Chapter 4

Operations on Graphs

4.1 Magic Labelings of Unions of Graphs

4.1.1 Main Result

This section contains a result by Figueroa et al., found in [13] that allows to generate infinite classes of disconnected magic and super magic bipartite and tripartite graphs with relative ease. This kind of result was unexpected since previously no such a technique was available for harmonious, sequential or cordial graphs.

Theorem 4.1. *Let G be a (super) magic bipartite or tripartite graph and let m be an odd integer. Then mG is also (super) magic.*

Proof.
The result is trivial for $m = 1$ so we assume, without loss of generality, that $m \geq 3$. Let G be a tripartite (p, q)-graph with partite sets U, V and W (let $W = \emptyset$ if G is bipartite). Then let $E(G) = UV \cup UW \cup VW$, where the juxtaposition of names of partite sets denotes the set of edges between those two sets. Also, let $f : V(G) \cup E(G) \to \{1, 2, \ldots, p + q\}$ be a (super) magic labeling of G. Now, we define mG to be such that

$$V(mG) = \bigcup_{i=1}^{m}(U_i \cup V_i \cup W_i)$$

and

$$E(mG) = \bigcup_{i=1}^{m}(UV_i \cup UW_i \cup VW_i),$$

where for every i we have that $x_i \in X_i$ if and only if $x \in X$, where X is one of the sets U, V, W, UV, UW or VW. Consider then $g : V(mG) \cup E(mG) \to \{1, 2, \ldots, m(p+q)\}$ such that

$$g(x_i) = \begin{cases} mf(x) - m + i, & \text{if } x \in W \cup UV \text{ and } 1 \leq i \leq m; \\ mf(x) - 2i + 1, & \text{if } x \in U \cup VW \text{ and } 1 \leq i \leq \frac{m-1}{2}; \\ mf(x) + m - 2i + 1, & \text{if } x \in U \cup VW \text{ and } \frac{m+1}{2} \leq i \leq m; \\ mf(x) - \frac{m-1}{2} + i, & \text{if } x \in V \cup UW \text{ and } 1 \leq i \leq \frac{m-1}{2}; \\ mf(x) - \frac{3m-1}{2} + i, & \text{if } x \in V \cup UW \text{ and } \frac{m+1}{2} \leq i \leq m. \end{cases}$$

Then, g is a (super) magic labeling of mG. To verify this, observe first that $g(x) + g(y) + g(xy) = mk + 3(1-m)/2$ for every $xy \in E(mG)$, where k is the valence of f. Next, to see that

$$g(V(mG) \cup E(mG)) = \{1, 2, \ldots, m(p+q)\},$$

notice that for every $x \in V(G) \cup E(G)$ we have that

$$\bigcup_{i=1}^{m} \{g(x_i)\} = \bigcup_{i=1}^{m} \{mf(x) - m + i\},$$

thus, the set

$$f(V(G) \cup E(G)) = \{1, 2, \ldots, p+q\}$$

is spread by the function g to the entirety of its range. □

The preceding result is best possible in the sense that m cannot be even for Kotzig and Rosa [27] have shown that the graph mP_2 is magic if and only if m is odd.

4.1.2 Results on 2-Regular Graphs

The results on this section can be found in [13] unless otherwise stated. The result on the previous section, makes it worthwhile to search for graphs which are (super) magic and either bipartite or tripartite. In this section, we thus, concentrate on the magic properties of some 2-regular graphs, which are certainly either bipartite or tripartite, depending on the length of the cycles that form them.

The following example is the kind of result that follows immediately from Theorem 4.1 with the help of the work done by Kotzig and Rosa in [27], and Enomoto et al. in [7].

Corollary 4.2. *For every odd m and positive integer n, the 2-regular graph mC_n is magic. Moreover, it is super magic if n is odd.*

Proof.

Kotzig and Rosa [27] have shown that C_n is magic for every n, and Enomoto et al. [7],proved that C_n is super magic if and only if n is odd. Therefore, our corollary follows immediately using Theorem 4.1.

□

With the previous theorem in hand, it is not hard to see that mC_n is super magic if and only if both $m \geq 1$ and $n \geq 3$ are odd. Thus, mC_n is magic if m is odd. For the case when m is even, we only have basically the following result.

Theorem 4.3. *The 2-regular graph $G \cong 2C_n$ is magic for $n \equiv 1, 5$ or $7 \pmod{12}$.*

Proof.

Assume that $n \equiv 1, 5$, or $7 \pmod{12}$, and let $G \cong 2C_n$ be the 2-regular graph with

$$V(G) = \{u_i, v_i \mid 1 \leq i \leq n\}$$

and

$$E(G) = \{u_1 u_n, v_1 v_n\} \cup \{u_i u_{i+1}, v_i v_{i+1} \mid 1 \leq i \leq n-1\}.$$

We will consider three cases.

Case 1: Let $n = 12k - 7$, where k is a positive integer, and define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 48k - 28\}$ such that

$$f(x) = \begin{cases} 24k - 3i - 10, & \text{if } x = u_{2i-1} \text{ and } 1 \leq i \leq 3k-1; \\ 6k + 3i - 5, & \text{if } x = u_{2i-1} \text{ and } 3k \leq i \leq 6k-3; \\ 12k - 3i - 5, & \text{if } x = u_{2i} \text{ and } 1 \leq i \leq 3k-2; \\ 3i - 6k + 3, & \text{if } x = u_{2i} \text{ and } 3k-1 \leq i \leq 6k-4; \\ 12k - 3i - 4, & \text{if } x = v_{2i-1} \text{ and } 1 \leq i \leq 3k-2; \\ 24k - 3i - 12, & \text{if } x = v_{2i} \text{ and } 1 \leq i \leq 3k-2; \\ 3k - 3i + 1, & \text{if } x = v_{6k+6i-9} \text{ and } 1 \leq i \leq k; \\ 15k - 3i - 5, & \text{if } x = v_{6k+6i-8} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 2, & \text{if } x = v_{6k+6i-7} \text{ and } 1 \leq i \leq k; \\ 15k - 3i - 7, & \text{if } x = v_{6k+6i-6} \text{ and } 1 \leq i \leq k-1; \\ 3k - 3i, & \text{if } x = v_{6k+6i-5} \text{ and } 1 \leq i \leq k-1; \\ 15k - 3i - 6, & \text{if } x = v_{6k+6i-4} \text{ and } 1 \leq i \leq k-1. \end{cases}$$

Case 2: Let $n = 12k - 5$, where k is a positive integer, and define the vertex labeling $f : V(G) \to \{1, 2, \ldots, 48k - 20\}$ such that

$$f(x) = \begin{cases} 24k - 3i - 6, & \text{if } x = u_{2i-1} \text{ and } 1 \leq i \leq 3k - 1; \\ 6k + 3i - 4, & \text{if } x = u_{2i-1} \text{ and } 3k \leq i \leq 6k - 2; \\ 12k - 3i - 3, & \text{if } x = u_{2i} \text{ and } 1 \leq i \leq 3k - 1; \\ 3i - 6k + 2, & \text{if } x = u_{2i} \text{ and } 3k \leq i \leq 6k - 3; \\ 12k - 3i - 2, & \text{if } x = v_{2i-1} \text{and } 1 \leq i \leq 3k - 1; \\ 24k - 3i - 8, & \text{if } x = v_{2i} \text{ and } 1 \leq i \leq 3k - 2; \\ 15k - 3i - 3, & \text{if } x = v_{6k+6i-8} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 2, & \text{if } x = v_{6k+6i-7} \text{ and } 1 \leq i \leq k - 1; \\ 15k - 3i - 2, & \text{if } x = v_{6k+6i-6} \text{ and } 1 \leq i \leq k; \\ 3k - 3i, & \text{if } x = v_{6k+6i-5} \text{ and } 1 \leq i \leq k - 1; \\ 15k - 3i - 4, & \text{if } x = v_{6k+6i-4} \text{ and } 1 \leq i \leq k - 1; \\ 3k - 3i + 1, & \text{if } x = v_{6k+6i-3} \text{ and } 1 \leq i \leq k - 1; \\ i, & \text{if } x = v_{12k+2i-9} \text{ and } 1 \leq i \leq 2. \end{cases}$$

Case 3: Let $n = 12k + 1$, where k is a positive integer, and define the vertex labeling $f : V(G) \rightarrow \{1, 2, \ldots, 48k + 4\}$ such that

$$f(x) = \begin{cases} 24k - 3i + 6, & \text{if } x = u_{2i-1} \text{ and } 1 \leq i \leq 3k + 1; \\ 6k + 3i - 1, & \text{if } x = u_{2i-1} \text{ and } 3k + 2 \leq i \leq 6k + 1; \\ 12k - 3i + 3, & \text{if } x = u_{2i} \text{ and } 1 \leq i \leq 3k; \\ 3i - 6k - 1, & \text{if } x = u_{2i} \text{ and } 3k + 1 \leq i \leq 6k; \\ 12k - 3i + 4, & \text{if } x = v_{2i-1} \text{and } 1 \leq i \leq 3k; \\ 24k - 3i + 4, & \text{if } x = v_{2i} \text{ and } 1 \leq i \leq 3k; \\ 3k - 3i + 3, & \text{if } x = v_{6k+6i-5} \text{ and } 1 \leq i \leq k; \\ 15k - 3i + 5, & \text{if } x = v_{6k+6i-4} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 4, & \text{if } x = v_{6k+6i-3} \text{ and } 1 \leq i \leq k; \\ 15k - 3i + 3, & \text{if } x = v_{6k+6i-2} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 2, & \text{if } x = v_{6k+6i-1} \text{ and } 1 \leq i \leq k - 1; \\ 15k - 3i + 4, & \text{if } x = v_{6k+6i} \text{ and } 1 \leq i \leq k; \\ i, & \text{if } x = v_{12k+2i-3} \text{ and } 1 \leq i \leq 2. \end{cases}$$

Therefore, f extends to a magic labeling of G with valence $5n + 2$. □

4.1.3 Results on Forests

This section is devoted to the study of magic and super magic properties of certain classes of forests; which complements the results in the previous section nicely, since these graphs are bipartite and hence can serve as seeds for creating other infinite classes of bipartite graphs by means of Theorem 4.1. They are also interesting since the forests referred to, in this section, have each two components and thus, show that bipartite graphs with an even number of components may be magic or super magic. For the results of this section see [13]

Theorem 4.4. *If m is a positive multiple of $n+1$, then the forest $F \cong K_{1,m} \cup K_{1,n}$ is super magic.*

Proof.
Let

$$V(F) = \{x, y\} \cup \{u_1, u_2, \ldots, u_m\} \cup \{v_1, v_2, \ldots, v_n\}$$

and

$$E(F) = \{xu_1, xu_2, \ldots, xu_m\} \cup \{yv_1, yv_2, \ldots, yv_n\}.$$

Consider a bijective function $f : V(F) \to \{1, 2, \ldots, m+n+2\}$ such that $f(x) = \alpha + 2, f(y) = 1$ and $f(v_i) = (i+1)(\alpha+1)+1$ for $i = 1, 2, \ldots, n$, where $\alpha = m/(n+1)$. Therefore, f is the canonical form of a super magic labeling of F with valence $\alpha + 2m + 2n + 4$. □

We actually suspect that the converse of the previous theorem also holds.

Conjecture 4.5. *The forest $K_{1,n} \cup K_{1,n+1}$ is super magic if and only if m is a positive multiple of $n+1$.*

The following remark and two theorems support the above conjecture. Notice that the forest $K_{1,1} \cup K_{1,n}$, $n > 1$, is super magic if and only if n is even by Lemma 2.1 and the previous theorem.

Theorem 4.6. *The forest $F \cong K_{1,2} \cup K_{1,n}$ is super magic if and only if n is a positive multiple of 3. Furthermore, there are essentially only two super magic labelings of F.*

Proof.
Let

$$V(F) = \{u\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{w_1, w_2, w_3\}$$

and

$$E(F) = \cup \{uv_1, uv_2, \ldots, uv_n\} \cup \{w_1w_2, w_1w_3\},$$

and $f : V(F) \to \{1, 2, \ldots, n+4\}$ be an arbitrary super magic labeling of F such that $f(u) = \alpha$ and $\{f(w_1), f(w_2), f(w_3)\} = \{i, j, k\}$. Notice then that α, i, j and k are different. Without loss of generality, we may assume that $i < j < k$. Let $S = \{f(x) + f(y) \mid xy \in E(F)\}$ and $L = \{\alpha+1, \alpha+2, \ldots, \alpha+n+4\}$, which are two sets of consecutive integers with $|S| = n+2$ and $|L| = n+4$. Observe then that

$$S - \{f(w_1) + f(w_2), f(w_1) + f(w_3)\} = L - \{2\alpha, \alpha+i, \alpha+j, \alpha+k\}.$$

Thus, $\{\alpha+1, \alpha+n+4\} \subset \{2\alpha, \alpha+i, \alpha+k\}$ since by removing 2α, $\alpha+i$, $\alpha+j$ and $\alpha+k$ from L, we obtain

$$S - \{f(w_1)+f(w_2), f(w_1)+f(w_3)\},$$

which is a set of consecutive integers minus two elements and $i < j < k$. This implies that $\{1, n+4\} \subset \{\alpha, i, k\}$.

Next, we show that $i = 1$ and $k = n+4$. In order to do this, it suffices to verify that $\alpha \notin \{1, n+4\}$. Let $\beta = f(w_1)$, then since $\deg w_1 = 2$, $\deg u = n$ and $f(u) = \alpha$, it follows from Lemma 2.3 that

$$\sum_{t=1}^{n+4} t + \alpha(n-1) + \beta = (n+2)s + \binom{n+2}{2},$$

where $s = \min S$. Hence,

$$s = \frac{3(n+3)+\alpha(n-1)+\beta}{n+2}.$$

Now, assume, to the contrary, that $\alpha = 1$. Then, $s = 4 + \beta/(n+2)$, so $n+2$ divides β, which implies that $\beta = n+2$. This, in turn, leads to conclude that $s = 5$. Furthermore, the vertex u which is labeled 1 cannot be adjacent to the vertices labeled 2 or 3; for otherwise $s = 3$ or 4. Therefore, $\{2, 3, n+2\} = \{i, j, k\}$, which is impossible.

Next, suppose, to the contrary, that $\alpha = n+4$. Then $s = n+4+(\beta-3)/(n+2)$ and, consequently, $n+2$ divides $\beta - 3$, which implies that $\beta - 3 = 0$ since $n+2 \geq 3$ and $1 \leq \beta \leq n+3$. Thus, $\beta = 3$ and $s = n+4$. Therefore, either $f(w_2) = 1$ or $f(w_3) = 1$, implying that $f(w_1)+f(w_2) = 4$ or $f(w_1)+f(w_3) = 4$ and $4 < s = n+4$, which is a contradiction. Finally, since the vertices w_2 and w_3 are indistinguishable, the following three cases remain.

Case 1: Suppose that $f(w_1) = 1, f(w_2) = n+4$ and $f(w_3) = j$. Then, we obtain $\{1+j, n+5\} = \{\alpha+j, 2\alpha\}$. Thus, $1+j = 2\alpha$ and $n+5 = \alpha+j$, which leads to $\alpha = (n/3)+2$, so n is a positive multiple of 3. Therefore by taking $f(u) = (n/3)+2, f(w_1) = 1$, $f(w_2) = n+4$ and $f(w_3) = (2n/3)+3$, we get a super magic labeling of F.

Case 2: Suppose that $f(w_1) = n+4, f(w_2) = 1$ and $f(w_3) = j$. Then, we obtain $\{n+5, j+n+4\} = \{\alpha+j, 2\alpha\}$. Hence, $n+5 = \alpha+j$ and $j+n+4 = 2\alpha$, which leads to $\alpha = (2n)/3+3$, so n is a positive multiple of 3. Now, it is easy to verify that if we take $f(u) = (2n)/3+3, f(w_1) = n+4, f(w_2) = 1$ and $f(w_3) = (n/3)+2$, then we attain a super magic labeling of F by assigning the remaining labels to all other vertices of F.

Case 3: Suppose that $f(w_1) = j, f(w_2) = 1$ and $f(w_3) = n + 4$. Then we obtain $\{1 + j, j + n + 4\} = \{\alpha + j, 2\alpha\}$. Since $\alpha > 1$, it follows that $1+j \neq \alpha+j$. Thus, $1+j = 2\alpha$ and $j+n+4 = \alpha+j$; hence $j = 2n+7 > n+4$, which is not possible.

The labelings provided in Cases 1 and 2 are unique (up to isomorphism), and therefore the proof is complete. □

The approach used in the previous proof can also be applied to establish the following theorem which we state without proof.

Theorem 4.7. *The forest $F \cong K_{1,3} \cup K_{1,n}$ is super magic if and only if n is a positive multiple of 4.*

The next characterization is the magic analogue to Conjecture 4.5.

Theorem 4.8. *For all positive integers m and $n, m \geq n$, the forest $F \cong K_{1,m} \cup K_{1,n}$ is magic if and only if m or n is even.*

Proof.
If F is magic, then m or n is even.

For the converse, without loss of generality, assume that n is even and let

$$V(F) = \{x, y\} \cup \{u_i \mid i = 1, 2, \dots, m\} \cup \{v_j \mid j = 1, 2, \dots, n\}$$

and

$$E(F) = \{xu_i \mid i = 1, 2, \dots, m\} \cup \{yv_j \mid j = 1, 2, \dots, n\}.$$

Then consider the vertex labeling $f : V(G) \to \{1, 2, \dots, m + n + 2\}$ such that

$$f(w) = \begin{cases} 2m + \frac{3n}{2} + 2, & \text{if } w = x; \\ m + \frac{n}{2} + 1, & \text{if } w = y; \\ i, & \text{if } w = u_i \text{ and } 1 \leq i \leq m; \\ m + i, & \text{if } w = v_i \text{ and } 1 \leq i \leq \frac{n}{2}; \\ m + i + 1, & \text{if } w = v_i \text{ and } \frac{n+1}{2} \leq i \leq n. \end{cases}$$

Therefore, f extends to a magic labeling of F with valence $4m + (5n)/2 + 4$. □

In [14], it is shown that the forest $mK_{1,n}$ is super magic if m is odd. Further, in light of the previous theorem, the forest $2K_{1,n}$ is magic if and only if n is even, which together with Theorem 4.1 leads to conclude that whenever $m \equiv 2 \pmod 4$, the forest $mK_{1,n}$ is magic if and only if n is even. Thus, the only instance that needs to be settled is when m is a positive multiple of 4. For this, we have found that the linear forest $4K_{1,2} \cong 4P_3$ is super magic with valence 30 by simply labeling the four disjoint copies of P_3 as follows: $1 - 9 - 2, 4 - 8 - 5$, $6 - 10 - 7$ and $11 - 3 - 12$. On the other hand, Kotzig and Rosa [27] determined that the linear forest $mK_{1,1} \cong mP_2$ is magic if and only if m is odd (Figueroa et al. recently showed in [14] that this result can be extended to state that the linear forest mP_2 is super magic if and only if m is odd).

We studied above the forests $K_{1,1} \cup K_{1,n}$ and $K_{1,2} \cup K_{1,n}$ as members of the class of forest $K_{1,m} \cup K_{1,n}$. However, they are also in the class of forests $P_m \cup K_{1,n}$. Therefore, the next theorem by Figueroa et al. [13] fits well into our theme. This generalizes the result found in [14] that $P_2 \cup P_m$ is super magic for $m \geq 3$.

Theorem 4.9. *The forest $F \cong P_m \cup K_{1,n}$ is super magic for every integer $m \geq 4$ and $n \geq 1$.*

Proof.

Let

$$V(F) = \{u_i \mid i = 1, 2, \ldots, m\} \cup \{v_j \mid j = 1, 2, \ldots, n\} \cup \{w\}$$

and

$$E(F) = \{u_i u_{i+1} \mid i = 1, 2, \ldots, m-1\} \cup \{v_j w \mid j = 1, 2, \ldots, n\}.$$

Suppose that $f : V(F) \to \{1, 2, \ldots, m+n+1\}$ is a vertex labeling of F. We will consider four cases.

Case 1: Let $m \equiv 0 \pmod 4$, then

$$f(x) = \begin{cases} \frac{m+2n+2}{2}, & \text{if } x = u_1; \\ \frac{m+2n+6}{2}, & \text{if } x = u_3; \\ n + 2i - 1, & \text{if } x = u_{4i} \text{ and } 1 \leq i \leq \frac{m}{4}; \\ \frac{m+2n+4i+6}{2}, & \text{if } x = u_{4i+1} \text{ and } 1 \leq i \leq \frac{m-4}{4}; \\ n + 2i + 2, & \text{if } x = u_{4i+2} \text{ and } 0 \leq i \leq \frac{m-4}{4}; \\ \frac{m+2n+4i+4}{2}, & \text{if } x = u_{4i+3} \text{ and } 1 \leq i \leq \frac{m-4}{4}; \\ i, & \text{if } x = v_i \text{ and } 1 \leq i \leq n; \\ \frac{m+2n+4}{2}, & \text{if } x = w. \end{cases}$$

Case 2: Let $m \equiv 1 \pmod 4$, then

$$f(x) = \begin{cases} n + 2i - 1, & \text{if } x = u_{4i} \text{ and } 1 \leq i \leq \frac{m-1}{4}; \\ \frac{m+2n+4i+1}{2}, & \text{if } x = u_{4i+1} \text{ and } 0 \leq i \leq \frac{m-1}{4}; \\ n + 2i + 2, & \text{if } x = u_{4i+2} \text{ and } 0 \leq i \leq \frac{m-5}{4}; \\ \frac{m+2n+4i+7}{2}, & \text{if } x = u_{4i+3} \text{ and } 0 \leq i \leq \frac{m-5}{4}; \\ i, & \text{if } x = v_i \text{ and } 1 \leq i \leq n; \\ \frac{m+2n+3}{2}, & \text{if } x = w. \end{cases}$$

Case 3: Let $m \equiv 2 \pmod 4$, then

$$f(x) = \begin{cases} m+n+1, & \text{if } x = u_1; \\ m+n-1, & \text{if } x = u_3; \\ \frac{m+2n-4i+2}{2}, & \text{if } x = u_{4i} \text{ and } 1 \le i \le \frac{m-2}{4}; \\ m+n-2i-1, & \text{if } x = u_{4i+1} \text{ and } 1 \le i \le \frac{m-2}{4}; \\ \frac{m+2n-4i-4}{2} & \text{if } x = u_{4i+2} \text{ and } 0 \le i \le \frac{m-6}{4}; \\ m+n-2i, & \text{if } x = u_{4i+3} \text{ and } 1 \le i \le \frac{m-6}{4}; \\ m+n, & \text{if } x = u_m; \\ i, & \text{if } x = v_i \text{ and } 1 \le i \le n; \\ \frac{m+2n+2}{2}, & \text{if } x = w. \end{cases}$$

Case 4: Let $m \equiv 3 \pmod 4$, then

$$f(x) = \begin{cases} \frac{m+2n+1}{2}, & \text{if } x = u_1; \\ \frac{m+2n+5}{2}, & \text{if } x = u_3; \\ n+2i-1, & \text{if } x = u_{4i} \text{ and } 1 \le i \le \frac{m-3}{4}; \\ \frac{m+2n+4i+5}{2}, & \text{if } x = u_{4i+1} \text{ and } 1 \le i \le \frac{m-3}{4}; \\ n+2i+2, & \text{if } x = u_{4i+2} \text{ and } 0 \le i \le \frac{m-7}{4}; \\ \frac{m+2n+4i+3}{2}, & \text{if } x = u_{4i+3} \text{ and } 1 \le i \le \frac{m-3}{4}; \\ \frac{m+2n-1}{2}, & \text{if } x = u_{m-1}; \\ i, & \text{if } x = v_i \text{ and } 1 \le i \le n; \\ \frac{m+2n+3}{2}, & \text{if } x = w. \end{cases}$$

Therefore, f is a canonical form of super magic labeling of F with valence

$$k = \begin{cases} \frac{5m}{2} + 3n + 2, & \text{if } m \equiv 2 \pmod 4; \\ \left\lfloor \frac{m}{2} \right\rfloor + 2m + 3n + 3, & \text{otherwise.} \end{cases}$$

□

The next class of forest that we study is $2P_n$.

Theorem 4.10. *The forest $F \cong 2P_n$ is super magic if and only if $n \neq 2$ or 3.*

Proof.

Assume that $n \geq 4$, and let $F \cong 2P_n$ be the forest with

$$V(F) = \{u_i, v_i \mid 1 \le i \le n\}$$

and

$$E(F) = \{u_i u_{i+1}, v_i v_{i+1} \mid 1 \le i \le n-1\}.$$

Then we proceed by cases according to the possible values of the integer n.

Case 1: If $n = 9$, then label the vertices of one P_9 with $10 - 17 - 14 - 4 - 13 - 6 - 16 - 9$, and the label the ones of the other P_9 with $8 - 18 - 5 - 15 - 1 - 11 - 2 - 12 - 3$ to obtain a super magic labeling of $2P_9$.

Case 2: If $n = 4k$, where k is a positive integer, then define the vertex labeling $f : V(F) \to \{1, 2, \ldots, 8k\}$ such that

$$f(x) = \begin{cases} 1, & \text{if } x = u_1; \\ 2k + i - 1, & \text{if } x = u_{2i-1} \text{ and } 2 \leq i \leq k; \\ 2k - i + 2, & \text{if } x = u_{2i-1} \text{ and } k + 1 \leq i \leq 2k; \\ 6k + i, & \text{if } x = u_{2i} \text{ and } 1 \leq i \leq k; \\ 6k - i + 1, & \text{if } x = u_{2i} \text{ and } k + 1 \leq i \leq 2k; \\ 3k + i - 1, & \text{if } x = v_{2i-1} \text{ and } 1 \leq i \leq k + 1; \\ 3k - i + 2, & \text{if } x = v_{2i-1} \text{ and } k + 2 \leq i \leq 2k; \\ 7k + i, & \text{if } x = v_{2i} \text{ and } 1 \leq i \leq k; \\ 7k - i + 1, & \text{if } x = v_{2i} \text{ and } k + 1 \leq i \leq 2k. \end{cases}$$

Case 3: if $n = 12k - 7$, where k is a positive integer, then define the vertex labeling $f : V(F) \to \{1, 2, \ldots, 24k - 14\}$ such that

$$f(x) = \begin{cases} 12k - 3i - 4, & \text{if } x = u_{2i-1} \text{ and } 1 \leq i \leq 3k - 1; \\ 3i - 6k + 1, & \text{if } x = u_{2i-1} \text{ and } 3k \leq i \leq 6k - 3; \\ 24k - 3i - 12, & \text{if } x = u_{2i} \text{ and } 1 \leq i \leq 3k - 2; \\ 6k + 3i - 4, & \text{if } x = u_{2i} \text{ and } 3k - 1 \leq i \leq 6k - 4; \\ 24k - 3i - 11, & \text{if } x = v_{2i-1} \text{ and } 1 \leq i \leq 3k - 2; \\ 12k - 3i - 6, & \text{if } x = v_{2i} \text{ and } 1 \leq i \leq 3k - 2; \\ 15k - 3i - 6, & \text{if } x = v_{6k+6i-9} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 1, & \text{if } x = v_{6k+6i-8} \text{ and } 1 \leq i \leq k; \\ 15k - 3i - 5, & \text{if } x = v_{6k+6i-7} \text{ and } 1 \leq i \leq k; \\ 3k - 3i - 1, & \text{if } x = v_{6k+6i-6} \text{ and } 1 \leq i \leq k - 1; \\ 15k - 3i - 7, & \text{if } x = v_{6k+6i-5} \text{ and } 1 \leq i \leq k - 1; \\ 3k - 3i, & \text{if } x = v_{6k+6i-4} \text{ and } 1 \leq i \leq k - 1. \end{cases}$$

Case 4: If $n = 12k - 6$, where k is a positive integer, then define the vertex labeling $f : V(F) \to \{1, 2, \ldots, 24k - 12\}$ such that

$$f(x) = \begin{cases} 12k - 3i - 3, & \text{if } x = u_{2i-1} \text{ and } 1 \le i \le 3k - 1; \\ 3i - 6k + 2, & \text{if } x = u_{2i-1} \text{ and } 3k \le i \le 6k - 3; \\ 24k - 3i - 11, & \text{if } x = u_{2i} \text{ and } 1 \le i \le 3k - 2; \\ 6k + 3i - 3, & \text{if } x = u_{2i} \text{ and } 3k \le i \le 6k - 3; \\ 24k - 3i - 10, & \text{if } x = v_{2i-1} \text{ and } 1 \le i \le 3k - 2; \\ 12k - 3i - 5, & \text{if } x = v_{2i} \text{ and } 1 \le i \le 3k - 2; \\ 15k - 3i - 5, & \text{if } x = v_{6k+6i-9} \text{ and } 1 \le i \le k; \\ 3k - 3i + 2, & \text{if } x = v_{6k+6i-8} \text{ and } 1 \le i \le k - 1; \\ 15k - 3i - 4, & \text{if } x = v_{6k+6i-7} \text{ and } 1 \le i \le k; \\ 3k - 3i, & \text{if } x = v_{6k+6i-6} \text{ and } 1 \le i \le k - 1; \\ 15k - 3i - 6, & \text{if } x = v_{6k+6i-5} \text{ and } 1 \le i \le k - 1; \\ 3k - 3i + 1, & \text{if } x = v_{6k+6i-4} \text{ and } 1 \le i \le k - 1; \\ i, & \text{if } x = v_{12k+2i-10} \text{ and } 1 \le i \le 2. \end{cases}$$

Case 5: If $n = 12k - 5$, where k is a positive integer, then define the vertex labeling $f : V(F) \to \{1, 2, \ldots, 24k - 10\}$ such that

$$f(x) = \begin{cases} 24k - 3i - 7, & \text{if } x = u_{2i-1} \text{ and } 1 \le i \le 3k - 1; \\ 6k + 3i - 5, & \text{if } x = u_{2i-1} \text{ and } 3k \le i \le 6k - 2; \\ 12k - 3i - 3, & \text{if } x = u_{2i} \text{ and } 1 \le i \le 3k - 1; \\ 3i - 6k + 2, & \text{if } x = u_{2i} \text{ and } 3k \le i \le 6k - 3; \\ 12k - 3i - 2, & \text{if } x = v_{2i-1} \text{ and } 1 \le i \le 3k - 1; \\ 24k - 3i - 9, & \text{if } x = v_{2i} \text{ and } 1 \le i \le 3k - 2; \\ 15k - 3i - 4, & \text{if } x = v_{6k+6i-8} \text{ and } 1 \le i \le k; \\ 3k - 3i + 2, & \text{if } x = v_{6k+6i-7} \text{ and } 1 \le i \le k - 1; \\ 15k - 3i - 3, & \text{if } x = v_{6k+6i-6} \text{ and } 1 \le i \le k; \\ 3k - 3i, & \text{if } x = v_{6k+6i-5} \text{ and } 1 \le i \le k - 1; \\ 15k - 3i - 5, & \text{if } x = v_{6k+6i-4} \text{ and } 1 \le i \le k - 1; \\ 3k - 3i + 1, & \text{if } x = v_{6k+6i-3} \text{ and } 1 \le i \le k - 1; \\ i, & \text{if } x = v_{12k+2i-9} \text{ and } 1 \le i \le 2. \end{cases}$$

Case 6: If $n = 12k - 2$, where k is a positive integer, then define the vertex labeling $f : V(F) \rightarrow \{1, 2, \ldots, 24k - 4\}$ such that

$$f(x) = \begin{cases} 12k - 3i + 1, & \text{if } x = u_{2i-1} \text{ and } 1 \leq i \leq 3k; \\ 3i - 6k - 1, & \text{if } x = u_{2i-1} \text{ and } 3k + 1 \leq i \leq 6k - 1; \\ 24k - 3i - 2, & \text{if } x = u_{2i} \text{ and } 1 \leq i \leq 3k - 1; \\ 6k + 3i - 1, & \text{if } x = u_{2i} \text{ and } 3k \leq i \leq 6k - 1; \\ 12k - 3i, & \text{if } x = v_{2i-1} \text{ and } 1 \leq i \leq 3k - 1; \\ 24k - 3i - 3, & \text{if } x = v_{2i} \text{ and } 1 \leq i \leq 3k - 1; \\ 3k - 3i + 2, & \text{if } x = v_{6k+6i-7} \text{ and } 1 \leq i \leq k; \\ 15k - 3i + 1, & \text{if } x = v_{6k+6i-6} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 3, & \text{if } x = v_{6k+6i-5} \text{ and } 1 \leq i \leq k; \\ 15k - 3i - 1, & \text{if } x = v_{6k+6i-4} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 1, & \text{if } x = v_{6k+6i-3} \text{ and } 1 \leq i \leq k; \\ 15k - 3i, & \text{if } x = v_{6k+6i-2} \text{ and } 1 \leq i \leq k. \end{cases}$$

Case 7: If $n = 12k - 1$, where k is a positive integer, then define the vertex labeling $f : V(F) \rightarrow \{1, 2, \ldots, 24k - 2\}$ such that

$$f(x) = \begin{cases} 12k - 3i + 2, & \text{if } x = u_{2i-1} \text{ and } 1 \leq i \leq 3k; \\ 3i - 6k - 2, & \text{if } x = u_{2i-1} \text{ and } 3k + 1 \leq i \leq 6k; \\ 24k - 3i, & \text{if } x = u_{2i} \text{ and } 1 \leq i \leq 3k; \\ 6k + 3i - 1, & \text{if } x = u_{2i} \text{ and } 3k + 1 \leq i \leq 6k - 1; \\ 24k - 3i + 1, & \text{if } x = v_{2i-1} \text{ and } 1 \leq i \leq 3k; \\ 12k - 3i, & \text{if } x = v_{2i} \text{ and } 1 \leq i \leq 3k - 1; \\ 3k - 1, & \text{if } x = v_{6k}; \\ 15k - 3i + 2, & \text{if } x = v_{6k+6i-5} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 3, & \text{if } x = v_{6k+6i-4} \text{ and } 1 \leq i \leq k; \\ 15k - 3i, & \text{if } x = v_{6k+6i-3} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 1, & \text{if } x = v_{6k+6i-2} \text{ and } 1 \leq i \leq k; \\ 15k - 3i + 1, & \text{if } x = v_{6k+6i-1} \text{ and } 1 \leq i \leq k; \\ 3k - 3i - 1, & \text{if } x = v_{6k+6i} \text{ and } 1 \leq i \leq k - 1. \end{cases}$$

Case 8: If $n = 12k + 1$, where k is a positive integer, then define the vertex labeling $f : V(F) \to \{1, 2, \dots, 24k + 2\}$ such that

$$f(x) = \begin{cases} 24k - 3i + 5, & \text{if } x = u_{2i-1} \text{ and } 1 \le i \le 3k + 1; \\ 6k + 3i - 2, & \text{if } x = u_{2i-1} \text{ and } 3k + 2 \le i \le 6k + 1; \\ 12k - 3i + 3, & \text{if } x = u_{2i} \text{ and } 1 \le i \le 3k; \\ 3i - 6k - 1, & \text{if } x = u_{2i} \text{ and } 3k + 1 \le i \le 6k; \\ 12k - 3i + 4, & \text{if } x = v_{2i-1} \text{ and } 1 \le i \le 3k; \\ 24k - 3i + 3, & \text{if } x = v_{2i} \text{ and } 1 \le i \le 3k; \\ 3k - 3i + 3, & \text{if } x = v_{6k+6i-5} \text{ and } 1 \le i \le k; \\ 15k - 3i + 4, & \text{if } x = v_{6k+6i-4} \text{ and } 1 \le i \le k; \\ 3k - 3i + 4, & \text{if } x = v_{6k+6i-3} \text{ and } 1 \le i \le k; \\ 15k - 3i + 2, & \text{if } x = v_{6k+6i-2} \text{ and } 1 \le i \le k; \\ 3k - 3i + 2, & \text{if } x = v_{6k+6i-1} \text{ and } 1 \le i \le k - 1; \\ 15k - 3i + 3, & \text{if } x = v_{6k+6i} \text{ and } 1 \le i \le k; \\ i, & \text{if } x = v_{12k+2i-3} \text{ and } 1 \le i \le 2. \end{cases}$$

Case 9: If $n = 12k + 2$, where k is a positive integer, then define the vertex labeling $f : V(F) \to \{1, 2, \dots, 24k + 4\}$ such that

$$f(x) = \begin{cases} 12k - 3i + 5, & \text{if } x = u_{2i-1} \text{ and } 1 \le i \le 3k + 1; \\ 3i - 6k - 3, & \text{if } x = u_{2i-1} \text{ and } 3k + 2 \le i \le 6k + 1; \\ 24k - 3i + 6, & \text{if } x = u_{2i} \text{ and } 1 \le i \le 3k; \\ 6k + 3i + 1, & \text{if } x = u_{2i} \text{ and } 3k + 1 \le i \le 6k + 1; \\ 12k - 3i + 4, & \text{if } x = v_{2i-1} \text{ and } 1 \le i \le 3k; \\ 24k - 3i + 5, & \text{if } x = v_{2i} \text{ and } 1 \le i \le 3k + 1; \\ 3k + i - 1, & \text{if } x = v_{6k+2i-1} \text{ and } 1 \le i \le 2; \\ 15k - 3i + 6, & \text{if } x = v_{6k+6i-2} \text{ and } 1 \le i \le k; \\ 3k - 3i + 1, & \text{if } x = v_{6k+6i-1} \text{ and } 1 \le i \le k; \\ 15k - 3i + 4, & \text{if } x = v_{6k+6i} \text{ and } 1 \le i \le k - 1; \\ 3k - 3i + 2, & \text{if } x = v_{6k+6i+1} \text{ and } 1 \le i \le k; \\ 15k - 3i + 2, & \text{if } x = v_{6k+6i+2} \text{ and } 1 \le i \le k - 1; \\ 3k - 3i, & \text{if } x = v_{6k+2i+3} \text{ and } 1 \le i \le k - 1; \\ 12k + i + 2, & \text{if } x = v_{12k+2i-2} \text{ and } 1 \le i \le 2. \end{cases}$$

Case 10: If $n = 12k + 3$, where k is a positive integer, then define the vertex labeling $f : V(F) \to \{1, 2, \ldots, 24k + 6\}$ such that

$$f(x) = \begin{cases} 12k - 3i + 6, & \text{if } x = u_{2i-1} \text{ and } 1 \leq i \leq 3k + 1; \\ 3i - 6k - 4, & \text{if } x = u_{2i-1} \text{ and } 3k + 2 \leq i \leq 6k + 2; \\ 24k - 3i + 8, & \text{if } x = u_{2i} \text{ and } 1 \leq i \leq 3k + 1; \\ 6k + 3i + 1, & \text{if } x = u_{2i} \text{ and } 3k + 2 \leq i \leq 6k + 1; \\ 24k - 3i + 9, & \text{if } x = v_{2i-1} \text{ and } 1 \leq i \leq 3k + 2; \\ 12k - 3i + 4, & \text{if } x = v_{2i} \text{ and } 1 \leq i \leq 3k; \\ 3k + i - 1, & \text{if } x = v_{6k+2i} \text{ and } 1 \leq i \leq 2; \\ 15k - 3i + 7, & \text{if } x = v_{6k+6i-1} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 1, & \text{if } x = v_{6k+6i} \text{ and } 1 \leq i \leq k; \\ 15k - 3i + 5, & \text{if } x = v_{6k+6i+1} \text{ and } 1 \leq i \leq k - 1; \\ 3k - 3i + 2, & \text{if } x = v_{6k+6i+2} \text{ and } 1 \leq i \leq k; \\ 15k - 3i + 3, & \text{if } x = v_{6k+6i+3} \text{ and } 1 \leq i \leq k - 1; \\ 3k - 3i, & \text{if } x = v_{6k+2i+4} \text{ and } 1 \leq i \leq k - 1; \\ 12k + i + 3, & \text{if } x = v_{12k+2i-1} \text{ and } 1 \leq i \leq 2. \end{cases}$$

Case 11: If $n = 12k + 9$, where k is a positive integer, then define the vertex labeling $f : V(F) \to \{1, 2, \ldots, 24k + 18\}$ such that

$$f(x) = \begin{cases} 12k - 3i + 12, & \text{if } x = u_{2i-1} \text{ and } 1 \leq i \leq 3k + 3; \\ 3i - 6k - 7, & \text{if } x = u_{2i-1} \text{ and } 3k + 4 \leq i \leq 6k + 5; \\ 24k - 3i + 20, & \text{if } x = u_{2i} \text{ and } 1 \leq i \leq 3k + 2; \\ 6k + 3i + 4, & \text{if } x = u_{2i} \text{ and } 3k + 3 \leq i \leq 6k + 4; \\ 24k - 3i + 21, & \text{if } x = v_{2i-1} \text{ and } 1 \leq i \leq 3k + 2; \\ 12k - 3i + 10, & \text{if } x = v_{2i} \text{ and } 1 \leq i \leq 3k + 3; \\ 15k + i - 10, & \text{if } x = v_{6k+2i+3} \text{ and } 1 \leq i \leq 2; \\ 3k - 3i + 5, & \text{if } x = v_{6k+6i+2} \text{ and } 1 \leq i \leq k + 1; \\ 15k - 3i + 12, & \text{if } x = v_{6k+6i+3} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 3, & \text{if } x = v_{6k+6i+4} \text{ and } 1 \leq i \leq k; \\ 15k - 3i + 13, & \text{if } x = v_{6k+6i+5} \text{ and } 1 \leq i \leq k; \\ 3k - 3i + 1, & \text{if } x = v_{6k+6i+6} \text{ and } 1 \leq i \leq k; \\ 15k - 3i + 11, & \text{if } x = v_{6k+2i+7} \text{ and } 1 \leq i \leq k; \\ 12k + i + 9, & \text{if } x = v_{12k+2i+5} \text{ and } 1 \leq i \leq 2. \end{cases}$$

Therefore, f is the canonical form of a super magic labeling of F with valence $5n$ when $n = 4k$ and $5n + 1$, otherwise. □

Theorem 4.11. *The forest $F \cong K_{1,m} \cup 2nK_2$, where m and n are positive integers, is super magic. Furthermore, if $m + 2n$ and $2n + 1$ are relatively prime then only the valences $2m + 9n + 4$ and $3m + 9n + 3$ are attained by the super magic labelings of F.*

Proof.

Let $F \cong K_{1,m} \cup 2nK_2$ be a (p, q)-forest such that

$$V(F) = \{u\} \cup \{v_1, v_2, \ldots, v_n\} \cup \{w_1, w_2, \ldots, w_{4n}\}$$

and

$$E(F) = \{uv_1, uv_2, \ldots, uv_m\} \cup \{w_1w_{2n+1}, w_2w_{2n+2}, \ldots, w_{2n}w_{4n}\}.$$

Then consider f and $g : V(F) \to \{1, 2, \ldots, p\}$ to be the vertex labelings of F such that

$$f(x) = \begin{cases} n+1, & \text{if } x = u; \\ 2n+i+1, & \text{if } x = v_i \text{ and } 1 \le i \le m; \\ i, & \text{if } x = w_i \text{ and } 1 \le i \le n; \\ i+1, & \text{if } x = w_i \text{ and } n+1 \le i \le 2n; \\ m+n+i+1, & \text{if } x = w_i \text{ and } 2n+1 \le i \le 3n; \\ m-n+i+1, & \text{if } x = w_i \text{ and } 3n+1 \le i \le 4n; \end{cases}$$

and

$$g(x) = \begin{cases} m+3n+1, & \text{if } x = u \\ i, & \text{if } x = v_i \text{ and } 1 \le i \le m; \\ m+2i, & \text{if } x = w_i \text{ and } 1 \le i \le n; \\ m-2n+2i-1, & \text{if } x = w_i \text{ and } n+1 \le i \le 2n; \\ m+5n-i+1, & \text{if } x = w_i \text{ and } 2n+1 \le i \le 3n; \\ m+7n-i+2, & \text{if } x = w_i \text{ and } 3n+1 \le i \le 4n. \end{cases}$$

Then, f and g are the canonical form of some super magic labeling of F with valences $2m + 9n + 4$ and $3m + 9n + 3$ respectively.

To see that, the above two valences are the only possible ones when $m+2n$ and $2n+1$ are relatively prime, let k be the valence of a super magic labeling f of F. Then

$$\begin{aligned} k &= \frac{(m-1)\,f(u) + \sum_{i=1}^{p+q} i}{q} \\ &= 2m + 8n + 3 + f(u) + \frac{(2n+1)(n+1-f(v))}{m+2n}. \end{aligned}$$

Then, there exists an integer α such that $\alpha(m+2n) = 1+n-f(v)$. Since $1 \le f(v) \le p$, we have that α is 0 or -1, values that lead to the valences $2m + 9n + 4$ and $3m + 9n + 3$ respectively. □

Notice that if we relax the hypothesis of the previous theorem to refer to just magic labelings, we have that another valence occurs as stated in the following corollary.

Corollary 4.12. *Let $F \cong K_{1,m} \cup 2nK_2$, where m and n are positive integers such that $m + 2n$ and $2n + 1$ are relatively prime. Then only the valences $2m + 9n + 4$, $3m + 9n + 3$ and $4m + 9n + 2$ are attained by the magic labelings of F.*

Proof.
To prove this we use the facts and notation of the proof of the previous theorem. First, notice that the vertex labeling $h : V(F) \to \{1, 2, \ldots, p\}$ such that $h(v) = p + q + 1 - f(v)$ extends to a magic labeling of F with valence $4m + 9n + 2$. Next, if we allow magic labelings of F, the value of α in the proof can also be -2, and thus, only one further valence is attained. □

4.2 Crown Products of Some Super Magic Graphs

4.2.1 General Results

Unless stated otherwise, the results on this section are due to Figueroa et al. [12]. In this section, we first provide a construction that shows that $G \odot \overline{K}_n$ is super magic whenever G is a graph of odd order at least 3 and admits certain super magic labelings.

Theorem 4.13. *Let G be a graph of odd order $p \geq 3$ for which there exists a super magic labeling f with the property that*

$$\max\{f(u) + f(v) \mid uv \in E(G)\} = \frac{3p+1}{2},$$

then, $G \odot \overline{K}_n$ is super magic for every positive integer n.

Proof.
Let f be a super magic labeling of G with valence k, and assume that f has the property that $f(v_i) = i$ for every integer i with $1 \leq i \leq p$, where $V(G) = \{v_i \mid 1 \leq i \leq p\}$. Further, let

$$V\left(G \odot \overline{K}_n\right) = V(G) \cup \left\{w_i^j \mid 1 \leq i \leq p \text{ and } 1 \leq j \leq n\right\}$$

and

$$E\left(G \odot \overline{K}_n\right) = E(G) \cup \left\{v_i w_i^j \mid 1 \leq i \leq p \text{ and } 1 \leq j \leq n\right\}.$$

Now, define the vertex labeling

$$g : V\left(G \odot \overline{K}_n\right) \to \{1, 2, \ldots, p(n+1)\}$$