\mathrm{t}} \lrcorner \mathbf{A} = \boldsymbol{\partial}_{\mathrm{s}} \lrcorner \mathbf{A} = 0$, where $\boldsymbol{\partial}_{\mathrm{t}}$ and $\boldsymbol{\partial}_{\mathrm{s}}$ respectively represent the time and space components of $\boldsymbol{\partial}$.

Considering an $(r-2)$-vector gauge field $\mathbf{G}$, the Lorenz gauge condition together with (34) imply that

$$\boldsymbol{\partial} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{G}) = (-1)^{r-2}(\boldsymbol{\partial} \cdot \boldsymbol{\partial})\mathbf{G} + \boldsymbol{\partial} \wedge (\boldsymbol{\partial} \lrcorner \mathbf{G}) = 0. \tag{50}$$

When $r = 2$, the interior derivative $\boldsymbol{\partial} \lrcorner \mathbf{G}$ of a scalar gauge field $\mathbf{G}$ vanishes and we find that $(\boldsymbol{\partial} \cdot \boldsymbol{\partial})\mathbf{G} = 0$, namely that the gauge field $\mathbf{G}$ must be a harmonic function. For larger values of r, however, one should consider the full equation (50).

3.3 Maxwell Equations in the Fourier Domain

It is instructive to express the Maxwell equations in the Fourier domain, especially to study the field propagation in the absence of sources. Let the Fourier variable be denoted by $\boldsymbol{\xi} = (\xi_0, \ldots, \xi_{k+n-1})$

and the Fourier transform $\hat{\mathbf{F}}(\boldsymbol{\xi})$ of the Maxwell field be

$$\hat{\mathbf{F}}(\boldsymbol{\xi}) = \int \cdots \int \mathrm{d}^{k+n}\mathbf{x}\, e^{-j2\pi\boldsymbol{\xi}\cdot\mathbf{x}}\mathbf{F}(\mathbf{x}). \tag{51}$$

We also have a similar expression for the Fourier transform $\hat{\mathbf{J}}(\boldsymbol{\xi})$ of the source. The Maxwell equations (42) and (43) adopt the algebraic form

$$j2\pi\boldsymbol{\xi} \lrcorner \hat{\mathbf{F}} = \hat{\mathbf{J}}, \tag{52}$$

$$\boldsymbol{\xi} \wedge \hat{\mathbf{F}} = 0. \tag{53}$$

For the Fourier transform of the vector potential $\hat{\mathbf{A}}(\boldsymbol{\xi})$, the identity $\mathbf{F} = \boldsymbol{\partial} \wedge \mathbf{A}$ implies that $\hat{\mathbf{F}}(\boldsymbol{\xi}) = j2\pi\boldsymbol{\xi} \wedge \hat{\mathbf{A}}(\boldsymbol{\xi})$.

In the absence of sources, the Maxwell equations (42) and (43) adopt the simpler form

$$\boldsymbol{\xi} \lrcorner \hat{\mathbf{F}} = 0, \tag{54}$$

$$\boldsymbol{\xi} \wedge \hat{\mathbf{F}} = 0. \tag{55}$$

The first equation would seem to require that $\hat{\mathbf{F}}$ is orthogonal to $\boldsymbol{\xi}$, while the second requires that $\hat{\mathbf{F}}$ is parallel to $\boldsymbol{\xi}$. The only non-trivial combination of these two possibilities is that $\boldsymbol{\xi}$ has zero norm, that is $\boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0$. This is possible only if both k and n are positive numbers. Using (34) and these two Maxwell equations in the Fourier domain, we have

$$0 = \boldsymbol{\xi} \lrcorner (\boldsymbol{\xi} \wedge \hat{\mathbf{F}}) = (-1)^r(\boldsymbol{\xi} \cdot \boldsymbol{\xi})\hat{\mathbf{F}} + \boldsymbol{\xi} \wedge (\boldsymbol{\xi} \lrcorner \hat{\mathbf{F}}) = (-1)^{r-1}(\boldsymbol{\xi} \cdot \boldsymbol{\xi})\hat{\mathbf{F}}, \tag{56}$$

from which we conclude that the Fourier transform of the fields is supported only in the set $\boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0$. With some abuse of notation, let $\hat{\mathbf{F}}$ denote this function with support only in the set $\boldsymbol{\xi} \cdot \boldsymbol{\xi} = 0$. We postulate that the inverse Fourier transform of the fields in free space is indeed given by

$$\mathbf{F}(\mathbf{x}) = \int \cdots \int \mathrm{d}^{k+n}\boldsymbol{\xi}\, e^{j2\pi\boldsymbol{\xi}\cdot\mathbf{x}}\hat{\mathbf{F}}(\boldsymbol{\xi})\delta(\boldsymbol{\xi} \cdot \boldsymbol{\xi}). \tag{57}$$

3.4 Integral Form of Maxwell Equations

The generalized Maxwell equations also admit an integral form obtained by using the Stokes theorem [14, Sect. 36.D]. Moreover, these integral expressions are associated with two natural operations on the Maxwell field, namely its circulation and its flux. These associations are particularly transparent in the exterior-algebra formulation presented in this paper, while being simultaneously valid for generic values of r, k, and n. From (39) and (43), and noting that the interior product is now a dot product, we find that the circulation of the Maxwell field $\mathbf{F}$ along the boundary of any $(r+1)$-dimensional space-time volume $\mathcal{V}^{r+1}$ is zero:

$$\int_{\partial\mathcal{V}^{r+1}} \mathrm{d}^r\mathbf{x} \cdot \mathbf{F} = \int_{\mathcal{V}^{r+1}} \mathrm{d}^{r+1}\mathbf{x} \cdot (\boldsymbol{\partial} \wedge \mathbf{F}) \tag{58}$$

$$= 0. \tag{59}$$

For $r = 2$, $k = 1$ and $n = 3$, (59) is a scalar equation and we obtain the usual integral forms of the pair of homogeneous Maxwell equations by considering two different hypersurfaces, all-space, and time-space, as verified in detail in [11, Sect. 4.3].

Similarly, from (41) and (42), the flux of the Maxwell field $\mathbf{F}$ across the boundary of any $(k+n-r+1)$-dimensional space-time volume is equal to the flux of the current density $\mathbf{J}$ across the $(k+n-r+1)$-dimensional space-time volume:

$$\int_{\partial\mathcal{V}^{k+n-r+1}} \mathrm{d}^{k+n-r}\mathbf{x}^{\mathcal{H}^{-1}} \cdot \mathbf{F} = \int_{\mathcal{V}^{k+n-r+1}} \mathrm{d}^{k+n-r}\mathbf{x}^{\mathcal{H}^{-1}} \cdot (\boldsymbol{\partial} \lrcorner \mathbf{F}) \tag{60}$$

$$= \int_{\mathcal{V}^{k+n-r+1}} \mathrm{d}^{k+n-r+1}\mathbf{x}^{\mathcal{H}^{-1}} \cdot \mathbf{J}. \tag{61}$$

As with the homogeneous Maxwell equations, for $r = 2$, $k = 1$, and $n = 3$ the scalar equation (61) yields the usual integral inhomogeneous equations by considering two different hypersurfaces $\mathcal{V}^3$ [11, Sect. 4.3], respectively all-space and time-space.

4 Lorentz Force and Stress-Energy-Momentum Tensor

As stated in the Introduction, the action of the electromagnetic field on the charges is described by the Lorentz force. In relativistic form and tensor notation, the four-dimensional Lorentz force density vector $\mathbf{f}$ is related to the electromagnetic field $F^{\alpha\beta}$ and the source density J^β as given in (9),

$$\mathrm{f}^\alpha = F^{\alpha\beta} J_\beta. \tag{62}$$

The interaction between fields and charges involves a transfer of energy and momentum between the former and the latter. This interaction is subject to a conservation law relating the Lorentz force density f^α and the (symmetric) stress-energy-momentum tensor $T^{\alpha\beta}$ of the electromagnetic field [1, Sect. 32] or [2, Sect. 12.10],

$$\mathrm{f}^\alpha + \partial_\beta T^{\alpha\beta} = 0. \tag{63}$$

For any values of k and n and irrespective of the value of r, both the Maxwell field and the source density carry energy-momentum. This generalized energy-momentum is a vector with $k+n$ components, the first k (resp. remaining n) of which represent temporal (resp. spatial) components.

As we discuss in Sect. 4.1, the generalized Lorentz force density remains a 1-vector with $k+n$ components. Integrated over a space-time region, this density characterizes the action of the field upon the source density. As explained in Sect. 4.2, this interaction is also subject to a conservation law similar to (63). The stress-energy-momentum tensor of the Maxwell field is found to be a symmetric bitensor of rank 2 for any values of r, k and n. Finally, as an application, Sect. 4.3 studies the flux of the stress-energy-momentum tensor across a $(k+n-1)$-dimensional slice of space-time and gives an expression for this flux in terms of the Fourier transform of the potential.

4.1 Lorentz Force Density

Energy-momentum can be transferred from the fields to the charges through a process modelled as a force acting on the charges. We refer to the Lorentz force, whose density was introduced in (9). In exterior-calculus form, the generalized Lorentz force density $\mathbf{f}$ is a vector of grade 1 with $k+n$ components given by

$$\mathbf{f} = \mathbf{J} \lrcorner \mathbf{F} = (\boldsymbol{\partial} \lrcorner \mathbf{F}) \lrcorner \mathbf{F}. \tag{64}$$

The volume integral of the Lorentz force density $\mathbf{f}$ over an $(k+n)$-dimensional hypervolume $\mathcal{V}^{k+n}$ quantifies the transfer of energy-momentum to the charges in that volume. In turn, from the discussion in Sect. 2.2, this volume integral is the flux of $\mathbf{f}$ over the volume. As the grade of $\mathbf{F}$ is r and that of $\mathbf{J}$ is $r-1$, and the left interior product lowers the grade, the result has indeed grade $r-(r-1)=1$. It is rather straightforward to verify that this Lorentz force coincides with the relativistic Lorentz force for $r=2$, $k=1$ and $n=3$ [11, Sect. 4.1], and with the forces upon charges or currents in electrostatics or magnetostatics.

4.2 Stress-Energy-Momentum Tensor

The flux or transfer of energy-momentum over space-time is described by means of the stress-energy-momentum tensor $\mathbf{T}(\mathbf{x})$, a bitensor with $\binom{k+n+1}{2}$ independent components at each point of space-time, regardless of the value of the grade r,

$$\mathbf{T} = \sum_{i\leq j} T_{ij}\mathbf{u}_{ij}. \tag{65}$$

Here, the bitensor basis elements are denoted by $\mathbf{u}_{ij}$, where (i,j) is an ordered list of two (possibly repeated) space and time indices. In the usual tensor algebra, this bitensor would be a symmetric tensor of rank 2 as we can identify $\mathbf{u}_{ij}$ as $\mathbf{u}_{ij} = \mathbf{e}_i \vee \mathbf{e}_j$, the symmetric tensor product of $\mathbf{e}_i$ and $\mathbf{e}_j$. A benefit of using exterior algebra is that the stress-energy-momentum tensor of the Maxwell field admits a rather simple formula valid for any triplet r, k, and n.

Before proceeding further, we define the left and right interior product of a vector $\mathbf{e}_i$ and a bitensor $\mathbf{u}_{jk}$, two bilinear operations giving a vector. These products are the extension by linearity of the product of unit basis vectors given by

$$\mathbf{e}_i \lrcorner \mathbf{u}_{jk} = \mathbf{u}_{jk} \llcorner \mathbf{e}_i = \begin{cases} \mathbf{e}_j \Delta_{ii}, & i = k, \\ \mathbf{e}_k \Delta_{ii}, & i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{66}$$

Let $\boldsymbol{\partial}\lrcorner$ denote the interior derivative, as in (66). We prove in A.2 the conservation law for the equivalent energy-momentum relating the Lorentz force (64) and the stress-energy-momentum tensor $\mathbf{T}$ of the Maxwell field $\mathbf{F}$

$$\mathbf{f} + \boldsymbol{\partial} \lrcorner \mathbf{T} = 0. \tag{67}$$

Here, the stress-energy-momentum tensor $\mathbf{T}$ of the Maxwell field $\mathbf{F}$ is given by the expression

$$\mathbf{T} = -(\mathbf{F} \odot \mathbf{F} + \mathbf{F} \owedge \mathbf{F}), \tag{68}$$

where $\mathbf{F} \odot \mathbf{F}$ and $\mathbf{F} \owedge \mathbf{F}$ are two bitensors with components respectively given by

$$\mathbf{F} \odot \mathbf{F}\big|_{ij} = \frac{1}{2}\Delta_{ii}\Delta_{jj}(\mathbf{e}_i \lrcorner \mathbf{F}) \cdot (\mathbf{F} \llcorner \mathbf{e}_j) \tag{69}$$

$$\mathbf{F} \owedge \mathbf{F}\big|_{ij} = \frac{1}{2}\Delta_{ii}\Delta_{jj}(\mathbf{e}_i \wedge \mathbf{F}) \cdot (\mathbf{F} \wedge \mathbf{e}_j). \tag{70}$$

In A.3 we compute the explicit expressions for T_{ij}, namely

$$T_{ii} = \frac{(-1)^r}{2}\Delta_{ii}\left(\sum_{I\in\mathcal{I}_r:i\in I} F_I^2\Delta_{II} - \sum_{I\in\mathcal{I}_r:i\notin I} F_I^2\Delta_{II}\right) \tag{71}$$

$$T_{ij} = -\sum_{L\in\mathcal{I}_{r-1}} \sigma(L,i)\sigma(j,L)F_{i+L}F_{j+L}\Delta_{LL}. \tag{72}$$

A tedious calculation would serve to verify that the tensor components in (69) and (70) indeed coincide with the well-known electromagnetic stress-energy tensor for $r = 2$, $k = 1$, and $n = 3$ [1, Sect. 32]. In general, the components of $\mathbf{T}$ come in two different forms. For $i = j$, the component T_{ii} in (71) is given by the sum of the squares of all $\binom{k+n}{r}$ components in the Maxwell field F_I, each weighted by either $+\frac{1}{2}$ or $-\frac{1}{2}$ depending of the values of i and I. For $i \neq j$, the component T_{ij} in (72) is given by the sum of $\binom{k+n-2}{r-1}$ terms, each of this is a product of two different components of the Maxwell field weighted by either +1 or -1.

The values of T_{ij} in (69) and (70) do not change if we swap the values of i and j so $\mathbf{T}$ indeed corresponds to a symmetric tensor. To verify this fact, we note that from (21), it holds that $\mathbf{e}_j \lrcorner \mathbf{F} = (-1)^{r+1}\mathbf{F} \llcorner \mathbf{e}_j$. This implies that

$$(\mathbf{e}_j \lrcorner \mathbf{F}) \cdot (\mathbf{F} \llcorner \mathbf{e}_i) = (\mathbf{F} \llcorner \mathbf{e}_i) \cdot (\mathbf{e}_j \lrcorner \mathbf{F}) \tag{73}$$

$$= (\mathbf{e}_i \lrcorner \mathbf{F})(-1)^{r+1} \cdot (\mathbf{F} \llcorner \mathbf{e}_j)(-1)^{r+1} \tag{74}$$

$$= (\mathbf{e}_i \lrcorner \mathbf{F}) \cdot (\mathbf{F} \llcorner \mathbf{e}_j). \tag{75}$$

Therefore the components of $\mathbf{F} \odot \mathbf{F}$ in (69) are symmetric. Concerning the components of $\mathbf{F} \wedge \mathbf{F}$ in (70), one proves that they are symmetric by using a similar reasoning exploiting that $\mathbf{e}_j \wedge \mathbf{F} = (-1)^r(\mathbf{F} \wedge \mathbf{e}_j)$ from (20).

The tensor trace $\operatorname{Tr}\mathbf{T} = \sum_i \Delta_{ii}T_{ii}$ is given for a generic triplet k, n, and r by

$$\operatorname{Tr}\mathbf{T} = \frac{(-1)^r}{2}\sum_i \Delta_{ii}^2\left(\sum_{I\in\mathcal{I}_r:i\in I} F_I^2\Delta_{II} - \sum_{I\in\mathcal{I}_r:i\notin I} F_I^2\Delta_{II}\right) \tag{76}$$

$$= \frac{(-1)^r}{2}\sum_{I\in\mathcal{I}_r} F_I^2\Delta_{II}\left(\sum_{i:i\in I} 1 - \sum_{i:i\notin I} 1\right) \tag{77}$$

$$= \frac{(-1)^{r+1}}{2}(k + n - 2r)\,\mathbf{F} \cdot \mathbf{F}, \tag{78}$$

where we swapped the summation order in (77) and extracted a common factor and then we used that $\mathbf{F} \cdot \mathbf{F} = \sum_I F_I^2\Delta_{II}$ and that $\sum_{i:i\in I} 1 = r$ and that $\sum_{i:i\notin I} 1 = n + k - r$ in (78). The tensor is traceless for $k + n = 2r$ or in the case that $\mathbf{F} \cdot \mathbf{F} = 0$. To any extent, the trace is a Lorentz invariant.

4.3 Flux of the Stress-Energy-Momentum Tensor

The energy-momentum conservation law also admits an integral form, which we derive next. First, the volume integral of the Lorentz force density $\mathbf{f}$ over an $(k+n)$-dimensional hypervolume $\mathcal{V}^{k+n}$ gives the transfer of energy-momentum to the charges in that volume. From the discussion in Sect. 2.2, this volume integral is the flux of $\mathbf{f}$ over $\mathcal{V}^{k+n}$, and from the conservation law in differential form (67), we obtain the following integral representation of the energy-momentum transfer

$$\int_{\mathcal{V}^{k+n}} \mathbf{f}\, \mathrm{d}x_{0,\cdots,k+n-1} = \int_{\mathcal{V}^{k+n}} \mathrm{d}^{k+n}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{f} = -\int_{\mathcal{V}^{k+n}} \mathrm{d}^{k+n}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, (\boldsymbol{\partial} \lrcorner\, \mathbf{T}). \tag{79}$$

A short extension to the proof of (41) in [11, Sect. 3.5], included in A.4 proves a Stokes theorem for bitensors: the flux of a bitensor field $\mathbf{T}$ across the boundary $\partial\mathcal{V}^{\ell}$ of an ℓ-dimensional hypersurface $\mathcal{V}^{\ell}$ is equal to the flux of the interior derivative of $\mathbf{T}$ across $\mathcal{V}^{\ell}$ for any ℓ, and in particular for $\ell = k+n$. Using this form of the Stokes theorem, we obtain

$$\int_{\mathcal{V}^{k+n}} \mathbf{f}\, \mathrm{d}x_{0,\cdots,k+n-1} = -\int_{\partial\mathcal{V}^{k+n}} \mathrm{d}^{k+n-1}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{T}. \tag{80}$$

As an example, and for some fixed x_ℓ and $\ell \in \{0, \ldots, k+n-1\}$, consider a $(k+n)$-dimensional space-time region of the form

$$\mathcal{V}_\ell^{k+n} = (-\infty,\infty) \times (-\infty,\infty) \cdots \times (-\infty, x_\ell) \times \cdots (-\infty,\infty). \tag{81}$$

This region is a half space-time with boundary a surface with constant space-time coordinate ℓ of value x_ℓ given by

$$\partial\mathcal{V}_\ell^{k+n} = (-\infty,\infty) \times (-\infty,\infty) \cdots \times \{x_\ell\} \times \cdots (-\infty,\infty). \tag{82}$$

In the computation of the flux of $\mathbf{T}$ across the boundary $\partial\mathcal{V}^{k+n}$ in (80), the infinitesimal vector element is given by

$$\mathrm{d}^{k+n-1}\mathbf{x}^{\mathcal{H}^{-1}} = \mathrm{d}x_{\ell^c}\sigma(\ell,\ell^c)\Delta_{\ell\ell}\mathbf{e}_\ell, \tag{83}$$

where the proper orientation carries an additional factor $\sigma(\ell,\ell^c)$, the normal vector pointing outside the integration region being $\mathbf{e}_\ell$. Using (65), the flux is an integration over all space-time dimensions other than the ℓ-th,

$$\begin{aligned}\int_{\partial\mathcal{V}_\ell^{k+n}} \mathrm{d}^{k+n-1}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{T}(\mathbf{x}) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}x_{\ell^c}\sigma(\ell,\ell^c)\Delta_{\ell\ell}\mathbf{e}_\ell \lrcorner\, \mathbf{T}(\mathbf{x}) &(84)\\ &= \sum_{i=0}^{k+n-1} \mathbf{e}_i \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathrm{d}x_{\ell^c}\sigma(\ell,\ell^c)T_{i\ell}(\mathbf{x}), &(85)\end{aligned}$$

where we used the symmetry of T_{ij} between i and j and the interior product in (66). This formula represents the fact that the component $T_{i\ell}(\mathbf{x})$ characterizes the flux of the i-th component of the energy-momentum vector across a surface with constant space-time coordinate ℓ, namely the boundary of $\mathcal{V}_\ell^{k+n}$. For instance, for $r=2$, $k=1$, $n=3$, and $\ell=0$ the integrals in (85) give the energy (for $i=0$) and the momentum (for $i=1,2,3$), in line with the fact that $T_{00}(\mathbf{x})$ represents the energy density U and $T_{0i}(\mathbf{x})$ the three components of the Poynting vector $\mathbf{S}$. In this case, the time component of (67) gives the standard conservation law $\partial_0 U + \nabla\cdot\mathbf{S} = -f_0$, f_0 being the work done on the charges by the field.

The flux (85) across the surface with constant space-time coordinate ℓ can also be expressed in a relatively compact form that involves the Fourier transform of the vector potential $\hat{\mathbf{A}}(\boldsymbol{\xi})$ in the Lorenz gauge introduced in Sect. 3.2. As we prove in A.5, this flux is given by

$$\int_{\partial\mathcal{V}_\ell^{k+n}} \mathrm{d}^{k+n-1}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{T} = (-1)^r 2\pi^2 \sigma(\ell,\ell^c) \int_{\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar{\ell}}\cdot\boldsymbol{\xi}_{\bar{\ell}} \leq 0} \mathrm{d}\xi_{\ell^c} \frac{\boldsymbol{\xi}_+}{\xi_{+,\ell}} \big|\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big|^2, \tag{86}$$

where $\boldsymbol{\xi}_+$ are frequency vectors satisfying $\boldsymbol{\xi}_+\cdot\boldsymbol{\xi}_+ = 0$ given by

$$\boldsymbol{\xi}_+ = (\xi_0, \ldots, \xi_{\ell-1}, +\sqrt{-\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar{\ell}}\cdot\boldsymbol{\xi}_{\bar{\ell}}}, \xi_{\ell+1}, \ldots, \xi_{k+n-1}), \tag{87}$$

$$\boldsymbol{\xi}_{\bar{\ell}} = (\xi_0, \ldots, \xi_{\ell-1}, 0, \xi_{\ell+1}, \ldots, \xi_{k+n-1}). \tag{88}$$

This expression is reminiscent of the one appearing in the second quantization of the electromagnetic field [15, Sect. 2], where the Hamiltonian, i. e. $i = \ell = 0$, is picked as the starting point for the quantization procedure, and the independent degrees of freedom of the vector potential are quantized. Eq. (86) suggests that quantization of the vector potential does not require starting with the Hamiltonian. Having said this, a possible quantization of the generalized Maxwell equations would have to properly deal with the gauge independence of the theory. There exist several methods of deriving gauge-independent quantum Maxwell equations, such as path-integral and BRST quantizations. Studying the applicability of these methods to the exterior-algebra multivectors is beyond the scope of this work.

We conclude this section with a brief discussion on the number of degrees of freedom in the Maxwell field in the absence of sources. Eq. (86) expresses the change of energy-momentum as a linear superposition of the squared modulus of the Fourier transform of the vector potential $\hat{\mathbf{A}}(\boldsymbol{\xi})$. If both k and n are positive, for any frequency vector $\boldsymbol{\xi}$, we may choose a transverse gauge where not only $\boldsymbol{\xi} \lrcorner \hat{\mathbf{A}} = 0$ is satisfied, but also $\boldsymbol{\xi}_{\mathrm{t}} \lrcorner \hat{\mathbf{A}} = \boldsymbol{\xi}_{\mathrm{s}} \lrcorner \hat{\mathbf{A}} = 0$, where $\boldsymbol{\xi}_{\mathrm{t}}$ and $\boldsymbol{\xi}_{\mathrm{s}}$ respectively represent the time and space components of $\boldsymbol{\xi}$. The Fourier vector potential $\hat{\mathbf{A}}(\boldsymbol{\xi})$ is thus perpendicular to the vectors $\boldsymbol{\xi}_{\mathrm{t}}$ and $\boldsymbol{\xi}_{\mathrm{s}}$, and the number of available dimensions of space-time where $\hat{\mathbf{A}}(\boldsymbol{\xi})$ lies is reduced to a total of $k+n-2$. The number of independent components of $\hat{\mathbf{A}}(\boldsymbol{\xi})$, or degrees of freedom, is therefore $\binom{k+n-2}{r-1}$, e. g. two polarizations for $r = 2$, $k = 1$, and $n = 3$. Moreover, these independent degrees of freedom have the form of propagating waves in space-time.

5 Conclusions and Future Work

In this paper, we have put forward exterior algebra and calculus as a natural setting for an abstract yet intuitively simple form of generalized Maxwell equations, Lorentz force density, and stress-energy tensor for generalized electromagnetic fields (or Maxwell fields) represented by multivectors of grade r in a space-time with an arbitrary number of dimensions. The source density is modeled as a multivector of grade $r-1$. The phenomenological description of the charges associated to these source densities will be done elsewhere.

The generalized Maxwell equations are given in terms of exterior-calculus operations. In differential form, the homogeneous Maxwell equation states that the exterior derivative of the Maxwell field is zero; the inhomogeneous Maxwell equation states that the interior derivative of the Maxwell field is the source density. In integral form, the homogeneous equation states that the circulation of the Maxwell field along the boundary of any $(r+1)$-dimensional space-time volume is zero; the inhomogeneous equation states that the flux of the Maxwell field across the boundary of any $(k+n-r+1)$-dimensional space-time hypervolume is equal to the flux of the current density across the same hypervolume. The Lorentz force density is given by the left interior product of the source density and the Maxwell field. Moreover, a conservation law relates this Lorentz force density with the interior derivative of the stress-energy-momentum tensor of the Maxwell field.

Among several applications, a simple expression for the flux of the stress-energy-momentum tensor across an slice of space-time with a constant coordinate is given in terms of the Fourier transform of the potentials. The description of Maxwell fields is classical, as energy and momentum carried by the fields are continuous rather than discrete, in contrast with the experimental observations in the ordinary space-time. There exist several methods of postulating gauge-independent quantum Maxwell equations, such as second or canonical quantization or path integral quantization. A study of the applicability of these methods to the generalized Maxwell equations is left open for future work.

A Proofs Related to the Stress-Energy-Momentum Tensor

A.1 Distributive Properties of the Interior Product

Proof of (22) To prove this equation, we first expand the vectors and multivectors in the left-hand side of (22) in terms of their components to get

$$(\mathbf{v}\wedge\mathbf{w})\cdot(\mathbf{w}'\wedge\mathbf{v}') = \left(\sum_{i,I:i\notin I} v_i w_I \sigma(i,I)\mathbf{e}_{i+I}\right)\cdot\left(\sum_{j,J:j\notin J} v'_j w'_J \sigma(J,j)\mathbf{e}_{j+J}\right) \tag{89}$$

$$= \sum_{\substack{i,I:i\notin I\\ j,J:j\notin J}} v_i w_I v'_j w'_J \sigma(i,I)\sigma(J,j)\Delta_{i+I,j+J}. \tag{90}$$

At this point, we separate the cases $i=j$ and $i\neq j$, namely

$$(\mathbf{v}\wedge\mathbf{w})\cdot(\mathbf{w}'\wedge\mathbf{v}') = \sum_{i,I} v_i v'_i w_I w'_I \sigma(i,I)\sigma(I,i)\Delta_{ii}\Delta_{ii} + \sum_{i\neq j,I,J} v_i v'_j w_I w'_J \sigma(i,I)\sigma(J,j)\Delta_{ii}\Delta_{jj}\Delta_{I\setminus j,J\setminus i}, \tag{91}$$

and it is easy to note that the first term can be written as

$$\sum_{i,I} v_i v'_i w_I w'_I \sigma^2(i,I)(-1)^r\Delta_{ii}\Delta_{ii} = (-1)^r\,(\mathbf{v}\cdot\mathbf{v}')\,(\mathbf{w}\cdot\mathbf{w}')\,. \tag{92}$$

Regarding the second term, we first prove the equality

$$\sigma(i,I)\sigma(J,j) = \sigma(I\setminus j,j)\sigma(i,J\setminus i), \tag{93}$$

which we can be rewritten in the form

$$\sigma(I\setminus j,j)\sigma(i,I) = \sigma(i,J\setminus i)\sigma(J,j). \tag{94}$$

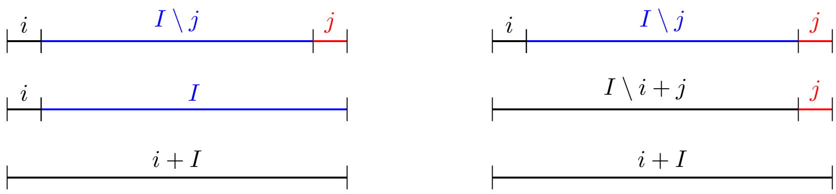

Figure 1: Visual aid for the identity among permutations in Equation (94).

In fact, with the visual help of Figure 1 it is easy to see how the sign of the permutation on the left-hand side of (94) is the same as that on the right-hand side. Then, the second term of the expression (91) can be written

$$\sum_{i\neq j,I,J} v'_j w_I \sigma(I\setminus j,j)\Delta_{jj}\, v_i w'_J \sigma(i,J\setminus i)\Delta_{ii}\,\Delta_{I\setminus j,J\setminus i} = (\mathbf{v}'\lrcorner\,\mathbf{w})\cdot(\mathbf{w}'\llcorner\,\mathbf{v}). \tag{95}$$

□

Proof of (24) To prove this identity we write separately the explicit expressions of the left and the right side of (24). On the left side we get

$$\mathbf{u}\lrcorner\,(\mathbf{w}\llcorner\,\mathbf{v}) = \left(\sum_a u_a\mathbf{e}_a\right)\lrcorner\left(\sum_{b\in I} v_b w_I\Delta_{b,b}\sigma(b,I\setminus b)\mathbf{e}_{I\setminus b}\right) \tag{96}$$

$$= \sum_{\substack{a,b\in I\\ a\neq b}} u_a v_b w_I \Delta_{a,a}\Delta_{b,b}\sigma(b,I\setminus b)\sigma(I\setminus b\setminus a,a)\mathbf{e}_{I\setminus b\setminus a}, \tag{97}$$

while on the right side we result is

$$(\mathbf{u} \lrcorner \mathbf{w}) \llcorner \mathbf{v} = \left(\sum_{a \in I} u_a w_I \Delta_{a,a} \sigma(I \setminus a, a) \mathbf{e}_{I \setminus a} \right) \llcorner \left(\sum_b v_b \mathbf{e}_b \right) \tag{98}$$

$$= \sum_{\substack{a,b \in I \\ a \neq b}} u_a v_b w_I \Delta_{a,a} \Delta_{b,b} \sigma(I \setminus a, a) \sigma(b, I \setminus a \setminus b) \mathbf{e}_{I \setminus a \setminus b}. \tag{99}$$

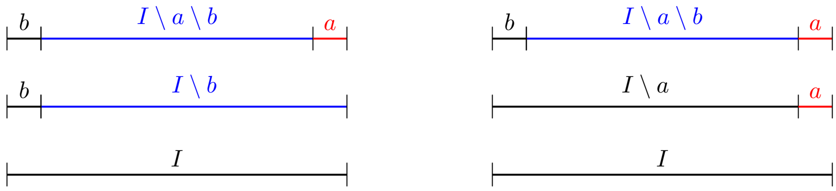

Figure 2: Visual aid for the identity among permutations in Equation (100).

For the expressions (97) and (99) to be identical, it is sufficient that

$$\sigma(I \setminus b \setminus a, a) \sigma(b, I \setminus b) = \sigma(b, I \setminus a \setminus b) \sigma(I \setminus a, a) \tag{100}$$

is satisfied. With the aid of Figure 2, we notice that both sides represent the signature of two possible permutations reordering the list $(b, I \setminus a \setminus b, a)$ into I. This proves (24). □

Proof of (27) We prove this relation by first writing separately the three terms of the equation. On the left-hand side, we get

$$(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = \left(\sum_{i,I} u_i v_I \sigma(i, I) \mathbf{e}_{i+I} \right) \cdot \left(\sum_J w_J \mathbf{e}_J \right) \tag{101}$$

$$= \sum_{\substack{i,I,J \\ i+I=J}} u_i v_I w_J \sigma(i, I) \Delta_{i+I,J}. \tag{102}$$

The central term is

$$\mathbf{v} \cdot (\mathbf{w} \llcorner \mathbf{u}) = \left(\sum_I v_I \mathbf{e}_I \right) \cdot \left(\sum_{i,J} u_i w_I \sigma(i, J \setminus i) \Delta_{ii} \mathbf{e}_{J \setminus i} \right) \tag{103}$$

$$= \sum_{\substack{i,I,J \\ J \setminus i = I}} u_i v_I w_J \sigma(i, I) \Delta_{ii} \Delta_{J \setminus i, I}. \tag{104}$$

Thus, it is easy to check that $\Delta_{i+I,J} = \Delta_{ii} \Delta_{J \setminus i, I}$, so that the first two terms of (27) coincide. Regarding the third term on the left hand side, we obtain

$$\mathbf{u} \cdot (\mathbf{v} \lrcorner \mathbf{w}) = \left(\sum_i u_i \mathbf{e}_i \right) \cdot \left(\sum_{I,J} v_I w_J \sigma(J \setminus I, I) \Delta_{II} \mathbf{e}_{J \setminus I} \right) \tag{105}$$

$$= \sum_{\substack{i,I,J \\ i = J \setminus I}} u_i v_I w_J \sigma(i, I) \Delta_{i, J \setminus I} \Delta_{II}, \tag{106}$$

which corresponds to the first two expressions since $\Delta_{i,J \setminus I} \Delta_{II} = \Delta_{i+I,J} = \Delta_{ii} \Delta_{J \setminus i, I}$. □

A.2 Interior Derivative of the Tensor

For the sake of compactness, we define the bitensors $\mathbf{T}_{\odot} = \mathbf{F} \odot \mathbf{F}$ and $\mathbf{T}_{\owedge} = \mathbf{F} \owedge \mathbf{F}$ to prove the following identities

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\odot} = (\boldsymbol{\partial} \lrcorner \mathbf{F}) \lrcorner \mathbf{F} \tag{107}$$

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\owedge} = (\boldsymbol{\partial} \wedge \mathbf{F}) \llcorner \mathbf{F}. \tag{108}$$

Using equations (13), (14), (16) and (17), we write $\mathbf{T}_\odot$ and $\mathbf{T}_\owedge$ explicitly in terms of components. That is, we obtain

$$\mathbf{T}_\odot = \frac{1}{2}\sum_{i\le j}\Delta_{ii}\Delta_{jj}(\mathbf{e}_i \rfloor \mathbf{F})\cdot(\mathbf{F} \lfloor \mathbf{e}_j)\,\mathbf{u}_{ij} \tag{109}$$

$$= \frac{1}{2}\sum_{i\le j}\sum_{I,J\in\mathcal{I}_r} F_I F_J\,\sigma(I\setminus i,i)\sigma(j,J\setminus j)\Delta_{I\setminus i,J\setminus j}\,\mathbf{u}_{ij} \tag{110}$$

and

$$\mathbf{T}_\owedge = \frac{1}{2}\sum_{i\le j}\Delta_{ii}\Delta_{jj}(\mathbf{e}_i \wedge \mathbf{F})\cdot(\mathbf{F} \wedge \mathbf{e}_j)\,\mathbf{u}_{ij} \tag{111}$$

$$= \frac{1}{2}\sum_{i\le j}\sum_{I,J\in\mathcal{I}_r} F_I F_J\,\sigma(i,I)\sigma(J,j)\Delta_{ii}\Delta_{jj}\Delta_{I+i,J+j}\,\mathbf{u}_{ij}. \tag{112}$$

We start by computing the interior derivative $\boldsymbol{\partial} \rfloor \mathbf{T}_\odot$ given by

$$\boldsymbol{\partial} \rfloor \mathbf{T}_\odot = \frac{1}{2}\sum_u\sum_{i\le j}\sum_{I,J}(F_I\partial_u F_J + F_J\partial_u F_I)$$
$$\sigma(I\setminus i,i)\sigma(j,J\setminus j)\Delta_{I\setminus i,J\setminus j}\Delta_{uu}\,\mathbf{e}_u \rfloor \mathbf{u}_{ij}. \tag{113}$$

After some mathematical manipulations, equation (113) is expanded as

$$\begin{aligned}\boldsymbol{\partial} \rfloor \mathbf{T}_\odot &= \sum_{i=j}\sum_{I,J} F_I\partial_i F_J\sigma(I\setminus i,i)\sigma(j,J\setminus j)\Delta_{I\setminus i,J\setminus j}\,\mathbf{e}_i \\ &+ \frac{1}{2}\sum_{i<j}\sum_{I,J}(F_I\partial_j F_J + F_J\partial_j F_I)\sigma(I\setminus i,i)\sigma(j,J\setminus j)\Delta_{I\setminus i,J\setminus j}\,\mathbf{e}_i \\ &+ \frac{1}{2}\sum_{i<j}\sum_{I,J}(F_I\partial_i F_J + F_J\partial_i F_I)\sigma(I\setminus i,i)\sigma(j,J\setminus j)\Delta_{I\setminus i,J\setminus j}\,\mathbf{e}_j.\end{aligned} \tag{114}$$

Since $\Delta_{I\setminus i,J\setminus j}$ is nonzero only if $J\setminus j = I\setminus i$, we can use this condition in the relation $\sigma(j,J\setminus j) = \sigma(j,I\setminus i) = \sigma(I\setminus i,j)(-1)^{r-1}$ such that (114) becomes

$$\begin{aligned}\boldsymbol{\partial} \rfloor \mathbf{T}_\odot &= \sum_{i=j}\sum_{I,J} F_I\partial_i F_J\sigma(I\setminus i,i)\sigma(I\setminus i,j)(-1)^{r-1}\Delta_{I\setminus i,J\setminus j}\,\mathbf{e}_i \\ &+ \frac{1}{2}\sum_{i<j}\sum_{I,J}(F_I\partial_j F_J + F_J\partial_j F_I)\sigma(I\setminus i,i)\sigma(I\setminus i,j)(-1)^{r-1}\Delta_{I\setminus i,J\setminus j}\,\mathbf{e}_i \\ &+ \frac{1}{2}\sum_{i>j}\sum_{I,J}(F_I\partial_j F_J + F_J\partial_j F_I)\sigma(I\setminus i,i)\sigma(I\setminus i,j)(-1)^{r-1}\Delta_{I\setminus i,J\setminus j}\,\mathbf{e}_i.\end{aligned} \tag{115}$$

Exchanging the indices i and j and the labels I and J, we get the following simplified expression

$$\boldsymbol{\partial} \rfloor \mathbf{T}_\odot = \sum_{i,j}\sum_{I,J} F_I\partial_j F_J\,\sigma(I\setminus i,i)\sigma(I\setminus i,j)\Delta_{I\setminus i,J\setminus j}(-1)^{r-1}\,\mathbf{e}_i. \tag{116}$$

A similar reasoning can be followed for the operation $\boldsymbol{\partial} \rfloor \mathbf{T}_\owedge$ given by

$$\boldsymbol{\partial} \rfloor \mathbf{T}_\owedge = \frac{1}{2}\sum_u\sum_{i\le j}\sum_{I,J}(F_I\partial_u F_J + F_J\partial_u F_I)\,\sigma(i,I)\sigma(J,j)$$
$$\Delta_{I+i,J+j}\Delta_{u,u}\Delta_{i,i}\Delta_{j,j}\,\mathbf{e}_u \rfloor \mathbf{u}_{ij}. \tag{117}$$

The former equation can be expanded as

$$\begin{aligned}\boldsymbol{\partial} \rfloor \mathbf{T}_\owedge &= \sum_{i=j}\sum_{I,J} F_I\partial_i F_J\sigma(i,I)\sigma(J,j)\Delta_{I+i,J+j}\,\mathbf{e}_i \\ &+ \frac{1}{2}\sum_{i<j}\sum_{I,J}(F_I\partial_j F_J + F_J\partial_j F_I)\sigma(i,I)\sigma(J,j)\Delta_{I+i,J+j}\Delta_{i,i}\Delta_{j,j}\,\mathbf{e}_i \\ &+ \frac{1}{2}\sum_{i<j}\sum_{I,J}(F_I\partial_i F_J + F_J\partial_i F_I)\sigma(i,I)\sigma(J,j)\Delta_{I+i,J+j}\Delta_{i,i}\Delta_{j,j}\,\mathbf{e}_j,\end{aligned} \tag{118}$$

and after a few manipulations using the properties of the signatures as done to obtain (115), from (118) we have

$$\begin{aligned}
\boldsymbol{\partial} \rfloor \mathbf{T}_{\oslash} = &\sum_{i=j} \sum_{\substack{I,J \\ I+i=J+j}} F_I \partial_i F_J \sigma(i,I)\sigma(j,J)(-1)^r \Delta_{I,I}\Delta_{j,j}\,\mathbf{e}_i \\
&+ \frac{1}{2}\sum_{i<j} \sum_{\substack{I,J \\ I+i=J+j}} (F_I \partial_j F_J + F_J \partial_j F_I)\sigma(i,I)\sigma(j,J)(-1)^r \Delta_{I,I}\Delta_{j,j}\,\mathbf{e}_i \\
&+ \frac{1}{2}\sum_{i>j} \sum_{\substack{I,J \\ I+i=J+j}} (F_I \partial_j F_J + F_J \partial_j F_I)\sigma(i,I)\sigma(j,J)(-1)^r \Delta_{I,I}\Delta_{j,j}\,\mathbf{e}_i.
\end{aligned} \tag{119}$$

Simplifying terms, we obtain

$$\boldsymbol{\partial} \rfloor \mathbf{T}_{\oslash} = \sum_{i,j} \sum_{\substack{I,J \\ I+i=J+j}} F_I \partial_j F_J\, \sigma(i,I)\sigma(j,J)\Delta_{I,I}\Delta_{j,j}(-1)^r\,\mathbf{e}_i. \tag{120}$$

We next derive explicit forms for the operations $(\boldsymbol{\partial} \rfloor \mathbf{F}) \rfloor \mathbf{F}$ and $(\boldsymbol{\partial} \wedge \mathbf{F}) \lfloor \mathbf{F}$. We start by noting that $\boldsymbol{\partial} \rfloor \mathbf{F}$ can be expanded in terms of component as

$$\boldsymbol{\partial} \rfloor \mathbf{F} = \left(\sum_j \Delta_{j,j}\partial_j \mathbf{e}_j\right) \rfloor \left(\sum_B F_B \mathbf{e}_B\right) \tag{121}$$

$$= \sum_{\substack{B,j \\ j\in B}} \partial_j F_B \sigma(B\setminus j, j)\,\mathbf{e}_{B\setminus j}. \tag{122}$$

As a consequence, we have that

$$(\boldsymbol{\partial} \rfloor \mathbf{F}) \rfloor \mathbf{F} = \sum_{\substack{A,B,j \\ (B\setminus j)\in A}} F_A(\partial_j F_B)\sigma(B\setminus j, j)\Delta_{B\setminus j, B\setminus j}\sigma(A\setminus(B\setminus j), B\setminus j)\,\mathbf{e}_{A\setminus(B\setminus j)}. \tag{123}$$

Noting that $A\setminus(B\setminus j)$ is a single element i such that $A\setminus i = B\setminus j$, we finally get

$$(\boldsymbol{\partial} \rfloor \mathbf{F}) \rfloor \mathbf{F} = \sum_{i,j} \sum_{\substack{A,B \\ A\setminus i=B\setminus j}} F_A(\partial_j F_B)\sigma(A\setminus i, j)\sigma(i, A\setminus i)\Delta_{A\setminus i, A\setminus i}\,\mathbf{e}_i \tag{124}$$

$$= \sum_{i,j} \sum_{\substack{A,B \\ A\setminus i=B\setminus j}} F_A(\partial_j F_B)\sigma(A\setminus i, j)\sigma(A\setminus i, i)(-1)^{r-1}\Delta_{A\setminus i, A\setminus i}\,\mathbf{e}_i. \tag{125}$$

Since (125) corresponds exactly to (116), we proved (107).

Similarly, we next write the operation $(\boldsymbol{\partial} \wedge \mathbf{F}) \lfloor \mathbf{F}$ and write it out in terms of components, i. e.,

$$(\boldsymbol{\partial} \wedge \mathbf{F}) \lfloor \mathbf{F} = \left(\sum_{\substack{j,B \\ j\notin B}} \Delta_{j,j}\partial_j F_B \sigma(j,B)\mathbf{e}_{j+B}\right) \lfloor \left(\sum_A F_A \mathbf{e}_A\right) \tag{126}$$

$$= \sum_{\substack{j,A,B \\ A\in j+B}} \Delta_{j,j} F_A \partial_j F_B \sigma(j,B)\Delta_{A,A}\sigma(A, j+B\setminus A)\,\mathbf{e}_{j+B\setminus A}. \tag{127}$$

Again noting that $j+B\setminus A = i$ and that $A+i = B+j$, from (127) we obtain

$$(\boldsymbol{\partial} \wedge \mathbf{F}) \lfloor \mathbf{F} = \sum_{i,j} \sum_{\substack{A,B \\ A+i=B+j}} F_A \partial_j F_B\, \sigma(i,A)\sigma(j,B)(-1)^r \Delta_{A,A}\Delta_{j,j}\,\mathbf{e}_i, \tag{128}$$

proving the equivalence with (120) and therefore proving (108).

Combining (107) and (108) and defining the stress-energy-momentum tensor $\mathbf{T}$ of the Maxwell field $\mathbf{F}$ as $\mathbf{T} = -(\mathbf{T}_{\odot} + \mathbf{T}_{\oslash})$, we find that the interior derivative of the tensor $\mathbf{T}$ satisfies the following formula

$$\boldsymbol{\partial} \rfloor \mathbf{T} + (\boldsymbol{\partial} \rfloor \mathbf{F}) \rfloor \mathbf{F} + (\boldsymbol{\partial} \wedge \mathbf{F}) \lfloor \mathbf{F} = 0. \tag{129}$$

Therefore, from the Maxwell equations, that is $\boldsymbol{\partial} \wedge \mathbf{F} = 0$ and $\boldsymbol{\partial} \rfloor \mathbf{F} = \mathbf{J}$, we recover the conservation law for energy-momentum relating the Lorentz force $\mathbf{f}$ (64) and the stress-energy-momentum tensor,

$$\mathbf{f} + \boldsymbol{\partial} \rfloor \mathbf{T} = 0. \tag{130}$$

A.3 Explicit Formulas for the Tensor Components for Generic r

Starting with (69), we note that

$$\mathbf{e}_i \,\lrcorner\, \mathbf{F} = \sum_{I \in \mathcal{I}_r} \Delta_{ii} \sigma(I \setminus i, i) F_I \mathbf{e}_{I \setminus i} \tag{131}$$

$$\mathbf{F} \,\llcorner\, \mathbf{e}_j = \sum_{J \in \mathcal{I}_r} \Delta_{jj} \sigma(j, J \setminus j) F_J \mathbf{e}_{J \setminus j}, \tag{132}$$

for any pair of i and j, and therefore

$$\mathbf{F} \odot \mathbf{F}\big|_{ij} = \frac{1}{2} \sum_{I,J \in \mathcal{I}_r} \sigma(I \setminus i, i)\sigma(j, J \setminus j) F_I F_J \mathbf{e}_{I \setminus i} \cdot \mathbf{e}_{J \setminus j} \tag{133}$$

$$= \frac{1}{2} \sum_{L \in \mathcal{I}_{r-1} : i,j \notin L} \sigma(L, i)\sigma(j, L) F_{i+L} F_{j+L} \Delta_{LL}, \tag{134}$$

where we have defined the set L such that $I \setminus i = J \setminus j = L$. The summation contains $\binom{k+n-2}{r-1}$ non-zero terms. For $i = j$, and using that $\sigma(L, i)\sigma(i, L) = (-1)^{r-1}$, it can be evaluated as

$$\mathbf{F} \odot \mathbf{F}\big|_{ii} = (-1)^{r-1} \frac{1}{2} \sum_{L \in \mathcal{I}_{r-1} : i \notin L} F_{i+L}^2 \Delta_{LL}, \tag{135}$$

Moving on to (70), we note that

$$\mathbf{e}_i \wedge \mathbf{F} = \sum_{I \in \mathcal{I}_r} F_I \sigma(i, I) \mathbf{e}_{i+I} \tag{136}$$

$$\mathbf{F} \wedge \mathbf{e}_j = \sum_{J \in \mathcal{I}_r} F_J \sigma(J, j) \mathbf{e}_{j+J}, \tag{137}$$

and we study the cases such that the lists $i + I$ and $j + J$ coincide. First, if $i = j$, we have to sum over $I = J$ such that $i \notin I$, that is

$$\mathbf{F} \owedge \mathbf{F}\big|_{ii} = \frac{1}{2} \sum_{I \in \mathcal{I}_r : i \notin I} F_I^2 \sigma(i, I)\sigma(I, i) \Delta_{ii} \Delta_{II} \tag{138}$$

$$= (-1)^r \frac{1}{2} \sum_{I \in \mathcal{I}_r : i \notin I} F_I^2 \Delta_{ii} \Delta_{II}, \tag{139}$$

because $\sigma(i, I)\sigma(I, i) = (-1)^r$. Second, if $i \neq j$, we can find an $L \in \mathcal{I}_{r-1}$ such that $L = I \setminus j = J \setminus i$ and $i, j \neq L$. Then,

$$\mathbf{F} \owedge \mathbf{F}\big|_{ij} = \frac{1}{2} \Delta_{ii} \Delta_{jj} \sum_{L \in \mathcal{I}_{r-1}} F_{L+j} F_{L+i} \sigma(i, L+j) \sigma(L+i, j) \mathbf{e}_{j+L+i} \cdot \mathbf{e}_{j+L+i} \tag{140}$$

$$= \frac{1}{2} \sum_{L \in \mathcal{I}_{r-1}} F_{L+j} F_{L+i} \sigma(i, L+j) \sigma(L+i, j) \Delta_{LL}. \tag{141}$$

The product $\sigma(i, L+j)\sigma(L+i, j)$ is equal to $\sigma(L, i)\sigma(j, L)$. To prove it, we write the relation $\sigma(i, L+j)\sigma(L+i, j) = \sigma(L, i)\sigma(j, L)$ and we first multiply both sides by $\sigma(L+i, j)\sigma(j, L)$ to obtain $\sigma(i, L+j)\sigma(j, L) = \sigma(L+i, j)\sigma(L, i)$, namely the permutations sorting the lists (i, j, L) and (L, i, j) respectively. Secondly, we note that $\sigma(i, j, L) = (-1)^{2(r-1)} \sigma(L, i, j) = \sigma(L, i, j)$.

Combining (134) and (141) into $\mathbf{T}\big|_{ij} = -\mathbf{F} \odot \mathbf{F}\big|_{ij} - \mathbf{F} \owedge \mathbf{F}\big|_{ij}$, we have

$$T_{ii} = \frac{(-1)^r}{2} \Delta_{ii} \left(\sum_{I \in \mathcal{I}_r : i \in I} F_I^2 \Delta_{II} - \sum_{I \in \mathcal{I}_r : i \notin I} F_I^2 \Delta_{II} \right) \tag{142}$$

$$T_{ij} = - \sum_{L \in \mathcal{I}_{r-1}} \sigma(L, i)\sigma(j, L) F_{i+L} F_{j+L} \Delta_{LL}. \tag{143}$$

A.4 Stokes Theorem for the Interior Derivative of a Bitensor

Considering a bitensor field

$$\mathbf{T} = \sum_i T_{ii}\mathbf{u}_{ii} + \sum_{i<j} T_{ij}\mathbf{u}_{ij}, \tag{144}$$

then the Stokes theorem we wish to prove states that

$$\int_{\partial\mathcal{V}^{n+k}} \mathrm{d}^{n+k-1}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{T} = \int_{\mathcal{V}^{n+k}} \left(\mathrm{d}^{n+k}x\right)^{\mathcal{H}^{-1}} \lrcorner\, (\partial \lrcorner\, \mathbf{T}). \tag{145}$$

The proof will follow the reasoning operating for the vector field [11, Sect. 3.5], so we start expanding the integrand on the right-hand side

$$\mathrm{d}^{n+k-1}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{T} = \left(\sum_{I_{n+k-1}:q\notin I} \mathrm{d}x_I \Delta_{qq}\sigma(q,I)\mathbf{e}_q\right) \lrcorner \left(\sum_i T_{ii}\mathbf{u}_{ii} + \sum_{i<j} T_{ij}\mathbf{u}_{ij}\right) \tag{146}$$

$$= \sum_{i,I_{n+k-1}:i\notin I} T_{ii}\,\mathrm{d}x_I\sigma(i,I)\,\mathbf{e}_i + \sum_{i<j}\sum_{I_{n+k-1}:j\notin I} T_{ij}\,\mathrm{d}x_I\sigma(j,I)\,\mathbf{e}_i + \sum_{i<j}\sum_{I_{n+k-1}:i\notin I} T_{ij}\,\mathrm{d}x_I\sigma(i,I)\,\mathbf{e}_j. \tag{147}$$

The last term can be rewritten by changing the indices $i \longleftrightarrow j$ and using the property of symmetry $T_{ji} = T_{ij}$ as

$$\sum_{i>j}\sum_{I_{n+k-1}:j\notin I} T_{ij}\,\mathrm{d}x_I\sigma(j,I)\,\mathbf{e}_i, \tag{148}$$

and finally, the three terms can be compacted in

$$\mathrm{d}^{n+k-1}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{T} = \sum_{i,j}\sum_{I_{n+k-1}:j\notin I} T_{ij}\,\mathrm{d}x_I\sigma(j,I)\,\mathbf{e}_i. \tag{149}$$

Then, taking the exterior derivative [14, Sect. 36.B], we get

$$\mathrm{d}\left(\mathrm{d}^{n+k-1}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, \mathbf{T}\right) = \sum_{i,j}\sum_{I_{n+k-1}:j\notin I} \partial_j T_{ij}\,\mathrm{d}x_{\varepsilon(j,I)}\sigma^2(j,I)\,\mathbf{e}_i \tag{150}$$

$$= \sum_{i,j}\sum_{L_{n+k}} \partial_j T_{ij}\,\mathrm{d}x_L\,\mathbf{e}_i. \tag{151}$$

On the other hand, regarding the right-hand side, we first evaluate

$$\partial \lrcorner\, \mathbf{T} = \left(\sum_h \Delta_{hh}\partial_h\mathbf{e}_h\right) \lrcorner \left(\sum_i T_{ii}\mathbf{u}_{ii} + \sum_{i<j} T_{ij}\mathbf{u}_{ij}\right) \tag{152}$$

$$= \sum_i \partial_i T_{ii}\mathbf{e}_i + \sum_{i<j}\partial_j T_{ij}\mathbf{e}_i + \sum_{i<j}\partial_i T_{ij}\mathbf{e}_j \tag{153}$$

$$= \sum_{i,j}\partial_j T_{ij}\mathbf{e}_i \tag{154}$$

and the differential

$$\left(\mathrm{d}^{n+k}\mathbf{x}\right)^{\mathcal{H}^{-1}} = \left(\sum_{L_{n+k}} \mathrm{d}x_L\mathbf{e}_L\right)^{\mathcal{H}^{-1}} = \sum_{L_{n+k}} \mathrm{d}x_L. \tag{155}$$

Then, we get

$$\left(\mathrm{d}^{n+k}\mathbf{x}\right)^{\mathcal{H}^{-1}} \lrcorner\, (\partial \lrcorner\, \mathbf{T}) = \sum_{L_{n+k}}\sum_{i,j}\partial_j T_{ij}\,\mathrm{d}x_L\,\mathbf{e}_i, \tag{156}$$

namely (151) and thereby proving the stated Stokes' Theorem.

A.5 Flux of the Stress-Energy-Momentum Tensor

The flux of the field (65) across the boundary $\partial\mathcal{V}_\ell^{k+n}$, denoted by $\Phi_{\partial\mathcal{V}_\ell^{k+n}}(\mathbf{T})$, is given by the integral in (85),

$$\Phi_{\partial\mathcal{V}_\ell^{k+n}}(\mathbf{T}) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \mathrm{d}x_{\ell^c}\,\Delta_{\ell\ell}\sigma(\ell,\ell^c)\,\mathbf{e}_\ell \,\lrcorner\, \mathbf{T}. \tag{157}$$

The r.h.s. of (157) is computed w.r.t. x_{ℓ^c}, being ℓ^c the set of indices excluding ℓ.

We next write the flux (157) in terms of the Fourier transform of $\mathbf{F}$, denoted as $\hat{\mathbf{F}}(\boldsymbol{\xi})$ as in (51). Assuming that the Fourier transform of $\mathbf{F}$ is supported only in the set $\boldsymbol{\xi}\cdot\boldsymbol{\xi} = 0$, as postulated in (57), we express $\mathbf{F}$ as

$$\mathbf{F}(\mathbf{x}) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \mathrm{d}^{k+n}\boldsymbol{\xi}\,\delta(\boldsymbol{\xi}\cdot\boldsymbol{\xi})\,e^{j2\pi\boldsymbol{\xi}\cdot\mathbf{x}}\,\hat{\mathbf{F}}(\boldsymbol{\xi}). \tag{158}$$

Inserting (158) in (68) and using the linearity properties of $\odot$ and $\oslash$, we obtain that the stress-energy-momentum tensor $\mathbf{T}$ can be written as

$$\mathbf{T} = -\frac{1}{2}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \mathrm{d}^{k+n}\boldsymbol{\xi}\,\mathrm{d}^{k+n}\boldsymbol{\xi}'\,\delta(\boldsymbol{\xi}\cdot\boldsymbol{\xi})\delta(\boldsymbol{\xi}'\cdot\boldsymbol{\xi}')\,e^{j2\pi(\boldsymbol{\xi}+\boldsymbol{\xi}')\cdot\mathbf{x}}\big(\hat{\mathbf{F}}(\boldsymbol{\xi})\odot\hat{\mathbf{F}}(\boldsymbol{\xi}')+\hat{\mathbf{F}}(\boldsymbol{\xi})\oslash\hat{\mathbf{F}}(\boldsymbol{\xi}')\big). \tag{159}$$

Using (159) back in (157), we obtain that

$$\begin{aligned}\Phi_{\partial\mathcal{V}_\ell^{k+n}}(\mathbf{T}) = -\frac{1}{2}\Delta_{\ell\ell}\sigma(\ell,\ell^c)\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} &\mathrm{d}^{k+n}\boldsymbol{\xi}\,\mathrm{d}^{k+n}\boldsymbol{\xi}'\,\mathrm{d}x_{\ell^c}\\ &\delta(\boldsymbol{\xi}\cdot\boldsymbol{\xi})\delta(\boldsymbol{\xi}'\cdot\boldsymbol{\xi}')\,e^{j2\pi(\boldsymbol{\xi}+\boldsymbol{\xi}')\cdot\mathbf{x}}\mathbf{e}_\ell\,\lrcorner\,\big(\hat{\mathbf{F}}(\boldsymbol{\xi})\odot\hat{\mathbf{F}}(\boldsymbol{\xi}')+\hat{\mathbf{F}}(\boldsymbol{\xi})\oslash\hat{\mathbf{F}}(\boldsymbol{\xi}')\big).\end{aligned} \tag{160}$$

Since the integration w.r.t. x_{ℓ^c} only acts on the exponential term in (160), interchanging the integration order and using the definition of the delta function, we can find the inverse Fourier transform of the exponential as

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \mathrm{d}x_{\ell^c}\,e^{j2\pi(\boldsymbol{\xi}+\boldsymbol{\xi}')\cdot\mathbf{x}} = \prod_{m\neq\ell}\delta(\xi_m+\xi'_m)e^{j2\pi(\xi_\ell+\xi'_\ell)x_\ell\Delta_{\ell\ell}}. \tag{161}$$

Plugging the r.h.s. of (161) in (160) and defining $\varphi(\boldsymbol{\xi},\boldsymbol{\xi}')$ as

$$\varphi(\boldsymbol{\xi},\boldsymbol{\xi}') = -\frac{1}{2}\Delta_{\ell\ell}\sigma(\ell,\ell^c)e^{j2\pi(\xi_\ell+\xi'_\ell)x_\ell\Delta_{\ell\ell}}\mathbf{e}_\ell\,\lrcorner\,\big(\hat{\mathbf{F}}(\boldsymbol{\xi})\odot\hat{\mathbf{F}}(\boldsymbol{\xi}')+\hat{\mathbf{F}}(\boldsymbol{\xi})\oslash\hat{\mathbf{F}}(\boldsymbol{\xi}')\big), \tag{162}$$

we write the flux as

$$\Phi_{\partial\mathcal{V}_\ell^{k+n}}(\mathbf{T}) = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \mathrm{d}^{k+n}\boldsymbol{\xi}\,\mathrm{d}^{k+n}\boldsymbol{\xi}'\,\delta(\boldsymbol{\xi}\cdot\boldsymbol{\xi})\delta(\boldsymbol{\xi}'\cdot\boldsymbol{\xi}')\prod_{m\neq\ell}\delta(\xi_m+\xi'_m)\varphi(\boldsymbol{\xi},\boldsymbol{\xi}'). \tag{163}$$

In order to solve the integration w.r.t. ξ_ℓ, we rewrite the condition $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=0$ as

$$\Delta_{\ell\ell}\xi_\ell^2 + \boldsymbol{\xi}_{\bar{\ell}}\cdot\boldsymbol{\xi}_{\bar{\ell}} = 0, \tag{164}$$

where $\boldsymbol{\xi}_{\bar{\ell}} = \boldsymbol{\xi} - \xi_\ell\mathbf{e}_\ell$. We can solve this equation for ξ_ℓ as long as $-\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar{\ell}}\cdot\boldsymbol{\xi}_{\bar{\ell}} \geq 0$, in which case we define χ_ℓ as the positive root of the equation

$$\chi_\ell^2 = -\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar{\ell}}\cdot\boldsymbol{\xi}_{\bar{\ell}}, \tag{165}$$

and we thus take for ξ_ℓ the two possible values $\xi_\ell = \pm\chi_\ell$. We similarly have the analogous versions of (164) and (165) for ξ'_ℓ.

Using [16, p. 184] w.r.t. the integration variables ξ_ℓ and ξ'_ℓ and the limitation in the integration range, equation (163) is expressed as

$$\Phi_{\partial\mathcal{V}_\ell^{k+n}}(\mathbf{T}) = \int_{\substack{\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar{\ell}}\cdot\boldsymbol{\xi}_{\bar{\ell}}\leq 0\\ \Delta_{\ell\ell}\boldsymbol{\xi}'_{\bar{\ell}}\cdot\boldsymbol{\xi}'_{\bar{\ell}}\leq 0}} \mathrm{d}\xi_{\ell^c}\,\mathrm{d}\xi'_{\ell^c}\,\frac{1}{4\chi_\ell\chi'_\ell}\prod_{m\neq\ell}\delta(\xi_m+\xi'_m)\sum_{\xi_\ell=\pm\chi_\ell,\xi'_\ell=\pm\chi'_\ell}\varphi(\boldsymbol{\xi},\boldsymbol{\xi}'). \tag{166}$$

Since the remaining Dirac delta function conditions imply that $\xi'_m = -\xi_m$ for every $m \neq \ell$, we also have that $\xi'_\ell = \pm\chi_\ell$. To further deal with the four terms in the summation in (166), we define the vectors

$$\boldsymbol{\xi}_+ = (\xi_0, \dots, \xi_{\ell-1}, \chi_\ell, \xi_{\ell+1}, \dots, \xi_{k+n-1}), \tag{167}$$

$$\boldsymbol{\xi}_- = (\xi_0, \dots, \xi_{\ell-1}, -\chi_\ell, \xi_{\ell+1}, \dots, \xi_{k+n-1}), \tag{168}$$

and the counterparts $\boldsymbol{\xi}'_+ = -\boldsymbol{\xi}_-$ and $\boldsymbol{\xi}'_- = -\boldsymbol{\xi}_+$. Using these definitions to solve the integration w.r.t. ξ'_{ℓ^c}, we obtain that

$$\Phi_{\partial\mathcal{V}_\ell^{k+n}}(\mathbf{T}) = \int_{\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar{\ell}}\cdot\boldsymbol{\xi}_{\bar{\ell}}\leq 0} \mathrm{d}\xi_{\ell^c}\, \frac{1}{4\chi_\ell^2}\big(\varphi(\boldsymbol{\xi}_+,\boldsymbol{\xi}'_+) + \varphi(\boldsymbol{\xi}_+,\boldsymbol{\xi}'_-) + \varphi(\boldsymbol{\xi}_-,\boldsymbol{\xi}'_+) + \varphi(\boldsymbol{\xi}_-,\boldsymbol{\xi}'_-)\big). \tag{169}$$

It remains to study the four summands in (169) by exploiting the properties of exterior algebra. We start by writing $\varphi(\boldsymbol{\xi}_+,\boldsymbol{\xi}'_+)$ from its definition in (162) and use the fact that $\hat{\mathbf{F}}(\boldsymbol{\xi}'_+) = \hat{\mathbf{F}}(-\boldsymbol{\xi}_-) = \hat{\mathbf{F}}^*(\boldsymbol{\xi}_-)$ to obtain

$$\varphi(\boldsymbol{\xi}_+,\boldsymbol{\xi}_-) = -\frac{1}{2}\Delta_{\ell\ell}\sigma(\ell,\ell^c)e^{j4\pi\chi_\ell x_\ell\Delta_{\ell\ell}}\mathbf{e}_\ell \lrcorner \mathbf{T}_1 \tag{170}$$

where for the sake of clarity we defined the tensor $\mathbf{T}_1$ as

$$\mathbf{T}_1 = \hat{\mathbf{F}}(\boldsymbol{\xi}_+) \odot \hat{\mathbf{F}}^*(\boldsymbol{\xi}_-) + \hat{\mathbf{F}}(\boldsymbol{\xi}_+) \owedge \hat{\mathbf{F}}^*(\boldsymbol{\xi}_-). \tag{171}$$

Using the definitions of $\odot$ and $\owedge$ in (69) and (70) respectively, and the identity (22), we may write the ij-th component of $\mathbf{T}_1$ as

$$\begin{aligned} T_{1,ij} &= \Delta_{ii}\Delta_{jj}\big((\mathbf{e}_i \lrcorner \hat{\mathbf{F}}(\boldsymbol{\xi}_+))\cdot(\hat{\mathbf{F}}^*(\boldsymbol{\xi}_-) \llcorner \mathbf{e}_j) + (\mathbf{e}_i \wedge \hat{\mathbf{F}}(\boldsymbol{\xi}_+))\cdot(\hat{\mathbf{F}}^*(\boldsymbol{\xi}_-) \wedge \mathbf{e}_j)\big) && (172)\\ &= \Delta_{ii}\Delta_{jj}\big(\underbrace{(\mathbf{e}_i \lrcorner \hat{\mathbf{F}}(\boldsymbol{\xi}_+))\cdot(\hat{\mathbf{F}}^*(\boldsymbol{\xi}_-) \llcorner \mathbf{e}_j)}_{\alpha_{1,ij}} + \underbrace{(-1)^r\Delta_{ij}\hat{\mathbf{F}}(\boldsymbol{\xi}_+)\cdot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_-)}_{\beta_{1,ij}}\\ &\quad + \underbrace{(\mathbf{e}_j \lrcorner \hat{\mathbf{F}}(\boldsymbol{\xi}_+))\cdot(\hat{\mathbf{F}}^*(\boldsymbol{\xi}_-) \llcorner \mathbf{e}_i)}_{\alpha_{1,ji}}\big). && (173) \end{aligned}$$

It will prove convenient to study equation (173) in terms of the $(r-1)$-vector potential $\mathbf{A}$, which is related to $\mathbf{F}$ as in (48), or in the Fourier domain,

$$\hat{\mathbf{F}}(\boldsymbol{\xi}) = 2\pi j\,\boldsymbol{\xi} \wedge \hat{\mathbf{A}}(\boldsymbol{\xi}), \tag{174}$$

where $\hat{\mathbf{A}}(\boldsymbol{\xi})$ denotes the Fourier transform of $\mathbf{A}$. Substituting (174) in the definitions of $\alpha_{1,ij}$ and $\beta_{1,ij}$ in (173) and using the identity (21), we obtain that

$$\alpha_{1,ij} = 4\pi^2(-1)^{r-1}\big(\mathbf{e}_i \lrcorner (\boldsymbol{\xi}_+ \wedge \hat{\mathbf{A}}(\boldsymbol{\xi}_+))\big)\cdot\big(\mathbf{e}_j \lrcorner (\boldsymbol{\xi}_- \wedge \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-))\big) \tag{175}$$

$$\beta_{1,ij} = 4\pi^2(-1)^r\Delta_{ij}(\boldsymbol{\xi}_+ \wedge \hat{\mathbf{A}}(\boldsymbol{\xi}_+))\cdot(\boldsymbol{\xi}_- \wedge \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)). \tag{176}$$

We start by expanding $\alpha_{1,ij}$. Using the identiy (26), we get

$$\begin{aligned} \alpha_{1,ij} = 4\pi^2(-1)^{r-1}&\Big((-1)^{r-1}(\mathbf{e}_i\cdot\boldsymbol{\xi}_+)\hat{\mathbf{A}}(\boldsymbol{\xi}_+) + \boldsymbol{\xi}_+ \wedge \big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\Big)\\ &\cdot\Big((-1)^{r-1}(\mathbf{e}_j\cdot\boldsymbol{\xi}_-)\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) + \boldsymbol{\xi}_- \wedge (\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-))\Big). \end{aligned} \tag{177}$$

Computing the products, rearranging terms and using the relations (22), (24)–(25) and (27) in various places, we obtain

$$\begin{aligned} \alpha_{1,ij} = 4\pi^2\Big(&(-1)^{r-1}\Delta_{ii}\Delta_{jj}\xi_{+,i}\xi_{-,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\\ &+ \Delta_{ii}\xi_{+,i}\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\cdot\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \llcorner \boldsymbol{\xi}_-\big)\\ &+ \Delta_{jj}\xi_{-,j}\big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\cdot\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \llcorner \boldsymbol{\xi}_+\big)\\ &+ (-1)^{r-1}(\boldsymbol{\xi}_+\cdot\boldsymbol{\xi}_-)\big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\cdot\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\\ &+ \big(\mathbf{e}_i \lrcorner \big(\boldsymbol{\xi}_- \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\big)\cdot\big(\mathbf{e}_j \lrcorner \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \llcorner \boldsymbol{\xi}_+\big)\big)\Big). \end{aligned} \tag{178}$$

We next simplify the terms of the form $\boldsymbol{\xi}_- \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)$ and $\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \llcorner \boldsymbol{\xi}_+$. To do so, we note that $\boldsymbol{\xi}_+$ and $\boldsymbol{\xi}_-$ respectively given in (167) and (168) are related as

$$\boldsymbol{\xi}_+ = \boldsymbol{\xi}_- + 2\chi_\ell \mathbf{e}_\ell. \tag{179}$$

Recalling that the gauge condition in the Fourier domain is given by

$$\boldsymbol{\xi}_+ \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+) = 0 = \boldsymbol{\xi}_- \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-), \tag{180}$$

equations (179) and (180) imply that

$$\boldsymbol{\xi}_+ \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) = \boldsymbol{\xi}_- \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) + 2\chi_\ell \mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) = 2\chi_\ell \mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-), \tag{181}$$

$$\boldsymbol{\xi}_- \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+) = \boldsymbol{\xi}_+ \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+) - 2\chi_\ell \mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+) = -2\chi_\ell \mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+). \tag{182}$$

Similar relations are obtained for the right interior product. Applying (181) and (182) into (178), we obtain for $\alpha_{1,ij}$ that

$$\begin{aligned}\alpha_{1,ij} = 4\pi^2 \Big(& (-1)^{r-1} \Delta_{ii}\Delta_{jj}\xi_{+,i}\xi_{-,j} \hat{\mathbf{A}}(\boldsymbol{\xi}_+) \cdot \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \\ & - 2\Delta_{ii}\xi_{+,i}\chi_\ell \big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big) \cdot \big(\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \llcorner \mathbf{e}_\ell\big) \\ & + 2\Delta_{jj}\xi_{-,j}\chi_\ell \big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \cdot \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \llcorner \mathbf{e}_\ell\big) \\ & + (-1)^{r-1}\big(-2\Delta_{\ell\ell}\chi_\ell^2\big)\big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \cdot \big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big) \\ & - 4\chi_\ell^2 \big(\mathbf{e}_i \lrcorner \big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\big) \cdot \big(\mathbf{e}_j \lrcorner \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \llcorner \mathbf{e}_\ell\big)\big)\Big). \end{aligned} \tag{183}$$

For $\beta_{1,ij}$, we first use (22) directly into (176) so that it is written as

$$\begin{aligned}\beta_{1,ij} &= 4\pi^2(-1)^r \Delta_{ij}(-1)^{r-1}\big((-1)^{r-1}(\boldsymbol{\xi}_+ \cdot \boldsymbol{\xi}_-)(\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \cdot \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)) \\ &\qquad + (\boldsymbol{\xi}_+ \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)) \cdot (\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \llcorner \boldsymbol{\xi}_-)\big) \qquad (184) \\ &= 4\pi^2(-1)^r \Delta_{ij}(\boldsymbol{\xi}_+ \cdot \boldsymbol{\xi}_-)\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \cdot \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big) \\ &\qquad - 4\pi^2 \Delta_{ij}\big(\boldsymbol{\xi}_+ \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big) \cdot \big(\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \llcorner \boldsymbol{\xi}_-\big). \qquad (185)\end{aligned}$$

In view of (179) and $\boldsymbol{\xi}_+ \cdot \boldsymbol{\xi}_+ = 0$, the $\boldsymbol{\xi}_+ \cdot \boldsymbol{\xi}_-$ term in the previous equation equals

$$\boldsymbol{\xi}_+ \cdot \boldsymbol{\xi}_- = \boldsymbol{\xi}_+ \cdot \boldsymbol{\xi}_+ - 2\chi_\ell\, \boldsymbol{\xi}_+ \cdot \mathbf{e}_\ell = -2\Delta_{\ell\ell}\chi_\ell^2. \tag{186}$$

Therefore, using (181), (182) and (186) we finally obtain

$$\beta_{1,ij} = 8\pi^2\chi_\ell^2\Delta_{ij}\Big((-1)^{r-1}\Delta_{\ell\ell}\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \cdot \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) + 2\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big) \cdot \big(\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \llcorner \mathbf{e}_\ell\big)\Big). \tag{187}$$

Although we derived expressions of $\alpha_{1,ij}$, $\alpha_{1,ji}$ and $\beta_{1,ij}$ in (183) and (187), needed to obtain $T_{1,ij}$ in (173) for arbitrary ij, we are only interested in such terms containing the ℓ-th component, since $\varphi(\boldsymbol{\xi}_+, \boldsymbol{\xi}'_+)$ in (170) involves computing the quantity

$$\mathbf{e}_\ell \lrcorner \mathbf{T}_1 = \mathbf{e}_\ell \lrcorner \sum_{i \le j} T_{1,ij}\mathbf{u}_{ij} = T_{1,\ell\ell}\Delta_{\ell\ell}\mathbf{e}_\ell + \sum_{i \ne \ell}(T_{1,i\ell} + T_{1,\ell i})\Delta_{\ell\ell}\mathbf{e}_i. \tag{188}$$

We start with the first case in which $i = \ell \ne j$. Using that $\mathbf{e}_\ell \lrcorner \mathbf{e}_\ell = 0$, from (183) we get

$$\begin{aligned}\alpha_{1,\ell j} = 4\pi^2 \Big(& (-1)^{r-1} \Delta_{\ell\ell}\Delta_{jj}\xi_{+,\ell}\xi_{-,j} \hat{\mathbf{A}}(\boldsymbol{\xi}_+) \cdot \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \\ & - 2\Delta_{\ell\ell}\xi_{+,\ell}\chi_\ell \big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big) \cdot \big(\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \llcorner \mathbf{e}_\ell\big) \\ & + 2\Delta_{jj}\xi_{-,j}\chi_\ell \big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \cdot \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \llcorner \mathbf{e}_\ell\big) \\ & + 2(-1)^r \Delta_{\ell\ell}\chi_\ell^2 \big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \cdot \big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\Big). \end{aligned} \tag{189}$$

Similarly, we also have

$$\begin{aligned}\alpha_{1,j\ell} = 4\pi^2 \Big(& (-1)^{r-1} \Delta_{\ell\ell}\Delta_{jj}\xi_{+,j}\xi_{-,\ell} \hat{\mathbf{A}}(\boldsymbol{\xi}_+) \cdot \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \\ & - 2\Delta_{jj}\xi_{+,j}\chi_\ell \big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big) \cdot \big(\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \llcorner \mathbf{e}_\ell\big) \\ & + 2\Delta_{\ell\ell}\xi_{-,\ell}\chi_\ell \big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \cdot \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \llcorner \mathbf{e}_\ell\big) \\ & + 2(-1)^r \Delta_{\ell\ell}\chi_\ell^2 \big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \cdot \big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\Big). \end{aligned} \tag{190}$$

Furthermore, the fact that $\Delta_{ij} = 0$ for $i \neq j$ implies from (187) that

$$\beta_{1,\ell j} = 0. \tag{191}$$

Combining (189), (190) and (191) in the initial expression of $T_{1,\ell j}$ in (173), using that $\xi_{\pm,\ell} = \pm\chi_\ell$, and writing the right interior products as left interior products, we get

$$\begin{aligned} T_{1,\ell j} = 4\pi^2 \Delta_{\ell\ell}\Delta_{jj}\Big(&(-1)^{r-1}\Delta_{\ell\ell}\Delta_{jj}\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\chi_\ell\,(\xi_{-,j}-\xi_{+,j}) \\ &+ 2(-1)^r\Delta_{jj}\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\cdot\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\chi_\ell(\xi_{-,j}-\xi_{+,j}) \\ &+ 2(-1)^r\Delta_{\ell\ell}\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\cdot\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)(\chi_\ell^2-\chi_\ell^2) \\ &+ 2(-1)^r\Delta_{\ell\ell}\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\cdot\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)(\chi_\ell^2-\chi_\ell^2)\Big). \end{aligned} \tag{192}$$

We note that the last two summands in the former equation trivially cancel out, whereas the remaining two also do so because $\xi_{-,j} = \xi_{+,j}$ for $j \neq \ell$. Hence,

$$T_{1,\ell j} = 0, \qquad j \neq \ell. \tag{193}$$

We continue with the second case $j = \ell \neq i$ and we have, as in the first case

$$\begin{aligned} \alpha_{1,i\ell} = 4\pi^2\Big(&(-1)^{r-1}\Delta_{ii}\Delta_{\ell\ell}\xi_{+,i}\xi_{-,\ell}\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \\ &- 2(-1)^r\Delta_{ii}\xi_{+,i}\chi_\ell\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\cdot\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \\ &+ 2(-1)^r\Delta_{\ell\ell}\xi_{-,\ell}\chi_\ell\big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\cdot\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big) \\ &+ 2(-1)^r\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\cdot\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\Big), \end{aligned} \tag{194}$$

$$\begin{aligned} \alpha_{1,\ell i} = 4\pi^2\Big(&(-1)^{r-1}\Delta_{ii}\Delta_{\ell\ell}\xi_{+,\ell}\xi_{-,i}\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \\ &- 2(-1)^r\Delta_{\ell\ell}\xi_{+,\ell}\chi_\ell\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\cdot\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \llcorner \mathbf{e}_\ell\big) \\ &+ 2(-1)^r\Delta_{ii}\xi_{-,i}\chi_\ell\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\cdot\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big) \\ &+ 2(-1)^r\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\cdot\big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\Big), \end{aligned} \tag{195}$$

and

$$\beta_{1,i\ell} = 0. \tag{196}$$

We rearrange (194), (195) and (196) in the initial expression (173), using $\xi_{\pm,\ell=\pm\chi_\ell}$ and that $\xi_{-,i} = \xi_{+,i}$ for $i \neq \ell$, we obtain It results to be zero since for $i \neq \ell$ we have $\xi_{-,i} = \xi_{+,i}$, namely

$$T_{1,i\ell} = 0, \qquad i \neq \ell. \tag{197}$$

Regarding the last case $i = j = \ell$ we evaluate $\alpha_{1,\ell\ell}$ from (183) writing all the right interior products as left interior products

$$\begin{aligned} \alpha_{1,\ell\ell} = 4\pi^2\Big(&(-1)^{r-1}\xi_{+,\ell}\xi_{-,\ell}\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) \\ &- 2(-1)^r\Delta_{\ell\ell}\xi_{+,\ell}\chi_\ell\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\cdot\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \\ &+ 2(-1)^r\Delta_{\ell\ell}\xi_{-,\ell}\chi_\ell\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\cdot\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big) \\ &+ 2(-1)^r\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\cdot\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\Big), \end{aligned} \tag{198}$$

and from (187)

$$\beta_{1,\ell\ell} = 8\pi^2(-1)^r\chi_\ell^2\Big(-\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-) + 2\Delta_{\ell\ell}\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)\big)\cdot\big(\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\Big). \tag{199}$$

We substitute $\alpha_{1,\ell\ell} = \beta_{1,\ell\ell}$ from (198) and $\gamma_{1,\ell\ell}$ from (198) into $T_{1,\ell\ell}$ (173) and considering that $\xi_{\pm_\ell} = \pm\chi_\ell$, we directly get

$$T_{1,\ell\ell} = 0. \tag{200}$$

In conclusion, from (188) we realize that

$$\mathbf{e}_\ell \lrcorner \mathbf{T}_1 = 0. \tag{201}$$

We continue studying the second summand in (169). We consider $\varphi(\boldsymbol{\xi}_+, \boldsymbol{\xi}'_-)$ using its definition in (162) joint with the fact that $\hat{\mathbf{F}}(\boldsymbol{\xi}'_-) = \hat{\mathbf{F}}(-\boldsymbol{\xi}_+) = \hat{\mathbf{F}}^*(\boldsymbol{\xi}_+)$ and we get

$$\varphi(\boldsymbol{\xi}_+, \boldsymbol{\xi}_+) = -\frac{1}{2}\Delta_{\ell\ell}\sigma(\ell, \ell^c)\mathbf{e}_\ell \lrcorner \mathbf{T}_2, \tag{202}$$

where the tensor $\mathbf{T}_2$ is defined as

$$\mathbf{T}_2 = \hat{\mathbf{F}}(\boldsymbol{\xi}_+) \odot \hat{\mathbf{F}}^*(\boldsymbol{\xi}_+) + \hat{\mathbf{F}}(\boldsymbol{\xi}_+) \owedge \hat{\mathbf{F}}^*(\boldsymbol{\xi}_+). \tag{203}$$

As for $\mathbf{T}_1$, we use the definitions of $\odot$ and $\owedge$ in (69) and (70) and the identity (22) and we spread out the ij-th component of $\mathbf{T}_2$, which is written

$$\begin{aligned} T_{2,ij} = \Delta_{ii}\Delta_{jj}\Big(&\underbrace{(\mathbf{e}_i \lrcorner \hat{\mathbf{F}}(\boldsymbol{\xi}_+)) \cdot (\hat{\mathbf{F}}^*(\boldsymbol{\xi}_+) \llcorner \mathbf{e}_j)}_{\alpha_{2,ij}} + \underbrace{(-1)^r\Delta_{ij}\hat{\mathbf{F}}(\boldsymbol{\xi}_+) \cdot \hat{\mathbf{F}}^*(\boldsymbol{\xi}_+)}_{\beta_{2,ij}} \\ &+ \underbrace{(\mathbf{e}_j \lrcorner \hat{\mathbf{F}}(\boldsymbol{\xi}_+)) \cdot (\hat{\mathbf{F}}^*(\boldsymbol{\xi}_+) \llcorner \mathbf{e}_i)}_{\alpha_{2,ji}}\Big), \end{aligned} \tag{204}$$

and we substitute the Maxwell field in terms of the potential in the Fourier domain thanks to (174), so that we find

$$\alpha_{2,ij} = 4\pi^2\big(\mathbf{e}_i \lrcorner \big(\boldsymbol{\xi}_+ \wedge \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\big) \cdot \big(\big(\boldsymbol{\xi}_+ \wedge \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)\big) \llcorner \mathbf{e}_j\big), \tag{205}$$

$$\beta_{2,ij} = 4\pi^2(-1)^r\Delta_{ij}\big(\boldsymbol{\xi}_+ \wedge \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \cdot \big(\boldsymbol{\xi}_+ \wedge \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)\big). \tag{206}$$

We start from (205) and we use the relation (26) after writing $\big(\boldsymbol{\xi}_+ \wedge \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)\big) \llcorner \mathbf{e}_j = (-1)^{r-1}\mathbf{e}_j \lrcorner \big(\boldsymbol{\xi}_+ \wedge \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)\big)$. Thus we get

$$\begin{aligned} \alpha_{2,ij} = 4\pi^2(-1)^{r-1}\big(&(-1)^{r-1}(\mathbf{e}_i \cdot \boldsymbol{\xi}_+)\hat{\mathbf{A}}(\boldsymbol{\xi}_+) + \boldsymbol{\xi}_+ \wedge \big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\big) \cdot \\ &\big((-1)^{r-1}(\mathbf{e}_j \cdot \boldsymbol{\xi}_+)\hat{\mathbf{A}}^*(\boldsymbol{\xi}_+) + \boldsymbol{\xi}_+ \wedge \big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)\big)\big). \end{aligned} \tag{207}$$

Again, carrying out all the products, and applying (22) and (26), we obtain

$$\begin{aligned} \alpha_{2,ij} = 4\pi^2\Big(&(-1)^{r-1}\Delta_{ii}\Delta_{jj}\xi_{+,i}\xi_{+,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \cdot \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+) \\ &+ \Delta_{ii}\xi_{+,i}\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)\big) \cdot \big(\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \llcorner \boldsymbol{\xi}_+\big) \\ &+ \Delta_{jj}\xi_{+,j}\big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \cdot \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_+) \llcorner \boldsymbol{\xi}_+\big) \\ &- \Big((-1)^{r-1}(\boldsymbol{\xi}_+ \cdot \boldsymbol{\xi}_+)\big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \cdot \big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)\big) \\ &+ \big(\boldsymbol{\xi}_+ \lrcorner \big(\mathbf{e}_i \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big)\big) \cdot \big(\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)\big) \llcorner \boldsymbol{\xi}_+\big)\Big)\Big). \end{aligned} \tag{208}$$

We note that (24) implies

$$\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)\big) \llcorner \boldsymbol{\xi}_+ = \mathbf{e}_j \lrcorner \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_+) \llcorner \boldsymbol{\xi}_+\big) = 0, \tag{209}$$

and we can simplify $\alpha_{2,ij}$ thanks to the facts that $\boldsymbol{\xi}_+ \cdot \boldsymbol{\xi}_+ = 0$ and to the gauge condition $\boldsymbol{\xi}_+ \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+) = 0$ from (180), obtaining

$$\alpha_{2,ij} = 4\pi^2(-1)^{r-1}\Delta_{ii}\Delta_{jj}\xi_{+,i}\xi_{+,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \cdot \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+). \tag{210}$$

Hence, as a consequence,

$$\alpha_{2,ji} = 4\pi^2(-1)^{r-1}\Delta_{ii}\Delta_{jj}\xi_{+,i}\xi_{+,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \cdot \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+). \tag{211}$$

Regarding $\beta_{2,ij}$, we write $\boldsymbol{\xi}_+ \wedge \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+) = (-1)^{r-1}\hat{\mathbf{A}}^*(\boldsymbol{\xi}_+) \wedge \boldsymbol{\xi}_+$ and we apply again (22) in (206) so that we immediately verify that it vanishes

$$\begin{aligned} \beta_{2,ij} = -4\pi^2\Delta_{ij}\Big(&(-1)^{r-1}(\boldsymbol{\xi}_+ \cdot \boldsymbol{\xi}_+)\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_+) \cdot \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)\big) \\ &+ \big(\boldsymbol{\xi}_+ \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big) \cdot \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_+) \llcorner \boldsymbol{\xi}_+\big)\Big) = 0. \end{aligned} \tag{212}$$

As a consequence, the result for $T_{2,ij}$ in (204) is

$$T_{2,ij} = 8\pi^2(-1)^{r-1}\xi_{+,i}\xi_{+,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_+) \tag{213}$$

and we finally evaluate the tensor $\mathbf{T}_2$ as

$$\mathbf{T}_2 = 8\pi^2\sum_{i\leq j}(-1)^{r-1}\xi_{+,i}\xi_{+,j}\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big|^2\mathbf{u}_{ij}. \tag{214}$$

We move on to the third term of (169). The equality $\hat{\mathbf{F}}(\boldsymbol{\xi}'_+) = \hat{\mathbf{F}}^*(\boldsymbol{\xi}_-)$ allows us to write $\varphi(\boldsymbol{\xi}_-, \boldsymbol{\xi}'_+)$ as

$$\varphi(\boldsymbol{\xi}_-, \boldsymbol{\xi}_-) = -\frac{1}{2}\Delta_{\ell\ell}\sigma(\ell,\ell^c)\mathbf{e}_\ell \lrcorner \mathbf{T}_3, \tag{215}$$

where we defined the tensor $\mathbf{T}_3$ as follow

$$\mathbf{T}_3 = \hat{\mathbf{F}}(\boldsymbol{\xi}_-)\odot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_-) + \hat{\mathbf{F}}(\boldsymbol{\xi}_-)\oslash\hat{\mathbf{F}}^*(\boldsymbol{\xi}_-). \tag{216}$$

Comparing (216) and (203), we note that the difference is only in the presence of $\boldsymbol{\xi}_-$ instead of $\boldsymbol{\xi}_+$. Thus, the mathematical steps are identical, including that the relation $\boldsymbol{\xi}_+\cdot\boldsymbol{\xi}_+ = 0$ has its counterpart $\boldsymbol{\xi}_-\cdot\boldsymbol{\xi}_- = 0$. The gauge conditions in (180) can also be written as

$$\boldsymbol{\xi}_- \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_-) = 0 = \boldsymbol{\xi}_+ \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+). \tag{217}$$

So, in analogy with the result obtained in (214), the final expression for $\mathbf{T}_3$ is

$$\mathbf{T}_3 = 8\pi^2\sum_{i\leq j}(-1)^{r-1}\xi_{-,i}\xi_{-,j}\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_-)\big|^2\mathbf{u}_{ij}. \tag{218}$$

We conclude the evaluation of the initial integral in (169) computing $\varphi(\boldsymbol{\xi}_-, \boldsymbol{\xi}'_-)$. From ts definition in (162) and using $\hat{\mathbf{F}}(\boldsymbol{\xi}'_-) = \hat{\mathbf{F}}^*(\boldsymbol{\xi}_+)$, we get

$$\varphi(\boldsymbol{\xi}_-, \boldsymbol{\xi}_+) = -\frac{1}{2}\Delta_{\ell\ell}\sigma(\ell,\ell^c)e^{-j4\pi\chi_\ell x_\ell\Delta_{\ell\ell}}\mathbf{e}_\ell \lrcorner \mathbf{T}_4, \tag{219}$$

defined, as in the previous cases,

$$\mathbf{T}_4 = \hat{\mathbf{F}}(\boldsymbol{\xi}_-)\odot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_+) + \hat{\mathbf{F}}(\boldsymbol{\xi}_-)\oslash\hat{\mathbf{F}}^*(\boldsymbol{\xi}_+). \tag{220}$$

We can further expand (220) in components which would appear

$$\begin{aligned} T_{4,ij} = \Delta_{ii}\Delta_{jj}\Big(&\underbrace{(\mathbf{e}_i \lrcorner \hat{\mathbf{F}}(\boldsymbol{\xi}_-))\cdot(\hat{\mathbf{F}}^*(\boldsymbol{\xi}_+)\llcorner \mathbf{e}_j)}_{\alpha_{4,ij}} + \underbrace{(-1)^r\Delta_{ij}\hat{\mathbf{F}}(\boldsymbol{\xi}_-)\cdot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_+)}_{\beta_{4,ij}} \\ &+ \underbrace{(\mathbf{e}_j \lrcorner \hat{\mathbf{F}}(\boldsymbol{\xi}_-))\cdot(\hat{\mathbf{F}}^*(\boldsymbol{\xi}_+)\llcorner \mathbf{e}_i)}_{\alpha_{4,ji}}\Big). \end{aligned} \tag{221}$$

As in the analysis of $T_{1,ij}$, we express $\alpha_{4,ij}$ and $\beta_{4,ij}$ in terms of the potential

$$\alpha_{4,ij} = 4\pi^2(-1)^{r-1}(\mathbf{e}_i \lrcorner (\boldsymbol{\xi}_- \wedge \hat{\mathbf{A}}(\boldsymbol{\xi}_-)))\cdot(\mathbf{e}_j \lrcorner (\boldsymbol{\xi}_+ \wedge \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+))), \tag{222}$$

$$\beta_{4,ij} = 4\pi^2(-1)^r\Delta_{ij}(\boldsymbol{\xi}_- \wedge \hat{\mathbf{A}}(\boldsymbol{\xi}_-))\cdot(\boldsymbol{\xi}_+ \wedge \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)). \tag{223}$$

We can now note that the differences between (220) and (171) are in $\boldsymbol{\xi}_+$ exchanged with $\boldsymbol{\xi}_-$, $\hat{\mathbf{F}}(\boldsymbol{\xi}_+)$ with $\hat{\mathbf{F}}(\boldsymbol{\xi}_-)$ and $\hat{\mathbf{F}}^*(\boldsymbol{\xi}_-)$ with $\hat{\mathbf{F}}^*(\boldsymbol{\xi}_+)$. So, the differences among (222) and (223) with respect to (175) and (176) are, in addition to the aforementioned, $\hat{\mathbf{A}}(\boldsymbol{\xi}_+)$ interchanged with $\hat{\mathbf{A}}(\boldsymbol{\xi}_-)$ and $\hat{\mathbf{A}}^*(\boldsymbol{\xi}_-)$ with $\hat{\mathbf{A}}^*(\boldsymbol{\xi}_+)$. We consider (179) and the gauge condition (217), we replace the conditions (181) and (182) with

$$\boldsymbol{\xi}_+ \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_-) = \boldsymbol{\xi}_- \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_-) + 2\chi_\ell\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_-) = 2\chi_\ell\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_-) \tag{224}$$

$$\boldsymbol{\xi}_- \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+) = \boldsymbol{\xi}_+ \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+) - 2\chi_\ell\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+) = -2\chi_\ell\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_+). \tag{225}$$

If we follow the same procedure applied to obtain $\mathbf{T}_1$ with the conditions (217), (224) and (225), we can rapidly state

$$\mathbf{e}_\ell \lrcorner \mathbf{T}_4 = 0. \tag{226}$$

Then, we write the integral for the flux in (169) substituting the definition (162). Removing the first and the last summands thanks to (201) and (226), it results

$$\Phi_{\partial\mathcal{V}_\ell^{k+n}}(\mathbf{T}) = -\frac{1}{8}\Delta_{\ell\ell}\sigma(\ell,\ell^c)\int_{\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar\ell}\cdot\boldsymbol{\xi}_{\bar\ell}\le 0} \mathrm{d}\xi_{\ell^c}\,\frac{1}{\chi_\ell^2}\mathbf{e}_\ell \lrcorner \big(\mathbf{T}_2+\mathbf{T}_3\big). \tag{227}$$

Using (214) and (218), we get

$$\Phi_{\partial\mathcal{V}_\ell^{k+n}}(\mathbf{T}) = (-1)^r\pi^2\Delta_{\ell\ell}\sigma(\ell,\ell^c)\mathbf{e}_\ell \lrcorner$$
$$\int_{\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar\ell}\cdot\boldsymbol{\xi}_{\bar\ell}\le 0} \mathrm{d}\xi_{\ell^c}\,\frac{1}{\chi_\ell^2}\sum_{i\le j}\Big(\xi_{+,i}\xi_{+,j}\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big|^2 + \xi_{-,i}\xi_{-,j}\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_-)\big|^2\Big)\mathbf{u}_{ij}. \tag{228}$$

We consider that $\chi_\ell^2 = \xi_{+,\ell}^2 = \xi_{-,\ell}^2$ and we expand the product $\mathbf{e}_\ell \lrcorner \mathbf{u}_{ij}$ thanks to (85) and then use that $\sum_i \xi_{\pm,i}\mathbf{e}_i = \boldsymbol{\xi}_\pm$, so that (228) is

$$\Phi_{\partial\mathcal{V}_\ell^{k+n}}(\mathbf{T}) = (-1)^r\pi^2\sigma(\ell,\ell^c)\int_{\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar\ell}\cdot\boldsymbol{\xi}_{\bar\ell}\le 0} \mathrm{d}\xi_{\ell^c}\left(\frac{\boldsymbol{\xi}_+}{\xi_{+,\ell}}\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big|^2 + \frac{\boldsymbol{\xi}_-}{\xi_{-,\ell}}\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_-)\big|^2\right). \tag{229}$$

As an aside, we may use [16, p. 184] and that $\chi_\ell = \xi_{+,\ell} = -\xi_{-,\ell}$ to undo the step leading to (166) to recover the Dirac delta function

$$\Phi_{\partial\mathcal{V}_\ell^{k+n}}(\mathbf{T}) = (-1)^r 2\pi^2\sigma(\ell,\ell^c)\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\mathrm{d}^{k+n}\boldsymbol{\xi}\,\operatorname{sgn}(\xi_\ell)\boldsymbol{\xi}\big|\hat{\mathbf{A}}(\boldsymbol{\xi})\big|^2\delta(\boldsymbol{\xi}\cdot\boldsymbol{\xi}). \tag{230}$$

Returning to (229), we split the integral into $I_{\ell,+} + I_{\ell,-}$, where

$$I_{\ell,\pm} = \int_{\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar\ell}\cdot\boldsymbol{\xi}_{\bar\ell}\le 0} \mathrm{d}\xi_{\ell^c}\,\frac{\boldsymbol{\xi}_\pm}{\xi_{\pm,\ell}}\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_\pm)\big|^2. \tag{231}$$

Taking into account that $\mathbf{A}(\mathbf{x})$ is real, we may express the squared modulus of $\hat{\mathbf{A}}(\boldsymbol{\xi}_-)$ as $\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_-)\big|^2 = \hat{\mathbf{A}}(-\boldsymbol{\xi}_-)\hat{\mathbf{A}}^*(-\boldsymbol{\xi}_-)$. Therefore the integral $I_{\ell,-}$ becomes

$$I_{\ell,-} = \int_{\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar\ell}\cdot\boldsymbol{\xi}_{\bar\ell}\le 0} \mathrm{d}\xi_{\ell^c}\,\frac{\boldsymbol{\xi}_-}{\xi_{-,\ell}}\hat{\mathbf{A}}(-\boldsymbol{\xi}_-)\hat{\mathbf{A}}^*(-\boldsymbol{\xi}_-). \tag{232}$$

Changing the integration variables according to $\boldsymbol{\xi}_{\bar\ell} \to \boldsymbol{\zeta}_{\bar\ell} = -\boldsymbol{\xi}_{\bar\ell}$, together with the definition

$$\boldsymbol{\zeta}_\pm = (\zeta_0,\dots,\zeta_{\ell-1},\pm\chi_\ell,\zeta_{\ell+1},\dots,\zeta_{n+k-1}),$$

yields

$$I_{\ell,-} = \int_{\Delta_{\ell\ell}\boldsymbol{\zeta}_{\bar\ell}\cdot\boldsymbol{\zeta}_{\bar\ell}\le 0} \mathrm{d}\zeta_{\ell^c}\,\frac{\boldsymbol{\zeta}_+}{\zeta_{+,\ell}}\hat{\mathbf{A}}(\boldsymbol{\zeta}_+)\hat{\mathbf{A}}^*(\boldsymbol{\zeta}_+). \tag{233}$$

Since (233) is formally equivalent to $I_{\ell,+}$, the flux can be rewritten as

$$\Phi_{\partial\mathcal{V}_\ell^{k+n}}(\mathbf{T}) = (-1)^r 2\pi^2\sigma(\ell,\ell^c)\int_{\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar\ell}\cdot\boldsymbol{\xi}_{\bar\ell}\le 0} \mathrm{d}\xi_{\ell^c}\,\frac{\boldsymbol{\xi}_+}{\xi_{+,\ell}}\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_+)\big|^2. \tag{234}$$

Bibliography

[1] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* ser. Course of Theoretical Physics, vol. 2, 4th ed., Butterworth-Heinemann, Oxford, 1987.

[2] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. Wiley, Hoboken, 1999.

[3] J. C. Maxwell, *A Treatise on Electricity & Magnetism*. Dover Publications, Mineola, 2003.

[4] A. Sommerfeld, "Zur Relativitätstheorie. I. Vierdimensionale Vektoralgebra," *Annalen der Physik*, vol. 337, no. 9, pp. 749–776, 1910.

[5] ——, "Zur Relativitätstheorie. II. Vierdimensionale Vektoranalysis," *Annalen der Physik*, vol. 338, no. 14, pp. 649–689, 1910.

[6] E. B. Wilson and G. N. Lewis, "The space-time manifold of relativity. the non-euclidean geometry of mechanics and electromagnetics," *Proceedings of the American Academy of Arts and Sciences*, vol. 48, no. 11, pp. 389–507, November 1912.

[7] E. Kähler, "Bemerkungen über die maxwellschen Gleichungen," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 12, no. 1, pp. 1–28, 1937.

[8] D. Hestenes, *Space-Time Algebra*. Birkhäuser, Basel, 2015.

[9] D. G. Edelen, "A metric free electrodynamics with electric and magnetic charges," *Annals of Physics*, vol. 112, no. 2, pp. 366–400, 1978.

[10] G. A. Deschamps, "Electromagnetics and differential forms," *Proc. IEEE*, vol. 69, pp. 676–696, June 1981.

[11] I. Colombaro, J. Font-Segura, and A. Martinez, "An introduction to space–time exterior calculus," *Mathematics*, vol. 7, pp. 564–583, 2019.

[12] H. Grassmann, *Extension Theory*, no. 19, ser. History of Mathematics Sources. American Mathematical Society, Providence, 2000.

[13] D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, no. 5, ser. Fundamental Theories of Physics. Springer, Berlin, 1987.

[14] V. I. Arnold, A. Weinstein, and K. Vogtmann, *Mathematical Methods Of Classical Mechanics*. Springer, Berlin, 1989.

[15] V. B. Berestetskii, L. D. Landau, E. M. Lifshitz, L. P. Pitaevskii, and J. B. Sykes, *Quantum Electrodynamics*, , vol. 4, 2nd ed., ser. Course of Theoretical Physics. Pergamon Press Ltd., Oxford, 1982.

[16] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, vol. 1. Academic Press Inc., Cambridge, 1964.

An Exterior Algebraic Derivation of the Euler–Lagrange Equations from the Principle of Stationary Action *

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Abstract

In this paper, we review two related aspects of field theory: the modeling of the fields by means of exterior algebra and calculus, and the derivation of the field dynamics, i.e., the Euler–Lagrange equations, by means of the stationary action principle. In contrast to the usual tensorial derivation of these equations for field theories, that gives separate equations for the field components, two related coordinate-free forms of the Euler–Lagrange equations are derived. These alternative forms of the equations, reminiscent of the formulae of vector calculus, are expressed in terms of vector derivatives of the Lagrangian density. The first form is valid for a generic Lagrangian density that only depends on the first-order derivatives of the field. The second form, expressed in exterior algebra notation, is specific to the case when the Lagrangian density is a function of the exterior and interior derivatives of the multivector field. As an application, a Lagrangian density for generalized electromagnetic multivector fields of arbitrary grade is postulated and shown to have, by taking the vector derivative of the Lagrangian density, the generalized Maxwell equations as Euler–Lagrange equations.

Keywords— Euler–Lagrange equations; exterior algebra; exterior calculus; tensor calculus; action principle; Lagrangian; electromagnetism; Maxwell equations

MSC— primary 37J05; secondary 15A75

1 Introduction

In classical mechanics, the action is a scalar quantity, with units of energy $\times$ time, that encodes the dynamical evolution of a given physical system; mathematically, the action is given by an integral functional of the trajectory or dynamical path (or an integral of the Lagrangian density for field theories) followed by the physical system over space-time. The principle of stationary action states that the actual dynamical path followed by the system, subject to some appropriate boundary constraints, possibly at infinity, corresponds to a stationary point of the action [1] (Ch. 19), [2] (Section 8). An application of the principle yields the Euler–Lagrange equations, which describe the dynamics of the system [3] (Section I.3), [4] (Section 3.1), [5] (Section 7.2). The historical development of the stationary-action principle—in essence, a far-reaching generalization of Fermat's principle—that states that light follows the shortest-time path between two points is described in detail in [6] (Section X).

This paper revisits the derivation of the Euler–Lagrange equations for field theories from the principle of stationary action from the point of view of exterior algebra and calculus. There exist several alternative

*This work has been funded in part by the Spanish Ministry of Science, Innovation and Universities under grants TEC2016-78434-C3-1-R and BES-2017-081360. Published in: ***Mathematics* 9, 2178 (2021). DOI:** 10.3390/math9182178.

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mathematical representations for the fields, ranging from the original vector calculus by Gibbs [7] and Heaviside to geometric and Clifford algebras [8], where vectors are replaced by multivectors and operations such as the cross and the dot products subsumed in the geometric product; a modern perspective on the use of geometric algbra in physics is given in [9]. Early in the 20th century, tensors such as the Faraday tensor in electromagnetism were quickly and almost universally adopted as the natural mathematical representation of fields in space-time [10] (pp. 135–144). In parallel, mathematicians such as Cartan generalized the fundamental theorems of vector calculus i.e., Gauss, Green, and Stokes, by means of differential forms [11]. Later on, differential forms were used in Hamiltonian mechanics, e.g., to calculate trajectories as vector field integrals [12] (pp. 194–198). Since differential forms may be seen as the circulation or flux over appropriate space-time regions of multivector fields, it may be preferable in some contexts to directly study the multivector fields. Therefore, we build our analysis on the exterior algebra originally developed by Grassmann [13], which has comparatively received little attention in the literature and leads to simple formulae that merge the simplicity and intuitiveness of standard vector calculus with the power of tensors and differential forms [14, 15].

In Section 2, we provide the necessary background on exterior algebra and calculus, including the important notion of multivector-valued derivative with respect to a vector $\mathbf{v}$. Then, we obtain in Section 3 two related coordinate-free forms of the Euler–Lagrange equations for the dynamics of a multivector field $\mathbf{a}$ of grade r as vector derivatives of the Lagrangian density $\mathcal{L}$. Our work is related to the geometric–algebraic multivectorial formulation of the Euler–Lagrange equations in [16] (Equations (4.7) and (4.8)). The first form in (39) is valid for a generic Lagrangian density that only depends on the first-order derivatives of the field, more specifically on the tensor derivative $\boldsymbol{\partial} \otimes \mathbf{a}$ in (27), and is given by

$$\partial_{\mathbf{a}}\mathcal{L} = \boldsymbol{\partial} \times \big(\partial_{\boldsymbol{\partial}\otimes\mathbf{a}}\mathcal{L}\big), \tag{1}$$

as a function of the vector and matrix derivatives $\partial_{\mathbf{a}}\mathcal{L}$ and $\partial_{\boldsymbol{\partial}\otimes\mathbf{a}}\mathcal{L}$ in (28) and (29), respectively. The $(k+n)$-dimensional differential operator $\boldsymbol{\partial}$ is defined in (20); together with the matrix product $\times$ defined in (19), the operation in the right-hand side generalizes the concept of the divergence of a field. The second form (47), expressed in exterior algebra notation, is specific to the case when the Lagrangian density depends only on exterior (denoted by $\boldsymbol{\partial}\wedge$; see (21)) and interior derivatives (denoted by $\boldsymbol{\partial}\lrcorner$; see (22)) of the multivector field, and is given by

$$\partial_{\mathbf{a}}\mathcal{L} = (-1)^{r-1}\boldsymbol{\partial}\lrcorner\big(\partial_{\boldsymbol{\partial}\wedge\mathbf{a}}\mathcal{L}\big) + (-1)^{r}\boldsymbol{\partial}\wedge\big(\partial_{\boldsymbol{\partial}\lrcorner\mathbf{a}}\mathcal{L}\big), \tag{2}$$

where r is the grade of the multivector field $\mathbf{a}$. A complementary analysis, which shows the invariance of the action to infinitesimal space–time translations in exterior algebra, was conducted in [17], where the stress–energy–momentum tensor is evaluated and profusely discussed. We conclude the paper in Section 4 with an application of our analysis to a Lagrangian density for generalized electromagnetic multivector fields that leads, by directly taking the vector derivative of the Lagrangian density, to the generalized Maxwell equations for multivector fields of grade r [15]. We also provide a short discussion, of independent interest, of a dual form of Maxwell equations where the exterior derivative is replaced by the interior derivative in the definition of the field from the potential.

2 Fundamentals of Exterior Algebra and Calculus: Notation, Definitions, and Operations

2.1 Multivector Fields

While our space-time has four space-time dimensions in relativistic terms, it will prove convenient to consider a generic flat space-time $\mathbf{R}^{k+n}$ with k temporal dimensions and n spatial dimensions, as this generality allows for a more natural description of the underlying algebraic structure of the equations and of their derivations. Points and position in space-time are denoted by $\mathbf{x}$, with components x_i in the canonical basis $\{\mathbf{e}_i\}_{i=0}^{k+n-1}$; by convention, the first k indices, i.e., $i = 0, \ldots, k-1$, correspond to time components while the indices $i = k, \ldots, k+n-1$ represent space components. We let space and time coordinates have the same units. Although we shall not make use of this fact, space-time vectors transform contravariantly under changes of coordinates.

In exterior algebra, one considers vector spaces whose basis elements $\mathbf{e}_I$ are indexed by lists $I = (i_1, \ldots, i_m)$ drawn from $\mathcal{I}_m$, the set of all ordered lists with m nonrepeated indices, with $m \in \mathcal{I} = \{0, 1\ldots, k+n\}$. Later on, in (6), we express the basis elements $\mathbf{e}_I$ in terms of the vectorial canonical basis $\mathbf{e}_i$, for an ordered list $i_1, \ldots, i_m$. These vectors, which we identify with fields, live in the tangent space and transform covariantly under changes of coordinates [18] (Ch. 2), [19] (Ch. V). We refer to elements of these vector field spaces as multivector fields of grade m. While multivector fields do not cover all relevant physical models, e.g., spinor fields or the tensor field in general relativity, they do model a number of interesting cases; for instance, a scalar field is represented by multivectors of grade 0, the electric field, the electromagnetic vector potential and source current by multivectors of grade 1, and the electromagnetic field by a multivector of grade 2. A multivector field $\mathbf{a}(\mathbf{x})$ of grade m, possibly a function of the position $\mathbf{x}$, with components $a_I(\mathbf{x})$ in the canonical basis $\{\mathbf{e}_I\}_{I\in\mathcal{I}_m}$ can be written as

$$\mathbf{a}(\mathbf{x}) = \sum_{I\in\mathcal{I}_m} a_I(\mathbf{x})\mathbf{e}_I. \tag{3}$$

We denote by $\mathrm{gr}(\mathbf{a})$ the operation that returns the grade of a vector $\mathbf{a}$ and by $|I|$ the length of a list I. The dimension of the vector space of all grade m multivectors is $\binom{k+n}{m}$, the number of lists in $\mathcal{I}_m$.

2.2 Operations on Index Lists

As the basis elements of multivector fields are indexed by lists I, it proves convenient to define some basic operations on such lists: permutations and their signatures, concatenations (mergers), and subtractions of lists.

First of all, if the list I is not ordered, let $\sigma(I)$ denote the signature of the permutation sorting the elements of I in increasing order. If the permutation is even (resp. odd), the signature is $+1$ (resp. -1). If the list I contains repeated indices, its signature is 0.

More generally, for two index lists I and J with respective lenghts $m = |I|$ and $m' = |J|$, let $(I, J) = \{i_1, \dots, i_m, j_1, \dots, j_{m'}\}$ be the concatenation of the two index lists I and J. We let $\sigma(I, J)$ denote the signature of the permutation sorting the concatenated list of $|I| + |J|$ indices, and let $I + J$, or $\varepsilon(I, J)$ if the notation $I + J$ is ambiguous in a given context, denote the sorted concatenated list, which we refer to as merged list.

In general, we view the lists as ordered sets, and apply standard operations on sets to the lists. For instance, I is contained in J, the list $J \setminus I$ is the result of removing from J all the elements in I, while keeping the order. As another example, we denote by I^c the complement of I, namely the ordered sequence of indices not included in I. We denote the empty list by $\emptyset$; it holds that $\sigma(\emptyset, K) = \sigma(K, \emptyset) = 1$ for an ordered list K, and that $\mathbf{e}_\emptyset = 1$.

2.3 Operations on Multivectors

We next define several operations acting on multivectors; our presentation loosely follows [14] (Sections 2 and 3) and [15] (Section 2) and is close in spirit and form to vector calculus. Introductions to exterior algebra from the perspective and language of differential forms can be found in [18, 19]. A geometric algebra perspective can be found in [9]. With no real loss of generality, we define the operations only for the canonical basis vectors, the operation acting on general multivectors being a mere extension by linearity of the former.

First, the dot product $\cdot$ of two arbitrary grade m basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ is defined as

$$\mathbf{e}_I \cdot \mathbf{e}_J = \Delta_{IJ} = \Delta_{i_1 j_1}\Delta_{i_2 j_2}\dots\Delta_{i_m j_m}, \tag{4}$$

where I and J are the ordered lists $I = (i_1, i_2, \dots, i_m)$ and $J = (j_1, j_2, \dots, j_m)$ and $\Delta_{ij} = 0$ if $i \neq j$, and we let time unit vectors $\mathbf{e}_i$ have negative metric $\Delta_{ii} = -1$ and space unit vectors $\mathbf{e}_i$ have positive metric $\Delta_{ii} = +1$. When $m = 0$, we interpret the dot product in (4) as 1 since $\mathbf{e}_\emptyset = 1$.

The following operations to be defined are the interior and exterior products, which subsume and generalize the operations of gradient, curl, and divergence of vector calculus to multivector fields. These operations transform pairs of multivectors into a multivector of a different grade, introducing in the process some signs, i.e., ±1. When these signs are related to the dot product in (4), we explicitly write the signs as quantities such as Δ_{IJ}. Other sign contributions arise from the signatures of permutations ordering lists of indices. A common practice in the literature to deal with these signatures is to write factors such as $(-1)^{|I|+|J|}$. However, it seems more convenient to explicitly keep track of the lists and write the permutation associated to this factor, e.g., $\sigma(I, J)$, as clearer connections between different formulae can be established by harnessing the power of group theory for permutations.

Let two basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ have grades $m = |I|$ and $m' = |J|$. As defined in Section 2.2, let $(I, J) = \{i_1, \dots, i_m, j_1, \dots, j_{m'}\}$ be the concatenation of the two index lists I and J, let $\sigma(I, J)$ denote the signature of the permutation sorting the elements of this concatenated list. Then, the exterior product of $\mathbf{e}_I$ and $\mathbf{e}_J$ is defined as

$$\mathbf{e}_I \wedge \mathbf{e}_J = \sigma(I, J)\mathbf{e}_{I+J}. \tag{5}$$

The exterior product is thus either zero or a multivector of grade $|I| + |J|$, since $\sigma(I, J) = 0$ when the lists I and J have elements in common. The unit scalar (multivector of grade 0) is an identity of the exterior product, as $1 \wedge \mathbf{e}_I = \mathbf{e}_I \wedge 1 = \mathbf{e}_I$. The exterior product provides a construction of the basis vector $\mathbf{e}_I$, with I an ordered list $I = (i_1, \dots, i_m)$, from the canonical basis vectors $\mathbf{e}_i$, namely

$$\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_m}. \tag{6}$$

When $I = \emptyset$, we adopt the usual convention that the right-hand side is 1.

We next define two generalizations of the dot product, the left and right interior products. Let $\mathbf{e}_I$ and $\mathbf{e}_J$ be two basis vectors of respective grades $|I|$ and $|J|$. The left interior product, denoted by $\lrcorner$, is defined as

$$\mathbf{e}_I \lrcorner \mathbf{e}_J = \begin{cases} \Delta_{II}\sigma(J \setminus I, I)\mathbf{e}_{J\setminus I}, & \text{if } I \subseteq J, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Although we might have overloaded the meaning of $\sigma(J \setminus I, I)$ to be zero when $I \nsubseteq J$, we prefer to list the separate cases in (7). The vector $\mathbf{e}_{J\setminus I}$ has grade $|J| - |I|$ and is indexed by the elements of J not in common with I. The use of the word left represents the fact that $\mathbf{e}_I$ acts from the left on $\mathbf{e}_J$ and removes the elements in I from J.

Analogously, the right interior product, denoted by $\llcorner$, of two basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ is defined as

$$\mathbf{e}_J \llcorner \mathbf{e}_I = \begin{cases} \Delta_{II}\sigma(I, J \setminus I)\mathbf{e}_{J\setminus I}, & \text{if } I \subseteq J, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

As in the previous case, the use of the word right represents the fact that $\mathbf{e}_I$ acts from the right on $\mathbf{e}_J$ and removes the elements in I from J. The unit scalar (multivector of grade 0) acting from the left (resp. right) is an identity of the left (resp. right) interior product, as $1 \lrcorner \mathbf{e}_I = \mathbf{e}_I \llcorner 1 = \mathbf{e}_I$.

It proves instructive to evaluate the left and right interior products between two multivectors of the same grade, i.e., if $|I| = |J|$. From (7) and (8), and taking into account that $\sigma(\emptyset, K) = \sigma(K, \emptyset) = 1$ for an ordered list K, and that $\mathbf{e}_\emptyset = 1$, we see that

$$\mathbf{e}_I \lrcorner \mathbf{e}_J = \mathbf{e}_J \llcorner \mathbf{e}_I = \mathbf{e}_I \cdot \mathbf{e}_J, \qquad \text{if } |I| = |J|, \tag{9}$$

supporting the idea that the interior products generalize the dot product. Both interior products are grade-lowering operations, as the interior product is either zero or a multivector of grade $|J| - |I|$.

Finally, we define the complement of a multivector. For a multivector $\mathbf{e}_I$ with grade m, its Grassmann or Hodge complement, denoted by $\mathbf{e}_I^{\mathcal{H}}$, is the unit $(k + n - m)$-vector

$$\mathbf{e}_I^{\mathcal{H}} = \Delta_{II}\sigma(I, I^c)\mathbf{e}_{I^c}, \tag{10}$$

where I^c is the complement of the list I and $\sigma(I, I^c)$ is the signature of the permutation sorting the elements of the concatenated list (I, I^c) containing all space-time indices. In other words, $\mathbf{e}_{I^c}$ is the basis multivector of grade $k + n - m$ whose indices are in the complement of I. In addition, we define the inverse complement transformation as

$$\mathbf{e}_I^{\mathcal{H}^{-1}} = \Delta_{I^c I^c}\sigma(I^c, I)\mathbf{e}_{I^c}. \tag{11}$$

The interior products are not independent operations from the exterior product, as they can be expressed in terms of the latter, the Hodge complement and its inverse:

$$\mathbf{e}_I \lrcorner \mathbf{e}_J = \left(\mathbf{e}_I \wedge \mathbf{e}_J^{\mathcal{H}}\right)^{\mathcal{H}^{-1}}, \tag{12}$$

$$\mathbf{e}_J \llcorner \mathbf{e}_I = \left(\mathbf{e}_J^{\mathcal{H}^{-1}} \wedge \mathbf{e}_I\right)^{\mathcal{H}}. \tag{13}$$

The vector calculus cross product between two vectors in $\mathbf{R}^3$ can be expressed in several alternative ways in terms of the interior, and exterior products and Hodge dual [14] (Equation (18)). This fact allows us to distinguish various roles that the cross product takes in Maxwell equations and lies at the origin of generalized electromagnetism described by multivectors in generic flat space-time [15].

2.4 Matrix Vector Spaces

We do not need to consider general tensor fields but rather the matrix field (vector) space whose basis elements can be represented as $\mathbf{w}_{I_1,I_2}$, where both I_1 and I_2 are ordered lists of nonrepeated ℓ_1 and ℓ_2 elements, respectively. We may identify these basis elements with the tensor product of two multivectors of grade ℓ_1 and ℓ_2, namely

$$\mathbf{w}_{I_1,I_2} = \mathbf{e}_{I_1} \otimes \mathbf{e}_{I_2}. \tag{14}$$

The dimension of the vector space spanned by these basis elements is $\binom{k+n}{\ell_1}\binom{k+n}{\ell_2}$; the elements of this vector space can be identified with matrices $\mathbf{A}$ whose rows and columns are indexed by lists, $I_1 \in \mathcal{I}_{\ell_1}$ and $I_2 \in \mathcal{I}_{\ell_2}$, respectively,

$$\mathbf{A} = \sum_{I_1 \in \mathcal{I}_{\ell_1}, I_2 \in \mathcal{I}_{\ell_2}} a_{I_1 I_2} \mathbf{w}_{I_1,I_2}. \tag{15}$$

The transpose of a matrix element $\mathbf{w}_{I_1,I_2}$, denoted as $\mathbf{w}^T_{I_1,I_2}$, is defined as $\mathbf{w}_{I_2,I_1}$. These matrices, the underlying vector space, and the operations that we describe next are fundamental in the study of changes of coordinates in space-time. However, consideration of these changes is beyond the scope of this paper. To any extent, this short section provides a perspective on matrices from the point of view of exterior algebra, highlighting the connections between multivectors and matrices, and bypassing the standard introduction of tensor fields.

As we did with multivectors, we consider the dot product $\cdot$ of two arbitrary matrix basis elements $\mathbf{w}_{I_1,I_2}$ and $\mathbf{w}_{J_1,J_2}$. This dot product is written

$$\mathbf{w}_{I_1,I_2} \cdot \mathbf{w}_{J_1,J_2} = \Delta_{I_1 J_1} \Delta_{I_2 J_2}. \tag{16}$$

The ordering within the pairs (I_1, I_2) and (J_1, J_2) is important in (16). This dot product, when applied to two matrices, is seen to give their Frobenius inner product, or equivalently, the square of the Frobenius norm (also known as the Hilbert–Schmidt norm) [20] when the product is of a matrix with itself.

We also define the matrix product $\times$ between two matrix basis elements $\mathbf{w}_{I,J}$ and $\mathbf{w}_{K,L}$ as

$$\mathbf{w}_{I,J} \times \mathbf{w}_{K,L} = \mathbf{w}_{I,L} \Delta_{JK}, \tag{17}$$

an operation that coincides with the standard product of two matrices for matrices labeled by spatial indices. For square matrices $\mathbf{A}$ indexed by grade m multivectors, it is natural to define the matrix inverse (whenever the inverse exists), denoted as $\mathbf{A}^{-1}$, such that $\mathbf{A}^{-1} \times \mathbf{A} = \mathbf{I}_m = \mathbf{A} \times \mathbf{A}^{-1}$, where the grade ℓ square identity matrix, denoted by $\mathbf{I}_\ell$, is given by

$$\mathbf{I}_\ell = \sum_{I \in \mathcal{I}_\ell} \Delta_{II} \mathbf{w}_{I,I}. \tag{18}$$

Last, we define the matrix product $\times$ between a matrix $\mathbf{w}_{I,J}$ and a multivector $\mathbf{e}_K$ (or between a multivector $\mathbf{e}_K$ and the matrix $\mathbf{w}_{J,I}$, i.e., the transpose of $\mathbf{w}_{I,J}$) as

$$\mathbf{w}_{I,J} \times \mathbf{e}_K = \mathbf{e}_K \times \mathbf{w}_{J,I} = \mathbf{e}_I \Delta_{JK}, \tag{19}$$

a generalization of the idea of multiplication of a row (or column) vector by a matrix.

2.5 Exterior and Matrix Calculus

In vector calculus, extensive use is made of the partial time derivative, ∂_t, and the nabla operator ∇ of partial space derivatives. In our case, we need the generalization to (k, n) space-time to the differential vector operator $\boldsymbol{\partial}$, defined as $(-\partial_0, -\partial_1, \dots, -\partial_{k-1}, \partial_k, \dots, \partial_{k+n-1})$, that is,

$$\boldsymbol{\partial} = \sum_{i \in \mathcal{I}} \Delta_{ii} \mathbf{e}_i \partial_i. \tag{20}$$

As was done in [14] (Section 3), we define the exterior derivative, $\boldsymbol{\partial} \wedge \mathbf{a}$, of a given multivector field $\mathbf{a}$ of grade m as

$$\boldsymbol{\partial} \wedge \mathbf{a} = \sum_{i \in \mathcal{I}, I \in \mathcal{I}_m :\, i \notin I} \Delta_{ii} \sigma(i, I) \partial_i a_I \, \mathbf{e}_{i+I}. \tag{21}$$

The grade of the exterior derivative of $\mathbf{a}$ is $m + 1$, unless $m = k + n$, in which case the exterior derivative is zero. In addition, we define the interior derivative, $\boldsymbol{\partial} \lrcorner \mathbf{a}$, of $\mathbf{a}$ as

$$\boldsymbol{\partial} \lrcorner \mathbf{a} = \sum_{i \in \mathcal{I}, I \in \mathcal{I}_m :\, i \in I} \sigma(I \setminus i, i) \partial_i a_I \, \mathbf{e}_{I \setminus i}. \tag{22}$$

The grade of the interior derivative of $\mathbf{a}$ is $m - 1$, unless $m = 0$, in which case the interior derivative is zero.

The formulae for the exterior and interior derivatives allow us to recover some standard formulae in vector calculus. For a scalar function ϕ, its gradient is given by its exterior derivative $\nabla\phi = \boldsymbol{\partial} \wedge \phi$, while for a vector field $\mathbf{v}$, its divergence $\nabla \cdot \mathbf{v}$ is given by its interior derivative $\nabla \cdot \mathbf{v} = \boldsymbol{\partial} \rfloor \mathbf{v}$. Also, for a vector fields $\mathbf{v}$ in $\mathbf{R}^3$, taking into account [14] (Equation (18)), the curl can be variously expressed as $\nabla \times \mathbf{v} = (\nabla \wedge \mathbf{v})^{\mathcal{H}^{-1}} = \nabla \rfloor \mathbf{v}^{\mathcal{H}^{-1}} = \nabla \rfloor \mathbf{v}^{\mathcal{H}}$, thereby generalizing both the cross product and the curl to grade m vector fields in space-time algebras with different dimensions. Specific vector calculus formulae such as that for the divergence of a gradient or the curl of the curl of a vector can be seen as instances of general exterior calculus formulae such as [14] (Equation (38)) and [15] (Equation (35)),

$$\boldsymbol{\partial} \rfloor (\mathbf{a} \wedge \mathbf{b}) = \mathbf{a}(\boldsymbol{\partial} \cdot \mathbf{b}) - (\boldsymbol{\partial} \cdot \mathbf{a})\mathbf{b}, \tag{23}$$

$$\boldsymbol{\partial} \cdot (\mathbf{a} \rfloor \mathbf{b}) = (\boldsymbol{\partial} \wedge \mathbf{a}) \cdot \mathbf{b} + (-1)^{\mathrm{gr}(\mathbf{a})}(\boldsymbol{\partial} \rfloor \mathbf{b}) \cdot \mathbf{a}, \tag{24}$$

where in (23), $\mathbf{a}$ and $\mathbf{b}$ are 1-vectors, while in (24), $\mathbf{a}$ and $\mathbf{b}$ are $(s-1)$-vector and s-vector, respectively.

The exterior and interior derivatives satisfy the property $\boldsymbol{\partial} \wedge (\boldsymbol{\partial} \wedge \mathbf{a}) = 0 = \boldsymbol{\partial} \rfloor (\boldsymbol{\partial} \rfloor \mathbf{a})$, for a general twice-differentiable multivector field $\mathbf{a}$. These identities imply the well-known facts that the curl of the gradient and the divergence of the curl are zero.

The circulation $\mathcal{C}(\mathbf{a}, \mathcal{V}^\ell)$ and the flux $\mathcal{F}(\mathbf{a}, \mathcal{V}^\ell)$ of a multivector field $\mathbf{a}$ over an ℓ-dimensional space-time hypervolume $\mathcal{V}^\ell$ are defined as integrals of interior products of the field with infinitesimal integration volumes:

$$\mathcal{C}(\mathbf{a}, \mathcal{V}^\ell) = \int_{\mathcal{V}^\ell} \mathrm{d}^\ell \mathbf{x} \llcorner \mathbf{a}, \tag{25}$$

$$\mathcal{F}(\mathbf{a}, \mathcal{V}^\ell) = \int_{\mathcal{V}^\ell} \mathrm{d}^\ell \mathbf{x}^{\mathcal{H}^{-1}} \rfloor \mathbf{a}. \tag{26}$$

As a specific example for (26), the flux of a field over an $(k+n)$-dimensional hypervolume is the volume integral of the field. For both of these operations, the interior product in the integrand is expressed as a differential form, which allows us to invoke the theory of differential forms to prove a Stokes theorem. This Stokes theorem relates the circulation (resp. the flux) of the field over the boundary of some hypervolume to the circulation (resp. flux) over the same hypervolume of the exterior (resp. interior) derivative of the multivector field.

We also define the tensor derivative of $\mathbf{a}$, $\boldsymbol{\partial} \otimes \mathbf{a}$, of a given vector field $\mathbf{a}$ of grade m as

$$\boldsymbol{\partial} \otimes \mathbf{a} = \sum_{i \in \mathcal{I}, I \in \mathcal{I}_m} \Delta_{ii} \partial_i a_I \, \mathbf{w}_{i,I}, \tag{27}$$

where $\mathbf{w}_{i,I}$ is a matrix vector space basis element.

To conclude this section, we define a derivative operator with respect to an element of a vector space, e.g., a multivector field or a matrix. A relevant example of vector derivative operator is $\boldsymbol{\partial}$, where the derivative is taken with respect to the position vector $\mathbf{x}$. In general, the vector derivative operator with respect to a multivector field $\mathbf{a}$ of grade m (resp. matrix $\mathbf{A}$ of dimensions $\ell_1 \times \ell_2$) is a multivector field (resp. matrix) denoted by $\partial_{\mathbf{a}}$ (resp. $\partial_{\mathbf{A}}$) [21] and given by

$$\partial_{\mathbf{a}} = \frac{\partial}{\partial \mathbf{a}} = \sum_{I \in \mathcal{I}_m} \Delta_{II} \mathbf{e}_I \frac{\partial}{\partial a_I}, \tag{28}$$

$$\partial_{\mathbf{A}} = \frac{\partial}{\partial \mathbf{A}} = \sum_{I \in \mathcal{I}_{\ell_1}, J \in \mathcal{I}_{\ell_2}} \Delta_{II} \Delta_{JJ} \mathbf{w}_{I,J} \frac{\partial}{\partial a_{I,J}}. \tag{29}$$

Specifically, we shall later need the exterior vector derivative of a scalar function $g(\mathbf{x})$, denoted by $\partial_{\mathbf{a}} \wedge g(\mathbf{x})$ or with some abuse of notation simply by $\partial_{\mathbf{a}}\, g(\mathbf{x})$, and given by

$$\partial_{\mathbf{a}} \wedge g(\mathbf{x}) = \partial_{\mathbf{a}}\, g(\mathbf{x}) = \sum_{I \in \mathcal{I}_m} \Delta_{II} \mathbf{e}_I \frac{\partial g(\mathbf{x})}{\partial a_I}, \tag{30}$$

and similarly for the matrix derivative. This exterior vector derivative is thus some form of generalized gradient. We shall need the derivative of a scalar function given by a quadratic form in the field and/or its interior or exterior derivatives. Let $\mathbf{a}$ and $\mathbf{b}$ represent two vectors of the same grade. Evaluation of the vector derivatives is straightforward and coincides with the infinitesimal calculus expressions [21]:

$$\partial_{\mathbf{a}} (\mathbf{a} \cdot \mathbf{a}) = 2\mathbf{a} \tag{31}$$

$$\partial_{\mathbf{a}} (\mathbf{a} \cdot \mathbf{b}) = \mathbf{b}. \tag{32}$$

3 Principle of Stationary Action: Derivation of the Euler–Lagrange Equations

3.1 General Case: Lagrangian Dependent on the Tensor Derivative

As we briefly reviewed in the Introduction, in classical mechanics, one defines the action $\mathcal{S}$ as a scalar quantity, with units of energy $\times$ time, that encodes the dynamical evolution of a physical system. Mathematically, the action $\mathcal{S}$ is an integral functional of the trajectory or path over space-time, or of the Lagrangian density $\mathcal{L}(\mathbf{x})$ for field theories, followed by the physical system. The principle of stationary action states the the path actually followed by the system, e.g., the field dynamics, corresponds to a stationary point of the action [1] (Ch. 19), [2]

(Section 8).

In general, the application of the principle of stationary action gives the EulerLagrange equations, which describe the dynamics of the system [3] (Section I.3), [4] (Section 3.1), [5] (Section 7.2). We start by reviewing how to obtain these equations in coordinate-free form with tensorial notation. Differently from the usual approach, that gives the dynamics for the individual components of the field, our coordinate-free derivation directly works with some twice-differentiable multivector field $\mathbf{a}$ of grade s and its tensor derivative $\boldsymbol{\partial} \otimes \mathbf{a}$ in (27).

For a given region $\mathcal{R}$ that comprises the physical system under consideration, let the action $\mathcal{S}(\mathbf{a})$ be given by

$$\mathcal{S}(\mathbf{a}) = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\, \mathcal{L}(\mathbf{a}, \boldsymbol{\partial} \otimes \mathbf{a}). \tag{33}$$

We assume that the region $\mathcal{R}$ is large enough to make the physical system closed, and that the fields decay fast enough over $\mathcal{R}$ so that the flux of the fields over the boundary of $\mathcal{R}$ is negligible. We note that the Lagrangian density $\mathcal{L}$ is a real-valued function of the $\binom{k+n}{s}$ components of $\mathbf{a}$ and the $(k+n)\binom{k+n}{s}$ components of $\boldsymbol{\partial} \otimes \mathbf{a}$, and the Lagrangian density does not depend explicitly on the space-time components.

Let the field $\mathbf{a}$ be infinitesimally perturbed by an amount $\mathbf{a}_\varepsilon$, possibly dependent on the space-time coordinates, so that the field is transformed as $\mathbf{a} \to \mathbf{a}+\mathbf{a}_\varepsilon$ and the tensor derivative is transformed as $\boldsymbol{\partial}\otimes\mathbf{a} \to \boldsymbol{\partial}\otimes\mathbf{a}+\boldsymbol{\partial}\otimes\mathbf{a}_\varepsilon$. We assume that $\mathbf{a}_\varepsilon$ is twice differentiable. We can expand the Lagrangian density in a first-order multivariate Taylor series, where the matrix of partial derivatives with respect to the $(k+n+1)\binom{k+n}{s}$ variables is a block matrix having along the diagonal the vector derivatives $\partial_{\mathbf{a}}\mathcal{L}$ and $\partial_{\boldsymbol{\partial}\otimes\mathbf{a}}\mathcal{L}$ of the density $\mathcal{L}$ with respect to the field $\mathbf{a}$ and its tensorial derivative $\partial_{\boldsymbol{\partial}\otimes\mathbf{a}}$, defined in (28) and (29), respectively. Neglecting terms of second and higher order in the perturbation $\mathbf{a}_\varepsilon$ and grouping terms in the Taylor series yields

$$\mathcal{S}(\mathbf{a}+\mathbf{a}_\varepsilon) = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\, \Big(\mathcal{L} + (\partial_{\mathbf{a}}\mathcal{L}) \cdot \mathbf{a}_\varepsilon + (\partial_{\boldsymbol{\partial}\otimes\mathbf{a}}\mathcal{L}) \cdot (\boldsymbol{\partial} \otimes \mathbf{a}_\varepsilon)\Big). \tag{34}$$

We may thus evaluate the first-order change of action $\delta\mathcal{S}$ as

$$\delta\mathcal{S} = \mathcal{S}(\mathbf{a}+\mathbf{a}_\varepsilon) - \mathcal{S}(\mathbf{a}) = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\, \Big((\partial_{\mathbf{a}}\mathcal{L}) \cdot \mathbf{a}_\varepsilon + (\partial_{\boldsymbol{\partial}\otimes\mathbf{a}}\mathcal{L}) \cdot (\boldsymbol{\partial} \otimes \mathbf{a}_\varepsilon)\Big), \tag{35}$$

always neglecting all the contributions of order $(\mathbf{a}_\varepsilon)^2$ or higher in the action change.

Next, we note the following Leibniz product rule, an equality between scalar quantities proved in Appendix A.1, for a multivector field $\mathbf{a}$ and a matrix field $\mathbf{B}$ with basis $\mathbf{w}_{i,I}$, involving the product $\times$ defined in (19),

$$\boldsymbol{\partial} \cdot (\mathbf{B} \times \mathbf{a}) = (\boldsymbol{\partial} \times \mathbf{B}) \cdot \mathbf{a} + \mathbf{B} \cdot (\boldsymbol{\partial} \otimes \mathbf{a}). \tag{36}$$

Choosing $\mathbf{a} = \mathbf{a}_\varepsilon$ and $\mathbf{B} = \partial_{\boldsymbol{\partial}\otimes\mathbf{a}}\mathcal{L}$ in (36), we can then rewrite Equation (35) as

$$\delta\mathcal{S} = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\, \Big(\partial_{\mathbf{a}}\mathcal{L} - \boldsymbol{\partial} \times \big(\partial_{\boldsymbol{\partial}\otimes\mathbf{a}}\mathcal{L}\big)\Big) \cdot \mathbf{a}_\varepsilon + \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\, \boldsymbol{\partial} \cdot \Big(\big(\partial_{\boldsymbol{\partial}\otimes\mathbf{a}}\mathcal{L}\big) \times \mathbf{a}_\varepsilon\Big). \tag{37}$$

We identify the second integrand with a flux (26) over an $(k+n)$-dimensional hypervolume $\mathcal{R}$ and use the Stokes theorem [14] (Section 3.5) to rewrite the flux of the interior derivative of a vector field as the flux of the field itself across the region boundary $\partial\mathcal{R}$. The second integral in (37) then vanishes if we assume that the field $\mathbf{a}$ and its perturbation $\mathbf{a}_\varepsilon$ vanish sufficiently fast at infinity. Under this assumption, if the change of action is zero for any perturbation of the field $\mathbf{a}_\varepsilon$, the integrand in the first summand of (37) must be identically zero. Setting to zero the quantity between parentheses in the integrand yields the coordinate-free form of the Euler–Lagrangian equations,

$$\partial_{\mathbf{a}}\mathcal{L} = \boldsymbol{\partial} \times (\partial_{\boldsymbol{\partial}\otimes\mathbf{a}}\mathcal{L}), \tag{38}$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{a}} = \boldsymbol{\partial} \times \left(\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\partial}\otimes\mathbf{a})}\right). \tag{39}$$

Both expressions in (38) and (39) are equivalent since they only differ in the notation for the vector derivative. It is also possible to recover a component form of the Euler–Lagrange equations from (39) [3] (Section I.3), [4] (Section 3.1), [5] (Section 7.2). Explicitly writing out the definitions in (28) and (29), we obtain the standard formula for each $I \in \mathcal{I}_m$,

$$\frac{\partial\mathcal{L}}{\partial a_I} = \sum_{i\in\mathcal{I}} \partial_i \left(\frac{\partial\mathcal{L}}{\partial(\partial_i a_I)}\right). \tag{40}$$

In general, the use of coordinate-free expression as in (39) is closer to the common practice of vector calculus and allows us to better identify the algebraic structure of the underlying equations, which gets obscured when the components are used. Moreover, expressions as (39) are better suited to generalizations, or more properly, particularizations, to exterior calculus when the Lagrangian depends on the exterior and interior derivatives of the field, rather than the tensor derivative. This case is explored and analyzed in the next subsection.

3.2 Derivation of the Euler–Lagrange Equations in Exterior Algebraic Form

For electromagnetism, the Lagrangian density $\mathcal{L}(\mathbf{x})$ is a function of the vector potential $\mathbf{A}$, the bivector field $\mathbf{F}$, and the source density vector $\mathbf{J}$. Expressed in exterior calculus notation, the Lagrangian density is given by

$$\mathcal{L}(\mathbf{x}) \quad = -\tfrac{1}{2}\mathbf{F}\cdot\mathbf{F} + \mathbf{J}\cdot\mathbf{A}. \tag{41}$$

Remark 3.1. If the field is represented by an antisymmetric tensor of rank 2, the factor before $\mathbf{F}\cdot\mathbf{F}$ becomes $-\frac{1}{4}$ to account for the repeated sum over pairs of indices [3] (Section I.5), [4] (Section 3.5), [2] (Section 27). ¶

The Lagrangian density depends on the field through the potential $\mathbf{A}$ and its exterior derivative $\mathbf{F} = \boldsymbol{\partial}\wedge\mathbf{A}$ [15] (Section 3). Instead of using (39), which was derived from the assumption that the Lagrangian density depends explicitly only the field and its tensor derivative, it is worth obtaining the Euler–Lagrange equations when the Lagrangian density is a function of a generic multivector field $\mathbf{a}$ of grade s, and its exterior and interior derivatives.

As in (33), for a given region $\mathcal{R}$ that comprises the physical system under consideration, and assumed to be large enough to make the physical system closed so that the fields decay fast enough over $\mathcal{R}$ and the flux of the fields over the boundary of $\mathcal{R}$ is arbitrarily small, the action $\mathcal{S}(\mathbf{a})$ is given by the integral

$$\mathcal{S}(\mathbf{a}) = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\,\mathcal{L}(\mathbf{a},\boldsymbol{\partial}\wedge\mathbf{a},\boldsymbol{\partial}\lrcorner\,\mathbf{a}). \tag{42}$$

Again, for an infinitesimal perturbation of the field $\mathbf{a}_\varepsilon$, and neglecting all the contributions of order $(\mathbf{a}_\varepsilon)^2$ or higher in the Taylor expansion of the Lagrangian density and the action, the first-order change in action $\delta\mathcal{S}$ is given by

$$\delta\mathcal{S} = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\Big((\partial_{\mathbf{a}}\mathcal{L})\cdot\mathbf{a}_\varepsilon + (\partial_{\boldsymbol{\partial}\wedge\mathbf{a}}\mathcal{L})\cdot(\boldsymbol{\partial}\wedge\mathbf{a}_\varepsilon) + (\partial_{\boldsymbol{\partial}\lrcorner\mathbf{a}}\mathcal{L})\cdot(\boldsymbol{\partial}\lrcorner\,\mathbf{a}_\varepsilon)\Big). \tag{43}$$

From (35) in [15], given a vector $\mathbf{a}$ and a vector $\mathbf{b}$ of grade $\mathrm{gr}(\mathbf{a})+1$, the Leibniz product rule in (24) holds. Choosing $\mathbf{a}=\mathbf{a}_\varepsilon$ and $\mathbf{b}=\partial_{\boldsymbol{\partial}\wedge\mathbf{a}}\mathcal{L}$ (resp. $\mathbf{a}=\partial_{\boldsymbol{\partial}\lrcorner\mathbf{a}}\mathcal{L}$ and $\mathbf{b}=\mathbf{a}_\varepsilon$) in the second (resp. third) summand inside the integral, substituting these values in (24) and the result back into (43), we obtain

$$\begin{aligned}\delta\mathcal{S} \quad &= \textstyle\int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\Big(\partial_{\mathbf{a}}\mathcal{L} + (-1)^{s+1}\boldsymbol{\partial}\lrcorner\,(\partial_{\boldsymbol{\partial}\wedge\mathbf{a}}\mathcal{L}) \quad - (-1)^{s-1}\boldsymbol{\partial}\wedge(\partial_{\boldsymbol{\partial}\lrcorner\mathbf{a}}\mathcal{L})\Big)\cdot\mathbf{a}_\varepsilon \\ &\quad + \textstyle\int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\,\boldsymbol{\partial}\cdot\Big(\mathbf{a}_\varepsilon\lrcorner\,(\partial_{\boldsymbol{\partial}\wedge\mathbf{a}}\mathcal{L}) \quad + (-1)^{s}(\partial_{\boldsymbol{\partial}\lrcorner\mathbf{a}}\mathcal{L})\lrcorner\,\mathbf{a}_\varepsilon\Big).\end{aligned} \tag{44}$$

In the second integrand, a flux over the $(k+n)$-dimensional region $\mathcal{R}$, the Stokes theorem [14] (Section 3.5) allows us to rewrite the flux of the interior derivative as the flux across the region boundary $\partial\mathcal{R}$. As both the field $\mathbf{a}$ and its perturbation $\mathbf{a}_\varepsilon$ vanish sufficiently fast at infinity, the first-order change in action is given by

$$\delta\mathcal{S} = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\left(\partial_{\mathbf{a}}\mathcal{L} + (-1)^{s+1}\boldsymbol{\partial}\lrcorner\,(\partial_{\boldsymbol{\partial}\wedge\mathbf{a}}\mathcal{L}) - (-1)^{s-1}\boldsymbol{\partial}\wedge(\partial_{\boldsymbol{\partial}\lrcorner\mathbf{a}}\mathcal{L})\right)\cdot\mathbf{a}_\varepsilon. \tag{45}$$

and the principle of stationary action, namely that the first-order change in action identically vanishes, leads to the coordinate-free form of the Euler–Lagrange equations, in one of the two equivalent forms:

$$\partial_{\mathbf{a}}\mathcal{L} = (-1)^s \boldsymbol{\partial} \lrcorner (\partial_{\boldsymbol{\partial}\wedge\mathbf{a}}\mathcal{L}) - (-1)^s \boldsymbol{\partial} \wedge (\partial_{\boldsymbol{\partial}\lrcorner\mathbf{a}}\mathcal{L}) \tag{46}$$

$$\frac{\partial\mathcal{L}}{\partial\mathbf{a}} = (-1)^s \boldsymbol{\partial} \lrcorner \left(\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\partial}\wedge\mathbf{a})}\right) - (-1)^s \boldsymbol{\partial} \wedge \left(\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\partial}\lrcorner\mathbf{a})}\right). \tag{47}$$

It might appear that the tensorial and multivectorial expressions in (39) and (47) differ. If the Lagrangian density depends on the tensor derivative only through the interior and exterior derivatives, we verify in Appendix A.2 that both expressions are indeed identical and the following identity holds:

$$\boldsymbol{\partial} \times \left(\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\partial}\otimes\mathbf{a})}\right) = (-1)^s \boldsymbol{\partial} \lrcorner \left(\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\partial}\wedge\mathbf{a})}\right) - (-1)^s \boldsymbol{\partial} \wedge \left(\frac{\partial\mathcal{L}}{\partial(\boldsymbol{\partial}\lrcorner\mathbf{a})}\right). \tag{48}$$

4 Application to Generalized Electromagnetism: Maxwell Equations

4.1 Generalized Maxwell Equations

As application of the methods derived in the previous section, we study the generalized Maxwell equations [15] and their associated fields. For a given natural number r, the Maxwell field $\mathbf{F}(\mathbf{x})$ and the generalized source density $\mathbf{J}(\mathbf{x})$ are respectively characterized by multivector fields of grade r and $r-1$ at every point $\mathbf{x}$ of the flat (k, n)-space-time [15] (Section 3). The potential field $\mathbf{A}(\mathbf{x})$ is a multivector field of grade $r-1$ such that

$$\mathbf{F} = \boldsymbol{\partial} \wedge \mathbf{A}. \tag{49}$$

If we replace the potential $\mathbf{A}$ by a new field $\mathbf{A}' = \mathbf{A} + \bar{\mathbf{A}} + \boldsymbol{\partial} \wedge \mathbf{G}$, where $\bar{\mathbf{A}}$ is a constant $(r-1)$-vector and $\mathbf{G}$ is an $(r-2)$-vector gauge field, the homogenous Maxwell Equation (51) is unchanged [15] (Section 3). For a given Maxwell field, there is therefore some unavoidable (gauge) ambiguity on the value of the vector potential.

Scalar fields are given by the vector potential by setting $r = 1$ in Minkowski space-time, namely $k = 1$ and $n = 3$. For classical electromagnetism ($r = 2$, $k = 1$, $n = 3$), the bivector field is usually expressed as an antisymmetric tensor of rank 2; electrostatics and magnetostatics are recovered for $k = 0$, $n = 3$, by setting $r = 1$ and $r = 2$, respectively. The generalized Maxwell equations for arbitrary values of r, k, and n are the following pair of coupled differential equations:

$$\boldsymbol{\partial} \lrcorner \mathbf{F} = \mathbf{J}, \tag{50}$$

$$\boldsymbol{\partial} \wedge \mathbf{F} = 0. \tag{51}$$

The interior derivative in (50) and the exterior derivative in (51) are respectively defined in (22) and (21). As we stated in Section 2.5, the interior derivative lowers the grade by one, while the exterior derivative increases the grade by one; therefore, Equation (50) is an identity of $(r-1)$-vectors while Equation (51) is an identity of $(r+1)$-vectors.

4.2 Lagrangian Density for Generalized Electromagnetism

For electromagnetism, the Lagrangian density $\mathcal{L}(\mathbf{x})$ is a function of the potential $\mathbf{A}$, the Maxwell field $\mathbf{F}$, and the source density $\mathbf{J}$. Expressed in exterior calculus notation, we postulate the generalized Lagrangian density to be

$$\mathcal{L}(\mathbf{x}) = \frac{(-1)^{r-1}}{2}\mathbf{F}\cdot\mathbf{F} + \mathbf{J}\cdot\mathbf{A}. \tag{52}$$

For classical electromagnetism ($r = 2$, $k = 1$, $n = 3$), if the field is expressed as an antisymmetric tensor of rank 2 the factor before $\mathbf{F}\cdot\mathbf{F}$ becomes $-\frac{1}{4}$, see Remark 1. In contrast, for electrostatics ($r = 1$, $k = 0$, $n = 3$), the Lagrangian density is given by $\mathcal{L} = \frac{1}{2}\mathbf{E}\cdot\mathbf{E} + \rho\phi$, where $\mathbf{E}$ is the electric field, ρ the charge density, and ϕ is the opposite in sign of the usual electric potential, that is, $\mathbf{E} = \boldsymbol{\partial} \wedge \phi = \nabla\phi$ [1] (Ch. 19).

While the Lagrangian in (52) leads to the generalized Maxwell Equations (50) and (51), as we shall see in the following section, it is not the most general Lagrangian associated to electromagnetism. Two terms that can be added to it respectively deal with the hypothetical mass of the photon, that is, the Proca term [4] (p. 107),

[22] (Section 12.8), and a gauge-fixing term that appears in the context of quantization of the electromagnetic field [3] (Section II.7), [4] (Section 7.1). This general Lagrangian density for electromagnetism is now given by

$$\mathcal{L}(\mathbf{x}) = \frac{(-1)^{r-1}}{2}\mathbf{F}\cdot\mathbf{F} + \mathbf{J}\cdot\mathbf{A} - \frac{1}{2}m^2\mathbf{A}\cdot\mathbf{A} + \frac{(-1)^{r-1}}{2\xi}(\boldsymbol{\partial}\,\lrcorner\,\mathbf{A})\cdot(\boldsymbol{\partial}\,\lrcorner\,\mathbf{A}), \tag{53}$$

where m is the hypothetical photon mass and ξ is a parameter that determines the so-called R_ξ gauge; for $\xi = 1$, we have the Feynman gauge, and in the limit $\xi \to 0$, we have the Landau gauge.

4.3 Euler–Lagrange Equations

For Lagrangian densities such as (52) or (53), which are essentially quadratic forms in the field and/or its interior or exterior derivatives, evaluation of the vector derivatives is straightforward, as the derivative has the same form as that obtained in infinitesimal calculus for the derivative of a polynomial (31) and (32). For the Lagrangian density in (52), evaluation of the derivatives in the Euler–Lagrange Equation (47) give

$$\partial_{\mathbf{A}}\mathcal{L} = \mathbf{J}, \tag{54}$$

$$\partial_{\boldsymbol{\partial}\wedge\mathbf{A}}\mathcal{L} = (-1)^{r-1}(\boldsymbol{\partial}\wedge\mathbf{A}), \tag{55}$$

from which the Euler–Lagrange equations themselves (46), with $s = r-1$, can be expressed as

$$\mathbf{J} = \boldsymbol{\partial}\,\lrcorner\,(\boldsymbol{\partial}\wedge\mathbf{A}) = \boldsymbol{\partial}\,\lrcorner\,\mathbf{F}, \tag{56}$$

namely the generalized nonhomogenous Maxwell Equation (50) for arbitrary r, k, and n. The homogeneous Maxwell Equation (51) is also satisfied as a consequence of the definition of $\mathbf{F} = \boldsymbol{\partial}\wedge\mathbf{A}$.

The exterior algebraic formulation of the Lagrangian and the Euler–Lagrange equations brings the advantage of allowing for a more direct derivation of the Maxwell equations, since evaluation of the vector derivatives mimicks more closely the steps carried out in usual differential calculus to evaluate the derivatives.

The factor $(-1)^{r-1}$ in the Lagrangian density is needed to compensate for the identical term $(-1)^{r-1}$ that appears in the Euler–Lagrange Equation (47). An alternative way of writing the Lagrangian density, without this sign factor, would involve replacing one of the exterior derivatives $\boldsymbol{\partial}\wedge\mathbf{A}$ by a right exterior derivative $\mathbf{A}\wedge\boldsymbol{\partial}$, where the partial derivative operator is understood to act from the right on the potential. In this case, the skew commutativity of the wedge product, $\boldsymbol{\partial}\wedge\mathbf{A} = (-1)^{r-1}(\mathbf{A}\wedge\boldsymbol{\partial})$ [14] (Section 2.2), cancels the sign in the Lagrangian density and results in a somewhat neater expression for it.

As for the Lagrangian density in (53), evaluation of the derivatives in the Euler–Lagrange Equation (47) give

$$\partial_{\mathbf{A}}\mathcal{L} = \mathbf{J} - m^2\mathbf{A}, \tag{57}$$

$$\partial_{\boldsymbol{\partial}\wedge\mathbf{A}}\mathcal{L} = (-1)^{r-1}(\boldsymbol{\partial}\wedge\mathbf{A}), \tag{58}$$

$$\partial_{\boldsymbol{\partial}\lrcorner\mathbf{A}}\mathcal{L} = (-1)^{r-1}\frac{1}{\xi}(\boldsymbol{\partial}\,\lrcorner\,\mathbf{A}), \tag{59}$$

from which the Euler–Lagrange Equation (46) with the Proca and quantization R_ξ-gauge terms become

$$\boldsymbol{\partial}\,\lrcorner\,(\boldsymbol{\partial}\wedge\mathbf{A}) + m^2\mathbf{A} = \mathbf{J} + \frac{1}{\xi}\boldsymbol{\partial}\wedge(\boldsymbol{\partial}\,\lrcorner\,\mathbf{A}). \tag{60}$$

Using the relationship (34) in [15], we may rewrite (60) in an alternative form with a wave equation,

$$(-1)^{r-1}(\boldsymbol{\partial}\cdot\boldsymbol{\partial})\mathbf{A} + m^2\mathbf{A} = \mathbf{J} + \left(\frac{1}{\xi} - 1\right)\boldsymbol{\partial}\wedge(\boldsymbol{\partial}\,\lrcorner\,\mathbf{A}), \tag{61}$$

which somewhat simplifies in the Feynman gauge, for which $\xi = 1$.

4.4 Dual Generalized Maxwell Equations

An interesting dual form of Maxwell equations is obtained by swapping the roles played by the interior and exterior derivatives in the Lagrangian density and the Maxwell equations themselves. Let the "potential" $\bar{\mathbf{A}}$ and "source density" $\bar{\mathbf{J}}$ be multivectors of grade s, and let us define a dual Maxwell field $\bar{\mathbf{F}}$ of grade $r = s-1$ by $\bar{\mathbf{F}} = \boldsymbol{\partial}\,\lrcorner\,\bar{\mathbf{A}}$. The Lagrangian density is now given by

$$\mathcal{L}(\mathbf{x}) = \frac{(-1)^r}{2}(\boldsymbol{\partial}\,\lrcorner\,\bar{\mathbf{A}})\cdot(\boldsymbol{\partial}\,\lrcorner\,\bar{\mathbf{A}}) + \bar{\mathbf{J}}\cdot\bar{\mathbf{A}} \tag{62}$$

$$= \frac{1}{2}(\boldsymbol{\partial}\,\lrcorner\,\bar{\mathbf{A}})\cdot(\bar{\mathbf{A}}\,\llcorner\,\boldsymbol{\partial}) + \bar{\mathbf{J}}\cdot\bar{\mathbf{A}}, \tag{63}$$

where we used the relationship between left and right interior derivatives $\boldsymbol{\partial} \lrcorner \bar{\mathbf{A}} = (-1)^{s+1}(\bar{\mathbf{A}} \llcorner \boldsymbol{\partial})$ to write (63). Direct evaluation of the Euler–Lagrange Equation (47), with $r = s - 1$, gives

$$\bar{\mathbf{J}} = \boldsymbol{\partial} \wedge \bar{\mathbf{F}}. \tag{64}$$

This nonhomogeneous "Maxwell" equation is complemented by a homogeneous equation $\boldsymbol{\partial} \lrcorner \bar{\mathbf{F}} = 0$, itself a consequence of the definition of $\bar{\mathbf{F}}$ as $\bar{\mathbf{F}} = \boldsymbol{\partial} \lrcorner \bar{\mathbf{A}}$.

As it happened with the generalized Maxwell equations, the exterior algebraic formulation of the Lagrangian and the Euler–Lagrange equations allows for a more direct derivation of the dual Maxwell equations. An interesting question, which we do not dwell upon as it lies beyond the scope of this paper, is whether the physics of the dual Maxwell equations is different from the usual Maxwell equations, or simply involves a transformation of the fields, potential, and source density, with no new phenomena. Along this direction, and leaving the details left as an exercise to the reader, it is relatively easy to verify that one obtains a wave equation relating $\bar{\mathbf{A}}$ and $\bar{\mathbf{J}}$ in a "Lorenz gauge" where $\boldsymbol{\partial} \wedge \bar{\mathbf{A}} = 0$. Solutions to this wave equation have several independent degrees of freedom or polarizations. The number of these polarizations is $\binom{k+n-2}{r-1}$, as for the standard Maxwell Equation [15] (Section 4.3); this number can be justified as the number of possible $(r+1)$-vectors where two dimensions, one temporal and one spatial are fixed, and the remaining $r + 1 - 2 = r - 1$ indices have to be filled with $k + n - 2$ possible values. We also have a "Lorentz force" density $\mathbf{f} = \bar{\mathbf{J}} \llcorner \bar{\mathbf{F}}$ such that a conservation law holds for the stress–energy-momentum tensor $\mathbf{T}_{\text{em}}$ of the field [15] (Appendix A.2), as for the usual Maxwell equations.

A

A.1 Proof of the Leibniz Product Rule in (36)

Let us consider a multivector field $\mathbf{a}$ of grade m and a matrix field $\mathbf{B}$ with basis $\mathbf{w}_{i,I} = \mathbf{e}_i \otimes \mathbf{e}_I$, where $|I| = m$. Using the definitions of dot and matrix product (4) and (19), the first term of the right-hand side of (36) is evaluated as

$$(\boldsymbol{\partial} \times \mathbf{B}) \cdot \mathbf{a} = \left(\sum_{i,j \in \mathcal{I}, J \in \mathcal{I}_m} \Delta_{ii} \partial_i b_{j,J}\, \mathbf{e}_i \times (\mathbf{e}_j \otimes \mathbf{e}_J) \right) \cdot \left(\sum_{I \in \mathcal{I}_m} a_I\, \mathbf{e}_I \right) \tag{65}$$

$$= \left(\sum_{i \in \mathcal{I}, J \in \mathcal{I}_m} \partial_i b_{i,J}\, \mathbf{e}_J \right) \cdot \left(\sum_{I \in \mathcal{I}_m} a_I\, \mathbf{e}_I \right) \tag{66}$$

$$= \sum_{i \in \mathcal{I}, I \in \mathcal{I}_m} \Delta_{II} a_I \partial_i b_{i,I}, \tag{67}$$

where in (65), we wrote the components of $\boldsymbol{\partial}\mathbf{B}$ and $\mathbf{a}$, in (66), we computed the matrix product and removed the j index, and in (67), we carried out the dot product and removed the J index.

In turn, the second term in the right-hand side of (36) can similarly be evaluated using the dot and times products in (16) and (19) as

$$\mathbf{B} \cdot (\boldsymbol{\partial} \otimes \mathbf{a}) = \left(\sum_{i \in \mathcal{I}, I \in \mathcal{I}_m} b_{i,I}\, \mathbf{e}_i \otimes \mathbf{e}_I \right) \cdot \left(\sum_{j \in \mathcal{I}, J \in \mathcal{I}_m} \Delta_{jj} \partial_j a_J\, \mathbf{e}_j \otimes \mathbf{e}_J \right) \tag{68}$$

$$= \sum_{i \in \mathcal{I}, I \in \mathcal{I}_m} \Delta_{II} b_{i,I} \partial_i a_I. \tag{69}$$

Next, using again the definitions of dot and matrix product (4) and (17), the left-hand side of (36) becomes

$$\boldsymbol{\partial} \cdot (\mathbf{B} \times \mathbf{a}) = \left(\sum_{i \in \mathcal{I}} \Delta_{ii} \mathbf{e}_i \frac{\partial}{\partial x_i} \right) \cdot \left(\sum_{I \in \mathcal{I}_m} \sum_{j \in \mathcal{I}, J \in \mathcal{I}_m} a_I b_{j,J} (\mathbf{e}_j \otimes \mathbf{e}_J) \times \mathbf{e}_I \right) \tag{70}$$

$$= \left(\sum_{i \in \mathcal{I}} \Delta_{ii} \mathbf{e}_i \frac{\partial}{\partial x_i} \right) \cdot \left(\sum_{j \in \mathcal{I}, I \in \mathcal{I}_m} a_I b_{j,I} \Delta_{II} \mathbf{e}_j \right) \tag{71}$$

$$= \sum_{i \in \mathcal{I}, I \in \mathcal{I}_m} \Delta_{II} \partial_i \big(a_I b_{i,I} \big). \tag{72}$$

Summing (67) and (69) and applying the rule for the derivative of a product yields the desired (36).

A.2 Identity between Tensorial and Exterior Algebraic Euler–Lagrange Equations

Since both the exterior and interior derivatives are surjective linear functions of the tensor derivative, the respective vector derivatives of the Lagrangian density are related. Each component of the exterior and interior derivatives (21) and (22) is a scalar (affine) function of several distinct components of the tensor derivative. The Lagrangian density depends on the components of the tensor derivative only through these scalar functions. We thus need to compute the derivative of a function $\mathcal{L}\big(g_1(\mathbf{z}),\dots,g_\ell(\mathbf{z})\big)$, where $\mathbf{z}$ stands for a vector with the $\ell' = (k+n)\binom{k+n}{s}$ components of the tensor derivative, and $(g_1,\dots,g_\ell)$ are the (differentiable) functions that give the components of the exterior (resp. interior) derivative from the tensor derivative, where $\ell = \binom{k+n}{s+1}$ (resp. $\ell = \binom{k+n}{s-1}$). By construction, a given $z_k = \partial_j a_J$ appears only in one $g_i(\mathbf{z})$, the I component of either the exterior or the interior derivative. In the former case, $I = j + J$, in the latter case $I = J \setminus j$. .

From the definition of partial derivative, and for any $i = 1,\dots,\ell$, we have the relation

$$\frac{\partial \mathcal{L}}{\partial g_i(\mathbf{z})} = \lim_{h\to 0} \frac{\mathcal{L}\big(g_1(\mathbf{z}),\dots,g_i(\mathbf{z})+h,\dots,g_\ell(\mathbf{z})\big) - \mathcal{L}\big(g_1(\mathbf{z}),\dots,g_\ell(\mathbf{z})\big)}{h}. \tag{73}$$

Then, assuming that $\frac{\partial g_i(\mathbf{z})}{\partial z_k} \neq 0$ and defining $h'_{ik} = \frac{h}{\partial g_i/\partial z_k}$, we can then write for every value of k such that the partial derivative $z_k = \partial_j a_J$ appears $g_i(\mathbf{z})$,

$$g_i(\mathbf{z}) + h = g_i(\mathbf{z}) + h'_{ik}\frac{\partial g_i(\mathbf{z})}{\partial z_k} \simeq g_i(z_1,\dots,z_k + h'_{ik},\dots,z_{\ell'}), \tag{74}$$

where we used the differentiablility of the function g_i. Substituting (74) back into (73) yields

$$\frac{\partial \mathcal{L}}{\partial g_i(\mathbf{z})} = \frac{1}{\partial g_i/\partial z_k} \lim_{h'_{ik}\to 0} \frac{\mathcal{L}\big(g_1(\mathbf{z}),\cdots,g_i(z_1,\cdots,z_k+h'_{ik},\cdots,z_{\ell'}),\cdots,g_\ell(\mathbf{z})\big) - \mathcal{L}\big(g_1(\mathbf{z}),\cdots,g_\ell(\mathbf{z})\big)}{h'_{ik}}. \tag{75}$$

Since the z_k component appears only in one of the functions g_i, the limit in (75) is the partial derivative of the Lagrangian density with respect to the kth component of the tensor derivative, for any k, that is,

$$\frac{\partial \mathcal{L}}{\partial g_i(\mathbf{z})} = \frac{1}{\partial g_i/\partial z_k}\frac{\partial \mathcal{L}}{\partial z_k}. \tag{76}$$

We now proceed to evaluate the vector derivative with respect to the exterior derivative. First, we have

$$\frac{\partial \mathcal{L}}{\partial(\boldsymbol{\partial}\wedge\mathbf{a})} = \sum_{I\in\mathcal{I}_{s+1}} \frac{\partial \mathcal{L}}{\partial(\boldsymbol{\partial}\wedge\mathbf{a})_I}\Delta_{II}\mathbf{e}_I, \tag{77}$$

where the Ith component of the exterior derivative, $(\boldsymbol{\partial}\wedge\mathbf{a})_I$ is the equivalent of $g_i(\mathbf{z})$ in (76). The equivalent of k is any pair of j and $J \in \mathcal{I}_s$ such that $I = j + J$ and the corresponding z_k is $\partial_j a_J$. The partial derivative $\partial g_i/\partial z_k$ in (76) is thus $\Delta_{jj}\sigma(j,J)$, of value ± 1, and we therefore have for any pair of i and $J \in \mathcal{I}_s$ such that $I = j + J$ that

$$\frac{\partial \mathcal{L}}{\partial(\boldsymbol{\partial}\wedge\mathbf{a})_I} = \frac{1}{\Delta_{jj}\sigma(j,J)}\frac{\partial \mathcal{L}}{\partial(\partial_j a_J)} = \Delta_{jj}\sigma(j,J)\frac{\partial \mathcal{L}}{\partial(\partial_j a_J)}. \tag{78}$$

Substituting (78) back in (77) yields

$$\frac{\partial \mathcal{L}}{\partial(\boldsymbol{\partial}\wedge\mathbf{a})} = \sum_{I\in\mathcal{I}_{s+1}} \frac{\partial \mathcal{L}}{\partial(\partial_j a_J)}\Delta_{jj}\Delta_{II}\sigma(j,J)\mathbf{e}_I, \tag{79}$$

where j and J are any pair such that $I = j + J$. Now, taking the interior derivative of (79), we obtain for any j, J such that $I = j + J$,

$$\boldsymbol{\partial} \lrcorner \left(\frac{\partial \mathcal{L}}{\partial(\boldsymbol{\partial}\wedge\mathbf{a})}\right) = \sum_{i\in\mathcal{I}}\Delta_{ii}\mathbf{e}_i\partial_i \lrcorner \left(\sum_{I\in\mathcal{I}_{s+1}} \frac{\partial \mathcal{L}}{\partial(\partial_j a_J)}\Delta_{jj}\Delta_{II}\sigma(j,J)\mathbf{e}_I\right) \tag{80}$$

$$= \sum_{i\in\mathcal{I}, I\in\mathcal{I}_{s+1}: i\in I} \Delta_{ii}\Delta_{II}\partial_i\left(\frac{\partial \mathcal{L}}{\partial(\partial_i a_{I\setminus i})}\right)\sigma(i, I\setminus i)\sigma(I\setminus i, i)\mathbf{e}_{I\setminus i}, \tag{81}$$

where we have selected $j = i$ and, therefore, $J = I \setminus i$. We also note that $\sigma(i, I\setminus i)\sigma(I\setminus i, i) = (-1)^s$.

We now evaluate the vector derivative with respect to the interior derivative in an analogous manner,

$$\frac{\partial \mathcal{L}}{\partial(\boldsymbol{\partial}\lrcorner\mathbf{a})} = \sum_{I\in\mathcal{I}_{s-1}} \frac{\partial \mathcal{L}}{\partial(\partial_j a_J)}\Delta_{II}\sigma(J\setminus j, j)\mathbf{e}_I, \tag{82}$$

where j and J are any pair such that $I = J \setminus j$. Now, taking the exterior derivative of (82), we obtain, for any j, J such that $I = J \setminus j$,

$$\boldsymbol{\partial} \wedge \left(\frac{\partial \mathcal{L}}{\partial(\boldsymbol{\partial} \lrcorner \mathbf{a})} \right) = \sum_{i \in \mathcal{I}} \Delta_{ii} \mathbf{e}_i \partial_i \wedge \left(\sum_{I \in \mathcal{I}_{s-1}} \frac{\partial \mathcal{L}}{\partial(\partial_j a_J)} \Delta_{II} \sigma(J \setminus j, j) \mathbf{e}_I \right) \tag{83}$$

$$= \sum_{i \in \mathcal{I}, I \in \mathcal{I}_{s-1} : i \notin I} \Delta_{ii} \Delta_{II} \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i a_{i+I})} \right) \sigma(I, i) \sigma(i, I) \mathbf{e}_{i+I}, \tag{84}$$

where we have selected $j = i$ and, therefore, $J = i + I$. We also note that $\sigma(I, i)\sigma(i, I) = (-1)^{s-1}$.

Putting (81) and (84), as well as the relationships between the product of permutation signatures, back into the right-hand side of (48) yields the expression

$$\sum_{i \in \mathcal{I}, I \in \mathcal{I}_{s+1} : i \in I} \Delta_{ii} \Delta_{II} \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i a_{I \setminus i})} \right) \mathbf{e}_{I \setminus i} + \sum_{i \in \mathcal{I}, I \in \mathcal{I}_{s-1} : i \notin I} \Delta_{ii} \Delta_{II} \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i a_{i+I})} \right) \mathbf{e}_{i+I}. \tag{85}$$

Since the basis elements are multivectors with s components, we may rewrite (85) as

$$\sum_{i \in \mathcal{I}, I \in \mathcal{I}_s : i \notin I} \Delta_{II} \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i a_I)} \right) \mathbf{e}_I + \sum_{i \in \mathcal{I}, I \in \mathcal{I}_s : i \in I} \Delta_{II} \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i a_I)} \right) \mathbf{e}_I. \tag{86}$$

The two summations in (85) might be further combined in a single summation over $I \in \mathcal{I}_s$. The resulting expression coincides with the left-hand side of (48), which can be expanded using the computation in (40) into

$$\sum_{i, I \in \mathcal{I}_s} \Delta_{II} \partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i a_I)} \right) \mathbf{e}_I. \tag{87}$$

Bibliography

[1] R. P. Feynman, R. B. Leighton, and M. Sands, The Feynman Lectures on Physics, Vol. II: Mainly Electromagnetism and Matter. Addison-Wesley, 1977.

[2] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields, 4th ed., ser. Course of Theoretical Physics. Butterworth-Heinemann, 1987, vol. 2.

[3] A. Zee, Quantum Field Theory in a Nutshell, ser. Nutshell handbook. Princeton, NJ: Princeton Univ. Press, 2003.

[4] M. Maggiore, A Modern Introduction to Quantum Field Theory, ser. Oxford Master Series in Statistical, Computational, and Theoretical Physics. Oxford: Oxford Univ. Press, 2005.

[5] S. Weinberg, The Quantum Theory of Fields, Volume 1: Foundations. Cambridge University Press, 2005.

[6] C. Lanczos, The Variational Principles of Mechanics, 4th ed. University of Toronto Press Toronto, 1970.

[7] J. W. Gibbs and E. B. Wilson, Vector Analysis. New Haven, Yale University Press, 1929.

[8] W. K. Clifford, Mathematical Papers. London: Macmillan, 1882.

[9] C. Doran and A. Lasenby, Geometric Algebra for Physicists. Cambridge: Cambridge University Press, 2003.

[10] M. M. G. Ricci and T. Levi-Civita, "Méthodes de calcul différentiel absolu et leurs applications," Mathematische Annalen, vol. 54, no. 1-2, pp. 125–201, 1900.

[11] E. Cartan, Les Systemes Differentiels Exterieurs Et Leurs Applications Geometriques. Hermann & Cie, 1945.

[12] V. I. Arnold, A. Weinstein, and K. Vogtmann, Mathematical Methods Of Classical Mechanics. Springer, 1989.

[13] H. Grassmann, Extension Theory, ser. History of Mathematics Sources. Providence, R.I.: American Mathematical Society, 2000, no. 19.

[14] I. Colombaro, J. Font-Segura, and A. Martinez, "An introduction to space–time exterior calculus," Mathematics, vol. 7, pp. 564–583, Jun. 2019.

[15] ——, "Generalized Maxwell equations for exterior-algebra multivectors in (k, n) space-time dimensions," European Physical Journal Plus, vol. 135, no. 305, Mar. 2020.

[16] A. Lasenby, C. Doran, and S. Gull, "A multivector derivative approach to lagrangian field theory," Found. Phys., vol. 23, no. 10, pp. 1295–1327, 1993.

[17] A. Martinez, J. Font-Segura, and I. Colombaro, "An exterior-algebraic derivation of the symmetric stress-energy-momentum tensor in flat space-time," European Physical Journal Plus, 2021.

[18] D. Lovelock and H. Rund, Tensors, Differential Forms, and Variational Principles. Dover Publications, 1989.

[19] H. Flanders, Differential Forms with Applications to the Physical Sciences. Dover Publications, 1989.

[20] R. A. Horn and C. R. Johnson, Matrix Analysis, 2nd ed. Cambridge: Cambridge University Press, 2013.

[21] P. S. Dwyer and M. S. Macphail, "Symbolic matrix derivatives," Ann. Math. Statist., vol. 19, no. 4, pp. 517–534, 12 1948.

[22] J. D. Jackson, Classical Electrodynamics, 3rd ed. John Wiley & Sons, 1999.

An exterior-algebraic derivation of the symmetric stress-energy-momentum tensor in flat space-time *

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Abstract

This paper characterizes the symmetric rank-2 stress-energy-momentum tensor associated with fields whose Lagrangian densities are expressed as the dot product of two multivector fields, e. g., scalar or gauge fields, in flat space-time. The tensor is derived by a direct application of exterior-algebraic methods to deal with the invariance of the action to infinitesimal space-time translations; this direct derivation avoids the use of the canonical tensor. Formulas for the tensor components and for the tensor itself are derived for generic values of the multivector grade s and of the number of time and space dimensions, k and n, respectively. A simple, coordinate-free, closed-form expression for the interior derivative (divergence) of the symmetric stress-energy-momentum tensor is also obtained: this expression provides a natural generalization of the Lorentz force that appears in the context of the electromagnetic theory. Applications of the formulas derived in this paper to the cases of generalized electromagnetism, Proca action, Yang-Mills fields and conformal invariance are briefly discussed.

Keywords— exterior calculus, exterior algebra, stress-energy-momentum tensor, conservation laws, Lagrangian, multivector fields

1 Introduction: derivations of the symmetric stress-energy-momentum tensor

The stress-energy-momentum (SEM) tensor, a symmetric, rank-2 tensor indexed by pairs of space-time indices, describes the flux of energy-momentum across regions of space-time; in a related manner, its divergence is used to express the conservation law for energy-momentum of isolated systems. In a different context, general relativity, the manifestly symmetric SEM tensor is also the source of the gravitational field. Both reasons amply support the importance of this tensor in mathematical physics. For electromagnetism, the SEM tensor combines the density of energy (component with time-time indices), the Poynting vector (components with space-time indices) and the Maxwell stress tensor (components with space-space indices) [1, Sect. 33], [2, Sect. 12.10]. Moreover, the divergence of the SEM tensor is (minus) the Lorentz force [1, Sect. 33],[2, Sect. 12.10] that describes the effect of transfer of energy-momentum from the field to the charges.

In the context of field theories, the usual derivation of the SEM builds on the Lagrangian density and relates the invariance of the action with respect to infinitesimal space-time translations via Noether's theorem to the existence of a conserved current, which is in turn identified with the SEM tensor [3, 4], [5, Sect. 3.2], [6, Sect. 2.5]. However, the canonical tensor obtained in this way is not necessarily symmetric and not necessarily gauge-independent for gauge theories. In order to symmetrize the canonical tensor, the Belinfante-Rosenfeld procedure is often used [3, 4], [6, Sect. 2.5]. An alternative method that directly leads to a symmetric SEM

*This work has been funded in part by the Spanish Ministry of Science, Innovation and Universities under grants TEC2016-78434-C3-1-R and BES-2017-081360. Published in: ***Eur. Phys. J. Plus*** **136, 212 (2021). DOI:** 10.1140/epjp/s13360-021-01192-7.

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tensor builds on the invariance of the action to variations in the Einstein-Hilbert space-time metric [3, 4]. In parallel, the study of generalized SEM tensors using the formalism of fiber bundles is a fertile area of research in mathematical physics [7, 8].

In this paper, we consider field theories whose Lagrangian densities can be expressed as linear combinations of the dot products of some pairs of multivector fields $\mathbf{a}$ and $\mathbf{b}$ of the same grade. We apply exterior-algebraic methods to deal with the invariance of the action of closed systems under infinitesimal space-time translations; Sect. 2 provides a brief introduction to the necessary concepts of exterior algebra, including the formulation of the action for several field theories in exterior-algebraic terms. By means of a direct derivation that avoids the canonical tensor and yields a symmetric SEM tensor, we provide in Sect. 3 several alternative, equivalent formulas for the SEM components and for the SEM tensor $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$ itself for generic values of the multivector grade s,

$$\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = (-1)^s\big(\mathbf{a}\odot\mathbf{b} + \mathbf{a}\oslash\mathbf{b}\big), \tag{1}$$

where the operations $\odot$ and $\oslash$ are defined in Sect. 2.4. A simple, coordinate-free, closed-form expression for the interior derivative (divergence) of the SEM tensor, expressed in terms of the interior and exterior derivatives of multivector fields, respectively, denoted by $\boldsymbol{\partial}\lrcorner$ and $\boldsymbol{\partial}\wedge$ and defined in Sect. 2.2, is also obtained:

$$\boldsymbol{\partial}\lrcorner\,\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = \mathbf{a}\lrcorner\,(\boldsymbol{\partial}\wedge\mathbf{b}) + \mathbf{b}\lrcorner\,(\boldsymbol{\partial}\wedge\mathbf{a}) - \mathbf{a}\llcorner\,(\boldsymbol{\partial}\lrcorner\,\mathbf{b}) - \mathbf{b}\llcorner\,(\boldsymbol{\partial}\lrcorner\,\mathbf{a}). \tag{2}$$

This formula provides a natural generalization of the electromagnetic Lorentz force. Applications of the formulas derived in this paper to the cases of generalized electromagnetism (including scalar fields and electromagnetism), Proca action, Yang-Mills fields and conformal invariance are briefly discussed in Sect. 4.

2 Fundamentals of exterior algebra: notation and definitions

2.1 Multivectors

We consider a flat space-time $\mathbf{R}^{k+n}$ with k temporal dimensions and n spatial dimensions. We represent the canonical basis of this (k,n)- or $(k+n)$-space-time by $\{\mathbf{e}_i\}_{i=0}^{k+n-1}$; we adopt the convention that the first k indices, i. e., $i = 0,\dots,k-1$, correspond to time components while the indices $i = k,\dots,k+n-1$ represent space components. Space and time have the same units and the speed of light is $c = 1$. Points and position in space-time are denoted by $\mathbf{x}$, with components x_i in the canonical basis $\{\mathbf{e}_i\}_{i=0}^{k+n-1}$.

In exterior algebra, one considers vector field spaces whose basis elements $\mathbf{e}_I$ are indexed by lists $I = (i_1,\dots,i_m)$ drawn from $\mathcal{I}_m$, the set of all ordered lists with non-repeated m indices, with $m \in \{0, 1\dots, k+n\}$. We refer to elements of these vector field spaces as multivector fields of grade m. A multivector field $\mathbf{a}(\mathbf{x})$ of grade m, possibly a function of the position $\mathbf{x}$, with components $a_I(\mathbf{x})$ in the canonical basis $\{\mathbf{e}_I\}_{I\in\mathcal{I}_m}$ can be written as

$$\mathbf{a}(\mathbf{x}) = \sum_{I\in\mathcal{I}_m} a_I(\mathbf{x})\mathbf{e}_I. \tag{3}$$

The dimension of the vector space of all grade-m multivectors is $\binom{k+n}{m}$, the number of lists in $\mathcal{I}_m$. We denote by $|I|$ the length of a list I and by $\mathrm{gr}(\mathbf{a})$ the operation that returns the grade of a vector $\mathbf{a}$.

2.2 Operations on multivectors

We next define several operations acting on multivectors; our presentation loosely follows [9, Sect. 2 and 3] and [10, Sect. 2]. Introductions to exterior algebra from the perspective and language of differential forms can be found in [11], [12]. With no real loss of generality, we define the operations only for the canonical basis vectors, the operation acting on general multivectors being a mere extension by linearity of the former. First, the dot product $\cdot$ of two arbitrary grade-m basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ is defined as

$$\mathbf{e}_I\cdot\mathbf{e}_J = \Delta_{IJ} = \Delta_{i_1j_1}\Delta_{i_2j_2}\cdots\Delta_{i_mj_m}, \tag{4}$$

where I and J are the ordered lists $I = (i_1, i_2,\dots,i_m)$ and $J = (j_1, j_2,\dots,j_m)$ and $\Delta_{ij} = 0$ if $i \neq j$, and we let time unit vectors $\mathbf{e}_i$ have negative metric $\Delta_{ii} = -1$ and space unit vectors $\mathbf{e}_i$ have positive metric $\Delta_{ii} = +1$.

Let two basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ have grades $m = |I|$ and $m' = |J|$. Let $(I,J) = \{i_1,\dots,i_m, j_1,\dots,j_{m'}\}$ be the concatenation of I and J, let $\sigma(I,J)$ denote the signature of the permutation sorting the elements of this concatenated list of $|I|+|J|$ indices, and let $I+J$, or $\varepsilon(I,J)$ if the notation $I+J$ is ambiguous in a given context, denote the resulting sorted list. For a generic list I, let $\sigma(I)$ denote the signature of the permutation sorting the elements of the list I. Then, the exterior product of $\mathbf{e}_I$ and $\mathbf{e}_J$ is defined as

$$\mathbf{e}_I\wedge\mathbf{e}_J = \sigma(I,J)\mathbf{e}_{I+J}. \tag{5}$$

The exterior product is thus either zero or a vector of grade $|I| + |J|$. The exterior product provides a construction of the basis vector $\mathbf{e}_I$, with I an ordered list $I = (i_1, \ldots, i_m)$, from the canonical basis vectors $\mathbf{e}_i$, namely

$$\mathbf{e}_I = \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_m}. \tag{6}$$

We next define two generalizations of the dot product, the left and right interior products. Let $\mathbf{e}_I$ and $\mathbf{e}_J$ be two basis vectors of respective grades $|I|$ and $|J|$. The left interior product, denoted by $\lrcorner$, is defined as

$$\mathbf{e}_I \lrcorner \mathbf{e}_J = \Delta_{II}\sigma(J \setminus I, I)\mathbf{e}_{J\setminus I}, \tag{7}$$

if I is a subset of J and zero otherwise. The vector $\mathbf{e}_{J\setminus I}$ has grade $|J| - |I|$ and is indexed by the elements of J not in common with I. The right interior product, denoted by $\llcorner$, of two basis vectors $\mathbf{e}_I$ and $\mathbf{e}_J$ is defined as

$$\mathbf{e}_I \llcorner \mathbf{e}_J = \Delta_{JJ}\sigma(J, I \setminus J)\mathbf{e}_{I\setminus J}, \tag{8}$$

if J is a subset of I and zero otherwise. Both interior products are grade-lowering operations, as the left (resp. right) interior product is either zero or a multivector of grade $|J| - |I|$ (resp. $|I| - |J|$).

We define the differential vector operator $\boldsymbol{\partial}$ as $(-\partial_0, -\partial_2, \ldots, -\partial_{k-1}, \partial_k, \ldots, \partial_{k+n-1})$, that is

$$\boldsymbol{\partial} = \sum_{i=0}^{k+n-1} \Delta_{ii}\mathbf{e}_i\partial_i. \tag{9}$$

As done in [9, Sect. 3], we define the exterior derivative of $\mathbf{v}$, $\boldsymbol{\partial} \wedge \mathbf{v}$, of a given vector field $\mathbf{v}$ of grade m as

$$\boldsymbol{\partial} \wedge \mathbf{v} = \sum_{i, I \in \mathcal{I}_m :\, i \notin I} \Delta_{ii}\sigma(i, I)\partial_i v_I \mathbf{e}_{i+I}. \tag{10}$$

In addition, we define the interior derivative of $\mathbf{v}$ as $\boldsymbol{\partial} \lrcorner \mathbf{v}$, namely

$$\boldsymbol{\partial} \lrcorner \mathbf{v} = \sum_{i, I \in \mathcal{I}_m :\, i \in I} \sigma(I \setminus i, i)\partial_i v_I \mathbf{e}_{I\setminus i}. \tag{11}$$

The exterior and interior derivatives satisfy the property $\boldsymbol{\partial} \wedge (\boldsymbol{\partial} \wedge \mathbf{v}) = 0 = \boldsymbol{\partial} \lrcorner (\boldsymbol{\partial} \lrcorner \mathbf{v})$.

2.3 Exterior-algebraic formulation of the Lagrangian densities and the action

We review how several Lagrangian densities may be written as the linear combination of dot products of pairs of multivector fields. First, the free Lagrangian density for generalized electromagnetism [10] is given by

$$\mathcal{L}_{\text{free-gem}} = \frac{(-1)^{r-1}}{2}\mathbf{F} \cdot \mathbf{F}, \tag{12}$$

where $\mathbf{F}$ is a grade-r multivector field related to the generalized potential $\mathbf{A}$, a multivector field of grade $r-1$, as $\mathbf{F} = \boldsymbol{\partial} \wedge \mathbf{A}$. This is a gauge theory, since replacing the potential $\mathbf{A}$ by a new field $\mathbf{A}' = \mathbf{A} + \bar{\mathbf{A}} + \boldsymbol{\partial} \wedge \mathbf{G}$, where $\bar{\mathbf{A}}$ is a constant $(r-1)$-vector and $\mathbf{G}$ is an $(r-2)$-vector gauge field, leaves $\mathbf{F}$ and therefore the Lagrangian unchanged.

The model in (12) subsumes a number of relevant cases. Setting $r = 1$, $k = 1$, $n = 3$, and $\mathbf{A} = \phi$, gives the scalar field Lagrangian [5, Sect. 3.3], [3, 4],

$$\mathcal{L}_{\text{free-scalar}} = \frac{1}{2}(\boldsymbol{\partial} \wedge \phi) \cdot (\boldsymbol{\partial} \wedge \phi) \tag{13}$$

$$= \frac{1}{2}\sum_i \Delta_{ii}(\partial_i\phi)^2. \tag{14}$$

For electrostatics, we set $r = 1$, $k = 0$, $n = 3$, and the Lagrangian density [13, Ch. 19] is given by

$$\mathcal{L}_{\text{free-es}} = \frac{1}{2}\mathbf{E} \cdot \mathbf{E}, \tag{15}$$

where $\mathbf{E}$ is the electric field and ϕ is the opposite in sign of the usual electric potential, that is $\mathbf{E} = \boldsymbol{\partial} \wedge \phi = \nabla\phi$. For electromagnetism, we set $r = 2$, $k = 1$, $n = 3$; in this case, when the Lagrangian is expressed in terms of the Faraday tensor, an antisymmetric tensor of rank 2, the factor before $\mathbf{F} \cdot \mathbf{F}$ becomes $-\frac{1}{4}$ to account for the repeated sum over pairs of indices [1, Sect. 27], [2, Sect. 12.10], [5, Sect. 3.5], [14, Sect. 1.5]. We have instead,

$$\mathcal{L}_{\text{free-em}} = -\frac{1}{2}\mathbf{F} \cdot \mathbf{F}. \tag{16}$$

In Yang-Mills theories in Minkowksi space-time, the multivector field takes values in a Lie algebra, and the free Lagrangian density [5, Sect. 10.2], [3, 4], is given by

$$\mathcal{L}_{\text{free-ym}} = -\frac{1}{2g^2}\operatorname{Tr}(\mathbf{F}\cdot\mathbf{F}) \tag{17}$$

$$= -\frac{1}{2g^2}\sum_a \mathbf{F}^a\cdot\mathbf{F}^a, \tag{18}$$

where g is a coupling parameter, (17) is the fundamental representation and (18) the form as an linear combination of real-valued multivector fields in a given representation of the Lie algebra.

In addition to the free Lagrangian density, the action may include interaction terms. For instance, we may have a source density, a vector $\mathbf{J}(\mathbf{x})$ of grade $(r-1)$ [10], such that the generalized-electromagnetism Lagrangian density [2, Sect. 12.7], is now

$$\mathcal{L}_{\text{gem}} = \frac{(-1)^{r-1}}{2}\mathbf{F}\cdot\mathbf{F} + \mathbf{J}\cdot\mathbf{A}. \tag{19}$$

For this density, the requirement that the action is gauge invariant leads to the continuity equation for the current density, $\boldsymbol{\partial}\,\lrcorner\,\mathbf{J} = 0$. As shown in [10, Sect. 4], the generalized Lorentz force density remains a grade-1 vector, whose value coincides with the opposite in sign of the divergence of the stress-energy-momentum tensor related to (19).

Alternatively, the free Lagrangian density in (16) may describe a generalized Proca equation [2, Sect. 12.8], [5, p. 107] with mass m and Lagrangian density

$$\mathcal{L}_{\text{free-proca}} = -\frac{1}{2}\mathbf{F}\cdot\mathbf{F} + \frac{1}{2}m^2\mathbf{A}\cdot\mathbf{A}. \tag{20}$$

In each of the cases we have described, namely (13), (15), (17), (19) and (20), one can construct an action $\mathcal{S}_{\text{sys}}$ from the Lagrangian densities by integrating the density over a region $\mathcal{R}$ that comprises the system under consideration. We shall assume that the region $\mathcal{R}$ is large enough to make the physical system closed, so that an infinitesimal translation of space-time coordinates is a symmetry of the system and the action is invariant under this translation. We shall also assume that the fields decay fast enough over $\mathcal{R}$ so that the flux of the fields over the boundary of $\mathcal{R}$ is arbitrarily small. In general, we may write the action $\mathcal{S}_{\text{sys}}$ as

$$\mathcal{S}_{\text{sys}} = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\left(\sum_{\mathbf{a},\mathbf{b}}\gamma_{\mathbf{a},\mathbf{b}}\,(\mathbf{a}\cdot\mathbf{b})\right), \tag{21}$$

namely an integral over $\mathcal{R}$ of a linear combination of dot products of properly selected pairs of multivector fields $\mathbf{a}$ and $\mathbf{b}$ of the same degree, with respective coefficients $\gamma_{\mathbf{a},\mathbf{b}}$. We exploit this linearity to express the stress-energy momentum tensor associated with the action (21) as a linear combination of tensors $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$, as we shall see in Sect. 3.

2.4 Symmetric tensor field spaces

In addition to exterior-algebraic vector spaces, we also need to consider symmetric tensor field (vector) spaces and matrix field (vector) spaces. The symmetric tensor field space has basis elements $\mathbf{u}_J$ indexed by lists $J = (j_1, \dots, j_r)$ drawn from $\mathcal{J}_r$, the set of all ordered lists of (possibly repeated) r indices. We refer to elements of these vector spaces as symmetric tensors of rank r. The dimension of the vector space of rank-r symmetric tensors is $\binom{k+n+r-1}{r}$, the number of lists in $\mathcal{J}_r$. The dot product $\cdot$ of two arbitrary rank-r basis vectors $\mathbf{u}_I$ and $\mathbf{u}_J$ is defined as

$$\mathbf{u}_I\cdot\mathbf{u}_J = \Delta_{IJ} = \Delta_{i_1j_1}\Delta_{i_2j_2}\cdots\Delta_{i_rj_r}, \tag{22}$$

where I and J are the ordered lists $I = (i_1, i_2, \dots, i_r)$ and $J = (j_1, j_2, \dots, j_r)$. Let two basis vectors $\mathbf{u}_I$ and $\mathbf{u}_J$ have respective grades $m = |I|$ and $m' = |J|$. Then, the symmetric tensor product of $\mathbf{u}_I$ and $\mathbf{u}_J$ is defined as

$$\mathbf{u}_I \vee \mathbf{u}_J = \mathbf{u}_{I+J}. \tag{23}$$

We may thus construct the basis vector $\mathbf{u}_I$, with $I \in \mathcal{J}_r$, from the canonical basis vectors elements $\mathbf{e}_i$ as

$$\mathbf{u}_I = \mathbf{e}_{i_1}\vee\mathbf{e}_{i_2}\vee\cdots\vee\mathbf{e}_{i_r}. \tag{24}$$

We may also define the interior product between a multivector and a symmetric tensor. Let $\mathbf{e}_I$ and $\mathbf{u}_J$ be two basis vectors of respective grade $|I|$ and rank $|J|$. The interior product, indistinctly denoted by $\lrcorner$ or $\llcorner$, is defined as

$$\mathbf{e}_I\,\lrcorner\,\mathbf{u}_J = \mathbf{u}_J\,\llcorner\,\mathbf{e}_I = \Delta_{II}\mathbf{u}_{J\setminus I}, \tag{25}$$

if I is a subset of J and zero otherwise. In this case, left and right interior products coincide.

We shall need to consider only rank-2 symmetric tensors, as the stress-energy-momentum tensor is a symmetric tensor field of rank 2, irrespective of the grade of the multivector fields and the number of space dimensions. In this case, it will prove convenient to have two operations, $\owedge$ and $\odot$, that generate a rank-2 symmetric tensor from a pair of multivectors of the same grade, $\mathbf{a}$ and $\mathbf{b}$. These operations are, respectively, defined as

$$\mathbf{a} \owedge \mathbf{b} = \sum_{i \leq j} (\Delta_{ii}\mathbf{e}_i \wedge \mathbf{a}) \cdot (\mathbf{b} \wedge \mathbf{e}_j \Delta_{jj}) \mathbf{u}_{ij}, \tag{26}$$

$$\mathbf{a} \odot \mathbf{b} = \sum_{i \leq j} (\Delta_{ii}\mathbf{e}_i \rfloor \mathbf{a}) \cdot (\mathbf{b} \lfloor \mathbf{e}_j \Delta_{jj}) \mathbf{u}_{ij}. \tag{27}$$

While the tensor field basis $\mathbf{u}_{ij}$ is symmetric in the pair (i,j), that is $\mathbf{u}_{ij} = \mathbf{u}_{ji}$, the tensors defined in (26) and (27) are not separately symmetric, as the coefficients multiplying $\mathbf{u}_{ij}$, for $i \neq j$, are not symmetric over the pair (i,j). However, for $i \neq j$, from [10, Eq. (22)], we have that

$$(\Delta_{ii}\mathbf{e}_i \wedge \mathbf{a}) \cdot (\mathbf{b} \wedge \mathbf{e}_j \Delta_{jj}) = (\Delta_{jj}\mathbf{e}_j \rfloor \mathbf{a}) \cdot (\mathbf{b} \lfloor \mathbf{e}_i \Delta_{ii}), \tag{28}$$

and therefore the (i,j) coefficient of $\mathbf{a} \owedge \mathbf{b}$, with $i < j$ (resp. $j > i$), coincides with the value of the coefficient (j,i) of $\mathbf{a} \odot \mathbf{b}$, with $j > i$ (resp. $j < i$). Therefore, the sum tensor $\mathbf{a} \owedge \mathbf{b} + \mathbf{a} \odot \mathbf{b}$ is indeed symmetric over the pair (i,j).

2.5 Matrix vector spaces

We do not need consider general tensor fields, but rather the matrix field (vector) space whose basis elements can be represented as $\mathbf{w}_{I_1,I_2}$, where both I_1 and I_2 are ordered lists of non-repeated ℓ_1 and ℓ_2 elements, respectively. We may identify these basis elements with the tensor product of two multivectors of grade ℓ_1 and ℓ_2, namely

$$\mathbf{w}_{I_1,I_2} = \mathbf{e}_{I_1} \otimes \mathbf{e}_{I_2}. \tag{29}$$

The dimension of the vector space spanned by these basis elements is $\binom{k+n}{\ell_1}\binom{k+n}{\ell_2}$; the elements of this vector space can be identified with matrices $\mathbf{A}$ whose rows and columns are indexed by lists of $I_i \in \mathcal{I}_{\ell_i}$,

$$\mathbf{A} = \sum_{I_1 \in \mathcal{I}_{\ell_1}, I_2 \in \mathcal{I}_{\ell_2}} a_{I_1 I_2} \mathbf{w}_{I_1,I_2}. \tag{30}$$

The transpose of a matrix element $\mathbf{w}_{I_1,I_2}$, denoted as $\mathbf{w}^T_{I_1,I_2}$, is defined as $\mathbf{w}_{I_2,I_1}$. For $\ell_1 = \ell_2 = \ell$, we also define the grade-ℓ square identity matrix, denoted by $\mathbf{1}_\ell$, as

$$\mathbf{1}_\ell = \sum_{I \in \mathcal{I}_\ell} \Delta_{II} \mathbf{w}_{I,I}. \tag{31}$$

The dot product $\cdot$ of two arbitrary matrix elements $\mathbf{w}_{I_1,I_2}$ and $\mathbf{w}_{J_1,J_2}$ is defined as

$$\mathbf{w}_{I_1,I_2} \cdot \mathbf{w}_{J_1,J_2} = \Delta_{I_1 J_1} \Delta_{I_2 J_2}. \tag{32}$$

We also define the dot product between symmetric tensor and matrix basis elements, $\mathbf{u}_I$ and $\mathbf{w}_{J_1,J_2}$, as

$$\mathbf{u}_I \cdot \mathbf{w}_{J_1,J_2} = \mathbf{w}_{J_1,J_2} \cdot \mathbf{u}_I = \begin{cases} \Delta_{I,I}, & \text{if } I = \varepsilon(J_1,J_2), \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

We define the matrix product $\times$ between two matrix basis elements $\mathbf{w}_{I,J}$ and $\mathbf{w}_{K,L}$ as

$$\mathbf{w}_{I,J} \times \mathbf{w}_{K,L} = \mathbf{w}_{I,L} \Delta_{JK}. \tag{34}$$

For square matrices $\mathbf{A}$ indexed by grade-m multivectors, it is natural to define the matrix inverse, denoted as $\mathbf{A}^{-1}$, as the matrix such that $\mathbf{A}^{-1} \times \mathbf{A} = \mathbf{1}_m = \mathbf{A} \times \mathbf{A}^{-1}$. And lastly, we define the matrix product $\times$ between a matrix $\mathbf{w}_{I,J}$ and a multivector $\mathbf{e}_K$ (or between a multivector $\mathbf{e}_K$ and the matrix $\mathbf{w}_{J,I}$, i.e., the transpose of $\mathbf{w}_{I,J}$) as

$$\mathbf{w}_{I,J} \times \mathbf{e}_K = \mathbf{e}_K \times \mathbf{w}_{J,I} = \mathbf{e}_I \Delta_{JK}. \tag{35}$$

3 Variational derivation of the symmetric energy-momentum tensor as conserved current with respect to infinitesimal space-time translations

3.1 Introduction

In all the systems described in Sect. 2.3, namely (13), (15), (17), (19) and (20), we could construct an action $\mathcal{S}_{\text{sys}}$ by integrating the Lagrangian density over a region $\mathcal{R}$ that comprises the physical system under consideration. As we stated earlier, we assume that the region $\mathcal{R}$ is large enough to make the physical system closed, so that an infinitesimal translation of space-time coordinates is a symmetry of the system and the action is invariant under this change. We also assume that the fields decay fast enough over $\mathcal{R}$ so that the flux of the fields over the boundary of $\mathcal{R}$ is arbitrarily small. The action for the physical system under consideration, $\mathcal{S}_{\text{sys}}$ was written in (21) as

$$\mathcal{S}_{\text{sys}} = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x} \left(\sum_{\mathbf{a},\mathbf{b}} \gamma_{\mathbf{a},\mathbf{b}} \, (\mathbf{a} \cdot \mathbf{b}) \right), \tag{36}$$

namely an integral over $\mathcal{R}$ of a linear combination of dot products of properly selected pairs of multivector fields $\mathbf{a}$ and $\mathbf{b}$ of the same degree, with respective coefficients $\gamma_{\mathbf{a},\mathbf{b}}$. We exploit this linearity to express the stress-energy momentum tensor associated with the action (36) as a linear combination, with coefficients $\gamma_{\mathbf{a},\mathbf{b}}$, of tensors $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$ associated with Lagrangian densities $\mathbf{a} \cdot \mathbf{b}$, namely

$$\mathbf{T}_{\text{sys}} = \sum_{\mathbf{a},\mathbf{b}} \gamma_{\mathbf{a},\mathbf{b}} \mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}, \tag{37}$$

where the tensor $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$ is given by $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = (-1)^s \big(\mathbf{a} \odot \mathbf{b} + \mathbf{a} \circledwedge \mathbf{b}\big)$ in (102), with components given in (82) and (83),

$$T^{\mathbf{a}\cdot\mathbf{b}}_{ii} = \Delta_{ii} \left(\sum_{K \in \mathcal{I}_s : i \notin K} \Delta_{KK} a_K b_K - \sum_{K \in \mathcal{I}_s : i \in K} \Delta_{KK} a_K b_K \right), \tag{38}$$

$$T^{\mathbf{a}\cdot\mathbf{b}}_{ij} = - \sum_{K \in \mathcal{I}_{s-1} : i,j \notin K} \Delta_{KK} \sigma\big(\varepsilon(i,K)_{i\leftrightarrow j}\big) \big(a_{\varepsilon(i,K)} b_{\varepsilon(j,K)} + b_{\varepsilon(i,K)} a_{\varepsilon(j,K)}\big), \tag{39}$$

where $I_{i\leftrightarrow j}$ is a list of indices where the index i in I is replaced by j, for a generic I. As we shall see, consideration of the possible internal structure of the fields $\mathbf{a}$ or $\mathbf{b}$, e. g., $\mathbf{a} = \boldsymbol{\partial} \wedge \mathbf{A}$ in generalized electromagnetism or an analogous formula for the Yang-Mills fields, may be circumvented to directly obtain the tensor components (38)–(39) in terms of the components of the fields $\mathbf{a}$ and $\mathbf{b}$ appearing in the Lagrangian density.

Moreover, we prove in (127) that the interior derivative (divergence) of $\mathbf{T}_{\text{sys}}$ is given by

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\text{sys}} = \sum_{\mathbf{a},\mathbf{b}} \gamma_{\mathbf{a},\mathbf{b}} \big(\mathbf{a} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{b}) + \mathbf{b} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{a}) - \mathbf{a} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{b}) - \mathbf{b} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{a})\big), \tag{40}$$

in terms of interior and exterior derivatives of $\mathbf{a}$ and $\mathbf{b}$. Assuming that infinitesimal space-time translations are a symmetry of the system under consideration and that the fields decay sufficiently fast at the boundary of the region $\mathcal{R}$, this interior derivative is zero,

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\text{sys}} = 0, \tag{41}$$

which yields a conservation law for the energy-momentum of the system.

3.2 Action for the Lagrangian density $\mathbf{a} \cdot \mathbf{b}$

The linearity of the action in (42) allows us to find the tensor and its interior derivative as a linear combination of quantities derived from the Lagrangian density $\mathbf{a} \cdot \mathbf{b}$ and its action $\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}}$, given by

$$\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}} = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x} \, (\mathbf{a} \cdot \mathbf{b}). \tag{42}$$

Let us shift the origin of coordinates by an infinitesimal perturbation $\boldsymbol{\varepsilon}$ and denote by $\{\mathbf{e}\}$ and $\{\mathbf{e}'\}$, respectively, the original and shifted basis elements, both expressed in the original basis. In general, and with some abuse of notation, we denote the components of a multivector $\mathbf{a}$ by $\mathbf{a}$ and $\mathbf{a}'$ in the original and new coordinates respectively. Along the i-th coordinate, the basis element $\mathbf{e}_i$ is perturbed to first order by an infinitesimal amount

$$\mathbf{e}_i \times (\boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}), \tag{43}$$

where the Jacobian partial-derivative matrix $\boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}$ is given by

$$\boldsymbol{\partial} \otimes \boldsymbol{\varepsilon} = \sum_{i,j} \Delta_{ii} \partial_i \varepsilon_j \mathbf{w}_{ij}. \tag{44}$$

The j-th column of the Jacobian matrix contains the exterior derivative (gradient) of the j-th component of the perturbation in the coordinates, ε_j. With the identity matrix in (31), which we simply denote as $\mathbf{1}$ since the grade of the vectors in the Jacobian matrix for $\boldsymbol{\varepsilon}$ is 1, and taking into account that $\mathbf{e}'_i = \mathbf{e}_i + \mathbf{e}_i \times (\boldsymbol{\partial} \otimes \boldsymbol{\varepsilon})$, the following relationship between the basis $\{\mathbf{e}\}$ and $\{\mathbf{e}'\}$ holds for all i,

$$\mathbf{e}'_i = \mathbf{e}_i \times (\mathbf{1} + \boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}). \tag{45}$$

In the new coordinates, the action in (42) becomes:

$$\mathcal{S}_{\mathbf{a}'\cdot\mathbf{b}'} = \int_{\mathcal{R}'} \mathrm{d}^{k+n}\mathbf{x}' \, (\mathbf{a}' \cdot \mathbf{b}'), \tag{46}$$

where $\mathcal{R}'$ is the new integration region and the new differential integration element is $\mathrm{d}^{k+n}\mathbf{x}'$.

In the following subsections, after expressing the multivectors and the action in the new coordinates, we shall first find the change in action $\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}}$ related to this change of coordinates. Then, we shall link the differential form expressing the change in action with the symmetric, rank-2 stress-energy-momentum tensor. In the process, we will obtain several equivalent, alternative expressions for this tensor and its components, as well as a closed-form formula for its interior derivative (divergence). Finally, we shall relate the vanishing of $\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}}$ induced by the closed nature to the system to the symmetry under the change of coordinates and to a conservation law in terms of the interior derivative of the stress-energy-momentum tensor.

3.3 Transformation of multivectors of grade s

While the basis vectors transform covariantly according to (45), the multivector field $\mathbf{a}$ transforms contravariantly, that is with the inverse transpose matrix representing the change of coordinates. When $\mathbf{a}$ is a scalar, i.e., with grade 0, it simply holds that

$$\mathbf{a}' = \mathbf{a}. \tag{47}$$

When $\mathbf{a}$ is a vector field with grade 1, we need two additional facts. First, multiplication by a matrix transpose from the right is equivalent to multiplication by the matrix (untransposed) from the left, as in (34); second, as the transformation $\boldsymbol{\varepsilon}$ is infinitesimal, we have $(\mathbf{1} + \boldsymbol{\partial} \otimes \boldsymbol{\varepsilon})^{-1} = \mathbf{1} - \boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}$. We therefore variously have

$$\mathbf{a}' = \mathbf{a} \times (\mathbf{1} + \boldsymbol{\partial} \otimes \boldsymbol{\varepsilon})^{-T} = \mathbf{a} \times (\mathbf{1} - \boldsymbol{\partial} \otimes \boldsymbol{\varepsilon})^{T} = (\mathbf{1} - \boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}) \times \mathbf{a}. \tag{48}$$

In passing, and for later use, we note that the derivative operator $\boldsymbol{\partial}$ transforms as

$$\boldsymbol{\partial}' = (\mathbf{1} - \boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}) \times \boldsymbol{\partial}. \tag{49}$$

We shall shortly find that an s-vector $\mathbf{a}$ transforms in general under this infinitesimal translation as

$$\mathbf{a}' = \mathbf{a} - \mathbf{G}^s_{\boldsymbol{\varepsilon}} \times \mathbf{a}, \tag{50}$$

where $\mathbf{G}^s_{\boldsymbol{\varepsilon}}$ is a matrix that varies with the grade s, with rows and columns indexed by s-tuples, and given by

$$\mathbf{G}^s_{\boldsymbol{\varepsilon}} = \sum_{I,J \in \mathcal{I}_s} G^s_{I,J} \mathbf{w}_{I,J}. \tag{51}$$

As in (47), we have $\mathbf{G}^0_{\boldsymbol{\varepsilon}} = 0$ for $s = 0$, as there is only one option for $I = J$, the empty set $\emptyset$ (in other words, $1' = 1$). Similarly, for $s = 1$, the matrix is given by $\mathbf{G}^1_{\boldsymbol{\varepsilon}} = \boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}$ as in (48), and therefore, $G^s_{I,J} = G_{ij} = \Delta_{ii} \partial_i \varepsilon_j$.

From (45), we find that the basis elements in the new coordinates can be variously written as

$$\mathbf{e}'_i = \mathbf{e}_i + \sum_j \partial_i \varepsilon_j \mathbf{e}_j \tag{52}$$

$$= \sum_j \tau_{i,j} \mathbf{e}_j, \qquad \text{with } \tau_{i,j} = \delta_{ij} + \partial_i \varepsilon_j. \tag{53}$$

Therefore, the basis element $\mathbf{e}'_I$ for the s-vector in the shifted coordinates, with $I = (i_1, \ldots, i_s)$, is in turn given by

$$\begin{aligned}
\mathbf{e}'_I &= \mathbf{e}'_{i_1} \wedge \mathbf{e}'_{i_2} \cdots \wedge \mathbf{e}'_{i_s} && (54)\\
&= \left(\sum_j \tau_{i_1,j}\mathbf{e}_j\right) \wedge \left(\sum_j \tau_{i_2,j}\mathbf{e}_j\right) \cdots \wedge \left(\sum_j \tau_{i_s,j}\mathbf{e}_j\right) && (55)\\
&= \sum_{J\in\mathcal{I}_s} \det \begin{pmatrix} \tau_{i_1,j_1} & \cdots & \tau_{i_1,j_s} \\ \vdots & & \vdots \\ \tau_{i_s,j_1} & \cdots & \tau_{i_s,j_s} \end{pmatrix} \mathbf{e}_J && (56)\\
&= \sum_{J\in\mathcal{I}_s} \det \tau_{I\otimes J}\, \mathbf{e}_J, && (57)
\end{aligned}$$

where we have used in (56) the relationship between the determinant and the wedge product [12, Sect. 2.2] and in (56) we introduced the (linear-algebraic) matrix $\tau_{I\otimes J}$, a submatrix of $\mathbf{1} + \boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}$, explicitly given by

$$\tau_{I\otimes J} = \begin{pmatrix} \tau_{i_1,j_1} & \cdots & \tau_{i_1,j_s} \\ \vdots & & \vdots \\ \tau_{i_s,j_1} & \cdots & \tau_{i_s,j_s} \end{pmatrix}. \tag{58}$$

In the computation of the determinant in (57), we only keep terms up to the first order in the derivatives of $\boldsymbol{\varepsilon}$. In order to remove unneeded terms, we first observe that the number of possible overlaps between I and J ranges from 0 to s. If there are strictly fewer than $s-1$ overlapping indices, the determinant is zero up to first order, as all summands in the determinant have at least two partial derivatives multiplied together. For each list I, we need thus consider only index lists J such that $I = J$ or $|I \cap J| = s - 1$:

$$\mathbf{e}'_I = \det \tau_{I\otimes I}\, \mathbf{e}_I + \sum_{J\in\mathcal{I}_s:|I\cap J|=s-1} \det \tau_{I\otimes J}\, \mathbf{e}_J. \tag{59}$$

For $I = J$, the only non zero contribution to the determinant comes from the main diagonal and we directly obtain

$$\det \tau_{I\otimes I} = 1 + \sum_{i\in I} \partial_i \varepsilon_i. \tag{60}$$

As for the case where $|I \cap J| = s - 1$, of which there is a total number of $s(k + n - s)$ possibilities, let us define a set $K = I \cap J$ and two indices i and j, $i = I \setminus K$ and $j = J \setminus K$, such that $I = \varepsilon(i, K)$ and $J = \varepsilon(j, K)$. We can directly infer from (55) that there is a single non zero contribution to the determinant in this case, namely

$$\sigma\big(I_{i\leftrightarrow j}\big)\, \tau_{i,j} \prod_{k\in K} \tau_{k,k}, \tag{61}$$

where the signature is that of the permutation that orders the list of indices $I_{i\leftrightarrow j}$ where i in I is replaced by j. Since $\tau_{i,j} = \partial_i \varepsilon_j$ and the terms $\tau_{k,k}$ contribute to the determinant with a 1, we have

$$\det \tau_{I\otimes J} = \sigma\big(I_{i\leftrightarrow j}\big)\, \partial_i \varepsilon_j. \tag{62}$$

As we prove in Appendix A, the permutation signature can be expressed various equivalent forms:

$$\sigma\big(I_{i\leftrightarrow j}\big) = \sigma\big(\varepsilon(i,K)_{i\leftrightarrow j}\big) = \sigma\big(\varepsilon(j,K)_{j\leftrightarrow i}\big) = \sigma\big(J_{j\leftrightarrow i}\big), \tag{63}$$

thereby proving that the permutation signature is indeed symmetric in the pair of indices (i, j). In the same Appendix, we prove the following identities relating the signature in (63),

$$\begin{aligned}
\sigma(K,i)\sigma(j,K) &= (-1)^{|K|}\sigma\big(\varepsilon(i,K)_{i\leftrightarrow j}\big), && (64)\\
\sigma(i, j+K)\sigma(i+K, j) &= (-1)^{|K|}\sigma\big(\varepsilon(i,K)_{i\leftrightarrow j}\big). && (65)
\end{aligned}$$

Combining (60) and (62) into (59) yields the following expression for the new coordinate basis elements:

$$\mathbf{e}'_I = \left(1 + \sum_{i\in I} \partial_i \varepsilon_i\right) \mathbf{e}_I + \sum_{K\in\mathcal{I}_{s-1}:i,j\notin K} \sigma\big(I_{i\leftrightarrow j}\big)\, \partial_i \varepsilon_j\, \mathbf{e}_{\varepsilon(j,K)}. \tag{66}$$

As we did in (45), we can write (66) as

$$\mathbf{e}'_I = \mathbf{e}_I \times (\mathbf{1}_s + \mathbf{G}^s_{\boldsymbol{\varepsilon}}), \tag{67}$$

where $\mathbf{1}_s$ is the identity matrix for grade-s multivectors, $\mathbf{1}_s = \sum_I \Delta_{II} \mathbf{w}_{I,I}$, and the matrix $\mathbf{G}^s_{\boldsymbol{\varepsilon}}$ is such that

$$\mathbf{e}_I \times \mathbf{G}^s_{\boldsymbol{\varepsilon}} = \left(\sum_{i\in I} \partial_i \varepsilon_i\right) \mathbf{e}_I + \sum_{K\in\mathcal{I}_{s-1}:i,j\notin K} \sigma\big(I_{i\leftrightarrow j}\big)\, \partial_i \varepsilon_j\, \mathbf{e}_{\varepsilon(j,K)}. \tag{68}$$

Taking into account the definition of the product $\times$, the matrix $\mathbf{G}^s_{\boldsymbol{\varepsilon}}$ is therefore given by

$$\mathbf{G}^s_{\boldsymbol{\varepsilon}} = \Delta_{II}\mathbf{e}_I \otimes \left(\sum_{i\in I}\partial_i\varepsilon_i\right)\mathbf{e}_I + \Delta_{II}\mathbf{e}_I \otimes \sum_{K\in\mathcal{I}_{s-1}:i,j\notin K} \sigma\big(I_{i\leftrightarrow j}\big)\,\partial_i\varepsilon_j\,\mathbf{e}_{\varepsilon(j,K)} \tag{69}$$

$$= \Delta_{II}\left(\sum_{i\in I}\partial_i\varepsilon_i\right)\mathbf{w}_{I,I} + \Delta_{II}\sum_{K\in\mathcal{I}_{s-1}:i,j\notin K} \sigma\big(I_{i\leftrightarrow j}\big)\,\partial_i\varepsilon_j\,\mathbf{w}_{I,\varepsilon(j,K)}. \tag{70}$$

For later use, it will prove convenient to express the matrix $\mathbf{G}^s_{\boldsymbol{\varepsilon}}$ in the equivalent form:

$$\mathbf{G}^s_{\boldsymbol{\varepsilon}} = \sum_i \Delta_{ii}\partial_i\varepsilon_i\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i) + \sum_{i,j:i\neq j}\Delta_{ii}\partial_i\varepsilon_j\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j), \tag{71}$$

where the matrices $\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i)$ and $\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j)$, for all i and j, with $j\neq i$, are, respectively, given by

$$\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i) = \sum_{K\in\mathcal{I}_{s-1}:i\notin K}\Delta_{KK}\mathbf{w}_{\varepsilon(i,K),\varepsilon(i,K)} \tag{72}$$

$$\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j) = \sum_{K\in\mathcal{I}_{s-1}:i,j\notin K}\Delta_{KK}\sigma\big(\varepsilon(i,K)_{i\leftrightarrow j}\big)\mathbf{w}_{\varepsilon(i,K),\varepsilon(j,K)}. \tag{73}$$

3.4 Transformation of the action: stress-energy-momentum tensor

We now return to the expression of the action in the new coordinates up to derivatives of first order in $\boldsymbol{\varepsilon}$. We first note that the differential $\mathrm{d}^{k+n}\mathbf{x}$ appearing in the action is expressed in the new coordinates using (60) with I including all space-time indices, that is $|I| = k+n$, so that we directly obtain

$$\mathrm{d}^{k+n}\mathbf{x}' = \mathrm{d}^{k+n}\mathbf{x}\big(1+\boldsymbol{\partial}\cdot\boldsymbol{\varepsilon}\big). \tag{74}$$

In the new coordinates, the integration region is denoted as $\mathcal{R}'$; for an infinitesimal translation, the regions $\mathcal{R}$ and $\mathcal{R}'$ differ only on the boundary of the former, which is located far from the origin. With the identity in (74), together with (50) for $\mathbf{a}$ and $\mathbf{b}$, and neglecting higher-order derivative terms, Eq. (46) becomes

$$\mathcal{S}_{\mathbf{a}'\cdot\mathbf{b}'} = \int_{\mathcal{R}'}\mathrm{d}^{k+n}\mathbf{x}'\,(\mathbf{a}'\cdot\mathbf{b}') \tag{75}$$

$$= \int_{\mathcal{R}}\mathrm{d}^{k+n}\mathbf{x}\big(1+\boldsymbol{\partial}\cdot\boldsymbol{\varepsilon}\big)\Big(\big((\mathbf{1}-\mathbf{G}^s_{\boldsymbol{\varepsilon}})\times\mathbf{a}\big)\cdot\big((\mathbf{1}-\mathbf{G}^s_{\boldsymbol{\varepsilon}})\times\mathbf{b}\big)\Big) \tag{76}$$

$$= \int_{\mathcal{R}}\mathrm{d}^{k+n}\mathbf{x}\,(\mathbf{a}\cdot\mathbf{b}) + \int_{\mathcal{R}}\mathrm{d}^{k+n}\mathbf{x}(\boldsymbol{\partial}\cdot\boldsymbol{\varepsilon})\,(\mathbf{a}\cdot\mathbf{b}) - \int_{\mathcal{R}}\mathrm{d}^{k+n}\mathbf{x}\,\big(\mathbf{G}^s_{\boldsymbol{\varepsilon}}\times\mathbf{a}\big)\cdot\mathbf{b}\big) - \int_{\mathcal{R}}\mathrm{d}^{k+n}\mathbf{x}\,\big(\mathbf{a}\cdot(\mathbf{G}^s_{\boldsymbol{\varepsilon}}\times\mathbf{b})\big) \tag{77}$$

$$= \mathcal{S}_{\mathbf{a}\cdot\mathbf{b}} + \delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}}, \tag{78}$$

where in (76) we used that the difference in integration regions $\mathcal{R}'$ and $\mathcal{R}$ lies far from the origin, coupled with the rapid decay of the fields, to replace $\mathcal{R}'$ by $\mathcal{R}$, and finally defined the change in action $\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}}$ is given by

$$\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}} = \int_{\mathcal{R}}\mathrm{d}^{k+n}\mathbf{x}(\boldsymbol{\partial}\cdot\boldsymbol{\varepsilon})\,(\mathbf{a}\cdot\mathbf{b}) - \int_{\mathcal{R}}\mathrm{d}^{k+n}\mathbf{x}\,\big((\mathbf{G}^s_{\boldsymbol{\varepsilon}}\times\mathbf{a})\cdot\mathbf{b}\big) - \int_{\mathcal{R}}\mathrm{d}^{k+n}\mathbf{x}\,\big(\mathbf{a}\cdot(\mathbf{G}^s_{\boldsymbol{\varepsilon}}\times\mathbf{b})\big). \tag{79}$$

We next prove the existence of a symmetric, rank-2 tensor $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$ such that

$$\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}} = \int_{\mathcal{R}}\mathrm{d}^{k+n}\mathbf{x}\,(\boldsymbol{\partial}\otimes\boldsymbol{\varepsilon})\cdot\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}. \tag{80}$$

The tensor $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$ may in fact be identified with the symmetric stress-energy momentum tensor associated with the Lagrangian density $\mathbf{a}\cdot\mathbf{b}$. In terms of the components of $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$, we have

$$\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = \sum_{i\leq j}T^{\mathbf{a}\cdot\mathbf{b}}_{ij}\mathbf{u}_{ij}, \tag{81}$$

where on- and off-diagonal components, $T^{\mathbf{a}\cdot\mathbf{b}}_{ii}$ and $T^{\mathbf{a}\cdot\mathbf{b}}_{ij}$, are respectively given by

$$T^{\mathbf{a}\cdot\mathbf{b}}_{ii} = \Delta_{ii}\left(\sum_{K\in\mathcal{I}_s:i\notin K}\Delta_{KK}a_Kb_K - \sum_{K\in\mathcal{I}_s:i\in K}\Delta_{KK}a_Kb_K\right), \tag{82}$$

$$T^{\mathbf{a}\cdot\mathbf{b}}_{ij} = -\sum_{K\in\mathcal{I}_{s-1}:i,j\notin K}\Delta_{KK}\sigma\big(\varepsilon(i,K)_{i\leftrightarrow j}\big)\big(a_{\varepsilon(i,K)}b_{\varepsilon(j,K)} + b_{\varepsilon(i,K)}a_{\varepsilon(j,K)}\big). \tag{83}$$

As befits a symmetric rank-2 tensor, the coefficient $T_{ij}^{\mathbf{a}\cdot\mathbf{b}}$ in (83) does not change under permutation of i and j: the factor $\big(a_{\varepsilon(i,K)}b_{\varepsilon(j,K)} + b_{\varepsilon(i,K)}a_{\varepsilon(j,K)}\big)$ is clearly symmetric and the permutation signature $\sigma\big(\varepsilon(i,K)_{i\leftrightarrow j}\big)$ is also symmetric in i, j, as shown in (63).

In order to prove (80), we shall establish the following identity of differential forms appearing in the integrand,

$$(\boldsymbol{\partial}\otimes\boldsymbol{\varepsilon})\cdot\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = (\boldsymbol{\partial}\cdot\boldsymbol{\varepsilon})\,(\mathbf{a}\cdot\mathbf{b}) - (\mathbf{G}^s_{\boldsymbol{\varepsilon}}\times\mathbf{a})\cdot\mathbf{b} - \mathbf{a}\cdot(\mathbf{G}^s_{\boldsymbol{\varepsilon}}\times\mathbf{b}). \tag{84}$$

Expanding the left-hand side of (84) and using the dot product formula for $\mathbf{w}_{ij}$ and $\mathbf{u}_{ij}$ in (33) yields

$$(\boldsymbol{\partial}\otimes\boldsymbol{\varepsilon})\cdot\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = \left(\sum_{i,j}\Delta_{ii}\partial_i\varepsilon_j\mathbf{w}_{ij}\right)\cdot\left(\sum_{i'\leq j'}T^{\mathbf{a}\cdot\mathbf{b}}_{i'j'}\mathbf{u}_{i'j'}\right) \tag{85}$$

$$= \sum_{i,j}\Delta_{ii}\partial_i\varepsilon_j\Delta_{ii}\Delta_{jj}T^{\mathbf{a}\cdot\mathbf{b}}_{\varepsilon(i,j)}. \tag{86}$$

At this point, if we expand the right-hand side of (84) into a summation whose terms are indexed by the pair (i, j) such that each of these summands has the form of a coefficient multiplying $\Delta_{ii}\partial_i\varepsilon_j$, we can directly read $T^{\mathbf{a}\cdot\mathbf{b}}_{\varepsilon(i,j)}$ from this multiplicative coefficient. More precisely, using the definition of $\boldsymbol{\partial}\cdot\boldsymbol{\varepsilon}$ together with (71) yields

$$\left(\sum_i\partial_i\varepsilon_i\right)(\mathbf{a}\cdot\mathbf{b}) - \left(\sum_i\Delta_{ii}\partial_i\varepsilon_i\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i)\times\mathbf{a} + \sum_{i,j:i\neq j}\Delta_{ii}\partial_i\varepsilon_j\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j)\times\mathbf{a}\right)\cdot\mathbf{b} -$$
$$\mathbf{a}\cdot\left(\sum_i\Delta_{ii}\partial_i\varepsilon_i\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i)\times\mathbf{b} + \sum_{i,j:i\neq j}\Delta_{ii}\partial_i\varepsilon_j\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j)\times\mathbf{b}\right). \tag{87}$$

It will prove convenient to distinguish the cases $i = j$ and $i \neq j$. First, for $i = j$ and writing the relationship between the diagonal terms in (86) and (87), we obtain

$$\Delta_{ii}\partial_i\varepsilon_i\Delta_{ii}\Delta_{ii}T^{\mathbf{a}\cdot\mathbf{b}}_{ii} = \partial_i\varepsilon_i\,(\mathbf{a}\cdot\mathbf{b}) - \big(\Delta_{ii}\partial_i\varepsilon_i\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i)\times\mathbf{a}\big)\cdot\mathbf{b} - \big(\Delta_{ii}\partial_i\varepsilon_i\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i)\times\mathbf{b}\big)\cdot\mathbf{a} \tag{88}$$

$$T^{\mathbf{a}\cdot\mathbf{b}}_{ii} = \Delta_{ii}(\mathbf{a}\cdot\mathbf{b}) - \big(\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i)\times\mathbf{a}\big)\cdot\mathbf{b} - \big(\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i)\times\mathbf{b}\big)\cdot\mathbf{a}. \tag{89}$$

Taking into account the definition of $\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i)$ in (72) and carrying out the matrix products in (89), we obtain

$$\mathbf{a}\cdot\big(\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i)\times\mathbf{b}\big) = \mathbf{a}\cdot\left(\sum_{K\in\mathcal{I}_{s-1}:i\notin K}\Delta_{ii}\mathbf{e}_{\varepsilon(i,K)}b_{\varepsilon(i,K)}\right) \tag{90}$$

$$= \sum_{K\in\mathcal{I}_{s-1}:i\notin K}\Delta_{KK}a_{\varepsilon(i,K)}b_{\varepsilon(i,K)}, \tag{91}$$

$$\mathbf{b}\cdot\big(\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,i)\times\mathbf{a}\big) = \sum_{K\in\mathcal{I}_{s-1}:i\notin K}\Delta_{KK}b_{\varepsilon(i,K)}a_{\varepsilon(i,K)}. \tag{92}$$

Using the definition of the dot product together with (91)–(92) back in (89) yields

$$T^{\mathbf{a}\cdot\mathbf{b}}_{ii} = \Delta_{ii}\sum_{K\in\mathcal{I}_s}\Delta_{KK}b_Ka_K - 2\sum_{K\in\mathcal{I}_{s-1}:i\notin K}\Delta_{KK}b_{\varepsilon(i,K)}a_{\varepsilon(i,K)} \tag{93}$$

$$= \Delta_{ii}\left(\sum_{K\in\mathcal{I}_s}\Delta_{KK}b_Ka_K - \sum_{K\in\mathcal{I}_s:i\in K}\Delta_{KK}b_Ka_K\right) \tag{94}$$

$$= \Delta_{ii}\left(\sum_{K\in\mathcal{I}_s:i\notin K}\Delta_{KK}b_Ka_K - \sum_{K\in\mathcal{I}_s:i\in K}\Delta_{KK}b_Ka_K\right), \tag{95}$$

namely the desired expression for $T^{\mathbf{a}\cdot\mathbf{b}}_{ii}$ in (82).

In an analogous manner, we write the relationship between the off-diagonal terms in (86) and (87), $i \neq j$, to obtain the following expressions for the off-diagonal tensor elements:

$$\Delta_{ii}\partial_i\varepsilon_j\Delta_{ii}\Delta_{jj}T^{\mathbf{a}\cdot\mathbf{b}}_{\varepsilon(i,j)} = -\big(\Delta_{ii}\partial_i\varepsilon_j\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j)\times\mathbf{a}\big)\cdot\mathbf{b} - \big(\Delta_{ii}\partial_i\varepsilon_j\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j)\times\mathbf{b}\big)\cdot\mathbf{a} \tag{96}$$

$$T^{\mathbf{a}\cdot\mathbf{b}}_{\varepsilon(i,j)} = -\Delta_{ii}\Delta_{jj}\big(\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j)\times\mathbf{a}\big)\cdot\mathbf{b} - \Delta_{ii}\Delta_{jj}\big(\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j)\times\mathbf{b}\big)\cdot\mathbf{a}, \tag{97}$$

where one should verify that the right-hand side of (97) is indeed symmetric in i and j, as we will shortly do. As before, using the definition of $\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j)$ in (73) and carrying out the matrix products in (97), we obtain

$$\mathbf{a}\cdot\bigl(\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j)\times\mathbf{b}\bigr) = \mathbf{a}\cdot\left(\sum_{K\in\mathcal{I}_{s-1}:i,j\notin K}\Delta_{jj}\sigma\bigl(\varepsilon(i,K)_{i\leftrightarrow j}\bigr)\mathbf{e}_{\varepsilon(i,K)}b_{\varepsilon(j,K)}\right) \tag{98}$$

$$= \sum_{K\in\mathcal{I}_{s-1}:i,j\notin K}\Delta_{jj}\Delta_{ii}\Delta_{KK}\sigma\bigl(\varepsilon(i,K)_{i\leftrightarrow j}\bigr)a_{\varepsilon(i,K)}b_{\varepsilon(j,K)}, \tag{99}$$

$$\mathbf{b}\cdot\bigl(\mathbf{G}^s_{\boldsymbol{\varepsilon}}(i,j)\times\mathbf{a}\bigr) = \sum_{K\in\mathcal{I}_{s-1}:i,j\notin K}\Delta_{jj}\Delta_{ii}\Delta_{KK}\sigma\bigl(\varepsilon(i,K)_{i\leftrightarrow j}\bigr)b_{\varepsilon(i,K)}a_{\varepsilon(j,K)}. \tag{100}$$

Again, replacing both (99) and (100) back in (97) leads to the desired expression for $T^{\mathbf{a}\cdot\mathbf{b}}_{ij}$, namely (83):

$$T^{\mathbf{a}\cdot\mathbf{b}}_{ij} = -\sum_{K\in\mathcal{I}_{s-1}:i,j\notin K}\Delta_{KK}\sigma\bigl(\varepsilon(i,K)_{i\leftrightarrow j}\bigr)\bigl(a_{\varepsilon(i,K)}b_{\varepsilon(j,K)}+b_{\varepsilon(i,K)}a_{\varepsilon(j,K)}\bigr). \tag{101}$$

In this equation, the factor $\bigl(a_{\varepsilon(i,K)}b_{\varepsilon(j,K)}+b_{\varepsilon(i,K)}a_{\varepsilon(j,K)}\bigr)$ clearly remains unchanged after permutation of i and j. As for $\sigma\bigl(\varepsilon(i,K)_{i\leftrightarrow j}\bigr)$, we proved in (63) that the permutation signature is indeed symmetric in i,j. Thus, the coefficient $T^{\mathbf{a}\cdot\mathbf{b}}_{ij}$ does not change under permutation of i and j, as befits a symmetric rank-2 tensor.

3.5 An alternative, coordinate-free, expression for the tensor

In this section, we provide an alternative, coordinate-free, expression for the symmetric stress-energy-momentum tensor $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$. This expression given in terms of the operations $\odot$ and $\circledwedge$ is defined in (27)–(26), as follows:

$$\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = (-1)^s\bigl(\mathbf{a}\odot\mathbf{b}+\mathbf{a}\circledwedge\mathbf{b}\bigr). \tag{102}$$

The discussion after the definitions of (26) and (27) proves that the tensor field in (102) is indeed symmetric.

In order to prove the expression in (102), we make use of the tensor components in (82)–(83). From the definitions in (26) and (27), we may, respectively, write the coefficients multiplying the basis element $\mathbf{u}_{ij}$ as:

$$(\Delta_{ii}\mathbf{e}_i\wedge\mathbf{a})\cdot(\mathbf{b}\wedge\mathbf{e}_j\Delta_{jj}) = \left(\sum_{I\in\mathcal{I}_s}\Delta_{ii}a_I\mathbf{e}_i\wedge\mathbf{e}_I\right)\cdot\left(\sum_{I\in\mathcal{I}_s}\Delta_{jj}b_I\mathbf{e}_I\wedge\mathbf{e}_j\right) \tag{103}$$

$$= \left(\sum_{I\in\mathcal{I}_s:i\notin I}\Delta_{ii}\sigma(i,I)a_I\mathbf{e}_{\varepsilon(i,I)}\right)\cdot\left(\sum_{I\in\mathcal{I}_s:j\notin I}\Delta_{jj}\sigma(I,j)b_I\mathbf{e}_{\varepsilon(j,I)}\right), \tag{104}$$

where we carried out the exterior products in each of the factors, and

$$(\Delta_{ii}\mathbf{e}_i\,\lrcorner\,\mathbf{a})\cdot(\mathbf{b}\,\llcorner\,\mathbf{e}_j\Delta_{jj}) = \left(\sum_{I\in\mathcal{I}_s}\Delta_{ii}a_I\mathbf{e}_i\,\lrcorner\,\mathbf{e}_I\right)\cdot\left(\sum_{I\in\mathcal{I}_s}\Delta_{jj}b_I\mathbf{e}_I\,\llcorner\,\mathbf{e}_j\right) \tag{105}$$

$$= \left(\sum_{I\in\mathcal{I}_s:i\in I}\sigma(I\setminus i,i)a_I\mathbf{e}_{I\setminus i}\right)\cdot\left(\sum_{I\in\mathcal{I}_s:j\in I}\sigma(j,I\setminus j)b_I\mathbf{e}_{I\setminus j}\right), \tag{106}$$

where we similarly carried out the interior products in each of the factors.

In order to evaluate the dot products in (104) and (106), it proves convenient to distinguish between the on- and off-diagonal components. First, for $i=j$ and (104), we obtain

$$(\Delta_{ii}\mathbf{e}_i\wedge\mathbf{a})\cdot(\mathbf{b}\wedge\mathbf{e}_i\Delta_{ii}) = \left(\sum_{I\in\mathcal{I}_s:i\notin I}\Delta_{ii}\sigma(i,I)a_I\mathbf{e}_{\varepsilon(i,I)}\right)\cdot\left(\sum_{I\in\mathcal{I}_s:i\notin I}\Delta_{ii}\sigma(I,i)b_I\mathbf{e}_{\varepsilon(i,I)}\right) \tag{107}$$

$$= \sum_{I\in\mathcal{I}_s:i\notin I}\Delta_{ii}\Delta_{II}\sigma(I,i)\sigma(i,I)a_Ib_I \tag{108}$$

$$= (-1)^s\sum_{I\in\mathcal{I}_s:i\notin I}\Delta_{ii}\Delta_{II}a_Ib_I, \tag{109}$$

where we used the identity $\sigma(i,I)\sigma(I,i)=(-1)^{|I|}=(-1)^s$, $i\notin I$. Similarly, for $i=j$ and (106), we obtain

$$(\Delta_{ii}\mathbf{e}_i\,\lrcorner\,\mathbf{a})\cdot(\mathbf{b}\,\llcorner\,\mathbf{e}_i\Delta_{ii}) = \left(\sum_{I\in\mathcal{I}_s:i\in I}\sigma(I\setminus i,i)a_I\mathbf{e}_{I\setminus i}\right)\cdot\left(\sum_{I\in\mathcal{I}_s:j\in I}\sigma(j,I\setminus j)b_I\mathbf{e}_{I\setminus j}\right) \tag{110}$$

$$= \sum_{I\in\mathcal{I}_s:i\in I}\Delta_{ii}\Delta_{II}\sigma(i,I\setminus i)\sigma(I\setminus i,i)a_Ib_I \tag{111}$$

$$= (-1)^{s-1}\sum_{I\in\mathcal{I}_s:i\in I}\Delta_{ii}\Delta_{II}a_Ib_I, \tag{112}$$

where we again used the identity $\sigma(i,K)\sigma(K,i) = (-1)^{s-1}$, for $i \notin K$. Combining (109) with (112) and (82) gives:

$$(\Delta_{ii}\mathbf{e}_i \wedge \mathbf{a}) \cdot (\mathbf{b} \wedge \mathbf{e}_i \Delta_{ii}) + (\Delta_{ii}\mathbf{e}_i \lrcorner \mathbf{a}) \cdot (\mathbf{b} \llcorner \mathbf{e}_i \Delta_{ii}) = (-1)^{s-1}\Delta_{ii}\left(\sum_{I\in\mathcal{I}_s : i\in I} \Delta_{II} a_I b_I - \sum_{I\in\mathcal{I}_s : i\notin I} \Delta_{II} a_I b_I\right) \tag{113}$$

$$= (-1)^s T^{\mathbf{a}\cdot\mathbf{b}}_{ii}, \tag{114}$$

Noticing that this expression is actually symmetric in $\mathbf{a}$ and $\mathbf{b}$, its average with itself with the roles of $\mathbf{a}$ and $\mathbf{b}$ interchanged gives the same expression, in correspondence with (102), as far as the on-diagonal terms are concerned.

Concerning the off-diagonal components, $i \neq j$, we start by evaluating (106) to directly obtain:

$$(\Delta_{ii}\mathbf{e}_i \lrcorner \mathbf{a}) \cdot (\mathbf{b} \llcorner \mathbf{e}_j \Delta_{jj}) = \sum_{K\in\mathcal{I}_{s-1} : i,j\notin K} \Delta_{KK}\sigma(K,i)\sigma(j,K) a_{\varepsilon(i,K)} b_{\varepsilon(j,K)}. \tag{115}$$

Similarly, we evaluate (104) and expand the dot product therein to obtain:

$$(\Delta_{ii}\mathbf{e}_i \wedge \mathbf{a}) \cdot (\mathbf{b} \wedge \mathbf{e}_j \Delta_{jj}) = \left(\sum_{I\in\mathcal{I}_s : i\notin I} \Delta_{ii}\sigma(i,I) a_I \mathbf{e}_{\varepsilon(i,I)}\right) \cdot \left(\sum_{I\in\mathcal{I}_s : j\notin I} \Delta_{jj}\sigma(I,j) b_I \mathbf{e}_{\varepsilon(j,I)}\right) \tag{116}$$

$$= \sum_{K\in\mathcal{I}_{s-1} : i,j\notin K} \Delta_{KK}\sigma(i,j+K)\sigma(i+K,j) a_{j+K} b_{i+K}. \tag{117}$$

Using (64) and (65), we combine (115) together with (117) and (83) to write

$$(\Delta_{ii}\mathbf{e}_i \lrcorner \mathbf{a}) \cdot (\mathbf{b} \llcorner \mathbf{e}_j \Delta_{jj}) + (\Delta_{ii}\mathbf{e}_i \wedge \mathbf{a}) \cdot (\mathbf{b} \wedge \mathbf{e}_j \Delta_{jj})$$

$$= (-1)^{s-1} \sum_{K\in\mathcal{I}_{s-1} : i,j\notin K} \Delta_{KK}\sigma\big(\varepsilon(i,K)_{i\leftrightarrow j}\big)(a_{i+K} b_{j+K} + a_{j+K} b_{i+K}) \tag{118}$$

$$= (-1)^s T^{\mathbf{a}\cdot\mathbf{b}}_{ij}, \tag{119}$$

once again in correspondence with the off-diagonal terms in (102).

3.6 Conservation law for energy-momentum

The next step is to apply a generalized Leibniz rule to rewrite the integrand in (80) in convenient way. For a vector $\boldsymbol{\varepsilon}$ of grade 1 and a symmetric tensor of rank 2, denoted by $\mathbf{T}$, the following generalized Leibniz rule holds:

$$\boldsymbol{\partial} \cdot (\boldsymbol{\varepsilon} \lrcorner \mathbf{T}) = \boldsymbol{\varepsilon} \cdot (\boldsymbol{\partial} \lrcorner \mathbf{T}) + (\boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}) \cdot \mathbf{T}, \tag{120}$$

where the interior product between a vector and a symmetric tensor is computed according to (25). This equation relates scalars, or zero-grade multivectors, on both sides. It is proved in Appendix B. Using (120) and the vanishing at infinity of $\boldsymbol{\varepsilon}$ to directly neglect the term in the left-hand side of (120), we express $\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}}$ in terms of the interior derivative (or divergence) of the symmetric stress-energy momentum tensor $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$ as

$$\delta\mathcal{S}_{\mathbf{a}\cdot\mathbf{b}} = -\int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\,(\boldsymbol{\partial} \lrcorner \mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}) \cdot \boldsymbol{\varepsilon}. \tag{121}$$

Assuming that infinitesimal space-time translations are a symmetry of the whole system and that the fields decay sufficiently fast at the boundary of the region $\mathcal{R}$, the fact that the variation of the action $\delta\mathbf{T}_{\text{sys}}$ must be zero for all infinitesimal perturbations $\boldsymbol{\varepsilon}$ implies that the interior derivative of the overall tensor $\mathbf{T}_{\text{sys}}$ in (37) is zero,

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\text{sys}} = 0, \tag{122}$$

which yields a conservation law for the energy-momentum of the system under consideration.

Using the definition of interior product in (25), we directly obtain:

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = \sum_{i,j} \partial_i T^{\mathbf{a}\cdot\mathbf{b}}_{\varepsilon(i,j)} \mathbf{e}_j \tag{123}$$

$$= \sum_i \left(\partial_i T^{\mathbf{a}\cdot\mathbf{b}}_{i,i} \mathbf{e}_i + \sum_{j\neq i} \partial_i T^{\mathbf{a}\cdot\mathbf{b}}_{\varepsilon(i,j)} \mathbf{e}_j\right), \tag{124}$$

where we may use the formulas for the tensor components (82) and (83) to write:

$$\partial_i T^{\mathbf{a}\cdot\mathbf{b}}_{i,i} = \Delta_{ii}\left(\sum_{K\in\mathcal{I}_s:i\notin K}\Delta_{KK}\partial_i(b_K a_K) - \sum_{K\in\mathcal{I}_s:i\in K}\Delta_{KK}\partial_i(b_K a_K)\right), \tag{125}$$

$$\partial_i T^{\mathbf{a}\cdot\mathbf{b}}_{\varepsilon(i,j)} = -\sum_{K\in\mathcal{I}_{s-1}:i,j\notin K}\Delta_{KK}\sigma\big(\varepsilon(i,K)_{i\leftrightarrow j}\big)\partial_i\big(a_{\varepsilon(i,K)}b_{\varepsilon(j,K)} + b_{\varepsilon(i,K)}a_{\varepsilon(j,K)}\big). \tag{126}$$

In addition to the identities in (124)–(126), we provide in this section a simple, coordinate-free, closed-form expression for the interior derivative of the stress-energy-momentum tensor $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$:

$$\boldsymbol{\partial} \rfloor \mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = \mathbf{a} \rfloor (\boldsymbol{\partial}\wedge\mathbf{b}) + \mathbf{b} \rfloor (\boldsymbol{\partial}\wedge\mathbf{a}) - \mathbf{a} \llcorner (\boldsymbol{\partial} \rfloor \mathbf{b}) - \mathbf{b} \llcorner (\boldsymbol{\partial} \rfloor \mathbf{a}). \tag{127}$$

As first step in our proof of (127), we use (102) in order to rewrite the interior derivative of $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$,

$$\boldsymbol{\partial} \rfloor \mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = (-1)^s\big(\boldsymbol{\partial} \rfloor (\mathbf{a} \oslash \mathbf{b}) + \boldsymbol{\partial} \rfloor (\mathbf{a}\odot\mathbf{b})\big), \tag{128}$$

and analyze the interior derivatives of $\mathbf{a} \oslash \mathbf{b}$ and $\mathbf{a}\odot\mathbf{b}$. Putting the definition of $\oslash$ in (26) into (124), we get:

$$\boldsymbol{\partial} \rfloor (\mathbf{a} \oslash \mathbf{b}) = \sum_{i,j}\Delta_{ii}\Delta_{jj}\partial_i\big((\mathbf{e}_{\min(i,j)}\wedge\mathbf{a})\cdot(\mathbf{b}\wedge\mathbf{e}_{\max(i,j)})\big)\mathbf{e}_j \tag{129}$$

$$= \sum_i\left(\partial_i\big((\mathbf{e}_i\wedge\mathbf{a})\cdot(\mathbf{b}\wedge\mathbf{e}_i)\big)\mathbf{e}_i\right) + \sum_i\sum_{j\neq i}\left(\Delta_{ii}\Delta_{jj}\partial_i\big((\mathbf{e}_{\min(i,j)}\wedge\mathbf{a})\cdot(\mathbf{b}\wedge\mathbf{e}_{\max(i,j)})\big)\mathbf{e}_j\right), \tag{130}$$

where we wrote the equations in a convenient way as a double summation over the pair of indices i and j.

We now use (109) to evaluate the first summation over i in (130),

$$\sum_i \partial_i\big((\mathbf{e}_i\wedge\mathbf{a})\cdot(\mathbf{b}\wedge\mathbf{e}_i)\big)\mathbf{e}_i = (-1)^s\sum_i\sum_{I\in\mathcal{I}_s:i\notin I}\Delta_{ii}\Delta_{II}\partial_i\big(a_I b_I\big)\mathbf{e}_i. \tag{131}$$

In a similar manner, we use (117) to evaluate the second summation, over i and $j\neq i$, in (130),

$$\sum_i\sum_{j\neq i}\Delta_{ii}\Delta_{jj}\partial_i\big((\mathbf{e}_{\min(i,j)}\wedge\mathbf{a})\cdot(\mathbf{b}\wedge\mathbf{e}_{\max(i,j)})\big)\mathbf{e}_j =$$

$$= \sum_i\sum_{j\neq i}\sum_{K\in\mathcal{I}_{s-1}:i,j\notin K}\Delta_{KK}\sigma\big(K,\min(i,j)\big)\sigma\big(\max(i,j),K\big)\partial_i\big(a_{\varepsilon(\max(i,j),K)}b_{\varepsilon(\min(i,j),K)}\big)\mathbf{e}_j \tag{132}$$

$$= (-1)^{s-1}\sum_i\sum_{j\neq i}\sum_{K\in\mathcal{I}_{s-1}:i,j\notin K}\Delta_{KK}\sigma(i,K)\sigma(j,K)\partial_i\big(a_{\varepsilon(\max(i,j),K)}b_{\varepsilon(\min(i,j),K)}\big)\mathbf{e}_j, \tag{133}$$

where we used that $\sigma(K,\ell) = (-1)^{|K|}\sigma(\ell,K)$, and that $\sigma\big(\min(i,j),K\big)\sigma\big(\max(i,j),K\big) = \sigma(i,K)\sigma(j,K)$ as $i\neq j$.

We now relate (130) to a closed-form, coordinate-free expression. First, let us evaluate $\mathbf{a} \rfloor (\boldsymbol{\partial}\wedge\mathbf{b})$, that is,

$$\mathbf{a} \rfloor (\boldsymbol{\partial}\wedge\mathbf{b}) = \left(\sum_{J\in\mathcal{I}_s}a_J\mathbf{e}_J\right) \rfloor \left(\sum_{i,I\in\mathcal{I}_s:i\notin I}\Delta_{ii}\sigma(i,I)\partial_i b_I\mathbf{e}_{\varepsilon(i,I)}\right) \tag{134}$$

$$= \sum_{i,I\in\mathcal{I}_s:i\notin I}\sum_{J\in\mathcal{I}_s}\Delta_{ii}\sigma(i,I)a_J\partial_i b_I\mathbf{e}_J \rfloor \mathbf{e}_{\varepsilon(i,I)} \tag{135}$$

$$= \sum_i\sum_{I,J\in\mathcal{I}_s:i\notin I,i\in J}\Delta_{ii}\sigma(i,I)a_J\partial_i b_I\mathbf{e}_J \rfloor \mathbf{e}_{\varepsilon(i,I)} + \sum_i\sum_{I,J\in\mathcal{I}_s:i\notin I,i\notin J}\Delta_{ii}\sigma(i,I)a_J\partial_i b_I\mathbf{e}_J \rfloor \mathbf{e}_{\varepsilon(i,I)}, \tag{136}$$

where we split the summation over J in two parts, depending on whether $i\in J$ or not. Let us first focus and evaluate the second summation over i in (136), namely

$$\sum_i\sum_{I,J\in\mathcal{I}_s:i\notin I,i\notin J}\Delta_{ii}\sigma(i,I)a_J\partial_i b_I\mathbf{e}_J \rfloor \mathbf{e}_{\varepsilon(i,I)} = \sum_i\sum_{I\in\mathcal{I}_s:i\notin I}\Delta_{ii}\sigma(i,I)a_I\partial_i b_I\mathbf{e}_I \rfloor \mathbf{e}_{\varepsilon(i,I)} \tag{137}$$

$$= \sum_i\sum_{I\in\mathcal{I}_s:i\notin I}\Delta_{ii}\Delta_{II}a_I\partial_i b_I\mathbf{e}_i, \tag{138}$$

where we used in (137) that the constraints that J is a subset of $\varepsilon(i,I)$ and $i \notin J$ enforce that $J = I$, and computed and simplified the interior product in (138). As for the first summand in (136), we similarly obtain

$$\sum_i \sum_{I,J\in\mathcal{I}_s : i\notin I, i\in J} \Delta_{ii}\sigma(i,I) a_J \partial_i b_I \mathbf{e}_J \lrcorner \mathbf{e}_{\varepsilon(i,I)} = \sum_i \sum_{j\neq i} \sum_{K\in\mathcal{I}_{s-1}:i,j\notin K} \Delta_{ii}\sigma(i,j+K) a_{i+K}\partial_i b_{j+K} \mathbf{e}_{i+K} \lrcorner \mathbf{e}_{i+j+K} \tag{139}$$

$$= \sum_i \sum_{j\neq i} \sum_{K\in\mathcal{I}_{s-1}:i,j\notin K} \Delta_{KK}\sigma(i,j+K)\sigma(j,i+K) a_{i+K}\partial_i b_{j+K} \mathbf{e}_j \tag{140}$$

$$= -\sum_i \sum_{j\neq i} \sum_{K\in\mathcal{I}_{s-1}:i,j\notin K} \Delta_{KK}\sigma(i,K)\sigma(j,K) a_{i+K}\partial_i b_{j+K} \mathbf{e}_j, \tag{141}$$

where we rewrote the summation over I and J in (139) in terms of an index j and a set $K \in \mathcal{I}_{s-1}$ such that $i,j \notin K$ and $I = \varepsilon(j,K)$ and $J = \varepsilon(i,K)$, computed the interior product in (140), and used the fact that $\sigma(i,j+K)\sigma(j,i+K) = -\sigma(i,K)\sigma(j,K)$ in (141); this latter identity follows from the fact that ordering the list (i,j,K) can be done in two ways, with respective signatures $\sigma(j,K)\sigma(i,j+K)$ and $-\sigma(i,K)\sigma(j,i+K)$, whose values should coincide. Substituting $\mathbf{b}$ for $\mathbf{a}$ in (136) and (141) and grouping terms, we thus obtain

$$\begin{aligned}\mathbf{a} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{b}) + \mathbf{b} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{a}) = &-\sum_i \sum_{j\neq i} \sum_{K\in\mathcal{I}_{s-1}:i,j\notin K} \Delta_{KK}\sigma(i,K)\sigma(j,K)\bigl(a_{i+K}\partial_i b_{j+K} + b_{i+K}\partial_i a_{j+K}\bigr)\mathbf{e}_j + \\ &+ \sum_i \sum_{I\in\mathcal{I}_s:i\notin I} \Delta_{ii}\Delta_{II}\partial_i(a_I b_I)\mathbf{e}_i.\end{aligned} \tag{142}$$

The second summation, over i and I, in (142) can be seen to coincide with (131), apart from a factor $(-1)^s$ which cancels out with the same factor present in (128). As for the first summation, over i, j, K, in (142), and again apart from common factors, each of the terms in the triple summation in (142) contains the term

$$a_{i+K}\partial_i b_{\ell+K} + b_{i+K}\partial_i a_{\ell+K}, \tag{143}$$

to be compared to the analogous term in the triple summation over i, j and K in (133), namely

$$\partial_i\bigl(a_{\varepsilon(\max(i,j),K)} b_{\varepsilon(\min(i,j),K)}\bigr) = a_{\varepsilon(\max(i,j),K)}\partial_i\bigl(b_{\varepsilon(\min(i,j),K)}\bigr) + b_{\varepsilon(\min(i,j),K)}\partial_i\bigl(a_{\varepsilon(\max(i,j),K)}\bigr). \tag{144}$$

Adding and subtracting some terms in (144), as well as observing that the pair of $\max(i,j)$ and $\min(i,j)$ is either (i,j) or (j,i), but always contains both i and j, we obtain

$$\begin{aligned}\partial_i\bigl(a_{\varepsilon(\max(i,j),K)} b_{\varepsilon(\min(i,j),K)}\bigr) &= a_{\varepsilon(\max(i,j),K)}\partial_i\bigl(b_{\varepsilon(\min(i,j),K)}\bigr) + b_{\varepsilon(\min(i,j),K)}\partial_i\bigl(a_{\varepsilon(\max(i,j),K)}\bigr) \\ &\quad + a_{\varepsilon(\min(i,j),K)}\partial_i\bigl(b_{\varepsilon(\max(i,j),K)}\bigr) - a_{\varepsilon(\min(i,j),K)}\partial_i\bigl(b_{\varepsilon(\max(i,j),K)}\bigr) \\ &\quad + b_{\varepsilon(\max(i,j),K)}\partial_i\bigl(a_{\varepsilon(\min(i,j),K)}\bigr) - b_{\varepsilon(\max(i,j),K)}\partial_i\bigl(a_{\varepsilon(\min(i,j),K)}\bigr)\end{aligned} \tag{145}$$

$$\begin{aligned}&= a_{\varepsilon(i,K)}\partial_i\bigl(b_{\varepsilon(j,K)}\bigr) + b_{\varepsilon(i,K)}\partial_i\bigl(a_{\varepsilon(j,K)}\bigr) \\ &\quad + a_{\varepsilon(j,K)}\partial_i\bigl(b_{\varepsilon(i,K)}\bigr) - a_{\varepsilon(\min(i,j),K)}\partial_i\bigl(b_{\varepsilon(\max(i,j),K)}\bigr) \\ &\quad + b_{\varepsilon(j,K)}\partial_i\bigl(a_{\varepsilon(i,K)}\bigr) - b_{\varepsilon(\max(i,j),K)}\partial_i\bigl(a_{\varepsilon(\min(i,j),K)}\bigr),\end{aligned} \tag{146}$$

where some of the terms in (146) coincide with those in (143). We thus conclude that

$$(-1)^s \boldsymbol{\partial} \lrcorner (\mathbf{a} \owedge \mathbf{b}) = \mathbf{a} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{b}) + \mathbf{b} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{a}) - R_s(\mathbf{a},\mathbf{b}), \tag{147}$$

where the function $R_s(\mathbf{a},\mathbf{b})$ is given by

$$\begin{aligned}R_s(\mathbf{a},\mathbf{b}) = \sum_i \sum_{j\neq i} \sum_{K\in\mathcal{I}_{s-1}:i,j\notin K} \Delta_{KK}\sigma(i,K)\sigma(j,K)\bigl(&a_{\varepsilon(j,K)}\partial_i\bigl(b_{\varepsilon(i,K)}\bigr) - a_{\varepsilon(\min(i,j),K)}\partial_i\bigl(b_{\varepsilon(\max(i,j),K)}\bigr) \\ &+ b_{\varepsilon(j,K)}\partial_i\bigl(a_{\varepsilon(i,K)}\bigr) - b_{\varepsilon(\max(i,j),K)}\partial_i\bigl(a_{\varepsilon(\min(i,j),K)}\bigr)\bigr)\mathbf{e}_j.\end{aligned} \tag{148}$$

We now get back to the remaining term in (128) and again put the definition of $\odot$ in (27) into (124) to obtain

$$\boldsymbol{\partial} \lrcorner (\mathbf{a} \odot \mathbf{b}) = \sum_{i,j} \Delta_{ii}\Delta_{jj}\partial_i\bigl((\mathbf{e}_{\min(i,j)} \lrcorner \mathbf{a}) \cdot (\mathbf{b} \llcorner \mathbf{e}_{\max(i,j)})\bigr)\mathbf{e}_j \tag{149}$$

$$= \sum_i \Bigl(\partial_i\bigl((\mathbf{e}_i \lrcorner \mathbf{a}) \cdot (\mathbf{b} \llcorner \mathbf{e}_i)\bigr)\mathbf{e}_i\Bigr) + \sum_i \sum_{j\neq i} \Bigl(\Delta_{ii}\Delta_{jj}\partial_i\bigl((\mathbf{e}_{\min(i,j)} \lrcorner \mathbf{a}) \cdot (\mathbf{b} \llcorner \mathbf{e}_{\max(i,j)})\bigr)\mathbf{e}_j\Bigr), \tag{150}$$

where we rewrote the double summation over the pair of indices i and j in a convenient way. In an analogous manner to the previous analysis, using (112) to evaluate the first summation over i in (150) yields,

$$\sum_i \partial_i\bigl((\mathbf{e}_i \lrcorner \mathbf{a}) \cdot (\mathbf{b} \llcorner \mathbf{e}_i)\bigr)\mathbf{e}_i = (-1)^{s-1} \sum_i \sum_{I\in\mathcal{I}_s:i\in I} \Delta_{ii}\Delta_{II}\partial_i\bigl(a_I b_I\bigr)\mathbf{e}_i. \tag{151}$$

In a similar manner, we use (115) to evaluate the second summation, over i and $j \neq i$, in (150),

$$\sum_{i \neq j} \Delta_{ii}\Delta_{jj}\partial_i\big((\mathbf{e}_{\min(i,j)} \lrcorner \mathbf{a}) \cdot (\mathbf{b} \llcorner \mathbf{e}_{\max(i,j)})\big)\mathbf{e}_j =$$

$$= \sum_i \sum_{j \neq i} \sum_{K \in \mathcal{I}_{s-1}: i,j \notin K} \Delta_{KK}\sigma\big(K, \min(i,j)\big)\sigma\big(\max(i,j), K\big)\partial_i\big(a_{\varepsilon(\min(i,j),K)}b_{\varepsilon(\max(i,j),K)}\big)\mathbf{e}_j \tag{152}$$

$$= (-1)^{s-1}\sum_i \sum_{j \neq i} \sum_{K \in \mathcal{I}_{s-1}: i,j \notin K} \Delta_{KK}\sigma(K,i)\sigma(K,j)\partial_i\big(a_{\varepsilon(\min(i,j),K)}b_{\varepsilon(\max(i,j),K)}\big)\mathbf{e}_j, \tag{153}$$

where we used that $\sigma(K,\ell) = (-1)^{|K|}\sigma(\ell,K)$, and that $\sigma\big(K,\min(i,j)\big)\sigma\big(K,\max(i,j)\big) = \sigma(K,i)\sigma(K,j)$ as $i \neq j$.

We proceed by relating (150) to a closed-form, coordinate-free expression. First, we evaluate $\mathbf{a} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{b})$,

$$\mathbf{a} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{b}) = \left(\sum_{J \in \mathcal{I}_s} a_J \mathbf{e}_J\right) \llcorner \left(\sum_{i, I \in \mathcal{I}_s : i \in I} \sigma(I \setminus i, i)\partial_i b_I \mathbf{e}_{I\setminus i}\right) \tag{154}$$

$$= \sum_{i, I \in \mathcal{I}_s : i \in I} \sum_{J \in \mathcal{I}_s} \sigma(I \setminus i, i) a_J \partial_i b_I \mathbf{e}_J \llcorner \mathbf{e}_{I\setminus i} \tag{155}$$

$$= \sum_i \sum_{I,J \in \mathcal{I}_s : i \in I, i \in J} \sigma(I \setminus i, i) a_J \partial_i b_I \mathbf{e}_J \llcorner \mathbf{e}_{I\setminus i} + \sum_i \sum_{I,J \in \mathcal{I}_s : i \in I, i \notin J} \sigma(I \setminus i, i) a_J \partial_i b_I \mathbf{e}_J \llcorner \mathbf{e}_{I\setminus i}, \tag{156}$$

where we split the summation over J in two parts, depending on whether $i \in J$ or not. Evaluating the first summation over i in (156) yields,

$$\sum_i \sum_{I,J \in \mathcal{I}_s : i \in I, i \in J} \sigma(I \setminus i, i) a_J \partial_i b_I \mathbf{e}_J \llcorner \mathbf{e}_{I\setminus i} = \sum_i \sum_{I \in \mathcal{I}_s : i \in I} \sigma(I \setminus i, i) a_I \partial_i b_I \mathbf{e}_I \llcorner \mathbf{e}_{I\setminus i} \tag{157}$$

$$= \sum_i \sum_{I \in \mathcal{I}_s : i \in I} a_I \partial_i b_I \mathbf{e}_i, \tag{158}$$

where we used in (157) that the constraints that $I\backslash i$ is a subset of J and $i \in J$ enforce that $J = I$, and computed and simplified the interior product in (158). As for the second summand in (156), we similarly obtain

$$\sum_i \sum_{I,J \in \mathcal{I}_s : i \in I, i \notin J} \sigma(I \setminus i, i) a_J \partial_i b_I \mathbf{e}_J \llcorner \mathbf{e}_{I\setminus i} = \sum_i \sum_{\ell \neq i} \sum_{K \in \mathcal{I}_{s-1} : i \notin K} \sigma(K,i) a_{\ell+K}\partial_i b_{i+K}\mathbf{e}_{\ell+K} \llcorner \mathbf{e}_K \tag{159}$$

$$= \sum_i \sum_{\ell \neq i} \sum_{K \in \mathcal{I}_{s-1} : i \notin K} \Delta_{KK}\sigma(K,i)\sigma(K,\ell) a_{\ell+K}\partial_i b_{i+K}\mathbf{e}_\ell \tag{160}$$

where we rewrote the summation over I and J in (159) in terms of an index j and a set $K \in \mathcal{I}_{s-1}$ such that $i, j \notin K$ and $I = \varepsilon(i,K)$ and $J = \varepsilon(j,K)$, and computed the interior product in (160).

Substituting $\mathbf{b}$ for $\mathbf{a}$ in (156) and (160) and grouping terms, we thus obtain

$$\mathbf{a} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{b}) + \mathbf{b} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{a}) = \sum_i \sum_{\ell \neq i} \sum_{K \in \mathcal{I}_{s-1} : i \notin K} \Delta_{KK}\sigma(K,i)\sigma(K,\ell)\big(a_{\ell+K}\partial_i b_{i+K} + b_{\ell+K}\partial_i a_{i+K}\big)\mathbf{e}_\ell$$
$$+ \sum_i \sum_{I \in \mathcal{I}_s : i \in I} \partial_i(a_I b_I)\mathbf{e}_i. \tag{161}$$

The second summation, over i and I, in (161) can be seen to coincide with (151), apart from a factor $(-1)^s$ that cancels out with the same factor present in (128) and a minus sign. As for the first summation, over i, j, K, in (161), and again apart from common factors, each of the terms in the triple summation in (161) contains the term

$$a_{\ell+K}\partial_i b_{i+K} + b_{\ell+K}\partial_i a_{i+K}, \tag{162}$$

to be compared to the analogous term in the triple summation over i, j and K in (153), namely,

$$\partial_i\big(a_{\varepsilon(\min(i,j),K)}b_{\varepsilon(\max(i,j),K)}\big) = a_{\varepsilon(\min(i,j),K)}\partial_i(b_{\varepsilon(\max(i,j),K)}) + b_{\varepsilon(\max(i,j),K)}\partial_i(a_{\varepsilon(\min(i,j),K)}). \tag{163}$$

Adding and subtracting some terms in (163), as well as observing that the pair of $\max(i,j)$ and $\min(i,j)$ is either (i,j) or (j,i), but always contains both i and j, we obtain

$$\begin{aligned}\partial_i\big(a_{\varepsilon(\min(i,j),K)}b_{\varepsilon(\max(i,j),K)}\big) &= a_{\varepsilon(\min(i,j),K)}\partial_i(b_{\varepsilon(\max(i,j),K)}) + b_{\varepsilon(\max(i,j),K)}\partial_i(a_{\varepsilon(\min(i,j),K)}) \\ &\quad + a_{\varepsilon(\max(i,j),K)}\partial_i(b_{\varepsilon(\min(i,j),K)}) - a_{\varepsilon(\max(i,j),K)}\partial_i(b_{\varepsilon(\min(i,j),K)}) \\ &\quad + b_{\varepsilon(\min(i,j),K)}\partial_i(a_{\varepsilon(\max(i,j),K)}) - b_{\varepsilon(\min(i,j),K)}\partial_i(a_{\varepsilon(\max(i,j),K)}) \end{aligned} \tag{164}$$

$$\begin{aligned} &= a_{\varepsilon(i,K)}\partial_i(b_{\varepsilon(j,K)}) + b_{\varepsilon(i,K)}\partial_i(a_{\varepsilon(j,K)}) \\ &\quad + a_{\varepsilon(j,K)}\partial_i(b_{\varepsilon(i,K)}) - a_{\varepsilon(\max(i,j),K)}\partial_i(b_{\varepsilon(\min(i,j),K)}) \\ &\quad + b_{\varepsilon(j,K)}\partial_i(a_{\varepsilon(i,K)}) - b_{\varepsilon(\min(i,j),K)}\partial_i(a_{\varepsilon(\max(i,j),K)}), \end{aligned} \tag{165}$$

where some of the terms in (165) coincide with those in (162). We may therefore conclude that

$$(-1)^s \boldsymbol{\partial} \lrcorner (\mathbf{a} \odot \mathbf{b}) = -\mathbf{a} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{b}) - \mathbf{b} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{a}) - Q_s(\mathbf{a}, \mathbf{b}), \tag{166}$$

where the function $Q_s(\mathbf{a}, \mathbf{b})$ is given by

$$\begin{aligned} Q_s(\mathbf{a}, \mathbf{b}) = \sum_i \sum_{j \neq i} \sum_{K \in \mathcal{I}_{s-1}: i, j \notin K} \Delta_{KK} \sigma(i, K) \sigma(j, K) \big(a_{\varepsilon(i,K)} \partial_i (b_{\varepsilon(j,K)}) - a_{\varepsilon(\max(i,j),K)} \partial_i (b_{\varepsilon(\min(i,j),K)}) \\ + b_{\varepsilon(i,K)} \partial_i (a_{\varepsilon(j,K)}) - b_{\varepsilon(\min(i,j),K)} \partial_i (a_{\varepsilon(\max(i,j),K)}) \big) \mathbf{e}_j. \end{aligned} \tag{167}$$

It remains to verify that $R_s(\mathbf{a}, \mathbf{b}) + Q_s(\mathbf{a}, \mathbf{b}) = 0$, where $R_s(\mathbf{a}, \mathbf{b})$ was given in (148). This condition is satisfied if

$$\begin{aligned} a_{\varepsilon(j,K)} \partial_i (b_{\varepsilon(i,K)}) - a_{\varepsilon(\min(i,j),K)} \partial_i (b_{\varepsilon(\max(i,j),K)}) + b_{\varepsilon(j,K)} \partial_i (a_{\varepsilon(i,K)}) - b_{\varepsilon(\max(i,j),K)} \partial_i (a_{\varepsilon(\min(i,j),K)}) \\ + a_{\varepsilon(i,K)} \partial_i (b_{\varepsilon(j,K)}) - a_{\varepsilon(\max(i,j),K)} \partial_i (b_{\varepsilon(\min(i,j),K)}) + b_{\varepsilon(i,K)} \partial_i (a_{\varepsilon(j,K)}) - b_{\varepsilon(\min(i,j),K)} \partial_i (a_{\varepsilon(\max(i,j),K)}) = 0, \end{aligned} \tag{168}$$

as can be verified for both possible orderings of the pair (i, j), that is $\max(i, j) = i$, $\max(i, j) = j$. Therefore, we may combine (147) and (166) into (128) into the final, closed-form, coordinate-free expression in (127), that is

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} = \mathbf{a} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{b}) + \mathbf{b} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{a}) - \mathbf{a} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{b}) - \mathbf{b} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{a}). \tag{169}$$

4 Examples and applications

4.1 Conformal invariance

Conformal invariance of a field theory is related to the vanishing of the trace of the stress-energy momentum tensor [6, Sect. 4.2]. From the definition of the trace, we may compute the trace of $\mathbf{T}_{\mathbf{a}\cdot\mathbf{b}}$ from (82):

$$\begin{aligned} \operatorname{Tr} \mathbf{T}_{\mathbf{a}\cdot\mathbf{b}} &= \sum_i \Delta_{ii} T_{ii}^{\mathbf{a}\cdot\mathbf{b}} && (170) \\ &= \sum_i \left(\sum_{K \in \mathcal{I}_s : i \notin K} \Delta_{KK} a_K b_K - \sum_{K \in \mathcal{I}_s : i \in K} \Delta_{KK} a_K b_K \right) && (171) \\ &= \sum_{K \in \mathcal{I}_s} \left(\sum_{i : i \notin K} \Delta_{KK} a_K b_K - \sum_{i : i \in K} \Delta_{KK} a_K b_K \right) && (172) \\ &= \sum_{K \in \mathcal{I}_s} \big((k + n - s) \Delta_{KK} a_K b_K - s \Delta_{KK} a_K b_K \big) && (173) \\ &= (k + n - 2s)(\mathbf{a} \cdot \mathbf{b}), && (174) \end{aligned}$$

where (170) follows from the definition of trace, in (171) we replaced the coefficient $T_{ii}^{\mathbf{a}\cdot\mathbf{b}}$ by its formula in (82), we reversed the summation order in (172), we counted the number of indices i appearing in each summation over i for fixed s-tuple K in (173), and we finally used the definition of $\mathbf{a} \cdot \mathbf{b}$ in (174). From (174), we obtain

$$\operatorname{Tr} \mathbf{T}_{\text{sys}} = \sum_{\mathbf{a},\mathbf{b}} \gamma_{\mathbf{a},\mathbf{b}} \big(k + n - 2 \operatorname{gr}(\mathbf{a}) \big) (\mathbf{a} \cdot \mathbf{b}), \tag{175}$$

for the action given in (37). The formula in (175) is a function of the fields appearing explicitly in the Lagrangian density only. In particular, if $\mathbf{a}$ or $\mathbf{b}$ have some internal structure, e. g., $\mathbf{a} = \boldsymbol{\partial} \wedge \mathbf{A}$ in generalized electromagnetism or Yang-Mills fields, this vector potential may be bypassed. The same principle holds for the following examples, and need not explicitly consider the internal structure of the Lagrangian density terms in our analysis.

4.2 Scalar field

In flat (k, n)-dimensional space-time, the Lagrangian density $\mathcal{L}_{\text{free-scalar}}$ of a free scalar field ϕ is given in (13), so we can make the identification $\mathbf{a} = \mathbf{b} = \boldsymbol{\partial} \wedge \phi = \sum_i \Delta_{ii} \partial_i \phi \, \mathbf{e}_i$, and $\operatorname{gr}(\mathbf{a}) = \operatorname{gr}(\mathbf{b}) = 1$. Taking into account the multiplicative factor $\frac{1}{2}$ in $\mathcal{L}$, the on-diagonal component of the tensor (81) in (82) can be directly evaluated as:

$$\begin{aligned} T_{ii}^{\text{free-scalar}} &= \frac{1}{2} \Delta_{ii} \left(-\Delta_{ii} a_i^2 + \sum_{j : i \neq j} \Delta_{jj} a_j^2 \right) && (176) \\ &= -\frac{1}{2} (\partial_i \phi)^2 + \frac{1}{2} \sum_{j : i \neq j} \Delta_{ii} \Delta_{jj} (\partial_j \phi)^2. && (177) \end{aligned}$$

As for the off-diagonal terms, using (83) and taking into account that $\mathcal{I}_0$ contains only the empty set $\varnothing$, we obtain

$$T_{ij}^{\text{free-scalar}} = -a_i a_j \tag{178}$$
$$= -\Delta_{ii}\Delta_{jj}(\partial_i\phi)(\partial_j\phi). \tag{179}$$

The interior derivative of the tensor $\mathbf{T}_{\text{scalar}}$ is computed from (127), that is

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\text{free-scalar}} = \mathbf{a} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{a}) - \mathbf{a} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{a}) \tag{180}$$
$$= -(\boldsymbol{\partial} \wedge \phi)\big(\boldsymbol{\partial} \lrcorner (\boldsymbol{\partial} \wedge \phi)\big) \tag{181}$$
$$= -\big((\boldsymbol{\partial} \cdot \boldsymbol{\partial})\phi\big)(\boldsymbol{\partial} \wedge \phi), \tag{182}$$

where we took into account the multiplicative factor $\frac{1}{2}$ in the Lagrangian density and used that $\mathbf{a} = \mathbf{b}$ in (180), used that $\boldsymbol{\partial} \wedge (\boldsymbol{\partial} \wedge \phi) = 0$ (cf. [10, Eq. (32)]) and that $\boldsymbol{\partial} \lrcorner \mathbf{a}$ is a scalar in (181), and the identity [10, Eq. (34)] in (182) to rewrite the term $\boldsymbol{\partial} \lrcorner (\boldsymbol{\partial} \wedge \phi)$ as $(\boldsymbol{\partial} \cdot \boldsymbol{\partial})\phi$.

4.3 Electromagnetism

In flat (k, n)-dimensional space-time, the Lagrangian density $\mathcal{L}_{\text{free-gem}}$ of a free generalized electromagnetic field $\mathbf{F}$, a multivector field of grade r, with vector potential $\mathbf{A}$, such that $\mathbf{F} = \boldsymbol{\partial} \wedge \mathbf{A}$, is given in (12). From (102), we may express the stress-energy-momentum tensor $\mathbf{T}_{\text{free-gem}}$ for any values of r, k and n as [1]

$$\mathbf{T}_{\text{free-gem}} = -\frac{1}{2}(\mathbf{F} \oslash \mathbf{F} + \mathbf{F} \odot \mathbf{F}), \tag{183}$$

with on- and off-diagonal components respective given by (38) and (39),

$$T_{ii}^{\text{free-gem}} = \frac{(-1)^r}{2}\Delta_{ii}\left(\sum_{I \in \mathcal{I}_r : i \in I} F_I^2 \Delta_{II} - \sum_{I \in \mathcal{I}_r : i \notin I} F_I^2 \Delta_{II}\right) \tag{184}$$
$$T_{ij}^{\text{free-gem}} = -\sum_{L \in \mathcal{I}_{r-1} : i,j \neq L} \sigma(L, i)\sigma(j, L) F_{i+L} F_{j+L} \Delta_{LL}, \tag{185}$$

in alignment with [10] and with the stress-energy tensor for standard electromagnetism with bivectors, or the Faraday tensor ($r = 2$, $k = 1$, $n = 3$) [1, Sect. 33], [2, Sect. 12.10].

While the physical interpretation remains open, we may apply (102) to find the stress-energy-momentum tensor of the interaction Lagrangian density $\mathbf{J} \cdot \mathbf{A}$, which denote as $\mathbf{T}_{\text{int-gem}}$, as

$$\mathbf{T}_{\text{int-gem}} = (-1)^{r-1}(\mathbf{J} \oslash \mathbf{A} + \mathbf{J} \odot \mathbf{A}), \tag{186}$$

with on- and off-diagonal components respective given by (38) and (39),

$$T_{ii}^{\text{int-gem}} = \Delta_{ii}\left(\sum_{K \in \mathcal{I}_s : i \notin K} \Delta_{KK} J_K A_K - \sum_{K \in \mathcal{I}_s : i \in K} \Delta_{KK} J_K A_K\right), \tag{187}$$
$$T_{ij}^{\text{int-gem}} = -\sum_{K \in \mathcal{I}_{s-1} : i,j \notin K} \Delta_{KK}\sigma\big(\varepsilon(i, K)_{i \leftrightarrow j}\big)\big(J_{\varepsilon(i,K)} A_{\varepsilon(j,K)} + A_{\varepsilon(i,K)} J_{\varepsilon(j,K)}\big). \tag{188}$$

The transfer of energy-momentum is described by the interior derivative of the stress-energy-momentum tensor. Applying (127) to $\mathbf{T}_{\text{free-gem}}$ and $\mathbf{T}_{\text{int-gem}}$, respectively, gives:

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\text{free-gem}} = (-1)^{r-1}\big(\mathbf{F} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{F}) - \mathbf{F} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{F})\big) \tag{189}$$
$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\text{int-gem}} = \mathbf{J} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{A}) + \mathbf{A} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{J}) - \mathbf{J} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{A}) - \mathbf{A} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{J}). \tag{190}$$

Setting now $\mathbf{F} = \boldsymbol{\partial} \wedge \mathbf{A}$, $\boldsymbol{\partial} \lrcorner \mathbf{F} = \mathbf{J}$ and $\boldsymbol{\partial} \wedge \mathbf{F} = 0$ in (189)–(190), and combining the resulting expressions yields

$$\boldsymbol{\partial} \lrcorner (\mathbf{T}_{\text{free-gem}} + \mathbf{T}_{\text{int-gem}}) = (-1)^r \mathbf{F} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{F}) + \mathbf{J} \lrcorner \mathbf{F} + \mathbf{A} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{J}) - \mathbf{J} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{A}) - \mathbf{A} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{J}) \tag{191}$$
$$= (-1)^r \mathbf{F} \llcorner \mathbf{J} + \mathbf{J} \lrcorner \mathbf{F} + \mathbf{A} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{J}) - \mathbf{J} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{A}) \tag{192}$$
$$= -\mathbf{J} \lrcorner \mathbf{F} + \mathbf{J} \lrcorner \mathbf{F} + \mathbf{A} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{J}) - \mathbf{J} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{A}) \tag{193}$$
$$= \mathbf{A} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{J}) - \mathbf{J} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{A}), \tag{194}$$

where we used in (192) the Maxwell equation $\boldsymbol{\partial} \lrcorner \mathbf{F} = \mathbf{J}$ and the continuity equation for the current $\boldsymbol{\partial} \lrcorner \mathbf{J} = 0$, and the identity $\mathbf{F} \llcorner \mathbf{J} = (-1)^{r-1}\mathbf{J} \lrcorner \mathbf{F}$ in (193). The generalized Lorentz force density $\mathbf{f} = \mathbf{J} \lrcorner \mathbf{F}$ cancels out from the interior derivative as $\mathbf{f} + \boldsymbol{\partial} \lrcorner \mathbf{T}_{\text{free-gem}} = 0$. However, the physical interpretation of the terms in (194), including their Gauge invariance and possible connection to the tensor of the matter fields, remains open.

[1] We have moved a factor $\frac{1}{2}$ from the definition of $\mathbf{F} \odot \mathbf{F}$ and $\mathbf{F} \oslash \mathbf{F}$ in [10]. Also, the proof in Appendix A.2 of [10] should be amended as of (147) and (166) in this paper; the final formula for the interior derivative given in Appendix A.2 of [10] remains unchanged.

4.4 Yang-Mills fields

In flat $(1,3)$-dimensional space-time, the Lagrangian density $\mathcal{L}_{\text{free-ym}}$ of a free Yang-Mills field $\mathbf{F}$, a Lie-algebra valued bivector field, with connection $\mathbf{A}$, is given in (17)–(18). From (102), we may directly express the stress-energy-momentum tensor $\mathbf{T}_{\text{free-ym}}$ as

$$\mathbf{T}_{\text{free-ym}} = -\frac{1}{2g^2}\operatorname{Tr}(\mathbf{F} \owedge \mathbf{F} + \mathbf{F} \odot \mathbf{F}) \tag{195}$$

$$= -\frac{1}{2g^2}\sum_a (\mathbf{F}^a \owedge \mathbf{F}^a + \mathbf{F}^a \odot \mathbf{F}^a), \tag{196}$$

with on- and off-diagonal components respective given by (38) and (39),

$$T_{ii}^{\text{free-ym}} = \frac{(-1)^r}{4g^2}\Delta_{ii}\sum_a\left(\sum_{I\in\mathcal{I}_r:i\in I}(F_I^a)^2\Delta_{II} - \sum_{I\in\mathcal{I}_r:i\notin I}(F_I^a)^2\Delta_{II}\right) \tag{197}$$

$$T_{ij}^{\text{free-ym}} = -\frac{1}{2g^2}\sum_a\sum_{L\in\mathcal{I}_{r-1}:i,j\neq L}\sigma(L,i)\sigma(j,L)F_{i+L}^a F_{j+L}^a\Delta_{LL}, \tag{198}$$

in agreement with [4, Eq. (2.11)] and [15, Eq. (46)]. The transfer of energy-momentum from the Yang-Mills field is described by the interior derivative of $\mathbf{T}_{\text{free-ym}}$ given in (127).

4.5 Proca field

The Lagrangian density $\mathcal{L}_{\text{proca}}$ of a Proca field $\mathbf{A}$ is given in (20), with $\mathbf{F} = \boldsymbol{\partial} \wedge \mathbf{A}$. We obtain its stress-energy-momentum tensor by adding to the tensor $\mathbf{T}_{\text{free-gem}}$ with $r = 2$ another tensor $\mathbf{T}_{\text{mass-proca}}$ for the mass terms,

$$\mathbf{T}_{\text{mass-proca}} = -\frac{1}{2}m^2(\mathbf{A} \owedge \mathbf{A} + \mathbf{A} \odot \mathbf{A}), \tag{199}$$

with on- and off-diagonal components respectively given by (38) and (39),

$$T_{ii}^{\text{mass-proca}} = \Delta_{ii}\frac{m^2}{2}\left(\sum_{K\in\mathcal{I}_s:i\notin K}\Delta_{KK}A_K^2 - \sum_{K\in\mathcal{I}_s:i\in K}\Delta_{KK}A_K^2\right), \tag{200}$$

$$T_{ij}^{\text{mass-proca}} = -m^2\sum_{K\in\mathcal{I}_{s-1}:i,j\notin K}\Delta_{KK}\sigma\big(\varepsilon(i,K)_{i\leftrightarrow j}\big)A_{\varepsilon(i,K)}A_{\varepsilon(j,K)}, \tag{201}$$

in agreement with [15, Eq. (62)]. Applying (127) to determine the interior derivative of the stress-energy-momentum tensor to the sum $\mathbf{T}_{\text{free-proca}} = \mathbf{T}_{\text{free-gem}} + \mathbf{T}_{\text{mass-proca}}$, together with (189) after setting $\boldsymbol{\partial} \wedge \mathbf{F} = 0$, gives

$$\boldsymbol{\partial} \lrcorner \mathbf{T}_{\text{free-proca}} = \mathbf{F} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{F}) + m^2\mathbf{A} \lrcorner (\boldsymbol{\partial} \wedge \mathbf{A}) - m^2\mathbf{A} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{A}) \tag{202}$$

$$= \mathbf{F} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{F}) + m^2\mathbf{A} \lrcorner \mathbf{F} - m^2\mathbf{A} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{A}) \tag{203}$$

$$= m^2\mathbf{F} \llcorner \mathbf{A} + m^2\mathbf{A} \lrcorner \mathbf{F} - m^2\mathbf{A} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{A}) \tag{204}$$

$$= -m^2\mathbf{A} \llcorner (\boldsymbol{\partial} \lrcorner \mathbf{A}), \tag{205}$$

where we used in (203) the definition $\mathbf{F} = \boldsymbol{\partial} \wedge \mathbf{A}$, in (204) the equation $\boldsymbol{\partial} \lrcorner \mathbf{F} = m^2\mathbf{A}$, and the identity $\mathbf{F} \llcorner \mathbf{A} = -\mathbf{A} \lrcorner \mathbf{F}$, obtained from the Euler-Lagrange equations of the system, in (205), as $\mathbf{F}$ is a bivector. While the interior derivative of the free (generalized) electromagnetic field tensor (189) vanishes in the absence of interaction with a current density $\mathbf{J}$, the interior derivative of the Proca field tensor does not vanish unless the Lorenz gauge condition, $\boldsymbol{\partial} \lrcorner \mathbf{A}$, holds, thereby breaking the gauge invariance.

4.6 Conclusions and future work

In this paper, we have provided an exterior-algebraic derivation of the symmetric stress-energy-momentum tensor that naturally appears in field theories from the invariance of the action of a closed physical system to infinitesimal space-time translations. Our focus lies on Lagrangian densities that are expressed as the dot product of two multivector fields, e. g., scalar or gauge fields, in flat space-time. The analysis covers a number of interesting cases, such as electromagnetic fields, Proca fields or Yang-Mills fields, and it leaves out the relevant case of spinor matter fields. An extension to spinor fields is left for future work and will be reported elsewhere. Our formalism allows us to calculate the tensor and its interior derivative, not only for free fields but also for the interaction terms appearing in the action. It would be interesting to relate the interior derivative of the stress-energy-tensor associated with spinor fields, e.g., that of the electron, to the interior derivative of the tensor associated with the interaction Lagrangian density.

Finally, while we have considered invariance of the action to space-time translations, it would be worthwhile considering the full Poincaré group, i.e., including Lorentz transformations. A study of this case would lead to an exterior-algebraic characterization of the angular momentum tensor, extending the analysis presented in this paper. Moreover, considering the conserved charges associated to the stress-energy-momentum or angular momentum tensors would lead to a redefinition of energy-momentum and angular momentum in general flat space-times.

A Proof of $\sigma\bigl(\varepsilon(i,K)_{i\leftrightarrow j}\bigr) = \sigma\bigl(\varepsilon(j,K)_{j\leftrightarrow i}\bigr)$ in (63) and related expressions

With no real loss of generality, assume that $i < j$ and let K be an index set that does not include i and j. We can then write the set K as the union of three disjoint subsets as depicted in Figure 1. From the graphic representation in 1, we can express some signatures as follows:

$$\sigma(K,i) = \sigma(K_3,i)\sigma(K_2,i), \tag{206}$$
$$\sigma(j,K) = \sigma(j,K_1)\sigma(j,K_2), \tag{207}$$
$$\sigma(i,j+K) = \sigma(i,K_1), \tag{208}$$
$$\sigma(i+K,j) = \sigma(K_3,j), \tag{209}$$
$$\sigma\bigl(\varepsilon(i,K)_{i\leftrightarrow j}\bigr) = \sigma(j,K_2). \tag{210}$$

As the permutations on the right-hand sides of (206)–(210) represent the indices i and j going through fixed subsets, we may evaluate the following signatures:

$$\sigma(i,K_1) = \sigma(j,K_1) = (-1)^{|K_1|}, \tag{211}$$
$$\sigma(j,K_2) = \sigma(K_2,i) = (-1)^{|K_2|}, \tag{212}$$
$$\sigma(K_3,j) = \sigma(K_3,i) = (-1)^{|K_3|}. \tag{213}$$

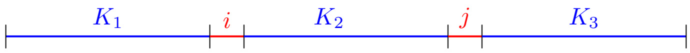

Figure 1: Characterization of the set K as union of three subsets, for $i < j$.

As the indices i and j are separated by the set K_2, the permutation that orders the set $\varepsilon(i,K)$ when i is replaced by j simply has to rearrange the set (j,K_2),

$$\sigma\bigl(\varepsilon(i,K)_{i\leftrightarrow j}\bigr) = \sigma(j,K_2) = (-1)^{|K_2|}. \tag{214}$$

Similarly, the permutation that orders the set $\varepsilon(j,K)$ when j is replaced by i has to rearrange the set (K_2,i),

$$\sigma\bigl(\varepsilon(j,K)_{j\leftrightarrow i}\bigr) = \sigma(K_2,i) = (-1)^{|K_2|}. \tag{215}$$

As Eqs. (214) and (215) coincide, and the reasoning is unaffected if $i > j$ and Eq. (63) is proved,

$$\sigma\bigl(\varepsilon(i,K)_{i\leftrightarrow j}\bigr) = \sigma\bigl(\varepsilon(j,K)_{j\leftrightarrow i}\bigr). \tag{216}$$

In fact, one can prove two alternative characterizations of $\sigma\bigl(\varepsilon(j,K)_{j\leftrightarrow i}\bigr)$, as we do next. Assume again with no loss of generality that $i < j$. First, we compute

$$(-1)^{|K|}\sigma\bigl(\varepsilon(i,K)_{i\leftrightarrow j}\bigr) = (-1)^{|K|}(-1)^{|K_2|} \tag{217}$$
$$= (-1)^{|K_1|}(-1)^{|K_3|}. \tag{218}$$

Then, we can verify by using (206)–(210), (211)–(213), and (218) that

$$\sigma(K,i)\sigma(j,K) = \sigma(K_3,i)\sigma(K_2,i)\sigma(j,K_1)\sigma(j,K_2) \tag{219}$$
$$= (-1)^{|K_3|}(-1)^{|K_2|}(-1)^{|K_1|}(-1)^{|K_2|} \tag{220}$$
$$= (-1)^{|K_3|}(-1)^{|K_1|} \tag{221}$$
$$= (-1)^{|K|}\sigma\bigl(\varepsilon(i,K)_{i\leftrightarrow j}\bigr). \tag{222}$$

And similarly, by using (206)–(210), (211)–(213) and (218) we find that

$$\sigma(i, j + K)\sigma(i + K, j) = \sigma(i, K_1)\sigma(K_3, j) \tag{223}$$
$$= (-1)^{|K_1|}(-1)^{|K_3|} \tag{224}$$
$$= (-1)^{|K|}\sigma\big(\varepsilon(i, K)_{i\leftrightarrow j}\big). \tag{225}$$

If we carry out the analysis for $i > j$, we obtain the same expressions in (222) and (225).

B Proof of Leibniz rule for mixed product

We prove (120) starting by the evaluation of the left-hand side

$$\boldsymbol{\partial} \cdot (\boldsymbol{\varepsilon} \lrcorner \mathbf{T}) = \Big(\sum_i \Delta_{ii}\partial_i \mathbf{e}_i\Big) \cdot \bigg(\Big(\sum_j \varepsilon_j \mathbf{e}_j\Big) \lrcorner \Big(\sum_{\ell\leq m} T_{\ell m}\mathbf{u}_{\ell m}\Big)\bigg) \tag{226}$$
$$= \Big(\sum_i \Delta_{ii}\partial_i \mathbf{e}_i\Big) \cdot \Big(\sum_{j,\ell} \varepsilon_j T_{\varepsilon(\ell,j)}\Delta_{jj}\mathbf{e}_\ell\Big) \tag{227}$$
$$= \sum_{i,j} \Delta_{jj}\partial_i\big(\varepsilon_j T_{\varepsilon(i,j)}\big). \tag{228}$$

Then, on the right-hand side of (120), the first term can be similarly expressed as

$$\boldsymbol{\varepsilon} \cdot (\boldsymbol{\partial} \lrcorner \mathbf{T}) = \Big(\sum_j \varepsilon_j \mathbf{e}_j\Big) \cdot \bigg(\Big(\sum_i \Delta_{ii}\partial_i \mathbf{e}_i\Big) \lrcorner \Big(\sum_{\ell\leq m} T_{\ell m}\mathbf{u}_{\ell m}\Big)\bigg) \tag{229}$$
$$= \Big(\sum_j \varepsilon_j \mathbf{e}_j\Big) \cdot \Big(\sum_{i,\ell} \partial_i T_{i\ell}\mathbf{e}_\ell\Big) = \sum_{i,j} \Delta_{jj}\varepsilon_j\partial_i T_{\varepsilon(i,j)}. \tag{230}$$

Finally, the second term on the right-hand side of (120) can be expanded using (33) as

$$(\boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}) \cdot \mathbf{T} = \Big(\sum_{i,j} \Delta_{ii}\partial_i\varepsilon_j \mathbf{e}_i \otimes \mathbf{e}_j\Big) \cdot \Big(\sum_{\ell\leq m} T_{\ell m}\mathbf{u}_{\ell m}\Big) \tag{231}$$
$$= \sum_{i,j} \Delta_{jj} T_{\min(i,j),\max(i,j)}\partial_i\varepsilon_j \tag{232}$$
$$= \sum_{i,j} \Delta_{jj} T_{\varepsilon(i,j)}\partial_i\varepsilon_j. \tag{233}$$

Combining the three expressions in (228), (230) and (233) proves the generalized Leibniz rule in (120).

Bibliography

[1] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields, 4th ed., ser. Course of Theoretical Physics. Butterworth-Heinemann, 1987, vol. 2.

[2] J. D. Jackson, Classical Electrodynamics, 3rd ed. John Wiley & Sons, 1999.

[3] M. Forger and H. Römer, "Currents and the energy-momentum tensor in classical field theory: a fresh look at an old problem," Annals of Physics, vol. 309, no. 2, pp. 306–389, Feb 2004. [Online]. Available: http://dx.doi.org/10.1016/j.aop.2003.08.011

[4] D. N. Blaschke, F. Gieres, M. Reboud, and M. Schweda, "The energy-momentum tensor(s) in classical gauge theories," Nuclear Physics B, vol. 912, pp. 192–223, Nov 2016. [Online]. Available: http://dx.doi.org/10.1016/j.nuclphysb.2016.07.001

[5] M. Maggiore, A modern introduction to quantum field theory, ser. Oxford Master Series in Statistical, Computational, and Theoretical Physics. Oxford: Oxford Univ. Press, 2005.

[6] P. Di Francesco, P. Mathieu, and D. Sénéchal, Conformal Field Theory, ser. Graduate texts in contemporary physics. New York, NY: Springer, 1997.

[7] M. J. Gotay and J. E. Marsden, "Stress-energy-momentum tensors and the belinfante-rosenfeld formula," Contemp. Math., vol. 132, pp. 367–392, 1992.

[8] N. Voicu, "Energy-momentum tensors in classical field theories – A modern perspective," International Journal of Geometric Methods in Modern Physics, vol. 13, p. 1640001, 2015.

[9] I. Colombaro, J. Font-Segura, and A. Martinez, "An introduction to space–time exterior calculus," Mathematics, vol. 7, pp. 564–583, Jun. 2019.

[10] ——, "Generalized Maxwell equations for exterior-algebra multivectors in (k, n) space-time dimensions," European Physical Journal Plus, vol. 135, no. 305, Mar. 2020.

[11] D. Lovelock and H. Rund, Tensors, Differential Forms, and Variational Principles. Dover Publications, 1989.

[12] H. Flanders, Differential Forms with Applications to the Physical Sciences. Dover Publications, 1989.

[13] R. P. Feynman, R. B. Leighton, and M. Sands, The Feynman Lectures on Physics, Vol. II: Mainly Electromagnetism and Matter. Addison-Wesley, 1977.

[14] A. Zee, Quantum Field Theory in a Nutshell, ser. Nutshell handbook. Princeton, NJ: Princeton Univ. Press, 2003.

[15] M. Montesinos and E. Flores, "Symmetric energy-momentum tensor in Maxwell, Yang-Mills, and Proca theories obtained using only Noether's theorem," https://arxiv.org/abs/hep-th/0602190.

On the Angular Momentum and Spin of Generalized Electromagnetic Field for r-Vectors in (k, n) Space-Time Dimensions *

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Abstract

This paper studies the relativistic angular momentum for the generalized electromagnetic field, described by r-vectors in (k, n) space-time dimensions, with exterior-algebraic methods. First, the angular-momentum tensor is derived from the invariance of the Lagrangian to space-time rotations (Lorentz transformations), avoiding the explicit need of the canonical tensor in Noether's theorem. The derivation proves the conservation law of angular momentum for generic values of r, k, and n. Second, an integral expression for the flux of the tensor across a $(k + n - 1)$-dimensional surface of constant ℓ-th space-time coordinate is provided in terms of the normal modes of the field; this analysis is a natural generalization of the standard analysis of electromagnetism, i. e. a three-dimensional space integral at constant time. Third, a brief discussion on the orbital angular momentum and the spin of the generalized electromagnetic field, including their expression in complex-valued circular polarizations, is provided for generic values of r, k, and n.

Keywords— Angular Momentum, Spin, Electromagnetism, Maxwell Equations, Exterior Algebra, Exterior Calculus, Tensor Calculus

1 Introduction: Preliminaries, Notation, and Main Results

1.1 Generalized Maxwell Equations

For a given natural number r, the generalized Maxwell field $\mathbf{F}(\mathbf{x})$ and source density $\mathbf{J}(\mathbf{x})$ are characterized by multivector fields of respective grades r and $r - 1$ at every point $\mathbf{x}$ of a flat (k, n)-space-time with k temporal and n spatial dimensions [1, Sec. 3]. For any $0 \leq s \leq k + n$, grade-s multivectors belong to a vector space with basis elements $\mathbf{e}_I$, where I is an ordered list of s non-repeated space-time indices; we represent space-time indices by Latin letters. We denote by $\mathcal{I}_s$ the set of all such ordered lists of s space-time indices; we let $I_0 = \emptyset$ and we write $\mathcal{I}$ for $\mathcal{I}_1$. Let $\Delta_{II} = \mathbf{e}_I \cdot \mathbf{e}_I$ for $I \in \mathcal{I}_s$ be the space-time metric, where $\cdot$ denotes the dot product [1, Eqs. (12)–(13)]. The temporal (resp. spatial) basis elements are $\mathbf{e}_0$ to $\mathbf{e}_{k-1}$ (resp. $\mathbf{e}_k$ to $\mathbf{e}_{k+n-1}$) and have metric -1 (resp. $+1$). The generalized Maxwell equations for arbitrary r, k, and n are the following pair of coupled differential equations:

$$\boldsymbol{\partial} \lrcorner \mathbf{F} = \mathbf{J}, \tag{1}$$

$$\boldsymbol{\partial} \wedge \mathbf{F} = 0, \tag{2}$$

*This work has been funded in part by the Spanish Ministry of Science, Innovation and Universities under grants TEC2016-78434-C3-1-R, BES-2017-081360 and PID2020-116683GB-C22. Alfonso Martinez is a Serra-Húnter Fellow. Published in: ***Eur. Phys. J. Plus* 136, 1047 (2021). DOI:** 10.1140/epjp/s13360-021-02023-5.

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in units such that $c = 1$. The interior derivative (or divergence), expressed with the left interior product ($\rfloor$) in (1), and the exterior derivative, expressed in terms of the wedge product ($\wedge$) in (2), are both defined in [1, Sec. 2] or [2, Sec. 2] and the operator $\boldsymbol{\partial}$ is given by $\boldsymbol{\partial} = \sum_{i\in\mathcal{I}} \Delta_{ii}\partial_i$. For $r = 2$, $k = 1$, and $n = 3$, Eqs (1)–(2) coincide with the standard Maxwell equations, with the identification of $\mathbf{F}$ as the (antisymmetric) Faraday tensor of the electromagnetic field, in contravariant form and $\boldsymbol{\partial}$ the four-gradient [3, Ch. 4], [4, Ch. 11].

The Maxwell equations can be derived by an application of the principle of stationary action [5, Ch. 19], [3, Sec. 8]. For a field theory, the action is a quantity given by the integral over a $(k+n)$-dimensional space-time of a scalar Lagrangian density $\mathcal{L}(\mathbf{x})$. For generalized electromagnetism, the basic field in this formulation is taken to be the vector potential $\mathbf{A}(\mathbf{x})$, a multivector field of grade $r-1$, such that

$$\mathbf{F} = \boldsymbol{\partial} \wedge \mathbf{A}. \tag{3}$$

The Lagrangian density $\mathcal{L}$ is expressed in terms of the multivector dot (scalar) product [1, Sec. 2] as the sum of two terms: a free-field density, $\mathcal{L}_{\text{em}} = \frac{(-1)^{r-1}}{2}\mathbf{F}\cdot\mathbf{F}$, and an interaction term, $\mathcal{L}_{\text{int}} = \mathbf{J}\cdot\mathbf{A}$, that is

$$\mathcal{L} = \frac{(-1)^{r-1}}{2}\mathbf{F}\cdot\mathbf{F} + \mathbf{A}\cdot\mathbf{J}. \tag{4}$$

The Euler-Lagrange equations for the Lagrangian density $\mathcal{L}$ in (4) give indeed the Maxwell equation (1) as vector derivatives of $\mathcal{L}$ with respect to the potential $\mathbf{A}$ and its exterior derivative $\boldsymbol{\partial} \rfloor \mathbf{A}$, namely [6, Sec. 3.2]

$$\partial_{\mathbf{A}}\mathcal{L} = (-1)^{r-1}\boldsymbol{\partial} \rfloor \left(\partial_{\boldsymbol{\partial}\wedge\mathbf{A}}\mathcal{L}\right). \tag{5}$$

If we replace the potential $\mathbf{A}$ by a new field $\mathbf{A}' = \mathbf{A} + \bar{\mathbf{A}} + \boldsymbol{\partial}\wedge\mathbf{G}$, where $\bar{\mathbf{A}}$ is a constant $(r-1)$-vector and $\mathbf{G}$ is an $(r-2)$-vector gauge field, the homogenous Maxwell equation (2) is unchanged [1, Sec. 3]. For a given Maxwell field, there is therefore some unavoidable (gauge) ambiguity on the value of the vector potential if $r \geq 2$. Of special interest for this work are the Coulomb-ℓ gauge and the Lorenz gauge. For a space-time index ℓ, let us define the differential operator $\boldsymbol{\partial}_{\bar{\ell}} = \sum_{i\in\mathcal{I}} \Delta_{ii}\partial_i$. In the Coulomb-$\ell$-gauge, the following two conditions are imposed:

$$\mathbf{e}_\ell \rfloor \mathbf{A} = 0, \tag{6}$$

$$\boldsymbol{\partial}_{\bar{\ell}} \rfloor \mathbf{A} = 0. \tag{7}$$

In classical electromagnetism, setting $\ell = 0$ recovers the Coulomb or radiation gauge. In the Coulomb-ℓ gauge, it also holds that $\boldsymbol{\partial} \rfloor \mathbf{A} = 0$. In the less restrictive Lorenz gauge, it simply holds that

$$\boldsymbol{\partial} \rfloor \mathbf{A} = 0. \tag{8}$$

The multivectorial equation in (8) has $\binom{k+n}{r-2}$ components, i. e. a scalar equation for $r = 2$.

1.2 Energy-Momentum Tensor and Lorentz Force

Energy-momentum can be transferred from the field to the source through a process modelled as a force acting on the source. The generalized Lorentz force density $\mathbf{f}$ is a grade-1 vector with $k+n$ components given by [1, Sec. 4]

$$\mathbf{f} = \mathbf{J} \rfloor \mathbf{F} = (\boldsymbol{\partial} \rfloor \mathbf{F}) \rfloor \mathbf{F}. \tag{9}$$

The volume integral of the Lorentz force density $\mathbf{f}$ over a $(k+n)$-dimensional hypervolume $\mathcal{V}^{k+n}$ quantifies the transfer of energy-momentum to the source in that volume. The conservation law relating the Lorentz force (9) and the stress-energy-momentum tensor $\mathbf{T}_{\text{em}}$ of the free Maxwell field $\mathbf{F}$ is given by [1, Sec. 4], [7, Sec. 4.3],

$$\mathbf{f} + \boldsymbol{\partial} \rfloor \mathbf{T}_{\text{em}} = 0, \tag{10}$$

where $\mathbf{T}_{\text{em}}$ is a symmetric rank-2 tensor for all values of r, k, and n. In analogy to the (antisymmetric) multivector basis elements $\mathbf{e}_I$, we denote the rank-s symmetric-tensor basis elements by $\mathbf{u}_I$, where $I \in \mathcal{J}_s$ is an ordered list of s, possibly repeated, space-time indices and $\mathcal{J}_s$ denotes the set of all such lists. The interior derivative (divergence) $\boldsymbol{\partial} \rfloor \mathbf{T}_{\text{em}}$ is computed according to the interior product [7, Eq. (25)], and indeed satisfies (10), cf. [7, Eq. (40)].

The tensor $\mathbf{T}_{\text{em}}$ is expressed in terms of the $\odot$ and $\oslash$ tensor products [7, Sec. 2.4]. Given two multivectors $\mathbf{a}$ and $\mathbf{b}$ of the same grade s, the $\mathbf{a}\odot\mathbf{b}$ and $\mathbf{a}\oslash\mathbf{b}$ are two rank-2 tensors [7, Sec. 2.4] with basis elements $\mathbf{w}_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j$ and respective (i,j)-th components given by

$$\mathbf{a}\odot\mathbf{b}\big|_{ij} = (\Delta_{ii}\mathbf{e}_i \rfloor \mathbf{a}) \cdot (\mathbf{b} \llcorner \Delta_{jj}\mathbf{e}_j), \tag{11}$$

$$\mathbf{a}\oslash\mathbf{b}\big|_{ij} = (\Delta_{ii}\mathbf{e}_i \wedge \mathbf{a}) \cdot (\mathbf{b} \wedge \Delta_{jj}\mathbf{e}_j), \tag{12}$$

where Δ_{ii} and Δ_{jj} are the space-time metric defined previously. In general, neither $\mathbf{a} \odot \mathbf{b}$ nor $\mathbf{a} \oslash \mathbf{b}$ are symmetric; however, the sum $\mathbf{a} \odot \mathbf{b} + \mathbf{a} \oslash \mathbf{b}$ is symmetric in its components [7, Sec. 2.4]. For all values of r, k, and n, the tensor $\mathbf{T}_{\mathrm{em}}$ is expressed in terms of the $\odot$ and $\oslash$ tensor products [1, Sec. 4.2], [7, Sec. 4.3], as

$$\mathbf{T}_{\mathrm{em}} = -\frac{1}{2}(\mathbf{F} \odot \mathbf{F} + \mathbf{F} \oslash \mathbf{F}). \tag{13}$$

The diagonal, T^{em}_{ii}, and off-diagonal, T^{em}_{ij} with $i < j$, components of $\mathbf{T}_{\mathrm{em}}$ are explicitly given by [7, Eqs (38)–(39)]

$$T^{\mathrm{em}}_{ii} = \frac{(-1)^{r-1}}{2}\Delta_{ii}\left(\sum_{L\in\mathcal{I}_r : i\notin L} \Delta_{LL}F_L^2 - \sum_{L\in\mathcal{I}_r : i\in L} \Delta_{LL}F_L^2\right), \tag{14}$$

$$T^{\mathrm{em}}_{ij} = -\sum_{L\in\mathcal{I}_{r-1} : i,j\notin L} \Delta_{LL}\sigma(L,i)\sigma(j,L)F_{\varepsilon(i,L)}F_{\varepsilon(j,L)}, \tag{15}$$

where for two disjoint lists I and J of non-repeated space-time indices, $\sigma(I,J)$ is the signature of the permutation that sorts the concatenated list (I,J), and $\varepsilon(I,J)$ is the sorted concatenated list (I,J). If the lists I and J are not disjoint, we adopt the convention that $\sigma(I,J) = 0$.

For later use, let us define the product $\boxslash$ between basis elements $\mathbf{e}_i$ and $\mathbf{u}_I$, $I = (i_1, i_2) \in \mathcal{J}_2$ as

$$\mathbf{e}_i \boxslash \mathbf{u}_I = \sum_{I^\pi \in I!} \sigma(i, i_2^\pi)\mathbf{w}_{i_1^\pi,\varepsilon(i,i_2^\pi)}. \tag{16}$$

Here $I!$ denotes the set of all permutations (not necessarily ordered) of I, and $I^\pi = (i_1^\pi, i_2^\pi)$ denotes one such permuted list. The condition $i_2^\pi \neq i$ is implicitly enforced by the permutation signature $\sigma(i, i_2^\pi)$.

Both the conservation law (10) and the formula for the symmetric tensor $\mathbf{T}_{\mathrm{em}}$ (13) can be derived by exterior-algebraic methods from the invariance of the free-field action with density $\mathcal{L}_{\mathrm{em}}$ to infinitesimal space-time translations [7]. This exterior-algebraic derivation directly gives a symmetric tensor, without recurring to the Belinfante-Rosenfeld procedure to symmetrize the canonical tensor that appears in a standard application of Noether's theorem to the invariance of the action [8, 9], [10, Sec. 3.2], [11, Sec. 2.5]. In Sec. 2 of this paper, we show how a formula for the relativistic angular-momentum tensor can be derived by exterior-algebraic methods from the invariance of the action for the free field with density $\mathcal{L}_{\mathrm{em}}$ to infinitesimal space-time rotations.

Generalizing the usual electromagnetic analysis of flux as a three-dimensional space integral at constant time, the energy-momentum flux $\mathbf{\Pi}^\ell$ across the $(k+n)$-dimensional half space-time $\mathcal{V}_\ell^{k+n}$ of fixed ℓ-th space-time coordinate x_ℓ, for $\ell \in \{0, \dots, k+n-1\}$, can be expressed in terms of the transverse normal modes of the field [1, Eq. (86)] as a multidimensional integral over $\mathbf{\Xi}_\ell$, the set of values of $\boldsymbol{\xi}_{\bar{\ell}}$ for which $\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar{\ell}} \cdot \boldsymbol{\xi}_{\bar{\ell}} \leq 0$, where $\boldsymbol{\xi}_{\bar{\ell}} = \boldsymbol{\xi} - \xi_\ell \mathbf{e}_\ell$, namely

$$\mathbf{\Pi}^\ell = 4\pi^2(-1)^r\sigma(\ell,\ell^c)\int_{\mathbf{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\, \boldsymbol{\xi}_{\bar{\ell},+}|\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+})|^2, \tag{17}$$

where $\mathrm{d}\xi_{\ell^c}$ is an infinitesimal element [2, Sec. 3.1] along all coordinates except the ℓ-th, the frequency χ_ℓ is given by $\chi_\ell = +\sqrt{-\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar{\ell}} \cdot \boldsymbol{\xi}_{\bar{\ell}}}$, and $\boldsymbol{\xi}_{\bar{\ell},+} = \boldsymbol{\xi}_{\bar{\ell}} + \chi_\ell \mathbf{e}_\ell$; the complex-valued normal field components are denoted by $\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+})$. In Sec. 3 of this paper, we provide an analogous formula for the angular-momentum flux and its split into center-of-motion, orbital angular momentum, and spin components, as described in the next section.

1.3 Relativistic Angular Momentum: Background and Summary of Main Results

In classical mechanics, the angular momentum $\mathbf{L}$ is an axial vector (or pseudovector) with three spatial components. The relativistic angular momentum $\mathbf{\Omega}$ is an antisymmetric tensor of rank 2, or a bivector, that combines the angular momentum $\mathbf{L}$ and the polar vector $\mathbf{N}$ for the velocity of the center-of-mass (also known as moment of energy). In fact, the way $\mathbf{\Omega}$ is constructed is the same as the way the electromagnetic field bivector $\mathbf{F}$ is constructed from the axial magnetic field and the polar electric field, that is $\mathbf{\Omega} = \mathbf{e}_0 \wedge \mathbf{N} + \mathbf{L}^{\mathcal{H}}$ [1, Sec. 3.1], where $\mathbf{L}^{\mathcal{H}}$ is the spatial Hodge dual of $\mathbf{L}$ [1, Eq. (18)], i. e. the bivector corresponding to the axial vector. In (k,n)-space-time, relativistic angular momentum $\mathbf{\Omega}$ is a grade-2 multivector with $\binom{k+n}{2}$ components.

In analogy to energy-momentum, a conservation law relates the transfer of angular momentum over a $(k+n)$-dimensional hypervolume $\mathcal{V}^{k+n}$ to the divergence of an angular-momentum tensor $\mathbf{M}_{\boldsymbol{\alpha}}$ with rotation center $\boldsymbol{\alpha}$. In contrast to $\mathbf{T}_{\mathrm{em}}$, the basis elements of $\mathbf{M}_{\boldsymbol{\alpha}}$ are of the form $\mathbf{w}_{i,I} = \mathbf{e}_i \otimes \mathbf{e}_I$, where $i \in \mathcal{I}$ and $I \in \mathcal{I}_2$. For classical electromagnetism, with $r = 2$, $k = 1$, and $n = 3$, this tensor is given in contravariant form as [4, Sec. 12.10.B]

$$\mathbf{M}_{\boldsymbol{\alpha}}^{\alpha\beta\gamma} = T^{\alpha\beta}(x^\gamma - \alpha^\gamma) - T^{\alpha\gamma}(x^\beta - \alpha^\beta), \tag{18}$$

where $T^{\alpha\beta}$ are the components of the symmetric stress-energy-momentum tensor. In our notation, $T^{\alpha\beta} = T^{\mathrm{em}}_{\varepsilon(\alpha,\beta)}$. The vectors $\mathbf{L}$ and $\mathbf{N}$ are given by volume integrals of some appropriate functions of $\mathbf{M}_{\boldsymbol{\alpha}}$. For instance, for $\boldsymbol{\alpha} = 0$, the spatial angular momentum vector $\mathbf{L}$ of the electromagnetic field is given [4, Prob. 7.27] in terms of the standard cross product of the spatial position vector $\mathbf{x}$ and electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ by:

$$\mathbf{L} = \int_{\mathbf{R}^3} \mathrm{d}x_{123} \,\big(\mathbf{x} \times (\mathbf{E} \times \mathbf{B})\big). \tag{19}$$

Since the spatial relativistic angular momentum bivector is the space-Hodge-dual $\mathbf{L}^{\mathcal{H}}$, using [1, Eq. (36)] we have

$$\mathbf{L}^{\mathcal{H}} = \int_{\mathbf{R}^3} \mathrm{d}x_{123} \,\big(\mathbf{x} \wedge (\mathbf{E} \times \mathbf{B})\big). \tag{20}$$

Moreover, the j-th component of the Poynting vector $\mathbf{E} \times \mathbf{B}$ coincides with T_{ij}^{em} in (15), with $i = 0$,

$$T_{0j}^{\mathrm{em}} = \sum_{m \in \mathcal{I}: m \neq 0, j} \sigma(j,m) F_{\varepsilon(0,m)} F_{\varepsilon(j,m)} \tag{21}$$

$$= (\mathbf{E} \times \mathbf{B})\big|_j, \tag{22}$$

where we have used that $r = 2$ to rewrite L as $m \in \mathcal{I}$, that $\Delta_{mm} = 1$ for the spatial indices, and that $\sigma(m,0) = -1$ for any spatial m, as well as the definition of the cross-product $\mathbf{E} \times \mathbf{B}$. The (i,j)-th component of $\mathbf{L}^{\mathcal{H}}$ in (19) is thus given by the volume integral of the quantity

$$x_i T_{0j}^{\mathrm{em}} \sigma(i,j) + x_j T_{0i}^{\mathrm{em}} \sigma(j,i), \tag{23}$$

which in turn can be identified with the component in $\mathbf{w}_{0,ij}$ of the product $\mathbf{x} \boxtimes \mathbf{T}_{\mathrm{em}}$ defined in (16). In Sec. 2, we prove that this is no coincidence, and that in general it holds that

$$\mathbf{M}_{\boldsymbol{\alpha}} = (\mathbf{x} - \boldsymbol{\alpha}) \boxtimes \mathbf{T}_{\mathrm{em}}. \tag{24}$$

The proof is built on the principle of invariance of the action to infinitesimal space-time rotations around $\boldsymbol{\alpha}$.

In Sec. 3, we provide a formula for the relativistic angular momentum $\boldsymbol{\Omega}_{\boldsymbol{\alpha}}^{\ell}$ of the generalized electromagnetic field, including $\mathbf{L}$ and the center-of-mass velocity $\mathbf{N}$, for any values of k, n, and r, as the flux of the tensor $\mathbf{M}_{\boldsymbol{\alpha}}$ across a $(k+n-1)$-dimensional surface of constant ℓ-th space-time coordinate (Eqs (58) and (63)), for any ℓ,

$$\boldsymbol{\Omega}_{\boldsymbol{\alpha}}^{\ell} = \int_{\partial \mathcal{V}^{k+n}} \mathrm{d}^{k+n-1} \mathbf{x}^{\mathcal{H}^{-1}} \times \mathbf{M}_{\boldsymbol{\alpha}} \tag{25}$$

$$= \sigma(\ell, \ell^c) \sum_{i,j \in \mathcal{I}} \sigma(i,j) \mathbf{e}_{\varepsilon(i,j)} \int_{\mathbf{R}^{k+n-1}} \mathrm{d}x_{\ell^c} (x_i - \alpha_i) T_{\varepsilon(\ell,j)}, \tag{26}$$

where the flux integral is carried out with respect to the inverse Hodge of the infinitesimal element $\mathrm{d}\mathbf{x}$ [2, Eq. (19)]. The total angular momentum $\boldsymbol{\Omega}_{\boldsymbol{\alpha}}^{\ell}$ can be decomposed as $\boldsymbol{\Omega}_{\boldsymbol{\alpha}}^{\ell} = \mathbf{N}^{\ell} + \mathbf{L}^{\ell} + \mathbf{S}^{\ell} - \boldsymbol{\alpha} \wedge \boldsymbol{\Pi}^{\ell}$, i.e. the center-of-mass component $\mathbf{N}^{\ell}$, the orbital angular momentum $\mathbf{L}^{\ell}$, and the spin $\mathbf{S}^{\ell}$. In terms of the transverse normal modes of the field, evaluated in the Coulomb-ℓ gauge, these three terms are, respectively, expressed (cf. Eqs (74)–(76)), as

$$\mathbf{N}^{\ell} = x_\ell \wedge \boldsymbol{\Pi}^{\ell} + j\pi(-1)^r \sigma(\ell,\ell^c) \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell} \chi_\ell \mathbf{e}_\ell \wedge \Big(\big(\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar\ell}} \otimes \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\big) \times \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) - \mathrm{cc} \Big), \tag{27}$$

$$\mathbf{L}^{\ell} = j\pi(-1)^r \sigma(\ell,\ell^c) \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell} \boldsymbol{\xi}_{\bar\ell} \wedge \Big(\big(\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar\ell}} \otimes \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\big) \times \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) - \mathrm{cc} \Big), \tag{28}$$

$$\mathbf{S}^{\ell} = -j2\pi \sigma(\ell,\ell^c) \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell} \Big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+}) \odot \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) - \mathrm{cc} \Big), \tag{29}$$

where cc stands for the complex conjugate. Expressions for the bivector components of $\mathbf{L}^{\ell}$ and $\mathbf{S}^{\ell}$ are given in (77) and (80). Of special interest are the circular-polarization-basis formulas for the orbital angular momentum and the spin, respectively given in (87) and (85). For the standard electromagnetic field, the spatial components of the orbital angular momentum and spin in (28)–(29), computed for $\ell = 0$, $r = 2$, $k = 1$, and $n = 3$, coincide with the well-known values [12, Eq. (16) in B$_{\mathrm{I}}$.2], respectively given in vector notation, rather than as a bivector, by

$$\mathbf{L} = -j\pi \int_{\mathbf{R}^3} \frac{\mathrm{d}\xi_{123}}{2\chi_0} \sum_{m=1}^{n} \boldsymbol{\xi}_{\bar\ell} \times \Big(\big(\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar\ell}} \hat{A}_m(\boldsymbol{\xi}_{\bar\ell,+})\big) \hat{A}_m^*(\boldsymbol{\xi}_{\bar\ell,+}) - \mathrm{cc} \Big), \tag{30}$$

$$\mathbf{S} = -j2\pi \int_{\mathbf{R}^3} \frac{\mathrm{d}\xi_{123}}{2\chi_0} \Big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+}) \times \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) - \mathrm{cc} \Big). \tag{31}$$

By construction, the components of the angular momentum and spin bivectors that include the index ℓ are zero.

The feasibility of the separation of angular momentum into orbital and spin parts in a gauge-invariant manner, as well as its possible operational meaning, have been subject to some discussion, particularly in a quantum context [13, 14, 15]. Since the consideration of quantum aspects is beyond the scope of this work, and it seems unlikely that statements about the generalized electromagnetic field can be supported by experimental observations to settle the issue, we do not dwell on this matter in this paper, apart from noting that we carry out our analysis in the Coulomb-ℓ gauge (or equivalently for the transverse normal modes of the field [12, Sec. $\mathrm{B_I}$]), the condition that has been found to be in best empirical agreement with observations for the standard electromagnetic field [15].

2 Angular-Momentum Conservation Law for the Free Generalized Electromagnetic Field

In this section, we exploit the invariance of the action with Lagrangian density $\mathcal{L}_{\mathrm{em}}$ to infinitesimal space-time rotations, e. g. Lorentz transformations, to derive a conservation law and an expression for the relativistic angular-momentum tensor by direct exterior-algebraic methods, avoiding the non-symmetric canonical tensor and the related currents in Noether's theorem. For the sake of notational compactness, we remove the subscript em in the tensor.

2.1 Conservation Law for Angular Momentum

Let us shift the origin of coordinates by an infinitesimal perturbation $\boldsymbol{\varepsilon}$. For a translation, each of the $k+n$ components is an independent function of space-time $\boldsymbol{\varepsilon}_{\mathrm{t}}$. For a space-time rotation (Lorentz transformation) around a center point $\boldsymbol{\alpha}$, and given an infinitesimal bivector $\boldsymbol{\varepsilon}_{\mathrm{r}}$ with $\binom{k+n}{2}$ components, it holds that

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\mathrm{r}} \llcorner (\mathbf{x} - \boldsymbol{\alpha}). \tag{32}$$

Let $\{\mathbf{e}'\}$ denote the rotated (perturbed) basis elements, expressed in the original basis $\{\mathbf{e}\}$. Along the i-th coordinate, the basis element $\mathbf{e}_i$ is perturbed to first order by an infinitesimal amount

$$\mathbf{e}'_i = \mathbf{e}_i \times (\mathbf{1} + \boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}), \tag{33}$$

where $\mathbf{1} = \sum_{i\in\mathcal{I}} \Delta_{ii}\mathbf{w}_{ii}$ is the identity matrix and the Jacobian partial-derivative matrix $\boldsymbol{\partial} \otimes \boldsymbol{\varepsilon}$ is given by

$$\boldsymbol{\partial} \otimes \boldsymbol{\varepsilon} = \sum_{i,j\in\mathcal{I}} \Delta_{ii}\partial_i\varepsilon_j\mathbf{w}_{ij}. \tag{34}$$

The j-th column of the Jacobian matrix contains the exterior derivative, i. e. gradient, of the j-th component of the perturbation in the coordinates, ε_j. As proved in [7, Sec. 3.3], a similar general expression holds for the transformation of multivector basis elements of grade s,

$$\mathbf{e}'_I = \mathbf{e}_I \times (\mathbf{1}_s + \mathbf{G}^s_{\boldsymbol{\varepsilon}}), \tag{35}$$

where $\mathbf{1}_s = \sum_{I\in\mathcal{I}_s} \Delta_{II}\mathbf{w}_{I,I}$ is the grade-s identity matrix and the matrix $\mathbf{G}^s_{\boldsymbol{\varepsilon}}$ is given by [7, Eq. (70)]

$$\mathbf{G}^s_{\boldsymbol{\varepsilon}} = (-1)^{s-1} \sum_{I\in\mathcal{I}_s} \sum_{i\in I} \sum_{j\in\mathcal{I}\setminus\{I\setminus i\}} \Delta_{II}\sigma(I\setminus i, i)\sigma(j, I\setminus i)\,\partial_i\varepsilon_j\,\mathbf{w}_{I,\varepsilon(j,I\setminus i)}. \tag{36}$$

Writing the action functional over a closed region $\mathcal{R}$ in the new perturbed coordinates involves changing the integrand and the differentials according to (33) and (35). For the Lagrangian density $\mathcal{L}_{\mathrm{em}}$, given by a scalar product of two multivectors, the full details are given in [7, Sec. 3.4–3.5]. Let us assume that the fields vanish at infinity sufficiently fast, e. g. the integral of $\boldsymbol{\varepsilon} \lrcorner \mathbf{T} = (\boldsymbol{\varepsilon}_{\mathrm{r}} \llcorner (\mathbf{x}-\boldsymbol{\alpha})) \lrcorner \mathbf{T}$ at infinity (the boundary of the volume in the action) vanishes. Then, the change of action $\delta\mathcal{S}_{\mathcal{L}_{\mathrm{em}}}$ is expressed in terms of the rank-2 manifestly symmetric tensor $\mathbf{T}$, the stress-energy-momentum tensor (13) of the free generalized electromagnetic field, as

$$\delta\mathcal{S}_{\mathcal{L}_{\mathrm{em}}} = \int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\,(\boldsymbol{\partial}\otimes\boldsymbol{\varepsilon})\cdot\mathbf{T} \tag{37}$$

$$= -\int_{\mathcal{R}} \mathrm{d}^{k+n}\mathbf{x}\,(\boldsymbol{\partial}\lrcorner\mathbf{T})\cdot\boldsymbol{\varepsilon}, \tag{38}$$

having assumed that the integration region $\mathcal{R}$ is large enough to make the physical system closed, and that the fields decay fast enough over $\mathcal{R}$ so that the flux of the fields over the boundary of $\mathcal{R}$ is negligible. This formula for the change of action (38) holds for arbitrary grades of the generalized electromagnetic field $\mathbf{F}$.

The integrand in (38) can be rewritten using (32) and [1, Eq. (27)] as

$$(\boldsymbol{\partial} \rfloor \mathbf{T}) \cdot \big(\boldsymbol{\varepsilon}_{\mathrm{r}} \lfloor (\mathbf{x}-\boldsymbol{\alpha})\big) = \big((\mathbf{x}-\boldsymbol{\alpha}) \wedge (\boldsymbol{\partial} \rfloor \mathbf{T})\big) \cdot \boldsymbol{\varepsilon}_{\mathrm{r}}. \tag{39}$$

Assuming that infinitesimal space-time rotations are a symmetry of the system and that the fields decay sufficiently fast, the fact that the variation of the action $\delta\mathcal{S}_{\mathcal{L}_{\mathrm{em}}}$ must be zero for all infinitesimal perturbations $\boldsymbol{\varepsilon}_{\mathrm{r}}$ implies that

$$(\mathbf{x}-\boldsymbol{\alpha}) \wedge (\boldsymbol{\partial} \rfloor \mathbf{T}) = 0. \tag{40}$$

This expression characterizes the conservation law related to angular momentum, in the absence of external currents. Differently from the condition $\boldsymbol{\partial} \rfloor \mathbf{T} = 0$ that appears in the context of invariance to translations and gives a the conservation law for the energy-momentum, invariance to infinitesimal rotations requires the interior derivative (divergence) of the stress-energy-tensor to be radial, or equivalently parallel to the relative-position vector $\mathbf{x}-\boldsymbol{\alpha}$.

In the following section, we provide an expression for a rank-3 angular-momentum tensor, valid for any number of space-time dimensions and grade of the electromagnetic field.

2.2 Relativistic Angular-Momentum Tensor

In this section, we prove that (40) can be expressed as the matrix derivative (divergence) of a rank-3 tensor, which we will identify with the relativistic angular-momentum tensor of the generalized electromagnetic field.

To start, we expand the bivector equation (40) in components as

$$(\mathbf{x}-\boldsymbol{\alpha}) \wedge (\boldsymbol{\partial} \rfloor \mathbf{T}) = \sum_{i\in\mathcal{I}} (x_i-\alpha_i)\mathbf{e}_i \wedge \left(\sum_{j,\ell\in\mathcal{I}} \partial_j T_{\varepsilon(j,\ell)}\mathbf{e}_\ell\right) \tag{41}$$

$$= \sum_{i,j,\ell\in\mathcal{I}} \sigma(i,\ell)(x_i-\alpha_i)\partial_j T_{\varepsilon(j,\ell)}\mathbf{e}_{\varepsilon(i,\ell)}. \tag{42}$$

Consider now a bivector of a similar form, where $(x_i-\alpha_i)$ and $T_{\varepsilon(j,\ell)}$ are swapped, i.e. $(x_i-\alpha_i)\partial_j T_{\varepsilon(j,\ell)}$ is replaced by $T_{\varepsilon(j,\ell)}\partial_j(x_i-\alpha_i)$. Since $\partial_j(x_i-\alpha_i) = \delta_{ji}$, this bivector can be evaluated as the zero bivector,

$$\sum_{i,j,\ell\in\mathcal{I}} \sigma(i,\ell)T_{\varepsilon(j,\ell)}\partial_j(x_i-\alpha_i)\mathbf{e}_{\varepsilon(i,\ell)} = \sum_{i,j,\ell\in\mathcal{I}} \sigma(i,\ell)T_{\varepsilon(j,\ell)}\delta_{ji}\mathbf{e}_{\varepsilon(i,\ell)} \tag{43}$$

$$= \sum_{i,\ell\in\mathcal{I}} \sigma(i,\ell)T_{\varepsilon(i,\ell)}\mathbf{e}_{\varepsilon(i,\ell)} \tag{44}$$

$$= \sum_{i,\ell\in\mathcal{I}:i<\ell} \big(\sigma(i,\ell)+\sigma(\ell,i)\big)T_{\varepsilon(i,\ell)}\mathbf{e}_{\varepsilon(i,\ell)}, \tag{45}$$

where we have used that $\sigma(i,i)=0$ to keep only the terms with $i\neq\ell$ and then split the summation into the disjoint cases $i<\ell$ and $\ell<i$ and interchanged the roles of i and ℓ in the latter case. Since $\sigma(i,\ell)=-\sigma(\ell,i)$, we verify that Eq. (45) is zero. Adding this zero bivector to (42) and applying the Leibniz rule for the derivative gives

$$(\mathbf{x}-\boldsymbol{\alpha}) \wedge (\boldsymbol{\partial} \rfloor \mathbf{T}) = \sum_{i,j,\ell\in\mathcal{I}} \sigma(i,\ell)\big((x_i-\alpha_i)\partial_j T_{\varepsilon(j,\ell)} + T_{\varepsilon(j,\ell)}\partial_j(x_i-\alpha_i)\big)\mathbf{e}_{\varepsilon(i,\ell)} \tag{46}$$

$$= \sum_{i,j,\ell\in\mathcal{I}} \sigma(i,\ell)\partial_j\big((x_i-\alpha_i)T_{\varepsilon(j,\ell)}\big)\mathbf{e}_{\varepsilon(i,\ell)}. \tag{47}$$

It remains to prove that (47) is the divergence of a suitably defined tensor field. Let $\mathbf{M}_{\boldsymbol{\alpha}} = (\mathbf{x}-\boldsymbol{\alpha}) \boxbslash \mathbf{T}$ be the angular-momentum tensor field, where the product $\boxbslash$ is defined in (16). The tensor field $\mathbf{M}_{\boldsymbol{\alpha}}$ is antisymmetric in the second and third components, as its basis elements are given by $\mathbf{w}_{i,I} = \mathbf{e}_i \otimes \mathbf{e}_I$. Expanding the product $(\mathbf{x}-\boldsymbol{\alpha}) \boxbslash \mathbf{T}$ with the definition in (16), the tensor field $\mathbf{M}_{\boldsymbol{\alpha}}$ is given by

$$\mathbf{M}_{\boldsymbol{\alpha}} = \sum_{i\in\mathcal{I}}\sum_{I\in\mathcal{J}_2} (x_i-\alpha_i)T_I\mathbf{e}_i \boxbslash \mathbf{u}_I \tag{48}$$

$$= \sum_{i\in\mathcal{I}}\sum_{I\in\mathcal{J}_2} (x_i-\alpha_i)T_I\left(\sum_{I^\pi\in I!} \sigma(i,i_2^\pi)\mathbf{w}_{i_1^\pi,\varepsilon(i,i_2^\pi)}\right) \tag{49}$$

$$= \sum_{i,j\in\mathcal{I}} (x_i-\alpha_i)T_{jj}\sigma(i,j)\mathbf{w}_{j,\varepsilon(i,j)} + \sum_{i,j,\ell\in\mathcal{I}:j<\ell} (x_i-\alpha_i)T_{j\ell}\big(\sigma(i,\ell)\mathbf{w}_{j,\varepsilon(i,\ell)} + \sigma(i,j)\mathbf{w}_{\ell,\varepsilon(i,j)}\big), \tag{50}$$

where we have split the summation over lists $I \in \mathcal{J}_2$ into two, the first one for the lists I of the form (j, j) and the second one for the lists of the form (j, ℓ), with $j < \ell$. Splitting further the second summation into two, and renaming j and ℓ as ℓ and j, respectively, we obtain

$$\mathbf{M}_{\boldsymbol{\alpha}} = \sum_{i,j \in \mathcal{I}} (x_i - \alpha_i) T_{jj} \sigma(i,j) \mathbf{w}_{j,\varepsilon(i,j)} + \sum_{i,j,\ell \in \mathcal{I}: j<\ell} (x_i - \alpha_i) T_{j\ell} \sigma(i,\ell) \mathbf{w}_{j,\varepsilon(i,\ell)} + \sum_{i,j,\ell \in \mathcal{I}: j>\ell} (x_i - \alpha_i) T_{\ell j} \sigma(i,\ell) \mathbf{w}_{j,\varepsilon(i,\ell)} \tag{51}$$

$$= \sum_{i,j \in \mathcal{I}} (x_i - \alpha_i) T_{jj} \sigma(i,j) \mathbf{w}_{j,\varepsilon(i,j)} + \sum_{i,j,\ell \in \mathcal{I}: j \neq \ell} (x_i - \alpha_i) T_{\varepsilon(j,\ell)} \sigma(i,\ell) \mathbf{w}_{j,\varepsilon(i,\ell)} \tag{52}$$

$$= \sum_{i,j,\ell \in \mathcal{I}} (x_i - \alpha_i) T_{\varepsilon(j,\ell)} \sigma(i,\ell) \mathbf{w}_{j,\varepsilon(i,\ell)}, \tag{53}$$

where we have combined in (52) the separate summations over $j < \ell$ and $j > \ell$ into one single summation over $j \neq \ell$, and then in (53) combined this result with the first summand, expressed as a double summation over j and ℓ such that $j = \ell$, into a triple summation over indices i, j, and ℓ.

Computing the matrix derivative [7, Eq. (34)] of $\mathbf{M}_{\boldsymbol{\alpha}}$, denoted by $\boldsymbol{\partial} \times \mathbf{M}_{\boldsymbol{\alpha}}$, we recover (47), that is

$$\boldsymbol{\partial} \times \mathbf{M}_{\boldsymbol{\alpha}} = \sum_{i,j,\ell \in \mathcal{I}} \partial_j \big((x_i - \alpha_i) T_{\varepsilon(j,\ell)} \big) \sigma(i,\ell) \mathbf{e}_{\varepsilon(i,\ell)} \tag{54}$$

$$= (\mathbf{x} - \boldsymbol{\alpha}) \wedge (\boldsymbol{\partial} \rfloor \mathbf{T}). \tag{55}$$

Substituting this expression in (39) and the result back in (38), we find that the change of action is given by

$$\delta \mathcal{S}_{\mathcal{L}_{\text{em}}} = - \int_{\mathcal{R}} \mathrm{d}^{k+n} \mathbf{x} \, (\boldsymbol{\partial} \times \mathbf{M}_{\boldsymbol{\alpha}}) \cdot \boldsymbol{\varepsilon}_{\mathrm{r}}. \tag{56}$$

The invariance of the action to rotations, $\delta \mathcal{S}_{\mathcal{L}_{\text{em}}} = 0$, implies (40) and equivalently that $\boldsymbol{\partial} \times \mathbf{M}_{\boldsymbol{\alpha}} = 0$. In the presence of sources, the divergence $\boldsymbol{\partial} \times \mathbf{M}_{\boldsymbol{\alpha}}$ can be seen as an angular-momentum density, and the volume integral of $\boldsymbol{\partial} \times \mathbf{M}_{\boldsymbol{\alpha}}$ across an $(k+n)$-dimensional hypervolume $\mathcal{V}^{k+n}$ gives the transfer of relativistic angular momentum from the field to the sources in the volume. In the next section, we characterize this transfer of angular momentum in terms of the flux of $\mathbf{M}_{\boldsymbol{\alpha}}$, and provide an expression for the flux in terms of the normal modes of the field.

3 Flux of the Angular-Momentum Tensor: Spin and Orbital Angular Momentum of the Generalized Electromagnetic Field

3.1 Integral Form of the Conservation Law and Angular-Momentum Flux

The angular-momentum conservation law admits an integral form, which we derive next. First, the volume integral of the divergence $\boldsymbol{\partial} \times \mathbf{M}_{\boldsymbol{\alpha}}$ over an $(k+n)$-dimensional hypervolume $\mathcal{V}^{k+n}$ gives the transfer of angular momentum from the field to the sources. This volume integral is the flux of the divergence over $\mathcal{V}^{k+n}$ [1, Eq. (40)],

$$\int_{\mathcal{V}^{k+n}} \mathrm{d}x_{0,\cdots,k+n-1} \, (\boldsymbol{\partial} \times \mathbf{M}_{\boldsymbol{\alpha}}) = \int_{\mathcal{V}^{k+n}} \mathrm{d}^{k+n} \mathbf{x}^{\mathcal{H}^{-1}} \rfloor (\boldsymbol{\partial} \times \mathbf{M}_{\boldsymbol{\alpha}}), \tag{57}$$

where the flux integral is carried out with respect to the inverse Hodge [2, Eq. (10)] of the infinitesimal element $\mathrm{d}^{k+n}\mathbf{x}$, i.e. $\mathrm{d}x_{0,\cdots,k+n-1}$. A short adaptation, included in Appendix A, of the analysis in [2, Sec. 3.5]proves a Stokes theorem for the angular-momentum tensor: the flux of $\mathbf{M}_{\boldsymbol{\alpha}}$ across the boundary $\partial \mathcal{V}^m$ of an m-dimensional hypersurface $\mathcal{V}^m$ is equal to the flux of the divergence of $\mathbf{M}_{\boldsymbol{\alpha}}$ across $\mathcal{V}^m$ for any $m \leq k+n$. For $m = k+n$, this Stokes theorem gives

$$\int_{\mathcal{V}^{k+n}} \mathrm{d}^{k+n} \mathbf{x}^{\mathcal{H}^{-1}} \rfloor (\boldsymbol{\partial} \times \mathbf{M}_{\boldsymbol{\alpha}}) = \int_{\partial \mathcal{V}^{k+n}} \mathrm{d}^{k+n-1} \mathbf{x}^{\mathcal{H}^{-1}} \times \mathbf{M}_{\boldsymbol{\alpha}}. \tag{58}$$

As an example, and for some fixed x_ℓ and $\ell \in \mathcal{I}$, consider the $(k+n)$-dimensional half space-time region

$$\mathcal{V}_\ell^{k+n} = (-\infty, \infty) \times (-\infty, \infty) \cdots \times (-\infty, x_\ell) \times \cdots (-\infty, \infty). \tag{59}$$

The boundary of this region is a surface of constant space-time coordinate ℓ of value x_ℓ, given by

$$\partial \mathcal{V}_\ell^{k+n} = (-\infty, \infty) \times (-\infty, \infty) \cdots \times \{x_\ell\} \times \cdots (-\infty, \infty). \tag{60}$$

Let $\boldsymbol{\Omega}_{\boldsymbol{\alpha}}^\ell$ denote the flux of the tensor field $\mathbf{M}_{\boldsymbol{\alpha}} = (\mathbf{x} - \boldsymbol{\alpha}) \boxtimes \mathbf{T}$ across the boundary $\partial \mathcal{V}_\ell^{k+n}$. In this case, the Hodge-dual infinitesimal vector element in the r. h. s. of (58) is given by [1, Eq. (83)]

$$\mathrm{d}^{k+n-1} \mathbf{x}^{\mathcal{H}^{-1}} = \mathrm{d}x_{\ell^c} \sigma(\ell, \ell^c) \Delta_{\ell\ell} \mathbf{e}_\ell, \tag{61}$$

where the factor $\sigma(\ell,\ell^c)$ arises from the orientation such that the normal vector $\mathbf{e}_\ell$ points outside the integration region. Using (53) in (58) and using (61), carrying out the matrix product, and rearranging the expression, yields

$$\boldsymbol{\Omega}^\ell_{\boldsymbol{\alpha}} = \int_{\mathbf{R}^{k+n-1}} \mathrm{d}x_{\ell^c}\sigma(\ell,\ell^c)\Delta_{\ell\ell}\mathbf{e}_\ell \times \left(\sum_{i,m,j\in\mathcal{I}}(x_i-\alpha_i)T_{\varepsilon(m,j)}\sigma(i,j)\mathbf{w}_{m,\varepsilon(i,j)}\right) \tag{62}$$

$$= \sigma(\ell,\ell^c)\sum_{i,j\in\mathcal{I}}\sigma(i,j)\mathbf{e}_{\varepsilon(i,j)}\int_{\mathbf{R}^{k+n-1}}\mathrm{d}x_{\ell^c}(x_i-\alpha_i)T_{\varepsilon(\ell,j)}. \tag{63}$$

An alternative, slightly more explicit, expression for (63) is the following

$$\boldsymbol{\Omega}^\ell_{\boldsymbol{\alpha}} = \sigma(\ell,\ell^c)\sum_{(i,j)\in\mathcal{I}_2}\int_{\mathbf{R}^{k+n-1}}\mathrm{d}x_{\ell^c}\big(\mathbf{e}_{ij}(x_i-\alpha_i)T_{\varepsilon(\ell,j)}+\mathbf{e}_{ji}(x_j-\alpha_j)T_{\varepsilon(\ell,i)}\big). \tag{64}$$

3.2 Normal Modes of the Field

Substituting in (62) the stress-energy-momentum tensor $\mathbf{T}$ by its expression in (13), the flux $\boldsymbol{\Omega}^\ell_{\boldsymbol{\alpha}}$ of the angular-momentum tensor a surface of constant space-time coordinate ℓ of value x_ℓ is given by the integral

$$\boldsymbol{\Omega}^\ell_{\boldsymbol{\alpha}} = -\frac{1}{2}\Delta_{\ell\ell}\sigma(\ell,\ell^c)\int_{\mathbf{R}^{k+n-1}}\mathrm{d}x_{\ell^c}\mathbf{e}_\ell \times \big((\mathbf{x}-\boldsymbol{\alpha})\boxtimes(\mathbf{F}\odot\mathbf{F}+\mathbf{F}\oslash\mathbf{F})\big). \tag{65}$$

The r. h. s. of (65) is computed w. r .t. x_{ℓ^c}, being ℓ^c the set of indices excluding ℓ. We let $\mathbf{x}_{\bar{\ell}} = \mathbf{x} - x_\ell\mathbf{e}_\ell$ and similarly $\boldsymbol{\xi}_{\bar{\ell}} = \boldsymbol{\xi} - \xi_\ell\mathbf{e}_\ell$ for the frequency vector defined below. We also let $\kappa_\ell = -\frac{1}{2}\Delta_{\ell\ell}\sigma(\ell,\ell^c)$.

In the absence of charges, the free field $\mathbf{F}$ satisfies the homogeneous wave equation and can be expressed as a linear superposition of complex exponentials $e^{j2\pi\boldsymbol{\xi}\cdot\mathbf{x}}$ such that $\boldsymbol{\xi}\cdot\boldsymbol{\xi} = 0$. Note that here $j=\sqrt{-1}$; the context will make it clear whether j refers to a coordinate label or to the imaginary number. Denoting the coefficient of each complex exponential by $\hat{\mathbf{F}}$, the Fourier transform of $\mathbf{F}$, and with the definition $\mathrm{d}^{k+n} = \mathrm{d}\xi_0\cdots\mathrm{d}\xi_{k+n-1}$, we have

$$\mathbf{F}(\mathbf{x}) = \int_{\mathbf{R}^{k+n}}\mathrm{d}^{k+n}\boldsymbol{\xi}\,\delta(\boldsymbol{\xi}\cdot\boldsymbol{\xi})\,e^{j2\pi\boldsymbol{\xi}\cdot\mathbf{x}}\,\hat{\mathbf{F}}(\boldsymbol{\xi}). \tag{66}$$

We resolve the Dirac delta by rewriting the condition $\boldsymbol{\xi}\cdot\boldsymbol{\xi} = 0$ in terms of $\boldsymbol{\xi}_{\bar{\ell}}$ as $\Delta_{\ell\ell}\xi_\ell^2 + \boldsymbol{\xi}_{\bar{\ell}}\cdot\boldsymbol{\xi}_{\bar{\ell}} = 0$. This equation has real solutions for ξ_ℓ only if $\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar{\ell}}\cdot\boldsymbol{\xi}_{\bar{\ell}} \leq 0$, namely the two possible values $\xi_\ell = \pm\chi_\ell$, where χ_ℓ is given by

$$\chi_\ell = +\sqrt{-\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar{\ell}}\cdot\boldsymbol{\xi}_{\bar{\ell}}}. \tag{67}$$

Let $\boldsymbol{\Xi}_\ell$ be the set of values of $\boldsymbol{\xi}_{\bar{\ell}}$ for which $\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar{\ell}}\cdot\boldsymbol{\xi}_{\bar{\ell}} \leq 0$. We define the pair of frequency vectors $\boldsymbol{\xi}_{\bar{\ell},\sigma}$ as

$$\boldsymbol{\xi}_{\bar{\ell},\sigma} = \boldsymbol{\xi}_{\bar{\ell}} + \sigma\chi_\ell\mathbf{e}_\ell. \tag{68}$$

for $\sigma\in\mathcal{S}=\{+1,-1\}$, respectively, shortened to $+$ and $-$. Using [16, p. 184], we can write the inverse Fourier transform (66) w. r. t. the integration variables ξ_{ℓ^c}, now with the appropriate constraints on the integration range so that χ_ℓ exists, in various equivalent forms as

$$\mathbf{F}(\mathbf{x}) = \int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\left(\sum_{\sigma\in\mathcal{S}}e^{j2\pi\boldsymbol{\xi}_{\bar{\ell},\sigma}\cdot\mathbf{x}}\,\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\right) \tag{69}$$

$$= \int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}e^{j2\pi\boldsymbol{\xi}_{\bar{\ell}}\cdot\mathbf{x}_{\bar{\ell}}}\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}}), \tag{70}$$

where we have factored out a common factor $e^{j2\pi\boldsymbol{\xi}_{\bar{\ell}}\cdot\mathbf{x}_{\bar{\ell}}}$ and defined the function $\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}})$ as

$$\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}}) = e^{j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},+}) + e^{-j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},-}). \tag{71}$$

We may rewrite the flux $\boldsymbol{\Omega}^\ell_{\boldsymbol{\alpha}}$ in terms of $\hat{\mathbf{F}}^\ell$ by substituting (70) in (65) as

$$\boldsymbol{\Omega}^\ell_{\boldsymbol{\alpha}} = \kappa_\ell\int_{\mathbf{R}^{k+n-1}}\mathrm{d}x_{\ell^c}\iint_{\boldsymbol{\Xi}_\ell\times\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\frac{\mathrm{d}\xi'_{\ell^c}}{2\chi'_\ell}\,e^{j2\pi(\boldsymbol{\xi}_{\bar{\ell}}+\boldsymbol{\xi}'_{\bar{\ell}})\cdot\mathbf{x}_{\bar{\ell}}}\mathbf{e}_\ell\times\Big((\mathbf{x}-\boldsymbol{\alpha})\boxtimes\big(\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}})\odot\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar{\ell}})+\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}})\oslash\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar{\ell}})\big)\Big). \tag{72}$$

3.3 Spin and Angular Momentum of the Generalized Electromagnetic Field

In Appendix B.1 we carry out the rather tedious evaluation of this integral in terms of the transverse normal modes in the Coulomb-ℓ gauge. Under the assumption that the various field components commute, we obtain the following formula for the angular momentum as a sum of four components, cf. Eq. (151),

$$\mathbf{\Omega}^{\ell}_{\boldsymbol{\alpha}} = \mathbf{N}^{\ell} + \mathbf{L}^{\ell} + \mathbf{S}^{\ell} - \boldsymbol{\alpha} \wedge \mathbf{\Pi}^{\ell}, \tag{73}$$

namely the center-of-mass velocity $\mathbf{N}^{\ell}$, the orbital angular momentum $\mathbf{L}^{\ell}$, and the spin $\mathbf{S}^{\ell}$, respectively, given by

$$\mathbf{N}^{\ell} = x_{\ell} \wedge \mathbf{\Pi}^{\ell} + j\pi(-1)^{r}\sigma(\ell,\ell^{c}) \int_{\mathbf{\Xi}_{\ell}} \frac{\mathrm{d}\xi_{\ell^{c}}}{2\chi_{\ell}}\, \chi_{\ell}\mathbf{e}_{\ell} \wedge \Big(\big(\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar{\ell}}} \otimes \hat{\mathbf{A}}^{*}(\boldsymbol{\xi}_{\bar{\ell},+})\big) \times \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\Big), \tag{74}$$

$$\mathbf{L}^{\ell} = j\pi(-1)^{r}\sigma(\ell,\ell^{c}) \int_{\mathbf{\Xi}_{\ell}} \frac{\mathrm{d}\xi_{\ell^{c}}}{2\chi_{\ell}}\, \boldsymbol{\xi}_{\bar{\ell}} \wedge \Big(\big(\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar{\ell}}} \otimes \hat{\mathbf{A}}^{*}(\boldsymbol{\xi}_{\bar{\ell},+})\big) \times \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\Big), \tag{75}$$

$$\mathbf{S}^{\ell} = -j2\pi\sigma(\ell,\ell^{c}) \int_{\mathbf{\Xi}_{\ell}} \frac{\mathrm{d}\xi_{\ell^{c}}}{2\chi_{\ell}} \Big(\hat{\mathbf{A}}^{*}(\boldsymbol{\xi}_{\bar{\ell},+}) \odot \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\Big), \tag{76}$$

where $\mathbf{\Pi}^{\ell}$ is the energy-momentum flux across the region in (17) and contributes to the angular momentum with a term dependent of the origin of coordinates $\boldsymbol{\alpha}$. The product $\odot$ could be replaced by $\oslash$ in (76) with an overall change of sign, since the off-diagonal transposed components of both products coincide [1, Eq. (22)], and the diagonal components vanish in the Coulomb-ℓ gauge defined in (6)–(7).

Using the various product definitions, the I-th component, where $I = (i,j) \in \mathcal{I}_2$ and $\ell \notin I$, of the orbital angular momentum and spin are, respectively, given by

$$\begin{aligned} L^{\ell}_{I} &= j\pi(-1)^{r}\sigma(\ell,\ell^{c}) \int_{\mathbf{\Xi}_{\ell}} \frac{\mathrm{d}\xi_{\ell^{c}}}{2\chi_{\ell}} \Big(\Delta_{jj}\xi_{i}\big(\partial^{*}_{\xi_{j}}\hat{\mathbf{A}}^{*}(\boldsymbol{\xi}_{\bar{\ell},+})\big)\cdot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) - \Delta_{ii}\xi_{j}\big(\partial^{*}_{\xi_{i}}\hat{\mathbf{A}}^{*}(\boldsymbol{\xi}_{\bar{\ell},+})\big)\cdot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\Big) & (77)\\ &= j\pi(-1)^{r}\sigma(\ell,\ell^{c}) \int_{\mathbf{\Xi}_{\ell}} \frac{\mathrm{d}\xi_{\ell^{c}}}{2\chi_{\ell}} \sum_{K\in\mathcal{I}_{r-1}} \Big(\Delta_{KK}\Big(\Delta_{jj}\xi_{i}\big(\partial_{\xi_{j}}\hat{A}^{*}_{K}(\boldsymbol{\xi}_{\bar{\ell},+})\big)\hat{A}_{K}(\boldsymbol{\xi}_{\bar{\ell},+}) - \Delta_{ii}\xi_{j}\big(\partial_{\xi_{i}}\hat{A}^{*}_{K}(\boldsymbol{\xi}_{\bar{\ell},+})\big)\hat{A}_{K}(\boldsymbol{\xi}_{\bar{\ell},+})\Big) - \mathrm{cc}\Big), & (78) \end{aligned}$$

and

$$S^{\ell}_{I} = -j2\pi\sigma(\ell,\ell^{c}) \int_{\mathbf{\Xi}_{\ell}} \frac{\mathrm{d}\xi_{\ell^{c}}}{2\chi_{\ell}} \Bigg(\sum_{L\in\mathcal{I}_{r-2}:i,j\notin L} \Delta_{LL}\sigma(L,i)\sigma(j,L)\hat{A}^{*}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},+})\hat{A}_{\varepsilon(j,L)}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\Bigg). \tag{79}$$

By construction, the subspace of vector potential components in (79) is restricted to those lists disjoint from I, with components different from ℓ (from the Coulomb-ℓ gauge condition in (6)), and orthogonal to $\boldsymbol{\xi}_{\bar{\ell},+}$ (from (7)). This leaves a total of $k+n-4$ space-time indices, to be distributed in lists of $r-2$ different elements. The dimension of this subspace is thus $\binom{k+n-4}{r-2}$. This dimension might be related to the classification of distinct pairs of spin-1 particles linked to the direction $\boldsymbol{\xi}_{\bar{\ell},+}$, a possibility to be studied elsewehere.

The feasibility of the separation of angular momentum into orbital and spin parts in a gauge-invariant manner, as well as its operational meaning, have long been subject to some level of discussion, particularly in a quantum context [13, 14, 15, 17, 18, 19]. As stated earlier in the paper, quantum aspects lie beyond the scope of this work and we do not dwell further on this matter, apart from noting that our analysis is done in the Coulomb-ℓ gauge (or equivalently for the transverse normal modes of the field [12, Sec. $\mathrm{B_I}$]), the condition that has been found to be in best empirical agreement with observations for the standard electromagnetic field [15].

As a complement, we include in Appendix C a "canonical" derivation of the spin components extended to the generalized multivectorial electromagnetic field. Ignoring the quantum aspects, we have used as a basis Sections 12 and 16 of Wentzel's treatise on quantum field theory [20], one of the first book treatments of the subject. Our analysis bypasses the canonical tensor that Wentzel makes use of, so the appropriate adaptations have been made. As expected, the final formulas obtained with this extended analysis coincide with (76) and (79).

3.4 Spin and Orbital Angular Momentum in a Complex-valued Circular Polarization Basis

From the definition of the $\odot$ product in (11), the I-th component S^{ℓ}_{I} of the spin bivector $\mathbf{S}^{\ell}$ in (76) is given by

$$S^{\ell}_{I} = -j2\pi\sigma(\ell,\ell^{c}) \int_{\mathbf{\Xi}_{\ell}} \frac{\mathrm{d}\xi_{\ell^{c}}}{2\chi_{\ell}} \Big(\big(\Delta_{ii}\mathbf{e}_{i} \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+})\big)^{*} \cdot \big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) \llcorner \mathbf{e}_{j}\Delta_{jj}\big) - \mathrm{cc}\Big), \tag{80}$$

where $I = (i, j)$. The component S^ℓ_I adopts a particularly transparent form in the complex-valued circular-polarization basis. For any φ, let the right- and left-handed basis elements, respectively denoted by $\mathbf{e}^I_+$ and $\mathbf{e}^I_-$, be given by

$$\mathbf{e}^I_+ = \cos\varphi\,\Delta_{ii}\mathbf{e}_i - j\sin\varphi\,\Delta_{jj}\mathbf{e}_j, \tag{81a}$$

$$\mathbf{e}^I_- = -\sin\varphi\,\Delta_{ii}\mathbf{e}_i - j\cos\varphi\,\Delta_{jj}\mathbf{e}_j. \tag{81b}$$

Note that the symbol j is used to represent both the imaginary unit and one of the components of I, a possible source of confusion in expressions as (81) and others below. These vectors satisfy the orthonormality relations ${\mathbf{e}^I_+}^* \cdot \mathbf{e}^I_+ = \cos^2\varphi\Delta_{ii} + \sin^2\varphi\Delta_{jj}$, ${\mathbf{e}^I_-}^* \cdot \mathbf{e}^I_- = \sin^2\varphi\Delta_{ii} + \cos^2\varphi\Delta_{jj}$ and ${\mathbf{e}^I_+}^* \cdot \mathbf{e}^I_- = \sin\varphi\cos\varphi(\Delta_{jj} - \Delta_{ii})$, as well as the relationships $\mathbf{e}^I_+ \wedge \mathbf{e}^I_+ = \mathbf{e}^I_- \wedge \mathbf{e}^I_- = 0$ and $j\mathbf{e}^I_+ \wedge \mathbf{e}^I_- = \Delta_{II}\mathbf{e}_I$. The transformation in (81) has determinant $\Delta_{ii}\Delta_{jj}$ and the inverse transformation is given by

$$\Delta_{ii}\mathbf{e}_i = \cos\varphi\,\mathbf{e}^I_+ - \sin\varphi\,\mathbf{e}^I_-, \tag{82a}$$

$$\Delta_{ii}\mathbf{e}_j = j(\sin\varphi\,\mathbf{e}^I_+ + \cos\varphi\,\mathbf{e}^I_-). \tag{82b}$$

The basis elements for $\varphi = \frac{\pi}{4}$ appears in the analysis of helicity and circular polarization [4, Problem 7.27]; for $\varphi = 0$, and apart from a factor $-j$, we recover the standard basis, i. e. linear polarization.

When we substitute these expressions for $\mathbf{e}_i$ and $\mathbf{e}_j$ in (80) we have to take into account that the complex-conjugate operation acting on the potential also affects the basis elements. For the standard space-time basis, this observation is irrelevant since the basis elements are real-valued. However, the polarization vectors are complex-valued and we need to use $\mathbf{e}^*_i$ rather than $\mathbf{e}_i$ in (82a). With this observation, the component S^ℓ_I is given by

$$S^\ell_I = -j2\pi\sigma(\ell,\ell^c)\int_{\Xi_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\Big(j\big((\cos\varphi\,\mathbf{e}^I_+ - \sin\varphi\,\mathbf{e}^I_-)^* \lrcorner\, \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\big)\cdot\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) \llcorner (\sin\varphi\,\mathbf{e}^I_+ + \cos\varphi\,\mathbf{e}^I_-)\big) - \mathrm{cc}\Big) \tag{83}$$

$$= -j2\pi\sigma(\ell,\ell^c)\int_{\Xi_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\Big(j\sin(2\varphi)\,\big({\mathbf{e}^I_+}^* \lrcorner\, \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\big)\cdot\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) \llcorner \mathbf{e}^I_+\big) - \big({\mathbf{e}^I_-}^* \lrcorner\, \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\big)\cdot\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) \llcorner \mathbf{e}^I_-\big) +$$

$$+ j\cos(2\varphi)\,\big({\mathbf{e}^I_+}^* \lrcorner\, \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\big)\cdot\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) \llcorner \mathbf{e}^I_-\big) + \mathrm{cc}\Big), \tag{84}$$

where we have grouped common terms under the assumption that the fields $\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})$ and $\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})$ commute, as it corresponds to a classical theory. For the choice $\varphi = \pi/4$, the basis elements satisfy ${\mathbf{e}^I_+}^* \cdot \mathbf{e}^I_+ = {\mathbf{e}^I_-}^* \cdot \mathbf{e}^I_- = \frac{1}{2}(\Delta_{ii} + \Delta_{jj})$ and ${\mathbf{e}^I_+}^* \cdot \mathbf{e}^I_- = \frac{1}{2}(\Delta_{jj} - \Delta_{ii})$, and the components S^ℓ_I adopt a particularly simple form,

$$S^\ell_I = 2\pi\sigma(\ell,\ell^c)\int_{\Xi_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\Big(\big({\mathbf{e}^I_+}^* \lrcorner\, \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\big)\cdot\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) \llcorner \mathbf{e}^I_+\big) - \big({\mathbf{e}^I_-}^* \lrcorner\, \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\big)\cdot\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) \llcorner \mathbf{e}^I_-\big)\Big). \tag{85}$$

This formula extends a similar result for the standard electromagnetic field [4, Problem 7.27], and expresses the spin as the sum of independent right- and left-handed components. For other values of φ, the basis components are mixed.

The I-th component L^ℓ_I, with $\ell \notin I$, of the orbital angular momentum bivector $\mathbf{L}^\ell$ is given by (77). For the basis change in (81) with $\varphi = \pi/4$, the frequency vector components transform are expressed as a function of ξ^I_+ and ξ^I_- in terms of the Hermitian inverse of the transformation matrix, i. e.

$$\xi_i = \frac{1}{\sqrt{2}}(\xi^I_+ - \xi^I_-), \tag{86a}$$

$$\xi_j = \frac{1}{\sqrt{2}}j(\xi^I_+ + \xi^I_-), \tag{86b}$$

and similarly for ∂_{ξ_i} and ∂_{ξ_j}. Again, the symbol j doubly represents a coordinate label in the left-hand side and the imaginary unit in the right-hand side of (86b). We therefore can express the orbital angular momentum component L^ℓ_I in (77) in terms of the coefficients in the circular-polarization basis in (86) as

$$L^\ell_I = j\pi(-1)^r\sigma(\ell,\ell^c)\Delta_{ii}\int_{\Xi_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\,\frac{1}{2}\Big(-j(\xi^I_+ - \xi^I_-)\big((\partial_{\xi^I_+} + \partial_{\xi^I_-})\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)^*\cdot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) +$$

$$- j(\xi^I_+ + \xi^I_-)\big((\partial_{\xi^I_+} - \partial_{\xi^I_-})\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)^*\cdot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) - \mathrm{cc}\Big) \tag{87}$$

$$= 2\pi(-1)^r\sigma(\ell,\ell^c)\Delta_{ii}\int_{\Xi_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\,\Re\Big(\xi^I_+\big(\partial_{\xi^I_+}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)^*\cdot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) - \xi^I_-\big(\partial_{\xi^I_-}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)^*\cdot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\Big), \tag{88}$$

a formula reminiscent of that of the spin for the standard electromagnetic field [4, Problem 7.27]. As we have seen throughout the previous pages, a large number of standard results in the analysis of angular momentum for free electromagnetic fields naturally extend to arbitrary number of space-time dimensions and multivector field grade. This brief discussion on the orbital angular momentum and the spin of the generalized electromagnetic field and their relationship to complex-valued circular polarizations, for generic values of r, k, and n, concludes the paper. The remainder is devoted to appendices with details or proofs of several results mentioned earlier in the paper.

A Proof of the Stokes Theorem

In this appendix, we prove the following statement: the flux of a tensor field $\mathbf{M}$, antisymmetric in the second and third components and with basis elements given by $\mathbf{w}_{i,I} = \mathbf{e}_i \otimes \mathbf{e}_I$, across the boundary $\partial\mathcal{V}^m$ of an m-dimensional hypersurface $\mathcal{V}^m$ is equal to the flux of the divergence of $\mathbf{M}$ across $\mathcal{V}^m$ for any $m \leq k+n$, and in particular for $m = k+n$. This Stokes theorem thus gives

$$\int_{\mathcal{V}^{k+n}} \mathrm{d}^{k+n}\mathbf{x}^{\mathcal{H}^{-1}} \lrcorner\, (\boldsymbol{\partial} \times \mathbf{M}) = \int_{\partial\mathcal{V}^{k+n}} \mathrm{d}^{k+n-1}\mathbf{x}^{\mathcal{H}^{-1}} \times \mathbf{M}. \tag{89}$$

We prove (89) thanks to the generalized Stokes theorem for differential forms [21, pp. 80],

$$\int_{\mathcal{V}} \mathrm{d}\omega = \int_{\partial\mathcal{V}} \omega, \tag{90}$$

where ω is a differential form and $\mathrm{d}\omega$ is its exterior derivative, corresponding to the operator

$$\mathrm{d} = \sum_{j\in\mathcal{I}} \mathrm{d}x_j \partial_j. \tag{91}$$

The procedure we follow is almost identical to what was done in [2, Sec. 3.4–3.5], and it starts by identifying ω with the integrand on the right-hand side of (89). Using (53) with $\mathbf{M} = \mathbf{M}_{\boldsymbol{\alpha}}$, we have

$$\omega = \sum_{m\in\mathcal{I}} \mathrm{d}x_{m^c}\Delta_{mm}\sigma(m,m^c)\mathbf{e}_m \times \sum_{i,j,\ell\in\mathcal{I}} (x_i - \alpha_i)T_{\varepsilon(j,\ell)}\sigma(i,\ell)\mathbf{w}_{j,\varepsilon(i,\ell)} \tag{92}$$

$$= \sum_{i,j,\ell\in\mathcal{I}} \mathrm{d}x_{j^c}\sigma(j,j^c)(x_i - \alpha_i)T_{\varepsilon(j,\ell)}\sigma(i,\ell)\mathbf{e}_{\varepsilon(i,\ell)}, \tag{93}$$

having applied $\mathbf{e}_m \times \mathbf{w}_{j,\varepsilon(i,\ell)} = \Delta_{mj}\mathbf{e}_{\varepsilon(i,\ell)}$. We then let the exterior derivative in (91) act on (93) to obtain

$$\mathrm{d}\omega = \sum_{m\in\mathcal{I}} \sum_{i,j,\ell\in\mathcal{I}} \mathrm{d}x_m\, \mathrm{d}x_{j^c}\sigma(j,j^c)\sigma(i,\ell)\partial_m\big((x_i - \alpha_i)T_{\varepsilon(j,\ell)}\big)\mathbf{e}_{\varepsilon(i,\ell)}. \tag{94}$$

Since $j^c \in \mathcal{I}_{k+n-1}$, we can identify m with j and write $\mathrm{d}x_{\varepsilon(j,j^c)} = \mathrm{d}x_j\, \mathrm{d}x_{j^c}\sigma(j,j^c)$ to obtain

$$\mathrm{d}\omega = \sum_{i,j,\ell\in\mathcal{I}} \mathrm{d}x_{\varepsilon(j,j^c)}\sigma(i,\ell)\partial_j\big((x_i - \alpha_i)T_{\varepsilon(j,\ell)}\big)\mathbf{e}_{\varepsilon(i,\ell)}. \tag{95}$$

In parallel, we identify $\mathrm{d}\omega$ in (90) with the integrand of the left-hand side of (89), which can be expanded as

$$\mathrm{d}\omega = \sum_{i,j,\ell\in\mathcal{I}} \mathrm{d}x_{\varepsilon(j,j^c)}\sigma(i,\ell)\partial_j\big((x_i - \alpha_i)T_{\varepsilon(j,\ell)}\big)\mathbf{e}_{\varepsilon(i,\ell)}, \tag{96}$$

namely the same expression as (95), therefore proving (89).

B Flux of the Angular-Momentum Tensor

B.1 Computation of the Angular-Momentum Flux

Writing $\mathbf{x} - \boldsymbol{\alpha} = (x_\ell - \alpha_\ell)\mathbf{e}_\ell + \mathbf{x}_{\bar{\ell}} - \boldsymbol{\alpha}_{\bar{\ell}}$, we may split the flux in (72) as a weighted sum, namely

$$\boldsymbol{\Omega}^\ell_{\boldsymbol{\alpha}} = \kappa_\ell(x_\ell - \alpha_\ell)\boldsymbol{\mathcal{I}}_\ell + \kappa_\ell\boldsymbol{\mathcal{I}}_{\bar{\ell},+} - \kappa_\ell\boldsymbol{\mathcal{I}}_{\bar{\ell},-}, \tag{97}$$

where the bivector-valued integrals $\boldsymbol{\mathcal{I}}_\ell$, $\boldsymbol{\mathcal{I}}_{\bar{\ell},+}$, and $\boldsymbol{\mathcal{I}}_{\bar{\ell},-}$ are, respectively, given by

$$\boldsymbol{\mathcal{I}}_\ell = \int_{\mathbf{R}^{k+n-1}} \mathrm{d}x_{\ell^c} \iint_{\boldsymbol{\Xi}_\ell\times\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\frac{\mathrm{d}\xi'_{\ell^c}}{2\chi'_\ell}\, e^{j2\pi(\boldsymbol{\xi}_{\bar{\ell}}+\boldsymbol{\xi}'_{\bar{\ell}})\cdot\mathbf{x}_{\bar{\ell}}}\mathbf{e}_\ell \times \Big(\mathbf{e}_\ell \mathbin{\boxed{\wedge}} \big(\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}}) \odot \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar{\ell}}) + \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}}) \mathbin{\bigcirc\!\!\!\!\!\wedge}\, \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar{\ell}})\big)\Big), \tag{98}$$

$$\boldsymbol{\mathcal{I}}_{\bar{\ell},+} = \int_{\mathbf{R}^{k+n-1}} \mathrm{d}x_{\ell^c} \iint_{\boldsymbol{\Xi}_\ell\times\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\frac{\mathrm{d}\xi'_{\ell^c}}{2\chi'_\ell}\, e^{j2\pi(\boldsymbol{\xi}_{\bar{\ell}}+\boldsymbol{\xi}'_{\bar{\ell}})\cdot\mathbf{x}_{\bar{\ell}}}\mathbf{e}_\ell \times \Big(\mathbf{x}_{\bar{\ell}} \mathbin{\boxed{\wedge}} \big(\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}}) \odot \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar{\ell}}) + \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}}) \mathbin{\bigcirc\!\!\!\!\!\wedge}\, \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar{\ell}})\big)\Big), \tag{99}$$

$$\boldsymbol{\mathcal{I}}_{\bar{\ell},-} = \int_{\mathbf{R}^{k+n-1}} \mathrm{d}x_{\ell^c} \iint_{\boldsymbol{\Xi}_\ell\times\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\frac{\mathrm{d}\xi'_{\ell^c}}{2\chi'_\ell}\, e^{j2\pi(\boldsymbol{\xi}_{\bar{\ell}}+\boldsymbol{\xi}'_{\bar{\ell}})\cdot\mathbf{x}_{\bar{\ell}}}\mathbf{e}_\ell \times \Big(\boldsymbol{\alpha}_{\bar{\ell}} \mathbin{\boxed{\wedge}} \big(\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}}) \odot \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar{\ell}}) + \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}}) \mathbin{\bigcirc\!\!\!\!\!\wedge}\, \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar{\ell}})\big)\Big). \tag{100}$$

We next evaluate these integrals, starting with $\boldsymbol{\mathcal{I}}_\ell$. Interchanging the integration order of frequency and space-time, we evaluate the integral of $e^{j2\pi(\boldsymbol{\xi}_{\bar\ell}+\boldsymbol{\xi}'_{\bar\ell})\cdot\mathbf{x}_{\bar\ell}}$ over space-time $\mathbf{R}^{k+n-1}$ as the $(k+n-1)$-multidimensional Dirac delta. After integration over ξ'_{ℓ^c} to remove this Dirac delta, we directly obtain

$$\boldsymbol{\mathcal{I}}_\ell = \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell^2}\,\mathbf{e}_\ell \times \Big(\mathbf{e}_\ell \boxtimes \big(\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \odot \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell}) + \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \oslash \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell})\big)\Big). \tag{101}$$

It will prove convenient to define an integral $\boldsymbol{\mathcal{I}}_m$ for $m \in \mathcal{I}$, replacing $\mathbf{e}_\ell$ inside the parentheses in (101) by $\mathbf{e}_m$,

$$\boldsymbol{\mathcal{I}}_m = \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell^2}\,\mathbf{e}_\ell \times \Big(\mathbf{e}_m \boxtimes \big(\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \odot \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell}) + \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \oslash \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell})\big)\Big). \tag{102}$$

The integral $\boldsymbol{\mathcal{I}}_{\bar\ell,-}$ in (100) can now be evaluated in a similar way to $\boldsymbol{\mathcal{I}}_\ell$ to obtain

$$\boldsymbol{\mathcal{I}}_{\bar\ell,-} = \sum_{t\in\mathcal{I}\setminus\ell} \alpha_t \boldsymbol{\mathcal{I}}_t, \tag{103}$$

where $\boldsymbol{\mathcal{I}}_t$ is given by (102) setting $m = t$.

As for the integral $\boldsymbol{\mathcal{I}}_{\bar\ell,+}$ in (99), we first rewrite the formula for $\boldsymbol{\mathcal{I}}_{\bar\ell,+}$ by interchanging the integration order of frequency and space-time, using linearity and making some minor rearrangements and algebraic manipulations, as

$$\boldsymbol{\mathcal{I}}_{\bar\ell,+} = \iint_{\boldsymbol{\Xi}_\ell\times\boldsymbol{\Xi}_\ell} \mathrm{d}\xi_{\ell^c}\,\mathrm{d}\xi'_{\ell^c}\,\mathbf{e}_\ell \times \left(\int_{\mathbf{R}^{k+n-1}} \mathrm{d}x_{\ell^c} e^{j2\pi(\boldsymbol{\xi}_{\bar\ell}+\boldsymbol{\xi}'_{\bar\ell})\cdot\mathbf{x}_{\bar\ell}}\mathbf{x}_{\bar\ell} \boxtimes \left(\frac{\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \odot \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar\ell}) + \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \oslash \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar\ell})}{4\chi_\ell\chi'_\ell}\right)\right). \tag{104}$$

Under the usual assumptions that the fields vanish sufficiently fast at infinity, the space-time integral in (104) can be evaluated by integration by parts in terms of a derivative of the Dirac delta as

$$\int_{\mathbf{R}^{k+n-1}} \mathrm{d}x_{\ell^c} e^{j2\pi(\boldsymbol{\xi}_{\bar\ell}+\boldsymbol{\xi}'_{\bar\ell})\cdot\mathbf{x}_{\bar\ell}}\mathbf{x}_{\bar\ell} = \frac{1}{j2\pi}\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar\ell}}\big(\delta(\boldsymbol{\xi}_{\bar\ell}+\boldsymbol{\xi}'_{\bar\ell})\big), \tag{105}$$

where the vector-derivative operator $\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar\ell}}$ is given by

$$\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar\ell}} = \sum_{t\in\mathcal{I}\setminus\ell} \Delta_{tt}\mathbf{e}_t\partial_{\xi_t}. \tag{106}$$

Now, an extension of the proof in [16, p. 26] to our multidimensional bivector-valued integrals in (104) shows that the derivative of the Dirac delta can be evaluated as

$$\boldsymbol{\mathcal{I}}_{\bar\ell,+} = \frac{1}{j2\pi}\iint_{\boldsymbol{\Xi}_\ell\times\boldsymbol{\Xi}_\ell} \mathrm{d}\xi_{\ell^c}\,\mathrm{d}\xi'_{\ell^c}\,\mathbf{e}_\ell \times \left(\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar\ell}}\big(\delta(\boldsymbol{\xi}_{\bar\ell}+\boldsymbol{\xi}'_{\bar\ell})\big) \boxtimes \left(\frac{\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \odot \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar\ell}) + \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \oslash \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar\ell})}{4\chi_\ell\chi'_\ell}\right)\right) \tag{107}$$

$$= -\frac{1}{j2\pi}\int_{\boldsymbol{\Xi}_\ell} \mathrm{d}\xi_{\ell^c}\,\mathbf{e}_\ell \times \left(\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar\ell}} \boxtimes \left(\frac{\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \odot \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar\ell}) + \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \oslash \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}'_{\bar\ell})}{4\chi_\ell\chi'_\ell}\right)\right)\Bigg|_{\boldsymbol{\xi}'_{\bar\ell}=-\boldsymbol{\xi}_{\bar\ell}}. \tag{108}$$

Using the definition in (106), we can express (108) as

$$\boldsymbol{\mathcal{I}}_{\bar\ell,+} = -\frac{1}{j2\pi}\sum_{t\in\mathcal{I}\setminus\ell} \Delta_{tt}\boldsymbol{\mathcal{I}}_{t,+}, \tag{109}$$

where the bivector-valued integrals $\boldsymbol{\mathcal{I}}_{t,+}$ are given by

$$\boldsymbol{\mathcal{I}}_{t,+} = \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\,\mathbf{e}_\ell \times \left(\mathbf{e}_t \boxtimes \left(\partial_{\xi_t}\left(\frac{\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell})}{2\chi_\ell}\right) \odot \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell}) + \partial_{\xi_t}\left(\frac{\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell})}{2\chi_\ell}\right) \oslash \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell})\right)\right). \tag{110}$$

Finally, we evaluate the derivative of $\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell})/(2\chi_\ell)$ as

$$\partial_{\xi_t}\left(\frac{\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell})}{2\chi_\ell}\right) = \frac{\partial_{\xi_t}\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell})}{2\chi_\ell} + \Delta_{\ell\ell}\Delta_{tt}\xi_t\frac{\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell})}{2\chi_\ell^3}. \tag{111}$$

Using this expression, we may therefore rewrite (110) as

$$\boldsymbol{\mathcal{I}}_{t,+} = \boldsymbol{\mathcal{I}}_{t,1} + \Delta_{\ell\ell}\Delta_{tt}\boldsymbol{\mathcal{I}}_{t,0}, \tag{112}$$

where $\boldsymbol{\mathcal{I}}_{t,1}$ and $\boldsymbol{\mathcal{I}}_{t,0}$ are, respectively, given by

$$\boldsymbol{\mathcal{I}}_{t,1} = \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell^2}\,\mathbf{e}_\ell \times \Big(\mathbf{e}_t \boxtimes \Big(\big(\partial_{\xi_t}\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell})\big) \odot \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell}) + \big(\partial_{\xi_t}\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell})\big) \oslash \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell})\Big)\Big) \tag{113}$$

$$\boldsymbol{\mathcal{I}}_{t,0} = \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell^4}\,\xi_t\,\mathbf{e}_\ell \times \Big(\mathbf{e}_t \boxtimes \big(\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \odot \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell}) + \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \oslash \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell})\big)\Big). \tag{114}$$

Substituting (101), (109), (112), and (103) back into (97), we obtain

$$\mathbf{\Omega}^{\ell}_{\boldsymbol{\alpha}} = \kappa_\ell x_\ell \boldsymbol{\mathcal{I}}_\ell - \frac{\kappa_\ell}{j2\pi} \sum_{t\in\mathcal{I}\setminus\ell} (\Delta_{tt}\boldsymbol{\mathcal{I}}_{t,1} + \Delta_{\ell\ell}\boldsymbol{\mathcal{I}}_{t,0}) - \kappa_\ell \sum_{m\in\mathcal{I}} \alpha_m \boldsymbol{\mathcal{I}}_m, \tag{115}$$

where $\boldsymbol{\mathcal{I}}_m$, for $m \in \mathcal{I}$ is given in (102), and $\boldsymbol{\mathcal{I}}_{t,1}$ and $\boldsymbol{\mathcal{I}}_{t,0}$ are, respectively, given by (113) and (114).

The three bivector-valued integrands in (102), (113) and (114) are of the form $\mathbf{e}_\ell \times \big(\mathbf{e}_m \boxed{\wedge}\, \mathbf{B}\big)$, for some index m and some symmetric rank-2 tensor $\mathbf{B}$. As for some indices $I = (i_1, i_2) \in \mathcal{J}_2$ it holds that $\mathbf{e}_\ell \times \big(\mathbf{e}_m \boxed{\wedge}\, \mathbf{u}_I\big) = 0$, only some components of the tensor $\mathbf{B}$ contribute to the integral. To determine which components of $\mathbf{B}$ contribute to the integral, we compute the double product $\mathbf{e}_\ell \times \big(\mathbf{e}_m \boxed{\wedge}\, \mathbf{B}\big)$ with the definition of the product $\boxed{\wedge}$ in (16),

$$\begin{aligned}
\mathbf{e}_\ell \times \big(\mathbf{e}_m \boxed{\wedge}\, \mathbf{B}\big) &= \sum_{I\in\mathcal{J}_2} B_I \, \mathbf{e}_\ell \times \big(\mathbf{e}_m \boxed{\wedge}\, \mathbf{u}_I\big) && (116)\\
&= \sum_{I\in\mathcal{J}_2} \sum_{I^\pi \in I!} B_I \sigma(m, i^\pi_2)\, \mathbf{e}_\ell \times \big(\mathbf{e}_{i^\pi_1} \otimes \mathbf{e}_{\varepsilon(m,i^\pi_2)}\big) && (117)\\
&= \Delta_{\ell\ell} \sum_{j\in\mathcal{I}\setminus m} B_{\varepsilon(\ell,j)} \sigma(m, j)\, \mathbf{e}_{\varepsilon(m,j)}, && (118)
\end{aligned}$$

where we have used that I and its permutation I^π must be such that $i^\pi_1 = \ell$, i. e. that I must be of the form $\varepsilon(\ell, j)$ for some $j \in \mathcal{I}$. This observation fixes also the permutation $I^\pi = (\ell, j)$. Besides, we can remove $j = m$ from the summation as $\sigma(m, m) = 0$. For each m, we need thus consider only the components $B_{\varepsilon(\ell,j)}$, where $j \neq m$.

Computation of $\boldsymbol{\mathcal{I}}_m$. In (101), the tensor $\mathbf{B}$ mentioned in the previous paragraph is given by

$$\mathbf{B} = \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \odot \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell}) + \hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar\ell}) \owedge \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar\ell}). \tag{119}$$

In Section B.2 we evaluate its components $B_{\varepsilon(\ell,j)}$ needed in (118). Substituting (181) in (118) gives

$$\mathbf{e}_\ell \times \big(\mathbf{e}_m \boxed{\wedge}\, \mathbf{B}\big) = 2\beta_r \Delta_{\ell\ell}\chi_\ell \sum_{j\in\mathcal{I}\setminus m} \sigma(m, j)\, \mathbf{e}_{\varepsilon(m,j)} \sum_{\sigma\in\mathcal{S}} \sigma\, \xi_{\bar\ell,\sigma,j} |\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})|^2, \tag{120}$$

where β_r is given by

$$\beta_r = 4\pi^2(-1)^{r-1}, \tag{121}$$

and with some abuse of notation, σ denotes in this equation both the signature of a permutation and a sign. Substituting (120) back in (101) gives

$$\boldsymbol{\mathcal{I}}_m = \beta_r \Delta_{\ell\ell} \sum_{j\in\mathcal{I}\setminus m} \sigma(m, j)\, \mathbf{e}_{\varepsilon(m,j)} \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell} \big(\xi_{\bar\ell,+,j} |\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})|^2 - \xi_{\bar\ell,-,j} |\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})|^2\big). \tag{122}$$

Assume that $j \neq \ell$, so that $\xi_{\bar\ell,\sigma,j} = \xi_j$, regardless of the value of σ. Then, splitting the integral in two, and making a change of variables $\boldsymbol{\zeta}_{\bar\ell} = -\boldsymbol{\xi}_{\bar\ell}$ in the integral with $\boldsymbol{\xi}_{\bar\ell,-}$ yields

$$\begin{aligned}
-\int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\, \xi_j |\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})|^2 &= -\int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\zeta_{\ell^c}}{2\chi_\ell}\, (-\zeta_j) |\hat{\mathbf{A}}(-\boldsymbol{\zeta}_{\bar\ell} - \chi_\ell \mathbf{e}_\ell)|^2 && (123)\\
&= \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\zeta_{\ell^c}}{2\chi_\ell}\, \zeta_j |\hat{\mathbf{A}}(\boldsymbol{\zeta}_{\bar\ell,+})|^2, && (124)
\end{aligned}$$

since $-\boldsymbol{\zeta}_{\bar\ell} - \chi_\ell \mathbf{e}_\ell = -\boldsymbol{\zeta}_{\bar\ell,+}$, and $|\hat{\mathbf{A}}(\boldsymbol{\zeta}_{\bar\ell,+})|^2 = \hat{\mathbf{A}}(\boldsymbol{\zeta}_{\bar\ell,+})\hat{\mathbf{A}}^*(\boldsymbol{\zeta}_{\bar\ell,+}) = \hat{\mathbf{A}}^*(-\boldsymbol{\zeta}_{\bar\ell,+})\hat{\mathbf{A}}(-\boldsymbol{\zeta}_{\bar\ell,+}) = |\hat{\mathbf{A}}(-\boldsymbol{\zeta}_{\bar\ell,+})|^2$ thanks to the hermiticity of $\hat{\mathbf{A}}(\boldsymbol{\zeta}_{\bar\ell,+})$. The second summand in the integral in (122) coincides with the first.

If $j = \ell$, then $\xi_{\bar\ell,\sigma,j} = \sigma\chi_\ell$, and the integral in (122) is given by

$$\int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell} \big(\chi_\ell |\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})|^2 + \chi_\ell |\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})|^2\big). \tag{125}$$

Splitting the integral in two, and making a change of variables $\boldsymbol{\zeta}_{\bar\ell} = -\boldsymbol{\xi}_{\bar\ell}$ in the integral with $\boldsymbol{\xi}_{\bar\ell,-}$ shows that the second integral in (125) coincides with the first one, as it happened in (124).

Substituting (124) and (125) back in (122) gives the final expression for $\boldsymbol{\mathcal{I}}_m$, namely

$$\begin{aligned}
\boldsymbol{\mathcal{I}}_m &= 2\beta_r \Delta_{\ell\ell} \sum_{j\in\mathcal{I}\setminus m} \sigma(m, j)\, \mathbf{e}_{\varepsilon(m,j)} \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\, \xi_{\bar\ell,+,j} |\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})|^2 && (126)\\
&= 2\beta_r \Delta_{\ell\ell} \mathbf{e}_m \wedge \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\, \boldsymbol{\xi}_{\bar\ell,+} |\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})|^2, && (127)
\end{aligned}$$

where we have used that $\mathbf{e}_{\varepsilon(m,j)} = \sigma(m,j)\mathbf{e}_m \wedge \mathbf{e}_j$, that $\mathbf{e}_m \wedge \mathbf{e}_m = 0$, and the decomposition $\boldsymbol{\xi}_{\bar{\ell},+} = \boldsymbol{\xi}_{\bar{\ell}} + \chi_\ell \mathbf{e}_\ell$.

With the definition $\kappa_\ell = -\frac{1}{2}\Delta_{\ell\ell}\sigma(\ell,\ell^c)$, the bivector-valued integral $\boldsymbol{\mathcal{I}}_m$ can be expressed in terms of the energy-momentum flux $\boldsymbol{\Pi}^\ell$ in (17) across the $(k+n)$-dimensional half space-time $\mathcal{V}_\ell^{k+n}$ of fixed x_ℓ, for $\ell \in \{0,\dots,k+n-1\}$, in (59) as

$$\boldsymbol{\mathcal{I}}_m = \frac{1}{\kappa_\ell}\mathbf{e}_m \wedge \boldsymbol{\Pi}^\ell. \tag{128}$$

Computation of $\boldsymbol{\mathcal{I}}_{t,0}$. In (114), $m = t$ while the tensor $\mathbf{B}$ is again given by (119). Substituting the components $B_{\varepsilon(\ell,j)}$ in (181) into (118) with $m = t$, and then back in (114) yields an analogous equation to (122), namely

$$\boldsymbol{\mathcal{I}}_{t,0} = \beta_r \Delta_{\ell\ell} \sum_{j\in\mathcal{I}\setminus t} \sigma(t,j)\,\mathbf{e}_{\varepsilon(t,j)} \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell^3}\,\xi_t\left(\xi_{\bar{\ell},+,j}|\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+})|^2 - \xi_{\bar{\ell},-,j}|\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},-})|^2\right). \tag{129}$$

It proves convenient to split the integral in two and separate the cases $j \neq \ell$ and $j = \ell$. In the first case, i. e. $j \neq \ell$, noting first that $\xi_{\bar{\ell},-,j} = \xi_{\bar{\ell},+,j} = \xi_j$, making a change of variables $\boldsymbol{\zeta}_{\bar{\ell}} = -\boldsymbol{\xi}_{\bar{\ell}}$ in the integral with $\boldsymbol{\xi}_{\bar{\ell},-}$ gives

$$\int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell^3}\,\xi_t\,\xi_j|\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},-})|^2 = \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\zeta_{\ell^c}}{2\chi_\ell^3}\,(-\zeta_t)\,(-\zeta_j)|\hat{\mathbf{A}}(\boldsymbol{\zeta}_{\bar{\ell},+})|^2 \tag{130}$$

$$= \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\zeta_{\ell^c}}{2\chi_\ell^3}\,\zeta_t\,\zeta_j|\hat{\mathbf{A}}(\boldsymbol{\zeta}_{\bar{\ell},+})|^2, \tag{131}$$

which cancels out with the integral $\boldsymbol{\xi}_{\bar{\ell},+}$ and the integral in (129) vanishes for $j \neq \ell$. If $j = \ell$, $\xi_{\bar{\ell},-,\ell} = -\xi_{\bar{\ell},+,\ell} = -\chi_\ell$, unaffected by the change of variables $\boldsymbol{\zeta}_{\bar{\ell}} = -\boldsymbol{\xi}_{\bar{\ell}}$. The integral with $\boldsymbol{\xi}_{\bar{\ell},-}$ gives thus

$$\int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell^3}\,\xi_t(-\chi_\ell)\,|\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},-})|^2 = -\int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\zeta_{\ell^c}}{2\chi_\ell^2}\,(-\zeta_t)\,|\hat{\mathbf{A}}(\boldsymbol{\zeta}_{\bar{\ell},+})|^2, \tag{132}$$

and the total integral in (129) vanishes too. Therefore,

$$\boldsymbol{\mathcal{I}}_{t,0} = 0. \tag{133}$$

Computation of $\boldsymbol{\mathcal{I}}_{t,1}$. In (113), the index m is again $m = t$, while the tensor $\mathbf{B}$ is now given by

$$\mathbf{B} = \left(\partial_{\xi_t}\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}})\right) \odot \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar{\ell}}) + \left(\partial_{\xi_t}\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}})\right) \oslash \hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar{\ell}}). \tag{134}$$

Substituting the double product (118) in (113) gives

$$\boldsymbol{\mathcal{I}}_{t,1} = \Delta_{\ell\ell}\sigma(t,\ell)\,\mathbf{e}_{\varepsilon(t,\ell)} \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell^2}\,B_{\ell\ell} + \Delta_{\ell\ell} \sum_{j\in\mathcal{I}\setminus\ell,t} \sigma(t,j)\,\mathbf{e}_{\varepsilon(t,j)} \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell^2}\,B_{\varepsilon(\ell,j)}. \tag{135}$$

In Section B.3 we evaluate the components $B_{\varepsilon(\ell,j)}$ needed in (118), namely $\ell = j$ and $\ell \neq j$. Substituting the expression of $B_{\ell\ell}$ in (211) into the first integral in (135), and expanding the sum over σ gives

$$\begin{aligned}\int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell^2}\,B_{\ell\ell} = \beta_r \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell^2}\Big(&-\underbrace{\Delta_{\ell\ell}\Delta_{tt}\xi_t\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+})\big|^2}_{\text{1a}} - \underbrace{\Delta_{\ell\ell}\Delta_{tt}\xi_t\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},-})\big|^2}_{\text{1b}} + 2\underbrace{\chi_\ell^2\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+})}_{\text{2a}} +\\ &+ 2\underbrace{\chi_\ell^2\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},-})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},-})}_{\text{2b}} - \underbrace{e^{j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\Delta_{\ell\ell}\Delta_{tt}\xi_t\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},-})\big)}_{\text{3a}} -\\ &- \underbrace{e^{-j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\Delta_{\ell\ell}\Delta_{tt}\xi_t\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},-})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+})\big)}_{\text{3b}}\Big).\end{aligned} \tag{136}$$

We split the integrand in (136) into three terms, respectively, indexed by 1, 2, and 3, each with consecutive pairs of summands labelled by a and b. In the integrand with label 1b, the change of variables $\boldsymbol{\zeta}_{\bar{\ell}} = -\boldsymbol{\xi}_{\bar{\ell}}$ has opposite sign to the contribution from 1a, so the first integral is zero. Then, each of the integrands with labels 3a and 3b is an odd function of the integration variable $\boldsymbol{\xi}_{\bar{\ell}}$, as can be verified with the change of variables $\boldsymbol{\zeta}_{\bar{\ell}} = -\boldsymbol{\xi}_{\bar{\ell}}$. Indeed, $\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+})$ transforms into $\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},-})$ and (resp. $\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},-})$) into $\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+})$) and therefore the third integral is zero too. It only remains the second integral, which can be expressed with the usual change of variables $\boldsymbol{\zeta}_{\bar{\ell}} = -\boldsymbol{\xi}_{\bar{\ell}}$ in 2b as

$$\int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell^2}\,B_{\ell\ell} = 4\pi^2(-1)^{r-1} \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\,\chi_\ell\Big(\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\Big), \tag{137}$$

where cc denotes the complex conjugate.

Proceeding in a similar manner, substituting the expression for $B_{\varepsilon(\ell,j)}$ in (212) into the second integral in (135), and splitting the integrand into three terms, respectively, indexed by 1, 2, and 3, each with consecutive pairs of summands labelled by a and b, gives

$$\begin{aligned}\int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell^2}\, B_{\varepsilon(\ell,j)} = 4\pi^2 \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell}\Big(&\Delta_{tt}\underbrace{\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\big|_{tj} - \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\big|_{jt}\Big)}_{1a} - \\ &- \Delta_{tt}\underbrace{\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,-})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})\big)\big|_{tj} - \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,-})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})\big)\big|_{jt}\Big)}_{1b} - \\ &- 2(-1)^r\underbrace{\xi_j\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})}_{2a} + 2(-1)^r\underbrace{\xi_j\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,-})}_{2b} - \\ &- \underbrace{e^{j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\Delta_{tt}\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,-})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\big|_{jt} + \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,-})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\big|_{tj}\Big)}_{3a} + \\ &+ \underbrace{e^{-j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\Delta_{tt}\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})\big)\big|_{jt} + \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})\big)\big|_{tj}\Big)}_{3b}.\end{aligned} \tag{138}$$

As before, we consider separately the integrands. If the quantities $\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})$ and $\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})$ commute, the change of variables $\boldsymbol{\zeta}_{\bar\ell} = -\boldsymbol{\xi}_{\bar\ell}$ in the integrand 1b gives the integrand 1a with an opposite sign; this sign cancels the minus sign in front. Besides, for commuting quantities, the second summand of 1a (and of 1b) is the complex conjugate of the first summand. The same change of variables $\boldsymbol{\zeta}_{\bar\ell} = -\boldsymbol{\xi}_{\bar\ell}$ shows that the integrand 2b (resp. 3b) is the complex conjugate of 2a (resp. 3a). Similarly, the same change of variables and commutativity assumption applied in the second integrand of 3b shows that the second summand coincides with the first one. We therefore have

$$\begin{aligned}\int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{4\chi_\ell^2}\, B_{\varepsilon(\ell,j)} = 4\pi^2 \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\Big(&\Delta_{tt}\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\big|_{tj} - \mathrm{cc}\Big) - (-1)^r\xi_j\Big(\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+}) - \mathrm{cc}\Big) + \\ &+ \Delta_{tt}\Big(e^{-j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})\big)\big|_{tj} - \mathrm{cc}\Big).\end{aligned} \tag{139}$$

Note that the rank-2 tensors with components $\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)\big|_{tj} -$ cc are actually antisymmetric in the indices t and j and can thus be seen as a bivector component with element basis $\mathbf{e}_{tj}$.

Combining (137) and (139) back into (135) gives

$$\begin{aligned}\boldsymbol{\mathcal{I}}_{t,1} = 4\pi^2\Delta_{\ell\ell}\Delta_{tt}&\sum_{j\in\mathcal{I}\setminus\ell,t}\sigma(t,j)\,\mathbf{e}_{\varepsilon(t,j)}\int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\big|_{tj} - \mathrm{cc}\Big) + \\ &+ \beta_r\Delta_{\ell\ell}\sigma(t,\ell)\,\mathbf{e}_{\varepsilon(t,\ell)}\int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\,\chi_\ell\Big(\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+}) - \mathrm{cc}\Big) + \\ &+ \beta_r\Delta_{\ell\ell}\sum_{j\in\mathcal{I}\setminus\ell,t}\sigma(t,j)\,\mathbf{e}_{\varepsilon(t,j)}\int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\,\xi_j\Big(\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+}) - \mathrm{cc}\Big) + \\ &+ 4\pi^2\Delta_{\ell\ell}\Delta_{tt}\sum_{j\in\mathcal{I}\setminus\ell,t}\sigma(t,j)\,\mathbf{e}_{\varepsilon(t,j)}\int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\Big(e^{-j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})\big)\big|_{tj} - \mathrm{cc}\Big).\end{aligned} \tag{140}$$

Let us denote by $\boldsymbol{\mathcal{I}}^1_{t,1}$ the first summand in (140); the second and third terms in (140) can be grouped into a single summation, denoted by $\boldsymbol{\mathcal{I}}^2_{t,1}$, over $j \in \mathcal{I}\setminus t$. We denote by $\boldsymbol{\mathcal{I}}^3_{t,1}$ the remaining summand in (140). As $\mathbf{e}_{\varepsilon(t,j)} = -\sigma(t,j)\mathbf{e}_j \wedge \mathbf{e}_t$ and $\mathbf{e}_t \wedge \mathbf{e}_t = 0$, the contribution of $\boldsymbol{\mathcal{I}}^1_{t,1}$ to the flux is given by

$$-\frac{\kappa_\ell}{j2\pi}\sum_{t\in\mathcal{I}\setminus\ell}\Delta_{tt}\boldsymbol{\mathcal{I}}^1_{t,1} = j2\pi\kappa_\ell\Delta_{\ell\ell}\sum_{t\in\mathcal{I}\setminus\ell}\sum_{j\in\mathcal{I}\setminus\ell,t}\sigma(t,j)\,\mathbf{e}_{\varepsilon(t,j)}\int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\big|_{tj} - \mathrm{cc}\Big) \tag{141}$$

$$= j2\pi\kappa_\ell\Delta_{\ell\ell}\sum_{t\in\mathcal{I}}\sum_{j\in\mathcal{I}\setminus t}\sigma(t,j)\,\mathbf{e}_{\varepsilon(t,j)}\int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\big|_{tj} - \mathrm{cc}\Big) \tag{142}$$

$$= j4\pi\kappa_\ell\Delta_{\ell\ell}\sum_{t,j\in\mathcal{I}:t<j}\mathbf{e}_{\varepsilon(t,j)}\int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+})\big)\big|_{tj} - \mathrm{cc}\Big), \tag{143}$$

where we have extended the summations to $t = \ell$ and $j = \ell$ since these added terms are zero in the Coulomb-ℓ gauge in (142) and noted that every ordered list of non-repeated index pairs appears twice in the summation in (143). Interpreting $\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) -$ cc as a bivector, we obtain

$$-\frac{\kappa_\ell}{j2\pi}\sum_{t\in\mathcal{I}\setminus\ell}\Delta_{tt}\boldsymbol{\mathcal{I}}^1_{t,1} = -j2\pi\sigma(\ell,\ell^c)\int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\Big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,+})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) - \mathrm{cc}\Big). \tag{144}$$

Proceeding in a similar manner, we can rewrite $\boldsymbol{\mathcal{I}}_{t,1}^2$ as

$$\boldsymbol{\mathcal{I}}_{t,1}^2 = \beta_r \Delta_{\ell\ell} \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\, \boldsymbol{\xi}_{\bar{\ell},+} \wedge \Big(\mathbf{e}_t \big(\partial_{\xi_t} \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+})\big) \cdot \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\Big) \tag{145}$$

$$= \beta_r \Delta_{\ell\ell} \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\, \boldsymbol{\xi}_{\bar{\ell},+} \wedge \Big(\big(\mathbf{e}_t \partial_{\xi_t} \otimes \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+})\big) \times \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\Big), \tag{146}$$

where we have also used that $\mathbf{e}_t \big(\partial_{\xi_t} \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+})\big) \cdot \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) = \big(\mathbf{e}_t \partial_{\xi_t} \otimes \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+})\big) \times \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+})$, as can be seen by direct computation. Getting back to (115), and summing over $t \in \mathcal{I} \setminus \ell$, this first term of (140) contributes to the flux as

$$-\frac{\kappa_\ell}{j2\pi} \sum_{t \in \mathcal{I}\setminus\ell} \Delta_{tt} \boldsymbol{\mathcal{I}}_{t,1}^2 = j\pi(-1)^r \sigma(\ell,\ell^c) \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\, \boldsymbol{\xi}_{\bar{\ell},+} \wedge \Big(\big(\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar{\ell}}} \otimes \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+})\big) \times \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\Big). \tag{147}$$

It is straightforward to verify that this expression is indeed a bivector, regardless of the value of r.

Analogously, the contribution of $\boldsymbol{\mathcal{I}}_{t,1}^3$ is given by

$$-\frac{\kappa_\ell}{j2\pi} \sum_{t \in \mathcal{I}\setminus\ell} \Delta_{tt} \boldsymbol{\mathcal{I}}_{t,1}^3 = -j\pi\sigma(\ell,\ell^c) \sum_{t \in \mathcal{I}\setminus\ell} \sum_{j \in \mathcal{I}\setminus\ell,t} \sigma(t,j)\, \mathbf{e}_{\varepsilon(t,j)} \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell} \Big(e^{-j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell} \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+}) \odot \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},-})\big)\big|_{tj} - \mathrm{cc}\Big). \tag{148}$$

Under the change of variable $\boldsymbol{\zeta}_{\bar{\ell}} = -\boldsymbol{\xi}_{\bar{\ell}}$ and the commutativity assumption, this equation becomes

$$-\frac{\kappa_\ell}{j2\pi} \sum_{t \in \mathcal{I}\setminus\ell} \Delta_{tt} \boldsymbol{\mathcal{I}}_{t,1}^3 = -j\pi\sigma(\ell,\ell^c) \sum_{t \in \mathcal{I}\setminus\ell} \sum_{j \in \mathcal{I}\setminus\ell,t} \sigma(t,j)\, \mathbf{e}_{\varepsilon(t,j)} \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell} \Big(e^{-j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell} \big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+}) \odot \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},-})\big)\big|_{jt} - \mathrm{cc}\Big). \tag{149}$$

Renaming now t as j' and j as t', this equation coincides with (148), except that $\sigma(t,j)$ picks a minus sign. The integrand is thus an odd function of $\boldsymbol{\xi}_{\bar{\ell}}$ and the integral in (148) is zero, and therefore

$$-\frac{\kappa_\ell}{j2\pi} \sum_{t \in \mathcal{I}\setminus\ell} \Delta_{tt} \boldsymbol{\mathcal{I}}_{t,1}^3 = 0. \tag{150}$$

Combining $\boldsymbol{\mathcal{I}}_m$ in (127) with (133), (147), (142), and (148) into (115) gives

$$\boldsymbol{\Omega}_{\boldsymbol{\alpha}}^\ell = (x_\ell \mathbf{e}_\ell - \boldsymbol{\alpha}) \wedge \boldsymbol{\Pi}^\ell + j\pi(-1)^r \sigma(\ell,\ell^c) \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\, \boldsymbol{\xi}_{\bar{\ell},+} \wedge \Big(\big(\boldsymbol{\partial}_{\boldsymbol{\xi}_{\bar{\ell}}} \otimes \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+})\big) \times \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\Big) -$$
$$- j2\pi\sigma(\ell,\ell^c) \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell} \Big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},+}) \odot \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\Big), \tag{151}$$

where $\boldsymbol{\Pi}^\ell$ is the energy-momentum flux across the region in (17).

B.2 Evaluation of the Tensor Components $B_\varepsilon(\ell,j)$ for $\boldsymbol{\mathcal{I}}_m$ and $\boldsymbol{\mathcal{I}}_{t,0}$ (Lorenz Gauge)

We start by listing some useful identities relating interior and wedge products [1, Sec. 2.2]. Given two vectors $\mathbf{u}$ and $\mathbf{v}$ and a r-vector $\mathbf{w}$, the following expressions hold

$$\mathbf{u} \lrcorner (\mathbf{v} \lrcorner \mathbf{w}) = -\mathbf{v} \lrcorner (\mathbf{u} \lrcorner \mathbf{w}), \tag{152}$$

and

$$\mathbf{u} \lrcorner (\mathbf{v} \wedge \mathbf{w}) = (-1)^r (\mathbf{u} \cdot \mathbf{v}) \mathbf{w} + \mathbf{v} \wedge (\mathbf{u} \lrcorner \mathbf{w}). \tag{153}$$

In addition, given two vectors $\mathbf{v}$ and $\mathbf{v}'$ and two r-vectors $\mathbf{w}$ and $\mathbf{w}'$, then it holds that

$$(\mathbf{v} \wedge \mathbf{w}) \cdot (\mathbf{v}' \wedge \mathbf{w}') + (\mathbf{v}' \lrcorner \mathbf{w}) \cdot (\mathbf{v} \lrcorner \mathbf{w}') = (\mathbf{v} \cdot \mathbf{v}') (\mathbf{w} \cdot \mathbf{w}'). \tag{154}$$

Also, for a vector $\mathbf{u}$, an $(r-1)$-vector $\mathbf{v}$ and an r-vector $\mathbf{w}$, it holds that

$$(\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{w} = (-1)^{r-1} (\mathbf{u} \lrcorner \mathbf{w}) \cdot \mathbf{v}. \tag{155}$$

In the tensor definition in (119) we need both $\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell}})$, given in (71), and $\hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar{\ell}})$. The latter is evaluated noting that the real-valuedness of the field implies that $\hat{\mathbf{F}}(-\boldsymbol{\xi}) = \hat{\mathbf{F}}^*(\boldsymbol{\xi})$ as follows,

$$\hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar{\ell}}) = e^{j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\, \hat{\mathbf{F}}(-\boldsymbol{\xi}_{\bar{\ell},+}) + e^{-j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\, \hat{\mathbf{F}}(-\boldsymbol{\xi}_{\bar{\ell},-}) \tag{156}$$

$$= e^{j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\, \hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},-}) + e^{-j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\, \hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},+}) \tag{157}$$

$$= \sum_{\sigma \in \mathcal{S}} e^{-j2\pi\Delta_{\ell\ell}\sigma\chi_\ell x_\ell}\, \hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma}). \tag{158}$$

Substituting the definition of $\hat{\mathbf{F}}^\ell$ given in (71) together with (158) in the tensor definition (119), we obtain

$$\mathbf{B} = \sum_{\sigma_1,\sigma_2\in\mathcal{S}} e^{j2\pi\Delta_{\ell\ell}(\sigma_1-\sigma_2)\chi_\ell x_\ell}\Big(\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\odot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})+\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\owedge\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\Big). \tag{159}$$

Let us define $\mathbf{B}^{\sigma_1\sigma_2}$ as $\mathbf{B}^{\sigma_1\sigma_2}=\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\odot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})+\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\owedge\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})$; we need to evaluate the components $B^{\sigma_1\sigma_2}_{\varepsilon(\ell,j)}$. Using the definition of the products $\odot$ and $\owedge$ in (11) and (12), the (i,j)-th component $B^{\sigma_1\sigma_2}_{ij}$ is given by

$$\begin{aligned}
B^{\sigma_1\sigma_2}_{ij} &= \Delta_{ii}\Delta_{jj}\Big((\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}))\cdot(\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\,\llcorner\,\mathbf{e}_j)+(\mathbf{e}_i\wedge\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}))\cdot(\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\wedge\mathbf{e}_j)\Big) && (160)\\
&= \Delta_{ii}\Delta_{jj}\Big((\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}))\cdot(\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\,\llcorner\,\mathbf{e}_j)+(-1)^r\Delta_{ij}\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})+(\mathbf{e}_j\,\lrcorner\,\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}))\cdot(\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\,\llcorner\,\mathbf{e}_i)\Big) && (161)\\
&= (-1)^{r-1}\Delta_{ii}\Delta_{jj}\Big(\underbrace{(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}))\cdot(\mathbf{e}_j\,\lrcorner\,\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}))}_{=\alpha^{\sigma_1\sigma_2}_{ij}}+\underbrace{(\mathbf{e}_j\,\lrcorner\,\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}))\cdot(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}))}_{=\alpha^{\sigma_1\sigma_2}_{ji}}-\Delta_{ij}\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\Big), && (162)
\end{aligned}$$

where we have applied (154) in (161) and the identity $\mathbf{v}\,\lrcorner\,\mathbf{w}=(-1)^{\mathrm{gr}(\mathbf{v})(\mathrm{gr}(\mathbf{w})+\mathrm{gr}(\mathbf{v}))}\mathbf{w}\,\llcorner\,\mathbf{v}$ [1, Eq. (21)] in (161) and (162), and have defined the quantities $\alpha^{\sigma_1\sigma_2}_{ij}$ for ease of presentation.

Substituting the potential in the Fourier domain, $\hat{\mathbf{F}}(\boldsymbol{\xi})=j2\pi\boldsymbol{\xi}\wedge\hat{\mathbf{A}}(\boldsymbol{\xi})$, and subsequently using (154) yields

$$\begin{aligned}
\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}) &= 4\pi^2\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\wedge\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big) && (163)\\
&= 4\pi^2\Big((\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\cdot\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)-\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\Big). && (164)
\end{aligned}$$

We note that the wave-equation condition $\boldsymbol{\xi}_{\bar{\ell},\sigma}\cdot\boldsymbol{\xi}_{\bar{\ell},\sigma}=0$, together with the identity $\boldsymbol{\xi}_{\bar{\ell},\sigma_1}=\boldsymbol{\xi}_{\bar{\ell},\sigma_2}+(\sigma_1-\sigma_2)\chi_\ell\mathbf{e}_\ell$, with $\sigma_1,\sigma_2\in\{+1,-1\}$, implies that

$$\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\cdot\boldsymbol{\xi}_{\bar{\ell},\sigma_2}=(\sigma_1\sigma_2-1)\Delta_{\ell\ell}\chi_\ell^2. \tag{165}$$

Also, the Lorenz-gauge condition in the Fourier domain, namely $\boldsymbol{\xi}_{\bar{\ell},\sigma}\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})=0=\boldsymbol{\xi}_{\bar{\ell},\sigma}\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})$, implies that

$$\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})=(\sigma_1-\sigma_2)\chi_\ell\mathbf{e}_\ell\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}), \tag{166}$$

$$\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})=(\sigma_1-\sigma_2)\chi_\ell\mathbf{e}_\ell\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}). \tag{167}$$

Substituting (165)–(167) back into (164) yields

$$\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})=4\pi^2\Big((\sigma_1\sigma_2-1)\Delta_{\ell\ell}\chi_\ell^2\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)+(\sigma_1-\sigma_2)^2\chi_\ell^2\big(\mathbf{e}_\ell\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\big(\mathbf{e}_\ell\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\Big). \tag{168}$$

Turning back our attention to $\alpha^{\sigma_1\sigma_2}_{ij}$, substitution of the potential in the Fourier domain, $\hat{\mathbf{F}}(\boldsymbol{\xi})=j2\pi\boldsymbol{\xi}\wedge\hat{\mathbf{A}}(\boldsymbol{\xi})$, followed by the use of (153), gives

$$\begin{aligned}
\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}) &= j2\pi\big((-1)^{r-1}(\mathbf{e}_i\cdot\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})+\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big) && (169)\\
&= -j2\pi\big((-1)^r\Delta_{ii}\xi_{\bar{\ell},\sigma_1,i}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})-\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big). && (170)
\end{aligned}$$

We therefore have for $\alpha^{\sigma_1\sigma_2}_{ij}$ (and similarly for $\alpha^{\sigma_1\sigma_2}_{ji}$), apart from a factor $4\pi^2$

$$\begin{aligned}
\alpha^{\sigma_1\sigma_2}_{ij} &\propto \Big((-1)^r\Delta_{ii}\xi_{\bar{\ell},\sigma_1,i}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})-\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\Big)\cdot\Big((-1)^r\Delta_{jj}\xi_{\bar{\ell},\sigma_2,j}\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})-\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\wedge\big(\mathbf{e}_j\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\big)\Big) && (171)\\
&= \Delta_{ii}\Delta_{jj}\xi_{\bar{\ell},\sigma_1,i}\xi_{\bar{\ell},\sigma_2,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})-(-1)^r\Delta_{ii}\xi_{\bar{\ell},\sigma_1,i}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\wedge\big(\mathbf{e}_j\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\big)\Big)\\
&\quad-(-1)^r\Delta_{jj}\xi_{\bar{\ell},\sigma_2,j}\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)+\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\wedge\big(\mathbf{e}_j\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\big). && (172)
\end{aligned}$$

Using (155) in the second and third summands and (154) in the fourth summand, we have apart from a factor $4\pi^2$

$$\begin{aligned}\alpha_{ij}^{\sigma_1\sigma_2} \propto\; &\Delta_{ii}\Delta_{jj}\xi_{\bar\ell,\sigma_1,i}\xi_{\bar\ell,\sigma_2,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})-\\ &-\Delta_{ii}\xi_{\bar\ell,\sigma_1,i}\big(\boldsymbol{\xi}_{\bar\ell,\sigma_2}\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)-\Delta_{jj}\xi_{\bar\ell,\sigma_2,j}\big(\boldsymbol{\xi}_{\bar\ell,\sigma_1}\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)\cdot\big(\mathbf{e}_i\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)\\ &+(\boldsymbol{\xi}_{\bar\ell,\sigma_1}\cdot\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big(\mathbf{e}_i\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)-\big(\boldsymbol{\xi}_{\bar\ell,\sigma_2}\lrcorner\,\big(\mathbf{e}_i\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)\big)\cdot\big(\boldsymbol{\xi}_{\bar\ell,\sigma_1}\lrcorner\,\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)\big)\end{aligned} \tag{173}$$

$$\begin{aligned}=\;&\Delta_{ii}\Delta_{jj}\xi_{\bar\ell,\sigma_1,i}\xi_{\bar\ell,\sigma_2,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})-\\ &-\Delta_{ii}\xi_{\bar\ell,\sigma_1,i}\big(\boldsymbol{\xi}_{\bar\ell,\sigma_2}\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)-\Delta_{jj}\xi_{\bar\ell,\sigma_2,j}\big(\boldsymbol{\xi}_{\bar\ell,\sigma_1}\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)\cdot\big(\mathbf{e}_i\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)\\ &+(\boldsymbol{\xi}_{\bar\ell,\sigma_1}\cdot\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big(\mathbf{e}_i\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)-\big(\mathbf{e}_i\lrcorner\,\big(\boldsymbol{\xi}_{\bar\ell,\sigma_2}\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)\big)\cdot\big(\mathbf{e}_j\lrcorner\,\big(\boldsymbol{\xi}_{\bar\ell,\sigma_1}\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)\big),\end{aligned} \tag{174}$$

where we have used (152) to swap the product order between $\boldsymbol{\xi}_{\bar\ell,\sigma_2}$ and $\mathbf{e}_i$ and between $\boldsymbol{\xi}_{\bar\ell,\sigma_1}$ and $\mathbf{e}_j$. Finally, substituting the identities (165)–(167) into (174) gives

$$\begin{aligned}\alpha_{ij}^{\sigma_1\sigma_2} \propto\; &\Delta_{ii}\Delta_{jj}\xi_{\bar\ell,\sigma_1,i}\xi_{\bar\ell,\sigma_2,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})-\Delta_{ii}\xi_{\bar\ell,\sigma_1,i}(\sigma_2-\sigma_1)\chi_\ell\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)-\\ &-\Delta_{jj}\xi_{\bar\ell,\sigma_2,j}(\sigma_1-\sigma_2)\chi_\ell\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)\cdot\big(\mathbf{e}_i\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)+(\sigma_1\sigma_2-1)\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_i\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)+\\ &+(\sigma_1-\sigma_2)^2\chi_\ell^2\big(\mathbf{e}_i\lrcorner\,\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma_1})\big)\big)\cdot\big(\mathbf{e}_j\lrcorner\,\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma_2})\big)\big).\end{aligned} \tag{175}$$

We continue our evaluation of $B_{ij}^{\sigma_1\sigma_2}$ by considering separately the cases $\sigma_1=\sigma_2$ and $\sigma_1\neq\sigma_2$. First, for $\sigma_1=\sigma_2=\sigma$, and combining (175), both for $\alpha_{ij}^{\sigma_1\sigma_2}$ and $\alpha_{ji}^{\sigma_1\sigma_2}$, and (168), we express $B_{ij}^{\sigma\sigma}$ in (162) as

$$\begin{aligned}B_{ij}^{\sigma\sigma} &= 4\pi^2(-1)^{r-1}\Delta_{ii}\Delta_{jj}2\Delta_{ii}\Delta_{jj}\xi_{\bar\ell,\sigma,i}\xi_{\bar\ell,\sigma,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\sigma}) &(176)\\ &= 8\pi^2(-1)^{r-1}\xi_{\bar\ell,\sigma,i}\xi_{\bar\ell,\sigma,j}|\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})|^2. &(177)\end{aligned}$$

For $\sigma_1=\sigma$, and $\sigma_2=-\sigma=\bar\sigma$, combining (175) and (168) apart from a factor $4\pi^2(-1)^{r-1}\Delta_{ii}\Delta_{jj}$, we express $B_{ij}^{\sigma\bar\sigma}$ in (162) as

$$\begin{aligned}B_{ij}^{\sigma\bar\sigma} \propto\; &\Delta_{ii}\Delta_{jj}\xi_{\bar\ell,\sigma,i}\xi_{\bar\ell,\bar\sigma,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})+2\Delta_{ii}\xi_{\bar\ell,\sigma,i}\sigma\chi_\ell\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)-\\ &-2\Delta_{jj}\xi_{\bar\ell,\bar\sigma,j}\sigma\chi_\ell\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)\cdot\big(\mathbf{e}_i\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)-2\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_i\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)+\\ &+4\chi_\ell^2\big(\mathbf{e}_i\lrcorner\,\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\big)\cdot\big(\mathbf{e}_j\lrcorner\,\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)\big)+\\ &+\Delta_{ii}\Delta_{jj}\xi_{\bar\ell,\sigma,j}\xi_{\bar\sigma,i}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})+2\Delta_{jj}\xi_{\bar\ell,\sigma,j}\sigma\chi_\ell\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\cdot\big(\mathbf{e}_i\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)-\\ &-2\Delta_{ii}\xi_{\bar\sigma,i}\sigma\chi_\ell\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)-2\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\cdot\big(\mathbf{e}_i\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)+\\ &+4\chi_\ell^2\big(\mathbf{e}_j\lrcorner\,\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\big)\cdot\big(\mathbf{e}_i\lrcorner\,\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)\big)+\\ &-\Delta_{ij}\Big(-2\Delta_{\ell\ell}\chi_\ell^2\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)+4\chi_\ell^2\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\cdot\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)\Big).\end{aligned} \tag{178}$$

We now set $i=\ell$, and note that $\mathbf{e}_\ell\lrcorner\,\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)=0$, $\xi_{\bar\ell,\sigma,\ell}=\sigma\chi_\ell$, and $\xi_{\bar\ell,\bar\sigma,\ell}=-\sigma\chi_\ell$, to simplify (178) as

$$\begin{aligned}B_{ij}^{\sigma\bar\sigma} \propto\; &\Delta_{\ell\ell}\Delta_{jj}\sigma\chi_\ell\xi_{\bar\ell,\bar\sigma,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})+2\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)-\\ &-2\Delta_{jj}\xi_{\bar\ell,\bar\sigma,j}\sigma\chi_\ell\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)\cdot\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)-2\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)+\\ &-\Delta_{\ell\ell}\Delta_{jj}\xi_{\bar\ell,\sigma,j}\sigma\chi_\ell\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})+2\Delta_{jj}\xi_{\bar\ell,\sigma,j}\sigma\chi_\ell\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\cdot\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)-\\ &+2\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)\cdot\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)-2\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_j\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\cdot\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)+\\ &-\Delta_{\ell j}\Big(-2\Delta_{\ell\ell}\chi_\ell^2\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)+4\chi_\ell^2\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\cdot\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)\Big).\end{aligned} \tag{179}$$

Now, cancelling several common terms, Eq. (179) becomes

$$\begin{aligned}B_{ij}^{\sigma\bar\sigma} \propto\; &\Delta_{\ell\ell}\Delta_{jj}\sigma\chi_\ell(\xi_{\bar\ell,\bar\sigma,j}-\xi_{\bar\ell,\sigma,j})\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})-2\Delta_{jj}(\xi_{\bar\ell,\bar\sigma,j}-\xi_{\bar\ell,\sigma,j})\sigma\chi_\ell\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)\cdot\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)-\\ &-\Delta_{\ell j}\Big(-2\Delta_{\ell\ell}\chi_\ell^2\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)+4\chi_\ell^2\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})\big)\cdot\big(\mathbf{e}_\ell\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar\ell,\bar\sigma})\big)\Big).\end{aligned} \tag{180}$$

At this point, we need to distinguish two separate possibilities: $j\neq\ell$ and $j=\ell$. If $j\neq\ell$, it holds that $\xi_{\bar\ell,\sigma,j}=\xi_{\bar\ell,\bar\sigma,j}$ and $\Delta_{\ell j}=0$, and therefore (180) vanishes. When $j=\ell$, then $\xi_{\bar\ell,\sigma,j}=-\xi_{\bar\ell,\bar\sigma,j}=\sigma\chi_\ell$, and $\Delta_{\ell j}=\Delta_{\ell\ell}$ and (180) vanishes too. We conclude that $B_{\varepsilon(\ell,j)}^{\sigma\bar\sigma}=0$ if $\sigma\neq\bar\sigma$, regardless of the value of j. Moreover, the same steps (179)–(180) similarly prove that $B_{\varepsilon(\ell,i)}^{\sigma\bar\sigma}=0$ if $\sigma\neq\bar\sigma$ for any i.

Since the only nonzero contribution is given by $B_{ij}^{\sigma\sigma}$ in (177), substituting this latter equation in (159) yields

$$B_{\varepsilon(\ell,j)} = 8\pi^2(-1)^{r-1}\chi_\ell\sum_{\sigma\in\mathcal{S}}\sigma\,\xi_{\bar\ell,\sigma,j}|\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,\sigma})|^2. \tag{181}$$

B.3 Evaluation of the Tensor Components $B_\varepsilon(\ell, j)$ for $\mathcal{I}_{t,1}$ (Coulomb-ℓ Gauge)

As this section follows similar steps to those in the previous one, the presentation is streamlined somewhat.

Substituting the expressions for $\hat{\mathbf{F}}^\ell(\boldsymbol{\xi}_{\bar{\ell},\sigma})$ and $\hat{\mathbf{F}}^\ell(-\boldsymbol{\xi}_{\bar{\ell},\sigma})$, respectively, given in (71) and (158) in the tensor definition (134), we obtain

$$\mathbf{B} = \sum_{\sigma_1,\sigma_2\in\mathcal{S}} e^{j2\pi\Delta_{\ell\ell}(\sigma_1-\sigma_2)\chi_\ell x_\ell}\mathbf{B}^{\sigma_1\sigma_2}, \tag{182}$$

where the rank-2 symmetric tensor $\mathbf{B}^{\sigma_1\sigma_2}$ is defined as

$$\mathbf{B}^{\sigma_1\sigma_2} = \big(\partial_{\xi_t}\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big) \odot \hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}) + \big(\partial_{\xi_t}\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big) \owedge \hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}). \tag{183}$$

Following the same steps as in (161)–(162), the (i,j)-th component $B_{ij}^{\sigma_1\sigma_2}$ is given by

$$B_{ij}^{\sigma_1\sigma_2} = (-1)^{r-1}\Delta_{ii}\Delta_{jj}\Big(\alpha_{ij}^{\sigma_1\sigma_2} + \alpha_{ji}^{\sigma_1\sigma_2} - \Delta_{ij}\big(\partial_{\xi_t}\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\Big), \tag{184}$$

where $\alpha_{ij}^{\sigma_1\sigma_2}$, and similarly $\alpha_{ji}^{\sigma_1\sigma_2}$, is given by

$$\alpha_{ij}^{\sigma_1\sigma_2} = \Big(\mathbf{e}_i \,\lrcorner\, \big(\partial_{\xi_t}\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\Big)\cdot\Big(\mathbf{e}_j \,\lrcorner\, \hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\Big). \tag{185}$$

Substituting the potential in the Fourier domain, we obtain

$$\partial_{\xi_t}\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}) = j2\pi\mathbf{e}_t\wedge\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}) + j2\pi\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big). \tag{186}$$

Using this identity together with (154) allows us to write the last term in (184), apart from a factor $4\pi^2$, as

$$\big(\partial_{\xi_t}\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}) \propto \big(\mathbf{e}_t\wedge\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\wedge\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big) + \big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\wedge\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big) \tag{187}$$

$$\begin{aligned}&= (\mathbf{e}_t\cdot\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big) - \big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\big(\mathbf{e}_t\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)+\\ &\quad + (\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\cdot\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big(\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big) - \big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\,\lrcorner\,\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big).\end{aligned} \tag{188}$$

In the Coulomb-ℓ-gauge, for which $\mathbf{e}_\ell\,\lrcorner\,\hat{\mathbf{A}} = 0$, the conditions (166) and (167) become

$$\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}) = 0, \tag{189}$$

$$\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}) = 0. \tag{190}$$

Substituting the wave equation condition (165) together with (189)–(190) into (188) gives, apart from a factor $4\pi^2$,

$$\big(\partial_{\xi_t}\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\hat{\mathbf{F}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}) \propto \Delta_{tt}\xi_{\bar{\ell},\sigma_2,t}\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big) + (\sigma_1\sigma_2-1)\Delta_{\ell\ell}\chi_\ell^2\big(\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big). \tag{191}$$

With a similar substitution of the potential in the Fourier domain followed by (153), we write

$$\begin{aligned}\mathbf{e}_i\,\lrcorner\,\big(\partial_{\xi_t}\hat{\mathbf{F}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big) &= j2\pi\Big(\mathbf{e}_i\,\lrcorner\,\big(\mathbf{e}_t\wedge\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big) + \mathbf{e}_i\,\lrcorner\,\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\Big) &(192)\\ &= j2\pi\Big((-1)^{r-1}(\mathbf{e}_i\cdot\mathbf{e}_t)\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}) + \mathbf{e}_t\wedge\big(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)+\\ &\qquad + (-1)^{r-1}\big(\mathbf{e}_i\cdot\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\big)\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big) + \boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\mathbf{e}_i\,\lrcorner\,\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\Big) &(193)\\ &= -j2\pi\Big((-1)^r\Delta_{it}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}) - \mathbf{e}_t\wedge\big(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)+\\ &\qquad + (-1)^r\Delta_{ii}\xi_{\bar{\ell},\sigma_1,i}\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big) - \boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\mathbf{e}_i\,\lrcorner\,\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\Big). &(194)\end{aligned}$$

Combining (194) with (170), we therefore have for $\alpha_{ij}^{\sigma_1\sigma_2}$ (and similarly for $\alpha_{ji}^{\sigma_1\sigma_2}$), apart from a factor $4\pi^2$,

$$\begin{aligned}\alpha_{ij}^{\sigma_1\sigma_2} &\propto \Big((-1)^r\Delta_{it}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}) - \mathbf{e}_t\wedge\big(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big) + (-1)^r\Delta_{ii}\xi_{\bar{\ell},\sigma_1,i}\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big) - \boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\mathbf{e}_i\,\lrcorner\,\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\Big)\cdot\\ &\quad\cdot\Big((-1)^r\Delta_{jj}\xi_{\bar{\ell},\sigma_2,j}\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}) - \big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\wedge\big(\mathbf{e}_j\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\big)\Big) &(195)\\ &= \Delta_{it}\Delta_{jj}\xi_{\bar{\ell},\sigma_2,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}) - (-1)^r\Delta_{it}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\wedge\big(\mathbf{e}_j\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\big)-\\ &\quad - (-1)^r\Delta_{jj}\xi_{\bar{\ell},\sigma_2,j}\big(\mathbf{e}_t\wedge\big(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}) + \big(\mathbf{e}_t\wedge\big(\mathbf{e}_i\,\lrcorner\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\wedge\big(\mathbf{e}_j\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\big)+\\ &\quad + \Delta_{ii}\Delta_{jj}\xi_{\bar{\ell},\sigma_1,i}\xi_{\bar{\ell},\sigma_2,j}\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}) - (-1)^r\Delta_{ii}\xi_{\bar{\ell},\sigma_1,i}\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\wedge\big(\mathbf{e}_j\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\big)-\\ &\quad - (-1)^r\Delta_{jj}\xi_{\bar{\ell},\sigma_2,j}\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\mathbf{e}_i\,\lrcorner\,\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})+\\ &\quad + \big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\wedge\big(\mathbf{e}_i\,\lrcorner\,\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\big)\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\wedge\big(\mathbf{e}_j\,\lrcorner\,\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\big). &(196)\end{aligned}$$

Using (155) in the second, third, sixth, and seventh summands and (154) in the fourth and eighth ones, together with (152) to swap the order of the interior products between $\boldsymbol{\xi}_{\bar{\ell},\sigma_2}$ and $\mathbf{e}_i$, $\boldsymbol{\xi}_{\bar{\ell},\sigma_1}$ and $\mathbf{e}_j$, and $\mathbf{e}_t$ and $\mathbf{e}_j$, we obtain

$$\begin{aligned}\alpha_{ij}^{\sigma_1\sigma_2} \propto{}& \Delta_{it}\Delta_{jj}\xi_{\bar{\ell},\sigma_2,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}) - \Delta_{it}\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)-\\ &-\Delta_{jj}\xi_{\bar{\ell},\sigma_2,j}\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\cdot\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)+\big(\mathbf{e}_t\cdot\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\big)\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)-\\ &-\big(\mathbf{e}_j\lrcorner\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\big)\cdot\big(\mathbf{e}_i\lrcorner\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)+\Delta_{ii}\Delta_{jj}\xi_{\bar{\ell},\sigma_1,i}\xi_{\bar{\ell},\sigma_2,j}\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})-\\ &-\Delta_{ii}\xi_{\bar{\ell},\sigma_1,i}\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\cdot\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\lrcorner\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)-\Delta_{jj}\xi_{\bar{\ell},\sigma_2,j}\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\big)\cdot\big(\mathbf{e}_i\lrcorner\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)+\\ &+\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\cdot\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\big)\big(\mathbf{e}_i\lrcorner\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\cdot\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)-\\ &-\big(\mathbf{e}_j\lrcorner\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\big)\big)\cdot\big(\mathbf{e}_i\lrcorner\big(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}\lrcorner\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\big).\end{aligned}\tag{197}$$

By taking the derivative of (189) with respect to ξ_t, with $t \neq \ell$, we have

$$\boldsymbol{\xi}_{\bar{\ell},\sigma_1}\lrcorner\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big) = -\mathbf{e}_t\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}).\tag{198}$$

Substituting (165), (198), and (189)–(190) back into (197), this equation simplifies to

$$\begin{aligned}\alpha_{ij}^{\sigma_1\sigma_2} \propto{}& \Delta_{it}\Delta_{jj}\xi_{\bar{\ell},\sigma_2,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})+(\sigma_1\sigma_2-1)\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_i\lrcorner\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\big)\cdot\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)+\\ &+\Delta_{ii}\xi_{\bar{\ell},\sigma_1,i}\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\cdot\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)-\Delta_{jj}\xi_{\bar{\ell},\sigma_2,j}\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)\cdot\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)+\\ &+\Delta_{tt}\xi_t\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2})\big)+\Delta_{ii}\Delta_{jj}\xi_{\bar{\ell},\sigma_1,i}\xi_{\bar{\ell},\sigma_2,j}\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma_1})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma_2}).\end{aligned}\tag{199}$$

As in the analysis of the tensor components for I_ℓ and $\boldsymbol{\mathcal{I}}_{t,0}$ in B.2, we continue our evaluation of $B_{ij}^{\sigma_1\sigma_2}$ by considering separately the cases $\sigma_1 = \sigma_2$ and $\sigma_1 \neq \sigma_2$. First, for $\sigma_1 = \sigma_2 = \sigma$, combining (199) for $\alpha_{ij}^{\sigma_1\sigma_2}$ and $\alpha_{ji}^{\sigma_1\sigma_2}$ with (191) and (198), we write the tensor component $B_{ij}^{\sigma\sigma}$ in (184) apart from a factor $4\pi^2(-1)^{r-1}\Delta_{ii}\Delta_{jj}$ as

$$\begin{aligned}B_{ij}^{\sigma\sigma} \propto{}& \Delta_{it}\Delta_{jj}\xi_{\bar{\ell},\sigma,j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})+\Delta_{jt}\Delta_{ii}\xi_{\bar{\ell},\sigma,i}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})+\\ &+\Delta_{ii}\xi_{\bar{\ell},\sigma,i}\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)-\Delta_{jj}\xi_{\bar{\ell},\sigma,j}\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)+\\ &+\Delta_{jj}\xi_{\bar{\ell},\sigma,j}\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)-\Delta_{ii}\xi_{\bar{\ell},\sigma,i}\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)-\\ &+\Delta_{tt}\xi_t\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)+\Delta_{tt}\xi_t\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)+\\ &+2\Delta_{ii}\Delta_{jj}\xi_{\bar{\ell},\sigma,i}\xi_{\bar{\ell},\sigma,j}\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})-\Delta_{ij}\Delta_{tt}\xi_t\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big).\end{aligned}\tag{200}$$

Evaluating (200) for $i = j = \ell$, and noting that $t \neq \ell$, gives

$$B_{\ell\ell}^{\sigma\sigma} = 4\pi^2(-1)^r\Big(\Delta_{\ell\ell}\Delta_{tt}\xi_t\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big|^2-2\chi_\ell^2\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\Big).\tag{201}$$

Similarly, evaluating (200) for $i = \ell$ and $j \neq \ell$, and noting that $t \neq \ell$ and $j \neq t$, gives

$$\begin{aligned}B_{\varepsilon(\ell,j)}^{\sigma\sigma} &= 4\pi^2(-1)^{r-1}\Delta_{jj}\Big(\sigma\chi_\ell\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)-\sigma\chi_\ell\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)+\\ &\qquad+2\Delta_{jj}\sigma\chi_\ell\xi_j\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\Big) &(202)\\ &= 4\pi^2\Big(\Delta_{tt}\sigma\chi_\ell\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\big|_{tj}-\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\big|_{jt}\Big)-2(-1)^r\sigma\chi_\ell\xi_j\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\Big), &(203)\end{aligned}$$

where we have used the definition of the $\odot$ product in (11) and the identity $\mathbf{e}_i\lrcorner\mathbf{A} = (-1)^r\mathbf{A}\llcorner\mathbf{e}_i$ [1, Eq. (21)].

For $-\sigma_2 = \sigma_1 = \sigma$, combining (199) for $\alpha_{ij}^{\sigma\bar{\sigma}}$ and $\alpha_{ji}^{\sigma\bar{\sigma}}$ with (191), we can write the tensor component $B_{ij}^{\sigma\bar{\sigma}}$ in (184) apart from a factor $4\pi^2(-1)^{r-1}\Delta_{ii}\Delta_{jj}$ as

$$\begin{aligned}B_{ij}^{\sigma\bar{\sigma}} \propto{}& \Delta_{it}\Delta_{jj}\xi_{\bar{\ell},\bar{\sigma},j}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})-2\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_i\lrcorner\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\big)\cdot\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)+\\ &+\Delta_{ii}\xi_{\bar{\ell},\sigma,i}\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\cdot\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)-\Delta_{jj}\xi_{\bar{\ell},\bar{\sigma},j}\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\cdot\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)-\\ &+\Delta_{tt}\xi_t\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)+\Delta_{ii}\Delta_{jj}\xi_{\bar{\ell},\sigma,i}\xi_{\bar{\ell},\bar{\sigma},j}\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})+\\ &+\Delta_{jt}\Delta_{ii}\xi_{\bar{\ell},\bar{\sigma},i}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})-2\Delta_{\ell\ell}\chi_\ell^2\big(\mathbf{e}_j\lrcorner\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\big)\cdot\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)+\\ &+\Delta_{jj}\xi_{\bar{\ell},\sigma,j}\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\cdot\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)-\Delta_{ii}\xi_{\bar{\ell},\bar{\sigma},i}\big(\mathbf{e}_t\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\cdot\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)-\\ &+\Delta_{tt}\xi_t\big(\mathbf{e}_j\lrcorner\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\big(\mathbf{e}_i\lrcorner\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)+\Delta_{ii}\Delta_{jj}\xi_{\bar{\ell},\sigma,j}\xi_{\bar{\ell},\bar{\sigma},i}\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})-\\ &-\Delta_{ij}\Delta_{tt}\xi_t\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)+2\Delta_{ij}\Delta_{\ell\ell}\chi_\ell^2\big(\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big).\end{aligned}\tag{204}$$

For $i=\ell$, using that $t\neq\ell$, the Coulomb-ℓ-gauge condition $\mathbf{e}_\ell \lrcorner \hat{\mathbf{A}}$ and its consequence $\mathbf{e}_\ell \lrcorner (\partial_{\xi_t}\hat{\mathbf{A}})=0$, yield

$$\begin{aligned}B^{\sigma\bar{\sigma}}_{\varepsilon(\ell,j)} \propto{}& \Delta_{\ell\ell}\sigma\chi_\ell\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\cdot\big(\mathbf{e}_t \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)+\Delta_{\ell\ell}\sigma\chi_\ell\big(\mathbf{e}_t \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\cdot\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)+\\&+\Delta_{\ell\ell}\Delta_{jj}\sigma\chi_\ell\xi_{\bar{\ell},\bar{\sigma},j}\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})-\Delta_{\ell\ell}\Delta_{jj}\xi_{\bar{\ell},\sigma,j}\sigma\chi_\ell\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})-\\&-\Delta_{jt}\Delta_{\ell\ell}\sigma\chi_\ell\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})-\Delta_{\ell j}\Delta_{tt}\xi_t\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)+2\Delta_{\ell j}\Delta_{\ell\ell}\chi_\ell^2\big(\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big).\end{aligned} \tag{205}$$

If we also consider $j=\ell$, using again the Coulomb-ℓ-gauge condition and that $t\neq\ell$ in (205) gives

$$B^{\sigma\bar{\sigma}}_{\ell\ell} = 4\pi^2(-1)^r\Delta_{\ell\ell}\Delta_{jj}\Big(2\chi_\ell^2\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})+\Delta_{\ell\ell}\Delta_{tt}\xi_t\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)-2\chi_\ell^2\big(\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\Big) \tag{206}$$

$$= 4\pi^2(-1)^r\Delta_{\ell\ell}\Delta_{tt}\xi_t\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big). \tag{207}$$

For $j\neq\ell$, noting that $t\neq\ell$ and $j\neq t$, we evaluate (205) as

$$\begin{aligned}B^{\sigma\bar{\sigma}}_{\varepsilon(\ell,j)} ={}& 4\pi^2(-1)^{r-1}\Delta_{\ell\ell}\Delta_{jj}\Big(\Delta_{\ell\ell}\sigma\chi_\ell\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\cdot\big(\mathbf{e}_t \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)+\Delta_{\ell\ell}\sigma\chi_\ell\big(\mathbf{e}_t \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\cdot\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)+\\&+\Delta_{\ell\ell}\Delta_{jj}\sigma\chi_\ell\xi_j\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})-\Delta_{\ell\ell}\Delta_{jj}\xi_j\sigma\chi_\ell\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\Big)\end{aligned} \tag{208}$$

$$= 4\pi^2(-1)^{r-1}\Delta_{jj}\Big(\sigma\chi_\ell\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\cdot\big(\mathbf{e}_t \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)+\sigma\chi_\ell\big(\mathbf{e}_t \lrcorner \hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\cdot\big(\mathbf{e}_j \lrcorner \hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\Big) \tag{209}$$

$$= -4\pi^2\Delta_{tt}\sigma\chi_\ell\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\big|_{jt}+\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\big|_{tj}\Big), \tag{210}$$

where we have used the definition of the $\odot$ product in (11) and the identity $\mathbf{e}_i \lrcorner \mathbf{A} = (-1)^r\mathbf{A} \llcorner \mathbf{e}_i$ [1, Eq. (21)]. Note that the product $\odot$ could be replaced by $\oslash$ in (76) with an overall change of sign, since the off-diagonal transposed components of both products coincide [1, Eq. (22)].

Finally, using (159) and, respectively, combining (201) and (207), and (203) and (210), we obtain

$$B_{\ell\ell} = 4\pi^2(-1)^r\sum_{\sigma\in\mathcal{S}}\Big(\Delta_{\ell\ell}\Delta_{tt}\xi_t\big|\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big|^2-2\chi_\ell^2\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})+e^{j4\pi\Delta_{\ell\ell}\sigma\chi_\ell x_\ell}\Delta_{\ell\ell}\Delta_{tt}\xi_t\big(\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\big)\Big), \tag{211}$$

and

$$\begin{aligned}B_{\varepsilon(\ell,j)} = 4\pi^2\sum_{\sigma\in\mathcal{S}}\Big(&\Delta_{tt}\sigma\chi_\ell\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\big|_{tj}-\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\big|_{jt}\Big)-2(-1)^r\sigma\chi_\ell\xi_j\big(\partial_{\xi_t}\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\cdot\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\sigma})\Big)\\&-e^{j4\pi\Delta_{\ell\ell}\sigma\chi_\ell x_\ell}\Delta_{tt}\sigma\chi_\ell\Big(\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\big|_{jt}+\big(\hat{\mathbf{A}}^*(\boldsymbol{\xi}_{\bar{\ell},\bar{\sigma}})\odot\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar{\ell},\sigma})\big)\big|_{tj}\Big).\end{aligned} \tag{212}$$

C Spin Components: "Canonical" Analysis

For the standard electromagnetic field, the intrinsic angular momentum is defined only for the spatial components of the angular momentum bivector $\boldsymbol{\Omega}^\ell_{\boldsymbol{\alpha}}$. For the sake of simplicity, let $\boldsymbol{\alpha}=0$. For generic ℓ, we study thus the components Ω^ℓ_I, with $I\in\mathcal{I}_2$, that do not include ℓ, i. e. $\ell\notin I$. From (64), and writing $I=(i,j)$, with $i,j\notin\ell$, we have to evaluate the following integral:

$$\Omega^\ell_I = \sigma(\ell,\ell^c)\int_{\mathbf{R}^{k+n-1}}\mathrm{d}x_{\ell^c}\big(x_iT_{\varepsilon(\ell,j)}-x_jT_{\varepsilon(\ell,i)}\big). \tag{213}$$

The following analysis is inspired by Sections 12 and 16 of Wentzel's book [20], which describe how to obtain the spin components from the canonical stress-energy-momentum tensor. Our analysis bypasses however the canonical tensor, and the appropriate adaptations have been made. Using the expression of the non-diagonal components of the stress-energy-momentum tensor in (15) in the integrand in (213) gives

$$-x_i\sum_{\bar{L}\in\mathcal{I}_{r-1}:\ell,j\notin\bar{L}}\Delta_{\bar{L}\bar{L}}\sigma(\bar{L},\ell)\sigma(j,\bar{L})F_{\varepsilon(\ell,\bar{L})}F_{\varepsilon(j,\bar{L})}+x_j\sum_{\bar{L}\in\mathcal{I}_{r-1}:\ell,i\notin\bar{L}}\Delta_{\bar{L}\bar{L}}\sigma(\bar{L},\ell)\sigma(i,\bar{L})F_{\varepsilon(\ell,\bar{L})}F_{\varepsilon(i,\bar{L})}. \tag{214}$$

Let us split the summations over $\bar{L}$ into the cases where i (resp. j) belongs to $\bar{L}$ and those where it does not, and focus of the former. When i (resp. j) belongs to $\bar{L}$, we may define L as a set in $\mathcal{I}_{r-2}$ such that $\ell,i,j\notin L$, so that the original set $\bar{L}$ is now given by $L\cup i$ (resp. $L\cup j$). We rewrite (214) accordingly as

$$-\sum_{L\in\mathcal{I}_{r-2}:\ell,i,j\notin L}x_i\Delta_{ii}\Delta_{LL}\sigma\big(\varepsilon(i,L),\ell\big)\sigma\big(j,\varepsilon(i,L)\big)\big)F_{\varepsilon(\ell,i,L)}F_{\varepsilon(i,j,L)}+\sum_{L\in\mathcal{I}_{r-2}:\ell,i,j\notin L}x_j\Delta_{jj}\Delta_{LL}\sigma\big(\varepsilon(j,L),\ell\big)\sigma\big(i,\varepsilon(j,L)\big)F_{\varepsilon(\ell,j,L)}F_{\varepsilon(i,j,L)} \tag{215}$$

$$=-\sum_{L\in\mathcal{I}_{r-2}:\ell,i,j\notin L}\Delta_{LL}\Big(\Delta_{ii}\sigma\big(\varepsilon(i,L),\ell\big)\sigma\big(j,\varepsilon(L,i)\big)x_iF_{\varepsilon(\ell,i,L)}F_{\varepsilon(i,j,L)}-\Delta_{jj}\sigma\big(\varepsilon(L,j),\ell\big)\sigma\big(i,\varepsilon(L,j)\big)x_jF_{\varepsilon(\ell,j,L)}F_{\varepsilon(i,j,L)}\Big). \tag{216}$$

From the definition of the field from the potential, we have

$$F_{\varepsilon(i,j,L)} = \Delta_{ii}\sigma\big(i,\varepsilon(j,L)\big)\partial_i A_{\varepsilon(j,L)} + \dots \tag{217}$$

$$F_{\varepsilon(i,j,L)} = \Delta_{jj}\sigma\big(j,\varepsilon(i,L)\big)\partial_j A_{\varepsilon(i,L)} + \dots . \tag{218}$$

Ignoring the terms in the dots, that do not contribute to the spin, and, respectively, substituting these two expressions in the two appearances of $F_{\varepsilon(i,j,L)}$ in (216) gives the following expression for each summand

$$-\Delta_{LL}\sigma\big(j,\varepsilon(L,i)\big)\sigma\big(i,\varepsilon(j,L)\big)\Big(\sigma\big(\varepsilon(i,L),\ell\big)x_i F_{\varepsilon(\ell,i,L)}\partial_i A_{\varepsilon(j,L)} - \sigma\big(\varepsilon(L,j),\ell\big)x_j F_{\varepsilon(\ell,j,L)}\partial_j A_{\varepsilon(i,L)}\Big). \tag{219}$$

We continue with the following manipulations,

$$x_i F_{\varepsilon(\ell,i,L)}\partial_i A_{\varepsilon(j,L)} = \partial_i\big(x_i A_{\varepsilon(j,L)} F_{\varepsilon(\ell,i,L)}\big) - A_{\varepsilon(j,L)} F_{\varepsilon(\ell,i,L)} - x_i A_{\varepsilon(j,L)}\partial_i F_{\varepsilon(\ell,i,L)}. \tag{220}$$

A similar expression holds for the second summand in (219). If we now neglect the third summand, as unrelated to the spin, and argue that the first is zero after integration in (220), substituting these back in (219) yields

$$\Delta_{LL}\sigma\big(j,\varepsilon(L,i)\big)\sigma\big(i,\varepsilon(j,L)\big)\Big(\sigma\big(\varepsilon(i,L),\ell\big)A_{\varepsilon(j,L)}F_{\varepsilon(\ell,i,L)} - \sigma\big(\varepsilon(L,j),\ell\big)A_{\varepsilon(i,L)}F_{\varepsilon(\ell,j,L)}\Big). \tag{221}$$

In the Coulomb-ℓ-gauge, we also have that

$$F_{\varepsilon(\ell,i,L)} = \Delta_{\ell\ell}\sigma\big(\ell,\varepsilon(i,L)\big)\partial_\ell A_{\varepsilon(i,L)}, \tag{222}$$

$$F_{\varepsilon(\ell,j,L)} = \Delta_{\ell\ell}\sigma\big(\ell,\varepsilon(j,L)\big)\partial_\ell A_{\varepsilon(j,L)}, \tag{223}$$

and substituting these expressions in (221), and using that $\sigma\big(\varepsilon(i,L),\ell\big)\sigma\big(\ell,\varepsilon(i,L)\big) = (-1)^{r-1}$, gives

$$\Delta_{\ell\ell}\Delta_{LL}\sigma\big(j,\varepsilon(L,i)\big)\sigma\big(i,\varepsilon(j,L)\big)\Big(\sigma\big(\varepsilon(i,L),\ell\big)\sigma\big(\ell,\varepsilon(i,L)\big)A_{\varepsilon(j,L)}\partial_\ell A_{\varepsilon(i,L)} - \sigma\big(\varepsilon(L,j),\ell\big)\sigma\big(\ell,\varepsilon(j,L)\big)A_{\varepsilon(i,L)}\partial_\ell A_{\varepsilon(j,L)}\Big) = \tag{224}$$

$$= (-1)^{r-1}\Delta_{\ell\ell}\Delta_{LL}\sigma\big(j,\varepsilon(L,i)\big)\sigma\big(i,\varepsilon(j,L)\big)\big(A_{\varepsilon(j,L)}\partial_\ell A_{\varepsilon(i,L)} - A_{\varepsilon(i,L)}\partial_\ell A_{\varepsilon(j,L)}\big) \tag{225}$$

$$= \Delta_{\ell\ell}\Delta_{LL}\sigma(L,i)\sigma(j,L)\big(A_{\varepsilon(j,L)}\partial_\ell A_{\varepsilon(i,L)} - A_{\varepsilon(i,L)}\partial_\ell A_{\varepsilon(j,L)}\big), \tag{226}$$

where we have used that $-(-1)^r\sigma\big(j,\varepsilon(i,L)\big)\sigma\big(i,\varepsilon(j,L)\big) = \sigma(L,i)\sigma(j,L)$. Indeed, using [7, Appendix A], we have $\sigma(L,i)\sigma(j,L) = \sigma\big(i,\varepsilon(j,L)\big)\sigma\big(\varepsilon(i,L),j\big)$ and it also holds that $(-1)^{r-1}\sigma\big(j,\varepsilon(i,L)\big) = \sigma\big(\varepsilon(i,L),j\big)$.

Getting back to the integral in (213), the ℓ-spin component S^ℓ_{ij} is then given by

$$S^\ell_{ij} = -\Delta_{\ell\ell}\sigma(\ell,\ell^c)\sum_{L\in\mathcal{I}_{r-2}:\ell,i,j\notin L}\Delta_{LL}\sigma(L,i)\sigma(j,L)\int_{\mathbf{R}^{k+n-1}}\mathrm{d}x_{\ell^c}\Big(A_{\varepsilon(i,L)}\partial_\ell A_{\varepsilon(j,L)} - A_{\varepsilon(j,L)}\partial_\ell A_{\varepsilon(i,L)}\Big). \tag{227}$$

For classical electromagnetism with $r = 2$, $k = 1$, $n = 3$, and $\ell = 0$, we have $L = \varnothing$, $\sigma(i,j)\sigma(j,i) = -1$, and the spin components are given by the standard formula (e. g. [10, Eq. (4.83)])

$$\int_{\mathbf{R}^3}\mathrm{d}x_{123}\big(A_i\partial_0 A_j - A_j\partial_0 A_i\big). \tag{228}$$

Continuing with the integral in (227), we express the potentials in terms of their normal-mode decomposition:

$$\mathbf{A}(\mathbf{x}) = \int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}e^{j2\pi\boldsymbol{\xi}_{\bar\ell}\cdot\mathbf{x}_{\bar\ell}}\big(e^{j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) + e^{-j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})\big) \tag{229}$$

$$\Delta_{\ell\ell}\partial_\ell\mathbf{A}(\mathbf{x}) = \int_{\boldsymbol{\Xi}_\ell} j2\pi\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}e^{j2\pi\boldsymbol{\xi}_{\bar\ell}\cdot\mathbf{x}_{\bar\ell}}\big(\chi_\ell e^{j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) - \chi_\ell e^{-j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-})\big), \tag{230}$$

where $\boldsymbol{\xi}_{\bar\ell} = \boldsymbol{\xi} - \boldsymbol{\xi}_\ell\mathbf{e}_\ell$ and similarly $\mathbf{x}_{\bar\ell} = \mathbf{x} - \mathbf{x}_\ell\mathbf{e}_\ell$, $\chi_\ell = +\sqrt{-\Delta_{\ell\ell}\boldsymbol{\xi}_{\bar\ell}\cdot\boldsymbol{\xi}_{\bar\ell}}$ and $\boldsymbol{\xi}_{\bar\ell,\pm} = \boldsymbol{\xi}_{\bar\ell} \pm \chi_\ell\mathbf{e}_\ell$. Writing

$$\hat{\mathbf{A}}^{\ell,\pm}(\boldsymbol{\xi}_{\bar\ell}) = e^{j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,+}) \pm e^{-j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{\mathbf{A}}(\boldsymbol{\xi}_{\bar\ell,-}), \tag{231}$$

we may thus evaluate the integral by using a multidimensional Dirac function as

$$\Delta_{\ell\ell}\int_{\mathbf{R}^{k+n-1}}\mathrm{d}x_{\ell^c}A_{\varepsilon(i,L)}\partial_\ell A_{\varepsilon(j,L)} = j2\pi\int_{\boldsymbol{\Xi}_\ell\times\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}\,\mathrm{d}\xi'_{\ell^c}}{4\chi_\ell\chi'_\ell}\chi'_\ell\int_{\mathbf{R}^{k+n-1}}\mathrm{d}x_{\ell^c}e^{j2\pi(\boldsymbol{\xi}_{\bar\ell}+\boldsymbol{\xi}'_{\bar\ell})\cdot\mathbf{x}_{\bar\ell}}\hat{A}^{\ell,+}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar\ell})\hat{A}^{\ell,-}_{\varepsilon(j,L)}(\boldsymbol{\xi}'_{\bar\ell}), \tag{232}$$

$$= j\pi\int_{\boldsymbol{\Xi}_\ell\times\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}\,\mathrm{d}\xi'_{\ell^c}}{2\chi_\ell}\delta(\boldsymbol{\xi}_{\bar\ell}+\boldsymbol{\xi}'_{\bar\ell})\hat{A}^{\ell,+}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar\ell})\hat{A}^{\ell,-}_{\varepsilon(j,L)}(\boldsymbol{\xi}'_{\bar\ell}) \tag{233}$$

$$= j\pi\int_{\boldsymbol{\Xi}_\ell}\frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell}\,\hat{A}^{\ell,+}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar\ell})\hat{A}^{\ell,-}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar\ell}). \tag{234}$$

Expanding the Fourier components of the potential in (234) yields

$$\hat{A}^{\ell,+}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell}}) = e^{j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},+}) + e^{-j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},-}), \tag{235}$$

and similarly

$$\hat{A}^{\ell,-}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell}}) = e^{j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell}} + \chi_\ell\mathbf{e}_\ell) - e^{-j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell}} - \chi_\ell\mathbf{e}_\ell) \tag{236}$$

$$= e^{j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},-}) - e^{-j2\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},+}). \tag{237}$$

Taking the product of (235) and (237) yields

$$\begin{aligned}&e^{j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},+})\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},-}) - \hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},+})\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},+}) +\\ &\quad + \hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},-})\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},-}) - e^{-j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},-})\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},+}).\end{aligned} \tag{238}$$

Proceeding analogously with the second summand in (234), $A_{\varepsilon(j,L)}\partial_\ell A_{\varepsilon(i,L)}$, gives

$$\begin{aligned}&e^{j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(j,L)}(\boldsymbol{\xi}_{\bar{\ell},+})\,\hat{A}_{\varepsilon(i,L)}(-\boldsymbol{\xi}_{\bar{\ell},-}) - \hat{A}_{\varepsilon(j,L)}(\boldsymbol{\xi}_{\bar{\ell},+})\,\hat{A}_{\varepsilon(i,L)}(-\boldsymbol{\xi}_{\bar{\ell},+}) +\\ &\quad + \hat{A}_{\varepsilon(j,L)}(\boldsymbol{\xi}_{\bar{\ell},-})\,\hat{A}_{\varepsilon(i,L)}(-\boldsymbol{\xi}_{\bar{\ell},-}) - e^{-j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(j,L)}(\boldsymbol{\xi}_{\bar{\ell},-})\,\hat{A}_{\varepsilon(i,L)}(-\boldsymbol{\xi}_{\bar{\ell},+}).\end{aligned} \tag{239}$$

With the change of variable $\boldsymbol{\xi}_{\bar{\ell}} \to -\boldsymbol{\xi}_{\bar{\ell}}$, and noting that $-\boldsymbol{\xi}_{\bar{\ell},\pm} = -(\boldsymbol{\xi}_{\bar{\ell}} \pm \chi_\ell\mathbf{e}_\ell) = -\boldsymbol{\xi}_{\bar{\ell}} \mp \chi_\ell\mathbf{e}_\ell \to +\boldsymbol{\xi}_{\bar{\ell},\mp}$, we thus have

$$\begin{aligned}&e^{j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},-})\,\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},+}) - \hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},-})\,\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},-}) +\\ &\quad + \hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},+})\,\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},+}) - e^{-j4\pi\Delta_{\ell\ell}\chi_\ell x_\ell}\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},+})\,\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},-}).\end{aligned} \tag{240}$$

Combining (238) and (240) with its corresponding -1 sign, cancelling common terms (assuming that the relevant quantities commute), and grouping common terms yields the following,

$$-2\big(\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},+})\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},+}) - \hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},-})\,\hat{A}_{\varepsilon(j,L)}(-\boldsymbol{\xi}_{\bar{\ell},-})\big) = -2\big(\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},+})\,\hat{A}^*_{\varepsilon(j,L)}(\boldsymbol{\xi}_{\bar{\ell},+}) - \hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},-})\,\hat{A}^*_{\varepsilon(j,L)}(\boldsymbol{\xi}_{\bar{\ell},-})\big). \tag{241}$$

With the change of variable $\boldsymbol{\xi}_{\bar{\ell}} \to -\boldsymbol{\xi}_{\bar{\ell}}$, and noting again that $\boldsymbol{\xi}_{\bar{\ell},\pm} \to -\boldsymbol{\xi}_{\bar{\ell},\mp}$, we thus have as final result

$$-2\big(\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},+})\,\hat{A}^*_{\varepsilon(j,L)}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\big). \tag{242}$$

Putting this equation back into (234) and then into (227) gives

$$S^\ell_{ij} = j2\pi\sigma(\ell,\ell^c) \sum_{L\in\mathcal{I}_{r-2}:\ell,i,j\notin L} \Delta_{LL}\sigma(L,i)\sigma(j,L) \int_{\boldsymbol{\Xi}_\ell} \frac{\mathrm{d}\xi_{\ell^c}}{2\chi_\ell} \big(\hat{A}_{\varepsilon(i,L)}(\boldsymbol{\xi}_{\bar{\ell},+})\,\hat{A}^*_{\varepsilon(j,L)}(\boldsymbol{\xi}_{\bar{\ell},+}) - \mathrm{cc}\big). \tag{243}$$

The (i,j)-th spin component S^ℓ_{ij} coincides with the corresponding bivector component (79).

Bibliography

[1] I. Colombaro, J. Font-Segura, A. Martinez, European Physical Journal Plus **135**(305) (2020)

[2] I. Colombaro, J. Font-Segura, A. Martinez, Mathematics **7**(6), 564 (2019)

[3] L.D. Landau, E.M. Lifshitz, The Classical Theory of Fields, Course of Theoretical Physics, vol. 2, 4th edn. (Butterworth-Heinemann, 1987)

[4] J.D. Jackson, Classical Electrodynamics (Wiley, 1999)

[5] R.P. Feynman, R.B. Leighton, M. Sands, The Feynman Lectures on Physics, Vol. II: Mainly Electromagnetism and Matter (Addison-Wesley, 1977)

[6] I. Colombaro, J. Font-Segura, A. Martinez, Mathematics **9**(18), 2178 (2021)

[7] A. Martinez, J. Font-Segura, I. Colombaro, European Physical Journal Plus **136**(212) (2021)

[8] M. Forger, H. Römer, Annals of Physics **309**(2), 306 (2004)

[9] D.N. Blaschke, F. Gieres, M. Reboud, M. Schweda, Nuclear Physics B **912**, 192 (2016)

[10] M. Maggiore, A Modern Introduction to Quantum Field Theory. Oxford Master Series in Statistical, Computational, and Theoretical Physics (Oxford Univ. Press, Oxford, 2005)

[11] P. Di Francesco, P. Mathieu, D. Sénéchal, Conformal Field Theory. Graduate texts in contemporary physics (Springer, New York, NY, 1997)

[12] C. Cohen-Tannoudji, J. Dupont-Roc, G. Grynberg, Photons and Atoms: Introduction to Quantum Electrodynamics (Wiley, 1997)

[13] C.G. Darwin, Proceedings of the Royal Society of London Series A **136**, 36 (1932)

[14] S.J. van Enk, G. Nienhuis, EPL (Europhysics Letters) **25**, 497 (1994)

[15] E. Leader, Physics Letters B **779**, 385 (2018)

[16] I.M. Gel'fand, G.E. Shilov, Generalized Functions, vol. 1 (Academic Press Inc., 1964)

[17] I. Bialynicki-Birula, Z. Bialynicka-Birula, Journal of Optics (IOPscience) **13**, 064014 (2011)

[18] S. van Enk, G. Nienhuis, Journal of Modern Optics **41**, 963 (1994)

[19] E. Leader, C. Lorcé, Physics Reports **541**, 163 (2014)

[20] G. Wentzel, Quantum Theory Of Fields (Interscience Publishers, 1949)

[21] E. Cartan, Les Systemes Differentiels Exterieurs Et Leurs Applications Geometriques (Hermann & Cie, 1945)