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Moment Spectrum and First Dirichlet Eigenvalue of Geodesic Balls in Riemannian Manifolds

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La joventut és una cosa de la qual cal usar i abusar:
no som joves dues vegades.

Joan Fuster i Ortells.

La vie n'est bonne qu'à deux choses:
à faire des mathématiques et à les professer.

Siméon Denis Poisson.

A mathematician is a blind man in a dark room
looking for a black cat which isn't there

Charles Robert Darwin.

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Resum

L'objectiu principal d'aquest treball és mostrar els resultats que hem obtingut en els nostres articles d'investigació *First eigenvalue of the Laplacian of a geodesic ball and area-base symmetrization of its metric tensor* i *First Dirichlet eigenvalue and exit time moment spectra comparisons* (vegeu [28] i [63]). A aquests articles estudiem la relació entre certes propietats geomètriques de les boles geodèsiques de varietats de Riemann i les solucions de certes equacions diferencials plantejades en aquestes boles geodèsiques.

En particular, les equacions diferencials que estudiarem estan plantejades utilitzant el *laplacià (és a dir, l'operador de Laplace-Beltrami).* El laplacià és un dels operadors diferencials que estan relacionats amb l'estructura mètrica de la varietat de Riemann.

Al llarg d'aquest treball, mostrarem les nostres estimacions per a les solucions dels *problemes amb valors en la frontera* coneguts, en la literatura, com el *problema de Poisson* i el *problema de valors propis de Dirichlet*, plantejats en una bola geodèsica d'una varietat de Riemann completa. Concretament, provarem les nostres comparacions (fites) per a la *funció de temps d'eixida mitjà*, la *rigidesa torsional*, la *jerarquia de Poisson*, l'*espectre de moments* i el *primer valor propi del laplacià per al problema de Dirichlet* plantejats en boles geodèsiques d'una varietat de Riemann (vegeu Seccions 3.2 i 4.1 per a les definicions d'aquests problemes i dels invariants geomètrics esmenats).

Una manera de trobar aquestes fites és estudiar el comportament i quines propietats satisfan aquests invariants geomètrics quan assumim que les curvatures de la varietat estan fitades, com la lectora o el lector pot trobar, per exemple, als resultats d'A. Hurtado, S. Markvorsen i

V. Palmer en [37] i [38], els resultats de S.Y. Cheng en [12] i [13] i els resultats de G.P. Bessa i J.F. Montenegro en [5].

En els darrers capítols d'aquest treball veurem que les nostres fites per als invariants geomètrics definits en boles geodèsiques d'una varietat de Riemann completa, prèviament esmenats, s'obtenen comparant-los amb aquests invariants geomètrics definits en les corresponents boles geodèsiques d'*espais model rotacionalment simètrics* per mitjà de diverses tècniques:

Al llarg del capítol 3, provarem les nostres comparacions per a la funció temps d'eixida mitjà, la rigidesa torsional, la jerarquia de Poisson i l'espectre de moments definits en boles geodèsiques, amb radi menor que el radi d'injectivitat del seu centre, d'una varietat de Riemann completa assumint fites per a la curvatura mitjana de les esferes geodèsiques contingudes en la bola geodèsica.

En particular provarem que, si les curvatures mitjanes de totes les esferes geodèsiques estan fitades inferiorment per les curvatures mitjanes de les corresponents esferes geodèsiques d'un espai model rotacionalment simètric, aleshores la funció de temps d'eixida mitjà definida en la bola geodèsica està fitada per dalt per la funció de temps d'eixida mitjà definida en la corresponent bola geodèsica de l'espai model rotacionalment simètric i, anàlogament, també obtindrem el cas contrari, és a dir, si les curvatures mitjanes estan fitades superiorment, aleshores la funció de temps d'eixida mitjà està fitada per baix.

A més a més, caracteritzarem el cas de la igualtat provant que la igualtat s'assoleix si i només si les curvatures mitjanes de les esferes geodèsiques de la varietat i de l'espai model amb el mateix radi coincideixen.

A partir d'aquesta comparació, obtindrem fites per al quocient isoperimètric, la jerarquia de Poisson i el promig dels elements de l'espectre de moments comparant-los amb els corresponents definits en espais model rotacionalment simètrics. Pel que fa a la rigidesa torsional també trobarem una comparació, però necessitarem assumir certa *condició de balanceig* en l'espai model i utilitzar una altra tècnica que anomenarem *simetrització de Schwarz.*

A més a més, com comentarem en el capítol 4 i en les conclusions al capítol 5, les desigualtats obtingudes són molt rígides perquè la igualtat en una qualsevol d'elles determina la igualtat en totes les altres.

D'altra banda, al capítol 4, provarem que el primer valor propi del laplacià per al problema de Dirichlet definit en boles geodèsiques, amb radi menor que el radi d'injectivitat del seu centre, de qualsevol varietat de Riemann està fitat per dalt per funcions que només depenen de la funció àrea de les esferes geodèsiques contingudes en la bola. A més a més, caracteritzarem la igualtat provant que aquesta s'assoleix si i només si la curvatura mitjana de totes les esferes geodèsiques contingues en la bola és una funció radial (només depén de la funció distància al centre).

A partir d'aquest resultat provarem que si el quocient entre la funció d'àrea de les esferes geodèsiques contingudes en la bola de la varietat i la funció d'àrea de les corresponents esferes geodèsiques d'un espai model rotacionalment simètric és decreixent, aleshores el primer valor propi del laplacià per al problema de Dirichlet definit en la bola geodèsica de la varietat està fitat per dalt pel primer valor propi per al problema definit en la corresponent bola geodèsica de l'espai model. A més a més, provarem que la igualtat s'assoleix si i només si les curvatures mitjanes de les esferes geodèsiques de la varietat i de l'espai model, amb el mateix radi, coincideixen.

Contents

Chapter 1

Introduction

1.1 Purpose of study and objectives

The study of the connection between the function theory on Riemannian manifolds and the geometric structure of the manifold is one of the main goals of the so-called *Geometric Analysis*. The functions under study usually come as the solutions of differential equations defined on the manifold. The differential operators for which the partial differential equations are posed should be related with the metric structure of the Riemannian manifold, for instance the *Laplacian (the Laplace-Beltrami operator)*. One way to study this relationships is to control the curvatures of the manifold and study which properties are satisfied by the solutions of the partial differential equations, and vice-versa. For instance, there are estimates for the Laplacian of the distance function when the Ricci curvatures of the Riemannian manifold are bounded from below, as the reader can see in the R.E. Greene and H. Wu's book [32].

Along this work, we are going to consider a geodesic ball $B_R(o)$ of a given complete n-dimensional Riemannian manifold (M, g), and then, we will study some partial differential equations posed on these geodesic balls using the *Laplacian*. In particular, we will focus on the *boundary valued problems* known, in the literature, as the *Poisson problem* and the *Dirichlet eigenvalue problem* (see Sections 3.2 and 4.1). In this context, the goal of this work consists in studying the behaviour of the solutions of these problems and its relationship with the geometric properties satisfied by the given Riemannian manifold. In particular, we will describe and contextualize the results that we have obtained in our research papers *First eigenvalue of the Laplacian of a geodesic ball and area-base symmetrization of its metric tensor* and *First Dirichlet eigenvalue and exit time moment spectra comparisons* (see [28] and [63]).

In Chapter 3 we show the results that we have obtained in [63]. In particular, we will find comparisons for the *mean exit time function* defined on the geodesic balls. This function measures the expectation of the time that a *Brownian particle*, whose movement starts inside $B_R(o)$, takes to leave the geodesic ball through its boundary for the first time. It is known that this function is a solution of a Poisson problem. Moreover, we will construct other geometric invariants defined on the geodesic ball (we refer as geometric invariant to a value or function associated to a geometric object which remains invariant under isometries). These geometric invariants are the *torsional rigidity*, the *Poisson hierarchy* and the *moment spectrum*, which are related to the mean exit time function, and moreover, we will also show comparisons for these invariants.

On the other hand, in Chapter 4, we will show the results that we have obtained in [28]. In particular, we will find upper bounds for the *first eigenvalue of the Laplacian for the Dirichlet problem* defined on the geodesic ball and, furthermore, we will show the relationship between this geometric invariant and the ones previously mentioned.

1.2 Approach and methodology

The techniques that we will use to prove our statements are based on the use of the *Schwarz symmetrization*, the comparison with the *rotationally symmetric model spaces* and the construction of the *rotationally symmetric metric tensor of comparison* of geodesic balls. We shall apply then classical results such as the *Strong Maximum Principle*, the *Divergence Theorem*, the *co-area formula*, the *Rayleigh's Theorem* and the *Barta's Lemma*.

The *rotationally symmetric model spaces* $(\mathbb{M}_\omega, g_\omega)$ are geometric constructions that generalize the surfaces of revolution. This kind of spaces will play an important rôle along this work. In fact, we will compare the solutions of the Poisson and Dirichlet problems defined on geodesic balls of a complete Riemannian manifold with the corresponding solutions for the problem posed on geodesic balls of rotationally symmetric model spaces with same radius. On the other hand, given a Riemannian manifold (M, g) and $o \in M$ a point of M, the *rotationally symmetric metric tensor of comparison* is a new metric tensor $\widetilde{g}$ on the geodesic ball $B_R(o)$ constructed in such a way that we have the equality between the volumes of the geodesic spheres contained in $B_R(o)$ with respect to the metrics g and $\widetilde{g}$. Namely, given the geodesic ball $B_R(o)$ of M with radius $R > 0$ centered at o, we have the equality $\mathrm{vol}_g(S_r(o)) = \mathrm{vol}_{\widetilde{g}}(S_r(o))$ for all $r \in [0, R)$ (see Section 4.3). With this new metric tensor, $\widetilde{g}$ the geodesic ball $B_R(o)$ becomes a rotationally symmetric model space $(\mathbb{M}_\omega, \widetilde{g} = g_\omega)$.

Let us consider a complete n-dimensional Riemannian manifold (M, g) and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point such that $\mathrm{inj}_g(o) \leq \mathrm{inj}_{g_\omega}(o_\omega)$ (where inj denotes the injectivity radius, see Subsection 2.1.2). It is known that the solutions of the second order partial differential equations posed on the geodesic balls of the model spaces are radial, namely, they depend only on the distance function to the center of this geodesic balls. Thus, we shall see that to bound the solutions of the partial differential equations in the Riemannian manifold we need to compare them with the radial solutions in the rotationally symmetric model spaces, and for that, we must control their second derivatives, namely, their *Hessian* and their *Laplacian*. In order to control the corresponding Laplacian of these functions we shall assume some bounds on the mean curvatures of the geodesic spheres centered at $o \in M$. In particular, we will ask that the mean curvatures of the geodesic spheres $S_r(o)$ and $S_r^\omega(o_\omega)$ of M and $\mathbb{M}_\omega$, respectively, to satisfy, for any radius $R < \mathrm{inj}_g(o) \leq \mathrm{inj}_{g_\omega}(o_\omega)$, the inequality

$$H_{S_r(o)} \geq H_{S_r^\omega(o_\omega)} \quad \text{for all} \quad r \in (0, R], \tag{1.1}$$

or inequality

$$H_{S_r(o)} \leq H_{S_r^\omega(o_\omega)} \quad \text{for all} \quad r \in (0, R]. \tag{1.2}$$

On the other hand, in Section 3.6, we must assume a *balance condition* and use the *Schwarz symmetrization* technique to establish our bounds for the torsional rigidity. The *balance condition* is a hypothesis on the rotationally symmetric model spaces which will let us to control the growth of the isoperimetric quotient $\operatorname{vol}_{g_\omega}(B^\omega_R(o_\omega)) / \operatorname{vol}_{g_\omega}(S^\omega_R(o_\omega))$ (see Subsection 2.2.2) and the *Schwarz symmetrization* is a technique which will let us to compare the integrals of the solutions of the boundary valued problems posed on the Riemannian manifold with the integrals of the solutions for the problems on the rotationally symmetric model spaces. This technique consists in symmetrizing a geodesic ball $B_R(o)$ in M using a geodesic ball in $\mathbb{M}_\omega$ in such a way that the volume of the geodesic ball $B_R(o)$ is preserved (see Subsection 2.2.3).

Furthermore, the *Strong Maximum Principle* will be used in order to prove the comparisons for the mean exit time function and the Poisson hierarchy, and moreover, to characterize the equality cases (see Theorem 2.1.64).

On the other hand, we will use the *Divergence Theorem* and the *co-area formula* to compare integrals of functions over domains on the Riemannian manifold and their corresponding domains in the rotationally symmetric model spaces (see Theorems 2.1.59 and 2.1.62).

Finally, the *Rayleigh's Theorem* and the *Barta's Lemma* are techniques that we will use in Sections 4.4 and 4.5 to find our upper bounds for the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls, characterizing also the equality cases. These results state how the first eigenvalue can be bounded by using functions that are not necessarily eigenfunctions associated to the first eigenvalue of the Laplacian for the Dirichlet problem. In fact, these results characterize the equality case by showing that the equality is attained when the function is an eigenfunction associated to the first eigenvalue.

1.3 Previous results and motivation

The establishment of comparisons for the mean exit time, for the torsional rigidity, for the Poisson hierarchy and for the moment spectrum of a geodesic ball, together the geometric characterization of the domains and the spaces where these bounds are attained, encompasses the use of isoperimetric conditions as P. McDonald did in [55]. In that paper, by using the Schwarz symmetrization technique, he proved that the moment spectrum of precompact domains Ω can be bounded from above when equality between the volume of Ω and its Schwarz symmetrization implies an inequality between the volumes of their perimeters. The key idea to do that is to use the comparison of the symmetrized solution of the Poisson problem with the solution of the symmetrized problem given by G. Szegö and G. Talenti (see the works of C. Bandle [2] and G. Talenti [71], for instance). In this case, the moment spectrum is bounded, term by term, by the moment spectrum of its Schwarz symmetrization.

On the other hand, A. Hurtado, S. Markvorsen and V. Palmer in [37] and [38] and S. Markvorsen and V. Palmer in [53] showed some extrinsic comparisons for the torsional rigidity and the moment spectrum by assuming bounds on the radial sectional curvatures of the given Riemannian manifold and the control of the mean curvature of the submanifold (see Theorems 3.3.3 and 3.3.4). Furthermore, G.P. Bessa and J.F Montenegro in [5] used some bounds on the mean curvatures of the geodesic spheres as hypothesis to find upper and lower bounds for the first eigenvalue of the Laplacian for the Dirichlet problem (they used conditions (1.1) and (1.2)). Hence, a natural question is: can the moment spectrum be bounded assuming bounds on the mean curvatures of the geodesic spheres? Along this work we will give an answer to this question, *i.e.*, we will compare the moment spectrum of geodesic balls $B_R(o)$ of a given Riemannian manifold (M, g) with the one of geodesic balls of a rotationally symmetric model space $(\mathbb{M}_\omega, g_\omega)$ when the mean curvatures of all the geodesic spheres contained in $B_R(o)$ are bounded from above or from below by the mean curvatures of the corresponding geodesic spheres of $\mathbb{M}_\omega$. We should remark at this point that the mean curvature pointed inward of the geodesic spheres in a Riemannian manifold is the Laplacian of the distance function from its center (see Definitions 2.1.47, 2.1.49 and 2.1.56). We will show our results on the mean exit time, the torsional rigidity, the Poisson hierarchy and the moment spectrum in Chapter 3.

On the other hand, upper and lower bounds for the first eigenvalue of the Laplacian for the Dirichlet problem on precompact domains Ω of a Riemannian manifold (M, g) have been widely studied in the literature. As above, to bound the first eigenvalue we must control the geometry of the Riemannian manifold. One way to look for comparisons of the first eigenvalue of a precompact connected domain in a Riemannian manifold is to bound the curvatures of the Riemannian manifold. S.Y Cheng in [12] and [13] and G.P. Bessa and J.F. Montenegro in [5] showed upper and lower bounds for the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls $B_R(o)$ of a Riemannian manifold (M, g) by assuming some bounds of the sectional, the Ricci and the mean curvatures of the geodesic spheres contained in $B_R(o)$, respectively. Moreover, in [12] and [13], S.Y. Cheng found that the equality between the first eigenvalue of a geodesic ball $B_R(o)$ and the corresponding first eigenvalue of a geodesic ball of a rotationally symmetric model space is attained if, and only if, the geodesic balls are isometric, whereas, G.P Bessa and J.F. Montenegro in [5] found that the equality is attained if, and only if, the mean curvatures of the geodesic spheres $S_R(o)$ contained in $B_R(o)$ coincide with the mean curvatures of their corresponding geodesic spheres in the rotationally symmetric model space (see Theorems 4.2.1, 4.2.2 and 4.2.4 for more details on these statements).

G. Faber in [25] and E. Krahn in [47] proved, assuming certain isoperimetric condition, that the first eigenvalue of a precompact domain Ω in M can be bounded from below by the first eigenvalue of a geodesic ball $B^{\omega}_{L(\Omega)}(o_\omega)$ of a rotationally symmetric model space $\mathbb{M}_\omega$ such that $\mathrm{vol}(\Omega) = \mathrm{vol}(B^{\omega}_{L(\Omega)}(o_\omega))$. Furthermore, the equality between the first eigenvalues implies isometry between the precompact domain Ω and its corresponding geodesic ball $B^{\omega}_{L(\Omega)}(o_\omega)$ (see Theorem 4.2.6).

In Chapter 4, given a Riemannian manifold (M, g), we will construct the rotationally symmetric tensor of comparison $\widetilde{g}$ (which preserves the volume of the geodesic spheres contained in a geodesic ball $B_R(o)$ of M) and then, we will look for bounds of the first eigenvalue of $B_R(o)$ by comparing it with the first eigenvalue of $B_R(o)$ with respect to this new metric tensor. In fact, using this technique, we will show some upper bounds for the first eigenvalue of geodesic balls (see Sections 4.3, 4.4 and 4.5).

On the other hand, G.P. Bessa, V. Gimeno and L. Jorge in [6], E.B. Dryden, J.J. Langford and P. McDonald in [21], A. Hurtado, S. Markvorsen and V. Palmer in [39] and P. McDonald and R. Meyers in [57], studied bounds for the first eigenvalue of the Laplacian for the Dirichlet problem on domains of Riemannian manifolds in terms of the mean exit time, the torsional rigidity, the Poisson hierarchy, the moment spectrum and the volume of the domains (see Theorems

4.2.7, 4.2.8, 4.2.10 and 4.2.11). Then, as a consequence of their results and of our comparison given in Section 4.4, we will obtain upper bounds for the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls $B_R(o)$ in terms of the area functions of the geodesic spheres contained in $B_R(o)$ (see Corollary 4.4.4). Moreover, in Section 4.6, we will show some relationships between these geometric invariants while assuming the bounds on the mean curvature of the geodesic spheres as in [5] (namely, under conditions (1.1) and (1.2)).

Chapter 2

Preliminaries

We start this work by summarizing some preliminary concepts of Riemannian geometry which we will use in the further Chapters 3 and 4. We also shall introduce some basic properties of the so-called rotationally symmetric model spaces.

First, in Section 2.1, we will give, briefly, some basic definitions in Riemannian geometry. In particular, Subsections 2.1.1, 2.1.2 and 2.1.3, are devoted to define: the *length of a curve*, the *geodesic curves*, the *exponential map*, the *injectivity radius* and the *cut locus*. Next, in Subsections 2.1.4 and 2.1.5, we will define some intrinsic and some extrinsic curvatures of Riemannian manifolds. In Subsection 2.1.6, we will give some definitions of the differential operators on Riemannian manifolds that will be useful along this work. In particular, we will study some properties of the *Laplace-Beltrami operator*, which will play a key rôle along this work. Finally, to end this section, in Subsection 2.1.7 we will construct the systems of coordinates called *normal and polar coordinates* and we will describe the *Riemannian volume element* expressed in those polar coordinates.

On the other hand, in Section 2.2, we will define the so-called *rotationally symmetric model spaces*, which we will use to find our comparisons for some geometric invariants in the further chapters. In particular, Subsection 2.2.1 is devoted to define this spaces and study their properties. In Subsection 2.2.2, we will define our *balance condition* and give some example of rotationally symmetric model spaces which satisfy this condition. Finally, to end this chapter, in Subsection

2.2.3 we will present the *Schwarz symmetrization* of a precompact domain in a Riemannian manifold.

2.1 Riemannian geometry

Along this work we shall denote as (M, g) a finite dimensional Riemannian manifold with Riemannian metric tensor g and by ∇ its Levi-Civita connection, namely, the unique metric and torsion free connection.

In order to study a more detailed background of the concepts stated in this section, and the definitions of a Riemannian manifold and Riemannian metric tensor, we refer to I. Chavel [10], M.P. Do Carmo [19], J.M. Lee [48], P. Petersen [64] or T. Sakai [68].

2.1.1 Length of curves and metric distance function

The main objective of this subsection is to define the so-called *metric distance function* which gives us the distance between two given points of a Riemannian manifold and show some of its properties, see Section 3 of Chapter 5 of [64] and Section 1 of Chapter II of [68] to more detailed explanation. But first, we must define the *length of curves* on a Riemannian manifold as follows.

Definition 2.1.1 (see [68]). *Let (M, g) be an n-dimensional Riemannian manifold. Given a smooth parametrized curve $\gamma : I \longrightarrow M$ defined on an interval $I := [a, b] \subseteq \mathbb{R}$, we define the* length of γ, $\ell_g(\gamma)$, *by*

$$\ell_g^I(\gamma) := \int_a^b \|\gamma'(\sigma)\|_g \, d\sigma,$$

and the arc-length *parameter of the curve, $s_{g,\gamma}(t)$, by*

$$s_{g,\gamma}(t) = \int_a^t \|\gamma'(\sigma)\|_g \, d\sigma,$$

where $\|\cdot\|_g$ denotes the norm defined using the Riemannian metric tensor g.

Definition 2.1.2 (see [68]). *Let (M, g) be a connected n-dimensional Riemannian manifold and let p, q be two points of M. Then, we say that $\gamma : [a, b] \longrightarrow M$* is a smooth curve joining p with q *if γ is a smooth parametrized curve such that $\gamma(a) = p$ and $\gamma(b) = q$.*

Definition 2.1.3 (see [68]). *Let* (M, g) *be a connected* n*-dimensional Riemannian manifold. Given two points* $p, q \in M$*, we define the* metric distance function from p to q, $\mathrm{dist}_g(p, q)$, *as the infimum of the length of all the smooth curves joining* p *with* q*. Namely*

$$\mathrm{dist}_g(p, q) := \inf \{\ell_g(\gamma) \, : \, \textit{where } \gamma \textit{ is a smooth curve joining } p \textit{ with } q\} . \qquad (2.1)$$

For a non-connected Riemannian manifold (M, g) *we define the distance between two points* p, q *belonging to different connected components of* M *as infinity,* i.e.*,* $\mathrm{dist}_g(p, q) = +\infty$.

The following proposition shows some fundamental properties of the metric distance function.

Proposition 2.1.4 (see Remark 2.3 of Chapter 1 and Proposition 2.6 of Chapter 7 of [19] and Proposition 1.1 of Chapter II of [68]). *Let* (M, g) *be an* n*-dimensional Riemannian manifold and let* dist_g *be the metric distance function on* M*. Then,* (M, dist_g) *is a metric space and its topology coincides with the manifold topology (*i.e.*, coincides with the topology determined by the differentiable structure of* M*). In particular, the function* $\mathrm{dist}_g : M \times M \longrightarrow \mathbb{R}$ *is continuous with respect to the manifold topology.*

From these properties we give the notion of *metric ball* of a Riemannian manifold as follows.

Definition 2.1.5 (see [68]). *Let* (M, g) *be an* n*-dimensional Riemannian manifold. Given a point* $p \in M$ *and given a positive real value* R*,* $R \in \mathbb{R}^*_+$*, we define the* metric ball $\mathcal{B}_R(p)$ of M with radius R centered at p *as the set*

$$\mathcal{B}_R(p) := \{q \in M \, : \, \mathrm{dist}_g(p, q) < R\} .$$

Remark 2.1.6. Note that, from Proposition 2.1.4, the metric ball $\mathcal{B}_R(p)$ is an open set of M and its closure is $\overline{\mathcal{B}_R(p)} := \{q \in M \, : \, \mathrm{dist}_g(p, q) \leq R\}$ (see Proposition 1.1 of Chapter II of [68]). Thus, we can define the notion of *metric sphere* as the boundary of the metric ball.

Definition 2.1.7 (see [68]). *Let* (M, g) *be an* n*-dimensional Riemannian manifold. Given a point* $p \in M$ *and given a metric ball* $\mathcal{B}_R(p)$ *of* M *with radius*

$R \in \mathbb{R}_+^*$ *centered at* p*, we define the* metric sphere $\mathcal{S}_R(p)$ of M with radius R centered at p *as the set*

$$\mathcal{S}_R(p) = \overline{\mathcal{B}_R(p)} - \mathcal{B}_R(p) = \{q \in M \, : \, \mathrm{dist}_g(p,q) = R\} \, .$$

2.1.2 Geodesic curves, exponential map and injectivity radius

In this subsection, we give brief definitions of the notions of *geodesic curve*, *exponential map*, *injectivity radius*, *geodesic balls* and *geodesic spheres*, and moreover, we show some properties and considerations on these notions. See Chapter 3 of [19], Chapter 5 of [64] and Section 2 of Chapter II of [68], for more detailed explanation about these concepts.

Definition 2.1.8 (see [19])**.** *Let* (M, g) *be an* n*-dimensional Riemannian manifold with Levi-Civita connection* ∇*. Given a parametrized curved* $\gamma : I \longrightarrow M$*, defined on an interval* $I \subseteq \mathbb{R}$*, we say that* γ is a geodesic curve *if, and only if,*

$$\nabla_{\gamma'(t)}\gamma'(t) = 0 \quad \textit{for all} \quad t \in I.$$

Given $p \in M$ *and* $u \in T_pM$*, we say that* a geodesic curve $\gamma_u : I \longrightarrow M$, with $0 \in I \subset \mathbb{R}$, passes through the point p with velocity u at the instant $t = 0$ *if* $\gamma_u(0) = p$ *and* $\gamma_u'(0) = u$*. These conditions are called* initial conditions for the geodesic curve.

And moreover, we refer as normalized geodesic curve *to a geodesic curve* $\gamma : I \longrightarrow M$ *such that* $\|\gamma'(t)\|_g = 1$ *for all* $t \in I$*.*

Remark 2.1.9. Observe that if $\gamma : [a,b] \longrightarrow M$ is a normalized geodesic curve then $\ell_g^{[a,b]}(\gamma) = b - a$.

Along this work we need Riemannian manifolds such that its geodesic curves extend for all values of its parameter. Such Riemannian manifolds are said to be *(geodesically) complete* and are defined as follows.

Definition 2.1.10 ([68])**.** *Given* (M, g) *an* n*-dimensional Riemannian manifold, we say that* M *is* (geodesically) complete *if, for all* $p \in M$ *and for all* $u \in T_pM$*, the geodesic curve* $\gamma_u(t)$ *starting from* p *is defined for all* $t \in \mathbb{R}$*.*

Furthermore, we show in the following theorem the equivalence between the geodesic completeness of a Riemannian manifold (M, g) and its metric completeness when we consider (M, dist_g) as a metric space. This result is due to H. Hopf and W. Rinow (see [36] for the original paper), and it also states that all the closed and bounded subsets of M are compact, and moreover, that any two points of a complete Riemannian manifold can be joined by a geodesic curve which minimize the arc-length.

Theorem 2.1.11 (Hopf-Rinow Theorem (see Theorem 2.8 of Chapter 7 of [19] and Theorem 16 and Corollary 5 of Chapter 5 of [64]))**.** *Let (M, g) be an n-dimensional Riemannian manifold. Then the following assertions are equivalent:*

1. *(M, g) satisfies the Heine-Borel property,* i.e., *every closed and bounded subset of M is compact.*
2. *(M, dist_g) is a complete metric space.*
3. *(M, g) is geodesically complete.*

And moreover, any of the above assertions implies that:

4. *Any two points of M can be joined by a normalized geodesic curve which minimizes the arc-length.*

Remark 2.1.12. For a background on geodesically complete Riemannian manifolds see Section 2 of Chapter 7 of [19] or Section 1 of Chapter III of [68].

Now we show the function which allows us to define the *geodesic balls* and state some of its properties. This function is the so-called *exponential map* and it is defined as follows.

Definition 2.1.13 (see [64])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold. Given $p \in M$, we define the* exponential map *as the function*

$$\begin{aligned} \exp_p : T_pM &\longrightarrow M \\ u &\longmapsto \exp_p(u) := \gamma_u(1), \end{aligned}$$

where γ_u is the geodesic curve starting from p with velocity u, i.e., *$\gamma_u(0) = p$ and $\gamma_u'(0) = u$.*

Remark 2.1.14. Note that, since (M, g) is complete, the exponential map $\exp_p$ is defined for all $u \in T_pM$, which is another equivalent way to define the (geodesic) completeness, and moreover, we have that $\exp_p(o_p) = p$, where $o_p := \vec{0} \in T_pM$. Let us also remark that, in general, the exponential map is defined locally in a neighbourhood of $o_p \in T_pM$, see for instance [19], [64] and [68], to check the notions just mentioned.

Definition 2.1.15. *Let (M, g) be a complete n-dimensional Riemannian manifold. Given a point $p \in M$ and given $R > 0$, we define the* geodesic ball $B_R(p)$ *of M with radius R centered at p as the image by the exponential map of the entire open ball $\mathbb{B}_R(o_p)$ of T_pM with radius R centered at $o_p \in T_pM$. Namely,*

$$B_R(p) := \exp_p\left(\mathbb{B}_R(o_p)\right).$$

The following theorem summarizes some results about the smoothness of the exponential map.

Theorem 2.1.16 (see Proposition 2.9, Lemma 3.5 and Theorem 3.7 of Chapter 3 of [19] and see page 32 and Proposition 2.3 of Chapter II of [68])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold and let $p \in M$. Then the following assertions hold:*

1. *The exponential map $\exp_p$ is differentiable in the entire T_pM. Moreover, the differential of the exponential map at the origin o_p of T_pM is the identity for all $u \in T_{o_p}(T_pM)$. Namely,*

   $$D\exp_p(o_p)u = u, \quad \text{for all} \quad u \in T_{o_p}(T_pM) \equiv T_pM,$$

   where we identify $T_{o_p}(T_pM) \equiv T_pM$ via the canonical identification.

2. *The exponential map satisfies that $\exp_p(tu) = \gamma_u(t)$ with $\exp_p(o_p) = \gamma_u(0) = p$ and $\gamma_u'(0) = u$ for all $u \in T_pM$ and for all $t \in \mathbb{R}$. Moreover, we have that $\exp_p(u) = \gamma_{\frac{u}{\|u\|}}(\|u\|)$ for any $u \in T_pM$.*

3. Gauss's Lemma: *for any $u \in T_pM$ and for any $t \in \mathbb{R}$ the following assertions hold:*

   (a) The differential $D\exp_p(tu)$ maps u to $\gamma_u'(t)$, i.e., $D\exp_p(tu)u = \gamma_u'(t)$.

(b) *Consider $\xi \in T_pM$ also as a vector in $T_{tu}(T_pM)$ via the canonical identification, then*

$$g\left(D\exp_p(tu)\xi, \gamma_u'(t)\right) = g\left(D\exp_p(tu)\xi, D\exp_p(tu)u\right) = g\left(\xi, u\right).$$

Therefore, if ξ is g-orthogonal to u then $D\exp_p(tu)\xi$ is g-orthogonal to $\gamma_u'(t)$. Moreover, we have that $\left\|D\exp_p(tu)u\right\|_g = \|u\|_g$, where $\|\cdot\|_g$ denotes the norm defined using the Riemannian metric tensor g.

4. *There exists $R > 0$ such that $\exp_p : \mathbb{B}_r(o_p) \longrightarrow B_r(p)$ is a diffeomorphism for all $r \in [0, R]$, where $\mathbb{B}_r(o_p)$ and $B_r(p)$ are, respectively, the open ball of T_pM with radius r centered at $o_p \in T_pM$ and the geodesic ball of M with radius r centered at p.*

Remark 2.1.17. From these results we have that, geometrically, $\exp_p u$ is a point of M obtained by going out, starting from p, a distance of $\|u\|_g$ along the unique geodesic curve that passes through p with velocity $u/\|u\|_g$ at p (see [19]). Furthermore, since $\gamma_{\frac{u}{\|u\|_g}} : [0, \|u\|_g] \longrightarrow M$ is a normalized geodesic curve, we have that

$$\ell_g^{[0,1]}(\gamma_u) = \ell_g^{[0,\|u\|_g]}\left(\gamma_{\frac{u}{\|u\|_g}}\right) = \|u\|_g.$$

Now, from assertion (4) of Theorem 2.1.16, we want to define a quantity which let us ensure that the exponential map is a diffeomorphism onto the geodesic ball with radius less than this quantity. Namely, we want to know how large the radius of a geodesic ball can be so that the exponential map is a diffeomorphism onto this geodesic ball. In order to have this control on the radius, we define the so-called in the literature as the *injectivity radius.*

Definition 2.1.18 (see [68])**.** *Let (M, g) be an n-dimensional Riemannian manifold. Given a point $p \in M$, we define the* injectivity radius $\mathrm{inj}_g(p)$ of p *as the following quantity*

$$\mathrm{inj}_g(p) := \sup\left\{R \geq 0 \,:\, \exp_p\big|_{\mathbb{B}_R(o_p)} \text{ is a diffeomorphism}\right\}.$$

where $\mathbb{B}_R(o_p)$ is the open ball of T_pM with radius R centered at $o_p \in T_pM$.

If $p \in M$ is such that $\exp_p$ is a diffeomorphism from the entire T_pM, we take $\mathrm{inj}_g(p) = +\infty$ *and we say that p is a* pole of M.

Moreover, we define the injectivity radius $\mathrm{inj}_g(M)$ of M *as the infimum of the injectivity radius of all* $p \in M$*. Namely,*

$$\mathrm{inj}_g(M) := \inf\left\{\mathrm{inj}_g(p)\,:\, p \in M\right\}.$$

Remark 2.1.19. Note that, if we assume that (M, g) is a complete Riemannian manifold then, from assertion (4) of Theorem 2.1.16, we have that $\mathrm{inj}_g(p) > 0$ for any $p \in M$. For more detailed background on the injectivity radius we refer to Sections 2 and 3 of Chapter 13 of [19] or Section 4 of Chapter III of [68].

The following proposition show that geodesic curves minimize the arc-length in geodesic balls with radius less than the injectivity radius.

Proposition 2.1.20 (see Proposition 3.6 of Chapter 3 of [19] and Lemma 2.7 of Chapter II of [68])**.** *Let (M, g) be an n-dimensional Riemannian manifold and $p \in M$. Suppose that $R < \mathrm{inj}_g(p)$ and let $B_R(p)$ be the geodesic ball of M with radius R centered at p. Let $q \in B_R(p)$, $q \neq p$, and let $\gamma : [a, b] \longrightarrow B_R(p)$ be a segment of a geodesic curve such that $\gamma(a) = p$ and $\gamma(b) = q$. Then, for any (possibly piecewise) differentiable curve $c : [a, b] \longrightarrow M$ joining p with q, we have that $\ell_g(\gamma) \leq \ell_g(c)$. Furthermore, $\ell_g(\gamma) = \ell_g(c)$ if, and only if, $\gamma([a, b]) = c([a, b])$.*

As a consequence of the above proposition and the fact that the exponential map is a diffeomorphism onto the geodesic balls of radius less than the injectivity radius, the following corollary shows that there is a relationship between the geodesic curves and the distance function.

Corollary 2.1.21 (see Remark 3.8 of Chapter 3 of [19] and Corollary 2.8 of Chapter II of [68])**.** *Let (M, g) be an n-dimensional Riemannian manifold and $p \in M$. Suppose that $R < \mathrm{inj}_g(p)$ and let $B_R(p)$ be the geodesic ball of M with radius R centered at p. Then, for any $q \in B_R(p) - \{p\}$, there exists a unique normalized geodesic curve $\gamma : [0, b] \longrightarrow B_R(p)$, with $\gamma(0) = p$ and $\gamma(b) = q$, which minimizes the arc-length. In particular, this geodesic curve is given by*

$$\gamma(t) := \exp_p\left(t\frac{\exp_p^{-1} q}{\left\|\exp_p^{-1} q\right\|_g}\right).$$

Furthermore, we have that $\mathrm{dist}_g(p, q) = \left\|\exp_p^{-1} q\right\|_g$ and that the geodesic ball $B_R(p)$ coincides with the metric ball $\mathcal{B}_R(p)$, i.e.,

$$B_R(p) = \mathcal{B}_R(p) = \{q \in M\,:\, \mathrm{dist}_g(p, q) < R\}.$$

To end this subsection we define the notion of *geodesic sphere* of a Riemannian manifold as follows.

Definition 2.1.22 (see [19])**.** *Let (M, g) be an n-dimensional Riemannian manifold. Given a point $p \in M$ and given a geodesic ball $B_R(p)$ of M with radius $R < \mathrm{inj}_g(p)$ centered at p, we define the* geodesic sphere $S_R(p)$ of radius R centered at p *as the boundary of the geodesic ball $B_R(p) = \exp_p(\mathbb{B}_R(o_p))$. Namely,*

$$S_R(p) := \partial B_R(p) = \{q \in M \,:\, \mathrm{dist}_g(p, q) = R\}.$$

Remark 2.1.23. Note that, from Remark 2.1.17 and Corollary 2.1.21, we have that

$$S_R(p) = \Big\{\exp_p(u) \,:\, u \in T_pM \text{ with } \|u\|_g = R\Big\} = \exp_p(\partial\mathbb{B}_R(o_p))$$

and that the geodesic sphere $S_R(p)$ coincides with the metric sphere $\mathcal{S}_R(p)$.

On the other hand, observe that the above results are not global, as we have previously mentioned. In fact, in some cases, if we consider a large segment of a geodesic curve, the geodesic curve cannot minimize the arc-length after some point, and moreover, there can exist many geodesic curves joining two given points which minimize the arc-length, contradicting uniqueness. For instance, geodesic curves on the sphere which start at some point p did not minimize the arc-length after passing through the antipode of p (see [19]). On the other hand, any great circle on the sphere that joins a point p with its antipode is a geodesic curve which minimizes the arc-length, which shows that the existence of an arc-length minimizing geodesic curve does not imply uniqueness (see [68]).

2.1.3 Cut locus and relationship with injectivity radius

In this subsection, we define the *cut locus* and show, briefly, its relationship with the *injectivity radius* (see Definition 2.1.18). For a more detailed background on the cut locus and the mentioned relationship with the injectivity radius, we refer to Section 2 of Chapter 3 of [10], Section 2 of Chapter 13 of [19] and Section 4 of Chapter III of [68], for instance.

Definition 2.1.24 (see [19])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold. Given a point $p \in M$, we say that $\gamma(t_0)$ is a* cut point of p along a normalized geodesic curve $\gamma : [0, +\infty) \longrightarrow M$ with $\gamma(0) = p$ *if* $\gamma(t)$ *minimizes the arc-length for all* $t \leq t_0$ *(that is* $\mathrm{dist}_g(\gamma(0), \gamma(t)) = \ell(\gamma(t)) = t$*), and not for* $t > t_0$*. If* γ *minimizes the arc-length for all* $t > 0$*, we say that such* cut point does not exists.

Definition 2.1.25 (see [19])**.** *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold. Given a point* $p \in M$*, the* cut locus of p, $\mathrm{Cut}(p)$, *is the union of the cut points of* p *along all the normalized geodesic curves emanating from* p*.*

Example 2.1.26 (Cut locus)**.** Here we show some of the examples for the cut locus that appear in [19]:

1. The cut locus of a point p in the sphere $\mathbb{S}^n$ is its antipodal point (*i.e.*, consists of the antipodal point of p).

2. The cut locus of a point p of a cylinder in $\mathbb{R}^3$ is the "generating line" of the cylinder (straight line that generates the cylinder) which is at the opposite part to that the generating line which passes through p.

Furthermore, we have the following result of existence.

Proposition 2.1.27 (see Corollary 2.8 of Chapter 13 of [19] and page 104 of [68])**.** *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold and* $p \in M$*. Then, for any* $q \in M - \mathrm{Cut}(p)$*, there exists a unique arc-length minimizing normalized geodesic curve joining* p *with* q*.*

It is well known that there exists a relationship between the cut locus and the injectivity radius of a point. In fact, it can be shown that the cut point along the unique normalized geodesic curve that minimizes the arc-length is attained exactly at a length equal to the injectivity radius.

Theorem 2.1.28 (see page 271 and Proposition 2.9 of Chapter 13 of [19] and Proposition 4.13 of Chapter III of [68])**.** *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold and* $p \in M$*. Then,* $\mathrm{inj}_g(p) = \mathrm{dist}_g(p, \mathrm{Cut}(p))$ *for all* $p \in M$*, and hence, for any open neighbourhood* $D \subseteq M - \mathrm{Cut}(p)$ *of* p *there exists a neighbourhood* V_D *of the origin* o_p *in* T_pM *such that the exponential map* $\exp_p : V_D \longrightarrow D$ *is a diffeomorphism. Furthermore,* $p \longmapsto \mathrm{inj}_g(p)$ *is a continuous function from* M *to* $\mathbb{R}_+ \cup \{+\infty\}$ *and if* $\mathrm{Cut}(p) = \emptyset$ *then* $\mathrm{inj}_g(p) = +\infty$*.*

2.1.4 Intrinsic curvatures

In this subsection, we present the notion of curvature in a Riemannian manifold. For a more detailed background on the notion of curvature, we refer the reader to check Chapter 4 and 6 of [19], Chapters 7, 8 and 11 of [48], Chapter 2 of [64], or Section 3 of Chapter II and Chapter V of [68], for instance. First, we give the definition of the *curvature tensor*.

Definition 2.1.29 (see [19])**.** *Let (M, g) be an n-dimensional Riemannian manifold with Levi-Civita connection ∇. Given $\mathfrak{X}(M)$ the family of smooth vector fields of M, we define the* curvature tensor of (M, g) for every pair $X, Y \in \mathfrak{X}(M)$ *as the map $R_{XY} : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ given by*

$$Z \longmapsto R_{XY}Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]}Z, \tag{2.2}$$

where $[X, Y]$ denotes the Lie-Bracket*.*

Remark 2.1.30. Observe that, since ∇ is the Levi-Civita connection, the Lie-Bracket can be expressed, for all $X, Y \in \mathfrak{X}(M)$, by $[X, Y] = \nabla_X Y - \nabla_Y X$. Moreover, one can easily compute the curvature tensor of the Euclidean space $\mathbb{R}^n$ and check that $R_{XY}Z = 0$ for any $X, Y, Z \in \mathfrak{X}(\mathbb{R}^n)$. Thus, the curvature is usually understood as a tensor which measure how much the given Riemannian manifold deviates from being the Euclidean space (see [19] for instance).

On the other hand, it need to be noted that the curvature tensor on the previous mentioned literature may differ by a sign. In fact, we choose the sing of the curvature tensor as in I. Chavel [10], M.P Do Carmo [19], A. Gray [31] and J. Milnor [59]. See, for instance, S. Kobayashi and K. Nomizu [46], J.M. Lee [48] or P. Petersen [64] for the definition with opposite sign.

From this we define the *Riemannian curvature tensor* as follows.

Definition 2.1.31 (see [19])**.** *Given (M, g) an n-dimensional Riemannian manifold with Levi-Civita connection ∇, we define* Riemannian curvature tensor for every $X, Y, Z \in \mathfrak{X}(M)$ *as the correspondence*

$$(X, Y, Z, W) \longmapsto R_{XYZ}W := g\left(R_{XY}Z, W\right). \tag{2.3}$$

Remark 2.1.32. It is well known that the values of the curvature tensor and the Riemannian curvature tensor at a point $p \in M$ only depend on the values of the vector fields X, Y, Z, W at p, see [19]. Moreover, note that we can define the curvature tensor with opposite sign as

$$\overline{R}_{XY}Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z.$$

Then, to preserve the sign of the Riemannian curvature tensor, it must be defined as the correspondence

$$(X, Y, Z, W) \longmapsto \overline{R}_{XYZ}W := g(\overline{R}_{ZW}Y, X)$$

see [46] for instance. In fact, using Proposition 2.5 of Chapter 4 of [19], we have

$$\begin{aligned} \overline{R}_{XYZ}W &= g\left(\overline{R}_{ZW}Y, X\right) = g\left(-R_{ZW}Y, X\right) = -g(R_{ZW}Y, X) \\ &= -g(R_{YX}Z, W) = g(R_{XY}Z, W) = R_{XYZ}W \end{aligned}$$

Now, we define the *sectional curvature* as follows.

Definition 2.1.33 (see [19])**.** *Let (M, g) be an n-dimensional Riemannian manifold and let $p \in M$ be a point of M. Given two linearly independent vectors $u, v \in T_pM$ of the tangent space at p, we define the* sectional curvature of the 2-dimensional subspace $\sigma_p(u, v) \subset T_pM$ generated by u and v *as the number*

$$\sec_g\left(\sigma_p(u, v)\right) := \frac{R_{uvu}v}{g\left(u, u\right) g\left(v, v\right) - g\left(u, v\right)}. \tag{2.4}$$

Remark 2.1.34. Observe that if $u, v \in T_pM$ are orthonormal then

$$\sec_g\left(\sigma_p(u, v)\right) = R_{uvu}v. \tag{2.5}$$

Note moreover that the definition of the sectional curvature does not depend of the chosen basis for the 2-dimensional subspace (see [19] for instance). Thus, we can choose an orthonormal basis in order to compute the sectional curvature by (2.5).

Definition 2.1.35. *Given (M, g) an n-dimensional Riemannian manifold, we say that M has* constant sectional curvature κ *if, for each point $p \in M$, the sectional curvature is constant and equal to κ for all 2-dimensional subspaces of*

T_pM. *And we say that the* sectional curvatures are bounded from above (resp. below) *by* κ *if*

$$\sec_g(\sigma_p(u,v)) \leq (\geq)\kappa$$

for all $\sigma_p(u,v) \subset T_pM$ *and for all* $p \in M$.

Example 2.1.36. Examples of Riemannian manifolds with constant sectional curvature are the *simply connected real space forms* $\mathbb{K}^n(\kappa)$, where $\mathbb{K}^n(0) = \mathbb{R}^n$, $\mathbb{K}^n(\kappa) = \mathbb{S}^n(\kappa)$ for $\kappa > 0$, and $\mathbb{K}^n(\kappa) = \mathbb{H}^n(\kappa)$ for $\kappa < 0$.

On the other hand, for a complete Riemannian manifold, we can define the *radial sectional curvature* as R.E. Greene and H.H. Wu did in [32].

Definition 2.1.37 (see [32])**.** *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold and let* $p \in M$ *be a point of* M. *Given a neighbourhood* $D \subset M - \mathrm{Cut}(p)$ *of* p *and given* $q \in D - \{p\}$, *we define the* unit radial vector field ∂r *as the unit vector tangent to the unique normalized geodesic curve which joins* p *with* q.

Moreover, given $\Pi \subset T_qM$ *a* 2*-dimensional subspace of* T_qM, *we say that* Π *is a* radial plane *if, and only if,* Π *contains the unit radial vector field* ∂r.

And we define the radial sectional curvature at $q \in D - \{p\} \subset M - \mathrm{Cut}(p)$ *as the restriction of the sectional curvature function to radial planes in* T_qM, i.e., *to the sectional curvature*

$$\sec_g\left(\sigma_q\left(\partial r|_q, u\right)\right), \quad \textit{for all} \quad u \in T_qD \subseteq T_qM.$$

Moreover, from the definition of the curvature tensor, we define the following curvature which is a sum of sectional curvatures of the given Riemannian manifold.

Definition 2.1.38 (see [19])**.** *Given* (M, g) *an* n*-dimensional Riemannian manifold and a point* $p \in M$, *we define the* Ricci curvature *as the trace of the curvature tensor* (2.2)*. Namely, the Ricci curvature is the function* $\mathrm{Ric}_g : T_pM \times T_pM \longrightarrow \mathbb{R}$ *given by*

$$(u, v) \longmapsto \mathrm{Ric}_g(u, v) := \mathrm{trace}\,(w \longmapsto R_{uw}v)\,.$$

Remark 2.1.39. Note that, choosing an orthonormal basis $\{e_i\}_{i=1}^n$ of T_pM and from the definition of the Riemannian curvature tensor (2.3), we have that

$$\mathrm{Ric}_g(u,v) = \sum_{i=1}^n g\left(R_{ue_i}v, e_i\right) = \sum_{i=1}^n R_{ue_iv}e_i.$$

On the other hand, we have that if we choose the opposite sing for the curvature tensor then the sign of the Ricci curvature remains the same.

Moreover, it is easy to check, from the properties of the curvature and metric tensors, that the Ricci curvature is a symmetric bilinear form and its expression does not depend of the chosen orthonormal basis (see [19] or [64], for instance). Thus, as the reader can check in Chapter 2 of [64], if $v \in T_pM$ is a unit vector and we choose the orthonormal basis $\{e_i\}_{i=1}^n$ such that $e_n = v$, from the expression of sectional curvature (2.5) on an orthonormal basis we have the quadratic form associated to the Ricci curvature

$$\mathrm{Ric}_g(v,v) = \sum_{i=1}^{n-1} R_{ve_iv}e_i + R_{vvv}v = \sum_{i=1}^{n-1} \sec_g\left(\sigma(v,e_i)\right).$$

Therefore, from this expression, if the Riemannian manifold (M,g) has constant sectional curvature κ we obtain that $\mathrm{Ric}_g(v,v) = (n-1)\kappa$. Moreover, when all the eigenvalues of the quadratic form, $\mathrm{Ric}_g(v,v)$, are greater or equal than some real value b, we denote this fact by $\mathrm{Ric}_g \geq b$ and we say that the *Ricci curvature is bounded from below by* b.

2.1.5 Extrinsic curvature

Now, we present the notion of *mean curvature of an isometric immersion.* But first, let us define the notions of *isometric immersion* and *second fundamental form of an isometric immersion.*

Definition 2.1.40 (see [48])**.** *Let $(\widetilde{M},\widetilde{g})$ be an m-dimensional Riemannian manifold and let (M,g) be an n-dimensional manifold with $m \geq n$. Given a smooth map $\varphi : M \longrightarrow \widetilde{M}$, we say that φ is an* immersion *if, for any $p \in M$,*

$$d\varphi_p = \varphi_{*_p} : T_pM \longrightarrow T_{\varphi(p)}\widetilde{M}$$

is non-singular.

Moreover, we say that $\varphi : M \longrightarrow \widetilde{M}$ *is an* isometric immersion *if* φ *is an immersion and, for any* $p \in M$ *and for any* $u, v \in T_pM$*, the Riemannian metric tensors* g *and* $\widetilde{g}$ *satisfy that*

$$g_p(u,v) = \widetilde{g}_{\varphi(p)}\left(\varphi_{*_p}(u), \varphi_{*_p}(v)\right),$$

Namely, φ *is an isometric immersion if the metric tensor* g *is given by the induced metric tensor* $\varphi^*\widetilde{g}$*,* i.e., $g = \varphi^*\widetilde{g}$*, where* φ^* *denotes the pullback of the immersion.*

Moreover, if $\varphi : M \longrightarrow \widetilde{M}$ *is an isometric immersion, we say that* $\varphi(M)$ *is a* submanifold immersed in $\widetilde{M}$ *and we refer to* $\widetilde{M}$ *as the* ambient manifold*.*

Remark 2.1.41. Note that, if M is a Riemannian submanifold of $\widetilde{M}$ then, at each $p \in M$, the tangent space $T_{\varphi(p)}\widetilde{M}$ of the ambient manifold splits into the direct sum

$$T_{\varphi(p)}\widetilde{M} = \varphi_{*_p}(T_pM) \oplus \left(\varphi_{*_p}(T_pM)\right)^{\perp} \equiv T_pM \oplus (T_pM)^{\perp}$$

by identifying $T_pM \equiv \varphi_{*_p}(T_pM) \hookrightarrow T_{\varphi(p)}\widetilde{M}$ and $(T_pM)^{\perp} \equiv \left(\varphi_{*_p}(T_pM)\right)^{\perp}$, where $(T_pM)^{\perp}$ is the orthogonal complement of T_pM in $T_{\varphi(p)}\widetilde{M}$ with respect to the inner product $\widetilde{g}$ on $T_{\varphi(p)}\widetilde{M}$.

Moreover, we can extend any local smooth vector fields $X, Y \in \mathfrak{X}(M)$ to local smooth vector fields $\widetilde{X}, \widetilde{Y}$ on the ambient manifold $\widetilde{M}$, and decompose the Levi-Civita connection $\widetilde{\nabla}$ on $\widetilde{M}$ as

$$\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} = \left(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}\right)^{\top} + \left(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}\right)^{\perp}.$$

From now on, when it is clear from the context, we identify the tangent space $T_{\varphi(p)}\widetilde{M}$ and the extensions $\widetilde{X}, \widetilde{Y}$ by $T_p\widetilde{M}$ and X, Y, respectively. For a background on these concepts see for instance Chapter 6 of [19] and Chapter 8 of [48].

Thus, we define the *second fundamental form* as follows.

Definition 2.1.42 (see [48])**.** *Let* $(\widetilde{M}, \widetilde{g})$ *be an* m*-dimensional Riemannian manifold and let* (M, g) *be an* n*-dimensional Riemannian manifold with* $m \geq n$*. Given an isometric immersion* $\varphi : M \longrightarrow \widetilde{M}$ *and a point* $p \in M$*, we define the* second fundamental form of M in $\widetilde{M}$ at p *as the map*

$$\begin{aligned} \mathrm{II}_p : T_pM \times T_pM &\longrightarrow T_pM^{\perp} \\ (u,v) &\longmapsto \mathrm{II}_p(u,v) := \left(\widetilde{\nabla}_{U_p}V\right)^{\perp}(p), \end{aligned}$$

where $U, V \in \mathfrak{X}(M)$ *such that* $U_p = u$ *and* $V_p = v$*.*

Remark 2.1.43. It can be proved that the second fundamental form is independent of the extensions $\widetilde{X}$ and $\widetilde{Y}$ of X and Y to $\mathfrak{X}(\widetilde{M})$, and that is bilinear and symmetric with respect to X and Y. Furthermore note that, for any $p \in M$,

$$\mathrm{II}_p(X_p, Y_p) = \left(\widetilde{\nabla}_{X_p} Y\right)^{\perp}(p) = \widetilde{\nabla}_{X_p} Y(p) - \left(\widetilde{\nabla}_{X_p} Y\right)^{\top}(p).$$

From now on, we refer as *normal vector fields* to the vector fields in $\mathfrak{X}(M)^{\perp}$.

Now, we show a property of the second fundamental form that we make use to compute the mean curvature of the geodesic spheres.

Proposition 2.1.44 (The Weingarten Equation (see Lemma 8.3 of [48]))**.** *Let $(\widetilde{M}, \widetilde{g})$ be an m-dimensional Riemannian manifold and let (M, g) be an n-dimensional Riemannian submanifold of $\widetilde{M}$. Let $X, Y \in \mathfrak{X}(M)$ and $N \in \mathfrak{X}(M)^{\perp}$. Then, when X, Y are extended to $\widetilde{M}$, the following equation holds at any $p \in M$*

$$\widetilde{g}\left(\mathrm{II}_p(X_p, Y_p), N_p\right) = \widetilde{g}\left(\left(\widetilde{\nabla}_{X_p} Y\right)^{\perp}(p), N(p)\right) = -\widetilde{g}\left(Y_p, \left(\widetilde{\nabla}_{X_p} N\right)(p)\right). \tag{2.6}$$

Proof. Let p be a point of M, let $X, Y \in \mathfrak{X}(M)$ and $N \in \mathfrak{X}(M)^{\perp}$. Then, since

$$\widetilde{g}_p(Y_p, N_p) = 0 = \widetilde{g}_p\left(\left(\widetilde{\nabla}_{X_p} Y\right)^{\top}(p), N_p\right),$$

we have that

$$\begin{aligned}
\widetilde{g}_p\left(\mathrm{II}_p(X_p, Y_p), N_p\right) &= \widetilde{g}_p\left(\left(\widetilde{\nabla}_{X_p} Y\right)^{\perp}(p), N_p\right) \\
&= \widetilde{g}_p\left(\widetilde{\nabla}_{X_p} Y(p) - \left(\widetilde{\nabla}_{X_p} Y\right)^{\top}(p), N_p\right) \\
&= \widetilde{g}_p\left(\widetilde{\nabla}_{X_p} Y(p), N_p\right) \\
&= X\widetilde{g}\left(Y, N\right)(p) - \widetilde{g}\left(Y, \widetilde{\nabla}_X N\right)(p) \\
&= -\widetilde{g}_p\left(Y_p, \nabla_{X_p} N(p)\right).
\end{aligned}$$

□

Definition 2.1.45. *Let $(\widetilde{M}, \widetilde{g})$ be an m-dimensional Riemannian manifold and let (M, g) be an n-dimensional Riemannian submanifold of $\widetilde{M}$. Given a point $p \in M$ and $N \in \mathfrak{X}(M)^{\perp}$, we define the* Weingarten map L_{N_p} *as*

$$\begin{aligned}
L_{N_p} : T_pM &\longrightarrow T_pM \\
X_p &\longmapsto L_{N_p}(X_p) := -\left(\widetilde{\nabla}_{X_p} N\right)^{\top}(p).
\end{aligned}$$

Remark 2.1.46. Note that, L_{N_p} is a self-adjoint operator which satisfies,

$$\widetilde{g}_p\left(\mathrm{II}_p\left(X_p, Y_p\right), N_p\right) = \widetilde{g}_p\left(L_{N_p}(X_p), Y_p\right)$$

for all $p \in M$ and for all $X_p, Y_p \in T_pM$ and $N_p \in (T_pM)^{\perp}$.

Thus, choosing an orthonormal basis $\{e_i\}_{i=1}^n$ of T_pM and an orthonormal basis $\{\xi_j\}_{j=n+1}^m$ of $(T_pM)^{\perp}$, we have an orthonormal basis $\{\widetilde{e}_i\}_{i=1}^m$ of $T_p\widetilde{M}$ where $\widetilde{e}_i = e_i$ for $i = 1, 2, \ldots, n$ and $\widetilde{e}_j = \xi_j$ for $j = n+1, n+2, \ldots, m$. Then, taking $X_p = Y_p = \widetilde{e}_i$ for any $i \in \{1, 2, \ldots, n\}$, we can express the second fundamental form as

$$\mathrm{II}_p(\widetilde{e}_i, \widetilde{e}_i) = \left(\widetilde{\nabla}_{\widetilde{e}_i}\widetilde{e}_i\right)^{\perp}(p) = \sum_{j=n+1}^{m} \widetilde{g}\left(\left(\widetilde{\nabla}_{\widetilde{e}_i}\widetilde{e}_i\right)^{\perp}(p), \xi_j\right)\xi_j,$$

and hence, applying the Weingarten equation (2.6) of Proposition 2.1.44 and from Definition 2.1.45 of the Weingarten map, we obtain, for all $i = 1, \ldots, n$, that

$$\mathrm{II}_p(\widetilde{e}_i, \widetilde{e}_i) = -\sum_{j=n+1}^{m} \widetilde{g}\left(\widetilde{\nabla}_{\widetilde{e}_i}\xi_j(p), \widetilde{e}_i\right)\xi_j = \sum_{j=n+1}^{m} \widetilde{g}\left(L_{\xi_j}(\widetilde{e}_i), \widetilde{e}_i\right)\xi_j. \tag{2.7}$$

On the other hand, we define the *mean curvature* as follows.

Definition 2.1.47 (see [46]). *Given $(\widetilde{M}, \widetilde{g})$ an m-dimensional Riemannian manifold and (M, g) an n-dimensional Riemannian submanifold of $\widetilde{M}$ with $n \leq m$, we define the* mean curvature vector $\vec{H}_M$ of M at a point $p \in M$ *as*

$$\vec{H}_M(p) := \operatorname{trace}_g \mathrm{II}_p. \tag{2.8}$$

The following proposition shows how the mean curvature vector can be expressed in terms of the Weingarten map.

Proposition 2.1.48. *Let $(\widetilde{M}, \widetilde{g})$ be an m-dimensional Riemannian manifold and let (M, g) be an n-dimensional Riemannian submanifold of $\widetilde{M}$. Let $p \in M$ and let $\{e_i\}_{i=1}^n$ and $\{\xi_j\}_{j=n+1}^m$ be, respectively, an orthonormal basis of T_pM and $T_pM^{\perp}$. Let $\{\widetilde{e}_i\}_{i=1}^m$ be an orthonormal basis of $T_p\widetilde{M}$ such that $\widetilde{e}_i = e_i$ for $i = 1, 2, \ldots, n$ and $\widetilde{e}_j = \xi_j$ for $j = n+1, n+2, \ldots, m$. Then, the mean curvature vector can be expressed as*

$$\vec{H}_M(p) = \sum_{j=n+1}^{m} \left(\operatorname{trace}_g L_{\xi_j}\right)\xi_j. \tag{2.9}$$

Proof. Applying equation (2.7), we have that the mean curvature vector of M can be expressed as

$$\begin{aligned}\vec{H}_M(p) = \operatorname{trace}_g \mathrm{II}_p &= \sum_{i=1}^{n} \sum_{j=n+1}^{m} \widetilde{g}\left(\left(\widetilde{\nabla}_{\widetilde{e}_i}\widetilde{e}_i\right)^{\perp}(p), \xi_j\right)\xi_j \\ &= -\sum_{i=1}^{n} \sum_{j=n+1}^{m} \widetilde{g}\left(\widetilde{\nabla}_{\widetilde{e}_i}\xi_j(p), \widetilde{e}_i\right)\xi_j \\ &= \sum_{i=1}^{n} \sum_{j=n+1}^{m} \widetilde{g}\left(L_{\xi_j}(\widetilde{e}_i), \widetilde{e}_i\right)\xi_j \\ &= \sum_{j=n+1}^{m} \left(\operatorname{trace}_g L_{\xi_j}\right)\xi_j .\end{aligned} \tag{2.10}$$

□

To end this subsection we define the *mean curvature pointing inward of the geodesic spheres*. But first note that, given (M, g) an n-dimensional Riemannian manifold, the geodesic sphere $S_R(p)$ of M with radius $R < \operatorname{inj}_g(p)$ centered at $p \in M$ is an $(n-1)$-dimensional submanifold of M, and moreover, $S_R(p)$ is orientable. Thus, for each $q \in S_R(p)$, the orthogonal complement $(T_q S_R(p))^{\perp}$ of $T_q S_R(p)$ in $T_q M$ has dimension equal to 1, and hence, we can choose an unit normal vector $N \in (T_q S_R(p))^{\perp}$ (*i.e*, N is orthogonal to $T_q S_R(p)$ and $\|N\|_g = 1$) such that N points inward $S_R(p)$. Then, we can define the mean curvature pointing inward of the geodesic spheres as follows.

Definition 2.1.49. *Let (M, g) be an n-dimensional Riemannian manifold and $p \in M$. Given $S_R(p)$ a geodesic sphere of M with radius $R < \operatorname{inj}_g(p)$ centered at p and a point $q \in S_R(p)$, we define the* mean curvature pointing inward of the geodesic sphere $S_R(p)$ at q *as the number*

$$H_{S_R(p)}(q) := g\left(\vec{H}_{S_R(p)}(q), N(q)\right), \tag{2.11}$$

where $N \in (T_q S_R(p))^{\perp}$ such that N points inward $S_R(p)$ and $\|N\|_g = 1$, namely $N(q) = -\left.\partial r\right|_q$ being ∂r the unit radial vector field on $p \in M$.

Remark 2.1.50. Throughout this work, we shall consider the *scalar mean curvature* defined using this orientation for the normal vector field N.

2.1.6 Differential operators in Riemannian manifolds

In this subsection, we define some classical differential operators on a Riemannian manifold. We refer to Chapter 1 of [8] and Section 11 of Chapter 3 of [60] for more detailed information about this operators. Let us begin this subsection by giving the notion of *directional derivative* on Riemannian manifolds.

Definition 2.1.51 (see [8]). *Let (M,g) be an n-dimensional Riemannian manifold. Let $p \in M$ be a point of M and let U be a neighbourhood of p. Given a differentiable real valued function f, $f \in C^1(U)$, we associate, for each $v \in T_pM$, the* directional derivative of f in the direction v, $v(f)$, *as*

$$v(f) = (f \circ \alpha)'(0),$$

where $\alpha(t)$ is any curve such that $\alpha(0) = p$ and $\alpha'(0) = v$.

Remark 2.1.52. It is well known that the definition of directional derivative does not depend of the chosen curve. If fact, given two curves α, β such that $\alpha(0) = \beta(0) = p$ and $\alpha'(0) = \beta'(0) = v$, $\alpha \neq \beta$, we have that

$$(f \circ \alpha)'(0) = f_{*_{\alpha(0)}}(\alpha'(0)) = f_{*_p}(v) = f_{*_{\beta(0)}}(\beta'(0)) = (f \circ \beta)'(0).$$

From this, we define the *gradient* operator as follows.

Definition 2.1.53 (see [8]). *Given (M,g) an n-dimensional Riemannian manifold, the* gradient of a scalar field f of M, $\nabla_g f$, *as the vector field which satisfies that*

$$g(\nabla_g f, X) = df(X) = X(f),$$

for any $X \in \mathfrak{X}(M)$ smooth vector field of M.

The gradient of the metric distance function on complete Riemannian manifolds satisfies the following property.

Theorem 2.1.54 (see Proposition 4.8 of Chapter III of [68]). *Let (M,g) be a complete n-dimensional Riemannian manifold and $p \in M$. Then, the metric distance function $\mathrm{dist}_g(p,\cdot)$ is smooth on $M - \{\mathrm{Cut}(p) \cup \{p\}\}$ and its gradient, for any $q \in M - \{\mathrm{Cut}(p) \cup \{p\}\}$, is given by*

$$\nabla_g \left(\mathrm{dist}_g(p,x)\right)\big|_{x=q} = \gamma'\left(\mathrm{dist}_g(p,q)\right)$$

where γ is the unique arc-length minimizing normalized geodesic curve joining p with q. In particular, $\left\| \nabla_g \left(\operatorname{dist}_g(p,x) \right)|_{x=q} \right\|_g = 1$.

Now, we define the *divergence* and the *Laplace-Beltrami* operators as follows.

Definition 2.1.55 (see [8]). *Let (M,g) be an n-dimensional Riemannian manifold with Levi-Civita connection ∇. Given $X \in \mathfrak{X}(M)$ a smooth vector field of M, we define the* divergence of X at a point $p \in M$ *as the function* $\operatorname{div}_g X : M \longrightarrow \mathbb{R}$ *given by*

$$\operatorname{div}_g X(p) := \operatorname{trace}_g \Big(Y(p) \longrightarrow \nabla_Y X(p) \Big) = \sum_{i=1}^{n} g\left(\nabla_{e_i} X(p), e_i \right),$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of T_pM.

Definition 2.1.56 (see [8]). *Let (M,g) be an n-dimensional Riemannian manifold. Given a smooth function $f : M \longrightarrow \mathbb{R}$, we define the* Laplacian of f, $\Delta_g f$, *as the divergence of the gradient of f,* i.e.,

$$\Delta_g f = \operatorname{div}_g \nabla_g f.$$

where the symbol Δ_g denotes the Laplace-Beltrami operator acting on smooth functions.

The following result shows the expression of the Laplacian for any given chart.

Proposition 2.1.57 (see page 5 of [8] and page 67 of [34]). *Let (M,g) be an n-dimensional Riemannian manifold and $p \in M$. Let (D,φ), $\varphi := (x^1,\dots,x^n)$, be a system of coordinates around p. Then, the Laplacian Δ_g with respect to the metric tensor g can be computed in these local coordinates by*

$$\Delta_g = \frac{1}{\sqrt{\det \mathfrak{G}}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\left(\sqrt{\det \mathfrak{G}} \right) \mathfrak{g}^{ij} \frac{\partial}{\partial x^j} \right). \tag{2.12}$$

where $\mathfrak{G} = (\mathfrak{g}_{ij})_{i,j\in\{1,\dots,n\}}$, is the matrix form of the metric tensor g with $\mathfrak{g}_{ij} = g\left(\partial/\partial x^i, \partial/\partial x^j\right)$ and $\mathfrak{g}^{ij}$ are the elements of the inverse matrix of $\mathfrak{G}$, i.e., $\mathfrak{G}^{-1} = (\mathfrak{g}^{ij})_{i,j\in\{1,\dots,n\}}$.

Now, we state some properties of these operators defined on Riemannian manifolds starting with the so-called *Divergence Theorem.* But first, let us define the following sets.

Definition 2.1.58. *Let (M, g) be an n-dimensional Riemannian manifold. We say that a subset $\Omega \subseteq M$ is a* domain in M *if Ω is an open subset in M with piecewise smooth boundary $\partial\Omega$.*

Theorem 2.1.59 (Divergence Theorem (see Theorem 5.11 of Chapter II of [68]))**.** *Let (M, g) be an n-dimensional Riemannian manifold and let $\Omega \subseteq M$ be a pre-compact domain in M. Let ν be the normal unit vector field along $\partial\Omega$ pointing outward. Then, for any C^1 vector field $X \in \mathfrak{X}(\Omega)$, we have*

$$\int_{\Omega} \operatorname{div}_g X \, dV = \int_{\partial\Omega} g\left(X, \nu\right) \, dA.$$

where dV and dA are, respectively, the volume elements on Ω and $\partial\Omega$.

On the other hand, we resume, in the following proposition, the properties satisfied by the gradient, the divergence and the Laplacian.

Proposition 2.1.60 (see pages 1-3 of [8], pages 139-142 of [10] and pages 59 and 69 of [34])**.** *Let (M, g) be an n-dimensional Riemannian manifold. Let $f, h : M \longrightarrow \mathbb{R}$ be two smooth functions and let X, Y be two smooth vector field of M, $X, Y \in \mathfrak{X}(M)$, then*

1. $\nabla_g(f + h) = \nabla_g f + \nabla_g h$.

2. $\nabla_g(fh) = h\nabla_g f + f\nabla_g h$.

3. $\operatorname{div}_g(X + Y) = \operatorname{div}_g X + \operatorname{div}_g Y$.

4. $\operatorname{div}_g(fX) = f \operatorname{div}_g X + g\left(\nabla_g f, X\right)$.

5. $\Delta_g(f + h) = \Delta_g f + \Delta_g h$.

6. $\Delta_g(fh) = h\Delta_g f + 2g\left(\nabla_g f, \nabla_g h\right) + f\Delta_g h$.

7. $\operatorname{div}_g\left(h\nabla_g f\right) = h\Delta_g f + g\left(\nabla_g h, \nabla_g f\right)$.

Now, we have the well known *Green's Formulas*, which are a consequence of the Divergence Theorem and the above properties.

Theorem 2.1.61 (Green's Formulas (see page 6 of [8])). *Let (M, g) be an n-dimensional Riemannian manifold and let $\Omega \subseteq M$ be a precompact domain in M. Let $h \in C^1(\Omega)$ and $f \in C^2(\Omega)$ be functions on Ω such that $h\left(\nabla^M f\right)$ has compact support on Ω. Then*

$$\int_\Omega \left(h\Delta_g f + g(\nabla_g h, \nabla_g f) \right) dV = 0. \tag{2.13}$$

Moreover, assuming further that $h \in C^2(\Omega)$ and that h and f have compact support, we have

$$\int_\Omega (h\Delta_g f - f\Delta_g h)\, dV = 0, \tag{2.14}$$

where dV is the volume element on Ω.

Now we are going to see the *co-area formula*, which will be the key to prove some of our results stated along this work. But first, let us give some considerations about the set of critical values and the set of regular values of a smooth function. Given (M, g) an n-dimensional Riemannian manifold and given $f : M \longrightarrow \mathbb{R}$ a proper smooth function defined on M, (by proper we mean that the inverse image of a compact set is a compact set), we have, by Sard's Theorem (see [11], [29] and [69]), that the set of critical values of f is a set with null measure in $\mathbb{R}$ and the set of regular values, R_f, is an open and dense subset of $\mathbb{R}$. Moreover, it is well known that if $t \in R_f$ then $f^{-1}(t)$ is a compact hypersurface of M, and that $\nabla_g f(p)$ (with $f(p) = t$) is perpendicular to $f^{-1}(t)$. Thus, we have that $g(\nabla_g f, X) = X_p f = 0$ for all $X \in T_p f^{-1}(t)$ (see [59], [68] and [74], for more detailed information about these results). Let us now define, from a function f with the above considerations, the subsets of M given by f as

$$\begin{aligned} \Omega_t &:= \{p \in M \,:\, f(p) < t\}, \\ \Gamma_t &:= \{p \in M \,:\, f(p) = t\}. \end{aligned} \tag{2.15}$$

Then, the *co-area formula* for any integrable function on this level sets is the following

Theorem 2.1.62 (Co-area Formula (see Theorem 5.8 of Chapter II of [68])). *Let (M, g) be an n-dimensional Riemannian manifold and let $f : M \longrightarrow \mathbb{R}_+$ be a proper smooth function on M. Let $u : M \longrightarrow \mathbb{R}_+$ be an integrable function defined on (M, g). Then, for the sets (2.15) given by f, the following assertions hold:*

1. *Denoting by g_t the metric tensor on Γ_t induced by g, we have*
$$\int_M u \left\|\nabla_g f\right\|_g dV_g = \int_{-\infty}^{\infty} \left(\int_{\Gamma_t} u\, dA_{g_t}\right) dt.$$
2. *Since the function $t \longmapsto \operatorname{vol}(\Omega_t)$ is a smooth function when $t \in R_f$ is a regular value of f such that $\operatorname{vol}(\Omega_t) < \infty$, we have*
$$\frac{d}{dt}\operatorname{vol}(\Omega_t) = \int_{\Gamma_t} \left\|\nabla_g f\right\|_g^{-1} dA_{g_t}.$$

where dV_g and dA_{g_t} are, respectively, the volume elements on M and Γ_t.

To end this section we define the notion of subharmonic functions and we gather *the strong maximum principle and the Hopf's boundary point lemma for subharmonic functions*, which will let us prove our comparisons of the mean exit time function, torsional rigidity, Poisson hierarchy and moment spectrum, and moreover, characterise the equality cases (we state and prove these mentioned results at Chapter 3). For more information about subharmonic functions and the strong maximum principle see [34] and [64], and for the *Hopf's boundary point lemma* see [27], for instance.

Definition 2.1.63 (see [64])**.** *Let (M, g) be an n-dimensional Riemannian manifold. Given a function $u : M \longrightarrow \mathbb{R}$, we say that u is a* subharmonic (superharmonic) function *if, and only if,*
$$\Delta_g u \geq (\leq)\, 0.$$

Moreover, we say that u is a harmonic function *if, and only if,*
$$\Delta_g u = 0.$$

Theorem 2.1.64 (Strong Maximum Principle and Hopf Boundary Point Lemma (see Theorems 2.2 and 3.5 and Lemma 3.4 of [27] and Theorem 66 of Section 9.3 of [64]))**.** *Let (M, g) be an n-dimensional Riemannian manifold and let $\Omega \subseteq M$ be a bounded and connected domain of M. Let $u : \Omega \longrightarrow \mathbb{R}$ be a subharmonic function,* i.e., *$\Delta_g u \geq 0$, such that $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$. Then, we have that:*

1. *If u achieves its maximum in Ω then u is constant.*
2. *If there is a point $p_0 \in \partial\Omega$ such that $u(p) < u(p_0)$ for any $p \in \Omega$ then $\nu(u) = g(\nabla_g u, \nu) > 0$, where ν denotes the unit normal vector along $\partial\Omega$ pointing outward.*

2.1.7 Normal and polar coordinates. Riemannian measure

In this subsection we define the *normal* and *polar coordinates* on the open neighbourhoods of a point of complete n-dimensional Riemannian manifolds such that the exponential map is a diffeomorphism (see Subsection 2.1.3 to check the existence of such neighbourhoods). Finally, to end the section, we will define the *Riemannian volume element* and show its expressions in normal and polar coordinates.

2.1.7.1 Normal coordinates

First note that, given (M, g) a complete n-dimensional Riemannian manifold and given a point $p \in M$, we know, from Theorem 2.1.28, that if $D \subseteq M - \mathrm{Cut}(p)$ is an open neighbourhood of p then there is a neighbourhood V_D of the origin o_p in T_pM such that the exponential map $\exp_p : V_D \longrightarrow D$ is a diffeomorphism. Thus, given an orthonormal basis $\{e_i\}_{i=1}^n$ of T_pM and following the construction given in Section 3 of Chapter 5 of [48], we may define a diffeomorphism $\zeta : D \longrightarrow V_D$ by assigning to each $q \in D$ the components of $\exp_p^{-1}(q) \in T_pM$ with respect to $\{e_i\}_{i=1}^n$. Moreover, these components, together the diffeomorphism, forms a system of coordinates (or chart) on D. Thus we define the following.

Definition 2.1.65 (see [68]). *With the previous setting, we define the* normal coordinate functions associated to the orthonormal basis $\{e_i\}_{i=1}^n$ with respect to g on D *as the functions* $\{x^i\}_{i=1}^n$ *given by*

$$\begin{aligned} x^i : D &\longrightarrow \mathbb{R} \\ q &\longmapsto x^i(q) := g\left(e_i, \exp_p^{-1}(q)\right), \end{aligned} \tag{2.16}$$

in such a way that $\exp_p^{-1}(q) = \sum_{i=1}^n x^i(q)e_i$ *for all* $q \in D$.

Moreover, we define the normal coordinates on D centered at p *as the system of coordinates (or chart)* (D, ζ) *given by the normal coordinate functions and the diffeomorphism*

$$\begin{aligned} \zeta : D &\longrightarrow V_D \subset T_pM \\ q &\longmapsto \zeta(q) := \left(x^1(q), \dots, x^n(q)\right). \end{aligned} \tag{2.17}$$

Remark 2.1.66. By Definition 2.1.18 and Proposition 2.1.16, if $R < \mathrm{inj}_g(p)$ then the exponential map $\exp_p$ defined from the open ball $\mathbb{B}_R(o_p)$ of T_pM with radius R centered at the center o_p of T_pM is a diffeomorphism onto the geodesic ball $B_R(p)$ of M with radius R centered at p. Therefore, by taking $D = B_R(p)$ and $V_D = \mathbb{B}_R(o_p)$ in the above definition, we can define normal coordinates $(B_R(p), \zeta)$ as in equation (2.17).

On the other hand, since (D, ζ) is a system of coordinates, the coordinate vectors, for any $q \in D$,

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_q, \dots, \left.\frac{\partial}{\partial x^n}\right|_q \right\}$$

forms a basis of the tangent space T_qM, and hence, we can define coordinate vector fields $\partial/\partial x^i$ which sends each $q \in D$ to $\partial/\partial x^i|_q$ (see Chapter 1 of [60], for instance). This observation allow us to remark the following.

Remark 2.1.67. Let (M, g) be a complete n-dimensional Riemannian manifold and $p \in M$. Given an open neighbourhood $D \subseteq M - \mathrm{Cut}(p)$ of p and given normal coordinates (D, ζ), we have the *normal coordinate one-forms* $\left\{dx_q^i\right\}$ as the dual one-forms given by

$$dx_q^i \left(\left.\frac{\partial}{\partial x^j}\right|_q \right) = \delta_{i,j}$$

where x^i are the normal coordinate functions (2.16).

Remark 2.1.68. Observe that, since the normal coordinates (D, ζ) are a system of coordinates, we have that the Riemannian metric g can be expressed at any point q of D as

$$g_q = \sum_{i,j=1}^{n} \mathfrak{g}_{ij}(q)\, dx_q^i \otimes dx_q^j,$$

where $\mathfrak{g}_{ij}(q) = g_q \left(\partial/\partial x^i|_q \,, \partial/\partial x^j|_q \right)$ for all $i, j \in \{1, \dots, n\}$ and $\left\{dx_q^i\right\}_{i=1}^n$ are the normal coordinate one-forms, and moreover, we obtain that the matrix

$$\mathfrak{G}(q) = \Big(\mathfrak{g}_{ij}(q) \Big)_{i,j \in \{1,\dots,n\}} \tag{2.18}$$

is a positive definite matrix.

2.1.7.2 Polar coordinates

Now, we define the *polar coordinates* on neighbourhoods $D \subseteq M - \mathrm{Cut}(p)$ of $p \in M$. The following concepts and results can be found in Chapter 5 and 6 of [48]. First, let us define the natural projection of a point given in normal coordinates onto $\mathbb{R}_+$ as follows.

Definition 2.1.69 (see [48]). *Let (M, g) be a complete n-dimensional Riemannian manifold. Given a point $p \in M$ and given normal coordinates (D, ζ) centered at p, we define the* radial distance function r_p *as the function*

$$\begin{aligned} r_p : D &\longrightarrow \mathbb{R}_+ \\ q &\longmapsto r_p(q) := \sqrt{\left(x^1(q)\right)^2 + \cdots + \left(x^n(q)\right)^2} \end{aligned} \tag{2.19}$$

where x^i are the normal coordinate functions (2.16).

The following result shows some important properties of the normal coordinates, the metric distance function, the radial distance function and the unit radial vector field (see Definitions 2.1.65, 2.1.3, 2.1.69 and 2.1.37). This statement can be deduced from the properties of the geodesics, the exponential map and the differential of the distance function (see Subsections 2.1.2, 2.1.3 and 2.1.6).

Proposition 2.1.70 (see Proposition 5.11 and Corollaries 6.9 and 6.11 of [48]). *Let (M, g) be a complete n-dimensional Riemannian manifold and $p \in M$. Let D be an open neighbourhood $D \subseteq M - \mathrm{Cut}(p)$ of p. Then, for normal coordinates (D, ζ) centered at p, normal coordinate functions x^i and radial distance function r_p, the following assertions hold:*

1. *For any $u = u^1\partial/\partial x^1 + \cdots + u^n\partial/\partial x^n \in T_pM$, the geodesic curve γ_u starting at p with velocity u is represented in normal coordinates by the radial line segment*
$$\gamma_u(t) = \zeta^{-1}\left(tu^1, \ldots, tu^n\right),$$
for all t such that $\gamma_u(t) \in D$.

2. *The normal coordinates at p are $(0, \ldots, 0)$, and hence, $r_p(p) = 0$.*

3. *The components of the metric at p are $\mathfrak{g}_{ij} = \delta_{i,j}$, and the first partial derivatives of $\mathfrak{g}_{ij}$ vanish at p.*

4. *The radial distance function r_p is exactly the metric distance function* $\operatorname{dist}_g(p,\cdot)$ *in D, and the vector field $\partial/\partial r_p$ on T_qM given by the radial distance function is the gradient of the distance function on D. Moreover, at $q \in D - \{p\}$, $\partial/\partial r_p$ is the tangent vector to the unique arc-length minimizing normalized geodesic curve joining p with q,* i.e., *$\partial/\partial r_p$ is the unit radial vector field ∂r, and hence, its norm is equal to 1. Namely, we have that r_p is a positive smooth function on $D - \{p\}$ such that*

$$r_p(q) = \|\zeta\|_g = \left\|\exp_p^{-1}(q)\right\|_g = \operatorname{dist}_g(p,q), \quad \textit{for all} \quad q \in D - \{p\},$$

and

$$\partial r = \frac{\partial}{\partial r_p} = \nabla_g \operatorname{dist}_g(p,q) = \nabla_g r_p \quad \textit{and} \quad \left\|\frac{\partial}{\partial r_p}\right\|_g = 1, \quad \textit{on} \quad D - \{p\}.$$

Moreover, the vector field $\partial/\partial r_p$ is g-orthogonal to the geodesic sphere $S_{r_p(q)}(p)$ of M with radius $r_p(q) = \operatorname{dist}_g(p,q)$ centered at p and it can be expressed in T_qM by normal coordinate vector fields $\{\partial/\partial x^i\}_{i=1}^n$ as

$$\left.\frac{\partial}{\partial r_p}\right|_q = \sum_{i=1}^n \frac{x^i(q)}{r_p(q)} \left.\frac{\partial}{\partial x^i}\right|_q. \tag{2.20}$$

On the other hand, from the definition of the radial distance function, we define the natural projection to the $(n-1)$-dimensional usual unit sphere $\mathbb{S}_1^{n-1}$ as follows.

Definition 2.1.71 (see [68]). *Let (M,g) be a complete n-dimensional Riemannian manifold and $p \in M$. Given an open neighbourhood $D \subseteq M - \operatorname{Cut}(p)$ of p and normal coordinates (D,ζ) centered at p, we define the* projection to the usual unit sphere *π_p as*

$$\begin{aligned} \pi_p : D - \{p\} &\longrightarrow \mathbb{S}_1^{n-1} \\ q &\longmapsto \pi_p(q) := \frac{\zeta(q)}{r_p(q)}, \end{aligned} \tag{2.21}$$

where r_p is the radial distance function on D.

Remark 2.1.72. Note that the projection π_p is not defined at p because, by assertion (2) of Proposition 2.1.70, $r_p(p) = 0$.

Thus, considering $\mathbb{S}_1^{n-1}$ as an $(n-1)$-dimensional Riemannian manifold and given a chart $(\mathbb{S}_1^{n-1}, \widetilde{\theta})$ with coordinate functions $\{\widetilde{\theta}^i\}_{i=1}^{n-1}$, we can define coordinate functions of a point $q \in D \subseteq M - \mathrm{Cut}(p)$ in $\mathbb{S}_1^{n-1}$ as $(\theta^1(q), \ldots, \theta^{n-1}(q))$ where each θ^i is given by the projection π and the ith-coordinate function $\widetilde{\theta}^i$ in $\mathbb{S}_1^{n-1}$ of $\pi(q)$, *i.e.*, for all $i = 1, 2, \ldots, n-1$,

$$\begin{array}{ccccc} D - \{p\} & \xrightarrow{\pi_p} & \mathbb{S}_1^{n-1} & \xrightarrow{\widetilde{\theta}^i} & \mathbb{R} \\ q & & \longmapsto & & \theta^i(q) := \widetilde{\theta}^i \circ \pi_p(q) \end{array}$$

and hence, we have that $\pi_p(q)$ can be represented in coordinates in $\mathbb{S}_1^{n-1}$ as

$$\theta(q) = \left(\theta^1(q), \ldots, \theta^{n-1}(q)\right) = \left(\widetilde{\theta}^1\left(\pi_p(q)\right), \ldots, \widetilde{\theta}^{n-1}\left(\pi_p(q)\right)\right) = \widetilde{\theta}\left(\pi_p(q)\right).$$

From this facts and from the Definition 2.1.69 of the radial distance function, we can define the following system of coordinates on D associated to the normal coordinates (2.17).

Definition 2.1.73 (see [68])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold and $p \in M$. Let an $D \subseteq M - \mathrm{Cut}(p)$ be an open neighbourhood of p and let V_D be a neighbourhood of the origin o_p in T_pM such that* $\exp_p : D \longrightarrow V_D$ *is a diffeomorphism. Given normal coordinates (D, ζ), we define the* polar coordinates on $D - \{p\}$ centered at p associated to the normal coordinates *as the system of coordinates (or chart) $(D - \{p\}, \psi)$ given by the diffeomorphism*

$$\begin{aligned} \psi : D - \{p\} &\longrightarrow V \subset \mathbb{R}^n \\ q &\longmapsto \psi(q) := (r_p(q), \theta^1(q), \ldots, \theta^n(q)). \end{aligned}$$

As noted for the normal coordinates, for any $q \in D - \{p\}$, the coordinate vectors

$$\left\{ \left.\frac{\partial}{\partial r_p}\right|_q, \left.\frac{\partial}{\partial \theta^1}\right|_q, \ldots, \left.\frac{\partial}{\partial \theta^{n-1}}\right|_q \right\}$$

form a basis of the tangent space T_qM, and hence, we can define the coordinate vector fields $\partial/\partial r_p, \partial/\partial\theta^i$ which sends each $q \in D$ to $\partial/\partial r_p|_q, \partial/\partial\theta^i|_q$, and moreover, we have the associated dual one-forms $\{dr_{p_q}, d\theta_q^1 \ldots, d\theta_q^{n-1}\} \in T_qM$. Therefore, the metric tensor g can be expressed in polar coordinates $(D - \{p\}, \psi)$ as

$$g = g_{rr} dr_p \otimes dr_p + \sum_{i=1}^{n-1} g_{ri} dr_p \otimes d\theta^i + \sum_{i=1}^{n-1} g_{ri} d\theta^i \otimes dr_p + \sum_{i,j=1}^{n-1} g_{ij} d\theta^i \otimes \theta^j,$$

where $\otimes$ denotes the tensor product and, for all $i, j \in \{1, \ldots, n-1\}$,

$$g_{rr} = g\left(\frac{\partial}{\partial r_p}, \frac{\partial}{\partial r_p}\right) = \left\|\frac{\partial}{\partial r_p}\right\|_g^2, \quad g_{ri} = g\left(\frac{\partial}{\partial r_p}, \frac{\partial}{\partial \theta^i}\right), \quad g_{ij} = g\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j}\right).$$

But, since by assertion (4) of Proposition 2.1.70 we know that the vector field $\partial/\partial r_p$ is an unitary vector field and that is g-orthogonal to the geodesic sphere $S_{r_p(q)}(p)$ of $M - \mathrm{Cut}(p)$ with radius $r_p(q)$ centered at p, then $g_{rr} = 1$ and $g_{ri} = 0$ for all $i \in \{1, \ldots, n-1\}$. Therefore, the metric tensor g in polar coordinates in $D - \{p\}$ is given by

$$g = dr_p \otimes dr_p + \sum_{i,j=1}^{n-1} g_{ij} d\theta^i \otimes d\theta^j. \tag{2.22}$$

Thus, the matrix form of the metric tensor g in polar coordinates is the positive definite matrix

$$\mathcal{G} := \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & G & \\ 0 & & & \end{array}\right) \tag{2.23}$$

where G is the matrix which elements are g_{ij}, *i.e.*, $G = (g_{ij})_{i,j\in\{1,\ldots,n-1\}}$, and hence, we have, for any point $q \equiv (r, \theta) \in D - \{p\}$, that

$$\sqrt{\det(\mathcal{G}(r,\theta))} = \sqrt{\det(G(r,\theta))}. \tag{2.24}$$

Moreover, if we consider the geodesic sphere $S_r(p)$ of $M - \mathrm{Cut}(p)$ with radius r centered at p as an $(n-1)$-dimensional Riemannian submanifold, we have that the system of coordinates $(S_r(p), \theta)$, with coordinate function θ^i, is a chart on $S_r(p)$, and hence, its metric tensor induced by g can be expressed as

$$g_{S_r(p)} = \sum_{i,j=1}^{n-1} g_{ij} d\theta^i \otimes d\theta^j. \tag{2.25}$$

Remark 2.1.74. From now on, when it is clear from the context, we denote r, $\partial/\partial r$, $\nabla_g r$, dr and π to refer to r_p, $\partial/\partial r_p$, $\nabla_g r_p$, dr_p and π_p, respectively.

2.1.7.3 Laplacian and mean curvature in polar coordinates

Using the expression of the Laplacian for any given system of coordinates and the expression of the metric tensor in polar coordinates $(D - \{p\}, \psi)$ on $M - \mathrm{Cut}(p)$ centered at p with coordinate functions r, θ^i (see equations (2.12) and (2.22)), we obtain, by a straightforward computation, that the Laplacian in polar coordinates is given by

$$\begin{aligned}\Delta_g = \frac{1}{\sqrt{\det G}} \Bigg(&\frac{\partial}{\partial r}\left(\sqrt{\det G}\, g^{rr} \frac{\partial}{\partial r}\right) + \sum_{i=1}^{n-1} \frac{\partial}{\partial r}\left(\sqrt{\det G}\, g^{ri} \frac{\partial}{\partial \theta^i}\right) \\ &+ \sum_{i=1}^{n-1} \frac{\partial}{\partial \theta^i}\left(\sqrt{\det G}\, g^{ir} \frac{\partial}{\partial r}\right) + \sum_{i,j=1}^{n-1} \frac{\partial}{\partial \theta^i}\left(\sqrt{\det G}\, g^{ij} \frac{\partial}{\partial \theta^j}\right)\Bigg).\end{aligned}$$

Therefore, since it is easy to check that $g^{rr} = 1$ and $g^{ri} = g^{ir} = 0$ for all $i \in \{1, \ldots, n-1\}$, and using the expression (2.12) of the Laplacian $\Delta_{g_{S_r(p)}}$ of the geodesic sphere $S_r(p)$ with respect to the chart $\{\theta^i\}_{i=1}^{n-1}$ in $S_r(p)$ (considered $S_r(p)$ as a submanifold of M), we have that the Laplacian in polar coordinates at a point $q \equiv (r, \theta) \in D - \{p\}$ can be computed as

$$\begin{aligned}\Delta_g &= \frac{1}{\sqrt{\det\left(G(r,\theta)\right)}} \left(\frac{\partial}{\partial r}\left(\sqrt{\det\left(G(r,\theta)\right)} \frac{\partial}{\partial r}\right) \right. \\ &\qquad\qquad \left. + \sum_{i,j=1}^{n-1} \frac{\partial}{\partial \theta^i}\left(\sqrt{\det\left(G(r,\theta)\right)}\, g^{ij} \frac{\partial}{\partial \theta^j}\right)\right) \\ &= \frac{1}{\sqrt{\det\left(G(r,\theta)\right)}} \frac{\partial}{\partial r}\left(\sqrt{\det\left(G(r_p,\theta)\right)} \frac{\partial}{\partial r}\right) + \Delta_{g_{S_r(p)}} \qquad (2.26) \\ &= \frac{\partial^2}{\partial r^2} + \frac{\frac{\partial}{\partial r}\sqrt{\det\left(G(r,\theta)\right)}}{\sqrt{\det\left(G(r,\theta)\right)}} \frac{\partial}{\partial r} + \Delta_{g_{S_r(p)}} \\ &= \left(\frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}\left(\ln \sqrt{\det\left(G(r,\theta)\right)}\right) \frac{\partial}{\partial r}\right) + \Delta_{g_{S_r(p)}}.\end{aligned}$$

We refer to [33] for more information about this expression of the Laplacian. Moreover, from this last expression of the Laplacian and using the expression (2.9) of the mean curvature vector, we can prove the following result which shows how we can compute the mean curvature pointing inward of geodesic spheres.

Proposition 2.1.75. *Let (M, g) be a complete n-dimensional Riemannian manifold with Levi-Civita connection ∇ and $p \in M$. Let $B_R(p)$ be a geodesic ball of M with radius R centered at p. Suppose that $R < \operatorname{inj}_g(p)$ and let $(B_R(p) - \{p\}, \psi)$ be polar coordinates in $B_R(p) - \{p\}$ centered at p with coordinate functions r, θ^i. Then, for all $0 < r \leq R$, the mean curvature pointing inward of the geodesic sphere $S_r(p)$ at a point $q \equiv (r, \theta) \in S_r(p)$ is given by*

$$H_{S_r(p)}(q) = \Delta_g r(q) = \frac{\frac{\partial}{\partial r}\sqrt{\det\left(G(r,\theta)\right)}}{\sqrt{\det\left(G(r,\theta)\right)}}. \tag{2.27}$$

Proof. Let $S_r(p)$ be a geodesic sphere with radius $0 < r \leq R < \operatorname{inj}_g(p)$ centered at p and let $q \equiv (r, \theta) \in S_r(p)$. Then, by assertion (4) of Proposition 2.1.70, we know that the gradient of the radial distance function, $\nabla_g r$ is the unit radial vector field in the direction of the geodesic curves emanating from p and that is g-orthogonal to $S_r(p)$. Therefore, choosing an orthonormal basis $\{e_i\}_{i=1}^{n-1}$ of $T_q S_r(p)$ and $e_n = \nabla_g r|_q$, we have that $\{e_i\}_{i=1}^{n}$ is an orthonormal basis of $T_q M$ (from now on we identify $\nabla_g r = \nabla_g r|_q$ to simplify the notation). Moreover, note that $\nabla_g r$ points outward the geodesic sphere $S_r(p)$.

Thus, since $g(\nabla_{\nabla_g r}\nabla_g r(q), \nabla_g r(q)) = 0$ (because $g(\nabla_g r, \nabla_g r) = 1$), and applying equation (2.9) of Proposition 2.1.48, we can express the mean curvature vector of the geodesic sphere $S_r(p)$ at the point q as

$$\begin{aligned}
\vec{H}_{S_r(p)}(q) &= (\operatorname{trace} L_{\nabla_g r})\nabla_g r = \left(\sum_{i=1}^{n-1} g\left(L_{\nabla_g r} e_i, e_i\right)\right)\nabla_g r \\
&= -\sum_{i=1}^{n-1} g\left(\nabla_{e_i}\nabla_g r, e_i\right)\nabla_g r = -\sum_{i=1}^{n} g\left(\nabla_{e_i}\nabla_g r, e_i\right)\nabla_g r \\
&= -\operatorname{div}_g\left(\nabla_g r\right)\nabla_g r = -\Delta_g r\,\nabla_g r.
\end{aligned}$$

Hence, since $N = -\nabla_g r = -\partial r$ is the unit normal vector to the geodesic sphere $S_r(p)$ pointing inward, we obtain that

$$H_{S_r(p)}(q) = g\left(-\Delta_g r\,\nabla_g r, -\nabla_g r\right) = \Delta_g r,$$

and hence, from the above expression of the Laplacian in polar coordinates (see equation (2.26)), the proposition follows.

□

2.1.7.4 Notes on the sectional curvature

Let (M^2, g) be a complete 2-dimensional Riemannian manifold and let $p \in M$ be a point in M. Taking polar coordinates $(D - \{p\}, \psi)$ on an open neighbourhood $D \subseteq M^2 - \mathrm{Cut}(p)$ of p, with coordinate functions r, θ, we have that the metric tensor at $D - \{p\}$ can be expressed as

$$g = dr \otimes dr + \varphi^2(r, \theta) d\theta \otimes d\theta, \tag{2.28}$$

for some positive smooth function $\varphi : D \longrightarrow \mathbb{R}_+$ (see Definition 2.1.73 and equation (2.22)). And moreover, choosing the orthogonal basis $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\}$ of the tangent space $T_q M^2$ we have that, for any point $q \in D - \{p\}$, there is only one 2-dimensional subspace tangent to q, $\sigma(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta})$, and hence, we have that the sectional curvature is, in this case, the radial sectional curvature (see Definitions 2.1.33 and 2.1.37). Furthermore, we have the following result which shows how the radial sectional curvature for any point $q \equiv (r, \theta) \in D - \{p\}$ at the unique 2-plane $\sigma(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta})$ can be computed as follows.

Proposition 2.1.76 (see page 46 of [64]). *Let (M^2, g) be a complete 2-dimensional Riemannian manifold and $p \in M$. Let $D \subset M - \mathrm{Cut}(p)$ be an open neighbourhood of p and let $(D - \{p\}, \psi)$ be polar coordinates on $D - \{p\}$ centered at p with coordinate functions r, θ. Then, for any $q \in D - \{p\}$, the sectional curvature at q of $\sigma_q(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}) = T_q M$ coincides with its radial sectional curvature and it can be computed as*

$$\sec_g\left(\sigma_q\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right)\right) = -\frac{\dfrac{\partial^2 \varphi}{\partial r^2}(r, \theta)}{\varphi(r, \theta)}.$$

where $\varphi : D - \{p\} \longrightarrow \mathbb{R}_+$ is the positive smooth function given by the metric tensor expressed in polar coordinates (2.28).

2.1.7.5 The volume element

To end this section, we are going to express the *Riemannian volume element* using normal and polar coordinates. For a more detailed background on volume elements on Riemannian manifolds, we refer to Section 3 of Chapter 15 and Chapter 16 of [49], for instance.

Given the normal coordinates (D, ζ) and the polar coordinates $(D - \{p\}, \psi)$ centered at a point p of an n-dimensional Riemannian manifold (M, g), the Riemannian volume element is, respectively, the following

$$\begin{aligned} dV_g &= \sqrt{\det(\mathfrak{G})}\, dx^1 \wedge \cdots \wedge dx^n \\ &= \sqrt{\det(G)}\, dr \wedge d\theta^1 \wedge \cdots \wedge d\theta^{n-1} \end{aligned}$$

where $\mathfrak{G}$ and G are, respectively, determined by the positive definite matrices (2.18) and (2.23) (with the consideration (2.24)).

Now, we present the following result which shows how the volumes of geodesic balls with radius less than the injectivity radius can be computed.

Theorem 2.1.77. *(see page 116 and Propositions 3.2 and 3.4 of [10]) Let (M, g) be a complete n-dimensional Riemannian manifold and $p \in M$. Let $B_R(p)$ be a geodesic ball of M with radius R centered at p. Suppose that $R < \mathrm{inj}_g(p)$ and let $(B_R(p), \zeta)$ and $(B_R(p) - \{p\}, \psi)$ be, respectively, normal coordinates with coordinate functions x^i and polar coordinates with coordinates functions r, θ^i. Then, the* $\mathrm{vol}(B_R(p))$ *and the* $\mathrm{vol}(S_R(p))$ *are smooth functions of $\mathbb{R}$, and moreover, these volumes can be computed by*

$$\begin{aligned} \mathrm{vol}(B_R(p)) &= \int_{B_R(p)} dV_g = \int_{B_R(p)} \sqrt{\det(\mathfrak{G}(\zeta(q))}\, dx^1 \wedge \cdots \wedge dx^n \\ &= \int_{B_R(p)} \sqrt{\det(G(r,\theta))}\, dr \wedge d\theta^1 \wedge \cdots \wedge d\theta^{n-1} \\ &= \int_0^R \left(\int_{S_{r_p}(p)} \sqrt{\det(G(r,\theta))}\, d\theta^1 \wedge \cdots \wedge d\theta^n \right) dr, \end{aligned}$$

$$\mathrm{vol}(S_R(p)) = \left. \frac{\partial \, \mathrm{vol}(B_r(p))}{\partial r} \right|_{r=R} = \int_{S_R(p)} \sqrt{\det(G(R,\theta))}\, d\theta^1 \wedge \cdots \wedge d\theta^n.$$

Remark 2.1.78. Note that, for a fixed $R > 0$, the metric tensor of the geodesic sphere $S_R(p)$ considered as an $(n-1)$-dimensional Riemannian manifold is given by (2.25). Therefore, its Riemannian volume element is given by

$$dV_{g_{S_R(p)}} = \sqrt{\det(G(R,\theta))}\, d\theta^1 \wedge \cdots \wedge d\theta^n.$$

To simplify the notation, we will denote the *Riemannian volume element of a geodesic sphere* $S_R(p)$ by

$$dA_g := dV_{g_{S_R(p)}} \quad \text{for any} \quad R < \mathrm{inj}_g(p).$$

Moreover, along this work, we will use the volume of the geodesic sphere as a functions depending only on the radius of the geodesic sphere and thus, in this sense, we will refer to its volume as the *area function of the geodesic sphere* $S_r(p)$ given by

$$A_g(r) := \int_{S_r(p)} dA_g = \int_{S_r(p)} \sqrt{\det\left(G(r,\theta)\right)}\, d\theta^1 \wedge \cdots \wedge d\theta^n, \tag{2.29}$$

Moreover, the Taylor expansion of the area function $A_g(r)$ about $r = 0$ can be expressed as follows.

Theorem 2.1.79 (see Theorem 3.1 and equation (11) of [30])**.** *Let (M, g) be an n-dimensional Riemannian manifold and let $p \in M$. Let $S_R(p)$ be the geodesic sphere of M with radius R centered at p. Suppose that $R < \operatorname{inj}_g(p)$. Then, for all $0 \leq r \leq R$, the expression of the Taylor expansion about $r = 0$ of the area function $A_g(r)$ is given by*

$$A_g(r) = a_0 r^{n-1} + a_2 r^{n+1} + a_4 r^{n+3} + \cdots\,,$$

for some constants $a_{2k} \in \mathbb{R}$, $k \in \mathbb{N}$, with $a_0 = \operatorname{vol}\left(\mathbb{S}_1^{n-1}\right)$.

2.2 Rotationally symmetric model spaces

In this section we define the *rotationally symmetric model spaces* and study the definition of its metric tensor. For a background on rotationally symmetric model spaces we refer to Section 8 of Chapter 1 of L.J Alías, P. Mastrolia and M. Rigoli [1], Chapter 2 of R.E. Greene and H.H. Wu [32], Chapter 3 of A. Grigor'yan [33], Sections 10, 11 and 12 of Chapter 7 of B. O'Neil [60], Sections 3 and 4 of Chapter 1 and Section 2 of Chapter 3 of P. Petersen [64], or more recently, Subsection 3 of Section 2 of A. Hurtado, V. Palmer and C. Rosales [41].

2.2.1 Rotationally symmetric model spaces

To define the rotationally symmetric model spaces we follow the notion of these kind of spaces given in, for instance, Section 8 of Chapter 1 of [1], Section 2 of Chapter 3 of [33] and, more recently, Subsection 3 of Section 2 of [41]. There are several other equivalent ways to define them, see Chapter 2 of [32] for instance.

Definition 2.2.1 (see [41]). *Given (M, g) an n-dimensional Riemannian manifold with $n \geq 2$, we say that (M, g) is a* rotationally symmetric model space with center $p \in M$ and model radius $\Lambda \in \mathbb{R}_+ \cup \{+\infty\}$ *if* $\exp_p : \mathbb{B}_\Lambda(o_p) \subset T_pM \longrightarrow M$ *is a diffeomorphism and the metric tensor g can be expressed on $M - \{p\}$ as*

$$g = dr_p \otimes dr_p + \left(\omega^2 \circ r_p\right) \pi_p^* \, g_{\mathbb{S}_1^{n-1}} \tag{2.30}$$

where ω is a positive smooth function $\omega : [0, \Lambda) \longrightarrow \mathbb{R}_+$ such that $\omega(t) > 0$ for all $t > 0$, r_p and π_p are, respectively, the radial distance function to p and the projection to $\mathbb{S}_1^{n-1}$ (see Definitions 2.1.69 and 2.1.71, respectively), and $\pi_p^* g_{\mathbb{S}_1^{n-1}}$ is the pullback by π_p of the canonical metric tensor $g_{\mathbb{S}_1^{n-1}}$ of $\mathbb{S}_1^{n-1}$.

Remark 2.2.2. We shall denote the rotationally symmetric model space (M, g) by $(\mathbb{M}_\omega, g_\omega)$, its center point p by o_ω and its geodesic balls and geodesic spheres with radius R centered at the center o_ω by $B_R^\omega(o_\omega)$ and $S_R^\omega(o_\omega)$, respectively, and moreover, when it is clear from the context, we denote the radial distance function r_{o_ω} to the center o_ω and the projection π_{o_ω} to $\mathbb{S}_1^{n-1}$ by r and π, respectively. Furthermore, we will refer to the metric tensor g_ω as the *rotationally symmetric metric tensor*. Note that, since exponential map is a diffeomorphism from the metric ball $\mathbb{B}_\Lambda(o_p)$ onto the entire Riemannian manifold M it can be checked, by Definition 2.1.18, that the model radius Λ coincides with injectivity radius of o_ω. Furthermore, if $(\mathbb{M}_\omega, g_\omega)$ is rotationally symmetric model space with model radius $\Lambda = +\infty$ then the exponential map is a diffeomorphism from the entire $T_{o_p}M$, and hence, o_ω is a pole of $\mathbb{M}_\omega$.

Moreover, observe that the expression (2.30) of the rotationally symmetric metric tensor implies that the expression of the metric tensor expressed in polar coordinates of a geodesic sphere $S_r^\omega(o_\omega) \subset \mathbb{M}_\omega$ of radius $r > 0$ centered at the center o_ω is obtained by scaling the canonical metric tensor of $\mathbb{S}_1^{n-1}$ by the factor $\omega^2(r)$ (see [33]), *i.e.*,

$$g_{S_r^\omega(o_\omega)} = \omega^2(r) \pi_{o_\omega}^* g_{\mathbb{S}_1^{n-1}}.$$

Hence, from Theorem 2.1.77 and Remark 2.1.78, the area function of the geodesic spheres $S_r^\omega(o_\omega) \subseteq \mathbb{M}_\omega$ and the volume of the geodesic balls $B_r^\omega(o_\omega) \subseteq \mathbb{M}_\omega$ are

$$\begin{aligned} A_{g_\omega}(r) = \operatorname{vol}\left(S_r^\omega(o_\omega)\right) &= \int_{\mathbb{S}_1^{n-1}} \omega^{n-1}(r) \, dV_{g_{\mathbb{S}_1^{n-1}}} = \omega^{n-1}(r) \operatorname{vol}\left(\mathbb{S}_1^{n-1}\right), \\ \operatorname{vol}\left(B_r(o_\omega)\right) &= \int_0^r \operatorname{vol}\left(S_t^\omega(o_\omega)\right) dt = \operatorname{vol}\left(\mathbb{S}_1^{n-1}\right) \int_0^r \omega^{n-1}(t) \, dt. \end{aligned} \tag{2.31}$$

Furthermore, by an straightforward computation using equation (2.12) of Proposition 2.1.57 and from Proposition 2.1.75, we have that the Laplacian Δ_{g_ω} and the mean curvature pointing inward $H_{S_r^\omega(o_\omega)}$ at a point $p \equiv (r, \theta) \in S_r^\omega(o_\omega)$ of the rotationally symmetric model spaces can be computed by

$$\Delta_{g_\omega} = \left.\frac{\partial^2}{\partial r_{o_\omega}^2}\right|_{r_{o_\omega}=r} + (n-1)\frac{\omega'(r)}{\omega(r)}\left.\frac{\partial}{\partial r_{o_\omega}}\right|_{r_{o_\omega}=r} + \Delta_{g_{S_r^\omega(o_\omega)}}, \tag{2.32}$$

$$H_{S_r^\omega(o_\omega)}(r,\theta) = \Delta_{g_\omega} r_{o_\omega}(r,\theta) = (n-1)\frac{\omega'(r)}{\omega(r)}, \tag{2.33}$$

where $\Delta_{g_{S_r^\omega(o_\omega)}}$ denotes the Laplacian of the geodesic sphere considered as a submanifold of $\mathbb{M}_\omega$.

To end this remark note that, if $(\mathbb{M}_\omega, g_\omega)$ is a rotationally symmetric model space with center o_ω and its radius $\Lambda < +\infty$, then $\mathbb{M}_\omega - \{o_\omega\}$ is isometric to the *warped product* $(0, \Lambda) \times_\omega \mathbb{S}_1^{n-1}$ with *warping function* ω, while if o_ω is a pole then the rotationally symmetric model space is isometric to the *warped product* $\mathbb{R}_+ \times_\omega \mathbb{S}_1^{n-1}$. From this fact, we will refer along this work to ω as the *warping function* (as its usually done in the literature, see [60] for instance).

Now, we study which conditions need to be satisfied by the warping function ω so that the rotationally symmetric metric tensor g_ω can be smoothly extended to the center o_ω of the rotationally symmetric model space $(\mathbb{M}_\omega, g_\omega)$. In fact, we prove, in the following theorem, that g_ω is smooth in the entire $\mathbb{M}_\omega$ if, and only if, ω satisfies that $\omega(0) = 0$, $\omega'(0) = 0$ and $\omega^{(2k)}(0) = 0$ for all $k \in \mathbb{N}^*$, where $\omega^{(2k)}$ denotes the even derivatives of ω.

Theorem 2.2.3. *Let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Then, the following assertions are equivalent:*

1. *The rotationally symmetric metric tensor g_ω defined on $\mathbb{M}_\omega - \{o_\omega\}$ can be smoothly extended to $\mathbb{M}_\omega$.*
2. *There is a positive smooth function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\omega : [0, R) \longrightarrow \mathbb{R}_+$ can be expressed as*
$$\omega(t) = t\left(1 + t^2\varphi(t^2)\right) \quad \textit{for all} \quad t \in [0, R).$$
3. *The warping function ω satisfies that $\omega(0) = 0$, $\omega'(0) = 1$ and $\omega^{(2k)}(0) = 0$ for all $k \in \mathbb{N}^*$, where $\omega^{(2k)}$ denotes all the even derivatives of ω.*

Proof. Let us begin this proof by showing that (2) and (3) are equivalent. Assume first that the warping function satisfies that $\omega(0) = 0$, $\omega'(0) = 0$ and $\omega^{(2k)}(0) = 0$ for all $k \in \mathbb{N}^*$. Let us define the function

$$\begin{aligned} f : \mathbb{R}_+ &\longrightarrow \mathbb{R}_+ \\ t &\longmapsto f(t) := \int_0^1 \omega'(ts)\, ds. \end{aligned}$$

Thus, we have that f is smooth and that

$$\omega(t) = tf(t). \tag{2.34}$$

In fact, since ω' is smooth and $\omega(0) = 0$ and using the change of variable $x = ts$ for a fixed t, we have, fixing t, that

$$tf(t) = t\int_0^1 \omega'(ts)\, ds = \int_0^1 \omega'(ts)t\, ds = \int_0^t \omega'(x)\, dx = \omega(t) - \omega(0) = \omega(t).$$

Moreover, using the *generalized Leibniz's rule* (see page 508 of [43]), we obtain that

$$\omega^{(m)}(t) = mf^{(m-1)}(t) + tf^{(m)}(t).$$

Thus, from the fact that $\omega'(0) = 1$ and $\omega^{(2k)}(0) = 0$ for all $k \in \mathbb{N}^*$, we have

$$\begin{aligned} f(0) &= \omega'(0) = 1, \\ f^{(2k-1)}(0) &= \omega^{(2k)}(0) = 0, \quad \text{for all} \quad k \in \mathbb{N}^*. \end{aligned} \tag{2.35}$$

Namely, $f(0) = 1$ and all the odd derivatives of f vanish at zero.

Now, let us define the even function $\widetilde{f} : \mathbb{R} \longrightarrow \mathbb{R}_+$ as

$$\widetilde{f}(t) := \begin{cases} f(t), & \text{if} \quad t \geq 0, \\ f(-t), & \text{if} \quad t < 0. \end{cases}$$

It is easy to check that $\widetilde{f}$ is continuous and differentiable at 0. Moreover, computing the consecutive derivatives of $\widetilde{f}$ and using equation (2.35), we have that $\widetilde{f}$ is an even smooth function of $\mathbb{R}$. Therefore, by Theorem 1 of [75], there exists a function $h \in C^\infty(\mathbb{R}_+)$ such that $\widetilde{f}(t) = h(t^2)$ for all $t \in \mathbb{R}$. Thus, we obtain that $f(t) = \widetilde{f}\Big|_{\mathbb{R}_+}(t) = h(t^2)$, and hence, from equation (2.34),

$$\omega(t) = th(t^2), \tag{2.36}$$

for all $t \in \mathbb{R}_+$. Note that $h(0) = 1$, because $f(0) = 1$. Now, to conclude this part of the proof, let us define the function

$$\begin{aligned} \phi : \mathbb{R}_+ &\longrightarrow \mathbb{R}_+ \\ t &\longrightarrow \phi(t) = h(t) - 1, \end{aligned} \tag{2.37}$$

which is a smooth and $\phi(0) = 0$. Then, from the same argument used to obtain (2.34), we know that there is a smooth function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\phi(t) = t\varphi(t)$. Therefore, from equations (2.37) and (2.36), we obtain that $h(t) = 1 + t\varphi(t)$, and hence, we have that the warping function can be expressed as

$$\omega(t) = th(t^2) = t\left(1 + t^2\varphi(t^2)\right) \quad \text{for all} \quad t \in \mathbb{R}_+,$$

showing that assertion (3) implies assertion (2).

On the other hand, assume now that there is a smooth function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\omega(t) = t(1 + t^2\varphi(t^2))$. Then, it is easy to check that $\omega(0) = 0$ and $\omega'(0) = 1$. To show that all the even derivatives of ω vanish at zero let us define the following function

$$\begin{aligned} F : \mathbb{R} &\longrightarrow \mathbb{R}_+ \\ t &\longmapsto F(t) := 1 + t^2\varphi(t^2). \end{aligned}$$

Thus we have that F is an even smooth function with $F(0) = 1$ and $\omega(t) = tF(t)$. Therefore, by applying *generalized Leibniz's rule*, we obtain that

$$\omega^{(m)}(t) = mF^{(m-1)}(t) + tF^{(m)}(t)$$

for all $m \in \mathbb{N}^*$ and for all $t \in [0, R)$. In particular, we obtain that the even derivatives of the warping function can be computed, for all $t \in [0, R)$, by

$$\omega^{(2k)}(t) = 2kF^{(2k-1)}(t) + tF^{(2k)}(t), \quad \text{for all} \quad k \in \mathbb{N}^*. \tag{2.38}$$

Therefore, since F is an even smooth function, computing the consecutive derivatives of F evaluated at 0 we know that all the odd derivates of F vanish at 0, *i.e.*, $F^{(2k-1)}(0) = 0$ for all $k \in \mathbb{N}^*$. To see this assertion, let us consider an even smooth function $\mathcal{F}$ and an odd smooth function $\mathcal{H}$, *i.e.*, $\mathcal{F}(x) = \mathcal{F}(-x)$ and $\mathcal{H}(x) = -\mathcal{H}(-x)$. Then, computing its first order derivatives using the chain rule for derivatives, we have that

$$\mathcal{F}'(x) = -\mathcal{F}'(-x) \quad \text{and} \quad \mathcal{H}'(x) = \mathcal{H}'(-x),$$

and hence, we obtain that, in general, the derivative of an even (resp. odd) function is an odd (resp. even) function. Therefore, we have that $\mathcal{F}'$ is odd, $\mathcal{F}''$ is even, $\mathcal{F}'''$ is odd, $\mathcal{F}^{IV}$ is even..., and hence, we obtain that all the odd derivatives of $\mathcal{F}$ are odd functions, *i.e.*, $\mathcal{F}^{(2k-1)}(t) = -\mathcal{F}^{(2k-1)}(-t)$ for all $k \in \mathbb{N}^*$. On the other hand, it is known that all the odd functions $\mathcal{H}$ vanish at 0. Indeed, since $\mathcal{H}(0) = -\mathcal{H}(0)$, we obtain that $\mathcal{H}(0) = 0$. Then, we obtain that $\mathcal{F}^{(2k-1)}(0) = 0$ for all $k \in \mathbb{N}^*$. In particular, since F is an even smooth function, we have that $F^{(2k-1)}(0) = 0$ for all $k \in \mathbb{N}^*$.

Therefore, from equation (2.38), we have that $\omega^{(2k)}(0) = 0$ for all $k \in \mathbb{N}^*$, which proves that assertion (2) implies assertion (3). Hence, we obtain that assertions (2) and (3) are equivalent.

The second part of this proof consists in show that assertion (1) and assertion (2) are equivalent. First, assuming that there exists a smooth function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\omega(t) = t(1 + t^2\varphi(t^2))$ for all $t \in [0, R)$ (*i.e.* (2)), we want to show that the rotationally symmetric metric tensor g_{ω} can be smoothly extended to $\mathbb{M}_\omega$, namely, to show that it is smooth at the center o_ω (where the radial distance function $r = 0$). Let us begin by computing the expression of the rotationally symmetric metric tensor g_{ω} in a system of normal coordinates.

Since, by Definition 2.2.1 of the rotationally symmetric model spaces, we have that the exponential map is a diffeomorphism on $\mathbb{M}_\omega$, we can define normal coordinates $(\mathbb{M}_\omega, \zeta, x^i)$ on $\mathbb{M}_\omega$ and we can express the canonical metric tensor $g_{\text{can}} \equiv \zeta^* g_{\text{can}}$ on $\mathbb{M}_\omega$ as $g_{\text{can}} = \sum_{i=1}^n dx^i \otimes dx^i$ (which is smooth on $\mathbb{M}_\omega$, taking the pullback). It is well known that

$$g_{\text{can}} = \sum_{i=1}^n dx^i \otimes dx^i = dr \otimes dr + r^2 \pi^* g_{\mathbb{S}_1^{n-1}} \tag{2.39}$$

in $\mathbb{M}_\omega - \{o_\omega\}$, where r and π are, respectively, the radial distance function to the center o_ω and the projection to $\mathbb{S}_1^{n-1}$ (see Section 9 of Chapter 3 of [34] for a proof of this fact). Then

$$\pi^* g_{\mathbb{S}_1^{n-1}} = \frac{\sum_{i=1}^n dx^i \otimes dx^i - dr \otimes dr}{r^2}. \tag{2.40}$$

Moreover, since $r(q) = \sqrt{(x^1(q))^2 + \cdots + (x^n(q))^2}$ for any $q \in \mathbb{M}_\omega$, we have that

$$dr = \sum_{i=1}^n \frac{\partial r}{\partial x^i}\, dx^i = \sum_{i=1}^n \frac{x^i}{r}\, dx^i.$$

Therefore,

$$dr \otimes dr = \left(\sum_{i=1}^{n} \frac{x^i}{r} \, dx^i \right) \otimes \left(\sum_{j=1}^{n} \frac{x^j}{r} \, dx^j \right) = \sum_{i,j=1}^{n} \frac{x^i x^j}{r^2} \, dx^i \otimes dx^j \tag{2.41}$$

On the other hand, expressing the rotationally symmetric metric tensor g_ω in normal coordinates in $\mathbb{M}_\omega - \{o_\omega\}$ and using (2.39), (2.40) and (2.41), we have that

$$\begin{aligned}
g_\omega &= dr \otimes dr + \omega^2(r) \pi^* g_{\mathbb{S}_1^{n-1}} \\
&= dr \otimes dr + r^2 \pi^* g_{\mathbb{S}_1^{n-1}} + \left(\omega^2(r) - r^2 \right) \pi^* g_{\mathbb{S}_1^{n-1}} \\
&= \sum_{i=1}^{n} dx^i \otimes dx^i + \left(\omega^2(r) - r^2 \right) \pi^* g_{\mathbb{S}_1^{n-1}} \\
&= \sum_{i=1}^{n} dx^i \otimes dx^i + \frac{\omega^2(r) - r^2}{r^2} \left(\sum_{i=1}^{n} dx^i \otimes dx^i - dr \otimes dr \right) \\
&= \sum_{i=1}^{n} dx^i \otimes dx^i + \frac{\omega^2(r) - r^2}{r^2} \left(\sum_{i=1}^{n} dx^i \otimes dx^i - \sum_{i,j=1}^{n} \frac{x^i x^j}{r^2} \, dx^i \otimes dx^j \right) \qquad (2.42) \\
&= \sum_{i=1}^{n} dx^i \otimes dx^i + \frac{\omega^2(r) - r^2}{r^2} \left(\sum_{i,j=1}^{n} \left(\delta_{i,j} - \frac{x^i x^j}{r^2} \right) dx^i \otimes dx^j \right) \\
&= \sum_{i=1}^{n} dx^i \otimes dx^i + \frac{\omega^2(r) - r^2}{r^4} \left(\sum_{i,j=1}^{n} \left(r^2 \delta_{i,j} - x^i x^j \right) dx^i \otimes dx^j \right) \\
&= \sum_{i,j=1}^{n} \left(\delta_{i,j} + \frac{\omega^2(r) - r^2}{r^4} \left(r^2 \delta_{i,j} - x^i x^j \right) \right) dx^i \otimes dx^j .
\end{aligned}$$

Finally, since we assume that $\omega(t) = t(1 + t^2 \varphi(t^2))$, replacing this expression of the warping function in the above equation we obtain that

$$\begin{aligned}
g_\omega &= \sum_{i,j=1}^{n} \left(\delta_{i,j} + \frac{r^2 \left(1 + r^2 \varphi(r^2)\right)^2 - r^2}{r^4} \left(r^2 \delta_{i,j} - x^i x^j \right) \right) dx^i \otimes dx^j \\
&= \sum_{i,j=1}^{n} \left(\delta_{i,j} + \left(2\varphi(r^2) + r^2 \varphi^2(r^2) \right) \left(r^2 \delta_{i,j} - x^i x^j \right) \right) dx^i \otimes dx^j .
\end{aligned}$$

Therefore, since x^i and r^2 are smooth functions from $\mathbb{M}_\omega$ to $\mathbb{R}$ and $\mathbb{R}_+$, respectively, and by hypothesis φ is a smooth function of $\mathbb{R}_+$, we have that g_ω is smooth on the entire $\mathbb{M}_\omega$, showing that (2) implies (1).

Now, to end this proof, we show that assertion (1) implies assertion (2). Assume that the rotationally symmetric metric tensor g_ω is smooth on the entire $\mathbb{M}_\omega$. As we compute above, we can express the rotationally symmetric metric tensor g_ω in normal coordinates centered at the center o_ω as (2.42), namely, for any point $q \in \mathbb{M}_\omega - \{o_\omega\}$,

$$g_\omega = \sum_{i,j=1}^{n} \mathfrak{g}_{ij}\, dx^i \otimes dx^j$$

where, for all $i, j = 1, \ldots, n$,

$$\mathfrak{g}_{ij}(x_1, \ldots, x_n) = \delta_{i,j} + \frac{\omega^2(r) - r^2}{r^4}\left(r^2\delta_{i,j} - x^i x^j\right).$$

Moreover, by assuming that g_ω is smooth at o_ω, we have that $\mathfrak{g}_{ij}$ is smooth at o_ω. In particular, for the direction $(x_1, x_2, \ldots, x_n) = (x, 0, \ldots, 0)$, $i = j = 2$ and since $r(x, 0, \ldots, 0) = |x|$, we have that

$$\mathfrak{g}_{22}(x, 0, \ldots, 0) = 1 + \frac{\omega^2(|x|) - x^2}{x^2},$$

Thus, $\mathfrak{g}_{22}(0, \ldots, 0) = 1$, and hence,

$$\lim_{x\to 0} \frac{\omega^2(|x|) - x^2}{x^2} = \lim_{x\to 0}\left(\mathfrak{g}_{22}(x, 0, \ldots, 0) - 1\right) = \mathfrak{g}_{22}(0, \ldots, 0) - 1 = 0.$$

Therefore, defining $F : \mathbb{R} \longrightarrow \mathbb{R}_+$, $F(x) := \left(\omega^2(|x|) - x^2\right)/x^2$, we have that F is an even smooth function with $F(0) = 0$. Then, applying [75] as above, we know that there is a smooth function $h : \mathbb{R} \longrightarrow \mathbb{R}_+$ such that $F(x) = h(x^2)$ and $h(0) = 0$. Thus, we obtain that,

$$\omega^2(t) = t^2\left(1 + h(t^2)\right) \quad \text{for all} \quad t \geq 0,$$

and hence,

$$\omega(t) = t f(t^2),$$

where $f(t) = \sqrt{1 + h(t)}$. Note that f is smooth, because ω is smooth by Definition 2.2.1, and $f(0) = 1$.

Finally, as previously did to obtain equation (2.34), since $f(0) - 1 = 0$, we know that there is a smooth function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $f(t) - 1 = t\varphi(t)$. Therefore, $f(t) = 1 + t\varphi(t)$, and hence, the warping function can be expressed as $\omega(t) = t\left(1 + t^2\varphi(t^2)\right)$ showing that assertion (1) implies (2), and the theorem follows.

$\square$

Now, we present some results concerning the sectional curvatures in the rotationally symmetric model spaces.

Proposition 2.2.4 (see Section 2.3 of Chapter 3 of [64])**.** *Let* $(\mathbb{M}_\omega, g_{\omega})$ *be an* n*-dimensional rotationally symmetric model space with center* o_ω*. Then the radial sectional curvature, for any point* $p \in \mathbb{M}_\omega - \{o_\omega\}$ *and for any tangent vector field* $X \in T_p S^\omega_{r_{o_\omega}(p)}(o_\omega)$*, is given by*

$$\sec_{g_\omega}\big(\sigma(\nabla_{g_\omega} r_{o_\omega}, X)(p)\big) = -\frac{\omega''\left(r_{o_\omega}(p)\right)}{\omega\left(r_{o_\omega}(p)\right)}, \tag{2.43}$$

where $\sigma(\nabla_{g_\omega} r_{o_\omega}, X)$ *is the radial plane tangent to* p *generated by* $\nabla_{g_\omega} r_{o_\omega}$ *and* X*.*

Some examples of the rotationally symmetric metric spaces are the simply connected real space forms $\mathbb{K}^n(\kappa)$ of constant sectional curvature κ. In fact, from the Hadamard-Cartan Theorem (see Theorem 4.1 of Chapter 5 of [68], for instance), it is well known that if $\kappa \leq 0$ then any point p of $\mathbb{K}^n(\kappa)$ is a pole, so $\mathbb{K}^n(\kappa)$ can be constructed as a rotationally symmetric model space $(\mathbb{M}_\omega, g_{\omega})$ with any given point of $\mathbb{K}^n(\kappa)$ considered as the center of $\mathbb{M}_\omega$ and a particular warping function ω_κ which depends on the value of κ. For $\kappa > 0$, for any $p \in \mathbb{K}^n(\kappa) = \mathbb{S}^{n-1}_\kappa$ and denoting by $\overline{p}$ the antipodal point of p in $\mathbb{S}^{n-1}_1$, we can construct $\mathbb{S}^{n-1}_\kappa - \{\overline{p}\}$ as a rotationally symmetric model space with center p and a particular warping function ω_κ. We show this construction at the following Proposition 2.2.5 by defining each warping function ω_κ.

Proposition 2.2.5. *(see pages 74-80 [34] and pages 12 and 69 of [64]) The simply connected real space forms* $\mathbb{K}^n(\kappa)$ *of constant sectional curvature* $\kappa \in \mathbb{R}$ *can be considered as rotationally symmetric model spaces* $\left(\mathbb{M}_{\omega_\kappa}, g_{\omega_\kappa}\right)$ *which warping function are given by*

$$\omega_\kappa(t) := \begin{cases} \dfrac{\sin\left(\sqrt{\kappa}t\right)}{\sqrt{\kappa}}, & \text{if} \quad \kappa > 0, \quad t \in \left[0, \dfrac{\pi}{\sqrt{k}}\right), \\ t, & \text{if} \quad \kappa = 0, \quad t \in [0, +\infty), \\ \dfrac{\sinh\left(\sqrt{-\kappa}t\right)}{\sqrt{-\kappa}}, & \text{if} \quad \kappa < 0, \quad t \in [0, +\infty). \end{cases} \tag{2.44}$$

2.2.2 Balance condition

In this subsection, we present a purely intrinsic condition on the rotationally symmetric model spaces $(\mathbb{M}_\omega, g_\omega)$ (see Section 3.1 of S. Markvorsen and V. Palmer [53] for more details on this condition), which will play a key role in Chapter 3 in order to find our comparisons for the *torsional rigidity* (see Definition 3.2.5). But first, let us define the following.

Definition 2.2.6. *Let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center o_ω and let $p \in \mathbb{M}_\omega - \{o_\omega\}$. Given, $B_r^\omega(o_\omega)$ and $S_r^\omega(o_\omega)$, the geodesic ball and the geodesic sphere of $\mathbb{M}_\omega$, respectively, with radius $r = r_{o_\omega}(p)$ centered at the center o_ω, we define the* normalized mean curvature pointing inward of the geodesic sphere at p *as*

$$\eta_\omega(r) = \frac{1}{(n-1)} H_{S_r^\omega(o_\omega)}(p) = \frac{\omega'(r)}{\omega(r)}, \tag{2.45}$$

and the isoperimetric quotient *as*

$$q_\omega(r) = \frac{\operatorname{vol}(B_r^\omega(o_\omega))}{\operatorname{vol}(S_r^\omega(o_\omega))} = \frac{\int_0^r \omega^{n-1}(s)\, ds}{\omega^{n-1}(r)}. \tag{2.46}$$

Remark 2.2.7. Note that, if g_ω can be smoothly extended to $\mathbb{M}_\omega$, we have that $\omega(0) = 0$ and $\omega'(0) = 1$, (see Proposition 2.2.3), and hence, applying *L'Hôpital's rule*, we obtain that

$$\lim_{r\to 0} q_\omega(r) = \lim_{r\to 0} \frac{\int_0^r \omega^{n-1}(s)\, ds}{\omega^{n-1}(r)} = \lim_{r\to 0} \frac{\omega^{n-1}(r)}{(n-1)\omega^{n-2}(r)\omega'(r)} = \lim_{r\to 0} \frac{\omega(r)}{(n-1)\omega'(r)} = 0.$$

We define now the intrinsic condition previously mentioned as follows.

Definition 2.2.8. *Given $(\mathbb{M}_\omega, g_\omega)$ an n-dimensional rotationally symmetric model space with center o_ω, we say that $\mathbb{M}_\omega$ is* balanced from above *if, and only if, the inequality*

$$q_\omega(r)\eta_\omega(r) \leq \frac{1}{n-1} \tag{2.47}$$

holds for all $0 < r < \operatorname{inj}_{g_\omega}(o_\omega)$.

Remark 2.2.9. Note that saying that the above inequality holds for all $r \in (0, \operatorname{inj}_{g_\omega}(o_\omega))$ is equivalent of saying that it holds for all $p \in \mathbb{M} - \{o_\omega\}$. Indeed,

q_ω and η_ω are radial functions and, from Remark 2.2.2, we have that the model radius Λ coincides with the injectivity radius of o_ω, and hence, by Definition 2.2.1, $\mathbb{M}_\omega = \exp_{o_\omega}(\mathbb{B}_\Lambda(o))$ where $\mathbb{B}_\Lambda(o)$ is the open ball of $T_{o_\omega}M$ with radius $\Lambda = \mathrm{inj}_{g_\omega}(o_\omega)$ centered at $o \in T_{o_\omega}M$. Thus, for any $p \in \mathbb{M}_\omega - \{o_\omega\}$, we have that $0 < r = r_{o_\omega}(p) < \mathrm{inj}_{g_\omega}(o_\omega)$. Therefore to ask condition (2.47) is equivalent to ask that $q_\omega(r_{o_\omega}(p))\eta_\omega(r_{o_\omega}(p)) \leq \dfrac{1}{n-1}$ for all $p \in \mathbb{M}_\omega - \{o_\omega\}$.

Furthermore, we have the following characterization of the balance from above condition.

Proposition 2.2.10 (see Observation 3.8 of [53]). *Let* $(\mathbb{M}_\omega, g_\omega)$ *be an n-dimension rotationally symmetric model space with center* o_ω. *Then, the following assertion are equivalent:*

1. *The rotationally symmetric model space* $\mathbb{M}_\omega$ *is balanced from above.*
2. *The isoperimetric quotient is non-decreasing,* i.e., $q'_\omega(r) \geq 0$ *for all radius* $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$.
3. *The inequality* $\omega^n(r) \geq (n-1)\omega'(r)\int_0^r \omega^{n-1}(s)\,ds$ *holds for all radius* $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$.

Proof. First we show that assertions (1) and (2) are equivalent. Computing the derivative of q_ω we have the following

$$q'_\omega(r) = \left(\frac{\mathrm{vol}\,(B^\omega_r(o_\omega))}{\mathrm{vol}\,(S^\omega_r(o_\omega))}\right)' = \frac{\mathrm{vol}\,(S^\omega_r(o_\omega))^2 - \mathrm{vol}\,(B^\omega_r(o_\omega))\,(\mathrm{vol}\,(S^\omega_r(o_\omega)))'}{\mathrm{vol}\,(S^\omega_r(o_\omega))^2}.$$

Thus, computing the derivative of the area function (2.31), we obtain that

$$\begin{aligned} q'_\omega(r) &= \frac{\mathrm{vol}\,(S^\omega_r(o_\omega))^2 - \mathrm{vol}\,(B^\omega_r(o_\omega))\,\mathrm{vol}\left(\mathbb{S}^{n-1}_1\right)(n-1)\omega^{n-2}(r)\omega'(r)}{\mathrm{vol}\,(S^\omega_r(o_\omega))^2} \\ &= 1 - \frac{\mathrm{vol}\,(B^\omega_r(o_\omega))}{\mathrm{vol}\,(S^\omega_r(o_\omega)(o_\omega))}\frac{\mathrm{vol}\left(\mathbb{S}^{n-1}_1\right)(n-1)\omega^{n-2}(r)\omega'(r)}{\mathrm{vol}\,(S^\omega_r(o_\omega))} \\ &= 1 - q_\omega(r)\frac{\mathrm{vol}\left(\mathbb{S}^{n-1}_1\right)(n-1)\omega^{n-2}(r)\omega'(r)}{\mathrm{vol}\left(\mathbb{S}^{n-1}_1\right)\omega^{n-1}(r)} \\ &= 1 - (n-1)q_\omega(r)\frac{\omega'(r)}{\omega(r)} = 1 - (n-1)q_\omega(r)\eta_\omega(r). \end{aligned}$$

Therefore, $q'_\omega(r) \geq 0$ for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$ if, and only if, $q_\omega(r)\eta_\omega(r) \leq \frac{1}{n-1}$ for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$. Namely, $q'_\omega(r) \geq 0$ for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$ if, and only if, $\mathbb{M}_\omega$ is balanced from above, which shows that (1) and (2) are equivalent.

On the other hand, to show that assertions (1) and (3) are equivalent, we have, by the expressions (2.46) and (2.45) of q_ω and η_ω (respectively), that $q_\omega(r)\eta_\omega(r) \leq \frac{1}{n-1}$ for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$ if, and only if,

$$\frac{\int_0^r \omega^{n-1}(s)\,ds}{\omega^{n-1}(r)} \frac{\omega'(r)}{\omega(r)} \leq \frac{1}{n-1} \quad \text{for all} \quad 0 < r < \mathrm{inj}_{g_\omega}(o_\omega),$$

which, at its time, since $\omega(r) > 0$ for all $r > 0$ by definition of ω, holds for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$ if, and only if,

$$(n-1)\omega'(r)\int_0^r \omega^{n-1}(s)\,ds \leq \omega^n(r) \quad \text{for all} \quad 0 < R < \mathrm{inj}_{g_\omega}(o_\omega),$$

showing the equivalence between (1) and (3), and the proposition follows.

□

Example 2.2.11. Now we show some examples of rotationally symmetric model spaces balanced from above. You can also find some other examples in [53].

1. The n-dimensional rotationally symmetric model space $(\mathbb{M}_\omega, g_\omega)$ with center o_ω and warping function $\omega : [0, \mathrm{inj}_{g_\omega}(o_\omega)) \longrightarrow \mathbb{R}_+$ given by $\omega(t) = t + t^3$ is balance from above. Indeed, by condition 3 of Proposition 2.2.10, we know that $\mathbb{M}_\omega$ is balanced from above if, and only if,

$$\omega^n(r) \geq (n-1)\omega'(r)\int_0^r \omega^{n-1}(s)\,ds \quad \text{for all} \quad 0 < r < \mathrm{inj}_{g_\omega}(o_\omega).$$

Thus, $\mathbb{M}_\omega$ is balanced from above if, and only if,

$$\left(r + r^3\right)^n \geq (n-1)\left(1 + 3r^2\right)\int_0^r \left(s + s^3\right)^{n-1} ds \quad \text{for all} \quad 0 < r < \mathrm{inj}_{g_\omega}(o_\omega),$$

which, in its turn, holds if, and only if,

$$\frac{(r + r^3)^n}{1 + 3r^2} - (n-1)\int_0^r \left(s + s^3\right)^{n-1} ds \geq 0 \quad \text{for all} \quad 0 < r < \mathrm{inj}_{g_\omega}(o_\omega),$$

because $1 + 3r^2 > 0$ for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$.

Now let us define the function $F : [0, \mathrm{inj}_{g_\omega}(o_\omega)) \longrightarrow \mathbb{R}$ given by

$$r \longmapsto F(r) := \frac{\left(r+r^3\right)^n}{1+3r^2} - (n-1)\int_0^r \left(s+s^3\right)^{n-1} ds.$$

Hence, we have that $\mathbb{M}_\omega$ is balance from above if, and only if, $F(r) > 0$ for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$. Let's study the sign of F. First, observe that $F(0) = 0$, therefore, if F is an increasing function we have that $F(r) > 0$ for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$, as wanted. In fact, by computing the derivative of F, applying the fundamental theorem of calculus, and simplifying, we obtain that

$$\begin{aligned}
F'(r) &= \frac{n\left(r+r^3\right)^{n-1}\left(1+3r^2\right)^2 - \left(r+r^3\right)^n 6r}{\left(1+3r^2\right)^2} - (n-1)(r+r^3)^{n-1} \\
&= n(r+r^3)^{n-1} - 6\,\frac{r\left(r+r^3\right)^n}{\left(1+3r^2\right)^2} - (n-1)\left(r+r^3\right)^{n-1} \\
&= (r+r^3)^{n-1} - 6\,\frac{r\left(r+r^3\right)^n}{\left(1+3r^2\right)^2} = \left(r+r^3\right)^{n-1}\left(1-6\,\frac{r\left(r+r^3\right)}{\left(1+3r^2\right)^2}\right).
\end{aligned}$$

Then, we have that $F'(0) = 0$ and, since $r + r^3 > 0$ and $1 + 3r^2 > 0$ for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$, we have that $F'(r) > 0$ for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$ if, and only if, for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$ we have that

$$0 < \left(1+3r^2\right)^2 - 6r\left(r+r^3\right) = 1 + 6r^2 + 9r^4 - 6r^2 - 6r^4 = 1 + 3r^4,$$

which is true. Therefore, we have that F is an increasing function, and hence, since $F(0) = 0$, we have that $F(r) > 0$ for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$, showing that $(\mathbb{M}_\omega, g_\omega)$ with $\omega(t) = t + t^3$ is an n-dimensional rotationally symmetric model space balanced from above.

2. Let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center o_ω such that its rotationally symmetric metric tensor g_ω is smooth in the entire $\mathbb{M}_\omega$ and such that all its radial sectional curvatures are non-positive, *i.e.*, such that, for all $p \in \mathbb{M}_\omega - \{o_\omega\}$, $\sec_{g_\omega}(\sigma_p(\nabla r|_p, u)) \leq 0$ for all $u \in T_pS^\omega_{r_{o_\omega}(p)}(o_\omega)$, where we denote the gradient $\nabla_{g_\omega} r_{o_\omega}$ by ∇r and $\sigma_p(\nabla r, u)$ is the 2-dimensional subspace of $T_pS_{r_{o_\omega}(p)}(o_\omega)$ generated by ∇r and u, to simplify the notation. Then, we are going to see that $\mathbb{M}_\omega$ is balanced from above if, and only if,

$$\sec_{g_\omega}\left(\sigma_p(\nabla r|_p, u)\right) \geq -\eta^2_\omega(r_{o_\omega}(p))$$

for all $p \in \mathbb{M}_\omega$ and for all $u \in T_p S^\omega_{r_{o_\omega}(p)}(o_\omega)$.

In fact, from condition 3 of Proposition 2.2.10 and denoting $r = r_{o_\omega}(p)$, we know that $\mathbb{M}_\omega$ is balanced from above if, and only if,

$$\omega^n(r) \geq (n-1)\omega'(r) \int_0^r \omega^{n-1}(s)\, ds.$$

Moreover, since $\omega(r) > 0$ for all $0 < r < \operatorname{inj}_{g_\omega}(o_\omega)$ (by definition) and $\sec_{g_\omega}(\sigma_p(\nabla r|_p, u))(p) \leq 0$ for all $p \in \mathbb{M}_\omega$, $u \in T_p S^\omega_r(o_\omega)$ (by hypothesis), we have, from the expression of the radial sectional curvature on rotationally symmetric model spaces (see equation (2.43)), that $\omega''(r) \geq 0$ for all $0 < r < \operatorname{inj}_{g_\omega}(o_\omega)$. Then, we have that ω' is an increasing function and, since $\omega'(0) = 1$ because g_ω is smooth at o_ω (see assertion (3) of Theorem 2.2.3), we obtain that $\omega'(r) \geq 1 > 0$ for all $0 \leq r < \operatorname{inj}_{g_\omega}(o_\omega)$. Then, dividing the above expression by ω', we obtain that $\mathbb{M}_\omega$ is balanced from above if, and only if,

$$\frac{\omega^n(r)}{\omega'(r)} - (n-1)\int_0^r \omega^{n-1}(s)\, ds \geq 0, \quad \text{for all} \quad 0 < r < \operatorname{inj}_{g_\omega}(o_\omega). \tag{2.48}$$

Now let us define, as we did before in the previous example, the function $F : [0, \operatorname{inj}_{g_\omega}(o_\omega)) \longrightarrow \mathbb{R}$ given by

$$r \longmapsto F(r) := \frac{\omega^n(r)}{\omega'(r)} - (n-1)\int_0^r \omega^{n-1}(s)\, ds. \tag{2.49}$$

Thus, $\mathbb{M}_\omega$ is balanced from above if, and only if, $F(r) \geq 0$ for all $0 < r < \operatorname{inj}_{g_\omega}(o_\omega)$. But, since $F(0) = 0$, we have that $F(r) \geq 0$ for all $0 < r < \operatorname{inj}_{g_\omega}(o_\omega)$ if, and only if, F is an increasing function, *i.e.*, $F'(r) \geq 0$ for all $0 < r < \operatorname{inj}_{g_\omega}(o_\omega)$. Then, we compute

$$F'(r) = \frac{n\omega^{n-1}(r)\,(\omega'(r))^2 - \omega^n(r)\omega''(r)}{(\omega'(r))^2} - (n-1)\omega^{n-1}(r).$$

Therefore, we have that that $F'(0) = 0$, and moreover, for all $0 < r < \operatorname{inj}_{g_\omega}(o_\omega)$, for all point $p \in \mathbb{M}_\omega$ such that $r_{o_\omega}(p) = r$ and for all $u \in T_p S^\omega_r(o_\omega)$, we obtain that

$$\begin{aligned}
F'(r) &= n\omega^{n-1}(r) - \frac{\omega^n(r)\omega''(r)}{\left(\omega'(rt)\right)^2} - (n-1)\omega^{n-1}(r) \\
&= \omega^{n-1}(r) - \frac{\omega^{n+1}(r)}{\left(\omega'(r)\right)^2}\frac{\omega''(r)}{\omega(r)} \\
&= \omega^{n-1}(r) + \frac{\omega^{n+1}(R)}{\left(\omega'(r)\right)^2}\sec_{g_\omega}\left(\sigma_p(\nabla r|_p, u)\right) \\
&= \omega^{n-1}(r)\left(1 + \frac{\omega^2(r)}{\left(\omega'(r)\right)^2}\sec_{g_\omega}\left(\sigma_p(\nabla r|_p, u)\right)\right) \\
&= \omega^{n-1}(r)\left(1 + \frac{\sec_{g_\omega}\left(\sigma_p(\nabla r|_p, u)\right)}{\eta_\omega^2(r)}\right).
\end{aligned} \tag{2.50}$$

Then, since $\omega(r) > 0$ for all $0 < r < \operatorname{inj}_{g_\omega}(o_\omega)$, we have that $F'(r) \geq 0$ for all $0 < r < \operatorname{inj}_{g_\omega}(o_\omega)$ if, and only if,

$$\sec_{g_\omega}\left(\sigma_p(\nabla r|_p, u)\right) \geq -\eta_\omega^2(r)$$

for all $0 < r < \operatorname{inj}_{g_\omega}(o_\omega)$, for any point $p \in \mathbb{M}_\omega$ such that $r_{o_\omega}(p) = r$ and for all $u \in T_pS_r^\omega(o_\omega)$.

Remark 2.2.12. Observe that, from the above Example (2) in 2.2.11, we obtain that a rotationally symmetric model space $(\mathbb{M}_\omega, g_\omega)$ with non-positive radial sectional curvatures $\sec_{g_\omega}(\sigma(\nabla r, \cdot)) \leq 0$ is balanced from above if, and only if,

$$0 \geq \sec_{g_\omega}\left(\sigma_p(\nabla r|_p, u)\right) = -\frac{\omega''(r)}{\omega(r)} \geq -\left(\frac{\omega'(r)}{\omega(r)}\right)^2 = -\eta_\omega^2(r)$$

for all $0 < r < \operatorname{inj}_{g_\omega}(o_\omega)$, for any point $p \in \mathbb{M}_\omega$ such that $r_{o_\omega}(p) = r$ and for all $u \in T_pS_r^\omega(o_\omega)$. Furthermore, this condition can be rewritten in terms of the warping function as

$$\omega''(r)\omega(r) \leq \left(\omega'(r)\right)^2 \quad \text{for all} \quad 0 < r < \operatorname{inj}_{g_\omega}(o_\omega).$$

Thus, not all the rotationally symmetric model spaces with non-positive radial sectional curvature are balanced from above. It is easy to check that our Example 2.2.11.(1) is a rotationally symmetric model space with negative sectional curvatures which satisfies the above equation, and hence, as we show at the example, is balanced from above.

3. The simply connected real space forms of constant sectional curvature $\kappa \leq 0$, considered as rotationally symmetric model spaces $\left(\mathbb{M}_{\omega_\kappa}, g_{\omega_\kappa}\right)$ with $\kappa \leq 0$, are balanced from above. In fact, for $\kappa = 0$ we know that $\omega_0 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is given by $\omega_0(t) = t$ (see Proposition 2.2.5), so

$$\omega_0''(r)\omega_0(r) = 0 \leq 1 = \left(\omega_0'(r)\right)^2 \quad \text{for all} \quad r > 0,$$

which shows, by the above remark, that $\mathbb{M}_{\omega_0}$ is balanced from above.

For $\kappa < 0$, we know that $\omega_\kappa : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is given by $\omega_\kappa(t) = \sinh\left(\sqrt{-\kappa}t\right)/\sqrt{-\kappa}$ (see Proposition 2.2.5). Then

$$\omega_\kappa''(r)\omega_\kappa(r) = \sqrt{-\kappa}\sinh\left(\sqrt{-\kappa}r\right)\frac{\sinh\left(\sqrt{-\kappa}rt\right)}{\sqrt{-\kappa}} = \sinh^2\left(\sqrt{-\kappa}r\right)$$

and

$$\left(\omega_\kappa'(r)\right)^2 = \cosh^2\left(\sqrt{-\kappa}r\right)$$

thus $\omega_\kappa''(r)\omega(r) \leq \left(\omega_\kappa'(r)\right)^2$ for all $r > 0$ if, and only if, $\sinh^2\left(\sqrt{-\kappa}r\right) \leq \cosh^2\left(\sqrt{-\kappa}r\right)$ for all $r > 0$, which is true. Therefore, we have shown that all $\mathbb{M}_{\omega_\kappa}$ with $k < 0$ are balanced from above.

Another way to prove that these examples are balanced from above consists in to see that $F'(r) \geq 0$ replacing $\omega_\kappa(r)$ by r or $\sinh\left(\sqrt{-\kappa}r\right)/\sqrt{-\kappa}$ in the balance condition. For $\kappa = 0$ it will be easy. But, for $\kappa < 0$, we may use the following inequality for the hyperbolic sinus that you can find in [61]

$$\int_0^r \sinh^{n-1}\left(\sqrt{-\kappa}s\right)\, ds \leq \frac{\sinh^n\left(\sqrt{-\kappa}r\right)}{\sqrt{-\kappa}(n-1)\cosh\left(\sqrt{-\kappa}r\right)}.$$

4. The simply connected real space forms $\mathbb{M}_{\omega_\kappa}$ of positive constant sectional curvature $\kappa > 0$ considered as rotationally symmetric model spaces $\left(\mathbb{M}_{\omega_\kappa}, g_{\omega_\kappa}\right)$ which warping function $\omega_\kappa : [0, \pi/\sqrt{\kappa}) \longrightarrow \mathbb{R}_+$ is given by $\omega_\kappa(t) = \sin\left(\sqrt{\kappa}t\right)/\sqrt{\kappa}$ are balanced from above. In fact, since $\omega_\kappa'(t) \leq 0$ for all $t \in [\pi/(2\sqrt{k}), \pi/\sqrt{k})$, we have that

$$\omega_\kappa^n(r) \geq 0 \geq (n-1)\omega_\kappa'(r)\int_0^r \omega_\kappa^{n-1}(s)\, ds, \quad \text{for all} \quad \frac{\pi}{2\sqrt{\kappa}} \leq r < \frac{\pi}{\sqrt{\kappa}}.$$

On the other hand, we have that $\omega_\kappa'(t) > 0$ for all $t \in [0, \pi/(2\sqrt{\kappa}))$. Then, following the same argument that in Example 2.2.11.(2) (*i.e.*, using equation

(2.48)), we have that $\mathbb{M}_{\omega_\kappa}$ is balanced from above if, and only if,

$$\frac{\omega_\kappa^n(r)}{\omega_\kappa'(r)} - (n-1)\int_0^r \omega_\kappa^{n-1}(s)\,ds \geq 0, \quad \text{for all} \quad 0 \leq r < \frac{\pi}{2\sqrt{\kappa}}.$$

Thus, defining a function F as in equation (2.49), *i.e.*,

$$r \longmapsto F(r) := \frac{\omega^n(r)}{\omega'(r)} - (n-1)\int_0^r \omega^{n-1}(s)\,ds, \tag{2.51}$$

we have that $F(0) = 0$ and that $\mathbb{M}_{\omega_\kappa}$ is balanced from above if, and only if, $F(r) \geq 0$ for all $0 < r < \mathrm{inj}_{g_\omega}(o_\omega)$. Thus, since $F(0) = 0$, $\mathbb{M}_{\omega_\kappa}$ is balanced from above if, and only if, F is an increasing function. Namely, from equation (2.50), $\mathbb{M}_{\omega_\kappa}$ is balanced from above if, and only if,

$$F'(r) = \omega_\kappa^{n-1}(r)\left(1 + \frac{\kappa}{\eta^2_{\omega_\kappa}(r)}\right) \geq 0$$

for all $0 \leq r < \pi/(2\sqrt{\kappa})$. Therefore, since $\kappa > 0$ (by hypothesis), we obtain that $F'(r) > 0$ for all $0 < r < \pi/(2\sqrt{\kappa})$. Thus, since $\mathbb{M}_{\omega_\kappa}$ with $\kappa > 0$ satisfies condition (3) of Proposition 2.2.10 for all $0 \leq r < \pi/\sqrt{\kappa}$, it is balanced from above. The reader can find another proof for this last example in [52].

2.2.3 Schwarz symmetrization

Along this work, we use the notion of *Schwarz symmetrization* as considered, *e.g.*, by C. Bandle in [2], by G. Pólya in [66] or, more recently, by I. Chavel in [9] and by P. McDonald in [55]. Here we review and show some facts about the *Schwarz symmetrization* in the context of Riemannian manifolds, as we can find in Section 4 of [37] and in Section 3.2 of [53].

Definition 2.2.13 (see [2], [9], [53] and [66])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold and let $D \subseteq M$ be a precompact connected domain in M. Given $(\mathbb{M}_\omega, g_\omega)$ an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$ we define, when it does exists, the* ω-rotationally symmetric model space symmetrization D^{*_ω} of D in $\mathbb{M}_\omega$ *as the unique geodesic ball of $\mathbb{M}_\omega$ with radius $L(D)$ centered at o_ω,*

$$D^{*_\omega} := B^\omega_{L(D)}(o_\omega),$$

satisfying

$$\operatorname{vol}(D) = \operatorname{vol}\left(B^{\omega}_{L(D)}(o_\omega)\right).$$

Remark 2.2.14. In the particular case that D is a geodesic ball $B_R(o)$ in M of radius $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$ centered at $o \in M$, then the radius $L\left(B_R(o)\right)$ is some increasing function $s(R,o) = L\left(B_R(o)\right)$ which depends on the geometry of M, so we can write

$$B_R(o)^{*_\omega} = B^{\omega}_{s(R,o)}(o_\omega)$$

and this symmetrization $B^{\omega}_{s(R)}(o_\omega)$ satisfies

$$\operatorname{vol}\left(B_R(o)\right) = \operatorname{vol}\left(B^{\omega}_{s(R,o)}(o_\omega)\right).$$

In fact, computing the derivatives of the volumes with respect to its radius (see Theorem 2.1.77), we have that

$$\operatorname{vol}\left(S_R(o)\right) = \operatorname{vol}\left(S^{\omega}_{s(R,o)}(o_\omega)\right)\frac{d}{dR}s(R,o),$$

and hence,

$$\frac{d}{dR}s(R,o) = \frac{\operatorname{vol}\left(S_R(o)\right)}{\operatorname{vol}\left(S^{\omega}_{s(R)}(o_\omega)\right)} > 0,$$

showing that $s(R,o)$ is an increasing function with respect to R.

From now on, when it is clear from the context, we will write D^* instead of D^{*_ω}, we will denote, respectively, $s(R)$ and $s'(R)$ to refer to $s(R,o)$ and $\frac{d}{dR}s(R,o)$, and we will refer to the ω-rotationally symmetric model space symmetrization of D simply as the *Schwarz symmetrization of* D, or simply as the *symmetrization of* D.

Now, given $f : D \longrightarrow \mathbb{R}_+$ a non-negative function defined on D, we introduce the notion of *ω-rotationally symmetric model space symmetrization of* f,

$$f^{*_\omega} : D^{*_\omega} \longrightarrow \mathbb{R}_+.$$

But first, we show some useful facts.

Definition 2.2.15 (see [53]). *Let (M,g) be a complete n-dimensional Riemannian manifold and let $D \subseteq M$ be a precompact connected domain in M. Given*

$f : D \longrightarrow \mathbb{R}_+$ a non-negative smooth function defined on D we define, for any $t \geq 0$, the subsets $D(t)$ and $\Gamma(t)$ of D *as*

$$D(t) := \{x \in D \,:\, f(x) \geq t\} \subseteq M$$

and

$$\Gamma(t) := \{x \in D \,:\, f(x) = t\}\,.$$

Remark 2.2.16. Note that the subsets defined above satisfy the following:

1. The boundary of $D(t)$ is $\partial D(t) = \Gamma(t) \subseteq D(t)$ for all $t \geq 0$.

2. Note too that $D(0) = D$ and, if $t_1 \leq t_2$, then $D(t_2) \subseteq D(t_1)$.

3. If $T := \sup_{x \in D} f(x)$ then $D(t) = \emptyset$ for all $t > T$, and hence, $\operatorname{vol}(D(t)) = 0$ for all $t > T$.

4. Therefore, we have a family of nested sets $\{D(t)\}_{t \in [0,T]}$ that covers D. Namely, $D = \bigcup_{t \in [0,T]} D(t)$.

Thus, from the definition of the above sets, we define the *ω-rotationally symmetric model space symmetrization of a function* as follows.

Definition 2.2.17 (see [53])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold, $D \subseteq M$ a precompact connected domain in M. Given*

$$f : D \longrightarrow \mathbb{R}_+$$

a non-negative smooth function defined on D and given $(\mathbb{M}_\omega, g_\omega)$ an n-dimensional rotationally symmetric model space, we define, when it does exists, the ω-rotationally symmetric model space symmetrization of f in $\mathbb{M}_\omega$ *as the function $f^{*_\omega} : D^{*_\omega} \longrightarrow \mathbb{R}_+$ defined for all $x^* \in D^{*_\omega}$ as*

$$f^{*_\omega}(x^*) := \sup\{t \geq 0 \,:\, x^* \in D(t)^{*_\omega}\}\,.$$

Remark 2.2.18. As we did before for the above definition, let us show some observations about the definition of *ω-symmetrization of f*:

1. Observe that the symmetrization of f^{*_ω} ranges on $[0, T]$. Namely,

$$f^{*_\omega} : D^{*_\omega} \longrightarrow [0, T],$$

where $T := \sup_{x \in D} f(x)$.

2. From now on, when it is clear from the context, we write f^* and $D(t)^*$, instead of f^{*_ω} and $D(t)^{*_\omega}$, respectively, and we will refer to the ω-rotationally symmetric model space symmetrization of f as the *Schwarz symmetrization of f*, or simply as the *symmetrization of f*.

3. By *Sard's Theorem* (see [11], [29], and [69]), denoting by $D_f \subseteq D$ the set of critical points of f, we know that the set $S_f = f(D_f) \subseteq [0,T]$ of critical values of f has null measure, and the set of regular values of f, $R_f = [0,T] - S_f$, is open and dense in $[0,T]$. In particular, for any $t \in R_f$, the set $\Gamma(t) = \{x \in D : f(x) = t\}$ is a smooth embedded hypersurface in D and $\left\|\nabla^M f\right\|$ does not vanish along $\Gamma(t)$.

With these observations in hand, we are able to define the following function.

Definition 2.2.19. *Let (M,g) be a complete n-dimensional Riemannian manifold and let $D \subseteq M$ be a precompact connected domain in M. Given $(\mathbb{M}_\omega, g_\omega)$ an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$ and given $f : D \longrightarrow \mathbb{R}_+$ a non-negative smooth function defined on D, we define, when it does exists, the function $\widetilde{r} : [0,T] \longrightarrow [0, L(D)]$ as the radius of the symmetrization*

$$D(t)^* = B^\omega_{\widetilde{r}(t)}(o_\omega)$$

satisfying

$$\operatorname{vol}(D(t)) = \operatorname{vol}\left(B^\omega_{\widetilde{r}(t)}(o_\omega)\right),$$

where $B^\omega_{\widetilde{r}(t)}(o_\omega)$ is a geodesic ball of $\mathbb{M}_\omega$ with radius $\widetilde{r}(t)$ centered at o_ω.

Remark 2.2.20. Note that, since $D(0) = D$, we have that $D(0)^* = D^*$. Then $\widetilde{r}(0) = L(D)$, where $L(D)$ is the radius defined on Definition 2.2.13, and $D^* = B^\omega_{\widetilde{r}(0)}(o_\omega)$. Furthermore, since $\operatorname{vol}(D(t)) = 0$ for all $t > T$, we have that $\widetilde{r}(t) = 0$ for all $t > T$. From now on, we will refer to the function $\widetilde{r}$ as the *symmetrized radius*.

Concerning this last definition, we have the following lemma which describes the behaviour of $\widetilde{r}$. This lemma will play a key rôle to prove Theorem 3.6.2 which we will be used, in its turn, to prove our comparisons for the *torsional rigidity*.

Lemma 2.2.21. *The function $\widetilde{r} : [0, T] \longrightarrow [0, L(D)]$, defined in the above definition, is non-increasing. In particular, for all regular values $t \in R_f$, the function $\widetilde{r}|_{R_f} : R_f \subseteq [0, T] \longrightarrow [0, L(D)]$ satisfies*

$$\widetilde{r}'(t) = -\frac{\int_{\partial D(t)} \|\nabla_g f\|_g^{-1} \, dA_g}{\operatorname{vol}\left(S^{\omega}_{\widetilde{r}(t)}(o_\omega)\right)} < 0.$$

Then, $\widetilde{r}$ is strictly decreasing in R_f, and hence, injective (and bijective onto its image).

Remark 2.2.22. Note that when $R_f = [0, T]$, then $\widetilde{r} : [0, T] \longrightarrow [0, L(D)]$ is bijective.

Proof of Lemma 2.2.21. Let us assume that $t_1 \leq t_2$. By assertion (2) of Remark 2.2.16, we know that $D(t_2) \subseteq D(t_1)$, and hence, $\operatorname{vol}(D(t_2)) \leq \operatorname{vol}(D(t_1))$. Thus, since $\operatorname{vol}(D(t_2)) = \operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t_2)}(o_\omega)\right)$ and $\operatorname{vol}(D(t_1)) = \operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t_1)}(o_\omega)\right)$, we have that $\operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t_2)}(o_\omega)\right) \leq \operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t_1)}(o_\omega)\right)$, and hence, $\widetilde{r}(t_2) \leq \widetilde{r}(t_1)$, because the volume of geodesic balls is an increasing function over its radius.

On the other hand, given $t \in R_f$, and denoting

$$V(t) = \operatorname{vol}(D(t)) = \operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t)}(o_\omega)\right),$$

we have, applying Theorem 2.1.77, that

$$V'(t) = \frac{d}{dt} \operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t)}(o_\omega)\right) = \operatorname{vol}\left(S^{\omega}_{\widetilde{r}(t)}(o_\omega)\right) \widetilde{r}'(t). \tag{2.52}$$

Moreover, defining $\Omega(t) = \{x \in D : f(x) < t\}$, we have that $\Omega(t) = D - D(t)$, and hence,

$$\operatorname{vol}(\Omega(t)) = \operatorname{vol}(D) - V(t)$$

Thus, since $\partial D(t) = \Gamma(t) = \{x \in D : f(x) = t\} = \partial\Omega(t)$ for all $t \in R_f$ and $d/dt \operatorname{vol}(D) = 0$, and applying the *Co-area Formula* (see Theorem 2.1.62), we obtain that

$$-V'(t) = \frac{d}{dt} \operatorname{vol}(\Omega(t)) = \int_{\partial\Omega(t) = \partial D(t)} \|\nabla_g f\|^{-1} \, dA_g,$$

Then, applying equation (2.52), we have that the function $\widetilde{r}(t)$ satisfies

$$\widetilde{r}'(t) = -\frac{\int_{\partial D(t)} \left\|\nabla^M f\right\|^{-1} dA_g}{\operatorname{vol}\left(S^{\omega}_{\widetilde{r}(t)}(o_\omega)\right)} < 0 \quad \text{for all} \quad t \in R_f$$

and the proposition follows.

□

Now we show, in the following theorem, some properties satisfied by the symmetrization of a function. In fact, we show that, given $f : D \subseteq M \longrightarrow \mathbb{R}_+$ a non-negative smooth function defined on the precompact domain D, the symmetrized function $f^{*_\omega} : D^{*_\omega} \longrightarrow \mathbb{R}_+$ is a radial function, and that f and f^{*_ω} are equimeasurable.

Theorem 2.2.23. *Let (M, g) be a complete n-dimensional Riemannian manifold, $D \subseteq M$ be a precompact connected domain in M and $f : D \longrightarrow \mathbb{R}_+$ be a non-negative smooth function defined on D. Let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Then, the symmetrized objects f^* and D^* satisfy the following properties:*

1. *The function f^* depends only on the radial distance function to the center o_ω of the ball D^* in $\mathbb{M}_\omega$,* i.e., *only depends on r, and it is non-increasing.*
2. *The functions f and f^* are equimeasurable in the sense that*

$$\operatorname{vol}\Big(\{x \in D\,:\, f(x) \geq t\}\Big) = \operatorname{vol}\Big(\{x^* \in D^*\,:\, f^*(x^*) \geq t\}\Big)$$

for all $t \geq 0$.

Proof. Let us begin this proof by showing assertion (1). Let $x_1^*, x_2^* \in D^* = B^\omega_{\widetilde{r}(0)}(o_\omega)$ such that $r(x_1^*) = r(x_2^*)$. Then, given $t_0 \in [0, T]$, it is evident that $x_1^* \in B^\omega_{\widetilde{r}(t)}(o_\omega)$ for all $t \in [0, t_0]$ if, and only if, $x_2^* \in B^\omega_{\widetilde{r}(t)}(o_\omega)$ for all $t \in [0, t_0]$. In fact, for all $t \in [0, t_0]$, $x_1^* \in B^\omega_{\widetilde{r}(t)}(o_\omega)$ if, and only if, $r(x_1^*) \leq \widetilde{r}(t)$, then $r(x_2^*) \leq \widetilde{r}(t)$ which, in its turn, is true if, and only if, $x_2^* \in B^\omega_{\widetilde{r}(t)}(o_\omega)$. Hence, by this last fact and by the definition of the symmetrization of f (see Definition 2.2.17),

$$f^*(x_1^*) = \sup\left\{t \geq 0\,:\, x_1^* \in B^\omega_{\widetilde{r}(t)(o_\omega)}\right\} = \sup\left\{t \geq 0\,:\, x_2^* \in B^\omega_{\widetilde{r}(t)}(o_\omega)\right\} = f^*(x_2^*)$$

which shows that f^* is a radial function. Therefore, f^* depends only on the geodesic distance to the center o_ω, *i.e.*, $f^*(x^*) = f^*\left(r(x^*)\right)$.

Now, to prove that f^* is non-increasing, let $x_1^*, x_2^* \in D^*$ such that $r(x_1^*) \leq r(x_2^*)$. We are going to prove that $t_1 := f^*(x_1^*) \geq t_2 := f^*(x_2^*)$. In fact, since from

Lemma 2.2.21 we know that $\widetilde{r}$ is a non-increasing function then, by Definition 2.2.17, we have that

$$f^*(x_2^*) = \sup\left\{t \geq 0 \,:\, x_2^* \in B^\omega_{\widetilde{r}(t)}\right\} = \sup\left\{t \geq 0 \,:\, r(x_2^*) \leq \widetilde{r}(t)\right\} = t_2.$$

Then, if $t \leq t_2$, we have that $x_2^* \in B^\omega_{\widetilde{r}(t)}(o_\omega)$, thus $r(x_2^*) \leq \widetilde{r}(t)$ for all $t \leq t_2$. In particular, $r(x_1^*) \leq r(x_2^*) \leq \widetilde{r}(t_2)$ which, in its turn, shows that $x_1^* \in B^\omega_{\widetilde{r}(t_2)}$, and hence, $t_1 = f^*(x_1^*) = \sup\{t \geq 0 \,:\, x_1^* \in B^\omega_{\widetilde{r}(t)}\} \geq t_2 = f^*(x_2^*)$, showing that f^* is non-increasing.

To prove assertion (2), note first that we have, by Definitions 2.2.17 and 2.2.19, that

$$D(t)^* = B^\omega_{\widetilde{r}(t)} = \{x^* \in D^* \,:\, f^*(x^*) \geq t\} \quad \text{for all} \quad t > 0.$$

In fact, if $x^* \in B^\omega_{\widetilde{r}(t_0)}$, then $f^*(x^*) = \sup\{t \geq 0 \,:\, x^* \in B^\omega_{\widetilde{r}(t)}\} \geq t_0$, and, conversely, if $f^*(x^*) = \sup\{t \geq 0 \,:\, x^* \in B^\omega_{\widetilde{r}(t)}\} \geq t_0$, then $x^* \in B^\omega_{\widetilde{r}(t_0)}$. Therefore, since $D(t) = \{x \in D \,:\, f(x) \geq t\}$, we obtain, for all $t \geq 0$, that

$$\begin{aligned}\operatorname{vol}\Big(\{x \in D \,:\, f(x) \geq t\}\Big) &= \operatorname{vol}(D(t)) = \operatorname{vol}(D(t)^*)\\ &= \operatorname{vol}\Big(\{x^* \in D^* \,:\, f^*(x^*) \geq t\}\Big)\end{aligned}$$

□

To end this chapter, we show a relationship between the integral of a function defined on a geodesic ball of a Riemannian manifold and the integral of its symmetrization by controlling the monotony of the function. In fact, this theorem below will play a key rôle in order to proof our comparison for the *torsional rigidity* (see Theorem 3.6.3). First, let us define the following.

Definition 2.2.24 (see [53])**.** *Let (M, g) a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ a point of M and let $B_R(o)$ be the geodesic ball of M with radius $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$ centered at o. Given $f : [0, R] \longrightarrow \mathbb{R}_+$ a non-negative, real valued, smooth function, we define the* transplanted function ψ of f into $B_R(o)$ *as the radial function defined as*

$$\begin{aligned}\psi : B_R(o) &\longrightarrow \mathbb{R}_+\\ p &\longmapsto \psi(p) := f\left(r_o(p)\right),\end{aligned}$$

where r_o is the radial distance function to o, the center of the geodesic ball $B_R(o)$ (see Definition 2.1.69).

Remark 2.2.25. Note that, since f is a radial function, we have that the transplanted function ψ satisfies that

$$\left.\frac{d}{dr}\right|_{r=r_o(p)} \psi = f'(r_o(p))$$

for all $p \in B_R(o) - \{o\}$. From now on, when it is clear from the context, we idenfity $\psi(p) = f(r_o(p))$ by $\psi(r)$ where $r = r_o(p)$, and its first derivative by $\psi'(r)$, to simplify the notation.

Theorem 2.2.26. *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_{_\omega})$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ be a geodesic ball of M with radius R centered at o. Let $f : [0, R] \longrightarrow \mathbb{R}_+$ be a non-negative, real valued, smooth function such that $f'(r) < 0$ for all $r \in (0, R]$, $f'(0) = 0$ and $f(R) = 0$. Suppose that $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$, and that there exists the Schwarz symmetrization $B^\omega_{s(R)}(o_\omega)$ of $B_R(o)$ in $\mathbb{M}_\omega$. Let $\psi : B_R(o) \longrightarrow \mathbb{R}_+$ be the transplanted function of f into $B_R(o)$,* i.e., *$\psi(p) := f\left(r_o(p)\right)$. Then, we have that*

$$\int_{B_R(o)} \psi \, dV_g = \int_{B^\omega_{s(R)}(o_\omega)} \psi^* dV_{g_\omega}, \tag{2.53}$$

where $\psi^* : B^\omega_{s(R)}(o_\omega) \longrightarrow \mathbb{R}_+$ is the symmetrization of the transplanted function ψ, dV_g and dV_{g_ω} are, respectively, the Riemannian volume elements in $B_R(o)$ and $B^\omega_{s(R)}(o_\omega)$, and r_o is the radial distance function to o in $B_R(o)$.

Proof. We are going to analyze first the symmetrization ψ^*. The transplanted function

$$\psi \,:\, B_R(o) \longrightarrow \mathbb{R}_+$$

satisfies, by definition, that $\psi \in C^\infty\left(B_R(o) - \{o\}\right) \cap C^0\left(\overline{B_R(o)}\right)$ and, moreover, that $\psi|_{S_R(o)} = 0$.

From Remark 2.2.25, we consider ψ defined as a radial function on the interval $[0, R]$. Let us denote by $T = \max_{[0,R]} \psi$. Then as, by hypothesis, f is monotone, we have that $\psi' < 0$ in $(0, R]$ and $\psi'(0) = 0$, and that $\psi : [0, R] \longrightarrow [0, T]$ is bijective with $\psi(0) = T$ and $\psi(R) = 0$.

On the other hand, from assertion (3) of Remark 2.2.18, we have that the set of critical values $S_\psi \subset [0,T]$ of ψ has null measure, and the set of regular values, $R_\psi = [0,T] - S_\psi$, is open and dense in $[0,R]$. In particular, for any $r \in R_\psi$, the set $\{p \in B_R(o) : \psi(p) = r\}$ is a smooth embedded hypersurface in $B_R(o)$ and $\|\nabla_g \psi\|_g$ does not vanish along $\{p \in B_R(o) : \psi(p) = r\}$. In fact, since $\|\nabla_g r_o\|_g = 1$ in $B_R(o)$ (see assertion (4) of Proposition 2.1.70), the transplanted function $\psi : B_R(o) \longrightarrow [0,T]$ satisfies, for all $p \in B_R(o) - \{o\}$ such that $r_o(p) = r$, that

$$\|\nabla_g \psi(p)\|_g = |f'(r)| \, \|\nabla_g r_o(p)\|_g = |f'(r)| \neq 0, \tag{2.54}$$

and hence, the set of regular values of ψ is $R_\psi = (0,T)$.

Now, let us define the function $a : [0,T] \longrightarrow [0,R]$ as $a(t) := \psi^{-1}(t) = f^{-1}(t)$, satisfying $a(0) = \psi^{-1}(0) = f^{-1}(0) = R$ and $a(T) = \psi^{-1}(T) = f^{-1}(t) = 0$. We know that

$$a'(t) = \frac{1}{\psi'(a(t))} < 0 \quad \text{for all} \quad t \in [0,T),$$

so, $a(t)$ is strictly decreasing in $[0,T]$.

On the other hand, and given $t \in [0,T]$, let us consider the sets $D(t)$ and $\Gamma(t)$ defined in Definition 2.2.15, *i.e.*,

$$\begin{aligned} D(t) &= \{p \in B_R(o) : \psi(p) \geq t\} = \{p \in B_R(o) : f(r_o(p)) \geq t\} \\ &= \left\{p \in B_R(o) : r_o(p) \leq f^{-1}(t)\right\} = \{p \in B_R(o) : r_o(p) \leq a(t)\} \\ &= B_{a(t)}(o) \end{aligned} \tag{2.55}$$

and

$$\begin{aligned} \Gamma(t) &= \{p \in B_R(o) : \psi(p) = t\} = \{p \in B_R(o) : f(r_o(p)) = t\} \\ &= \left\{p \in B_R(o) : r_o(p) = f^{-1}(t)\right\} = \{p \in B_R(o) : r_o(p) = a(t)\} \\ &= S_{a(t)}(o). \end{aligned} \tag{2.56}$$

Moreover, we have that $D(0) = B_{a(0)}(o) = B_R(o)$ and $D(T) = B_{a(T)}(o) = \{o\}$, where o is the center of the geodesic ball $B_R(o)$.

Now, we consider the symmetrization in $\mathbb{M}_\omega$ of the sets $D(t) = B_{a(t)}(o) \subseteq B_R(o) \subseteq M$, namely, the geodesic balls $D(t)^* = B^\omega_{\widetilde{r}(t)}(o_\omega)$ in $\mathbb{M}_\omega$ such that

$$\operatorname{vol}(D(t)) = \operatorname{vol}\left(B^\omega_{\widetilde{r}(t)}(o_\omega)\right),$$

where, for each $t \in [0,T]$, $\widetilde{r}(t)$ denotes the symmetrized radius (see Definition 2.2.19. Then, in this particular context and from Lemma 2.2.21, we have that $\widetilde{r} : [0,T] \longrightarrow [0, s(R)]$ is strictly decreasing, and hence, bijective, where $s(R)$ is the radius of symmetrization of $B_R(o)$, *i.e.*, the symmetrization of $B_R(o)$ in $\mathbb{M}_\omega$ is $B^\omega_{s(R)}(o_\omega)$. In fact, note that if $t_1, t_2 \in [0,T]$ such that $t_1 < t_2$, then, since $a(t)$ is strictly decreasing, $a(t_1) > a(t_2)$, thus

$$\operatorname{vol}\left(B^\omega_{\widetilde{r}(t_1)}(o_\omega)\right) = \operatorname{vol}\left(B_{a(t_1)}(o)\right) > \operatorname{vol}\left(B_{a(t_2)}(o)\right) = \operatorname{vol}\left(B^\omega_{\widetilde{r}(t_2)}(o_\omega)\right),$$

and hence, $\widetilde{r}(t_1) > \widetilde{r}(t_2)$. Furthermore, applying Lemma 2.2.21, for all t in the set of the regular values of ψ, *i.e.*, for all $t \in R_\psi = (0,T)$, we have that

$$\widetilde{r}'(t) = -\frac{1}{\operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right)} \int_{\Gamma(t)} \|\nabla_g \psi\|_g^{-1} \, dA_g, \tag{2.57}$$

where dA_g is the Riemannian volume element in $\Gamma(t)$ with respect to the metric tensor g (see Subsection 2.1.7.5).

On the other hand, the inverse of $\widetilde{r}$ is the decreasing function

$$\phi : [0, s(R)] \longrightarrow [0,T], \quad \phi(\ell) := (\widetilde{r})^{-1}(\ell),$$

such that $\phi'(\widetilde{r}(t)) = \frac{1}{\widetilde{r}'(t)}$ for all $t \in [0,T]$, $\phi(0) = T$ and $\phi(s(R)) = 0$.

With all this background, we can say now that there exists a smooth function $\widetilde{\psi} : [0, s(R)] \longrightarrow \mathbb{R}$ such that the symmetrization of $\psi : B_R(o) \longrightarrow \mathbb{R}$ is a radial function $\psi^* : B^\omega_{s(R)}(o_\omega) \longrightarrow \mathbb{R}$, $\psi^* := \widetilde{\psi}(r_o(p^*))$ which satisfies, for all $p^* \in B^\omega_{s(R)}(o_\omega)$, the following equality

$$\psi^*(p^*) = \widetilde{\psi}\left(r_{o_\omega}(p^*)\right) = t_0 = \phi\left(\widetilde{r}(t_0)\right). \tag{2.58}$$

Therefore, for all $t \in (0,T)$, we have, applying equation (2.57), that

$$\left.\frac{d}{dr_{o_\omega}}\psi^*\right|_{r_{o_\omega}(p^*) = \widetilde{r}(t)} = \widetilde{\psi}'\left(\widetilde{r}(t)\right) = \phi'\left(\widetilde{r}(t)\right) = \frac{1}{\widetilde{r}'(t)}. \tag{2.59}$$

Now, let us make the following abuse of notation: since ψ^* is a radial function, we will identify $\psi^*(p^*)$ by $\psi^*\left(r_{o_\omega}(p^*)\right)$ for all $p^* \in B^\omega_{s(R)}(o_\omega)$, and hence, considering r_{o_ω} as a parameter $r \in [0, s(R)]$, we identify

$$\psi^* \equiv \psi^*(r_{o_\omega}) \equiv \psi^*(r) = \widetilde{\psi}(r) \quad \text{for all} \quad r \in [0, s(R)].$$

Moreover, considering $\widetilde{r}(t)$ ($t \in [0,T]$) as a parameter $\widetilde{r} \in [0, s(R)]$, we have, from equation (2.58), that $\widetilde{\psi}(r) = \phi(\widetilde{r})$. Therefore, we obtain that

$$\psi^* \equiv \psi^*(r_{o_\omega}) \equiv \psi^*(r) = \widetilde{\psi}(r) = \phi(\widetilde{r}),$$

where $r \in [0, s(R)]$ and $\widetilde{r} \in [0, s(R)]$.

Then, using the Co-area formula, the identification $r_{o_\omega}(p^*) = r$ for any $p^* \in B^\omega_{s(R)}(o_\omega)$ and from the expressions (2.31) for the volumes of the geodesic balls and for the geodesic spheres of rotationally symmetric model spaces, we obtain that

$$\begin{aligned}
\int_{B^\omega_{s(R)}(o_\omega)} \psi^*(r) dV_{g_\omega} &= \int_{B^\omega_{s(R)}(o_\omega)} \phi(\widetilde{r})\, dV_{g_\omega} \\
&= \int_0^{s(R)} \left(\int_{S^\omega_{\widetilde{r}}(o_\omega)} \phi(\widetilde{r}) \frac{1}{\left\|\nabla_{g_\omega} \widetilde{r}\right\|_{g_\omega}} dV_{g_{S^\omega_{\widetilde{r}}(o_\omega)}} \right) d\widetilde{r} \\
&= \int_0^{s(R)} \phi(\widetilde{r}) \operatorname{vol}\left(S^\omega_{\widetilde{r}}(o_\omega)\right) d\widetilde{r}.
\end{aligned}$$

Now, since the symmetrized radius $\widetilde{r} : [0,T] \longrightarrow [0, s(R)]$ is a strictly decreasing and bijective function with $\widetilde{r}(0) = s(R)$ and $\widetilde{r}(T) = 0$, we obtain, using the identification $\widetilde{r}(t) = \widetilde{r}$, and hence, $\widetilde{r}'(t)dt = d\widetilde{r}$, and applying equations (2.57) and (2.58), that

$$\begin{aligned}
\int_{B^\omega_{s(R)}(o_\omega)} \psi^* dV_{g_\omega} &= \int_0^{s(R)} \phi(\widetilde{r}) \operatorname{vol}\left(S^\omega_{\widetilde{r}}(o_\omega)\right) d\widetilde{r} \\
&= \int_T^0 \phi(\widetilde{r}(t)) \operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right) \widetilde{r}'(t)\, dt \\
&= \int_0^T t \left(\int_{\Gamma(t)} \left\|\nabla_g \psi(q)\right\|_g^{-1} dA_g \right) dt.
\end{aligned}$$

Finally since, by definition of $\Gamma(t)$, $\psi|_{\Gamma(t)} = t$ for all $t \in [0,T]$ and by applying again the Co-area formula, we have that

$$\begin{aligned}
\int_{B^{\omega}_{s(R)}(o_\omega)} \psi^* dV_{g_\omega} &= \int_0^T t \left(\int_{\Gamma(t)} \|\nabla_g \psi(q)\|_g^{-1} \, dA_g \right) dt \\
&= \int_0^T \psi(t) \left(\int_{\Gamma(t)} \|\nabla_g \psi(q)\|_g^{-1} \, dA_g \right) dt \\
&= \int_0^T \left(\int_{\Gamma(t)} \psi(t) \, \|\nabla_g \psi(q)\|_g^{-1} \, dA_g \right) dt \\
&= \int_{B_R(o)} \psi(p) \, \|\nabla_g \psi(p)\|_g^{-1} \, \|\nabla_g \psi(p)\|_g \, dV_g \\
&= \int_{B_R(o)} \psi \, dV_g .
\end{aligned}$$

□

Chapter 3

Mean exit time, torsional rigidity, Poisson hierarchy and moment spectrum comparisons on geodesic balls

In this chapter, we are going to bound from above and from below the *mean exit time*, the *torsional rigidity*, the *Poisson hierarchy* and the *moment spectrum* defined on geodesic balls $B_R(o)$ of a given complete Riemannian manifold (M, g) by the ones defined on geodesic balls $B_R^\omega(o_\omega)$ with the same radius of rotationally symmetric model spaces $(\mathbb{M}_\omega, g_\omega)$ by assuming that mean curvatures of the geodesic spheres contained in $B_R(o)$ are bounded from below (respectively from above) by the mean curvatures of the geodesic spheres contained in $B_R^\omega(o_\omega)$ with the same radius. All our results presented along this chapter can be found in our research paper [63].

These geometric invariants are related with the *Brownian motion*. In particular, the mean exit time function defined on a bounded region D with boundary ∂D measures the expectation of the time that a *Brownian particle*, whose movement starts inside D, takes to leave D through its boundary ∂D for the first time. Furthermore, the torsional rigidity, the Poisson hierarchy and the moment spectrum defined on D are constructed in terms of the mean exit time function of D.

We start this chapter by giving a brief definition of the *Brownian motion* in Section 3.1. Then, in Section 3.2, we will present the preliminary concepts, *i.e.*, we will give detailed definitions of these mentioned invariants, and moreover, we will show some of its properties in rotationally symmetric model spaces. Next, in Section 3.3, we will show the different directions that research has taken in this area, as well as some of the results obtained along the last years. Furthermore, we will explain why we have chosen our hypothesis. Section 3.4 is devoted to deal with the properties of the mean exit time function defined on geodesic balls of a complete Riemannian manifold satisfying our hypotheses, its relationship with its volume, and the isoperimetric inequalities satisfied by these domains. Next, in Section 3.5, we will prove our bounds for the Poisson hierarchy and the *averaged moment spectrum* of a geodesic ball under our restrictions. Finally, in Section 3.6, we will bound the torsional rigidity of a geodesic ball by means of its Schwarz symmetrization.

3.1 Brownian motion

The first complete mathematical description of the *Brownian motion* is due to A. Einstein. In [23] he explained the ceaseless erratic motion of pollen grains on a water surface observed for the first time by the botanist R. Brown. This movement is the result of the irregular collisions with the molecules of water experienced by these particles in suspension. This physical phenomenon was called *Brownian motion*. In the cited paper, Einstein showed the stochastic nature of the Brownian motion by proving that the displacement of a Brownian particle is governed by a probability distribution which satisfies a diffusion equation (see the survey [33] of A. Grigor'yan for more background on the origin of the Brownian motion).

The simplest example of a mathematical model for the Brownian motion is a random walk on the lattice $\mathbb{Z}^n$. This model consists in a particle which moves on the nodes of $\mathbb{Z}^n$ by choosing the node to jump randomly, at each step, among one of the $2n$ neighbouring nodes, with equal probability $\frac{1}{2n}$ (as the reader can see in Figure 3.1 which we have taken from [33]).

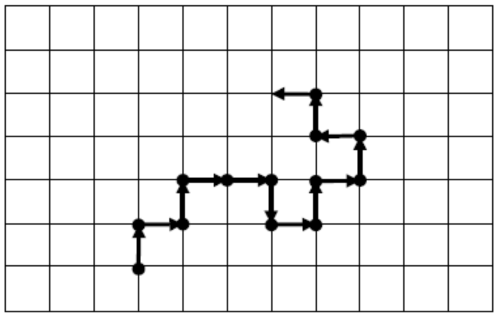

Figure 3.1: The random walk on $\mathbb{Z}^2$.

The natural question that appears is following: what happens with the trajectory of a Brownian particle when the number of steps goes to infinity? On these question, G. Pólya in [65] showed that the Brownian particle does not necessarily pass through each node infinitely many times as one could expect in view of the homogeneous and isotropic character of the movement. In fact, Pólya showed that it depends on the dimension of $\mathbb{Z}^n$: he proved that for $n \leq 2$ the Brownian particle does pass through each node infinitely many times with probability 1 but, for $n > 2$, it passes through each node only finitely many times with probability 1 too.

The same phenomenon and questions take place in the continuous case in the *Euclidean space* $\mathbb{R}^n$. For the continuous case, N. Wiener in [76] constructed a continuous model for the Brownian motion which is considered as the standard model for the Brownian motion and it is usually called as the *Wiener process* (the reader can see an image which represents the Wiener process in $\mathbb{R}^2$ in Figure 3.2 which we have taken from [33]).

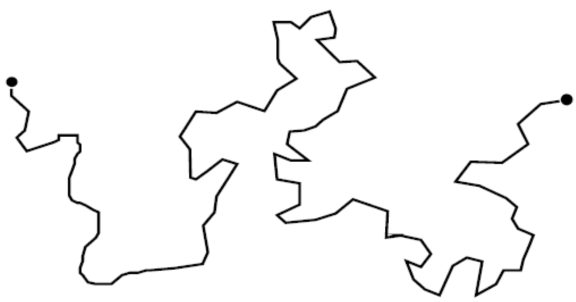

Figure 3.2: The Brownian motion on $\mathbb{R}^2$.

Moreover, the continuous Brownian motion can be constructed using as the underlying space a Riemannian manifold. We say that the Brownian motion defined on a Riemannian manifold is *recurrent* if any Brownian particle visits any open set at arbitrary large times with probability 1 and otherwise is said to be *transient*. There are several papers which study what geometric properties of the Riemannian manifold ensure that the Brownian motion is recurrent or transient, we refer to [33] for a background on results in that direction. In fact, this is a second natural question that arises in the study of the Brownian motion. Namely: what geometric properties of the underlying space where the movement takes place ensure that a Brownian particle which starts its movement inside any bounded region D on the Riemannian manifold returns to this bounded region at arbitrarily large times?

On the other hand, there appears a third question on which our research will be focused throughout this chapter. This question is following: given a bounded domain of a Riemannian manifold and a Brownian particle that starts its displacement at a point inside this bounded domain, which is the time required so that the particle leave this bounded domain? In this sense, we study the *mean exit time function* that is a function which assigns, to each point in a bounded domain $p \in D$, the average of the times that a Brownian particle starting at p takes to leave the bounded domain through its boundary (see Definition 3.2.1). Moreover, from the bounds on the mean exit time function defined on a geodesic ball in a Riemannian manifold (M, g), we find comparisons for the *torsional rigidity*, the *Poisson hierarchy* and the *moment spectrum* of this geodesic ball in M (see Definition 3.2.5, 3.2.7 and 3.2.10).

The detailed description of the Wiener process goes beyond the objectives of this work. Nevertheless, let us show briefly the construction of this Wiener process in terms of the *heat kernel* following [33]. In fact, given (M, g) a Riemannian manifold, let $p(t, x, y)$ be the heat kernel defined in M, where $t > 0$ is a time and x, y are points of M. Then, the probability that the Brownian motion starting at the point $x \in M$ lies at a measurable domain $\Omega \subseteq M$ at the time t is:

$$P^x(\Omega, t) = \int_\Omega p(t, x, y)dV(y).$$

Then, as stated by A. Grigor'yan in [33], from the properties of the heat kernel, we can construct a (sub)Markov process X_t with the transition density p by using the standard probabilistic tools as it has been done by K.L Chung and Z. Zhao in [15] and by E.B. Dynkin in [22]. This process X_t is the *Wiener process* on a complete Riemannian manifold M. Therefore, we refer to the process X_t as the *Brownian motion on* M, and moreover, given a point $x \in M$ we denote the family of probability measures in the space of Brownian path emanating from x as $\mathbb{P}^x$.

There are other ways to construct the Brownian motion on manifold (or even on more general spaces), we refer to K.D. Elworthy [24], M. Fukushima, Y. Oshima and M. Takeda [26], H.P. McKean [58], or more recently P. Malliavin [50], for instance. For a more detailed background on the heat kernel, we refer to J. Dodziuk [20] and A. Grigor'yan [33] and [34], for instance.

3.2 Mean exit time, torsional rigidity, Poisson hierarchy and moment spectrum

In the first part of this section we define the geometric invariants on geodesic balls of a given Riemannian manifold on which we shall find comparisons between them and the ones defined on geodesic balls of a rotationally symmetric model space with the same radius by controlling the mean curvatures of the geodesic spheres and using the *Schwarz symmetrization* technique (see Subsection 2.2.3 for more details on the Schwarz symmetrization). Moreover, in the second part of this section, we will show some interesting properties of these invariants on rotationally symmetric model spaces, which we use in order to prove our statements. For a background on classical definitions see E.B. Dryden, J.J. Langford and P. McDonald [21], E.B. Dynkin[22], A. Hurtado, S. Markvorsen and V. Palmer [37], [38] and [39], K. Kinateder and P. McDonald [44], K. Kinateder, P. McDonald and D. Miller [45], S. Markvorsen [51], S. Markvorsen and V. Palmer [53] and P. McDonald [55] and [56].

Definition 3.2.1 (see [21])**.** *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold. Given* $\Omega \subseteq M$ *a precompact domain, the* first exit time from Ω *is given* *by the quantity*

$$\tau_\Omega := \inf\{t \geq 0 \,:\, X_t \notin \Omega\}\,.$$

where X_t is the Brownian motion defined on M.

Moreover, given $x \in \Omega$, the mean exit time function from x *is the function $E_\Omega : \Omega \longrightarrow \mathbb{R}$ that assigns to the point x the expectation of the value of the first exit time τ_Ω with respect to $\mathbb{P}^x$, $E_\Omega(x)$.*

Remark 3.2.2. Note that $E_\Omega \geq 0$ on Ω.

Furthermore, we have the following characterization due to E.B. Dynkin of the mean exit time function E_Ω as a solution of a second order partial differential with Dirichlet boundary data. This problem is known as the *Poisson problem* (see [21] for instance).

Proposition 3.2.3 (see [22]). *Let (M, g) be a complete n-dimensional Riemannian manifold and $\Omega \subseteq M$ be a precompact domain. Then, the mean exit time function E_Ω on Ω is the solution of the boundary valued problem*

$$\begin{aligned} \Delta_g E_\Omega + 1 = 0, \quad &on \quad \Omega, \\ E_\Omega|_{\partial\Omega} = 0, \end{aligned} \tag{3.1}$$

where Δ_g denotes the Laplacian on (M, g) with respect to the metric tensor g.

Remark 3.2.4. From now on, when the precompact domain Ω is a geodesic ball $B_R(o)$ of M with radius R centered at $o \in M$, we denote the mean exit time function $E_{B_R(o)}$ simply as E_R.

From the definition of the mean exit time function we define the *torsional rigidity* as follows.

Definition 3.2.5 (see [44], [45], [66]). *Let (M, g) be a complete n-dimensional Riemannian manifold. Given $\Omega \subseteq M$ a precompact domain and given E_Ω the mean exit time function on Ω, the* torsional rigidity of Ω *is the integral*

$$\mathcal{A}(\Omega) := \int_\Omega E_\Omega(x)\, dV_g,$$

where dV_g is the Riemannian volume element in Ω (see Subsection 2.1.7.5).

Remark 3.2.6. The name *torsional rigidity* comes form the fact that, when $\Omega \subseteq \mathbb{R}^2$ is a plane domain, the quantity $\mathcal{A}(\Omega)$ represents the torque required when twisting an elastic beam of uniform cross section Ω (see [2] and [67]).

On the other hand, the mean exit time function is the first in a sequence of functions $\{u_k\}_{k=1}^{\infty}$ defined on $\Omega \subseteq M$ inductively as in the following definition (see [35] and [66]). This sequence of functions is known as the *Poisson hierarchy* (see [21] or [38], for instance).

Definition 3.2.7 (see [38])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold. Given $\Omega \subseteq M$ a precompact domain, the* Poisson hierarchy of Ω *is the sequence $\{u_{k,\Omega}\}_{k=1}^{\infty}$ of solutions of the following recurrence of boundary valued problems*

$$\begin{aligned} \Delta_g u_{1,\Omega} + 1 = 0, &\quad on \quad \Omega, \\ u_{1,\Omega}\big|_{\partial\Omega} = 0, & \end{aligned} \tag{3.2}$$

and, for $k \geq 2$,

$$\begin{aligned} \Delta_g u_{k,\Omega} + k u_{k-1,\Omega} = 0, &\quad on \quad \Omega, \\ u_{k,\Omega}\big|_{\partial\Omega} = 0. & \end{aligned} \tag{3.3}$$

where Δ_g denotes the Laplacian on M with respect to the metric tensor g.

Remark 3.2.8. Note that the boundary valued problems (3.1) and (3.2) are the same, and hence, the mean exit time function and the first element of the Poisson hierarchy of Ω are equal, *i.e.*, $u_{1,\Omega} = E_\Omega$ on Ω. From now on, when the precompact domain Ω is a geodesic ball $B_R(o)$ of M with radius R centered at $o \in M$, we denote the Poisson hierarchy $\{u_{k,B_R(o)}\}_{k=1}^{\infty}$ simply as $\{u_{k,R}\}_{k=1}^{\infty}$.

Let us also remark that sometimes the Poisson hierarchy is defined from $k = 0$ with $u_{0,\Omega} = 1$ on a precompact domain Ω and for $k \geq 1$ from the recurrence of the boundary valued problem (3.3). Note that, for the case $k = 1$, we have the boundary valued problem (3.2), *i.e.*, $\Delta_g u_{1,\Omega} = -u_{0,\Omega} = -1$ and, on the other hand, we have that $\int_\Omega u_{0,\Omega}\, dV_g = \mathrm{vol}(\Omega)$ (see [39], for instance).

The elements of the Poisson hierarchy transfer some of its properties from one to another. For instance, since $u_{1,\Omega} \geq 0$, then all the elements of the Poisson hierarchy are non-negative. On the other hand, given two precompact domains $\Omega_2 \subset \Omega_1 \subseteq M$ of a complete Riemannian manifold, there is a relationship between the Poisson hierarchies of Ω_1 and Ω_2, as we can see in the following result.

Proposition 3.2.9. *Let (M, g) be a complete n-dimensional Riemannian manifold. Let Ω be a precompact domain of M. Then, the following assertions hold:*

1. *The elements of the Poisson hierarchy of Ω are positive on Ω,* i.e., *$u_{k,\Omega} \geq 0$ on Ω for all $k \geq 1$.*
2. *For any precompact doman $\widetilde{\Omega} \subset \Omega \subseteq M$, we have inequalities $u_{k,\Omega} \geq u_{k,\widetilde{\Omega}}$ on $\widetilde{\Omega}$ for all $k \geq 1$.*

where $\{u_{k,\Omega}\}_{k=1}^{\infty}$ and $\{u_{k,\widetilde{\Omega}}\}_{k=1}^{\infty}$ are the Poisson hierarchies of Ω and $\widetilde{\Omega}$, respectively.

Proof. To prove assertion (1) we proceed by an induction argument. From Remark 3.2.2, we know that $u_{1,\Omega} = E_\Omega \geq 0$ on Ω. Thus, assuming that $u_{k,\Omega} \geq 0$ on Ω, we have that

$$\Delta_g u_{k+1,\Omega} = -(k+1)u_{k,\Omega} \leq 0 \quad \text{on} \quad \Omega.$$

Thus, we have that $\Delta_g(-u_{k+1,\Omega}) \geq 0$, and hence, $-u_{k+1,\Omega}$ is a subharmonic function. Then, if there is a point $p_0 \in \Omega$ where $-u_{k+1,\Omega}$ attains its maximum, *i.e.*, $-u_{k+1,\Omega}(p) \leq -u_{k+1,\Omega}(p_0)$ for all $p \in \Omega$, we obtain, by applying the Strong Maximum Principle (see Theorem 2.1.64), that $-u_{k+1,\Omega} = c$ on Ω for some constant $c \in \mathbb{R}$. Hence, since $-u_{k+1,\Omega} = 0$ on $\partial\Omega$ we obtain, by continuity, that $c = 0$, and hence, $u_{k+1,\Omega} = 0$ on Ω. Otherwise, if $-u_{k+1,\Omega}$ attains its maximum at $q \in \partial\Omega$, we have that $-u_{k+1,\Omega}(p) \leq -u_{k+1,\Omega}(q) = 0$ for all $p \in \Omega$. Therefore, we obtain that $u_{k+1,\Omega} \geq 0$ on Ω, and hence, $u_{k,\Omega} \geq 0$ on Ω for all $k \geq 1$, showing statement (1).

Now, to prove assertion (2), we also proceed by an induction argument. First, let us study the case $k = 1$ where $u_{1,\widetilde{\Omega}} - u_{1,\Omega}$ is defined in $\widetilde{\Omega}$. From the definition of the Poisson hierarchy, we obtain that

$$\Delta_g \left(u_{1,\widetilde{\Omega}} - u_{1,\Omega} \right) = \Delta_g u_{1,\widetilde{\Omega}} - \Delta_g u_{1,\Omega} = -1 + 1 = 0 \quad \text{on} \quad \widetilde{\Omega}.$$

Thus, we have that $u_{1,\widetilde{\Omega}} - u_{1,\Omega}$ is a harmonic function. Therefore, if there exists $p_0 \in \widetilde{\Omega}$ where $u_{1,\widetilde{\Omega}} - u_{1,\Omega}$ attains its maximum, we have, by applying the Strong

Maximum Principle, that $u_{1,\widetilde{\Omega}} - u_{1,\Omega} = c$ on $\widetilde{\Omega}$ for some constant $c \in \mathbb{R}$. Moreover, we have, by definition, that $u_{1,\widetilde{\Omega}} = 0$ on $\partial\widetilde{\Omega}$ and, from assertion (1), that $u_{1,\Omega} \geq 0$ on Ω. Hence, by continuity, we obtain that $c = u_{1,\widetilde{\Omega}} - u_{1,\Omega} = -u_{1,\Omega} \leq 0$ on $\partial\widetilde{\Omega}$. Thus, we have that $u_{1,\widetilde{\Omega}} - u_{1,\Omega} = c \leq 0$ on $\widetilde{\Omega}$. Otherwise, if $u_{1,\widetilde{\Omega}} - u_{1,\Omega}$ attains its maximum at $q \in \partial\widetilde{\Omega}$, we have that

$$\left(u_{1,\widetilde{\Omega}} - u_{1,\Omega}\right)(p) \leq \left(u_{1,\widetilde{\Omega}} - u_{1,\Omega}\right)(q) = -u_{1,\Omega}(q) \leq 0 \quad \text{for all} \quad p \in \widetilde{\Omega}.$$

Therefore, $u_{1,\Omega} \geq u_{1,\widetilde{\Omega}}$ on $\widetilde{\Omega}$.

Now, assuming that $u_{k,\Omega} \geq u_{k,\widetilde{\Omega}}$ on $\widetilde{\Omega}$, we obtain that

$$\begin{aligned}
\Delta_g\left(u_{k+1,\widetilde{\Omega}} - u_{k+1,\Omega}\right) &= \Delta_g u_{k+1,\widetilde{\Omega}} - \Delta_g u_{k+1,\Omega} \\
&= -(k+1)u_{k,\widetilde{\Omega}} + (k+1)u_{k,\Omega} \\
&= (k+1)(u_{k,\Omega} - u_{k,\widetilde{\Omega}}) \geq 0
\end{aligned}$$

on $\widetilde{\Omega}$. Then, we have that $u_{k+1,\widetilde{\Omega}} - u_{k+1,\Omega}$ is subharmonic in $\widetilde{\Omega}$. Finally, applying the Strong Maximum Principle as in the case $k = 1$, we have that if its maximum is attained in $\widetilde{\Omega}$ then $u_{k+1,\widetilde{\Omega}} - u_{k+1,\Omega} = c$ on $\widetilde{\Omega}$ for some constant $c \in \mathbb{R}$. Thus, since $u_{k+1,\widetilde{\Omega}} = 0$ on $\partial\Omega$ and $u_{k+1,\Omega} \geq 0$ on Ω, we obtain, by continuity, that $c = u_{k+1,\widetilde{\Omega}} - u_{k+1,\Omega} = -u_{k+1,\Omega} \leq 0$ on $\widetilde{\Omega}$. Hence, $u_{k+1,\widetilde{\Omega}} - u_{k+1,\Omega} = c \leq 0$ on $\widetilde{\Omega}$. Otherwise, if its maximum is attained at $q \in \partial\Omega$, we have that

$$\left(u_{k+1,\widetilde{\Omega}} - u_{k+1,\Omega}\right)(p) \leq \left(u_{k+1,\widetilde{\Omega}} - u_{k+1,\Omega}\right)(q) = -u_{k+1,\Omega} \leq 0 \quad \text{for all} \quad p \in \widetilde{\Omega}.$$

Therefore, $u_{k+1,\Omega} \geq u_{k+1,\widetilde{\Omega}}$ on $\widetilde{\Omega}$ and the proposition follows.

□

The L^p*-moment spectrum* of a precompact domain Ω is defined from its Poisson hierarchy as follows.

Definition 3.2.10 (see [38]). *Let (M, g) be a complete n-dimensional Riemannian manifold. Given $\Omega \subseteq M$ a precompact domain and given $\{u_{k,\Omega}\}_{k=1}^{\infty}$ the Poisson hierarchy of Ω, the L^p-*moment spectrum of Ω *is the sequence of integrals $\{\mathcal{A}_{p,k}\}_{k=1}^{\infty}$ given by*

$$\mathcal{A}_{p,k}(\Omega) := \left(\int_\Omega (u_{k,\Omega}(x))^p \, dV_g\right)^{1/p}, \quad \textit{for} \quad k = 1, 2, \ldots, \infty,$$

where dV_g is the Riemannian volume element in Ω.

In particular, we refer as the moment spectrum of Ω *to the L^1-moment spectrum of Ω, and we denote it by $\{\mathcal{A}_k\}_{k=1}^{\infty}$. Namely, the moment spectrum of Ω is the sequence of integrals given by*

$$\mathcal{A}_k(\Omega) := \mathcal{A}_{1,k}(\Omega) = \int_\Omega u_{k,\Omega}\, dV_g.$$

Remark 3.2.11. In this work, we focus our study in the *moment spectrum of* Ω and, in particular, in the *torsional rigidity of* Ω, which is the first value of the moment spectrum of Ω. In fact, the torsional rigidity of Ω is defined as the integral

$$\mathcal{A}(\Omega) = \mathcal{A}_1(\Omega) = \int_\Omega u_{1,\Omega}\, dV_g$$

where $u_{1,\Omega} = E_\Omega$ is the mean exit time function on Ω (see Definitions 3.2.5 and 3.2.1).

To end this section, we show some properties of the *mean exit time function* and the *Poisson hierarchy* for geodesic balls of rotationally symmetric model spaces. In fact, Propositions 3.2.13 and 3.2.16 shows, respectively, which are the solutions to the boundary valued problem (3.1) and to the recurrence of boundary valued problems given by (3.2) and (3.3) (see Definitions 3.2.1 and 3.2.7). This properties will play a key rôle in order to prove our comparisons for the geometric invariants defined along this section. But first, let us clarify our notation for these invariants on a rotationally symmetric model space.

Remark 3.2.12. Along this work, we shall denote by E_R^ω and $\{u_{k,R}^\omega\}_{k=1}^\infty$, respectively, the mean exit time function and the Poisson hierarchy of a geodesic ball $B_R^\omega(o_\omega)$ of a rotationally symmetric model space $(\mathbb{M}_\omega, g_\omega)$ with radius $R < \operatorname{inj}_{g_\omega}(o_\omega)$ centered at the center o_ω of $\mathbb{M}_\omega$. Namely, $\{u_{k,R}^\omega\}_{k=1}^\infty$ is a sequence of solutions of the recurrence of boundary valued problems

$$\begin{aligned} \Delta_{g_\omega} u_{1,R}^\omega + 1 &= 0, \quad \text{on} \quad B_R^\omega(o_\omega), \\ u_{1,R}^\omega\big|_{S_R^\omega(o_\omega)} &= 0, \end{aligned}$$

and, for $k \geq 2$,

$$\begin{aligned} \Delta_{g_\omega} u_{k,R}^\omega + k u_{k-1,R}^\omega &= 0, \quad \text{on} \quad B_R^\omega(o_\omega), \\ u_{k,R}^\omega\big|_{\partial S_R^\omega(o_\omega)} &= 0. \end{aligned}$$

where Δ_{g_ω} denotes the Laplacian on $\mathbb{M}_\omega$ with respect to the rotationally symmetric metric tensor g_ω (see equation (2.32)). Observe that $u^\omega_{1,R} = E^\omega_R$, *i.e.*, $u^\omega_{1,R}$ is the mean exit time function on $B^\omega_R(o_\omega)$.

On the other hand, it is known that all the elements of the Poisson hierarchy of geodesic ball in rotationally symmetric model spaces are non-increasing, positive, radial functions, as we show in the following results.

Proposition 3.2.13 (see [53]). *Let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in M$ and $B^\omega_R(o_\omega)$ be a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \operatorname{inj}_{g_\omega}(o_\omega)$ and let E^ω_R be the mean exit time function on $B^\omega_R(o_\omega)$. Then, there is a non-increasing, positive, real valued function $\mathcal{E}^\omega_R : [0, R] \longrightarrow \mathbb{R}_+$ given by*

$$\mathcal{E}^\omega_R(r) = \int_r^R q_\omega(t)\, dt$$

such that

$$E^\omega_R = \mathcal{E}^\omega_R \circ r_{o_\omega} \quad on \quad \overline{B^\omega_R(o_\omega)},$$

where r_{o_ω} is the radial distance function to the center point o_ω and q_ω is the isoperimetric quotient in $B^\omega_R(o_\omega)$ (see Definitions 2.1.69 and 2.2.6). Moreover,

$$\mathcal{E}^{\omega\prime}_R(0) = 0 \quad and \quad \mathcal{E}^{\omega\prime}_R(r) < 0 \quad for\ all \quad r \in (0, R].$$

Hence, E^ω_R attains its maximum at o_ω.

Remark 3.2.14. Observe that, since $E^\omega_R(p) = \mathcal{E}^\omega_R \circ r_{o_\omega}(p)$ for all $p \in B^\omega_R(o_\omega)$, we know that E^ω_R is a radial function. In fact, given $p_1, p_2 \in B^\omega_R(o_\omega)$ such that $r_{o_\omega}(p_1) = r_{o_\omega}(p_2)$ we have that $E^\omega_R(p_1) = \mathcal{E}^\omega_R(r_{o_\omega}(p_1)) = \mathcal{E}^\omega_R(r_{o_\omega}(p_2)) = E^\omega_R(p_2)$. Therefore, considering r_{o_ω} as a parameter $r \in [0, R]$, we can identify $E^\omega_R(p) \equiv E^\omega_R(r_{o_\omega}(p)) \equiv E^\omega_R(r) = \mathcal{E}^\omega_R(r)$. Thus, we can say that the mean exit time function E^ω_R attains its maximum at $r = 0$.

Proof of Proposition 3.2.13. Using the expression (2.32) of the Laplacian for rotationally symmetric model spaces, it is straightforward to check that the function $(\mathcal{E}^\omega_R \circ r_{o_\omega})(p) = \int_{r_{o_\omega}(p)}^R q_\omega(t)\, dt$ satisfies equation

$$\Delta_{g_\omega} \mathcal{E}^\omega_R = -1 \quad \text{on} \quad B^\omega_R(o_\omega),$$

and that $\mathcal{E}_R^\omega \circ r_{o_\omega}(q) = 0$ for all $q \in S_R^\omega(o_\omega)$. Then, $E_R^\omega = \mathcal{E}_R^\omega \circ r_{o_\omega}$ is a solution of the boundary valued problem (3.1).

Now, suppose that there exists another function F that is also a solution of the Poisson problem $\Delta_{g_\omega} F = -1$ on $B_R^\omega(o_\omega)$ with $F = 0$ on $S_R^\omega(o_\omega)$. Then, the function $E_R^\omega - F$ is a harmonic function on $B_R^\omega(o_\omega)$. Therefore, applying the *Strong Maximum Principle* to $E_R^\omega - F$ as in the proof of Proposition 3.2.9, it is easy to check that $E_R^\omega \leq F$ on $B_R^\omega(o_\omega)$. Analogously, for $F - E_R^\omega$, we obtain that $E_R^\omega \geq F$ on $B_R^\omega(o_\omega)$, and hence, $E_R^\omega = F$ on $B_R^\omega(o_\omega)$, showing that $E_R^\omega = \mathcal{E}_R^\omega \circ r_{o_\omega}$ is the unique solution of the Poisson problem on $B_R^\omega(o_\omega)$. Therefore, $E_R^\omega = \mathcal{E}_R^\omega \circ r_{o_\omega}$ is the mean exit time function from $B_R^\omega(o_\omega)$.

Moreover, since $q_\omega(r) > 0$ for all $r \in (0, R]$, we have that E_R^ω is a positive radial function, and moreover, since $\lim_{t\to 0} q_\omega(t) = 0$ (see Remark 2.2.7) and applying the fundamental theorem of calculus, we obtain that

$$\mathcal{E}_R^{\omega\prime}(0) = 0 \quad \text{and} \quad \mathcal{E}_R^{\omega\prime}(r) < 0$$

for all $r \in (0, R]$, and hence, we have that $E_R^\omega(p) \equiv E_R^\omega(r_{o_\omega}(p)) \equiv E_R^\omega(r) = \mathcal{E}_R^\omega(r)$ is a non-increasing function that attains its maximum at the center o_ω (when $r = 0$). $\square$

Remark 3.2.15. From now on, for the sake of simplifying the notation, we will identify $\mathcal{E}_R^\omega$ by E_R^ω and its derivatives $\mathcal{E}_R^{\omega\prime}$ and $\mathcal{E}_R^{\omega\prime\prime}$ by $E_R^{\omega\prime}$ and $E_R^{\omega\prime\prime}$, respectively. Namely, we are identifying r_{o_ω} with its value r as a parameter of E_R^ω.

Moreover, observe that, since $(\mathbb{M}_\omega, g_\omega)$ is a rotationally symmetric model space then, from equation (2.46), we have, for all $r \in [0, R]$, that

$$E_R^\omega(r) = \int_r^R \frac{\operatorname{vol}(B_s^\omega(o_\omega))}{\operatorname{vol}(S_s^\omega(o_\omega))} = \int_r^R \frac{\int_0^s \omega^{n-1}(\sigma)d\sigma}{\omega^{n-1}(s)} ds \tag{3.4}$$

and

$$E_R^{\omega\prime}(r) = -\frac{\int_0^r \omega^{n-1}(s)ds}{\omega^{n-1}(r)}.$$

Proposition 3.2.16 (see [38])**.** *Let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$ and $B_R^\omega(o_\omega)$ be a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \operatorname{inj}_{g_\omega}(o_\omega)$ and let $\{u_{k,R}^\omega\}_{k=1}^\infty$ the Poisson hierarchy of $B_R^\omega(o_\omega)$. Then, there is a family $\{\mathrm{u}_{k,R}^\omega\}_{k=1}^\infty$ of non-increasing,* *positive, real valued, integrable functions* $\mathfrak{u}^{\omega}_{k,R} : [0,R] \longrightarrow \mathbb{R}_+$ *given by*

$$\mathfrak{u}^{\omega}_{1,R}(r) = \int_r^R q_\omega(s)\,ds \quad \textit{and} \quad \mathfrak{u}^{\omega}_{k,R}(r) = k\int_r^R \frac{\int_0^s \omega^{n-1}(\sigma)\mathfrak{u}^{\omega}_{k-1,R}(\sigma)d\sigma}{\omega^{n-1}(s)}ds$$

for all $k \geq 2$*, such that*

$$u^{\omega}_{k,R} = \mathfrak{u}^{\omega}_{k,R} \circ r_{o_\omega} \quad \textit{on} \quad \overline{B^{\omega}_R(o_\omega)},$$

where r_{o_ω} *is the radial distance function to the center point* o_ω *and* q_ω *is the isoperimetric quotient in* $B^{\omega}_R(o_\omega)$*. Moreover, for all* $k \geq 1$*,*

$$\mathfrak{u}^{\omega}_{k,R}{}'(0) = 0 \quad \textit{and} \quad \mathfrak{u}^{\omega}_{k,R}{}'(r) = -k\frac{\int_0^{r_{o_\omega}(p)} \omega^{n-1}(s)\mathfrak{u}^{\omega}_{k-1,R}(s)\,ds}{\omega^{n-1}(r_{o_\omega(p)})}$$

for all $r \in (0,R]$ *(considering* $\mathfrak{u}^{\omega}_{0,R} = 1$*). Hence, for all* $k \geq 1$*,* $u_{k,R}$ *is a non-increasing function that attains its maximum at* o_ω*.*

Remark 3.2.17. As in Remark 3.2.14 and by virtue of the equation $u^{\omega}_{k,R}(p) = \mathfrak{u}^{\omega}_{k,R} \circ r_{o_\omega}(p)$ for all $p \in B^{\omega}_R(o_\omega)$, we have that $u^{\omega}_{k,R}$ is a radial function. Thus, considering r_{o_ω} as a parameter $r \in [0,R]$, we can identify $u^{\omega}_{k,R}(p) \equiv u^{\omega}_{k,R}\left(r_{o_\omega}(p)\right) \equiv u^{\omega}_{k,R}(r) = \mathfrak{u}^{\omega}_{k,R}(r)$.

Proof of Proposition 3.2.16. As in the proof of Proposition 3.2.13, using the expression (2.32) of the Laplacian for rotationally symmetric model spaces, it is straightforward to check that, for all $k \geq 2$,

$$u^{\omega}_{k,R}(p) \equiv u^{\omega}_{k,R}(r_{o_\omega}(p)) = k\int_{r_{o_\omega}(p)}^R \frac{\int_0^s \omega^{n-1}(\sigma)u^{\omega}_{k-1,R}(\sigma)d\sigma}{\omega^{n-1}(s)}ds$$

satisfies the following equation

$$\Delta_{g_\omega} u^{\omega}_{k,R} = -u^{\omega}_{k-1,R} \quad \text{on} \quad B^{\omega}_R(o_\omega), \tag{3.5}$$

and that $u^{\omega}_{k,R}(R) = 0$ for all $q \in S^{\omega}_R(o_\omega)$. Then, $u^{\omega}_{k,R} = \mathfrak{u}^{\omega}_{k,R} \circ r_{o_\omega}$ is a solution of the boundary valued problem (3.1), and hence, using the *Strong Maximum Principle* as in the proof of Proposition 3.2.13, we have that $u^{\omega}_{k,R} = \mathfrak{u}^{\omega}_{k,R} \circ r_{o_\omega}$ is the unique solution of (3.5) for all $k \geq 2$.

Therefore, since $u^\omega_{1,R}(p) = E^\omega_R(p) \geq 0$ for all $p \in \overline{B^\omega_R(o_\omega)}$, we have that $u^\omega_{2,R}$ is a positive radial function, and the same occurs recursively for $u^\omega_{k,R}$ for all $k \geq 2$. Moreover, since

$$u^{\omega}_{k,R}{}'(r) = -k\frac{\int_0^r \omega^{n-1}(s)u^\omega_{k,R}(s)ds}{\omega^{n-1}(r)},$$

applying the fundamental theorem of calculus and Proposition 3.2.13, we obtain, for all $k \geq 2$, that

$$u^{\omega}_{k,R}{}'(0) = 0 \quad \text{and} \quad u^{\omega}_{k,R}{}'(r_{o_\omega}(p)) < 0$$

for all $r \in (0, R]$, and hence, we have that $u^\omega_{k,R}(p) \equiv u_{k,R}\left(r_{o_\omega}(p)\right) \equiv u_{k,R}(r) = \mathfrak{u}^\omega_{k,R}(r)$ is a non-increasing function that attains its maximum at the center o_ω for all $k \geq 1$ (when $r = 0$). The case $k = 1$ comes from Proposition 3.2.13 and the proposition follows.

□

3.3 Some background

A natural question that appears while studying the moment spectrum consists in to optimize the quantity of the torsional rigidity among all the domains which have the same area/volume in a fixed space or under some other geometrical setting. This problem is known as a *Saint-Venant type problem*. In particular, the study of this variational problem on Riemannian manifolds (M, g) involves the establishment of bounds on the torsional rigidity of a given domain $D \subseteq M$, together the determination of the domains and the spaces where these bounds are attained. The techniques used in this analysis encompasses the use of the Schwarz symmetrization (see Subsection 2.2.3), as well as the isoperimetric inequalities satisfied by the domains in question.

From the intrinsic viewpoint, bounds for the L^p-moment spectrum (and, in particular, for the moment spectrum) and the study of the relationship between the torsional rigidity of a domain $D \subset M$ in a Riemannian manifold (M, g) and its Dirichlet spectrum have been widely studied along the last years (see, among others, [21], [37], [38], [39], [44], [45], [53], [55], [56], [57], [73], and the references therein). Related with this issue and in the line of the classical *Kac's question*, we have the isospectrality problem, namely, to see to what extent the moment spectrum of a domain determines it up to isometry (see [16] and [17], for instance). Along this chapter we will focus on finding bounds for the mean exit time, the Poisson hierarchy, the torsional rigidity and the moment spectrum and, on the other hand, in Chapter 4 we will study the relationship between the moment spectrum and the first eigenvalue of the Laplacian for the Dirichlet problem (see Section 4.6).

These bounds were given by the corresponding values for the torsional rigidity of the Schwarz symmetrization of the geodesic balls in rotationally symmetric model spaces, and were obtained from isoperimetric inequalities previously established. For instance, P. McDonald proved in [55] the following result.

Theorem 3.3.1 (see Theorem 1.2 of [55])**.** *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold and let* $(\mathbb{M}_{\omega_\kappa}, g_{\omega_\kappa})$ *be an* n*-dimensional simply connected space form of constant sectional curvature* κ *with center* o_{ω_κ}*. Let* Ω *be a precompact domain* $\Omega \subseteq M$ *and let* $B^{\omega_\kappa}_{L(\Omega)}$ *be the Schwarz symmetrization of* Ω *in* $\mathbb{M}_{\omega_\kappa}$*. Suppose that for all precompact domains* $\Omega \subseteq M$ *we have that the volume equality implies the following inequality between the volumes of its perimeters*

$$\operatorname{vol}(\Omega) = \operatorname{vol}\left(B^{\omega_\kappa}_{L(\Omega)}\right) \Longrightarrow \operatorname{vol}(\partial\Omega) \geq \operatorname{vol}\left(\partial B^{\omega_\kappa}_{L(\Omega)}\right). \tag{3.6}$$

Then, for all precompact domains $\Omega \subset M$*, each element of the moment spectrum* $\{\mathcal{A}_k(\Omega)\}_{k=1}^{\infty}$ *of* Ω *is bounded from above by*

$$\mathcal{A}_k(\Omega) \leq \mathcal{A}_k\left(B^{\omega_\kappa}_{L(\Omega)}\right), \quad \textit{for all} \quad k \geq 1,$$

where $\left\{\mathcal{A}_k\left(B^{\omega_\kappa}_{L(\Omega)}\right)\right\}_{k=1}^{\infty}$ *is the moment spectrum of the Schwarz symmetrization* $B^{\omega_\kappa}_{L(\Omega)}$ *of* Ω *in* $\mathbb{M}_{\omega_\kappa}$*.*

Remark 3.3.2. The isoperimetric condition (3.6) is the hypothesis used in *Faber-Krahn inequality* (see [25] and [47]). In fact, they proved that the first eigenvalue of the Laplacian for the Dirichlet problem can be bounded from below by assuming this mentioned condition (see Theorem 4.2.6). Moreover, observe that in the above theorem there is still a question remaining: what happens with the equality case?

From the submanifold theory approach, the establishment of upper and lower bounds for the moment spectrum of extrinsic balls can be found in A. Hurtado, S. Markvorsen and V. Palmer [37] and [38] and S. Markvorsen and V. Palmer [53], for instance. In this mentioned papers, given an ambient Riemannian manifold $\widetilde{M}^m$ with a pole p, the moment spectrum of extrinsic domains of submanifolds M^n is bounded from above or from below by imposing, respectively, that the sectional curvatures of the ambient Riemannian manifold $\widetilde{M}^m$ are bounded from above and from below. Moreover, they characterize which geometric properties have the Riemannian manifold when the equality with the bounds is attained.

On the other hand, in these papers, it also were given too intrinsic upper and lower bounds for the torsional rigidity of geodesic balls of the ambient manifold when considering the submanifold as the entire Riemannian manifold, so the extrinsic distance became the intrinsic distance, and assuming bounds on the radial sectional curvatures of the ambient Riemannian manifold. To summarize the intrinsic results obtained in [37], [38] and [53], let us first recall that given a point $p \in M$ of a complete Riemannian manifold (M, g) and r_p the radial distance function from p in M, we have that the radial sectional curvatures are the sectional curvatures computed on those planes that contains the radial distance vector $\nabla_g r_p$ (see Subsections 2.1.4 and 2.1.7). Then, it can be stated the following results:

Theorem 3.3.3 (see Proposition 6.1 of [37], Theorem 1.3 of [38] and Corollary 8.1 of [54])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ be a geodesic ball of M with radius R centered at o. Suppose that $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$ and that the radial sectional curvatures of M, $\sec_g$, are bounded from below (above) by the radial sectional curvatures of* $\mathbb{M}_\omega$, i.e.,

$$\sec_g\left(\sigma\left(\nabla_g r_o, \cdot\right)\right) \geq (\leq) \sec_{g_\omega}\left(\sigma\left(\nabla_{g_\omega} r_{o_\omega}, \cdot\right)\right). \tag{3.7}$$

Then, we have the isoperimetric inequality

$$\frac{\operatorname{vol}\left(B_R(o)\right)}{\operatorname{vol}\left(S_R(o)\right)} \leq (\geq) \frac{\operatorname{vol}\left(B^\omega_R(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_R(o_\omega)\right)} \tag{3.8}$$

and that the averaged moment spectrum $\left\{\frac{\mathcal{A}_k(B_R(o))}{\mathrm{vol}(B_R(o))}\right\}_{k=1}^{\infty}$ *is bounded by*

$$\frac{\mathcal{A}_k\left(B_R(o)\right)}{\mathrm{vol}\left(S_R(o)\right)} \geq (\leq) \frac{\mathcal{A}_k\left(B_R^{\omega}(o_\omega)\right)}{\mathrm{vol}\left(S_R^{\omega}(o_\omega)\right)}, \quad \textit{for all} \quad k \geq 1, \tag{3.9}$$

where $B_R^{\omega}(o_\omega)$ is the geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω, and $\left\{\mathcal{A}_k\left(B_R(o)\right)\right\}_{k=1}^{\infty}$ and $\left\{\mathcal{A}_k\left(B_{s(R)}^{\omega}(o_\omega)\right)\right\}_{k=1}^{\infty}$ are, respectively, the moment spectrum of $B_R(o)$ and $B_R^{\omega}(o_\omega)$. Moreover, equality in any of the inequalities (3.7), (3.8) *and* (3.9) *implies that $B_R(o)$ and $B_R^{\omega}(o_\omega)$ are isometric, and hence, equalities $\mathcal{A}_k\left(B_R(o)\right) = \mathcal{A}_k\left(B_{s(R)}^{\omega}(o_\omega)\right)$ for all $k \geq 1$.*

Theorem 3.3.4 (see Theorem 6.2 of [37] and Corollary 2.4 of [53])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_{\omega})$ be an n-dimensional rotationally symmetric model space balanced from above with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ be a geodesic ball of M with radius R centered at o. Suppose that $R < \mathrm{inj}_g(o) \leq \mathrm{inj}_{g_\omega}(o_\omega)$, that there exists the Schwarz symmetrization $B_{s(R)}^{\omega}(o_\omega)$ of $B_R(o)$ in $\mathbb{M}_\omega$ and that the sectional curvatures of M, $\sec_g$, are bounded from below by the sectional curvatures of $\mathbb{M}_\omega$,* i.e.,

$$\sec_g\left(\sigma\left(\nabla_g r_o, \cdot\right)\right) \geq \sec_{g_\omega}\left(\sigma\left(\nabla_{g_\omega} r_{o_\omega}, \cdot\right)\right). \tag{3.10}$$

Then, the torsional rigidity $\mathcal{A}_1\left(B_R(o)\right)$ of the geodesic ball $B_R(o)$ is bounded from below (above) by

$$\mathcal{A}_1\left(B_R(o)\right) \geq \mathcal{A}_1\left(B_{s(R)}^{\omega}(o_\omega)\right), \tag{3.11}$$

where $\mathcal{A}_1\left(B_{s(R)}^{\omega}(o_\omega)\right)$ is the torsional rigidity of $B_{s(R)}^{\omega}(o_\omega)$. Moreover, equality in any of the inequalities (3.11) *implies that $s(R) = R$ and that $B_R(o)$ and $B_R^{\omega}(o_\omega)$ are isometric.*

Remark 3.3.5. Let us remark that the other direction of inequality (3.11) is obtained by assuming that $\sec_g\left(\sigma\left(\nabla_g r_o, \cdot\right)\right) \leq \sec_{g_\omega}\left(\sigma\left(\nabla_{g_\omega} r_{o_\omega}, \cdot\right)\right)$ and that the rotationally symmetric model space $(\mathbb{M}_\omega, g_{\omega})$ is *totally balanced* (see Definition 2.10 and Theorem 6.3 of [37] for the definition of totally balanced and to see the mentioned upper bound for the torsional rigidity).

Moreover, it can be proved too, from these results, that the torsional rigidity and the moment spectrum of geodesic balls in a Riemannian manifold are *determined* by the first eigenvalue of the Laplacian for the Dirichlet problem in the sense that, under the above hypothesis, the equality between the first eigenvalues implies that the moment spectrums of $B_R(o)$ and $B^\omega_R(o_\omega)$ are equal and, reciprocally, that the moment spectrum of the geodesic balls determines the first eigenvalue of these balls (see Sections 4.2 and 4.6).

Along this chapter, we will consider a complete n-dimensional Riemannian manifold (M, g) and an n-dimensional rotationally symmetric model space $(\mathbb{M}_\omega, g_{_\omega})$ with center o_ω, and we shall assume, given a point $o \in M$, that the injectivity radius of o satisfies $\operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o)$. Moreover, fixing $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$, we will control the mean curvatures of the geodesic spheres centered at o by assuming that

$$H_{S_r(o)} \geq (\leq) H_{S^\omega_r(o_\omega)} \quad \text{for all} \quad r \in (0, R],$$

instead of controlling the radial sectional curvatures of the Riemannian manifold as in Theorems 3.3.3 and 3.3.4. Namely, we are going to show that the bounds established in [37], [38] and [53], can be obtained under a "weaker condition". The bounds that we impose on the mean curvatures of the geodesic spheres are the same as those imposed by P. Bessa and J.F. Montenegro in [5] to obtain bounds for the first eigenvalue of the Laplacian for the Dirichlet problem. They showed that S.Y. Cheng's results, based on bounds on the Ricci and the sectional curvatures (see [12] and [13]), also works by assuming bounds on the mean curvature of the geodesic spheres but only up to the injectivity radius (see Section 4.2 for the complete statements on the first eigenvalue). By saying that this assumption is a "weaker condition" than the corresponding bounds on the sectional curvatures, we mean first the following:

It can be proved that bounds on the radial sectional curvatures implies bounds for the mean curvature of the geodesic spheres (see R.E. Greene and H.H. Wu [32], A. Hurtado, S. Markvorsen, M. Min-Oo and V. Palmer [40] and V. Palmer [62]). Namely, if (M, g) is a complete Riemannian manifold with radial sectional curvatures satisfying

$$\sec_g\left(\sigma\left(\nabla_g r_o, \cdot\right)\right) \geq (\leq) \sec_{g_\omega}\left(\sigma\left(\nabla_{g_\omega} r_{o_\omega}, \cdot\right)\right) = -\frac{\omega''(r)}{\omega(r)},$$

then we have

$$H_{S_r(o)} \leq (\geq) H_{S_r^\omega(o_\omega)} = \frac{\omega'(r)}{\omega(r)} \quad \text{for all} \quad r \in (0, R].$$

The proof of this implication relies in the fact that the mean curvature pointing inward of the geodesic spheres is the Laplacian of the radial distance from its center $o \in M$ (see Proposition 2.1.75), together the Hessian comparison analysis of the distance function. But the converse is not true. In fact, in Example 3.1 of Section 3 of [5], G.P. Bessa and J.F. Montenegro showed that there exist smooth complete and rotationally symmetric metrics g on the Euclidean space $\mathbb{R}^n$ with radial sectional sectional curvatures bounded from below outside a compact set, $\sec_g(\sigma(\nabla_g r_o, \cdot)) \geq \kappa$, such that the mean curvature of its geodesic spheres satisfies that $H_{S_r(o)} \geq H_{S_r^{\omega_\kappa}(o_{\omega_\kappa})}$. Moreover, we are going to present a new example which shows that bounds on the mean curvature of the geodesic spheres does not imply bounds on the radial sectional curvatures of the Riemannian manifold.

Example 3.3.6. Let $(\mathbb{R}^2, g)$ be a Riemannian manifold such that its metric tensor expressed in a system of polar coordinates $(\mathbb{R}^2, \psi = (r, \theta))$ is given by

$$g = dr \otimes dr + \varphi^2(r, \theta) d\theta \otimes d\theta,$$

where $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a positive smooth function given by

$$\varphi(r, \theta) = r\left(1 + \frac{r^2}{1 + r^2\cos^2(\theta)}\right).$$

Observe that g is smooth. Indeed, taking the system of normal coordinates $\{x, y\}$ given by $x = r\cos(\theta)$ and $y = r\sin(\theta)$, it is easy to see that the metric tensor g can be expressed as

$$\begin{aligned} g = dr \otimes dr + \varphi^2(r,\theta) d\theta \otimes \theta = & \left(1 + \frac{2y^2}{1+x^2} + \frac{r^2 y^2}{(1+x^2)^2}\right) dx \otimes dx \\ & - \left(\frac{2xy}{1+x^2} + \frac{r^2 xy}{(1+x^2)^2}\right)(dx \otimes dy + dy \otimes dx) \\ & + \left(1 + \frac{2x^2}{1+x^2} + \frac{r^2 x^2}{(1+x^2)^2}\right) dy \otimes dy, \end{aligned}$$

and hence, since r^2, x^2, y^2 and xy are smooth from $\mathbb{R}^2$ to $\mathbb{R}$, we have that g is smooth on the entire $\mathbb{R}^2$.

On the other hand, we consider the simply connected real space form $(\mathbb{R}^2, g_{\text{can}})$ of constant sectional curvature $\kappa = 0$ as a 2-dimensional rotationally symmetric model space $(\mathbb{R}^2, g_{\omega_0})$ for which, from equation (2.33), we have $H_{S_r^{\omega_0}}(\vec{0}) = 1/r$.

Then, we are going to see that the mean curvatures of the geodesic spheres $S_r(\vec{0})$ of $(\mathbb{R}^2, g)$ with radius r centered at $\vec{0}$, are bounded from below by the mean curvatures of the geodesic spheres $S_r^{\omega_0}(\vec{0})$ of $(\mathbb{R}^2, g_{\omega_0})$ with the same radius R centered at $\vec{0}$, namely, that

$$H_{S_r(\vec{0})} \geq H_{S_r^{\omega_0}(\vec{0})} = \frac{1}{r}.$$

From equations (2.26) and (2.27), we know that $H_{S_r(o)} = \frac{\frac{\partial}{\partial r}\varphi(r,\theta)}{\varphi(r,\theta)}$. Thus, since

$$\frac{\partial \varphi}{\partial r}(r, \theta) = 1 + \frac{r^2}{1 + r^2\cos^2(\theta)} + \frac{2r^2}{\left(1 + r^2\cos^2(\theta)\right)^2},$$

we obtain that

$$H_{S_r(\vec{0})}(r, \theta) = \frac{\frac{\partial}{\partial r}\varphi(r, \theta)}{\varphi(r, \theta)} = \frac{1}{r} + \frac{2r}{\left(1 + \frac{r^2}{1+r^2\cos^2(\theta)}\right)\left(1 + r^2\cos^2(\theta)\right)^2},$$

for all $(r, \theta) \in (0, +\infty) \times [0, 2\pi)$. But

$$\frac{2r}{\left(1 + \frac{r^2}{1+r^2\cos^2(\theta)}\right)\left(1 + r^2\cos^2(\theta)\right)^2} \geq 0, \quad \text{for all} \quad (r, \theta) \in (0, +\infty) \times [0, 2\pi),$$

and hence, we have that

$$H_{S_r(\vec{0})}(r, \theta) \geq \frac{1}{r} = H_{S_r^{\omega_0}(\vec{0})}, \quad \text{for all} \quad (r, \theta) \in (0, +\infty) \times [0, 2\pi).$$

Now, given a point $(r, \theta) \in \mathbb{R}^2$, let us consider the unique 2-plane tangent to (r, θ) generated by the coordinate vector fields $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right\}$. We are going to compute the sectional curvature of $(\mathbb{R}^2, g)$ at (r, θ) and we will see that it is not bounded by the corresponding sectional curvature of $(\mathbb{R}^2, g_{\omega_0})$, *i.e.*, we will show that $\sec_g\left(\sigma\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right)\right)(r, \theta)$ is not bounded either from above or below by 0. Indeed, from Proposition 2.1.76, we know that $\sec_g\left(\sigma\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right)\right)(r, \theta) = -\frac{\frac{\partial^2 \varphi}{\partial r^2}(r,\theta)}{\varphi(r,\theta)}$. Then, by an straightforward computation, we obtain that

$$\sec_g\left(\sigma\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right)\right)(r, \theta) = -\frac{\frac{\partial^2 \varphi}{\partial r^2}(r, \theta)}{\varphi(r, \theta)} = \frac{2\left(r^2\cos^2(\theta) - 3\right)}{\left(1 + r^2\cos^2(\theta)\right)^2\left(1 + r^2 + r^2\cos^2(\theta)\right)}.$$

Thus, for $\theta = 0$, we have that

$$\sec_g \left(\sigma \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \right) \right) (r, 0) = \frac{2\,(r^2 - 3)}{(1 + r^2)^2\,(1 + 2r^2)}.$$

Then, there are points $(r, \theta) \in \mathbb{R}^2$ where the sectional curvature of $(\mathbb{R}^2, g)$ is positive and points $(r, \theta) \in \mathbb{R}^2$ where the sectional curvature is negative, which shows that the sectional curvature is not bounded either from above or from below by 0.

On the other hand, assuming the bounds on the radial sectional curvatures of the manifold as hypothesis, we have that the equality between one of the geometric invariants defined on a geodesic ball $B_R(o)$ of a Riemannian manifold (M, g) and this invariant defined on geodesic balls $B_R^\omega(o_\omega)$ of the rotationally symmetric model space $(\mathbb{M}_\omega, g_\omega)$ with the same radius implies that the geodesic balls are isometric. Namely, the equality between the Poisson hierarchies, the averaged moment spectrums, or the torsional rigidities, implies that $B_R(o)$ is isometric to $B_R^\omega(o_\omega)$. However, assuming the bounds on the mean curvatures, we have that the equality between one of the geometric invariants defined on $B_R(o)$ and this invariant defined on $B_R^\omega(o_\omega)$ does not imply the isometry among the geodesic balls. In fact, in the Example 5.3 of [6], G.P. Bessa, V. Gimeno and L. Jorge construct a 4-dimensional geodesic ball $B_R(o)$ which is not isometric to the geodesic ball of the 4-dimensional hyperbolic space $\mathbb{M}^4_{\omega_1} = \mathbb{H}^4(1)$ of constant sectional curvature 1 and show that the mean curvatures of the geodesic spheres $S_R(o) \subset B_R(o)$ are equal to the mean curvatures of the geodesic spheres $S_R^{\omega_1}(o_{\omega_1})$ in $\mathbb{H}^4(1)$ with the same radius. In this case, we will prove, throughout the remainder of this chapter, that the equality between the mean exit time functions, the Poisson hierarchies, the averaged moment spectrums and the torsional rigidities of $B_R(o)$ and $B_R^{\omega_1}(o_{\omega_1})$ is attained.

Therefore, the results that we present along this work (in particular, along this chapter) are inspired, by one hand, by the intrinsic bounds for the torsional rigidity and the moment spectrum of the geodesic balls obtained by A. Hurtado, S. Markvorsen and V. Palmer in [37, 38] and by S. Markvorsen and V. Palmer in [53], and on the other hand, by the weaker restrictions on the mean curvatures of the geodesic spheres assumed by G.P. Bessa and J.F. Montenegro in [5].

3.4 Mean exit time comparison

In this section we show our comparison of the mean exit time function on geodesic balls of complete Riemannian manifolds (M, g) with the mean exit time function on geodesic balls of rotationally symmetric model spaces $(\mathbb{M}_\omega, g_\omega)$. In order to prove these comparisons, we need to transplant the mean exit time function on geodesic balls of $\mathbb{M}_\omega$ into the geodesic balls of M with the same radius as S. Markvorsen and V. Palmer did in [53].

Definition 3.4.1 (see [53])**.** *Let (M, g) a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Given $o \in M$ a point of M such that* $\operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$ *and given $B_R(o)$ the geodesic ball of M with radius $R <$* $\operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$ *centered at o, we define the* transplanted mean exit time function $\mathbb{E}_R^\omega$ *as the radial function defined as*

$$\begin{aligned} \mathbb{E}_R^\omega : B_R(o) &\longrightarrow \mathbb{R} \\ p &\longmapsto \mathbb{E}_R^\omega(p) := \left(E_R^\omega \circ r_o\right)(p) = E_R^\omega\left(r_o(p)\right), \end{aligned}$$

where r_o is the radial distance function to o, the center of the geodesic ball $B_R(o)$ (see Definition 2.1.69), and E_R^ω is the mean exit time function on $B_R^\omega(o_\omega)$.

Remark 3.4.2. Note that, from Proposition 3.2.13, we have that $E_R^\omega = E_R^\omega(r)$ is a radial function and $E_R^{\omega\prime}(r) < 0$ for all $r \in (0, R]$, and hence, the transplanted mean exit time function is also a radial function $\mathbb{E}_R^\omega = \mathbb{E}_R^\omega(r) \equiv E_R^\omega(r)$ which satisfies that

$$\frac{d}{dr}\mathbb{E}_R^\omega(r) = E_R^{\omega\prime}(r) < 0,$$

for all $r \in (0, R]$. From now on, when it is clear from the context, we idenfity $\mathbb{E}_R^\omega(p) = E_R^\omega(r_o(p))$ by $\mathbb{E}_R^\omega(r)$ where $r = r_o(p)$, and its first and second derivatives by $\mathbb{E}_R^{\omega\prime}(r)$ and $\mathbb{E}_R^{\omega\prime\prime}(r)$, respectively, to simplify the notation.

Now, we show our results which compare the transplanted mean exit time function $\mathbb{E}_R^\omega$ defined in a geodesic ball $B_R(o)$ with the mean exit time function E_R corresponding with this geodesic ball. The first result in this regard consists in to characterize the equality between E_R and $\mathbb{E}_R^\omega$ for all $p \in B_R(o)$ as follows.

Proposition 3.4.3. *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ be a geodesic ball of M with radius R centered at o. Suppose that $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$. Then, the following assertions are equivalent:*

1. *$E_R = \mathbb{E}_R^\omega$ on $B_R(o)$.*

2. *$H_{S_r(o)} = H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$,*

where $H_{S_r(o)}$ denotes the mean curvature of the geodesic sphere $S_r(o) \subseteq M$ of radius r centered at o and $H_{S_r^\omega(o_\omega)}$ is the corresponding mean curvature of the geodesic sphere $S_r^\omega(o_\omega) \subseteq \mathbb{M}_\omega$ with same radius r centered at o_ω.

Proof. Let us first assume that $E_R = \mathbb{E}_R^\omega$ on $B_R(o)$. Then, since $\mathbb{E}_R^\omega$ and E_R^ω are radial functions, and E_R and E_R^ω are solutions of the Poisson problem (3.1) on $B_R(o)$ and $B_R^\omega(o_\omega)$, respectively, we have that

$$\Delta_g \mathbb{E}_R^\omega(r) = \Delta_g E_R(r) = -1 = \Delta_{g_\omega} E_R^\omega(r), \quad \text{for all} \quad r \in (0, R]. \tag{3.12}$$

On the other hand, using a system of polar coordinates in the geodesic ball $B_R(o) - \{o\}$ of M, by the expressions of the Laplacian and the mean curvature in polar coordinates in $B_R(o) - \{o\}$, and since $\Delta_{g_{S_r(o)}} \mathbb{E}_R^\omega = 0$ because $\mathbb{E}_R^\omega$ is radial, and hence, constant in $S_r(o)$ for all $r \in (0, R]$, we have, for all $r \in (0, R]$ and for all $\theta \in \mathbb{S}_1^{n-1}$, that

$$\begin{aligned}\Delta_g \mathbb{E}_R^\omega(r) &= \mathbb{E}_R^{\omega\prime\prime}(r) + \frac{\frac{\partial}{\partial r_o}\Big|_{r_o = r} \sqrt{\det G(r_o, \theta)}}{\sqrt{\det G(r, \theta)}} \mathbb{E}_R^{\omega\prime}(r) \\ &= \mathbb{E}_R^{\omega\prime\prime}(r) + H_{S_r(o)}(r, \theta) \mathbb{E}_R^{\omega\prime}(r).\end{aligned} \tag{3.13}$$

Furthermore, using polar coordinates too in the geodesic ball $B_R^\omega(o_\omega) - \{o_\omega\}$ of $\mathbb{M}_\omega$ with radius R centered at the center o_ω, by the expressions of the Laplacian and the mean curvature in polar coordinates in rotationally symmetric model spaces and since E_R^ω is a radial function, we have that

$$\begin{aligned}\Delta_{g_\omega} E_R^\omega(r) &= E_R^{\omega\prime\prime}(r) + (n-1)\frac{\omega'(r)}{\omega(r)} E_R^{\omega\prime}(r) \\ &= E_R^{\omega\prime\prime}(r) + H_{S_r^\omega(o_\omega)}(r) E_R^{\omega\prime}(r),\end{aligned} \tag{3.14}$$

for all $r \in (0, R]$.

Thus, from equality (3.12) and as, by Definition 3.4.1, $\mathbb{E}_R^\omega(r) = E_R^\omega(r)$ for all $r \in [0, R]$, and $\mathbb{E}_R^{\omega\prime}(r) = E_R^{\omega\prime}(r) > 0$ and $\mathbb{E}_R^{\omega\prime\prime}(r) = E_R^{\omega\prime\prime}(r)$ for all $r \in (0, R]$, we obtain that $H_{S_r(o)} = H_{S_r(o_\omega)}$ for all $r \in (0, R]$, showing that (1) implies (2).

Let us assume now that $H_{S_R(o)} = H_{S_R^\omega(o_\omega)}$ for all $r \in (0, R]$. From the above expressions of the Laplacian (3.13) and (3.14), and since E_R and E_R^ω are solutions of the Poisson problem (3.1), we have that

$$\Delta_g \mathbb{E}_R^\omega(r_o(p)) = \Delta_{g_\omega} E_R^\omega(r) = -1 = \Delta_g E_R(p) \quad \text{for all} \quad p \in B_R(o) - \{o\}.$$

We can extend the above equality to $B_R(o)$ because E_R is smooth on $B_R(o)$ and $\mathbb{E}_R^\omega$ has the same asymptotic behaviour as E_R^ω at $r = 0$. Then, we obtain that $\Delta_g (E_R - \mathbb{E}_R^\omega)(p) = 0$ for all $p \in B_R(o)$. Thus, we have that $E_R - \mathbb{E}_R^\omega$ is a subharmonic function in $B_R(o)$ which vanish for any point $S_R(o)$, *i.e.*,

$$\begin{cases} \Delta_g (E_R - \mathbb{E}_R^\omega) \geq 0, & \text{on} \quad B_R(o), \\ E_R - \mathbb{E}_R^\omega = 0, & \text{on} \quad S_R(o). \end{cases} \tag{3.15}$$

Therefore, by the Strong Maximum Principle Theorem (see Theorem 2.1.64), we have that $E_R \leq \mathbb{E}_R^\omega$ in $B_R(o)$. In fact, suppose first that there is a point $p_0 \in B_R(o)$ where $E_R - \mathbb{E}_R^\omega$ reach its maximum. Then, by the Strong Maximum Principle, we know that $E_R - \mathbb{E}_R^\omega$ is constant in $B_R(o)$, *i.e*, there exists a real value c such that $(E_R - \mathbb{E}_R^\omega)(p) = c$ for all $p \in B_R(o)$. Thus, by continuity, we have that $(E_R - \mathbb{E}_R^\omega)(q) = c$ for all $q \in S_R(o)$, and hence, since by equation (3.15) we have that $(E_R - \mathbb{E}_R^\omega)(q) = 0$ for all $q \in S_R(o)$, we obtain that $c = 0$, and therefore, that $E_R(p) = \mathbb{E}_R^\omega(p)$ for all $p \in B_R(o)$.

Otherwise, suppose that $E_R - \mathbb{E}_R^\omega$ does not reach its maximum in $B_R(o)$. Then, since $B_R(o)$ is a precompact domain, there exists a point $q_0 \in S_R(o)$ such that $(E_R - \mathbb{E}_R^\omega)(p) \leq (E_R - \mathbb{E}_R^\omega)(q_0) = 0$ for all $p \in B_R(o)$. Thus, we obtain that $E_R \leq \mathbb{E}_R^\omega$ in $B_R(o)$. Therefore, in any case (whether or not the maximum is reached in $B_R(o)$), we have that $E_R \leq \mathbb{E}_R^\omega$ in $B_R(o)$.

Finally, since $\Delta_g (\mathbb{E}_R^\omega - E_R) = 0$, we have, with the same argument, that $\mathbb{E}_R^\omega - E_R$ is also a subharmonic function in $B_R(o)$. Then, arguing as above, we obtain opposite inequalites, and hence, we have that $E_R \geq \mathbb{E}_R$ in $B_R(o)$, and the proposition follows.

□

Now, we show our first comparison result for the mean exit time function on geodesic balls of Riemannian manifolds.

Theorem 3.4.4. *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ be a geodesic ball of M with radius R centered at o. Suppose that $R < \mathrm{inj}_g(o) \leq \mathrm{inj}_{g_\omega}(o_\omega)$ and suppose moreover that*

$$H_{S_r(o)} \geq (\leq) H_{S_r^\omega(o_\omega)} \quad \text{for all} \quad r \in (0, R],$$

where $H_{S_r(o)}$ denotes the mean curvature of the geodesic sphere $S_r(o) \subseteq M$ of radius r centered at o and $H_{S_r^\omega(o_\omega)}$ is the corresponding mean curvature of the geodesic sphere $S_r^\omega(o_\omega) \subseteq \mathbb{M}_\omega$ with radius the same radius r centered at o_ω.

Then, we have the inequality

$$E_R \leq (\geq) \mathbb{E}_R^\omega \quad \text{on} \quad B_R(o), \tag{3.16}$$

where E_R is the mean exit time function on $B_R(o)$ and $\mathbb{E}_R^\omega$ is the transplanted mean exit time function on $B_R(o)$.

Furthermore, if there exists a point $p \in B_R(o)$ such that $E_R(p) = \mathbb{E}_R^\omega(p)$, then

$$E_R = \mathbb{E}_R^\omega \quad \text{on} \quad B_R(o),$$

and hence,

$$H_{S_r(o)} = H_{S_r^\omega(o_\omega)} \quad \text{for all} \quad r \in (0, R].$$

Proof. To prove the first assertion, we shall assume inequality $H_{S_r(o)} \geq H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$, and let us consider a system of polar coordinates $(B_R(o) - \{o\}, \psi)$, $\psi = (r_o, \theta)$, with $R < \mathrm{inj}_g(o) \leq \mathrm{inj}_{g_\omega}(o_\omega)$. From Definition 3.4.1 of $\mathbb{E}_R^\omega$ and Proposition 3.2.13, we have that this radial function satisfies that

$$\mathbb{E}_R^{\omega\prime}(r) = E_R^{\omega\prime}(r) < 0, \quad \text{for all} \quad r \in (0, R] \tag{3.17}$$

and, since E_R^ω is a solution to the Poisson problem (3.1) on $B_R^\omega(o_\omega)$, *i.e.*, $\Delta_{g_\omega} E_R^\omega = -1$ on $B_R^\omega(o_\omega)$, and by the expression of the Laplacian on geodesic balls of rotationally symmetric model spaces (see equation (2.32)), we have that

$$\Delta_{g_\omega} E_R^\omega(r) = E_R^{\omega\prime\prime}(r) + (n-1)\frac{\omega'(r)}{\omega(r)} E_R^{\omega\prime}(r) = -1, \quad \text{for all} \quad r \in (0, R].$$

Then, for all $r \in (0, R]$, we have, using equation (2.33), that

$$\mathbb{E}_R^{\omega}{}''(r) = E_R^{\omega}{}''(r) = -1 - (n-1)\frac{\omega'(r)}{\omega(t)}E_R^{\omega}{}'(r) = -1 - H_{S_r^{\omega}(o_\omega)}E_R^{\omega}{}'(r).$$

Therefore, from this expression and using equations (2.26) and (2.27)), we have, for all $p \equiv (r, \theta) \in B_R(o) - \{o\}$, that

$$\begin{aligned}\Delta_g \mathbb{E}_R^{\omega}(p) &= \mathbb{E}_R^{\omega}{}''(r) + H_{S_r(o)}(r,\theta)\mathbb{E}_R^{\omega}{}'(r)\\ &= -1 - H_{S_r^{\omega}(o_\omega)}E_R^{\omega}{}'(r) + H_{S_r(o)}(r,\theta)\mathbb{E}_R^{\omega}{}'(r)\\ &= -1 + \left(H_{S_r(o)}(r,\theta) - H_{S_r^{\omega}(o_\omega)}(r)\right)\mathbb{E}_R^{\omega}{}'(r).\end{aligned} \tag{3.18}$$

Then, from equations (3.18) and (3.17), and assuming inequality $H_{S_r(o)}(r,\theta) \geq H_{S_r^{\omega}(o_\omega)}(r)$ for all $(r, \theta) \in (0, R] \times \mathbb{S}_1^{n-1}$, we obtain that

$$\Delta_g \mathbb{E}_R^{\omega}(p) \leq -1 = \Delta_g E_R(p), \quad \text{for all} \quad p \equiv (r,\theta) \in B_R(o) - \{o\}. \tag{3.19}$$

Hence, from the above inequality, by continuity at $r = 0$ (arguing as in the proof of Proposition 3.4.3), and since $E_R(R, \theta) = \mathbb{E}_R^{\omega}(R) = E_R^{\omega}(R) = 0$ for all $\theta \in \mathbb{S}_1^{n-1}$, we have that

$$\begin{cases}\Delta_g\left(E_R - \mathbb{E}_R^{\omega}\right) \geq 0, & \text{on} \quad B_R(o),\\ E_R - \mathbb{E}_R^{\omega} = 0, & \text{on} \quad S_R(o).\end{cases} \tag{3.20}$$

Now, we make use of the Strong Maximum Principle Theorem 2.1.64, and then, arguing as in the proof of Proposition 3.4.3, we obtain that $E_R \leq \mathbb{E}_R^{\omega}$ on $B_R(o)$, which shows one of the directions of inequality (3.16).

To prove the equality case, suppose that there is a point $p_0 \in B_R(o)$ such that $E_R(p_0) = \mathbb{E}_R^{\omega}(p_0)$ and assume as above that $H_{S_r(o)}(r,\theta) \geq H_{S_r^{\omega}(o_\omega)}(t)$ for all $(r,\theta) \in (0, R] \times \mathbb{S}_1^{n-1}$. Then, since $E_R - \mathbb{E}_R^{\omega} \leq 0$ on $B_R(o)$, equality $\left(E_R - \mathbb{E}_R^{\omega}\right)(p_0) = 0$ implies $E_R - \mathbb{E}_R^{\omega}$ reach its maximum at $p_0 \in B_R(o)$. Therefore, applying the Strong Maximum Principle (see Theorem 2.1.64), we obtain that $E_R - \mathbb{E}_R^{\omega}$ is constant and, by continuity, we have that $E_R = \mathbb{E}_R^{\omega}$ on $B_R(o)$. And finally, from Proposition 3.4.3, we have that $H_{S_r(o)}(r,\theta) = H_{S_r^{\omega}(o_\omega)}(r)$ for all $(r,\theta) \in (0, R] \times \mathbb{S}_1^{n-1}$.

Assuming $H_{S_r(o)} \leq H_{S_r^{\omega}}(o_\omega)$, we obtain opposite inequalities and then, arguing as above, the theorem follows. The equality discussion is the same than above, *mutatis mutandis.* $\square$

This comparison for the mean exit time function of Theorem 3.4.4 lead us to prove an isoperimetric inequality for the volume of the geodesic balls and its boundaries in (M, g) and $(\mathbb{M}_\omega, g_{_\omega})$, and moreover, we obtain, as a consequence, a comparison between these volumes.

Corollary 3.4.5. *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_{_\omega})$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ and $B^\omega_R(o_\omega)$ be, respectively, a geodesic ball of M with radius R centered at o and a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_{_\omega}}(o_\omega)$ and suppose moreover that*

$$H_{S_r(o)} \geq (\leq) H_{S^\omega_r(o_\omega)} \quad \textit{for all} \quad r \in (0, R].$$

Then, we have the isoperimetric inequalities

$$\frac{\operatorname{vol}(B_r(o))}{\operatorname{vol}(S_r(o))} \leq (\geq) \frac{\operatorname{vol}(B^\omega_r(o_\omega))}{\operatorname{vol}(S^\omega_r(o_\omega))} \quad \textit{for all} \quad r \in (0, R]. \tag{3.21}$$

Furthermore, equality in inequalities (3.21) *for some radius $r_0 \in (0, R]$ implies that*

$$H_{S^M_t(o)} = H_{S^\omega_t(o_\omega)} \quad \textit{for all} \quad t \in (0, r_0].$$

As a consequence of inequalities (3.21)*, we have, for all $r \in (0, R]$, that*

$$\begin{aligned} \operatorname{vol}(B_r(o)) &\geq (\leq) \operatorname{vol}(B^\omega_r(o_\omega)), \\ \operatorname{vol}(S_r(o)) &\geq (\leq) \operatorname{vol}(S^\omega_r(o_\omega)). \end{aligned} \tag{3.22}$$

Finally, equality

$$\operatorname{vol}(B_{r_0}(o)) = \operatorname{vol}\left(B^\omega_{r_0}(o_\omega)\right)$$

for some $r_0 \in (0, R]$ implies that

$$H_{S_t(o)} = H_{S^\omega_t(o_\omega)} \quad \textit{for all} \quad t \in (0, r_0].$$

Proof. To prove this result, we are going to follow the lines of the proof of Theorem 1.1 and Corollary 1.2 of [61] adapting it to this intrinsic context and using the new hypothesis.

First, let us assume that $H_{S_r(o)} \geq H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$ where $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_{_\omega}}(o_\omega)$. If we fix $r \in (0, R]$, then we have, in particular, that

$H_{S_t(o)} \geq H_{S_t^\omega(o_\omega)}$ for all $t \in (0, r]$. Moreover, let us consider the geodesic ball $B_r(o)$ of M with radius r centered at $o \in M$ and let us consider polar coordinates $(B_r(o) - \{o\}, \psi)$, $\psi = (r, \theta)$. We can apply the same argument as in the proof of Theorem 3.4.4 to obtain

$$-1 = \Delta_g E_r \geq \Delta_g \mathbb{E}_r^\omega \quad \text{on} \quad B_r(o). \tag{3.23}$$

Therefore, since $\|\nabla_g r_o\| = 1$ and $\mathbb{E}_r^\omega$ is a radial function, and using the Divergence Theorem 2.1.59, we have that

$$\begin{aligned}
\operatorname{vol}(B_r(o)) &= \int_{B_r(o)} dV_g = \int_{B_r(o)} -\Delta_g E_r \, dV_g \leq \int_{B_r(o)} -\Delta_g \mathbb{E}_r^\omega \, dV_g \\
&= -\int_{B_r(o)} \operatorname{div}(\nabla_g \mathbb{E}_r^\omega) \, dV_g = -\int_{S_r(o)} g(\nabla_g \mathbb{E}_r^\omega, \nabla_g r_o) \, dA_g \qquad (3.24) \\
&= -\mathbb{E}_r^{\omega\prime}(r) \int_{S_r(o)} dA_g = -\mathbb{E}_r^{\omega\prime}(r) \operatorname{vol}(S_r(o)) .
\end{aligned}$$

Thus, since by Proposition 3.2.13 we have that $\mathbb{E}_r^{\omega\prime}(t) = -q_\omega(t)$ for all $t \in [0, r]$, where q_ω is the isoperimetric quotient on $\mathbb{M}_\omega$, we obtain that

$$\operatorname{vol}(B_r(o)) \leq -\mathbb{E}_r^{\omega\prime}(r) \operatorname{vol}(S_r(o)) = q_\omega(r) \operatorname{vol}(S_r(o)) = \frac{\operatorname{vol}(B_r^\omega(o_\omega))}{\operatorname{vol}(S_r^\omega(o_\omega))} \operatorname{vol}(S_r(o)),$$

and hence,

$$\frac{\operatorname{vol}(B_r(o))}{\operatorname{vol}(S_r(o))} \leq \frac{\operatorname{vol}(B_r^\omega(o_\omega))}{\operatorname{vol}(S_r^\omega(o_\omega))}.$$

Therefore, since the above inequality its satisfied for all fixed $r \in (0, R]$ with $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$, we have obtain one inequality of the statement (3.21). Namely,

$$\frac{\operatorname{vol}(B_r(o))}{\operatorname{vol}(S_r(o))} \leq \frac{\operatorname{vol}(B_r^\omega(o_\omega))}{\operatorname{vol}(S_r^\omega(o_\omega))} \quad \text{for all} \quad r \in (0, R].$$

Now, we are going to discuss the equality assertion while we are still assuming that $H_{S_r(o)} \geq H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$. If there exists $r_0 \in (0, R]$ such that we have

$$\frac{\operatorname{vol}(B_{r_0}(o))}{\operatorname{vol}(S_{r_0}(o))} = \frac{\operatorname{vol}\left(B_{r_0}^\omega(o_\omega)\right)}{\operatorname{vol}\left(S_{r_0}^\omega(o_\omega)\right)},$$

then all inequalities in (3.24) become equalities for the radius r_0. In particular,

$$\operatorname{vol}(B_{r_0}(o)) = \int_{B_{r_0}(o)} -\Delta_g \mathbb{E}_{r_0}^\omega dV_g,$$

and hence,

$$\int_{B_{r_0}(o)} \left(1 + \Delta_g \mathbb{E}^\omega_{r_0}\right) dVg = 0.$$

Therefore, as from inequality (3.23) we know that $1 + \Delta_g \mathbb{E}^\omega_{r_0} \leq 0$, then we have that $1 + \Delta_g \mathbb{E}^\omega_{r_0} = 0$ on $B_{r_0}(o)$, and hence, for any $p \in B_{r_0}(o)$, we obtain that

$$\Delta_g \left(E_{r_0} - \mathbb{E}^\omega_{r_0}\right)(p) = \Delta_g E_{r_0}(p) - \Delta_g \mathbb{E}^\omega_{r_0}(p) = -1 - \Delta_g \mathbb{E}^\omega_{r_0}(p) = 0.$$

Thus, by the above equation and since $E_{r_0}(q) = \mathbb{E}^\omega_{r_0}(q) = 0$ for all $q \in S_{r_0}(o)$, we have that $E_{r_0} - \mathbb{E}^\omega_{r_0}$ is a harmonic function on $B_{r_0}(o)$ and $\left(E_{r_0} - \mathbb{E}^\omega_{r_0}\right)(q) = 0$ for all $q \in S_{r_0}(o)$, *i.e.*,

$$\begin{cases} \Delta_g \left(E_{r_0} - \mathbb{E}^\omega_{r_0}\right) = 0, & \text{on} \quad B_{r_0}(o), \\ E_{r_0} - \mathbb{E}^\omega_{r_0} = 0, & \text{on} \quad S_{r_0}(o). \end{cases} \tag{3.25}$$

Therefore, applying the Strong Maximum Principle, we obtain that $E_{r_0} = \mathbb{E}_{r_0}$ on $B_{r_0}(o)$, and hence, that $H_{S_{r_0}(o)}(t, \theta) = H_{S^\omega_{r_0}(o_\omega)}(t)$ for all $(t, \theta) \in (0, r_0] \times \mathbb{S}^{n-1}_1$.

When we assume that $H_{S_r(o)} \leq H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$, we argue as before, inverting all the inequalities, to conclude the opposite isoperimetric inequality. The equality discussion is the same, *mutatis mutandis*.

To prove statement (3.22), and as in Corollary 1.2 in [61], let us define, given $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$, the function $G : [0, R] \longrightarrow \mathbb{R}$ as

$$G(r) := \begin{cases} \ln\left(\dfrac{\operatorname{vol}\left(B_r(o)\right)}{\operatorname{vol}\left(B^\omega_r(o_\omega)\right)}\right), & \text{if} \quad r > 0 \\ 0, & \text{if} \quad r = 0. \end{cases}$$

Then, assuming that $H_{S_r(o)} \geq H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$, we have, applying statement (3.21), that

$$G'(r) = \frac{\operatorname{vol}\left(S_r(o)\right)}{\operatorname{vol}\left(B_r(o)\right)} - \frac{\operatorname{vol}\left(S^\omega_r(o_\omega)\right)}{\operatorname{vol}\left(B^\omega_r(o_\omega)\right)} \geq 0, \tag{3.26}$$

for all $r \in (0, R]$. Hence, G is non-decreasing in $(0, R]$.

On the other hand, using the asymptotic expansion around $r = 0$ for the volume of a geodesic ball (see Theorem 9.12 in [31]), we can conclude with a straightforward computation that $\lim_{r\to 0} G(r) = 0 = G(0)$. Therefore, $G(r)$ is continuous and $G(r) \geq G(0)$ for all $r \in (0, R]$. Thus, since the exponential map is strictly increasing, we have

$$\mathrm{vol}\,(B_r(o)) \geq \mathrm{vol}\,(B_r^\omega(o_\omega)) \quad \text{for all} \quad r \in (0, R].$$

Moreover, isoperimetric inequality (3.21), together the above inequality, implies that

$$\frac{\mathrm{vol}\,(B_r(o))}{\mathrm{vol}\,(S_r^\omega(o_\omega))} \geq \frac{\mathrm{vol}\,(B_r^\omega(o_\omega))}{\mathrm{vol}\,(S_r^\omega(o_\omega))} \geq \frac{\mathrm{vol}\,(B_r(o))}{\mathrm{vol}\,(S_r(o))} \quad \text{for all} \quad r \in (0, R],$$

and hence,

$$\mathrm{vol}\,(S_r(o)) \geq \mathrm{vol}\,(S_r^\omega(o_\omega)) \quad \text{for all} \quad r \in (0, R].$$

which shows one of the directions of inequality (3.22).

Finally, we are going to discuss the second equality assertion while we are still assuming that $H_{S_r(o)} \geq H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$. If there exists $r_0 \in (0, R]$ such that $\mathrm{vol}\,(B_{r_0}(o)) = \mathrm{vol}\left(B_{r_0}^\omega(o_\omega)\right)$, then $G(0) = G(r_0) = 0$ and, since G is non-decreasing, we have, for all $t \in (0, r_0]$, that

$$0 = G(0) \leq G(t) \leq G(r_0) = 0.$$

Thus $G(t) = 0$ for all $t \in (0, r_0]$, and therefore, $G'(t) = 0$ for all $t \in (0, r_0]$. This implies that the isoperimetric inequality (3.21) become an equality for all $t \in (0, r_0]$, and hence, by the equality case of the first statement, we obtain that $H_{S_t(o)} = H_{S_t^\omega(o_\omega)}$ for all $t \in (0, r_0]$.

When we assume that $H_{S_r(o)} \leq H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$, we argue as before, inverting all the inequalities, to conclude that G is non-increasing in $(0, R]$. The equality discussion is the same than above, *mutatis mutandis.*

□

Corollary 3.4.6. *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ and $B_R^\omega(o_\omega)$ be, respectively, a geodesic ball of M with radius R centered at o and a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \mathrm{inj}_g(o) \leq \mathrm{inj}_{g_\omega}(o_\omega)$ and suppose moreover that*

$$H_{S_r(o)} \geq (\leq) H_{S_r^\omega(o_\omega)} \quad \textit{for all} \quad r \in (0, R].$$

Then, if there exists $p \in B_R(o)$ such that $E_R(p) = \mathbb{E}^\omega_R(p)$ then the following assertions hold:

1. *The equalities $E_r = \mathbb{E}^\omega_r$ on $B_r(o)$ for all $r \in (0, R]$.*
2. *The isoperimetric equalities*
$$\frac{\operatorname{vol}(B_r(o))}{\operatorname{vol}(S_r(o))} = \frac{\operatorname{vol}(B^\omega_r(o_\omega))}{\operatorname{vol}(S^\omega_r(o_\omega))}$$
for all $r \in (0, R]$.
3. *And the volume equalities $\operatorname{vol}(B_r(o)) = \operatorname{vol}(B^\omega_r(o_\omega))$ and $\operatorname{vol}(S_r(o)) = \operatorname{vol}(S^\omega_r(o_\omega))$ for all $r \in (0, R]$.*

Proof. First of all, assuming one of the directions of the hypothesis on the mean curvatures, for instance $H_{S_r(o)} \geq H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$, if there exists a point $p \in B_R(o)$ such that $E_R(p) = \mathbb{E}^\omega_R(p)$ then, by the equality case of Theorem 3.4.4, we have that $E_R = \mathbb{E}^\omega_R$ on $B_R(o)$ and $H_{S_r(o)} = H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$. Therefore, fixing $r_0 \in (0, R]$, we have that $H_{S_t(o)} = H_{S^\omega_t(o_\omega)}$ for all $t \in (0, r_0]$. Then, applying Proposition 3.4.3, we have that $E_{r_0} = \mathbb{E}^\omega_{r_0}$ on $B_{r_0}(o)$. Thus, for any $r \in (0, R]$, we obtain that $E_r = \mathbb{E}^\omega_r$ on $B_r(o)$, proving the first statement.

On the other hand, fixing $r_0 \in (0, R]$ and since we just proved that $E_{r_0} = \mathbb{E}^\omega_{r_0}$ on $B_{r_0}(o)$, we have that $\Delta_g \mathbb{E}^\omega_{r_0} = \Delta_g E_{r_0} = -1$ on $B_{r_0}(o)$. Therefore, since all the inequalities of (3.24) become equalities, we have that

$$\operatorname{vol}(B_{r_0}(o)) = \int_{B_{r_0}(o)} -\Delta_g E_{r_0}\, dV_g = \int_{B_{r_0}(o)} -\Delta_g \mathbb{E}^\omega_{r_0}\, dV_g = -{\mathbb{E}^\omega_{r_0}}'(r_0) \operatorname{vol}(S_{r_0}(o))$$

and hence, since by Proposition 3.2.13 we have that ${\mathbb{E}^\omega_{r_0}}'(t) = -q_\omega(t)$ for all $t \in [0, r_0]$, where q_ω is the isoperimetric quotient on $\mathbb{M}_\omega$, we obtain that

$$\frac{\operatorname{vol}(B_{r_0}(o))}{\operatorname{vol}(S_{r_0}(o))} = \frac{\operatorname{vol}\left(B^\omega_{r_0}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{r_0}(o_\omega)\right)}.$$

Thus, since by the first statement we have, for all $r \in (0, R]$, that $E_r = \mathbb{E}^\omega_r$ on B_r, we obtain the isoperimetric equalities

$$\frac{\operatorname{vol}(B_r(o))}{\operatorname{vol}(S_r(o))} = \frac{\operatorname{vol}(B^\omega_r(o_\omega))}{\operatorname{vol}(S^\omega_r(o_\omega))} \quad \text{for all} \quad r \in (0, R].$$

Finally, defining the function $G:(0,R]\longrightarrow\mathbb{R}$ as

$$G(r):=\begin{cases}\ln\left(\dfrac{\operatorname{vol}\left(B_r(o)\right)}{\operatorname{vol}\left(B^\omega_r(o_\omega)\right)}\right), & \text{if} \quad r>0\\ 0, & \text{if} \quad r=0,\end{cases}$$

we have, from equation (3.26) and the above isoperimetric equalities, that $G'(r)=0$ for all $r\in(0,R]$, and hence, $G(r)=0$ for all $r\in(0,R]$. Thus, we obtain that $\operatorname{vol}\left(B_r(o)\right)=\operatorname{vol}\left(B^\omega_r(o_\omega)\right)$ for all $r\in(0,R]$, and then, using the above isoperimetric equalities, we finally obtain that $\operatorname{vol}\left(S_r(o)\right)=\operatorname{vol}\left(S^\omega_r(o_\omega)\right)$ for all $r\in(0,R]$. □

3.5 Poisson hierarchy and moment spectrum comparison

In this section, we apply the comparison for the mean exit time function obtained in Section 3.4 to obtain estimates of the Poisson hierarchy and the moment spectrum of a geodesic ball of a Riemannian manifold by controlling the behaviour of the mean curvature of the geodesic spheres as before (see Definitions 3.2.7 and 3.2.10).

But first, as we did in Definition 3.4.1 for the mean exit time function, we transplant the Poisson hierarchy of the geodesic ball of a model space into the geodesic ball of a Riemannian manifold with the same radius in the following way.

Definition 3.5.1. *Let (M,g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega,g_{\omega})$ be an n-dimensional rotationally symmetric model space with center $o_\omega\in\mathbb{M}_\omega$. Let $o\in M$ be a point in M. Given $R<\operatorname{inj}_g(o)\leq\operatorname{inj}_{g_\omega}(o_\omega)$ and $B_R(o)$ a geodesic ball of M with radius R centered at o, and given $B^\omega_R(o_\omega)$ a geodesic ball of $\mathbb{M}_\omega$ with the same radius R centered at o_ω, we define the* transplanted Poisson hierarchy $\{\widetilde{u}^\omega_{k,R}\}_{k=0}^\infty$ *as the radial functions obtained by transplanting the Poisson hierarchy $\{u^\omega_{k,R}\}_{k=1}^\infty$ for $B^\omega_R(o_\omega)$ to $B_R(o)$ as*

$$\widetilde{u}^\omega_{k,R}:B_R(o)\longrightarrow\mathbb{R},\quad \widetilde{u}^\omega_{k,R}(p):=u^\omega_{k,R}\left(r_o(p)\right),$$

for all $k\geq 1$, where r_o is the radial distance function to o, the center of the geodesic ball $B_R(o)$ (see Definition 2.1.69).

Remark 3.5.2. Note that, from Proposition 3.2.16, we have, for all $k \geq 1$, that $u^{\omega}_{k,R}$ is a radial function and that ${u^{\omega}_{k,R}}'(r) < 0$ for all $r \in (0, R]$, and hence, we can define the transplanted Poisson hierarchy as above and we have that ${\widetilde{u}^{\omega}_{k,R}}'(r) < 0$ for all $k \geq 1$ and for all $r \in (0, R]$.

We have the following Theorem which gives a comparison for the Poisson hierarchy of a geodesic ball of a Riemannian manifold with the transplanted Poisson hierarchy from a rotationally symmetric model space to this Riemannian manifold.

Theorem 3.5.3. *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ and $B^{\omega}_R(o_\omega)$ be, respectively, a geodesic ball of M with radius R centered at o and a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$ and suppose moreover that*

$$H_{S_r(o)} \geq (\leq) H_{S^{\omega}_r(o_\omega)} \quad \text{for all} \quad r \in (0, R].$$

Let $\{u_{k,R}\}_{k=1}^{\infty}$ and $\left\{\widetilde{u}^{\omega}_{k,R}\right\}_{k=1}^{\infty}$ be, respectively, the Poisson hierarchy of $B_R(o) \subseteq M$ and the transplanted Poisson hierarchy from $B^{\omega}_R(o_\omega)$ to $B_R(o)$. Then, for all $k \geq 1$, we have that

$$u_{k,R} \leq (\geq)\, \widetilde{u}^{\omega}_{k,R} \quad \text{on} \quad B_R(o). \tag{3.27}$$

Furthermore, if there exists $p \in B_R(o)$ and $k_0 \geq 1$ such that we have the equality $u_{k_0,R}(p) = \widetilde{u}^{\omega}_{k_0,R}(p)$, then we have:

1. *The equalities $H_{S_r(o)} = H_{S^{\omega}_r(p_\omega)}$ for all $r \in (0, R]$.*

2. *The equalities $u_{k,r} = \widetilde{u}^{\omega}_{k,r}$ on $B_r(o)$ for all $k \geq 1$ and for all $r \in [0, R]$.*

3. *The volume equalities $\operatorname{vol}(B_r(o)) = \operatorname{vol}(B^{\omega}_r(o_\omega))$ and the volume equalities $\operatorname{vol}(S_r(o)) = \operatorname{vol}(S^{\omega}_r(o_\omega))$ for all $r \in (0, R]$.*

4. *The equalities $\mathcal{A}_k(B_r(o)) = \mathcal{A}_k(B^{\omega}_r(o_\omega))$ for all $k \geq 1$ and for all $r \in (0, R]$.*

where $\{\mathcal{A}_k(B_r(o))\}_{k=1}^{\infty}$ and $\{\mathcal{A}_k(B^{\omega}_r(o_\omega))\}_{k=1}^{\infty}$ are, respectively, the moment spectrum of $B_r(o)$ and $B^{\omega}_r(o_\omega)$ (see Definition 3.2.10).

Proof. We proceed using an induction argument, as it is done in [38]. First, note that $u_{1,R} = E_R$ and $\widetilde{u}^\omega_{1,R} = \mathbb{E}^\omega_R$ on $B_R(o)$ and $B^\omega_R(o_\omega)$, respectively (see Remark 3.2.8 and Definitions 3.4.1 and 3.5.1). Then, from Theorem 3.4.4, we have that, if $H_{S_r(o)} \geq (\leq) H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$, then $u_{1,R} \leq (\geq) \widetilde{u}^\omega_{1,R}$ on $B_R(o)$.

Now, let us assume that $H_{S_r(o)} \geq H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$ and let us suppose that $u_{k,R} \leq \widetilde{u}^\omega_{k,R}$ on $B_R(o)$. We are going to see that $u_{k+1,R} \leq \widetilde{u}^\omega_{k+1,R}$ on $B_R(o)$. Since ${\widetilde{u}^\omega_{k,R}}'(r) = {u^\omega_{k,R}}'(r) \leq 0$ for all $r \in (0, R]$ and for all $k \geq 2$ (see Proposition 3.2.16), we obtain, for all $(r, \theta) \in (0, R] \times \mathbb{S}^{n-1}_1$ and for all $k \geq 2$, that

$$ {\widetilde{u}^\omega_{k,R}}'(r) H_{S_r(o)}(r, \theta) \leq {\widetilde{u}^\omega_{k,R}}'(r) H_{S^\omega_r(o_\omega)}(r) = {u^\omega_{k,R}}'(r) H_{S^\omega_r(o_\omega)}(r), $$

and hence, since ${\widetilde{u}^\omega_{k,R}}''(r) = {u^\omega_{k,R}}''(r)$ for all $r \in (0, R]$ and for all $k \geq 2$, using equations (2.26), (2.27), (2.32) and (2.33), we have, for all $k \geq 2$ and for all $(r, \theta) \in (0, R] \times \mathbb{S}^{n-1}_1$, that

$$ \begin{aligned} \Delta_g \widetilde{u}^\omega_{k,R}(r) &= {\widetilde{u}^\omega_{k,R}}''(r) + H_{S_r(o)}(r, \theta) {\widetilde{u}^\omega_{k,R}}'(r) \\ &\leq {u^\omega_{k,R}}''(r) + H_{S^\omega_r(o_\omega)}(r) {u^\omega_{k,R}}'(r) \\ &= \Delta_{g_\omega} u^\omega_{k,R}(r) = -k u^\omega_{k-1,R}(r) = -k \widetilde{u}^\omega_{k-1,R}(r). \end{aligned} \tag{3.28} $$

Then, since we suppose that $u_{k,R} \leq \widetilde{u}^\omega_{k,R}$ on $B_R(o)$ and using equation (3.28), we obtain that

$$ \Delta_g u_{k+1,R} = -(k+1) u_{k,R} \geq -(k+1) \widetilde{u}^\omega_{k,R} \geq \Delta_g \widetilde{u}^\omega_{k+1,R} \quad \text{on} \quad B_R(o). \tag{3.29} $$

Thus, $\Delta_g \left(u_{k+1,R} - \widetilde{u}^\omega_{k+1,R} \right) \geq 0$ on $B_R(o)$ and as $\left(u_{k+1,R} - \widetilde{u}^\omega_{k+1,R} \right)\big|_{S_R(o)} = 0$ then, arguing as in the proof of Proposition 3.4.3, by applying the Strong Maximum Principle, we obtain that $u_{k+1,R} \leq \widetilde{u}^\omega_{k+1,R}$ on $B_R(o)$.

Now, we prove the equality case. Suppose that $H_{S_r(o)} \geq H_{S^\omega_r(o_\omega)}$ for all radius $r \in (0, R]$ and that there is a point $p_0 \in B_R(o)$ and $k_0 \geq 1$ such that

$$ u_{k_0,R}(p_0) = \widetilde{u}^\omega_{k_0,R}(p_0). $$

Let us first show the equality between the mean curvatures, *i.e.*, assertion (1). We know, for all $k \geq 1$, that $u_{k,R} \leq \widetilde{u}^\omega_{k,R}$ and that $u_{k,R} - \widetilde{u}^\omega_{k,R}$ is subharmonic on $B_R(o)$. In particular, for k_0 we have, from equation (3.29), that

$$ u_{k_0,R} - \widetilde{u}^\omega_{k_0,R} \leq 0 \quad \text{and} \quad \Delta_g \left(u_{k_0,R} - \widetilde{u}^\omega_{k_0,R} \right) \geq 0, \quad \text{on} \quad B_R(o). $$

Thus, we obtain that $u_{k_0,R} - \widetilde{u}^{\omega}_{k_0,R}$ reach its maximum in $B_R(o)$. In fact, since there is a point $p_0 \in B_R(o)$ such that $\left(u_{k_0,R} - \widetilde{u}^{\omega}_{k_0,R}\right)(p_0) = 0$, we have that for all $p \in B_R(o)$

$$\left(u_{k_0,R} - \widetilde{u}^{\omega}_{k_0,R}\right)(p) \leq 0 = \left(u_{k_0,R} - \widetilde{u}^{\omega}_{k_0,R}\right)(p_0).$$

Therefore, by applying the Strong Maximum Principle and by continuity, we obtain that $u_{k_0,R} = \widetilde{u}^{\omega}_{k_0,R}$ on $B_R(o)$.

On the other hand, since $u_{k_0-1,R} \leq \widetilde{u}^{\omega}_{k_0-1,R}$ on $B_R(o)$, we have that

$$\Delta_g \widetilde{u}^{\omega}_{k_0,R} = \Delta_g u_{k_0,R} = -k_0 u_{k_0-1,R} \geq -k_0 \widetilde{u}^{\omega}_{k_0-1,R} = -k_0 u^{\omega}_{k_0-1,R} = \Delta_{g_\omega} u^{\omega}_{k_0,R}$$

on $B_R(o)$. Thus, using equations (2.26), (2.27), (2.32) and (2.33), we have, for all $(r,\theta) \in (0,R] \times \mathbb{S}^{n-1}_1$, that

$$\widetilde{u}^{\omega}_{k_0,R}{}''(r) + H_{S_r(o)}(r,\theta)\widetilde{u}^{\omega}_{k_0,R}{}'(r) \geq u^{\omega}_{k_0,R}{}''(r) + H_{S^{\omega}_r(o_\omega)}(r) u^{\omega}_{k_0,R}{}'(r).$$

Then, since $\widetilde{u}^{\omega}_{k_0,R}{}''(r) = u^{\omega}_{k_0,R}{}''(r)$ and $\widetilde{u}^{\omega}_{k_0,R}{}'(r) = u^{\omega}_{k_0,R}{}'(r)$ for all $r \in (0,R]$, we conclude that

$$H_{S_r(o)}(r,\theta) u^{\omega}_{k_0,R}{}'(r) \geq H_{S^{\omega}_r(o_\omega)}(r) u^{\omega}_{k_0,R}{}'(r) \quad \text{for all} \quad (r,\theta) \in (0,R] \times \mathbb{S}^{n-1}_1,$$

and hence, since $u^{\omega}_{k_0,R}{}'(r) < 0$ for all $r \in (0,R]$, we obtain that

$$H_{S_r(o)} \leq H_{S^{\omega}_r(o_\omega)} \quad \text{for all} \quad r \in (0,R].$$

Therefore, as we assumed that $H_{S_r(o)} \geq H_{S^{\omega}_r(o_\omega)}$ for all $R \in (0,R]$, we finally have that

$$H_{S_r(o)} = H_{S^{\omega}_r(o_\omega)} \quad \text{for all} \quad r \in (0,R],$$

which shows assertion (1).

Now, to prove that $u_{k,r} = \widetilde{u}^{\omega}_{k,r}$ on $B_r(o)$, for all $k \geq 1$ and for all $r \in [0,R]$, we argue as follows: fix $r \in [0,R]$, since we know that $H_{S_s(o)} = H_{S^{\omega}_s(o_\omega)}$ for all $s \in (0,r]$, applying Proposition 3.4.3, we have that $u_{1,r} = \widetilde{u}^{\omega}_{1,r}$ on $B_r(o)$, and we proceed by induction. Suppose that $u_{k,r} = \widetilde{u}^{\omega}_{k,r}$ on $B_r(o)$ and let us show that $u_{k+1,r} = \widetilde{u}^{\omega}_{k+1,r}$ on $B_r(o)$. For that, we compute on $B_r(o)$:

$$\begin{aligned}
\Delta_g u_{k+1,r} &= -(k+1)u_{k,r} = -(k+1)\widetilde{u}^{\omega}_{k,r} \\
&= -(k+1)u^{\omega}_{k,r} = \Delta_{g_\omega} u^{\omega}_{k+1,r} \\
&= u^{\omega}_{k+1,r}{}'' + H_{S^{\omega}_s(o_\omega)} u^{\omega}_{k+1,r}{}' \\
&= \widetilde{u}^{\omega}_{k+1,r}{}'' + H_{S_s(o)} \widetilde{u}^{\omega}_{k+1,r}{}' = \Delta_g \widetilde{u}^{\omega}_{k+1,r}
\end{aligned}$$

for all $p \equiv (s,\theta) \in B_r(o)$. Then, $\Delta_g\left(u_{k+1,r} - \widetilde{u}^\omega_{k+1,r}\right) = 0$ on $B_r(o)$ and since $u_{k+1,r} - \widetilde{u}^\omega_{k+1,r} = 0$ on $S_r(o)$, applying the Strong Maximum Principle again as in the proof of Proposition 3.4.3, we conclude that $u_{k+1,r} = \widetilde{u}^\omega_{k+1,r}$ on $B_r(o)$. Therefore, for all $k \geq 1$ and for all $r \in [0,R]$, we obtain that $u_{k,r} = \widetilde{u}^\omega_{k,r}$ on $B_r(o)$ showing assertion (2).

Moreover, since we just proved that $u_{1,r} = \widetilde{u}^\omega_{1,r}$ on $B_r(o)$ for all $r \in [0,R]$, we have that there is a point $p \in B_r(o)$ such that $E_r(p) = \mathbb{E}^\omega_r(p)$, and hence, from Corollary 3.4.6, we obtain the equalities $\operatorname{vol}(B_r(o)) = \operatorname{vol}(B^\omega_r(o_\omega))$ and $\operatorname{vol}(S_r(o)) = \operatorname{vol}(S^\omega_r(o_\omega))$ for all $r \in (0,R]$, namely, assertion (3).

Finally, to prove assertion (4), let us first do the following computation. Fixing $r \in [0,R]$, since $\Delta_{g_\omega} u^\omega_{k+1,r} = -(k+1)u^\omega_{k,r}$ on $B^\omega_r(o_\omega)$ for all $k \geq 1$, by applying the Divergence Theorem 2.1.59, using that $u^\omega_{k+1,r}$ is a radial function (see Proposition 3.2.16) and that

$$\nabla_{g_\omega} u^\omega_{k+1,r} = {u^\omega_{k+1,r}}' \, \nabla_{g_\omega} r_{o_\omega}$$

on $B^\omega_r(o_\omega) - \{o_\omega\}$ for all $k \geq 1$, and since $\left\|\nabla_{g_\omega} r_{o_\omega}(q)\right\|_{g_\omega} = 1$ and $r_{o_\omega}(q) = r$ for any $q \in S^\omega_r(o_\omega)$, we obtain, for all $k \geq 1$, that

$$\begin{aligned}
\mathcal{A}_k\left(B^\omega_r(o_\omega)\right) &= \int_{B^\omega_r(o_\omega)} u^\omega_{k,r}\, dV_{g_\omega} = -\frac{1}{k+1}\int_{B^\omega_r(o_\omega)} \Delta_{g_\omega} u^\omega_{k+1,r}\, dV_{g_\omega} \\
&= -\frac{1}{k+1}\int_{S^\omega_r(o_\omega)} g_\omega\left(\nabla_{g_\omega} u^\omega_{k+1,r}, \nabla_{g_\omega} r_{o_\omega}\right) dA_{g_\omega} \\
&= -\frac{1}{k+1}\int_{S^\omega_r(o_\omega)} {u^\omega_{k+1,r}}'\, g_\omega\left(\nabla_{g_\omega} r_{o_\omega}, \nabla_{g_\omega} r_{o_\omega}\right) dA_{g_\omega} \\
&= -\frac{1}{k+1}{u^\omega_{k+1,r}}'(r)\int_{S^\omega_r(o_\omega)} dA_{g_\omega} \\
&= -\frac{1}{k+1}{u^\omega_{k+1,r}}'(r)\, \operatorname{vol}\left(S^\omega_r(o_\omega)\right).
\end{aligned}$$

Therefore, for all $k \geq 1$ and for all $r \in [0,R]$, we have that

$$-\frac{1}{k+1}{u^\omega_{k+1,r}}'(r) = \frac{\mathcal{A}_k\left(B^\omega_r(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_r o_\omega\right)}. \tag{3.30}$$

On the other hand, since $\Delta_g u_{k+1,r} = -(k+1)u_{k,r}$ on $B_r(o)$ for all $k \geq 1$ and for all $r \in [0,R]$, and we have proved assertion (2), *i.e.*, $u_{k,r} = \widetilde{u}^\omega_{k,r}$ on $B_r(o)$ for all $k \geq 1$ and for all $r \in [0,R]$, we have that all the inequalities in equations (3.28) and (3.29) become equalities, and hence, $\Delta_g u_{k+1,r} = \Delta_g \widetilde{u}^\omega_{k+1,r}$ on $B_r(o)$ for all

$k \geq 1$ and for all $r \in [0, R]$. Then, fixing $r \in [0, R]$, by applying the Divergence Theorem 2.1.59, using that $\widetilde{u}^{\omega}_{k,r}$ is a radial function, in fact, $\widetilde{u}^{\omega}_{k,r}(s) = u^{\omega}_{k,r}(s)$ for all $s \in [0, r]$ and for all $k \geq 1$, using that $\nabla_g \widetilde{u}^{\omega}_{k,r} = \widetilde{u}^{\omega}_{k,r}{}' \, \nabla_g r_o$ on $B_r(o) - \{o\}$ for all $k \geq 1$, and since $\|\nabla_g r_o(q)\|_g = 1$ and $r_o(q) = r$ for all $q \in S_r(o)$, we obtain, for all $k \geq 1$, that

$$\begin{aligned}
\mathcal{A}_k\left(B_r(o)\right) &= \int_{B_R(o)} u_{k,r}\, dV_g = \frac{1}{k+1} \int_{B_r(o)} \Delta_g u_{k+1,r}\, dV_g \\
&= -\frac{1}{k+1} \int_{B_r(o)} \Delta_g \widetilde{u}^{\omega}_{k+1,r}\, dV_g \\
&= -\frac{1}{k+1} \int_{S_r(o)} g\left(\nabla_g \widetilde{u}^{\omega}_{k+1,r}, \nabla_g r_o\right) dA_g \\
&= -\frac{1}{k+1} \int_{S_r(o)} \widetilde{u}^{\omega}_{k+1,r}{}' \, g\left(\nabla_g r_o(q), \nabla_g r_o(q)\right) dA_g \\
&= -\frac{1}{k+1} \widetilde{u}^{\omega}_{k+1,r}{}'(r) \int_{S_r(o)} dA_g = -\frac{1}{k+1} u^{\omega}_{k+1,r}{}'(r) \operatorname{vol}\left(S_r(o)\right).
\end{aligned} \tag{3.31}$$

Therefore, from equation (3.30), we finally obtain, for all $k \geq 1$ and for all $r \in [0, R]$, that

$$\mathcal{A}_k\left(B_r(o)\right) = \frac{\mathcal{A}_k\left(B^{\omega}_r(o_\omega)\right)}{\operatorname{vol}\left(S^{\omega}_r(o_\omega)\right)} \operatorname{vol}\left(S_r(o)\right),$$

and hence, since we proved that $\operatorname{vol}\left(S_r(o)\right) = \operatorname{vol}\left(S^{\omega}_r(o_\omega)\right)$ for all $r \in (0, R]$ (*i.e.*, from assertion (3)), we have that $\mathcal{A}_k\left(B_r(o)\right) = \mathcal{A}_k\left(B^{\omega}_r(o_\omega)\right)$ for all $k \geq 1$ and for all $r \in (0, R]$, which shows assertion (4).

When we assume that $H_{S_r(o)} \leq H_{S^{\omega}_r(o_\omega)}$ for all $r \in (0, R]$, the argument is exactly the same but inverting all the inequalities, to conclude the opposite inequality for the Poisson hierarchy and the transplanted Poisson hierarchy, *namely*, we obtain that $u_{k,R} \geq \widetilde{u}^{\omega}_{k,R}$ in $B_R(o)$ for all $k \geq 1$. The equality discussion is the same, *mutatis mutandis*.

□

Remark 3.5.4. Note that, from the equality case of the above theorem, we have that one value of $u_{k,R}$ for some $k \geq 1$ determines the Poisson hierarchy, the volume and the moment spectrum of all the geodesic balls $B_r(o)$ with radius $r \in [0, R]$.

As a consequence of Theorem 3.5.3 we have the following result concerning the *"averaged" moment spectrum*, *i.e.*, concerning $\mathcal{A}_k\left(B_R(o)\right) / \operatorname{vol}\left(B_R(o)\right)$.

Corollary 3.5.5. *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ and $B_R^\omega(o_\omega)$ be, respectively, a geodesic ball of M with radius R centered at o and a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \mathrm{inj}_g(o) \le \mathrm{inj}_{g_\omega}(o_\omega)$ and moreover that*

$$H_{S_r(o)} \ge (\le) H_{S_r^\omega(o_\omega)} \quad \textit{for all} \quad r \in (0, R].$$

Then, for all $k \ge 1$, we have that

$$\frac{\mathcal{A}_k\left(B_R(o)\right)}{\mathrm{vol}\left(S_R(o)\right)} \le (\ge) \frac{\mathcal{A}_k\left(B_R^\omega(o_\omega)\right)}{\mathrm{vol}\left(S_R^\omega(o_\omega)\right)}. \tag{3.32}$$

Furthermore, equality in inequality (3.32) for some $k \ge 1$ implies:

1. *The equalities $H_{S_r(o)} = H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$.*
2. *The equalities $u_{k,r} = \widetilde{u}_{k,r}^\omega$ on $B_r(o)$ for all $k \ge 1$ and for all $r \in [0, R]$.*
3. *The volume equalities $\mathrm{vol}(B_r(o)) = \mathrm{vol}\left(B_r^\omega(o_\omega)\right)$ and the volume equalities $\mathrm{vol}\left(S_r(o)\right) = \mathrm{vol}\left(S_r^\omega(o_\omega)\right)$ for all $r \in (0, R]$.*
4. *The equalities $\mathcal{A}_k\left(B_r(o)\right) = \mathcal{A}_k\left(B_r^\omega(o_\omega)\right)$ for all $k \ge 1$ and for all $r \in (0, R]$.*

where $\{u_{k,r}\}_{k=1}^\infty$ and $\left\{\widetilde{u}_{k,r}^\omega\right\}_{k=1}^\infty$ are, respectively, the Poisson hierarchy of $B_r(o)$ and the transplanted Poisson hierarchy from $B_r^\omega(o_\omega)$ to $B_r(o)$, and $\{\mathcal{A}_k\left(B_r(o)\right)\}_{k=1}^\infty$ and $\{\mathcal{A}_k\left(B_r^\omega(o_\omega)\right)\}_{k=1}^\infty$ are, respectively, the moment spectrum of $B_r(o)$ and $B_r^\omega(o_\omega)$.

Proof. By applying the Divergence Theorem 2.1.59 and computing as in equation (3.30), since $\Delta_{g_\omega} u_{k+1,R}^\omega = -(k+1)u_{k+1,R}^\omega$ on $B_R^\omega(o_\omega)$ for all $k \ge 1$, we obtain for all $k \ge 1$, that

$$-\frac{1}{k+1} {u_{k+1,R}^\omega}'(R) = \frac{\mathcal{A}_k\left(B_R^\omega(o_\omega)\right)}{\mathrm{vol}\left(S_R^\omega(o_\omega)\right)}. \tag{3.33}$$

Assuming now that $H_{S_r(o)} \ge H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$ we know, by Theorem 3.5.3, that $u_{k,R} \le \widetilde{u}_{k,R}^\omega$ on $B_R(o)$ for all $k \ge 1$. Then, from inequality (3.28), we obtain, for all $k \ge 1$, that

$$\Delta_g u_{k+1,R} = -(k+1)u_{k,R} \ge -(k+1)\widetilde{u}_{k,R}^\omega \ge \Delta_g \widetilde{u}_{k+1,R}^\omega \quad \text{on} \quad B_R(o). \tag{3.34}$$

Therefore, we obtain that equalities in (3.31) become inequalities. Indeed, applying the Divergence Theorem 2.1.59 and that $\widetilde{u}^{\omega}_{k+1,R}$ is a radial function on $B_R(o)$ for all $k \geq 1$, we have that

$$\begin{aligned}
\mathcal{A}_k\left(B_R(o)\right) &= \int_{B_R(o)} u_{k,R}\, dV_g \\
&= -\frac{1}{k+1}\int_{B_R(o)} \Delta_g u_{k+1,R}\, dV_g \\
&\leq -\frac{1}{k+1}\int_{B_R(o)} \Delta_g \widetilde{u}^{\omega}_{k+1,R}\left(r\right) dV_g \\
&= -\frac{1}{k+1}\int_{S_R(o)} g\left(\nabla_g \widetilde{u}^{\omega}_{k+1,R}\left(r\right), \nabla_g r\right) dA_g \\
&= -\frac{1}{k+1}\widetilde{u}^{\omega}_{k+1,R}{}'(R)\, \mathrm{vol}\left(S_R(o)\right) \\
&= -\frac{1}{k+1}u^{\omega}_{k+1,R}{}'(R)\, \mathrm{vol}\left(S_R(o)\right).
\end{aligned} \tag{3.35}$$

Hence, from equation (3.33), we finally obtain, for all $k \geq 1$, that

$$\mathcal{A}_k\left(B_R(o)\right) \leq \frac{\mathcal{A}_k\left(B^{\omega}_R(o_\omega)\right)}{\mathrm{vol}\left(S^{\omega}_R(o_\omega)\right)}\mathrm{vol}\left(S_R(o)\right),$$

which shows one of the directions of inequality (3.32).

Now, we discuss the equality case assuming that $H_{S_r(o)} \geq H_{S^{\omega}_r(o_\omega)}$ for all $r \in (0, R]$. Suppose that there exists $k_0 \geq 1$ such that

$$\frac{\mathcal{A}_{k_0}\left(B_R(o)\right)}{\mathrm{vol}\left(S_R(o)\right)} = \frac{\mathcal{A}_{k_0}\left(B^{\omega}_R(o_\omega)\right)}{\mathrm{vol}\left(S^{\omega}_R(o_\omega)\right)}.$$

Therefore, all the inequalities in (3.35) become equalities and in particular, for this fixed $k_0 \geq 1$, we obtain that

$$\mathcal{A}_{k_0}\left(B_R(o)\right) = -\frac{1}{k_0+1}\int_{B_R(o)} \Delta_g u_{k_0+1,R}\, dV_g = -\frac{1}{k_0+1}\int_{B_R(o)} \Delta_g \widetilde{u}^{\omega}_{k_0+1,R}\, dV_g$$

and hence,

$$\int_{B_R(o)} \Delta_g\left(u_{k_0+1,R} - \widetilde{u}^{\omega}_{k_0+1,R}\right) dV_g = 0. \tag{3.36}$$

Thus, since the integral of $\Delta_g\left(u_{k_0+1,R} - \widetilde{u}^{\omega}_{k_0+1,R}\right)$ over $B_R(o)$ vanish and since, from inequality (3.34), $\Delta_g\left(u_{k_0+1,R} - \widetilde{u}^{\omega}_{k_0+1,R}\right) \geq 0$ on $B_R(o)$, we obtain that

$\Delta_g \left(u_{k_0+1,R} - \widetilde{u}^\omega_{k_0+1,R}\right) = 0$ on $B_R(o)$. Moreover, we know that, by definition, $u_{k_0+1,R} = \widetilde{u}^\omega_{k_0+1,R} = 0$ on $S_R(o)$. Then, arguing as in the proof of Proposition 3.4.3, by applying the Strong Maximum Principle (see 2.1.64), we conclude that $u_{k_0+1,R} = \widetilde{u}_{k_0+1,R}$ on $B_R(o)$.

Finally, since there is some $k_1 = k_0 + 1 \geq 1$ and a point $p_0 \in B_R(o)$ such that $u_{k_1,R}(p_0) = \widetilde{u}^\omega_{k_1,R}(p_0)$ (note that, in this case, we have the equality for all $p \in B_R(o)$), we obtain assertions (1), (2), (3) and (4) by applying the equality case of Theorem 3.5.3.

When we assume that $H_{S_r(o)} \leq H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$, the argument is exactly the same but inverting all the inequalities, to conclude the opposite inequality for the averages of the moment spectrums for $B_R(o)$ and $B^\omega_R(o_\omega)$. The equality discussion is the same, *mutatis mutandis.*

□

Remark 3.5.6. Note that one value of $\mathcal{A}_k\left(B_R(o)\right) / \operatorname{vol}\left(S_R(o)\right)$ for some $k \geq 1$ determines the Poisson hierarchy, the volume and the moment spectrum of all the geodesic balls $B_r(o)$ with radius $r \in [0, R]$.

3.6 Torsional rigidity comparison

In this section we compare the torsional rigidity of a geodesic ball $B_R(o)$ of a complete Riemannian manifold (M, g) in Theorem 3.6.3, assuming that the mean curvature of the geodesic spheres in $B_R(o)$ is bounded from below or from above by the corresponding mean curvature point inward of the geodesic spheres in a rotationally symmetric space $(\mathbb{M}_\omega, g_\omega)$ which is balanced from above (see Definition 2.2.8 for the notion of being balanced from above). This result can be considered as a continuation of the intrinsic comparison done by A. Hurtado, S. Markvorsen and V. Palmer in Section 6 of [37]. In that paper it were obtained upper and lower bounds for the torsional rigidity of a geodesic ball $B_R(o)$ of a Riemannian manifold (M, g) with a pole $o \in M$ under more restrictive conditions, namely, assuming that the radial sectional curvatures were from below or from above by the corresponding radial sectional curvatures of a suitable rotationally symmetric model space.

To show our comparison result, we need to consider the symmetrization given by $\mathbb{E}_R^{\omega *} : B_{s(R)} \longrightarrow \mathbb{R}$ of the transplanted mean exit time function on $B_R^\omega(o_\omega)$ to $B_R(o)$. From this consideration, we have Theorems 3.6.1 and 3.6.2 which show some properties that the Schwarz symmetrization of the transplated mean exit time function $\mathbb{E}_R^\omega$ satisfies and which will be used to prove our mentioned comparison for the torsional rigidity. Namely, we consider a rotationally symmetric model space rearrangement of the geodesic ball $B_R(o)$ of a given complete Riemannian manifold (M, g), *i.e.*, a Schwarz symmetrization of the geodesic ball $B_R(o)$ that is a geodesic ball $B_{s(R)}^\omega(o_\omega)$ in a rotationally symmetric model space $(\mathbb{M}_\omega, g_\omega)$ such that $\operatorname{vol}(B_R(o)) = \operatorname{vol}\left(B_{s(R)}^\omega(o_\omega)\right)$, together the symmetrization $\mathbb{E}_R^{\omega *} : B_{s(R)}^\omega(o_\omega) \longrightarrow \mathbb{R}$ of the transplanted mean exit time function $\mathbb{E}_R^\omega : B_R(o) \longrightarrow \mathbb{R}$, $\mathbb{E}_R^\omega(p) := E_R^\omega(r_o(p))$, where E_R^ω is the mean exit time function on $B_R^\omega(o_\omega)$ and r_o is the radial distance function to o in $B_R(o)$ (see Subsection 2.2.3 for more information about the Schwarz symmetrization).

The first result is an intrinsic version of Theorem 4.4 in [37], and it follows directly from this result (see too Section 6 in [37]). Moreover, the following Theorem makes sense for those geodesic balls which poses a Schwarz symmetrization $B_{s(R)}^\omega(o_\omega)$.

Theorem 3.6.1. *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ and $B_R^\omega(o_\omega)$ be, respectively, a geodesic ball of M with radius R centered at o and a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$ and that there exists the Schwarz symmetrization $B_{s(R)}^\omega(o_\omega)$ of $B_R(o)$ in $\mathbb{M}_\omega$. Then, we have that*

$$\int_{B_R(o)} \mathbb{E}_R^\omega \, dV_g = \int_{B_{s(R)}^\omega(o_\omega)} \mathbb{E}_R^{\omega *} \, dV_{g_\omega}. \tag{3.37}$$

Proof. From Proposition 3.2.13, we know that the mean exit time E_R^ω is a positive radial function that is strictly decreasing and which attains its maximum at the center o_ω and vanish at $S_R^\omega(o_\omega)$. Then, the theorem follows from applying Theorem 2.2.26.

□

The second result about the symmetrization of the transplanted mean exit time function $\mathbb{E}_R^{\omega *}$ consist in a comparison between $\mathbb{E}_R^{\omega *}$ and the mean exit time function $E_{s(R)}^{\omega}$ of the Schwarz symmetrization $B_{s(R)}^{\omega}(o_\omega)$ of $B_R(o)$ which we show in the following proposition. Its proof follows closely the lines of the proof of Propositions 5.2 and 5.4 in [37], and we have included it because the changes due to its intrinsic character, the different assumptions on the curvatures we have assumed along this chapter and the new analysis of the equality which we present in this case. As in Theorem 3.6.1, the following theorem makes sense for those geodesic balls which poses a Schwarz symmetrization $B_{s(R)}^{\omega}(o_\omega)$.

Theorem 3.6.2. *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_{_\omega})$ be an n-dimensional rotationally symmetric model space balanced from above with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ and $B_R^{\omega}(o_\omega)$ be, respectively, a geodesic ball of M with radius R centered at o and a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$, that there exists the Schwarz symmetrization $B_{s(R)}^{\omega}(o_\omega)$ of $B_R(o)$ in $\mathbb{M}_\omega$, and moreover that*

$$H_{S_r(o)} \geq (\leq) H_{S_r^{\omega}(o_\omega)} \quad \textit{for all} \quad r \in (0, R].$$

Then, we have that

$$\mathbb{E}_R^{\omega *\prime}(\widetilde{r}) \geq (\leq) {E_{s(R)}^{\omega}}'(\widetilde{r}) \quad \textit{for all} \quad \widetilde{r} \in (0, s(R)), \tag{3.38}$$

and hence,

$$\mathbb{E}_R^{\omega *}(\widetilde{r}) \leq (\geq) E_{s(R)}^{\omega}(\widetilde{r}) \quad \textit{for all} \quad \widetilde{r} \in [0, s(R)], \tag{3.39}$$

where $s(R)$ is the symmetrized radius of the Schwarz symmetrization $B_{s(R)}^{\omega}(o_\omega)$ of $B_R(o)$ in the rotationally symmetric model space $\mathbb{M}_\omega$ and $E_{s(R)}^{\omega}$ is the mean exit time function on $B_{s(R)}^{\omega}(o_\omega)$.

Furthermore, equality in inequality (3.39) for all $t \in [0, s(R)]$ implies:

1. *The equality among the radius $s(R) = R$.*

2. *The volume equalities $\operatorname{vol}(B_r(o)) = \operatorname{vol}(B_r^{\omega}(o_\omega))$ and the volume equalities $\operatorname{vol}(S_r(o)) = \operatorname{vol}(S_r^{\omega}(o_\omega))$ for all $r \in [0, R]$.*

3. *The equality $H_{S_r(o)} = H_{S_r^{\omega}(o_\omega)}$ for all $r \in (0, R]$.*

Proof. The first part of the proof follows the argument used in the proof of Theorem 2.2.26 replacing ψ by $\mathbb{E}_R^\omega$. We are going to analyze first the symmetrization $\mathbb{E}_R^{\omega *}$. The transplanted function

$$\mathbb{E}_R^\omega : B_R(o) \longrightarrow \mathbb{R}$$

satisfies, by definition, that $\mathbb{E}_R^\omega \in C^\infty\left(B_R(o) - \{o\}\right) \cap C^0\left(\overline{B_R(o)}\right)$ and, moreover, that $\mathbb{E}_R^\omega|_{S_R(o)} = 0$. We know too that $\mathbb{E}_R^\omega = E_R^\omega \circ r_o$ on $B_R(o)$, where E_R^ω is the mean exit time function defined on $B_R^\omega(o_\omega)$.

From Proposition 3.2.13 we know that the mean exit time function E_R^ω in $B_R^\omega(o_\omega)$ is a strictly decreasing radial function. Thus, let us consider E_R^ω as the radial function defined on the interval $[0, R]$ in equation (3.4) of Remark 3.2.15. Let us denote by $T = \max_{[0,R]} E_R^\omega$. Then, as E_R^ω is monotone, we have that $E_R^{\omega\prime} < 0$ in $B_R^\omega(o_\omega)$ and that $E_R^\omega : [0, R] \longrightarrow [0, T]$ is bijective with $E_R^\omega(0) = T$ and $E_R^\omega(R) = 0$.

Now, let us define the function $a : [0, T] \longrightarrow [0, R]$ as $a(t) := (E_R^\omega)^{-1}(t)$, satisfying $a(0) = (E_R^\omega)^{-1}(0) = R$ and $a(T) = (E_R^\omega)^{-1}(T) = 0$. We know that

$$a'(t) = \frac{1}{E_R^{\omega\prime}\left(a(t)\right)} < 0 \quad \text{for all} \quad t \in [0, T),$$

so, $a(t)$ is strictly decreasing in $[0, T]$.

On the other hand, by definition, we have that the transplanted mean exit time function $\mathbb{E}_R^\omega$ from $B_R^\omega(o_\omega)$ to $B_R(o)$ is the radial function $\mathbb{E}_R^\omega(p) = E_R^\omega(r_o(p))$, where r_o is the radial distance function to o in $B_R(o)$. Then, we have that $\mathbb{E}_R^\omega\left(B_R(o)\right) = E_R^\omega\left([0, R]\right) = [0, T]$. Thus, since $\left\|\nabla_g r_o\right\|_g = 1$ in $B_R(o)$, the transplanted mean exit time function $\mathbb{E}_R^\omega : B_R(o) \longrightarrow [0, T]$ satisfies, for all $p \in B_R(o) - \{o\}$ such that $r_o(p) = r$, that

$$\left\|\nabla_g \mathbb{E}_R^\omega(p)\right\|_g = \left|E_R^{\omega\prime}(r)\right| \left\|\nabla_g r_o\right\|_g = \left|E_R^{\omega\prime}(r)\right| \neq 0, \tag{3.40}$$

and hence, the set of regular values of $\mathbb{E}_R^\omega$ is $R_{\mathbb{E}_R^\omega} = (0, T)$.

On the other hand, and given $t \in [0, T]$, let us consider the sets $D(t)$ and $\Gamma(t)$ defined in Definition 2.2.15, *i.e.*,

$$\begin{aligned} D(t) &= \{p \in B_R(o) \,:\, \mathbb{E}_R(p) \geq t\} = \{p \in B_R(o) \,:\, E_R^\omega\left(r_o(p)\right) \geq t\} \\ &= \left\{p \in B_R(o) \,:\, r_o(p) \leq (E_R^\omega)^{-1}(t)\right\} = \{p \in B_R(o) \,:\, r_o(p) \leq a(t)\} \\ &= B_{a(t)}(o) \end{aligned} \tag{3.41}$$

and

$$\begin{aligned}\Gamma(t) &= \{p \in B_R(o) \,:\, \mathbb{E}_R^\omega(p) = t\} = \{p \in B_R(o) \,:\, E_R^\omega\left(r_o(p)\right) = t\} \\ &= \left\{p \in B_R(o) \,:\, r_o(p) = (E_R^\omega)^{-1}(t)\right\} = \{p \in B_R(o) \,:\, r_o(p) = a(t)\} \\ &= S_{a(t)}(o).\end{aligned} \tag{3.42}$$

Moreover, we have that $D(0) = B_{a(0)}(o) = B_R(o)$ and $D(T) = B_{a(T)}(o) = \{o\}$, where o is the center of the geodesic ball $B_R(o)$.

Now, we consider the symmetrization in $\mathbb{M}_\omega$ of the sets $D(t) = B_{a(t)}(o) \subseteq B_R(o) \subseteq M$, namely, the geodesic balls $D(t)^* = B^\omega_{\widetilde{r}(t)}(o_\omega)$ in $\mathbb{M}_\omega$ such that

$$\operatorname{vol}\left(D(t)\right) = \operatorname{vol}\left(B^\omega_{\widetilde{r}(t)}(o_\omega)\right),$$

where, for each $t \in [0, T]$, $\widetilde{r}(t)$ denotes the symmetrized radius (see Definition 2.2.19. Then, in this particular context and from Lemma 2.2.21, we have that $\widetilde{r} : [0, T] \longrightarrow [0, s(R)]$ is strictly decreasing, and hence, bijective, where $s(R)$ is the radius of symmetrization of $B_R(o)$, *i.e.*, the symmetrization of $B_R(o)$ in $\mathbb{M}_\omega$ is $B^\omega_{s(R)}(o_\omega)$. In fact, note that if $t_1, t_2 \in [0, T]$ such that $t_1 < t_2$, then, since $a(t)$ is strictly decreasing, $a(t_1) > a(t_2)$, thus

$$\operatorname{vol}\left(B^\omega_{\widetilde{r}(t_1)}(o_\omega)\right) = \operatorname{vol}\left(B_{a(t_1)}(o)\right) > \operatorname{vol}\left(B_{a(t_2)}(o)\right) = \operatorname{vol}\left(B^\omega_{\widetilde{r}(t_2)}(o_\omega)\right),$$

and hence, $\widetilde{r}(t_1) > \widetilde{r}(t_2)$. Furthermore, applying Lemma 2.2.21, for all t in the set of the regular values of $\mathbb{E}_R^\omega$, *i.e.*, for all $t \in R_{\mathbb{E}_R^\omega} = (0, T)$, we have that

$$\widetilde{r}'(t) = -\frac{1}{\operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right)} \int_{\Gamma(t)} \|\nabla_g \mathbb{E}_R^\omega\|_g^{-1}\, dA_g, \tag{3.43}$$

where dA_g is the Riemannian volume element in $\Gamma(t)$ with respect to the metric tensor g (see Subsection 2.1.7.5).

On the other hand, the inverse of $\widetilde{r}$ is the decreasing function

$$\phi : [0, s(R)] \longrightarrow [0, T], \quad \phi(\ell) := (\widetilde{r})^{-1}(\ell),$$

such that $\phi'\left(\widetilde{r}(t)\right) = \frac{1}{\widetilde{r}'(t)}$ for all $t \in [0, T]$, $\phi(0) = T$ and $\phi\left(s(R)\right) = 0$.

With all this background, we can say now, in accordance with Definition 2.2.17 of a symmetrization of a function and its properties given in Theorem 2.2.23, that the symmetrization of $\mathbb{E}_R^\omega \,:\, B_R(o) \longrightarrow \mathbb{R}$ in $\mathbb{M}_\omega$ is a radial function

$\mathbb{E}_R^{\omega *} : B^\omega_{s(R)}(o_\omega) \longrightarrow \mathbb{R}$ which satisfies, for all $p^* \in B^\omega_{s(R)}(o_\omega)$, the following equality

$$\mathbb{E}_R^{\omega *}(p^*) = \mathbb{E}_R^{\omega *}\left(r_{o_\omega}(p^*)\right) = t_0 = \phi\left(\widetilde{r}(t_0)\right).$$

Therefore, for all $t \in (0, T)$, we have, applying equation (3.43), that

$$\begin{aligned} \frac{d}{dr_{o_\omega}} \mathbb{E}_R^{\omega *}\Big|_{r_{o_\omega}(p^*)=\widetilde{r}(t)} &= \mathbb{E}_R^{\omega *\prime}\left(\widetilde{r}(t)\right) = \phi'\left(\widetilde{r}(t)\right) = \frac{1}{\widetilde{r}'(t)} \\ &= -\frac{\operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right)}{\int_{\Gamma(t)} \|\nabla_g \mathbb{E}_R^\omega\|_g^{-1} dA_g}. \end{aligned} \tag{3.44}$$

But since, from equations (3.40) and (3.42), we have that

$$\begin{aligned} \int_{\Gamma(t)} \|\nabla_g \mathbb{E}_R^\omega\left(r_o(q)\right)\|_g^{-1} dA_g &= \int_{S_{a(t)}(o)} |E_R^{\omega\prime}\left(r_o(q)\right)|^{-1} dA_g \\ &= \frac{1}{|E_R^{\omega\prime}\left(a(t)\right)|} \int_{S_{a(t)}(o)} dA_g \\ &= \frac{1}{|E_R^{\omega\prime}\left(a(t)\right)|} \operatorname{vol}\left(S_{a(t)}(o)\right), \end{aligned} \tag{3.45}$$

and hence, using that $E_R^{\omega\prime}\left(a(t)\right) = -q_\omega\left(a(t)\right)$ (see Proposition (3.2.13)) and equation (3.45), we obtain, for all $t \in [0, T]$, that equation (3.44) becomes

$$\begin{aligned} \mathbb{E}_R^{\omega *\prime}(\widetilde{r}(t)) &= -\frac{\operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right)}{\int_{\Gamma(t)} \|\nabla_g \mathbb{E}_R^\omega\|_g^{-1} dA_g} = -\left|E_R^{\omega\prime}\left(a(t)\right)\right| \frac{\operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right)}{\operatorname{vol}\left(S_{a(t)}(o)\right)} \\ &= -\frac{\operatorname{vol}\left(B^\omega_{a(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{a(t)}(o_\omega)\right)} \frac{\operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right)}{\operatorname{vol}\left(S_{a(t)}(o)\right)}. \end{aligned} \tag{3.46}$$

On the other hand, assuming that $H_{S_r(o)} \geq H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$, by Corollary 3.4.5, we know that $\operatorname{vol}\left(B_r(o)\right) \geq \operatorname{vol}\left(B^\omega_r(o_\omega)\right)$ for all $r \in (0, R]$. Therefore, since $B^\omega_{\widetilde{r}(t)}(o_\omega)$ is the Schwarz symmetrization of $D(t) = B_{a(t)}(o)$ (see equation (3.41)), we have that

$$\operatorname{vol}\left(B^\omega_{\widetilde{r}(t)}(o_\omega)\right) = \operatorname{vol}\left(B_{a(t)}(o)\right) \geq \operatorname{vol}\left(B^\omega_{a(t)}(o_\omega)\right), \quad \text{for all} \quad t \in (0, T). \tag{3.47}$$

Then, since $\operatorname{vol}\left(B^\omega_r(o_\omega)\right)$ is an increasing function (indeed, from Proposition 2.1.77, $\frac{\partial}{\partial r}\left(B^\omega_r(o_\omega)\right) = \operatorname{vol}\left(S^\omega_r(o_\omega)\right) \geq 0$), we have that

$$\widetilde{r}(t) \geq a(t) \quad \text{for all} \quad t \in (0, T). \tag{3.48}$$

Thus, since $\mathbb{M}_\omega$ is balanced from above, *i.e.*, $q_\omega{}'(r) \geq 0$, we obtain that

$$\frac{\operatorname{vol}\left(B^\omega_{\widetilde{r}(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right)} \geq \frac{\operatorname{vol}\left(B^\omega_{a(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{a(t)}(o_\omega)\right)} \quad \text{for all} \quad t \in (0,T). \tag{3.49}$$

Therefore, since $\operatorname{vol}\left(B_{a(t)}(o)\right) = \operatorname{vol}\left(B^\omega_{\widetilde{r}(t)}(o_\omega)\right)$, we have that

$$\frac{\operatorname{vol}\left(B_{a(t)}(o)\right)}{\operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right)} = \frac{\operatorname{vol}\left(B^\omega_{\widetilde{r}(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right)} \geq \frac{\operatorname{vol}\left(B^\omega_{a(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{a(t)}(o_\omega)\right)} \quad \text{for all} \quad t \in (0,T),$$

and hence,

$$\operatorname{vol}\left(B_{a(t)}(o)\right) \geq \operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right) \frac{\operatorname{vol}\left(B^\omega_{a(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{a(t)}(o_\omega)\right)} \quad \text{for all} \quad t \in (0,T).$$

Then, from equation (3.46), we obtain, for all $t \in (0,T)$, that

$$\mathbb{E}^{\omega\,*\prime}_R\left(\widetilde{r}(t)\right) = -\frac{\operatorname{vol}\left(B^\omega_{a(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{a(t)}(o_\omega)\right)} \frac{\operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right)}{\operatorname{vol}\left(S_{a(t)}(o)\right)} \geq -\frac{\operatorname{vol}\left(B_{a(t)}(o)\right)}{\operatorname{vol}\left(S_{a(t)}(o)\right)} \tag{3.50}$$

Therefore, using the isoperimetric inequality (3.21) of Corollary 3.4.5, and using the fact that $\widetilde{r}(t) \geq a(t)$ and $q'_\omega(t) \geq 0$ (*i.e*, using equation (3.49)), we finally obtain, for all $t \in (0,T)$, that

$$\begin{aligned}\mathbb{E}^{\omega\,*\prime}_R\left(\widetilde{r}(t)\right) &\geq -\frac{\operatorname{vol}\left(B_{a(t)}(o)\right)}{\operatorname{vol}\left(S_{a(t)}(o)\right)} \geq -\frac{\operatorname{vol}\left(B^\omega_{a(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{a(t)}(o_\omega)\right)} \\ &\geq -\frac{\operatorname{vol}\left(B^\omega_{\widetilde{r}(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{\widetilde{r}(t)}(o_\omega)\right)} = -q_\omega(\widetilde{r}(t)) = E^\omega_{s(R)}{}'\left(\widetilde{r}(t)\right),\end{aligned} \tag{3.51}$$

and hence, we obtain that $\mathbb{E}^{\omega\,*\prime}_R\left(\widetilde{r}\right) \geq E^\omega_{s(R)}{}'\left(\widetilde{r}\right)$ for all $\widetilde{r} \in (0, s(R))$. Furthermore, we have, integrating along $[0, s(R)]$ by considering the radius of the symmetrization $\widetilde{r}$ as a parameter, and taking into account that $\mathbb{E}^{\omega\,*}_R\left(s(R)\right) = E^\omega_{s(R)}\left(s(R)\right) = 0$, that

$$-\mathbb{E}^{\omega\,*}_R\left(\widetilde{r}\right) = \int_{\widetilde{r}}^{s(R)} \mathbb{E}^{\omega\,*\prime}_R(x)\,dx \geq \int_{\widetilde{r}}^{s(R)} E^\omega_{s(R)}{}'(x)\,dx = -E^\omega_{s(R)}\left(\widetilde{r}\right),$$

and hence, $\mathbb{E}_R^{\omega *}(\widetilde{r}) \leq E^{\omega}_{s(R)}(\widetilde{r})$ for all $\widetilde{r} \in [0, s(R)]$, which shows one of the directions of inequalities (3.38) and (3.39).

Now, we are going to study the equality case by assuming that $H_{S_r(o)} \geq H_{S^{\omega}_r(o_\omega)}$ for all $r \in (0, R]$. Suppose that equality $\mathbb{E}_R^{\omega *}(\widetilde{r}) = E^{\omega}_{s(R)}(\widetilde{r})$ holds for all $\widetilde{r} \in [0, s(R)]$, then we have equality $\mathbb{E}_R^{\omega *\prime}(\widetilde{r}) = E^{\omega\,\prime}_{s(R)}(\widetilde{r})$ for all $\widetilde{r} \in (0, s(R))$, which in its turn implies that inequalities in (3.51) and hence, inequalities (3.50) and (3.49), become equalities for all $t \in (0, T)$. In particular, from equality in (3.49) and inequality (3.47) (namely, $\operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t)}(o_\omega)\right) \geq \operatorname{vol}\left(B^{\omega}_{a(t)}(o_\omega)\right)$ for all $t \in (0, T)$), we deduce that

$$\frac{\operatorname{vol}\left(S^{\omega}_{a(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^{\omega}_{\widetilde{r}(t)}(o_\omega)\right)} = \frac{\operatorname{vol}\left(B^{\omega}_{a(t)}(o_\omega)\right)}{\operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t)}(o_\omega)\right)} \leq 1 \quad \text{for all} \quad t \in (0, T). \tag{3.52}$$

We are going to show the equality among the radius $s(R) = R$, *i.e*, assertion (1). In fact, we show that $\widetilde{r}(t) \leq a(t)$ for all $t \in (0, T)$. First, using again equality in inequality (3.49) and by Corollary (3.4.5) (assuming that $H_{S_r(o)} \geq H_{S^{\omega}_r(o_\omega)}$ for all $r \in (0, R]$), we obtain that

$$\frac{\operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^{\omega}_{\widetilde{r}(t)}(o_\omega)\right)} = \frac{\operatorname{vol}\left(B^{\omega}_{a(t)}(o_\omega)\right)}{\operatorname{vol}\left(S^{\omega}_{a(t)}(o_\omega)\right)} \geq \frac{\operatorname{vol}\left(B_{a(t)}(o)\right)}{\operatorname{vol}\left(S_{a(t)}(o)\right)} \quad \text{for all} \quad t \in (0, T),$$

and hence, as $\operatorname{vol}\left(B_{a(t)}(o)\right) = \operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t)}(o_\omega)\right)$, using moreover equation (3.52), we have that

$$\operatorname{vol}\left(S_{a(t)}(o)\right) \geq \operatorname{vol}\left(S^{\omega}_{\widetilde{r}(t)}(o_\omega)\right) \geq \operatorname{vol}\left(S^{\omega}_{a(t)}(o_\omega)\right) \quad \text{for all} \quad t \in (0, T). \tag{3.53}$$

Now, differentiating the equality $\operatorname{vol}\left(B_{a(t)}(o)\right) = \operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t)}(o_\omega)\right)$ by using Theorem 2.1.77, for all $t \in (0, T)$, we obtain that

$$\operatorname{vol}\left(S_{a(t)}(o)\right) a'(t) = \operatorname{vol}\left(S^{\omega}_{\widetilde{r}(t)}(o_\omega)\right) \widetilde{r}'(t) \quad \text{for all} \quad t \in [0, T],$$

and hence, using inequality (3.53),

$$\frac{\widetilde{r}'(t)}{a'(t)} = \frac{\operatorname{vol}\left(S_{a(t)}(o)\right)}{\operatorname{vol}\left(S^{\omega}_{\widetilde{r}(t)}(o_\omega)\right)} \geq 1 \quad \text{for all} \quad t \in [0, T].$$

Thus, $\widetilde{r}'(t) \geq a'(t)$ for all $t \in (0,T)$, and therefore, since $\widetilde{r}(T) = a(T)$, we finally obtain, integrating along $[0,T]$, that $\widetilde{r}(t) \leq a(t)$ for all $t \in [0,T]$. Hence, as we know, by inequality (3.48), that $\widetilde{r}(t) \geq a(t)$ for all $t \in [0,T]$, we obtain that

$$\widetilde{r}(t) = a(t) \quad \text{for all} \quad t \in [0,T].$$

Therefore, $s(R) = \widetilde{r}(0) = a(0) = R$ which shows assertion (1), and moreover, since $B^{\omega}_{\widetilde{r}(t)}(o_\omega)$ is the Schwarz symmetrization of $B_{a(t)}(o)$ and $\widetilde{r}(t) = a(t)$ for all $t \in [0,T]$, we have that

$$\operatorname{vol}\left(B_{\widetilde{r}(t)}(o)\right) = \operatorname{vol}\left(B^{\omega}_{\widetilde{r}(t)}(o_\omega)\right) \quad \text{for all} \quad t \in [0,T],$$

and hence, $\operatorname{vol}(B_r(o)) = \operatorname{vol}(B^{\omega}_r(o_\omega))$ for all $r \in [0,R]$ and, differentiating over r, $\operatorname{vol}(S_r(o)) = \operatorname{vol}(S^{\omega}_r(o_\omega))$ for all $r \in [0,R]$, showing assertion (2).

Finally, applying the second equality case of Corollary 3.4.5, we conclude that $H_{S_r(o)} = H_{S^{\omega}_r(o_\omega)}$ for all $r \in (0,R]$, which is assertion (3).

When we assume that $H_{S_r(o)} \leq H_{S^{\omega}_r(o_\omega)}$ for all $r \in (0,R]$, the argument is exactly the same but inverting all the inequalities, to conclude, for all $r \in (0, s(R))$, that

$$\begin{aligned} \mathbb{E}^{\omega\,*\prime}_R(\widetilde{r}) &\leq {E^{\omega}_{s(R)}}'(\widetilde{r}), \\ \mathbb{E}^{\omega\,*}_R(\widetilde{r}) &\geq E^{\omega}_{s(R)}(\widetilde{r}). \end{aligned}$$

The equality discussion is the same, *mutatis mutandis*. □

As a consequence of Theorems 3.6.1 and 3.6.2 we have the following result which shows our mentioned comparison for the torsional rigidity for geodesic balls of Riemannian manifolds.

Theorem 3.6.3. *Let (M,g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_{_\omega})$ be an n-dimensional rotationally symmetric model space balanced from above with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ and $B^{\omega}_R(o_\omega)$ be, respectively, a geodesic ball of M with radius R centered at o and a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$, that there exists the Schwarz symmetrization $B^{\omega}_{s(R)}(o_\omega)$ of $B_R(o)$ in $\mathbb{M}_\omega$, and moreover that*

$$H_{S_r(o)} \geq (\leq) H_{S^{\omega}_r(o_\omega)} \quad \textit{for all} \quad r \in (0,R].$$

Then, we have that

$$\mathcal{A}_1\left(B_R(o)\right) \leq (\geq) \mathcal{A}_1\left(B^{\omega}_{s(R)}(o_\omega)\right), \tag{3.54}$$

where $\mathcal{A}_1\left(B_R(o)\right)$ *and* $\mathcal{A}_1\left(B^{\omega}_{s(R)}(o_\omega)\right)$ *are, respectively, the torsional rigidity for* $B_R(o)$ *and for* $B^{\omega}_{s(R)}(o_\omega)$.

Furthermore, equality in inequality (3.54) *implies:*

1. *The equality among the radius* $s(R) = R$.
2. *The volume equalities* $\operatorname{vol}(B_r(o)) = \operatorname{vol}(B^{\omega}_r(o_\omega))$ *and* $\operatorname{vol}(S_r(o)) = \operatorname{vol}(S^{\omega}_r(o_\omega))$ *for all* $r \in (0, R]$.
3. *The equality* $H_{S_r(o)} = H_{S^{\omega}_r(o_\omega)}$ *for all* $r \in (0, R]$.
4. *The equalities* $u_{k,r} = \widetilde{u}^{\omega}_{k,r}$ *on* $B_r(o)$ *for all* $k \geq 1$ *and for all* $r \in [0, R]$.
5. *The equalities* $\mathcal{A}_k\left(B_r(o)\right) = \mathcal{A}_k\left(B^{\omega}_r(o_\omega)\right)$ *for all* $k \geq 1$ *and for all* $r \in [0, R]$.

Proof. Let us consider a rotationally symemtric model space rearrangement of the metric ball $B_R(o)$ as it has ben described in Definitions 2.2.13 and 2.2.17, namely, a symmetrization of $B_R(o)$ which is a geodesic ball $B^{\omega}_{s(R)}(o_\omega)$ of $\mathbb{M}_\omega$ with radius $s(R)$ centered at o_ω such that $\operatorname{vol}(B_R(o)) = \operatorname{vol}\left(B^{\omega}_{s(R)}(o_\omega)\right)$, together the symmetrization $\mathbb{E}^{\omega\,*}_R : B^{\omega}_{s(R)}(o_\omega) \longrightarrow \mathbb{R}$ of the transplanted mean exit time function $\mathbb{E}^{\omega}_R : B_R(o) \longrightarrow \mathbb{R}$.

Then, from Theorems 3.4.4, 3.6.1 and 3.6.2, we have that

$$\begin{aligned} \mathcal{A}_1\left(B_R(o)\right) &= \int_{B_R(o)} E_R \, dV_g \leq (\geq) \int_{B_R(o)} \mathbb{E}^{\omega}_R \, dV_g = \int_{B^{\omega}_{s(R)}(o_\omega)} \mathbb{E}^{\omega\,*}_R \, dV_{g_\omega} \\ &\leq (\geq) \int_{B^{\omega}_{s(R)}(o_\omega)} E^{\omega}_{s(R)} \, dV_{g_\omega} = \mathcal{A}_1\left(B^{\omega}_{s(R)}(o_\omega)\right), \end{aligned} \tag{3.55}$$

which shows inequality (3.54).

Now, we study the equality case by assuming that $H_{S_r(o)} \geq H_{S^{\omega}_r(o_\omega)}$ for all $r \in (0, R]$. Suppose that $\mathcal{A}_1\left(B_R(o)\right) = \mathcal{A}_1\left(B^{\omega}_{s(R)}(o_\omega)\right)$, *i.e.*, equality in inequality (3.54). Then, all the inequalities of (3.55) become equalities. In particular, we have that $\int_{B_R(o)} E_R \, dV_g = \int_{B_R(o)} \mathbb{E}^{\omega}_R \, dV_g$ and that $\int_{B^{\omega}_{s(R)}(o_\omega)} \mathbb{E}^{\omega\,*}_R \, dV_{g_\omega} = \int_{B^{\omega}_{s(R)}(o_\omega)} E^{\omega}_{s(R)} \, dV_{g_\omega}$.

From this second equality we have that $\int_{B^\omega_{s(R)}(o_\omega)}\left(\mathbb{E}^{\omega\,*}_R - E^\omega_{s(R)}\right) dV_{g_\omega} = 0$ and since, by inequality (3.39) of Theorem 3.6.2, $\left(\mathbb{E}^{\omega\,*}_R - E^\omega_{s(R)}\right)(p) \leq 0$ for all $p \in B^\omega_{s(R)}(o_\omega)$, we have that $\mathbb{E}^{\omega\,*}_R = E^\omega_{s(R)}$ in $B^\omega_{s(R)}(o_\omega)$. Therefore, from assertions (1), (2) and (3) of the equality case of Theorem 3.6.2, we deduce that $s(R) = R$, $\operatorname{vol}(B_r(o)) = \operatorname{vol}(B^\omega_r(o_\omega))$ and $\operatorname{vol}(S_r(o)) = \operatorname{vol}(S^\omega_r(o_\omega))$ for all $r \in (o, R]$, and that $H_{S_r(o)} = H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$, showing assertions (1), (2) and (3).

Finally, since $H_{S_r(o)} = H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$, by Proposition 3.4.3, we have that $E_R = \mathbb{E}^\omega_R$ on $B_R(o)$ and then, by assertions (2) and (4) of the equality case of Theorem 3.5.3, we obtain equalities $u_{k,r} = \widetilde{u}^\omega_{k,r}$ on $B_r(o)$ and equalities $\mathcal{A}_k(B_r(o)) = \mathcal{A}_k(B^\omega_r(o_\omega))$ for all $k \geq 1$ and for all $r \in [0, R]$, which shows assertions (4) and (5). Another way to see this consists, as we did above, in deduce that $E_R = \mathbb{E}^\omega_R$ from equality $\int_{B_R(o)} E_R\, dV_g = \int_{B_R(o)} \mathbb{E}^\omega_R\, dV_g$ and apply Theorem 3.5.3.

When we assume that $H_{S_r(o)} \leq H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$, the equality discussion is the same, *mutatis mutandis.*

□

This theorem give the following consequence.

Corollary 3.6.4. *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be a n-dimensional rotationally symmetric model space balanced from above with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ and $B^\omega_R(o_\omega)$ be, respectively, a geodesic ball of M with radius R centered at o and a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$, that there exists the Schwarz symmetrization $B^\omega_{s(R)}(o_\omega)$ of $B_R(o)$ in $\mathbb{M}_\omega$, and moreover that*

$$H_{S_r(o)} \geq H_{S^\omega_r(o_\omega)} \quad \text{for all} \quad r \in (0, R].$$

Then, we have that

$$\mathcal{A}_1(B_R(o)) \leq E^\omega_{s(R)}(0) \operatorname{vol}(B_R(o)). \tag{3.56}$$

where $E^\omega_{s(R)}$ is the mean exit time function on $B^\omega_{s(R)}(o_\omega)$.

Proof. Assuming that $H_{S_r(o)} \geq H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$, we use inequality (3.55) of the proof of Theorem 3.6.3, and the fact that $E^\omega_{s(R)}(r) \leq E^\omega_{s(R)}(0)$ for all

$r \in (0, s(R)]$ (the mean exit time function of a geodesic ball of a rotationally symmetric model space is strictly decreasing, see Proposition 3.2.13), to obtain

$$\begin{aligned}\mathcal{A}_1\left(B_R(o)\right) &= \int_{B_R(o)} E_R(p)\, dV_g \leq \int_{B_R(o)} \mathbb{E}_R^\omega\left(r_o(p)\right) dV_g \\ &= \int_{B^\omega_{s(R)}(o_\omega)} \mathbb{E}_R^{\omega *}\left(r_{o_\omega}(q)\right) dV_{g_\omega} \leq \int_{B^\omega_{s(R)}(o_\omega)} E^\omega_{s(R)}\left(r_{o_\omega}(q)\right) dV_{g_\omega} \\ &\leq \int_{B^\omega_{s(R)}(o_\omega)} E^\omega_{s(R)}(0)\, dV_{g_\omega} = E^\omega_{s(R)}(0) \operatorname{vol}\left(B_R(o)\right).\end{aligned}$$

□

Remark 3.6.5. Since $(\mathbb{M}_\omega, g_\omega)$ is balanced from above, then $\frac{\partial}{\partial r} q_\omega(r)\big|_{r=t} \geq 0$, thus $q_\omega(r)$ is non-decreasing with r (see Proposition 2.2.10). Then, since $E^\omega_{s(R)}(r) = \int_r^{s(R)} q_\omega(t)\, dt$ (see Proposition 3.2.13), we have that

$$\begin{aligned}E^\omega_{s(R)}(0) &= \int_0^{s(R)} q_\omega(t)\, dt \leq \int_0^{s(R)} q_\omega\left(s(R)\right) dt \\ &= s(R) q_\omega\left(s(R)\right) = s(R) \frac{\operatorname{vol}\left(B^\omega_{s(R)}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{s(R)}(o_\omega)\right)}\end{aligned}$$

and hence, from inequality (3.56), we obtain that

$$\mathcal{A}_1\left(B_R(o)\right) \leq \mathbb{E}^\omega_{s(R)}(0) \operatorname{vol}\left(B_R(o)\right) \leq s(R) \frac{\operatorname{vol}\left(B^\omega_{s(R)}(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_{s(R)}(o_\omega)\right)} \operatorname{vol}\left(B_R(o)\right).$$

Example 3.6.6. To end this chapter, let us apply our comparisons for the mean exit time function and for the torsional rigidity to the Riemannian manifold of our Example 3.3.6, that is $(\mathbb{R}^2, g)$ with metric tensor expressed in a system of polar coordinates $(\mathbb{R}^2, \psi = (r, \theta))$ given by

$$g = dr \otimes dr + \varphi^2(r, \theta) d\theta \otimes d\theta,$$

where $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a positive smooth function given by

$$\varphi(r, \theta) = r \left(1 + \frac{r^2}{1 + r^2 \cos^2(\theta)}\right).$$

On the other hand, let us consider the simply connected real space form $(\mathbb{R}^2, g_{\text{can}})$ as a 2-dimensional rotationally symmetric model space $(\mathbb{R}^2, g_{\omega_0})$ where $\omega_0(r) = r$. Then, as we showed in Example 3.3.6, we have that g is smooth in the entire $\mathbb{R}^2$ and that the mean curvatures of the geodesic spheres $S_r(\vec{0})$ of $(\mathbb{R}^2, g)$ with radius r centered at $\vec{0}$ are bounded from below by the mean curvatures of the geodesic spheres $S_r^{\omega_0}(\vec{0})$ of $(\mathbb{R}^2, g_{\omega_0})$ with the same radius r centered at $\vec{0}$, namely, that

$$H_{S_r(\vec{0})} \geq H_{S_r^{\omega_0}(\vec{0})} = \frac{1}{r}.$$

Then, applying our comparison given in Theorem 3.4.4, we have that the mean exit time function E_R defined on geodesic ball $B_R(\vec{0})$ of $(\mathbb{R}^2, g)$ with radius $R > 0$ centered at $\vec{0}$ is bounded from above by the transplanted $\mathbb{E}_R^{\omega_0}$ of the mean exit time function $E_R^{\omega_0}$ defined on a geodesic ball of $(\mathbb{R}^2, g_{\omega_0})$ with the same radius R centered at 0. Namely,

$$\begin{aligned} E_R(p) \leq \mathbb{E}_R^{\omega_0}(p) = E_R^{\omega_0}(r(p)) &= \int_{r(p)}^{R} \frac{\int_0^s \omega_0(\sigma) d\sigma}{\omega_0(s)} \, ds \\ &= \int_{r(p)}^{R} \frac{\int_0^s \sigma d\sigma}{s} \, ds = \frac{R^2 - r^2(p)}{4}, \end{aligned}$$

where r is the radial distance function on $B_R(\vec{0})$ to $\vec{0}$. Furthermore, since $(\mathbb{R}^2, g_{\omega_0})$ is balanced from above (see Example 2.2.11.(3)), we can apply Theorem 3.6.3 to compare the torsional rigidity $\mathcal{A}_1(B_R(\vec{0}))$ of $B_R(\vec{0})$ with the torsional rigidity $\mathcal{A}_1(B_{s(R)}^{\omega_0}(\vec{0}))$ of its Schwarz symmetrizations $B_{s(R)}^{\omega_0}(\vec{0})$ in $(\mathbb{R}^2, g_{\omega_0})$. But first note that, since

$$\operatorname{vol}(B_R(\vec{0})) = \operatorname{vol}(B_{s(R)}^{\omega_0}(\vec{0})) = \pi \left(s(R)\right)^2,$$

we have that the symmetrized radius is

$$s(R) = +\sqrt{\frac{\operatorname{vol}(B_R(\vec{0}))}{\pi}}.$$

Then, applying Theorem 3.6.3, we have that

$$\begin{aligned} \mathcal{A}_1(B_R(\vec{0})) \leq \mathcal{A}_1(B_{s(R)}^{\omega_0}(\vec{0})) &= \int_0^R \int_0^{2\pi} E_{s(R)}^{\omega_0}(r) d\theta \, dr \\ &= \int_0^{s(R)} \int_0^{2\pi} \frac{(s(R))^2 - r^2}{4} \, d\theta dr = \frac{\pi}{3}(s(R))^3 = \frac{\operatorname{vol}(B_R(\vec{0}))^{3/2}}{3\sqrt{\pi}}. \end{aligned}$$

Finally, note that, since the sectional curvatures of $(\mathbb{R}^2, g)$ are not bounded by the sectional curvatures of $(\mathbb{R}^2, g_{\omega_0})$ (see Example 3.3.6), then this upper bound for the torsional rigidity $\mathcal{A}_1(B_R(\vec{0}))$ can not be obtained by using the comparisons of A. Hurtado, S. Markvorsen and V. Palmer in [37] (see Theorem 3.3.4).

Chapter 4

Comparisons for the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls

Along this chapter, we are going to establish comparisons for the geometric invariant called *first eigenvalue of the Laplacian for the Dirichlet problem* in geodesic balls of a given Riemannian manifold, and moreover, we will study its relationship with the moment spectrum. All our results presented along this chapter can be found in [28] and [63]. The importance of studying the first eigenvalue of the Laplacian for the Dirichlet problem is due, for instance, to the following:

The set of all the *eigenvalues of the Laplacian* takes part in the mathematical description of some properties of physical phenomena as the light, heat, sound, fluids and atomic phenomena. For instance, as a consequence of the heat equation, the *eigenvalues* of the Laplacian give the rates of temporal decay of its *eigenfunctions* over time. For a *drum* with certain given shape, the *eigenvalues of the Laplacian for the Dirichlet problem* are the fundamental modes of vibration of the drum. In fact, if we think of a drum as an elastic membrane (planar domain with fixed boundary), there appear the following problem: given the eigenvalues of the Laplacian for the Dirichlet problem on the drum, what characteristics of the shape of the drum can be deduced? This problem became famous thanks to

M. Kac's paper [42] and is often stated as the title of his paper: can one hear the shape of a drum?

We start this chapter by presenting some preliminary concepts, giving the definition of the *first eigenvalue of the Laplacian for the Dirichlet problem* posed on a precompact domain Ω in a Riemannian manifold M, and moreover, studying its properties in a complete Riemannian manifold and, in particular, in rotationally symmetric model spaces, see Section 4.1. Next, in Section 4.2, we will show some of the different directions that research has taken in this area, as well as some of the results obtained along the last years. Then, in Section 4.3, we will describe our new technique which we will use to prove our comparison results. In fact, we are going to symmetrize the metric tensor in such a way that the area functions coincide. Sections 4.4 and 4.5 are devoted to state and prove our upper bounds for the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls of a Riemannian manifold. Finally, in Section 4.6, we will show some relationships between the first eigenvalue and the moment spectrum.

4.1 First eigenvalue of the Laplacian for the Dirichlet problem on Riemannian manifolds

Let us start this section by defining the *Dirichlet eigenvalue problem* on precompact connected domains of Riemannian manifolds. For further information about the concepts defined along this section we refer to Sections 3 and 5 of Chapter 1 of I. Chavel [8] and Section 3 of Chapter VI of T. Sakai [68].

Definition 4.1.1 (see [8])**.** *Let (M, g) be an n-dimensional Riemannian manifold. Given $\Omega \subseteq M$ a precompact connected domain, the* Dirichlet eigenvalue problem *on Ω consists in to find all the real numbers λ such that there exists a non-trivial function $\phi \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$ that is a solution for the boundary valued problem*

$$\begin{aligned} \Delta_g \phi + \lambda \phi = 0, \quad &on \quad \Omega, \\ \phi|_{\partial\Omega} = 0. \end{aligned} \tag{4.1}$$

where Δ_g is the Laplacian operator with respect to the metric tensor g (see equation (2.12)*).*

The real numbers λ are called the eigenvalues *of Δ_g, their corresponding solutions for the problem, ϕ, are called the* eigenfunctions associated to λ*, the set L of all the eigenfunctions associated to one eigenvalue λ is called* eigenspace associated to λ *and the dimension of the eigenspace L associated to λ is called the* multiplicity of λ.

It can be proved the following result, which describes the set of all the eigenvalues and the eigenspaces.

Theorem 4.1.2 (see Theorem 1 of Chapter 1 of [8], Theorem 3.21 of [10], and Theorem 3.7 of Chapter VI of [68])**.** *Let (M, g) be an n-dimensional Riemannian manifold and let $\Omega \subseteq M$ be a precompact connected domain. Then, all the eigenvalues of the Laplacian for the Dirichlet problem are positive, the set of all the eigenvalues is discrete and their multiplicities are finite. Moreover, the eigenvalues can be ordered as*

$$0 < \lambda_{1,g}(\Omega) < \lambda_{2,g}(\Omega) < \lambda_{3,g}(\Omega) < \cdots ,$$

Therefore, $\{\lambda_{i,g}(\Omega)\}_{i=1}^{\infty}$ is discrete and $\lim_{i \mapsto \infty} \lambda_{i,g}(\Omega) = +\infty$. Furthermore, the eigenspaces L_i, $i = 1, 2, \ldots$, associated to each $\lambda_{i,g}(\Omega)$ are orthogonal in $L^2(\Omega)$ to each other with respect to the usual inner product $(f, h) = \int_\Omega f h \, dV_g$ on $L^2(\Omega)$, and their direct sum is dense in $L^2(\Omega)$. Moreover, each eigenfunction is smooth in Ω.

Thus, since all the eigenvalues of the Laplacian for the Dirichlet problem can be ordered as in the previous theorem, it makes sense to talk about the first eigenvalue as follows.

Definition 4.1.3 (see [68])**.** *Let (M, g) be an n-dimensional Riemannian manifold. Given $\Omega \subseteq M$ a precompact connected domain, the* first eigenvalue of the Laplacian for the Dirichlet problem in Ω *is the smallest of the positive real values such that there is a non-trivial function $\phi \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ that is a solution for the Dirichlet problem* (4.1)*. Moreover, we will denote it as $\lambda_{1,g}(\Omega)$.*

Now, let us show some results that characterize the eigenfunctions associated with the first eigenvalue of the Laplacian for the Dirichlet problem.

Lemma 4.1.4 (see Lemma 3.10 of Chapter VI of [68]). *Let (M, g) be an n-dimensional Riemannian manifold an let $\Omega \subseteq M$ be a precompact connected domain. Let $\phi \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ be an eigenfunction associated to $\lambda_{1,g}(\Omega)$,* i.e., *the first eigenvalue of the Laplacian for the Dirichlet problem in Ω. Then ϕ is either always a positive function or always a negative function in Ω.*

Corollary 4.1.5 (see Corollary 2 of Chapter I of [8] and Corollary 3.11 of Chapter VI and page 270 of [68]). *Let (M, g) be an n-dimensional Riemannian manifold an let $\Omega \subseteq M$ be a precompact connected domain. Then, the first eigenvalue of the Laplacian for the Dirichlet problem in Ω is simple, namely, the multiplicity of $\lambda_{1,g}(\Omega)$ is equal to 1, and moreover, $\lambda_{1,g}(\Omega)$ is characterized as being the only eigenvalue with associated eigenfunctions of constant sign.*

Remark 4.1.6. From now on, we shall refer as *first eigenfunction* to an eigenfunction associated to the first eigenvalue $\lambda_{1,g}(\Omega)$ and we denote it by ϕ_1.

Observe that, from Lemma 4.1.4, we can assume that $\phi_1 > 0$. In fact, if we choose $\phi_1 < 0$, we can define a function $\widetilde{\phi}_1 = -\phi_1$ which is also an eigenfunction associated to $\lambda_{1,g}(\Omega)$. And moreover, from Corollary 4.1.5, we know that the first eigenfunctions are the unique eigenfunctions that do not change sign in Ω.

Now, we are going to state two results which we will use to find our upper bounds for the first eigenvalue of the Laplacian for the Dirichlet problem at Theorems 4.4.1 and 4.5.1. The first one is the *Rayleigh's Theorem* and the second is the so-called *Barta's Lemma*, for more detailed information about this results see [8] and [68], for instance.

Theorem 4.1.7 (Rayleigh's Theorem, see page 16 of [8]). *Let (M, g) be an n-dimensional Riemannian manifold and let $\Omega \subseteq M$ be a precompact connected domain. Let $\lambda_{1,g}(\Omega)$ be the first eigenvalue of the Laplacian for the Dirichlet problem in Ω. Then, for any non-trivial $\phi \in C_0^\infty(\Omega) \cap C^0(\overline{\Omega})$, we have that*

$$\lambda_{1,g}(\Omega) \leq \frac{\int_\Omega g\left(\nabla^M \phi, \nabla^M \phi\right)\, dV_g}{\int_\Omega \phi^2\, dV_g}, \tag{4.2}$$

Furthermore, equality in inequality (4.2) *is attained if, and only if, ϕ is a first eigenfunction ϕ_1 associated to $\lambda_{1,g}(\Omega)$.*

Lemma 4.1.8 (Barta's Lemma, see Lemma 1 of Chapter III of [8] (or see J. Barta [4] for the original paper))**.** *Let (M, g) be an n-dimensional Riemannian manifold and let $\Omega \subseteq M$ be a precompact connected domain. Let $\lambda_{1,g}(\Omega)$ be the first eigenvalue of the Laplacian for the Dirichlet problem in Ω and let $\phi \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that $\phi > 0$ in Ω and $\phi|_{\partial\Omega} = 0$. Then*

$$\inf_{\Omega}\left(\frac{-\Delta^M \phi}{\phi}\right) \leq \lambda_{1,g}(\Omega) \leq \sup_{\Omega}\left(\frac{-\Delta^M \phi}{\phi}\right). \tag{4.3}$$

Furthermore, equality in some of the inequalities (4.3) *is attained if, and only if, ϕ is a first eigenfunction ϕ_1 associated to $\lambda_{1,g}(\Omega)$, and hence, equality in one of the inequalities implies equality in the other.*

4.1.1 First eigenvalue of the Laplacian for the Dirichlet problem on rotationally symmetric model spaces

It can be proved that the first eigenfunction on geodesic balls of rotationally symmetric model spaces have, besides being positive, some more properties.

Proposition 4.1.9. *Let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $B^\omega_R(o_\omega)$ be a geodesic ball with radius R centered at o_ω. Suppose that $R < \mathrm{inj}_{g_\omega}(o_\omega)$. Then, any positive first eigenfunction ϕ_1 of the Laplacian Δ_{g_ω} for the Dirichlet problem in $B^\omega_R(o_\omega)$ is a radial function,* namely, *there is a positive smooth function $f_1 : [0, R) \longrightarrow \mathbb{R}_+$, which is continuous at R with $f(R) = 0$, such that $\phi_1(p) = f_1(r(p))$ for all $p \in B^\omega_R(o_\omega)$, where r is the radial distance function to the center o_ω (see Definition 2.1.69).*

Furthermore, f_1 satisfies that $f_1'(0) = 0$ and $f_1'(t) < 0$ for all $t \in (0, R)$.

Proof. Let $\phi_1 \in C^\infty_0(B^\omega_R(o_\omega)) \cap C^0(\overline{B^\omega_R(o_\omega)})$ be any positive first eigenfunction of the Laplacian Δ_{g_ω} for the Dirichlet problem on a geodesic ball $B^\omega_R(o_\omega))$ of $\mathbb{M}_\omega$ with radius $R < \mathrm{inj}_{g_\omega}(o_\omega)$ centered at $o_\omega \in \mathbb{M}_\omega$. Then, since the metric tensor g_ω is invariant under a rotation around the center o_ω, *i.e.*, remains invariant under the action of the orthogonal group (see [32]), and applying Lemma 7 of [14], we have that ϕ_1 is a radial function. Namely, there exists a positive real valued smooth function $f : [0, R] \longrightarrow \mathbb{R}_+$, $f_1 \in C^\infty([0, R)) \cap C^0([0, R])$, with $f(R) = 0$ such that we can rewrite the first eigenfunction as $\phi_1(p) = f_1\left(r(p)\right)$ for all $p \in B^\omega_R(o_\omega)$.

Moreover, by using that ϕ_1 is a radial function, it is know that $f_1'(0) = 0$. Indeed, since

$$\nabla_{g_\omega}\phi_1(p) = \nabla_{g_\omega} f_1(r(p)) = f_1'(r(p))\nabla_{g_\omega} r(p) \quad \text{for all} \quad p \in B_R^\omega(o_\omega).$$

Therefore, given normal coordinates $(B_R^\omega(o_\omega), \zeta)$ with normal coordinate functions $\{x^i\}_{i=1}^n$, we have that $\{\partial/\partial x^i\}_{i=1}^n$ forms an orthonormal basis of T_pM, and hence, from assertion (4) of Proposition 2.1.70, we obtain that

$$\nabla_{g_\omega}\phi_1(p) = f_1'\left(r(p)\right)\sum_{i=1}^n \frac{x^i}{r(p)}\left.\frac{\partial}{\partial x^i}\right|_p \qquad \text{for all} \quad p \in B_R^\omega(o_\omega).$$

Thus, for the curve $\gamma(t) = (t\cos\theta, t\sin\theta, 0, \ldots, 0)$, we have that $r(p) = t$ for all $\theta \in [0, 2\pi)$, and hence, we obtain that

$$\nabla_{g_\omega}\phi_1\left(\gamma(t)\right) = f_1'(t)\left(\cos\theta \left.\frac{\partial}{\partial x^1}\right|_p + \sin\theta \left.\frac{\partial}{\partial x^2}\right|_p\right).$$

In particular, for $\gamma(0) = o_\omega$, we have that

$$\nabla_{g_\omega}\phi_1(o_\omega) = f_1'(0)\left(\cos\theta \left.\frac{\partial}{\partial x^1}\right|_{o_\omega} + \sin\theta \left.\frac{\partial}{\partial x^2}\right|_{o_\omega}\right).$$

Therefore, since $\nabla_{g_\omega}\phi_1(o_\omega)$ does not depend on the chosen $\theta \in [0, 2\pi)$ and since

$$\begin{cases} \nabla_{g_\omega}\phi_1(o_\omega) & = f_1'(0)\left.\dfrac{\partial}{\partial x^1}\right|_{o_\omega}, \quad \text{if} \quad \theta = 0, \\[2ex] \nabla_{g_\omega}\phi_1(o_\omega) & = {f_1}'(0)\left.\dfrac{\partial}{\partial x^2}\right|_{o_\omega}, \quad \text{if} \quad \theta = \dfrac{\pi}{2}, \end{cases}$$

we obtain that f_1' must vanish at 0, namely, $f_1'(0) = 0$.

On the other hand, from the expression (2.32) of the Laplacian in rotationally symmetric model spaces, an easy computation leads to

$$\Delta_{g_\omega}\phi_1(p) = f_1''\left(r(p)\right) + (n-1)\frac{\omega'\left(r(p)\right)}{\omega\left(r(p)\right)} f_1'\left(r(p)\right) = -\lambda_{1,g_\omega}\left(B_R^\omega(o_\omega)\right) f_1\left(r(p)\right),$$

for all $p \in B_R^\omega(o_\omega)$, where $\lambda_{1,g_\omega}\left(B_R^\omega(o_\omega)\right) > 0$ is the first eigenvalue of the Laplacian for the Dirichlet problem in $B_R^\omega(o_\omega)$. Hence, for any $r \in [0, R]$,

$$f_1''(r) + (n-1)\frac{\omega'(r)}{\omega(r)} f_1'(r) = -\lambda_{1,g_\omega}\left(B_R^\omega(o_\omega)\right) f_1(r). \tag{4.4}$$

Furthermore, suppose that there is a $r_0 \in (0,R)$ such that $f_1'(r_0) = 0$ then, from (4.4) and since $\lambda_{1,g_\omega}\left(B_R^\omega(o_\omega)\right) > 0$ and $f_1(r) > 0$ for all $r \in [0,R)$, we obtain that

$$f_1''(r_0) = -\lambda_{1,g_\omega}\left(B_R(o_\omega)\right) f_1(r_0) < 0.$$

Therefore, we obtain that f_1 have a relative maximum at r_0, and hence, all the critical points of f_1 are relative maximums.

Finally, since for a real valued smooth function we know that between two relative maximums there is at least one relative minimum, we have that f_1 has only one relative maximum in $[0,R)$. Then, since $f_1'(0) = 0$, we have that 0 is the only critical point of f_1 and it is a relative maximum. Therefore, f_1 is a decreasing function in $(0,R)$, *i.e.*, $f_1'(r) < 0$ for all $r \in (0,R)$.

□

4.2 Some Background

Upper and lower bounds for the first eigenvalue of the Laplacian for the Dirichlet problem on precompact domains Ω of a Riemannian manifold (M,g) have been widely studied in the literature. Let us first enumerate some of these known results.

S.Y. Cheng in [12] and [13] obtained upper (respectively, lower) bounds for the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls of Riemannian manifolds with Ricci curvatures bounded from below by the Ricci curvatures of the simply connected real space forms $\mathbb{K}^n(\kappa)$ with constant sectional curvature κ (respectively, sectional curvatures bounded from above), by comparing it with the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls, with the same radius, of $\mathbb{K}^n(\kappa)$.

Theorem 4.2.1 (see Theorem 1.1 of [13]). *Let (M,g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_{\omega_\kappa}, g_{\omega_\kappa})$ be an n-dimensional simply connected real space form of constant sectional curvature κ with center o_{ω_κ}. Suppose that the Ricci curvatures of M,* Ric_g*, are bounded from below by the Ricci curvatures of* $\mathbb{M}_{\omega_\kappa}$, i.e.,

$$\mathrm{Ric}_g \geq (n-1)\kappa,$$

then the first eigenvalue $\lambda_{1,g}(B_R(o))$ *of the Laplacian for the Dirichlet problem of a geodesic ball* $B_R(o)$ *of* M *with radius* R *centered at any point* $o \in M$ *is bounded from above by*

$$\lambda_{1,g}(B_R(o)) \leq \lambda_{1,g_{\omega_\kappa}}(B_R^{\omega_\kappa}), \tag{4.5}$$

where $\lambda_{1,g_{\omega_\kappa}}(B_R^{\omega_\kappa})$ *is the first Dirichlet eigenvalue of a geodesic ball* $B_R^{\omega_\kappa}$ *of* $\mathbb{M}_{\omega_\kappa}$ *with radius* R *centered at the center* o_{ω_κ}*. Moreover, equality in* (4.5) *is attained if, and only if,* $B_R(o)$ *is isometric to* $B_R^{\omega_\kappa}$*.*

Theorem 4.2.2 (see Theorem 3.6 of [12])**.** *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold and let* $(\mathbb{M}_{\omega_\kappa}, g_{\omega_\kappa})$ *be an* n*-dimensional simply connected real space form of constant sectional curvature* κ *with center* o_{ω_κ}*. Suppose that the sectional curvatures of* M*,* $\sec_g$*, are bounded from above by the sectional curvatures of* $\mathbb{M}_{\omega_\kappa}$*,* i.e.,

$$\sec_g \leq \kappa,$$

then the first eigenvalue $\lambda_{1,g}(B_R(o))$ *of the Laplacian for the Dirichlet problem of a geodesic ball* $B_R(o)$ *of* M *with radius* $R < \min\{\mathrm{inj}_g(o), \pi/\sqrt{\kappa}\}$ *(where* $\pi/\sqrt{\kappa}$ *is replaced by* $+\infty$ *if* $\kappa \leq 0$*) centered at any point* $o \in M$ *is bounded from below by*

$$\lambda_{1,g}(B_R(o)) \geq \lambda_{1,g_{\omega_\kappa}}(B_R^{\omega_\kappa}), \tag{4.6}$$

where $\lambda_{1,g_{\omega_\kappa}}(B_R^{\omega_\kappa})$ *is the first Dirichlet eigenvalue of a geodesic ball* $B_R^{\omega_\kappa}$ *of* $\mathbb{M}_{\omega_\kappa}$ *with radius* R *centered at the center* o_{ω_κ}*. Moreover, equality in* (4.6) *is attained if, and only if,* $B_R(o)$ *is isometric to* $B_R^{\omega_\kappa}$*.*

Remark 4.2.3. Note that inequalities (4.5) and (4.6) are sharp because equality is attained in both inequalities if, and only if, $B_R(o)$ is isometric to the geodesic ball of radius R in $\mathbb{M}_{\omega_\kappa}$.

More recently, A. Hurtado, S. Markvorsen and V. Palmer generalized in [39] S.Y. Cheng's result by proving that if $\sec_g(\sigma(\nabla_g r_o, \cdot)) \leq (\geq) \sec_{g_\omega}(\sigma(\nabla_{g_\omega} r_{o_\omega}))$ then $\lambda_{1,g}(B_R(o)) \geq (\leq) \lambda_{1,g_\omega}(B_R^\omega(o_\omega))$ where $B_R(o_\omega)$ is a geodesic ball of a rotationally symmetric model space $(\mathbb{M}_\omega, g_\omega)$ with center $o_\omega \in \mathbb{M}_\omega$. In particular, they showed bounds for the first eigenvalue on extrinsic balls and, from these bounds, they recover the intrinsic case (see Theorems 8 and 9 of [39]).

On the other hand, G.P Bessa and J.F Montenegro in [5] have obtained the same upper and lower bounds for the first eigenvalue of the Laplacian for the Dirichlet problem of geodesic balls of Riemannian manifolds. But they use the same control on the behaviour of mean curvatures of the geodesic sphere that we have used along Chapter 3.

Theorem 4.2.4 (see Theorem 1.1 of [5])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_{\omega_\kappa}, g_{\omega_\kappa})$ be an n-dimensional simply connected real space form of constant sectional curvature κ with center o_{ω_κ}. Suppose that the mean curvatures $H_{S_r(o)}$ of the geodesic spheres $S_r(o)$ of M with radius r centered at any point $o \in M$ are bounded from below (resp. from above) by the mean curvatures $H_{S_r^{\omega_\kappa}}$ of the geodesic spheres $S_r^{\omega_\kappa}$ of $\mathbb{M}_{\omega_\kappa}$ with the same radius r,* i.e.,

$$H_{S_r(o)}(q) \geq (\leq) H_{S_r^{\omega_\kappa}}(r)$$

for all point $q \in S_r(o)$ and for all $r \in (0, R]$. Then the first eigenvalue $\lambda_{1,g}(B_R(o))$ of the Laplacian for the Dirichlet problem of a geodesic ball $B_R(o)$ of M with radius $R < \min\{\mathrm{inj}_g(o), \pi/\sqrt{\kappa}\}$ (where $\pi/\sqrt{\kappa}$ is replaced by $+\infty$ if $\kappa \leq 0$) centered at $o \in M$ is bounded from below (resp. above) by

$$\lambda_{1,g}(B_R(o)) \geq (\leq) \lambda_{1,g_{\omega_\kappa}}(B_R^{\omega_\kappa}) \tag{4.7}$$

Moreover, equality in (4.7) is attained if, and only if, $H_{S_r(o)}(q) = H_{S_r^{\omega_\kappa}}(r)$ for all $q \in S_r(o)$ and for all $r \in (0, R]$.

Remark 4.2.5. Note that in this case, as in the case where is assumed lower bounds for the Ricci or upper bounds for the sectional curvatures, inequality (4.7) is sharp. But now, instead of an isometry between geodesic balls, equality is attained in (4.7) if, and only if, $H_{S_r(o)} = H_{S_r^{\omega_\kappa}}$ for all $r \in (0, R]$. Observe that the conclusion of equality between the mean curvatures of geodesic spheres does not imply isometry between the geodesic balls. Indeed, in Example 5.3 of [6], G.P. Bessa, V. Gimeno and L. Jorge showed a 4-dimensional geodesic ball non-isometric to the geodesic ball of $\mathbb{M}^4_{\omega_\kappa}$, but with $H_{S_r(o)} = H_{S_r^{\omega_\kappa}}$ for all $r \in (0, R]$. See too Example 3.3.6 and the argument above the example.

An alternative way to obtain bounds for the first eigenvalue of the Laplacian for the Dirichlet problem makes use of the isoperimetric inequalities and the so-called symmetrizations. In fact, the following results make use of the *Schwarz* *symmetrization* which we have already defined and studied in Subsection 2.2.3 (and used to prove our statements along Chapter 3).

The first result in this direction is due to G. Faber and E. Krahn which in [25, 47] showed a lower bound for the first eigenvalue of the Laplacian for the Dirichlet problem on a general kind of domains of Riemannian manifolds (see [8]) by comparing it with the first eigenvalue of the Laplacian for the Dirichlet posed on the Schwarz symmetrization of those domains. We state this result for precompact and connected domains in a Riemannian manifold.

Theorem 4.2.6 (see Theorem 2 of Chapter IV of [8] (or see [25, 47] for the original papers))**.** *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_{\omega_\kappa}, g_{\omega_\kappa})$ be an n-dimensional simply connected space form of constant sectional curvature κ with center o_{ω_κ}. Let Ω be a precompact connected domain $\Omega \subseteq M$ and let $B^{\omega_\kappa}_{L(\Omega)}(o_{\omega_\kappa})$ be the Schwarz symmetrization of Ω in $\mathbb{M}_{\omega_\kappa}$. Suppose that for all precompact connected domains $\Omega \subseteq M$ we have that the volume equality implies the following inequality between the volumes of its perimeters*

$$\operatorname{vol}(\Omega) = \operatorname{vol}\left(B^{\omega_\kappa}_{L(\Omega)}(o_{\omega_\kappa})\right) \Longrightarrow \operatorname{vol}(\partial\Omega) \geq \operatorname{vol}\left(\partial B^{\omega_\kappa}_{L(\Omega)}(o_{\omega_\kappa})\right).$$

Then, for all precompact connected domains $\Omega \subset M$, the first eigenvalue $\lambda_{1,g}(\Omega)$ of the Laplacian for the Dirichlet problem of Ω is bounded from below by

$$\lambda_{1,g}(\Omega) \geq \lambda_{1,g_{\omega_\kappa}}\left(B^{\omega_\kappa}_{L(\Omega)}(o_{\omega_\kappa})\right), \tag{4.8}$$

Moreover, equality in (4.8) *is attained if, and only if, Ω is isometric to $B^{\omega_\kappa}_{L(\Omega)}(o_{\omega_\kappa})$.*

The first eigenvalue of a precompact domain in a Riemannian manifold can be computed by means of the *moment spectrum* of D. In [57], P. McDonald and R. Meyers proved that the moment spectrum of a precompact domain Ω can be used to compute the first eigenvalue of the Laplacian for the Dirichlet problem on Ω as follows.

Theorem 4.2.7 (see equation (3.1) of [57])**.** *Let (M, g) be a complete n-dimensional Riemannian manifold and let $\Omega \subset M$ be a precompact connected domain. Let $\{\mathcal{A}_k(\Omega)\}_{k=1}^{\infty}$ be the moment spectrum of Ω. Then, the first eigenvalue of the Laplacian for the Dirichlet problem on Ω satisfies*

$$\lambda_{1,g}(\Omega) = \sup\left\{\eta \geq 0 \,:\, \lim_{k\to\infty}\sup\left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k(\Omega)}{\Gamma(k+1)} < \infty\right\}.$$

More recently, in the particular case when the precompact connected domain is a geodesic ball $B^\omega_R(o_\omega)$ of a rotationally symmetric model space $(\mathbb{M}_\omega, g_\omega)$ with radius R centered a the center $o_\omega \in \mathbb{M}_\omega$, A. Hurtado, S. Markvorsen and V. Palmer showed in [39] that the first eigenvalue of the Laplacian for the Dirichlet problem on $B^\omega_R(o_\omega)$ can be bounded, and computed, in terms of the Poisson hierarchy and the moment spectrum of $B^\omega_R(o_\omega)$ as in the following theorem.

Theorem 4.2.8 (see Proposition 2, Corollary 1 and Theorem A of [39])**.** *Let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional model space with center $o_\omega \in \mathbb{M}_\omega$ and let $B^\omega_R(o_\omega)$ be a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \operatorname{inj}_{g_\omega}(o_\omega)$ and let $\left\{u^\omega_{k,R}\right\}_{k=1}^\infty$ and $\{\mathcal{A}_k\left(B^\omega_R(o_\omega)\right)\}_{k=1}^\infty$ be, respectively, the Poisson hierarchy and the moment spectrum of $B^\omega_R(o_\omega)$. Then, for all $k \geq 0$, the first eigenvalue of the Laplacian for the Dirichlet problem on $B^\omega_R(o_\omega)$ is bounded from above, and from below, by*

$$(k+1)\frac{u^\omega_{k,R}(0)}{u^\omega_{k+1,R}(0)} \leq \lambda_{1,g_\omega}\left(B^\omega_R(o_\omega)\right) \leq (k+1)\frac{\mathcal{A}_k\left(B^\omega_R(o_\omega)\right)}{\mathcal{A}_{k+1}\left(B^\omega_R(o_\omega)\right)}. \tag{4.9}$$

In particular, when $k = 0$, we have that

$$\frac{1}{E^\omega_R(0)} = \frac{1}{\int_0^R q_\omega(r)dr} \leq \lambda_{1,g_\omega}\left(B^\omega_R(o_\omega)\right) \leq \frac{\operatorname{vol}\left(B^\omega_R(o_\omega)\right)}{\mathcal{A}_1\left(B^\omega_R(o_\omega)\right)}, \tag{4.10}$$

where E^ω_R is the mean exit time function on $B^\omega_R(o_\omega)$ and q_ω is the isoperimetric quotient

And moreover, the first eigenvalue can be computed by

$$\lambda_{1,g_\omega}\left(B^\omega_R(o_\omega)\right) = \lim_{k\to\infty}(k+1)\frac{u^\omega_{k,R}(0)}{u^\omega_{k+1,R}(0)} = \lim_{k\to\infty}(k+1)\frac{\mathcal{A}_k\left(B^\omega_R(o_\omega)\right)}{\mathcal{A}_{k+1}\left(B^\omega_R(o_\omega)\right)}. \tag{4.11}$$

Remark 4.2.9. The lower bound in equality (4.10) was proved first by C. Betz, G. Cámera and H. Gzyl for geodesic balls of the n-dimensional unit sphere $\mathbb{S}^n(1)$ (see Theorem 2.1 of [7]) and several years later it was generalized by C.S. Barroso and G.P. Bessa for geodesic balls of rotationally symmetric model spaces $\mathbb{M}_\omega$ (see Theorem 1.1. of [3]). Note moreover that, since we have, from Theorem 3.2.13, that $E^\omega_R(0) = \max_{B^\omega_R(o_\omega)}(E^\omega_R)$, we have that the left part of inequality (4.10) is related with the work of G. Del Grosso and F. Marchetti. In fact, they proved in

[18] that, given a precompact domain $\Omega \subset M$ of a Riemannian manifold (M, g), the first eigenvalue is lower bounded by

$$\lambda_{1,g}(\Omega) \geq \frac{1}{\max_\omega(E_\Omega)}.$$

On the other hand, the upper bound in inequality (4.9) for domains in the Euclidean space $\mathbb{R}^n$ is the classical G. Pólya's inequality which relates, taking $k = 1$, the first eigenvalue of the Laplacian for the Dirichlet problem and the torsional rigidity of the domain (see Section 5.2 of [67] for results on planar regions of $\mathbb{R}^n$ of the classical inequality and see Proposition 2.3 of [72] for the proof on precompact domains of $\mathbb{R}^n$). Furthermore, the upper bound in inequality (4.9) was generalized by E.B. Dryden, J.J. Langford and P. McDonald in [21] for a general bounded domain for k being even, and moreover, they also established upper bounds for the first eigenvalue on bounded domains Ω of a Riemannian manifold (M, g) by using the moment spectrum, the volume and the variance $\mathrm{Var}_k(\Omega) = \int_\Omega \left(u_{2k,\Omega} - u_{k,\omega}^2\right) dV_g$. We state this mentioned bounds in the following theorem.

Theorem 4.2.10 (see Theorems 1.1 and 1.2 and Corollary 3.1 of [21]). *Let (M, g) be a complete n-dimensional Riemannian manifold and let $\Omega \subset M$ be a bounded domain. Let $\{\mathcal{A}_k(\Omega)\}_{k=1}^\infty$ be the moment spectrum of Ω and let $\mathrm{Var}_k(\Omega) = \int_\Omega (u_{2k,\Omega} - u_{k,\Omega}^2) dV_g$. Then, the first eigenvalue of the Laplacian for the Dirichlet problem on Ω is bounded from above, for all $k \geq 0$, by*

$$\lambda_{1,g}(\Omega) \leq \frac{(2k+1)\mathcal{A}_{2k}(\Omega)}{A_{2k+1}(\Omega)},$$

$$\lambda_{1,g}(\Omega) \leq \frac{((k+1)!)^2}{(2k+1)!} \frac{\mathcal{A}_{2k+1}(\Omega)}{\mathcal{A}_{k+1}^2(\Omega)} \operatorname{vol}(\Omega),$$

$$\lambda_{1,g}(\Omega) \leq \frac{(2k+2)! - ((k+1)!)^2}{(2k+1)!} \frac{\mathcal{A}_{2k+1}(\Omega)}{\mathrm{Var}_{k+1}(\Omega)}.$$

Finally, in [6], G.P Bessa, V. Gimeno and L. Jorge generalized the equalities (4.11) of the above Theorem 4.2.8 to precompact domains Ω by showing that the first eigenvalue of the Laplacian for the Dirichlet problem of Ω can be computed in terms of its moment spectrum as above, and moreover, that it also can be computed in terms of the L^2-norm on Ω of the elements of its Poisson hierarchy as follows.

Theorem 4.2.11 (see Theorem 2.1 and Corollary 3.3 of [6]). *Let (M, g) be a complete n-dimensional Riemannian manifold and let $\Omega \subset M$ be a precompact connected domain. Let $\{u_{k,\Omega}\}_{k=1}^{\infty}$ and let $\{\mathcal{A}_k(\Omega)\}_{k=1}^{\infty}$ be, respectively, the Poisson hierarchy and the moment spectrum of Ω. Then, the first eigenvalue of the Laplacian for the Dirichlet problem on Ω can be computed by*

$$\lambda_{1,g}(\Omega) = \lim_{k\to\infty}(k+1)\frac{\mathcal{A}_k(\Omega)}{\mathcal{A}_{k+1}(\Omega)} = \lim_{k\to\infty}(k+1)\frac{\|u_{k,\Omega}\|_2}{\|u_{k+1,\Omega}\|_2},$$

where $\|\cdot\|_2$ denotes the L_2-norm on Ω.

In Section 4.4, given any complete Riemannian manifold (M, g), we will make use of the above Theorems 4.2.8, 4.2.10 and 4.2.11 to establish comparisons for the first eigenvalue of the Laplacian on geodesic balls of M. Moreover, we will show a S.Y. Cheng-type comparison in Section 4.5. Finally, in Section 4.6, we will make use of the above Theorems 4.2.8 and 4.2.11 to show that some of the equality cases of our comparisons for the Poisson hierarchy and the moment spectrum (which we proved along Chapter 3) characterizes the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls. Moreover, we use Theorem 4.2.7 to show that the first eigenvalue determines the Poisson hierarchy and the moment spectrum of geodesic balls. In fact, in Theorem 4.6.3, we give an alternative proof of Theorem 4.2.4 by using Theorem 4.2.7.

4.3 Volume-based rotational symmetrization of the metric tensor

In Sections 4.4 and 4.5, in order to prove or upper bounds for the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls, we shall define, given a Riemannian manifold (M, g) and a fixed point $o \in M$, a rotationally symmetric metric tensor $\widetilde{g}$ on M in such a way that the volumes of the geodesic spheres with respect to the metrics g and $\widetilde{g}$ are equal. Namely, given the geodesic ball $B_R(o)$ of M with radius $0 < R < \operatorname{inj}_g(o)$ centered at o, we shall have the equality

$\mathrm{vol}_g(S_r(o)) = \mathrm{vol}_{\widetilde{g}}(S_r(o))$ for all $r \in [0, R)$, *i.e.*, the area functions with respect to g and $\widetilde{g}$ satisfy $A_g(r) = A_{\widetilde{g}}(r)$ for all $r \in [0, R)$ (see equation (2.29) Remark 2.1.78 for the definition of the area function). In fact, this rotationally symmetric metric tensor will allow us to find a comparison for the Dirichlet eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls but without controlling the behaviour of the curvatures as in Cheng's and Bessa-Montenegro's Theorems 4.2.1, 4.2.2 and 4.2.4, and neither using an isoperimetric condition as hypothesis as in Faber and Krahn Theorem 4.2.6. Furthermore, in Theorems 4.4.1 and 4.5.1, we will use this technique to characterize the equality case under some condition on the mean curvatures of the geodesic spheres as in Bessa-Montenegro's Theorem 4.2.4 (in Theorem 4.6.3 we give an alternative proof of Bessa-Montenegro's result).

Along this section, we construct the rotationally symmetric metric tensor $\widetilde{g}$ which we called *rotationally symmetric metric tensor of comparison*, and moreover, we show some of its properties.

Definition 4.3.1. *Let (M, g) a complete n-dimensional Riemannian manifold and let o be a point of M. Given $B_R(o)$ the geodesic ball of radius $R < \mathrm{inj}_g(o)$ centered at o, we define the* rotationally symmetric metric tensor of comparison $\widetilde{g}$ associated to g on $B_R(o)$ *as the metric tensor given by*

$$\widetilde{g} = \begin{cases} dr \otimes dr + \left(\omega_g^2 \circ r\right) \pi^* g_{\mathbb{S}_1^{n-1}}, & on \quad B_R(o) - \{o\}, \\ \displaystyle\sum_{i=1}^{n} dx^i \otimes dx^i, & on \quad o, \end{cases} \tag{4.12}$$

where $\{x^i\}_{i=1}^n$ are the normal coordinate functions of a system of normal coordinates $(B_R(o), \zeta)$ (see Definition 2.1.65), r and π are, respectively, the radial distance function to o and the projection to $\mathbb{S}_1^{n-1}$ (see Definitions 2.1.69 and 2.1.71, respectively), and $\omega_g : [0, R) \longrightarrow \mathbb{R}_+$ is the positive function given by

$$t \longrightarrow \omega_g(r) := \left(\frac{A_g(r)}{\mathrm{vol}\left(\mathbb{S}_1^{n-1}\right)} \right)^{\frac{1}{n-1}} \tag{4.13}$$

where $A_g(r)$ is the area function of the geodesic spheres $S_r(o)$ of M with radius r centered at o.

Remark 4.3.2. Observe that $B_R(o)$ is rotationally symmetric with respect to $\widetilde{g}$. Namely, $(B_R(o), \widetilde{g})$ is a rotationally symmetric model space with center o, radius $\Lambda = R$ and warping function ω_g given by (4.13), and hence, the geodesic ball $B_R(o)$ together the rotationally symmetric metric tensor of comparison $\widetilde{g}$ have all the properties of the rotationally symmetric model spaces (see Section 2.2 to check these properties) including all the results on the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls of rotationally symmetric model spaces (see Section 4.2 to check this mentioned results), we will show these properties and results along this section.

Thus, try to find comparisons between geometric invariants of $(B_R(o), g)$ and $(B_R(o), \widetilde{g})$ is equivalent to looking for comparisons between these invariants of $(B_R(o), g)$ and geodesic balls of rotationally symmetric model spaces.

On the other hand, to show that the metric tensor $\widetilde{g}$ is well defined, we study, in the following Theorem 4.3.3, the smoothness of this new metric tensor by proving that $\widetilde{g}$ satisfy one of the smoothness conditions for the metric tensor of a model space of Theorem 2.2.3.

Theorem 4.3.3. *Let (M, g) be a complete n-dimensional Riemannian manifold and $o \in M$. Let $B_R(o)$ be the geodesic ball of M with radius R centered at $o \in M$. Suppose that $R < \operatorname{inj}_g(o)$. Then, the rotationally symmetric metric tensor of comparison $\widetilde{g}$ associated to g is smooth in $B_R(o)$.*

Proof. To prove the smoothness of $\widetilde{g}$ at o, we show that $\omega_g(r)$ can be rewritten as

$$\omega_g(r) = r\left(1 + r^2\varphi(r^2)\right)$$

with some positive smooth function φ, and hence, the theorem follows by applying Theorem 2.2.3.

First, from Theorems 2.1.77 and 2.1.79, we know that, for all $0 \leq r \leq R$, the area function $A_g(r)$ is smooth and it has Taylor expansion about $r = 0$ given by

$$A_g(r) = a_0 r^{n-1} + a_2 r^{n+1} + a_4 r^{n+3} + \cdots$$

for some constants $a_{2k} \in \mathbb{R}$, $k \in \mathbb{N}$, with $a_0 = \operatorname{vol}\left(\mathbb{S}_1^{n-1}\right)$. In particular, $A_g(0) = 0$. Moreover, since the Taylor expansion about 0 of a smooth function f is of the form

$$f(0) + \frac{f'(0)}{1!}r + \frac{f''(0)}{2!}r^2 + \frac{f'''(0)}{3!}r^3 + \cdots,$$

we have that the coefficients of the first $n-2$ terms and the ones of the terms with even order in the Taylor expansion of A_g are vanished, and hence, we obtain that the derivatives $A_g^{(k)}(0) = 0$ for $k = 1, \dots, n-2$, and the derivatives $A_g^{(n+2k)}(0) = 0$ for $k \in \mathbb{N}$. Since every derivative of $A_g(r)$ vanishes up to $n-1$ order and since $A_g(r)$ is a smooth function up to $r = \mathrm{inj}_g(o)$, then we can use, for all $r \in [0, \mathrm{inj}_g(o)]$, the Taylor expansion with integral form of the remainder (see [70] for instance) and we can rewrite $A_g(r)$ as

$$A_g(r) = \frac{1}{(n-2)!}\int_0^r (r-x)^{n-2} A_g^{(n-1)}(x)\,dx.$$

Moreover, by using the change of variable $x = sr$ in the above expression, we obtain that

$$A_g(r) = a_0 r^{n-1} f(r), \quad \text{where} \quad f(r) := \frac{1}{a_0(n-2)!}\int_0^1 (1-s)^{n-2} A_g^{(n-1)}(sr)\,ds.$$

We note that $f(r)$ is a positive smooth function with

$$f^{(k)}(r) = \frac{1}{a_0(n-2)!}\int_0^1 (1-s)^{n-2} s^k A_g^{(n-1+k)}(sr)\,ds.$$

In particular, $f(0) = 1$ and, since $A_g^{(n+2k)}(0) = 0$ for $k \in \mathbb{Z}$, the odd order derivatives of $f(r)$ vanish at 0. In fact,

$$f^{(2k+1)}(0) = \frac{1}{a_0(n-2)!}\int_0^1 (1-s)^{n-2} s^k A_g^{(n+2k)}(0)\,ds = 0$$

for all $k \in \mathbb{N}$. Then, as in the proof of Theorem 2.2.3, the function f can be extended to a smooth even function $\widetilde{f}(t)$ with $\widetilde{f}(0) = 1$, given by

$$\widetilde{f}(r) := \begin{cases} f(r), & \text{if} \quad r \geq 0, \\ f(-r), & \text{if} \quad r < 0. \end{cases}$$

Now, also following the same reasoning as in the proof of Theorem 2.2.3, since $\widetilde{f}$ is a smooth even function we have that it can be expressed as

$$\widetilde{f}(r) = h(r^2)$$

where h is a positive smooth function (see [75]). Note that $h(0) = f(0) = 1$. We can therefore express the area function as

$$A_g(r) = a_0 r^{n-1} h(r^2) = \mathrm{vol}\left(\mathbb{S}_1^{n-1}\right) r^{n-1} h(r^2).$$

Now, let us define the function $F(r) := (h(r))^{\frac{1}{n-1}}$. Then, since $h(r) > 0$ for all $0 < r \leq R$ and $h(0) = 1$, we have that F is smooth for all $r \in [0, R]$ with $F(0) = 1$. Hence, from (4.13), we obtain that

$$\omega_g(r) = rF(r^2).$$

On the other hand, since F is a positive smooth function with $F(0) = 1$ then, applying the fundamental theorem of calculus (see [70]), we can express F as

$$F(r) = 1 + \int_0^r F'(x)\, dx = 1 + r \int_0^1 F'(sr)\, ds.$$

Thus, we can rewrite F as

$$F(r) = 1 + r\varphi(r), \quad \varphi(r) := \int_0^1 F'(sr)\, ds.$$

This implies that $F(r^2) = 1 + r^2\varphi(r^2)$, and moreover, we obtain that

$$\omega_g(r) = r\left(1 + r^2\varphi(r^2)\right) \tag{4.14}$$

for some positive smooth function φ. This means that the rotationally symmetric metric tensor of comparison $\widetilde{g}$ satisfies assertion 2 of Theorem 2.2.3, and hence, the theorem follows. $\square$

Furthermore, since $(B_R(o), \widetilde{g})$ is a rotationally symmetric model space, we have the following proposition which summarizes some properties satisfied by the radial distance function, the unit radial tangent vector, the area function and the Laplacian in $B_R(o)$ with respect to the rotationally symmetric metric tensor of comparison $\widetilde{g}$.

Proposition 4.3.4. *Let (M, g) be a complete n-dimensional Riemannian manifold and $o \in M$. Let $B_R(o)$ be the geodesic ball with radius R centered at $o \in M$. Suppose that $R < \mathrm{inj}_g(o)$. Let $\{x^i\}_{i=1}^n$ be normal coordinates functions on $B_R(o)$, let r be the radial distance function to the center o in $B_R(o)$ and let $\widetilde{g}$ be the rotationally symmetric metric tensor of comparison associated to g defined on $B_R(o)$. Then, we have:*

1. *The equality between the gradients of the radial distance function r with respect to $\widetilde{g}$ and g,* i.e.*, for any $p \in B_R(o)$, we have*

$$\nabla_{\widetilde{g}} r(p) = \nabla_g r(p) = \partial r(q) = \sum_{i=1}^{n} \frac{x^i(q)}{r(p)} \left.\frac{\partial}{\partial x^i}\right|_p .$$

2. *The equality $\widetilde{g}\left(\nabla_{\widetilde{g}} r(p), \nabla_{\widetilde{g}} r(p)\right) = g\left(\nabla_g r(p), \nabla_g r(p)\right) = 1$ for any $p \in B_R(o)$, and hence, $\|\nabla_{\widetilde{g}} r(p)\|_{\widetilde{g}} = 1$ for any $p \in B_R(o)$.*

3. *The equality between the distance functions with respect to $\widetilde{g}$ and g,* i.e.*, for any $p \in B_R(o)$*

$$\mathrm{dist}_{\widetilde{g}}(o,p) = r(p) = \mathrm{dist}_g(o,p),$$

   and hence, the geodesic balls $(B_R(o), \widetilde{g})$ and $(B_R(o), g)$ are the same subset of M.

4. *The equality of the area function*

$$A_{\widetilde{g}}(r) = \mathrm{vol}_{\widetilde{g}}\left(S_r(o)\right) = \mathrm{vol}\left(\mathbb{S}_1^{n-1}\right)\omega_g^{n-1}(r) = A_g(r) \quad \textit{for all} \quad r \in [0,R).$$

5. *For any smooth function $f : B_R(o) \longrightarrow \mathbb{R}$, the Laplacian $\Delta_{\widetilde{g}}$ can be computed at any $p \in B_R(o)$ as*

$$\begin{aligned} \Delta_{\widetilde{g}} f(p) = {} & \frac{\partial^2 f}{\partial r^2}(p) + (n-1)\frac{\omega_g'\left(r(p)\right)}{\omega_g\left(r(p)\right)}\frac{\partial f}{\partial r}(p) \\ & + \frac{1}{\omega_g^2(r(p))}\Delta_{g_{\mathbb{S}_1^{n-1}}}\left(f \circ \pi^{-1}\right) \circ \pi(p) \end{aligned} \tag{4.15}$$

where $\Delta_{g_{\mathbb{S}_1^{n-1}}}$ denotes the Laplacian of the $(n-1)$-dimensional usual unit sphere.

Proof. Let begin this proof by showing assertion (1). Let $(B_R(o), \zeta)$ be normal coordinates centered at o with normal coordinate functions $\{x^i\}_{i=1}^n$ and let $(B_R(o) - \{o\}, \psi)$ be polar coordinates on M centered at o with coordinate functions r, θ^i, where r is the radial distance function to o and, given a chart $(\mathbb{S}_1^{n-1}, \widetilde{\theta})$ with coordinate functions $\{\widetilde{\theta}^i\}_{i=1}^{n-1}$, $\theta^i = \widetilde{\theta}^i \circ \pi$ with π the projection to $\mathbb{S}_1^{n-1}$ (see Definition 2.1.73 and equations (2.19) and (2.21). Then, we know that, given $p \in B_R(o) - \{o\}$, the coordinate vector fields $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta^1}, \dots, \frac{\partial}{\partial \theta^{n-1}}\right\}$ form a basis of the tangent space T_pM and that the metric tensor g can be expressed in polar coordinates as

$$g = dr \otimes dr + \sum_{i,j=1}^{n-1} g_{ij}\, d\theta^i \otimes d\theta^j \quad \text{with} \quad g_{ij} = g\left(\frac{\partial}{\partial\theta^i}, \frac{\partial}{\partial\theta^i}\right),$$

where $\left\{dr_p, d\theta_p^1, \ldots, d\theta_p^{n-1}\right\}$ are the associated dual forms in T_pM.

On the other hand, from Definition 4.3.1, we know that the rotationally symmetric tensor of comparison $\widetilde{g}$ on $B_R(o) - \{o\}$ is given by

$$\widetilde{g} = dr \otimes dr + (\omega_g \circ r)^2\, \pi^* g_{\mathbb{S}_1^{n-1}}$$

with warping function ω_g given by (4.13). Hence, we can choose $\{\partial/\partial r, e_1, \ldots, e_{n-1}\}$ an orthonormal basis of T_pM with respect to $\widetilde{g}$, where $\partial/\partial r$ is the unit radial vector field and $\{e_i\}_{i=1}^{n-1}$ is an orthonormal basis of the orthogonal complement $(\partial/\partial r)_{\widetilde{g}}^{\perp}$ of $\partial/\partial r$ in T_pM such that

$$\operatorname{span}\{e_1, \ldots, e_{n-1}\} = \operatorname{span}\left\{\frac{\partial}{\theta^1}, \ldots, \frac{\partial}{\partial\theta^{n-1}}\right\}.$$

Then, expressing the gradient of the radial distance function in this basis, we obtain that

$$\begin{aligned}\nabla_{\widetilde{g}} r &= \widetilde{g}\left(\nabla_{\widetilde{g}} r, \frac{\partial}{\partial r}\right)\frac{\partial}{\partial r} + \sum_{i=1}^{n-1} \widetilde{g}\left(\nabla_{\widetilde{g}} r, e_i\right) e_i \\ &= \frac{\partial r}{\partial r}\frac{\partial}{\partial r} + \sum_{i=1}^{n-1} e_i(r)\, e_i = \frac{\partial}{\partial r},\end{aligned}$$

and hence, by applying assertion (4) of Proposition 2.1.70, we obtain the equality $\nabla_{\widetilde{g}} r = \frac{\partial}{\partial r} = \nabla_g r = \sum_{i=1}^n \frac{x^i}{r}\frac{\partial}{\partial x^i}$, which shows assertion (1).

Furthermore, from assertion (4) of Proposition 2.1.70, we also know that $\|\nabla_g r\|_g = 1$. Then, applying assertion (1), we obtain that

$$\widetilde{g}\left(\nabla_{\widetilde{g}} r, \nabla_{\widetilde{g}} r\right) = \widetilde{g}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = dr\left(\frac{\partial}{\partial r}\right) dr\left(\frac{\partial}{\partial r}\right) = 1 = g\left(\nabla_g r, \nabla_g r\right),$$

and hence, $\|\nabla_{\widetilde{g}} r\|_{\widetilde{g}} = \|\nabla_g r\|_g = 1$, proving assertion (2).

To prove assertion (3) let us first remark the following: as we know that $\left\{\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta^1}, \ldots, \frac{\partial}{\partial\theta^{n-1}}\right\}$ is a basis of T_pM and since

$$\operatorname{span}\{e_1, \ldots, e_{n-1}\} = \operatorname{span}\left\{\frac{\partial}{\partial\theta^1}, \ldots, \frac{\partial}{\partial\theta^{n-1}}\right\},$$

we have that $\left\{\frac{\partial}{\partial\theta^i}\right\}_{i=1}^{n-1}$ forms a basis of the orthonormal complement $(\partial/\partial r)^{\perp}_{\widetilde{g}}$ of $\partial/\partial r$ in T_pM with respect to $\widetilde{g}$.

Now, we are going to use *Koszul's formula*

$$\begin{aligned}2g\left(\nabla_X Y, Z\right) &= Xg(Y,Z) + Yg(X,Z) - Zg(X,Y)\\ &\quad + g\left([X,Y],Z\right) - g\left([X,Z],Y\right) - g([Y,Z],X)\end{aligned}$$

to show that $\nabla_{\widetilde{g}} r$ is a geodesic vector field with respect to the metric tensor $\widetilde{g}$. Let $X = Y = Z = \nabla_{\widetilde{g}} r$ then, using Koszul's formula, it is easy to check that $\widetilde{g}\left(\nabla_{\nabla_{\widetilde{g}} r}\nabla_{\widetilde{g}} r, \nabla_{\widetilde{g}} r\right) = 0$. On the other hand, taking $X = Y = \nabla_{\widetilde{g}} r$ and, for a fixed $i = 1, \ldots, n-1$, $Z = \frac{\partial}{\partial\theta^i}$, using Koszul's formula, we have that

$$\begin{aligned}2\widetilde{g}\left(\nabla_{\nabla_{\widetilde{g}} r}\nabla_{\widetilde{g}} r, \frac{\partial}{\partial\theta^i}\right) &= 2\nabla_{\widetilde{g}} r\, \widetilde{g}(\nabla_{\widetilde{g}} r, \frac{\partial}{\partial\theta^i}) - \frac{\partial}{\partial\theta^i}\, \widetilde{g}(\nabla_{\widetilde{g}} r, \nabla_{\widetilde{g}} r)\\ &\quad + \widetilde{g}\left([\nabla_{\widetilde{g}} r, \nabla_{\widetilde{g}} r], \frac{\partial}{\partial\theta^i}\right) - 2\widetilde{g}\left(\left[\nabla_{\widetilde{g}} r, \frac{\partial}{\partial\theta^i}\right], \nabla_{\widetilde{g}} r\right).\end{aligned}$$

Thus, since $\frac{\partial}{\partial\theta^i} \in (\partial r)^{\perp}_{g}$ and $\left[\nabla_{\widetilde{g}} r, \frac{\partial}{\partial\theta^i}\right] = 0$ because $\nabla_{\widetilde{g}} r = \frac{\partial}{\partial r}$, applying assertion (2), we obtain that $\widetilde{g}\left(\nabla_{\nabla_{\widetilde{g}} r}\nabla_{\widetilde{g}} r, \frac{\partial}{\partial\theta^i}\right) = 0$ for all $i = 1, \ldots, n-1$. Then $\widetilde{g}\left(\nabla_{\nabla_{\widetilde{g}} r}\nabla_{\widetilde{g}} r, e^i\right) = 0$ for all $i = 1, \ldots, n-1$, and hence, $\nabla_{\nabla_{\widetilde{g}} r}\nabla_{\widetilde{g}} r = 0$.

Therefore, the integral curves of $\nabla_{\widetilde{g}} r = \partial r = \frac{\partial}{\partial r}$ are geodesic curves in $B_R(o)$ with respect to $\widetilde{g}$. Namely, given $p \in B_R(o) - \{o\}$, the segment given by $\gamma : [0, r(p)] \longrightarrow B_R(o)$, $\gamma(t) := (t, \theta^1, \ldots, \theta^{n-1})$, is a geodesic curve with respect to $\widetilde{g}$ joining o with p and we have that $\gamma'(t) = \frac{\partial}{\partial r}\big|_{\gamma(t)} = \nabla_{\widetilde{g}} r(\gamma(t))$.

Finally, we are going to show that the geodesic curve γ defined as above minimize the arc-length with respect to $\widetilde{g}$. First, note that $r(p) = \ell_{\widetilde{g}}(\gamma) \geq \mathrm{dist}_{\widetilde{g}}(o,p)$ because $\|\gamma'(r)\|_{\widetilde{g}} = \|\nabla_{\widetilde{g}} r\|_{\widetilde{g}} = 1$. On the other hand, let $\ell = \mathrm{dist}_{\widetilde{g}}(o,p)$ and let $\beta : [0,\ell] \longrightarrow M$ be a normalized curve joining o with p, *i.e.*, $\beta(0) = o$, $\beta(\ell) = p$ and $\|\beta'\|_{\widetilde{g}} = 1$. Note that $\ell_{\widetilde{g}}(\beta) = \ell = \mathrm{dist}_{\widetilde{g}}(o,p)$. Then, applying assertion (2), we have that

$$\begin{aligned}\mathrm{dist}_{\widetilde{g}}(o,p) = \ell = \ell_{\widetilde{g}}(\beta) &= \int_0^{\ell} \|\beta'(s)\|_{\widetilde{g}}\, ds \geq \int_0^{\ell} \|\beta'(s)\|_{\widetilde{g}}\, \|\nabla_{\widetilde{g}} r(s)\|_{\widetilde{g}} \cos(\theta) ds\\ &= \int_0^{\ell} \widetilde{g}\left(\nabla_{\widetilde{g}} r(s), \beta'(s)\right) ds = \int_0^{\ell} \left(r\circ\beta\right)'(s) ds = \left(r\circ\beta\right)(\ell) - \left(r\circ\beta\right)(0)\\ &= r(p) - r(o) = r(p) = \ell_{\widetilde{g}}(\gamma) \geq \mathrm{dist}_{\widetilde{g}}(o,p),\end{aligned}$$

and hence, $\mathrm{dist}_{\widetilde{g}}(o,p) = r(p) = \mathrm{dist}_g(o,p)$ for all $p \in B_R(o) - \{o\}$, which shows assertion (3).

On the other hand, since $(B_R(o), \widetilde{g})$ is a rotationally symmetric metric tensor, we have, from equation (2.31), that $A_{\widetilde{g}}(r) = \mathrm{vol}\left(\mathbb{S}_1^{n-1}\right) \omega_g^{n-1}(r)$, and hence, from the definition of the warping function ω_g (see equation (4.13) of Definition 4.3.1), we obtain assertion (4).

Finally, assertion 5 comes by a straightforward computation using the expression (2.32) of the Laplacian on rotationally symmetric model spaces.

□

4.4 Upper bounds for the first eigenvalue on geodesic balls computed by the area function of the geodesic spheres

In this section we prove the first of our claims regarding the first eigenvalue of the Laplacian for geodesic balls and, as a consequence, we show some upper bounds for the first eigenvalue by applying the known results stated along Section 4.2. But first, observe that in the statement of the following theorem there are no conditions on the Ricci or sectional curvatures (as in the hypothesis of Cheng in Theorems 4.2.1 and 4.2.2), neither on the mean curvature of the geodesic spheres (as in the hypothesis of Bessa and Montenegro in Theorem 4.2.4). But the equality in inequality (4.16) is attained when the geodesic spheres have radial mean curvature as in the result of Bessa and Montenegro.

Theorem 4.4.1. *Let (M,g) be a complete n-dimensional Riemannian manifold and $o \in M$. Let $B_R(o)$ be the geodesic ball of M with radius R centered at o. Suppose that $R < \mathrm{inj}_g(o)$. Then, the first eigenvalue $\lambda_{1,g}\left(B_R(o)\right)$ of the Laplacian Δ_g for the Dirichlet problem in $B_R(o)$ with respect to g is bounded from above by*

$$\lambda_{1,g}\left(B_R(o)\right) \leq \lambda_{1,\widetilde{g}}\left(B_R(o)\right). \tag{4.16}$$

where $\widetilde{g}$ is the rotationally symmetric metric tensor of comparison associated to the metric tensor g and $\lambda_{1,g}$ is the first eigenvalue of the Laplacian $\Delta_{\widetilde{g}}$ for the Dirichlet problem on $B_R(o)$ with respect to $\widetilde{g}$.

Furthermore, equality in (4.16) *is attained if, and only if, there exists a smooth function* $h(r)$ *such that*

$$H_{S_r(o)} = h(r) \quad \textit{for all} \quad 0 < r < R.$$

Proof. We are going to use the Rayleigh's Theorem 4.1.7 as follows: let $\phi_1 \in C_0^\infty(B_R(o)) \cap C^0(\overline{B_R(o)})$ be a positive first eigenfunction associated to the first eigenvalue $\lambda_{1,\widetilde{g}}(B_R(o))$ of the Laplacian $\Delta_{\widetilde{g}}$ for the Dirichlet problem in $B_R(o)$. Then, from Rayleigh's Theorem, we know that ϕ_1 and $\lambda_{1,\widetilde{g}}(B_R(o))$ are related by the Rayleigh quotient with respect to $\widetilde{g}$ as follows

$$\lambda_{1,\widetilde{g}}(B_R(o)) = \frac{\int_{B_R(o)} \widetilde{g}\left(\nabla_{\widetilde{g}}\phi_1(p), \nabla_{\widetilde{g}}\phi_1(p)\right) dV_{\widetilde{g}}}{\int_{B_R(o} \phi_1^2(p)\, dV_{\widetilde{g}}}, \tag{4.17}$$

where $dV_{\widetilde{g}}$ is the Riemannian volume element in $B_R(o)$ with respect to the rotationally symmetric metric tensor of comparison $\widetilde{g}$.

On the other hand, let $\lambda_{1,g}(B_R(o))$ be the first eigenvalue of the Laplacian Δ_g for the Dirichlet problem in $B_R(o)$. Then, since $\phi_1 \in C_0^\infty(B_R(o)) \cap C^0(\overline{B_R(o)})$ and ϕ_1 is a non-trivial function, we have, by applying again Rayleigh's Theorem 4.1.7, that

$$\lambda_{1,g}(B_R(o)) \leq \frac{\int_{B_R(o)} g\left(\nabla\phi_1(p), \nabla\phi_1(p)\right) dV_g}{\int_{B_R(o)} \phi_1^2(p)\, dV_g}, \tag{4.18}$$

where dV_g is the Riemmanian volue element in $B_R(o)$ with respect to the metric tensor g.

Thus, our upper bound for $\lambda_{1,g}(B_R(o))$ in (4.16) is obtained by showing that

$$\frac{\int_{B_R(o)} g\left(\nabla\phi_1(p), \nabla\phi_1(p)\right) dV_g}{\int_{B_R(o)} \phi_1^2(p)\, dV_g} = \frac{\int_{B_R(o)} \widetilde{g}\left(\nabla_{\widetilde{g}}\phi_1(p), \nabla_{\widetilde{g}}\phi_1(p)\right) dV_{\widetilde{g}}}{\int_{B_R(o} \phi_1^2(p)\, dV_{\widetilde{g}}}. \tag{4.19}$$

In fact, taking into account that $\phi_1(p) = f_1(r(p))$ is a decreasing radial function where r is the radial distance function to the center o and f_1 is a positive smooth function such that $f_1(R) = 0$, $f_1'(0) = 0$ and $f_1'(r) < 0$ for all $r \in (0, R)$ (see Proposition 4.1.9), using that $\|\nabla_g r\|_g = \|\nabla_{\widetilde{g}} r\|_{\widetilde{g}} = 1$ and that $A_g(r) = A_{\widetilde{g}}(r)$ for all $0 \leq r < R$ (see assertions (2) and (4) of Proposition (4.3.4)), and applying the co-area formula (see Theorem 2.1.62), we obtain that

$$
\begin{aligned}
\int_{B_R(o)} \phi_1^2(p)\, dV_g &= \int_{B_R(o)} \frac{f_1^2\left(r(p)\right)}{\left\|\nabla_g r(p)\right\|_g} \left\|\nabla_g r(p)\right\|_g \, dV_g \\
&= \int_0^R \left(\int_{\{p\in M\,:\, r(p)=r\}} \frac{f_1^2\left(r(p)\right)}{\left\|\nabla_g r(p)\right\|_g} \, dA_g \right) dr \\
&= \int_0^R f_1^2(r) \left(\int_{S_r(o)} dA_g \right) dr = \int_0^R f_1^2(r) A_g(r)\, dr \\
&= \int_0^R f_1^2(r) A_{\widetilde{g}}(r)\, dr = \int_0^R f_1^2(r) \left(\int_{S_r(o)} dA_{\widetilde{g}} \right) dr \\
&= \int_0^R \left(\int_{\{p\in M\,:\, r(p)=r\}} \frac{f_1^2\left(r(p)\right)}{\left\|\nabla_{\widetilde{g}} r(p)\right\|_{\widetilde{g}}} \, dA_{\widetilde{g}} \right) dr \\
&= \int_{B_R(o)} \frac{f_1^2\left(r(p)\right)}{\left\|\nabla_{\widetilde{g}} r(p)\right\|_{\widetilde{g}}} \left\|\nabla_{\widetilde{g}} r(p)\right\|_{\widetilde{g}} \, dV_{\widetilde{g}} \\
&= \int_{B_R(o)} \phi_1^2(p)\, dV_{\widetilde{g}}.
\end{aligned}
$$

Moreover, as $\nabla_g \phi_1(p) = \nabla_g f_1\left(r(p)\right) = f_1'\left(r(p)\right) \nabla_g r(p)$ for all $p \in B_R(o)$, then

$$
g\left(\nabla_g \phi_1(p), \nabla_g \phi_1(p)\right) = \left|f_1'\left(r(p)\right)\right|^2 g\left(\nabla_g r(p), \nabla_g r(p)\right) = \left(f_1'\left(r(p)\right)\right)^2
$$

is also a radial function for all $p \in B_R(o)$. Hence, the numerator of the Rayleigh's quotient with respect to the metric tensor g satisfies that

$$
\int_{B_R(o)} g\left(\nabla_g \phi_1(p), \nabla_g \phi_1(p)\right) dV_g = \int_{B_R(o)} \left(f_1'\left(r(p)\right)\right)^2 dV_g.
$$

Analogously, since $\widetilde{g}\left(\nabla_{\widetilde{g}} \phi_1(p), \nabla_{\widetilde{g}} \phi_1(p)\right) = f_1'(r(p)) \widetilde{g}\left(\nabla_{\widetilde{g}} r(p), \nabla_{\widetilde{g}} r(p)\right) = f_1'(r(p))$, we have that

$$
\int_{B_R(o)} \widetilde{g}\left(\nabla_{\widetilde{g}} \phi_1(p), \nabla_{\widetilde{g}} \phi_1(p)\right) dV_{\widetilde{g}} = \int_{B_R(o)} \left(f_1'\left(r(p)\right)\right)^2 dV_{\widetilde{g}}.
$$

Finally, using the same reasoning we used to obtain the equality between the denominators of the Rayleigh's quotients, we have that

$$\begin{aligned}\int_{B_R(o)} \left(f_1'\left(r(p)\right)\right)^2 dV_g &= \int_{B_R(o)} \frac{\left(f_1'\left(r(p)\right)\right)^2}{\left\|\nabla_g r(p)\right\|_g} \left\|\nabla_g r(p)\right\|_g dV_g \\ &= \int_0^R \left(\int_{\{p\in M\,:\,r(p)=r\}} \frac{\left(f_1'\left(r(p)\right)\right)^2}{\left\|\nabla_g r(p)\right\|_g} dA_g \right) dr \\ &= \int_0^R \left(f_1'\left(r\right)\right)^2 \left(\int_{S_r(o)} dA_g \right) dr = \int_0^R \left(f_1'\left(r\right)\right)^2 A_g(r)\, dr \\ &= \int_0^R \left(f_1'\left(r\right)\right)^2 A_{\widetilde{g}}(r)\, dr = \int_0^R \left(f_1'\left(r\right)\right)^2 \left(\int_{S_r(o)} dA_{\widetilde{g}} \right) dr \\ &= \int_0^R \left(\int_{\{p\in M\,:\,r(p)=r\}} \frac{\left(f_1'\left(r(p)\right)\right)^2}{\left\|\nabla_{\widetilde{g}} r(p)\right\|_{\widetilde{g}}} dA_{\widetilde{g}} \right) dr \\ &= \int_{B_R(o)} \frac{\left(f_1'\left(r(p)\right)\right)^2}{\left\|\nabla_{\widetilde{g}} r(p)\right\|_{\widetilde{g}}} \left\|\nabla_{\widetilde{g}} r(p)\right\|_{\widetilde{g}} dV_{\widetilde{g}} \\ &= \int_{B_R(o)} \left(f_1'\left(r(p)\right)\right)^2 dV_{\widetilde{g}},\end{aligned}$$

which shows equality (4.19). Therefore, by inequality (4.18) and by equalities (4.19) and (4.17), we obtain that

$$\begin{aligned}\lambda_{1,g}\left(B_r(o)\right) &\leq \frac{\int_{B_R(o)} g\left(\nabla_g \phi_1(p), \nabla_g \phi_1(p)\right) dV_g}{\int_{B_R(o)} \phi_1^2\, dV_g} \\ &= \frac{\int_{B_R(o)} \widetilde{g}\left(\nabla_{\widetilde{g}} \phi_1(p), \nabla_{\widetilde{g}} \phi_1(p)\right) dV_{\widetilde{g}}}{\int_{B_R(o} \phi_1^2(p)\, dV_{\widetilde{g}}} = \lambda_{1,\widetilde{g}}\left(B_R(o)\right)\end{aligned} \tag{4.20}$$

Before begin to prove the equality case, let us remember that, from equations (2.29) and (2.31), the area function of the geodesic sphere $S_r(o)$ of radius r centered at o in a system of polar coordinates $(B_R(o), \psi)$ with respect to the metric tensor g (see Definition 2.1.73) and with respect to the rotationally symmetric metric tensor $\widetilde{g}$ is, respectively,

$$\begin{aligned}A_g(r) &= \int_{\mathbb{S}_1^{n-1}} \sqrt{\det\left(G(r,\theta)\right)}\, d\theta^1 \wedge \cdots \wedge d\theta^{n-1}, \\ A_{\widetilde{g}}(r) &= \omega_g^{n-1}(r)\, \mathrm{vol}\left(\mathbb{S}_1^{n-1}\right).\end{aligned}$$

And moreover, from Proposition 2.1.75, we have, for any $p \equiv (r,\theta) \in S_r(o)$, that

$$H_{S_r(o)}(p) = \frac{\frac{\partial}{\partial r}\sqrt{\det\left(G(r,\theta)\right)}}{\sqrt{\det\left(G(r,\theta)\right)}} = \frac{\partial}{\partial r} \ln \sqrt{\det\left(G(r,\theta)\right)}.$$

Now, to prove the equality case, let us first assume equality in inequality (4.16). Then, all the inequalities in equation (4.20) became equalities, which implies, in particular, that

$$\lambda_{1,g}\left(B_R(o)\right) = \frac{\int_{B_R(o)} g\left(\nabla_g\phi_1(p), \nabla_g\phi_1(p)\right) dV_g}{\int_{B_R(o)} \phi_1^2(p) dV_g},$$

and hence, from Rayleigh's Theorem 4.1.7, we obtain that ϕ_1 is also a first positive eigenfunction associated to $\lambda_{1,g}\left(B_R(o)\right)$. Therefore, from the expression of Δ_g in polar coordinates, we have, for any $p \in B_R(o)$, that

$$\begin{aligned}\Delta_g\phi_1(p) &= f_1''\left(r(p)\right) + \frac{\partial}{\partial r}\left(\ln\sqrt{\det\left(G(r,\theta)\right)}\right)\Big|_{r(p)} f_1'\left(r(q)\right) \\ &= -\lambda_{1,g}\left(B_R(o)\right) f_1\left(r(p)\right).\end{aligned}$$

Then, for any point $p \in S_r(o)$, $r(p) = r$, we obtain that

$$f_1''\left(r\right) + \lambda_{1,g}\left(B_R(o)\right) f_1\left(r\right) = -f_1'\left(r\right) H_{S_r(o)}(p). \tag{4.21}$$

Furthermore, since ϕ_1 is a first positive eigenfunction associated to $\lambda_{1,\widetilde{g}}\left(B_R(o)\right)$, by equation (4.15), we have, for any $p \in B_R(o)$, that

$$\Delta_{\widetilde{g}}\phi_1(p) = f_1''\left(r(p)\right) + (n-1)\frac{\omega_g'(r(p))}{\omega_g(r(q))} f_1'\left(r(p)\right) = -\lambda_{1,g}\left(B_R(o)\right) f_1\left(r(p)\right),$$

and hence, for any $p \in S_r(o)$, $r(p) = r$,

$$f_1''(r) + \lambda_{1,g}\left(B_R(o)\right) f_1(r) = -(n-1)\frac{\omega_g'(r)}{\omega_g(r)} f_1'(r).$$

Therefore, from (4.21) and taking into account that $f_1'(r) < 0$ for all $r \in (0, R)$, we obtain that the mean curvature of the geodesic sphere $S_r(o)$ computed at any point $p \in S_r(o)$ is

$$H_{S_r(o)}(p) = (n-1)\frac{\omega_g'(r)}{\omega_g(r)},$$

namely, $H_{S_r(o)}$ is a radial function as stated.

To end this proof, let us show that if the mean curvature of the geodesic spheres is a radial function, *i.e.*, $\vec{H}_{S_r(o)}(p) = -h(r)\nabla_g r$, then the equality in inequality (4.16) is attained. More precisely, we show that $\lambda_{1,g}\left(B_R(o)\right) =$

$\lambda_{1,\widetilde{g}}\left(B_R(o)\right)$. Indeed, we can prove that ϕ_1 is a positive eigenfunction of Δ_g because

$$\begin{aligned}\Delta_g\phi_1(p) &= f_1''\left(r(p)\right) + \frac{\partial}{\partial r}\left(\ln\sqrt{\det\left(G(r,\theta)\right)}\right)\bigg|_{r(p)} f_1'\left(r(p)\right) \\ &= f_1''\left(r(p)\right) + h(r(p))f_1'\left(r(p)\right).\end{aligned} \tag{4.22}$$

But since

$$\begin{aligned}(n-1)\frac{\omega_g'(r)}{\omega_g(r)} &= \frac{1}{\omega_g^{n-1}(r)\operatorname{vol}\left(\mathbb{S}_1^{n-1}\right)}\frac{d}{dr}\left(\omega_g^{n-1}(r)\operatorname{vol}\left(\mathbb{S}_1^{n-1}\right)\right) \\ &= \frac{d}{dr}\ln\left(\omega_g^{n-1}(r)\operatorname{vol}\left(\mathbb{S}_1^{n-1}\right)\right) = \frac{d}{ds}\ln A_{\widetilde{g}}(s)\bigg|_{s=r} = \frac{d}{ds}\ln A_g(s)\bigg|_{s=r} \\ &= \frac{1}{A_g(r)}\int_{\mathbb{S}_1^{n-1}}\frac{\partial}{\partial s}\sqrt{\det\left(G(s,\theta)\right)}\bigg|_{s=r} d\theta^1\wedge\cdots\wedge d\theta^{n-1} \\ &= \frac{1}{A_g(r)}\int_{\mathbb{S}_1^{n-1}} H_{S_r(o)}(r,\theta)\sqrt{\det\left(G(r,\theta)\right)}\, d\theta^1\wedge\cdots\wedge d\theta^{n-1} \\ &= \frac{1}{A_g(r)}\int_{\mathbb{S}_1^{n-1}} h(r)\sqrt{\det\left(G(r,\theta)\right)}\, d\theta^1\wedge\cdots\wedge d\theta^{n-1} \\ &= \frac{h(r)}{A_g(r)}\int_{\mathbb{S}_1^{n-1}} \sqrt{\det\left(G(r,\theta)\right)}\, d\theta^1\wedge\cdots\wedge d\theta^{n-1} \\ &= \frac{h(r)}{A_g(r)}A_g(r) = h(r),\end{aligned}$$

from equations (4.22) and (4.15), we have that

$$\Delta_g\phi_1(p) = f_1''\left(r(p)\right) + (n-1)\frac{\omega_g'(r(p))}{\omega_g(r(p))}f_1'\left(r(p)\right) = \Delta_{\widetilde{g}}\phi_1 = -\lambda_{1,\widetilde{g}}\left(B_R(o)\right)\phi_1(p).$$

Hence, ϕ_1 is a positive eigenfunction of the Laplacian Δ_g and then, since a first eigenfunction is the only eigenfunction which does not change sign (see Corollary 4.1.5 and Remark 4.1.6), we have that ϕ_1 is a first eigenfunction of the Laplacian Δ_g and the Theorem follows.

□

Remark 4.4.2. Observe that the function $h(r)$ of Theorem 4.4.1 is

$$h(r) = (n-1)\frac{\omega_g'(r)}{\omega_g(r)}.$$

On the other hand, note that in the classical symmetrization results (see Section 4.2), the symmetrized object minimizes the first eigenvalue of the Laplacian for the Dirichlet problem, but in the above result our symmetrized object maximizes the first eigenvalue. Namely, our symmetrized object ($\widetilde{g}$) provides an upper bound instead of a lower bound.

As a consequence of Theorem 4.4.1, we can obtain bounds $\lambda_{1,g}(B_R(o))$ by applying Theorems 4.2.8, 4.2.10 and 4.2.11 to obtain upper bounds for $\lambda_{1,\widetilde{g}}(B_R(o))$ where $\widetilde{g}$ is the rotationally symmetric tensor of comparison, as we will show in the following Corollary 4.4.4. But first, given a rotationally symmetric model space $(\mathbb{M}_\omega, g_{_\omega})$ with center o_ω and a geodesic ball $B^\omega_R(o_\omega)$ of M with radius $R < \mathrm{inj}_g(o)$ centered at $o \in M$, let us define the family of functions $\{T_k\}_{i=1}^\infty$ such that, for all $k \geq 1$, $T_k := [0, R] \longrightarrow \mathbb{R}_+$ is given by

$$T_k(r) := \frac{u^\omega_{k,R}(r)}{k!}, \tag{4.23}$$

where $\{u^\omega_{k,R}\}_{k=1}^\infty$ is the Poisson hierarchy of $B^\omega_R(o_\omega)$. Therefore, from equation (4.23) and applying the co-area formula (see Theorem 2.1.62), we have, for all $k \geq 0$, that

$$\mathcal{A}_k(B^\omega_R(o_\omega)) = \int_{B^\omega_R(o_\omega)} u^\omega_{k,R}\, dV_{g_\omega} = k! \int_0^R T_k(r) A_{g_\omega}(r) dr. \tag{4.24}$$

Remark 4.4.3. Observe that the family of functions $\{T_k\}_{k=1}^\infty$ only depends on the area function A_{g_ω} of the geodesic spheres. Indeed, from Proposition 3.2.16 and equation (2.31), we know that

$$u^\omega_{k,R}(r) = k \int_r^R \frac{\int_0^s u^\omega_{k-1,R}(\sigma) A_{g_\omega}(\sigma) d\sigma}{A_{g_\omega}(s)} ds,$$

and hence, for all $k \geq 1$, we have that T_k, $u^\omega_{k,R}$ and $\mathcal{A}_k(B^\omega_R(o_\omega))$ are functions that only depends on the area function.

In particular, given a Riemannian manifold (M, g) and given $\widetilde{g}$ the rotationally symmetric tensor of comparison defined on a geodesic ball $B_R(o)$ of M with radius $R < \mathrm{inj}_g(o)$ centered at a point $o \in M$ then, since $(B_R(o), \widetilde{g})$ is a rotationally symmetric model space where $B_R(o)$ is a geodesic ball of M, we can bound from above $\lambda_{1,\widetilde{g}}(B_R(o))$ in terms of functions $\{T_k\}_{k=1}^\infty$ by applying some of the results that we showed in Sections 4.2 and obtain the following comparisons.

Corollary 4.4.4. *Let (M, g) be a complete n-dimensional Riemannian manifold. Let $o \in M$ and let $B_R(o)$ be a geodesic ball of M with radius R centered at o. Suppose that $R < \operatorname{inj}_g(o)$. Let $\{T_k\}_{k=1}^{\infty}$ be the family of functions* (4.23) *that only depends on the area functions A_g with respect to the metric tensor g. Then, the first eigenvalue $\lambda_{1,g}(B_R(o))$ of the Laplacian for the Dirichlet problem on $B_R(o)$ is bounded from above, for all $k \geq 0$ by*

$$\begin{aligned}
\lambda_{1,g}(B_R(o)) &\leq \frac{\int_0^R T_k(r)A_g(r)dr}{\int_0^R T_{k+1}(r)A_g(r)dr}, \\
\lambda_{1,g}(B_R(o)) &\leq \frac{\int_0^R T_{2k+1}(r)A_g(r)dr}{\left(\int_0^R T_{k+1}(r)A_g(r)dr\right)^2}, \\
\lambda_{1,g}(B_R(o)) &\leq \frac{((2k+2)! - ((k+1)!)^2)\int_0^R T_{2k+1}(r)A_g(r)dr}{\int_0^R \left((2k+2)!T_{2k+2}(r) - ((k+1)!)^2 T_{k+1}^2(r)\right)A_g(r)dr}, \\
\lambda_{1,g}(B_R(o)) &\leq \frac{\operatorname{vol}(B_R(o))}{\int_0^R T_1(r)A_g(r)dr},
\end{aligned} \tag{4.25}$$

and moreover, it can be sharply computed by

$$\begin{aligned}
\lambda_{1,g}(B_R(o)) \leq \lim_{k\to\infty} \frac{T_k(0)}{T_{k+1}(0)} &= \lim_{k\to\infty} \frac{\int_0^R T_k(r)A_g(r)dr}{\int_0^R T_{k+1}(r)A_g(r)dr} \\
&= \lim_{k\to\infty} \left(\frac{\int_0^R T_k^2(r)A_g(r)dr}{\int_0^r T_{k+1}^2(r)A_g(r)dr}\right)^{1/2}
\end{aligned} \tag{4.26}$$

in the sense that equality in inequality (4.26) *is attained if, and only if, there exists a smooth function $h(r)$ such that $H_{S_r(o)} = h(r)$ for all $0 < r < R$.*

Proof. Let $B_R(o)$ be a geodesic ball of a complete n-dimensional Riemannian manifold (M, g) with radius R centered at a point $o \in M$. Then, assuming that $R < \operatorname{inj}_g(o)$, we know, from Section 4.3, that there exists a smooth metric tensor $\widetilde{g}$ (the rotationally symmetric metric tensor of comparison) such that $(B_R(o), \widetilde{g})$ is a rotationally symmetric model space and that the area function with respect to the metric tensor g of the geodesic spheres contained in $B_R(o)$ is preserved, *i.e.*, $A_g(r) = A_{\widetilde{g}}(r)$ for all $r \in [0, R]$. Moreover, by Theorem 4.4.1, we know that

$$\lambda_{1,g}(B_R(o)) \leq \lambda_{1,\widetilde{g}}(B_R(o)),$$

where $\lambda_{1,g}(B_R(o))$ and $\lambda_{1,\widetilde{g}}(B_R(o))$ are, respectively, the first eigenvalues of the Laplacian for the Dirichlet problem on $B_R(o)$ with respect to metric tensor g and the rotationally symmetric metric tensor of comparison $\widetilde{g}$.

On the other hand, let $\left\{u_{k,R}^{\widetilde{g}}\right\}_{k=1}^{\infty}$ and $\{\widetilde{\mathcal{A}}_k(B_R(o))\}$ be, respectively, the Poisson hierarchy and the moment spectrum of $(B_R(o), \widetilde{g})$. Then, since $(B_R(o), \widetilde{g})$ is a rotationally symmetric model space, we can define the family of functions $\{T_k\}_{k=1}^{\infty}$ given by (4.23), and moreover, from Remark 4.4.3, we have, for all $k \geq 1$, that T_k, $u_{k,R}^{\widetilde{g}}$ and $\widetilde{A}_k(B_R(o))$ only depends on the area functions $A_{\widetilde{g}}$. Therefore, since $(B_R(o), \widetilde{g})$ is a rotationally symmetric model space, by applying Theorems 4.2.8, 4.2.10 and 4.2.11 to $\lambda_{1,\widetilde{g}}(B_R(o))$, replacing $u_{k,R}^{\widetilde{g}}$ and $\widetilde{\mathcal{A}}_k(B_R(o))$ by its expressions (4.23) and (4.24) and using that $A_g(r) = A_{\widetilde{g}}(r)$ for all $r \in [0, R]$, we obtain inequalities (4.25) and (4.26). Moreover, the equality case for inequality (4.26) comes from Theorems 4.4.1, 4.2.8 and 4.2.11.

$\square$

Example 4.4.5. To end this subsection we show an example where we apply the above theorem to find a bound for the first eigenvalue of the Laplacian for the Dirichlet problem. And moreover, we show that, in this case, our bound can not be obtained by the classical comparison results (see Section 4.2).

Let (r, θ) be a system of polar coordinates in $\mathbb{R}^2$ around $\vec{0} \in \mathbb{R}^2$. Let us endow $\mathbb{R}^2$ with the following metric

$$g = dr \otimes dr + (r + \varphi(r)\cos(\theta))^2\, d\theta \otimes d\theta,$$

where φ is the non-negative, real valued, smooth function given by

$$\varphi : [0, +\infty) \longrightarrow [0, +\infty), \quad t \longmapsto \varphi(t) := \begin{cases} 0, & \text{if } t \leq 2, \\ e^{-\frac{1}{(t-2)^2}}, & \text{if } t > 2. \end{cases}$$

Observe that g is smooth on the entire $\mathbb{R}^2$ because φ is smooth and we have, for any $r \leq 2$, that g is the rotationally symmetric tensor g_{ω_0} on $\mathbb{R}^2$ with warping function $\omega_0(r) = r$, and hence, since $\omega_0(0) = 0$, $\omega_0'(0) = 1$ and $\omega_0^{(2k)}(0) = 0$ for all $k \in \mathbb{N}$, we have, from Theorem 2.2.3, that g is smooth. Thus, computing the area function for the geodesic spheres of $(\mathbb{R}^2, g)$, we have that

$$A_g(t) = \int_0^{2\pi} (t + \varphi(t)\cos(\theta))\, d\theta = 2\pi t.$$

Therefore, from Definition 4.3.1, the warping function ω_g of the rotationally symmetric metric tensor of comparison $\widetilde{g}$ is

$$\omega_g(t) = \frac{A_g(t)}{\mathrm{vol}\left(\mathbb{S}^1\right)} = \frac{2\pi t}{2\pi} = t.$$

Thus, the rotationally symmetric tensor of comparison is

$$\widetilde{g} = dr \otimes dr + r^2 d\theta \otimes d\theta,$$

which is the canonical metric tensor g_{can} of $\mathbb{R}^2$ expressed in polar coordinates (see Section 9 of Chapter 3 of [34]). Then, by using Theorem 4.4.1, we conclude that the first eigenvalue of the Laplacian Δ_g for the Dirichlet problem in a geodesic ball $B_R(\vec{0})$ of $\mathbb{R}^2$ with radius R centered at $\vec{0}$ is bounded from above by

$$\lambda_{1,g}\left(B_R(\vec{0})\right) \leq \lambda_{1,g_{\mathrm{can}}}\left(B_R(\vec{0})\right) = \frac{j_0^2}{R^2} \approx \frac{7,78319}{R^2}, \tag{4.27}$$

where j_0 is the first zero of the Bessel function J_0. (see [6] and [8]).

On the other hand, from Remark 2.1.39 and Proposition 2.1.76, we have that the Ricci curvature is given by

$$\mathrm{Ric}_g = \begin{cases} 0, & \text{for } r \leq 2, \\ \dfrac{2\left(3(r-4)r+10\right)\cos(\theta)}{(r-6)^2\left(\cos(\theta)+e^{\frac{1}{(r-2)^2}}r\right)}, & \text{for } r > 2. \end{cases}$$

Thus, for $r > 2$ there are regions where $\mathrm{Ric}_g < 0$. Hence, for a geodesic ball with radius $R > 2$, the upper bound (4.27) can not be obtained by using the comparison of Cheng with the Ricci curvature assumption, $\mathrm{Ric}_g \geq \mathrm{Ric}_{g_{\mathrm{can}}} = 0$ (see Theorem 4.2.1).

Moreover, from Proposition 2.1.75, we have the mean curvature of the geodesic spheres $S_r(\vec{0})$ of $(\mathbb{R}^2, g)$ with radius r centered at $\vec{0}$, for any point $q \in S_r(\vec{0})$, is given by

$$H_{S_r(\vec{0})}(q) = \begin{cases} \dfrac{1}{r}, & \text{for } r \leq 2, \\ \dfrac{1}{r} - \dfrac{(r-4)\left((r-2)r+2\right)\cos(\theta)}{(r-2)^3 r\left(\cos(\theta)+e^{\frac{1}{(r-2)^2}}r\right)}, & \text{for } r > 2. \end{cases}$$

Thus, for $r > 2$ there are point in the geodesic sphere $S_r(\vec{0})$ where $H_{S_r(\vec{0})} > 1/r$. Hence, for a geodesic ball with radius $R > 2$, the upper bound (4.27) can not be obtained by using the comparison of Bessa and Montenegro with the mean curvature assumption, $H_{S_R(\vec{0})} \leq 1/r$ for all $0 < r < R$ (see Theorem 4.2.4).

Note that, since for $R > 2$ the mean curvature of the geodesic spheres $S_r(\vec{0})$ is not a radial function for any $2 < r < R$, then equality in inequality (4.27) can not be attained, namely

$$\lambda_{1,g}\left(B_R(\vec{0})\right) < \lambda_{1,g_{\text{can}}}\left(B_R(\vec{0})\right), \quad \text{for all} \quad R > 2. \tag{4.28}$$

This upper bound for the first eigenvalue $\lambda_{1,g}(B_R(\vec{0}))$ allows us to state that there exists a precompact connected domain Ω in $(\mathbb{R}^2, g)$ with symmetrized radius $L(\Omega) > 0$ with respect to the Euclidean space $(\mathbb{R}^2, g_{\text{can}})$ considered as the rotationally symmetric model space $(\mathbb{M}_{\omega_0}, g_{\omega_0})$, *i.e.*,

$$\operatorname{vol}(\Omega) = \operatorname{vol}\left(B^{\omega_0}_{L(\Omega)}(\vec{0})\right) = \pi L^2(\Omega),$$

such that

$$\operatorname{vol}(\partial\Omega) < \operatorname{vol}\left(S^{\omega_0}_{L(\Omega)}(\vec{0})\right) = 2\pi L(\Omega).$$

Because otherwise, if for any precompact connected domain Ω we have that $\operatorname{vol}(\partial\Omega) \geq 2\pi L(\Omega)$ then, by the Faber-Krahn Theorem 4.2.6, we have that $\lambda_{1,g}(B_R(\vec{0}))$ should be greater or equal to $\lambda_{1,g_{\text{can}}}(B^{\omega_0}_{L(\Omega)}(\vec{0}))$, but this is a contradiction with inequality (4.28).

4.5 Upper bound for the first eigenvalue on geodesic balls by controlling the behaviour of the area function

In this section we show our second upper bound for the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls of Riemannian manifolds (M, g) by comparing it with the first eigenvalue on geodesic balls of with respect a rotationally symmetric metric tensor g_W. In order to show our result, we need to impose that the quotient A_g/A_{g_W} is decreasing.

Observe that when the Ricci curvatures of a Riemannian manifold (M, g) are bounded from below by the Ricci curvatures of a simply connected real space form $(\mathbb{M}_{\omega_\kappa}, g_{\omega_\kappa})$ of constant sectional curvature κ, *i.e.*, $\mathrm{Ric}_g \geq (n-1)\kappa$ (as the hypothesis of Cheng in Theorem 4.2.1), the function

$$r \longmapsto \frac{A_g(r)}{A_{g_{\omega_\kappa}}(r)}$$

is a decreasing function (see I. Chavel [10] for more details on this statement). This monotonicity condition of the area function will be our hypothesis in the following Theorem 4.5.1. But first let us remark that, to characterize the equality, S.Y. Cheng shows that the equality is attained if, and only if, the geodesic ball $B_R(o)$ of M is isometric to the ball with the same radius of $\mathbb{M}_{\omega_\kappa}$. However, with our weaker hypothesis, equality is attained if, and only if, we have the equality between the mean curvature of the geodesic spheres of M with the same radius and $\mathbb{M}_{\omega_\kappa}$. Moreover, we prove our result by comparing the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls of Riemannian manifolds with the first eigenvalue on rotationally symmetric model spaces.

Theorem 4.5.1. *Let (M, g) be a complete n-dimensional Riemannian manifold and $o \in M$. Let $B_R(o)$ be the geodesic ball of M with radius R centered at o. Let $W : [0, R] \longrightarrow \mathbb{R}$ be a non-negative smooth function such that the metric tensor*

$$g_W = dr \otimes dr + \left(W^2 \circ r\right) \pi^* g_{\mathbb{S}_1^{n-1}}$$

is smooth on $B_R(o)$. Suppose that $R < \mathrm{inj}_g(o)$ and that for any $r < R$ the function

$$r \mapsto \frac{A_g(r)}{A_{g_W}(r)}$$

is a decreasing function. Then, the first eigenvalue $\lambda_{1,g}(B_R(o))$ of the Laplacian Δ_g for the Dirichlet problem in $B_R(o)$ of radius R centered at o is bounded from above by the first eigenvalue of the Laplacian Δ_{g_W} for the Dirichlet problem in $B_R(o)$, namely,

$$\lambda_{1,g}(B_R(o)) \leq \lambda_{1,g_W}(B_R(o)). \tag{4.29}$$

Furthermore, equality in inequality (4.29) *is attained if, and only if, the mean curvature $H_{S_r(o)}$ of the geodesic sphere $S_r(o)$ is*

$$H_{S_r(o)}(p) = (n-1)\frac{W'(r)}{W(r)}$$

for all $0 < r < R$ *and for all* $p \in S_r(o)$, $r(p) = r$.

Proof. Let $\widetilde{g}$ be the rotationally symmetric metric tensor of comparison in $B_R(o)$ (see Definition 4.3.1). Hence, by using the same reasoning of the proof of Theorem 4.4.1, we have that

$$\lambda_{1,g}\left(B_R(o)\right) \leq \lambda_{1,\widetilde{g}}\left(B_R(o)\right). \tag{4.30}$$

Moreover, from assertion (4) of Proposition 4.3.4, we have that $A_{\widetilde{g}}(r) = A_g(r)$, and hence, by hypothesis, the function

$$r \mapsto \frac{A_{\widetilde{g}}(r)}{A_{g_W}(r)}$$

is assumed to be a decreasing function. Therefore, for all $r \in [0, R)$, we have

$$0 \geq \frac{d}{ds}\left(\frac{A_{\widetilde{g}}(s)}{A_{g_W}(s)}\right)\Bigg|_{s=r} = \frac{\frac{d}{ds}\left(A_{\widetilde{g}}(s)\right)\big|_{s=r} A_{g_W}(r) - A_{\widetilde{g}}(r)\,\frac{d}{ds}\left(A_{g_W}(s)\right)\big|_{s=r}}{A^2_{g_W}(r)}$$

$$= \frac{\frac{d}{dr}\left(\omega_g^{n-1}(r)\operatorname{vol}\left(\mathbb{S}_1^{n-1}\right)\right) W^{n-1}(r)\operatorname{vol}\left(\mathbb{S}_1^{n-1}\right)}{\left(W^{n-1}(r)\operatorname{vol}\left(\mathbb{S}_1^{n-1}\right)\right)^2}$$

$$- \frac{\omega_g^{n-1}(r)\operatorname{vol}\left(\mathbb{S}_1^{n-1}\right)\frac{d}{dr}\left(W^{n-1}(r)\operatorname{vol}\left(\mathbb{S}_1^{n-1}\right)\right)}{\left(W^{n-1}(r)\operatorname{vol}\left(\mathbb{S}_1^{n-1}\right)\right)^2}$$

$$= \frac{(n-1)\omega_g'(r)\omega_g^{n-2}(r)W^{n-1}(r) - \omega_g^{n-1}(r)(n-1)W'(r)W^{n-2}(r)}{W^{2(n-1)}(r)}$$

$$= \frac{(n-1)\omega_g^{n-2}(r)W^{n-2}(r)}{W^{2(n-1)}(r)}\left(\omega_g'(r)W(r) - \omega_g(r)W'(r)\right).$$

Thus, since $\omega_g(r)$ and $W(r)$ are greater or equal than 0 for all $0 \leq r < R$, we obtain that

$$\frac{\omega_g'(r)}{\omega_g(r)} \leq \frac{W'(r)}{W(r)}, \quad \text{for all} \quad r \in [0, R). \tag{4.31}$$

On the other hand, let us denote by $\phi_{1,W}(p) = f_{1,W}\left(r(p)\right)$ a positive first eigenfunction of the Laplacian Δ_{g_W} for the Dirichlet problem in $B_R(o)$ ($\phi_{1,W}$ is radial by Proposition 4.1.9). Then, from the expression of the Laplacian (2.32), we have, for any $p \in S_r(o)$, $r(p) = r$, that

$$\Delta_{g_W}\phi_{1,W}(p) = f_{1,W}''(r) + (n-1)\frac{W'(r)}{W(r)}f_{1,W}'(r) = -\lambda_{1,g_W}\left(B_R(o)\right)f_{1,W}(r).$$

Hence by (4.31), for any $q \in B_R(o)$ with $r(p) = r$, we obtain that

$$\begin{aligned}
-\frac{\Delta_{\widetilde{g}}\phi_{1,W}(p)}{\phi_{1,W}(p)} &= \frac{-f''_{1,W}(r) - (n-1)\frac{\omega'_g(r)}{\omega_g(r)} f'_{1,W}(r)}{f_{1,W}(r)} \\
&\leq \frac{-f''_{1,W}(r) - (n-1)\frac{W'(r)}{W(r)} f'_{1,W}(r)}{f_{1,W}(r)} \\
&= \frac{-\Delta_{g_W}\phi_{1,W}(q)}{f_{1,W}(r)} = \frac{\lambda_{1,g_W}\left(B_R(o)\right) f_{1,W}(r)}{f_{1,W}(r)} = \lambda_{1,g_W}\left(B_R(o)\right)
\end{aligned}$$

Finally, by (4.30), applying Barta's Lemma 4.1.8, and by the above inequality, we conclude that

$$\begin{aligned}
\lambda_{1,g}\left(B_R(o)\right) \leq \lambda_{1,\widetilde{g}}\left(B_R(o)\right) &\leq \sup_{B_R(o)}\left(-\frac{\Delta_{\widetilde{g}}\phi_{1,W}(p)}{\phi_{1,W}(p)}\right) \\
&\leq \sup_{B_R(o)} \lambda_{1,W}\left(B_R(o)\right) = \lambda_{1,W}\left(B_R(o)\right).
\end{aligned} \tag{4.32}$$

Now, for the equality case, let us first assume $\lambda_{1,g}\left(B_R(o)\right) = \lambda_{1,g_W}\left(B_R(o)\right)$. The equality of the first eigenvalues implies that all the inequalities in equation (4.32) became equalities, then we have that

$$\lambda_{1,\widetilde{g}}\left(B_R(o)\right) = \sup_{B_R(o)}\left(-\frac{\Delta_{\widetilde{g}}\phi_{1,W}(p)}{\phi_{1,W}(p)}\right).$$

Hence, by Barta's Lemma 4.1.8, we have that $\phi_{1,W}$ is a first positive eigenfunction of $\Delta_{\widetilde{g}}$. Therefore, following the same reasoning as in the proof of the equality of Theorem 4.4.1, the equality of the first eigenvalues implies that $\phi_{1,W}$ is also a first positive eigenfunction of Δ_g. Therefore, from the expression of the Laplacian Δ_g in a system of polar coordinates $(B_R(o), \psi)$ (see equation (2.26)), we have, for any $p \in B_R(o)$, that

$$\begin{aligned}
\Delta_g\phi_{1,W}(p) &= f''_{1,W}\left(r(p)\right) + \frac{\partial}{\partial r}\left(\ln\sqrt{\det\left(G(r,\theta)\right)}\right)\Big|_p f_{1,W}\left(r(p)\right) \\
&= -\lambda_{1,g}\left(B_R(o)\right) f_{1,W}\left(r(p)\right) = -\lambda_{1,g_W}\left(B_R(o)\right) f_{1,W}\left(r(p)\right) \\
&= f''_{1,W}\left(r(p)\right) + (n-1)\frac{W'\left(r(p)\right)}{W\left(r(p)\right)} f_{1,W}\left(r(p)\right) = \Delta_{g_W}\phi_{1,W}(p).
\end{aligned}$$

Then, for any point $p \in S_r(o)$, $r(p) = r$, we obtain that

$$H_{S_r(o)} f'_{1,W}(r) = (n-1)\frac{W'(r)}{W(r)} f'_{1,W}(r)$$

and hence, since $f'_{1,W}(r) < 0$ (see Proposition 4.1.9), we have

$$H_{S_r(o)} = (n-1)\frac{W'(r)}{W(r)} \tag{4.33}$$

showing the first direction of the equality case.

Another way for proving this consist in show that inequality (4.31) became an equality when $\lambda_{1,g}\left(B_R(o)\right) = \lambda_{1,W}\left(B_R(o)\right)$, by proving that $\Delta_{\widetilde{g}}\phi_{1,W} = \Delta_{g_W}\phi_{1,W}$, and then, using Remark 4.4.2, we obtain (4.33).

To end this proof, let us assume that $H_{S_r(o)} = \dfrac{W'(r)}{W(r)}$ for all $r \in (0, R)$. Then, for any $p \in B_R(o)$, we have that

$$\begin{aligned}\Delta_g\phi_{1,W}(p) = f''_{1,W}\left(r(p)\right) + (n-1)\frac{W'\left(r(p)\right)}{W\left(r(p)\right)} f_{1,W}\left(r(p)\right) = \Delta_{g_W}\phi_{1,W}(q)\\ = -\lambda_{1,g_W}\left(B_R(o)\right) f_{1,W}\left(r(p)\right).\end{aligned} \tag{4.34}$$

Hence, $\phi_{1,W}$ is a positive eigenfunction of Δ_g and then, since a first eigenfunction is the only eigenfunction which does not change sign (see Remark 4.1.6), we have that $\phi_{1,W}$ is a first eigenfunction of Δ_g and the Theorem follows.

□

Remark 4.5.2. Observe that, for the equality case of the above theorem, it is not sufficient to assume that the mean curvature is radial. If fact, if $H_{S_r(o)}$ is radial, we can ensure that it is equal to $\omega'_g(r)/\omega_g(r)$, following the proof of Theorem 4.4.1. But, in this case, it does not necessarily have to be equal to $W'(r)/W(r)$, and hence, we can not obtain equality (4.34).

Moreover note that, if the equality in inequality 4.29 is attained, then $\omega_g(r) = W(r)$ for all $r \in [0, R)$, and hence, g_W is the rotationally symmetric metric tensor of comparison $\widetilde{g}$.

4.6 Mean exit time, Poisson hierarchy, torsional rigidity, moment spectrum and first eigenvalue comparisons on geodesic balls

Concerning the analysis of the equality cases of the results shown along this work, there appears an important notion which is the concept of *determination* of a Riemannian invariant defined on the geodesic balls by its Poisson hierarchy, its moment spectrum, its averaged moment spectrum or its torsional rigidity, in a way which, although it is not exactly the same, it has been directly inspired by the notion of *determination* of a Riemannian invariant by the moment spectrum given by P. McDonald in [56]. In that paper, he presented this notion of *determination* as follows: *"it is said that the moment spectrum* $\{\mathcal{A}_k(\Omega)\}_{k=1}^{\infty}$ determines *the Riemannian invariant* $I(\Omega)$ *if, and only if, when* $\mathcal{A}_k(\Omega) = \mathcal{A}_k(\Omega')$ *for all* $k \in \mathbb{N}$ *then* $I(\Omega) = I(\Omega')$*"*.

Definition 4.6.1. *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold and let* $(\mathbb{M}_\omega, g_\omega)$ *be an* n*-dimensional rotationally symmetric model space with center* $o_\omega \in \mathbb{M}_\omega$*. Let* $o \in M$ *be a point in* M*. Given* $R < \mathrm{inj}_g(o) \leq \mathrm{inj}_{g_\omega}$ *and* $B_R(o)$ *a geodesic ball of* M *with radius* R *centered at* o*, and given* $B_R^\omega(o_\omega)$ *a geodesic balls of* $\mathbb{M}_\omega$ *with the same radius* R *centered at* o_ω*, we say that:*

1. *The Poisson hierarchy* $\{u_{k,R}\}_{k=1}^{\infty}$ *of* $B_R(o)$ determines *the first eigenvalue* $\lambda_{1,g}(B_R(o))$ *of the Laplacian for the Dirichlet problem on* $B_R(o)$ *if, and only if, the equalities* $u_{k,R} = \widetilde{u}_{k,R}^{\omega}$ *on* $B_R(o)$ *for all* $k \geq 1$ *implies that* $\lambda_{1,g}(B_R(o)) = \lambda_{1,g_\omega}(B_R^\omega(o_\omega))$*.*

2. *The moment spectrum* $\{\mathcal{A}_k(B_R(o))\}_{k=1}^{\infty}$ *of* $B_R(o)$ determines $\lambda_{1,g}(B_R(o))$ *if, and only if, the equalities* $\mathcal{A}_k(B_R(o)) = \mathcal{A}_k(B_R^\omega(o_\omega))$ *for all* $k \geq 1$ *implies that* $\lambda_{1,g}(B_R(o)) = \lambda_{1,g_\omega}(B_R^\omega(o_\omega))$*.*

3. *The torsional rigidity* $\mathcal{A}_1(B_R(o))$ *of* $B_R(o)$ determines $\lambda_{1,g}(B_R(o))$ *if, and only if, equality* $\mathcal{A}_1(B_R(o)) = \mathcal{A}_1(B_R^\omega(o_\omega))$ *implies that* $\lambda_{1,g}(B_R(o)) = \lambda_{1,g_\omega}(B_R^\omega(o_\omega))$*.*

4. *The first eigenvalue* $\lambda_{1,g}(B_R(o))$ determines *the Poisson hierarchy* $\{u_{k,R}\}_{k=1}^{\infty}$*, the moment spectrum* $\{\mathcal{A}_k(B_R(o))\}_{k=1}^{\infty}$*, the torsional rigidity* $\mathcal{A}_1(B_R(o))$ *and the volume* $\mathrm{vol}(B_R(o))$ *of the geodesic ball, respectivley, if, and only if, the equality* $\lambda_{1,g}(B_R(o)) = \lambda_{1,g_\omega}(B_R^\omega(o_\omega))$ *implies equalities* $u_{k,R} = \widetilde{u}_{k,R}^{\omega}$ *on* $B_R(o)$ *for all* $k \geq 1$*, equalities* $\mathcal{A}_k(B_R(o)) = \mathcal{A}_k(B_R^\omega(o_\omega))$ *for all* $k \geq 1$*, equality* $\mathcal{A}_1(B_R(o)) = \mathcal{A}_1(B_R^\omega(o_\omega))$ *and equality* $\mathrm{vol}(B_R(o)) = \mathrm{vol}(B_R^\omega(o_\omega))$*.*

where $\left\{\widetilde{u}^{\omega}_{k,R}\right\}_{k=1}^{\infty}$, $\left\{\mathcal{A}_k\left(B^{\omega}_R(o_\omega)\right)\right\}_{k=1}^{\omega}$, $\mathcal{A}_1\left(B^{\omega}_R(o_\omega)\right)$ *and* $\lambda_{1,g_\omega}\left(B^{\omega}_R(o_\omega)\right)$ *are, respectively, the transplanted Poisson hierarchy from* $B^{\omega}_R(o_\omega)$ *to* $B_R(o)$*, the moment spectrum of* $B^{\omega}_R(o_\omega)$*, the torsional rigidity of* $B^{\omega}_R(o_\omega)$ *and the first eigenvalue of the Laplacian for the Dirichlet problem on* $B^{\omega}_R(o_\omega)$ *(see Definitions 3.2.7, 3.5.1, 3.2.10, 3.2.5 and 4.1.3).*

From this definition, we are going to explore some relationships between all the invariants for which we find comparison results throughout this work. First, we show that the mean exit time function, the Poisson hierarchy, the torsional rigidity and the moment spectrum of a geodesic ball determines the first eigenvalue of the Laplacian for the Dirichlet problem on the geodesic ball by assuming bounds on the mean curvature of the geodesic spheres included in the geodesic ball.

Theorem 4.6.2. *Let* (M, g) *be a complete* n*-dimensional Riemannian manifold and let* $(\mathbb{M}_\omega, g_\omega)$ *be an* n*-dimensional rotationally symmetric model space with center* $o_\omega \in \mathbb{M}_\omega$*. Let* $o \in M$ *be a point of* M *and let* $B_R(o)$ *and* $B^{\omega}_R(o_\omega)$ *be, respectively, a geodesic ball of* M *with radius* R *centered at* o *and a geodesic ball of* $\mathbb{M}_\omega$ *with radius* R *centered at* o_ω*. Suppose that* $R < \operatorname{inj}_g(o) < \operatorname{inj}_{g_\omega}(o_\omega)$ *and suppose moreover that*

$$H_{S_r(o)} \geq (\leq) H_{S^{\omega}_r(o_\omega)} \quad \text{for all} \quad r \in (0, R].$$

Let $\{u_{k,R}\}_{k=1}^{\infty}$ *and* $\left\{\widetilde{u}^{\omega}_{k,R}\right\}_{k=1}^{\infty}$ *be, respectively, the Poission hierarchy for* $B_R(o)$ *and the transplanted Poisson hierarchy from* $B^{\omega}_R(o_\omega)$ *to* $B_R(o)$*. And let* $\left\{\mathcal{A}_k\left(B_R(o)\right)\right\}_{k=1}^{\infty}$ *and* $\left\{\mathcal{A}_k\left(B^{\omega}_R(o_\omega)\right)\right\}$ *be, respectively, the moment spectrum of* $B_R(o)$ *and* $B^{\omega}_R(o_\omega)$*.*

Then, if any of the following assertions is satisfied:

1. *There exists a point* $p \in B_R(o)$ *and* $k_0 \geq 1$ *such that* $u_{k_0,R}(p) = \widetilde{u}^{\omega}_{k_0,R}(p)$*.*

2. *There exists some* $k_0 \geq 1$ *such that we have*
$$\frac{\mathcal{A}_{k_0}\left(B_R(o)\right)}{\operatorname{vol}\left(S_R(o)\right)} = \frac{\mathcal{A}_{k_0}\left(B^{\omega}_R(o_\omega)\right)}{\operatorname{vol}\left(S^{\omega}_R(o_\omega)\right)}.$$

3. *The rotationally symmetric model space* $\mathbb{M}_\omega$ *is balanced from above and we have the equality of the torsional rigidities* $\mathcal{A}_1\left(B_R(o)\right) = \mathcal{A}_1\left(B^{\omega}_{s(R)}(o_\omega)\right)$*, where* $B^{\omega}_{s(R)}(o_\omega)$ *is the Schwarz symmetrization of* $B_R(o)$ *in* $\mathbb{M}_\omega$*.*

We have the equalities

$$\lambda_{1,g}\left(B_r(o)\right)=\lambda_{1,g_\omega}\left(B_r^\omega(o_\omega)\right),\quad \textit{for all}\quad r\in(0,R],$$

where $\lambda_{1,g}\left(B_r(o)\right)$ *and* $\lambda_{1,g_\omega}\left(B_r^\omega(o_\omega)\right)$ *are, respectively, the first eigenvalue of the Laplacian for the Dirichlet problem on* $B_r(o)$ *and* $B_r^\omega(o_\omega)$*.*

Proof. From the equality cases of Theorem 3.5.3, Corollary 3.5.5 and Theorem 3.6.3, we have, respectively, that any assertion (1), (2) and (3), implies the equality between the moment spectrum of $B_r(o)$ and $B_r^\omega(o_\omega)$ for all $r\in(0,R]$, *i.e.*, $\mathcal{A}_k\left(B_r(o)\right)=\mathcal{A}_k\left(B_r^\omega(o_\omega)\right)$ for all $k\geq 1$ and for all $r\in(0,R]$. Hence, applying Theorems 4.2.11 and 4.2.8, we obtain, for all $r\in(0,R]$, that

$$\lambda_{1,g}\left(B_r(o)\right)=\lim_{k\to\infty}k\frac{\mathcal{A}_{k-1}\left(B_r(o)\right)}{\mathcal{A}_k\left(B_r(o)\right)}=\lim_{k\to\infty}k\frac{\mathcal{A}_{k-1}\left(B_r^\omega(o_\omega)\right)}{\mathcal{A}_k\left(B_r^\omega(o_\omega)\right)}=\lambda_{1,g_\omega}\left(B_r^\omega(o_\omega)\right).$$

□

Finally, as a consequence of the results proved throughout Chapter 3, we have in following Theorem 4.6.3 a comparison of the first Dirichlet Eigenvalue of a geodesic ball by controlling the behaviour of the mean curvatures. This result is the comparison proved by G.P. Bessa and J.F. Montenegro in [5] (see Theorem 4.2.4) but we give an alternative proof, and moreover, we summarize some implications of the equality between the first eigenvalues. On the other hand, in Corollary 4.6.4, we have been able to show that, under our hypothesis, the first Dirichlet eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls determines its Poisson hierarchy and its moment spectrum.

Theorem 4.6.3. *Let* (M,g) *be a complete* n*-dimensional Riemannian manifold and let* $(\mathbb{M}_\omega,g_\omega)$ *be an* n*-dimensional rotationally symmetric model space with center* $o_\omega\in\mathbb{M}_\omega$*. Let* $o\in M$ *be a point of* M *and let* $B_R(o)$ *be a geodesic ball of* M *with radius* R *centered at* o*. Suppose that* $R<\mathrm{inj}_g(o)\leq\mathrm{inj}_{g_\omega}(o_\omega)$ *and moreover that*

$$H_{S_r(o)}\geq(\leq)H_{S_r^\omega(o_\omega)}\quad\textit{for all}\quad r\in(0,R].$$

Then,

$$\lambda_{1,g}\left(B_R(o)\right)\geq(\leq)\lambda_{1,g_\omega}\left(B_R^\omega\left(o_\omega\right)\right), \tag{4.35}$$

where $\lambda_{1,g}(B_R(o))$ *and* $\lambda_{1,g}(B^\omega_R(o_\omega))$ *are, respectively, the first eigenvalue of the Laplacian for the Dirichlet problem on* $B_R(o)$ *and* $B^\omega_R(o_\omega)$*.*

Furthermore, equality in inequality (4.35) *implies that*

$$H_{S_r(o)} = H_{S^\omega_r(o_\omega)} \quad \textit{for all} \quad r \in (0, R],$$

and hence, we have:

1. *The equality* $u_{k,r} = \widetilde{u}^\omega_{k,r}$ *on* $B_r(o)$ *for all* $k \geq 1$ *and for all* $r \in [0, R]$*.*
2. *The volume equalities* $\operatorname{vol}(B_r(o)) = \operatorname{vol}(B^\omega_r(o_\omega))$ *and the volume equalities* $\operatorname{vol}(S_r(o)) = \operatorname{vol}(S^\omega_r(o_\omega))$ *for all* $r \in (0, R]$*.*
3. *The equalities* $\mathcal{A}_k(B_r(o)) = \mathcal{A}_k(B^\omega_r(o_\omega))$ *for all* $k \geq 1$ *and for all* $r \in [0, R]$*.*

Namely, the first Dirichlet eigenvalue determines the Poisson hierarchy, the volume, and the moment spectrum of the geodesic balls $B_r(o)$ *for all* $r \in [0, R]$*.*

Proof. This proof follows the lines of the proof of Theorem 6 and 7 in [39]. This technique is based in the description of the first Dirichlet eigenvalue of smooth precompact domain D in a Riemannian manifold given by P. McDonald and R. Meyers in [57]. Thus, from Theorem 4.2.7 when $D = B_R(o)$, we have

$$\lambda_{1,g}(B_R(o)) = \sup\left\{\eta \geq 0 \ : \ \limsup_{k\to\infty} \left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k(B_R(o))}{\Gamma(k+1)} < \infty\right\}, \tag{4.36}$$

where Γ is the gamma function.

Assuming that $H_{S_r(o)} \geq H_{S^\omega_r(o_\omega)}$ for all $r \in (0, R]$ we have, by Corollary 3.5.5, that

$$\frac{\mathcal{A}_k(B_R(o))}{\operatorname{vol}(S_R(o))} \leq \frac{\mathcal{A}_k(B^\omega_R(o_\omega))}{\operatorname{vol}(S^\omega_R(o_\omega))} \quad \text{for all} \quad k \geq 1.$$

Then,

$$\mathcal{A}_k(B_R(o)) \leq \frac{\mathcal{A}_k(B^\omega_R(o_\omega))}{\operatorname{vol}(S^\omega_R(o_\omega))} \operatorname{vol}(S_R(o)) \quad \text{for all} \quad k \geq 1,$$

and hence,

$$\left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k(B_R(o))}{\Gamma(k+1)} \leq \left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k(B^\omega_R(o_\omega))}{\Gamma(k+1)} \frac{\operatorname{vol}(S_R(o))}{\operatorname{vol}(S^\omega_R(o_\omega))} \quad \text{for all} \quad k \geq 1, \tag{4.37}$$

On the other hand, by Corollary 3.4.5, we have that

$$\operatorname{vol}(S_R(o)) \geq \operatorname{vol}(S^\omega_R(o_\omega))$$

and hence,

$$\frac{\operatorname{vol}(S_R(o))}{\operatorname{vol}(S_R^\omega(o_\omega))} \geq 1. \tag{4.38}$$

Now, let us define the sets

$$\begin{aligned} D_1 &:= \left\{\eta \geq 0 \,:\, \limsup_{k\to\infty} \left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k\left(B_R^\omega(o_\omega)\right)}{\Gamma(k+1)} < \infty\right\}, \\ D_2 &:= \left\{\eta \geq 0 \,:\, \limsup_{k\to\infty} \left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k\left(B_R^\omega(o_\omega)\right)}{\Gamma(k+1)} \frac{\operatorname{vol}(S_R(o))}{\operatorname{vol}(S_R^\omega(o_\omega))} < \infty\right\}. \end{aligned}$$

Thus, using inequality (4.37), we obtain that if $\eta \in D_2$ then $\eta \in D_1$. Therefore, D_2 is included in D_1, *i.e.*, $D_2 \subseteq D_1$, and hence, $\sup D_1 \geq \sup D_2$. Then, applying inequality (4.38) and equation (4.36), we have that

$$\begin{aligned} \lambda_{1,g}\left(B_R(o)\right) &= \sup\left\{\eta \geq 0 \,:\, \limsup_{k\to\infty} \left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k\left(B_R(o)\right)}{\Gamma(k+1)} < \infty\right\} \\ &\geq \sup\left\{\eta \geq 0 \,:\, \limsup_{k\to\infty} \left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k\left(B_R^\omega(o_\omega)\right)}{\Gamma(k+1)} \frac{\operatorname{vol}(S_R(o))}{\operatorname{vol}(S_R^\omega(o_\omega))} < \infty\right\} \\ &= \frac{\operatorname{vol}(S_R(o))}{\operatorname{vol}(S_R^\omega(o_\omega))} \sup\left\{\eta \geq 0 \,:\, \limsup_{k\to\infty} \left(\frac{\eta}{2}\right)^k \frac{\mathcal{A}_k\left(B_R^\omega(o_\omega)\right)}{\Gamma(k+1)} < \infty\right\} \\ &= \frac{\operatorname{vol}(S_R(o))}{\operatorname{vol}(S_R^\omega(o_\omega))} \lambda_{1,g_\omega}\left(B_R^\omega(o_\omega)\right) \geq \lambda_{1,g_\omega}\left(B_R^\omega(o_\omega)\right). \end{aligned} \tag{4.39}$$

Now, we discuss the equality case by assuming that $H_{S_r(o)} \geq H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$. Then, equality between the first eigenvalues, $\lambda_{1,g}\left(B_R(o)\right) = \lambda_{1,g_\omega}\left(B_R^\omega(o_\omega)\right)$, implies that all the inequalities in (4.39) become equalities. In particular, we have that $\operatorname{vol}(S_R(o)) = \operatorname{vol}(S_R^\omega(o_\omega))$, and hence, we have the equality of the isoperimetric quotients

$$\frac{\operatorname{vol}(B_R(o))}{\operatorname{vol}(S_R(o))} = \frac{\operatorname{vol}(B_R^\omega(o_\omega))}{\operatorname{vol}(S_R^\omega(o_\omega))}.$$

Therefore, from the equality case of Corollary 3.4.5, we have that $H_{S_r(o)} = H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$. Then, from Proposition 3.4.3, we have the equality for the mean exit time function on $B_R(o)$ and the transplanted mean exit time function from $B_R^\omega(o_\omega)$ to $B_R(o)$, *i.e.*, $E_R = \mathbb{E}_R^\omega$ on $B_R(o)$, and hence, from the equality case of Theorem 3.5.3, we obtain that assertions (1), (2) and (3) hold.

When we assume that $H_{S_r(o)} \leq H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$, we argue as above but inverting all the inequalities to obtain the opposite inequality, *i.e.*, $\lambda_{1,g}\left(B_R(o)\right) \leq \lambda_{1,g_\omega}\left(B_R^\omega(o_\omega)\right)$. The equality discussion is the same, *mutatis mutandis.*

$\square$

We finish this work with a consequence of Theorems 4.6.2 and 3.5.3, which summarizes the relationship between the first eigenvalue of the Laplacian for the Dirichlet problem, the Poisson hierarchy and the moment spectrum on geodesic balls of Riemannian manifolds which satisfies our hypothesis on the behaviour of the mean curvatures of the geodesic spheres.

Corollary 4.6.4. *Let (M, g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ be a geodesic ball of M with radius R centered at o. Suppose that $R < \mathrm{inj}_g(o) \leq \mathrm{inj}_{g_\omega}(o_\omega)$ and moreover that*

$$H_{S_r(o)} \geq (\leq) H_{S_r^\omega(o_\omega)} \quad \textit{for all} \quad r \in (0, R].$$

Then, the following assertions are equivalent:

1. $\lambda_{1,g}\left(B_R(o)\right) = \lambda_{1,g_\omega}\left(B_R^\omega(o_\omega)\right)$.
2. $\mathcal{A}_k\left(B_R(o)\right) = \mathcal{A}_k\left(B_R^\omega(o_\omega)\right)$ *for all* $k \geq 1$.
3. $u_{k,R} = \widetilde{u}_{k,R}^\omega$ *on* $B_R(o)$ *for all* $k \geq 1$.

Furthermore, equality $H_{S_r(o)} = H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$ implies any (and hence, all) of the equalities (1), (2) *and* (3).

Proof. Let us assume that $H_{S_t(o)} \geq H_{S_t^\omega(o_\omega)}$ for all $t \in (0, R]$. When we assume that $H_{S_t(o)} \leq H_{S_t^\omega(o_\omega)}$ for all $t \in (0, R]$, the argument is exactly the same, *mutatis mutandis.*

First, from the equality case of Theorem 4.6.3, we have that equality (1) implies equalities (2) and (3).

Now, assuming equality (2) and arguing as in the proof of Theorem 4.6.2 we obtain equality (1), and hence, from the equality case of Theorem 4.6.3, we have equality (3).

Finally, assuming equality (3) we have, from Theorem 3.5.3, equality (2) and, from Theorem 4.6.2, we obtain (1) and the corollary follows.

□

Remark 4.6.5. Observe that we have shown, under the hypothesis $H_{S_r(o)} = H_{S_r^\omega(o_\omega)}$ for all $r \in (0, R]$, that the equality of the first eigenvalues of the Laplacian for the Dirichlet problem on the geodesic ball $B_R(o)$ determines the Poisson hierarchy (and hence, the mean exit time), the volume and the moment spectrum of $B_R(o)$. And moreover, that the Poisson hierarchy and the moment spectrum of $B_R(o)$ determine the first eigenvalue of $B_R(o)$. In fact, we have that one value: of the mean exit time at a point of $B_R(o)$, or of the Poisson hierarchy of $B_R(o)$ for some $k_0 \geq 1$, or of the averaged moment spectrum of $B_R(o)$ for some $k_0 \geq 1$, determines the Poisson hierarchy, the volume, the moment spectrum and the first eigenvalue of the geodesic ball $B_R(o)$.

The last natural question that remains is: what happens with the torsional rigidity? From Theorem 4.6.2 we know that if the rotationally symmetric model spaces is balanced from above and we have the equality between the torsional rigidity of a geodesic ball $B_R(o)$ and the torsional rigidity of its Schwarz symmetrization $B^\omega_{s(R)}(o_\omega)$, *i.e.*, $\mathcal{A}_1\left(B_R(o)\right) = \mathcal{A}_1(B^\omega_{s(R)}(o_\omega))$, then we have the equality of the first eigenvalue, and moreover, from the equality case of Theorem 3.6.3, we have that $\mathcal{A}_1\left(B_R(o)\right) = \mathcal{A}_1\left(B^\omega_R(o_\omega)\right)$ and the equalities for the Poisson hierarchy, the volume and the entire moment spectrum. Furthermore, since the equality of the first eigenvalues determines the moment spectrum, we have, in particular, the equality between the torsional rigidities.

But for now, note that we do not know, under our hypothesis, if the equality for the torsional rigidities of the geodesic balls with the same radius, $\mathcal{A}_1\left(B_R(o)\right) = \mathcal{A}_1\left(B^\omega_R(o_\omega)\right)$, determines the first eigenvalue, the Poisson hierarchy, the volume and the moment spectrum of the geodesic ball.

Chapter 5

Conclusions

We are going to present in the following statements of Theorems 5.1, 5.2 and 5.3 summarized versions of our results concerning bounds on the Poisson hierarchy and the moment spectrum of geodesic balls and its relationship with the first eigenvalue of the Laplacian for the Dirichlet problem (see Sections 3.5, 3.6 and 4.6 for more details on these results).

We shall see in Theorems 5.1 and 5.2 that if the mean curvatures of the geodesic spheres contained in the geodesic ball $B_R(o)$ of a Riemannian manifold (M, g) are bounded from below or from above by the mean curvatures of the corresponding geodesic spheres contained in the geodesic ball $B^{\omega}_R(o_{\omega})$ with the same radius of a rotationally symmetric model space, then the torsional rigidity $\mathcal{A}_1\left(B_R(o)\right)$ or any individual averaged moment $\mathcal{A}_{k_0}\left(B_R(o)\right)/\operatorname{vol}\left(S_R(o)\right)$ determines the Poisson hierarchy, the volume, the moment spectrum and the first eigenvalue of the Laplacian for the Dirichlet problem, in the sense that:

When $\mathcal{A}_1\left(B_R(o)\right) = \mathcal{A}_1(B^{\omega}_{s(R)}(o_{\omega}))$, or when there exists $k_0 \geq 1$ such that

$$\frac{\mathcal{A}_{k_0}\left(B_R(o)\right)}{\operatorname{vol}\left(S_R(o)\right)} = \frac{\mathcal{A}_{k_0}\left(B^{\omega}_R(o_{\omega})\right)}{\operatorname{vol}\left(S^{\omega}_R(o_{\omega})\right)},$$

then $s(R) = R$ and the Poisson hierarchy, the volume, the moment spectrum and the first eigenvalue of the Laplacian for the Dirichlet problem are the same than the corresponding values for the geodesic ball $B^{\omega}_R(o_{\omega})$ (see Definition 4.6.1 to check what we mean by saying that a geometric invariant determines another one). We refer to Section 3.2 of Chapter 3 for the definitions of Poisson hierarchy and moment spectrum of a geodesic ball $B_R(o)$, and to Section 4.1 of Chapter 4 for the definition of first eigenvalue of the Laplacian for the Dirichlet problem on $B_R(o)$.

Theorem 5.1 (see Corollary 3.5.5 and Theorem 4.6.2). *Let (M,g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ and $B^\omega_R(o_\omega)$ be, respectively, a geodesic ball of M with radius R centered at o and a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \operatorname{inj}_g(o) \leq \operatorname{inj}_{g_\omega}(o_\omega)$ and moreover that*

$$H_{S_r(o)} \geq (\leq) H_{S^\omega_r(o_\omega)} \quad \text{for all} \quad r \in (0,R].$$

Then, for all $k \geq 1$, we have that

$$\frac{\mathcal{A}_k\left(B_R(o)\right)}{\operatorname{vol}\left(S_R(o)\right)} \leq (\geq) \frac{\mathcal{A}_k\left(B^\omega_R(o_\omega)\right)}{\operatorname{vol}\left(S^\omega_R(o_\omega)\right)}. \tag{5.1}$$

Furthermore, equality in inequality (5.1) for some $k \geq 1$ is attained if, and only if, we have that any of the following assertions holds:

1. *The equalities $H_{S_r(o)} = H_{S^\omega_r(o_\omega)}$ for all $r \in (0,R]$.*
2. *The equalities $u_{k,r} = \widetilde{u}^\omega_{k,r}$ on $B_r(o)$ for all $k \geq 1$ and for all $r \in [0,R]$.*
3. *The volume equalities $\operatorname{vol}(B_r(o)) = \operatorname{vol}\left(B^\omega_r(o_\omega)\right)$ and the volume equalities $\operatorname{vol}\left(S_r(o)\right) = \operatorname{vol}\left(S^\omega_r(o_\omega)\right)$ for all $r \in (0,R]$.*
4. *The equalities $\mathcal{A}_k\left(B_r(o)\right) = \mathcal{A}_k\left(B^\omega_r(o_\omega)\right)$ for all $k \geq 1$ and for all $r \in [0,R]$.*
5. *The equalities $\lambda_{1,g}\left(B_r(o)\right) = \lambda_{1,g_\omega}\left(B^\omega_r(o_\omega)\right)$ for all $r \in (0,R]$.*

The second result is a comparison for the torsional rigidity of $B_R(o)$ and, in this case, we need the rotationally symmetric model space to be balanced from above. We refer to Section 3.2 of Chapter 3 for the definition of torsional rigidity of geodesic balls, and to Subsections 2.2.2 and 2.2.3 of Chapter 2 for the definitions of balance condition and Schwarz symmetrization of geodesic balls, respectively.

Theorem 5.2 (see Theorem 3.6.3 and 4.6.2)**.** *Let (M,g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space balanced from above with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ and $B_R^\omega(o_\omega)$ be, respectively, a geodesic ball of M with radius R centered at o and a geodesic ball of $\mathbb{M}_\omega$ with radius R centered at o_ω. Suppose that $R < \mathrm{inj}_g(o) \leq \mathrm{inj}_{g_\omega}(o_\omega)$, that there exists the Schwarz symmetrization $B^\omega_{s(R)}(o_\omega)$ of $B_R(o)$ in $\mathbb{M}_\omega$, and moreover that*

$$H_{S_r(o)} \geq (\leq) H_{S_r^\omega(o_\omega)} \quad \textit{for all} \quad r \in (0,R].$$

Then, we have that

$$\mathcal{A}_1\left(B_R(o)\right) \leq (\geq) \mathcal{A}_1\left(B^\omega_{s(R)}(o_\omega)\right), \tag{5.2}$$

where $\mathcal{A}_1\left(B_R(o)\right)$ and $\mathcal{A}_1\left(B^\omega_{s(R)}(o_\omega)\right)$ are, respectively, the torsional rigidity for $B_R(o)$ and for $B^\omega_{s(R)}(o_\omega)$.

Furthermore, equality in inequality (5.2) *is attained if, and only if, we have that any of the following assertions holds:*

1. *The equality among the radius $s(R) = R$.*
2. *The volume equalities* $\mathrm{vol}\left(B_r(o)\right) = \mathrm{vol}\left(B_r^\omega(o_\omega)\right)$ *and the volume equalities* $\mathrm{vol}\left(S_r(o)\right) = \mathrm{vol}\left(S_r^\omega(o_\omega)\right)$ *for all $r \in (0,R]$.*
3. *The equalities $H_{S_r(o)} = H_{S_r^\omega(o_\omega)}$ for all $r \in (0,R]$.*
4. *The equalities $u_{k,r} = \widetilde{u}^\omega_{k,r}$ on $B_r(o)$ for all $k \geq 1$ and for all $r \in [0,R]$.*
5. *The equalities $\mathcal{A}_k\left(B_r(o)\right) = \mathcal{A}_k\left(B_r^\omega(o_\omega)\right)$ for all $k \geq 1$ and for all $r \in [0,R]$.*
6. *The equalities $\lambda_{1,g}\left(B_r(o)\right) = \lambda_{1,g_\omega}\left(B_r^\omega(o_\omega)\right)$ for all $r \in (0,R]$.*

Furthermore, as a consequence of the proof of Theorem 1.1 in P. McDonald and R. Meyers [57], Theorem 5.1 and the volume inequalities that we showed in Corollary 3.4.5, we have the following S.Y. Cheng-type Dirichlet eigenvalue comparison, following the work of G.P. Bessa and J.F. Montenegro in [5]. In this case, we have proved that the first Dirichlet eigenvalue of the geodesic ball $B_R(o)$ determines its Poisson hierarchy, its volume and its moment spectrum.

Theorem 5.3 (see Theorem 4.6.3). *Let (M,g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_\omega, g_\omega)$ be an n-dimensional rotationally symmetric model space with center $o_\omega \in \mathbb{M}_\omega$. Let $o \in M$ be a point of M and let $B_R(o)$ be a geodesic ball of M with radius R centered at o. Suppose that $R < \mathrm{inj}_g(o) \leq \mathrm{inj}_{g_\omega}(o_\omega)$ and moreover that*

$$H_{S_r(o)} \geq (\leq) H_{S_r^\omega(o_\omega)} \quad \text{for all} \quad r \in (0,R].$$

Then,

$$\lambda_{1,g}\left(B_R(o)\right) \geq (\leq) \lambda_{1,g_\omega}\left(B_R^\omega(o_\omega)\right), \tag{5.3}$$

where $\lambda_{1,g}\left(B_R(o)\right)$ and $\lambda_{1,g}\left(B_R^\omega(o_\omega)\right)$ are, respectively, the first eigenvalue of the Laplacian for the Dirichlet problem on $B_R(o)$ and $B_R^\omega(o_\omega)$.

Furthermore, equality in inequality (5.3) is attained if, and only if, we have that any of the following assertions holds:

1. *The equalities $H_{S_r(o)} = H_{S_r^\omega(o_\omega)}$ for all $r \in (0,R]$.*
2. *The equality $u_{k,r} = \widetilde{u}_{k,r}^\omega$ on $B_r(o)$ for all $k \geq 1$ and for all $r \in [0,R]$.*
3. *The volume equalities $\mathrm{vol}\left(B_r(o)\right) = \mathrm{vol}\left(B_r^\omega(o_\omega)\right)$ and the volume equalities $\mathrm{vol}\left(S_r(o)\right) = \mathrm{vol}\left(S_r^\omega(o_\omega)\right)$ for all $r \in (0,R]$.*
4. *The equalities $\mathcal{A}_k\left(B_r(o)\right) = \mathcal{A}_k\left(B_r^\omega(o_\omega)\right)$ for all $k \geq 1$ and for all $r \in [0,R]$.*

Namely, the first Dirichlet eigenvalue determines the Poisson hierarchy, the volume, and the moment spectrum of the geodesic balls $B_r(o)$ for all $r \in [0,R]$.

On the other hand, given a complete n-dimensional Riemannian manifold (M,g) we know, from Section 4.3, that there exists a *rotationally symmetric model space of comparison* $(\mathbb{M}_{\omega_g}, g_{\omega_g})$ associated to M such that the volumes of the geodesic spheres of M coincide with volumes of the geodesic spheres of $\mathbb{M}_{\omega_g}$ with the same radius, *i.e.*, given $R < \mathrm{inj}_o \leq \mathrm{inj}_{g_{\omega_g}}(o_{\omega_g})$ we have that $\mathrm{vol}(S_r(o)) = \mathrm{vol}(S_r^{\omega_g}(o_{\omega_g}))$ for all $r \leq R$. Then, applying Theorem 4.4.1, we have that the first eigenvalue of the Laplacian for the Dirichlet problem on geodesic balls $B_R(o)$ of M is bounded from above by the first eigenvalue of the Laplacian for the Dirichlet problem on the corresponding geodesic balls in $\mathbb{M}_{\omega_g}$. In the following Theorem 5.4, we present our mentioned upper bound for the first eigenvalue of the Laplacian for the Dirichlet problem posed on geodesic balls, and moreover, we show that if the mean curvatures of the geodesic spheres of M coincide with the ones of the geodesic spheres of $\mathbb{M}_{\omega_g}$ with the same radius, then we have the equality between the first eigenvalues (and vice versa), and hence, from Corollary 4.6.4, we have that the first eigenvalue determines the Poisson hierarchy, the volume and the moment spectrum.

Theorem 5.4 (see Theorem 4.4.1 and Corollary 4.6.4)**.** *Let (M,g) be a complete n-dimensional Riemannian manifold and let $(\mathbb{M}_{\omega_g}, g_{\omega_g})$ be the rotationally symmetric model space of comparison associated to M with center $o_{\omega_g} \in \mathbb{M}_{\omega_g}$. Let $o \in M$ be a point of M and let $B_R(o)$ and $B_R^{\omega_g}(o_{\omega_g})$ be, respectively, a geodesic ball of M with radius R centered at o and a geodesic ball of $\mathbb{M}_{\omega_g}$ with radius R centered at o_{ω_g}. Suppose that $R < \mathrm{inj}_g(o) \leq \mathrm{inj}_{g_{\omega_g}}(o_{\omega_g})$. Then, the first eigenvalue $\lambda_{1,g}(B_R(o))$ of the Laplacian Δ_g for the Dirichlet problem in $B_R(o)$ is bounded by*

$$\lambda_{1,g}(B_R(o)) \leq \lambda_{1,g_{\omega_g}}(B_R^{\omega_g}(o_{\omega_g})), \tag{5.4}$$

where $\lambda_{1,g_{\omega_g}}(B_R^{\omega_g}(o_{\omega_g}))$ is the first eigenvalue of the Laplacian for the Dirichlet problem on $B_R^{\omega_g}(o_{\omega_g})$.

Furthermore, equality in inequality (5.4) *is attained if, and only if, we have that any of the following assertions holds:*

1. *The equalities $H_{S_r(o)} = H_{S_r^{\omega_g}(o_{\omega_g})}$ for all $r \in (0,R]$.*
2. *The equality $u_{k,r} = \widetilde{u}_{k,r}^{\omega_g}$ on $B_r(o)$ for all $k \geq 1$ and for all $r \in [0,R]$.*

Moreover, any of the conditions (1) *and* (2) *implies the equalities $\mathcal{A}_k(B_r(o)) = \mathcal{A}_k(B_r^{\omega_g}(o_{\omega_g}))$ for all $k \geq 1$ and for all radius $r \in [0,R]$.*

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