Chapter 2

Magic and Super Magic Labelings; Definitions, Examples and Basic Results

2.1 Introduction

Recently, new life has been injected into the subject of magic labelings of graphs, mainly because of a paper by Ringel and Lladó [34]. However this concept was first introduced by Kotzig and Rosa [27] in 1970. It is worthwhile to mention that Kotzig and Rosa introduced this concept, under the name of magic valuations, and in [34] Ringel and Lladó redefined the same concept but this time under the name of edge-magic labelings. However, through this thesis we will use the names magic labeling and magic graphs for the sake of simplicity.

A *magic labeling* of a (p,q)-graph G is a bijective function $f : V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$ such that for any edge xy of $E(G)$, the sum $f(x)+f(xy)+f(y)=k$ is a constant, called the valence of the labeling f. If G admits a magic labeling, then such a graph is called a *magic graph*. Next, we will conduct a little survey about what was already known about magic graph until the point when we began our research. In 1970, Kotzig and Rosa [27] proved that the cycle C_n is magic for every $n \in \mathbb{N}$. They also proved that the graph nK_2 is magic if and only if n is odd and that the complete bipartite graph $K_{m,n}$ is always magic. Later, in 1974, Kotzig and Rosa [28] completely characterized the complete graphs that are magic as follows. The complete graph K_p is magic if and only if $p = 1,2,3,5,6$. At this point the research in this field stopped and it was not until 1996 when G. Ringel and A. Lladó [34], unaware of the work done by Kotzig and Rosa introduced once again the same concept, and proved several of the results already known by Kotzig and Rosa. In spite of this, some new results in the field were proved, as for instance the result that states that all caterpillars are magic. Also a very interesting result that appears in this paper by Ringel and Lladó is the one we now state.

Lemma 2.1. *Let G be a (p,q)-graph such that the degree of every vertex is odd, q is even, and $p+q \equiv 2 \pmod 4$. Then, G is not a magic graph.*

This lemma shows that graphs that are not magic are not hard to find and allows to construct infinite many graphs that are not magic.

Not only results were presented in these papers, but also problems and conjectures that are still open up to now can be found therein. For instance, Kotzig and Rosa asked if it is possible to characterize the set of magic 2-regular graphs, and they asked whether all trees admit magic labelings [27], [28]. Later on, Ringel and Lladó conjectured that all trees admit magic labelings [34]. This last conjecture has become very popular and it seems to be an extremely challenging problem. Motivated by the concept of magic labelings, Enomoto et al. [7], defined in 1992 the concept of super magic labelings as follows.

A *super magic labeling* of a (p,q)-graph G is a bijective function $f : V(G) \cup E(G) \to \{1,2,\ldots,p+q\}$ such that in addition of being a magic labeling of G, it satisfies the extra property that

$$f(V(G)) = \{1,2,\ldots,p\}.$$

If a graph admits a super magic labeling, then is called a *super magic graph*. In this same paper they proved that the cycle C_n is super magic if and only if n is odd. They also gave an interesting necessary condition for a graph to admit a super magic labeling, that we now state.

Lemma 2.2. *If a (p,q)-graph G is super magic, then $q \leq 2p-3$.*

It is interesting to mention that the previous lemma implies that the minimum degree of any super magic graph is at most three. This fact was first observed by Figueroa et al. in [17].

Although, the way the labeling was defined looks like a total labeling, thanks to a result by Figueroa et al. [17], we can redefine the concept of super magic labeling in such a way that only the vertices of the graphs are involved (and hence transform super magic labelings into vertex labelings). We do this in the next lemma.

Lemma 2.3. *A (p,q)-graph G is super magic if and only if there exists a bijective function $\overline{f}: V(G) \to \{1,2,\ldots,p\}$ such that the set*

$$S = \{\overline{f}(x) + \overline{f}(y) \mid xy \in E(G)\}$$

consist of q consecutive integers. In such a case, $\overline{f}$ extends to a super magic labeling of G with valence $k = p+q+\min S$.

Proof.

Let G be a super magic (p,q)-graph and f any super magic labeling of G with valence k. Then,

$$\begin{aligned} S &= \{k - f(xy) \mid xy \in E(G)\}, \\ &= \{k-(p+1), k-(p+2), \ldots, k-(p+q)\}. \end{aligned}$$

Thus, let $\overline{f} = f\mid_{V(G)}$.
Let G be any (p,q)-graph that admits a function $\overline{f}$ with the properties described in the statement of this lemma. Next, let $xy \in E(G)$ such that $f(x)+f(y) = \min S$. Then $\overline{f}$ extends to a super magic labeling f of G in the following manner. Let $f(x'y') = p+q+\min S - f(x') - f(y')$ for any edge $x'y' \in E(G)$. Thus $f(E(G)) = \{p+1, p+2, \ldots, p+q\}$, and therefore f is a super magic labeling of G. □

¿From now on, if f is any super magic labeling of any graph, we will call the function $\overline{f}$ described in the proof of Lemma 2.3, the canonical form of f.

E. Tesar and D. Craft informed us through personal communication they had proved that for any magic labeling f of a given (p,q)-graph G, the labeling $f': V(G) \cup E(G) \to \{1,\ldots,q\}$ defined by the rule

$$f'(w) = p+q+1-f(w) \text{ for all } w \in V(G) \cup E(G)$$

is also a magic labeling of G. A similar result can be obtained for super magic labelings using the canonical form.

Theorem 2.4. *If f is any super magic labeling of any (p,q)-graph G, then the function $f': V(G) \cup E(G) \to \{1,\ldots,p+q\}$ defined by the rule*

$$f'(w) = \begin{cases} 2p+q+1-f(w), & \text{if } w \in E(G); \\ p+1-f(w), & \text{if } w \in V(G); \end{cases}$$

is also a super magic labeling of G.

The canonical form of a super magic labeling has proven to be a very useful tool in order to study super magic labelings and super magic graphs, and many new results have been obtained using this form. Good examples are the following results obtained by Figueroa et al. in [13], and [17] respectively. The first of these results is basically a continuation of Lemma 2.2.

Lemma 2.5. *If G is a super magic bipartite (p,q)-graph, then $q \leq 2p-5$.*

The validity of this lemma will be clear, after we state and prove the following two results, also found in Figueroa et al. [17].

Theorem 2.6. *Let G be a super magic (p,q)-graph with $p \geq 4$ and $q \geq 2p-4$. Then G contains triangles.*

Proof.
Assume to the contrary that G contains no triangles. Let $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ be a super magic labeling of G, and let $V(G) = \{v_1, v_2, \ldots, v_p\}$ be such that $f(v_i) = i$ for every $1 \leq i \leq p$.

Observe first that since $q \geq 2p-4$ we have that either v_1 and v_2 or v_p and v_{p-1} are adjacent, as the numbers 3 and $2p-1$ can be expressed uniquely as the sums of distinct integers in the range 1 through p. So suppose, without loss of generality, that v_1 and v_2 are adjacent. Then v_1 and v_3 are adjacent also, as the sum 4 can also be expressed uniquely as $1+3$ distinct with integers in the permitted range. This in turn implies that v_2 and v_3 cannot be adjacent since G contains no triangles, and thus v_1 and v_4 are adjacent. Continuing to avoid triangles in this manner we conclude that v_1 is adjacent to the vertices v_1 and v_{d+1} where $d = \deg(v_1)$, and none of this vertices are adjacent to one another. We have thus accounted for the sums 3 through $d+2$. Now, if $d = p-1$ we are done since there is no way for us to obtain the sum $d+3$ avoiding triangles. Otherwise, if $d < p-1$ then with the remaining options the smallest sum possible is $d+4$ (joining v_2 with v_{d+2}) and we would have no way of obtaining the sum $d+3$. Therefore, in either case we have arrived to a contradiction. □

The converse of the previous theorem provides the desired improved bound, since bipartite graphs contain no odd cycles.

Corollary 2.7. *Let G be a triangle free super magic (p,q)-graph with $p \geq 4$. Then $q \leq 2p-5$.*

This bound is sharp as it is not hard to find bipartite super magic (p,q)-graphs with $p \geq 8$ and $q = 2p-5$.

The second result is stated next.

Lemma 2.8. *Let G be a super magic graph of size q and f a super magic labeling of G. Then, there are exactly $\lfloor q/2 \rfloor$ or $\lceil q/2 \rceil$ edges between V_e and V_o, where*

$$V_e = \{v \in V(G) \mid f(v) \text{ is even}\}$$

and

$$V_o = \{v \in V(G) \mid f(v) \text{ is odd}\}$$

Proof.

Since f is a super magic labeling of G, it follows that the set

$$S = \{f(u) + f(v) \mid uv \in E(G)\}$$

consists of q consecutive integers. Then $\lfloor q/2 \rfloor$ or $\lceil q/2 \rceil$ of the elements in S are odd and each of these has to be the result of adding the label of an element in V_e to the label of an element in V_o. □

As a corollary to Lemma 2.5, we obtain the following less powerful, although easier to use result also observed by Figueroa et al. in [17].

Corollary 2.9. *If G is any (p, q)-graph such that $\deg(v)$ is even for all $v \in V(G)$ and $q \equiv 2 \pmod 4$ then G is not super magic.*

Proof. Since the degree of any vertex of G is even, it follows that all components of G are eulerian. Thus, we can decompose G into a union of edge disjoint cycles. Any partition of $V(G)$ into sets V_1 and V_2 induces partitions on each of the vertex sets of each cycle in our decomposition. If C is any cycle then any partition of $V(C)$ into two sets V_1' and V_2' will produce an even number of edges joining the vertices of V_1' with the vertices of V_2'. Thus, there is an even number of edges joining the vertices of V_1 and V_2. But $q/2$ is odd. Therefore, G cannot be super magic.

□

Finally, before concluding this section, we will introduce a result by Figueroa et al. [17], which is particularly useful when we need to show that a regular graph is not super magic.

Theorem 2.10. *If G is an r-regular super magic (p, q)-graph, where $r > 0$, then q is odd and the valence of any super magic labeling of G is*

$$(4p + q + 3)/2.$$

Proof.

The valence of any super magic labeling of G is

$$\frac{1}{q}\left\{r\sum_{i=1}^{p} i + \sum_{i=p+1}^{p+q} i\right\} = \frac{1}{2}(4p + q + 3)$$

which implies that q is odd □

2.2 Theorems and Examples About Magic and Super Magic Graphs

In this section, we will provide some theorems that allow us to construct new magic and super magic graphs from known magic and super magic graphs. We also will provide several examples of graphs that admit magic and super magic labelings, in order to familiarize the reader a little bit more with these concepts. The next two theorems are due to Figueroa et al. [14].

Theorem 2.11. *Let G be a magic graph, f a magic labeling of G, and $u, v \in V(G)$ such that $f(u) + f(v) = k$, where k is the valence of f, then $G + uv$ is magic.*

Proof.
Notice that if $f(u) + f(v) = k$, then $uv \notin E(G)$; for otherwise $f(uv) = 0$. Therefore, we immediately obtain a magic labeling g of $G + uv$ by letting $g(x) = f(x) + 1$ for every $x \in (V(G) \cup E(G)) - \{uv\}$ and $g(uv) = 1$. □

The previous result cannot be used if G is connected and f is a super magic labeling. To see why, let G be a super magic graph with a super magic labeling f. By Lemma 2.3, the valence of f is

$$k = p + q + \min\{f(u) + f(v) \mid uv \in E(G)\}.$$

Thus, $k \geq p + q + 1 + 2 \geq 2p + 2$ since G is connected so that $q \geq p - 1$. Now,

$$\max\{f(u) + f(v) \mid uv \in E(G)\} \leq p + (p - 1) = 2p - 1$$

since f is a super magic labeling. Therefore,

$$f(u) + f(v) \leq 2p - 1 < 2p + 2 \leq k.$$

Whereas the previous theorem concerns the addition of an edge, our next theorem involves the deletion of an edge.

Theorem 2.12. *If G is a magic graph and f is a magic labeling of G for which there exits $e \in E(G)$ such that $f(e) = 1$, then $G - e$ is magic.*

Proof.
We immediately obtain a magic labeling g of $G - e$ by letting $g(x) = f(x) - 1$ for every $x \in (V(G) \cup E(G)) - \{e\}$. □

It is obvious that, the previous theorem cannot be used for super magic graphs. However, using the canonical labeling we can find similar results that involve the addition and deletion of edges of super magic graphs.

Theorem 2.13. *Let G be a super magic graph of order p and let $f : V(G) \to \{1, 2, \ldots, p\}$ be the canonical form of some super magic labeling of G. Let $u, v, u', v' \in V(G)$ such that*

$$f(u) + f(v) = \max\{f(x) + f(y) \mid xy \in E(G)\} + 1$$

and

$$f(u') + f(v') = \min\{f(x) + f(y) \mid xy \in E(G)\} - 1$$

then, $G + uv$ and $G + u'v'$ are super magic.

Theorem 2.14. *Let G be a super magic graph of order p and let $f : V(G) \to \{1, 2, \ldots, p\}$ be the canonical form of some super magic labeling of G. Let $e = uv$ and $e' = uv$ be two edges of G such that*

$$f(u) + f(v) = \max\{f(x) + f(y) \mid xy \in E(G)\}$$

and

$$f(u') + f(v') = \min\{f(x) + f(y) \mid xy \in E(G)\}$$

then, $G + e$ and $G - e'$ are super magic.

Next, we will provide some examples of magic and super magic graphs.

In [7], it was shown that the complete bipartite graph $K_{m,n}$ is super magic if and only if $m = 1$ or $n = 1$. The next theorem by Figueroa et al. [14] partially extends their result by determining all magic and super magic labelings of the star $K_{1,n}$.

Theorem 2.15. *Every star $K_{1,n}$ is super magic. Moreover, there are exactly $3{\cdot}2^n$ distinct magic labeling of $K_{1,n}$ of which only two are super magic labelings up to isomorphisms.*

Proof.

First, notice that the order of $K_{1,n}$ is $n + 1$ and its size is n. Next, define the star $G \cong K_{1,n}$ as follows, $V(G) = \{u\} \cup \{v_i \mid 1 \le i \le n\}$ and $E(G) = \{e_i = uv_i \mid 1 \le i \le n\}$. Assume that there is a magic labeling f of G and let k be its valence. Then

$$\left(\sum_{i=1}^{n} (f(v_i) + f(e_i))\right) + nf(u) = nk.$$

Thus, n divides $\sum_{i=1}^{n} (f(v_i) + f(e_i))$.

Now,

$$\left(\sum_{i=1}^{n}(f(v_i)+f(e_i))\right)+f(u)=1+\cdots+(2n+1)=2n^2+3n+1$$

hence,

$$\sum_{i=1}^{n}(f(v_i)+f(e_i))=2n^2+3n+(1-f(u)).$$

Therefore, n divides $f(u)-1$, but $1\leq f(u)\leq 2n+1$, which implies that $f(u)$ is $1, n+1$ or $2n+1$. Since

$$nk=2n^2+3n+1+(n-1)f(u),$$

it follows that $k=2n+4$, $3n+3$ or $4n+2$, which correspond to $f(u)=1, n+1, 2n+1$ respectively.

It suffices now to exhibit labelings with each of the three possible valences, and then describe how to obtain all of the other labelings from them. Let f_1, f_2, and f_3 be magic labeling of G defined as follows,

$$\begin{array}{lll} f_1(u)=1, & f_1(v_i)=i+1, & f_1(uv_i)=2n+2-i \\ f_2(u)=n+1, & f_2(v_i)=i, & f_2(uv_i)=2n+2-i \\ f_3(u)=2n+1, & f_3(v_i)=i, & f_3(uv_i)=2n+1-i \end{array}$$

where $1\leq i\leq n$. Then the valences of f_1, f_2, and f_3 are $2n+4$, $3n+3$, and $4n+2$, respectively. Note that all other magic labelings of G can be obtained by permuting the labels of uv_i and v_i for any i with $1\leq i\leq n$, and that of these $3\cdot 2^n$ possible permutations, only f_1 and f_2 are super magic labelings of G. □

The following corollary is an immediate consequence of the proof of the preceding theorem. It is interesting since Godbold and Slater [20] have conjectured that, for sufficiently large cycles, there are no gaps between the set of possible valences.

Corollary 2.16. *For every integer $n\geq 2$, there exists a super magic graph G such that $|k_1-k_2|\geq n-1$ where k_1 and k_2 are two possible distinct valences of G.*

The next corollary discovered by Figueroa et al. [14], describes how new super magic graphs can be found from known super magic graphs.

Corollary 2.17. *For every positive integer n, the graph K_2+nK_1 is super magic.*

Proof.

Let $G \cong K_{1,n}$ be defined as in the proof of the previous theorem and consider the following magic labeling g of G defined by $g(u) = n+1, g(v_i) = 2(n+1) - i$ and $g(uv_i) = i$ for $1 \leq i \leq n$. Notice that the valence k is $3n+3$ and $g(v_1) + g(v_i) = 4n + 3 - i$ for $2 \leq i \leq n$. Then define the graph $H \cong K_2 + nK_1$ as follows, $V(H) = V(G)$ and $E(H) = E(G) \cup \{v_1 v_i \mid 2 \leq i \leq n\}$; and consider the magic labeling of H with valence $6n$ such that $f(v) = g(v) + n - 1$ for any vertex v of H, $f(uv_i) = g(uv_i) + n - 1$ for $1 \leq i \leq n$ and $f(v_1 v_i) = i - 1$ for $2 \leq i \leq n$.

Finally, observe that f is a super magic labeling of this graph since $f(v) > f(e)$ for any vertex v and edge e of H.

□

Notice that the above corollary establishes the sharpness of Lemma 2.2.

We remark that, from the preceding proof, we can obtain a sequence of super magic graphs as follows. Take the labeling f employed for $K_2 + nK_1$ in the proof and then remove the edge labeled 1 from it and decrease all labels by 1. Continue in this fashion until arriving to $K_{1,n}$. Every such labeling of each graph is magic by Lemma 2.12 and its complementary labeling is super magic.

The above corollary also allows to characterize all the super magic complete m-partite graphs [14].

Theorem 2.18. *The only super magic complete m-partite graphs are $K_{1,n}$ and $K_{1,1,n}$, for $n \geq 1$.*

Proof.

Enomoto et al. [7] proved that the star $K_{1,n}$, $n \geq 1$, is the only super magic complete bipartite graph. Furthermore, the complete tripartite graph $K_{1,1,n} \cong K_2 + nK_1$ is super magic by Corollary 2.17.

In order to see that $K_{1,n}$ and $K_{1,1,n}$ are the unique complete partite graphs, note that m has to be less than or equal to 4 ; for otherwise, the minimum degree would be greater than 3, which is impossible. Thus, it remains to be shown that $K_{1,1,n}$ is the unique super magic tripartite graph and that there are no complete 4-partite super magic graphs.

For the uniqueness of $K_{1,1,n}$, let $G \cong K_{n_1, n_2, n_3}$ be a complete tripartite graph with $n_1 \geq n_2 \geq n_3 \geq 1$. Then, assume, to the contrary, that $n_2 \geq 2$ and G is super magic. The order of G is $n_1 + n_2 + n_3$ and its size is $n_1 n_2 + n_1 n_3 + n_2 n_3$. By Lemma 2.2,

$$n_1 n_2 + n_1 n_3 + n_2 n_3 \leq 2(n_1 + n_2 + n_3) - 3,$$

which implies that $n_1 n_3 \leq 2n_2 - 3$ since $n_2 \geq 2$ and $n_3 \geq 2$.

Hence, $n_2 n_3 \leq 2n_2 - 3$, so $2 - n_3 > 0$ from which we conclude that $n_3 = 1$. Therefore, if we apply Lemma 2.2 again, we get that $n_1 \leq 1$, producing a contradiction.

To show that there are no super magic complete 4-partite graphs, observe that $K_{1,1,n}$ is not super magic by Lemma 2.2 and all remaining graphs have minimum degrees greater than 3, completing the proof. □

Next, we will study the magic and super magic properties of some important families of graphs; for instance, the fans, ladders, generalized prisms, books, and the Petersen graph.

The following theorem by Figueroa et al. [17] is also interesting since it analyzes an infinite family of (p, q)-graphs for which $q = 2p - 3$.

Theorem 2.19. *The fan $f_n \cong P_n + K_1$ is magic for every positive integer n, and is super magic if and only if $n \in \{1, 2, 3, 4, 5, 6\}$.*

Proof.

First, we prove that f_n is magic for every positive integer n. Let f_n be the fan with

$$V(f_n) = \{u\} \cup \{v_i \mid 1 \leq i \leq n\}$$

and

$$E(f_n) = \{uv_i \mid 1 \leq i \leq n\} \cup \{v_i v_{i+1} \mid 1 \leq i \leq n-1\}.$$

Now, construct the function $f : V(f_n) \cup E(f_n) \rightarrow \{1, 2, \ldots, 3n\}$ as follows,

$$f(x) = \begin{cases} 1 & \text{if } x = u; \\ \frac{1-5(-1)^i+6i}{4} & \text{if } x = v_i \text{ and } 1 \leq i \leq n; \\ \frac{12n+7+5(-1)^i-6i}{4} & \text{if } x = uv_i \text{ and } 1 \leq i \leq n; \\ 3n - 3i + 1 & \text{if } x = v_i v_{i+1} \text{ and } 1 \leq i \leq n-1. \end{cases}$$

Notice that $f(x) + f(y) + f(xy) = 3n + 3$ for any edge xy of f_n. Also, observe that

$$\begin{aligned} \left\{f(v_{2i+1}) \mid 0 \leq i \leq \left\lfloor \tfrac{n-1}{2} \right\rfloor\right\} &= \left\{3i \mid 1 \leq i \leq \left\lfloor \tfrac{n+1}{2} \right\rfloor\right\}, \\ \left\{f(uv_{2i}) \mid 1 \leq i \leq \left\lceil \tfrac{n-1}{2} \right\rceil\right\} &= \left\{3i \mid \left\lfloor \tfrac{n+1}{2} \right\rfloor + 1 \leq i \leq n\right\}, \\ \{f(v_i v_{i+1}) \mid 1 \leq i \leq n-1\} &= \{3i + 1 \mid 1 \leq i \leq n-1\}, \\ \left\{f(v_{2i}) \mid 1 \leq i \leq \left\lceil \tfrac{n-1}{2} \right\rceil\right\} &= \left\{3i + 2 \mid 0 \leq i \leq \left\lfloor \tfrac{n-2}{2} \right\rfloor\right\}, \\ \left\{f(uv_{2i+1}) \mid 0 \leq i \leq \left\lfloor \tfrac{n-1}{2} \right\rfloor\right\} &= \left\{3i + 2 \mid \left\lfloor \tfrac{n-2}{2} \right\rfloor + 1 \leq i \leq n-1\right\}, \end{aligned}$$

and $f(u) = 1$, so all the integers 1 through $3n$ are used exactly once. Therefore, f is a magic labeling of f_n with valence $3n + 3$.

Next, we prove that f_n is super magic if $n \in \{1, 2, 3, 4, 5, 6\}$. The graphs $f_1 \cong K_2$ and $f_2 \cong K_3$ are obviously super magic. For $n = 3, 4, 5, 6$, label K_1

with 4 and the vertices of P_n with $3-1-2$, $5-3-1-2$, $6-5-3-1-2$ and $6-7-5-3-1-2$, respectively. Then, these labelings are the canonical form for super magic labelings of the graphs we are considering.

Finally, we will show that if $n \notin \{1,2,3,4,5,6\}$ then f_n is not super magic. Assume to the contrary that f_n is super magic with super magic labeling g for some integer $n \geq 7$. Define $p = n+1$, $q = 2n-1$, and $V(f_n) = \{v_i \mid g(v_i) = i\}$. Now, since f_n is super magic, it follows from Lemma 2.3 that $S = \{g(u) + g(v) \mid uv \in E(f_n)\}$ is a set of $q = 2p-3$ consecutive integers, implying that $S = \{3, 4, \ldots, 2p-1\}$. Since $n \geq 7$, the vertices $v_1, v_2, v_3, v_4, v_{p-3}, v_{p-1}$ and v_p are all distinct.

Observe next that each of 3, 4, $2p-2$ and $2p-1$ can be expressed uniquely as sums of two distinct elements in the set $L = \{1, 2, \ldots, p\}$, namely, $3 = 1+2$, $4 = 1+3$, $2p-2 = p+(p-2)$ and $2p-1 = p+(p-1)$. Therefore, $\{v_1v_2, v_1v_3, v_{p-2}v_p, v_{p-1}v_p\} \subseteq E(f_n)$. Also, notice that the integers 5 and $2p-3$ can be expressed each in exactly two ways as sums of distinct elements of L. Thus, $\{v_1v_4, v_pv_{p-3}\}$, $\{v_1v_4, v_{p-1}v_{p-2}\}$, $\{v_2v_3, v_pv_{p-3}\}$ and $\{v_2v_3, v_{p-1}v_{p-2}\}$ are four mutually exclusive possibilities for subsets of $E(f_n)$. Finally, by adding any of these four pairs of edges to the four edges that are necessarily in the edge set of the fan, we obtain a forbidden subgraph of the fan, namely, either $2K_{1,3}, K_{1,3} \cup K_3$ or $2K_3$.

□

The next two results about ladders and generalized prisms have been found independently by Enomoto and Yokomura (personal comunication) and by Figueroa et al. [17].

Theorem 2.20. *The ladder $L_n \cong P_n \times P_2$ is super magic , if n is odd.*

Proof.

Let L_n be the ladder with

$$V(L_n) = \{u_i, v_i \mid 1 \leq i \leq n\}$$

and

$$E(L_n) = \{u_iu_{i+1}, v_iv_{i+1}, u_jv_j \mid 1 \leq i \leq n-1, 1 \leq j \leq n\}.$$

Now, consider the following function

$$f : V(L_n) \to \{1, 2, \ldots, 2n\},$$

defined by the rule

$$f(x)=\begin{cases}\frac{i+1}{2}, & x=u_i \text{ ,} i \text{ odd and } 1\le i\le n;\\ \frac{n+i+1}{2}, & x=u_i \text{ ,} i \text{ even and } 1\le i\le n;\\ \frac{3n+i}{2}, & x=v_i \text{ ,} i \text{ odd and } 1\le i\le n;\\ \frac{2n+i}{2}, & x=v_i \text{ ,} i \text{ even and } 1\le i\le n.\end{cases}$$

We conclude that f is the canonical form of some super magic labeling of the ladder l_n, with valence $(11n+1)/2$.

□

The converse of the previous theorem is not true. Although the graph $L_2 \cong C_4$ is not super magic, we have found super magic labelings for $n=4$ and $n=6$. See Figure 2.1.

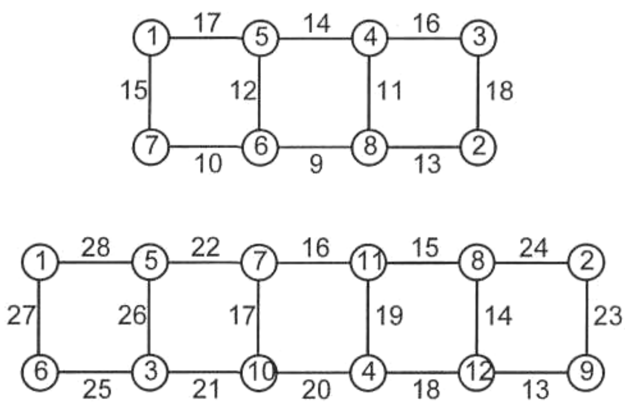

Figure 2.1

Theorem 2.21. *The generalized prisms $C_m \times P_n$ is super magic if m is odd and $n \geq 2$.*

Proof.
Define the generalized prism $G \cong C_m \times P_m$ as follows,

$$V(G)=\{v_{i,j} \mid 1\le i\le m, 1\le j\le n\}$$

and

$$\begin{aligned} E(G) &= \{v_{i,j}v_{i+1,j} \mid 1\le i\le m, 1\le j\le n\}\\ &\quad \cup \{v_{i,j}v_{i,j+1} \mid 1\le i\le m, 1\le j\le n-1\}\end{aligned}$$

where i is taken modulo m (replacing 0 by m).
Consider the following function $f : V(G) \to \{1, 2, \ldots, mn\}$, where

$$f(v_{i,j}) = \begin{cases} \frac{i+1}{2}, & \text{if } 1 \le i \le m \text{ is odd and } j = 1; \\ \frac{i+m+1}{2}, & \text{if } 1 \le i \le m \text{ is even and } j = 1; \\ \frac{i+m(2j-2)}{2}, & \text{if } 1 \le i \le m \text{ is even and } 2 \le j \le n; \\ \frac{i+m(2j-1)}{2}, & \text{if } 1 \le i \le m \text{ is odd } 2 \le j \le n. \end{cases}$$

We conclude that f is the canonical form of some magic labeling of G whose valence is $(6mn - m + 3)/2$. □

It is important to notice that the converse of the previous result is an immediate consequence of Theorem 2.10 when $n = 2$. The case where $n = 2$ and m is odd is interesting since it presents examples of 3-regular super magic graphs, which is best possible because $r = 0, 1, 2$, or 3 for r-regular super magic graphs.

The next three results about the book $B_n \cong K_{1,n} \times K_2$ where first introduced by Figueroa et al. [17].

Theorem 2.22. *If the book B_n is super magic with a super magic labeling f such that*

$$s = \min\{f(x) + f(y) \mid xy \in E(G)\},$$

then the following conditions are satisfied:

(1) if n is odd, then $n \equiv 5 \pmod 8$ and

$$s \in \left\{\frac{n+27}{8}, \frac{3n+25}{8}, \frac{5n+23}{8}, \frac{7n+21}{8}, \frac{9n+19}{8}\right\}$$

unless $n = 5$, in which case, s can also be 3;

(2) if n is even, then $s = (n/2) + 3$ unless $n = 2$, in which case, s can be 3.

Proof. The book B_n has order $p = 2n + 2$ and size $3n + 1$. Now, if x and y represent the labels of the two vertices of degree $n + 1$ of B_n, then

$$2\sum_{i=1}^{2n+2} i + (x+y)(n-1) = (3n+1)s + \frac{3n(3n+1)}{2}$$

thus,

$$x + y = \frac{n^2 + 6sn - 17n + 2s - 12}{2n - 2}; \tag{2.1}$$

however, $x + y \le p + (p - 1) = 4n + 3$. Consequently,

$$3 \le s \le \frac{7}{6}n + \frac{19}{9} + \frac{8}{27n + 9} \le \frac{7}{6}n + \frac{7}{3} \tag{2.2}$$

since $n \geq 1$.

If n is even, then $n = 2k$ for some integer k, so

$$x + y = k + 3s - 8 + \frac{4s - 14}{2k - 1} \tag{2.3}$$

by (2.1) and hence $2k - 1$ divides $2s - 7$ for $k \geq 2$, that is, there exists an integer m such that

$$\frac{m\,(2k - 1) + 7}{2} = s.$$

Then, from (6.9), we obtain $-1 \leq m \leq 2$, implying that $m = 1$ since s is an integer and $k \geq 2$. Hence, $s = (n/2) + 3$. For $n = 2$, notice that $s = 3$ or $s = 4$ by (6.9).

For the cases where n is odd, if $n \equiv 3 \pmod 4$, then every vertex of B_n is even and $q \equiv 2 \pmod 4$, so B_n is not super magic by Corollary 2.9. On the other hand, if $n \equiv 1 \pmod 4$, then $n = 4k + 1$ for some integer k and

$$2\,(x + y) = 4k + 6s - 15 + \frac{2s - 7}{k}$$

by (2.3), which means that k divides $2s - 7$.

Now, if $n = 8k - 1$ for some integer k, then $2k$ divides $2s - 7$, which is not possible. Therefore, when n is odd, there exists an integer k such that

$$\frac{m\,(2k + 1) - 1}{2} = s.$$

Then, from (6.9), we obtain $-1 \leq m \leq 9$; so $m \in \{-1, 1, 3, 5, 7, 9\}$ since s is an integer. Therefore,

$$s \in \left\{ \frac{-n + 29}{8}, \frac{n + 27}{8}, \frac{3n + 25}{8}, \frac{5n + 23}{8}, \frac{7n + 21}{8}, \frac{9n + 19}{8} \right\}.$$

Finally, notice that $s = (-n + 29)/8$ only when $n = 5$, which completes the proof. □

Conjecture 2.23. *For every integer $n \geq 5$, the book B_n is super magic, if and only if n is even or $n \equiv 5 \pmod 8$.*

Although books are sometimes super magic, they are always magic as the following theorem demonstrates.

Theorem 2.24. *The book B_n is magic for any positive integer n.*

Proof.

Let B_n be the book defined as follows,
$V(B_n) = \{u, v\} \cup \{u_i, v_i \mid 1 \le i \le n\}$

$$V(B_n) = \{u, v\} \cup \{u_i, v_i \mid 1 \le i \le n\}$$

and

$$E(B_n) = \{uv\} \cup \{uu_i, vv_i, u_iv_i \mid 1 \le i \le n\}.$$

Then consider the function

$$f : V(B_n) \cup E(B_n) \to \{1, 2, \ldots, 5n+3\},$$

where

$$f(x) = \begin{cases} 1, & \text{if } x = u; \\ 5n+3, & \text{if } x = v; \\ 2n+2, & \text{if } x = uv; \\ 2n+i+2, & \text{if } x = u_i \text{ and } 1 \le i \le n; \\ 2n-2i+2, & \text{if } x = v_i \text{ and } 1 \le i \le n; \\ 5n-i+3, & \text{if } x = uu_i \text{ and } 1 \le i \le n; \\ 3n+i+2, & \text{if } x = u_iv_i \text{ and } 1 \le i \le n; \\ 2i+1, & \text{if } x = vv_i \text{ and } 1 \le i \le n. \end{cases}$$

Finally, observe that f is a magic labeling of B_n having valance $7n+6$.
□

In order to conclude this chapter, we will study the super magicness of probably the most famous graph in the whole subject of graph theory: the Petersen graph.

Theorem 2.25. *The Petersen graph is super magic.*

Proof. See Figure 2.2. □

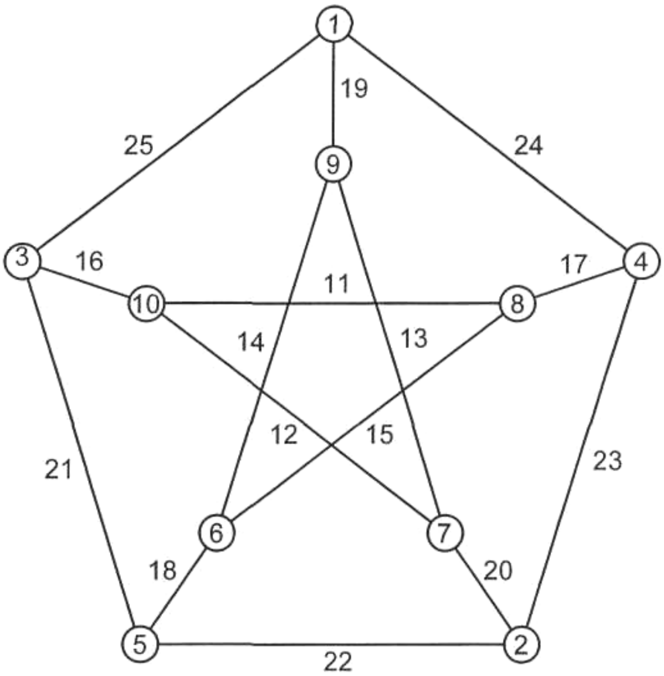

Figure 2.2