UNIVERSITAT DE BARCELONA

Essays on Dynamic Games, Sustainable Endogenous Growth and Regime Shifts

Carles Mañó-Cabello

PhD in Economics

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Als meus pares Lola i Boro,
i al meu germà Natxo

"In 1940, when Samuelson came to write his doctoral dissertation, he sensed that, although he was studying apparently diverse fields, he kept encountering and solving the same problems over and over. The economics might differ in each subfield, but the mathematical structure of the problems was the same. They were all about choosing the best possible outcome given the constraints facing people."

Roger Backhouse

Acknowledgments

As the great financial crisis of 2008 unfolded, I had little understanding of the trajectory and intricacies of economic science. At the time, I was attending high school in my hometown, Gandia, València. Those formative years were marked by a constant influx of economic news and uncertainty about macroeconomic developments. The "economic consequences" of the crisis, coupled with a reduction in the welfare state, sparked my curiosity about the daily events I saw, read, and heard. How could the collapse of banks in the United States of America and the falling prices of assets that "never go down" affect the working class in Southern European countries so profoundly? Like most of my friends, this crisis accelerated our nascent political awareness. In the summer of 2012, as Southern European countries faced challenges and such news dominated the media, I had to choose my Bachelor's. Motivated by curiosity and an engaging curriculum at the Universitat de València, I embarked on a journey to answer the many questions occupying my mind. During my first months at the university, I fell in love with the questions and methods of economics. I am grateful to Dr. Maria Paz Coscolla for guiding us through the initial steps of this adventure and igniting my passion for this field of knowledge. Thank you, Dr. Loles Alepúz, for helping me fall in love with intermediate microeconomics, and Dr. Jose Emilio Boscá for making me appreciate advanced and modern macroeconomics. Additionally, I am thankful to Dr. Maria Ángeles Tortosa for giving me the opportunity to ask questions and reflect "outside the box", and to Dr. Bernardí Cabrer Borràs for teaching us more advanced parts of econometrics and supporting me in my adventure of growth models with econometrics in my Final Degree Project (Treball Final de Grau). I extend my deepest gratitude to the Universitat de València for providing the opportunity to learn from exceptional professors and offering me the chance to study at the Johannes Gutenberg-Universität Mainz in Germany, thanks to the European Union's Erasmus grant. This adventure completely changed my life and the way I see the world. I am happy I met incredible people and got great friends such as Román, Rubén Rodríguez, Ruben, Marina, Manu, Sofia, Antonio, Yoan, Nieves, Juancar, Laura, Daniel, Valentina, and Gonzalo.

Moreover, I have been able to write this doctoral thesis and will soon begin my postdoc at KU Leuven (Belgium), thanks to the unwavering love and support of my Mother, Father and Brother, who became my primary motivation to continue studying. They have been my moral, familial, and financial support, even during difficult economic times, always investing in our future and providing their children with a university education they never had the chance to enjoy. Thank you for your immense effort and sacrifice. I will always be grateful for teaching me to think,

and I will forever appreciate the set of values you have instilled in me. Thank you for allowing us to experience countless wonderful activities and letting us choose our path without conditions. I am also grateful to my "güelo" and "güela" and my "yayo". I miss you all. I hope to enjoy the stories and love of my "yaya" in these last coming years.

Returning to my academic endeavors (and consequently, a part of my life journey), after completing my degree, I needed to continue acquiring knowledge and studying to have better tools for answering the questions that arose before studying economics. After living a year in Germany, I was eager to go to Barcelona. Having been accepted into master's programs at the Universitat Autònoma de Barcelona, Universitat Pompeu Fabra, and the Universitat de Barcelona. I chose the latter, where I received modest scholarships to cover my studies and had a public education fee. Jordi Roca would become the guardian angel for anyone applying to our master's program. Thank you, Jordi, for all your dedication, effort, and time. You always made us feel that we were part of a big project.

My time at the Universitat de Barcelona allowed me to study fascinating topics for long hours, providing me with a solid theoretical foundation that would prove crucial in my subsequent stages. I am grateful to the outstanding professors who introduced me to and made me think about amazing questions, for whom I hold a fond affection. From our Macroeconomics classes with Dr. Marc Teignier to the wonders of Game Theory with Dr. Marina Núñez, these are fundamental subjects in my doctoral thesis. Thank you, Marina, for making me fall in love with Game Theory! Additionally, I am grateful to the excellent Mathematics professors and now colleagues from the "Departament de Matemàtica Econòmica, Financera i Actuarial" Dr. Mikel Álvarez Mozos and Dr. Javier Martínez, from whom I learned so much. Thank you very much Dr. Carmen Ribas for giving me the opportunity to work in a place with natural light! I would also like to extend my gratitude to Dr. Ata Atay and Dr. Oriol Tejada for their kindness and invaluable advice, which I greatly appreciate. Additionally, I would like to express my appreciation to Dr. Pere Calleja and Dr. Francesc Llerena for their kindness.

In my second year of the master's program, I obtained another Erasmus grant to study at the Toulouse School of Economics (TSE), where I would meet wonderful people and study at one of the most renowned institutions in the field of economics. Thank you, Lluís and Julia, for following our plans in Barcelona, and thank you, Phillip, for our great time together.

Upon completing my second and final year of the master's program, I had the chance to undertake an internship at the OECD in Paris during the first months of my doctoral studies. Motivated by the precarious lack of funding in the initial months of the doctorate programs in Spain (the doctoral program begins in September, and

funding starts in April of the following year) and the uncertainty of where I would end up, I decided to apply for the Aquae Foundation and OECD scholarship, which seemed interesting, at the center of decision-making for developed countries. There, I had the opportunity to become familiar with Circular Economy, working closely with Oriana Romano. I will always remember my time in Paris fondly, where I met a group of “Jóvenes Talentos” who made my experience in Paris truly special. Thanks to Pierre and Hubert, who later came to Barcelona and shared an apartment with me, and to Andrés, Claudia, Cristina, Fabiola, Felipe, Jonathan, Lizeth, Sonia, and Xavi. Special thanks to my wonderful friends and brilliant minds, Matteo Schleicher and Jose Carlos Ortega Regalado, for providing me with so many great moments.

Upon returning to Barcelona in March 2019, I had to begin my doctoral research. I was eager to immerse myself in academic subjects again. I started my PhD with two of the most important people in my life during these last years. I would like to express my most sincere gratitude to my supervisors, and members of the “Departament de Matemàtica Econòmica, Financera i Actuarial”, Dr. Jesús Marín-Solano and Dr. Jorge Navas, who introduced me to the wonderful world of economic applications of Optimal Control Theory and Dynamic Programming, first as professors in my master’s program and later as PhD supervisors. Thank you for your patience, dedication, and guidance at all times, and for allowing me to experiment and satisfy my curiosity in my research. You have allowed me to learn from you and enjoy your conversations. It has been a pleasure to have you both as my supervisors. Additionally, special thanks go to my young colleague from the Dynamic Games group, with whom I had the pleasure of sharing teaching responsibilities, Dr. Julia De Frutos Cachorro, for her kindness and constant willingness to help me. Thank you for all your support!

Furthermore, I must thank the University of Barcelona for giving me the opportunity to conduct my research with an APIF grant, although it was only for three years, contrary to the Statute of Research Personnel in Training (EPIF), Real Decreto 103/2019, and for which “Doctorandes en Lluita” demands a fourth year.

During my PhD, I had the opportunity to undertake a research stay at McGill University in Montreal (Canada) with Professor Hassan Benchekroun, who is incredible researcher and a wonderfully exceptional human being. Thank you, Hassan, for all our hours together, walks, and meals. Thank you for inviting me into your home. I would also like to thank McGill professors Francisco Alvarez-Cuadrado, Markus Poschke, and Mayssun El-Attar for making me feel at home. This research stay would have been impossible without the financial support of the “Societat Econòmica Barcelonesa d’Amics del País”, with their “mobility grants for young researchers”. My time at McGill allowed me to explore fascinating concepts in a field I had been interested in for years, Mean Field Games, and Hassan provided the necessary touch for me to dive into a highly demanding and complicated world. In Montreal, I was also lucky to find a group of young people who made me feel at home. Thank you, Rodrigo, for our basketball games and talks about your post-doc in Philosophy at UdeM. Thank you, (almost Dr.) Jorge Agulló and (recently Dr.) Nuria Morales Puerto, for our wonderful moments and conversations about our lives, linguistics, philosophy, neuroscience, politics, and other interesting topics. The Zipi and Zape of Montreal will always be a marvelous duo. Thanks also to Claudia for her fantastic personality and great humor, and to Anthony for welcoming us to Canada.

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I would also like to express my gratitude to Dr. Sareh Vosooghi from KU Leuven and Dr. Rick van der Ploeg from the University of Oxford for their trust and for offering me the opportunity to join them as a postdoctoral researcher. I am confident that my involvement in projects under their expert guidance will leave an enduring impact on my future academic pursuits.

I cannot forget my dear colleague Gerard Domènech (always a potential comedian in the shadow), who has given me magnificent moments of humor, and thanks to my colleague in the math department, Joakim, for our time together and our conversations. Thanks to my dear friends Vincent, Pere, Jeff, Claudia, Guillem, and Filippo, for providing me with moments of happiness and an exciting intellectual environment. I would also like to thank Lili, Marianna, Mattia, Abhinav, Marta, Carlos, Candan, Kaan, Dani, Ghizlen, Ceci, Alejandra, and Marie, for all our conversations and the great moments we have shared. Also thank you to the new PhD students. I hope we all enjoy this incredible journey!

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I would also like to give special thanks to seemingly abstract concepts, the welfare state, and public investment. This entire journey has been possible because of public education and the investment and effort made by taxpayers. Thank you for educating us, taking care of our health, and funding our research. My gratitude extends to all the teachers at IES Maria Enriquez de Gandia for their patience with us and for teaching us how to think – a necessary condition to responsibly engage as citizens. Particularly, I am deeply grateful to Enric Castanyer for helping me appreciate history and recognize its true significance in contemporary society, and to Pepe Llorens for inspiring a profound love for mathematics within me and revealing its inherent beauty. I also wish to express my sincere thanks to Pura for leading us out of the cave and helping us appreciate philosophy, something I am, of course, deeply grateful for years later. I must also acknowledge all of my teachers at the CEIP Roíç de Corella, who guided us and provided us with the essential foundation for becoming good citizens and fostering curiosity during that crucial developmental stage. Thank you very much Paqui Mascarell Miñana! Thank you to "l'Escola Pública".

Finally, thank you to all those who have helped me to be happy in this life. After all, that is what economics is all about, seeing how people can be happier. And I apologize for the times when I have been absent and have not given back the corresponding love.

Carles Mañó-Cabello
Barcelona, 2023

Contents

1 Introduction

The pressing need to address the intertwined challenges of economic growth, environmental sustainability, and strategic interactions among economic agents has never been more urgent. Our world faces unprecedented environmental and economic challenges, such as climate change, depletion of natural resources, and growing income inequalities driven by varying growth impacts. To navigate these complex issues and design effective policies, it is essential to understand the dynamic and strategic nature of economic systems. This thesis highlights the crucial role that dynamic games, environmental economics, and the study of economic growth play in enhancing our comprehension of these challenges. In doing so, it establishes the groundwork for the development of effective and well-informed policies. Moreover, the thesis integrates valuable insights from behavioral economics, encompassing time-inconsistencies and status concerns, and interweaves ideas of regime switching to further enrich the analysis.

Time inconsistency occurs when the preferences of a decision-maker evolve over time, deviating from their original intentions or plans. This phenomenon typically emerges when individuals or policymakers must weigh short-term benefits against long-term goals. As time progresses, their guiding preferences may shift, resulting in less-than-ideal outcomes or the abandonment of previously set long-term goals. Grasping the concept of time-inconsistency is vital for understanding a range of economic issues, as it can compromise policy efficacy and contribute to market failures. On the other hand, status concern preferences refer to situations where individuals or participants in an economic context derive satisfaction not only from their own actions but also from their relative position compared to others. In such scenarios, players may engage in strategic behaviors to enhance or preserve their comparative status, leading to economic outcomes that differ from those in cases without status concerns. Comprehending status concern preferences is essential for examining various economic phenomena, as it can impact consumption behaviors, labor market choices, and investment decisions, as well as the formulation and efficacy of public policies targeting social and economic inequalities. Moreover, endogenous regime switching occurs when economic, financial, or environmental systems transition between distinct states or regimes as a direct result of the strategic choices made by the agents involved. In these scenarios, agents actively decide when to switch between regimes based on their assessments of costs, benefits, and the prevailing circumstances.

The interdisciplinary approach employed in this research enables us to synthesize insights from various fields of study, fostering a holistic understanding of the complex issues under consideration. Throughout the subsequent paragraphs, we will try to demonstrate the critical significance of this research by offering a persuasive justification for exploring the captivating realm of dynamic games, environmental economics, and economic growth while incorporating invaluable knowledge and perspectives gleaned from advancements in psychological science over recent decades.

Dynamic games offer a powerful framework for analyzing strategic interactions and dynamic processes in multi-agent systems. By capturing the complex interdependencies and feedback effects that arise in economic systems, dynamic games provide valuable insights into the behavior of economic agents and the potential consequences of their decisions. Environmental economics, on the other hand, is a vital field that seeks to harmonize economic activities with environmental sustainability. It recognizes the indispensable role of natural resources and ecosystems in sustaining human welfare and fostering responsible economic practices. This discipline systematically assesses the complex connections between the economy and the environment, striving to find a balance that ensures both human prosperity and ecological preservation. By integrating the perspectives and methodologies of dynamic games and environmental economics, this thesis endeavors to cultivate a more sophisticated comprehension of the multifaceted relationships among human activities, strategic behavior, and environmental sustainability. Thus, studying environmental economics is essential for shaping effective policies and solutions that safeguard our natural world without compromising human well-being.

Moreover, the importance of understanding human decision-making and its influence on future events cannot be overstated, as it has profound implications for the evolution of humankind. Examples of such decisions include determining the extent of pollution today in exchange for immediate wealth versus a future where humans suffer the consequences of past pollution and climate damage, or weighing the benefits of extracting more natural resources against the drawbacks of depleting those resources. Additionally, there is the dilemma of choosing between immediate gratification through consumption today and saving for consumption tomorrow to reap the benefits of compound interest. Noting the prevalence of human "misbehaving", authors like Richard Thaler propose plans to "nudge" humans toward strategies that are in their best interest (Thaler and Sunstein, 2008; Thaler, 2015). Building upon the ideas of the previous author, and incorporating ideas presented in Daniel Kahneman's book "Thinking, Fast and Slow", we also emphasize the need to incorporate psychological factors in economic models for a more accurate understanding of decision-making processes. As Kahneman (2011) wrote in chapter 25 of his book:

"The field had a theory, expected utility theory, which was the foundation of the rational-agent model and is to this day the most important theory in the social sciences. Expected utility theory was not intended as a psychological model; it was a logic of choice, based on elementary rules (axioms) of rationality. [...] The mathematician John von Neumann, one of the giant intellectual figures of the twentieth century, and the economist Oskar Morgenstern had derived their theory of rational choice between gambles from a few axioms. Economists adopted expected utility theory in a dual role: as a logic that prescribes how decisions should be made, and as a description of how Econs make choices. Amos and I were psychologists, however, and we set out to understand how Humans actually make risky choices, without assuming anything about their rationality."

By interweaving elements from behavioral economics, this research aims to enhance the examination of dynamic games and environmental economics, facilitating a more profound grasp of the intricate connections among human decisions, strategic interactions, and environmental sustainability.

It is essential to recognize that the models in this doctoral thesis should be viewed as frames of reference or tools for thought experiments rather than direct mappings of reality. By using these models, researchers can reflect on complex issues and draw conclusions that would otherwise be highly costly or nearly impossible to attain. This approach to modeling economic behavior aligns with the ongoing paradigm shift in the field of economics, where scholars are progressively integrating insights from behavioral economics into their analyses, even without explicitly identifying them as behavioral concepts. This convergence of ideas reflects a growing recognition of the value and relevance of incorporating behavioral perspectives into the study of economic systems and decision-making processes.

In what follows, we will walk through the origins of game theory, its importance, its connection with optimal control theory, and how differential games emerged when game theory met optimal control theory, revealing the captivating interplay between these two remarkable fields. We will also explain the importance of including non-constant discounting and examining regime shifts, as both elements contribute to a profound understanding of the nuanced challenges investigated in this dissertation.

1.1 Game Theory: Theoretical Foundations and Applications

Game theory, a mathematical framework for modeling and analyzing situations of strategic interaction among decision-makers, has become an essential tool for understanding various aspects of economics, including environmental economics. We will now examine the origins of game theory, its evolution, its applications in economics, and its connections to optimal control theory, and differential games. We will also provide an overview of its historical development and significant research in the field.

The foundations of game theory can be traced back to the early 20th century, with the pioneering work of mathematicians such as Ernst Zermelo, Émile Borel, John von Neumann, and economist Oskar Morgenstern. Zermelo's work published in 1913 on chess showed that in a finite two-player game with perfect information and no chance, if a tie cannot happen, then one player has a winning strategy Zermelo (1913).[1] Later, mathematician Émile Borel published a series of papers from 1921 to 1927 that defined games of strategy Borel (1921, 1924, 1927).[2] However, years later, John von Neumann contended that Borel's definition of games of strategy was flawed, as he did not accept the minimax theorem (Von Neumann and Fréchet, 1953). Nevertheless, other authors claimed that the work by Cournot (1838), Bertrand (1883), and Edgeworth (1881) on oligopoly pricing and production were the first studies of games in economics but "were seen as special models that did little to change the way economists thought about most problems" (Fudenberg and Tirole, 1991).

In the 1920s, John von Neumann made significant contributions to the development of game theory, most notably the minimax theorem for zero-sum games (von Neumann, 1928). Building on this foundational work, von Neumann and Oskar Morgenstern published their seminal book "Theory of Games and Economic Behavior" in 1944, marking the beginning of modern game theory and laying the groundwork for its subsequent development and application across various fields (von Neumann and Morgenstern, 1944). The book also laid the foundations for various solution concepts, such as the Nash equilibrium, which was later introduced by John Nash in a concise, one-page paper in 1950 (Nash, 1950), and further de-

[1] The paper was published in German with the title *Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels* (On an Application of Set Theory to the Theory of the Game of Chess) https://web.archive.org/web/20131217224959if_/http://www.socio.ethz.ch/publications/spieltheorie/klassiker/Zermelo_Uber_eine_Anwendung_der_Mengenlehre_auf_die_Theorie_des_Schachspiels.pdf.

[2] https://www.britannica.com/science/function-mathematics

veloped in Nash (1951). For an interesting historical perspective on the creation of game theory see Leonard (1995) and the chapter "Introduction to the Theory of Games" in Basar and Zaccour (2018).

Over the years, game theory has evolved and expanded, with numerous scholars contributing to the development of the field. Among the most influential contributors are several Nobel laureates in economics, such as John Nash, who shared the 1994 Nobel Prize with Reinhard Selten, and John Harsanyi for their pioneering work on game theory as a tool for analyzing strategic interactions. Furthermore, in 2005, Thomas Schelling and Robert Aumann were awarded the Nobel Prize for their research on conflict and cooperation through game theory. Two years later, three more game theorists, Leonid Hurwicz, Eric S. Maskin, and Roger B. Myerson were awarded the Nobel Prize for their work on mechanism design. Elinor Ostrom's 2009 Nobel Prize can also be considered a recognition within the field of game theory, as her analysis of economic governance, particularly the study of the commons and non-market institutions such as natural resources managed by common property and firms, has become a vibrant area of research in dynamic games nowadays. In 2012, two more game theorists, Alvin E. Roth and Lloyd S. Shapley were recognized with the Nobel Prize for studying stable allocations and market designs. French economist Jean Tirole received the same honor in 2014 for his analysis of market power and regulation. In 2016, two other game theorists, Oliver Hart, and Bengt Holmström were recognized for their work on contract theory. Recently, in 2020 Paul R. Milgrom and Robert B. Wilson were also awarded the Nobel Prize for their work in auction theory and inventions of new auction formats. While Douglas Diamond and Philip Dybvig may not be considered traditional game theorists, their innovative model of bank runs, developed in 1983, ultimately earned them the Nobel Prize in 2022.

Game theory has significantly contributed to our understanding of human behavior in competitive and cooperative settings across a wide array of fields, including economics, political science, and biology. The diverse models encompassed by game theory, ranging from static games with complete information to dynamic games with incomplete information, have found applications in numerous areas, reflecting the versatility and adaptability of game theory as an analytical tool (Myerson, 1997). The work of the aforementioned Nobel laureates, along with other scholars, has led to the development of several key concepts and refinements within game theory. Among these are the concepts of Nash equilibrium, developed by John Nash in 1950, and subgame perfect equilibrium, introduced by Reinhard Selten (Selten, 1965). These refinements of the equilibrium concept have proven instrumental in guiding our understanding of strategic behavior across a myriad of contexts, such as oligopolistic competition, contract theory, mechanism design, auctions, and bargaining (Tirole, 1988).

In the broader economic landscape, game theory has proven invaluable for understanding how firms compete in markets, how countries engage in international trade, and how individuals make choices in strategic situations. By synergistically combining game theory with insights from behavioral economics, researchers have been able to delve deeper into the decision-making processes of individuals across various contexts (Camerer, 2011). This interdisciplinary fusion has not only broadened the scope of game theory but also enriched our understanding of complex economic phenomena, leading to more effective policies and strategies.

Environmental economics, in particular, offers a fascinating and policy-relevant application of game theory. In this field, strategic interactions among agents play a crucial role in determining the exploitation of natural resources and the provision of public goods, such as clean air and water. By employing game-theoretic models, researchers have successfully analyzed and proposed solutions to complex environmental challenges, including the tragedy of the commons postulated by Hardin (1968), the management of transboundary pollution (Barrett, 1994b), and the negotiation of international climate agreements (Barrett, 1994a; Rubio and Ulph, 2006; Barrett, 2016; Battaglini and Harstad, 2016; Harstad, 2016; Harstad et al., 2019). This integration of game theory into environmental economics demonstrates its versatility and ongoing relevance in addressing contemporary economic issues.

In conclusion, game theory has come a long way since its inception in the early 20th century, playing a critical role in shaping modern economic research, particularly in the realm of environmental economics. The pioneering work of mathematicians and economists, alongside the contributions of numerous Nobel laureates, has shaped the field and demonstrated its wide-ranging applicability across various disciplines. As an interdisciplinary tool, game theory has greatly influenced the way researchers approach and study complex environmental issues, offering valuable insights and strategies for managing scarce resources and fostering sustainable development. As we continue to face global challenges, the insights provided by game theory will be crucial in guiding research, informing policy, and fostering cooperation among diverse stakeholders. In essence, game theory has proven to be an invaluable and indispensable resource in understanding and navigating the complex strategic interactions that define our world and addressing the contemporary challenges that face our global society.

Good books on game theory are, for instance, Fudenberg and Tirole (1991), Osborne (2004), Osborne and Rubinstein (1994), among many others. In the absence of any time dependence, a game Γ in its normal form game is characterized by a *set of players* $\mathscr{N}$, the *set of strategies* (or actions) $\mathscr{A}^i$ for each player $i \in \mathscr{N}$, with its *payoff function* $\mathscr{U}^i : \mathscr{A} \to \mathbb{R}$ for each player $i \in \mathscr{N}$. One defines $\mathscr{A} := \prod_{i \in \mathscr{N}} \mathscr{A}^i$

as the product of the strategies (or actions) available to player $\mathscr{A}^i$, and $a = (a^i)_{i\in\mathscr{N}}$ as the set of (pure) strategy profiles. Therefore, the normal form game can be represented as

$$\Gamma = \left(\mathscr{N}, \mathscr{A}, \{\mathscr{U}^i(\cdot)\}_{i\in\mathscr{N}}\right),$$

which describes a static game with complete information. Moreover, it is also assumed that each player fulfills the assumptions of "rationality" and "common knowledge". The first concept means that each player chooses her strategy (or action) a^i to maximize her payoff $\mathscr{U}^i(a^i, a^{-i})$, which is a function of her own strategy a^i and what all the other players will do $a^{-i} := \left(a^j\right)_{j\in\mathscr{N}\setminus\{i\}}$. The second concept, common knowledge, implies that each player is aware of the rules of the game, and knows that the other players know the rules of the game, and that those other players know that she knows the rules of the game and so one and so forth *ad infinitum*.

One can define a Nash Equilibrium in the game Γ as the strategy profile $a^* = (a^{i*})_{i\in\mathscr{N}} \in \mathscr{A}$ as the strategy with the property that for every player $i \in \mathscr{N}$, we have

$$U^i\left(a^{i*}, a^{-i*}\right) \geq U^i\left(a^i, a^{-i*}\right), \quad \forall a^i \in \mathscr{A}^i, \tag{1.1}$$

capturing the idea that a Nash Equilibrium is a strategy profile a^* where each player maximizes her own payoff given that all the other agents are playing their equilibrium strategies. Put differently, no player has an incentive to deviate. This is one of the most important results in game theory.

As we move to explore the connections between game theory and other mathematical frameworks, we will first delve into the calculus of variations, a field that has played a critical role in the development of optimal control theory.

1.2 From Calculus of Variations to Optimal Control Theory

The history of the calculus of variations, a branch of mathematical analysis concerned with the optimization of functionals (functions that map functions to scalars), is deeply rooted in the quest for solving optimization problems and can be traced back to the ancient world. Among the earliest instances of these optimization problems are the concepts of minimum distance in Euclidean space and the famous isoperimetric problem. These mathematical ideas laid the foundation for the development of the calculus of variations and continue to be relevant in modern research. The concept of the minimum distance in Euclidean space is a fundamental notion that originated from the study of geometry. The idea is simple: in a two-dimensional plane, the shortest distance between two points is a straight line.

Another significant optimization problem from antiquity is the isoperimetric problem, which can be traced back to the legendary tale of Queen Dido, the founder of Carthage (modern Republic of Tunisia). The story goes that Queen Dido wanted to build a new city on the coast of North Africa and was given as much land as she could enclose with a single oxhide. To maximize the area of her new domain, Dido cleverly cut the oxhide into thin strips and arranged them into a circle, forming the largest possible area enclosed by a given perimeter. This tale highlights the essence of the isoperimetric problem: finding the shape with the largest possible area for a given boundary length. The solution to this problem, as hypothesized by Queen Dido, is the circle.

From the intuitive understanding of the minimum distance and the isoperimetric problem in antiquity, the calculus of variations has evolved into a sophisticated mathematical tool that allows for the optimization of a wide array of problems. Among the most famous problems in the calculus of variations is the brachistochrone problem, first posed by Johann Bernoulli in 1696. The term "brachistochrone" is derived from the Greek words "brachistos" (shortest) and "chronos" (time), aptly reflecting the nature of the problem. The Brachistochrone problem seeks to determine the curve of fastest descent, that is, the path taken by a particle moving under the influence of gravity along a curve, connecting two points in the plane, which minimizes the time of travel.

Bernoulli's challenge to the mathematical community to solve the Brachistochrone problem in 1696 attracted the attention of prominent mathematicians. The solution, as independently discovered by Newton, Jakob Bernoulli, Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus, and Guillaume de l'Hôpital, is the cycloid, a curve traced by a point on the circumference of a circle as it rolls along a straight line without slipping. The significance of this problem lies not only in the solution itself but also in the techniques and concepts that emerged as a result of its investigation.

Serving as a catalyst for the development of the calculus of variations, the Brachistochrone problem introduced the notion of a functional and the concept of minimizing a functional subject to constraints. This new field began to take shape in the late 17th and early 18th centuries, building upon early optimization problems such as Newton's minimal resistance problem in 1687 and Johann Bernoulli's Brachistochrone curve problem in 1696. The techniques and concepts derived from these problems found applications in numerous fields, such as physics, engineering, and economics, and laid the foundation for the further development of the calculus of variations. One of the most important developments in this field was the introduction of the Euler-Lagrange equation, which helped in finding extrema of functionals.

As a result, the Brachistochrone problem not only highlighted the power and relevance of the calculus of variations in the study of optimization problems but also established its vital role in modern mathematical analysis.

The contributions of Leonhard Euler and Joseph-Louis Lagrange in the 18th century marked a turning point in the development of the calculus of variations. Euler's groundbreaking work on the Euler-Lagrange equation provided a systematic method for solving variational problems, greatly expanding the scope and applicability of the field. The simplest variational problem is

$$\max \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))dt, \quad \text{subject to} \quad x(t_0) = x_0, \quad x(t_1) = x_1. \tag{1.2}$$

The equation allowing to search for the solution (of the extremums, which may not be maximums) is the so-called Euler-Lagrange equation, or the fundamental lemma of calculus of variations. One gets extremum only if the Euler-Lagrange differential equation is satisfied. It was already discovered by the Swiss mathematician Leonhard Euler in 1744 and is given by[3]

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0, \tag{1.3}$$

where $x(t)$ and $\dot{x}(t)$ represent a function that evolves over time and its first derivative, respectively. This equation has become a cornerstone of modern calculus of variations, as it allows researchers to find the extrema of functionals subject to various constraints.

The 19th and 20th centuries saw further advancements in the calculus of variations, as mathematicians like Carl Gustav Jacobi, William Rowan Hamilton, and Richard Courant contributed to the refinement and generalization of the theory. In particular, the development of the Hamiltonian formalism and the Pontryagin Maximum Principle extended the applicability of variational techniques to a wide range of fields. For instance, in economics, some of its first applications were by Ramsey (1928), where he studied an optimal saving problem, and by Hotelling (1931), where he studied a problem of finding the optimal extraction of a non-renewable resource, which we use in Chapter 4.

The foundations of the calculus of variations paved the way for the development of optimal control theory.

[3] See, for instance, Sydsæter et al. (2008).

1.3 On Optimal Control Theory and its Impact on Economics

Optimal control theory has established itself as a fundamental and indispensable tool in a wide range of disciplines. The theory's core tenet is the systematic determination of the best possible control for a given dynamic system in order to achieve a desired objective, subject to certain constraints. Next, we will explore the applications of optimal control theory, focusing on its importance and influence in the field of economics. This dissertation will employ the framework of continuous-time methods, reflecting the pertinence of these techniques in our investigation.

The genesis of optimal control theory can be traced back to the work of mathematicians Richard Bellman and Lev Pontryagin in the 1950s. Bellman (1957) introduced the concept of dynamic programming, a method for solving complex problems by breaking them down into smaller, more manageable subproblems.[4] The second major advance was made by Pontryagin and his colleagues in 1956, who developed the Maximum Principle, a necessary condition for an optimal solution (see Pontryagin et al. (1962) for a version in English). These two groundbreaking contributions laid the foundation for the modern framework of optimal control theory.

Optimal control theory has evolved significantly since its inception, encompassing an array of techniques and methodologies, each tailored to address specific classes of problems. The rich and diverse nature of this field has enabled it to find applications in fields such as aerospace, robotics, finance, and, of course, economics. The application of this theory in economics has a long history, with seminal contributions from researchers such as David Cass, and Tjalling Koopmans. These pioneers applied the principles of optimal control to a variety of economic problems, such as growth theory, resource allocation, and intertemporal consumption, paving the way for its widespread adoption in economic research. Building on the foundation laid by pioneers like Ramsey (1928), Cass (1965), and Koopmans (1965), the application of optimal control theory in economics has significantly influenced the study of optimal economic growth.[5] These groundbreaking models used optimal control to optimally allocate resources between consumption and investment, thereby maximizing intertemporal welfare. The scope of this research

[4] Although as mentioned in Weber (2011), "[t]he idea of dynamic programming precedes Bellman's work: for example, von Neumann and Morgenstern (1944, ch. 15) used backward induction to solve sequential decision problems in perfect-information games."

[5] Tjalling C. Koopmans shared the Nobel Prize in Economics in 1975 with Leonid Kantorovich for their contributions to the theory of optimum allocation of resources https://www.nobelprize.org/prizes/economic-sciences/1975/press-release/

has expanded over time, incorporating elements such as endogenous technological progress, environmental constraints, and various other factors, which will be thoroughly discussed in Chapter 4. In addition to growth theory, optimal control theory has found applications in monetary and fiscal policy, international trade, environmental economics, and finance, among others. In monetary and fiscal policies, one can include optimal inflation targeting and optimal taxation. In international trade models, one can study the optimal tariff policies. In environmental economics, one can study the optimal extraction of a renewable resource, the amount of emissions, and climate change mitigation strategies.[6]

In this dissertation, we will employ the tools of optimal control theory, highlighting its applications in economics. First, we just state the basic problem. By doing so, we aim to provide a comprehensive understanding of this powerful analytical tool and its continued relevance in the ever-evolving landscape of economic research.

Let us consider a general optimal control problem, which can be formulated as follows:

$$\underset{\{u(s)\}_{s\in[t_0,T]}}{\text{Opt}} \quad J(x,u,t) = \int_{t_0}^{T} L(s,x(s),u(s))ds \tag{1.4}$$

$$\text{subject to} \quad \dot{x}^i(s) = f^i(s,x(s),u(s)), \tag{1.5}$$

$$x(t_0) = x_0, \quad \text{for} \quad i = 1,...,n. \tag{1.6}$$

In the problem formulation above, the vector of control variables is formed by $u(s) = (u^1(s),...,u^m(s)) \in U \subset \mathbb{R}^m$, representing the decision-making processes that influence the system, i.e., what to do at any moment in time. Observe that the decision-maker has m different decisions to choose, which are $u^1(s), u^2(s), ..., u^j(s)$, up to $u^m(s)$. This general formulation allows the agent to choose many different important variables in the system. The vector of state variables of the system at time s is denoted by $x(s) = (x^1(s),...,x^n(s)) \in X \subset \mathbb{R}^n$. Observe that now the system has n different dynamics or variables that evolve in that system. The dynamics of the system are described by a set of differential equations, one equation per state variable, $\dot{x}^i(s) = f^i(s,x(s),u(s))$, with the corresponding initial condition $x^i(t_0) = x_0^i$, which describes how the system starts at the initial time t_0. Moreover, $L : [t_o,T] \times X \times U \to \mathbb{R}$ represents the discounted utility function at time s. One should consider

[6] Central banks, for instance, frequently employ optimal control methods to determine the best course of action for achieving macroeconomic stability. In each of these contexts, the elegance and versatility of optimal control theory have proven invaluable in providing rigorous, quantitative solutions to complex economic problems.

$L(s,x(s),u(s))$ to be a general function (including the discount function if needed).

To address such an optimal control problem, various techniques can be employed. Two of the most prominent approaches are Bellman's dynamic programming and Pontryagin's Maximum Principle. Dynamic programming is well-suited for both discrete and continuous time systems and relies on the principle of optimality, which states that an optimal trajectory can be decomposed into an initial suboptimal trajectory followed by an optimal trajectory. Pontryagin's Maximum Principle, on the other hand, provides a set of necessary conditions for optimality in continuous-time systems. The principle introduces the concept of an adjoint variable, in this case will be the vector $\lambda(s) = (\lambda^1(s),...,\lambda^n(s))$, and a Hamiltonian function, $H(s,x(s),u(s),\lambda(s))$, defined as:

$$H(s,x(s),u(s),\lambda(s)) = \lambda_0 L(s,x(s),u(s)) + \lambda(s) f(s,x(s),u(s)). \tag{1.7}$$

The Maximum Principle[7] states that for an optimal control trajectory $u^*(s)$ and a corresponding optimal state trajectory $x^*(s)$, there exists a constant λ_0, with $\lambda_0 = 0$ or $\lambda_0 = 1$, and a continuous and piecewise differentiable adjoint vector function $\lambda(s) = (\lambda^1(s),...,\lambda^n(s))$, such that for all time $s \in [t_0,T]$, one has $(\lambda_0,\lambda(s)) \neq (0,\mathbf{0})$, and:

1. The control function $u^*(s)$ maximizes the Hamiltonian $H(s,x^*(s),u,\lambda(s))$ for $u \in U$, that is,

$$H(s,x^*(s),u,\lambda(s)) \leq H(s,x^*(s),u^*(s),\lambda(s)) \quad \text{for all } u \text{ in } U, \tag{1.8}$$

2. Whenever $u^*(s)$ is continuous, the adjoint function satisfy

$$\frac{d\lambda^i(s)}{ds} = -\frac{\partial}{\partial x^i}\left[H(s,x^*(s),u^*(s),\lambda(s))\right] \quad \text{for } i = 1,...,n \tag{1.9}$$

3. Given different terminal conditions, it should be that

   a) When $x^i(T) = x^i_1$, there are no conditions for $\lambda^i(T)$

   b) When $x^i(T) \geq x^i_1$, then $\lambda^i(T) \geq 0$ ($\lambda^i(T) = 0$ if $x^{i,*} > x^i_1$)

   c) When $x^i(T)$ is free, $\lambda^i(T) = 0$.

These conditions, when combined with the system dynamics, provide a set of equations that can be solved to obtain the optimal control and state trajectories. See Sydsæter et al. (2008), Weber (2011), Sethi (2019), Kamien and Schwartz (1991) and Liberzon (2011).

[7] We follow Sydsæter et al. (2008).

Moreover, suppose that $(x^*(s), u^*(s))$ is an optimal pair with a corresponding adjoint function $\lambda(s)$, together with $\lambda_0 = 1$, such that the necessary conditions are met. If the control region U is convex and $H(s,x,u,\lambda(s))$ is concave with respect to (x,u) for every time $s \in [t_0, T]$, together with the partial derivatives $\partial L/\partial u^j$ and $\partial f^i/\partial u^j$ existing, then $(x^*(s), u^*(s))$ is an optimal pair. This is known as the Mangasarian sufficiency conditions.[8] For the infinite horizon problem, see the previous references.

Throughout this dissertation, we will further explore the nuances and intricacies of optimal control theory within the realm of economics. For a captivating account of the history of the calculus of variations and optimal control theory, see Goldstine (1980) and the previous references.

1.4 Differential Games: Bridging Game Theory and Optimal Control Theory

Differential games is an area of research that brings together the concepts and tools from game theory and optimal control theory to study strategic interactions in dynamic settings. It harmonizes these two previous fields by modeling scenarios where agents make decisions over time, considering the actions and reactions of others. It was first introduced by Rufus Isaacs in the 1950s as a means to analyze pursuit-evasion problems in the context of military operations (Isaacs, 1965).

Since then, differential games has evolved into a vital instrument for comprehending the strategic behavior of economic agents in dynamic settings. Consequently, in today's increasingly interconnected world, understanding the dynamics of strategic interactions between such economic agents is crucial. Thus, this is where differential game theory equips economists with a powerful tool to model and analyze these interactions, shedding light on optimal decision-making in complex and uncertain environments. This approach enables economists to better grasp the intricacies of the global economy, anticipate future trends, and inform policymaking. An example of the application of differential games in economics is the study of dynamic competition between firms, where firms choose their strategies over time to maximize their profits while taking into account the actions of their competitors. In these settings, differential games can help to analyze issues such as entry and exit, pricing, and investment decisions. In particular, environmental economics has greatly benefited from the implementation of differential games techniques. The dynamic nature of environmental problems, such as resource ex-

[8] One could also consider Arrow's sufficient conditions.

traction and pollution control, makes them well-suited for analysis using this framework. For example, researchers have used differential games to study the strategic behavior of countries in addressing transboundary pollution problems, where each country's actions affect the others' environments. Differential games have also been employed to analyze the optimal management of renewable resources, such as fisheries and forests, where the actions of multiple agents can significantly impact the long-term sustainability of these resources. A comprehensive exploration of differential games with economics applications can be found in Dockner et al. (2000), while Lambertini (2018) offers a detailed analysis of applications in industrial organization. Additionally, for insights into the applications of dynamic games in the economics and management of pollution, refer to Jørgensen et al. (2010) and the cited references therein. The authors thoroughly discuss challenges associated with pollution control instruments, such as quotas, taxes, subsidies, and tradable emission permits. Furthermore, they address issues in transboundary pollution scenarios and macroeconomic concerns, incorporating aspects such as economic and population growth, climate change, income and technology transfers, as well as the pursuit of sustainable development.

As a result of combining insights from game theory, optimal control theory, and differential games, researchers in environmental economics have been able to better understand the complex interactions between economic agents and the environment. This understanding has, in turn, informed the design of policies and institutions that promote the sustainable use and management of environmental resources.

1.5 On Discouting

Discounting is a fundamental aspect of economics and intertemporal decision-making, playing a crucial role in determining the value of costs and benefits that occur at different points in time. The importance of discounting cannot be overstated, particularly when it comes to evaluating long-term projects and policies, such as climate change mitigation, natural resource management, public infrastructure investments, and research and development initiatives. As a result, discounting will hold a central position in this dissertation, featuring prominently in the problem statement of Chapters 2 and 4. In the subsequent section, we will explain the motivation underlying our research problem and its importance.

As clearly expressed in Myerson et al. (2001), "[d]iscounting is a pervasive phenomenon in decision making by humans and nonhuman animals. The results of a large number of experiments using delayed rewards have shown that the subjective value of a delayed reward is less than the value of an immediate reward of the same nominal amount (e.g., Green, Fry, & Myerson, 1994; Kirby, 1997; Mazur, 1987; Myerson & Green, 1995; Rachlin, 1989). More specifically, the value of a reward has been shown to decrease as a function of delay, and this phenomenon is termed temporal discounting." The idea of discounting and self-control can be traced back to the work published by the Scottish philosopher Adam Smith (2010 [1759]).[9]

Additionally, how human beings compare current and future events has also been the subject of extensive research in psychology and economics since the nineteenth century. Pioneering studies on the study of intertemporal decisions can go back to the work by Rae (1834), Senior (1836), von Böhm-Bawerk (1889) and Jevons (1879 [1871]), where the last author highlighted, "[t]he intensity of present anticipated feeling must, to use a mathematical expression, be some function of the future actual feeling and of the intervening time, and it must increase as we approach the moment of realisation. The change, again, must be less rapid the farther we are from the moment, and more rapid as we come nearer to it. An event which is to happen a year hence affects us on the average about as much one day as another; but an event of importance, which is to take place three days hence, will probably affect us on each of the intervening days more acutely than the last". Fisher (1930), who used the term "impatience", argued that such time preferences showed a lack of foresight and self-control. Moreover, (Ramsey, 1928, p. 543) preferred to use a discount rate equal to zero.[10] In addition, Pigou (2017 [1920]) also thought that discounting was a sign of the poor functioning of our imagination.[11] Nevertheless, despite the reluctance of previous authors, it was in the work by Samuelson (1937) when the Discounted Utility (DU) Model was popularized. In his model, decision-makers are characterized by having a constant discount rate, and time preferences are time-consistent. This is the well-known exponential discounting problem, which has as discount function

$$\theta(s-t) = e^{-\rho(s-t)}, \tag{1.10}$$

where $\theta(s-t)$ is the discount function, ρ is the constant discount rate, time t is when the agent takes the decisions, and time s is when she experiences such a decision

[9] Adam Smith wrote in his first book "*The theory of moral sentiments*" that "[t]he pleasure which we are to enjoy ten years hence interests us so little in comparison with that which we may enjoy to-day, the passion which the first excites, is naturally so weak in comparison with that violent emotion which the second is apt to give occasion to, that the one could never be any balance to the other".

[10] His justification was that "we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination."

[11] Pigou (2017 [1920]) wrote: "[b]ut this preference for present pleasures does not - the idea is self-contradictory - imply that a present pleasure of given magnitude is any greater than a future pleasure of the same magnitude. It implies only that our telescopic faculty is defective, and that we, therefore, see future pleasures, as it were, on a diminished scale."

(number of years into the future until the action occurs).[12]

The exponential discounting approach, characterized by a constant discount rate, has been widely adopted due to its simplicity and leading to time-consistency. However, it has been criticized for its lack of empirical and theoretical foundations, with evidence suggesting that individuals often rely on hyperbolic discounting (Frederick et al., 2002). A rich body of literature supports the time varying discount rate framework, extensively surveyed by the previously mentioned authors and DellaVigna (2009) providing seminal insights into this area.

Time inconsistency, the discrepancy between the choices made by an individual at different points in time, has also been widely studied in Strotz (1955), Phelps and Pollak (1968) and Laibson (1997). This has led to the consideration of alternative discounting models, such as quasi-hyperbolic discounting, which can better capture the observed behavioral patterns in intertemporal decision-making. Strotz (1955) showed that exponential discounting with constant rates of time preferences was the only discount function that generated time-consistency, and Phelps and Pollak (1968) introduced the (quasi)hyperbolic (or quasigeometric) discount functions in discrete time.

Some authors give validity to the hypothesis that individuals are simply not aware of their future impatience (Caliendo and Aadland, 2007; Findley and Caliendo, 2014). However, others argue that humans behave in a sophisticated, or time-consistent manner, as Barro (1999), Karp (2007), Karp and Tsur (2011), Tsoukis et al. (2017) or Cabo et al. (2020a). The first author of this latter group initiated the literature on non-constant discounting and economic growth by studying the logarithmic and power utilities. He concluded that for a neoclassical growth model with log utility, exponential discounting is observationally equivalent to quasi-hyperbolic (non-constant) discounting. This is precisely what we get for our procrastinator agent under an endogenous growth model with natural resources in Chapter 4. Nonetheless, this equivalence does not hold when we consider elasticities of substitution different from one (Pollak, 1968; Marín-Solano and Navas, 2009; De-Paz et al., 2014, 2013; Farzin and Wendner, 2014).

Moreover, Marín-Solano and Patxot (2012) examined a distinct form of time-inconsistent preferences by introducing heterogeneous discounting in a deterministic setting. In their finite horizon model, the utility payments derived from consump-

[12]Even Paul Samuelson, the proponent of the exponential formulation, harbored doubts regarding its suitability as a representation of an individual's preferences and as a representation of the collective preferences of individuals, writing "[i]t is completely arbitrary to assume that the individual behaves so as to maximize an integral of [this] form" (p.159), and further arguing that "any connection between utility as discussed here and any welfare concept is disavowed" (p.161). However, as Frederick et al. (2002) clearly state, "despite Samuelson's manifest reservations, the simplicity and elegance of this formulation was irresistible".

tion during the planning horizon were discounted at a rate (ρ_1) different from that of the final function (ρ_2), which could represent savings to be enjoyed post-retirement. This approach recognizes that it may be unrealistic to assume that the enjoyment of various goods should be discounted at the same rate, as the final function can be viewed as a distinct type of good.

In one of the problems studied in Chapter 2, we incorporate different (constant) discount rates in our regime-switch model. This can be justified in several ways. For instance, given that $e^{-\rho_2(T-t)} = e^{-\rho_1(T-t)} \cdot e^{(\rho_1-\rho_2)(T-t)}$, when $\rho_2 > \rho_1$, this function increases with time t. As t approaches the final time T, the agent assigns greater value to the final term, representing the adoption of new technology in our model. Consequently, the decision-maker exhibits a present bias that diminishes as the technology adoption approaches. This aligns with the psychological perceptions of many decision-makers, who often assign increasing value to a regime change as it approaches. Moreover, we also consider general non-constant discount functions as in Karp (2007) and Marín-Solano and Navas (2009).

Additionally, it is claimed in the literature that individuals invest or save too little when they discount the future hyperbolically and are subject to self-control problems (Laibson, 1997, 1998). This highlights the fact that individuals are more patient concerning decisions in the distant future and highly impatient when decisions take place in the near future. Recent work analyzing present bias in consumption-saving models are Maxted (2020) and Laibson et al. (2021).

As it has been studied in the neoclassical literature, rational agents come to play an important role. Nevertheless, as expressed by Kahneman (2011), "[r]ational agents are expected to know their tastes, both present and future, and they are supposed to make good decisions that will maximize these interests".

Additionally, the choice of the discount rate has significant implications for long-term decision-making and policy, as evidenced by the disagreement between Nobel Prize laureate William Nordhaus and Nicolas Stern on climate policy, following the Stern (2007) Review. As clearly highlighted in Harstad (2020):

"Over the past decades, our profession has settled on employing exponential discounting, partly because preferences are then likely to be time-consistent. Apart from the convenience, however, there are few reasons to impose exponential discounting as a reasonable model of decision-making. The lack of empirical and theoretical foundations for exponential discounting will be reviewed [later], suggesting that individuals often rely on hyperbolic discounting. I also explain why, even if every individual and voter applies constant discount factors, policymakers who rotate being in office will evaluate investment projects using discount factors that increase (i.e., discount rates decrease) in relative time. Intuitively, even if everyone wants a future government to invest for the future, those ending up in office may

rather prefer perks. This time-inconsistency problem turns out to be particularly severe for investment projects that are associated with externalities, such as climate change."

Time preferences are defined generally by the discount function $\theta(s-t) \geq 0$, which is not just a function of the time when the control (consumption) is enjoyed, i.e., time s, but a function of the time distance from the present t (when the decision is made). This time distance between when consumption is enjoyed and when the decision is made will be defined as $j \equiv s-t$. The discount function $\theta(j)$ satisfies the properties $\theta(j) > 0$, $\dot{\theta}(j) < 0$, for all $j > 0$ and $\theta(0) = 1$. The *instantaneous discount rate* $\rho(j) \equiv -\dot{\theta}(j)/\theta(j)$ does not have to be constant and decreases with the time distance from the present, i.e., $\rho(j) > 0$, $\dot{\rho}(j) \leq 0$, for all $j > 0$ (Laibson, 1997; Barro, 1999). One might conceptualize it as the (negative) growth rate of the discount function. Intuitively, this means that the further an event is in the future, the "less important" it is for the present agent at time t. In contrast, under the standard time distance exponential discounting popularized in Samuelson (1937), where $\theta(s-t) = e^{-\rho(s-t)} = e^{-\rho j} = \theta(j)$, one can observe that $\rho(j) = -\dot{\theta}(j)/\theta(j) = -\dfrac{(-\rho)e^{-\rho j}}{e^{-\rho j}} = \rho \in \mathbb{R}_{++}$ is constant in comparison to a general discount function. Put differently, the logarithmic rate of change of the discount is constant, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}j}\left[ln\left(\theta(j)\right)\right] = \frac{\mathrm{d}}{\mathrm{d}j}\left[ln\left(e^{-\rho(j)}\right)\right] = \frac{\mathrm{d}}{\mathrm{d}j}\left[-\rho j\right] = -\rho,$$

which ensures time-consistency in standard inter-temporal problems (Strotz, 1955). The problem of time-inconsistencies has deep historical roots, even appearing in the ancient epic, The Odyssey:

"but you must bind me hard and fast, so that I cannot stir from the spot where you will stand me... and if I beg you to release me, you must tighten and add to my bonds."[13]

Therefore, understanding the intricacies of discounting is crucial for designing effective policies that address long-term challenges. Time-inconsistency can motivate political measures such as subsidies or taxes, normally reserved for traditional market failures. Furthermore, the interaction between time-inconsistency and traditional market failures, such as spillovers and externalities between countries, can have important implications for international environmental policy and participation in international environmental agreements.

By examining the interplay between discounting, time-inconsistency, and policy, we seek to contribute to the understanding of how discounting influences long-term decision-making and the design of effective policy interventions.

[13] See the quote in (Strotz, 1955).

1.6 On Switching

Economists should pay close attention to switching problems and optimal switching regimes for several compelling reasons. First and foremost, real-world economic systems frequently involve multiple states or regimes, with agents needing to determine when to transition from one state to another. By investigating these switching problems, economists can acquire a deeper understanding of how decision-makers optimize their actions when confronted with diverse scenarios, ultimately improving outcomes in various sectors.

There are many real-world examples that demonstrate the importance of studying optimal switching problems. In the energy sector, utility companies can use these models to determine the ideal transition points between different power generation sources, such as fossil fuels, renewable energy, and nuclear power, while balancing energy production costs, environmental impact, and supply reliability. Similarly, the agricultural sector benefits from understanding optimal switching models, as it would help farmers decide the optimal time to transition between different crops or cultivation techniques, taking into account factors like climate, soil conditions, and market demand to maximize yield and profit. In financial markets, investors and portfolio managers can use optimal switching models to decide when to transition between different investment strategies or asset classes, optimizing risk-return profiles for improved investment outcomes. Climate policy also benefits from these models, as they assist governments and policymakers in determining the optimal timing and conditions for implementing various climate change mitigation strategies, enabling the identification of the most effective and cost-efficient policy mix.

Furthermore, switching problems are particularly pertinent in the context of technological advancements and evolving market conditions. As new technologies emerge and market dynamics shift, comprehending the optimal timing and conditions for adopting novel methods or transitioning between market strategies becomes critical for economic agents to maintain competitiveness and secure long-term profitability. Manufacturing firms facing the challenge of deciding when to upgrade production technologies or switch between different production processes can rely on optimal switching models to identify the most cost-effective times and conditions for transitioning. The study of optimal switching problems and regimes seamlessly integrates also into diverse contexts, such as environmental economics, where adopting greener technologies can enhance efficiency and reduce emissions. Moreover, there are interesting applications in the field of international economics where countries may decide whether to join a free trade agreement or a climate change agreement. One can also find exciting macroeconomic applications such as adopting a new currency and giving up their monetary policy (for instance, European countries adopting the Euro). In addition, there are direct applications to health economics, such as deciding when to stop smoking, when to start going to the gym, or even in transport economics, deciding when to switch to a more efficient but more expensive electric car. The case of adopting a more efficient technology will be studied in Chapter 2.

The extensive applicability and adaptability of optimal switching models in diverse economic contexts highlight their crucial importance in modern economic research. By enabling more informed, strategic decision-making, these models contribute significantly to the efficient allocation of resources, the identification of optimal policy interventions, and the overall enhancement of economic welfare. As the global economy faces ongoing evolution and new challenges, the insights derived from optimal switching models prove indispensable in fostering innovation, advancing sustainable development, and deepening our understanding of the intricate dynamics shaping the economic landscape. Moreover, the study of optimal switching problems bolsters the creation of robust economic models that account for the inherent complexities characteristic of real-world decision-making processes. Integrating switching mechanisms into economic models allows economists to deliver more accurate forecasts and policy recommendations, ultimately promoting improved economic outcomes for society as a whole.

1.7 Conclusion

In conclusion, the interdisciplinary fusion of game theory and optimal control theory that led to the development of differential games has had a profound impact on economics, particularly in the field of environmental economics. The development of these sophisticated mathematical frameworks and their applications, which allow us to unravel strategic behavior between agents, the impact of dynamic competition, and the implications for natural resource extraction, have provided invaluable insights that have helped to shape new policies and improve our understanding of the challenges and opportunities facing the economy in the coming decades and centuries. This confluence of theoretical and applied knowledge underlines the crucial role of these interdisciplinary approaches in shaping a more sustainable future.

Therefore, by acknowledging the historical context of our research, we can better appreciate the importance of the theories and methodologies we use. As we continue to explore the applications of game theory, optimal control theory, and differential games, we will be following the steps of pioneers like Von Neumann, Nash, Bellman, Pontryagin, and Isaacs, who helped shape our understanding of complex dynamic systems and the ways in which individuals and organizations interact with

	1 agent	n-agents/players
Partial Equilibrium	2^{nd} Chapter	3^{rd} Chapter
General Equilibrium	$\nexists$	4^{th} Chapter

Table 1.1: Evolution of the doctoral thesis.

one another.

The structure of the thesis is as follows. In Chapter 2, "*Time to Switch: Different Regimes and Competitive Management of Natural Resources with Technology Adoption under Time Inconsistent Preferences*", we investigate an optimal switching point problem with time-inconsistent preferences. In this case, a time-consistent (sophisticated) decision-maker chooses the time of switching between two consecutive regimes. The corresponding dynamic programming equations are presented, and conditions for deriving the switching time by agents with different degrees of sophistication are studied. In Chapter 3, "*What is my Neighbor Doing? Heterogeneous agents under Free Trade with Renewable Resources*", we study a dynamic (differential) game where two countries extract a renewable resource when they show "status concern" preferences, and they are "keeping up with the Joneses", which means that they also care about the performance of other players. We compare the implications of staying under autarky or having free trade between countries and how they extract the resource. We also analyze the welfare implications of the different games. Finally, in Chapter 4, entitled "*Being Human: Endogenous Growth, Pollution and Natural Resources under Time Inconsistent Preferences*", we examine the implications of agents having time-inconsistent behavior (procrastination), an intrinsic part of being humans (Thaler, 2015), in an endogenous growth model with non-renewable resources and pollution. Pollution is a by-product of economic activity and the extraction of an exhaustible resource, which negatively affects agents. Curiously, we show that when decision-makers have "human behaviors", they can enjoy higher levels of "well-being" (sum of discounted utilities) than those who are time-consistent under the strong observational equivalence principle. By modeling time preferences in a general setting, we contribute to the behavioral macroeconomics debate and shed light on its social implications.

In order to provide a visual representation of the overall structure of the thesis, Table 1.1 is presented as an intuitive guide. The second chapter will present a comprehensive model that involves a single agent in a partial equilibrium setting. The third chapter will study a two-player game under partial equilibrium conditions. Finally, in the fourth chapter, we will investigate the complex dynamics that arise when multiple agents interact and make decisions in a general equilibrium framework.

In conclusion, the research presented in this thesis may have important implications for our understanding of the complex challenges that confront our world today. By delving into the intricacies of dynamic games, environmental economics, behavioral economics, switching problems, and economic growth, we can develop the intellectual tools and insights necessary to inform the design of effective policies and institutions that promote economic prosperity, social equity, and environmental sustainability. We hope that our work will contribute to the ongoing development and application of these powerful mathematical frameworks in the pursuit of a better understanding of our world and the improvement of our societies.

2 Time to Switch: Different Regimes and Competitive Management of Natural Resources with Technology Adoption under Time Inconsistent Preferences[1]

Abstract

A two-stage non-standard optimal control problem with time-inconsistent preferences is studied. In an infinite horizon setting, a time-consistent (sophisticated) decision-maker chooses the time of switching between two consecutive regimes. The second regime corresponds to the implementation of a new technology, and a cost must be paid at the switching time. Although the problem is formulated for a general discount function, special attention is devoted to models with nonconstant discounting and heterogeneous discounting. The problem is solved by transforming it into a problem in finite horizon and free terminal time. The corresponding dynamic programming equations are presented, and conditions for the derivation of the switching time by decision-makers with different degrees of sophistication are studied. A model of extraction of a nonrenewable resource with technology adoption is solved in detail. Effects of the adoption of different discount functions are illustrated numerically.

Keywords: Resource Management; Regime Shift; Switching Time; Non-constant Discounting; Heterogeneous Discounting

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2.1 Introduction

In the optimal management of a natural resource, one problem of interest is whether or not it is profitable to change to a new technology and, in the affirmative case, when to do it. This is of particular interest in the case of nonrenewable natural resources, since if the new technology implies a more efficient extraction or exploitation, we can extend the actual availability of the resource.

From a formal perspective, the former question is a controlled endogenous regime shift or two-stage (or multiple-stage) optimal control problem. While there is a rich literature on regime shifts (some recent contributions on the topic are Gromov and Gromova (2017) for switches in differential games, Gromov et al. (2022) for optimal control problems with infinite switches, or the different chapters in Haunschmied et al. (2021) for a recent good overview of applications to Economics), there are less papers focusing on the optimal timing of switching. Some papers studying this last problem, for the case of one decision-maker, are Tomiyama (1985) and Amit (1986), who derived necessary conditions for the optimal switching time in a finite time horizon, while Makris (2001) focused on the infinite time horizon case. The extension to problems with more than one agent has been studied, for instance, in Dawid and Gezer (2021) and in Van Long et al. (2017). In all these models, the switching time involves a trade-off between immediate costs and potential future benefits.

Regarding the study of biases in intertemporal decision processes, variable rates of time preference have received considerable attention in recent years. These biases, leading to decisions not totally rational from an axiomatic point of view, are supported by experimental evidence pointing out the fact that decision-makers are more impatience in their short term choices than in their long when they face similar decisions (see, for instance, Thaler (1981)) . Thus, payoffs in the near future used to be discounted at higher instantaneous rates than payoffs in the long run, for instance, by using a discount function of the type

$$\theta(s-t) = e^{-\int_t^s \rho(\tau - t)d\tau}, \tag{2.1}$$

where $\rho'(\cdot) \neq 0$. The case when $\rho'(\cdot) \leq 0$ describes a situation in which decision-makers are more impatience for short term decisions, whereas for $\rho'(\cdot) \geq 0$ the effect is the opposite. For $\rho'(\cdot) = 0$, we recover the standard case with a constant discount rate. However, since the work by Strotz (1955), it is well known that when the instantaneous rate of discount depends on the position of the decision-maker, as in (2.1), standard optimization techniques fail in providing time-consistent solutions. The main reason is that, since preferences change with time, as long as the decision-maker goes over the planning horizon, they will depend on the instant t at which solutions are obtained. Because of this, a t'-agent, in general, will not find optimal the solutions computed by the t-agent, for any t and t' in the time horizon. Note that in the literature of models with general time preferences, by non-constant or variable discount rates we refer to the case where temporal preferences depend explicitly on the current position of the decision-maker in a way that it can not be removed from the optimization problem, and not only on the future calendar time at which utilities will be enjoyed. This dependence will imply that preferences become time-inconsistent.

Karp (2007) faced the analysis of dynamic optimization problems in a continuous time setting with non-constant discount rates, and obtained, in the infinite time horizon case, a dynamic programming equation (or modified Hamilton-Jacobi-Bellman equation) that characterizes time-consistent solutions in this framework. Later on, Marín-Solano and Navas (2009) extended the approach to the finite horizon case and studied the application to a nonrenewable resource problem with non-constant discounting.

A different type of time-inconsistent preferences was analyzed in Marín-Solano and Patxot (2012), that introduced and studied a problem with heterogeneous discounting in a deterministic setting. In that paper, in a finite horizon setting, payments of utilities derived from consumption enjoyed during the planning horizon are discounted at a rate (ρ_1) different to that (ρ_2) of the final function, representing, e.g., savings to be enjoyed after retirement. This can be justified in the sense that it seems questionable to assume that the enjoyment of different goods should be discounted at the same rate. In that model, the final function can be seen as a function of a good that is somehow different.

The introduction of different (constant) discount rates can be justified in different ways in our model with a regime switch. First, note that $e^{-\rho_2(T-t)} = e^{-\rho_1(T-t)} \cdot e^{(\rho_1-\rho_2)(T-t)}$. For $\rho_2 > \rho_1$, this is an increasing function in t. This means that, as time t approaches T, the t-agent assigns a higher value to the final term, which will be in our model the moment in which the new technology is adopted. Hence, the decision-maker has a bias towards the present, but this bias goes down as the moment of technology adoption is nearer. This is in agreement with psychological perceptions of many decision-makers, according to which they can assign an increasing value to a change in regime when it is nearer in the future. For example, if a decision-maker placed at time 0 exploits a natural resource and has the option to introduce a new technology at a future date T, but there exists some uncertainty on the actual effectiveness of the new technology, this uncertainty can be internalized by the decision-maker at time 0 by applying a discount rate to payoffs obtained after T different to that applied to current payoffs before that time. There are other potential justifications for introducing different discount rates. If, after the regime switch, the firm is more efficient, it could have access to better financial conditions, and this could impact the discount rate by reducing it ($\rho_2 < \rho_1$ in this case). An opposite effect ($\rho_2 > \rho_1$) could be present if we introduce mortality rates (of the business) in the long term (i.e., after T), maybe due to stopping in the use of the resource (oil, natural gas...) by the society.

The objective of this paper is to combine the above ideas, by extending previous results in standard optimal control problems with two regimes, in which the switching time is a decision variable, to a framework with time-inconsistent preferences. Special attention is paid to the case of non-constant discounting (an area of increasing interest in Economics) and to the case of heterogeneous discounting. In order to solve the problem, we transform the infinite horizon problem with a switching time into a finite horizon problem with free terminal time. Then we find necessary conditions on the terminal time to be satisfied by decision-makers with different degrees of sophistication (or rationality). The procedure proposed to solve the problem is then applied to a model of management of a natural resource, in which the agent has to decide when to implement a new technology.

The paper is organized as follows. Section 2.2 presents the main problem for an arbitrary discount function. Three particular classes of discount functions are described. Section 2.3 collects and derives some theoretical results that will be used in the paper. A procedure for solving the problem is presented in Section 2.4. In Section 2.5, we solve in detail a resource extraction model with technology adoption. Numerical illustrations showing the effects of introducing the different discount functions are presented in Section 2.6. Section 2.7 concludes the paper.

2.2 The Problem: Regime Switching with Time-Inconsistent Preferences

In this section we will state the general problem for the case of one decision-maker with time-inconsistent preferences. First, we introduce the general model in which future utility streams are discounted through a general discount function. Later on, we present some specifications for this discount function. The more relevant one is that of non-constant discounting (Problem A), a model that has been widely explored in a continuous time setting during the last fifteen years. A modified version of non-constant discounting is presented later on (Problem B). As we will see in Section 2.4, this modified version simplifies the resolution of the problem. Although it is less realistic and departs from the standard model with non-constant discounting (somehow it is in the middle point between non-constant discounting and standard exponential discounting), it will serve us to illustrate the difficulties of the problem due to the introduction of non-constant discounting. As a final specification, we will present a third problem (Problem C) in which the decision-maker discounts the future by using constant discount rates, but these discount rates can be different for the different utilities and costs.

2.2.1 The General Model

First, we state the general model. For simplicity, we will restrict our analysis to the one-dimensional case, so that there is just one state variable $x \in \mathbb{R}$ and one control variable $u \in \mathbb{R}$. The extension to multidimensional problems is straightforward.

The decision-maker maximizes a flow of utilities enjoyed along an infinite planning horizon $[0,\infty)$, but has the possibility to change to a better technology at any moment $T \in [0,\infty)$. This change can modify the state dynamics, improve the payoffs, or can have both effects. The utility function is given by

$$F(x,u) = \begin{cases} F_1(x,u) & \text{if} \quad t < T \ , \\ F_2(x,u) & \text{if} \quad t \geq T \ , \end{cases} \tag{2.2}$$

and the state dynamics is driven by

$$\dot{x} = f(x,u) = \begin{cases} f_1(x,u) & \text{if} \quad t < T \ , \\ f_2(x,u) & \text{if} \quad t \geq T \ . \end{cases} \tag{2.3}$$

Finally, the agent incurs a cost $\Omega(x(T),T)$ at the moment T in which she implements the new technology.

The objective of the decision-maker is to maximize

$$\int_0^T d_1(s,0)F_1(x,u)\,ds \; + \int_T^\infty d_2(s,0)F_2(x,u)\,ds - d_3(T,0)\Omega(x(T),T) \ . \tag{2.4}$$

Functions F_i and f_i, for $i \in \{1,2\}$, and d_j, for $j \in \{1,2,3\}$, are assumed to be, at least, continuously differentiable in all their arguments. In addition, we will assume that the second integral converges.

In the previous expression, functions $d_j(s,t)$, for $j \in \{1,2,3\}$, represent the way the agent at time t (the t-agent) discounts the different utilities (profits and costs) enjoyed at a future time s. Hence, it is natural to assume that $\dfrac{\partial d_i(s,t)}{\partial s} < 0$ (later enjoyments are valued less than more recent ones) and $\lim_{s\to\infty} d(s,t) = 0$ (decision-makers do not value utilities located in the very distant future). Unless all these functions coincide and $d_i(s,t) = e^{-\rho(s-t)}$, with ρ a (positive) constant number, time preferences become time-inconsistent, in the sense that what is optimal for the agent at time t is no longer optimal for the agent at a future time $t' > t$. In order to find time-consistent decision rules, we have to solve a sequential game with a continuous set of players, described by all of the t-agents. In the literature of non-constant discounting, these agents are said to be sophisticated.

Remark 2.1. *In economic models and, in particular, in the resource model studied in Section 2.5 in this paper, the utility functions depend just on the control variable (representing consumption, extraction rate, harvest rate...). In that case, it is common to assume that $F_i(u)$, for $i \in \{1,2\}$, is continuously differentiable, strictly increasing, and concave. In addition, $f_i(x,u) = g_i(x) - u$, where $g_i(x)$ is a continuously differentiable and concave (possibly linear) production function. These conditions facilitate the fulfillment of the conditions in Benveniste and Scheinkman (1979) for the concavity and differentiability of the value function.*

Remark 2.2. *The extension of the theoretical results in the paper to multidimensional problems with $x \in X \subset \mathbb{R}^n$ and $u \in U \subset \mathbb{R}^m$ is straightforward, provided that the value functions are sufficiently smooth.*

In order to solve the problem (2.2)–(2.4), we can proceed backward in time. For $t > T$, i.e., once the new technology has been adopted, the agent at time t aims to maximize in the control variable u the payoff function

$$J_2(x,u,t) = \int_t^\infty d_2(s,t) F_2(x,u)\, ds \tag{2.5}$$

given the dynamics

$$\dot{x} = f_2(x,u) \quad \text{with initial condition} \quad x(t) = x\,. \tag{2.6}$$

In order to find time-consistent decision rules (or time-consistent policies) followed by sophisticated agents, we can apply the nowadays well-established procedures described in, e.g., Karp (2007), Ekeland and Lazrak (2010), Marín-Solano and Shevkoplyas (2011) or Yong (2011), among others.

Later on, for $t < T$, the t-agent maximizes the general payoff function

$$\int_t^T d_1(s,t) F_1(x,u)\, ds + \int_T^\infty d_2(s,t) F_2(x,u)\, ds - d_3(T,t)\,\Omega(x(T),T)\,, \tag{2.7}$$

given the dynamics (2.3). In this problem, for $s > T$, the control decision rule $u(s) = \phi(x(s), s)$ is taken as given, and is that calculated in the resolution of Problem (2.5)–(2.6). Hence, we have to compute the decision rule $u(s)$ for the initial period $s \in [t, T]$. In addition, we must derive the switching time T.

2.2.2 Particular Cases

Although we will study how to solve the general problem stated in Section 2.2.1, in the present paper we will pay special attention to some particular cases that arise in economic applications and, in particular, in the management of a natural resource.

Problem A: Non-Constant Discounting

The standard procedure in economics is to assume that the discount function depends on the time distance between the moment t in which a decision is taken and the moment s in which utility derived from that decision will be enjoyed. In that case, $d_j(s,t) = \theta_j(s-t)$, for $j \in \{1,2,3\}$. Functions $\theta_j(\tau)$, $\tau \in [0,\infty)$, are assumed to be continuously differentiable. The corresponding instantaneous discount rates are given by

$$\rho_j(\tau) = -\frac{\theta_j'(\tau)}{\theta_j(\tau)} \, .$$

As usual, we assume that $\rho_j(\tau) > 0$, for all τ, and $\lim_{\tau\to\infty} \rho_j(\tau) > 0$. Present-biased preferences are represented by a nonincreasing discount rate ($\rho'(s) \leq 0$).

In addition, it is commonly assumed that the discount rate is unique, so that $\theta_j(\tau) = \theta(\tau)$ and $\rho_j(\tau) = \rho(\tau)$, for all $\tau \in [0,\infty)$. As a result, the intertemporal utility function (2.7) becomes

$$\int_t^T \theta(s-t) F_1(x,u)\, ds \; + \int_T^\infty \theta(s-t) F_2(x,u)\, ds - \theta(T-t)\, \Omega(x(T),T) \, . \tag{2.8}$$

Problem A consists in looking for time-consistent strategies maximizing (2.8) subject to (2.3) and to the future behavior of the agent. If the discount rate is constant and given by $\rho > 0$, then the discount function is an exponential, $\theta(\tau) = e^{-\rho\tau}$, and we recover the so-called Discounted Utility model that has been widely used in Economics. In that case, time preferences are time-consistent and we simply have to find the optimal switching time for a standard optimal control problem. However, the problem becomes much more complicated in the case of (time-distance) non-constant discounting.

Problem B: Modified Non-Constant Discounting

Problem A describes the standard model of non-constant discounting. Note that the decision-maker, at time $t < T$, discounts future enjoyments at time $s > T$ by taking as a reference point the initial time t, so that $d_2(s,t) = \theta(s-t)$. This is, we think, the natural approach in a setting with non-constant discounting in which the agent discounts the future by using the same discount rate. In Problem B we make a slight modification of this approach, by writing $d_2(s,t) = \theta(T-t)\cdot\theta(s-T)$. Then, the intertemporal utility function (2.7) becomes

$$\int_t^T \theta(s-t)F_1(x,u)\,ds \; + \theta(T-t)\int_T^\infty \theta(s-T)F_2(x,u)\,ds - \theta(T-t)\,\Omega(x(T),T)\,. \tag{2.9}$$

Although this approach is, we think, questionable, it will serve us to better understand the differences between non-constant and constant discounting. Note that, if the discount rate is constant, the discount function in Problems A and B is the same, $\theta(s-t) = \theta(T-t)\cdot\theta(s-T) = e^{-\rho(s-t)}$, and both problems become equivalent.

Problem C: Heterogeneous Discounting

As a third particular case, we consider a situation in which the decision-maker has a constant discount rate, but it is non-unique. In the present paper, Problem C is represented by the use of two different discount functions. More precisely, $d_1(s,t) = e^{-\rho_1(s-t)}$, $d_2(s,t) = e^{-\rho_2(s-t)}$ and $d_3(T,t) = e^{-\rho_2(T-t)}$, with $\rho_1, \rho_2 > 0$. Hence, the intertemporal utility function (2.7) becomes

$$\int_t^T e^{-\rho_1(s-t)}F_1(x,u)\,ds + \int_T^\infty e^{-\rho_2(s-t)}F_2(x,u)\,ds - e^{-\rho_2(T-t)}\,\Omega(x(T),T) = \tag{2.10}$$

$$\int_t^T e^{-\rho_1(s-t)}F_1(x,u)\,ds \; + e^{-\rho_2(T-t)}\left[\int_T^\infty e^{-\rho_2(s-T)}F_2(x,u)\,ds - \Omega(x(T),T)\right]\,.$$

Several justifications on the employment of heterogeneous discount rates in the problem of extraction of a natural resource with a regime switch (due to the implementation of a new technology) were presented in the Introduction. We refer to, e.g., Marín-Solano and Patxot (2012) and De-Paz et al. (2013) for the discussion of the rationale and quantitative and qualitative implications of the introduction of these time preferences in more general problems. As illustrated in those papers, there are some relevant qualitative effects appearing in real life situations that can be explained by the use of heterogeneous discount functions for goods of a different nature.

In the present paper, for simplicity, we will assume that both the cost of implementing the new technology and future utilities are discounted at the same discount rate ρ_2. For the derivation of the theoretical results we will not make assumptions on the sign of $\rho_1 - \rho_2$.

2.3 Preliminary Results

The standard switching conditions for our problem in standard optimal control theory are usually formulated in terms of the Hamiltonian functions (in present or current value forms) corresponding to the two regimes. Unfortunately, there is no easily manageable version of the Pontryagin maximum principle in problems with non-constant discounting.[2] In the present paper we will follow an alternative approach. The idea consists of transforming the problem with a switching time into a finite horizon problem with free terminal time and time-inconsistent preferences. More precisely, we will divide the problem into several steps, described in Section 2.4.1. In these steps, we will need to make use of conditions for finding strategies in both the control variable u and the terminal time T. In this section we collect the main theorems that will be used at the different steps.

2.3.1 Dynamic Programming Equation in Infinite Horizon

We summarize in this section some results presented in Marín-Solano and Shevkoplyas (2011). Let us consider the problem with an intertemporal utility function

$$J = \int_t^\infty d(s,t)\, F(x(s),u(s),s)\, ds \tag{2.11}$$

subject to

$$\dot{x}(s) = f(x(s),u(s),s)\ , \quad \text{with} \quad x(t) = x\ . \tag{2.12}$$

Functions $d(s,t)$, $F(x,u,s)$ and $f(x,u,s)$ are assumed to be continuously differentiable in all their arguments.

If $u^*(s) = \phi(x(s),s)$ is a decision rule, then the corresponding payoff is given by

$$V(x,t) = \int_t^\infty d(s,t)\, F(x(s),\phi(x(s),s),s)\, ds\ . \tag{2.13}$$

[2] As illustrated in Karp (2007), a problem with non-constant discounting can be rewritten as a standard problem with a constant discount rate by introducing an auxiliary term in the Hamiltonian function. However, such a term incorporates the solution to the problem in feedback form.

Following Ekeland and Lazrak (2010), for $\varepsilon > 0$, let us define

$$u_\varepsilon(s) = \begin{cases} v & \text{if} \quad s \in [t, t+\varepsilon) \, , \\ \phi(x(s), s) & \text{if} \quad s \geq t+\varepsilon \, . \end{cases} \tag{2.14}$$

If the t-agent can precommit her behavior during the period $[t, t+\varepsilon)$, the payoff along the perturbed control path u_ε is given by

$$V_\varepsilon(x,t) = \max_{\{v\}} \left\{ \int_t^{t+\varepsilon} d(s,t) \, F(x(s), v, s) \, ds + \right.$$

$$\left. \int_{t+\varepsilon}^{\infty} d(s,t) \, F(x(s), \phi(x(s), s), s) \, ds \right\} .$$

If we expand $V_\varepsilon(x,t)$ in ε, we obtain $V_\varepsilon(x,t) = V(x,t) + P(x, \phi, v, t)\varepsilon + o(\varepsilon)$, i.e.,

$$P(x, \phi, v, t) = \lim_{\varepsilon \to 0^+} \frac{V_\varepsilon(x,t) - V(x,t)}{\varepsilon} \, . \tag{2.15}$$

Definition 2.1. *A decision rule $u^*(s) = \phi(x(s), s)$ is called an equilibrium rule if function $P(x, \phi, \bar{c})$ given by (2.15) attains its maximum for $v = \phi(x,t)$. Alternatively, equilibrium rules are characterized by the condition $P(x, \phi, v, t) \leq 0$.*

From Theorem 6 in Marín-Solano and Shevkoplyas (2011), let the value function be given by (2.13), with $\phi(x(s), s)$ as the equilibrium rule. If the value function is of class C^1, then the solution $u = \phi(x,t)$ to

$$\max_{\{u\}} \{ F(x,u,t) + \nabla_x V(x,t) \cdot f(x,u,t) \} \tag{2.16}$$

is an equilibrium rule. Alternatively, the solutions to

$$-\frac{\partial V(x,t)}{\partial t} + \int_t^{\infty} \frac{\partial d(s,t)}{\partial t} F(x(s), \phi(x(s), s), s) \, ds = \tag{2.17}$$

$$\max_{\{u\}} \{ F(x,u,t) + \nabla_x V(x,t) \cdot f(x,u,t) \}$$

are equilibrium rules.

2.3.2 Dynamic Programming Equation in Finite Horizon

Next, let us consider the problem of a sophisticated agent maximizing

$$J = \int_t^T d(s,t) \, F(x(s), u(s), s) \, ds + d(T,t) \, G(x(T), t, T) \tag{2.18}$$

subject to (2.12), with functions $d(s,t)$, $F(x,u,s)$, $f(x,u,s)$, and $G(x,t,T)$ continuously differentiable in all their arguments. This problem is similar to the one studied in Section 4.1 in Marín-Solano and Shevkoplyas (2011), but now function G can also depend explicitly on t.

For a decision rule $u^*(s) = \phi(x(s),s)$, let

$$V(x,t) = \int_t^T d(s,t)\, F(x(s),\phi(x(s),s),s)\, ds + d(T,t)\, G(x(T),t,T)\,. \tag{2.19}$$

As above, for $\varepsilon > 0$, let us consider the variations (2.14). If the t-agent can precommit her behavior during the period $[t,t+\varepsilon)$, the valuation along the perturbed control path u_ε is given by

$$V_\varepsilon(x,t) = \max_{\{v\}} \left\{ \int_t^{t+\varepsilon} d(s,t)\, F(x(s),v,s)\, ds + \int_{t+\varepsilon}^T d(s,t)\, F(x(s),\phi(x(s),s),s)\, ds + d(T,t)\, G(x(T),t,T) \right\}\,. \tag{2.20}$$

Then, equilibrium rules for the problem (2.18)–(2.12) are defined as in Definition 2.1.

Proposition 2.1. *If the value function $V(x,t)$ is of class C^1 in all their arguments, then it satisfies the functional equation*

$$\frac{1}{d(T,t)} \frac{\partial d(T,t)}{\partial t} V(x,t) - \frac{\partial V(x,t)}{\partial t} + d(T,t) \frac{\partial G(x,t,T)}{\partial t} + \int_t^T \left[\frac{\partial d(s,t)}{\partial t} - \frac{d(s,t)}{d(T,t)} \frac{\partial d(T,t)}{\partial t} \right] F(x(s),\phi(x(s),s),s)\, ds = [F(x,\phi(x,t),t) + \nabla_x V(x,t) \cdot f(x,\phi(x,t),t)]\,.$$

Proof: See Appendix 2.8. ■

In the previous proposition we have assumed that the equilibrium rule is already given. Next, we prove that the equilibrium rule can be obtained by solving the right-hand term of the functional equation

$$\frac{1}{d(T,t)} \frac{\partial d(T,t)}{\partial t} V(x,t) - \frac{\partial V(x,t)}{\partial t} + \int_t^T \left[\frac{\partial d(s,t)}{\partial t} - \frac{d(s,t)}{d(T,t)} \frac{\partial d(T,t)}{\partial t} \right] F(x(s),\phi(x(s),s),s)\, ds + d(T,t) \frac{\partial G(x,t,T)}{\partial t} = \max_{\{u\}} [F(x,\phi(x,t),t) + \nabla_x V(x,t) \cdot f(x,\phi(x,t),t)]\,. \tag{2.21}$$

Equation (2.21) is the Dynamic Programming Equation for the problem (2.18)–(2.12).

Proposition 2.2. *If the value function is of class C^1, then the solution $u = \phi(x,t)$ to the dynamic programming Equation (2.21) is an equilibrium rule.*

Proof: See Appendix 2.8. ■

2.3.3 A Free Terminal Time Problem

Finally, we study the problem with intertemporal utility function (2.18) subject to the dynamics (2.12), but now we consider that the final time T is also a decision variable. We will analyze the problem under different degrees of sophistication of the decision-maker.

First, let us consider that the terminal time T can be decided by the agent at initial time, according to her time-preferences. Although this means that the terminal time can be precommited by the 0-agent and this is not in the spirit of looking for time-consistent decision rules, we will start with this simple approach to center the problem.

For $t \leq T$, let $V^T(x,t)$ denote the valuation along the equilibrium rule of the t-agent starting at initial state $x(t) = x$ with terminal time T. If the 0-agent can decide the terminal time, she will simply maximize in T the function $V^T(x,0)$. For this standard optimization problem, it is rather straightforward to adapt the proof in Hartl and Sethi (1983) for ordinary optimal control problems to our setting with a general discount function.

Proposition 2.3. *Let us consider Problem (2.18)–(2.12) for a time-consistent (or sophisticated) agent, with the terminal time T free. If the agent can decide the terminal time at $t = 0$, then a necessary condition for the optimality of T^* from the perspective of the 0-agent is*

$$\left[F(x,\phi(x(t),t)) + \frac{\partial G(x,0,T)}{\partial x} \cdot f(x,u) + \right. \tag{2.22}$$

$$\left.\left. \frac{1}{d(T,0)} \frac{\partial d(T,0)}{\partial T} \cdot G(x,0,T) + \frac{\partial G(x,0,T)}{\partial T} \right] \right|_{x=x(T^*),T=T^*} = 0\,.$$

Proof: It is similar to the proof of Proposition 4 in Marín-Solano and Navas (2009) for the case of non-constant discounting. ■

Under no commitment in the terminal time, each T-agent—who can be seen, for every T, as a different player in our setting with time-inconsistent preferences—will have to decide if it is convenient for her to stop the problem (so the terminal time is T) or to continue. In order to make this decision, the T-agent has to compare the payment received if she finishes the problem at time T, with the payment received in the future moment at which she will stop if she decides to continue at time T. Next we formalize this idea.

Definition 2.2. *A terminal strategy for sophisticates is a set $I \subset [0,\infty)$ defined as follows: $\tau \in I$ if, and only if, $V^{\tau}(x,\tau) \geq V^{\tau'}(x,\tau)$, where $\tau' = \inf\{s \in I \mid s > \tau\}$.*

The idea is that elements $\tau \in I$ are the terminal times at which the agent at time τ decides to stop if the problem has not finished previously. Then, the final time for a sophisticated agent T^* is characterized as follows: $T^* = \inf\{\tau \in I\}$.

Assume that T^* is the terminal time. Then, for every $s \in [t, T^*)$, every s-agent obtains higher profits by finishing the problem at time T^* compared with finishing the problem at time s, i.e., $V^s(x(s),s) < V^{T^*}(x(s),s)$.

Proposition 2.4. *If $T^* \in (0,\infty)$ is the final time for a sophisticated agent, then*

$$\left[F(x,u) + \frac{\partial G(x,t,T)}{\partial x} \cdot f(x,u) - \frac{\partial d(T,t)}{\partial t} \cdot G(x,t,T) + \right. \tag{2.23}$$

$$\left. \frac{\partial G(x,t,T)}{\partial T} \right]_{x=x(T^*),t=T^*,T=T^*} \geq 0\,.$$

Proof: See Appendix 2.8. ■

Next, if there exists $\varepsilon > 0$ such that the interval $[T^*, T^* + \varepsilon) \subset I$, and if the problem does not finish at time T^*, it will finish immediately later. Since T^* is the terminal time then, for all $\tau \in (T^*, T^* + \varepsilon)$, $V^{T^*}(x(T^*),T^*) \geq V^{\tau}(x(T^*),T^*)$. This suggests the following definition.

Definition 2.3. *We say that the agent is ε-sophisticated if candidates to the terminal time T satisfy the following conditions: There exists $\delta > 0$ such that:*

1. *For all $\tau \in (T-\delta, T)$, $V^{\tau}(x(\tau),\tau) < V^{T}(x(\tau),\tau)$, and*
2. *For all $\tau' \in (T, T+\delta)$, $V^{T}(x(T),T) \geq V^{\tau'}(x(T),T)$.*

If U is the set of points verifying these conditions, ε-sophisticated agents finish the problem at time $T^* = \inf\{T \in U\}$.

ε-sophisticated agents are partially myopic, in the sense that they analyze if it is convenient for them to stop at time T^* or to continue during a very short time period. The following proposition provides a necessary condition for a terminal time for ε-sophisticated agents.

Proposition 2.5. *If $T^* \in [0,\infty)$, $T^* < \infty$, is the final time for an ε-sophisticated agent, then*

- *If $T^* > 0$,*

$$\left[F(x,u) + \frac{\partial G(x,t,T)}{\partial x} \cdot f(x,u) - \frac{\partial d(T,t)}{\partial t} \cdot G(x,t,T) + \right. \tag{2.24}$$

$$\left. \frac{\partial G(x,t,T)}{\partial T} \right]_{x=x(T^*),t=T^*,T=T^*} = 0\,.$$

- *If $T^* = 0$, then*

$$\left[F(x,u) + \frac{\partial G(x,t,T)}{\partial x} \cdot f(x,u) - \frac{\partial d(T,t)}{\partial t} \cdot G(x,t,T) + \right. \tag{2.25}$$

$$\left. \frac{\partial G(x,t,T)}{\partial T} \right]_{x=x(T^*),t=T^*,T=T^*} \leq 0\,.$$

Proof: See Appendix 2.8. ■

Concerning the search of the terminal time for (fully) sophisticated agents, if T^* is the terminal time for ε-sophisticated agents and there exists $\varepsilon > 0$ such that the interval $[T^*, T^* + \varepsilon) \subset I$, then T^* is also the terminal time for sophisticated agents. This is indeed the situation that we find in the numerical resolution of the model of Section 2.5. In that model, if T^* is the terminal time (corresponding to the switching time in the original model) for an ε-sophisticated agent, for problems with an initial state lower than $x(T^*)$ we obtain a corner solution (condition (2.25) is satisfied), so that the agent decides to implement the new technology at initial time. Since the state dynamics $x(s)$ is (strictly) decreasing, every T-agent, for all $T > T^*$, will choose to stop at time T.

2.4 Solving the Model: Decision Rules and Switching Times

In this section we present a general procedure to solve Problem (2.7) subject to the state dynamics (2.3). The underlying idea consists in applying first the results presented in Section 2.3.1 in order to solve the problem for $t \geq T$. Later on, the original problem with a regime switch is transformed into a finite horizon problem with final function and free terminal time. For that problem, we compute first the decision rule for arbitrary T, and finally find the switching time to be chosen by

ε-sophisticated agents (we refer to the discussion in Section 2.3.3). We will assume that the regime switch can take place just one time. This will be the case, indeed, in our setting in which the decision-maker (e.g., a firm) has to decide when to change to a new and better technology. Once the firm has paid the cost of implementing the new technology, it will be profitable to maintain the improvement along the remaining whole planning horizon. After presenting the general procedure, we will make some remarks of some particularities that appear in our problems with non-constant discounting (Problem A), modified non-constant discounting (Problem B), and heterogeneous discounting (Problem C).

2.4.1 The General Model

We will solve the problem in several steps.

Step 1. The first step consists in solving the problem for $t > T$. Hence, we must solve the problem with intertemporal utility function (2.5) subject to (2.6). By applying the results presented in Section 2.3.1, the equilibrium decision rule can be derived by solving (2.16) or (2.17). Let $u^*(s) = \phi_2(x(s), s)$ denote the equilibrium decision rule for $t \geq T$, and let

$$V_2(x,t) = \int_t^\infty d_2(s,t) F_2(x(s), \phi_2(x(s), s))\, ds \tag{2.26}$$

be the corresponding value function.

Step 2. Once we have derived the equilibrium rule $u(s) = \phi_2(x(s), s)$, for $s \geq T$, we can solve the corresponding dynamical equation with initial condition $x(T) = x_T$, i.e.,

$$\dot{x}(s) = f_2(x(s), \phi_2(x(s), s)) \ , \quad x(T) = x_T \ , \quad \text{for} \quad s \geq T \ .$$

Let $x^*(s) = \varphi_2(x_T, s)$ be its solution, and define $\bar{\phi}_2(x_T, s) = \phi_2(\varphi_2(x_T, s), s)$. By substituting in (2.7), along this trajectory, the payoff function can be rewritten as

$$\int_t^T d_1(s,t) F_1(x,u)\, ds + \int_T^\infty d_2(s,t) F_2(\varphi_2(x_T, s), \bar{\phi}_2(x_T, s))\, ds -$$

$$d_3(T,t)\, \Omega(x_T, T) = \int_t^T d_1(s,t) F_1(x,u)\, ds +$$

$$d_1(T,t) \left[\int_T^\infty \frac{d_2(s,t)}{d_1(T,t)} F_2(\varphi_2(x_T, s), \bar{\phi}_2(x_T, s))\, ds - \frac{d_3(T,t)}{d_1(T,t)}\, \Omega(x_T, T) \right] .$$

By defining

$$G(x_T,t,T) = \int_T^\infty \frac{d_2(s,t)}{d_1(T,t)} F_2(\varphi_2(x_T,s),\bar{\phi}_2(x_T,s))\,ds - \frac{d_3(T,t)}{d_1(T,t)}\Omega(x_T,T)\,, \quad (2.27)$$

the intertemporal utility function can be rewritten as

$$J_1 = \int_t^T d_1(s,t)F_1(x(s),u(s))\,ds \; + d_1(T,t)G(x_T,t,T) \quad (2.28)$$

subject to

$$\dot{x}(s) = f_1(x(s),u(s))\,, \quad \text{with} \quad x(t) = x\,. \quad (2.29)$$

Step 3. Problem (2.28)–(2.29) is a non-standard optimal control problem with time-inconsistent preferences in a finite planning horizon. Hence, we can solve it for an arbitrary "final" time T. Unlike the problem studied by Marín-Solano and Shevkoplyas (2011), the present problem exhibits a "calendar effect", in the sense that the final function depends explicitly on t. By applying Proposition 2.2 we know that equilibrium rules $u(s) = \phi_1(x(s),s)$ for $s \in [0,T)$ are the solutions to

$$\max_u \left[F_1(x,u) + \frac{\partial V_1(x,t)}{\partial x} \cdot f_1(x,u) \right]\,. \quad (2.30)$$

For the calculation of the value function $V_1(x,t)$ we can solve the dynamic programming Equation (2.21). Alternatively, if we can solve explicitly (in closed form) the differential equation given by the state dynamics along the equilibrium rule, it can be more convenient to proceed as follows. Given the equilibrium rule $u(s) = \phi_1(x(s),s)$, for $s < T$, let $x(s) = \varphi_1(x,s)$ be the solution to

$$\dot{x}(s) = f_1(x(s),\phi_1(x(s),s))\,, \quad x(t) = x\,, \quad \text{for} \quad t < s < T\,.$$

By defining $\bar{\phi}_1(x,s) = \phi_1(\varphi_1(x,s),s)$ and substituting in (2.28), we obtain

$$V_1(x,t) = \int_t^T d_1(s,t)F_1(\varphi_1(x,s),\bar{\phi}_1(x,s))\,ds \; + d_1(T,t)G(x_T,t,T)\,, \quad (2.31)$$

where $x_T = \varphi_1(x,T)$. In practice, for the computation of the equilibrium rule and the corresponding value function, equations (2.30) and (2.31) have to be solved jointly.

Step 4. It remains to compute the switching time T^*. Note that we have transformed the problem of finding the switching point into that of looking for the "optimal" terminal time in a finite horizon problem with a final function. Hence, we can use Proposition 2.5 to solve the problem for ε-sophisticated decision-makers, as defined in Definition 2.3.

2.4.2 Particular Cases

Under non-constant discounting (Problem A), the decision rule after the switching point (i.e., for $t \geq T$) is stationary. Hence, since the problem is autonomous, we can restrict our attention to stationary convergent Markovian strategies, i.e., strategies $u(s) = \phi(x(s))$ for which there exists $x_\infty < \infty$ and a neighborhood U of x_∞ such that, for every $x_T \in U$, the solution to (2.6) along $u(s) = \phi(x(s))$ converges to x_∞. For stationary convergent strategies, the integral (2.5) converges. Later on, in the implementation of Step 2, the final function $G(x_T, t, T)$ depends explicitly on t and T. This fact can complicate some calculations, such as those related to the derivation of the terminal time corresponding to the switching point. Before the implementation of the new technology ($t < T$) the decision rules are non-stationary, in general. However, in the model of the following Section we will present a situation (with a constant cost function) in which the equilibrium strategies are stationary along the whole planning horizon.

In Problem B, the decision rule after the switching point (i.e., for $t \geq T$) coincides with that in Problem A and is, therefore, stationary. Concerning the final function, it is independent from t. If, in addition, the cost function Ω does not depend explicitly on T, then the final function G is also independent from T, simplifying in this way the search of the terminal or switching time. Later on, in the implementation of Step 2, as in the case of nonconstant discounting, the final function $G(x_T, t, T)$ depends explicitly on t and T. Before the implementation of the new technology ($t < T$) the decision rules are non-stationary, also for the case of constant (or null) cost.

Under heterogeneous discounting (Problem C) things are similar to Problem B. Equilibrium decision rules are stationary for $t > T$ and non-stationary for $t < T$. Although the final function (2.27) depends explicitly on t and T for this model, its dependence is such that it can be removed, as we show in Section 2.5.3 when we solve a resource extraction model with technology adoption.

2.5 A Resource Extraction Model with Technology Adoption

In this section we illustrate the previous results by applying them to the management of a natural resource whose owner has to decide when to adopt a new technology improving the extraction process. In the model, we assume that the utility function in both periods is the same, so that $F_1(x, u) = F_2(x, u)$. In particular, we take $F_1 = F_2 = \ln u$. Concerning the state dynamics, it is given by

$$\dot{x}(s) = \begin{cases} ax - \gamma_1 u & \text{if} \quad t < T \\ ax - \gamma_2 u & \text{for} \quad t \geq T \end{cases}$$

For $a = 0$ we recover the simplest model of the extraction of a nonrenewable resource, which is probably the most interesting case in our setting. If $a > 0$, the production function presents constant returns to scale. However, it implies an exponential, unlimited, growth of the resource, a property that is ecologically unrealistic in the setting of natural resources. In any case, we will solve the model for an arbitrary $a \geq 0$ (it can be easily checked that, under the assumptions made regarding Problems A, B, and C, the integrals converge). Concerning the remaining parameters, the improvement in the technology is represented by taking $\gamma_1 > \gamma_2 > 0$. For a two-player game discounting the future at constant (and unique) discount rates (and $a = 0$), this problem was studied in Van Long et al. (2017).

The cost function is assumed to be $\Omega(x) = \alpha \ln x + \beta$, with $\alpha, \beta \geq 0$. This choice can be justified economically in several ways. As we will illustrate later when we solve the model, it could correspond to a situation in which the cost is paid in units of the resource. Since, as we will show, the expression of the value function for time $t \geq T$ is $V(x_T) = A \ln x_T + B$, with A a positive number, if a fraction $\delta \in (0,1)$ of the resource is paid in order to implement the technology, then the valuation becomes $V((1-\delta)x_T) = A \ln x_T + A \ln(1-\delta) + B$. In this case, $\alpha = 0$ and $\beta = -A \ln(1-\delta)$. For our model with logarithmic utilities, paying in units of the resource implies that the cost is constant and independent from the amount of the resource. Probably, it would be more realistic to pay a fraction δ of the sum of discounted utilities after the implementation of the improvement, represented by the value function at time T. In that case, after paying the cost, the valuation would be $(1-\delta)V(x_T) = A \ln x_T + B - \delta A \ln x_T - \delta B$. Therefore, $\alpha = \delta A$ and $\beta = \delta B$ in this setting.

In the following we will solve the above problem for the three discount functions described in Section 2.2: nonconstant discounting, modified nonconstant discounting, and heterogeneous discounting. In the final step, we will derive the conditions for interior solutions, by making use of (2.24). Conditions for corner solutions can be written in a similar way.

2.5.1 Problem A

First, we will solve the problem stated in Section 2.2.2, corresponding to nonconstant discounting. This is indeed the most interesting case. We proceed according to the following steps.

Solution for $t \geq T$. In that case, the t-agent has to solve the problem with the intertemporal utility function given by

$$\int_t^{\infty} \theta(s-t) \ln u \, ds \tag{2.32}$$

subject to

$$\dot{x}(s) = ax - \gamma_2 u \,, \quad \text{with} \quad x(t) = x_t \,. \tag{2.33}$$

This problem has been already addressed in several papers (see, e.g., Marín-Solano and Navas (2009)). It can be easily shown that a stationary linear decision rule exists for this problem, and it is given by

$$u(s) = \phi(x(s)) = \frac{1}{\gamma_2 \int_0^{\infty} \theta(s)\, ds} x(s) \,. \tag{2.34}$$

By substituting (2.34) in (2.33) and solving the differential equation, we obtain

$$x(s) = \varphi(x_t, s) = e^{\left(a - \frac{1}{\int_0^{\infty} \theta(s)\, ds}\right)(s-t)} x_t \,.$$

Therefore,

$$u(s) = \bar{\phi}(x_t, s) = \phi(\varphi(x_t, s)) = \frac{1}{\gamma_2 \int_0^{\infty} \theta(s)\, ds} e^{\left(a - \frac{1}{\int_0^{\infty} \theta(s)\, ds}\right)(s-t)} x_t \,. \tag{2.35}$$

Transforming the switching time problem into a finite horizon problem. From (2.8), the payoff function of the t-agent at time $t < T$ is given by

$$J = \int_t^T \theta(s-t) \ln u \, ds + \int_T^{\infty} \theta(s-t) \ln \phi(x(s))\, ds - \theta(T-t)\, \Omega(x(T)) =$$

$$\int_t^T \theta(s-t) \ln u \, ds + \theta(T-t) \left[\int_T^{\infty} \frac{\theta(s-t)}{\theta(T-t)} \ln \bar{\phi}(x_T, s)\, ds - \alpha \ln x_T - \beta \right] .$$

By taking (2.35) for $t = T$, substituting and simplifying, the functional above can be written as

$$J = \int_t^T \theta(s-t) \ln u \, ds + \theta(T-t)\, G(x_T, t, T) \,, \tag{2.36}$$

where

$$G(x_T, t, T) = \left(\int_T^{\infty} \frac{\theta(s-t)}{\theta(T-t)} ds - \alpha \right) \ln x_T + \tag{2.37}$$

$$\left(a - \frac{1}{\int_0^\infty \theta(s)\,ds}\right) \int_T^\infty \frac{\theta(s-t)}{\theta(T-t)}(s-T)\,ds - \ln\left(\gamma_2 \int_0^\infty \theta(s)\,ds\right) \int_T^\infty \frac{\theta(s-t)}{\theta(T-t)}\,ds - \beta\,.$$

Finally, the dynamics for $s < T$ is given by

$$\dot{x}(s) = ax - \gamma_1 u\,, \quad \text{with} \quad x(t) = x_t\,. \tag{2.38}$$

Solving the problem for $t < T$. From Proposition 2.2 for the case $d(s,t) = \theta(s-t)$, first we solve

$$\max_{\{u\}} \left\{ \ln u + \frac{\partial V_1(x,t)}{\partial x}(ax - \gamma_1 u) \right\}\,,$$

hence,

$$\frac{1}{u} = \gamma_1 \frac{\partial V_1(x,t)}{\partial x}\,.$$

By guessing $V_1(x,t) = g(t)\ln x + h(t)$, then

$$u(s) = \phi(x(s),s) = \frac{x(s)}{\gamma_1 g(s)}\,. \tag{2.39}$$

By solving (2.38) for $u(s) = \phi(x(s))$ given as in (2.39), we obtain

$$x(s) = x_t e^{\int_t^s \left(a - \frac{1}{g(\tau)}\right) d\tau}\,. \tag{2.40}$$

Therefore,

$$u(s) = \bar{\phi}(x_t,s) = \frac{e^{\int_t^s \left(a - \frac{1}{g(\tau)}\right) d\tau}}{\gamma_1 g(s)} x_t\,. \tag{2.41}$$

By substituting (2.41) in (2.36), taking $s = T$ in (2.40) and substituting

$$\ln x_T = \int_t^T \left(a - \frac{1}{g(\tau)}\right) d\tau + \ln x_t$$

in (2.37), we obtain

$$V_1(x,t) = \left(\int_0^\infty \theta(s)\,ds - \theta(T-t)\alpha\right)\ln x - \ln\left(\gamma_2 \int_0^\infty \theta(s)\,ds\right) \int_T^\infty \theta(s-t)\,ds +$$
$$\int_t^T \theta(s-t)\left[\int_t^s \left(a - \frac{1}{g(\tau)}\right) d\tau - \ln(\gamma_1 g(s))\right] ds +$$
$$\left(a - \frac{1}{\int_0^\infty \theta(s)\,ds}\right) \int_0^\infty (s-T)\,\theta(s-t)\,ds +$$

$$\left(\int_T^\infty \theta(s-t)\,ds\right)\left(\int_t^T \left(a-\frac{1}{g(\tau)}\right)d\tau\right)\left(\int_T^\infty \theta(s-t)\,ds-\theta(T-t)\lambda\right)-\theta(T-t)\beta\,.$$

Therefore,

$$g(t)=\int_0^\infty \theta(s)\,ds-\theta(T-t)\alpha$$

and, from (2.39) and (2.41), the decision rule becomes

$$u(s)=\phi(x(s),s)=\frac{x(s)}{\gamma_1\left(\int_0^\infty \theta(\tau)\,d\tau-\theta(T-s)\alpha\right)}\,, \tag{2.42}$$

i.e.,

$$u(s)=\bar{\phi}(x_t,s)=\frac{\exp\int_t^s\left(a-\frac{1}{\left(\int_0^\infty \theta(\tau)\,d\tau-\theta(T-s)\alpha\right)}\right)d\tau}{\gamma_1\left(\int_0^\infty \theta(\tau)\,d\tau-\theta(T-s)\alpha\right)}x_t\,. \tag{2.43}$$

Note that if the cost is paid in units of resource, so that $\alpha=0$ (the cost is constant), the decision rule is stationary.

Derivation of the switching time. It remains to compute the switching time for ε-sophisticated agents. We apply the results in Section 2.3.3 to problem (2.36)–(2.37) for the case in which the discount function is $d(s,t)=\theta(s-t)$. In Problem A, the terminal condition becomes

$$\left[\ln u+\frac{\partial G(x,t,T)}{\partial x}\cdot(ax-\gamma_1 u)-\rho(0)\cdot G(x,t,T)+\frac{\partial G(x,t,T)}{\partial T}\right]_{x=x(T^*),t=T^*,T=T^*}=0\,. \tag{2.44}$$

It remains to compute the four terms appearing in Equation (2.44).

First, note that, taking $t=0$ and $s=T^*$ in (2.43),

$$\ln u\big|_{x=x(T^*),t=T^*,T=T^*}= \tag{2.45}$$

$$\int_0^{T^*}\left(a-\frac{1}{\left(\int_0^\infty \theta(\tau)\,d\tau-\alpha\right)}\right)d\tau-\ln\left[\gamma_1\left(\int_0^\infty \theta(\tau)\,d\tau-\alpha\right)\right]+\ln x_0\,.$$

Concerning the second term, since

$$\left.\frac{\partial G(x,t,T)}{\partial x}\right|_{x=x(T^*),t=T^*,T=T^*}=\frac{\int_0^\infty \theta(s)\,ds-\alpha}{x(T^*)}$$

and

$$ax-\gamma_1 u\big|_{x=x(T^*),t=T^*,T=T^*}=\left(a-\frac{1}{\int_0^\infty \theta(s)\,ds-\alpha}\right)x(T^*)\,,$$

therefore

$$\left.\frac{\partial G(x,t,T)}{\partial x}\cdot(ax-\gamma_1 u)\right|_{x=x(T^*),t=T^*,T=T^*}=a\left(\int_0^\infty\theta(s)\,ds-\alpha\right)-1\,. \qquad (2.46)$$

Next,

$$\rho(0)\cdot G(x,t,T)|_{x=x(T^*),t=T^*,T=T^*}=$$
$$\rho(0)\left(\int_0^\infty\theta(s)\,ds-\alpha\right)\ln x(T^*)+\rho(0)\left(a-\frac{1}{\int_0^\infty\theta(s)\,ds}\right)\int_0^\infty s\,\theta(s)\,ds-$$
$$\rho(0)\ln\left(\gamma_2\int_0^\infty\theta(s)\,ds\right)\int_0^\infty\theta(s)\,ds-\rho(0)\beta=\rho(0)\left(\int_0^\infty\theta(s)\,ds-\alpha\right)\ln x_0+$$
$$\rho(0)\left(\int_0^\infty\theta(s)\,ds-\alpha\right)\int_0^{T^*}\left(a-\frac{1}{\int_0^\infty\theta(s)\,ds-\theta(T^*-\tau)\alpha}\right)d\tau+$$
$$\rho(0)\left(a-\frac{1}{\int_0^\infty\theta(s)\,ds}\right)\int_0^\infty s\,\theta(s)\,ds-\rho(0)\ln\left(\gamma_2\int_0^\infty\theta(s)\,ds\right)\int_0^\infty\theta(s)\,ds-\rho(0)\beta\,. \qquad (2.47)$$

Finally, after several calculations, the fourth term in Equation (2.44) is given by

$$\left.\frac{\partial G(x,t,T)}{\partial T}\right|_{x=x(T^*),t=T^*,T=T^*}=\left[\rho(0)\left(\int_0^\infty\theta(s)\,ds-\alpha\right)-1\right]\ln x_0+$$
$$\left[\rho(0)\left(\int_0^\infty\theta(s)\,ds-\alpha\right)-1\right]\int_0^{T^*}\left(a-\frac{1}{\int_0^\infty\theta(s)\,ds-\theta(T^*-\tau)\alpha}\right)d\tau+$$
$$\rho(0)\left(a-\frac{1}{\int_0^\infty\theta(s)\,ds}\right)\int_0^\infty s\,\theta(s)\,ds-\rho(0)\ln\left(\gamma_2\int_0^\infty\theta(s)\,ds\right)\int_0^\infty\theta(s)\,ds-\rho(0)\beta-$$
$$\left(a-\frac{1}{\int_0^\infty\theta(s)\,ds}\right)\int_0^\infty\theta(s)\,ds+\ln\left(\gamma_2\int_0^\infty\theta(s)\,ds\right)\,. \qquad (2.48)$$

By substituting (2.45)–(2.48) in (2.44), the switching condition is derived.

2.5.2 Problem B

Next, let us solve the problem stated in Section 2.2.2, corresponding to a modified version of nonconstant discounting. We proceed as in the previous case.

Solution for $t \geq T$. The t-agent has to solve the problem with payments given by (2.32) and dynamics (2.33), whose solution is (2.34)–(2.35). In addition, for $t \geq T$, the value function is given by

$$V_2(x)=\left(\int_0^\infty\theta(s)\,ds\right)\ln x+\left(a-\frac{1}{\int_0^\infty\theta(s)\,ds}\right)\int_0^\infty s\,\theta(s)\,ds- \qquad (2.49)$$

$$\ln\left(\gamma_2\int_0^\infty \theta(s)\,ds\right)\int_0^\infty \theta(s)\,ds\,.$$

Transforming the switching time problem into a finite horizon problem. From (2.9) and (2.49), the intertemporal utility function of the t-agent at time $t<T$ is given by

$$\int_t^T \theta(s-t)\ln u\,ds\;+\theta(T-t)\,G(x_T)\,, \tag{2.50}$$

where

$$G(x_T)=\left(\int_0^\infty \theta(s)\,ds-\alpha\right)\ln x+ \tag{2.51}$$
$$\left(a-\frac{1}{\int_0^\infty \theta(s)\,ds}\right)\int_0^\infty s\,\theta(s)\,ds-\ln\left(\gamma_2\int_0^\infty \theta(s)\,ds\right)\int_0^\infty \theta(s)\,ds-\beta\,.$$

Solving the problem for $t<T$. As in the previous case, by applying Proposition 2.2 and guessing $V_2(x,t)=g(t)\ln x+h(t)$, we obtain (2.39)–(2.41). By following the same procedure as in Problem A we easily derive

$$V_2(x,t)=\left[\int_t^T \theta(s-t)\,ds+\theta(T-t)\left(\int_0^\infty \theta(s)\,ds-\alpha\right)\right]\ln x+$$
$$\int_t^T \theta(s-t)\left[\int_t^s\left(a-\frac{1}{\gamma_1 g(\tau)}\right)d\tau-\ln(\gamma_1 g(s)\right]ds+$$
$$\theta(T-t)\left[\left(\int_0^\infty \theta(s)\,ds-\alpha\right)\int_t^T\left(a-\frac{1}{\gamma_1 g(\tau)}\right)d\tau+\left(a-\frac{1}{\int_0^\infty \theta(s)\,ds}\right)\int_0^\infty s\,\theta(s)\,ds-\right.$$
$$\left.\ln\left(\gamma_2\int_0^\infty \theta(s)\,ds\right)\left(\int_o^\infty \theta(s)\,ds\right)-\beta\right]\,.$$

Therefore,

$$g(t)=\int_t^T \theta(s-t)\,ds+\theta(T-t)\left(\int_0^\infty \theta(s)\,ds-\alpha\right)$$

and from (2.39) and (2.41), the decision rule becomes

$$u(s)=\phi(x(s),s)=\frac{x(s)}{\gamma_1\left(\int_s^T \theta(\tau-s)\,d\tau+\theta(T-s)\,(\int_0^\infty \theta(\tau)\,d\tau-\alpha)\right)}\,, \tag{2.52}$$

i.e.,

$$u(s)=\bar{\phi}(x_t,s)=\frac{\exp\int_t^s\left(a-\frac{1}{\int_s^T \theta(\tau-s)\,d\tau+\theta(T-s)(\int_0^\infty \theta(\tau)\,d\tau-\alpha)}\right)}{\gamma_1\left(\int_s^T \theta(\tau-s)\,d\tau+\theta(T-s)\,(\int_0^\infty \theta(\tau)\,d\tau-\alpha)\right)}x_t\,. \tag{2.53}$$

Derivation of the switching time. For the calculation of the switching time for ε-sophisticated agents, note that in Problem B the final function depends just on the state variable. Hence, the terminal condition simplifies to

$$\left[\ln u + \frac{\partial G(x)}{\partial x}\cdot(ax-\gamma_1 u)\right]_{x=x(T^*),t=T^*,T=T^*} = [\rho(0)\cdot G(x)]_{x=x(T^*),t=T^*,T=T^*} \,. \tag{2.54}$$

Next we compute the three terms appearing in Equation (2.54).

Taking $t=0$ and $s=T^*$ in (2.53),

$$\ln u|_{x=x(T^*),t=T^*,T=T^*} = \int_0^{T^*}\left(a-\frac{1}{\int_\tau^{T^*}\theta(s-\tau)\,ds+\theta(T^*-\tau)\left(\int_0^\infty\theta(s)\,ds-\alpha\right)}\right)d\tau - \tag{2.55}$$

$$\ln\left[\gamma_1\left(\int_0^{T^*}\theta(s)\,ds+\left(\int_0^\infty\theta(s)\,ds-\alpha\right)\theta(T^*)\right)\right]+\ln x_0\,.$$

Next,

$$\left.\frac{\partial G(x)}{\partial x}\right|_{x=x(T^*),t=T^*,T=T^*} = \frac{\int_0^\infty\theta(s)\,ds-\alpha}{x(T^*)}$$

and

$$ax-\gamma_1 u|_{x=x(T^*),t=T^*,T=T^*} = \left(a-\frac{1}{\int_0^\infty\theta(s)\,ds-\alpha}\right)x(T^*)\,,$$

hence,

$$\left.\frac{\partial G(x)}{\partial x}\cdot(ax-\gamma_1 u)\right|_{x=x(T^*),t=T^*,T=T^*} = a\left(\int_0^\infty\theta(s)\,ds-\alpha\right)-1\,. \tag{2.56}$$

Finally,

$$\rho(0)\cdot G(x)|_{x=x(T^*),t=T^*,T=T^*} = \rho(0)\left(\int_0^\infty\theta(s)\,ds-\alpha\right)\ln x_0 + \tag{2.57}$$

$$\rho(0)\left(\int_0^\infty\theta(s)\,ds-\alpha\right)\int_0^{T^*}\left(a-\frac{1}{\int_\tau^{T^*}\theta(s-\tau)\,ds+\theta(T^*-\tau)\left(\int_0^\infty\theta(s)\,ds-\alpha\right)}\right)d\tau +$$

$$\rho(0)\left(a-\frac{1}{\int_0^\infty\theta(s)\,ds}\right)\int_0^\infty s\,\theta(s)\,ds-\rho(0)\ln\left(\gamma_2\int_0^\infty\theta(s)\,ds\right)\int_0^\infty\theta(s)\,ds-\rho(0)\beta\,.$$

By substituting (2.55)–(2.57) in (2.54), we obtain the switching condition.

2.5.3 Problem C

Finally, we solve the problem with heterogeneous discounting presented in Section 2.2.2.

Solution for $t \geq T$. In this case, the optimal decision rule for the problem (2.32)–(2.33) with $\theta(s) = e^{-\rho_2 s}$ is $u(s) = \phi(x(s)) = \rho_2 x(s)$, i.e., $u(s) = \bar{\phi}(x_T, s) = e^{(a-\rho_2)(s-T)} x_T$, and the corresponding value function is

$$V_2(x) = \frac{1}{\rho_2} \ln x + \frac{1}{\rho_2} \left(\frac{a}{\rho_2} - 1 - \ln \frac{\gamma_2}{\rho_2} \right) . \tag{2.58}$$

Transforming the switching time problem into a finite horizon problem. From (2.10) and (2.58), the payoff function of the t-agent at time $t < T$ can be written as

$$\int_t^T e^{-\rho_1(s-t)} \ln u \, ds + e^{-\rho_2(T-t)} \bar{G}(x_T) , \tag{2.59}$$

where

$$\bar{G}(x_T) = \left(\frac{1}{\rho_2} - \alpha \right) \ln x_T + \frac{1}{\rho_2} \left(\frac{a}{\rho_2} - 1 - \ln \frac{\gamma_2}{\rho_2} \right) - \beta . \tag{2.60}$$

Solving the problem for $t < T$. By proceeding as in the previous cases, if the value function is $V_1(x,t) = g(t) \ln x + h(t)$, we obtain (2.39)–(2.41). By substituting these expressions in (2.59) and (2.60), we obtain

$$g(t) = \frac{1}{\rho_1} \left(1 - e^{-\rho_1(T-t)} \right) + \left(\frac{1}{\rho_2} - \alpha \right) e^{-\rho_2(T-t)} , \tag{2.61}$$

$$h(t) = \int_t^T e^{-\rho_1(s-t)} \left[\int_t^s \left(a - \frac{1}{g(\tau)} \right) d\tau - \ln(\gamma_1 g(s)) \right] ds +$$
$$e^{-\rho_2(T-t)} \left[\left(\frac{1}{\rho_2} - \alpha \right) \int_t^T \left(a - \frac{1}{g(\tau)} \right) d\tau + \frac{1}{\rho_2} \left(\frac{a}{\rho_2} - 1 - \ln \frac{\gamma_2}{\rho_2} \right) - \beta \right] .$$

Derivation of the switching time. For the calculation of the switching time for ε-sophisticated agents, in order to apply the results in Section 2.3.3, we can write

$$e^{-\rho_2(T-t)} \bar{G}(x_T) = e^{-\rho_1(T-t)} G(x_T, t, T) ,$$

with

$$G(x_T, t, T) = e^{(\rho_1 - \rho_2)(T-t)} \bar{G}(x_T) ,$$

for $\bar{G}(x_T)$ given as in (2.60). Alternatively, we can apply Proposition 3 in Marín-Solano and Patxot (2012) to function $\bar{G}(x_T)$. It is straightforward to check that both procedures are equivalent. Indeed, note that, in the switching time T^*,

$$\left[\ln u + \frac{\partial G(x,t,T)}{\partial x}\cdot(ax-\gamma_1 u) - \rho_1 \cdot G(x,t,T) + \frac{\partial G(x,t,T)}{\partial T}\right]_{x=x(T^*),t=T^*,T=T^*} = 0\,, \tag{2.62}$$

where

$$\left[\frac{\partial G(x,t,T)}{\partial x}\cdot(ax-\gamma_1 u)\right]_{x=x(T^*),t=T^*,T=T^*} = \left[\frac{\partial \bar{G}(x)}{\partial x}\cdot(ax-\gamma_1 u)\right]_{x=x(T^*),t=T^*,T=T^*},$$

$$\left[\rho_1 \cdot G(x,t,T) - \frac{\partial G(x,t,T)}{\partial T}\right]_{x=x(T^*),t=T^*,T=T^*} = \left[\rho_2 \cdot \bar{G}(x)\right]_{x=x(T^*),t=T^*,T=T^*}$$

and

$$\left[\frac{\partial G(x,t,T)}{\partial x}\cdot(ax-\gamma_1 u)\right]_{x=x(T^*),t=T^*,T=T^*} = \left[\frac{\partial \bar{G}(x)}{\partial x}\cdot(ax-\gamma_1 u)\right]_{x=x(T^*),t=T^*,T=T^*}.$$

Since the decision rule is given by (2.41) with $g(\tau)$ given by (2.61), taking $t=0$ and $s=T^*$,

$$\ln u|_{x=x(T^*),t=T^*,T=T^*} = \int_0^{T^*}\left(a-\frac{1}{g(\tau)}\right)ds - \ln(\gamma_1 g(T^*)) + \ln x_0\,. \tag{2.63}$$

In a similar way,

$$\frac{\partial \bar{G}(x)}{\partial x}\cdot(ax-\gamma_1 u)\bigg|_{x=x(T^*),t=T^*,T=T^*} = a\left(\frac{1}{\rho_2}-\alpha\right)-1 \tag{2.64}$$

and

$$\rho_2 \cdot \bar{G}(x)\big|_{x=x(T^*),t=T^*,T=T^*} = \tag{2.65}$$

$$(-\rho_2\alpha)\ln x_0 + (1-\rho_2\alpha)\int_0^{T^*}\left(a-\frac{1}{g(\tau)}\right)d\tau + \left(\frac{a}{\rho_2}-1-\ln\frac{\gamma_2}{\rho_2}\right) - \rho_2\beta\,.$$

From (2.63)–(2.65) we derive the switching condition.

2.6 Numerical Illustration

Next, we illustrate numerically some of the previous results for the case of a nonrenewable natural resource ($a=0$) by focusing on the two main settings corresponding to the non-constant discounting case (Problem A) and to the heterogeneous discounting case (Problem C). Additionally, we will include the case of standard exponential discounting, where temporal preferences are time-consistent

(Problem S), which can be obtained from any of the other analyzed cases by eliminating the temporal bias. In the case of non-constant discounting, we take as a discount function a convex linear combination of two exponential functions, i.e., $\theta(\tau) = \nu e^{-\rho_1 \tau} + (1-\nu)e^{-\rho_2 \tau}$, with $\nu \in (0,1)$, and $\rho_1 < \rho_2$, for which the instantaneous discount rate is given by

$$r(\tau) = -\frac{\theta'(\tau)}{\theta(\tau)} = \frac{\nu \rho_1 e^{-\rho_1 \tau} + (1-\nu)\rho_2 e^{-\rho_2 \tau}}{\nu e^{-\rho_1 \tau} + (1-\nu)e^{-\rho_2 \tau}},$$

that decreases from $r(0) = \nu\rho_1 + (1-\nu)\rho_2$ to $\rho_1 = \lim_{\tau \to +\infty} r(\tau)$. Regarding the heterogeneous discounting case (Problem C), we take as discount functions $\theta_1(t-s) = e^{-\rho_1(s-t)}$ for the instantaneous utility before the introduction of the innovation and $\theta_2(t-s) = e^{-\rho_2(s-t)}$, $\rho_1 \neq \rho_2$, for utility after the regime switch. In our benchmark case, we take the values of the parameters $\nu = 0.5$, $\rho_1 = 0.05$, $\rho_2 = 0.15$ defining the temporal preference of the decision-maker. Regarding the efficiency in the exploitation process, we assume $\gamma_1 = 1.3$ and $\gamma_2 = 1.1$. Note that parameters γ_1 and γ_2 determine the efficiency in extraction before and after the introduction of the innovation, respectively. The lower the value of γ_i, $i \in \{1,2\}$, the more efficient is the extraction process. Regarding the cost of innovation, we assume that it is a fraction $\delta\%$ of the value of the project (given by the value function) for the decision-maker at the switching time T^*, and in particular we set $\delta = 0.045$. Moreover, as initial resource stock, we take $x_0 = 1000$. Finally, for the standard discounting case (Problem S), we will use as a discount function $\theta(\tau) = e^{-\hat{\rho}\tau}$, where $\hat{\rho} = \rho 1 \rho 2/(\rho 1 - \nu\rho_1 + \nu\rho_2)$ is obtained as the solution of

$$\int_0^\infty \left\{ \nu e^{-\rho_1 \tau} + (1-\nu)e^{-\rho_2 \tau} \right\} d\tau = \int_0^\infty e^{-\hat{\rho}\tau} d\tau. \tag{2.66}$$

The intuition behind (2.66) is to find a constant rate of time preference, $\hat{\rho}$, that shows a similar overall level of impatience to the one given by the non-constant discount function, an idea that was proposed in Strulik (2015).

Table 2.1 collects the switching times and the resource stock left at that time for non-constant, heterogeneous and standard discounting cases. We can observe that the existence of some bias in the temporal preferences negatively affects the early adoption of the new technology, especially under non-constant discounting. In that case, the introduction of the innovation lasts almost twice compared with the standard case. Looking now at Figure 2.1, it is interesting to observe that the evolution of the resource stock under non-constant and standard discounting is very similar. However, despite this coincidence in the extraction rates, note that since the decision-maker in Model S introduces the innovation at a significant earlier time, she will consume more from that moment up to the time at which a decision-maker with non-constant discounting preferences will do it. This can be easily seen in the plot of the evolution of the consumption rate at Figure 2.2. In the case of heterogeneous discounting, due to the particular bias in this setting, we can observe that at initial periods the decision-maker undervalues all of the payoffs she will earn after the regime shift, so there is a significant overconsumption during these initial periods, which can be observed in the consumption rate. As the switching time approaches, this undervaluation decreases, and disappears at T^*. Consequently, the time-consistent consumption rule will coincide with that of a decision-maker with standard discounting at a rate of time preference of ρ_2.

	Problem A	Problem C	Problem S
T^*	12.39	8.96	6.86
$x(T^*)$	384.68	293.40	586.81

Table 2.1: Switching time and resource stock at the time of the innovation

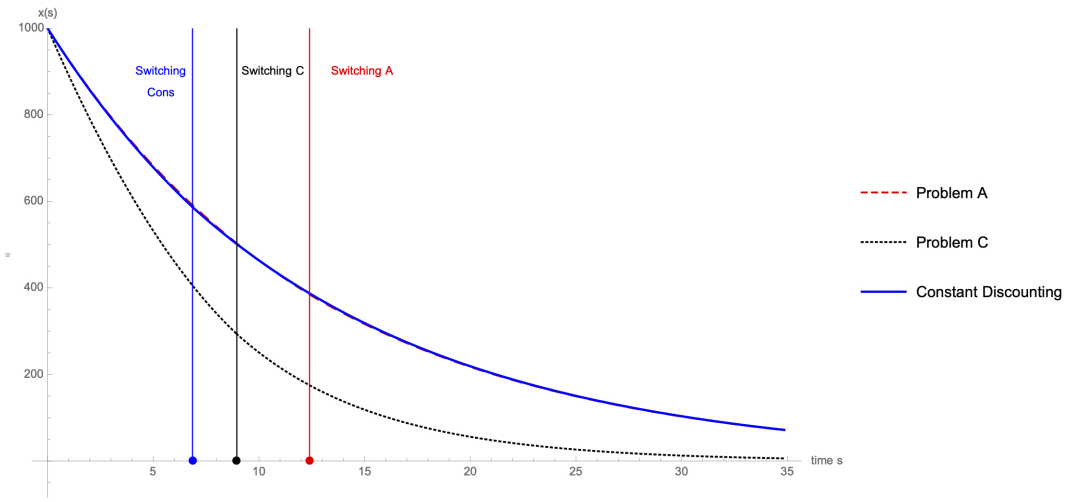

Figure 2.1: Evolution of the resource stock.

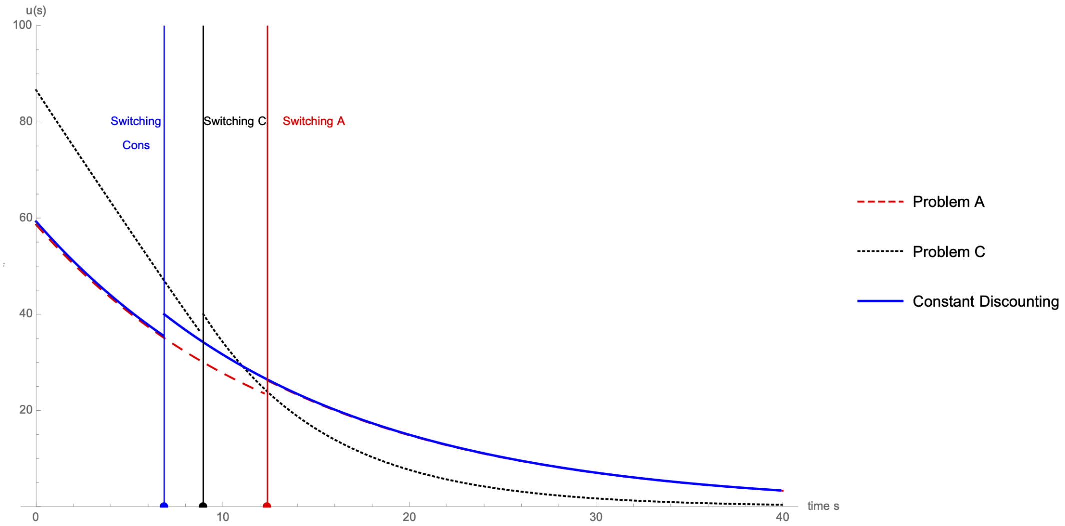

Figure 2.2: Evolution of the consumption.

Finally, we analyze results from Table 2.2, where a sensibility analysis with respect to some parameter values is included. In the setting of non-constant discounting, higher values of ρ_2 are associated with a higher impatience for short run decisions, while in the heterogeneous discounting setting it implies an overvaluation of payoffs before the introduction of the innovation compared with payoffs after T^*. Similarly, in the case of standard discounting, with an overall constant impatience rate, the level of impatience increases with ρ_2, although in this last case there is no particular bias in the temporal preferences. In all three settings we can observe that an increase of ρ_2 negatively affects the timing of the innovation, especially in the case of Problem A. Moreover, note that with non-constant discounting, the long term rate of time preference is always the same ($\rho_1 = 0.05$), so all of the resulting delays in T^* can be attributed to the increase in the impatience degree in the short term. With regards to changes in the efficiency improvement associated with the innovation, lower values of γ_2 represent larger improvements in efficiency. When this happens, in the three cases we can see a reduction in the timing of the innovation. On the contrary, by increasing the cost of the innovation (augmenting the value of δ) the effect is the opposite, and in all the cases the decision-maker will delay the regime shift. In conclusion, in terms of sustainability of the resource, it is clear that the sooner an improvement in the exploitation process is introduced, the larger the saving in wasted resources (note that one unit of consumption requires γ_i units of the resource).

		Problem A		Problem C		Problem S ($\hat{\rho}$)	
		T^*	$x(T^*)$	T^*	$x(T^*)$	T^*	$x(T^*)$
ρ_2	0.075	6.06	685.58	6.94	586.80	4.96	733.51
	0.10	8.74	547.19	8.29	440.10	6.00	660.16
	0.25	16.32	249.29	8.95	176.04	7.40	528.12
γ_2	1.25	38.87	52.02	46.95	19.47	42.64	39.08
	1.20	30.40	98.44	32.09	46.29	31.17	92.61
	1.15	21.57	191.71	19.06	114.23	19.21	228.45
δ	5%	16.07	290.61	12.49	201.88	11.69	403.75
	7.5%	26.91	124.79	26.40	65.39	26.22	130.79
	10%	32.25	81.41	35.05	36.94	33.42	74.00

Table 2.2: Sensitivity Analysis.

2.7 Conclusions

In this paper we have studied the switching conditions between two different regimes, characterized by a possible change in the objective function and/or in the system dynamics, when the decision-maker shows time-inconsistent temporal preferences. In particular, we have focused on the cases of non-constant discounting and heterogeneous discounting. Each of these two settings induce a different bias in the temporal preferences. The main objective has been to analyze this framework from the perspective of a sophisticated agent, by transforming our original infinite horizon problem with a switching time into a finite horizon problem with free terminal time. After this, we derived the necessary conditions on the terminal time to be satisfied by decision-makers with different degrees of sophistication (or rationality). Finally, the proposed procedure has been applied to a natural resource extraction model in which the decision-maker has the option of implementing a more efficient exploitation technology.

There are several possible extensions of this work. In our resource extraction model we have focused on the case of log utility and a linear natural growth function. The extension to general isoelastic utilities or to non-linear growth functions would allow a richer analysis of the resource management problem. Another extension that we consider of special interest is the case of two agents where only one or both can decide on a regime shift.

2.8 Appendix

Proof of Proposition 2.1: When the equilibrium decision rule $u^*(s) = \phi(x(s),s)$ is applied for $s \in [t,t+\varepsilon)$, the state variable changes to $x(t+\varepsilon) = x_{t+\varepsilon}$. From the definition of the value function,

$$V(x_{t+\varepsilon},t+\varepsilon) = \int_{t+\varepsilon}^{T} d(s,t+\varepsilon)\,F(s,x(s),\phi(x(s),s))\,ds + d(T,t+\varepsilon)G(x(T),t,T)\,. \tag{A.1}$$

By performing a Taylor expansion in ε, we obtain

$$V(x_{t+\varepsilon},t+\varepsilon) = V(x,t) - \int_{t}^{T} d(s,t)\,F(x(s),\phi(x(s),s),s)\,ds +$$
$$\left[\frac{\partial V(x,t)}{\partial x}\cdot\frac{\partial x_{t+\varepsilon}}{\partial \varepsilon}\bigg|_{\varepsilon=0^+} + \frac{\partial V(x,t)}{\partial t} + F(x,\phi(x,t),t) - \right.$$
$$\left.\int_{t}^{T}\frac{\partial d(s,t)}{\partial t}F(x(s),\phi(x(s),s),s)\,ds\right]\varepsilon + o(\varepsilon) = d(x(T),t)\,G(x(T),t,T) +$$
$$\left[\frac{\partial d(T,t)}{\partial t}G(x(T),t,T) + d(T,t)\frac{\partial G(x(T),t,T)}{\partial t}\right]\varepsilon + o(\varepsilon)\,.$$

By dividing by ε and taking the limit $\varepsilon \to 0^+$ we obtain

$$\frac{\partial V(x,t)}{\partial x}f(x,\phi(x,t),t) + \frac{\partial V(x,t)}{\partial t} + F(x,\phi(x,t),t) - \int_{t}^{T}\frac{\partial d(s,t)}{\partial t}F(x(s),\phi(x(s),s),s)\,ds$$
$$= \frac{\partial d(T,t)}{\partial t}G(x(T),t,T) + d(T,t)\frac{\partial G(x(T),t,T)}{\partial t}\,. \tag{A.2}$$

From (2.19),

$$G(X(T),t,T) = \frac{1}{d(T,t)}\left[V(x,t) - \int_{t}^{T} d(s,t)\,F(x(s),\phi(x(s),s),s)\,ds\right]$$

and substituting in (A.2), the result follows. ■

Proof of Proposition 2.2: It is very similar to the proof of Theorem 4 in Marín-Solano and Shevkoplyas (2011). By proceeding as in that paper, after several calculations we obtain

$$P(x,\phi,v,t) = \lim_{\varepsilon\to 0^+}\frac{V_\varepsilon(x,t) - V(x,t)}{\varepsilon} =$$

$$[F(x,v(t),t) + \nabla_x V(x,t)\cdot f(x,v(t),t)] - [F(x,\phi(x,t),t) + \nabla_x V(x,t)\cdot f(x,\phi(x,t),t)]$$

and, from Definition 2.1, the result follows. ■

Proof of Proposition 2.4: Assume that T^* is the terminal time. Then, for every $s \in [t, T^*)$, every s-agent obtains higher profits by finishing the problem at time T^* compared with finishing the problem at time s, i.e., $V^s(x(s), s) < V^{T^*}(x(s), s)$. In particular, for $\varepsilon > 0$, the $(T^* - \varepsilon)$-agent will decide to continue until T^*. Therefore,

$$V^{T^*-\varepsilon}(x(T^*-\varepsilon), T^*-\varepsilon) < V^{T^*}(x(T^*-\varepsilon), T^*-\varepsilon) .$$

Note that

$$V^{T^*-\varepsilon}(x(T^*-\varepsilon), T^*-\varepsilon) = G(x(T^*-\varepsilon), T^*-\varepsilon, T^*-\varepsilon) . \tag{A.3}$$

In addition, if $u = \phi(x(s), s)$ is the equilibrium rule and $x(s)$ is the corresponding path of the state variable,

$$V^{T^*}(x(T^*-\varepsilon), T^*-\varepsilon) = \tag{A.4}$$

$$\int_{T^*-\varepsilon}^{T^*} d(s, T^*-\varepsilon)\, F(x(s), \phi(x(s), s))\, ds + d(T^*, T^*-\varepsilon) G(x(T^*), T^*-\varepsilon, T^*) .$$

Next, for sufficiently small ε, from (A.3),

$$V^{T^*-\varepsilon}(x(T^*-\varepsilon), T^*-\varepsilon) = G(x(T^*), T^*, T^*) - \tag{A.5}$$

$$\varepsilon \cdot \left[\frac{\partial G(x,t,T)}{\partial x} \cdot f(x(s), \phi(x(s), s)) + \frac{\partial G(x,t,T)}{\partial t} + \frac{\partial G(x,t,T)}{\partial T} \right]_{x=x(T^*), t=T^*, T=T^*} + o(\varepsilon)$$

and, from (A.4), -4.6cm0cm

$$V^{T^*}(x(T^*-\varepsilon), T^*-\varepsilon) = G(x(T^*), T^*, T^*) - \tag{A.6}$$

$$\varepsilon \cdot \left[-L(x(t), \phi(x(t), t)) + \frac{\partial d(T,t)}{\partial t} \cdot G(x(t), t, T) + \frac{\partial G(x,t,T)}{\partial t} \right]_{x=x(T^*), t=T^*, T=T^*} + o(\varepsilon) .$$

By identifying (A.5) and (A.6), dividing by ε and taking the limit $\varepsilon \to 0^+$, condition (2.23) follows. ■

Proof of Proposition 2.5: Let $0 < \varepsilon < \delta$. From condition 1 in Definition 2.3, by taking $\tau = T - \varepsilon$, we can replicate the proof of Proposition 2.4 to obtain the inequality (2.23) if $T^* > 0$. From condition 2, if we write $\tau' = T + \varepsilon$, then $V^{T^*}(x(T^*), T^*) \geq V^{T^*+\varepsilon}(x(T^*), T^*)$, i.e.,

$$G(x(T^*), T^*, T^*) \geq \int_{T^*}^{T^*+\varepsilon} d(T^*+\varepsilon, T^*) F(x(s), \phi(x(s), s))\, ds +$$

$$d(T^*+\delta,T^*)G(x(T^*+\varepsilon),T^*,T^*+\varepsilon) = G(x(T^*),T^*,T^*)+$$
$$\varepsilon\cdot\left[L(x(t),\phi(x(t),t)+\frac{\partial d(T,t)}{\partial t}\cdot G(x(t),t,T)+\frac{\partial G(x,t,T)}{\partial x}\cdot f(x(s),\phi(x(s),s)+\right.$$
$$\left.\frac{\partial G(x,t,T)}{\partial T}\right]_{x=x(T^*),t=T^*,T=T^*}+o(\varepsilon)\;.$$

By simplifying, dividing by ε and taking the limit $\varepsilon \to 0^+$, Equation (2.25) is derived. If $T > 0$, (2.24) follows from (2.23) and (2.25). ■

3 What is my Neighbor Doing? Heterogeneous agents under Free Trade with Renewable Resources

Abstract

This chapter delves into the intricate relationship between status concerns, renewable resource extraction, and the strategic implications of autarky and free trade, under a dynamic game framework. We examine situations where agents care about their relative position and compare themselves to others, incorporating behavioral factors such as relative extraction and profit comparison. By analyzing symmetric and heterogeneous agents representing countries, we offer novel insights into the role of status concerns in shaping strategic decision-making. Our analysis highlights the importance of considering these behavioral factors when developing policies related to renewable resource extraction, trade, and sustainable development. By exploring the strategic interactions among nations, this research emphasizes the need to understand status-driven behavior in crafting effective policies that balance sustainability and competition in global resource management and international economics.

Keywords: Renewable Resources; Differential Games; Environment and Trade; Autarky; Free Trade; Heterogeneous agents; Status Concern; Social Status; Relative Performance.

JEL Codes: C73; D9; Q56

3.1 Introduction

In the past years, the proper management and extraction of natural resources, which will surely be an essential part of our green and sustainable economy in a not-too-distant future, has significantly influenced economic research. One of the most powerful approaches researchers have in order to analyze the impact of natural resources in the economy with different actors is the one of dynamic games. This framework enables us to explore the strategic interactions between different players over time, i.e., to study how different agents can extract and consume a natural resource and how their actions affect other agents' payoffs as well as the environment.

In the era of the Sustainable Development Goals (SDGs)[1] driven by the United Nations and widely promoted by the OECD (2016), the knowledge and correct management of natural resources would help to directly or indirectly achieve several objectives.[2] The tragedy of the commons, a concept well-established in the literature since Lloyd (1833) and further expounded by ecologist Hardin (1968), occurs when access to a common pool resource leads to its overexploitation. Common goods are characterized by their rivalry in consumption and non-excludability. The rivalry aspect implies that when an agent consumes, extracts, or harvests one unit of the resource, it reduces the stock available to all other agents (the group), which will turn out to have a negative effect. This negative externality arises from the combination of open access and resource depletion through use. The primary explanation for the tragedy of the commons lies in the strategic behavior of individuals in the game, wherein each agent considers their own private marginal costs of use, leading to excessive resource extraction without accounting for the impact on others. The aggregate behavior of all agents in this environment may lead to overexploitation and eventual exhaustion of the resource. Consequently, the use of a methodology that introduces the interaction between the dynamic strategies of the agents is required. This need arises from two key factors: first, current behavior has instantaneous effects on other agents; second, today's actions significantly impact future generations (Nordhaus, 2019). Considering all of the above, in this chapter we study a differential game, first with two symmetric players, and later we investigate the asymmetric game, in which agents exploit a renewable natural resource. We consider that agents have social status or relative performance preferences, i.e., they care about and are influenced by what their neighbors are doing. Moreover,

[1] https://sdgs.un.org/

[2] These are objectives 6, Clean Water and Sanitation; 7, Affordable and Clean Energy; 8, Decent Work and Economic Growth; 11, Sustainable Cities and Communities; 12, Responsible Consumption and Production; 13, Climate Action; 14, Life Below Water and 15, Life On Land.

we study the implications of moving from Autarky to Free Trade, where agents sell and buy under an integrated common market.

Recently, Benchekroun et al. (2020) analyzed the implications of multiple symmetric agents exploiting a common property renewable resource under an oligopoly framework. The authors study the differences between *autarky*, and when they allow countries to *trade* in a common market. They use a renewable resource of the form studied first in Benchekroun (2003), where a piecewise reproduction function for the renewable resource was introduced, and later further developed in Benchekroun (2008). The findings of the recent work by Benchekroun et al. (2020) offer surprising insights from an international economics perspective, as they show that "*free trade may lead to a lower discounted sum of consumer surplus and of social welfare than autarky. [...] A priori, this finding is not straightforward; a move from Autarky to Free Trade causes industry output to first increase and then decrease over time*". However, will this result hold if players present status concern preferences, and compare their actions with other players?

Additionally, countries are not only concerned about their consumption/extraction of a natural resource, but also consider their relative position compared to other actors in international geopolitics. Relatively recent work trying to understand these consequences is the work by Benchekroun and Long (2016), where they study how agents with status concern extract a common pool renewable resource (their utilities do not take just their consumption or profits into account, but also the relative consumption or profits with respect to other agents in the game). Moreover, it is well known in the literature that agents' utility is also influenced by other agents' performance, wealth, income, or consumption. These positional externalities are also known as "Catching Up with the Joneses" (Abel, 1990). The main idea is that agents care about their consumption relative to that of their neighbor (see, for instance, Galí (1994), Katayama and Van Long (2010), Ljungqvist and Uhlig (2000), Chan and Kogan (2002), Alvarez-Cuadrado et al. (2004), and Alonso-Carrera et al. (2005)).

According to Benchekroun and Long (2016), "*because of status concern, private decisions on consumption or asset accumulation generate externalities, and as a result, one can no longer presume that a competitive equilibrium is Pareto efficient*". Thus, one may wonder how the status concern will affect trade between players and if they could act strategically to influence other countries' welfare, which affects the utility of that country. This raises questions about how status concerns affect trade between countries, their competitiveness in the autarky scenario, and their extraction and consumer and producer surplus. Countries may act strategically to influence the welfare of others and, consequently, their own extraction and utility. To investigate these issues, we combine the framework proposed by Benchekroun et al. (2020), in which agents trade a common property renewable resource, with a model where countries care about their relative performance as in Benchekroun and Long (2016). Our findings demonstrate that introducing status concerns can lead to different outcomes than previous research, and agents can achieve higher discounted sums of consumer and producer surplus under free trade due to this behavioral component.

In this chapter, we present two players (first symmetric and later asymmetric), representing two countries, considering the consequences of joining or staying in a commonly integrated market versus staying under autarky. Each country is assumed to have a monopolist that exploits a natural resource. This could be thought of as a country per se extracting the asset. Thus, under autarky, only the monopolist in such a country is supplying that internal market. However, under free trade firms compete in quantities under an integrated common market, and choose their extraction time path $\{q_i(t)\}_{t\in\mathbb{R}_+}$. To capture the fact that firms' extraction strategy is influenced by the current state of the resource stock, we employ Markovian (feedback/closed loop) strategies. We focus on finding the Markov Perfect Nash Equilibrium of the game, which is subgame perfect.[3]

A real-life example of this problem can be found in the management of fisheries. The extraction and management of fish stocks are a classic example of a common-pool renewable resource. In this context, multiple countries have access to the resource (fish stocks), and their actions in extracting the resource have consequences for other countries as well. Countries may have a status concern, comparing their fishery industry's extraction levels and profits to those of other nations. This comparison can lead to overfishing as countries try to maintain or improve their relative status, resulting in the depletion of fish stocks and long-term negative consequences for the industry and environment. This situation illustrates the complex interplay between renewable resource extraction, trade policies, and status concerns among nations.

Trade policies, such as bilateral or multilateral agreements, can play a role in this context. For example, countries can decide to keep their fishing industries separate (autarky), or they may establish agreements to manage fish stocks jointly and engage in trade (free trade). The European Union's Common Fisheries Policy is one such example of an attempt to manage shared fish stocks through joint management and trade agreements among member countries. Incorporating the dynamic behavior of countries with status concerns in the analysis of fishery management and trade policies can provide valuable insights into the real-world complexity of renewable resource extraction and management.

[3] See, for instance, Dockner et al. (2000) and Van Long (2010).

One prominent real-life example is the dispute between Canada and the European Union over Atlantic cod fishing in the Grand Banks of Newfoundland in the early 1990s, which was part of a larger issue of overfishing that led to the collapse of the Atlantic northwest cod fishery. In 1992, John Crosbie, the Canadian Federal Minister of Fisheries and Oceans, announced a moratorium on the Northern Cod fishery. This fishery had been a central driver of life and community development along the eastern coast of Canada for the previous half-millennium.[4] Since the early 1990s, Atlantic cod populations in the waters off the northeast coast of Newfoundland have been in a critical state.[5] The dispute between Canada and Spain over fishing rights for turbot in international waters off Canada's east coast began in 1995. Eventually, direct negotiations between the EU and Canada resumed, culminating in an agreement on April 5th of that same year. Despite this, Spain dismissed the deal and sought more favorable conditions. In response to the threat of forcibly removing Spanish fishing vessels, the EU persuaded Spain to reach a compromise on April 15th.[6]

Another example of resource conflict is the "Scallop Wars" between French and British fishermen in 2018. The dispute arose over access to scallop-rich waters in the English Channel near the Bay of Seine. While British fishermen were allowed to fish for scallops year-round, French law restricted their fishermen to a season that runs from October 1st to May 15th to protect the scallop population. Tensions escalated as French and British fishing vessels clashed at sea, with reports of stone-throwing, flare-shooting, and aggressive maneuvers. The situation led to diplomatic talks between the two countries, resulting in an agreement that provided UK vessels larger than 15 meters with compensation for staying out of the disputed area during the closed French scallop fishing season. Smaller British boats retained their right to fish in the area year-round.[7]

The potential of this study to shape future discussions on sustainable development and resource conservation is significant and may capture the attention of policymakers. By examining real-world examples such as the management of fisheries, or any renewable resource, the analysis presented in this chapter sheds light on the complexities of natural resource extraction, trade policies, and status concerns among nations.

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[4] See https://en.wikipedia.org/wiki/Collapse_of_the_Atlantic_northwest_cod_fishery

[5] https://www.cbc.ca/news/canada/newfoundland-labrador/cod-return-1.5992916

[6] See https://en.wikipedia.org/wiki/Turbot_War and https://www.britannica.com/topic/exclusive-economic-zone#ref1308083

[7] See https://www.bbc.com/news/world-europe-45337091 and https://en.wikipedia.org/wiki/English_Channel_scallop_fishing_dispute

This chapter is organized as follows. In Section 3.2, we introduce the general models for both autarky and free trade scenarios. In Section 3.3 we study the game when players are symmetric (both autarky vs free trade). Later in section 3.4, we depart from the symmetric extraction costs and explore the heterogeneous agent case under autarky and free trade. Questions related to welfare analysis are developed in Section 3.5. Finally, Section 3.6 concludes the chapter.

3.2 The Model

Building upon Benchekroun et al. (2020), we modify the utility function to analyze how different players/countries are influenced by the performance of other players in the game, incorporating the status concern *à la* Benchekroun and Long (2016). We first study the symmetric scenario studied in the previous papers, and further extend it to the asymmetric case. Next, for the heterogeneous case, we will follow the ideas of the asymmetric game of agents extracting a renewable resource as in Benchekroun et al. (2014).

We consider a common pool of renewable resources where two players have access to it. We first analyze the symmetric game where the marginal extraction costs are the same for both players, but may differ under different regimes. We will call these our "symmetric" games. In contrast, Benchekroun et al. (2020) consider that the costs are symmetric and identical under *autarky* (A) and *free trade* (FT). In this chapter, we will distinguish two economic regimes/cases. We always study first the case where countries do not trade with each other, known as autarky, which means that countries harvest the resource and sell it at *home*. Later, we will allow them to trade under a common market, labeled as free trade. Furthermore, we later depart from the symmetric assumption and consider the asymmetric game, where there can be a big (b) and efficient player, and a small (s) and inefficient agent extracting both the resource, so we allow "*efficiency differences*" for the extraction between players. Thus, the marginal extraction cost for player i under regime k is given by $c_i^k \in \mathbb{R}_+$. Note, that we also depart from the assumption that a given player has the same marginal cost under autarky as under free trade, which is the reason why the label k appears in the marginal cost. One could think of this as an improvement one country experiences from the efficiency point of view when they switch to free trade, due to, for instance, improvements in market efficiency processes, increased competition, access to new technology by opening its borders, etc. The idea could be that given that players are selling the resource in a common market (FT), they have access to new and better technology, which would be, for example, internationally available due to a free trade agreement.

The set of players is $N = \{i, j\}$. We assume that there is one firm per country; intuitively, one could think of it as the country exploiting the resource. We define $q_i^k(t) \in \mathbb{R}_+$ as player i's extraction at time t under the regime $k \in \{A, FT\}$. Therefore, the total extraction of the resource at time t under regime k (autarky or free trade) is given by the aggregate extraction, defined as

$$Q^k(t) = q_i^k(t) + q_j^k(t).$$

As a result, we define two types of demand, one for each case (A and FT). The main reason for this is that under autarky, the extraction of the resource can be "sold just at home". If countries are allowed to trade, they face a "global demand", where all the players offer in the same integrated market. Having different demands for autarky and free trade, one should notice that the price under autarky reflects the price of a given market i, which is influenced just by the extraction of that player. However, under free trade, the price is influenced by the extraction of both players, as they offer their harvest in a common market. Thus,

$$\begin{aligned} P_i^A &= B^A - bq_i^A(t) \\ P^{FT} &= B^{FT} - \frac{b}{2}Q^{FT}(t), \end{aligned} \tag{3.1}$$

where $Q^{FT}(t)$ is the total extraction, and the demands are the standard forms in the literature.

According to the demand structure in each regime, instantaneous profits for agent i at time t are given by

$$\begin{aligned} \pi_i^A(q_i^A(t)) &= \left[a_i^A - bq_i^A(t)\right] q_i^A(t), \\ \pi_i^{FT}(q_1^{FT}(t), q_2^{FT}(t)) &= \left[a_i^{FT} - \frac{b}{2}\left(q_i^{FT}(t) + q_j^{FT}(t)\right)\right] q_i^{FT}(t), \end{aligned} \tag{3.2}$$

where we define $a_i^k \equiv B^k - c_i^k$, for all $k \in \{A, FT\}$. In each scenario, it is evident that the disparities arising from players' asymmetries have a direct impact on their immediate gains. When a player exhibits greater efficiency, characterized by a reduced marginal cost, this advantage translates into an elevated payoff. Moreover, due to the varying profits across different regimes, players' utilities will differ based on the specific regime in which they are engaged.

The utility for player i under *autarky* is given by:

$$u_i^A(q_i^A, q_j^A) = \left(a_i^A - bq_i^A\right) q_i^A + \theta^A \left(q_i^A - q_j^A\right)$$
$$+\gamma^A \left[\left(a_i^A - bq_i^A\right) q_i^A - \left(a_j^A - bq_j^A\right) q_j^A\right],$$

where q_j^k defines the consumption of the other player. Utilities in both autarky and free trade are composed of three terms. The first one is the utility agent i gets from her own profits. The second term with θ^k reflects the fact that agent i is comparing her extraction/harvest with the other player. The third term with γ^k corresponds to the case where each player compares her profits with the profit of another agent in the game. In this two players case, they compare their extraction and their profits with the other player.

The utility for player i under *free trade* is given by:

$$u_i^{FT}(q_i^{FT}, q_j^{FT}) = \left[a_i^{FT} - \frac{b}{2}\left(q_i^{FT}(t) + q_j^{FT}(t)\right)\right] q_i^{FT}(t) + \theta^{FT}\left(q_i^{FT} - q_j^{FT}\right)$$
$$+\gamma^{FT}\left[\left(a_i^{FT} - \frac{b}{2}\left(q_i^{FT}(t) + q_j^{FT}(t)\right)\right) q_i^{FT}(t) - \left(a_j^{FT} - \frac{b}{2}\left(q_j^{FT}(t) + q_i^{FT}(t)\right)\right) q_j^{FT}(t)\right].$$

Furthermore, in this chapter we will use the piecewise differential equation first introduced in Benchekroun (2003), where the growth of the renewable resource is firstly linearly increasing, until it arrives to the maximum sustainable yield of the asset S_y, and then decreases until it crosses the horizontal axis from above, meaning that a stable steady state is reached without harvest. This reproduction function of the resource is given by

$$F(S(t)) = \begin{cases} \delta S(t) & for \;\; S(t) \leq S_y \\ \delta S_y \left(\dfrac{\bar{S} - S(t)}{\bar{S} - S_y}\right) & for \;\; S(t) > S_y, \end{cases} \tag{3.3}$$

where $\delta > 0$ reflects the intrinsic growth rate of the population. When the resource arrives to the level S_y, the growth rate starts to decline. For this reason, level δS_y is called the maximum sustainable yield (MSY).

Agent i will solve the following game under autarky and free trade, i.e. $k \in \{A, FT\}$:

$$\begin{aligned} \underset{\{q_i^k(t)\}_{i\in N}}{\text{Max}} \quad & \int_0^\infty e^{-\rho t} u_i^k(q_i^k, q_j^k)dt \\ \text{s.t.} \quad & \dot{S}(t) = F(S(t)) - q_i^k - q_j^k, \\ & q_i^k(t) \geq 0, \\ & S(0) = S_0, \end{aligned} \tag{3.4}$$

where ρ is the discount factor, and the amount of resource at the initial time is given by S_0. In this chapter, we begin by examining a symmetric game, followed by an analysis of an asymmetric scenario in which both agents employ Markovian strategies to extract the resources. Having previously discussed the utility functions for autarky and free trade, we will now explore these market structures in-depth, delving into their respective impacts on agent behavior and welfare outcomes.

3.3 Symmetric Game

In this section, we initiate our investigation into the symmetric game, characterized by equal marginal extraction costs for both players. Our approach deviates from the assumption in Benchekroun et al. (2020), which posits identical extraction costs for both players across both regimes—specifically, $c_i^A = c_j^A = c_i^{FT} = c_j^{FT}$. Instead, we preserve symmetry only within the extraction costs of a specific regime, while allowing for distinct extraction costs between autarky and free trade, represented as $c_i^A = c_j^A \neq c_i^{FT} = c_j^{FT}$. Additionally, Benchekroun and Long (2016) examines a differential game focused on status concerns, wherein countries extract renewable resources and sell them in a common market, corresponding to our free trade framework. We introduce regime-specific status concern parameters, which remain symmetric across both players: $\gamma_i^k = \gamma_j^k = \gamma^k$ and $\theta i^k = \theta j^k = \theta^k$.

3.3.1 Symmetric Autarky

To study the feedback equilibrium, we will focus on Markovian strategies, commonly used in continuous-time dynamic games. For the 2 symmetric players game, we define the vector of control (decision) variables under autarky as $\boldsymbol{\phi}^A := (\phi_1^{Sym,A*}, \phi_2^{Sym,A*}) \in \mathbb{R}_+^2$, which form a Subgame Perfect Markov Nash Equilibrium (SPMNE). The state variable in this chapter is the renewable resource, $S(t) \in \mathbb{R}_+$. Then, the Hamilton-Jacobi-Bellman equation for player i associated with problem (3.4) is given by:

$$\rho V_i^{Sym,A}(S) = \underset{\{q_i^{Sym,A}\}}{\text{Max}} \left(\begin{array}{c} \left(a_i^{Sym,A} - b\, q_i^{Sym,A}\right) q_i^{Sym,A} + \theta^A \left(q_i^{Sym,A} - \phi_j^{Sym,A}\right) \\ \\ +\gamma^A \left[\left(a_i^{Sym,A} - b\, q_i^{Sym,A}\right) q_i^{Sym,A} - \left(a_j^{Sym,A} - b\, \phi_j^{Sym,A}\right) \phi_j^{Sym,A}\right] \\ \\ +\dfrac{\partial V_i^{Sym,A}(S)}{\partial S}\left[F(S) - q_i^{Sym,A} - \phi_j^{Sym,A}\right] \end{array} \right). \tag{3.5}$$

Symmetric players extract the renewable resource according to the strategy shown in Proposition 3.1.

Proposition 3.1. *The vector of symmetric strategies* $(\phi^{Sym,A*}, \phi^{Sym,A*})$ *constitutes a Subgame Perfect Markov Nash Equilibria for each agent* $i \in N$*, where*

$$\phi^{Sym,A*} = \begin{cases} 0 & if\, S \in [0, S_1^{Sym,A}) \\ \\ w^{Sym,A} + z^{Sym,A} S(t) & if\, S \in [S_1^{Sym,A}, S_2^{Sym,A}] \\ \\ q^{Sym,Cou,A} = \frac{a^{Sym,A}(1+\gamma^A)+\theta^A}{2b(1+\gamma^A)} & if\, S \in (S_2^{Sym,A}, \infty) \end{cases} \tag{3.6}$$

with

$$w^{Sym,A} = \frac{a^{Sym,A}(1+\gamma^A)\left[2(1+\gamma^A)\rho-\delta\right]+\theta^A(\delta+2\gamma^A\rho+\rho)}{2b\delta(1+\gamma^A)(4\gamma^A+3)},\ z^{Sym,A} = \frac{(1+\gamma^A)(2\delta-\rho)}{4\gamma^A+3},$$

and the "endogenously threshold levels of stock" are given by

$$S_1^{Sym,A} = \frac{a^{Sym,A}(1+\gamma^A)\left[2(1+\gamma^A)\rho-\delta\right]+\theta^A(\delta+2\gamma^A\rho+\rho)}{2b\delta(1+\gamma^A)^2(\rho-2\delta)},$$

$$S_2^{Sym,A} = \frac{2a^{Sym,A}(1+\gamma^A)^2+\theta^A(1+2\gamma^A)}{2b\delta(1+\gamma^A)^2}.$$

Proof. See Appendix 3.7.1. ■

Assumption 3.1. *We assume that the intrinsic growth rate is sufficiently large*

$$\delta > \ max \left\{ \frac{\rho}{2}, \frac{2a^{Sym,A}(1+\gamma^A)^2 + \theta^A(1+2\gamma^A)}{2b(1+\gamma^A)^2 S_y} \right\}, \tag{3.7}$$

and the status concern parameter is small enough,

$$\theta^A \leq \frac{a^{Sym,A}(1+\gamma^A)\left[\delta-2\rho(1+\gamma^A)\right]}{[\delta+\rho(1+2\gamma^A)]}. \tag{3.8}$$

The first condition (equation 3.7) in the previous assumption ensures that the slope of the extraction is positive when agents extract using a linear strategy for $S \in [S_1^{Sym,A}, S_2^{Sym,A}]$, which is defined as the affine part. This captures the fact that the more resource there is, the more the players extract. Moreover, this also guarantees the existence of a positive steady state. This can be seen from $z^{Sym,A} = \dfrac{(1+\gamma^A)(2\delta-\rho)}{4\gamma^A+3}$, which is the slope of the strategy. As γ^A is a positive parameter, the condition that must hold for z^A to be positive is $2\delta > \rho$. The second part of the inequality ensures that the second switching point $S_2^{Sym,A}$ is smaller than the maximum-sustainable-yield stock, S_y, which is commonly assumed in the literature. Inequality (3.8) ensures that the threshold $S_1^{Sym,A}$ is nonnegative.

Additionally, the distance between the two thresholds is:

$$S_2^{Sym,A} - S_1^{Sym,A} = \frac{(3+4\gamma^A)\left[\theta^A + a^{Sym,A}(1+\gamma^A)\right]}{2b(1+\gamma^A)^2(2\delta-\rho)}. \tag{3.9}$$

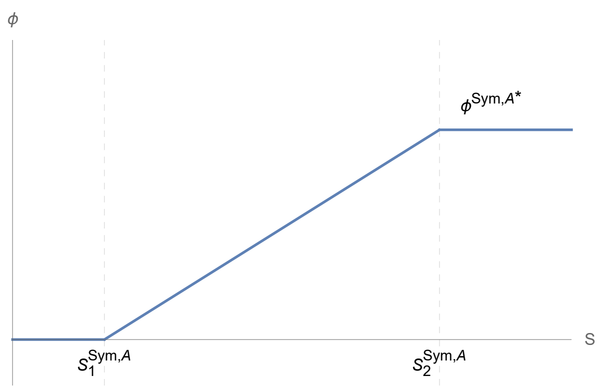


Figure 3.1: Symmetric Extraction Strategy under Autarky.

As in Benchekroun (2003), the strategy consists of three parts. In the first one, for $S(t) \in [0, S_1^{A,Sym})$, both players extract zero when there is too little resource. It can be interpreted that the best strategy for both agents is not to extract, since they are “investing” in the resource for the future. Waiting when there is too little resource is a good strategy because if they were to extract a positive amount, the resource would be exhausted.[8] The second term, when $S(t) \in [S_1^{A,Sym}, S_2^{A,Sym}]$ reflects the increasing linear strategy, i.e., the affine part of the agent’s strategy, reflecting

[8]See Benhabib and Radner (1992) and Fershtman and Kamien (1987), where players also cease consumption and production respectively when the stock or price is below a given threshold.

that when there is more resource, agents extract more. When resources are abundant, $S(t) \in (S_2^{A,Sym}, \infty)$, the last horizontal part of the strategy corresponds to the Cournot equilibrium, as if agents would play a one-shot game duopoly. This can be understood as if there were no resource constraints, players would extract as much as possible to maximize their profit or utility. If they extract too much, they would pump the market with a lot of resources, and the price would be so low that their profit would be even lower.[9]

The value function of the present problem for each player is,

$$V^{Sym,A}(S) = \begin{cases} \left(\frac{S}{S_1^{Sym,A}}\right)^{\frac{\rho}{\delta}} W(S_1^{Sym,A}; \gamma^A, \theta^A)^{Sym,A} & \text{if } S \in [0, S_1^{Sym,A}), \\ W(S; \gamma^A, \theta^A)^{Sym,A} & \text{if } S \in [S_1^{Sym,A}, S_2^{Sym,A}], \\ \frac{(a^{Sym,A})^2(1+\gamma^A)^2 - (\theta^A)^2}{4b\rho(1+\gamma^A)^2} & \text{if } S \in (S_2^{Sym,A}, \infty), \end{cases} \tag{3.10}$$

where $W(S; \gamma^A, \theta^A)^{Sym,A} = \frac{\alpha^{Sym,A}}{2} S^2 + \beta^{Sym,A} S + \mu^{Sym,A}$. The coefficients of the value function and its derivations are developed in Appendix 3.7.1.

Now we will study the sensitivity analysis of the status-concern parameters, first analyzing the effects of changing the weight of importance in relative extraction (θ^A), and later the effects of changing the weight of importance in relative profits (γ^A).

Effect of changing the weight of the relative extraction under the symmetric autarky game (θ^A)

As we incorporate the status concern to both players, an interesting question is how a key parameter of the model influences the behavior of both agents. One can show that the extraction strategies of players move upwards in parallel when we increase θ^A (see Figure 3.2). To see this, observe that the slope of the strategy does not change,

[9] As hightailed by Benchekroun (2003), "*When the asset is "abundant" $S \geq S_2$ firms can simply adopt the production they would adopt under a static Cournot game. Therefore, when firms play the equilibrium strategies of a static Cournot game, it does not necessarily mean that it is due to short-term foresight: it can be the outcome of a subgame perfect equilibrium of an infinite horizon game*".

$$\frac{\partial}{\partial \theta^A}\left[\underbrace{z^{Sym,A}}_{Slope}\right] = 0, \tag{3.11}$$

and the Cournot extraction is higher,

$$\frac{\partial q^{Sym,Cou,A}}{\partial \theta^A} = \frac{1}{2b(1+\gamma^A)} > 0. \tag{3.12}$$

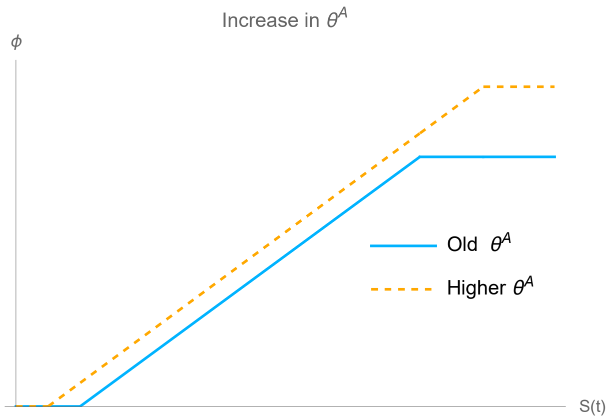


Figure 3.2: Extraction Strategy for both players under Autarky when θ^A increases.

To show that the affine part moves upward, given that the slope does not change, we have to show that S_1^A decreases and S_2^A increases,

$$\frac{\partial S_1^{Sym,A}}{\partial \theta^A} = -\frac{\delta + \rho(1+2\gamma^A)}{2b\delta(1+\gamma^A)^2(2\delta-\rho)} < 0, \tag{3.13}$$

which means it goes to the left, and

$$\frac{\partial S_2^{Sym,A}}{\partial \theta^A} = \frac{1+2\gamma^A}{2b\delta(1+\gamma^A)^2} > 0, \tag{3.14}$$

which means it goes to the right. Thus, it is straightforward to show that the distance $S_2^A - S_1^A$ in (3.9) increases,

$$\frac{\partial}{\partial \theta^A}\left[S_2^{Sym,A} - S_1^{Sym,A}\right] = \frac{3+4\gamma^A}{2b(1+\gamma^A)^2(2\delta-\rho)} > 0. \tag{3.15}$$

For this reason, in Figure 3.2, we show how the extraction strategies for both players move upwards when we increase θ^A ($\partial_{\theta^A}\phi^{A*}(S) > 0$). This means that now agents care more about their relative extraction, which results in players extracting more resources at any given point for all $S(t) > S_1^{Sym,A,new\,\theta}$. When agents care more about their relative extraction, both start extracting sooner, i.e., S_1^A decreases. On the contrary, both agents move to the right their second switching point S_2^A, which means they now wait a little longer before they start playing the Cournot strategy.

This behavior is driven by the fact that the relative importance of the profits in the utility has decreased. Thus, agents now care relatively more about the difference in extraction than the profits themselves, and have the incentive to harvest more because it increases their utility. Thus, one can see that the strategy moves upwards,

$$\frac{\partial}{\partial \theta^A}\left[\phi^{Sym,A*}\right] = \begin{cases} 0 & \text{if } S(t) \in [0, S_1^{Sym,A,New\,\theta}) \\ \frac{\delta+\rho(1+2\gamma^A)}{b\delta\left[8(\gamma^A)^2+14\gamma^A+6\right]} > 0 & \text{if } S(t) \in [S_1^{Sym,A,New\,\theta}, S_2^{Sym,A,New\,\theta}] \\ \frac{1}{2b(1+\gamma^A)} > 0 & \text{if } S(t) \in (S_2^{Sym,A,New\,\theta}, \infty) \end{cases} \tag{3.16}$$

Finally, one should note that there is the trade-off effect that keeps their extraction down to “preserve the resource” and pushes up to “maximize their profits”.

Effect of changing the weight of the relative profit under the symmetric autarky game (γ^A)

When agents care more about their relative profits (higher γ^A), we observe they start extracting with the Cournot strategy sooner (S_2^A moves to the left), and S_1^A can move to the left or to the right,

$$\frac{\partial}{\partial \gamma^A}\left[S_1^{Sym,A}\right] = \frac{2\theta^A(\delta+\gamma^A\rho) - a^{Sym,A}\delta(1+\gamma^A)}{2b\delta(1+\gamma^A)^3(2\delta-\rho)} \lesseqgtr 0,$$

$$\frac{\partial}{\partial \gamma^A}\left[S_2^{Sym,A}\right] = -\frac{\gamma^A\theta^A}{b\delta(1+\gamma^A)^3} < 0.$$

The first switching point could increase or decrease. One can see that $S_1^{Sym,A}$ moves to the left if $2\theta^A(\delta+\gamma^A\rho) < a^{Sym,A}\delta(1+\gamma^A)$. The distance between the two thresholds $S_2^{Sym,A} - S_1^{Sym,A}$ could be positive or negative,

$$\frac{\partial}{\partial \gamma^A}\left[S_2^{Sym,A} - S_1^{Sym,A}\right] = \frac{a^{Sym,A}(1+\gamma^A) - 2\theta^A(1+2\gamma^A)}{2b(1+\gamma^A)^3(2\delta-\rho)} \lesseqgtr 0. \tag{3.17}$$

The distance between S_2^A and S_1^A increases in the free trade scenario studied in

Benchekroun and Long (2016)[10] when players care more about the relative profits. However, in our model under autarky, we can get both situations. Moreover, the slope of the affine strategy goes down,

$$\frac{\partial}{\partial \gamma^A}\left[\frac{(1+\gamma^A)(2\delta-\rho)}{4\gamma^A+3}\right] = -\frac{2\delta-\rho}{(4\gamma^A+3)^2} < 0, \tag{3.18}$$

which is negative, as $2\delta > \rho$ by assumption (3.7). The fact that $S_2^{Sym,A}$ moves to the left, and $S_1^{Sym,A}$ can go both directions, is different from the free trade result in Benchekroun and Long (2016). This is mainly due to the fact that we are now exploring the autarky case while they study the free trade scenario (integrated market). Besides, the Cournot part of both players goes down when agents care more about the relative profits,

$$\frac{\partial}{\partial \gamma^A}\left[q^{Sym,Cou,A} = \frac{a^{Sym,A}(1+\gamma^A)\theta^A}{2b(1+\gamma^A)}\right] = -\frac{\theta^A}{2b(1+\gamma^A)^2} < 0, \tag{3.19}$$

unlike the result under free trade analyzed by the previous authors. The symmetric case for a $2-players$ scenario in their paper,[11] compares to our extraction under the Cournot part in a particular term. The strong effect comes from the demand parameter b, which captures the slope of the demand in the market. This parameter is also in the recent investigation carried out by Benchekroun et al. (2020), where the Cournot strategy under the autarky scenario is given by $q_i^{A,Cournot} = \frac{a}{2b}$, while in our model we see the effect of the status concern parameters under autarky γ^A and θ^A in the third term of the extraction, $q^{Sym,Cou,A} = \frac{a^{Sym,A}(1+\gamma^A)+\theta^A}{2b(1+\gamma^A)}$. Increasing γ^A in our case has a contrary effect to the one obtained by the mentioned authors in terms of the Cournot strategy. This behavior can be seen from the derivative of the Cournot strategy part (3.19), which is negative. This shows the reduction for the Cournot part of the strategy analytically. Moreover, if one eliminates the effects of the status concert, i.e., $\gamma^A = \theta^A = 0$, we recover the same Cournot strategy as in Benchekroun et al. (2020). In the following table, we show the results in a compact form.

Figure 3.3 shows the graphical analysis of a change in this parameter for the case when $S_1^{Sym,A}$ moves to the left. The graphical representation when $S_1^{Sym,A}$ increases is shown in Figure 3.4, and agents will always be extracting fewer resources. This case will happen when $a^{Sym,A}$ is small enough, i.e., when agents have large marginal

[10] In their paper, this result follows immediately, as S_1^A decreases and S_2^A increases.

[11] For comparison, the Cournot strategy in the paper by Benchekroun and Long (2016) is $q^{Ben.,Long} = \frac{a(1+\gamma)}{1+2(1+\gamma)}$.

extraction costs.

	S_1^A	S_2^A	$q_i^{Sym,Cou,A}$	Slope
Benchekroun and Long (2016) (FT)	$\Downarrow$	$\Uparrow$	$\Uparrow$	$\Downarrow$
Our Model under Autarky	$\Uparrow$ × or $\Downarrow$ ✓	$\Downarrow$ ×	$\Downarrow$ ×	$\Downarrow$ ✓

Table 3.1: Changes in the switching point in comparison when γ increases.

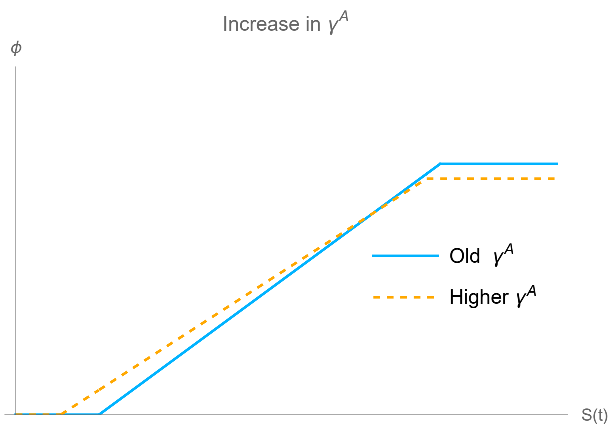


Figure 3.3: Extraction Strategy under Autarky when γ^A increases and $S_1^{Sym,A}$ decreases.

Symmetric Autarky Steady States

In order to study the symmetric autarky steady states of the game, we define $\Phi^{Sym,A*} := \phi_1^{Sym,A}(S) + \phi_2^{Sym,A}(S) = 2\phi^{Sym,A}(S)$ as the total extraction of the resource. Considering $\Phi^{Sym,A*}$ and using the reproduction function of the resource given byequation (3.3), we obtain an extra steady state compared to Benchekroun et al. (2020), where the authors obtain two equilibria, the first one stable from the left and unstable from the right, and a second steady state which is stable. In our model, we obtain three steady states as shown in Figure 3.5. The first steady state is stable, corresponding to the point where the extraction is linearly increasing and cutting the increasing part of the reproduction function. The second steady state corresponds to the intersection of the Cournot part of the extraction strategy and the section where the resource grows exponentially. Finally, the third steady state under autarky is given by the locus where the Cournot part cuts the decreasing part of the reproduction function. This extra steady state emerges purely from the status concern parameters γ^A and θ^A.

Under autarky, one can see that the first two steady states emerge from the one in Benchekroun et al. (2020), which was stable from the left and unstable from the

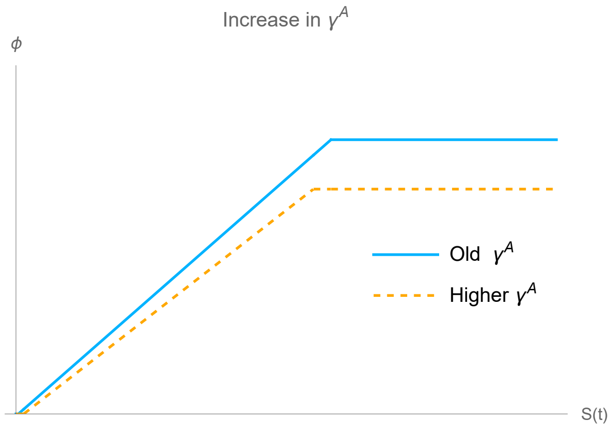


Figure 3.4: Extraction Strategy under Autarky when γ^A increases and $S_1^{Sym,A}$ increases.

right, and the extraction strategy was just touching the differential equations and never crossing it. If we set all the status concerns parameters to 0 such that we recover the results in Benchekroun et al. (2020), one can see that the first two steady states collapse to the one in their study. Thus, we obtain an extra equilibrium with our framework. The study of the stability of the equilibria can be easily seen in the evolution of the resource, which is the figure on the right. This figure shows the distance between the reproduction of the resource and the total extraction. Therefore, if the image of the function is positive, it means that the resource is moving to the right and increasing. The steady state is stable if the function crosses the horizontal axis from above. An interesting novel result is driven by the fact that now we obtain an extra steady state, which we can define as a "*Natural Resource Poverty Trap (NRPT)*". This concept borrowed from the economic growth/development literature can be extrapolated to our framework.[12] As one can see, starting from the first stable steady state, if there is a small perturbation to the right in the resource, it will come back to the left, as agents are extracting more than what the resource can regenerate itself. However, if policy-makers, for instance, forbid the extraction of the resource for a certain period such that the resource can regenerate until the point where it passes the NRPT, i.e., pushing further point $S_{2,\infty}$, then the "*poverty trap*" will have been overcome, since for any value of $S(t) \in (S_{2,\infty}, S_{3,\infty})$, the resource will regenerate faster than what agents extract, increasing the stock until it reaches the stationary state $S_{3,\infty}$. Thus, this "big push", or "do not extract for a while", will allow the resource to grow until it arrives to a stationary state considered as the "rich equilibrium".

As one can see, the linearly increasing part of the total extraction that intersects

[12] See, for instance, Banerjee et al. (2011) and Banerjee and Duflo (2005)

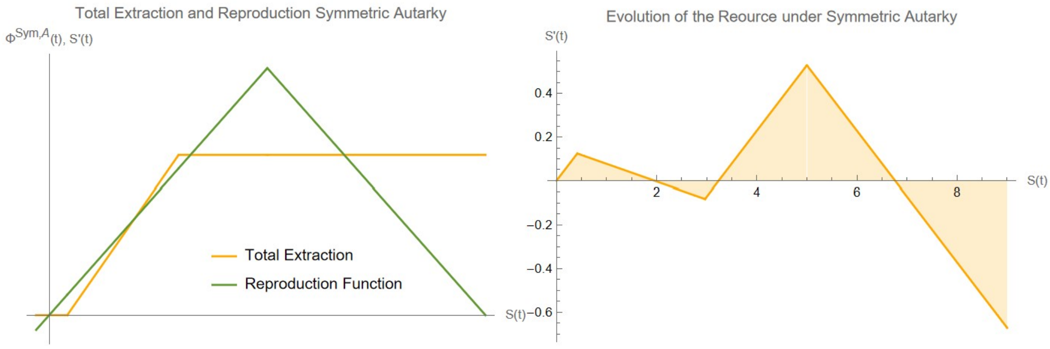


Figure 3.5: Total Extraction and Evolution of the Resource under Autarky.

the reproduction function is the one given by the sum of both linearly increasing parts of the strategy of both players (the part where the strategies are affine). Thus, this "vertical integration" is no more than the sum of the linear parts which is given by

$$\underbrace{w_1^A +^A S(t)}_{\text{affine strategy for P1}} + \underbrace{w_2^A + z^A S(t)}_{\text{affine strategy for P2}} = 2w^{Sym,A} + 2z^{Sym,A} S(t).$$

Therefore, the intersection between $2w^{Sym,A} + 2z^{Sym,A} S(t)$ and $\delta S(t)$ gives the first steady state $S_{1,\infty}^{Sym,A}$. Furthermore, the second steady state is given by the intersection of the sum of both Cournot strategies ($q_1^{Sym,Cou,A} + q_2^{Sym,Cou,A} = q^{Sym,Cou,A}$), and the increasing part of the reproduction function of the resource, i.e., $\delta S(t)$, defining this equilibrium as $S_{2,\infty}^A$. Finally, the last steady state can be found by the intersection of the sum of both Cournot strategies and the decreasing part of the reproduction function, which gives $S_{3,\infty}^A$. Easily described, one can find stable equilibria when the graph on the right cuts the horizontal axis from above, and unstable states when it cuts from below.

Remark 3.1. *Under the Autarky regime and agents playing their MPNE strategies, there are three positive steady states:*

$$S_{1,\infty}^A = \frac{2w^{Sym,A}}{\delta - 2z^{Sym,A}}, \qquad S_{2,\infty}^A = \frac{2q^{Sym,Cou,A}}{\delta},$$

$$S_{3,\infty}^A = \bar{S} - \frac{(\bar{S} - S_y)(2q^{Sym,Cou,A})}{\delta S_y}.$$

3.3.2 Symmetric Free Trade

In the following section, we study the effect of changing one rule of the game, i.e., allowing players to trade between them in a common market. In order to see how this "*institutional change*" affects the discounted utilities of both countries, we study first how free trade affects their extraction strategy. Thus, our main target is to compare when we allow countries to trade with the base case scenario (autarky). In the case with two symmetric agents, the common market consists of two oligopolistic firms who supply their extraction to an integrated market.

As in the autarky case, we need the following assumption.

Assumption 3.2. *We assume that the intrinsic growth rate is sufficiently large*

$$\delta > \max\left\{\frac{\rho}{2}, \frac{2a^{Sym,FT}\left[5+4\gamma^{FT}(2+\gamma^{FT})\right]+4\theta^{FT}(1+4\gamma^{FT})}{b(2\gamma^{FT}+3)^2 S_y}\right\}, \tag{3.20}$$

and the status concern parameter is small enough,

$$\theta^{FT} \leq \frac{a^{Sym,FT}(2\delta-\rho\left[5+4\gamma^{FT}(2+\gamma^{FT})\right])}{2\left[2\delta+\rho(1+2\gamma^{FT})\right]}. \tag{3.21}$$

Condition 3.1. *If the reproduction rate of the resource is big enough,*

$$\begin{aligned}\delta > & \frac{\rho}{8a^{Sym,FT}(1+\gamma^A)^2 - a^A(1+\gamma^A)(3+2\gamma^{FT})^2 - 16(1+\gamma^A)^2\theta^{FT} + (3+2\gamma^{FT})^2\theta^A} \times \\ & \{4a^{Sym,FT}(1+\gamma^A)^2[4(2+\gamma^{FT})\gamma^{FT}+5] - 2a^{Sym,A}(\gamma^A+1)^2(3+2\gamma^{FT})^2 \\ & +8(1+\gamma^A)^2(1+2\gamma^{FT})\theta^{FT} - (1+2\gamma^A)(3+2\gamma^{FT})^2\theta^A\},\end{aligned} \tag{3.22}$$

then, $S_1^{Sym,FT} < S_1^{Sym,A}$.

The first part of Assumption 3.20 ensures that the slope of the affine strategy is positive (as in the autarky case), and the second part guarantees that the second switching point $S_2^{Sym,FT}$, is smaller than the maximum-sustainable-yield stock, S_y. Inequality (3.21) ensures that the threshold $S_1^{Sym,A}$ is nonnegative. If Condition 3.1 holds, it will implies that when agents trade, they start extracting the resource sooner, i.e., $S_1^{Sym,FT} < S_1^{Sym,A}$, and one would obtain the same behavior as in Benchekroun et al. (2020). However, this is not a straightforward result in our model, and one should impose conditions. Our numerical simulations show that this is true when the efficiency parameters $a^{Sym,A}$ and $a^{Sym,FT}$ are closed enough, while the opposite is true if a large change in extraction efficiencies is allowed when the two countries trade the natural resource.

As before, we now solve problem (3.4), where player i chooses her Markovian strategies given the strategies of the other player. In order to solve the optimization problem, we solve the following Hamilton-Jacobi-Bellman equation for player i is,

$$\rho V_i^{FT}(S) = \max_{\{q_i^{Sym,FT}\}} \left(\begin{array}{c} \left(a^{Sym,FT} - \frac{b}{2}(q_i^{Sym,FT}(t) + \phi_j^{Sym,FT}(t))\right) q_i^{Sym,FT}(t) + \theta^{FT}\left(q_i^{Sym,FT} - \phi_j^{FT}\right) \\ \\ +\gamma^{FT}\left[\left(a^{Sym,FT} - \frac{b}{2}(q_i^{Sym,FT}(t) + \phi_j^{Sym,FT}(t))\right) q_i^{FT}(t) \right. \\ \left. -\left(a^{Sym,FT} - \frac{b}{2}(q_i^{Sym,FT}(t) + \phi_j^{Sym,FT}(t))\right)\phi_j^{FT}(t)\right] \\ \\ +\frac{\partial V_i^{FT}(S)}{\partial S}\left[F(S) - q_i^{FT} - \phi_j^{FT}\right] \end{array} \right) \tag{3.23}$$

Defining $\boldsymbol{\phi}^{FT*} := (\phi_i^{FT*}, \phi_j^{FT*}) \in \mathbb{R}_+^2$ as the stationary Markovian strategy, and the vector of Subgame Perfect Markov Nash Equilibrium, we can obtain an analytical expression for the strategies of the players. The symmetric vector of strategies $(\phi^{Sym,FT*}, \phi^{Sym,FT*})$ is shown in the following proposition.

Proposition 3.2. *The following vector of symmetric strategies under free trade (ϕ^{FT*}, ϕ^{FT*}) constitutes a Subgame Perfect Markov Nash Equilibria for each agent i, where*

$$\phi^{Sym,FT*} = \begin{cases} 0 & \text{if } S \in [0, S_1^{Sym,FT}) \\ \\ w^{Sym,FT} + z^{Sym,FT} S(t) & \text{if } S \in [S_1^{Sym,FT}, S_2^{Sym,FT}] \\ \\ q^{Sym,Cou,FT} = \frac{2(\theta^{FT} + a^{Sym,FT}(1+\gamma^{FT}))}{b(3+2\gamma^{FT})} & \text{if } S \in (S_2^{Sym,FT}, \infty) \end{cases} \tag{3.24}$$

with

$$w^{Sym,FT} = \frac{\rho a^{Sym,FT}\left[4\gamma^{FT}(2+\gamma^{FT})+5\right]+2\theta^{FT}(2\delta+2\gamma^{FT}\rho+\rho)-2\delta a^{Sym,FT}}{4b\delta(1+\gamma^{FT})(3+2\gamma^{FT})},$$

$$z^{Sym,FT} = \frac{(3+2\gamma^{FT})(2\delta-\rho)}{8(1+\gamma^{FT})},$$

and the "endogenously threshold levels of stock" are given by

$$S_1^{Sym,FT} = \frac{4\delta a^{Sym,FT}-2\rho a^{Sym,FT}\left[5+4\gamma^{FT}(2+\gamma^{FT})\right]-4\theta^{FT}\left[2\delta+\rho(1+2\gamma^{FT})\right]}{b\delta(3+2\gamma^{FT})^2(2\delta-\rho)},$$

$$S_2^{Sym,FT} = \frac{2a^{Sym,FT}\left[5+4\gamma^{FT}(2+\gamma^{FT})\right]+4\theta^{FT}(1+4\gamma^{FT})}{b\delta(2\gamma^{FT}+3)^2}.$$

Proof. See Appendix 3.7.2. ∎

The value function of the symmetric free trade game is,

$$V^{Sym,FT}(S) = \begin{cases} \left(\frac{S}{S_1^{Sym,FT}}\right)^{\frac{\rho}{\delta}} W(S_1^{Sym,FT};\gamma^{FT},\theta^{FT})^{Sym,FT} & \text{if } S \in [0, S_1^{Sym,FT}) \\ W(S;\gamma^{FT},\theta^{FT})^{Sym,FT} & \text{if } S \in [S_1^{Sym,FT}, S_2^{Sym,FT}] \\ \frac{2(a^{Sym,FT}-2\theta^{FT})[a^{Sym,FT}(1+\gamma^{FT})+\theta^{FT}]}{b\rho(3+2\gamma^{FT})^2} & \text{if } S \in (S_2^{Sym,FT}, \infty) \end{cases} \tag{3.25}$$

where $W(S;\gamma^{FT},\theta^{FT})^{Sym,FT} = \frac{\alpha^{Sym,FT}}{2}S^2 + \beta^{Sym,FT}S + \mu^{Sym,FT}$. The coefficients of the value function and its derivations are developed in Appendix 3.7.2.

As under the autarky game, when we now allow players to trade, the strategies of both players are defined by a piecewise function composed of three terms (eq. 3.24). The first one represents where agents harvest nothing when there is not enough resource. Thus, they allow the asset to regenerate until a certain threshold $S_1^{Sym,FT}$. The second term reflects the affine strategy, linearly increasing, which emulates the idea that the more there is, the more they extract. Finally, they use the Cournot strategy.[13] Keeping the same parameters as before,[14] i.e., same cost structure and status concern as in the autarky case, we can compare both extraction strategies under autarky vs. free trade in Figure 3.6.[15] Plotting them together help us visually understand the free trade effect. As one can observe, allowing trade between players

[13]See the autarky section for the economic interpretation.

[14]$b = 7, \theta^{FT} = 0.2, \gamma^{FT} = 0.08, \delta = 0.25, r = 0.02, a_1^{FT} = 5, a_2^{FT} = 4$

[15]We intentionally kept this combination of parameters to isolate and see (*only*) the effect of allowing players to trade.

has a significant change in their strategies. When players are allowed to trade, they start extracting the resource when there is less of it, i.e., $S_1^{Sym,FT}$ moves to the left by assumption (3.2).

Moreover, we can see that if we generalize and impose the following condition, we get the same result as in Benchekroun et al. (2020), where the second switching point moves to the right when players are allowed to trade (when they switch from the affine to the Cournot strategy) i.e., $S_2^{Sym,A} < S_2^{Sym,FT}$,

Condition 3.2. *If the status concern parameter θ^{FT} is big enough,*

$$\theta^{FT} \geq \frac{1}{8(1+\gamma^A)^2(1+2\gamma^{FT})} \times \{2a^{Sym,A}(1+\gamma^A)^2(3+2\gamma^{FT})^2$$

$$-4a^{Sym,FT}(1+\gamma^A)^2[5+4\gamma^{FT}(\gamma^{FT}+2)] + (1+2\gamma^A)(3+\gamma^{FT})^2\theta^A\}, \tag{3.26}$$

then, $S_2^{Sym,A} < S_2^{Sym,FT}$.

If Condition 3.2 holds, then we get $S_2^{Sym,A} < S_2^{Sym,FT}$, which is the same result as in Benchekroun et al. (2020). However, either if the status concern parameter θ^A is big enough, or $a^{Sym,A} >> a^{Sym,FT}$, we would obtain a different result from the one obtained by the above-mentioned authors. An example of this new result driven by the status concern parameter θ can be seen in Figure 3.7. Moreover, also the Cournot extraction could be higher under autarky. These new results are driven by the status concern parameters and extraction costs that we allow to be different under autarky and free trade.

As status concern behavior was not present in Benchekroun et al. (2020), they can prove that $S_1^{Sym,A} > S_1^{Sym,FT}$ and $S_2^{Sym,A} < S_2^{Sym,FT}$ without any further assumption as we have done (see their Proposition 3). If we would not consider Condition 3.2, any result could be possible (see Figure 3.7).

While in Benchekroun and Long (2016) the authors analyze the effect of the status concern in the players' strategy and welfare, we also include the possibility of moving from autarky to free trade *à la* Benchekroun et al. (2020). Furthermore, in contrast to the approach followed by the former authors, where they analyze in isolation each status concern force, we study the problem with both status parameters at the same time, i.e., the relative output gap (our γ^k) and the relative profit gap (our θ^k) interacting at the same time.

Turning back to the consequences of a regime change, one can see that both players always extract more under free trade if, and only if, Condition 3.2 holds. One can see that the gap (distance between the free trade and autarky strategies) increases as there is more resource available. This implies that,

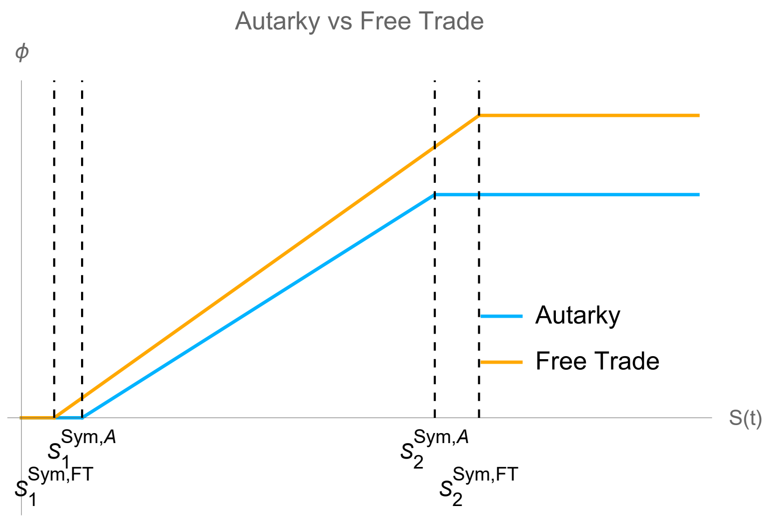


Figure 3.6: Extraction Strategy under Autarky and Free Trade as in Benchekroun et al. (2020).

Remark 3.2. *The slope under free trade will be higher than the slope under autarky if, and only if,*

$$z^{FT} > z^A \Leftrightarrow \frac{(3+2\gamma^{FT})}{8(1+\gamma^{FT})} > \frac{(1+\gamma^A)}{(3+4\gamma^A)} \iff \gamma^A > \frac{2\gamma^{FT}-1}{4}. \tag{3.27}$$

Remark 3.2 shows that the slope of the strategy is higher when players switch to free trade. This could be interpreted as the agents being more aggressive in their extraction.

Remark 3.3. *When players have the same status concern under autarky and free trade, i.e., $\gamma^A = \gamma^{FT} = \gamma$, the slope under free trade will be higher than the slope under autarky,*

$$z^{FT} > z^A \Leftrightarrow 2\gamma + 1 > 0. \tag{3.28}$$

Additionally, the distance between the two switching points is given by

$$S_2^{Sym,FT} - S_1^{Sym,FT} = \frac{16(1+\gamma^{FT})[\theta^{FT} a^{Sym,FT}(1+\gamma^{FT})]}{b(3+2\gamma^{FT})^2(2\delta-\rho)}. \tag{3.29}$$

Effect of changing the weight of the relative extraction under symmetric free trade (θ^{FT})

Following the same structure as in the autarky regime, we now expose the case when agents care more about their relative extraction. This results in an increase in

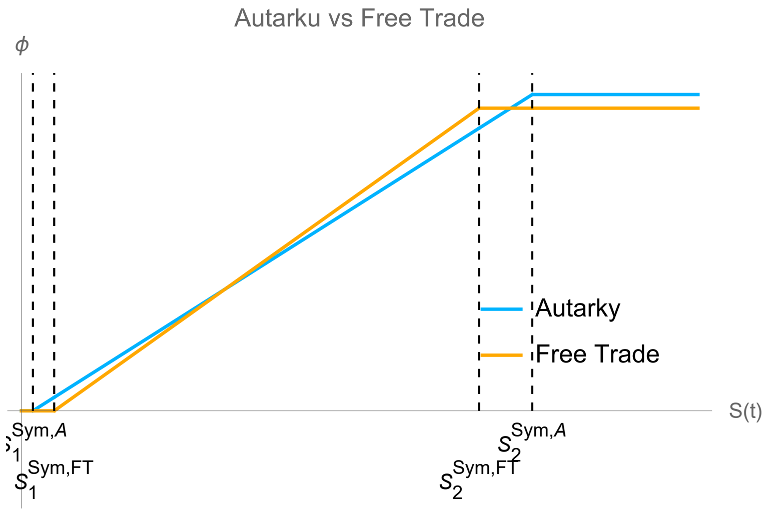


Figure 3.7: Extraction Strategy under Autarky and Free Trade when Condition 3.2 is not considered. All results are changed, $S_1^{Sym,A} < S_1^{Sym,FT}$, and $S_2^{Sym,A} > S_2^{Sym,FT}$, and $q^{Sym,Cou,FT} < q^{Sym,Cou,A}$. This is caused by having different status concern parameters and different marginal costs in autarky and free trade.

θ^{FT}. As in the autarky case, both players extract more, which means $\dfrac{\partial \phi_i^{FT}}{\partial \theta^{FT}} \geq 0$, for all $S(t)$ as seen in Figure 3.8. As shown earlier, this increase in the parameter captures the fact that players now care more about relative extraction, which results in players extracting more resources. Looking at the switching values of the strategies, we observe that S_1^{FT} moves to the left, capturing the idea that both players start extracting earlier. Moreover, as in the previous section, S_2^{FT} moves to the right, implying that they prefer to wait until there are more resources to adopt the Cournot strategy. The economic causes of this behavior were exposed in the autarky case and hold in the free trade scenario (Section 3.3.1). Moreover, as pointed out in the autarky regime, we see that the slope of the strategies do noes change since z^{FT} does not depend on θ^{FT}, as in Benchekroun et al. (2020),

$$\frac{\partial}{\partial \theta^{FT}} \left[\underbrace{z^{Sym,FT}}_{Slope} \right] = 0, \tag{3.30}$$

and the Cournot extraction is higher,

$$\frac{\partial q^{Sym,Cou,FT}}{\partial \theta^{FT}} = \frac{2}{b(3+2\gamma^{FT})} > 0. \tag{3.31}$$

To show that the affine part moves upward, given that the slope does not change, we have to show that S_1^{FT} decreases and S_2^{FT} increases,

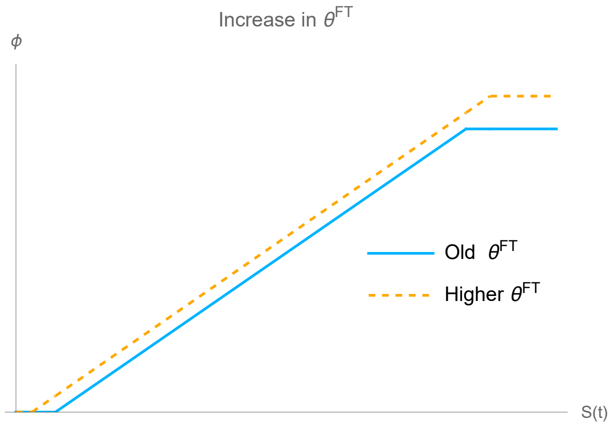


Figure 3.8: Extraction Strategy for both player under Autarky when θ^{FT} increases.

$$\frac{\partial S_1^{Sym,FT}}{\partial \theta^{FT}} = -\frac{4[2\delta + \rho(1+2\gamma^{FT})]}{b\delta(3+2\gamma^{FT})^2(2\delta-\rho)} < 0, \tag{3.32}$$

which means it goes to the left, and

$$\frac{\partial S_2^{Sym,FT}}{\partial \theta^{FT}} = \frac{4+8\gamma^{FT}}{b\delta(3+2\gamma^{FT})^2} > 0, \tag{3.33}$$

which means it goes to the right. Thus, it is straightforward to show that the distance $S_2^{FT} - S_1^{FT}$ in (3.29) increases,

$$\frac{\partial}{\partial \theta^{FT}} \left[S_2^{Sym,FT} - S_1^{Sym,FT} \right] = \frac{16(1+\gamma^{FT})}{b(3+2\gamma^{FT})^2(2\delta-\rho)} > 0. \tag{3.34}$$

For this reason, one can see in Figure 3.8 how the extraction strategies move upwards when we increase θ^{FT}. This means that agents now care more about their relative extraction, resulting in players extracting more resources at any given point for all $S(t) > S_1^{Sym,A,new\ \theta}$. When agents care more about their relative extraction, both start extracting sooner, i.e., $S_1^{Sym,FT}$ decreases. On the contrary, both agents move to the right their second switching point S_2^{FT}, which means they now wait a little longer before they start playing the Cournot strategy.

This behavior is driven by the fact that the relative importance of the profits in the utility has decreased. Thus, they now care relatively more about the difference in extraction than profits itself, having incentives to harvest more as it will increase their utility. Thus, one can see that the strategy moves upward,

$$\frac{\partial}{\partial \theta^{FT}}\left[\phi^{Sym,A*}\right] = \begin{cases} 0 & \text{if } S(t) \in [0, S_1^{Sym,FT,New\ \theta}) \\ \frac{2\delta+\rho(1+2\gamma^{FT})}{2b\delta[2(\gamma^{FT})^2+5\gamma^{FT}+3]} > 0 & \text{if } S(t) \in [S_1^{Sym,FT,New\ \theta}, S_2^{Sym,FT,New\ \theta}] \\ \frac{2}{b(3+2\gamma^{FT})} > 0 & \text{if } S(t) \in (S_2^{Sym,FT,New\ \theta}, \infty) \end{cases} \tag{3.35}$$

As in the autarky case, there is the trade-off effect that keeps their strategy down in order to "preserve the resource" and to "maximize their profits". The more they extract, the more happiness they get from extracting more than the other player, but profits could be lower if they sell too much.

In Table 3.2 we show the comparison between symmetric autarky and free trade scenarios when θ^k increases.

	S_1^k	S_2^k	$q_i^{Sym,Cou,k}$	Slope z^k
Autarky	$\Downarrow$	$\Uparrow$	$\Uparrow$	=
Free Trade	$\Downarrow$	$\Uparrow$	$\Uparrow$	=
Free Trade in Benchekroun and Long (2016)	$\Downarrow$	$\Uparrow$	$\Uparrow$	=

Table 3.2: Changes when θ^k increases.

Effect of changing the weight of the relative profit under symmetric free trade (γ^{FT})

Considering now the effect of an increase in γ^{FT} on players' strategies, we obtain an interesting result and different behavior from the one we saw under autarky. When players care more about relative profits under free trade (Figure 3.9), they behave differently with respect to the autarky regime when they pay more attention to their relative profits. One can see that players now extract more under the Cournot part if the following condition holds.

Condition 3.3. *The Cournot strategy moves upwards when players care more about the relative profits (increase in γ^{FT}) if, and only if,*

$$a^{Sym,FT} > 2\theta^{Sym,FT}. \tag{3.36}$$

Therefore, Condition 3.3 implies that agents extract more under the Cournot part when γ^{FT} increases as in Benchekroun and Long (2016). However, if θ^{FT} is large enough so that the above assumption is not met, we see a different result than the previous authors.

$$\frac{\partial}{\partial \gamma^{FT}} \left[q^{Sym,Cou,FT} = \frac{2(\theta^{FT} + a^{Sym,FT}(1+\gamma^{FT}))}{b(3+2\gamma^{FT})} \right] = \frac{2(a^{Sym,FT} - 2\theta^{FT})}{b(3+2\gamma^{FT})^2} \lesseqgtr 0. \tag{3.37}$$

The behavior of the switching points and the distance between $S_1^{Sym,FT}$ and $S_2^{Sym,FT}$ are

$$\frac{\partial}{\partial \gamma^{FT}} \left[S_1^{Sym,FT} \right] = -8 \cdot \frac{a^{Sym,FT}[2\delta + \rho(1+2\gamma^{FT})] + \theta^{FT}(\rho - 4\delta - 2\gamma^{FT}\rho)}{b\delta(3+2\gamma^{FT})^3(2\delta - \rho)},$$

$$\frac{\partial}{\partial \gamma^{FT}} \left[S_2^{Sym,FT} \right] = 8 \cdot \frac{a^{Sym,FT}(1+2\gamma^{FT}) + \theta^{FT}(1-2\gamma^{FT})}{b\delta(3+2\gamma^{FT})^3}.$$

Condition 3.4. *The first switching point $S_1^{Sym,FT}$ will move to the left, and the second switching point $S_2^{Sym,FT}$ will move to the right, when players care more about the relative profits (increase in γ^{FT}) if, and only if,*

$$\theta^{FT} < \min \left\{ \frac{a^{Sym,FT}[2\delta + \rho(1+2\gamma^{FT})]}{4\delta + 2\gamma^{FT}\rho - \rho}, \frac{a^{Sym,FT}(1+2\gamma^{FT})}{2\gamma^{FT} - 1} \right\}, \tag{3.38}$$

where the first element is derived from the condition that $S_1^{Sym,FT}$ moves to the left, and the second element is obtained imposing the condition that $S_2^{Sym,FT}$ moves to the right.

The distance between the first and the second switching points when agents care more about the relative profits is,

$$\frac{\partial}{\partial \gamma^{FT}} \left[S_2^{Sym,FT} - S_1^{Sym,FT} \right] = -16 \frac{\theta^{FT}(1+2\gamma^{FT}) - 2a^{Sym,FT}(1+\gamma^{FT})}{b(3+2\gamma^{FT})^3(2\delta - \rho)}, \tag{3.39}$$

which will be positive when using Condition 3.4. Moreover, the slope of the affine strategy goes down,

$$\frac{\partial}{\partial \gamma^{FT}} \left[z^{Sym,FT} = \frac{(3+2\gamma^{FT})(2\delta - \rho)}{8(1+\gamma^{FT})} \right] = -\frac{2\delta - \rho}{(1+8\gamma^{FT})^2} < 0, \tag{3.40}$$

since $2\delta > \rho$.

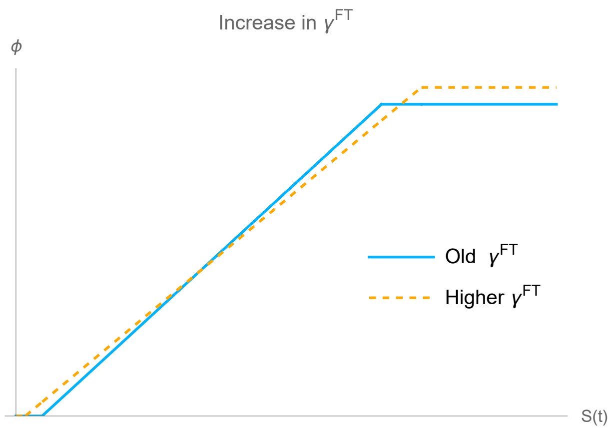


Figure 3.9: Extraction Strategy under Free Trade when γ^{FT} increases.

When agents care more about the relative profits (increasing γ^k), it reduces the slope of both agents under autarky and free trade. Under autarky, the strategy with higher γ^A crosses the old strategy just once, in the increasing part (see Figure 3.3) or never crosses (see Figure 3.4). However, under free trade, the strategy with higher γ^{FT}, crosses the old strategy (lower γ^{FT}) twice, one in the affine part, and another in the Cournot part (see Figure 3.9), or it could just cross one, in the Cournot strategy, as $S_1^{Sym,FT}$ could increase as in Figure 3.4. Intuitively, this means that under free trade, due to the pressure of catching up with the profits of the other agent (higher γ^{FT}), or even increasing the gap (which directly reports more utility), both decision-makers extract more in the first part of the affine strategy. Later on, in the last part of the affine strategy, they harvest less in comparison to a lover γ^{FT}, before they switch to the Cournot extraction. When agents play the Cournot strategy, with higher γ^{FT}, they now extract more. In contrast, under autarky when players care more about their relative profits (higher γ^A), they extract less in the Carnot part with respect to strategies with lower values of γ^A. This free trade behavior is exactly the one obtained in Benchekroun and Long (2016), when the parameter capturing the relative profit increases. However, the behavior under autarky is different. Thus, one could say that the result the previous researchers obtained comes from the free trade force. The results are summarized in Table 3.3.[16]

Symmetric Free Trade Steady States

As we proceed before, we first define $\Phi^{Sym,FT*} := \phi_1^{FT}(S) + \phi_2^{FT}(S) = 2\phi^{Sym,FT}(S)$ as the total extraction under free trade. Both total extractions under autarky and free

[16]We compare our results with the ones obtained in Benchekroun and Long (2016), shown as BL (2016) for space constraints.

	S_1^k	S_2^k	$q_i^{Sym,Cou,k}$	Slope z^k
Autarky	$\Uparrow\times$ or $\Downarrow\checkmark$	$\Downarrow\times$	$\Downarrow\times$	$\Downarrow\checkmark$
Free Trade	$\Uparrow\times$ or $\Downarrow\checkmark$	$\Uparrow\checkmark$ or $\Downarrow\times$	$\Uparrow\checkmark$ or $\Downarrow\times$	$\Downarrow\checkmark$
Free Trade in BL (2016)	$\Downarrow$	$\Uparrow$	$\Uparrow$	$\Downarrow$

Table 3.3: Changes when γ^k increases.

trade are shown in Figure 3.10. comparing the total extractions under both regimes, it becomes evident that players harvest more resources in the free trade scenario than in autarky, using Condition 3.1. Nevertheless, the fact that the gap between the two strategies grows as the resource increases (as seen in the figure, which aligns with the findings in **?**), requires additional assumptions not previously considered by the aforementioned authors due to the introduction of status concern parameters. Theoretically, one can have a higher, the same, or a lower slope. In order to get a higher slope under free trade, see Remark 3.2. To have the same slope, one should need a certain relationship between γ^A and γ^{FT} shown in Remark 3.4.

Remark 3.4. *The affine strategies under the symmetric autarky and free trade would have the same slope iff,*

$$z^{Sym,FT} = z^{Sym,A} \iff \gamma^A = \frac{2\gamma^{FT} - 1}{4}. \tag{3.41}$$

Note that we need new extra conditions to ensure that the slope under free trade is higher, in contrast to the immediate result obtained by the previous authors. If both status parameters would be the same under autarky and free trade, $\gamma^A = \gamma^{FT} = \gamma$, it is easy to see that the slope of free trade is higher as $2\gamma + 1 > 0$. Thus, when we add up the strategies of both players to study the total extraction, we observe that the free trade one increases at a higher rate $\left(\frac{\partial \Phi^{FT}}{\partial S(t)} > \frac{\partial \Phi^A}{\partial S(t)}\right)$ under Remark 3.2.

Having analyzed the shape of the total extraction under free trade, we should study whether this strategy has stable points or not. In order to obtain these interesting points, in Figure 3.11 we first show the total extraction with the reproduction of the resource on the left-hand side, and the evaluation of the resource under such strategy on the right-hand side. Each time the total extraction crosses the reproduction function, we obtain a steady state. We observe that where there is little resource, the reproduction function is above both autarky and free trade extraction strategy, thus, allowing the natural resource to reproduce and grow until a certain level of stock is reached.

As explained under the autarky scenario, we observe three steady states also under autarky, two of them stable ($S_{1,\infty}^{FT}$ and $S_{3,\infty}^{FT}$) and one unstable ($S_{2,\infty}^{FT}$). The two

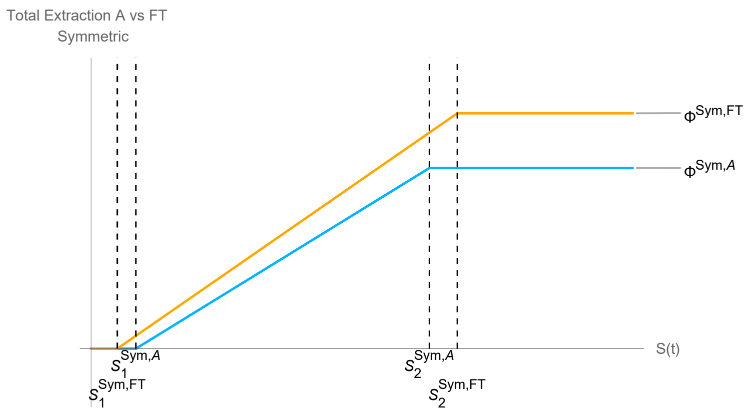


Figure 3.10: Symmetric Total Extraction under Autarky vs Free Trade

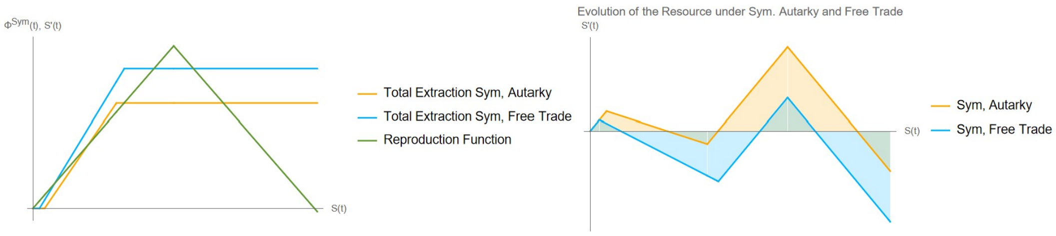


Figure 3.11: Total Extraction and Evolution of the Resource under Symmetric Autarky and Free Trade

stable equilibrium points are easily recognizable in both plots. They can be obtained when the slope of the extraction is higher than the one of the reproduction function on the plot of the left $\left(\frac{\partial \Phi^k}{\partial S(t)} > \frac{\partial F\left(S(t)\right)}{\partial S(t)}\right)$, or when the evolution of the resource (right-hand side) crosses the horizontal axes from above (with negative slope).

As before we define the "total affine strategy" under free trade as the sum of both affine strategies:

$$\underbrace{w_1^{FT} + z^{FT} S(t)}_{\text{affine strategy for P1}} + \underbrace{w_2^{FT} + z^{FT} S(t)}_{\text{affine strategy for P2}} = 2w^{Sym,FT} + 2z^{Sym,FT} S(t).$$

The intersection of $2w^{Sym,FT} + 2z^{Sym,FT} S(t)$ and $\delta S(t)$ shows the first steady state $S_{1,\infty}^{FT}$. The seconds steady state ($S_{2,\infty}^{A}$), which is unstable, is the intersection of the total symmetric Cournot strategies ($q_1^{Cou,FT} + q_2^{Cou,FT} = 2q^{Sym,Cou,FT}$) and the increasing part of the reproduction function ($\delta S(t)$). The third steady state, which is stable ($S_{3,\infty}^{FT}$), can be characterized as the intersection of the sum of both Cournot strategies and the decreasing part of the reproduction function. All the equilibrium points are defined in the following Remark.

Remark 3.5. *Under the Free Trade regime and agents playing their MPNE strategies, there are three positive steady states:*

$$S_{1,\infty}^{Sym,FT} = \frac{2w^{Sym,FT}}{\delta - 2z^Sym,FT}, \qquad S_{2,\infty}^{FT} = \frac{2q^{Sym,Cou,FT}}{\delta},$$

$$S_{3,\infty}^{Sym,FT} = \bar{S} - \frac{(\bar{S} - S_y)2q^{Sym,Cou,FT}}{\delta S_y}.$$

Once we have studied the behavior of both agents in the game and its equilibrium points, we can compare it with the autarky case, to see how changing one rule of the game (allowing them to trade) has effects on the evolution of the resource and the equilibria of the game (see Figure 3.11). Starting at the highest steady state under autarky ($S_{3,\infty}^{A}$), we study the effect of allowing trade between players. While in the short-run, agents move upwards from autarky to free trade (extract more under the symmetric Cournot in free trade), which is translated as an increase in the immediate extraction, one can observe that this point is no longer an equilibrium. This means that players were extracting $S_{3,\infty}^{A}$ using their *free trade total extraction strategy* at the instant they immediately moved to free trade. Thus, as the extraction is higher than the reproduction under the free trade scenario, this will reduce the resource until it reaches the free trade highest steady state $S_{3,\infty}^{FT}$. With this reasoning, we observe that both agents are extracting more in the short-run with abundant resource, but in the long-run, end up extracting more at a lower stock of the resource. For this reason, the impact of allowing players to trade will be determined by the discounted sum of producer and consumer surplus, which will take into account the increase in their utility in the short-run, and the impact of their utility in the long-run.

3.4 Asymmetric Game

As shown in Benchekroun et al. (2014), where the authors study the effects of asymmetries in a common pool natural resource with the dynamics given by equation (3.3), having different marginal costs leads to a "drastic effect on the nature of the equilibria that may be expected as compared to the identical cost case" (see Propositions 1 and 2 in their paper). In their paper, the strategy for player i consists of four distinct intervals. Initially, there is a period during which no firm engages in extraction. Following this, only the more efficient (big) firm proceeds to extract resources in a nonlinear manner. Subsequently, both agents extract resources using affine strategies within the endogenous thresholds of the less efficient (small) player, denoted as $S_{1,s}$ and $S_{2,s}$. Lastly, the authors prove that a stock larger than $S_{2,s}$ is not sustainable. Therefore, for the region where the resource is more abundant, as the authors write, "playing the static Cournot strategies beyond some endogenously determined interval of the stock over which linear strategies are played is not sustainable as an equilibrium." Following their approach, we define a big player with marginal cost c_b^k and a small player with marginal cost c_s^k under both regimes $k \in \{A, FT\}$, and assume that

$$c_s^k > c_b^k,$$

capturing the idea that big firms have a cost advantage over small firms. By the reason given above, in this chapter we will focus on the region where both agents play their affine strategies (Proposition 1 in Benchekroun et al. (2014), that is in our case, for all $S \in [S_{1,s}^k, S_{2,s}^k]$.

3.4.1 Asymmetric Autarky

For the case of 2-asymmetric players, the vector of control (decision) variables is $\boldsymbol{\phi}^A := (\phi_s^{A*}, \phi_b^{A*}) \in \mathbb{R}_+^2$. The corresponding Hamilton-Jacobi-Bellman equation for player i associated with the asymmetric problem (3.4) is given by:

$$\rho V_i^A(S) = \underset{\{q_i^A\}}{\text{Max}} \left(\begin{array}{c} \left(a_i^A - bq_i^A\right) q_i^A + \theta^A \left(q_i^A - \phi_j^A\right) \\ \\ + \gamma^A \left[\left(a_i^A - bq_i^A\right) q_i^A - \left(a_j^A - b\phi_j^A\right) \phi_j^A \right] \\ \\ + \dfrac{\partial V_i^A(S)}{\partial S} \left[F(S) - q_i^A - \phi_j^A \right] \end{array} \right). \tag{3.42}$$

The interior solution of the right-hand side must satisfy,

$$q_i^A = \frac{a_i^A(1+\gamma^A) - \theta^A - \partial_S V_i^A(S)}{2b(1+\gamma^A)}, \qquad \text{for } i \in \{s, b\}. \tag{3.43}$$

With the corresponding guessing for the value function for player i of the form $V_i^A(S) = \frac{\alpha_i^A}{2} S(t)^2 + \beta_i^A S + \mu_i^A$, one gets the following result:

Proposition 3.3. *The vector* $(\phi_s^{A*}, \phi_b^{A*})$ *given by the following affine strategies, constitutes a Subgame Perfect Markov Nash Equilibria for each agent* $i \in \{s,b\}$, *where*

$$\phi_i^{A*} = w_i^A + z^A S(t), \qquad for\ \ S(t) \in [S_{1,s}^A, S_{2,s}^A], \tag{3.44}$$

with

$$w_i^A = \frac{a_i^A(1+\gamma^A)[\delta(1+2\gamma^A)+\rho(1+\gamma^A)] - a_j^A(1+\gamma^A)^2(2\delta-\rho)+\theta^A[\delta+\rho(1+2\gamma^A)]}{2b\delta(1+\gamma^A)(3+4\gamma^A)},$$

$$z_i^A = z^A = z^{Sym,A} = \frac{(1+\gamma^A)(2\delta-\rho)}{4\gamma^A+3},$$

and the "endogenously threshold levels of stock" are given by

$$S_{1,s}^A = -\frac{a_s^A(1+\gamma^A)[\delta(1+2\gamma^A)+\rho(1+\gamma^A)] - a_b^A(1+\gamma^A)^2(2\delta-\rho)+\theta^A[\delta+\rho(1+2\gamma^A)]}{2b\delta(1+\gamma^A)^2(2\delta-\rho)},$$

$$S_{2,s}^A = \frac{(1+\gamma^A)^2(a_s^A+a_b^A)+\theta^A(1+2\gamma^A)}{2b\delta(1+\gamma^A)^2} = S_{2,b}^A.$$

Proof. See Appendix 3.7.3. ■

Observe that the slope of the strategy of the asymmetric players is the same as the slope for symmetric agents, together with the fact that $S_{2,s}^A = S_{2,b}^A = S_2^A$.

Assumption 3.3. *We assume that the intrinsic growth rate under the autarky asymmetric game is large enough,*

$$\delta > \ max\left\{\frac{\rho}{2}, \frac{(1+\gamma^A)^2(a_s^A+a_b^A)+\theta^A(1+2\gamma^A)}{2b(1+\gamma^A)^2 S_y}\right\}, \tag{3.45}$$

and the status concern parameter is small enough,

$$\theta^A \leq \frac{(1+\gamma^A)\left\{a_b^A(1+\gamma^A)(2\delta-\rho) - a_s^A[\delta(1+2\gamma^A)+\rho(1+\gamma^A)]\right\}}{\delta+\rho(1+2\gamma^A)}. \tag{3.46}$$

The first condition (3.45) in Assumption 3.3 ensures that the slope of the extraction is positive, which means, the more resource there is, the more they extract. Moreover, this also guarantees the existence of a positive steady state, where $z^A = \dfrac{(1+\gamma^A)(2\delta-r)}{(3+4\gamma^A)} > 0$ is the slope of the strategy. As γ^A is a positive parameter, the conditions that must hold for z^A to be positive is $2\delta > r$. The second condition

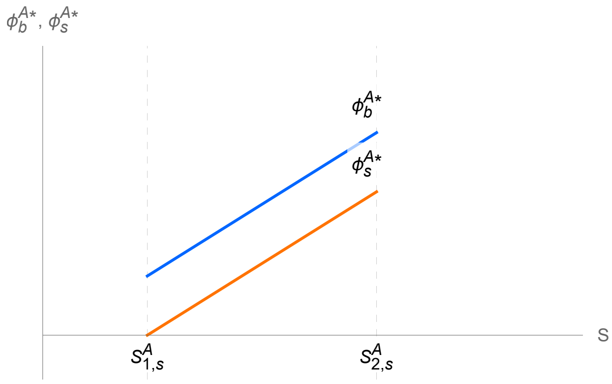


Figure 3.12: Extraction of two asymmetric players under autarky.

of inequality (3.45) ensures that the second switching point $S^A_{2,s}$ is smaller than the maximum-sustainable-yield stock, S_y, as explained in the symmetric case. Condition (3.46) ensures that the threshold $S^A_{1,s}$ is nonnegative.

Remark 3.6. *As the right-hand side of (3.46) is greater or equal to zero, it implies that a^A_b should be sufficiently large, or*

$$\frac{a^A_b}{a^A_s} \geq \frac{\delta(1+2\gamma^A)+\rho(1+\gamma^A)}{(1+\gamma^A)(2\delta-\rho)}.$$

The distance between the two switching thresholds is:

$$S^A_{2,s} - S^A_{1,s} = \frac{(3+4\gamma^A)\left[\theta^A + a^A_s(1+\gamma^A)\right]}{2b(1+\gamma^A)^2(2\delta-\rho)}, \tag{3.47}$$

which is very similar to the symmetric case, except that now the marginal cost is player specific. The extraction of both players can be seen in Figure 3.12.

The value function of the autarky game under asymmetries for $S \in [S^A_{1,s}, S^A_{2,s}]$ is

$$V^A_i(S) = \frac{\alpha^A_i}{2}S^2 + \beta^A_i S + \mu^A_i, \tag{3.48}$$

with the coefficients shown in Appendix 3.7.3.

Effect of changing the weight of the relative extraction under the asymmetric autarky game (θ^A)

As in the symmetric case, we now study the effect when agents care more about relative extraction. First, one should notice that the slope of the strategy does not change (as in the symmetric case),

$$\frac{\partial}{\partial \theta^A} \left[\underbrace{z^A}_{Slope} \right] = 0. \tag{3.49}$$

To show that the affine part moves upward, one can see that the following result is positive,

$$\frac{\partial \phi_i^{A*}}{\partial \theta^A} = \frac{\delta + \rho(1 + 2\gamma^A)}{2b\delta \left[4(\gamma^A)^2 + 7\gamma^A + 3\right]} > 0. \tag{3.50}$$

Moreover, $S^A_{1,s}$ decreases, and $S^A_{2,s}$ increases, moving in the same direction as in the symmetric case,

$$\frac{\partial S^A_{1,s}}{\partial \theta^A} = -\frac{\delta + \rho(1 + 2\gamma^A)}{2b\delta(1 + \gamma^A)^2(2\delta - \rho)} < 0, \tag{3.51}$$

$$\frac{\partial S^A_{2,s}}{\partial \theta^A} = \frac{1 + 2\gamma^A}{2b\delta(1 + \gamma^A)^2} > 0. \tag{3.52}$$

Thus, the distance $S^A_{2,s} - S^A_{1,s}$ increases,

$$\frac{\partial}{\partial \theta^A} \left[S^A_{2,s} - S^A_{1,s} \right] = \frac{3 + 4\gamma^A}{2b(1 + \gamma^A)^2(2\delta - \rho)} > 0. \tag{3.53}$$

Note that the effect of θ^A is the same for symmetric and asymmetric players. For an economic interpretation, see the symmetric case.

Effect of changing the weight of the relative profit under the asymmetric autarky game (γ^A)

When agents care more about their relative profits (higher γ^A), we observe that they move $S^A_{2,s}$ to the left, and $S^A_{1,s}$ can move in both directions as in the symmetric game,

$$\frac{\partial}{\partial \gamma^A} \left[S^A_{1,s} \right] = -\frac{a^A_s \delta(1 + \gamma^A) - 2\theta^A(\delta + \gamma^A \rho)}{2b\delta(1 + \gamma^A)^3(2\delta - \rho)} \lesseqgtr 0,$$

$$\frac{\partial}{\partial \gamma^A} \left[S^A_{2,s} \right] = -\frac{\gamma^A \theta^A}{b\delta(1 + \gamma^A)^3} < 0.$$

The distance between the two thresholds $S^A_{2,s} - S^A_{1,s}$ could be positive or negative,

$$\frac{\partial}{\partial \gamma^A}\left[S_{2,s}^A - S_{1,s}^A\right] = \frac{a_s^A(1+\gamma^A) - 2\theta^A(1+2\gamma^A)}{2b(1+\gamma^A)^3(2\delta-\rho)} \lesseqgtr 0. \tag{3.54}$$

The behavior of both switching points is the same as in the symmetric case. $S_{2,s}^A$ moves to the left, and the first switching point $S_{1,s}^A$ can also go in both directions, which is different to the result in Benchekroun and Long (2016). This is mainly due to the fact that we are exploring now the autarky case, while they study the free trade game. Regarding the question of what happens to the distance between the points of change when γ^A increases, the previous authors obtain an increase in the distance between the new second and first switching point. However, as one can see in 3.54, we could get the opposite result. To obtain the same result as the previous authors one should consider the following condition.

Condition 3.5. *The distance between the first and second switching points will increase when players care more about the relative profits (increase in γ^A) if, and only if,*

$$a_s^A > 2\theta^A\left(\frac{1+2\gamma^A}{1+\gamma^A}\right), \tag{3.55}$$

that is, when the marginal cost is small enough (big a_s^A) or when the parameter capturing the importance of relative extraction is low enough.

Additionally, the slope of the affine strategy goes down,

$$\frac{\partial}{\partial \gamma^A}\left[z^A\right] = -\frac{2\delta-\rho}{(4\gamma^A+3)^2} < 0, \tag{3.56}$$

(recall that $2\delta > \rho$), which is the same behavior as in the symmetric autarky case. The affine strategy acts as in symmetric cases in Figures 3.3 or 3.4 in the symmetric autarky case.

Asymmetric Autarky Steady States

As in the previous sections, we define $\Phi^{A*} := \phi_b^A(S) + \phi_s^A(S)$ as the total extraction of the resource carried out by both players. Thus,

$$\Phi^{A*} = (w_i^A + w_j^A) + 2z^A S(t).$$

Considering the total extraction and using the reproduction function of the resource given by equation (3.3), we obtain a stable steady state, in comparison to the unstable steady state in Benchekroun et al. (2020) when affine strategies are played.

The steady state of the previous authors is stable from the left and unstable from the right.

An interesting result is driven by the fact that now we obtain a steady state that is stable from both sides. We could define it as a "*Natural Resource Poverty Trap*" (NRPT). This concept borrowed from the economic growth/development literature can be extrapolated to our framework.[17] This is the unique stable steady state, which would correspond to a level of low natural resources. When a minor perturbation to the right occurs in the resource, it reverts to the left, since agents extract more than the resource can regenerate.

Although policymakers could temporarily ban resource extraction to allow regeneration beyond the NRPT, our analysis, which focuses on the region where both players use affine strategies, indicates that this hypothetical second steady state remains unattainable. Consequently, no higher steady state exists with heterogeneous agents, and players are trapped in the low NRPT. Implementing policies like the "big push" will not lead to a richer equilibrium, presenting a paradoxical situation where a higher steady state could be reached under symmetric agents instead.

The intersection between $(w_i^A + w_j^A) + 2z^A S(t)$ and $\delta S(t)$ gives the stable steady state $S_{1,\infty}^A$. In the heterogeneous case, there is just one steady state, losing the other two that were derived by the Cournot strategies in the symmetric case (as proved in Benchekroun et al. (2014), where the Cournot part does not belong to the Markov Perfect Nash Equilibrium strategy).

Corollary 3.1. *Under the asymmetric autarky regime when agents play their MPNE strategies, there is just one positive steady state for $S(t) \in \left[S_{1,s}^{FT}, S_{2,s}^{FT}\right]$:*

$$S_{1,\infty}^A = \frac{w_i^A + w_j^A}{\delta - 2z^A}.$$

3.4.2 Asymmetric Free Trade

Regarding the study of heterogeneous agents under free trade, we solve the following Hamilton-Jacobi-Bellman equation for each player $i \in \{s, b\}$,

[17] See, for instance, Banerjee et al. (2011) and Banerjee and Duflo (2005).

$$\rho V_i^{FT}(S) = \underset{\{q_i^{FT}\}}{\text{Max}} \left(\begin{array}{c} \left(a_i^{FT} - \frac{b}{2}(q_i^{FT}(t) + \phi_j^{FT}(t))\right) q_i^{FT}(t) + \theta^{FT}\left(q_i^{FT} - \phi_j^{FT}\right) \\ \\ +\gamma^{FT}\left[\left(a_i^{FT} - \frac{b}{2}(q_i^{FT}(t) + \phi_j^{FT}(t))\right) q_i^{FT}(t) \right. \\ \left. -\left(a_j^{FT} - \frac{b}{2}(q_i^{FT}(t) + \phi_j^{FT}(t))\right) \phi_j^{FT}(t)\right] \\ \\ +\frac{\partial V_i^{FT}(S)}{\partial S}\left[F(S) - q_i^{FT} - \phi_j^{FT}\right] \end{array} \right). \tag{3.57}$$

The interior solution of the right-hand side must satisfy the system of equations

$$q_i^{FT} = \frac{a_i^{FT}(1+\gamma^{FT}) + \theta^{FT} - \partial_S V_i^{FT}(S)}{b(1+\gamma^{FT})} + \frac{1}{2(1+\gamma^{FT})}\phi_j^{FT}, \qquad \text{for } i \in \{s,b\}, i \neq j, \tag{3.58}$$

which gives

$$q_i^{FT} = 2 \cdot \frac{(1+\gamma^{FT})\left[2a_i^{FT}(1+\gamma^{FT}) - a_j^{FT}\right] + \theta^{FT}(1+\gamma^{FT}) - 2(1+\gamma^{FT})\partial_S V_i^{FT} + \partial_S V_j^{FT}}{b\left[4(\gamma^{FT})^2 + 8\gamma^{FT} + 3\right]}. \tag{3.59}$$

With the corresponding guessing for the value function for player i of the form $v_i^{FT}(S) = \frac{\alpha_i^{FT}}{2} S(t)^2 + \beta_i^{FT} S + \mu_i^A$, one gets the following result.

Proposition 3.4. *The vector $(\phi_s^{FT*}, \phi_b^{FT*})$ given by the following affine strategies, constitutes a Subgame Perfect Markov Nash Equilibria for each agent $i \in \{s,b\}$ under the asymmetric free trade game, where*

$$\phi_i^{FT*} = w_i^{FT} + z^{FT} S(t), \qquad \textit{for } S(t) \in [S_{1,s}^{FT}, S_{2,s}^{FT}], \tag{3.60}$$

with

$$w_i^{FT} = \frac{1}{8b\delta(1+\gamma^{FT})(r-\delta)[4(\gamma^{FT}+2)\gamma^{FT}+3]} \times$$

$$\underbrace{\times \left(\begin{array}{c} a_i^{FT}\left\{(1+2\gamma^{FT})(4\gamma^{FT}(2+\gamma^{FT})+5)\rho^2-4\delta^2(1+\gamma^{FT})[1+4\gamma^{FT}(2+\gamma^{FT})] \right. \\ \left. +2\delta\rho(1+2\gamma^{FT})[(2\gamma^{FT}+7)\gamma^{FT}+4]\right\} \\ \\ +a_j^{FT}\left\{4\delta^2[\gamma^{FT}(4\gamma^{FT}(3+\gamma^{FT})+11)+2]+(1+2\gamma^{FT})\rho^2[5+4\gamma^{FT}(2+\gamma^{FT})] \right. \\ \left. -2\delta\rho(1+2\gamma^{FT})[3\gamma^{FT}(2\gamma^{FT}+5)+11]\right\} \\ \\ +4\theta^{FT}(\rho-\delta)(1+2\gamma^{FT})(2\delta+2\gamma^{FT}\rho+\rho) \end{array} \right),}_{:=M^{FT}} \tag{3.61}$$

$$z_i^{FT} = z^{FT} = z^{Sym,FT} = \frac{(1+\gamma^{FT})(2\delta-\rho)}{4\gamma^{FT}+3},$$

and the "endogenously threshold levels of stock" are given by

$$S_{1,s}^{FT} = -\frac{1}{b\delta(1+2\gamma^{FT})(3+2\gamma^{FT})^2(2\delta-\rho)(\rho-\delta)} \times M^{FT},$$

$$S_{2,s}^{FT} = \frac{1}{b\delta(1+2\gamma^{FT})(3+2\gamma^{FT})^2(r-\delta)} \times$$

$$\left\{a_s^{FT}(1+2\gamma^{FT})\rho[4\gamma^{FT}(2+\gamma^{FT})+5]-2\delta a_s^{FT}(1+\gamma^{FT})[4\gamma^{FT}(2+\gamma^{FT})+7]\right.$$

$$-2\delta a_b^{FT}[\gamma^{FT}(4\gamma^{FT}(2+\gamma^{FT})+3)-2]+a_b^{FT}(1+\gamma^{FT})\rho[4\gamma^{FT}(2+\gamma^{FT})+5]$$

$$\left.+4(1+2\gamma^{FT})^2\theta^{FT}(\rho-\delta)\right\}.$$

Proof. See Appendix 3.7.4. ■

Assumption 3.4. *We assume that the intrinsic growth rate is sufficiently large*

$$\delta > \max\left\{\frac{\rho}{2}, C_{2,\delta}^{FT}\right\}, \tag{3.62}$$

and the status concern parameter is small enough,

$$\theta^{FT} \leq \frac{1}{4(1+2\gamma^{FT})(\rho-\delta)[2\delta+\rho(1+2\gamma^{FT})]} \times \{$$

$$4\delta^2\left\{a_s^{FT}(1+\gamma^{FT})(4\gamma^{FT}(2+\gamma^{FT})+1)-a_b^{FT}[2+\gamma^{FT}(4\gamma^{FT}(3+\gamma^{FT})+11)]\right\}$$

$$-\rho^2(a_s^{FT}+a_b^{FT})(1+2\gamma^{FT})[4\gamma^{FT}(2+\gamma^{FT})+5]$$

$$-2\rho\delta(1+2\gamma^{FT})[a_s^{FT}(\gamma^{FT}(2\gamma^{FT}+7)+4)-a_b^{FT}(3\gamma^{FT}(2\gamma^{FT}+5)+11)]\}. \tag{3.63}$$

Parameter $C_{2,\delta}^{FT}$ is defined as

$$C_{2,\delta}^{FT} = \frac{1}{b\delta(1+2\gamma^{FT})(3+2\gamma^{FT})^2(r-\delta)} \times$$

$$\{\pm\left[\{2a_s^{FT}(1+\gamma^{FT})[4\gamma^{FT}(2+\gamma^{FT})+7]+2a_b^{FT}[\gamma^{FT}(4\gamma^{FT}(2+\gamma^{FT})+3)-2]\right.$$

$$+(1+2\gamma^{FT})\left[b\rho S_y(3+2\gamma^{FT})^2+4\theta^{FT}(1+2\gamma^{FT})\right]\}^2$$

$$\left.-4b\rho S_y[4\gamma^{FT}(2+\gamma^{FT})+3]^2\left[(4\gamma^{FT}(2+\gamma^{FT})+5)(a_s^{FT}+a_b^{FT})+4\theta^{FT}(1+2\gamma^{FT})\right]\right]^{1/2}$$

$$+2a_s^{FT}(1+\gamma^{FT})[4\gamma^{FT}(2+\gamma^{FT})+7]+8a_b^{FT}(\gamma^{FT})^3+16a_b^{FT}(\gamma^{FT})^2+6a_b^{FT}\gamma^{FT}-4a_b^{FT}$$

$$+b\rho S_y\left[8(\gamma^{FT})^3+28(\gamma^{FT})^2+30\gamma^{FT}+9\right]+4\theta^{FT}(1+2\gamma^{FT})\}. \tag{3.64}$$

The first and second elements in (3.62) capture, as before, that the slope of the strategy is positive, and that the second switching point happens before the maximum sustainable yield. Inequality (3.63) ensures that the threshold $S_{1,s}^{A}$ is nonnegative. It is clear that the study of heterogeneous agents involves dealing with very cumbersome expressions. The previous assumptions are feasible with reasonable parameter values.

The distance between the two switching thresholds is

$$S_{2,s}^{FT} - S_{1,s}^{FT} = \frac{16(1+\gamma^{FT})\left[2a_s^{FT}(1+\gamma^{FT})^2 - a_b^{FT}(1+\gamma^{FT}) + \theta^{FT}(1+2\gamma^{FT})\right]}{b(3+2\gamma^{FT})^2(2\delta-\rho)(1+2\gamma^{FT})}, \tag{3.65}$$

which is quite different from the symmetric case.

The value function of the asymmetric free trade game under for all $S \in [S_{1,s}^{FT}, S_{2,s}^{FT}]$ is,

$$V_i^{FT}(S) = \frac{\alpha_i^{FT}}{2}S^2 + \beta_i^{FT}S + \mu_i^{FT}, \tag{3.66}$$

with the coefficients shown in Appendix 3.7.4.

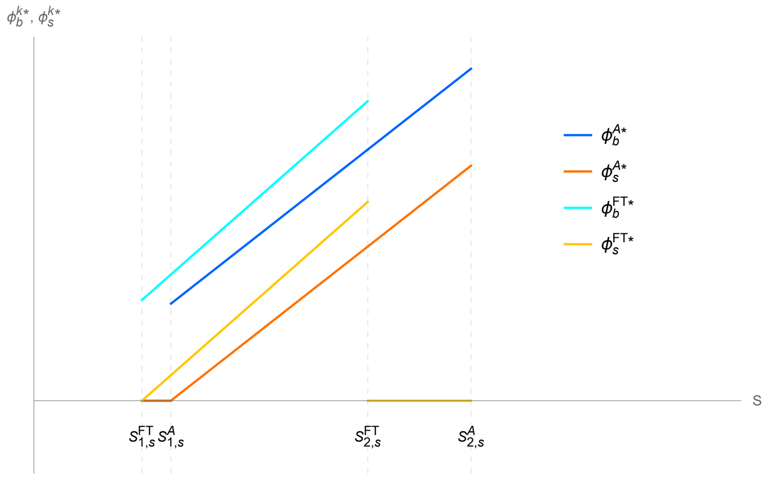


Figure 3.13: Affine strategies of two heterogeneous players under autarky and free trade.

Having presented how player's strategies look like under autarky and free trade, we can plot them together to visually understand the free trade effect in Figure 3.13. As one can observe, allowing trade between players has a significant change in their strategies. The big (efficient) player always harvests more than the small (inefficient) player, both under autarky and free trade. When players switch to free trade, the slope of the affine strategy increases.

As in the symmetric free case, we have a more general result regarding the evolution of the second switching point. Benchekroun et al. (2020) observe that when agents trade the resource, they move the second switching point to the left (see their Proposition 3). However, in our heterogeneous free trade game, we would need extra conditions to obtain this result.

Condition 3.6. *If the status concern parameter θ^{FT} is big enough, that is,*

$$\theta^{FT} \geq \frac{1}{8(1+2\gamma^{FT})^2} \times \Bigg\{ \frac{(1+2\gamma^{FT})(3+2\gamma^{FT})^2\left[(1+\gamma^A)^2(a_i^A+a_j^A)+\theta^A(1+2\gamma^A)\right]}{(1+\gamma^A)^2} + \frac{4\delta\left\{a_i^A(1+\gamma^{FT})[4\gamma^{FT}(2+\gamma^{FT})+7]+a_j^A(\gamma^{FT}[4\gamma^{FT}(2+\gamma^{FT})+3)-2]\right\}}{r-\delta} - \frac{2\rho(a_i^A+a_i^A)(1+2\gamma^{FT})[42\gamma^{FT}(2+2\gamma^{FT})+5]}{r-\delta}\Bigg\}, \tag{3.67}$$

then, $S_{2,s}^A < S_{2,s}^{FT}$.

If inequality (3.67) in the previous condition does not hold, then we get a different result than the one obtained by Benchekroun et al. (2020). The conditions for $S_{1,s}^{FT} <$

$S^A_{1,s}$ are several pages long for the asymmetric free trade game, making it impractical to write it in this chapter. However, one should notice that $S^{FT}_{1,s}$ could be higher, lower, or equal to $S^A_{1,s}$ if one finds the suitable parameters, which is a more general result than the one obtained by the previous authors (see their Proposition 3). The economic intuition is the same as in the symmetric free trade section, where we compared the movement of both switching points when agents move from autarky to free trade. This difference in their behavior is driven by having introduced status concern.

	$S_1^{A\to FT}$	$S_2^{A\to FT}$
Benchekroun et al. (2020) Symmetric	$\Downarrow$	$\Uparrow$
Our Model Asymmetric	$\Uparrow$ × or $\Downarrow$ ✓	$\Uparrow$ × or $\Downarrow$ ✓

Table 3.4: Changes in the switching point when free trade is allowed.

As in the symmetric game, when one studies the consequences of a regime change, one can see that both players extract more under free trade if the parameter capturing the importance of the relative profits under autarky (γ^A) is bigger than a given threshold (see Remark 3.2).

Effect of changing the weight of the relative extraction under the asymmetric free trade game (θ^{FT})

As in the symmetric case, we now study the effect when agents care more about relative extraction. First, one should notice that the slope of the strategy does not change (as in the symmetric case),

$$\frac{\partial}{\partial \theta^{FT}}\left[\underbrace{z^{FT}}_{Slope}\right] = 0, \tag{3.68}$$

To show that the affine part moves upward, one can see that the following result is positive,

$$\frac{\partial \phi_i^{FT*}}{\partial \theta^{FT}} = \frac{2\delta + 2\rho(1+\gamma^{FT})}{2b\delta\left[2(\gamma^{FT})^2 + 5\gamma^{FT} + 3\right]} > 0. \tag{3.69}$$

Moreover, $S^{FT}_{1,s}$ decreases,

$$\frac{\partial S^{FT}_{1,s}}{\partial \theta^{FT}} - \frac{4[2\delta + \rho(1+2\gamma^{FT})]}{b\delta(2\delta-\rho)(3+2\gamma^{FT})^2} = \frac{\partial S_1^{Sym,FT}}{\partial \theta^{FT}} < 0, \tag{3.70}$$

and $S_{2,s}^{FT}$ increases,

$$\frac{\partial S_{2,s}^{FT}}{\partial \theta^{FT}} = \frac{4+8\gamma^{FT}}{b\delta(3+2\gamma^{FT})^2} = \frac{\partial S_2^{Sym,FT}}{\partial \theta^{FT}} > 0, \tag{3.71}$$

which give exactly the same algebraic expressions (for both $S_{2,s}^{FT}$ and $S_{2,s}^{FT}$) as in the symmetric free trade game.

The distance $S_{2,s}^{FT} - S_{1,s}^{FT}$ increases,

$$\frac{\partial}{\partial \theta^{FT}}\left[S_{2,s}^{FT} - S_{1,s}^{FT}\right] = \frac{16(1+\gamma^{FT})}{b(3+2\gamma^{FT})^2(2\delta-\rho)} = \frac{\partial}{\partial \theta^{FT}}\left[S_2^{Sym,FT} - S_1^{Sym,FT}\right] > 0. \tag{3.72}$$

Note that the effect of θ^{FT} is the same for symmetric and asymmetric players.

Effect of changing the weight of the relative profit under the asymmetric free trade game (γ^{FT})

When agents care more about their relative profits (higher γ^{FT}), we observe that now $S_{2,s}^{FT}$ can move to the right or to the left (in comparison to the symmetric free trade game where it moved to the left), and $S_{1,s}^{FT}$ can also move in both directions as in the symmetric game, but now we obtain a very complicated expression,

$$\begin{gathered}\frac{\partial}{\partial \gamma^{FT}}\left[S_{1,s}^{FT}\right] = -\frac{2}{b\delta(3+2\gamma^{FT})^3(2\delta-\rho)(1+2\gamma^{FT})^2(r-\delta)}\times \\ \left\{a_s^{FT}\left[2\rho^2(1+2\gamma^{FT})^3 - 2\delta^2(2\gamma^{FT}[2\gamma^{FT}(9+2\gamma^{FT})+21]+17) - \delta\rho(2\gamma^{FT}-5)(1+2\gamma^{FT})^2\right]\right. \\ +a_b^{FT}\left[2\delta^2(2\gamma^{FT}[2\gamma^{FT}(5+2\gamma^{FT})+13]+13) + 2\rho^2(1+\gamma^{FT})^3 - \delta\rho(1+6\gamma^{FT})(1+2\gamma^{FT})^2\right] \\ \left.-4\theta^{FT}(1+2\gamma^{FT})^2(r-\delta)[4\delta+\rho(2\gamma^{FT}-1)]\right\} \lesseqgtr 0,\end{gathered}$$

and

$$\begin{gathered}\frac{\partial}{\partial \gamma^{FT}}\left[S_{2,s}^{FT}\right] = -\frac{2}{b\delta(3+2\gamma^{FT})^3(1+2\gamma^{FT})^2(r-\delta)}\times \\ \left\{\delta a_s^{FT}\left[2\gamma^{FT}\left(-4(\gamma^{FT})^2+6\gamma^{FT}+21\right)+25\right] + 2\rho a_s^{FT}(1+2\gamma^{FT})^3\right. \\ -\delta a_b^{FT}\left[6\gamma^{FT}(2\gamma^{FT}(5+2\gamma^{FT})+11)+29\right] \\ \left.+2\rho a_b^{FT}(1+2\gamma^{FT})^3 - 4\theta^{FT}(2\gamma^{FT}-1)(1+2\gamma^{FT})^2(r-\delta)\right\} \lesseqgtr 0.\end{gathered}$$

Thus, when we allow agents to care about what their neighbors and consider heterogeneity, we do not get a clear result, and all behaviors in terms of the switching points are feasible.

The distance between the two thresholds $S_{2,s}^{FT} - S_{1,s}^{FT}$ could be positive or negative,

$$\frac{\partial}{\partial\gamma^{FT}}\left[S_{2,s}^{FT}-S_{1,s}^{FT}\right]=\frac{16(1+\gamma^{FT})\left\{2a_s^{FT}(1+\gamma^{FT})^2-a_b^{FT}(1+\gamma^{FT})+\theta^{FT}(1+2\gamma^{FT})\right\}}{b(1+2\gamma^{FT})(3+2\gamma^{FT})^2(2\delta-\rho)}\lesseqgtr 0. \tag{3.73}$$

Remark 3.7. *The distance between the first and second switching points will increase when players care more about the relative profits (increase in γ^{FT}) if, and only if,*

$$a_s^{FT}>\frac{a_b^{FT}}{2(1+\gamma^{FT})}-\frac{\theta^{FT}(1+2\gamma^{FT})}{2(1+\gamma^{FT})^2}. \tag{3.74}$$

Asymmetric Free Trade Steady States

As in the previous sections, we define $\Phi^{FT*}:=\phi_b^{FT}(S)+\phi_s^{FT}(S)$ as the total extraction of the resource carried out by both players under the asymmetric free trade game. Thus,

$$\Phi^{FT*}=(w_i^{FT}+w_j^{FT})+2z^{FT}S(t).$$

As we are working with the affine strategies, there is one steady state, which will be obtained when the total extraction intersects the reproduction function of the resource.

Corollary 3.2. *Under the asymmetric free trade regime when agents play their MPNE strategies, there is just one positive steady state for $S(t)\in\left[S_{1,s}^{FT},S_{2,s}^{FT}\right]$:*

$$S_{1,\infty}^{FT}=\frac{w_i^{FT}+w_j^{FT}}{\delta-2z^{FT}}.$$

3.5 Welfare Analysis

In the previous sections, we studied the behavior of both players under different regimes, i.e., how they harvest the resource under autarky and free trade. Due to their asymmetries, it has been seen how they extract differently. Departing from autarky to free trade can cause an increase in the utility in the short-run, but a decrease in the utility in the long-run. This is explained by the fact that players jump immediately upwards and extract more under free trade in the short-run, which is greater than the reproduction function, moving to a lower steady state in the long-run. For this reason, in this section we study the discounted sum of instantaneous aggregate consumer and producer surplus, allowing us to see if overall, players are better off or if one of them is worse off.

3.5.1 Aggregate Consumer Surplus

In our model, policy recommendations will be driven by welfare variations, and therefore, we need to study the change in consumer and producer surplus. Consumer surplus for player i is defined as the area bounded by the demand curve and the equilibrium price. Thus, due to our linear inverse demand function, this takes the form of a triangle, with base ϕ_i^A for the autarky case and Φ^{FT} for the free trade scenario.[18] Moreover, the height of the triangle will also be different in each regime for each player, i.e., $b\phi_i^A$ under autarky, and $\frac{b}{2}(\phi_s^{FT*}+\phi_b^{FT*})$ under free trade as one can see in the figures below. This is driven by the fact that there are two different market structures in each regime as we show in Figures 3.14 and 3.15. Thus, the area corresponding to the consumer surplus of player i under autarky is given by

$$\frac{b\cdot h}{2}=\frac{\phi_i^{A*}\cdot\left(B^A-(B^A-b\phi_i^A)\right)}{2}=\frac{\phi_i^{A*}\cdot\left(b\phi_i^A\right)}{2}=\frac{b\left(\phi_i^A\right)^2}{2},$$

which can bee easily seen in Figure 3.14.

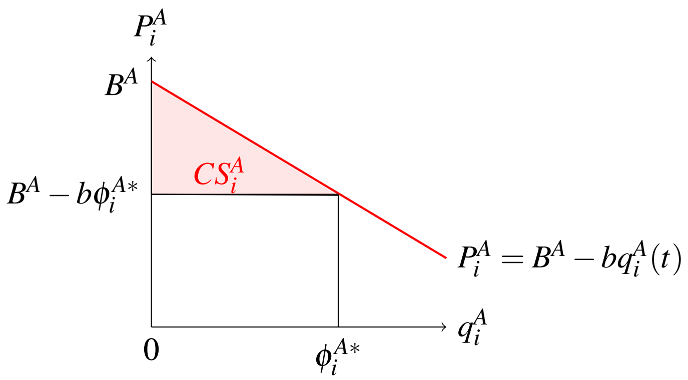


Figure 3.14: The Marshallian measure of Consumer Surplus under Autarky

[18] Remember that what drives the demand for a given market under autarky is the amount supplied to such a market by one player, while under free trade, what drives the demand is the total extraction sold. Therefore, we should use the total extraction Φ^{FT}.

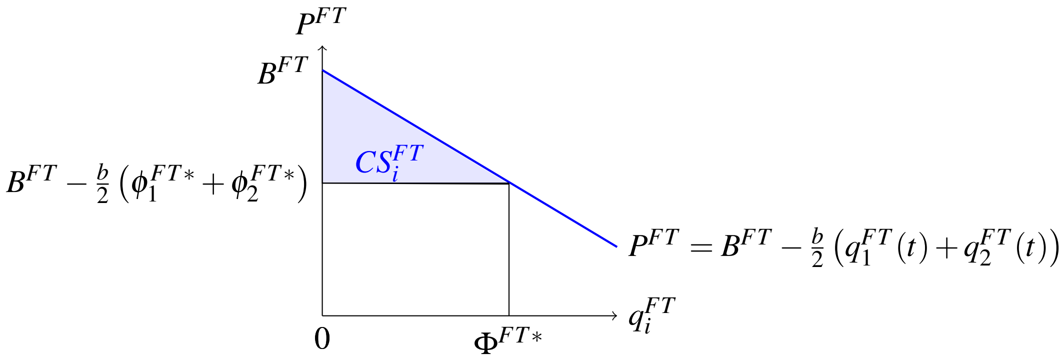


Figure 3.15: The Marshallian measure of Consumer Surplus under Free Trade

Figure 3.15, illustrates the demand that consumers in both countries experience within the integrated market. When countries engage in trade, it can potentially lead to lower prices for consumers in both nations, as a result of increased competition brought forth by the integration of their respective markets. However, it is important to note that this analysis does not yet account for the impact on producers, which will be examined later. The area corresponding to the consumer surplus of both countries under free trade is given by

$$\frac{b\cdot h}{2} = \frac{\Phi^{FT*}\cdot\left(B^{FT} - P^{FT*}(\Phi^{FT*})\right)}{2} = \frac{\Phi^{FT*}\cdot\left(B^{FT} - (B^{FT} - \frac{b}{2}\left(\phi_1^{FT*} + \phi_2^{FT*}\right))\right)}{2} =$$

$$\frac{\Phi^{FT*}\cdot\left(\frac{b}{2}\Phi^{FT}\right)}{2} = \frac{b}{4}\Phi^{FT*}.$$

An interesting difference between our model and Benchekroun et al. (2020) in the study of the consumer surplus under free trade is that their area takes the form of $\frac{bh}{2} = \frac{n\cdot\phi^{FT}*(\frac{b}{n}\cdot(n\phi^{FT}))}{2} = n\frac{\phi^{FT}\cdot(b\phi^{FT})}{2}$ and given that all the players are symmetric each player consumer surplus is $\frac{\phi^{FT}\cdot(b\phi^{FT})}{2}$.[19] Thus, we define the consumer surplus for player i in regime $k \in \{A, FT\}$ as

$$CS_i^k := \int_0^{\infty} \frac{b}{2}\left[q_i^k(S(t))\right]^2 e^{-\rho t} dt, \tag{3.75}$$

where $S(t)$ solves

$$\dot{S}(t) = F(S) - q_1^k - q_2^k$$

[19] Interested readers can see this in Appendix 3.7.5.

$$S(0) = S_0.$$

We will proceed as in Benchekroun et al. (2020), focusing on the Markovian extraction strategy of the form $q_i^k(S)$.Thus, the CS_i satisfies the following differential equation:

$$\rho CS_i^k = \frac{b}{2}\left(q_i^k(S)\right)^2 + \frac{\partial CS_i^k}{\partial S(t)}\left[F(S) - q_i^k - q_j^k\right].$$

Following the previous authors, we study the region characterized by affine strategie, i.e., when $q_i^k = w_i^k + z^k S(t)$, and the function CS_i^k takes the form $CS_i^k = A_i^k S^2 + B_i^k S + G_i^k$ for both players under both regimes.

The symmetric games $k \in \{A, FT\}$ have

$$A^{Sym,k} = \frac{b(z^{Sym,k})^2}{2\rho - 4\delta + 8z^{Sym,k}},$$

$$B^{Sym,k} = \frac{bw^{Sym,k}z^{Sym,k}(\rho + 2z^{Sym,k} - 2\delta)}{(\rho + 4z^{Sym,k} - 2\delta)(\rho + 2z^{Sym,k} - \delta)},$$

$$G^{Sym,k} = \frac{b(w^{Sym,k})^2\left(2\delta^2 + \rho^2 - 3\delta\rho + 2\rho z^{Sym,k}\right)}{2\rho(\rho + 4z^{Sym,k} - 2\delta)(\rho + 2z^{Sym,k} - \delta)}.$$

Under the symmetric games $k \in \{A, FT\}$ one gets

$$A_i^k = \frac{b(z^k)^2}{2\rho - 4\delta + 8z^k} = A^{Sym,k},$$

$$B_i^k = \frac{bz^k[w_i^k(r + 3z^k - 2\delta) - w_j^k z^k]}{(r + 4z^k - 2\delta)(r + 2z^k - \delta)},$$

$$G_i^k = b\frac{\rho^2(w_i^k)^2 + rw_i^k(4w_i^k z^k - 3\delta w_i^k - 2w_j^k z^k) + 2[w_i^k(\delta - z^k) + w_j^k z^k]^2}{2\rho(\rho + 4z^k - 2\delta)(\rho + 2z^k - \delta)}.$$

If we impose symmetry we recover the results in Benchekroun et al. (2020) for $n = 2$ players. Using the expressions above one can compute consumer surplus under both regimes for both players when they are playing their affine strategies, by plugging (w_i^A, z^A) for autarky or (w_i^{FT}, z^{FT}) for free trade. For the symmetric game,[20] this leads to the following results under autarky

[20] The asymmetric case gives very long value function parameters.

$$A^{Sym,A} = \frac{b(3+2\gamma^A)^2(2\delta-\rho)}{8\gamma^A+6}, \tag{3.76}$$

$$B^{Sym,A} = \frac{(1+2\gamma^{FT})(\rho-2\delta)\left\{a^A(1+\gamma^A)[2\rho(1+\gamma^A)-\delta]+\theta^A[\delta+\rho(1+2\gamma^A)]\right\}}{2\delta(3+4\gamma^A)[\delta+\rho(1+2\gamma^A)]}, \tag{3.77}$$

$$C^{Sym,A} = -\frac{[\rho(1+2\gamma^A)-\delta(4\gamma^A+3)][a^A(1+\gamma^A)[2\rho(\gamma^A+1)-\delta]+\theta^A(\delta+\rho(1+2\gamma^A))]^2}{8b\rho\delta^2(\gamma^A+1)^2(4\gamma^A+3)[\delta+\rho(1+2\gamma^A)]}. \tag{3.78}$$

Under the symmetric free trade game, one gets

$$A^{Sym,FT} = \frac{b(3+2\gamma^{FT})^2(2\delta-\rho)}{64(1+\gamma^{FT})}, \tag{3.79}$$

$$B^{Sym,FT} = -\frac{(1+2\gamma^{FT})(2\delta-\rho)}{16\delta(1+\gamma^{FT})[2\delta+\rho(1+22\gamma^{FT})]}\times$$
$$\left\{\rho a^{FT}[4\gamma^{FT}(2+\gamma^{FT})+5]-2\delta a^{FT}+2\theta^{FT}[2\delta+\rho(1+22\gamma^{FT})]\right\} \tag{3.80}$$

$$C^{Sym,FT} = -\frac{[\rho(1+2\gamma^{FT})-4(\gamma^{FT}+1)\delta]}{16b\delta^2\rho(1+\gamma^{FT})(2\gamma^{FT}+3)^2(2\delta+2\gamma^{FT}\rho+\rho)}\times$$
$$[a^{FT}\rho(4(\gamma^{FT}+2)\gamma^{FT}+5)-2a^{FT}\delta+2\theta^{FT}(2\delta+2\gamma^{FT}\rho+\rho)]^2. \tag{3.81}$$

As we now allow status concerns to be regime specific ($\theta^A, \theta^{FT}, \gamma^A, \gamma^{FT}$), and agents can have different extraction costs under autarky and free trade, the consumer surplus in our model is not that straightforward. Consequently, we can get a novel result where agents can be better off under free trade. In contrast, Benchekroun et al. (2020) claimed that "[w]hen n =2 we can show analytically that for $S \in [S_1^A, S_2^A]$ we have $CS^A(S) > CS^{FT}(S)$: while Free Trade always results in an increase (a decrease) in instantaneous consumer surplus in the short-run (long-run), the overall impact of Free Trade on consumer surplus when the transition dynamics is taken into account is to decrease aggregate consumer surplus. Numerical simulations for $n > 2$ yield the same conclusion." Theoretically, one could obtain a value of γ^A such that the opposite is true, so that consumer surplus is higher under free trade. However, we do not include here such expression, as it is several pages long.

To determine whether consumers are better off under autarky or free trade, one can compare $CS^A(S)$ and $CS^{FT}(S)$ using the inequality $CS^A(S) \lesseqgtr CS^{FT}(S)$, which yields

$$CS^A(S) \lesseqgtr CS^{FT}(S) \Leftrightarrow A^{Sym,A}S^2 + B^{Sym,A}S + G^{Sym,A} \lesseqgtr A^{FT}S^2 + B^{Sym,FT}S + G^{Sym,FT}.$$

Collecting terms, one can write

$$(A^{Sym,A} - A^{Sym,FT})S^2 + (B^{Sym,A} - B^{Sym,FT})S + (G^{Sym,A} - G^{Sym,FT}) \lesseqgtr 0 \Leftrightarrow$$

$$\frac{b}{2}\left(\frac{(z^{Sym,A})^2}{4z^{Sym,A} - (2\delta - \rho)} - \frac{(z^{Sym,FT})^2}{4z^{Sym,FT} - (2\delta - \rho)}\right) \cdot S^2$$

$$+b\left(\frac{w^{Sym,A}z^{Sym,A}[2z^{Sym,A} - (2\delta - \rho)]}{(4z^{Sym,A} - (2\delta - \rho))(2z^{Sym,A} + \rho - \delta)} - \frac{w^{Sym,FT}z^{Sym,FT}(2z^{Sym,FT} - (2\delta - \rho))}{(4z^{Sym,FT} - (2\delta - \rho))(2z^{Sym,FT} + \rho - \delta)}\right) \cdot S$$

$$+\frac{b}{2\rho}\left(\frac{(w^{Sym,A})^2(2\delta^2 + \rho^2 - 3\delta\rho + 2\rho z^{Sym,A})}{(4z^{Sym,A} - (2\delta - \rho))(2z^{Sym,A} + \rho - \delta)} - \frac{(w^{Sym,FT})^2(2\delta^2 + \rho^2 - 3\delta\rho + 2\rho z^{Sym,FT})}{(4z^{Sym,FT} - (2\delta - \rho))(2z^{Sym,FT} + \rho - \delta)}\right) \lesseqgtr 0. \tag{3.82}$$

One can find values of the status concern parameters γ^A and θ^A for a very particular case, such that both strategies under symmetric autarky and free trade would have the same slope $z^{Sym,A} = z^{Sym,FT}$ and same intersection point $w^{Sym,A} = w^{Sym,FT}$ (see equation (3.41) in Remark 3.4). Under those specific circumstances, they will have the same consumer surplus under autarky and free trade. It is straightforward to see that expression 3.82 collapses to zero and consequently, $CS^A = CS^{FT}$.

Plugging in the slopes ($z^{Sym,k}$) and intersections in the vertical axes ($w^{Sym,A}$) from their optimal strategies into the respective consumer surplus (see Propositions 3.1 and 3.2), we can plot ($CS^{Sym,k}$) for both regimes. Taking into account Assumptions 3.1 and 3.2, from autarky and free trade respectively, one can see that depending on the values of the status concern, agents can have higher consumer surplus under autarky (Figure 3.16) or under free trade (Figure 3.17). For values in Table (3.5), one gets the result in Figure 3.16 where consumer surplus under autarky is higher, i.e., $CS^A(S) > CS^{FT}(S)$ for $S \in [\max\{S_1^A, S_1^{FT}\}, \min\{S_2^A, S_2^{FT}\}]$.

One can see that Assumptions 3.1 and 3.2 hold,

$$\delta > \max \left\{ \frac{\rho}{2}, \frac{2a^{Sym,A}(1+\gamma^A)^2 + \theta^A(1+2\gamma^A)}{2b(1+\gamma^A)^2 S_y}, \right.$$
$$\left. \frac{2a^{Sym,FT}\left[5+4\gamma^{FT}(2+\gamma^{FT})\right] + 4\theta^{FT}(1+4\gamma^{FT})}{b(2\gamma^{FT}+3)^2 S_y} \right\}$$
$$\Leftrightarrow 0.25 > \max\{0.1,\ 0.127273,\ 0.164763\},$$

together with the inequality (3.8) under autarky

$$\theta^A \leq \frac{a^{Sym,A}(1+\gamma^A)\left[\delta - 2\rho(1+\gamma^A)\right]}{\left[\delta + \rho(1+2\gamma^A)\right]} \Leftrightarrow 0.1 \leq 3.30803,$$

and the inequality (3.21) under free trade,

$$\theta^{FT} \leq \frac{a^{Sym,FT}(2\delta - \rho\left[5+4\gamma^{FT}(2+\gamma^{FT})\right])}{2\left[2\delta + \rho(1+2\gamma^{FT})\right]} \Leftrightarrow 0.2 \leq 1.84771.$$

However, if agents now have a higher value of θ^A, for instance, $\theta^A = 1.68$, then consumer surplus under free trade is higher, $CS^A(S) < CS^{FT}(S)$ for $S \in [\max\{S_1^A, S_1^{FT}\}, \min\{S_2^A, S_2^{FT}\}]$, (Figure 3.17). The conditions still hold, $\delta = 0.25 > \max\{0.1,\ 0.151896,\ 0.164763\}$, $\theta^A \leq 3.30803$ and $\theta^{FT} \leq 1.84771$. Interestingly, there is also a value of $\tilde{\theta}^A$ where both consumer surpluses intersect, which means that for lower values of resource, consumer surplus is higher under free trade, and for higher levels of resource, consumer surplus is higher under autarky (see Figure 3.18). For instance, for a value of $\tilde{\theta}^A = 1.2$, the assumptions also hold, $0.25 > \max\{0.1,\ 0.144416,\ 0.164763\}$, $\theta^A \leq 3.30803$ and $\theta^{FT} \leq 1.84771$.

Although it seems demanding to have such a high level of importance in the relative extraction, the economic intuition suggests that agents or nations should attach considerable importance to their comparative extraction levels under autarky. This scenario might arise in contexts of elevated geopolitical tensions, where outperforming the extraction of a systemic competitor becomes an imperative objective, regardless of the costs involved.

We also get the same numerical result for the asymmetric games, where agents under free trade can have higher levels of consumer surplus than those under autarky, in contrast to Benchekroun et al. (2020). We use $a_b^{FT} = 5, a_s^{FT} = 4, a_b^A =$

b	δ	ρ	θ^A	θ^{FT}	γ^A	γ^{FT}	$a^{Sym,A}$	$a^{Sym,FT}$
7	0.25	0.02	0.1	0.2	0.1	0.08	4	5

Table 3.5: Parameters.

$4.5, a_s^A = 3.2$ (remember that player s is the small agent, with higher extraction costs, thus, smaller values of a). Obviously, the corresponding Assumptions 3.3 and 3.4 hold. We first show the case when both players have higher consumer surplus under autarky (Figure 3.19). However, it is also feasible that both agents have higher levels of consumer surplus under free trade as shown in Figure 3.20. As in the symmetric case, one would need a high level of θ^A. Finally, as in the symmetric case, there exists also a value of $\hat{\theta}^A = 1.39$, where consumer surplus for player i under free trade is higher for low levels of resource, and for high levels of S, consumer surplus is higher under autarky. Again, we have seen that introducing status concerns generalize the result in Benchekroun et al. (2020) and allows us to get different results, such as agents having higher consumer surplus under free trade, in contrast to the result obtained by the previous authors. Therefore, the novelty lies in the fact that CS_i^A can be lower or higher that CS_i^{FT}, depending on the status concern of the players.

Finally, one interesting result shown in Figure 3.21 is that consumers in the efficient (big) country are mostly happier under autarky (higher consumer surplus), while consumers in the inefficient (small) country would prefer to stay under free trade most of the part. Observe how in most of the region, $CS_b^A > CS_b^{FT}$ and $CS_s^A < CS_s^{FT}$. This is a new result caused by the heterogeneity of the players.

3.5.2 Aggregate Producer Surplus

Producer surplus consists of the discounted sum of instantaneous producer surplus over the infinite horizon, as represented by the value function of each respective game (see equations 3.10, 3.25, 3.48, 3.66). Following Benchekroun et al. (2020), we also study the region for $S \in [S_1^k, S_2^k]$.

For the symmetric case, we compare $V^{Sym,A}(S)$ and $V^{Sym,FT}(S)$. Subtracting

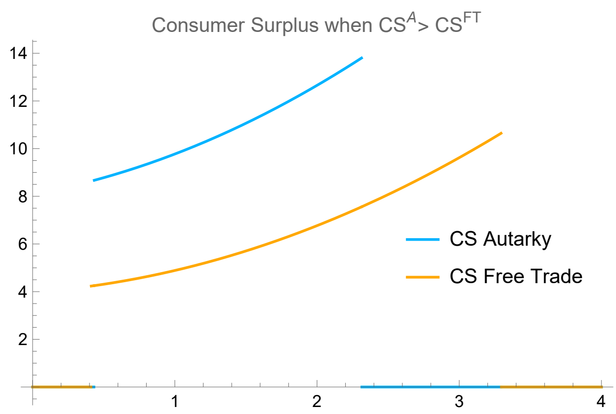


Figure 3.16: Consumer Surplus when $CS^A > CS^{FT}$.

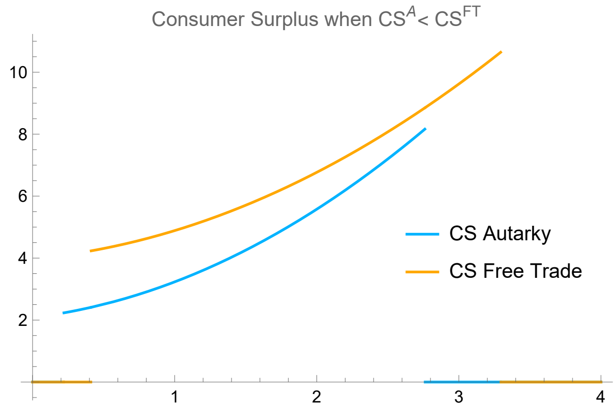


Figure 3.17: Consumer Surplus when $CS^A < CS^{FT}$.

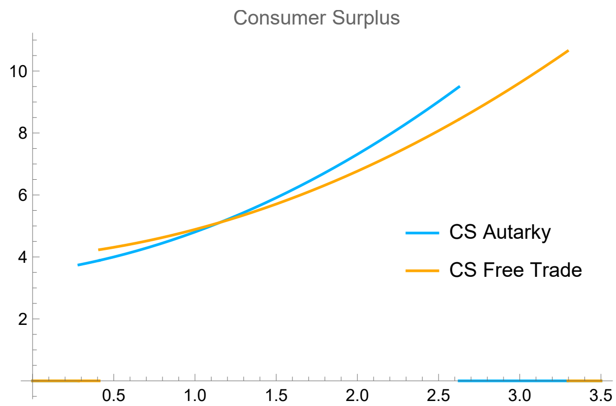


Figure 3.18: Consumer Surplus when $CS^A < CS^{FT}$ for low values of S, and $CS^A > CS^{FT}$ for high values of S .

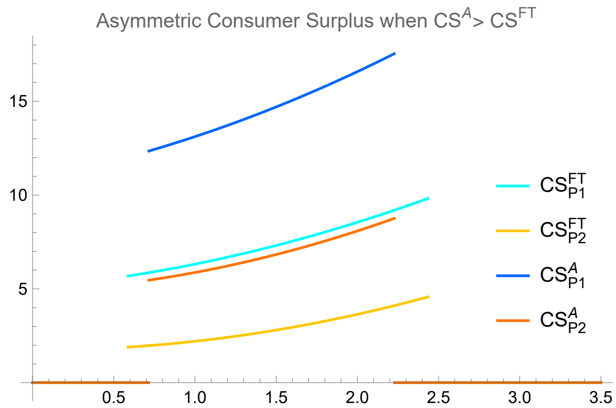


Figure 3.19: Asymmetric Consumer Surplus when both asymmetric players have $CS_i^A > CS_i^{FT}$.

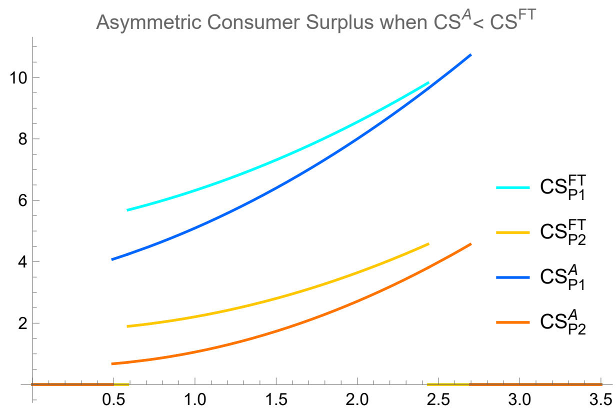


Figure 3.20: Asymmetric Consumer Surplus when both asymmetric players have $CS_i^A < CS_i^{FT}$.

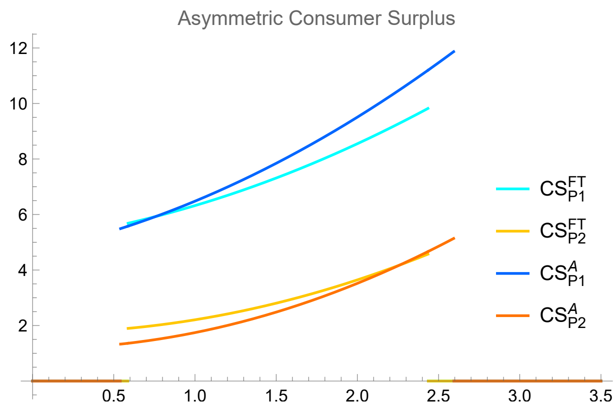


Figure 3.21: Asymmetric Consumer Surplus for asymmetric players.

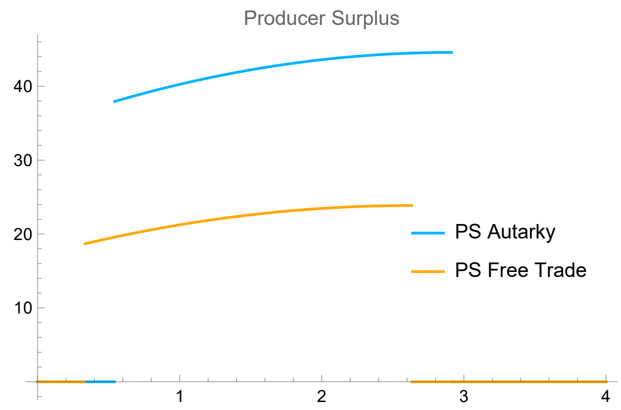


Figure 3.22: Symmetric Producer Surplus when $PS_i^A > PS_i^{FT}$.

$V^{Sym,FT}(S)$ to $V^{Sym,A}(S)$, and collecting terms leads to

$$(\alpha^{Sym,A} - \alpha^{Sym,FT})S^2 + (\beta^{Sym,A} - \beta^{Sym,FT})S + (\mu^{Sym,A} - \mu^{Sym,FT}) \lessgtr 0,$$

which can be positive or negative and where $(\alpha^{Sym,A} - \alpha^{Sym,FT})$, $(\beta^{Sym,A} - \beta^{Sym,FT})$ and $(\mu^{Sym,A} - \mu^{Sym,FT})$ are very long expressions shown in Appendix 3.7.6. As the expressions for the symmetric surplus are already too complicated, we will analyze this problem numerically, checking that all assumptions are satisfied and keeping the same parameters as before. One can observe in Figure 3.22 that producer surplus is higher under autarky. Moreover, if we keep the same parameters under autarky and free trade, $\gamma^{FT} = \gamma^A$, $\theta^{FT} = \theta^{FT}$, and $a^{FT} = a^A$, we still see the same result, where producer surplus is higher under autarky.

As before, we see that Assumptions 3.1 and 3.2 hold,

$$\delta > \max \left\{ \frac{\rho}{2}, \frac{2a^{Sym,A}(1+\gamma^A)^2 + \theta^A(1+2\gamma^A)}{2b(1+\gamma^A)^2 S_y}, \right.$$
$$\left. \frac{2a^{Sym,FT}\left[5+4\gamma^{FT}(2+\gamma^{FT})\right] + 4\theta^{FT}(1+4\gamma^{FT})}{b(2\gamma^{FT}+3)^2 S_y} \right\}$$
$$\Leftrightarrow 0.25 > \max\{0.1,\ 0.127273,\ 0.164763\},$$

together with the inequality (3.8) under autarky

$$\theta^A \leq \frac{a^{Sym,A}(1+\gamma^A)\left[\delta - 2\rho(1+\gamma^A)\right]}{[\delta + \rho(1+2\gamma^A)]} \Leftrightarrow 0.1 \leq 3.30803,$$

and the inequality (3.21) under free trade,

$$\theta^{FT} \leq \frac{a^{Sym,FT}(2\delta - \rho\left[5+4\gamma^{FT}(2+\gamma^{FT})\right])}{2\left[2\delta + \rho(1+2\gamma^{FT})\right]} \Leftrightarrow 0.2 \leq 1.84771.$$

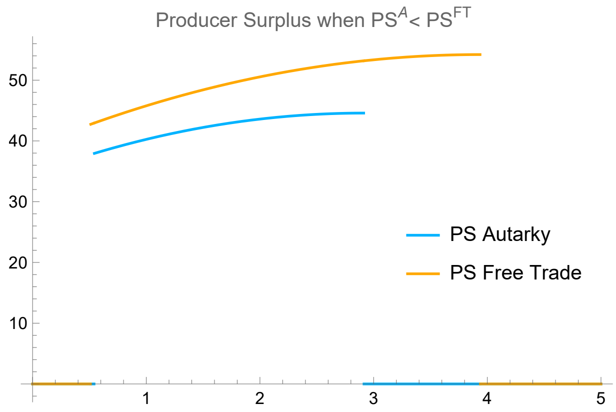


Figure 3.23: Symmetric Producer Surplus when $PS_i^A > PS_i^{FT}$.

Interestingly, as in our model we allow for different extraction costs under autarky and free trade, in contrast to the same extraction cost under autarky and free trade in Benchekroun et al. (2020), we can obtain a different result to that obtained by the authors cited above. One specific case arises when $a^A > a^{FT}$, which implies that countries have lower marginal costs under autarky for any possible reason. For instance, for a value of $a^A = 6$, one gets $V^{Sym,FT}(S) > V^{Sym,A}(S)$ shown in Figure 3.23. The assumptions remain valid for $a^A > a^{FT}$, where $\delta = 0.25 > \max\{0.1,\, 0.19013,\, 0.164763\}$, $\theta^A = 0.1 \leq 4.96204$, and $\theta^{FT} = 0.1 \leq 1.84771$.

When $a^A > a^{FT}$, but the difference is relatively small, such as $a^A = 5.55$, we observe a scenario where producer surplus is higher under autarky when resources are scarce, and as the resource availability increases, producer surplus is higher (see Figure 3.24). This outcome may also arise if players assign significant weight to relative extraction under free trade, i.e., large θ^{FT}, in conjunction with lower marginal costs. All the aforementioned results are consistent with the conditions specified in the assumptions.

Finally, we should mention that similar results are obtained for the asymmetric games.

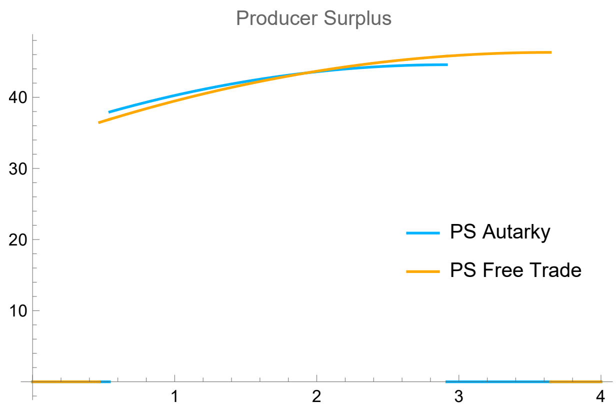


Figure 3.24: Symmetric Producer Surplus when $PS_i^A > PS_i^{FT}$.

3.6 Conclusions

In this chapter, we have studied a dynamic game model where agents extract renewable resources from a common pool. Players have status concern preferences, i.e., they see what their neighbor is doing, and it affects them. In addition to caring about their own profits, we also account for two types of status concerns simultaneously: players' comparison of relative extraction levels and players' comparison of relative profits.

Our model examines two players/countries that can extract the resource and sell it domestically (known as the autarky scenario) or trade with the other country (known as free trade). We have analyzed first the symmetric case (identical extraction cost), and later the heterogeneous/asymmetric case. In this last case, we have seen that the efficient (big) player extracts more resources than the inefficient (small) and has higher levels of consumer and producer surplus.

Our study has revealed that incorporating behavioral factors such as status concerns leads to more nuanced and general conclusions that augment and broaden the findings of previous studies (e.g., Benchekroun et al. (2020)). Incorporating status concern preferences into our analysis enables us to identify an additional steady state in the symmetric autarky game, which is fully stable. Moreover, we found that the impact of status concerns on the behavior of players can result in nuanced effects on the strategic outcomes of the game. Specifically, increasing the importance of relative extraction (θ^k) has similar effects on the switching points, Cournot strategy, and slope of the affine strategy under both free trade and autarky, consistent with the findings of the free trade model analyzed in Benchekroun and Long (2016). However, when players care more about relative profits (γ^k), we obtain more general results that differ from the previous authors' work. In such cases, the switching points and Cournot strategy may move in either direction, depending on the status concern parameters of the players, while the slope of the affine strategy decreases in both free trade and autarky, consistent with previous findings.

Additionally, our analysis reveals that both consumer and producer surplus can be higher or lower under autarky or free trade depending on the behavioral characteristics of the players. By allowing for status concern preferences, our study provides more generalizable outcomes, and it is not possible to conclude that producers or consumers will always have higher levels of surplus under a particular regime. Our analysis yielded additional results where consumer (producer) surplus was initially higher under free trade (autarky) when natural resources were scarce. However, as the availability of natural resources increased, consumer (producer) surplus became higher under autarky (free trade) instead. This finding suggests that the optimal trade policy for maximizing surplus may depend on the level of natural resource availability, as well as the behavioral aspects such as status concerns, that influence how agents compare their strategies. Our research highlights the complex and dynamic interplay between natural resource availability, trade policies, and behavioral factors, underscoring the need for a multifaceted approach when formulating policies in this context.

3.7 Appendix

3.7.1 Autarky Symmetric

Proof. For $S \in [0, S_1^{Sym,A})$, with zero extraction one gets an instantaneous value of zero. However, when the initial stock is lower than $S_1^{Sym,A}$, one knows that not extracting will allow the resource to grow exponentially (first linearly increasing part of the reproduction function) until it reaches the first threshold $S_1^{Sym,A}$. Thus the evolution of the resource for $S \in [0, S_1^{Sym,A})$ is $S(t) = S_0 e^{\delta t}$. One obtains the time $t_1 = \ln\left(\frac{S_1}{S_0}\right)^{\frac{1}{\delta}}$ when the resources reaches $S_1^{Sym,A}$. Therefore, the value for any initial amount of resource $S \in [0, S_1^{Sym,A})$ is

$$V^{Sym,A}(S) = e^{-\rho \ln\left(\frac{S_1}{S}\right)^{\frac{1}{\delta}}} W(S;\gamma^A,\theta^A)^{Sym,A} = \left(\frac{S}{S_1^{Sym,A}}\right)^{\frac{\rho}{\delta}} W(S;\gamma^A,\theta^A)^{Sym,A}. \tag{A.1}$$

For the affine strategy, when $S \in [S_1^{Sym,A}, S_2^{Sym,A}]$, plugging the maximized expression equation (3.6) into the HBJ (eq. (3.5)) and applying the undetermined coefficient technique, one gets the value function

$$W(S;\gamma^A,\theta^A)^{Sym,A} = \frac{\alpha^{Sym,A}}{2} S^2 + \beta^{Sym,A} S + \mu^{Sym,A}, \tag{A.2}$$

where

$$\alpha^{Sym,A} = -\frac{2b(1+\gamma^A)^2(2\delta-\rho)}{4\gamma^A+3}, \tag{A.3}$$

$$\beta^{Sym,A} = \frac{(2\delta-\rho)\left(2a^{Sym,A}(1+\gamma^A)^2 + \theta^A(1+2\gamma^A)\right)}{\delta(4\gamma^A+3)} \tag{A.4}$$

and

$$\begin{aligned}\mu^{Sym,A} &= [a^{Sym,A}(1+\gamma^A)(\delta - 2(1+\gamma^A)\rho) - \theta^A(\delta + 2\gamma^A\rho + \rho)] \\ &\times \frac{a^{Sym,A}(1+\gamma^A)[(4\gamma^A+3)\delta - 2(1+\gamma^A)\rho] + \theta^A((4\gamma^A+3)\delta - 2\gamma^A\rho - \rho)}{4b\delta^2\rho(1+\gamma^A)^2(4\gamma^A+3)}. \end{aligned} \tag{A.5}$$

For $S \in [S_2^{Sym,A}, \infty)$, one gets the corresponding part of the value function integrating the discounted stream of profits using the Cournot strategy,

$$\int_0^\infty e^{-\rho t}\pi^{Sym,Cou,A}dt = \frac{(a^{Sym,A})^2(1+\gamma^A)^2-\theta^A}{4b\rho(1+\gamma^A)^2}.$$

We must prove now that (i) the value function is continuously differentiable with respect to S, and that (ii) the policy function $\phi^{Sym,A*}$ in (3.6) is a solution to the HJB equation (3.5) where $V_1(S)=V_2(S)=V(S)$.

(i) Proof that $V^{Sym,A}(S)$ is continuously differentiable in S:

It is straightforward to see that the value function is clearly continuously differentiable over the intervals $[0,S_1^{Sym,A})$, $[S_1^{Sym,A},S_2^{Sym,A}]$, $[S_2^{Sym,A},\infty)$. We now show that it is also continuously differentiable at the critical points $S_1^{Sym,A}$ and $S_2^{Sym,A}$.

We first observe that the function is continuous at the switching points:

$$\lim_{S\to S_1^{Sym,A}-} V^{Sym,A}(S) = W(S_1^{Sym,A};\gamma^A,\theta^A)^{Sym,A} = \lim_{S\to S_1^{Sym,A}+} V^{Sym,A}(S),$$

where

$$\lim_{S\to S_1^{Sym,A}-} V^{Sym,A}(S) = W(S_1^{Sym,A};\gamma^A,\theta^A)^{Sym,A}$$

$$\Longleftrightarrow \underbrace{\left(\frac{S_1^{Sym,A}}{S_1^{Sym,A}}\right)^{\frac{\rho}{\delta}}}_{=1} W(S_1^{Sym,A};\gamma^A,\theta^A)^{Sym,A} = W(S_1^{Sym,A};\gamma^A,\theta^A)^{Sym,A},$$

and for the second limit condition, one can also check that the two sides are the same,

$$\frac{\alpha^{Sym,A}}{2}(S_2^{Sym,A})^2+\beta^{Sym,A}S_2^{Sym,A}+\mu^{Sym,A} = \frac{(a^{Sym,A})^2(1+\gamma^A)^2-\theta^A}{4b\rho(1+\gamma^A)^2}.$$

This shows that the value function $V^{Sym,A}(S)$ is continuous at $S_1^{Sym,A}$ and $S_2^{Sym,A}$.

We need to show now that $\partial_S V^{Sym,A}(S)$, which is the derivative of the value function with respect to the state S is continuous. Obviously, the function is differentiable over the intervals $[0,S_1^{Sym,A})$, $[S_1^{Sym,A},S_2^{Sym,A}]$, $[S_2^{Sym,A},\infty)$, as we have polynomials,

$$\partial_S V^{Sym,A}(S) = \begin{cases} \frac{\rho}{\delta \cdot S_1^{Sym,A}} \left(\frac{S}{S_1^{Sym,A}} \right)^{\frac{\rho}{\delta}-1} W(S_1^{Sym,A}; \gamma^A, \theta^A)^{Sym,A} & \text{if } S \in [0, S_1^{Sym,A}) \\ \partial_S W(S; \gamma^A, \theta^A)^{Sym,A} & \text{if } S \in [S_1^{Sym,A}, S_2^{Sym,A}] \\ 0 & \text{if } S \in (S_2^{Sym,A}, \infty) \end{cases} \tag{A.6}$$

For the first switching point, we can check

$$\lim_{S \to S_1^{Sym,A}-} \frac{\partial V^{Sym,A}(S)}{\partial S} = \left. \frac{\partial W(S; \gamma^A, \theta^A)^{Sym,A}}{\partial S} \right|_{S = S_1^{Sym,A}} = \lim_{S \to S_1^{Sym,A}+} \frac{\partial V^{Sym,A}(S)}{\partial S}.$$

The first equality leads to

$$\frac{\rho}{\delta \cdot S_1^{Sym,A}} W(S_1^{Sym,A}; \gamma^A, \theta^A)^{Sym,A} = \alpha^{Sym,A} \cdot S_1^{Sym,A} + \beta^{Sym,A},$$

which is true, showing that the value function is differentiable at $S_1^{Sym,A}$. Moreover, for the second switching point $S_2^{Sym,A}$,

$$\lim_{S \to S_2^{Sym,A}-} \frac{\partial V^{Sym,A}(S)}{\partial S} = \underbrace{\lim_{S \to S_2^{Sym,A}+} \frac{\partial V^{Sym,A}(S)}{\partial S}}_{=0},$$

which gives,

$$\alpha^{Sym,A} \cdot S_2^{Sym,A} + \beta^{Sym,A} = 0,$$

which is true, showing that the value function is differentiable at $S_2^{Sym,A}$. Therefore, the value function $V^{Sym,A}(S)$ is is continuously differentiable for all $S \in [0, \infty)$.

(*ii*) We now prove that the policy function $\phi^{Sym,A*}$ in (3.6) is a solution to the HJB equation (3.5) where the value function for both players are symmetric, $V_1(S) = V_2(S) = V(S)$. For positive amounts of extraction of the resource $S > S_1^{Sym,A}$, the HJB admits an interior solution. From the first order condition one obtains the function $\phi_i^{Sym,A\,*}$, which gives for the symmetric equilibrium

$$\phi_i^{Sym,A} = \phi_j^{Sym,A} = \phi^{Sym,A} = \frac{a^{Sym,A}(1+\gamma^A) + \theta^A - \partial_S V^{Sym,A}(S)}{2b(1+\gamma^A)}. \tag{A.7}$$

Substituting into the HJB (3.5) gives

$$\rho V^{Sym,A}(S) = \left(a^{Sym,A} - b\,\phi^{Sym,A}\right)\phi^{Sym,A} + \frac{\partial V^{Sym,A}(S)}{\partial S}\left[F(S) - 2\phi^{Sym,A}\right]. \quad \text{(A.8)}$$

It can be checked that the proposed value function satisfies the differential equation for $S \geq S_1^{Sym,A}$. Substituting $\partial_S V(.)$ into (A.7) yields exactly $\phi^{Sym,A}$ in (3.6).

For $S < S_1^{Sym,A}$, the HJB has a corner solution with zero extraction, i.e., $\phi^{Sym,A*} = 0$. It is straightforward to obtain this result by substituting the value function in (3.10) and $\partial_S V(.)$ in (A.6) into the HJB and see that $\phi^{Sym,A*} = 0$ is a solution of the differential equation.

Finally, for $S > S_2^{Sym,A}$, plugging $q^{Sym,Cou,A}$ from (3.6) into the HJB and using the value function and its derivative, one gets that the value function $V^{Sym,A}(S)$ satisfies the differential equation above for all $S > S_2^{Sym,A}$. This concludes the proof.

■

3.7.2 Free Trade Symmetric

Proof. The structure of the proof is the same as the symmetric autarky case. For $S \in [0, S_1^{Sym,FT})$, with zero extraction one gets an instantaneous value of zero. However, when the initial stock is lower than $S_1^{Sym,FT}$, one knows that not extracting will allow the resource to grow exponentially (first linearly increasing part of the reproduction function) until it reaches the first threshold $S_1^{Sym,FT}$. Thus the evolution of the resource for $S \in [0, S_1^{Sym,FT})$ is $S(t) = S_0 e^{\delta t}$. One obtains the time $t_1 = \ln\left(\frac{S_1}{S_0}\right)^{\frac{1}{\delta}}$ when the resources reaches $S_1^{Sym,FT}$. Therefore, the value for any initial amount of resource $S \in [0, S_1^{Sym,FT})$ is

$$\begin{aligned} V^{Sym,FT}(S) &= e^{-\rho \ln\left(\frac{S_1}{S}\right)^{\frac{1}{\delta}}} W(S;\gamma^{FT},\theta^{FT})^{Sym,FT} \\ &= \left(\frac{S}{S_1^{Sym,FT}}\right)^{\frac{\rho}{\delta}} W(S;\gamma^{FT},\theta^{FT})^{Sym,FT}. \end{aligned} \quad \text{(A.9)}$$

For the affine strategy, when $S \in [S_1^{Sym,FT}, S_2^{Sym,FT}]$, plugging the maximized expression equation (3.24) into the HBJ (equation (3.23)) and applying the undetermined coefficient technique, one gets the value function

$$W(S;\gamma^{FT},\theta^{FT})^{Sym,FT} = \frac{\alpha^{Sym,FT}}{2}S^2 + \beta^{Sym,FT}S + \mu^{Sym,FT}, \quad \text{(A.10)}$$

where

$$\alpha^{Sym,FT} = -\frac{b(3+2\gamma^{FT})^2(2\delta-\delta)}{16(1+\gamma^{FT})}, \tag{A.11}$$

$$\beta^{Sym,FT} = \frac{(2\delta-\rho)\left[2\theta^{FT}(1+2\gamma^{FT})+a^{Sym,FT}(5+8\gamma^{FT}+4(\gamma^{FT})^2\right]}{8\delta(1+\gamma^{FT})} \tag{A.12}$$

and

$$\begin{aligned}\mu^{Sym,FT} &= \left[a^{Sym,FT}\cdot\rho(5+4\gamma^{FT}(2+\gamma^{FT}))-2a^{Sym,FT}\delta+2\theta^{FT}(2\delta+2\gamma^{FT}\rho+\rho)\right] \\ &\times\frac{\rho\cdot a^{Sym,FT}[5+4\gamma^{FT}(2+\gamma^{FT})]-8\delta a^{Sym,FT}(1+\gamma^{FT})^2+2\theta^{FT}(2\rho\gamma^{FT}+\rho-4\delta(1+\gamma^{FT}))}{8b\delta^2\rho(1+\gamma^{FT})(3+2\gamma^{FT})^2}.\end{aligned} \tag{A.13}$$

For $S \in [S_2^{Sym,FT}, \infty)$, one gets the corresponding part of the value function integrating the discounted stream of profits using the Cournot strategy,

$$\int_0^\infty e^{-\rho t}\pi^{Sym,Cou,FT}dt = \frac{2(a^{Sym,FT}-2\theta^{FT})[a^{Sym,FT}(1+\gamma^{FT})+\theta^{FT}]}{b\rho(3+2\gamma^{FT})^2}.$$

We must prove now that (i) the value function is continuously differentiable with respect to S, and that (ii) the policy function $\phi^{Sym,FT*}$ in (3.24) is a solution to the HJB equation (3.23) where $V_1(S) = V_2(S) = V(S)$.

(i) Proof that $V^{Sym,FT}(S)$ is continuously differentiable in S:

It is straightforward to see that the value function is clearly continuously differentiable over the intervals $[0, S_1^{Sym,FT})$, $[S_1^{Sym,FT}, S_2^{Sym,FT}]$, $[S_2^{Sym,FT}, \infty)$. We now show that it is also continuously differentiable at the critical points $S_1^{Sym,FT}$ and $S_2^{Sym,FT}$.

We first observe that the function is continuous at the switching points:

$$\begin{aligned}\lim_{S\to S_1^{Sym,FT}-} V^{Sym,FT}(S) &= W(S_1^{Sym,FT};\gamma^{FT},\theta^{FT})^{Sym,FT} \\ &= \lim_{S\to S_1^{Sym,FT}+} V^{Sym,FT}(S),\end{aligned}$$

where

$$\lim_{S\to S_1^{Sym,FT}-} V^{Sym,FT}(S) = W(S_1^{Sym,FT};\gamma^{FT},\theta^{FT})^{Sym,FT}$$

$$\Longleftrightarrow \underbrace{\left(\frac{S_1^{Sym,FT}}{S_1^{Sym,FT}}\right)^{\frac{\rho}{\delta}}}_{=1} W(S_1^{Sym,FT};\gamma^{FT},\theta^{FT})^{Sym,FT} = W(S_1^{Sym,FT};\gamma^{FT},\theta^{FT})^{Sym,FT},$$

and for the second limit condition, one can also check that the two sides are the same,

$$\frac{\alpha^{Sym,FT}}{2}(S_2^{Sym,FT})^2 + \beta^{Sym,FT} S_2^{Sym,FT} + \mu^{Sym,FT}$$
$$= \frac{2(a^{Sym,FT} - 2\theta^{FT})[a^{Sym,FT}(1+\gamma^{FT}) + \theta^{FT}]}{b\rho(3+2\gamma^{FT})^2}.$$

This shows that the value function $V^{Sym,FT}(S)$ is continuous at $S_1^{Sym,FT}$ and $S_2^{Sym,FT}$.

We need to show now that $\partial_S V^{Sym,FT}(S)$, which is the derivative of the value function with respect to the state S is continuous. Obviously, the function is differentiable over the intervals $[0, S_1^{Sym,FT})$, $[S_1^{Sym,FT}, S_2^{Sym,FT}]$, $[S_2^{Sym,FT}, \infty)$, as we have polynomials,

$$\partial_S V^{Sym,FT}(S) =$$

$$\begin{cases} \frac{\rho}{\delta \cdot S_1^{Sym,FT}} \left(\frac{S}{S_1^{Sym,FT}}\right)^{\frac{\rho}{\delta}-1} W(S_1^{Sym,FT};\gamma^{FT},\theta^{FT})^{Sym,FT} & \text{if } S \in [0, S_1^{Sym,FT}) \\ \partial_S W(S;\gamma^{FT},\theta^{FT})^{Sym,FT} & \text{if } S \in [S_1^{Sym,FT}, S_2^{Sym,FT}] \\ 0 & \text{if } S \in (S_2^{Sym,FT}, \infty) \end{cases} \tag{A.14}$$

For the first switching point, we can check

$$\lim_{S \to S_1^{Sym,FT}-} \frac{\partial V^{Sym,FT}(S)}{\partial S} = \left.\frac{\partial W(S;\gamma^{FT},\theta^{FT})^{Sym,FT}}{\partial S}\right|_{S=S_1^{Sym,FT}}$$
$$= \lim_{S \to S_1^{Sym,FT}+} \frac{\partial V^{Sym,FT}(S)}{\partial S}.$$

The first equality leads to

$$\frac{\rho}{\delta \cdot S_1^{Sym,FT}} W(S_1^{Sym,FT};\gamma^{FT},\theta^{FT})^{Sym,FT} = \alpha^{Sym,FT} \cdot S_1^{Sym,FT} + \beta^{Sym,FT},$$

which is true, showing that the value function is differentiable at $S_1^{Sym,FT}$. Moreover, for the second switching point $S_2^{Sym,FT}$,

$$\lim_{S\to S_2^{Sym,FT}-} \frac{\partial V^{Sym,FT}(S)}{\partial S} = \underbrace{\lim_{S\to S_2^{Sym,FT}+} \frac{\partial V^{Sym,FT}(S)}{\partial S}}_{=0},$$

which gives,

$$\alpha^{Sym,FT} \cdot S_2^{Sym,FT} + \beta^{Sym,FT} = 0,$$

which is true, showing that the value function is differentiable at $S_2^{Sym,FT}$. Therefore, the value function $V^{Sym,FT}(S)$ is is continuously differentiable for all $S \in [0,\infty)$.

(*ii*) We now prove that the policy function $\phi^{Sym,FT*}$ in (3.24) is a solution to the HJB equation (3.23) where the value function for both players are symmetric, $V_1(S) = V_2(S) = V(S)$. For positive amounts of extraction of the resource $S > S_1^{Sym,FT}$, the HJB admits an interior solution. From the first order condition one obtains the function $\phi_i^{Sym,FT\,*}$, which gives for the symmetric equilibrium,

$$\phi_i^{Sym,FT} = \phi_j^{Sym,FT} = \phi^{Sym,FT} = \frac{2[\theta^{FT} + a^{Sym,FT}(1+\gamma^{FT}) - \partial_S V^{Sym,FT}(S)]}{b(3+2\gamma^{FT})}. \tag{A.15}$$

Substituting into the HJB (3.23) gives

$$\rho V^{Sym,FT}(S) = \left(a^{Sym,FT} - \frac{b}{2}\,\phi^{Sym,FT}\right)\phi^{Sym,FT} + \frac{\partial V^{Sym,FT}(S)}{\partial S}\left[F(S) - 2\phi^{Sym,FT}\right]. \tag{A.16}$$

It can be checked that the proposed value function satisfies the differential equation for $S \geq S_1^{Sym,FT}$. Substituting $\partial_S V(.)$ into (A.15) yields exactly $\phi^{Sym,FT}$ in (3.24).

For $S < S_1^{Sym,FT}$, the HJB has a corner solution with zero extraction, i.e., $\phi^{Sym,FT\,*} = 0$. It is straightforward to obtain this result by substituting the value function in (3.25) and $\partial_S V(.)$ in (A.14) into the HJB and see that $\phi^{Sym,FT\,*} = 0$ is a solution of the differential equation.

Finally, for $S \geq S_2^{Sym,FT}$, plugging $q^{Sym,Cou,FT}$ from (3.24) into the HJB and using the value function and its derivative, one gets that the value function $V^{Sym,FT}(S)$ satisfies the differential equation above for all $S \geq S_2^{Sym,FT}$. This concludes the proof.

■

3.7.3 Autarky Asymmetric

Proof. The structure of the proof is the same as the symmetric autarky case. For $S \in [S^A_{1,s}, S^A_{2,s}]$, plugging the maximized expression (3.43) into the HBJ (3.42) and applying the undetermined coefficient technique, one gets the value function

$$V_i^A(S) = \frac{\alpha_i^A}{2} S^2 + \beta_i^A S + \mu_i^A, \qquad \text{if } S \in [S^A_{1,s}, S^A_{2,s}] \tag{A.17}$$

where

$$\alpha_i^A = -\frac{2b(1+\gamma^A)^2(2\delta-\rho)}{4\gamma^A+3} = \alpha^{Sym,A}, \tag{A.18}$$

$$\beta_i^A = \frac{(2\delta-\rho)\left[(a_i^A + a_j^A)(1+\gamma^A)^2 + \theta^A(1+2\gamma^A)\right]}{\delta(4\gamma^A+3)} \tag{A.19}$$

and

$$\mu_i^A = \frac{1}{4b(1+\gamma^A)^2\rho} \times \{(a_i^A)^2(1+\gamma^A)^3 - 2a_i^A(1+\gamma^A)^2(\beta_i^A - \theta^A) - (a_j^A)^2\gamma^A(1+\gamma^A)^2$$
$$-2a_j^A(1+\gamma^A)^2(\beta_i^A+\theta^A) + (\beta_i^A)^2\gamma^A + (\beta_i^A)^2 + 2\beta_i^A\beta_j^A\gamma^A + 2\beta_i^A\beta_j^A - 4\beta_i^A\gamma^A\theta^A - 4\beta_i^A\theta^A +$$
$$(\beta_i^A)^2\gamma^A + 2\beta_j^A\theta^A - (\theta^A)^2\}. \tag{A.20}$$

We must prove now that (i) the value function is continuously differentiable with respect to S, and that (ii) the policy function ϕ_i^{A*} in (3.43) is a solution to the HJB equation (3.42) .

(i) Proof that $V_i^A(S)$ is continuously differentiable in S: It is straightforward to see that the value function is clearly continuously differentiable over the intervals $[S^A_{1,s}, S^A_{2,s}]$, since it is a polynomial of degree 2.

(ii) We now prove that the policy function ϕ^{A*} in (3.43) is a solution to the HJB equation (3.42). From the first order condition one obtains the function $\phi_i^{A\,*}$, which gives,

$$\phi_i^A = \frac{a_i^A(1+\gamma^A) - \theta^A - \partial_S V_i^A(S)}{2b(1+\gamma^A)}. \tag{A.21}$$

Substituting into the HJB (3.42) leads to

$$\rho V_i^A(S) = \begin{pmatrix} \left(a_i^A - b\phi_i^A\right)\phi_i^A + \theta^A\left(\phi_i^A - \phi_j^A\right) \\ \\ +\gamma^A\left[\left(a_i^A - b\phi_i^A\right)\phi_i^A - \left(a_j^A - b\phi_j^A\right)\phi_j^A\right] \\ \\ +\dfrac{\partial V_i^A(S)}{\partial S}\left[F(S) - \phi_i^A - \phi_j^A\right] \end{pmatrix}. \tag{A.22}$$

After a series of tedious calculations, one can see that the proposed value function satisfies the differential equation. Therefore, substituting $\partial_S V_i^A(S)$ into (A.21) solves the HJB.

■

3.7.4 Free Trade Asymmetric

Proof. The structure of the proof is the same as the symmetric free trade case. For $S \in [S_{1,s}^{FT}, S_{2,s}^{FT}]$, plugging the maximized expression (3.60) into the HBJ (3.57), and applying the undetermined coefficient technique, one gets the value function

$$V_i^{FT}(S) = \frac{\alpha_i^{FT}}{2}S^2 + \beta_i^{FT}S + \mu_i^{FT}, \qquad \text{if } S \in [S_{1,s}^{FT}, S_{2,s}^{FT}], \tag{A.23}$$

where

$$\alpha_i^{FT} = -\frac{b(3+2\gamma^{FT})^2(2\delta - \delta)}{16(1+\gamma^{FT})} = \alpha^{Sym,FT}, \tag{A.24}$$

$$\beta_i^{FT} = \frac{2\delta - \rho}{16(1+\gamma^{FT})\delta(\rho - \delta)} \times$$
$$\{\rho a_i^{FT}[4(2+\gamma^{FT})\gamma^{FT} + 5] - 2\delta a_i^{FT}[(2\gamma^{FT}+5)\gamma^{FT} + 4] - 2\delta a_j^{FT}(1+\gamma^{FT})(1+2\gamma^{FT})$$
$$+a_j^{FT}\rho[4\gamma^{FT}(2+\gamma^{FT}) + 5] + 4\theta^{FT}(1+2\gamma^{FT})(r-\delta)\}, \tag{A.25}$$

and

$$\begin{aligned}\mu_i^{FT} = \frac{2}{b\rho(1+2\gamma^{FT})(3+2\gamma^{FT})^2} &\times \{(a_i^{FT})^2(1+\gamma^{FT})^2[\gamma^{FT}(2\gamma^{FT}+5)+4] - a_i^{FT}(1+\gamma^{FT}) \\ &\times \{a_j^{FT}[(2\gamma^{FT}+5)\gamma^{FT}+4] + \beta_i^{FT}[4\gamma^{FT}(2+\gamma^{FT})+5] - 4\beta_j^{FT}(1+\gamma^{FT}) - [4\gamma^{FT}(2+\gamma^{FT})+7]\theta^{FT}\} \\ &- (a_j^{FT})^2(1+\gamma^{FT})^2[\gamma^{FT}(3+2\gamma^{FT})-1] \\ &- a_j^{FT}\{\beta_i^{FT}[(4\gamma^{FT}(3+\gamma^{FT})+11)\gamma^{FT}+2] + 2\beta_j^{FT}(1+\gamma^{FT}) + (\gamma^{FT}[4\gamma^{FT}(4+\gamma^{FT})+19]+8)\theta^{FT}\} \\ &+ (1+\gamma^{FT})(1+2\gamma^{FT})(\beta_i^{FT}+\beta_j^{FT})^2 - 2\theta^{FT}\beta_i^{FT}[4\gamma^{FT}(2+\gamma^{FT})+5] \\ &+ 8\theta^{FT}\beta_j^{FT}(1+\gamma^{FT}) - 2(1+2\gamma^{FT})(\theta^{FT})^2\}. \end{aligned} \tag{A.26}$$

We must prove now that (*i*) the value function is continuously differentiable with respect to S, and that (*ii*) the policy function ϕ_i^{FT*} in (3.60) is a solution to the HJB equation (3.57).

(*i*) Proof that $V(S,\gamma^{FT},\theta^{FT})^{FT}$ is continuously differentiable in S: It is straightforward to see that the value function is clearly continuously differentiable over the intervals $[S_{1,s}^{FT}, S_{2,s}^{FT}]$, since it is a polynomial of degree 2.

(*ii*) We now prove that the policy function ϕ_i^{FT*} in (3.60) is a solution to the HJB equation (3.57). From the first order condition and solving the system formed by the strategy of both players, one obtains the function $\phi_i^{FT\,*}$, which gives,

$$\phi_i^{FT} = 2\cdot\frac{(1+\gamma^{FT})\left[2a_i^{FT}(1+\gamma^{FT}) - a_j^{FT}\right] + \theta^{FT}(1+\gamma^{FT}) - 2(1+\gamma^{FT})\partial_S V_i^{FT} + \partial_S V_j^{FT}}{b\left[4(\gamma^{FT})^2+8\gamma^{FT}+3\right]}. \tag{A.27}$$

Substituting into the HJB (3.57) leads to

$$\rho V_i^{FT}(S) = \left(\begin{array}{c} \left(a_i^{FT} - \frac{b}{2}(\phi_i^{FT}(t)+\phi_j^{FT}(t))\right)\phi_i^{FT}(t) + \theta^{FT}\left(\phi_i^{FT}-\phi_j^{FT}\right) \\ \\ +\gamma^{FT}\left[\left(a_i^{FT} - \frac{b}{2}(\phi_i^{FT}(t)+\phi_j^{FT}(t))\right)\phi_i^{FT}(t)\right. \\ \left. -\left(a_j^{FT} - \frac{b}{2}(\phi_i^{FT}(t)+\phi_j^{FT}(t))\right)\phi_j^{FT}(t)\right] \\ \\ +\frac{\partial V_i^{FT}(S)}{\partial S}\left[F(S)-\phi_i^{FT}-\phi_j^{FT}\right] \end{array}\right). \tag{A.28}$$

After a series of tedious calculations, one can see that the proposed value function satisfies the differential equation. Therefore, substituting $\partial_S V(.)$ into (A.27) solves the HJB. ■

3.7.5 Consumer Surplus

Under the asymmetric case scenario the result does not seem that straightforward. The integrated market structure of different demands is the horizontal sum of the demand of the form $Q(P)$, due to the rivalry of the good, where there could be continuous but not differentiable parts, showing segments where just one demand was in play.[21] In our case, the free trade demand could be rewritten as $Q^{FT}(P) = \frac{2B^{FT}}{b} - \frac{2}{b}P$.

Thus, as the integrated demand (FT) is the horizontal sum of demands, and there is not a point such that the demand is not differentiable (as it may happen when adding horizontal demands), that implies that the intercept in the vertical axes of both country demands is the same. Therefore, as under free trade, the two countries merge their demand, one could write it as follows:

$$\left.\begin{aligned} q_1 = \alpha - \beta_1 P \\ q_2 = \alpha - \beta_1 2 \end{aligned}\right\} \Rightarrow q_1 + q_2 = Q^{FT}$$

$$\alpha - \beta_1 P + \alpha - \beta_2 p = \frac{2B}{b} - \frac{2}{b}P \Rightarrow \left.\begin{aligned} 2\alpha = \tfrac{2B}{b} \\ \beta_1 + \beta_2 = \tfrac{2}{b} \end{aligned}\right\},$$

where α and β are the intercept and slope of the demand respectively. Remember that both demand functions under free trade cut the vertical axes at the same point, from where we obtain the slope $\alpha = \frac{B}{b}$. Moreover, as under autarky, both players share the same slope of the demand, when they merge they still share the same parameter, i.e., b, which implies that under free trade both players also have the same slope, i.e. $\beta = \frac{1}{b}$. Furthermore, it should be pointed out that both countries are populated by a large number of consumers, and the market power comes from the producer, who is extracting the resource. One can easily see now that if we sum both demands horizontally we get our free trade demand for the integrated market,

$$Q^{FT} = q_1^{FT} + q_2^{FT} = \alpha - \beta P + \alpha - \beta P = 2\alpha - 2\beta P = 2\left(\frac{B}{b}\right) - 2\left(\frac{1}{b}\right)P,$$

which is exactly the free trade demand. Thus, the consumer surplus for country i is the area enclosed by the inverse demand $P = B^{FT} - bq_i^{FT}$.

[21] See, for instance, pag. 194-199 in Pindyck and Rubinfeld (2018)

3.7.6 Producer Surplus

The difference $V^{Sym,A}(S) - V^{Sym,FT}(S)$ leads to

$$(\alpha^{Sym,A} - \alpha^{Sym,FT}) \times S^2 + (\beta^{Sym,A} - \beta^{Sym,FT}) \times S + (\mu^{Sym,A} - \mu^{Sym,FT}),$$

with

$$(\alpha^{Sym,A} - \alpha^{Sym,FT}) = -\left(\frac{b(2\delta-\rho)[4\gamma^A - 2\gamma^{FT} + 1][8\gamma^A(1+\gamma^{FT}) + 6\gamma^{FT} + 5]}{32(3+4\gamma^A)(1+\gamma^{FT})}\right),$$

$$(\beta^{Sym,A} - \beta^{Sym,FT}) = -\frac{(2\delta-\rho)}{8\delta(4\gamma^A+3)\left(\gamma^{FT}+1\right)}\left\{\mathrm{a}^{Sym,FT}(4\gamma^A+3)[4\gamma^{FT}\left(2+\gamma^{FT}\right)+5]\right.$$
$$\left. -16\mathrm{a}^{Sym,A}\left(1+\gamma^A\right)^2\left(1+\gamma^{FT}\right) - 8\theta^A(1+2\gamma^A)\left(1+\gamma^{FT}\right) + 2\theta^{FT}(3+4\gamma^A)\left(1+2\gamma^{FT}\right)\right\},$$

and

$$(\mu^{Sym,A} - \mu^{Sym,FT}) = \frac{1}{8b\delta^2\rho(4\gamma^A+3)(2\gamma^{FT}+3)^2(1+\gamma^A)^2(1+\gamma^{FT})} \times$$
$$\left\{2(a^A)^2(\gamma^A+1)^2(\gamma^{FT}+1)(2\gamma^{FT}+3)^2[2\rho(\gamma^A+1)-\delta][2\rho(\gamma^A+1)-(4\gamma^A+3)\delta]\right.$$
$$-8(2\delta-\delta)a^A\theta^A\rho(2\gamma^A+1)(\gamma^A+1)^2(\gamma^{FT}+1)(2\gamma^{FT}+3)^2$$
$$+(a^{FT})^2(4\gamma^A+3)(\gamma^A+1)^2\left(-16(\gamma^{FT}+1)^2\delta^2 + (4(\gamma^{FT}+2)\gamma^{FT}+5)^2(-\rho^2) + 2(4(\gamma^{FT}+2)\gamma^{FT}+5)^2\delta\rho\right)$$
$$-4a^{FT}(4\gamma^A+3)(\gamma^A+1)^2(2\gamma^{FT}+1)\theta^{FT}\left[-4(\gamma^{FT}+1)\delta^2 + (4(\gamma^{FT}+2)\gamma^{FT}+5)\rho^2 - 2\delta\rho(4(\gamma^{FT}+2)\gamma^{FT}+5)\right]$$
$$+2(\gamma^{FT}+1)(2\gamma^{FT}+3)^2\theta^{A2}(\delta+2\gamma^A\rho+\rho)(-(4\gamma^A+3)\delta+2\gamma^A\rho+\rho)$$
$$\left. -4(4\gamma^A+3)(\gamma^A+1)^2\theta^{FT2}(2\delta+2\gamma^{FT}\rho+\rho)(-4(\gamma^{FT}+1)\delta+2\gamma^{FT}\rho+\rho)\right\}.$$

4 Being Human: Endogenous Growth, Pollution and Natural Resources under Time Inconsistent Preferences

Abstract

How does being time-inconsistent affect economic growth, the extraction of natural resources, and pollution? In this chapter we study an endogenous growth model of the expanding variety class, with exhaustible natural resources and pollution under non-constant discounting. We study the naive agent case, who is time-inconsistent under a general discount function and tends to procrastinate. We compare the solutions obtained with a general discount function versus the canonical time-consistent exponential discounting. This self-control component leads the analysis to a Behavioral Macroeconomics problem (willpower and the planner-doer). A firm in the resource sector extracts the non-renewable natural resource needed to produce the final good. Final producer uses labor, non-renewable resource, and a different number of intermediate inputs (machines) produced by a continuum of monopolists. Both economic activity and the extraction of the resource generate pollution, which negatively affects households. We then compare the behaviors under different discount functions and the implications on the sum of discounted utilities ("welfare") under the strong observational equivalence principle. We show that time-consistent agents with constant elasticity of intertemporal substitution (CEIS) lower than one have higher levels of economic growth. However, if households have a CEIS bigger than one, the economy with time-inconsistent decision-makers has higher growth rates. Paradoxically, we find that for any CEIS level, agents behaving time-inconsistently have higher discounted utilities than time-consistent agents. This gap becomes more significant as the CEIS level increases. Finally, we observe that time-consistent agents have higher levels of economic growth with a CEIS lower than one. However, with CEIS bigger than one, time-inconsistent agents have higher growth rates.

Keywords: Endogenous growth; Behavioral Macroeconomics; Non-Renewable Resource; Non-constant Discounting; Pollution.

JEL Codes: O30; O44; C60; D91; Q30, Q50

4.1 Introduction

Questions related to the study of economic growth have been of interest to social scientists since its inception. Unsurprisingly, the field has become highly specialized. Nevertheless, it is widely assumed in Macroeconomic models that agents have time-consistent preferences, i.e., when they decide on their future plan today, they will behave accordingly to such a plan. The main contribution of this chapter is the introduction of a general discount function into a model of endogenous growth with exhaustible natural resources and pollution. Pollution is a by-product of economic activity that negatively affects agents. Does being time-inconsistent make decision-makers better off? Curiously, we observe that when individuals have tendencies to procrastinate, i.e., human behaviors according to Thaler (2015), time-inconsistent agents with a general discount function have higher levels of well-being than time-consistent ones (who discount the future under the canonical exponential framework). This result is obtained under the strong observational equivalence principle, also known as "assumption of identical overall impatience" (Strulik, 2015; Cabo et al., 2015, 2020a). Would time-consistent agents influence the resource extraction firm to extract the resource more gently than time-inconsistent players? Who will pollute more, an economy populated by time-consistent or time-inconsistent agents? Who will experience lower growth rates and end up at lower levels of development? A priori, one could expect that rational agents who are time-consistent pollute less, extract the resource more sustainably, and have higher growth rates. However, as we will show in the chapter, this is not the case. Therefore, modeling time preferences in a general setting allows us to contribute to the Behavioral Macroeconomics literature and its consequences on the development of different economies.

The economic growth literature experienced a new momentum after major breakthroughs in the 1980s and 1990s when the revolution of endogenous growth models started. Moreover, the interaction and compatibility of exhaustible resources and economic growth have been the subject of several studies. Pioneering works in this area are Solow (1974), Stiglitz (1974) and Dasgupta and Heal (1974). The story behind the big impetus and the beginning of the growth-resource literature in the 1970s was a response by Stiglitz, Solow, and other economists to the book *The Limits to Growth* published by ecologists Meadows et al. (1972), where a pessimistic Malthusian view was presented. As a matter of course, the economists mentioned above asked the natural question of what the technological conditions would be such that we avoid a decrease in per capita consumption in the long-run, as the non-renewable resource would inevitably decline. The answer given by this group of economists was founded on three pillars that showed that the decline in the exhaustible resource could be counterbalanced by *i)* input substitution, *ii)* increasing returns to scale and *iii)* resource augmenting technical progress. Nevertheless, it was in the work by Benchekroun and Withagen (2011) where the authors characterized the closed-form solution to the Dasgupta-Heal-Solow-Stiglitz (DHSS) model and explicitly derived the dynamics along the optimal trajectory of all the variables in the model and from all possible initial values of the stocks. Besides, the authors show that for some initial states, consumption may rise, reach a peak, and finally decrease. The constant positive discount rate drives this behavior.

However, the previous authors worked under a framework of exogenous growth models. One of the first researchers trying to reduce this gap was Suzuki (1976), where the author attempted to introduce endogenous technical progress together with non-renewable resources and R&D activities that absorb part of the output, which ultimately is the driver of technical progress. Later on, Chiarella (1980) built upon the previously mentioned paper endogenizing the consumption-saving decision. However, these papers studied the market solutions without monopolistic competition or externalities. An attempt to improve on the existing growth literature to date was the work published by Judd (1985), where a model of expanding product variety was first suggested. In the same decade, a set of papers changed the macroeconomics research and gave huge momentum to the new theory of endogenous growth. The pioneering work of Romer (1986), Lucas (1988) and Rebelo (1991) used previous ideas in Arrow (1962), Sheshinski (1967) and Uzawa (1965).[1] Growth theories using the incentives of R&D and imperfect competitions started with Romer (1987, 1990) together with Aghion and Howitt (1992) and Grossman and Helpman (1991)[Chapters 3-4], where the R&D activity was the main driver of the creation of new ideas (technological advances). The incentives to invest in this sector were the future monopolistic reward of a given blueprint. In our chap-

[1] As written by Helpman (2004), "[t]he key argument advanced by Arrow was that information, unlike ordinary goods, can be repeatedly used by individuals and business firms without being depleted, and that individuals and business firms cannot be excluded from the use of information that becomes public. For this reason, the benefits of new knowledge are not limited to its original creators: hence the externality."

ter, we will follow the expanding variety approach. As exemplified in Helpman (2004), "[f]or some versions of this model [expanding variety] the reduced-form equations - which describe the links between an economy's features and its rate of growth - are almost identical to a version of Romer's model, despite the differences in approach. For this reason, Romer's model of expanding product variety exhibits similar dynamics to the expanding quality [Schumpeterian] models of Grossman and Helpman and of Aghion and Howitt." In Romer (1990), he developed a model where firms invest in R&D to develop new products. A system of patents protects the details of how to produce such innovations. Thus, the inventors get monopoly power, allowing them to get higher profits. Consequently, such additional profits create incentives to invest in R&D.

A paper combining both previous ideas, endogenous growth, and exhaustible resources is the work by Scholz and Ziemes (1999), where the authors highlight "[...] indeterminacy of equilibrium trajectories arises when the Romer (1990) model is extended for exhaustible resources. Two types of inefficiencies are responsible for this result: inefficiencies owing to (i) monopolistic behavior and (ii) information spillovers." That paper is very close to our chapter on the production side of the economy.

Regarding the study of biases in intertemporal decision processes, the significance of discounting has been underscored in the introduction of this dissertation (Chapter 1). In this chapter, we investigate a naive/time-inconsistent agent under a general non-constant discount function, which according to the definitions in the introduction of the thesis, would not fit into the rational agent paradigm. For this reason, we compare the result with the exponential discounting agent (time-consistent), which would fall into the definition of rational agents.[2]

Recent papers joining both previous fields on non-constant discounting and endogenous growth are Strulik (2015), Cabo et al. (2015), Cabo et al. (2016), Cabo et al. (2020a), and Cabo et al. (2020b), where a basic AK model is used in the production side. Our framework follows the approach of the aforementioned *Ex-*

[2]As stated in the book by Kahneman (2011), "[t]he evidence presents a profound challenge to the idea that humans have consistent preferences and know how to maximize them, a cornerstone of the rational-agent model. An inconsistency is built into the design of our minds". Furthermore, as highlighted by Thaler (2015), "since my time as a graduate student, I have been preoccupied by these kinds of stories about the myriad ways in which people depart from the fictional creatures that populate economic models. It has never been my point to say that there is something wrong with people; we are all just human beings-[H]omo [S]apiens. Rather, the problem is with the model being used by economists, a model that replaces homo sapiens with a fictional creature called [H]omo [E]conomicus, which I like to call an Econ for short. Compared to this fictional world of Econs, Humans do a lot of misbehaving, and that means that economic models make a lot of bad predictions[...]." Bulging upon Thaler's comment, in this chapter we study the naive solution, this is, the human solution using his notation.

panding Variety Models.[3] In contrast, Strulik (2015) follows an approach where intermediate good firms operate under perfect competition, whereas we here consider monopolistic competition and an additional sector (resource extraction). This implies that firms can charge their markup due to their monopolistic power and the resource sector takes into account the dynamics of the resource. Furthermore, pollution in Cabo et al. (2020b) is extended to be a byproduct of the production of the final good and the extraction of the exhaustible resource. Additionally, Dugan and Trimborn (2020) study the effects of a declining social discount rate on the optimal extraction of non-renewable resources and economic growth with hyperbolic discounting. In their work, they show that resource use is more conservative under hyperbolic discounting in the medium and long-run. However, they study the Dasgupta-Heal-Solow-Stiglitz (DHSS), which is an exogenous growth model and do not consider pollution. We generalize this idea, and will obtain different results depending on the value of the constant elasticity of intertemporal substitution (CEIS). In our analysis of medium and long term behaviors, we find contrasting results for agents with a constant elasticity of intertemporal substitution (CEIS) lower than one compared to those with a CEIS greater than one. Notably, agents with non-constant discounting and a CEIS lower than one begin extracting resources more aggressively in the short-run, which differs from their long-run behavior. This increased short term extraction implies a reduced availability of natural resources for the medium and long term.

Thus, combining the previous knowledge in the literature, we study an endogenous growth model with monopolistic competition, natural resources, and pollution in conjunction with non-constant discounting. We allow pollution to be a byproduct of the economic activity and extracting the exhaustible resource. Interestingly, if one allows having a CEIS bigger than one, all the results obtained for values lower than one are reversed.

Finally, the limited amount of natural resources and pollution problems derived from economic activity provide the basis for fundamental questions of human development. These issues concern the existence of sustainable development, the growth of such (in per capita welfare terms), and how to define (and consequently measure) the good evolution of the economy as a whole. Nonetheless, before even trying to answer these questions and related inquiries, one should recall the idea of *"sustainable development"*, which was described by the Brundtland Commission, a report by the UN (1987) as "development that meets the needs of present generations without compromising the ability of future generations to meet theirs". In canonical economic terms, this is translated as a time path over which per capita

[3] See, for instance , Romer (1987, 1990) or chapter 13 inAcemoglu (2009).

"welfare" is not-decreasing along generations forever. Following Smulders (1995), "welfare" should be understood broadly as "quality of life, "living conditions", or "well-being". Thus, questions of high interest are *i)* Is sustainable development possible with non-renewable resources? or *ii)* How could we design environmental policies to improve the prospects of sustainable development or even sustainable economic growth?

This chapter is organized as follows. Section 4.2 describes the production side of the economy. Section 4.3 studies how households behave. Section 4.4 show the General (Market) Equilibrium. The numerical illustrations are carried out in Section 4.5 and the sum of discounted utilities (welfare) implications are investigated in Section 4.6.

4.2 Production Sector

The production side of our model closely follows the work by Scholz and Ziemes (1999), where the expanding variety endogenous model in Romer (1990) is extended and a new sector of non-renewable resources is introduced.

4.2.1 Representative firm of the Resource Sector

In this chapter, time s is continuous and defined by the unbounded set $s \in \mathbb{T} \equiv [t_0, \infty)$, where $t_0 \leq t \leq s$. We consider the natural resource to be an asset with well-defined property rights, abstracting from the problems of common pool resources. We follow the work by Scholz and Ziemes (1999), the resource sector initially buys the existing stocks from households and maximizes its profits. Later on, these profits generated by the single firm in the resource sector will be acquired by households, who are the owners of this sector. This extraction is later supplied to the final-good sector at a price $p_R(s)$. The dynamics of the non-renewable resource $S(t)$ at time s is

$$\frac{dS(s)}{ds} = -R(s), \tag{4.1}$$

which gives the intuition that the resource will be all extracted in the whole time horizon, which is the sum of all the extractions $R(s)$ over time,

$$\int_{t_0}^{\infty} R(s)ds \leq S(t_0). \tag{4.2}$$

As noted here, t_0 will be the initial time at which the resource sector starts extracting the exhaustible natural resource. There are no extraction costs.[4] The discounted profit of the resource extraction firm is defined as

$$\Pi^{RS} \equiv \int_{t_0}^{\infty} e^{-\int_{t_0}^{s} r(h)dh} p_R(s)R(s)ds, \tag{4.3}$$

where $r(s)$ denotes the interest rate at time s and $p_R(s)$ is the price of the resource. One can see why the firm in the resource sector does not extract the whole resource today and get rich immediately, as it has to think about its future selves and leave resources underground to extract tomorrow and keep generating profits. Therefore, the optimization problem yields the equilibrium condition

$$\gamma_{P_R} \equiv \frac{\dot{P}_R(s)}{P_R(s)} = r(s), \tag{4.4}$$

which is the well-known Hotelling rule, stating that the growth rate of the price of the resource is equal to the interest rate (see Appendix 4.8.1 for its derivation). Thus, with a given initial price at the initial time $s = t_0$, the price evolution will be given by

$$P_R(s) = P_{R,t_0} \cdot e^{\int_{t_0}^{s} r(\xi)d\xi}. \tag{4.5}$$

4.2.2 Representative Firm of the Final-Good Sector

The unique final good[5] is produced competitively at time s with the following production function:

$$Y(s) = \frac{1}{1-\beta_1-\beta_2}\left(\int_0^{N(s)} x(\nu,s)^{1-\beta_1-\beta_2}d\nu\right) L_F(s)^{\beta_1} R(s)^{\beta_2}, \tag{4.6}$$

where $L_F(s) \in (0,1)$ is the labor used to produce the final good, $R(s) \in (0, S(s))$ is the amount of resource used by the final-good producer and supplied by the resource sector, $N(s)$ is the different number varieties of inputs, and $x(\nu,s)$ is the amount of input of variety $\nu \in [0, N(s)]$ used at time s. Moreover, the elasticities of labor and exhaustible resource used in the final production are β_1 and β_2 respectively.[6] The

[4]However, this could be interpreted as a reduced price. If there were a constant marginal cost c_0 and a price P_R^{cost}, the instantaneous profit would be $(P_R^{cost} - c_0)R(s)$. One could define this difference as the price we use.

[5]In the book *Big Ideas in Macroeconomics: A Nontechnical Wiew*, Kartik B. Athreya writes, "[...] an industry of price-taking, profit-maximizing firms will *look as if* it had set out to solve the profit maximization of a fictitious single firm that embodies the entire economy's production capabilities. Of course, no one is doing any".

[6]This follows ideas in Acemoglu (2002), where the author writes "[n]otice that given [N(s)], the production functions [...] exhibit[s] constant returns to scale. There will be aggregate increasing

final good production function can be rewritten as,

$$Y(s) = \frac{1}{1-\beta_1-\beta_2}\tilde{\mathbf{X}}(s)^{1-\beta_1-\beta_2} L_F(s)^{\beta_1} R(s)^{\beta_2}, \tag{4.7}$$

where

$$\tilde{\mathbf{X}}(s) \equiv \left(\int_0^{N(s)} x(\nu,s)^{\frac{\varepsilon_\beta -1}{\varepsilon_\beta}} d\nu\right)^{\frac{\varepsilon_\beta}{\varepsilon_\beta -1}}, \tag{4.8}$$

with $\varepsilon_\beta \equiv \frac{1}{\beta_1+\beta_2}$ being the elasticity of substitution between machines. The main reason for writing the production function (4.7) in such a form is that it is possible to appreciate both parallelisms between the Dixit-Stiglitz model and the constant returns to scale of the production function, i.e., the production function is homogeneous of degree 1. The parameter ε_β can be interpreted as the elasticity of substitution between different varieties, and we assume $\varepsilon_\beta > 1$.

The exact demand of machine $x(\nu,s)$ of variety $\nu \in [0,N(s)]$ is obtained by maximising net aggregate profits of the final-good producer. The intermediate monopolist supplies such variety of machine and sets a price $p^x(\nu,s)$ at time s of machine ν to maximize her profits. This could be seen as a "rental price" or the cost of that machine ν. The final producer will also use labor with a wage rate $w(s)$, and the extracted exhaustible resource $R(s)$ for which she will pay p_R. We normalise the price of the final good at every time s to 1.

The instantaneous profits of the final producer at time s are defined by

$$\begin{aligned}\Pi^{FP} \equiv \frac{1}{1-\beta_1-\beta_2}\left(\int_0^{N(s)} x(\nu,s)^{1-\beta_1-\beta_2} d\nu\right) L_F(s)^{\beta_1} R(s)^{\beta_2} \\ - \int_0^{N(s)} p^x(\nu,s) x(\nu,s) d\nu - w(s) L_F(s) - p_R R(s).\end{aligned} \tag{4.9}$$

Thus, the final producer will maximize the discounted present value of future cash flows,

$$\begin{aligned}\max_{\left\{[x(\nu,s)]_{\nu\in[0,N(s)]}, L_F(s), R(s)\right\}} \int_{t_0}^{\infty} e^{-\int_{t_0}^{s} r(h)dh} \\ \times \left\{\frac{1}{1-\beta_1-\beta_2}\left(\int_0^{N(s)} x(\nu,s)^{1-\beta_1-\beta_2} d\nu\right) L_F(s)^{\beta_1} R(s)^{\beta_2}\right. \\ \left. - \int_0^{N(s)} p^x(\nu,s) x(\nu,s) d\nu - w(s) L_F(s) - p_R R(s)\right\} ds.\end{aligned} \tag{4.10}$$

returns, however, when [N(s) is] endogenized."

However, given that this problem does not include intertemporal elements, as it has no adjustment costs and no goods can be accumulated, the maximization of this problem is equivalent to the maximization of current profits at each point in time. Therefore, the optimization problem is given by:

$$\max_{\left\{[x(\nu,s)]_{\nu\in[0,N(s)]},L_F(s),R(s)\right\}} \Pi^{FP} \equiv \frac{1}{1-\beta_1-\beta_2}\left(\int_0^{N(s)} x(\nu,s)^{1-\beta_1-\beta_2}d\nu\right)L_F(s)^{\beta_1}R(s)^{\beta_2} - \int_0^{N(s)} p^x(\nu,s)x(\nu,s)d\nu - w(s)L_F(s) - p_R(s)R(s). \tag{4.11}$$

The first order conditions yield

$$\frac{\partial\Pi^{FP}}{\partial x(\nu,s)} \Leftrightarrow x(\nu,s)^{-\beta_1-\beta_2}L_F(s)^{\beta_1}R(s)^{\beta_2} = p^x(\nu,s), \tag{4.12}$$

$$\frac{\partial\Pi^{FP}}{\partial L_F(s)} = 0 \Leftrightarrow \beta_1\frac{Y(s)}{L_F(s)} = w(s), \text{ and} \tag{4.13}$$

$$\frac{\partial\Pi^{FP}}{\partial R(s)} = 0 \Leftrightarrow \beta_2\frac{Y(s)}{R(s)} = p_R(s), \tag{4.14}$$

where equation (4.12) is the demand for machine $\nu \in [0,N(s)]$ by the final good producer. This demand can be expressed in the following isoelastic form:

$$x(\nu,s) = p^x(\nu,s)^{-1/(\beta_1+\beta_2)}L_F(s)^{\beta_1/(\beta_1+\beta_2)}R(s)^{\beta_2/(\beta_1+\beta_2)}. \tag{4.15}$$

Therefore, the demand for machinery ν depends negatively on the price of the machine, and positively on labor and the amount of natural resources used in equilibrium, but not on the wage, the price of the natural resource or the total amount of machines used $N(s)$. Intuitively, this demand shows directly the elasticity of demand for different intermediate inputs, i.e., $\varepsilon_\beta \equiv \frac{1}{\beta_1+\beta_2}$, which would allow us to rewrite the previous expression as $x(\nu,s) = p^x(\nu,s)^{\varepsilon_\beta}L_F(s)^{\beta_1\varepsilon_\beta}R(s)^{\beta_2\varepsilon_\beta}$.

Pollution is generated by the previous two sectors, i.e., by the extraction of the non-renewable resource $R(s)$ and the production of final good $Y(s)$,

$$P(R(s),Y(s)) = R(s)^{\mu_1}\cdot Y(s)^{\mu_2}. \tag{4.16}$$

In Cabo et al. (2020a) and Cabo et al. (2020b), pollution is positively related only to the input capital K, while in our setting, we consider that extracting a non-renewable resource and producing the final good generates pollution with intensities

μ_1 and μ_2 respectively. Although one might be tempted to interpret the parameters as the elasticities of pollution, i.e., when production increases in 1%, pollution increases by μ_1%, and when the extraction of the resource increases in 1%, pollution increases by μ_2%, this will not be accurate. Note that an increase in the extraction of the resource will directly affect the final production and therefore, it will also affect pollution. Later, we will disentangle the effect of the extraction of the natural resource on pollution. The reason why we have both behaviors affecting pollution, i.e., how much the natural resource sector extracts, and how much the final producer produces, reflects the fact that pollution comes directly from the production of the final good, and from the extraction of each unit of natural resource. It is uncontroversial to assume that the mere activity of extracting a natural resource generates pollution per se. Note, however, that if one is only interested in studying the direct effect of the final production on pollution, we should "turn off" the parameter μ_2. In this chapter we decide to study the general case where extracting the natural resource and producing the final good generates pollution.

4.2.3 Representative firm of the Intermediate-Good Sector

In this subsection, we present the intermediate-good sector, responsible for the supply of machines or capital of type $\nu \in [0, N(s)]$ used by the final sector. Such a machine of variety $\nu \in [0, N(s)]$ is assumed to fully depreciate after use. This assumption ensures that "the amount of theses machines used in the past are not additional state variables"(Acemoglu, 2009, p.434), considerably simplifying the model. Thus, $p^x(\nu, s)$ could be seen as the cost of using this machine or a "rental price". Intermediate good $x(\nu, s)$ is only produced by firm ν and the design for such goods is invented/produced in the R&D sector.

Each intermediate input $\nu \in [0, N(s)]$ is produced by a monopolist who charges a price $p^x(\nu, s)$. The present discounted value of owning the blueprint of a machine type ν is given by

$$V(\nu, s) = \int_s^\infty \exp\left(-\int_s^\tau r(\xi) d\xi\right) \pi(\nu, \tau) d\tau, \tag{4.17}$$

where $r(\xi)$ is the discount rate risk-free interest rate of the monopolist. The profit of monopolist ν at time s is defined as

$$\pi(\nu, s) := p^x(\nu, s) x(\nu, s) - r x(\nu, s). \tag{4.18}$$

In this case, we assume that one unit of machine ν can be produced at marginal cost $r > 0$ units of the final good as in Scholz and Ziemes (1999). Thus, any monopolist $\nu \in [0, N(s)]$ maximizes its profits defined in equation (4.18). We also assume that the firm that discovers a new blueprint will receive a perpetual patent for this new invention, as it is normally assumed in the literature (Romer, 1990; Acemoglu, 2002). Using the demand for a machine of type v in equation (4.15) and plugging it into the monopolist profit function gives the optimal price condition:

$$p^x(v,s) = \frac{r}{1-\beta_1-\beta_2} \equiv p^x \in \mathbb{R}_+ \text{ for all } v \text{ and } s. \tag{4.19}$$

Thus, any monopolist $v \in [0, N(s)]$ charges a constant rental price which is equal to a markup over the marginal cost r. Alternatively, one can write the first order condition of the monopolist v as

$$\frac{\partial \pi(v,s)}{\partial x(v,s)} = 0 \Leftrightarrow r = (1-\beta_1-\beta_2)x(v,s)^{-\beta_1-\beta_2} L_F(s)^{\beta_1} R(s)^{\beta_2}. \tag{4.20}$$

Therefore, the profits of the monopolist producing input v can be written as

$$\begin{aligned} \pi(v,s) &= r\frac{(\beta_1+\beta_2)}{1-\beta_1-\beta_2} \cdot x(v,s) \\ &= (\beta_1+\beta_2) \cdot \underbrace{p^x}_{\in \mathbb{R}_+} \cdot x(v,s), \end{aligned} \tag{4.21}$$

which is the same for all intermediate producers. Alternatively, one could write the profits of the intermediate firm v at time s as

$$\pi(v,s) = \underbrace{(\beta_1+\beta_2)\left(\frac{r}{1-\beta_1-\beta_2}\right)^{-\frac{1-\beta_1-\beta_2}{\beta_1+\beta_2}}}_{\equiv B_0 \in \mathbb{R}_+} L_F(s)^{\beta_1/(\beta_1+\beta_2)} R(s)^{\beta_2/(\beta_1+\beta_2)}. \tag{4.22}$$

Before entering the intermediate-good market, the monopolist v needs to obtain a patent that will allow her and only her to produce good $x(v,s)$. However, the entry decision will take the form of a two-step decision process. First, the monopolist will decide if she enters the intermediate-good sector, and second, if she decides to enter, she will choose the optimal price for the blueprint v. Thus, monopolist v buys a patent at the initial time $t_0 < s$ from a firm in the R&D sector of infinite duration at a price of p_{RD} that allows her to produce the intermediate-good $x(v)$. Monopolist v issues bonds at the capital market for a price p_{RD} to finance the acquisition of a patent, which yields an interest r at each time s. If the intermediate firm v wants to produce the monopolistic machinery, she has to pay the entry cost p_{RD}. Thus, the threshold for entry into the intermediate-good market for monopolist ν is given by the condition when the price she will pay for a patent is equal to the discounted flow of profits:

$$\underbrace{\int_{t_0}^{\infty} \pi(\nu,s) e^{-\int_{t_0}^{s} r(\xi)d\xi} ds}_{V^m(\nu,s)=\text{Value Function of the Monopolist } \nu \text{ at initial time } t_0} = \underbrace{p_{RD}(t_0).}_{\text{Entry cost / Patent price}} \tag{4.23}$$

Alternatively, one could write the previous expression as

$$r(t_0)\ p_{RD}(t_0) - \frac{d}{dt_0}[p_{RD}(t_0)] = \pi(\nu,t_0), \tag{4.24}$$

which leads the free entry conditions

$$r(t_0) = \gamma_{p_{RD}} + \frac{\pi(\nu,t_0)}{p_{RD}(t_0)}. \tag{4.25}$$

Therefore, plugging the optimal profits of the monopolist ν (equation 4.22) into equation (4.17), we get the net present value of profit at time s,

$$\begin{aligned} V^m(\nu,s) &= \int_s^{\infty} e^{-\int_t^s r(\xi)d\xi} (\beta_1+\beta_2) \underbrace{\left(\frac{\psi}{1-\beta_1-\beta_2}\right)^{-\frac{1-\beta_1-\beta_2}{\beta_1+\beta_2}}}_{\equiv B_0 \in \mathbb{R}_+} \\ &\times L_F(\tau)^{\beta_1/(\beta_1+\beta_2)} R(\tau)^{\beta_2/(\beta_1+\beta_2)} d\tau, \\ &= (\beta_1+\beta_2) B_0 \int_s^{\infty} e^{-\int_t^s r(\xi)d\xi} L_F(\tau)^{\beta_1/(\beta_1+\beta_2)} R(\tau)^{\beta_2/(\beta_1+\beta_2)} d\tau. \end{aligned} \tag{4.26}$$

4.2.4 Representative firm of the R&D Sector

Before developing how new ideas are generated in this sector and how technological progress evolves, we need to explain how this process takes place. In the economics of ideas, technology or recipes are very different goods from raw materials or finished tangible goods such as a MacBook Air or a table. By technology, one should understand the “formula" or knowledge that allows companies to “combine" labor and capital to produce a product. Ideas are particular goods in economics, as they are non-rival and partially excludable as Romer (1990) pointed out. Here we follow the “growth with knowledge spillovers” approach by the previous author. However, it is crucial to bear in mind, as Scholz and Ziemes (1999) emphasize, that the introduction of exhaustible resources in the former paper generates indeterminacy of equilibrium trajectories.

In this framework, scientists and researchers (labor force L_R) are the key creators of R&D. This implies a "scarce factor" used in the development and research of new intermediate goods. Therefore, economic growth under this specification will be driven by knowledge spillovers from past R&D. This could be seen as the well-known "standing on the shoulders of Giants". Mathematically, the former (innovation possibilities frontier) takes the expression

$$\dot{N}(s) = \eta N(s) L_{RD}(s), \tag{4.27}$$

where $\eta > 0$ is constant over time and $L_{RD}(s)$ is the amount of labor assigned to R&D in a broader sense. The spillover effect is captured by the right-hand side (RHS) of the expression, as the greater the stock of ideas, the more ideas will be created. Put differently, the more knowledge available to a worker in the R&D sector, the more productive she will be.[7] This expression will be the key factor in understanding the source of endogenous growth since the spillover effects are linear or proportional.

In this chapter, we assume that the regular labor force is employed in both sectors, i.e., in the final good sector and the R&D sector. Total labor is normalized to one without loss of generality. Therefore, the corresponding market-clearing condition is

$$L_{RD}(s) + L_F(s) \leq L \equiv 1,$$

where $L_{RD}(s)$ is the amount of labor employed in the R&D sector and $L_F(s)$ is the labor used in the final sector.

Furthermore, the behavior of the R&D sector is characterized by the maximization of the following profit function

$$\pi^{RD} = p_{RD}(s) \cdot \eta N(s) \cdot L_{RD}(s) - w(s) L_{RD}(s), \tag{4.28}$$

where households supply labor to this sector and are compensated by wages $w(s)$. The price this sector charges for a produced patent is $p_{RD}(s)$.The previous profit of the R&D gives the following equilibrium condition,

$$w(s) = \eta \cdot p_{RD}(s) N(s). \tag{4.29}$$

[7] However, in this context, we should understand this creation of new ideas in a broader sense. As pointed out by Bhide (2003) in his book *The Origin and Evolution of New Business*, 72% of the ideas come from workers, 20% from people such as students and professors, and only 7% come directly from the R&D. Additionally, recent empirical papers study if new ideas are harder to find (Bloom et al., 2020) in contrast to the scheme previously exposed. This extension is left for future research.

4.3 Households

In this Section, we present the behavior of the agents in the economy. It is assumed that households have the following separable[8] CRRA instantaneous utility function,

$$U\left(c(s),P(s)\right)=\begin{cases}\frac{c(s)^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}}-\psi\frac{P(s)^{1+b}-1}{1+b} & \text{if } \sigma\neq 1\wedge b\neq -1,\\ \ln c(s)-\psi\ln P(s) & \text{if } \sigma=1\wedge b=-1,\end{cases} \tag{4.30}$$

where $c(s)$ is the level of consumption at time s and $P(s)$ is the emission of pollutant particles in the economy at time s. Additionally, $\sigma\in\mathbb{R}_+$ represents the constant elasticity of intertemporal substitution (CEIS), $\psi\in\mathbb{R}_+$ captures the environmental concern or how strongly pollution influences (damages) utility, and b shows the curvature degree of pollution. Alternatively, one could interpret this pollution/emissions as the extra pollution generated by human beings due to the economic growth experienced after the Industrial Revolution. A classical way of studying this problem in models of climate change in economics is to introduce a damage function in the utility where the damage is bigger the larger the difference between current and pre-industrial pollution levels (see Nordhaus and Boyer (2003) and Golosov et al. (2014)). Intuitively, it could be thought of as a deviation from the pre-industrial level, i.e., the anthropogenic extra pollution.

In this context, households will decide how much to consume and save. As one can observe, the more polluted the environment is, the worse for the households. Furthermore, households hold capital and bonds. Due to the structure of the model, and following Scholz and Ziemes (1999), there exist $N(s)$ monopolists, which implies that households' wealth/assets, denoted as $a(s)$, consists of N bonds multiplied by their price p_{RD}, in addition to the total demand for the capital of the N monopolists, i.e.,

$$a(s)\equiv N(s)\cdot p_{RD}(s)+N(s)\cdot\underbrace{k(\nu,s)}_{\equiv x(\nu,s)}. \tag{4.31}$$

Besides, it is assumed that capital and bonds are perfect substitutes "as stores of

[8] Grimaud and Rougé (2005) and Aghion and Howitt (1998, Chapter 5) also use separable utilities in the framework of endogenous growth models and pollution. If one considers non-separable utility, pollution will be included in the modified Ramsey rule with the time distance and we would not be able to obtain analytical results. For pedagogical reasons, we believe that it is convenient to obtain as many mathematical expressions as possible and thus be able to derive theoretical results from them.

value", implying that they must yield the same interest rate. Moreover, monopolists use capital stock to produce intermediate goods. Each monopolist ν issues bonds to finance the purchase of a patent. Consequently, total household income comes from holding bonds ($rN(s) \cdot p_{RD}$), capital income $rK(s) \equiv rN(s)k(s)$, income from the monopolistic resource sector ($p_R(s) \cdot R(s)$), and labor income, which comes from the final income sector $wL_F(s)$ and from the research and development sector $wL_{RD}(s)$. Thus, as the labor force is normalized to one, total labor income is given by wage w, i.e., $wL_F(s) + wL_R(s) = w$. On the other side, all inflow of income can be saved or consumed. Households save in capital $K(s)$ and bonds.[9]

As explained in the introduction of the dissertation, the discount function $\theta(s-t) \geq 0$, is not just a function of the time when the control (consumption) is enjoyed, i.e., at time s, but a function of the time distance from the present t (when the decision is made). Let $j \equiv s-t$ be the time distance between when consumption is enjoyed and when the decision is made. The discount function $\theta(j)$ has the following properties: $\theta(j) > 0$, $\dot{\theta}(j) < 0$ for all $j > 0$, and $\theta(0) = 1$. It is also assumed that the *instantaneous discount rate* $\rho(j) \equiv -\dot{\theta}(j)/\theta(j)$ is non-increasing with the time distance from the present (Laibson, 1997; Barro, 1999).[10] When $\rho(j)$ is constant, this ensures time-consistency in standard inter-temporal problems (Strotz, 1955). In Figure 4.1 we show several discount functions. First, we have the classical exponential discount function popularized in Samuelson (1937),

$$\theta^{Exp.}(j) = e^{-\hat{\rho} j}, \tag{4.32}$$

with a constant instantaneous discount rate $\hat{\rho} \in \mathbb{R}_{++}$.

Later we show the discount function used in Tsoukis et al. (2017), Luttmer and Mariotti (2003) and Cabo et al. (2020a)

$$\theta^{Tsouk.}(j) = (1+\delta j)^{-\varphi/\delta} e^{-\rho j}, \tag{4.33}$$

where $\rho > 0$, $0 < \varphi < 1$, $\delta > 0$ and $\varphi/\delta < 1$. It combines a "generalized hyperbolic discount function" with the exponential discount.[11]

[9] See Scholz and Ziemes (1999) for a discussion of the budget constrain.

[10] As mentioned in the introduction of the thesis, It can also be defined as the (negative) growth rate of the discount function. Intuitively, this means that the further an event is in the future, the "less important" it is for the present agent at time t. In contrast, under the standard time distance exponential discounting popularized in Samuelson (1937), where $\theta(s-t) = e^{-\rho(s-t)} = e^{-\rho j} \equiv \theta(j)$, one can observe that $\rho(j) = -\dot{\theta}(j)/\theta(j) = -\dfrac{(-\rho)e^{-\rho j}}{e^{-\rho j}} = \rho \in \mathbb{R}_{++}$ is constant in comparison to a general discount function. Put differently, the logarithmic rate of change of the discount is constant and equal to $-\rho$.

[11] As explained in Luttmer and Mariotti (2003), "[i]t generalizes the hyperbolic discount function proposed by Ainslie (1975) to interpret experiments that indicate reversals over time of preferences

One can observe that the discount function $\theta(j)$ decreases with the time distance j, $\frac{\partial}{\partial j}\left[(1+\delta j)^{-\varphi/\delta}e^{-\rho j}\right] = \rho + \frac{\varphi}{1+\delta j} < 0$ and $\frac{\partial^2}{\partial j^2}\left[(1+\delta j)^{-\varphi/\delta}e^{-\rho j}\right] < 0$. Furthermore, if φ tends to 0, this discount function converges to the exponential discount function. The instantaneous discount rate is given by $\rho(j) = \rho + \frac{\varphi}{1+\delta j}$. Rising ρ increases the discount rate at all horizons, and rising φ increases the subjective discount rate more in the short-run than at long horizons.

The third discount function showed is a "convex linear combination of two exponentials", as for instance in Karp (2007),

$$\theta^{Conv.Exp}(j) = ze^{-\rho_1 j} + (1-z)e^{-\rho_2 j}, \tag{4.34}$$

where $z \in (0,1)$ and $\rho_1 < \rho_2$, with a decreasing instantaneous discount rate $\rho^{Conv.Exp}(j) = \frac{z\rho_1 e^{-\rho_1 j} + (1-z)\rho_2 e^{-\rho_2 j}}{ze^{-\rho_1 j} + (1-z)e^{-\rho_2 j}}$. With this discount function, ρ_1 captures the level of impatience in the long-run, and ρ_2 influences the level of impatience in the short-run. Consequently, higher values of ρ_2 imply higher impatience in the short-run. This function decreases from $\rho(0) = z\rho_1 + (1-z)\rho_2$ to its level of impatience in the long-run $\lim_{j\to\infty}\rho^{Conv.Exp}(j) = \rho_1$, as $\rho 1 < \rho_2$.

However, in order to compare the standard exponential discounting with non constant-discounting, and following the ideas in Myerson et al. (2001),[12] we impose that discounting parameters are such that the discounted infinite stream gives the same present value,

$$\int_0^\infty e^{-\hat{\rho} j} dj \stackrel{!}{=} \int_0^\infty \theta(j)dj, \quad \Longleftrightarrow \quad \hat{\rho} = \left[\int_0^\infty \theta(j)dj\right]^{-1}. \tag{4.35}$$

This is also known as the "strong observational equivalence principle" or the "assumption of identical overall impatience". The previous discount functions are shown in Figure 4.1, where as pointed out in Cabo et al. (2020a), "under non-constant discounting the short-run is less valued and the long-run more valued than under exponential discounting". This result comes from the fact that the instantaneous discount rate $\rho(j)$ is large at first and decreases with the time distance, jointly with the equivalence principle in (4.35). From now on, when we refer to the exponential discount rate, we refere to $\hat{\rho}$. Moreover, they also claim that equation (4.35) is important, as "in order to compare non-constant discounting with the standard results with exponential discounting, we need to guarantee that the dissimilarities are not due to different degrees of impatience [see, for example, Strulik (2015); Cabo et al. (2015)]."

for rewards at different horizons."

[12]See also Strulik (2015), Cabo et al. (2015), Cabo et al. (2020a) and Mañó-Cabello et al. (2021).

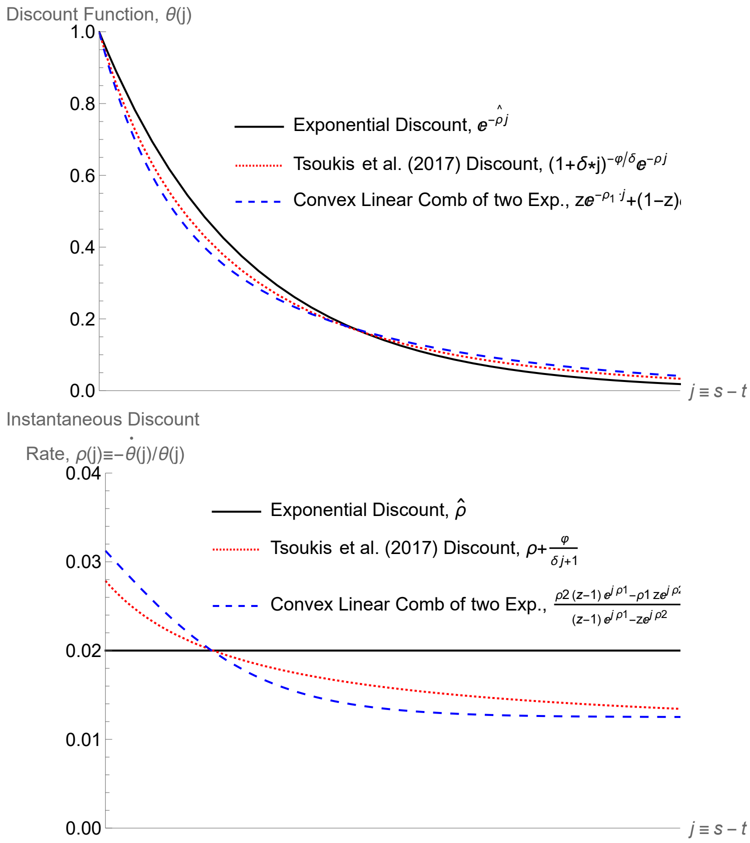


Figure 4.1: Discount Functions $\theta(j)$ and Instantaneous Discount Rate $\rho(j) \equiv -\dot{\theta}(j)/\theta(j)$.

The naive agent makes the decision at every (initial) time t. The optimization problem is,

$$\begin{aligned} &\underset{\{c_t(s)\}}{\text{Max}} && \int_t^{\infty} \theta(s-t)\, U\left(c_t(s), P_t(s)\right) ds \\ &\text{s.t.} && \dot{a}_t(s) = r_t(s) \cdot a_t(s) + w_t(s) + p_{R_t}(s) \cdot R_t(s) - c_t(s) \\ & && a_t(t) = a_t \in \mathbb{R}_+, \end{aligned} \tag{4.36}$$

where wealth or savings $a(s)$ was defined in equation (4.31). We now compute the Naive Solution of the problem. This is also known in the literature as t-agent problem. As pointed out by Marín-Solano and Navas (2009), "[n]aive t-agent will solve [the] problem [...] as a standard optimal control problem, using Pontryagin's Maximum Principle".[13] Henceforth, superscript N will denote naive consumers.

[13] See also Strulik (2015) and Cabo et al. (2015).

4.3.1 CES utility function

The Hamiltonian for the general CES function ($\sigma \neq 1$) is given by:

$$\begin{aligned}\mathscr{H}^N(s,t) &= \theta(s-t)\left[\frac{c(s)^{1-\frac{1}{\sigma}}-1}{1-\frac{1}{\sigma}} - \psi\frac{P(s)^{1+b}-1}{1+b}\right]\\ &+\lambda_t(s)\left[r_t(s)a_t(s)+w_t(s)+p_{R_t}(s)*R_t(s)-c_t(s)\right]\end{aligned} \tag{4.37}$$

The optimal consumption behavior is (see Appendix (4.8.2) for its derivation),

$$c_t^N(t) = \left(\frac{a_t+\int_t^\infty [w_t(s)+p_R(s)*R(s)]\cdot\exp\{-\int_t^s r_t(\xi)d\xi\}ds}{\int_t^\infty \theta(s-t)^\sigma\cdot\exp\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\}ds}\right). \tag{4.38}$$

The solution is written in feedback form. This result is similar to the one obtained in Cabo et al. (2015), but we now consider a broader income that also comes from the resource sector. Expression (4.38) represents the optimal consumption of short-sighted households under non-constant discounting. The numerator captures the total wealth (physical and human) $TW(s)$, and the denominator represents the inverse of the propensity to consume (MPc) out of total wealth (Farzin and Wendner, 2014). Thus, the consumption strategy looks very simple, $c_t^N(t) = \frac{1}{MPc}\times TW(s)$. Implicitly, one can see the linear Markovian strategy, i.e., consumption (control) is a linear function of the wealth (state). Taking natural logs on both sides, differentiating with respect to initial time t (when the decision is made), and making use of the Leibniz rule, one gets the following modified Ramsey rule for naive agents:

$$\frac{\dot{c}^N(t)}{c^N(t)} = \sigma\left[r_t - \frac{\int_t^\infty \theta(s-t)^\sigma \overbrace{\left[-\frac{\frac{d\theta(s-t)}{dt}}{\theta(s-t)}\right]}^{\equiv\rho(j)}\cdot\exp\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\}ds}{\int_t^\infty \theta(s-t)^\sigma \exp\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\}ds}\right]. \tag{4.39}$$

Taking into account the the temporal distance between when the action occurs (s) and when the decision is made (t), is denoted as $j\equiv s-t$, and considering that the instantaneous discount rate is $\rho(j)\equiv -\dot{\theta}(j)/\theta(j)$, the previous expression can be reformulated as

$$\frac{\dot{c}^N(t)}{c^N(t)} = \sigma\left[r_t - \frac{\int_0^\infty \rho(j)\theta(j)^\sigma \exp\left\{-(1-\sigma)\int_t^{t+j} r_t(\xi)d\xi\right\}dj}{\int_0^\infty \theta(i)^\sigma \exp\left\{-(1-\sigma)\int_t^{t+i} r_t(\xi)d\xi\right\}di}\right]. \tag{4.40}$$

Hence,

$$\gamma_c^{Na,ces} \equiv \frac{c^{\dot{Na},ces}(t)}{c^{Na,ces}(t)} = \sigma\left[r_t - \lambda_{Na,\theta}^{ces}\right], \tag{4.41}$$

where

$$\lambda_{Na,\theta}^{ces} = \frac{\int_0^\infty \rho(j)\theta(j)^\sigma \exp\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\}dj}{\int_0^\infty \theta(i)^\sigma \exp\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\}di}. \tag{4.42}$$

This term $\lambda_{Na,\theta}^{ces}$ is the *effective rate of time preference* and it is constant. It could be interpreted as a "weighted mean of the instantaneous discount rates $\rho(j)$" (Cabo et al., 2015), or as in Barro (1999), "weighted average of the instantaneous rates of time preference".[14] It is worth noting that effective rates of time preference $\lambda_{Na,\theta}^{ces}$ are specific to each discount function $\theta(s-t)$. The weights are captured by $\omega_N(j)$ with $\int_0^\infty \omega_N(j)dj = 1$, which can be seen as

$$\lambda_{Na,\theta}^{ces} = \int_0^\infty \rho(j)\underbrace{\frac{\theta(j)^\sigma \exp\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\}}{\int_0^\infty \theta(i)^\sigma \exp\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\}di}}_{\equiv \omega_N^{ces}(j)}dj. \tag{4.43}$$

Thus, the effective rate of time preference is a weighted mean with expression

$$\lambda_{Na,\theta}^{ces} = \int_0^\infty \rho(j)\omega_N^{ces}(j)dj. \tag{4.44}$$

If one plugs in an exponential discount function, we recover the canonical Ramsey rule where $\lambda_{Na,\theta}^{ces} = \hat{\rho}$. Furthermore, if one considers a constant interest rate, the household problem in Cabo et al. (2015) is recovered (even if one considers pollution and income from the resource sector). Furthermore, one could interpret equation (4.41) rearranging it as follows,

$$\lambda_{Na,\theta}^{ces} + \frac{1}{\sigma}\frac{\dot{c}^N(t)}{c^N(t)} = r_t, \tag{4.45}$$

where the left-hand side (LHS) captures the benefit derived by consumption and $\lambda_{Na,\theta}^{ces}$ expresses the increase in utility derived by consuming now and not postpon-

[14] Luttmer and Mariotti (2003) also call this the "weighted average of the subjective discount rates".

ing consumption to the "next period". Moreover, the term $\dfrac{1}{\sigma}\dfrac{\dot{c}^N(t)}{c^N(t)}$ captures the desire by households to smooth consumption.[15] Furthermore, the RHS term is the yield or net benefit obtained from saving. Thus, this expression shows the threshold (margin) where agents would be indifferent between consuming or saving.

Proposition 4.1. *Under CES utility, for $\sigma < 1$ ($\sigma > 1$) we have $\lambda^{ces}_{Na,\theta} > \hat{\rho}$ ($\lambda^{ces}_{Na,\theta} < \hat{\rho}$).*

Proof. See Appendix 4.8.4. ■

This shows that working with a general utility function of the CES form leads to rich and different results, capturing the fact that agents are time-inconsistent.

4.3.2 Logarithmic utility function

Finally, for the logarithmic utility function case, following the same procedure, one gets the consumption strategy $c_t^{N,log}(t)$,

$$c_t^{N,log}(t) = \left(\frac{a_t + \int_t^\infty [w_t(s) + p_R(s) \cdot R(s)] \cdot \exp\{-\int_t^s r_t(\xi)d\xi\} ds}{\int_t^\infty \theta(s-t)ds}\right), \tag{4.46}$$

which gives place to the following modified Ramsey rule for the logarithmic case in consumption and pollution,

$$\frac{\dot{c}^{Na,log}(t)}{c^{Na,log}(t)} = r_t - \underbrace{\frac{\int_0^\infty \rho(j)\theta(j)dj}{\int_0^\infty \theta(i)di}}_{\equiv \lambda^{log}_{Na,\theta}}. \tag{4.47}$$

Observe that the modified Ramsey rule for the logarithmic case could also be interpreted as a weighted sum since $\lambda^{log}_{Na,\theta}$ can be expressed as

$$\lambda^{log}_{Na,\theta} = \int_0^\infty \rho(j) \underbrace{\frac{\theta(j)}{\int_0^\infty \theta(i)di}}_{\omega_N^{log}(j)} dj = \int_0^\infty \rho(j)\omega_N^{log}(j)dj, \tag{4.48}$$

which gives the opportunity to rewrite expression (4.47) as

[15] When $\frac{\dot{c}^N(t)}{c^N(t)} > 0$, households would prefer to increase consumption today. Since households have the desire to smooth consumption, if they know that in the future they will consume more ($\dot{c}^N(t)/c^N(t) > 0$), driven by the behavior force to smooth their consumption, they will have incentives to increase consumption today.

$$\gamma_c^{N,log} \equiv \frac{\dot{c}^{N,log}(t)}{c^{N,log}(t)} = r_t - \lambda_{Na,\theta}^{log}. \tag{4.49}$$

As pointed out in Barro (1999), if agents have logarithmic utility, then, $\lambda_{Na,\theta}^{log}$, the "weighted mean of the instantaneous discount rates $\rho(j)$", or "weighted average of the instantaneous rates of time preference" for any general discount rate $\theta(j)$ is exactly $\left[\int_0^\infty \theta(i)di\right]^{-1} = \hat{\rho}$, where $\hat{\rho}$ is given by the equivalent present value expression (4.35). Thus, under a logarithmic utility function, the problem with constant and non-constant discounting collapses to the same result (Marín-Solano and Navas, 2010; De-Paz et al., 2014, 2013).

Lemma 1. *Under logarithmic utility ($\sigma = 1$),*

$$\lambda_{Na,\theta}^{log} = \hat{\rho}. \tag{4.50}$$

Proof. See Appendix 4.8.3. ■

This result shows that under logarithmic utilities, the effective rate of time preference collapses to the constant discount parameter of the exponential function. Thus, time-consistent (exponential) and time-inconsistent agents behave the same. This is what Barro (1999) called "observationally equivalent" albeit within the context of a neoclassical growth model that does not incorporate endogenous growth. Consequently, all the rich and interesting behavior that occurs under general CES utilities is lost when working with logarithmic utilities.

On the contrary, the effective rate of time preference $\lambda_{Na,\theta}^{ces}$ can be bigger or smaller than the constant discount rate of the exponential function $\hat{\rho}$ under a general CES utility, as shown in Proposition 4.1.

4.4 General (Market) Equilibrium

In the economy described so far, an allocation is defined by the time paths of consumption, total number of varieties, and the extraction of the natural resource, $[C(s), N(S), R(s)]_{s=t}^{\infty}$; time paths of quantities of each intermediate good and prices, $[x(\nu,s), p^x(\nu,s)]_{s=t}^{\infty}$; and the time paths of wages, interest rate, and price of the natural resource, $[w(s), r(s), p_R(S)]_{s=t}^{\infty}$.

Furthermore, the equilibrium $\mathscr{E}$ in this economy is defined by an allocation previously defined where all monopolists choose their quantities and price of the intermediate machine $[x(\nu,s), p^x(\nu,s)]_{s=t}^{\infty}$ to maximize their discounted profits; the evolution of prices clean the market $[w(s), r(s), p_R(S)]_{s=t}^{\infty}$; the evolution of the number of varieties $[N(s)]_{s=t}^{\infty}$ is determined by the free entry condition; and the evolution of $[C(s), R(s)]_{s=t}^{\infty}$ is consistent with the household and resource problem.[16] Thus, one could write the equilibrium as

$$\mathscr{E} = [x(\nu,s), p^x(\nu,s), w(s), r(s), p_R(S), N(s), C(s), N(S), R(s)]_{s=t}^{\infty}. \tag{4.51}$$

Following Scholz and Ziemes (1999), we assume that monopolists have the following capital production

$$x(\nu,s) = k(\nu,s) \text{ for all } \nu \in [0, N(s)] \text{ and } s, \tag{4.52}$$

which describes a linear relationship between output of the monopolist ν, $x(\nu,s)$, and the input man-made capital, $k(\nu,s)$. Substituting the demand for machinery (4.15), machine prices and the production function of the monopolist (4.52) into the production function (4.6) yields

$$Y(s) = \frac{1}{1-\beta_1-\beta_2}\left(\frac{r}{\underbrace{1-\beta_1-\beta_2}_{p^x\in\mathbb{R}_+}}\right)^{-\frac{1-\beta_1-\beta_2}{\beta_1+\beta_2}} N(s)L_F(s)^{\frac{\beta_1}{\beta_1+\beta_2}}\, R(s)^{\frac{\beta_2}{\beta_1+\beta_2}}. \tag{4.53}$$

Alternatively, the final production function can be rewritten as

$$Y(s) = \frac{1}{1-\beta_1-\beta_2} K(s)^{1-\beta_1-\beta_2}\,[N(s)L_F(s)]^{\beta_1}\; [N(s)R(s)]^{\beta_2}, \tag{4.54}$$

where $K(s) \equiv N(s)\, x(\nu,s) = N(s)\, k(\nu,s)$. From this expression, it can be seen that when $N(s)$ increases, the productivity of labor and the productivity of the resource increase.

4.4.1 Properties of the Balanced Growth Path

We now study the Balanced Growth Path (BGP). Using Hotelling's rule (4.4) from the natural resource sector, the resource FOC of the final producer equation (4.14), and the FOC of the monopolist ν equation (4.20), together with the growth rate for the prices of the resource from (4.14) and then equating the resulting expression with the Hotelling rule, one gets

[16] Due to the market power of the monopolist, the equilibrium is not competitive.

$$\underbrace{r(s) = \gamma_{P_R}}_{\text{Hotelling equation (4.4)}} = \gamma_Y - \gamma_R.$$

Making use of equation (4.20), and the definition of $K(s)$, the following condition holds,

$$\underbrace{\gamma_Y - \gamma_R}_{r} = \underbrace{(1-\beta_1-\beta_2)}_{\text{Inefficiency terms}} K(s)^{-\beta_1-\beta_2}\left[N(s)L_F(s)\right]^{\beta_1}\left[N(s)R(s)\right]^{\beta_2}. \tag{4.55}$$

From the final production function (4.54), one gets the expression for the economy's marginal productivity of total capital,

$$\frac{\partial Y(s)}{\partial K(s)} = K(s)^{-\beta_1-\beta_2}\left[N(s)L_F(s)\right]^{\beta_1}\left[N(s)R(s)\right]^{\beta_2}. \tag{4.56}$$

As pointed out by the Scholz and Ziemes (1999, p.176), the market solution violates the Solow-Stiglitz condition, that is, $\gamma_Y - \gamma_R = K(s)^{-\beta_1-\beta_2}\left[N(s)L_F(s)\right]^{\beta_1}\left[N(s)R(s)\right]^{\beta_2}$, where the RHS would correspond to the interest rate, and marginal productivity of capital in the social planner problem. One can notice that there is a mismatch between the efficient solution and condition (4.55), diminished by the term $(1-\beta_1-\beta_2)$. As discussed by the mentioned authors, "[t]his inefficiency is caused by the monopolistic sector that pays an interest rate lower than the marginal productivity of capital". Thus,

$$\underbrace{\gamma_{P_R} = r = \gamma_Y - \gamma_R}_{\text{Marginal Productivity of the Resource}} < \underbrace{K(s)^{-\beta_1-\beta_2}\left[N(s)L_F(s)\right]^{\beta_1}\left[N(s)R(s)\right]^{\beta_2}}_{\text{Marginal Productivity of Capital}} \tag{4.57}$$

The economic interpretation of equation (4.57) is as follows. The LHS represents the additional production obtained by not extracting one additional unit of the resource today and extracting it in the future. The RHS shows the additional production that could be reached in the future by obtaining an additional unit of the resource today. This extra extraction would be used in production, and adding this extra unit to the capital stock will increase the output in the future.

Proposition 4.2. *The equilibrium conditions of the BGP for the Naive agent with CES utility are given by*

$$\gamma_{\frac{c}{K}}^{Na,ces} = \frac{Y(s)}{K(s)}\left[\sigma(1-\beta_1-\beta_2)^2 - 1\right] - \sigma\lambda_{Na,\theta}^{ces} + \frac{c^{Na,ces}(s)}{K(s)}, \tag{4.58}$$

$$\gamma_R^{Na,ces} = (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c^{Na,ces}(s)}{K(s)} + \eta(1-L_F(s))\frac{\beta_2}{1-\beta 1-\beta_2} + \eta L_F(s)(\beta_1+\beta_2), \tag{4.59}$$

$$\gamma_{L_F}^{Na,ces} = (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c^{Na,ces}(s)}{K(s)} + \eta(1-L_F(s))\left[\frac{\beta_2-(1-\beta_1-\beta_2)}{1-\beta_1-\beta_2}\right] + \eta L_F(s)\frac{(\beta_1+\beta_2)(1-\beta_2)}{\beta_1}, \tag{4.60}$$

$$\gamma_{\frac{Y}{K}}^{Na,ces} = (\beta_1+\beta_2)\eta L_F(s) - (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{K(s)} + \frac{\beta_2}{1-\beta_1-\beta_2}\eta\,(1-L_F(s)). \tag{4.61}$$

Proof. See Appendix 4.8.5. ∎

Corollary 4.1. *The equilibrium conditions of the BGP for the Naive agent with LOG utility are given by*

$$\gamma_{\frac{c}{K}}^{Na,log} = \frac{Y(s)}{K(s)}\left[(1-\beta_1-\beta_2)^2-1\right] - \lambda_{Na,\theta}^{log} + \frac{c^{Na,log}(s)}{K(s)}, \tag{4.62}$$

$$\gamma_R^{Na,log} = (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c^{Na,log}(s)}{K(s)} + \eta(1-L_F(s))\frac{\beta_2}{1-\beta 1-\beta_2} + \eta L_F(s)(\beta_1+\beta_2), \tag{4.63}$$

$$\gamma_{L_F}^{Na,log} = (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c^{Na,log}(s)}{K(s)} + \eta(1-L_F(s))\left[\frac{\beta_2-(1-\beta_1-\beta_2)}{1-\beta_1-\beta_2}\right] + \eta L_F(s)\frac{(\beta_1+\beta_2)(1-\beta_2)}{\beta_1}, \tag{4.64}$$

$$\gamma_{\frac{Y}{K}}^{Na,log} = (\beta_1+\beta_2)\eta L_F(s) - (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{K(s)} + \frac{\beta_2}{1-\beta_1-\beta_2}\eta\,(1-L_F(s)). \tag{4.65}$$

Proof. It is straightforward to derive the results when $\sigma \to 1$ and together with its corresponding effective rate of time preference (or weighted average of the instantaneous rates of time preference) $\lambda_{Na,\theta}^{log}$, which collapses to $\hat{\rho}$ for the logarithmic case. ■

At every moment, the BGP for this economy is described by the system of differential equations just described for both utilities $h \in \{log, ces\}$. Jointly solved, these equations give information of the steady state and properties of its stability.

4.4.2 Properties of the Steady State

In the steady state, consumption $c_N^h(s)$, final output $Y(s)$ and aggregate capital $K(s)$ grow at the same constant rate, γ.[17] Labor used in the final production sector L_F and capital productivity remains constant, i.e., their growth rates are zero in the steady state. One could write the amount of the labor force used in the final sector in the steady state as $\lim_{s\to\infty} L_F(s) \equiv L_F^*$. Thus,

$$\gamma = \gamma_K = \gamma_c = \gamma_Y. \tag{4.66}$$

Proposition 4.3. *In the steady state, when agents are Naive and have CES utility, the equilibrium labor force used in the final good sector $L_F^{Na,ces}$, the growth rate of the extraction of the exhaustible resource $\gamma_R^{Na,ces}$, the growth rate of the intermediate inputs $\gamma_N^{Na,ces}$, and the growth rate of the entire economy $\gamma^{Na,ces}$ are given by*

$$L_F^{Na,ces*} = \frac{\beta_1 \left[\eta + \lambda_{Na,\theta}^{ces} - \left(\frac{\sigma-1}{\sigma}\right)\gamma^{Na,ces}\right]}{\eta\left[\beta_1 + (\beta_1+\beta_2)(1-\beta_1-\beta_2)\right]}, \tag{4.67}$$

$$\gamma_R^{Na,ces} = \eta(1-L_F^{Na,ces*}) - \eta L_F^{Na,ces*}\frac{(\beta_1+\beta_2)(1-\beta_1-\beta_2)}{\beta_1}, \tag{4.68}$$

$$\gamma_N^{Na,ces} = \eta - \left(\frac{\beta_1 \left[\eta + \lambda_{Na,\theta}^{ces} - \left(\frac{\sigma-1}{\sigma}\right)\gamma^{Na,ces}\right]}{\beta_1 + (1-\beta_1-\beta_2)(\beta_1+\beta_2)}\right), \tag{4.69}$$

$$\gamma^{Na,ces} = \sigma \cdot \frac{\lambda_{Na,\theta}^{ces}\left\{\beta_1^2(1-\beta_2) + \beta_1\beta_2(3-2\beta_2) + (1-\beta_2)\beta_2^2\right\} - \eta(1-\beta_1-\beta_2)(\beta_1+\beta_2)^2}{\beta_1^2\left[\beta_1\sigma + \beta_2(2\sigma+1) - (\sigma+1)\right] + \beta_1\beta_2\left[\beta_2(\sigma+2)-3\right] - \beta_2^2(1-\beta_2)}. \tag{4.70}$$

Proof. See Appendix 4.8.6. ■

Corollary 4.2. *In the steady state, when agents are Naive and have LOG utility, the equilibrium labor force used in the final good sector $L_F^{Na,log}$, the growth rate of the*

[17]This was noted in Kaldor (1961), where he observed that output per capita increases, while the ratio capital/output remains practically constant. This is known as the Kaldor facts.

extraction of the exhaustible resource $\gamma_R^{Na,log}$*, the growth rate of the intermediate inputs* $\gamma_N^{Na,log}$*, and the growth rate of the entire economy* $\gamma^{Na,log}$ *are given by*

$$L_F^{Na,log*} = \frac{\beta_1 \left[\eta + \lambda_{Na,\theta}^{log}\right]}{\eta \left[\beta_1 + (\beta_1 + \beta_2)(1 - \beta_1 - \beta_2)\right]}, \tag{4.71}$$

$$\gamma_R^{Na,log} = \eta(1 - L_F^{Na,log*}) - \eta L_F^{Na,log*} \frac{(\beta_1 + \beta_2)(1 - \beta_1 - \beta_2)}{\beta_1}, \tag{4.72}$$

$$\gamma_N^{Na,log} = \eta - \left[\frac{\beta_1 \left[\eta + \lambda_{Na,}^{log}\right]}{\beta_1 + (\beta_1 + \beta_2)(1 - \beta_1 - \beta_2)}\right], \tag{4.73}$$

$$\gamma^{Na,log} = \eta + \frac{\beta_1(\eta + \lambda_{Na,}^{log})}{\beta_1^2 - 2\beta_1(1 - \beta_2) - (1 - \beta_2)\beta_2} - \frac{\beta_2 \lambda_{Na,}^{log}}{\beta_1 + \beta_2}. \tag{4.74}$$

Proof. It is easy to prove it when $\sigma \to 1$ and together with its corresponding effective rate of time preference (or weighted average of the instantaneous rates of time preference) $\lambda_{Na,\theta}^{log}$. ■

Thus, plugging in the corresponding growth rates $\gamma^{Na,h}$ into $L_F^{Na,h*}$, $\gamma_N^{Na,h}$ and $\gamma_R^{Na,h}$, for $h \in \{ces, log\}$, gives their corresponding steady state values. However, it is not possible to make a general statement on the sign of the growth rate of the economy as a whole.

Furthermore, since the interest rate is constant in the steady state, one can take logs in expression (4.20) and differentiate with respect to time, and noting that the growth rate of the labor force used in the final good sector in the steady state is zero, it leads to

$$(\beta_1 + \beta_2) \cdot \gamma_{x(\nu,s)} = \beta_2 \cdot \gamma_R, \tag{4.75}$$

which is valid for both types of utilities $h \in \{ces, log\}$. From this expression, one can notice that the growth rate of the extraction will decline in a framework with exhaustible resources (i.e., the RHS will decline over time), which, in turn, implies a decline in the LHS. Consequently, the growth rate of the demand for a specific intermediate good $\nu \in [0, N]$ will decline over time. It is essential to note that this decline pertains only to the particular demand for a given intermediate input, rather than the total quantity of intermediate inputs, as the latter will increase over time.

Remark 4.1. *The declining in the extraction rate of the resource (expressed by* γ_R*) is proportional to the decline in the demand for intermediate good* $\nu \in [0, N]$

(expressed by γ_x),

$$\left(\frac{\beta_2}{\beta_1+\beta_2}\right)\gamma_R = \gamma_x. \tag{4.76}$$

Therefore, all monopolists ν, knowing that their demand for their production will decline, will decrease their production to keep prices $p^x(\nu,s) = p^x \in \mathbb{R}_+$ constant over time, as shown in equation (4.19). Plugging (4.75) into the growth rate of the final production given in (A.29), we get the growth rate of the economy $\gamma^{i,h}$ as the sum of the growth rate of the intermediate inputs γ_N, which is positive, and the growth rate of the production of the monopolist ν, $\gamma_{x(\nu,s)}$, which is negative:

$$\gamma^h = \underbrace{\gamma_N^h}_{>0} + \underbrace{\gamma_{x(\nu,s)}^h}_{<0} \text{ for all } h \in \{ces, log\}. \tag{4.77}$$

Assumption 4.1. *The growth rate of the entire economy in the steady state will be positive.*

Assumption 4.1 implies that the rate of new entrants (new monopolists) in the market ($\gamma_N^{j,h}$) is greater than the rate of decrease in the supply of a monopolist. Consequently, there will be growth as long as new monopolists enter the market to compensate for the decrease in supply by the old monopolists. If this will not be the case, then the growth rate in the BGP will be negative and the economic activity will disappear in the long-run. From equation 4.77,

$$\gamma^h > 0 \iff \underbrace{\gamma_N^h}_{>0} > -\underbrace{\gamma_{x(\nu,s)}^h}_{<0} \text{ for all } h \in \{ces, log\}. \tag{4.78}$$

We now focus on the evolution of the natural resource. Knowing that the growth rate of the extraction $R(s)$ is given by expressions (4.68) for the naive agent with CES utility and (4.72) with log utility, we can solve the corresponding differential equations. For the sake of simplicity, let us define the RHD of both equations as the constants $\gamma_R^{Na,ces} = \eta(1-L_F^{Na,ces*}) - \eta L_F^{Na,ces*}\dfrac{(\beta_1+\beta_2)(1-\beta_1-\beta_2)}{\beta_1} < 0$ and $\gamma_R^{Na,log} = \eta(1-L_F^{Na,log*}) - \eta L_F^{Na,log*}\dfrac{(\beta_1+\beta_2)(1-\beta_1-\beta_2)}{\beta_1} < 0$, which are negative due to the dynamics of the exhaustible resource. Thus, the corresponding solutions of the differential equations lead to

$$R_{Na}^h(s) = \underbrace{R_0}_{\in\mathbb{R}_+} \exp\left\{\underbrace{\gamma_R^{Na,h}}_{<0} \cdot s\right\}, \text{ for } h \in \{ces, log\}. \tag{4.79}$$

Having the evolution of the extraction strategy $R(s)$, we can plug it into the dynamics of the non-renewable resource (4.1) and solve the corresponding differential equation. This process results in the evolution of the natural resource, with an initial quantity of $S(0) = S_0$ given. Then,

$$S^h_{Na}(s) = \underbrace{\left(\frac{\gamma_R^{Na,h} \cdot S_0 + R_0}{\gamma_R^{Na,h}}\right)}_{\text{constant}} - \frac{R_0}{\gamma_R^{Na,h}} e^{\left(\gamma_R^{Na,h} \cdot s\right)}. \tag{4.80}$$

In order to fulfill the transversality condition (TVC) of the natural resource, i.e., that the resource will be all exhausted over the whole time horizon, $\lim_{s\to\infty} S(s) = 0$, one gets the condition

$$R_0 = \underbrace{-\gamma_R^{Na,h} \cdot S_0}_{>0}. \tag{4.81}$$

Thus the exhaustible resource evolves as

$$S^h_{Na}(s) = S_0 e^{\gamma_R^{Na,h} \cdot s} \text{ for } h \in \{log, ces\}. \tag{4.82}$$

Consequently, the extraction rate $R(s)/S(s)$ is

$$\frac{R^h_{Na}(s)}{S^h_{Na}(s)} = -\underbrace{\gamma_R^{Na,h}}_{<0} \text{ for } h \in \{log, ces\}. \tag{4.83}$$

Remark 4.2. *The extraction of the resource follows a Markovian strategy,*[18]

$$\phi^h_{Na}(S(s)) \equiv R^h_{Na}(s) = -\gamma_R^{Na,h} S^h_{Na}(s), \quad \textit{for } h \in \{log, ces\}. \tag{4.84}$$

See the simulations in the qualitative analysis Section for an easy visualization of the results. It can be proved that for the logarithmic case, the negative instantaneous discount rate ($\hat{\rho}$) of the exponential discount function coincides with the growth rate of the extraction of the resource $\gamma_R^{Na,log}$, which also coincides with the negative "effective rate of time preference" $\lambda^{log}_{Na,\theta}$. However, this is not the case for the CES utility scenario.

Proposition 4.4. *For the naive agent, the growth rate of the extraction of the resource under different utilities and any arbitrary discount function $\theta(j)$ along the*

[18]However, as highlighted by Groth (2015), "along any economic development path, the aggregate input of non-renewable resources must in the long-run asymptotically approach zero. From a physical point of view, however, there must be some minimum amount of the resource below which it can not fulfill its role as a productive input. Thus, strictly speaking, sustainability requires that in the "very long run", non-renewable resources become inessential".

BGP is given by

$$\gamma_R^{Na,h} = \gamma^{Na,h} \cdot \left(\frac{\sigma_h - 1}{\sigma_h}\right) - \lambda_{Na,\theta}^h \,, \tag{4.85}$$

where

$$\sigma_h = \begin{cases} \sigma & \text{if } h = ces, \\ 1 & \text{if } h = log. \end{cases} \tag{4.86}$$

Proof. See equation (A.30) in the Appendix. ■

Remark 4.3. *For the naive agent with logarithmic utility and any discount function $\theta(s-t)$ along the BGP, the Markovian extraction rate collapses to the same strategy,*

$$\phi_{Na}^{log}(S(s)) \equiv R_{Na}^{log}(s) = -\gamma_R^{Na,log} S_{Na}^{log}(s) = \hat{\rho} \cdot S_{Na}^{log}(s). \tag{4.87}$$

Proof. All that needs to be proved is that $\gamma_R^{Na,log} = -\hat{\rho}$. See Appendix (4.8.7). ■

Next, we prove that agents have a higher (lower) extraction rate under CES (log) utility with a CEIS $\sigma < 1$ ($\sigma = 1$). However, if we consider the CES case with CEIS $\sigma > 1$, we get lower extraction rate than under the log case ($\sigma = 1$). Therefore, this captures the fact that the greater the CEIS, the lower the extraction rate.

Proposition 4.5. *For the naive agent, if $\sigma < 1$ ($\sigma > 1$) she has a higher (lower) extraction rate under the general CES utility function than under the log case ($\sigma = 1$),*

$$\begin{aligned} &\textit{For } \sigma < 1 \;\Leftrightarrow \frac{R_{Na}^{log}(s)}{S_{Na}^{log}(s)} < \frac{R_{Na}^{ces}(s)}{S_{Na}^{ces}(s)} \\ &\textit{For } \sigma > 1 \Leftrightarrow \frac{R_{Na}^{log}(s)}{S_{Na}^{log}(s)} > \frac{R_{Na}^{ces}(s)}{S_{Na}^{ces}(s)} \end{aligned} \tag{4.88}$$

Proof. See Appendix 4.8.8. ■

One can prove that for $\sigma < 1$, the extraction rate of time-consistent agents (exponential discounting) is lower than those with a general time-inconsistent discount $\theta(s-t)$.

Proposition 4.6. *For $\sigma < 1$ ($\sigma > 1$), agents employing exponential discounting exhibit a lower (higher) extraction rate compared to those utilizing a general discount function $\theta(s-t)$.*

$$\frac{R_{Na,Exp}^{Ces,\sigma<1}(s)}{S_{Na,Exp}^{Ces,\sigma<1}(s)} < \frac{R_{Na,\theta}^{Ces,\sigma<1}(s)}{S_{Na,\theta}^{Ces,\sigma<1}(s)} \tag{4.89}$$

Proof. See Appendix (4.8.9). ■

Moreover, from equation (4.83) one can show that along the BGP, the extraction rate $R(s)/S(s)$ will be constant, i.e., agent will have the same extraction rate.

4.4.3 Stability of the Steady State

In order to study the stability properties of the steady state, we use the system of differential equations that describe the BGP. However, before driving into the technical details, the ratio of extraction over the amount of resource left $R(s)/S(s)$ has its own growth rate. From the dynamics of the non-renewable resource (4.1), one gets

$$\gamma_{\frac{R}{S}} = \gamma_R + \frac{R(s)}{S(s)}. \tag{4.90}$$

See Appendix 4.8.10 for its derivation.

Working now with the system of differential equations that characterize the BGP and plugging in the previously derived equation (4.90) into the growth rate of the resource γ_R, one gets

$$\gamma_{\frac{c}{K}}^{Na,ces} = \frac{Y(s)}{K(s)}\left[\sigma(1-\beta_1-\beta_2)^2 - 1\right] - \sigma\lambda_{Na,\theta}^{ces} + \frac{c^{Na,ces}(s)}{K(s)}, \tag{4.91}$$

$$\gamma_{\frac{R}{S}}^{Na,ces} = \frac{R(s)}{S(s)} + \underbrace{(\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c^{Na,ces}(s)}{K(s)} + \eta(1-L_F(s))\frac{\beta_2}{1-\beta 1-\beta_2} + \eta L_F(s)(\beta_1+\beta_2)}_{=\gamma_R^{Na,ces}}, \tag{4.92}$$

$$\gamma_{L_F}^{Na,ces} = (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c^{Na,ces}(s)}{K(s)} + \eta(1-L_F(s))\left[\frac{\beta_2-(1-\beta_1-\beta_2)}{1-\beta_1-\beta_2}\right] + \eta L_F(s)\frac{(\beta_1+\beta_2)(1-\beta_2)}{\beta_1}, \tag{4.93}$$

$$\gamma_{\frac{Y}{K}}^{Na,ces} = (\beta_1+\beta_2)\eta L_F(s) - (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{K(s)} + \frac{\beta_2}{1-\beta_1-\beta_2}\eta(1-L_F(s)). \tag{4.94}$$

Solving the steady state of the system, one gets the values in equilibrium shown in Appendix (4.8.11) due to space constraints. Furthermore, as the ratio extraction over the amount of resources left, $(R(s)/S(s))$, just appears in the first differential equation, the system is partially recursive. The evolution of the whole system is described jointly by all the differential equations. However, the dynamics of $c(s)/K(s), L_F(s)$ and $Y(s)/K(s)$ are captured by the other three differential equations (all except the one of the growth rate of $R(s)/S(s)$). Therefore, in order to study the stability properties we will work with the linearized system around the steady state.

Let $\mathbf{m}(s)$ be the column vector of the difference between variables and their steady state values (with a star),

$$\mathbf{m}(s) = \begin{pmatrix} \frac{Y(s)}{K(s)} - \left(\frac{Y}{K}\right)^* \\ \frac{c^{Na,ces}(s)}{K(s)} - \left(\frac{c^{Na,ces}}{K}\right)^* \\ L_F(s) - L_F^* \end{pmatrix}.$$

Thus, the linearized system around the steady state can be written as

$$\frac{d}{dt}\mathbf{m}(s) = \underbrace{\begin{pmatrix} \frac{\partial\left(\frac{\dot{Y}}{K}\right)}{\partial(Y/K)} & \frac{\partial\left(\frac{\dot{Y}}{K}\right)}{\partial(c/K)} & \frac{\partial\left(\frac{\dot{Y}}{K}\right)}{\partial(L_F)} \\ \frac{\partial\left(\frac{\dot{c}}{K}\right)}{\partial(Y/K)} & \frac{\partial\left(\frac{\dot{c}}{K}\right)}{\partial(c/K)} & \frac{\partial\left(\frac{\dot{c}}{K}\right)}{\partial(L_F)} \\ \frac{\partial(\dot{L}_F)}{\partial(Y/K)} & \frac{\partial(\dot{L}_F)}{\partial(c/K)} & \frac{\partial(\dot{L}_F)}{\partial(L_F)} \end{pmatrix}}_{\equiv \mathbf{J}\left(\left(\frac{Y}{K}\right)^*, \left(\frac{c^{Na,ces}}{K}\right)^*, L_F^*\right)} \begin{pmatrix} \frac{Y(s)}{K(s)} - \left(\frac{Y}{K}\right)^* \\ \frac{c^{Na,ces}(s)}{K(s)} - \left(\frac{c^{Na,ces}}{K}\right)^* \\ L_F(s) - L_F^* \end{pmatrix}, \tag{4.95}$$

Since $\mathbf{J}\left(\left(\frac{Y}{K}\right)^*, \left(\frac{c^{Na,ces}}{K}\right)^*, L_F^*\right)$ is the Jacobian matrix evaluated at the fixed point,

$$\mathbf{J}(.) = \begin{pmatrix} -\left(\frac{Y}{K}\right)^*(1-\beta_1-\beta_2)(\beta_1+\beta_2) & 0 & \left(\beta_1+\beta_2-\frac{\beta_2}{1-\beta_1-\beta_2}\right)\eta\left(\frac{Y}{K}\right)^* \\ \left(\frac{c_{Na}^{ces}}{K}\right)^*\left[\sigma(1-\beta_1-\beta_2)^2-1\right] & \left(\frac{c_{Na}^{ces}}{K}\right)^* & 0 \\ L_F^*(\beta_1+\beta_2) & -L_F^* & \eta L_F^*\left(\frac{(1-\beta_2)(\beta_1+\beta_2)}{\beta_1} - \frac{\beta_2}{1-\beta_1-\beta_2}+1\right) \end{pmatrix}. \tag{4.96}$$

where the trace of the matrix $\mathbf{J}$ is the sum of its eigenvalues ζ_i in the diagonal, $Trace(\mathbf{J}) = \sum_{i=1}^n \zeta_i$. Thus, we have

$$
\begin{aligned}
Trace(\mathbf{J}) &= \sum_{i=1}^{3} \zeta_i \\
&= \left(\frac{c_{Na}^{ces}}{K}\right)^* - \left(\frac{Y}{K}\right)^* (1-\beta_1-\beta_2)(\beta_1+\beta_2) \\
&+ \eta L_F^* \left(\frac{(1-\beta_2)(\beta_1+\beta_2)}{\beta_1} - \frac{\beta_2}{1-\beta_1-\beta_2} + 1\right).
\end{aligned}
\tag{4.97}
$$

As in a stable dynamical system all the eigenvalues of the Jacobian matrix should have real negative parts, i,e., $Re(\zeta_i) < 0$, it implies that the trace, which is the sum of the eigenvalues, should be negative in order to show a convergent and stable equilibrium. However, no general statement can be made about the sign of the trace. The reason is that the last term, $\left(\frac{(1-\beta_2)(\beta_1+\beta_2)}{\beta_1} - \frac{\beta_2}{1-\beta_1-\beta_2} + 1\right)$, can be positive or negative depending on the combination of parameters.

As the determinant is the product of all its eigenvalues $|\mathbf{J}| = \prod_{i=1}^{n} \zeta_i$, then,

$$
\begin{aligned}
Det(\mathbf{J}) &= \prod_{i=1}^{3} \zeta_i, \\
&= -\eta \left(\frac{Y}{K}\right)^* \left(\frac{c_{Na}^{ces}}{K}\right)^* L_F^* \\
&\times \left\{(1-\beta_1-\beta_2)(\beta_1+\beta_2)\left[\frac{(1-\beta_2)(\beta_1+\beta_2)}{\beta_1} - \frac{\beta_2}{1-\beta_1-\beta_2} + 1\right]\right. \\
&+ \left.\left(\beta_1+\beta_2 - \frac{\beta_2}{1-\beta_1-\beta_2}\right)\left[\left(\sigma(1-\beta_1-\beta_2)^2 - 1\right) + \beta_1 + \beta_2\right]\right\}.
\end{aligned}
\tag{4.98}
$$

In the steady state, the values of $\left(\frac{Y}{K}\right)^* \left(\frac{c_{Na}^{ces}}{K}\right)^* L_F^*$ will be positive. Thus, the first term $-\eta \left(\frac{Y}{K}\right)^* \left(\frac{c_{Na}^{ces}}{K}\right)^* L_F^*$ is negative. Therefore, if one is interested in getting all the real parts of the eigenvalues negative, it implies that $\prod_{i=1}^{3} \zeta_i <$ 0. If one could get such a condition, it should imply that the term in brackets would be positive. However, no general statement can be made about the sign of the determinant. As before, the terms $\left(\frac{(1-\beta_2)(\beta_1+\beta_2)}{\beta_1} - \frac{\beta_2}{1-\beta_1-\beta_2} + 1\right)$ and $\left(\beta_1+\beta_2 - \frac{\beta_2}{1-\beta_1-\beta_2}\right)$ can be negative or positive depending on the combination of parameters.

This is why we will study the stability properties numerically by analyzing the sign of the eigenvalues. Nonetheless, we can state that we obtain the same qualitative results as Scholz and Ziemes (1999), with one negative real eigenvalue and two positive real eigenvalues. Nevertheless, our result is much more general, as households contemplate any general non-constant discount function. The system has a saddle path behavior. The unique real and negative eigenvalue will correspond to the labor force (see Section 4.5).

4.4.4 Pollution

In the steady state, taking logs and differentiating with respect to time for the final production (4.54) gives

$$\gamma^h \equiv \gamma_Y^h = \gamma_K^h = \gamma_N^h + \frac{\beta_2}{\beta_1 + \beta_2}\gamma_R^h, \tag{4.99}$$

which is translated to expression (4.70) for the CES case, and to expression (4.74) for the log case. As pollution (equation (4.16)) is generated by both final good production and by the extraction of the exhaustible resource, which takes the form $P(s) = Y(s)^{\mu_1} R(s)^{\mu_2}$, we can express its growth rate in two forms.

- The first one comes directly from log-differentiating the expression,

$$\gamma_P = \mu_1 \gamma + \mu_2 \gamma_R. \tag{4.100}$$

One should keep in mind that the first term $\mu_1 \gamma$ already includes the force of the decline in the extraction strategy $R(s)$ implicitly (as final production is a function of the extraction). This is the reason why the growth rate of the economy is a function of the growth rate of the resource, $\gamma = \gamma(\gamma_R)$. Thus, the evolution of the pollution $P(s)$ is given by

$$P(s) = \underbrace{Y(0)^{\mu_1} R(0)^{\mu_2}}_{\equiv P(0)} e^{[\mu_1 \gamma + \mu_2 \gamma_R]\cdot s} \tag{4.101}$$

- The second form is much clearer, as one can disentangle the specific driving forces of pollution. Using the final production (4.53), which is also a function of the extraction $R(s)$ and plugging it into the pollution expression, it leads to

$$P(s) = Y(s)^{\mu_1} R(s)^{\mu_2}$$

$$= \left[\frac{1}{1-\beta_1-\beta_2} \left(\underbrace{\frac{r}{1-\beta_1-\beta_2}}_{p^x \in \mathbb{R}_+} \right)^{-\frac{1-\beta_1-\beta_2}{\beta_1+\beta_2}} N(s) L_F(s)^{\frac{\beta_1}{\beta_1+\beta_2}} \right]^{\mu_1} \cdot R(s)^{\left(\mu_1 \frac{\beta_2}{\beta_1+\beta_2} + \mu_2\right)}. \tag{4.102}$$

Proposition 4.7. *As in the steady state $\gamma_{L_F} = 0$, the growth rate of pollution is*

$$\gamma_P = \mu_1 \cdot \gamma_N + \left(\mu_1 \frac{\beta_2}{\beta_1+\beta_2} + \mu_2 \right) \cdot \gamma_R. \tag{4.103}$$

Proof. It is straightforward by taking logs and differentiating expression (4.102) with respect to time s. ■

We now have three different scenarios. Depending on the parameters of the model, pollution could increase, stay constant, or decrease over time:

- UN/ Humankind Goal: $\gamma_P < 0$,
- Business as Usual (BAU): $\gamma_P = 0$,
- Crazy Humans will Kill Themselves: $\gamma_P > 0$.

Assuming that humankind succeeds in achieving its pollution targets, set by various international agreements, one would expect that all pollution derived from economic activity would decrease and that only the so-called "natural pollution" would exist. One might consider pre-industrial pollution, for instance, generated by volcanic eruptions or by the environment itself. This would be the effect of decreasing pollution until it converges to its natural earth level or its pre-industrial level. Thus, the fact that pollution decreases towards zero in our model should be interpreted as converging to its natural level or no extra pollution from human beings. To get this behavior, we need $\gamma_P < 0$. There are three potential scenarios.

Scenario 4.1. *Human-made pollution decreases, $\gamma_P < 0$.*

In order to reduce human-made pollution (extra pollution), we need

$$\gamma_P < 0 \Longleftrightarrow \mu_1 \cdot \gamma_N < -\left(\mu_1 \frac{\beta_2}{\beta_1+\beta_2} + \mu_2 \right) \cdot \underbrace{\gamma_R}_{<0}. \tag{4.104}$$

Scenario 4.2. *Pollution is constant over time, $\gamma_P = 0$.*

In order to have constant pollution, we need

$$\gamma_P = 0 \Longleftrightarrow \mu_1 \cdot \gamma_N = -\left(\mu_1 \frac{\beta_2}{\beta_1 + \beta_2} + \mu_2\right) \cdot \underbrace{\gamma_R}_{<0}. \tag{4.105}$$

Scenario 4.3. *Pollution increases over time, $\gamma_P > 0$.*

In order to have increasing pollution, we need

$$\gamma_P > 0 \Longleftrightarrow \mu_1 \cdot \gamma_N > -\left(\mu_1 \frac{\beta_2}{\beta_1 + \beta_2} + \mu_2\right) \cdot \underbrace{\gamma_R}_{<0}. \tag{4.106}$$

For all the cases, the evolution of pollution $P(s)$ with any discount function[19] is

$$P(s) = \underbrace{Y(0)^{\mu_1} R(0)^{\mu_2}}_{\equiv P(0)} e^{\gamma_P \cdot s}, \tag{4.107}$$

where γ_P is given by expression (4.103).

Before moving to the numerical results, one should note that although the mathematical structure of the expanding variety model and the AK model (where the economy grows at a constant rate) are similar, the economics of both models differ considerably. In the expanding variety model, research firms spend their funds on developing (inventing) new machines. The incentive behind this behavior is motivated by the search for profit, guaranteed by their patents and their monopolistic power, which allows them to sell their machines to the final good producer. Therefore, motivated by this search for profit, companies impulse R&D, and it is this R&D that ultimately generates economic growth. Thus, as pointed out by Romer (1994), "monopoly profits motivate innovation". Consequently, in contrast to the AK model, here we have a change in the technology frontier driven by incentives.

4.5 Quantitative Analysis

In this section we parameterize the model for the US economy. The elasticity of the exhaustible resource is $\beta_2 = 0.04$, in line with the work of Golosov et al. (2014) (see their parameter $\nu = 0.04$ in Table 1 of their paper), and Hassler and Krusell (2012). Furthermore, we set the elasticities of the intermediate machines $\tilde{\mathbf{X}}(s)$ in equation (4.7) to $1 - \beta_1 - \beta_2 = 0.4$ (see for instance Englander and Gurney (1994), and Maddison (1987)). The previous implies an elasticity of the labor force used in the final sector of $\beta_1 = 0.56$.

[19] This is why we do not write a discount function θ in the growth rates to simplify notation.

Focusing now on the calibration of the intertemporal elasticity of substitution, we consider two cases. First, we try to contribute to the current debate on the actual value of the CEIS. As noted in Cabo et al. (2020b), it is normal to assume in theoretical models of economic growth a log-utility, as it simplifies the analysis when one studies "complexities like non-constant time preferences". However, whether the CEIS value σ should be greater or lower than one has recently been questioned. As pointed out in Thimme (2017), the default assumption in macroeconomic models of using a CEIS between 0.5 and 1 (prevailed by Hall (1988)'s early CEIS estimates close to zero) has been challenged by recent literature. For instance, Havranek et al. (2015) and Gruber (2013) mention values of the constant elasticity of intertemporal substitution σ greater than one. Values greater than one are correlated with higher wealth, education and are country-specific (see Havranek et al. (2015), Ben-Gad (2012) Thimme (2017)). For this reason, we study both scenarios, i.e., $\sigma = 0.5$ and $\sigma = 1.2$. Interestingly enough, it turns out that this has big implications. For $\sigma < 1$, log agents (that is $\sigma = 1$) will have higher sum of discounted consumer utilities than CES agents, but for $\sigma > 1$, log agents will be worst than CES agents. For the case of $\sigma = 0.5$, we follow the meta-analysis carried out by Havranek et al. (2015),[20]where they collect 2735 estimates of the elasticity of intertemporal substitution in consumption from 169 published studies for 104 countries during different periods.

Regarding the instantaneous discount rate of the exponential discount function (4.32), we choose a value of $\hat{\rho} = 2\%$, which is the "Medium Future" (26 to 75 years) value in the data by Weitzman (2001). Furthermore, we set the parameter $\eta = 0.12$ (which is a parameter of the innovation possibilities frontier). All parameters are summarized in Table 4.1.

We work with different discount functions to see their effects on how agents make decisions and the impact on economic growth. We also investigate how agents extract exhaustible resources and delve into numerous aspects of our model. First, we show the results under the discount function (4.33), which appears in Tsoukis et al. (2017) and Cabo et al. (2020a). Thus, we need to find the right parameters for the non-constant discounting as in expression (4.35), where the discounted infinite stream of both discount functions gives the same present value. This last expression will be used to compare all the different discount functions to be equivalent to the exponential discount.

- For the discount function used in Tsoukis et al. (2017), this implies the following relationship

[20] Dugan and Trimborn (2020) also use this meta-analysis for their parametrization.

Parameters	Description	Value
β_1	Output Elasticity of Final Labor	0.56
β_2	Output Elasticity of Resource	0.04
$1-\beta_1-\beta_2$	Output Elasticity of Intermediate Machines	0.4
σ	Intertemporal elasticity of substitution	0.5 and 1.2
$\hat{\rho}$	Instantaneous Discount Rate for the Exponential	0.02
η	Parameter of the Innovation Possibilities Frontier	0.12
$\mu_{1,\gamma_P<0}$	Final Production Intensity in Pollution	0.2
$\mu_{1,\gamma_P=0}^{\theta,log,\sigma=1}$	Final Production Intensity in Pollution	0.00483871
$\mu_{1,\gamma_P=0}^{Exp,Ces,\sigma<1}$	Final Production Intensity in Pollution	0.0135484
$\mu_{1,\gamma_P=0}^{Exp,Ces,\sigma>1}$	Final Production Intensity in Pollution	0.0033871
$\mu_{1,\gamma_P=0}^{Ts,Ces,\sigma<1}$	Final Production Intensity in Pollution	0.0151788
$\mu_{1,\gamma_P=0}^{Ts,Ces,\sigma>1}$	Final Production Intensity in Pollution	0.00283772
$\mu_{1,\gamma_P=0}^{ConEx}$	Final Production Intensity in Pollution	0.0162283
$\mu_{1,\gamma_P=0}^{ConEx,Ces,\sigma>1}$	Final Production Intensity in Pollution	0.00268484
$\mu_{1,\gamma_P>0}$	Final Production Intensity in Pollution	0.3
$\mu_{2,\gamma_P<0}$	Resource Extraction Intensity in Pollution	0.5
$\mu_{2,\gamma_P=0}$	Resource Extraction Intensity in Pollution	0.005
$\mu_{2,\gamma_P>0}$	Resource Extraction Intensity in Pollution	0.005
ψ	Environmental concern (pollution)	0.005
b	Curvature degree of Pollution in the utility	0.001

Table 4.1: Structural Parameter Values

	$e^{-\hat{\rho}}$	$e^{-\rho j}(1+\delta j)^{-\varphi/\delta}$			$ze^{-\rho_1 j}+(1-z)e^{-\rho_2 j}$		
Parameters	$\hat{\rho}$	ρ	δ	φ	z	ρ_2	ρ_1
Values	0.02	0.001	0.021	0.0178607	0.5	0.05	0.0125

Table 4.2: Parameter values for different Discount Functions $\theta(j)$.

$$\int_0^\infty e^{-\hat{\rho}j}dj \stackrel{!}{=} \int_0^\infty (1+\delta j)^{-\varphi/\delta}e^{-\rho j}dj, \quad \Longleftrightarrow \quad \hat{\rho} = \frac{e^{-\frac{\rho}{\delta}}\delta^{\frac{\varphi}{\delta}}\rho^{1-\frac{\varphi}{\delta}}}{\Gamma\left(1-\frac{\varphi}{\delta},\frac{\rho}{\delta}\right)}, \tag{4.108}$$

where $\Gamma(a,b) = \int_b^\infty x^{a-1}e^{-x}dx$ is the incomplete gamma function, which is easy to manipulate nowadays with algebraic software. The intuition behind this expression is to capture a constant rate of time preferences $\hat{\rho}$ such that it gives the same overall level of impatience with the exponential discount (time-consistent) as the overall level of impatience under a non-constant discounting (see e.g., Barro (1999), Strulik (2015) or Cabo et al. (2020a)).

Following the Stern Review (Stern, 2007, Box 6.3, p. 184), we set the parameter $\rho = 0.1\%$ in the Tsoukis discount function. For the corresponding parameters φ and δ, we follow the approach in Dugan and Trimborn (2020), where the parameters of the discount function are driven by the data in Weitzman (2001), in which the author studies a numerical example with data from a survey based on the opinions of 2,160 economists (see Table 2 in the mentioned work to see the different discount rates for different distant times).[21] With that data, Dugan and Trimborn (2020) obtain what would be in our work a value of $\delta = 0.021$. Together with $\rho = 0.001$, we can solve for φ in equation (4.108), where $\hat{\rho} = 0.02$, obtaining a value of $\varphi = 0.0178607$. It is worth noting that without specifying a parameter, a continuum of solutions would exist for the pair of parameters. All parameters related to the discounting functions are provided in Table 4.2. As previously emphasized, it becomes evident that both integrals yield the same discounted value

$$\int_0^\infty e^{-\hat{\rho}}jdj = \int_0^\infty (1+\delta j)^{-\varphi/\delta}e^{-\rho j}dj$$
$$\Leftrightarrow \frac{1}{\hat{\rho}} = e^{\rho/\delta}\delta^{-\frac{\varphi}{\delta}}\rho^{\frac{\varphi}{\delta}-1}\Gamma\left(1-\frac{\varphi}{\delta},\frac{\rho}{\delta}\right) = 50 \text{ (with the given parameters).}$$

- For the "convex linear combination of two exponentials" discount function (4.34), we have the following relationship

[21] The author asked the question "Taking all relevant considerations into account, what real interest rate do you think should be used to discount over time the (expected) benefits and (expected) costs of projects being proposed to mitigate the possible effects of global climate change?".

$$\int_0^\infty e^{-\hat{\rho} j} dj \stackrel{!}{=} \int_0^\infty \left[z e^{-\rho_1 j} + (1-z) e^{-\rho_2 j} \right] dj \quad \Longleftrightarrow \quad \hat{\rho} = \frac{\rho_1 \rho_2}{\rho_1 - z\rho_1 + z\rho_2}, \tag{4.109}$$

with $z = 0.5$, i.e. an equal weight for the long-run and the short-run impatience.[22] The short-run impatience is governed by $\rho_2 = 0.05$ and the long-run impatience by $\rho_1 = 0.0125$. As before, given $\hat{\rho}$ from the exponential discount, the weight z and ρ_1, one should look for the parameter governing the short-run impatience ρ_2 to give the same stream of the integral with an exponential discount, as it is shown in (4.109). One can easily check that both integrals give the same value using the parameters in Table 4.2.

Following Dugan and Trimborn (2020), we use their estimates to calibrate the initial amount of capital K_0 to 56 trillion dollars. Furthermore, they also get data from the U.S. Energy Information Administration (EIA). Natural resources at initial time S_0, are defined as the sum of crude oil and natural gas reserves, with a given value of 5 trillion dollars. Normalizing S_0, and adjusting proportionally K_0, one gets the values of $S_0 = 1$ and $K_0 = 11$.

An interesting feature of this model with both utilities (log and CES) and with any discount function $\theta(j)$, is that the interest rate in the equilibrium r is determined endogenously in the model. One should notice that it is not possible to solve for all discounts algebraically. Certainly, one should find the interest rate that satisfies the BGP conditions of $\gamma = \gamma_Y = \gamma_c$. This condition solves r implicitly. In this section, we solve it numerically for all the discount functions we consider and both utilities, which give rise to different interest rates in equilibrium. For the logarithmic case one should solve for r using equations (4.74), which is the growth rate of the economy, and (4.49), which gives the growth rate of consumption from the modified Ramsey rule, where $\lambda^{log}_{Na,\theta}$ is given by expression (4.48) for any different discount $\theta(j)$. For the CES case, one should solve for r using equations (4.70) and (4.41) where $\lambda^{ces}_{Na,\theta}$ is given the expression (4.43) for any different discount $\theta(j)$. An interesting result is that for any discount function, all the interest rates for the logarithmic case collapse to the same one. This is straightforward, as for any discount function, with log utilities we have the same consumer behavior (as all the $\lambda^{log}_{Na,\theta}$ are the same regardless of the discount function, see Lemma 1). All the equilibrium interest rates are shown in Table 4.3. It can be seen that the higher the desire to smooth consumption (CEIS), the lower the interest rate.

Focusing on the stationary equilibrium analysis, i.e., when the economy evolves along its equilibrium path, we use Proposition 4.3 and 4.2 to illustrate the dynamics

[22] See also Mañó-Cabello et al. (2021).

Parameters	Description	Value
$r_{Exp}^{Ces\sigma<1}$	Equilibrium Interest rate, CES ($\sigma < 1$) & Exp	0.0433962
r_{Exp}^{log}	Equilibrium Interest rate, Log & Exp	0.0406667
$r_{Exp}^{Ces\sigma>1}$	Equilibrium Interest rate, CES ($\sigma > 1$) & Exp	0.0397452
$r_{Tsoukis}^{ces,\sigma<1}$	Equilibrium Interest rate, CES ($\sigma < 1$) & Tsoukis	0.0436637
$r_{Tsoukis}^{log}$	Equilibrium Interest rate, Log & Tsoukis	0.0406667
$r_{Tsoukis}^{ces,\sigma>1}$	Equilibrium Interest rate, CES ($\sigma > 1$) & Tsoukis	0.0393219
$r_{ConvexExp}^{ces,\sigma<1}$	Equilibrium Interest rate, CES ($\sigma < 1$) & ConvexExp	0.0438156
$r_{ConvexExp}^{log}$	Equilibrium Interest rate, Log & ConvexExp	0.0406667
$r_{ConvexExp}^{ces,\sigma>1}$	Equilibrium Interest rate, CES ($\sigma > 1$) & ConvexExp	0.0391952

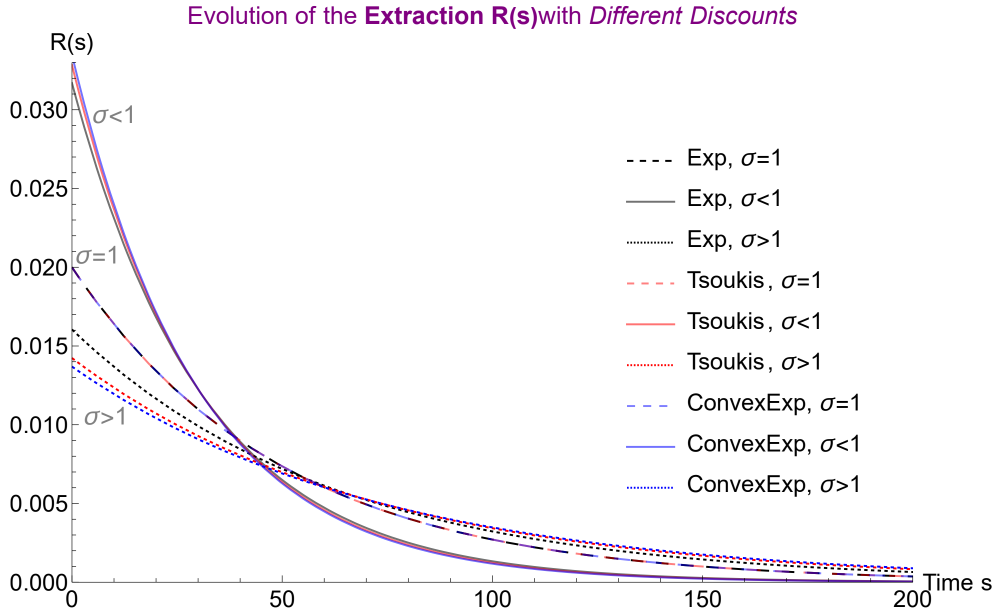
Table 4.3: Equilibrium Interest Rates

of the parameterized model using our simulations. In Figure 4.2, one can see the evolution of the extraction strategy of the economy $R(s)$ and the corresponding evolution of the resource $S(s)$.

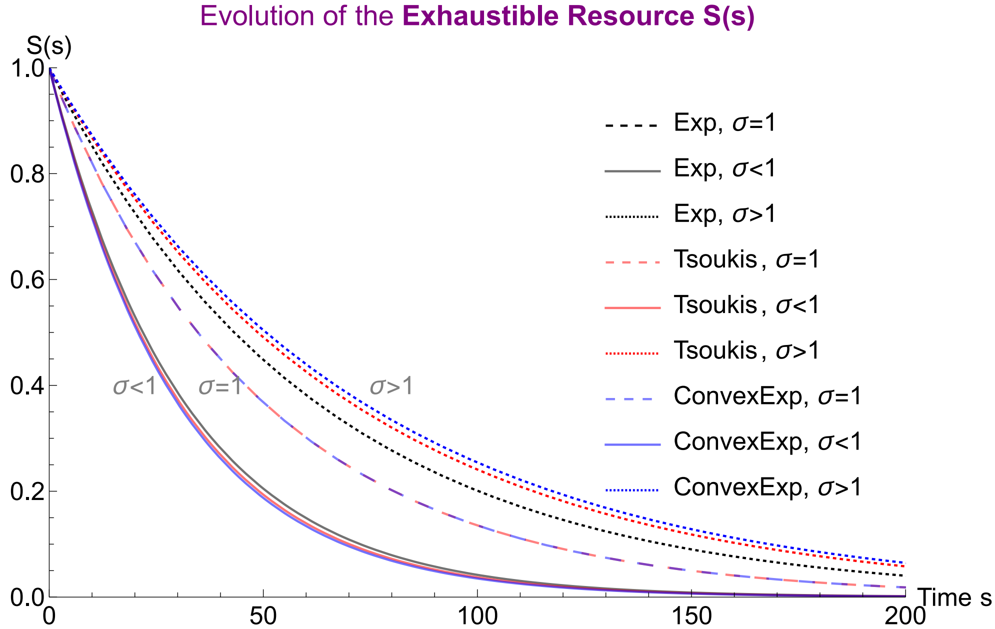
It is shown that with a constant elasticity of intertemporal substitution of $\sigma = 0.5$, which is equivalent to the elasticity of marginal utility $\{[-u''(c) \cdot c] / [u'(c)]\} = 1/\sigma = 2$, agents are more impatient and extract the resource more aggressively (see how they extract more in the near future).[23] As $\sigma < 1$, it can be seen that the lower the σ, the lower the willingness of households to substitute inter-temporally. Consequently, naive agents with $\sigma < 1$ start extracting more first in comparison to the logarithmic case, and later on when there are less resources, they extract less. Therefore, the stock of the resource for $\sigma < 1$ decreases faster at first, and there will always be more resource for the logarithmic agent at every point in time. The opposite is true for $\sigma > 1$, where agents extract fewer resources in the short term and as they have behaved more conservatively, they have access to more resource in the long-run.

The extraction rate $R(s)/S(s)$ given in equation (4.83) is shown in Figure 4.3 under all the different discount functions, where one can see that agents always extract a constant rate of the resource. This could be seen as an agent extracting a constant percentage of the resource left. It can be clearly seen why agents extract differently

[23] A lower σ means agents are less patient



(a) Evolution of the Extraction Strategy $R(s)^{Na,h}$ for $h \in \{Log, CES\}$.



(b) Evolution of the Resource Stock $S(s)^{Na,h}$ for $h \in \{Log, CES\}$.

Figure 4.2: Evolution of the Extraction $R(s)$ and dynamics of the Natural Resource $S(s)$ for the Naive agent.

under logarithmic and constant elasticity of substitution utilities. Notice that under the log utility case, all the extraction under different discounts functions collapse to the same strategy $R(s)/S(s) = -\gamma_R^{Na,log}$, which in turn coincides with the instantaneous discount rate of the exponential function $\hat{\rho}$ as shown in Proposition 4.3. One can see its effects on Figure 4.2a, where all different discount functions for the log case are the same, whereas for the cases with more general utility, each discount function leads to different extraction strategies (note in Proposition 4.4 that $\gamma_R^{Na,h} = \gamma_{Na}^h \cdot \left(\frac{\sigma_h - 1}{\sigma_h}\right) - \lambda_{Na,\theta}^h$, for $h \in \{log, ces\}$). The fact that under different discount functions $\theta(s-t)$, agents extract differently is driven by their different $\lambda_{Na,\theta}^h$, i.e., the *effective rate of time preference* or weighted mean of instantaneous discount rates (see Proposition 4.4). At first, households with $\sigma < 1$ ($\sigma > 1$) extract more (less) in comparison to the log case, while in the future agents extract less (more) under CES utility with $\sigma < 1$ ($\sigma > 1$). This behavior is driven by the "*Self-Control problem*"(Thaler, 2015, Chapters 11-12), where agents have a huge desire to consume now, and do not look after their future selves. The corresponding "effective rate of time preference" for each non-constant discount function is given in Table 4.4. It can be observed that the higher the constant elasticity of intertemporal substitution (CEIS, σ), the lower the effective rate of time preference ($\lambda_{Na,\theta}^h$). This is intuitive (they represent both sides of the same coin), as an increase in the level of patience (increase in σ) means that agents are less impatient (decrease in $\lambda_{Na,\theta}^h$). Generally, it was proved in Proposition 4.5 that agents extract more (less) under a CES utility with $\sigma < 1$ ($\sigma > 1$) in comparison to the log utility. These differences in extractions are appreciated in Figure 4.3, where the discount function plays a crucial role and it can be immediately seen that agents under CES utility extract differently if they discount the future differently.

	Exponential	Tsoukis	ConvexExp
CES ($\sigma < 1$)	0.02	0.0220253	0.0231754
Log ($\sigma = 1$)	0.02	0.02	0.02
CES ($\sigma > 1$)	0.02	0.0184177	0.0179438

Table 4.4: Effective Rate of time Preference, $\lambda_{Na,\theta}^h$ for both utilities under different discount functions.

For CES utility with $\sigma < 1$ ($\sigma > 1$), one can see that with exponential discounting one extracts less (more) than under Tsoukis discounting, and less (more) under Tsoukis than under a convex combination of exponentials. Under our specific parameterized model, for the logarithmic case, agents extract $-\gamma_R^{Na,} = \hat{\rho} = 2\%$ of the resource (see Proposition 4.3), while for agents with $\sigma < 1$, they extract more, at a 3.17%, 3.28%, and 3.35% for the Exponential Discounting, Tsoukis et al. (2017) and Linear Convex Combination of Exponentials respectively. This shows that for $\sigma < 1$, agents discounting exponentially have a lower extraction rate (3.17%) than agents with a non-constant discount function (3.28%, and 3.35%). This numerical

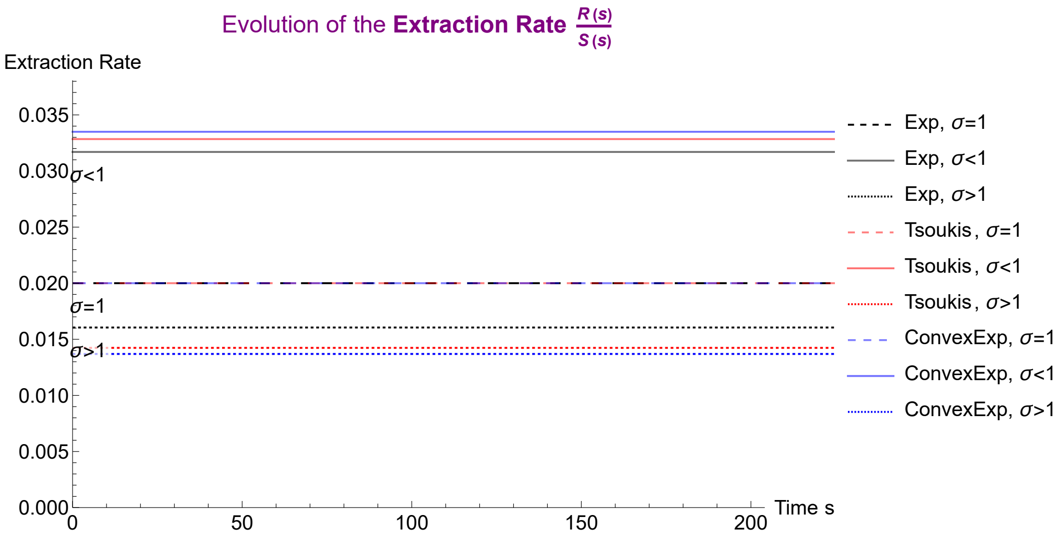


Figure 4.3: Evolution of the Extraction Rate with all different Discount under both utilities $\{Log, CES\}$.

result is a nice representation of Lemma 4.6. Furthermore, one can summarize all the important growth rates derived in Propositions (4.3) and (4.2) in Table 4.5.

		Exponential	Tsoukis	ConvexExp
CES $\sigma < 1$	γ	1.17%	1.08%	1.03%
	L_F^*	88.49%	89.15%	89.54%
	γ_R	-3.17%	-3.28%	-3.35%
	γ_N	1.38%	1.30%	1.26%
LOG $\sigma = 1$	γ	2.07%	=	=
	L_F^*	81.68%	=	=
	γ_R	-2%	=	=
	γ_N	2.2%	=	=
CES $\sigma > 1$	γ	2.37%	2.51 %	2.56%
	L_F^*	79.36%	78.31 %	77.89%
	γ_R	-1.61 %	-1.42%	-1.35%
	γ_N	2.48%	2.60%	2.65%

Table 4.5: Relevant Growth Rates of the Economy.

Concerning the evolution of pollution, captured by equation (4.107), one needs to calculate first the initial extraction and initial production under all different discounts to get the initial amount of pollution $P(0)$. A feasible set of parameters that satisfy Scenarios 4.1, 4.2 and 4.3 is shown in Table 4.1. One can notice that in order to have the BAU scenario ($\gamma_P = 0$) in Remark (4.2), a specific set of parameters should be considered. Given $\mu^{\theta}_{2,\gamma_P=0}$, we should find the parameter $\mu^{\theta}_{1,\gamma_P=0}$ that satisfies such Remark/condition. One can notice that the logarithmic utility ($\sigma = 1$), all $\mu^{\theta,log,\sigma=1}_{1,\gamma_P=0}$ collapse to the same number. However, for different levels of the CEIS different than one ($\sigma \neq 1$) and different discount functions, we have different values of $\mu^{\theta,ces,\sigma\neq 1}_{1,\gamma_P=0}$.

In Table 4.6 we show the corresponding values of the interesting variables and the corresponding growth rates of pollution. As before, all the discount cases under a logarithmic utility collapse to the same solution (in blue). However, under CES utility, agents behave differently depending on how they discount the future.

It is interesting to see that under non-constant discounting with $\sigma < 1$ ($\sigma > 1$), agents start extracting more (less) resource $R(0)$ and producing more (less) $Y(0)$ which implies higher (lower) pollution levels at first $P(0)$. However, when time passes by, the non-constant discounting agents (with a general $\theta(s-t)$) extract less (more) (as they have already extracted a lot (little) in the beginning), and this implies that they are producing less (more) in the future with lower (higher) pollution levels. This can be clearly seen in Figures 4.2a, 4.4 and 4.5. It is worth mentioning the previous behavior is driven by the fact that in the medium term non-constant discounting agents with $\sigma < 1$ ($\sigma > 1$) have less (more) resource to extract, which helps to directly reduce (increase) pollution.

When households have a higher CEIS (higher σ), they experience higher economic growth (and higher growth of the intermediate machines γ_N, which is the main driver of economic growth). Furthermore, the higher the CEIS, the less aggressively the resource is harvested. Moreover, if agents have a human behavior (in Thaler (2015)'s notation) where they procrastinate, with $\sigma < 1$ ($\sigma > 1$), the labor allocation to the final sector L_F^* is higher (lower) if they have a general discount function $\theta(s-t)$.

Finally, Figure 4.6 shows various scenarios with the different discount functions. Large variations in the performance of the economy can be observed, derived from the behavior of individuals, how patient they are, and how they make inter-temporal decisions. The best scenario is achieved when agents have a higher constant elasticity of intertemporal substitution. However, if households have CES utility with $\sigma > 1$, the best outcome is obtained when they are time-inconsistent. However, if $\sigma < 1$, exponential and time-consistent agents get higher levels of capital, production, varieties of inputs, and consumption.

		CES $\sigma < 1$			Log			CES $\sigma > 1$		
		$\gamma_P < 0$	$\gamma_P = 0$	$\gamma_P > 0$	$\gamma_P < 0$	$\gamma_P = 0$	$\gamma_P > 0$	$\gamma_P < 0$	$\gamma_P = 0$	$\gamma_P > 0$
Exponential	$R(0)$	0.0317	0.0317	0.0317	0.02	0.02	0.02	0.0161	0.0161	0.0161
	$Y(0)$	15.579	15.579	15.579	14.638	14.638	14.638	14.262	14.262	14.262
	$P(0)$	0.308	1.020	2.240	0.242	0.993	2.194	0.216	0.988	2.174
	γ_P	-1.351%	0%	0.335%	-0.587%	0%	0.61%	-0.329%	0%	0.703%
Tsoukis	$R(0)$	0.033	0.033	0.033	0.02	0.02	0.02	0.014	0.014	0.014
	$Y(0)$	15.662	15.662	15.662	14.638	14.638	14.638	14.072	14.072	14.072
	$P(0)$	0.314	1.025	2.244	0.242	0.993	2.194	0.202	0.986	2.164
	γ_P	-1.426%	0%	0.308 %	-0.587%	0%	0.61%	-0.210%	0%	0.745%
ConvexExp	$R(0)$	0.033	0.033	0.033	0.02	0.02	0.02	0.014	0.014	0.014
	$Y(0)$	15.709	15.709	15.709	14.638	14.638	14.638	14.013	14.013	14.013
	$P(0)$	0.317	1.028	2.246	0.242	0.993	2.194	0.198	0.986	2.161
	γ_P	-1.468%	0%	0.293%	-0.587%	0%	0.61%	-0.175%	0%	0.762%

Table 4.6: Values of initial extraction, production, pollution and growth rate of pollution.

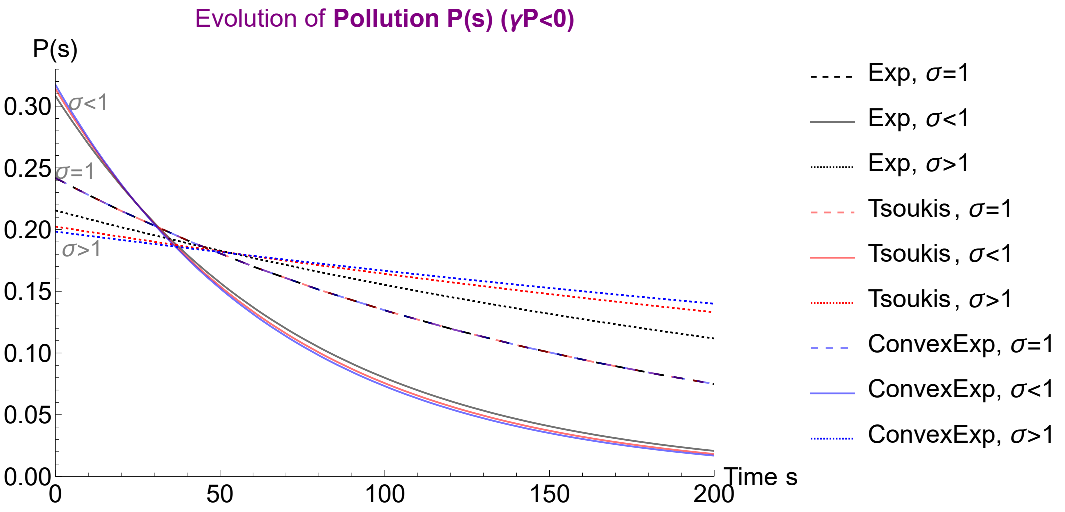


Figure 4.4: Evolution of pollution when $\gamma_P < 0$ with all different discount functions under both utilities.

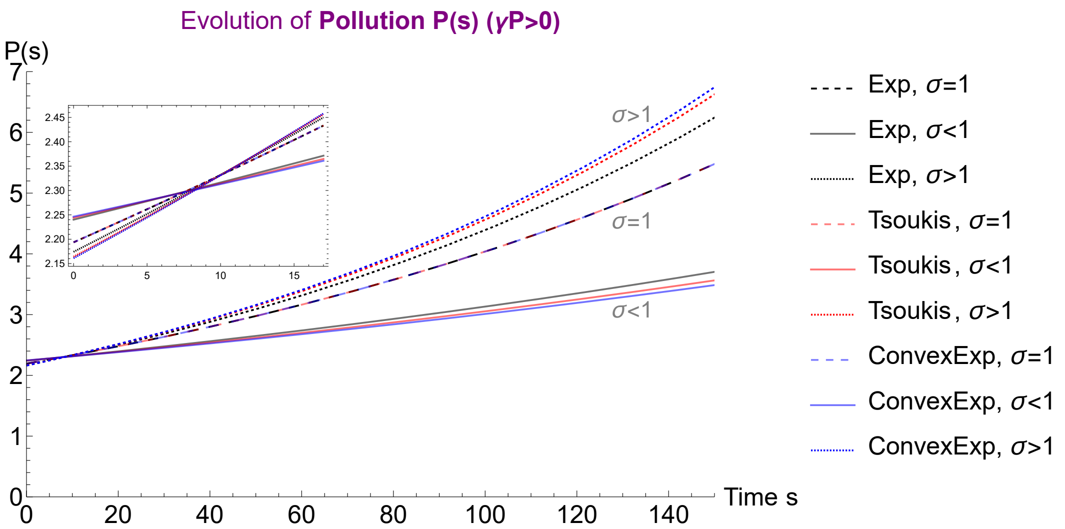
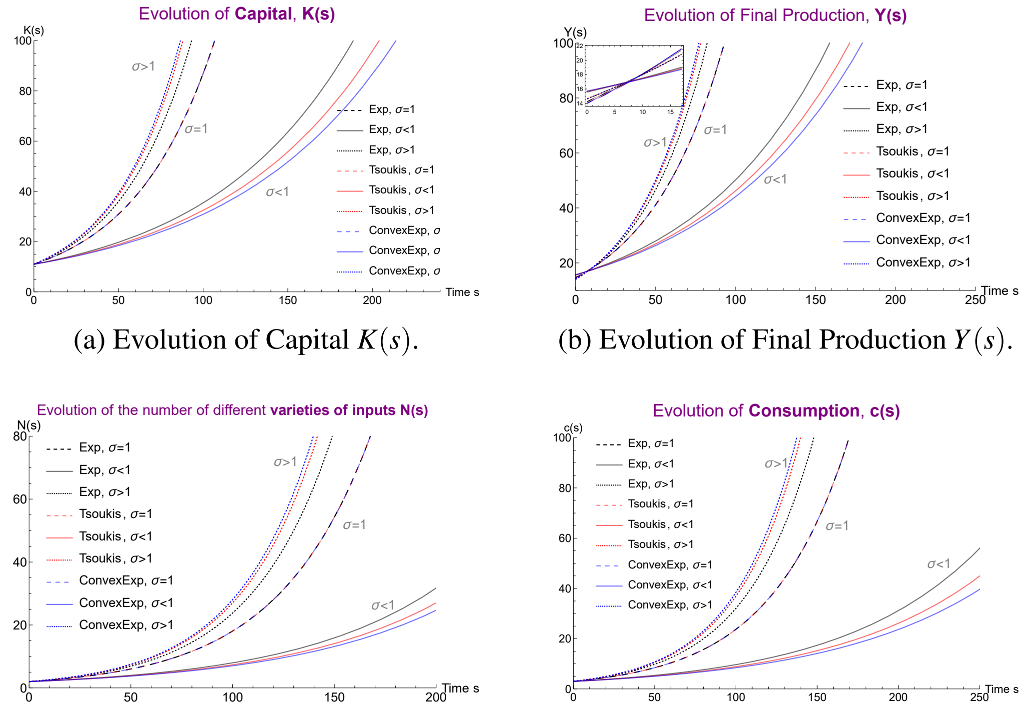


Figure 4.5: Evolution of pollution when $\gamma_P > 0$ with all different discount functions under both utilities.

Regarding the analysis of the stability of the steady state, one can focus on the linearized system (4.95). However, before the linearization of the system, it is worth noting that the system of equations (4.92), (4.93) and (4.94) form a 3D vector field (see Section 4.4.3 for why just studying the three differential equations is enough, as the system is partially recursive). We first show the Exponential-CES case. Later, we will show the cases for non-constant discount functions and see how we get different steady states driven by the fact that agents now are time-inconsistent.[24]

[24]Remember that for naive agents, the unique time-consistent behavior comes from an exponen-



(a) Evolution of Capital $K(s)$.

(b) Evolution of Final Production $Y(s)$.

(c) Evolution of new Blueprints $N(s)$.

(d) Evolution of Consumption $c(s)$.

Figure 4.6: Evolution of relevant variables on the model under different Discount Functions.

In Figure A.1 shown in Appendix (4.8.13), we show the 3D vector field. It can be clearly seen that many arrows are pointing in the opposite direction of the steady state. This is a visual indicator of an unstable equilibrium. However, to appreciate such behavior, one must analyze the eigenvalues of the Jacobian matrix evaluated at the steady state. Such eigenvalues confirm the unstable behavior seen in 3D vector field (see Figure A.1). In Table 4.7 we show different values of the steady state with different discount and utility functions (different CEIS) and its corresponding eigenvalues.

Observe that for the log case ($\sigma = 1$), all the equilibrium outcomes collapse to the same steady state and the same eigenvalues. This is driven by the fact that we are using a particular utility function, i.e., the logarithmic function. Thus, here the discount function does not matter. However, when we consider a more general utility function with $\sigma \neq 1$, we depart from the result of all steady states collapsing to just one equilibrium. We now get different equilibria for different discount functions. Working with a general utility function gives rise to different findings depending on how individuals discount the future. It can be seen that for $\sigma < 1$, with a non-constant discount function $\theta(s-t)$ agents have higher levels of production

tial discount. Using a general discount function $\theta(s-t)$ generates time-inconsistent behaviors.

		CES ($\sigma < 1$)	LOG ($\sigma = 1$)	CES ($\sigma > 1$)
Exponential	$\left(\frac{Y}{K}\right)^*$	0.271226	0.254167	0.248408
	$\left(\frac{c}{K}\right)^*$	0.259528	0.2335	0.224713
	L_F^*	0.884906	0.816667	0.793631
	$\left(\frac{R}{S}\right)^*$	0.0316981	0.02	0.016051
	$\lambda_{Y/K}$	0.336876	0.304151	0.292996
	$\lambda_{c/K}$	0.118878	0.114573	0.113257
	λ_L	-0.0565277	-0.0572236	-0.0574892
Tsoukis	$\left(\frac{Y}{K}\right)^*$	0.272898	0.254167	0.245762
	$\left(\frac{c}{K}\right)^*$	0.262079	0.2335	0.220677
	L_F^*	0.891593	0.816667	0.783048
	$\left(\frac{R}{S}\right)^*$	0.0328445	0.2	0.0142369
	$\lambda_{Y/K}$	0.339763	0.304151	0.288507
	$\lambda_{c/K}$	0.120001	0.114573	0.111339
	λ_L	-0.0568405	-0.0572236	-0.0569323
ConvexExp	$\left(\frac{Y}{K}\right)^*$	0.273848	0.254167	0.24497
	$\left(\frac{c}{K}\right)^*$	0.263528	0.2335	0.219468
	L_F^*	0.895391	0.816667	0.779879
	$\left(\frac{R}{S}\right)^*$	0.0334955	0.02	0.0136935
	$\lambda_{Y/K}$	0.341403	0.304151	0.287163
	$\lambda_{c/K}$	0.120639	0.114573	0.110765
	λ_L	-0.0570182	-0.0572236	-0.0567656

Table 4.7: Steady State and eigenvalues under different discounting and values of Constant Elasticity of Intertemporal Substitution, σ (CEIS).

per capital, consumption per capital, and higher extraction rates (as shown also in Figure 4.3) in comparison to agents discounting the future exponentially. However, for higher values of the CEIS with $\sigma > 1$, the opposite is true. Focusing now on the eigenvalues of the system, which show if the system is stable or not, one can observe that just the eigenvalue associated with the labor allocation to the final producer has negative (and real) value. The other two eigenvalues associated with output per capital and consumption per capital are positive. This implies that we obtain a saddle point. For values of $\sigma < 1$, $|\lambda_{L,\theta}^{\sigma<1}| > |\lambda_{L,exp}^{\sigma<1}|$, which means that the trajectory approaches the steady state along the saddle path quicker along the eigenvalue of a general discount function space. However, for values of $\sigma > 1$, time-consistent agents (those discounting exponentially) reach the steady state faster along the saddle path as $|\lambda_{L,\theta}^{\sigma>1}| < |\lambda_{L,exp}^{\sigma>1}|$. This is normally known as the speed of convergence of a dynamical system. It can be seen that, simply because agents discount the future differently and in a non-constant way, we obtain different equilibria in the steady state.

In Figure 4.7 we show the 3D vector field with the three steady states, one per each discount function when households have a CEIS $\sigma < 1$. It can be seen that the steady states of the exponential discounting is far away from that of the two non-constant discount functions. This is a numerical/graphical manner of realizing the effect of discounting the future on a non-constant basis. Interestingly, if one keeps increasing the level of the constant elasticity of intertemporal substitution and getting closer to one, all the steady states will collapse to the same one as shown in Table 4.7.

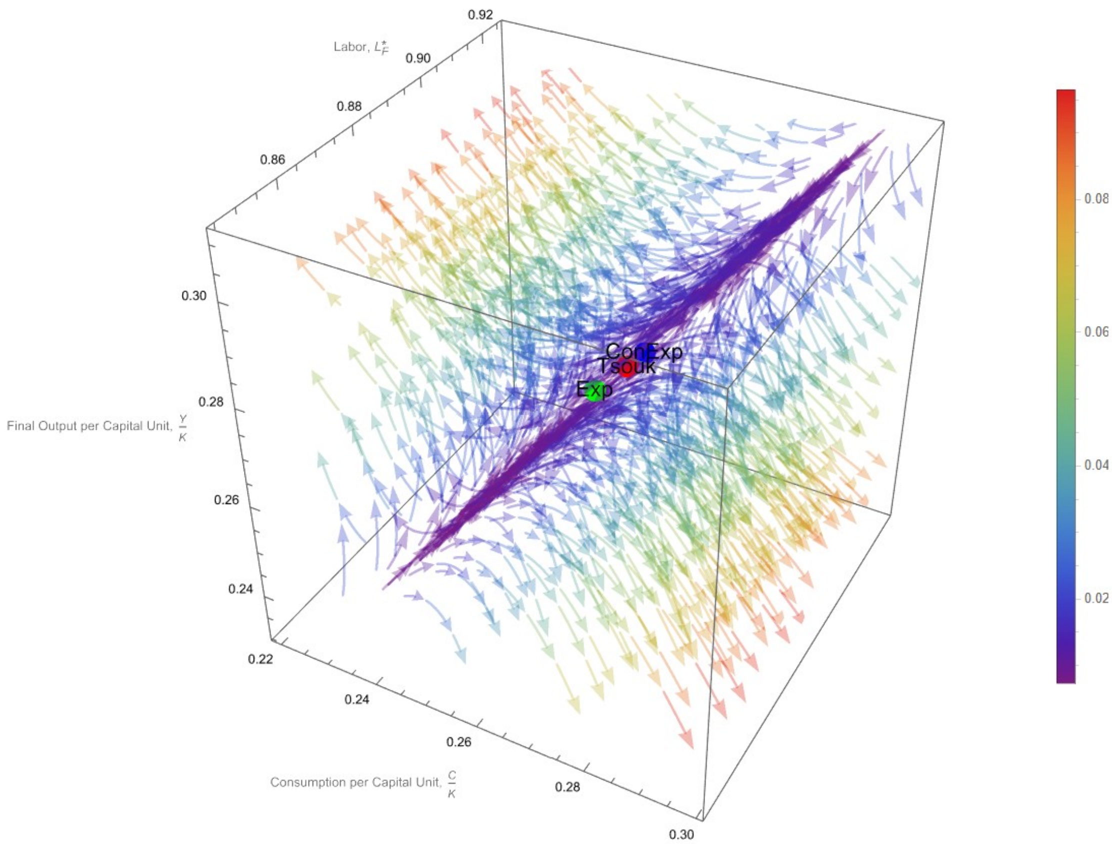


Figure 4.7: 3D Vector Field for Exponential, Tsoukis and Convex combination of exponential Discounting for $\sigma < 1$.

4.6 Sum of Discounted Utilities (Welfare)

In this section, we will focus on the implications of the sum of discounted utilities of households, that is, those who can behave time-consistently (discounting exponentially, i.e. $\theta(s-t) = e^{-\hat{\rho}(s-t)}$) or time-inconsistently, discounting the future with a general discount function $\theta(s-t)$. Furthermore, we also study the implication of having different degrees of elasticities of intertemporal substitution lower, greater or equal to one. Note that the lower or greater CEIS scenario corresponds to a general CES utility, while the case of the CEIS being one is the particular logarithmic function. The consumer's discounted utilities for the stationary equilibrium, that is, along the balanced growth path, will be no more than the value function, i.e., the discounted sum of the future stream of payoffs along the optimal paths,

$$W_\theta^h = \int_t^\infty \theta(s-t)U\left(c_t^*(s), P_t^*(s)\right) ds, \tag{4.110}$$

Using the optimal path of consumption along the BGP, which is in $c(s) = c_0 e^{\gamma_\theta^h \cdot s}$ and pollution (4.107) for any discount function $\theta(s-t)$ and utility $h \in \{log, ces\}$, and using the strong observational equivalence (4.35),[25] one can rewrite the previous expression as

$$W_\theta^{ces} = \frac{1}{\hat{\rho}}\left[\frac{\sigma}{1-\sigma} + \frac{\psi}{1+b}\right] - \frac{\sigma}{1-\sigma} c_0^{-\frac{1-\sigma}{\sigma}} \int_t^\infty \theta(s-t) e^{-\gamma_\theta^{ces}\cdot\left(\frac{1-\sigma}{\sigma}\right)s} ds$$
$$-\frac{\psi}{(1+b)} P_0^{(1+b)} \int_t^\infty \theta(s-t) e^{(1+b)\cdot\gamma_{P,\theta}^{ces}\cdot s} ds, \tag{4.111}$$

for the general CES case, and

$$W_\theta^{log} = \frac{1}{\hat{\rho}}\left[ln(c_0) - \psi \cdot lnP_0\right] + \left[\gamma_\theta^{log} - \psi \cdot \gamma_{P,\theta}^{log}\right] \int_t^\infty \theta(s-t)\cdot s\, ds, \tag{4.112}$$

for the logarithmic case. In Appendix 4.8.14 we show the results of the corresponding integrals for different discount functions. One should notice that all the integrals converge. Obtaining the corresponding closed-form solutions for consumers' sum of discounted utilities, one can observe that it leads to big differences depending on how agents discount the future. In Table 4.8 we show the numerical results of the sum of discounted utilities for different values of the constant elasticity of intertemporal substitution (σ) and different discount functions $\theta(s-t)$. One can immediately see that the higher level of patience (σ) the greater the sum of discounted utilities. It was seen in Table 4.4 that agents with higher CEIS (σ) had lower levels of impatience (lower $\lambda_{Na,\theta}^h$).[26] Therefore, more patient agents can save more today so they will enjoy a higher consumption tomorrow. This leads to better economic

[25] Recall that this concept is also referred to as the assumption of identical overall impatience.

[26] Remember that these were two sides of the same coin. The higher the level of patience (increase in σ) means that agents are obviously less impatient (decrease in $\lambda_{Na,\theta}^h$).

performance. This question of whether patient individuals become wealthier has been recently studied for instance in Epper et al. (2020). As the authors explain "*[w]hy some people are rich while others are poor is of fundamental interest in social science. Standard savings theory predicts that people who place a larger weight on future payoffs will be wealthier throughout the life cycle than more impatient people because of differences in savings behavior. Macroeconomic research suggests that this relationship between time discounting and wealth inequality can be quantitatively important and help explain why wealth inequality greatly exceeds income inequality (Krusell and Smith 1998, Quadrini and Ríos-Rull 2015, Carrollet al. 2017).*"

It is therefore interesting to see how individuals have a higher level of well-being with higher desire to smooth consumption (CEIS). This result is determined by the fact that with CES utility with constant elasticity of intertemporal substitution of $\sigma = 0.5$, individuals extract much more just in the short-run (see previous sections), but much less in the middle and long-run. Is has been shown (for instance in Figure 4.2a) that the higher the level of patience (higher σ), the more conservatively agents start extracting the natural resource. The economic intuition is that since they are more patient, they can wait and leave more natural resources for their future selves. It is clear that for low levels of impatience, agents extract a lot of resources at the beginning, which implies that they have access to fewer resources in the future. Additionally, as with more patient agents ($\sigma > 1$) the extraction of the resource is lower, we have also seen that levels of pollution are lower just within the first few years. In the long-run, however, pollution levels are higher (in all cases, when pollution has positive, negative or zero growth rates). One should keep in mind that the channel through which this mechanism is propagated is not direct. A consumer-specific behavior parameter (σ) affects the extraction of the resource carried out by the representative firm of the resource sector. However, this effect is not direct and comes from the general equilibrium property of the model.

The policy implications of the idea "the more patient you are, the better off you are" follow interestingly. If a nation is better off when its society has a high desire to smooth consumption (high CEIS), this leads to higher levels of economic well-being and living standards. Nonetheless, questions related to how to help society being more impatient are beyond the scope of this chapter. However, one could be interested in the policy implications of such results. It has been obvious that the higher the constant elasticity of intertemporal substitution, the higher sum of discounted household utilities.

Focusing now on the role of different discount functions (how individuals discount the future), it can be observed that with discount functions other than time-consistent (exponential), agents have a higher level of discounted utilities ("wel-

		CES ($\sigma < 1$)	LOG ($\sigma = 1$)	CES ($\sigma > 1$)
Exponential	$\gamma_P < 0$	39.688 $<$	107.025 $<$	149.125
	$\gamma_P = 0$	39.4791 $<$	106.599 $<$	148.925
	$\gamma_P > 0$	38.0611 $<$	106.325 $<$	148.33
Tsoukis	$\gamma_P < 0$	40.0111 $<$	120.031 $<$	197.133
	$\gamma_P = 0$	39.798 $<$	119.586 $<$	196.931
	$\gamma_P > 0$	39.3531 $<$	119.292 $<$	196.03
ConvexExp	$\gamma_P < 0$	40.1418 $<$	125.652 $<$	215.675
	$\gamma_P = 0$	39.9264 $<$	125.199 $<$	215.473
	$\gamma_P > 0$	39.4771 $<$	124.897 $<$	214.492

Table 4.8: Sum of Discounted Utilities under different *utility* functions, *discount functions* $\theta(j)$ and pollution environments.

fare”). This may sound paradoxical and counterintuitive. How can having time-inconsistent preferences lead to a higher sum of discounted utilities? The intuition is as follows. For values of a CEIS $\sigma < 1$, we have seen that capital, final production, consumption, and the number of varieties are higher when agents discount the future exponentially. However, for more patient agents with a CEIS $\sigma > 1$, the opposite is true, i.e., agents discounting the future exponentially have lower levels of capital, production, consumption, and the number of varieties. To understand why the sum of discounted consumer utilities is lower under exponential discounting we should look at the other variable affecting “welfare”, i.e., pollution. For the case with $\sigma < 1$, the levels of pollution for the exponential discounting are lower at first, but higher in the long-run. Thus, the total effect in the long-run if agents discount exponentially will be that they are exposed to larger levels of pollution, and will be harmed the most. However, for more patient agents with $\sigma > 1$, again the opposite is true, and exponential discounting agents pollute more in the short-run but in the long-run, they are exposed to lower levels of pollution, suffering less harm. This argument holds because the period of time in the short term before the situation is reversed is limited compared to the long term situation. Consequently, for $\sigma < 1$ the harm caused by pollution under exponential discounting effect dominates, while for $\sigma > 1$ the fact that agents can consume more exceeds the harm caused by pollution.

Hence, it is true that exponential agents (time-consistent), have a higher level of discounted utilities in the short-run, but lower levels in the long-run. This intuitive story can be appreciated in Figure 4.8, where for any value of the CEIS (σ), time-consistent agents (exponential) have higher “sums of discounted utilities” in the short-run but lower levels in the long-run. We plot the discounted utility over time. Thus, the image of the function in a future time τ will be the “welfare” at τ for today’s agent. Therefore, the sum of discounted consumer utilities will be no more than the area (integral) under each curve (see equation 4.110). It can be seen that the higher the level of patience (σ), the higher the levels of discounted utilities at any point in time. One should compare each discount function with themselves. Thus, $W_\theta(\text{with } \sigma < 1) < W_\theta(\text{with } \sigma > 1)$. Interestingly, agents discounting exponentially have higher discounted utilities at first but in the long-run, they have lower levels of discounting utility. Therefore, as the sum of discounted consumer utilities is defined by the area under the convergent functions, and the exponential instantaneous welfare in the future declines dramatically, the sum (integral) will be lower for exponential agents. This effect is more evident the higher the CEIS (σ). This is the intuition behind why agents are "better off" under non-constant discounting.

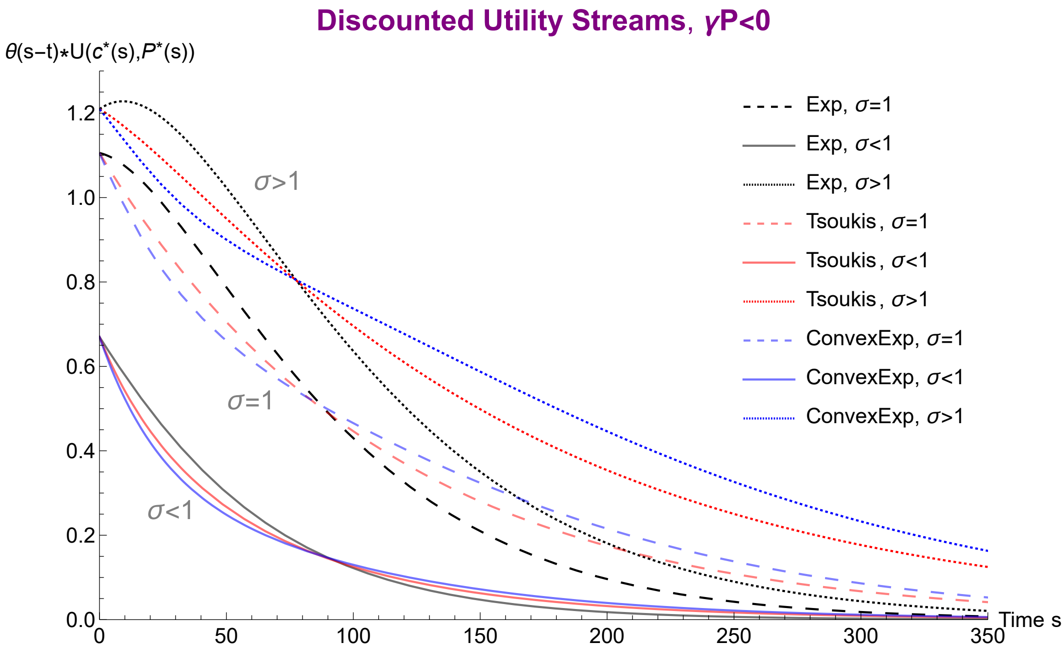


Figure 4.8: Stream of Utilities. The sum of discounted consumer utilities is the area under the curve.

4.7 Conclusions

In this chapter we have analyzed an endogenous growth model with exhaustible resources where agents have non-constant discounting preferences. Furthermore, we study the effect of pollution and its dynamic behavior, as well as its detrimental impact on households' welfare. Introducing pollution into the utility captures the damage function observed in classical climate change models, which could be considered as the deviation of pollution from pre-industrial levels, attributable to the extra anthropogenic pollution.

Incorporating this behavioral economics component, where agents exhibit greater impatience in the short-run but claim to be more patience between decisions concerning the future, yields a diverse array of results that are not observed in the standard endogenous growth models with exponential discounting. These new results come from the fact that agents are time-inconsistent, as they discount the future in a non-constant manner. One obtains a non-canonical Ramsey rule where the "effective rate of time preference" or "weighted average of the instantaneous rates of time preferences" $\lambda^{ces}_{Na,\theta}$ emerges as a generalization of the classical instantaneous discount rate of the exponential discounting $\hat{\rho}$. Thus, if agents discount exponentially, we recover the standard instantaneous discount rate $\hat{\rho}$. If agents have logarithmic utilities, the effective rate of time preference $\lambda^{log}_{Na,\theta}$ collapses to the same canonical instantaneous discount rate $\hat{\rho}$. Thus, working with a more general CES utility function together with a non-constant discount function leads to different results.

We have also shown the characteristics of the balanced growth path and the steady state properties. Furthermore, due to the general equilibrium property, the extraction of the exhaustible resource follows a feedback strategy, where a constant proportion of the available resource is extracted. Moreover, it has been proven that households with logarithmic utilities influence the production sector to have a lower (higher) extraction rate in comparison to households with a CES function featuring a CEIS less (greater) than one, across all distinct discount functions.

Regarding pollution, it has been considered that the extraction of an exhaustible resource, such as oil or natural gas, together with the final production, generate pollution. Analyzing the growth rate of such contamination, it has been observed that agents with CES utilities with a CEIS smaller (greater) than one, coupled with distinct discount functions, pollute much more (less) in the short term compared to logarithmic agents possessing a CEIS equal to one. This higher (lower) pollution levels under log agents in the short-run is derived from a greater (smaller) extraction of natural resources, since more extraction of an exhaustible resource generates more pollution. However, if pollution declines over time, it declines faster for agents with CEIS lower than one. This is translated into less contamination for CES agents with $\sigma < 1$ than with log agents in the long term. Consequently, in the long term, CES agents with $\sigma < 1$ ($\sigma > 1$) will suffer less (more) from the harmful effects of being exposed to these pollutants. This is also true when the levels of pollution stay constant or increase over time. This last effect has huge consequences for the well-being of the agents. An economy with less pollution in the future, where agents "enjoy a cleaner environment", will benefit the most and will be better off than those who continue to suffer from high levels of pollution. Therefore, being less patient has a positive side in the long-run, since there will be fewer resources to extract, and it will lead to lower levels of pollution.

We have analyzed the stability of all the different steady states. We show that having a non-constant discount leads to different equilibrium outcomes. Furthermore, we have seen that the equilibrium is a saddle path. Paradoxically, we find that agents behaving time-inconsistently are "better off" (higher sum of discounted utilities) in comparison to time-consistent agents (who discount the future exponentially). In Richard Thaler (2015)'s story, this means that Humans have a higher sum of discounted utilities than Econs. Moreover, we also show how agents with a small desire to smooth consumption over time (higher constant elasticity of intertemporal substitution) lead to a higher level of "welfare".

It has been shown that how agents discount the future has enormous implications for how nations develop. Therefore, it is critical to take these types of issues into account when designing public policies. Future developments include the study of pollution as in Rubio and Escriche (2001), where the dynamics of such is related to the production, and emissions are irreversible. Then, the model would not require the incorporation of an additional state variable. Other extensions would include the derivation of time-consistent equilibria with a general discount function as in Karp (2007) and Marín-Solano and Navas (2009). Introducing stochastic movements to the evolution of the wealth/savings of the agents in the setting of this chapter will help to study business cycle properties in the short-run. Moreover, if we extend the framework to one in which we have heterogeneity of agents, that is, each with stochastic dynamics of their own wealth, we could study Mean Field Games as in Achdou et al. (2022) or Laibson et al. (2021). This will allow us to see how the distribution of wealth evolves over time and its implications on inequality when agents behave like humans.

4.8 Appendix

4.8.1 Problem of the Representative firm of the Resource Sector

The Hamiltonian of problem (4.1), with the dynamics constraint (4.3) is given by,

$$\mathscr{H}^{nr} = \exp\left(-\int_t^s r(h)dh\right) P_R(s)R(s) + \lambda(s)\left[-R(s)\right], \tag{A.1}$$

with the correspondent F.O.C:

$$\frac{\partial \mathscr{H}^{nr}}{\partial R(s)} = 0 \Leftrightarrow \exp\left(-\int_t^s r(h)dh\right) P_R(s) = \lambda(s),$$

$$\dot{\lambda}(s) = -\frac{\partial \mathscr{H}^{nr}}{\partial S(s)} \Leftrightarrow \dot{\lambda}(s) = 0 \Leftrightarrow \lambda(s) = k_1,\ \forall k_1 \in \mathbb{R}.$$

Combining both previous equations gives $P_R(s) = k_1 \exp\left(\int_t^s r(h)dh\right)$. In order to get the growth rate of the prices as in the Hotelling rule, we firsts take natural logs and then differentiate with respect to time s, giving equation 4.4:

$$\gamma_{P_R} \equiv \frac{\dot{P}_R(s)}{P_R(s)} = r(s). \tag{A.2}$$

4.8.2 Derivation of the optimal consumption rule

Making use of the maximum principle, and following the approach in Strulik (2015) and Cabo et al. (2015), the first order condition for a maximum and the co-state variable give:

$$\frac{\partial \mathscr{H}^N}{\partial c_t(s)} \stackrel{!}{=} 0 \iff c_t(s) = \left[\frac{\theta(s-t)}{\lambda_t(s)}\right]^{\sigma}, \tag{A.3}$$

$$\dot{\lambda}_t(s) \stackrel{!}{=} -\left(\frac{\partial \mathscr{H}^N}{\partial a_t(s)}\right) \iff \dot{\lambda}_t(s) = -r_t(s)\lambda_t(s), \tag{A.4}$$

with the transversality condition (TVC) $\lim_{s\to\infty} \lambda(s)a(s) = 0$. Solving the differential equation (A.4), we obtain

$$\lambda_t(s) = \underbrace{\lambda_t(t)}_{\equiv\lambda(t)\in\mathbb{R}} \exp\left\{-\int_t^s r_t(\xi)d\xi\right\}. \tag{A.5}$$

Now, plugging equation (A.5) into the FOC of consumption (A.3) gives,

$$c_t(s) = \left[\frac{\theta(s-t)}{\lambda_t}\right]^{\sigma} \exp\left[\sigma \int_t^s r_t(\xi)d\xi\right]. \tag{A.6}$$

Inserting equations (A.6) into the budget constraint, and solving the corresponding differential equation for all $s \in [t,T)$, we get:

$$\begin{aligned} a_t(T) - \underbrace{a_t(t)}_{\equiv a_t} e^{\int_t^T r_t(\xi)d\xi} &= \int_t^T [w_t(\tau) + p_R(\tau) * R(\tau)]\, e^{\int_\tau^T r_t(\xi)d\xi} d\tau \\ &- \frac{1}{(\lambda_t)^{\sigma}} \int_t^T \theta(\tau-t)^{\sigma} \exp\left\{\sigma \int_t^{\tau} r_t(\xi)d\tau + \int_\tau^T r_t(\xi)d\xi\right\} d\tau. \end{aligned} \tag{A.7}$$

Dividing both sides of the equation by $e^{\int_t^T r_t(\xi)d\xi}$, one gets:

$$\begin{aligned} \frac{a_t(T)}{e^{\int_t^T r_t(\xi)d\xi}} &= a_t(t) + \int_t^T [w_t(\tau) + p_R(\tau) * R(\tau)]\, \frac{e^{\int_\tau^T r_t(\xi)d\xi}}{e^{\int_t^T r_t(\xi)d\xi}} d\tau \\ &- \frac{1}{(\lambda_t)^{\sigma}} \int_t^T \theta(\tau-t)^{\sigma}\, \frac{\exp\left\{\sigma \int_t^{\tau} r_t(\xi)d\xi + \int_\tau^T r_t(\xi)d\xi\right\}}{\exp\left\{\int_t^T r_t(\xi)d\xi\right\}} d\tau. \end{aligned} \tag{A.8}$$

Taking the limit when time $T \to \infty$, using the TVC $\lim_{T\to\infty}\left[a(T)\lambda_t e^{-\int_t^T r_t(\xi)d\xi}\right] = 0$, and simplifying the exponential terms, one gets,

$$\begin{aligned} 0 &= a_t + \int_t^{\infty} [w_t(\tau) + p_R(\tau) * R(\tau)] \cdot e^{-\int_t^{\tau} r_t(\xi)d\xi} d\tau \\ &- \frac{1}{(\lambda_t)^{\sigma}} \cdot \int_\tau^{\infty} \theta(\tau-t)^{\sigma} \cdot \exp\left\{-(1-\sigma) \int_t^{\tau} r_t(\xi)d\xi\right\} d\tau. \end{aligned}$$

Solving for λ_t leads to

$$\lambda_t = \left[\frac{\int_t^{\infty} \theta(\tau-t)^{\sigma} \cdot \exp\left\{-(1-\sigma)\int_t^{\tau} r_t(\xi)d\xi\right\} d\tau}{a_t + \int_t^{\infty} [w_t(\tau) + p_R(\tau) * R(\tau)] \cdot e^{-\int_t^{\tau} r_t(\xi)d\xi} d\tau}\right]^{\frac{1}{\sigma}}. \tag{A.9}$$

Plugging equation (A.9) into the consumption expression (A.6), we get the following consumption expression,

$$c_t(s) = \theta(s-t)^{\sigma} \cdot e^{\sigma \int_t^s r_t(\xi)d\xi} \cdot \left(\frac{a_t + \int_t^{\infty} [w_t(\tau) + p_R(\tau) * R(\tau)] \cdot \exp\left\{-\int_t^{\tau} r_t(\xi)d\xi\right\} d\tau}{\int_t^{\infty} \theta(\tau-t)^{\sigma} \cdot \exp\left\{-(1-\sigma)\int_t^{\tau} r_t(\xi)d\xi\right\} d\tau}\right). \tag{A.10}$$

As we are interested in the study of the naive agent, this means $s = t$ which helps with $\theta(s-t)^{\sigma} = \theta(0)^{\sigma} = 1^{\sigma} = 1$ to get,[27]

$$c_t^N(t) = \left(\frac{a_t + \int_t^{\infty} [w_t(\tau) + p_R(\tau) * R(\tau)] \cdot \exp\left\{-\int_t^{\tau} r_t(\xi) d\xi\right\} d\tau}{\int_t^{\infty} \theta(\tau - t)^{\sigma} \cdot \exp\left\{-(1-\sigma)\int_t^{\tau} r_t(\xi) d\xi\right\} d\tau} \right). \tag{A.11}$$

Observe that in the numerator, one could make use of the change of variable $\tau = s$, as time s does not appear and will be the decisions made in the future. This result is similar to the one obtained in Cabo et al. (2015), but we now consider a broader income that also comes from the resource sector,

$$c_t^N(t) = \left(\frac{a_t + \int_t^{\infty} [w_t(s) + p_R(s) * R(s)] \cdot \exp\left\{-\int_t^{s} r_t(\xi) d\xi\right\} ds}{\int_t^{\infty} \theta(s - t)^{\sigma} \cdot \exp\left\{-(1-\sigma)\int_t^{s} r_t(\xi) d\xi\right\} ds} \right). \tag{A.12}$$

4.8.3 Proof of Lemma 1

Proof. Starting with the definition of $\lambda_{Na,\theta}^{log}$,

$$\begin{aligned}
\lambda_{Na,\theta}^{log} &= \int_0^{\infty} \rho(j) \frac{\theta(j)}{\int_0^{\infty} \theta(i) di} dj, \quad \text{as } \rho(j) = \left[-\frac{\dot{\theta}(j)}{\theta(j)}\right], \\
&= \int_0^{\infty} \left[-\frac{\dot{\theta}(j)}{\theta(j)}\right] \frac{\theta(j)}{\int_0^{\infty} \theta(i) di} dj, \quad \text{as } \int_0^{\infty} \theta(i) di \text{ is not a function of } j, \\
&= \frac{1}{\int_0^{\infty} \theta(i) di} \int_0^{\infty} \left[-\frac{d\theta(j)}{dj}\right] dj, \\
&= \frac{1}{\int_0^{\infty} \theta(i) di} \left[-\theta(j)\Big|_0^{\infty}\right] = \frac{1}{\int_0^{\infty} \theta(i) di},
\end{aligned}$$

Making use of the equivalent present value (4.35) where $\hat{\rho} = [\int_0^{\infty} \theta(j) dj]^{-1}$one gets

$$\lambda_{Na,\theta}^{log} = \hat{\rho}.$$

■

[27]When the Naive agent makes a plan at the beginning t, she does not know that she will "procrastinate" or not follow the plan she had in mind. Thus, when thinking about the optimal path at time t, she sets the plan, but later on, at $s > t$ she will recalculate, and this new time will again be the initial time of the planning horizon. This is why we set $t = s$.

4.8.4 Proof of Proposition 4.1

Proof. We prove here the case for $\sigma < 1$. The study of $\sigma > 1$ is analogous. We will prove that for $\sigma < 1 \iff \lambda^{ces}_{Na,\theta} > \hat{\rho}$. From the definition of $\lambda^{ces}_{Na,\theta}$, in 4.42, and making use of Lemma 1 we can write the former as

$$\lambda^{ces}_{Na,\theta} = \frac{\int_0^\infty \rho(j)\theta(j)^\sigma \exp\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\}dj}{\int_0^\infty \theta(i)^\sigma \exp\{-(1-\sigma)\int_t^s r_t(\xi)d\xi\}di} > \frac{\int_0^\infty \rho(j)\theta(j)dj}{\int_0^\infty \theta(i)di} = \hat{\rho}$$

Knowing that the integral on the exponents will be finite, and writing it making us of the First Mean Value Theorem for definite integrals, where $r_t : [t,s] \to \mathbb{R}_{++}$ is a continuous function over the interval $[t,s]$, then there exist an image of r_t, $r(c) = \bar{r} \in \mathbb{R}_{++}$ such that $t < c < s$ where $\int_t^s r_t(\xi)d\xi = \bar{r}\cdot(s-t) = \bar{r}\cdot j$. Using this, we can write the previous expression as

$$\frac{\int_0^\infty \rho(j)\theta(j)^\sigma \exp\{-(1-\sigma)\bar{r}\cdot j\}dj}{\int_0^\infty \theta(j)^\sigma \exp\{-(1-\sigma)\bar{r}\cdot j\}di} > \frac{\int_0^\infty \rho(j)\theta(j)dj}{\int_0^\infty \theta(j)di}.$$

First of all, one should mention that the integrals of all the previous expressions do converge. One approach to prove the previous inequality could be to study which function is bigger and then compare the ratios. As the image of $\rho(j) \in (0,1)$, any function multiplied by this will be smaller than the given function. Therefore, the function $\rho(j)\theta(j)^\sigma \exp\{-(1-\sigma)\bar{r}\cdot j\}$ is smaller than $\theta(j)^\sigma \exp\{-(1-\sigma)\bar{r}\cdot j\}$. This means that the integral (area) of the function in the numeration of the LHS is lower than the integral of the function in the denominator of the LHS. Therefore, the ratio on the LHS is lower than one. However, the same is true for the RHS, where the function in the numerator is smaller than the one on the denominator. Thus, the ratio of the integrals is also lower than one but positive. Since this is not enough argument to see that the ratio on the LHS is bigger than the ratio on the RHS, we study which function in the numerator and denominator has the largest distance. If the distance between two functions is big, then the area of the above function will be much bigger than the area of the below function.

The distance between the function inside the integral in the numerator on the LHS and the function inside the integral in the denominator on the LHS is

$$\rho(j)\theta(j)^\sigma \exp\{-(1-\sigma)\bar{r}\cdot j\} - \theta(j)^\sigma \exp\{-(1-\sigma)\bar{r}\cdot j\} =$$

$$\theta(j)^\sigma \exp\{-(1-\sigma)\bar{r}\cdot j\}(\rho(j)-1).$$

Furthermore, the distance between the function inside the integral in the numerator on the RHS and the function inside the integral in the denominator on the RHS is

$$\rho(j)\theta(j) - \theta(j) = \theta(j)\left(\rho(j) - 1\right).$$

Therefore, it can be shown that the distance between the two functions on the LHS is higher than the distance between the two functions on the RHS,

$$\theta(j)^{\sigma} \exp\{-(1-\sigma)\bar{r}\cdot j\}\left(\rho(j) - 1\right) > \theta(j)\left(\rho(j) - 1\right).$$

As the image of $\rho(j) \in (0,1)$, it implies that $(\rho(j) - 1)$ is negative. Thus, multiplying both sides of the inequality by the negative term $\dfrac{1}{(\rho(j)-1)}$, which changes the inequality direction, shows that

$$\theta(j)^{\sigma} \exp\{-(1-\sigma)\bar{r}\cdot j\} < \theta(j),$$

which holds for all j. Thus, this shows that the sign of the inequality in the beginning was the correct one. Therefore, the ratio of the areas (integrals) numerator-denominator on the LHS is higher than the ratio of the areas (integrals) numerator-denominator on the RHS. One can see that for the case $\sigma > 1$, we get the inequality reversed, as now $\theta(j)^{\sigma} \exp\{-(1-\sigma)\bar{r}\cdot j\} > \theta(j)$.

■

4.8.5 BGP for the Naive Agent with CES Utility

From the FOC of the intermediate monopolist (4.20) and final production (4.54) notice that we can write

$$r = (1-\beta_1-\beta_2)^2 \frac{Y(s)}{K(s)}, \tag{A.13}$$

which will be constant over time. Thus, one can write it in terms of initial time t or future time s. Consequently, the modified Ramsey rule (4.40) or (4.41), can be rewritten as

$$\dot{c^{Na,ces}}(s) = c^{Na,ces}(s)\,\sigma\left[(1-\beta_1-\beta_2)^2 \frac{Y(s)}{K(s)} - \lambda^{ces}_{Na,\theta}\right]. \tag{A.14}$$

From the definition of the aggregate capital accumulation

$$\dot{K}(s) = \underbrace{\frac{1}{1-\beta_1-\beta_2} K(s)^{1-\beta_1-\beta_2}\left[N(s)L_F(s)\right]^{\beta_1}\left[N(s)R(s)\right]^{\beta_2}}_{=Y(s)} - c^{Na,ces}(s), \tag{A.15}$$

and from the previous Ramsey rule, we get,

$$\gamma_{\frac{c}{K}}^{Na,ces} = \frac{Y(s)}{K(s)}\left[\sigma(1-\beta_1-\beta_2)^2-1\right] - \sigma\lambda_{Na,\theta}^{ces} + \frac{c^{Na,ces}(s)}{K(s)}. \tag{A.16}$$

Noting that the law of motion of the innovation possibilities frontier, γ_N, is given by equation (4.27), we can derive a law of motion for the extraction of the natural resource $R(s)$. Plugging the finial production function (4.54) into (4.55) gives,

$$(1-\beta_1-\beta_2)\gamma_K + (\beta_1+\beta_2)\gamma_N + \beta_1\gamma_{L_F} - (1-\beta_2)\gamma_R = (1-\beta_1-\beta_2)^2\frac{Y(s)}{K(s)}. \tag{A.17}$$

Substituting the law of motion of aggregate capital equation (A.15) and the law of motion of the innovation possibilities frontier equation (4.27) into the previous expression (A.17), we get,

$$\begin{aligned}(1-\beta_2)\gamma_R^{Na,ces} = {} & (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{K(s)} - (1-\beta_1-\beta_2)\frac{c^{Na,ces}(s)}{K(s)} \\ & + \eta(\beta_1+\beta_2)(1-L_F(s)) + \beta_1\gamma_{L_F}.\end{aligned} \tag{A.18}$$

Now, in order to get the law of motion of the labor used in the final sector, γ_{L_F}, we proceed as follows. First, from the FOC of the final producer with respect to labor equation (4.13) and the equilibrium condition from the R&D sector equation (4.29) leads to

$$p_{RD} = \beta_1\frac{Y(s)}{\eta N(s)L_F(s)}. \tag{A.19}$$

Taking logs and differentiating with respect to time one obtains,

$$\gamma_{p_{RD}} = \gamma_Y - \gamma_N - \gamma_{L_F}. \tag{A.20}$$

Now, plugging the free entry conditions of the representative firm of the Intermediate-Good Sector (4.25), the FOC of the monopolist ν (4.20), the final production (4.54) and the law of motion of the innovation possibilities frontier (4.27), into the previous expression A.20 gives

$$\begin{aligned}(1-\beta_1-\beta_2)^2\frac{Y(s)}{K(s)} - \frac{\pi(\nu,s)}{p_{RD}(s)} = {} & (1-\beta_1-\beta_2)\gamma_K - (1-\beta_1-\beta_2)\eta(1-L_F(s)) \\ & -(1-\beta_1)\gamma_{L_F} + \beta_2\gamma_R.\end{aligned} \tag{A.21}$$

From the FOC of final good producer (4.12), which is the demand for machine $\nu \in [0, N(s)]$ and the profits of the monopolist producing input ν, equation (4.21) we get

$$\pi(\nu,s) = (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{N(s)}. \tag{A.22}$$

Dividing the previous profits expression by the prices charged by the representative firm of the R&D sector $p_{RD}(s)$, i.e., price of a patent, equation (A.19) leads to

$$\frac{\pi(\nu,s)}{p_{RD}(s)} = \eta L_F(s)\frac{(1-\beta_1-\beta_2)(\beta_1+\beta_2)}{\beta_1}. \tag{A.23}$$

Substituting the growth rate of aggregate capital in (A.15) and the previous expression (A.23) into (A.21) results in

$$\begin{aligned}(1-\beta_1)\gamma_{L_F} &= (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{K(s)} - (1-\beta_1-\beta_2)\frac{c^{Na,ces}(s)}{K(s)} \\ &= -(1-\beta_1-\beta_2)\eta(1-L_F(s)) + \beta_2\gamma_R + \eta L_F(s)\frac{(1-\beta_1-\beta_2)(\beta_1+\beta_2)}{\beta_1}.\end{aligned} \tag{A.24}$$

We can now write a system of equations formed by (A.18) and (A.24) that will define the growth rate of the extraction γ_R (equation 4.92) and the growth rate of the amount of labor used to produce the final good γ_{L_F} (equation 4.93):

$$\begin{aligned}\gamma_R^{Na,ces} &= (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c^{Na,ces}(s)}{K(s)} \\ &\quad + \eta(1-L_F^{Na,ces}(s))\frac{\beta_2}{1-\beta 1-\beta_2} + \eta L_F^{Na,ces}(s)(\beta_1+\beta_2),\end{aligned} \tag{A.25}$$

$$\begin{aligned}\gamma_{L_F}^{Na,ces} &= (\beta_1+\beta_2)\frac{Y(s)}{K(s)} - \frac{c^{Na,ces}(s)}{K(s)} + \eta(1-L_F^{Na,ces}(s))\left[\frac{\beta_2-(1-\beta_1-\beta_2)}{1-\beta_1-\beta_2}\right] \\ &\quad + \eta L_F^{Na,ces}(s)\frac{(\beta_1+\beta_2)(1-\beta_2)}{\beta_1}.\end{aligned} \tag{A.26}$$

Finally, in order to get the growth rate of output per capital, $\gamma_{\frac{Y}{K}}$ we proceed as follows. From the final production (4.54), the law of motion of the aggregate capital (A.15), the dynamics of the innovation possibilities frontier (4.27), and growth rate of the extraction of the natural resources (A.25) and the growth rate of the labor used in the final good production (A.26), one can derive an expression for $\gamma_{\frac{Y}{K}}$. Dividing (4.54) by $K(s)$ taking logs and differentiating with respect to time s gives

$$\gamma_{\frac{Y}{K}} = (\beta_1+\beta_2)\gamma_N - (\beta_1+\beta_2)\gamma_K + \beta_1\gamma_{L_F} + \beta_2\gamma_R. \tag{A.27}$$

Plugging in equations (A.15), (4.27), (A.25) and (A.26 into the previous expression (A.27), we finally get

$$\gamma_{\frac{Y}{K}} = (\beta_1+\beta_2)\eta L_F(S) - (1-\beta_1-\beta_2)(\beta_1+\beta_2)\frac{Y(s)}{K(s)} + \frac{\beta_2}{1-\beta_1-\beta_2}\eta(1-L_F(s)). \tag{A.28}$$

4.8.6 Properties of the Steady State for the Naive Agent with CES Utility

In the steady state, taking logs and differentiation with respect to time for the final production (4.54) gives,

$$\gamma^{i,h} = \gamma_N^{i,h} + \frac{\beta_2}{\beta_1+\beta_2}\gamma_R^{i,h}, \tag{A.29}$$

where this expression will be valid for agent $i \in \{Na\}$ with utility $h \in \{ces,\}$.

Making use of the modified Euler equation (A.14), which gives the growth rate of consumption and (4.55), i.e., the differences in growth rates between the final output and the extraction of the resource, gives the growth rate of the extraction of the natural resource

$$\gamma_R^{Na,ces} = \gamma\left(\frac{\sigma-1}{\sigma}\right) - \lambda_{Na,\theta}^{ces}. \tag{A.30}$$

Plugging in A.30 into the previous A.29 gives

$$\gamma^{Na,ces} = \left[(\beta_1+\beta_2)\gamma_N^{Na,ces} - \beta_2\lambda_{Na,\theta}^{ces}\right]\left[\frac{\sigma}{\beta_2+\sigma\beta_2}\right]. \tag{A.31}$$

Moreover, from the dynamics of the innovation possibilities frontier (4.27), and noting that $L_{R\&D}(s) = 1-L_F(s)$, it follows that in the steady state[28]

$$\gamma_N^{Na,ces} - \eta = -\eta L_F^{Na,ces*}. \tag{A.32}$$

[28] Observe that in the steady state, labor used in the final sector and the R&D sector will be stationary. This is why we will use the * notation.

Using now the growth rates of the extraction of the resource (4.92) and the growth rate of the labor used in the final output production (4.93), together with the fact that in the steady state $\gamma_{L_F} = 0$, one gets,

$$\gamma_R^{Na,ces} = \eta(1 - L_F^{Na,ces*}) - \eta L_F^{Na,ces*} \frac{(\beta_1 + \beta_2)(1 - \beta_1 - \beta_2)}{\beta_1}. \tag{A.33}$$

Using the previous expression together with A.30 gives the value of the labor force used in the final sector in the steady state

$$L_F^{Na,ces*} = \frac{\beta_1 \left[\eta + \lambda_{Na,\theta}^{ces} - \left(\frac{\sigma-1}{\sigma}\right)\gamma^{Na,ces}\right]}{\eta\left[\beta_1 + (\beta_1 + \beta_2)(1 - \beta_1 - \beta_2)\right]}. \tag{A.34}$$

Plugging now this expression (A.34) into A.32, leads to the growth rate of the new intermediate machines,

$$\gamma_N^{Na,ces} = \eta - \left(\frac{\beta_1 \left[\eta + \lambda_{Na,\theta}^{ces} - \left(\frac{\sigma-1}{\sigma}\right)\gamma^{Na,ces}\right]}{\beta_1 + (1 - \beta_1 - \beta_2)(\beta_1 + \beta_2)}\right). \tag{A.35}$$

Finally, in order to get the growth rate of the economy, we substitute (A.30) and (A.35) into (A.29), which gives

$$\gamma^{Na,ces} = \sigma \cdot \frac{\lambda_{Na,\theta}^{ces}\left\{\beta_1^2(1-\beta_2) + \beta_1\beta_2(3 - 2\beta_2) + (1-\beta_2)\beta_2^2\right\} - \eta(1-\beta_1-\beta_2)(\beta_1+\beta_2)^2}{\beta_1^2\left[\beta_1\sigma + \beta_2(2\sigma+1) - (\sigma+1)\right] + \beta_1\beta_2\left[\beta_2(\sigma+2) - 3\right] - \beta_2^2(1-\beta_2)}. \tag{A.36}$$

4.8.7 Proof of Proposition 4.3

Proof. One just need to prove that $\gamma_R^{Na,} = -\hat{\rho}$. From Proposition 4.2, the growth rate of the extraction of the resource $\gamma_R^{Na,}$ is given by equation (4.72). Substituting the value of the labor force used in the final labor in the steady state equation (4.71) and simplifying leads to

$$\gamma_R^{Na,} = -\lambda_{Na,}^{log}.$$

Therefore, all that remains now is to prove that $\lambda_{Na,}^{log} = \hat{\rho}$. This has been proved in Lemma 1. ∎

4.8.8 Proof of Proposition 4.5

Proof. Here we will show the case for $\sigma < 1$. The case for $\sigma > 1$ is obtained similarly.

$$\frac{R_{Na}^{log}(s)}{S_{Na}^{log}(s)} = \underbrace{-\underbrace{\gamma_R^{Na,log}}_{<0}}_{>0} < \frac{R_{Na}^{ces}(s)}{S_{Na}^{ces}(s)} = \underbrace{-\underbrace{\gamma_R^{Na,ces}}_{<0}}_{>0},$$

and using Propositions 4.3 and 4.4,

$$\hat{\rho} < \lambda_{Na,\theta}^{ces} - \gamma_{Na}^{ces} \cdot \left(\frac{\sigma_h - 1}{\sigma_h}\right).$$

As for $\sigma < 1$, $\hat{\rho} < \lambda_{Na,\theta}^{ces}$ (see Proposition 4.1),

$$\underbrace{\hat{\rho} - \lambda_{Na,\theta}^{ces}}_{<0} < \underbrace{-\gamma_{Na}^{ces} \cdot \underbrace{\left(\frac{\sigma_h - 1}{\sigma_h}\right)}_{<0}}_{>0}.$$

which proofs that the LHS is smaller than the RHS. ■

4.8.9 Proof of Lemma 4.6

Proof. In this proof, one should interpret a general discount function $\theta(s-t)$ different than the exponential function. From

$$\frac{R_{Na,Exp}^{Ces,\sigma<1}(s)}{S_{Na}^{Ces,\sigma<1}(s)} < \frac{R_{Na,\theta}^{Ces,\sigma<1}(s)}{S_{Na}^{Ces,\sigma<1}(s)},$$

and using the Markovian strategy Remark (4.2), one can write

$$-\gamma_R^{Exp,Ces,\sigma<1} < -\gamma_R^{\theta,Ces,\sigma<1},$$

$$\gamma_R^{\theta,Ces,\sigma<1} < \gamma_R^{Exp,Ces,\sigma<1}.$$

Using Proposition 4.4 involving the growth rate of the economy $\gamma_{Na,\theta}^{Ces}$, it leads to

$$\gamma_{Na,\theta}^{Ces,\sigma<1} \cdot \left(\frac{\sigma-1}{\sigma}\right) - \lambda_{Na,\theta}^{Ces,\sigma<1} < \gamma_{Na,Exp}^{Ces,\sigma<1} \cdot \left(\frac{\sigma-1}{\sigma}\right) - \lambda_{Na,Exp}^{Ces,\sigma<1}.$$

Using the fact that when agents have exponential discounting $\lambda_{Na,Exp}^{Ces}$ collapses to $\hat{\rho}$, we can write

$$\hat{\rho} - \lambda_{Na,\theta}^{Ces,\sigma<1} < \gamma_{Na,Exp}^{Ces,\sigma<1} \cdot \left(\frac{\sigma-1}{\sigma}\right) - \gamma_{Na,\theta}^{Ces,\sigma<1} \cdot \left(\frac{\sigma-1}{\sigma}\right)$$

$$\Leftrightarrow \hat{\rho} - \lambda_{Na,\theta}^{Ces,\sigma<1} < \left(\frac{\sigma-1}{\sigma}\right) \left[\gamma_{Na,Exp}^{Ces,\sigma<1} - \gamma_{Na,\theta}^{Ces,\sigma<1}\right].$$

From (4.70), one can rewrite the growth rate of the economy $\gamma_{Na,\theta}^{Ces}$ as $\gamma_{Na,\theta}^{Ces} = \frac{\sigma}{A_0}\left(\lambda_{Na,\theta}^{Ces} \cdot B_0 - C_0\right)$, where A_0, B_0 and C_0 are shared positive constant. Thus,

$$\Leftrightarrow \hat{\rho} - \lambda_{Na,\theta}^{Ces,\sigma<1} < (\sigma-1) \cdot \frac{B_0}{A_0} \left[\lambda_{Na,Exp}^{Ces,\sigma<1} - \lambda_{Na,\theta}^{Ces,\sigma<1}\right].$$

Using as before $\lambda_{Na,Exp}^{Ces} = \hat{\rho}$, one gets

$$\Leftrightarrow \hat{\rho} - \lambda_{Na,\theta}^{Ces,\sigma<1} < (\sigma-1) \cdot \frac{B_0}{A_0} \left[\hat{\rho} - \lambda_{Na,\theta}^{Ces,\sigma<1}\right].$$

Using Proposition 4.1 for the case $\sigma < 1$, we know that $\lambda_{Na,\theta}^{Ces,\sigma<1} > \hat{\rho}$. Thus, the LHS is negative. As on the RHD one has the negative term $\left[\hat{\rho} - \lambda_{Na,\theta}^{Ces,\sigma<1}\right]$ multiplied by a the negative $(\sigma-1)$, one gets a positive RHD. Thus, we have proved the inequality. The argument follows the same logic for the case of $\sigma > 1$ with the inequality will reverse.

■

4.8.10 Derivation of the growth rate of Extraction over Resource left

We now derive equation (4.90). From $\frac{R(s)}{S(s)}$, the time derivative of this quotient is given by

$$\dot{\left(\frac{R(s)}{S(s)}\right)} \equiv \frac{d}{ds}\left[\frac{R(s)}{S(s)}\right] = \frac{\dot{R}(s)S(s) - R(s)\dot{S}(s)}{S(s)^2} = \frac{\dot{R}(s)}{S(s)} - \frac{R(s)}{S(s)}\frac{\dot{S}(s)}{S(s)}.$$

From the dynamics of the resource, dividing both sides by $S(s)$ leads to $\frac{\dot{S}(s)}{S(s)} = -\frac{R(s)}{S(s)}$. Substituting it in the previous expressions gives

$$\frac{d}{ds}\left[\frac{R(s)}{S(s)}\right] = \frac{\dot{R(s)}}{S(s)} + \frac{R(s)}{S(s)}\frac{R(s)}{S(s)}.$$

Thus, the growth rate of $\frac{R(s)}{S(s)}$ is given by

$$\gamma_{\frac{R}{S}} \equiv \frac{\frac{d}{ds}\left[\frac{R(s)}{S(s)}\right]}{\frac{R(s)}{S(s)}} = \underbrace{\frac{\dot{R}(s)}{R(s)}}_{\equiv \gamma_R} + \frac{R(s)}{S(s)}. \tag{A.37}$$

4.8.11 The Equilibrium Values of the Steady State

Solving the corresponding system of differential equations (4.91), (4.92), (4.93) and (4.94) gives the steady state solution (with an asterisk $*$)

$$\left(\frac{c}{K}\right)^* = -\frac{(\beta_1+\beta_2)\left(\sigma\cdot\lambda^{ces}_{Na,\theta}\left(\beta_1^3(1-\beta_2)-3\beta_1^2(1-\beta_2)^2+\beta_1(((8-3\beta_2)\beta_2-8)\beta_2+2)-\beta_2(\beta_2-1)^3\right)+\eta\sigma(\beta_1+\beta_2)(\beta_1+\beta_2-1)^3+\eta(\beta_1+\beta_2)(1-\beta_1-\beta_2)\right)}{(\beta_1+\beta_2-1)^2\left(\beta_1^3\sigma+\beta_1^2(2\beta_2\sigma+\beta_2-\sigma-1)+\beta_1\beta_2(\beta_2(\sigma+2)-3)+(\beta_2-1)\beta_2^2\right)}, \tag{A.38}$$

$$\left(\frac{R}{S}\right)^* = \frac{(\beta_1+\beta_2)\left[\lambda^{ces}_{Na,\theta}\sigma\left(\beta_1^2-2\beta_1(1-\beta_2)-(1-\beta_2)\beta_2\right]+\eta\sigma(1-\beta_1-\beta_2)(\beta_1+\beta_2)-\eta(1-\beta_1-\beta_2)(\beta_1+\beta_2)\right)}{\beta_1^3\sigma+\beta_1^2\left[\beta_2(1+2\sigma)-(1+\sigma)\right]+\beta_1\beta_2(\beta_2(\sigma+2)-3)-(1-\beta_2)\beta_2^2}, \tag{A.39}$$

$$(L_F)^* = -\frac{\beta_1\left[\beta_1(\eta+\lambda^{ces}_{Na,\theta}\cdot\sigma)-\beta_2\eta(\sigma-2)+\beta_2\lambda^{ces}_{Na,\theta}\cdot\sigma\right]}{\eta\left(\beta_1^3\sigma+\beta_1^2\left[\beta_2(1+2\sigma)-(1+\sigma)\right]+\beta_1\beta_2(\beta_2(\sigma+2)-3)-(1-\beta_2)\beta_2^2\right)}, \tag{A.40}$$

$$\left(\frac{Y}{K}\right)^* = \frac{\beta_1\lambda^{ces}_{Na,\theta}\cdot\sigma\left(2\beta_1\beta_2-(1-\beta_1)\beta_1+\beta_2^2\right)-\eta(1-\beta_1-\beta_2)(\beta_1+\beta_2)^2}{(\beta_1+\beta_2-1)^2\left(\beta_1^3\sigma+\beta_1^2\left[\beta_2(1+2\sigma)-(1+\sigma)\right]+\beta_1\beta_2(\beta_2(\sigma+2)-3)-(1-\beta_2)\beta_2^2\right)}. \tag{A.41}$$

4.8.12 The Jacobian matrix of the dynamical system

The Jacobian matrix of the dynamical system is given by

$$J(.)=\begin{pmatrix} -\left(\frac{Y}{K}\right)^{*}(1-\beta_1-\beta_2)(\beta_1+\beta_2) & 0 & \left(\beta_1+\beta_2-\frac{\beta_2}{1-\beta_1-\beta_2}\right)\eta\left(\frac{Y}{K}\right)^{*} \\ \left(\frac{c_{Na}^{ces}}{K}\right)^{*}\left[\sigma(1-\beta_1-\beta_2)^2-1\right] & \left(\frac{c_{Na}^{ces}}{K}\right)^{*} & 0 \\ L_F^{*}(\beta_1+\beta_2) & -L_F^{*} & \eta L_F^{*}\left(\frac{(1-\beta_2)(\beta_1+\beta_2)}{\beta_1}-\frac{\beta_2}{1-\beta_1-\beta_2}+1\right) \end{pmatrix} \tag{A.42}$$

In order to study the stability of the system, one should look at the behavior of the Jacobian matrix analyzed at the equilibrium points. However, if one tries to plug in all the equilibrium values (A.38- A.41) in the former matrix it gets extremely demanding. This is why we proceed and study the sign of the eigenvalues of the matrix numerically.[29]

4.8.13 3D Vector Field

Here we show the three dimensional vector field of the system of three differential equations (4.91), (4.93) and (4.94).

In the following link, we have a video of the 3D vector field rotating, so the 3D aspect of the field is much appreciated, `https://drive.google.com/open?id=1EthR72_Yvz--4o3Yi5tcYP7OxIeVPiMp&authuser=carbas15540gmail.com&usp=drive_fs`

4.8.14 Closed form expressions of consumer welfare

The expressions for the welfare under different discount functions $\theta(s-t)$ is given by (4.111) for the general CES utility function and by (4.112) for the particular logarithmic case. One should notice that all the integrals converge. We now show the closed-form expression of the welfare for the different discounts and different utility functions. We start with the log case under exponential discounting, and with the condition $\{\hat{\rho}\geq 0\}$ for the integral to converge, we have

$$W_{exp}^{log}=\frac{1}{\hat{\rho}}\left[ln\left(c_0\right)-\psi\cdot lnP_0\right]+\left[\gamma_{exp}^{log}-\psi\cdot\gamma_{P,Exp}^{log}\right]\left(\frac{1+\hat{\rho}}{\hat{\rho}^2}\right).$$

The log consumer welfare under Tsoukis discounting with $\{\delta\geq 0\}\wedge\{\rho>0\}$ gives

[29] However, theoretically one could get the eigenvalues algebraically, but it will require a few pages of writing per eigenvalue, which we avoid for obvious practical reasons.

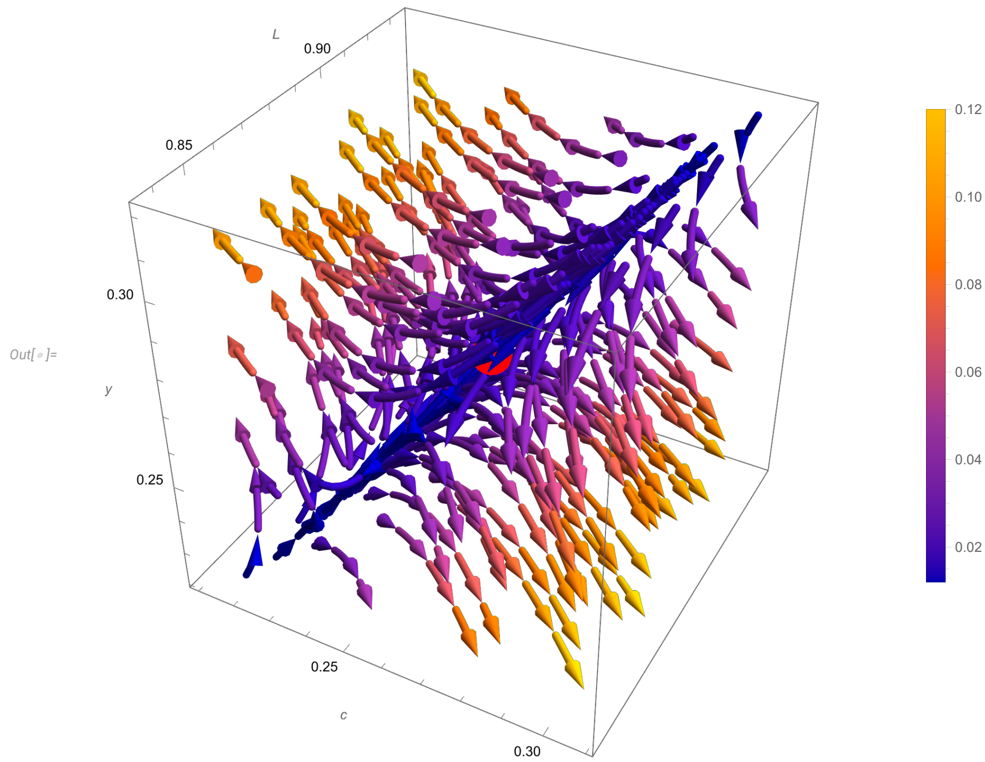


Figure A.1: 3D Vector Field under Exponential Discounting

$$W_{Ts}^{log} = \frac{1}{\hat{\rho}}\left[ln\left(c_0\right) - \psi \cdot lnP_0\right] + \left[\gamma_{Ts}^{log} - \psi \cdot \gamma_{P,Ts}^{log}\right]$$

$$\times \frac{\delta^{-\frac{\varphi}{\delta}-3}}{\rho^2 \Gamma\left(\frac{\varphi}{\delta}\right)} \left\{(2\delta-\varphi)\Gamma\left(\frac{\varphi}{\delta}-2\right)\left[\rho^2\delta^{\varphi/\delta} + \delta e^{\rho/\delta}\rho^{\varphi/\delta}(\rho+\varphi-\delta)\Gamma\left(2-\frac{\varphi}{\delta}\right)\right.\right.$$

$$\left. -\delta e^{\rho/\delta}\rho^{\varphi/\delta}(\rho+\varphi-\delta)\Gamma\left(2-\frac{\varphi}{\delta},\frac{\rho}{\delta}\right)\right]$$

$$-\pi\delta^2\rho^{\frac{\varphi}{\delta}}e^{\frac{\rho}{\delta}}(\rho+\varphi-\delta)\csc\left(\frac{\pi\varphi}{\delta}\right)$$

$$+\delta^2 t e^{\rho/\delta}\rho^{\frac{\delta+\varphi}{\delta}}\left[(\varphi-\delta)\Gamma\left(\frac{\varphi}{\delta}-1\right)\Gamma\left(1-\frac{\varphi}{\delta},\frac{\rho}{\delta}\right)\right.$$

$$\left.\left. +\pi\delta\csc\left(\frac{\pi\varphi}{\delta}\right) + (\delta-\varphi)\Gamma\left(\frac{\varphi}{\delta}-1\right)\Gamma\left(1-\frac{\varphi}{\delta}\right)\right]\right\}.$$

The consumer welfare for log utility under the convex combination of two exponentials with $\{\rho_1 > 0\} \wedge \{\rho_2 > 0\}$ is,

$$W^{log}_{ConExp} = \frac{1}{\hat{\rho}}\left[ln\left(c_0\right) - \psi \cdot lnP_0\right]$$

$$+\left[\gamma^{log}_{ConExp} - \psi \cdot \gamma^{log}_{P,ConExp}\right] \cdot \left[\frac{z}{\rho_1^2} + \frac{(1-z)(1+\rho_2 \cdot t)}{\rho_2^2} + \frac{z}{\rho_1}t\right].$$

We now focus on the general CES utility function. Studying the exponential discount function (time-consistent), with $-\gamma^{ces}_{exp} \cdot \left(\frac{1-\sigma}{\sigma}\right) < \hat{\rho}$ for the first integral term to converge and with the condition $(1+b) \cdot \gamma^{ces}_{P,exp} < \hat{\rho}$ for the second integral involving the pollution growth rate we get

$$W^{ces}_{exp} = \frac{1}{\hat{\rho}}\left(\frac{\sigma}{1-\sigma} + \frac{\psi}{1+b}\right) - \frac{\sigma}{1-\sigma} c_0^{-\left(\frac{1-\sigma}{\sigma}\right)}\left[\frac{\sigma e^{\left(\frac{1-\sigma}{\sigma}\right)\cdot \gamma^{ces}_{exp}\cdot t}}{(1-\sigma)\gamma^{ces}_{exp} + \sigma\hat{\rho}}\right]$$

$$+\psi \frac{P_0^{(1+b)}}{1+b}\left[\frac{\exp\left\{(1+b)\cdot \gamma^{ces}_{P,exp}\cdot t\right\}}{(1+b)\cdot \gamma^{ces}_{P,exp} - \hat{\rho}}\right].$$

For the Tsoukis discount function with the condition $\{\delta \geq 0\} \wedge \left\{-\gamma^{ces}_{Ts}\left(\frac{1-\sigma}{\sigma}\right) < \rho\right\}$ for the first integral and $\{\delta \geq 0\} \wedge \left\{(1+b) \cdot \gamma^{ces}_{P,Ts} < \rho\right\}$ for the second integral, we get

$$W^{ces}_{Ts} = \frac{1}{\hat{\rho}}\left(\frac{\sigma}{1-\sigma} + \frac{\psi}{1+b}\right)$$

$$-\frac{\sigma}{1-\sigma} c_0^{-\left(\frac{1-\sigma}{\sigma}\right)}\left[\frac{\sigma \delta^{-\frac{\varphi}{\delta}}\left[\gamma^{ces}_{Ts}\cdot\left(\frac{1-\sigma}{\sigma}\right)+\rho\right]^{\varphi/\delta} \exp\left\{\frac{\rho + \frac{\gamma^{ces}_{Ts}\cdot(1-\sigma)(1-\delta t)}{\sigma}}{\delta}\right\}\Gamma\left(1-\frac{\varphi}{\delta}, \frac{(1-\sigma)\cdot\gamma^{ces}_{Ts}+\rho}{\delta\sigma}\right)}{\gamma^{ces}_{Ts}(1-\sigma)+\sigma\rho}\right]$$

$$-\psi \frac{P_0^{(1+b)}}{1+b}\left[\delta^{-\frac{\varphi}{\delta}}\left[\rho - (1+b)\gamma^{ces}_{P,Ts}\right]^{\frac{\varphi}{\delta}-1} e^{\frac{(1+b)\cdot\gamma^{ces}_{P,Ts}\cdot(\delta t-1)+\rho}{\delta}}\Gamma\left(1-\frac{\varphi}{\delta}, \frac{\rho-(1+b)\cdot\gamma^{ces}_{P,Ts}}{\delta}\right)\right].$$

For the linear convex combination of exponentials, with the conditions $\left\{-\gamma^{ces}_{ConExp}\cdot\left(\frac{1-\sigma}{\sigma}\right) < \rho_1\right\} \wedge \left\{-\gamma^{ces}_{ConExp}\cdot\left(\frac{1-\sigma}{\sigma}\right) < \rho_2\right\}$ together with the pollution integral condition $\left\{(1+b)\cdot\gamma^{ces}_{P,ConExp} < \rho_1\right\} \wedge \left\{(1+b)\cdot\gamma^{ces}_{P,ConExp} < \rho_2\right\}$ one gets

$$W^{ces}_{ConExp} = \frac{1}{\hat{\rho}}\left(\frac{\sigma}{1-\sigma} + \frac{\psi}{1+b}\right)$$

$$-\frac{\sigma}{1-\sigma} c_0^{-\left(\frac{1-\sigma}{\sigma}\right)}\left[\sigma e^{-\frac{(1-\sigma)}{\sigma}\cdot\gamma^{ces}_{ConEx}\cdot t} \times \left[\frac{(1-\sigma)\gamma^{ces}_{ConEx} + (1-z)\rho_1\sigma + z\rho_2\sigma}{\left[(1-\sigma)\cdot\gamma^{ces}_{ConEx}+\rho_1\sigma\right]\cdot\left[(1-\sigma)\gamma^{ces}_{ConEx}+\rho_2\sigma\right]}\right]\right]$$

$$-\psi \frac{P_0^{(1+b)}}{1+b} \left[e^{(1+b)(\gamma^{ces}_{P,ConEx})\cdot t} \times \frac{(1-z)\rho_1 + z\rho_2 - (1-b)\cdot\gamma^{ces}_{P,ConEx}}{\left[(1+b)\cdot\gamma^{ces}_{P,ConEx} - \rho_1\right]\cdot\left[(1+b)\cdot\gamma^{ces}_{P,ConEx} - \rho_2\right]} \right].$$

5 Concluding remarks

The research presented in this doctoral thesis has explored the multifaceted relationships between dynamic games, environmental economics, economic growth, and behavioral economics. It has highlighted the important role these fields play in understanding the pressing challenges of our time, such as resource depletion, economic growth and its environmental impacts, adaptation to more efficient technologies, and pollution-related issues. By adopting an interdisciplinary approach and incorporating insights from behavioral economics, we have demonstrated the importance of addressing time-inconsistencies, status concerns, and regime shifts, paving the way for trying to get a more comprehensive and nuanced understanding of the strategic interactions among economic agents. This thesis has not only attempted to lay a foundation for future research, but has also sought to highlight the importance of taking human behavior into account in the development of effective and sustainable economic policies. Our findings emphasize the transformative potential of integrating behavioral components into economic models, seeking to provide a more realistic representation of the decision-making processes and capturing the essence of why this research may be of paramount importance to policymakers, stakeholders, and researchers alike. In this final chapter, we highlight the key findings and contributions of our research, discuss their potential implications for policymakers and stakeholders, and offer suggestions for future research avenues.

5.1 Key Findings and Contributions

The following key findings and contributions have emerged from our research:

Optimal switching and time-inconsistency: Our analysis of the optimal switching problem in Chapter 2 demonstrated the importance of considering time-inconsistency in decision-making processes. By accounting for agents with different degrees of sophistication, we were able to derive conditions for the optimal switching time and shed light on the consequences of time-inconsistency. Our theorems were applied to cases involving natural resource management and technology adoption, but the potential applications of these technical tools extend far beyond these contexts. Optimal switching models offer a diverse array of applications across numerous domains, helping decision-makers identify the ideal conditions and moments for transitioning between different processes or policies. For instance, manufacturing companies can apply optimal switching to determine the best time to upgrade production technologies or switch production methods, ensuring they remain competitive while maximizing profits. In environmental economics, optimal switching models can inform the adoption of eco-friendly technologies that boost efficiency and reduce emissions. These models also have intriguing applications in international economics, such as guiding countries' decisions on joining free trade agreements or climate change treaties. Furthermore, macroeconomic contexts like adopting a new currency and giving up control over monetary policy (e.g., European countries transitioning to the Euro) can also benefit from optimal switching analysis. Health economics is another field that can leverage optimal switching, with applications such as deciding when to quit smoking or begin a fitness lifestyle. Additionally, transport economics can employ optimal switching models to inform decisions regarding the adoption of more efficient, albeit costly, electric vehicles. All of the problems above have a transversal idea: incurring a cost (psychological or financial) right now, in anticipation of potentially improved future outcomes.

Therefore, recognizing the impact of time-inconsistency on decision-making processes can help policymakers design more effective policies and interventions. By considering the evolving preferences of agents and their potential deviations from long-term goals, policymakers can develop strategies that better align short-term incentives with long-term objectives.

Status concerns: In Chapter 3, we studied the strategic behavior of first symmetric players, and subsequently heterogeneous agents, in a dynamic game involving the extraction of renewable resources. Our examination of autarky and free trade scenarios revealed the profound impact of status concerns on resource exploitation, and its welfare implications. The insights gleaned from our analysis may have far-reaching implications for a diverse range of contexts, such as environmental policies, and international trade. In environmental policies, understanding the role of status concerns can lead to more sustainable resource extraction practices, improve our understanding of emissions by different actors, and facilitate the transition to green technologies. This understanding has the potential to help policymakers design better conservation strategies that account for the strategic behavior of competing agents. In international trade, addressing status concerns can help reduce trade imbalances and foster more equitable trading relationships between countries. In light of the Sustainable Development Goals (SDGs) driven by the United Nations, incorporating the role of status concerns in the analysis of renewable resource management and trade policies can provide valuable insights into the complex interplay between economic incentives, competition, and sustainability.

Consequently, understanding the role of status concerns in economic behavior can assist policymakers in designing policies that address social and economic inequalities. By recognizing the strategic behavior driven by status concerns, policymakers can create targeted interventions that reduce the negative consequences of status-seeking behaviors and promote economic prosperity and environmental sustainability.

Endogenous growth and human behavior: In Chapter 4, we explored the implications of time-inconsistency on an endogenous growth model with non-renewable resources and pollution. By incorporating human behaviors such as procrastination, we found that agents with time-inconsistent preferences could achieve higher levels of well-being than time-consistent agents under the observational equivalence principle. This result has tried to contribute to the ongoing debate in behavioral macroeconomics and offered valuable insights into the social implications of time-inconsistency. Specifically, we show that time-inconsistent agents with a constant elasticity of intertemporal substitution (CEIS) bigger than one have higher levels of economic growth. However, if households have a CEIS lower than one, the economy with time-consistent decision-makers has higher growth rates. Counterintuitively, we find that for any CEIS level, agents behaving time-inconsistently have higher discounted utilities than time-consistent agents. This gap becomes more significant as the CEIS level increases.

This doctoral thesis highlights the importance of incorporating psychological factors such as status concern, procrastination and time-inconsistency into economic models. By considering those factors that influence decision-making, we can better understand the implications of such behaviors on economic outcomes, improve our ability to make informed policy decisions, design more effective interventions, and better promote economic growth, social equity, environmental sustainability, and societal well-being.

The combination of dynamic games, environmental economics, and behavioral economics in our research allowed us to better understand the complex issues we investigated. By integrating these fields, we have tried to offer a more nuanced analysis of the strategic interactions among economic agents, as well as the implications of human behavior on economic growth and environmental sustainability.

5.2 Future Research Directions

Our interdisciplinary research has delved into the intersections of dynamic games, environmental economics, behavioral economics, and economic growth. However, as with any research, there are several potential avenues for future inquiry that can further enhance our understanding of these fields.

Expanding the analysis of status concerns: Future research could examine the role of status concerns in a broader range of economic contexts, such as agents deciding wealth accumulation and comparing with other agents in the macroeconomy, or agents comparing with their peers the quantity and quality of education they acquire, thus having relevance in contexts in labor markets and the study of endogenous growth models. Another potential application is the systemic competition and race in indispensable investment to develop new ideas and technology that keeps geopolitical powers in the lead and try to overtake their systemic rival. This could provide additional insights into the consequences of status-seeking behaviors and inform the design of more effective policies to address social and economic inequalities.

Studying the interplay between time-inconsistency and status concerns: Investigating the combined effects of time-inconsistency and status concerns on economic behavior and strategic interactions would provide a more comprehensive understanding of the complex dynamics at play in economic systems. This research could inform the development of policies that simultaneously address both time-inconsistency and status concerns, thereby promoting more sustainable and equitable outcomes. This research could also shed light on the underlying mechanisms driving behavior in various contexts, such as financial decision-making or the adoption of sustainable practices. By considering the ways in which individuals are influenced by their desire to keep up with others and their tendency towards time-inconsistency, we can gain a more nuanced understanding of how and why saving decisions are made. This knowledge can, in turn, inform the development of policies that promote greater financial stability and equal opportunities, by considering the complex interplay of individual behavior, social norms, and economic outcomes. Therefore, investigating the role of status concerns and time-inconsistency in saving decisions could represent an interesting area of research for those seeking to understand and address economic inequalities in today's societies.

Incorporating additional behavioral insights: Building on our interdisciplinary approach, future research could integrate further insights from behavioral economics, such as loss aversion, bounded rationality, or social preferences, into the analysis of dynamic games and environmental economics. This would offer a more complete picture of the psychological factors influencing economic behavior and could inform the design of policies that better account for human decision-making processes.

Another promising area for future research is the study of Mean Field Games with status concerns, which explores the implications of wealth evolution among agents in an economy. By examining the interplay between status concerns and wealth dynamics, this research could deepen our understanding of the dynamics of inequality and inform the design of effective policies to mitigate its negative effects in models with heterogeneous agents. Additionally, the development of climate change games with climate coalitions, in conjunction with international trade models, represents another area of study that I plan to pursue during my postdoctoral research at KU Leuven. The incorporation of political economy aspects into these frameworks could contribute to our understanding of the choices made by heterogeneous countries in response to climate change, and may subsequently guide the development of more effective climate policies.

In conclusion, our research has tried to contribute to the understanding of the complex challenges faced by our world today, through the integration of dynamic games, environmental economics, economic growth, and behavioral economics. By building on these findings and exploring the suggested avenues for future research, we can continue to advance our knowledge of the strategic interactions among economic agents and the implications for policy design, ultimately promoting a more sustainable and equitable future.

Finally, it is important to bear in mind that this PhD thesis is merely the beginning of a research career and lays the foundation and seeds for future ideas. I hope that my contributions can help answer the questions of the young student who witnessed the evolution and economic consequences of the 2008 crisis, and sparked an interest in the study of economics, without losing sight of the big questions and challenges that humanity will face in the coming decades.

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